Mathematical Simulations in Topology and Their Role in Mathematics Education

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Abstract

This thesis presents and discusses several software projects related to the learning of mathematics in general and topological concepts in particular, collecting the results from several publications in this field. It approaches mathematics education by construction of mathematical learning environments, which can be used for the learning of mathematics, as well as by contributing insights gained during the development and use of these learning environments. It should be noted that the presented software environments were not built for the use in schools or other settings, but to provide proofs of concepts and to act as a basis for research into mathematics and its education and communication.

The first developed and analyzed environment is Ariadne, a software for the interactive visualization of dots, paths, and homotopies of paths. Ariadne is used as an example of a "mathematical simulation", capable of supporting argumentation in a way that may be characterized as proving. The software was extended from two to three dimensions, making possible the investigation of two-dimensional manifolds, such as the torus or the sphere, using virtual reality.

Another extension, KnotPortal, enables the exploration of three-dimensional manifolds represented as branched covers of knots, after an idea by Bill Thurston to portray these branched covers of knots as knotted portals between worlds. This software was the motivation for and was used in an investigation into embodied mathematics learning, as this virtual reality environment challenges users to determine the structure of the covering by moving their body.

Also presented are some unpublished projects that were not completed during the doctorate. This includes work on concept images in topology as well as software for various purposes. One such software was intended for the construction of closed orientable surfaces, while another was focused on the interactive visualization of the uniformization theorem.

The thesis concludes with a meta-discussion on the role of design in mathematics education research. While design plays an important role in mathematics education, designing seems to not to be recognized as research in itself, but only as part of theory building or, in most cases, an empirical study. The presented argumentation challenges this view and points out the dangers and obstacles involved.

Zusammenfassung

In dieser Dissertation werden mehrere Projekte vorgestellt und diskutiert, wobei sich die Ergebnisse aus mehreren Veröffentlichungen aus diesem Gebiet zusammensetzen. Bei den Projekten handelt es sich um entwickelte Software, die sich auf das Erlernen von Mathematik im Allgemeinen und von topologischen Konzepten im Besonderen bezieht. Diese Arbeit nähert sich der Mathematikdidaktik zum einen durch die Konstruktion von mathematischen Lernumgebungen, die für das Erlernen von Mathematik verwendet werden können, und zum anderen durch das Einbringen von Erkenntnissen, die während der Entwicklung und Nutzung dieser Lernumgebungen gewonnen wurden. Es ist darauf hinzuweisen, dass die vorgestellten Lernumgebungen nicht für den Einsatz in Schulen oder anderen Einrichtungen entwickelt wurden, sondern um als „proofs of concepts“ zu dienen und eine Grundlage für die Erforschung der Mathematik sowie ihrer Didaktik und Kommunikation zu bilden.

Die erste dieser Umgebungen ist Ariadne, eine Software für die interaktive Visualisierung von Punkten, Wegen und Homotopien von Wegen. Ariadne wird als Beispiel für eine „mathematische Simulation“ verwendet, die in der Lage ist Argumentationen in einer Weise zu unterstützen, die man als beweisend bezeichnen kann. Die Software wurde von zwei auf drei Dimensionen erweitert, wodurch die Untersuchung zweidimensionaler Mannigfaltigkeiten, wie beispielsweise des Torus oder der Sphäre, mithilfe von Virtual Reality ermöglicht wird.

Eine zusätzliche Erweiterung, KnotPortal, ermöglicht die Untersuchung von dreidimensionalen Mannigfaltigkeiten, die als verzweigte Überlagerungen von Knoten dargestellt werden. Die Software entstand nach einer Idee von Bill Thurston, verzweigte Überlagerungen als verknotete Portale zwischen Welten zu sehen. Diese Umgebung war die Motivation für eine Studie zum „Embodied Learning“ in der Mathematik und wurde auch dort verwendet, da die Virtual Reality-Umgebung von KnotPortal Nutzer dazu anregt, die Struktur der verzweigten Überlagerung durch Bewegung des Körpers zu erkennen.

Außerdem werden einige unveröffentlichte Projekte vorgestellt, die während der Promotion nicht abgeschlossen wurden. Dazu gehören Arbeiten zu Grundvorstellungen in der Topologie sowie Software für verschiedene Zwecke. Eine solche Software war für die Konstruktion geschlossener orientierbarer Oberflächen vorgesehen, während sich eine andere auf die interaktive Visualisierung des Uniformisierungssatzes konzentrierte.

Abschließend enthält die Arbeit eine Meta-Diskussion über die Rolle von „Design“ in der mathematikdidaktischen Forschung. Obwohl Design in der Mathematikdidaktik eine wichtige Rolle spielt, scheint dieses nicht als Forschung an sich anerkannt zu sein, sondern nur als Teil von Theoriebildung oder meistens von empirischen Studien. Die vorgelegte Argumentation stellt diese Sichtweise in Frage und weist auf die damit verbundenen Gefahren und Hindernisse hin.

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1 Introduction

There's a tiresome young man in Bay Shore.
When his fiancée cried, 'I adore
The beautiful sea',
He replied, 'I agree,
It's pretty, but what is it for?'

-Morris Bishop

Mathematics can be described as one long conversation stretching over millennia (Mazur, 2013), but who can actually take part in it? Mathematical ideas are expressed mostly through concepts that are compressed into formalized writing, resulting in a very efficient yet obscure way of communication.

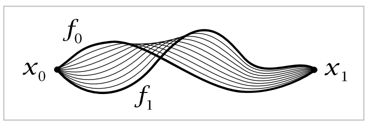
But while it is efficient, and somewhat universal within mathematical subfields, the language used by mathematicians also has its drawbacks. Firstly, it is *hard to learn*. Even people studying mathematics need several years to understand the way mathematics is communicated through this language containing a mixture of terms, symbols, and text.

Secondly, and more important, such a formalized language is just one way of representing mathematics, and may not be equally well suited for every kind of idea. There are many ways of representing knowledge, and no reason why one should always be preferred to the others. There are also many categorization systems for such modes of understanding or representing, an overview is given in Victor (2014). Examples are the Enactive-Iconic-Symbolic model of Bruner (1966) derived from Piaget, or the more sophisticated Vygotskian Somatic-Mythic-Romantic-Philosophic-Ironic model of Egan (1997). There is also a cognition theoretical angle to representation forms, which is tackled by the theory of embodied cognition (Abrahamson & Bakker, 2016). Embodiment proposes the repealing of the Cartesian dualism of mind and body as separate entities and views cognition as a process that is shaped by our body and interactions with the outside world. This implies that mathematics can be represented in an embodied way. Irrespective of the categorization system used, the main point is that there is a large variety of ways to represent mathematics, besides the "usual" way of symbols written on paper or blackboard.

This is well-known in mathematics, where some ideas are explained using *pictures* in different roles. Sometimes, they are accompanying the text and the formulas (see the definition of a homotopy in Fig. 1). In other cases, they are self-contained, such as "proofs without words" (Nelsen, 1993). This is of course very much dependent of the subfield, the particular problem discussed, the author's style and preferences, and the intended audience of a certain piece of mathematics. Furthermore, there exists a long standing *distrust* of images, to quote Littlewood (1953): "A heavy warning used to be given that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety." There exist wrong pictures as well as wrong formulas; the form neither safeguards against nor fosters errors suggested by an argument.

By a **path** in a space X we mean a continuous map $f: I \to X$ where I is the unit interval $[0,1]$. The idea of continuously deforming a path, keeping its endpoints fixed, is made precise by the following definition. A **homotopy** of paths in X is a family $f_t: I \to X$, $0 \le t \le 1$, such that

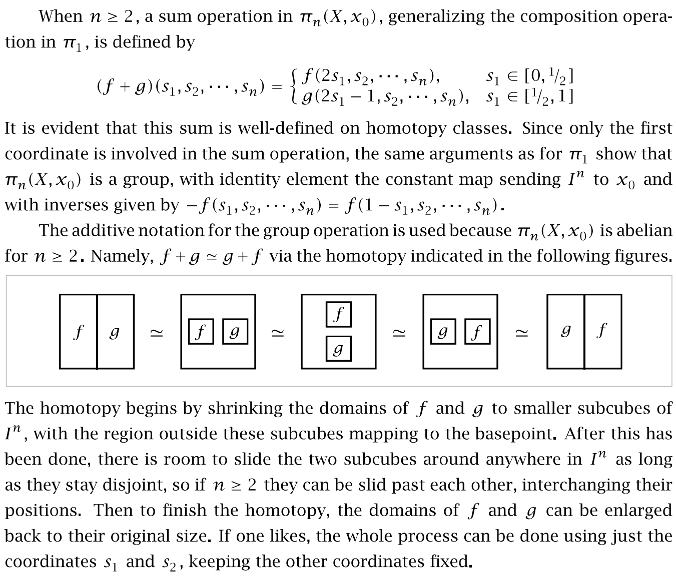
(1) The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.
(2) The associated map $F: I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous.



When two paths f_0 and f_1 are connected in this way by a homotopy f_t, they are said to be **homotopic**. The notation for this is $f_0 \simeq f_1$.

Figure 1: An image accompanying a formal definition (Hatcher, 2000).

A typical example of a field with many visual representation aides is the field of topology, where drawings often play a central role in communicating ideas, sometimes even going beyond the role of an aide, being used as a stand-alone argument or definition as in Fig. 2, challenging the view of an image being “not rigorous.”



When $n \geq 2$, a sum operation in $\pi_n(X, x_0)$, generalizing the composition operation in π_1, is defined by

$$(f+g)(s_1, s_2, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n), & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \cdots, s_n), & s_1 \in [1/2, 1] \end{cases}$$

It is evident that this sum is well-defined on homotopy classes. Since only the first coordinate is involved in the sum operation, the same arguments as for π_1 show that $\pi_n(X, x_0)$ is a group, with identity element the constant map sending I^n to x_0 and with inverses given by $-f(s_1, s_2, \cdots, s_n) = f(1 - s_1, s_2, \cdots, s_n)$.

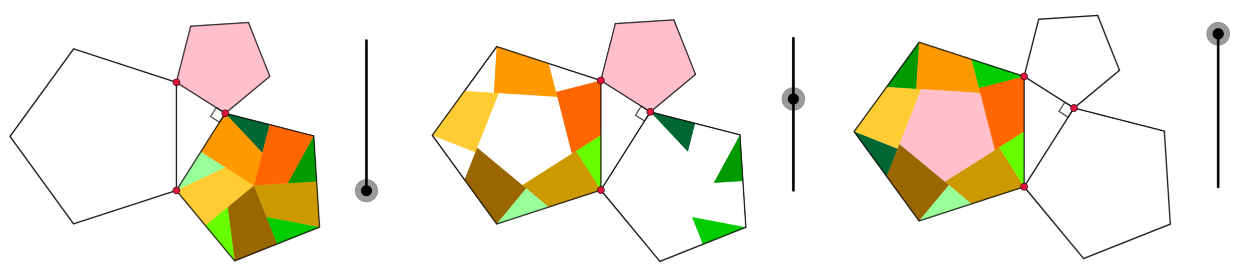
The additive notation for the group operation is used because $\pi_n(X, x_0)$ is abelian for $n \geq 2$. Namely, $f + g \simeq g + f$ via the homotopy indicated in the following figures.

The homotopy begins by shrinking the domains of f and g to smaller subcubes of I^n, with the region outside these subcubes mapping to the basepoint. After this has been done, there is room to slide the two subcubes around anywhere in I^n as long as they stay disjoint, so if $n \geq 2$ they can be slid past each other, interchanging their positions. Then to finish the homotopy, the domains of f and g can be enlarged back to their original size. If one likes, the whole process can be done using just the coordinates s_1 and s_2, keeping the other coordinates fixed.

Figure 2: The well-known visual proof that the higher homotopy groups are abelian, accompanied by a text (Hatcher, 2000). This is an application of the so-called “little cubes operad.”

As can be well observed in the case of the definition of a homotopy in Fig. 1, there is a mismatch between the definition, or better the concept, of a homotopy and the image representing it. There is a parameter t in the definition, denoting a sort of *time* parameter, as the definition speaks of a function “continuously deforming.” The image one is thus trying to evoke in the readers mind is a moving one, which cannot be displayed on paper.

This is where technology comes into play. It gave us the tools to create *animations*, which in the above case can capture the kind of movement the author intended, and may in this case be better suited than a static image.



(a) (b) (c)

Figure 3: Screenshots from an animation of the Pythagorean theorem for pentagons (Phelps, n.d.). The user can drag the slider, which moves the pieces on the screen. It is disputed if such an animated “proof without words,” or “PWW 2.0” (Doyle et al., 2014), actually constitutes a proof; or even if such pictures in general have the capacity to be a proof.

The shift induced by technology does, however, not stop here. Computers give us not only the possibility to *show* moving images, but also to *manipulate* them. This is not an affordance technology offers that we then implement for its own sake. It is a feature changing the dogma of information flow going from the author to the recipient via a medium. The recipient can become an active *user*, manipulating the medium and in this way receiving answers to questions the author of the medium may not have preconceived. The medium now functions as a canvas, where users can realize their ideas, externalizing some of their thoughts. These projected thoughts are then subject to the reflection of the users, but also reflection of others, making the medium one for *communication*. Such mediums are thus not to be seen as extensions of pictures, which can now be animated, but as extensions of blank paper and pencil, making them able to respond to input.

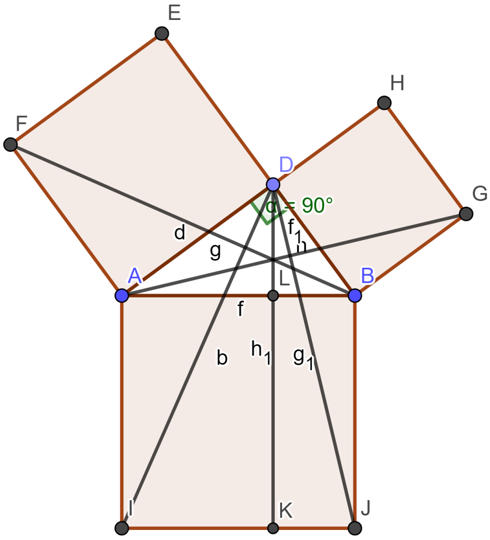


Figure 4: A simulation, in this case GeoGebra (Hohenwarter, 2006), to let the user construct and explore a classical proof of the Pythagorean theorem. It is, of course, not at all obvious from the image why this would be a proof; the user would have to use his or her knowledge on triangle geometry to deduce the theorem.

We call this sort of environment a *mathematical simulation*. It is important to note that mathematical simulations go beyond animations by the method with which the medium reacts to the input of the user. If the reaction is made through a hard-coded, case-by-case decision making, "built-in" system, then it is an animation; while a simulation is defined through "basic principle"-based reactions, therefore not needing a case-by-case analysis (compare Richter-Gebert (2013)). A definition and discussion on the relationship between these concepts as well as examples, can be found in Sec. 2.4 concerning the paper Mathematics in the Digital Age: The Case of Simulation-Based Proofs.

The nature of visual arguments also has the potential of making mathematics more accessible. Numerous "proofs without words" bear witness to the fact that one does not need the elaborate language of mathematical formalisms to *see* that an argument is true. There is no conceptual barrier to learn "advanced" subjects in mathematics, such as topology, as a member of an audience less acquainted to mathematical formalisms than the usual one. An accessible representation does in itself not give motivation for a subject, which may, together with the limits of a visual versus a formal-symbolic representation, lead to topology remaining a topic of study only for higher education, it does open and widen the discussion of when or what to learn or to teach.

This line of thought, connecting the learning of mathematics with computer interactions, basically follows the theories of Papert (1980). In his book *Mindstorms,* he describes a vision of what education could look like using computers. His focus is on the usage of the computer as a programmable tool. This has many implications, the most important being the externalization of ones thoughts by having to write them down as a computer program. This program can be debugged, which then in turns leads to the user learning this meta-way of thinking about his thoughts.

This approach relies on language, but in another sense than in "natural" language or the "language" of mathematical formalism. Papert's LOGO environment is on a computer, where language can be used not only as a carrier of information, but also as a tool to produce some output. Nevertheless, language is not the only way we think. This is how the idea for Ariadne was born, a learning environment where mathematics can be explored without language. In this case, mathematics is being represented by topology and the example of paths and homotopies, a subject easy to visualize. Users in Ariadne should be able to construct their own objects and manipulate them, resulting in a kind of dialog with the software, leading to insights on this new concept. It should be noted that topology does not only serve as the basis for the design of a learning environment, but learning about topology is meaningful for its own sake.

Paperts goal, and also mine, is to provide an environment that focuses on learning and not teaching of mathematics. However, I follow a different approach in the construction of the environment itself. The LOGO language and the Turtle environment of Papert show an environment as a tool that requires the learning of a language to express the users ideas. This automatically means a bottleneck in the expression of thought, as one first has to formulate the idea as a computer program.

There is a common trade-off between power and ease of use in computer interfaces. This puts programming languages at one end of the spectrum, and visualizations of mathematics on the other. But this is not a law of nature, and there is no reason why something as powerful as a programming language without its complexity shouldn't exist.

What I tried to build is an environment presenting the concepts in a format closer to the mathematics itself, by providing an opportunity to work with visual representations of the mathematical objects instead of words describing them. This approach also has its drawbacks, as it is by far not as powerful as a programming language, at least at this time.

Consequently, people using this environment do not learn about the symbols and words associated with topology in a textbook. What they do learn is "the language of mathematics itself"; not the formalism, but the "mathematical way of thinking." In a natural language, words are assembled to form sentences to encode meaning. In mathematics, and in this environment, objects are assembled to structures.

A first step to realize this vision was to implement the software capable of the features described above. The result is described in the first paper, Ariadne – A Digital Topology Environment, published in *The International Journal for Technology in Mathematics Education.*

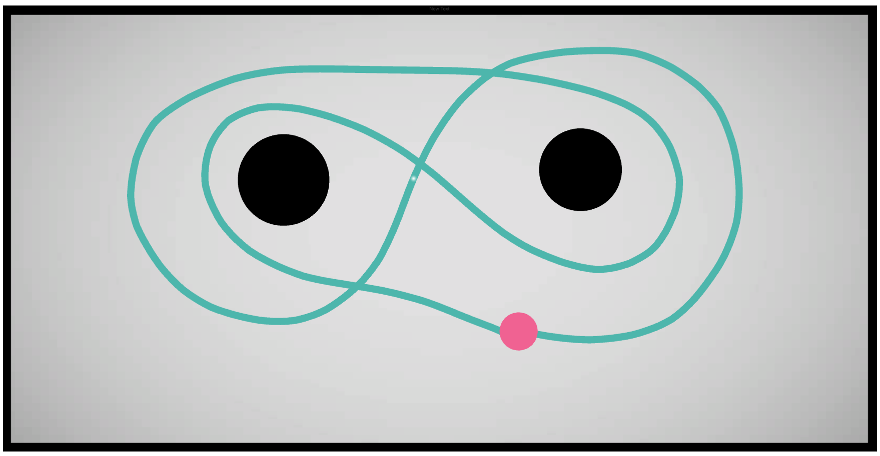


Figure 5: The Pochhammer contour, a non-nullhomotopic path with winding number zero around both punctures, drawn in Ariadne.

The paper gives an introduction into some topological concepts and their implementation, as well as what educational principles are the foundation for the design choices made. In this process, it classifies the software in a framework by Richter-Gebert (2013) for visualization software. It also provides some details of the implementation, describing how users interact with ARIADNE to construct and investigate topological objects. The paper was written quite early in the course of my dissertation and I was still unsure about the justifications I had to present to be able to pursue this kind of research designing topological learning software. This manifests not only in the introduction, where ties of topology to school use are discussed, but also towards the end of the paper, in a section presenting "questions to ask students," and in the outlook, which proposes work on concept images and definitions to test the environment empirically.

Despite its publication in a mathematics education journal and the aforementioned attempts to situate the environment in a school- or at least empirical child-oriented setting, the nature of this work was often questioned, challenging its classification as research in general, and as research in mathematics education in particular. This was expressed by categorizing it as "not research," as it does not involve empirical studies, or "not mathematics education" but rather mathematics, for a perceived lack of relevance to school curricula. The strong ties of mathematics education research to the school system made a software tool not intended for use seem futile.

Nevertheless, I continued to present my work at international conferences. As an example for the use of ARIADNE for a talk at CERME 11 in 2019 (see Sümmermann (2019c) or appendix A.2), I chose to present a proof done in the software. This led to thoughts about the nature of proof in mathematics education and research, and the possible impact on such *mathematical simulations* on it. These thoughts were written in collaboration with Daniel Sommerhoff and Benjamin Rott as Mathematics in the Digital Age: The Case of Simulation-Based Proofs.

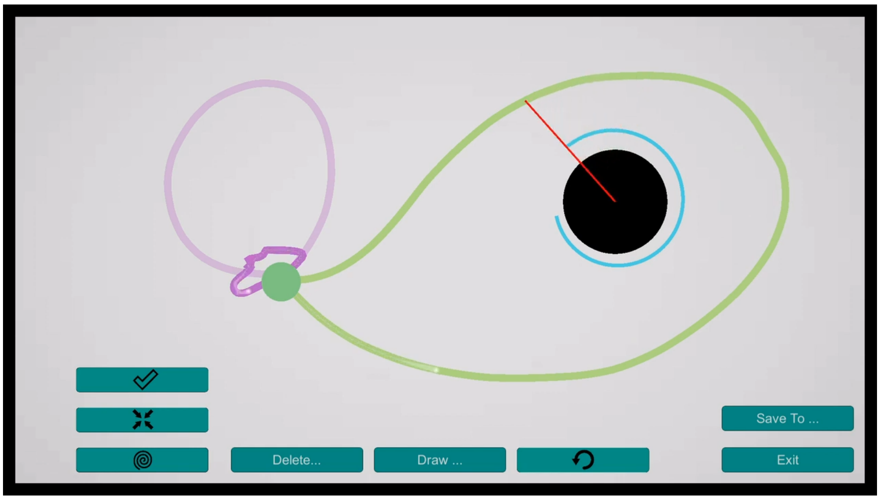


Figure 6: A screenshot from (Sümmermann, 2019d), showing a video of a simulation-based proof of the non-existence of a homotopy between two paths with different winding numbers.

The article first gives an overview on the role and, in particular, the controversies around the concept of "proof." The concept of a "mathematical simulation" is introduced and distinguished from "mathematical animations," referring to Richter-Gebert (2013) and Papert (1980). After these clarifications of terms, the paper continues to systematically expose as to why so-called simulation-based proofs may be seen as genuine proofs in the sense of proofs as a cluster concept (see Weber (2014)). This is done by analyzing user interaction patterns with a simulation following Vérillon (2000), followed by the categorization of extant proving environments and proof types in the dimensions of interactivity and formality. Finally, it is shown that simulation-based proofs fulfill the roles of a proof based on a framework of De Villiers (1990) consisting of the categories explanation, systematization, discovery, verification, and communication. These were extended by splitting up verification into relative and absolute conviction following (Weber & Mejia Ramos, 2015), as well as adding trust in the technology, level of detail, and limits of the representation to the relative conviction category.

The paper concludes with an outlook on the impact of such considerations on mathematics, especially mathematics education, as the combination of limited use of simulations at the moment, together with relative ease of implementation of such tools compared to research-level mathematics, promises to have the potential for considerable impact.

The conception of this paper happened over a longer period of time, during which the development of Ariadne continued. An inherent shortcoming in Ariadne was apparent, as already noted in the outlook of Ariadne – A Digital Topology Environment. The user interface of the software is a touch-based screen, which gives two-dimensional controls. The fundamental group of a n-punctured surface is quite uninteresting once one has figured it out, as it is just the free group on n generators. All the more interesting examples are in three dimensions, so the next step was thus to implement paths on two-dimensional surfaces embedded in three dimensions, such as a sphere or a torus. But paths are difficult to draw on such surfaces using a two-dimensional screen.

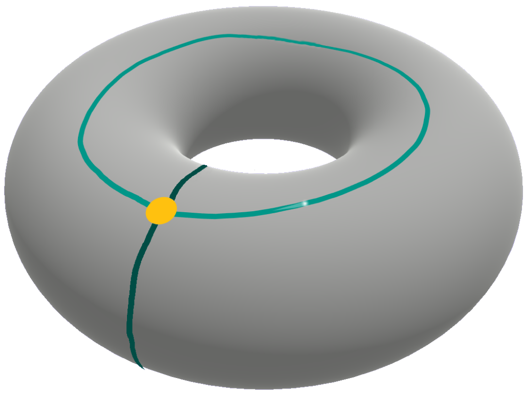


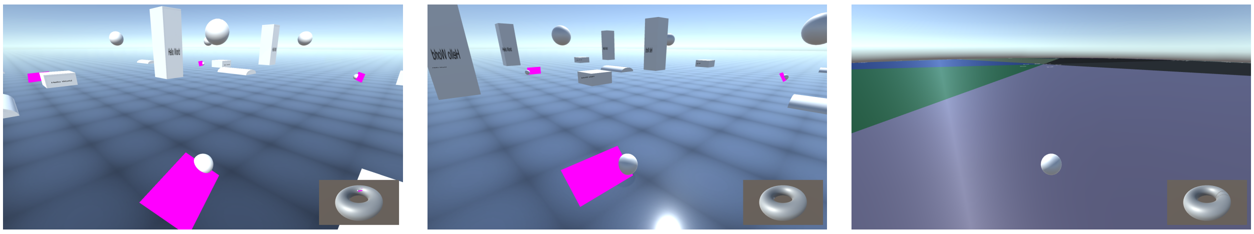
Figure 7: The two generators of the fundamental group of the torus, as drawn in the three-dimensional version of Ariadne.

There are two approaches to solve this problem, technologically or mathematically.

In the *technological* line of thought, the first approach consisted of letting users rotate the surface to enable the construction of paths by pulling the dot. This does not work well, as users have to fiddle around trying to coordinate movement of the dot and rotation of the surface. Another was to put the user in first-person perspective in relation to the dot, controlling its movement on and around the surface through keyboard controls. The path is then defined as the trace left by the dot, similar to the case of touch controls. This was again not feasible, as it left the user without orientation on his position or previous positions, making the purposeful construction of paths nearly impossible.

I then turned to using virtual reality equipment, circumventing the problem of 2D controls versus 3D environment. This gave rise to Ariadne3D, described in Sec. 3.2 and presented at the Imaginary conference in Montevideo, Uruguay, in late 2018.

The other approach was to solve the problem *mathematically*. Under the uniformization theorem, every simply connected two-dimensional surface is equivalent to either the plane, the sphere, or the hyperbolic plane.[1] Using this theorem and recycling the previous idea of a keyboard controlled first-person dot, the idea was to design a software letting users control the movement of a ball on the (hyperbolic) plane, whilst another ball draws a path on the corresponding surface.



(a) The euclidean plane, tiled by squares as fundamental regions of the torus. Note the small ball on the embedded torus in the corner, which is a copy of the user-controlled ball, mirroring its movement.

(b) The euclidean plane, tiled by squares as fundamental regions of the Klein bottle. The torus is representative for the Klein bottle.

(c) The plane with the $\{4,6\}$-tiling on the hyperbolic plane in the Klein-Beltrami disc model. The torus in the corner is representative for the surface of higher genus.

Figure 8: Some screenshots from the ball rolling around on different planes, from the prototype of SurfaceWalker. Further explanations are in Sec. 3.4.

This project was, however, never finished, as it took a very geometrical flavor and was not in line with (algebraic) topology set as an overall theme of the dissertation. The project is further discussed in Sec. 3.4.

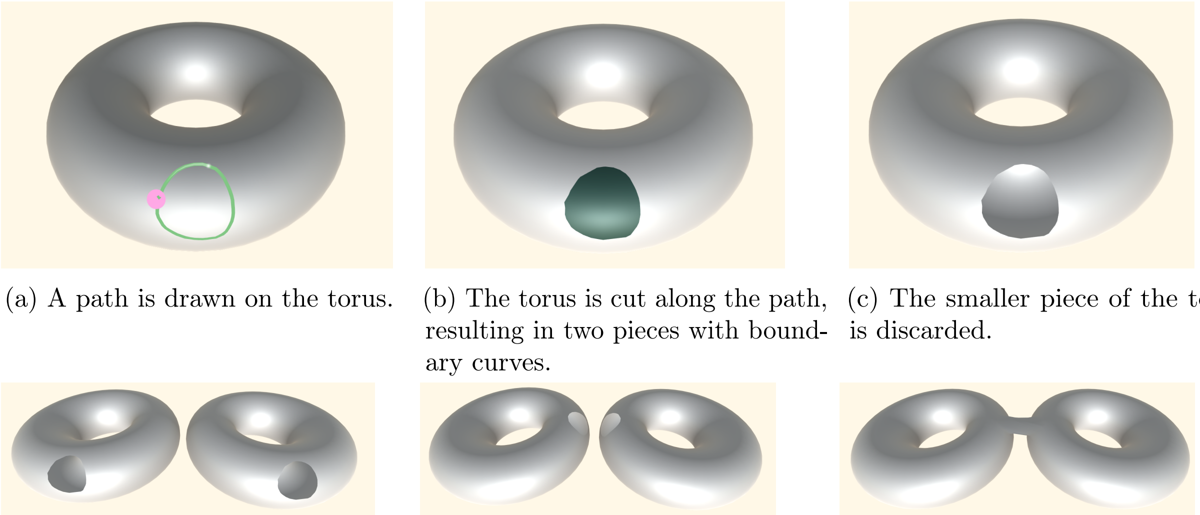
Instead, another project emerged. It occurred to me that if users would explore different surfaces with paths, the motivation for doing so might be disturbed by a lack of understanding on what these surfaces are, where they come from. But the construction of orientable surfaces is quite straightforward by the theorem on the classification of closed surfaces; they are all sums of tori (or the sphere). So it would be interesting to have a software letting the user construct his or her own surfaces through cut-and-paste methods, by starting with a surface as simple as possible. Recalling the animations bending a sheet of paper first in the form of a cylinder and then deforming[2] it into a torus, I tried to implement an interactive version of this. The user should be able to take the torus and, through cutting and gluing along paths drawn on it, retrace the genesis of all closed orientable surfaces (with boundary curves). Unfortunately, due to algorithmic and computational constraints (see Sec. 3.3), only the latter step of cutting and gluing could be achieved; this was realized and exhibited (see Fig. 9) at the workshop on “Illustrating Geometry and Topology” at the Institute for Computational and Experimental Research in Mathematics in 2019 (see appendix A.3).

Following the generalization process of the main idea, from the exploration of two-dimensional surfaces embedded in two-dimensional space in Ariadne, to two-dimensional surfaces embedded in three-dimensional space in Ariadne3D, the “logical” next step was to model three-dimensional “surfaces,” which can also be explored by paths.

Around the same time, I came upon a video by Bill Thurston introducing the idea of imagining branched coverings of knots as knotted portals to other worlds (Thurston, 2012). The footage of Thurston showing him actually stepping through such an imagined portal represented by a

[1] This is a very roughly stated, non-technical variant of the uniformization theorem. A more complete description would be “every simply connected Riemann surface is conformally equivalent to either the hyperbolic plane, the complex plane, or the Riemann sphere.”

[2] This is not bending anymore, following Gauss’ Theorema Egregium, there is no smooth embedding of the flat torus in $\mathbb{R}^3$ due to the curvature of the torus. But one must mention the amazing images of a C^1-embedding of a torus in Borrelli, Jabrane, Lazarus, and Thibert (2012) at this point.



(a) A path is drawn on the torus.

(b) The torus is cut along the path, resulting in two pieces with boundary curves.

(c) The smaller piece of the torus is discarded.

(d) The torus is duplicated.

(e) The tori are rotated, so that the holes are facing each other.

(f) The tori are glued together along their boundary curves.

Figure 9: The steps of the creation of a double torus, screenshots from ARIADNE3D. More information on this project is in Sec. 3.3.

wire coil in the form of a knot brought into my mind to use the virtual reality gear used in the ARIADNE3D-project to build a software where users can walk through this portal into other worlds.

In a mathematical way, this is a natural extension of ARIADNE3D, as 3-manifolds can be described as branched coverings of the 3-sphere with a knot as the branch set. The resulting product is described in Sümmermann (2020b). The article gives a self-contained explanation of KNOTPORTAL.

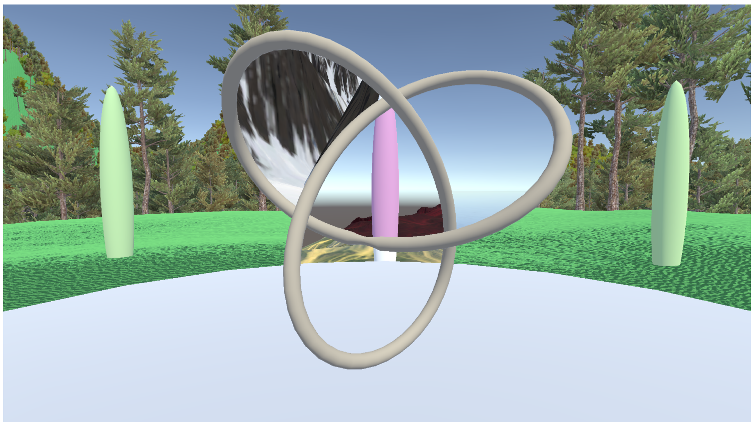


Figure 10: The four portals generated by the trefoil knot, a screenshot from KNOTPORTAL. More information on this project in Sec. 2.5.

This includes an introduction into the concept of branched coverings of knots, along with some interesting examples, such as a map projection by Charles Peirce. A small recapitulation of the history of branched coverings of knots gives motivation and completes the picture. An introduction into the attempts to visualize branches of the covering as worlds, and knots as portals between them as put forward by Thurston (2012), is also presented. This is followed by a description of the software, with mathematical and technical details of the implementation. Examples of knots and the structure of their branched coverings are given, as to flesh out the technical descriptions given before, and to equip the reader with the necessary tools to add their own knots, if desired.

While this article describes the project from a mathematical perspective, seeing Thurston's joy in moving around the room, stepping through the portal, reminded me of the theory of "embodied cognition." I was introduced to this theory in email correspondence with Liz de Freitas, whom I met at the CERME11 poster session.

This connection between embodiment and movement through the knotted portals of Thurston led to an investigation of embodied learning in KnotPortal, resulting in the article Embodied Mathematics: Forming Concepts in Topology by Moving Through Virtual Reality, written in collaboration with Benjamin Rott.

The article links a theoretical investigation into the affordances of KnotPortal for embodied learning of mathematics with a case study with two participants. The article presents several ways in how embodied learning is made possible with the software. Firstly, by letting users explore a mathematical world through footpaths, which align perfectly with the mathematical concept of paths. Secondly, contrasting the usual representation of mathematical knots as knot diagrams drawn on paper, the knot is now static and large, and is inspected by manipulating the position and rotation of the observer and not of the knot. The case study then revealed another aspect, the use of "embodied heuristics." This is the case if users show a behavior where they perform certain actions to solve a problem, not supported deductively but heuristically.

This sort of content-based analysis of an environment is contrasted with the use of general frameworks for classifying research in embodied learning, giving no insights to the usefulness of a specific environment. This is pointed out by the example of "immersion," a quality sought out in particular in virtual reality environments in the context of embodied learning, which in the case of KnotPortal led to a loss of overview in the participants. This again stresses the importance of a content-specific analysis and the dangers of trying to identify general "good practices" in the design of such environments.

As with Ariadne, the KnotPortal project was criticized from mathematics education researchers, again on the grounds of a lack in relevance for standard school systematized education and more importantly a want for empirical studies being done using the software, proving some capabilities of the software or using the software to demonstrate some theory on learners abilities.

As there was an ongoing discussion at the time in the journal "For the Learning of Mathematics" initiated by an article by Mogens Niss discussing the dangers of mathematics education becoming too narrow in focus and form, I wrote an article with Benjamin Rott furthering the discussion by defending the role of *design* in mathematics education research. The article discusses the role and standing of design in the research community, and in this way reflects on my personal experiences of the last few years. As this is a discussion article, it will be presented in the discussion in Sec. 4.

2 Research in publication

It should be noted that these versions were changed to comply with the formatting of this dissertation, and may differ in form and content from the published or submitted versions. Major changes in content are marked with a footnote.

2.1 Author contribution in publications

1. Ariadne – A Digital Topology Environment
   In this publication, I was the only author.

2. Mathematics in the Digital Age: The Case of Simulation-Based Proofs
   This article is a collaboration between Daniel Sommerhoff, Benjamin Rott and me. Daniel Sommerhoff and Benjamin Rott contributed to the conception of many ideas of the article and provided critical revisions of the article. I contributed the main ideas, wrote the article, and am thus listed as first author.

3. Knotted Portals in Virtual Reality
   In this publication, I was the only author.

4. Embodied Mathematics: Forming Concepts in Topology by Moving Through Virtual Reality
   This article is a collaborated work by Benjamin Rott and me. Benjamin Rott contributed through conception of many ideas of the article and provided critical revisions. I contributed the main ideas, wrote the article, and am thus listed as first author.

5. On the Future of Design in Mathematics Education Research
   This article is a collaborated work by Benjamin Rott and me. Benjamin Rott contributed through conception of many ideas of the article and provided critical revisions. I contributed the main ideas, wrote the article, and am thus listed as first author.

2.2 Code availability

All software code for the projects can be found publicly on GitHub, at `github.com/mosuem`. This encompasses four repositories with a total of ca. 10.000 lines of written code. The ready-to-use builds of the software projects themselves are available at `imaginary.org/users/moritz-summermann`. Videos of the software are available on YouTube at `https://www.youtube.com/channel/UCoUBUB-3HbaZzzb1m6Pq0Rg`.

2.3 Ariadne – A Digital Topology Environment

Author: Moritz L. Sümmermann
Published in *The International Journal for Technology in Mathematics Education* as Sümmermann (2019a).

The article presents ARIADNE, a software for interaction with objects such as dots, paths, and homotopies of paths, in this way enabling learning of basic concepts of topology. Besides a description of the software, the article presents the rationale of the design, as well as a set of questions suitable to assess learning of students using ARIADNE.

Ariadne – A Digital Topology Learning Environment

Moritz L. Sümmermann

Abstract

Ariadne is a touch-based program for the learning of homotopies of paths, without the use of formalism, by building mental models. Using Ariadne, the user can construct points, paths by dragging points and homotopies by dragging paths as well as compute winding numbers of paths, all on a variety of surfaces, through touch gestures. Ariadne provides surfaces in two and three dimensions and an optional number of punctures.
This environment enables the user to tackle questions regarding the equivalence of points by paths or paths by homotopies, because it allows only mathematically valid operations, i.e., paths and homotopies cannot pass through punctures on the surface.
Ariadne is designed to let students of all ages and prior states of knowledge approach problems ranging from the construction of a path connecting points to the classification of all paths up to homotopy on a punctured plane, effectively calculating the fundamental group of a sphere.

1 Introduction

People encounter topological ideas and principles every day. For example, the shape of a child's blanket can change through deformations, but it still remains the same blanket; it is "invariant" under these deformations. A doughnut illustrates another example of topological ideas in everyday life; for a doughnut to be a doughnut, it has to have a hole in the middle, and neither the size of the hole nor the size of the doughnut are relevant. These elementary mathematical thoughts are specified in topology.

Topology is a field of mathematics concerned with the study of spaces with a structure allowing

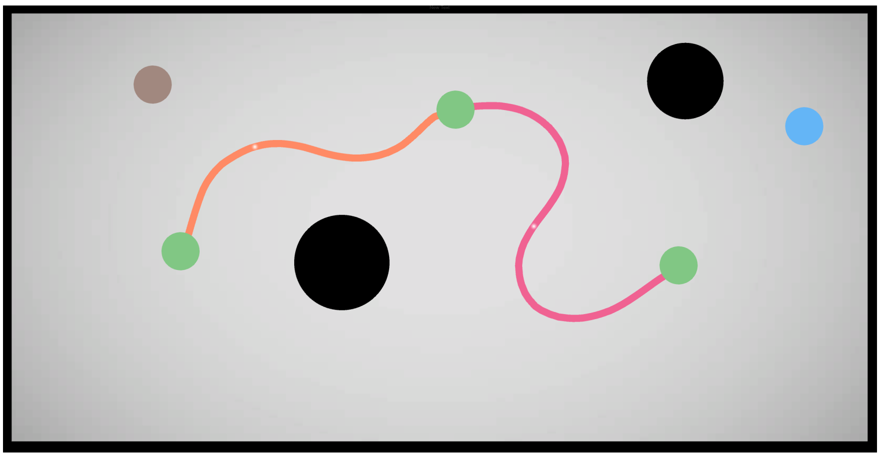


Figure 1: Dots, paths and obstacles on the plane in Ariadne

continuous maps. Two such spaces are regarded as equal if there exists a continuous map with a continuous inverse between them. These so-called homeomorphisms can be thought of as continuous stretching and bending of the space. Some deformations, such as the deformation of a line to a point, do not correspond to homeomorphisms, which makes the visualization difficult. A more general concept is then obtained by allowing all kinds of continuous deformations, which is

formalized by so-called homotopies. This idea is of great importance in mathematics, which has not found its way into the modern school curriculum. The omission is not unfounded, as a rigorous formal introduction to topology is quite demanding and uses a great deal of set theory. Historically, homotopies arose from questions of complex analysis (Eynde, 1992), but this approach is also out of reach for students. There were some attempts to introduce topology in the classroom during the "New Math" era, a movement which ultimately failed in its entirety for various reasons, leading to topology being dropped from the curriculum.
What can, however, be taught and learned is the idea behind the definition of homotopies and its application to simple examples. This has been achieved for isotopies, a special case of homotopies, by teaching isotopies of knots using ropes (Strohecker, 1996). There are also some digital approaches to knot theory (Shimizu, 2012; Scharein, 1998; Culler, Dunfield, Goerner, & Weeks, n.d.). When dealing with the more general concept of homotopies, malleable but elastic objects with the ability of self-intersection would be needed. As no real-world object with all these properties exists (cf. (Sugarman, 2014)), this is an opportunity to use computers to simulate such malleable objects. ARIADNE is a learning environment designed to this purpose of letting the user construct and deform such objects by touch gestures.
The generality of topological concepts necessitates a reduction to representative special cases. In ARIADNE, the only spaces being considered are the plane $\mathbb{C}$, the two-sphere $\mathbb{S}^2$, the two-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, and punctured versions thereof. The only objects constructed are points, homotopies of points, i.e. paths, and homotopies of paths. This application of homotopies to points encapsulates the concept of connectivity, "one of the most important properties of topological spaces" (Encyclopedia of Mathematics, 2018). The examination of homotopies of paths, i.e. homotopies of homotopies of points, is a generalization of connectivity to a higher dimension. Although paths are quite elementary functions, they are certainly rich enough to provide insights and intuition for the general notion of homotopies.
Finally, it is valuable in itself to give students the possibility to explore a topic of *modern* mathematics and the important place of topology in it, as demonstrated for example by the Poincaré Conjecture (Mackenzie, 2006).

2 Availability of Ariadne

Ariadne is available for download at the website of the Imaginary group imaginary.org (`https://imaginary.org/program/ariadne-drawing-topology`) for Windows and Android operating systems, together with installation manuals. The Android version is also available at the Google Play Store.

3 Mathematical Background

This section can only give a brief overview; for a detailed account we refer the reader to standard textbooks on algebraic topology, for example (Bredon, 1993; Hatcher, 2000). For the following definitions, the reader is assumed to be familiar with the notions of topological spaces and continuity.

Plane By *plane* we denote the space $\mathbb{C}$, which is topologically isomorphic to $\mathbb{R}^2$. It is also isomorphic to the *bounded plane* $(-1, 1)^2$.

Puncture To *puncture* a space means to remove a point from the space. An example is the *punctured plane* $\mathbb{C} \setminus \{0\}$.

Path A *path* is a continuous map from the closed interval $[0, 1]$ to a topological space X. The choice of the interval as domain is justified by arguments on connectivity using category theory (Wofsey, 2016; Leinster, 2012, 2010). A path is called *constant* if its image is a point. Given a path γ, its *inverse* is defined by

$$\forall t \in [0,1] : \gamma^{-1}(t) := \gamma(1-t)$$

, the path traveling along γ in reverse.

Homotopy Given two topological spaces X and Y and continuous functions $f, g : X \to Y$, a *homotopy* from f to g is a continuous map $H : X \times [0,1] \to Y$ such that

$$H(x,0) = f(x) \tag{1}$$
$$H(x,1) = g(x) \tag{2}$$

. If such a map exists, f and g are said to be *homotopic* and we write $f \Rightarrow_H g$. If f and g are paths, we will always assume that the homotopy is *based*, i.e. $H(1,t) = f(1) = g(1)$ for all $t \in [0,1]$. Examples of homotopies are

1. $X = \bullet$, $Y = \mathbb{C}$
   If X is just a single point, then f and g are maps picking a point from $\mathbb{C}$. As $\bullet \times [0,1]$ is naturally isomorphic to $[0,1]$, a homotopy from f to g is a map $H : [0,1] \to \mathbb{C}$, which is the definition of a *path* from $f(\bullet) \in \mathbb{C}$ to $g(\bullet) \in \mathbb{C}$.

2. $X = \mathbb{S}^1 := \{x \in \mathbb{C} \mid |x| = 1\}$, $Y = \mathbb{C}$
   This describes the homotopy of *closed* paths, paths for which the start- and endpoint are the same. The choice of $X = \mathbb{S}^1$ instead of the interval is justified by the fact that if f is a closed path, i.e. $f(0) = f(1)$, the universal property of the quotient space gives a restriction f from $[0,1]/_{0 \sim 1} = \mathbb{S}^1$ to $\mathbb{C}$.

3. $X = \mathbb{S}^1 := \{x \in \mathbb{C} \mid |x| = 1\}$, $f : X \to \bullet \in Y$
   If f is a *constant path*, i.e. a path staying at the point $\bullet$, and g homotopic to f, then g is called *null-homotopic*. Note that the constant path is also closed.

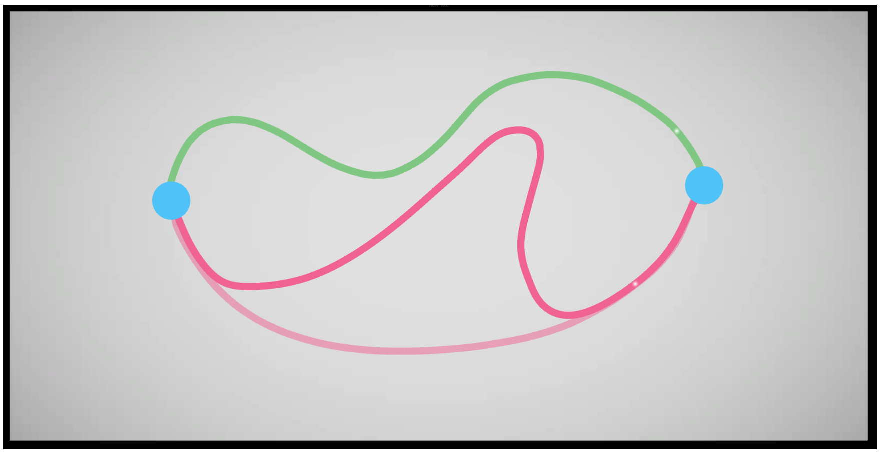


Figure 2: Homotopies in Ariadne

Winding Number Let $\gamma : [0,1] \to \mathbb{C}$ be a path. Define the *winding number* around a point $p \in \mathbb{C}$ not in the image of γ by first writing the path in polar coordinates as $t \mapsto p + r(t)e^{i\theta(t)}$ for some functions r and θ. Define the function $W_{\gamma,p}(t)$ by

$$W_{\gamma,p}(t) = \frac{\theta(t) - \theta(0)}{2\pi}$$

. Then the winding number of γ around p is given by $W_{\gamma,p}(1)$, as $W_{\gamma,p}(t)$ measures the normalized change in angle at time t of the path with respect to the point p.
The winding number can be shown to depend only on the homotopy class of a path.

Fundamental Group Given a topological space X and a point $x_0 \in X$, define

$$\pi_1(X, x_0) = \{\gamma\colon \mathbb{S}^1 \to X \mid \gamma(1) = x_0\} /_{\sim}$$

where

$$\gamma_0 \sim \gamma_1 :\iff \exists H\colon \gamma_0 \Rightarrow_H \gamma_1$$

. This set of closed paths from the *base point* x_0 up to homotopy comes with a group structure through concatenation of paths, with identity element the constant path at x_0 and the inverse γ^{-1} of a path γ given through the reversed path $\gamma^{-1}(t) = \gamma(1-t)$.

4 Design and Educational Concepts

Based on the artefact-centric activity theory (Ladel & Kortenkamp, 2013), Ariadne is seen as an artifact through which the subject, i.e. the user of the program, can internalize mathematics internalized in turn by the program.
To ease the internalization of the object properties through the artifact, the program interface is kept simple. This means not using any avatars or other decorations which are sometimes used in learning environment design to allow easier access to the artifact by children, as in this case the easier access would come at the expense of more difficulty in internalization of the mathematics.
By externalizing a mathematical object, there is an error introduced, i.e. some design choices do not represent mathematical properties. As the intention is to teach mathematics and not specifics of the program used (Richter-Gebert, 2013), a goal of the design process is to minimize this error. Examples of such artificialities include points being round and having an area, or lines having a width. Unfortunately, these cannot be avoided by any known means. Some dynamic geometry programs such as GeoGebra (Hohenwarter et al., 2013) indicate the infinite thinness of a line or a point by depicting these as being scale-independent, but Ariadne does not support a zoom feature, as this is a fundamentally geometrical idea.
Emphasis is put on making every internalized property of the objects externalizable to the subject, i.e. the user should not have to remember previous steps of manipulation to understand the mathematical properties of an object. An example of this is the enabling of the subject to cycle through the drawing order of the drawn paths to ensure that no path can obstruct the view of another, which unfortunately cannot solve the problem of a path obstructing parts of itself.

Although Ariadne gives the possibility to answer questions by manual construction, its purpose is the building of mental images in the subject to develop a shift from constructions on the screen to mental constructions. This follows the thoughts of (Swoboda & Vighi, 2016) on dynamic geometry software: “The emphasis here is set not so much on observing objects in motion nor on the final results of manipulation, but on the ability to predict the result of the transformation”. Also, memorization is not a feasible strategy in this environment, as the objects constructed are always different *geometrically*.

The concept of a homotopy formalizes the idea of continuity as in continuous transformation. By using paths and homotopies of paths, continuity is understood in respect to *time and motion*. A path not being allowed to traverse walls for the same reason as a homotopy is not allowed to let paths traverse punctures of the plane, giving the sense of a coherent environment with only fluent movement. The lack of coordinate systems and function expressions contrasts this approach to the usual notion of continuity developed by investigating functions (Hanke & Schäfer, 2017).

An interesting aspect arising from the study of homotopies in the special case of the punctured plane is the isomorphism $\pi_1(\mathbb{C} \setminus \{0\}, 1) \cong \mathbb{Z}$, which can enable the subject to understand the integers as paths with different winding numbers and addition as concatenation of paths. This is an alternative representation of the integers, in contrast with the usual one using the expansion to the left of the number ray. The orientation of a path is a basic notion for the sign of an integer.
An implication is that a number can, so to speak, be *constructed* as a *continuous* object. A major downside is however that one cannot identify at a glance the winding number of a path[1], unlike the cardinality of a set in the usual representation as a cardinal number.

Richter-Gebert identifies several "areas of tension" in the design of visualization software (Richter-Gebert, 2013), using which Ariadne can be classified. Some of these are

1. "Simulation vs. animation"
   Ariadne was programmed using basic principles, meaning that a result of an interaction with Ariadne is the consequence of these rules and not predefined by the programmer. It is thus a simulation and not an animation.

2. "Demonstration vs. self-study" and "Experiment vs. user guidance"
   The program can be used as a demonstration as well as for self study, as it features tools for the automated correctness checking of a construction by the correct approach. However, not all questions can be posed in such a constrained fashion, so a certain level of user guidance is always recommended.

3. "Freedom vs. restriction"
   To allow a piecewise digestion of the mathematical content presented with Ariadne, several "levels" are implemented, each enabling different features. In the simplest level, only points can be set, in the next, only paths between points etc. There are however no additional limits such as a maximal number of points constructible.

4. "Click vs. touch"
   Here Ariadne is obviously on the "touch" side. A great advantage of mouse over touch is the accuracy, which is not crucial in the construction of objects, so there is no reason to use the mouse in this project. Also, using a touch interface supports the impression of the physical manipulation of objects and gives a bridge between iconic and haptic representation forms. The use of multi-touch gives a wider range of possible commands by letting the user specify multiple objects simultaneously (Kortenkamp & Dohrmann, 2010).

5 Implementation

Ariadne was programmed using C$\sharp$ and Unity3D, a game engine with widespread use for mobile game applications.
The program contains two modes, one for constructions in two- and one for three dimensions. In both cases, the subject is first presented with an empty gray canvas. In the two-dimensional version, the canvas is a bounded plane. For the three-dimensional mode, the subject has the choice between a 2-sphere and a 2-torus. He can then construct different objects through touch interaction on the surface of the canvas. By a touch on the screen, a point is created at the corresponding location on the canvas. By dragging a point with a touch gesture, a new point is created which follows the movement of the subjects finger. This new point leaves a trace behind, which represents a path (see Fig. 1). If the dragging stops at the location of another point, the two points are connected by this path. If, on the other hand, the dragging stops at a location with no point, the path vanishes. This feature supports deliberate construction of paths as objects with a start and an end, and prevents

[1]Incidentally, this is the mechanism behind the "Fast and Loose" con.

thoughtless dragging of points over the screen as well as emphasizing that paths connect points. The construction of homotopies follows the same pattern, only in one dimension higher. By dragging a path with a touch gesture, a new path is created which follows the movement of the subjects finger. It does not leave a "trail" behind, as this trail would be a surface, which cannot give much information on the homotopy. For this reason, and after some preliminary tests, this feature was not implemented.
To make the underlying spaces more interesting, the subject (or the instructor) can construct "obstacles", corresponding to punctures in the topological sense. These obstacles are variable in size, which does not make a difference topologically but cognitively. The use of punctures makes more interesting examples possible, as a puncture leads to the possibility of non null-homotopic paths. In the two-dimensional version, it is also possible to construct "walls" to create several connected components of the plane, enabling the investigation of π_0.

In the case of the surface being a punctured plane, Ariadne has a feature to visualize the winding number $W_{\gamma,p}$. A touch on the associated button, followed by a touch on a path γ and finally a touch on the obstacle corresponding to the puncture p for which the winding number should be computed. The program thus follows the "Action-Object" approach (Kortenkamp & Dohrmann, 2010) by first specifying the action, and then the object on which the action should be performed. The winding number is then visualized by displaying a ray from p to $\gamma(\frac{t}{10})$, where the parameter t indicates the time in seconds after pressing the button. This means that the endpoint of the ray follows the path for a total duration of 10 s, while leaving the starting point fixed at p. It leaves behind a trace along an arithmetic spiral. After t seconds, the number of revolutions is $W_{\gamma,p}(t/10)$. For a closed path, this number is always an integer. The direction of rotation gives the sign of the winding number.

There are two built-in functions for the verification of results. One is a button which, given two paths to be tested on homotopicness, lets a line slide down the two paths simultaneously, stretching out if elongated. If the two paths are too far away from each other, the line is elongated too far and snaps. This can show if a homotopy between the two paths has been found.
The other feature is activated by touching a button and then a path. This path is gradually tightened until it is taut, which results in the path vanishing iff it is null-homotopic.

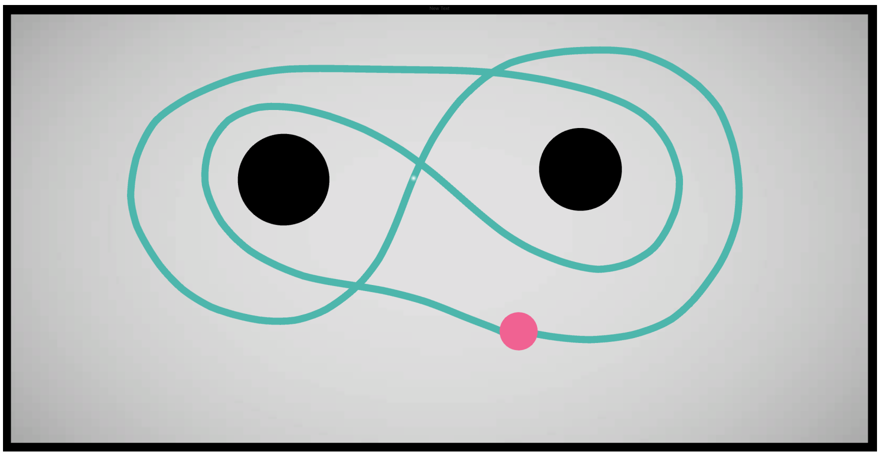


Figure 3: The Pochhammer Contour, a not null-homotopic path with winding number zero around both obstacles visualized in Ariadne.

6 Questions to ask students

When using Ariadne, two different goals can be distinguished. One is to *explain* the definitions and concepts involved. The other is then to *verify* the understanding reached by the subject by letting them solve problems.
The problems posed have to be chosen carefully as to reflect the understanding of the mathematical objects by the subject. The identification of the concept definitions that have to be present in the user to reach an understanding of the definitions is subject of separate and ongoing studies. In the following, some exemplary problems are presented, classified by the type of objects involved.
The language of these questions, in particular the specific terminology used, will be modified so as to be adequate for the target audience. When working with children, the replacement of words like "null-homotopic" with self-chosen terms may be advisable.

6.1 Dots

This is the simplest type of question: Where can a dot be placed? As the answer is "Anywhere but on obstacles", this problem can be solved quite easily and without gaining much mathematical insight. Also, to dwell longer on this topic would encourage the subject to find other properties of points, which would most likely be characteristics of the program, and not actual features relating to the mathematical object of a point.

6.2 Paths

Questions about Paths concern the existence and construction of paths connecting points. Examples are "Can these two points be connected?" or "Where could a dot be placed which is cannot be connected to this other dot?". In the second example, even if the question is about the positioning of a dot, this is still a question about paths as the existence of a path is the underlying problem.
Mathematically, these questions ask about connected components of the surface. This implies that non-trivial problems can only be posed if the surface is not path-connected, which in turn can be achieved by placing obstacles subdividing the plane.

6.3 Homotopies

In this category, questions encompass a wide range, but center on the relationship between paths. The uniqueness of paths is now well-defined, as homotopies establish the notion of difference of paths. Examples of questions are

1. "Are these paths homotopic?"
2. "Can an obstacle be placed such this path is not null-homotopic?"
3. "Are there two *different* paths connecting these points?" Where "different" is to be understood in the sense of "not homotopic".
4. "How many different paths can be drawn on this surface from this point to itself?"

6.4 Winding Number

As the Winding Number is a number assigned to a path, questions in this category involve this number and some constructed paths. Examples are

1. "What is the winding number of this path? Of its inverse?"
2. "Why is the winding number of a closed path an integer?"

3. “Do all homotopic paths have the same winding number?”

4. “Are all paths with the same winding number homotopic?”

Question 1 addresses both the readability of the externalization of the winding number as an arithmetic spiral in Ariadne, as well as the fact that the winding number comes with a sign. Questions 3 and 4 both contribute to the classification of paths through the winding number.

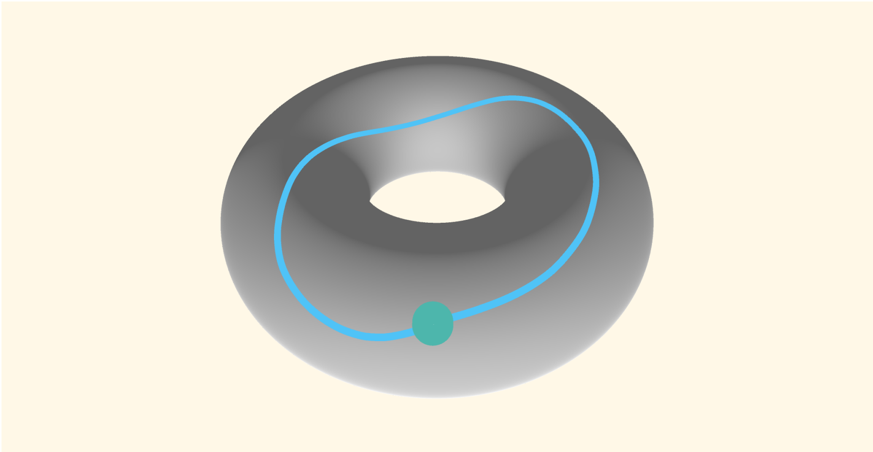


Figure 4: A path on the torus visualized in Ariadne.

7 Outlook

Ariadne has to undergo further rigorous testing to ensure the usefulness as a learning environment. For this, the problems posed to the subjects have to be constructed methodically, which relies on ongoing research on concept definitions and images in topology.
Also, the use of advanced reality could be integrated to ease the use of the three-dimensional mode. A virtual reality/mixed reality mode could further enhance the scope of Ariadne by improving interaction capabilities with the surfaces, for example enabling the subject to construct an arbitrary orientable surface by adding handles to a sphere. At the moment, such features could not be implemented as the two-dimensional interface of a tablet poses difficulties even for drawing paths on a surface in three-dimensional space such as a torus.

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2.4 Mathematics in the Digital Age: The Case of Simulation-Based Proofs

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In this paper, we advocate for a different view on the usage of a class of mathematics environments called "mathematical simulations" in the context of proving. While traditionally, these environments were restrained to activities such as exploration and conjecturing, we assert that they may be used to actually construct proofs. This claim is strengthened through the presentation of examples and a line of argumentation based on a framework for proof functions.

Mathematics in the Digital Age:
The Case of Simulation-Based Proofs

Moritz L. Sümmermann, Daniel Sommerhoff, Benjamin Rott

Abstract

Digital transformation has made possible the implementation of environments in which mathematics can be experienced in interplay with the computer. Examples are dynamic geometry environments or interactive computational environments, for example GeoGebra or Jupyter Notebook, respectively.

We argue that a new possibility to construct and experience proofs arises alongside this development, as it enables the construction of environments capable of not only showing predefined animations, but actually allowing user interaction with mathematical objects and in this way supporting the construction of proofs. We precisely define such environments and call them "mathematical simulations." Following a theoretical dissection of possible user interaction with these mathematical simulations, we categorize them in relation to other environments supporting the construction of mathematical proofs along the dimensions of "interactivity" and "formality." Furthermore, we give an analysis of the functions of proofs that can be satisfied by simulation-based proofs. Finally, we provide examples of simulation-based proofs in Ariadne, a mathematical simulation for topology.

The results of the analysis show that simulation-based proofs can in theory yield most functions of traditional symbolic proofs, showing promise for the consideration of simulation-based proofs as an alternative form of proof, as well as their use in this regard in education as well as in research. While a theoretical analysis can provide arguments for the functions of proof, their actual use and, in particular, their acceptance is of course subject to the sociomathematical norms of the respective communities and will be decided in the future.

1 Introduction

Proofs can come in many forms, ranging from a system of logical deductions done in a formalistic symbolic way, as in the *Principia Mathematica* (Whitehead & Russell, 1910), over plain text argumentations supplemented by formulas, which are the standard form in many research papers and textbooks, to so-called "Proofs Without Words" given by an image only (Nelsen, 1993).

Computers have added to this variety by giving rise to computable proofs, i.e. proofs that can be checked by a computer (Voevodsky, 2015), or even proofs executed by a program such as the much debated (Tymoczko, 1979) proof of the Four-Color Theorem (Appel & Haken, 1977), too long to ever be completely reviewed by a human.

The rise of computers and with it digital transformation of all aspects of human activities has also made environments possible, that enable to do mathematics in an interactive way. These environments can come in different representation modes, as some are more formal-symbolic in nature, such as Mathematica or Jupyter Notebook, while others have rather informal visual representations, such as GeoGebra for geometry or Ariadne for topology[1]. In the same way as programming stimulated computational thinking (Papert, 1980), we claim that mathematical simulations nurture mathematical thinking; the focus in both cases lies on human thought and development and not on technology.

[1] Ariadne is available for download at `https://imaginary.org/program/ariadne-drawing-topology`

Undoubtedly, these programs allow for mathematical activities such as exploration or checking special cases of conjectures (Borwein & Devlin, 2008). These "experimental mathematics" (Borwein, 2011) activities can be regarded as being equally important as "rigorous proving" in mathematics (Jaffe & Quinn, 1993). However, we argue from a theoretical perspective that they can also provide new means to formulate proofs, that is mathematical arguments fulfilling the roles of proofs pointed out, for example, by De Villiers (1990) (see Sec. 2.1).

Research on the impact of these new technologies on the concept of proof, which plays a central role in mathematics, is not very developed, despite their well-known impact on the understanding of mathematics in general (Hoyles & Lagrange, 2010). Research is focusing more on the role technology plays in establishing conviction of a fact, which may be even counterproductive in justifying the necessity of proof (Bolite Frant & Rabello de Castro, 2000; Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2004; Hoyles & Noss, 2003; Marrades & Gutiérrez, 2000), and efforts to remediate this (Hadas, Hershkowitz, & Schwarz, 2000; Jones, 2000), see Sinclair and Robutti (2013) for an overview. This focus on technology establishing conviction is related to the use of many of these technologies primarily in educational contexts. In curricula, proofs are well-established almost only in geometry (Hanna & de Bruyn, 1999; Stylianides, 2007), where students do proofs by geometric constructions in the spirit of Euclid (Mogetta, Olivero, & Jones, 1999). Some go as far as to say that the central reason for the existence of geometry in the curriculum is for serving as a paradigm for deductive proof (Hanna, 1998). Overall, research investigating possible evolutions of the concept of proof, such as GeoGebra, when digital tools are used is not well developed and focuses on their use for such constructions in Euclidean geometry (Marrades & Gutiérrez, 2000).

Moreover, the effect of using digital geometry environments in educational contexts is controversial (Hanna, 1998), as "proofs" in such environments are often seen as providing evidence for the truth of a statement by making available a large, by continuity seemingly infinite, number of examples (Hanna, 1998), without actually being a *real* proof (Nam, 2012). An example would be dragging the corners of a triangle while keeping track of the sum of the internal angles, which can convince a student of the fact that their sum is always 180° without giving any sort of explanation of this fact and its relation to the parallel axiom, and without allowing to be certain that no case exists in which the sum is different. The scope of such software is thus often seen as limited to *exploration* (Christou et al., 2004) and *conjecturing* based on this exploration (Mogetta et al., 1999; Venema, 2013).

This deeply rooted view on such technology as being there for exploration and conjecturing, and on proofs as being formal in their representation, led to attempts on doing proofs by integrating such technology and formal proofs in digital environments. This manifests in, for example, the implementation of proof assistants in dynamic geometry environments (Albano, Dello Iacono, & Mariotti, 2019; Hanna, Reid, & de Villiers, 2019; Kovács, 2015; Miyazaki, Fujita, Jones, & Iwanaga, 2017; Nam, 2012).

We argue that these limitations and separation in informal/exploration – formal/proving is mostly due to the nature of geometrical constructions and the role of Dynamic Geometry Environment (DGE) software in proof and technology related activities. To emphasize that these limitations are not inherent to digital environments in general, but to the practice of use of those from geometry, we present several proofs[2] done in a different type of software, a Dynamic *Topology* Environment called Ariadne (Sümmermann, 2019a). In Ariadne, the user cannot only explore basic concepts of topology including points, paths, and homotopies of paths, but can also formulate proofs of non-existence using invariants such as the winding number of a path. Examples of such proofs are given in Appendix A. They highlight the features of simulation-based proofs in such a digital environment, which are not limited to exploration, experiments, and constructions, but can satisfy the same functions as traditional proofs. In our argumentation, Ariadne serves as an

[2]Examples can be found on Youtube (Sümmermann, 2019b, 2019c, 2019d)

example; it can be exchanged for any simulation with similar characteristics.

Our first findings in this theoretical study are the necessary conditions that the broader class of environments and tools representing mathematics have to fulfill, in order to possibly allow the construction of proofs in such environments. These conditions define the new concept of a *mathematical simulation* and with it the term *simulation(-based) proofs*. An emphasis must be put on the word *necessary* in the previous formulation, as there are examples of mathematical simulations such as MATLAB or MAPLE, in which proofs that go beyond calculations are harder to achieve due to the focus on numerical manipulations.

Following an analysis of this kind of simulation proofs, we present a classification of proofs along the lines of *interactivity* and *formality*, showing the place of simulation-based proof in the context of more traditional or alternative forms of proof. We also present an analysis of different functions of simulation proofs, following and extending the framework created for proofs in general by De Villiers (1990). This leads to the conclusion that simulation proofs are of particular interest in mathematics education, with some caveats.

Going beyond the familiar debate of the non-surveyability of computer-generated proofs (Tymoczko, 1979) such as the proofs of the Four-Color Theorem and the Kepler conjecture, we introduce *technological reasons* as a category of acceptance criteria for proofs in the context of simulations.

1.1 Article organization

Roughly speaking, we will give definitions, examples, and intricacies of the use of simulation-based proofs. Then follows a classification, first external (in relation to other types of proof) and then internal (highlighting functions of proof).

More precisely, in Sec. 2, we will start by explaining more closely what we mean when talking about proofs. We then define the notion of a *mathematical simulation*, in particular distinguishing simulations from animations and microworlds. This is followed by a general analysis of the user interaction process with a simulation. As errors occurring in the interaction process are vital in establishing trust in the simulation, we give a categorization and examples of different types of error that may arise in the process.

This is followed by Sec. 3, in which various forms of proof and proving environments are classified along the dimensions of interactivity and formality. This aids in understanding the place of simulation proofs and mathematical simulations in relation to other forms of proof and mathematical environments.

Subsequently, Sec. 4 employs De Villiers' 1990 framework to discuss the functions of simulation proofs. In particular, this also encompasses new categories regarding the conviction of the proof recipient particular to simulation proofs.

Finally, Sec. 5 contains conclusions and implications for mathematics education, and mathematics research, followed by some descriptions of simulation proofs in appendix A.

2 The proof process in a simulation

2.1 The status of proof in mathematics and mathematics education

While proofs are at the center of mathematics, their nature is highly contested and comprises a wide range of different objects. The spectrum of proofs begins with "formal proofs" in a mathematical logic theory sense, as chains of formalized deductions, which can at least in theory be checked by a computer. As Krabbe (2008) states, these "are a logician's gadget," and do not exist in practice (Aberdein, 2008). The view that formal proofs are practically non-existent is challenged by new generations of formal proof theories and resulting computational advances, leading to formalized proofs as feasible tools for mathematicians work (Voevodsky, 2015), but it is still certainly true for "most" mathematicians and almost all of mathematics in educational contexts.

The next "step" are formal proofs in the sense of axiomatic, symbolic mathematics. Such proofs contain elements of an *argumentation* and are the type of proof that is most commonly employed in mathematics contexts. That mathematical proofs can be argumentations (Aberdein, 2008), i.e. let room for debate, is again a controversial notion and rejected by some (Johnson, 2012), as it conflicts with the view of proofs providing absolute conviction of truth *beyond doubt* (Krabbe, 2008).

Proofs in educational settings, such as schools or undergraduate courses, are then again different, as they adhere to forms of reasoning and are communicated with forms of expression "that are valid and known to, or within the conceptual reach of, the classroom community" (Stylianides, 2007). This can include other forms than the stricter representation in communications of the mathematical community.

What we propose, for the purpose of this article, is more of an implicit definition of proof, giving a list of attributes defining a proof in the sense of Lakoff (1987) and Weber (2014): By showing that simulation-based proofs can fulfill the roles and functions of proofs as specified by De Villiers (1990), they may be regarded as such. In this way, we adopt a view of proofs as a cluster concept in the sense of Weber (2014).

It should to be noted that when we argue that "simulation-based argumentations" can be regarded as proofs, it is not our intent to somehow bypass the sociomathematical norms (in the sense of Yackel and Cobb (1996)), and certainly not to set them; we cannot define externally what constitutes a proof, this has to be done by the mathematics community and time.

2.2 Mathematical simulations

Generally, a *simulation* can be defined as any attempt to mimic a real or imaginary environment or system (Rieber, 1996). Based on this, we define a *mathematical simulation* (MS) to be a simulation mimicking mathematics by following the *mechanisms of action* immanent to the mathematics being simulated, creating the representation as a consequence of general underlying rules (see Richter-Gebert, 2013). A mathematical simulation comes equipped with a certain representation of the mathematical content, in general defined by the person who built the simulation, and the capability to allow user interaction with this representation, be it through manipulation of objects, images, symbols, or other such modes of representation (see Sec. 3.1) yet to be conceived. We will call proofs based on a simulation either *simulation-based proofs* or simply *simulation proofs*.

Simulations are fundamentally different from animations, even if it may not be easy to distinguish them from another as a user. An animation is defined to be a software following a predetermined stimulus response mechanism (Richter-Gebert, 2013). While animations are defined by *what* is being presented, simulations are defined by *how* the presentation is made.

An example borrowed from physics may help to clarify the differences between animations and simulations. Two tablet apps are given, both allow the movement of a ball on the screen via dragging. If let go, the ball will fall to the ground, bouncing a few times. Behind the visual representation, this could be realized by either implementing the *mechanisms of action* of the ball, such as gravity and the spring force calculated from the kinetic energy. This would be a (physics) *simulation*. Another possibility would be to simply let the ball go down pixel by pixel until its coordinates reach a certain value, which then triggers a predefined movement along some curve, giving the "illusion" of the ball bouncing; this is an *animation*. Although they may not look different, at least at first sight, the simulation may allow the exploration of physical phenomena even beyond the intent of the programmer, while the animation does not. In the scenario of the bouncing ball, this may mean the deformation of the ball on impact, which may not have been intended by the programmer of the simulation but may nevertheless be observed, while it is only visible in the animation if the programmer has explicitly thought of it.

A mathematical example may be a software generating graphs of quadratic functions, in the case of a simulation really plotting the function, in the case of an animation presenting a predefined image from a collection of graphs best fitting the entered parameters. As the number of different graphs on a screen with a finite resolution is finite, the difference between the two cases would be virtually impossible to identify for a user.[3]

These examples showcase the predefined nature of animations, which certainly makes the construction of an animation easier than that of a simulation and may also have some advantages. By their very definition, they are, however, useless for the exploration or representation of mathematics beyond the build-in set of cases.

A term closely related to the one of a mathematical simulation is "microworld," a concept introduced by Papert (1980) describing a subset of reality or a constructed reality whose structure matches that of a given cognitive mechanism so as to provide an environment where the latter can operate effectively (Papert, 1980, p. 204), allowing learning. Hoyles, Noss, and Noss (1996) also stress the importance of *interactivity* for microworlds, stating "Software which fails to provide the learner with a means of *expressing mathematical ideas* also fails to open any window on the processes of mathematical learning. A student working with even the very best simulation, is intent on grasping what the simulation is *demonstrating* rather than attempting to articulate the relationships involved." (Hoyles et al., 1996, p. 54) [emphasis added].

Simulations can, however, be distinguished from these microworlds through design choices made in the design of the latter as to fit the environment to the learners' cognitive state (Rieber, 1996). The aims of a microworld go beyond those of a simulation as it not only represents a mathematical object faithfully, but also specifically strives to enable learning, making microworlds a subclass of simulations. In a microworld, the objects can be manipulated by the user "with the purpose of inducing or discovering their properties and the functioning of the system as a whole" (Edwards, 1995, p. 144).

Mathematical simulations can serve as a framework for formulating proofs, as in addition to the capability of representing mathematics in a certain way, which can also be said for Proofs Without Words or even writing on a blank sheet of paper, the user can interact with the simulation in a meaningful way. The interaction is meaningful, as conjectures and arguments can be made in response to the behavior of the simulation, because the simulation reacts according to mathematical laws. Furthermore, this central feature of simulations makes it possible to not only replicate known results, but also to discover new results as the grounding in underlying mathematical rules allows the user to go beyond the simulation designer's imagination and intentions.

The simulation used to demonstrate some examples of simulation-based proofs in this paper is Ariadne[4], a dynamic topology environment developed by Sümmermann (2019a). In Ariadne, the user can explore different mathematical spaces including points, paths, and homotopies of paths using touch gestures, constructing points by a touch, paths by dragging points, and homotopies by dragging paths. Mathematically speaking, the "dragging" feature of many DGEs represents a homotopy, so this exploration of spaces via homotopies goes to the core of this "most central tool of DGEs" (Sinclair & Robutti, 2013).

[3]Note that although most examples of animations or simulations that come to mind are visualizations, they do not have to be; any representation mode is possible.

[4]Although Ariadne can also be seen as a microworld in the sense outlined above, we use it in its capability as a mathematical simulation.

2.3 The interaction process with a mathematical simulation

When arguing and possibly even proving based on a mathematical simulation, trust of the user in the simulation is crucial, for example, trust in the physics engine in the bouncing-ball simulation mentioned above. Errors in the interaction with a mathematical simulation could undermine the trust of the user. To determine the kind of errors that can arise in the interaction, we must analyze the interaction process with such a simulation. This analysis is itself independent of the purpose of the interaction, be it exploration or argumentation. The interaction process follows the same principles as the interaction with the software. It can be described using a Subject-Object-Artefact triangle, derived from situated instrumented activity (Vérillon, 2000); our model is specialized for the case of user interaction with mathematical simulations.

In an interaction process with a simulation, the users are in a cycle of adjusting their knowledge of mathematics to the observation of the simulation's behavior, leading to the formulation of new actions to be taken, resulting in new output of the simulation, which is again observed by the users. Each step of the proof involving an interaction with the software leads to such an adjustment. The general nature of the relationship between the user, the software, and the underlying mathematics can be described using the triangle in Fig. 1.

During the interaction process, the relations in the triangle are traversed by repeating several steps:

1. The users decide their next action as a consequence of **their knowledge** of mathematics (*influenced* by **mathematics**), taking form of an *input* to the mathematical simulation.

2. The simulation internally *computes* a new **representation** as an *output*, *influenced* by the **mathematics** implemented in it.

3. The users interpret the new **representation** and *adjusts* their expectations on the simulation's behavior and possibly also **their knowledge** of mathematics.

4. With new expectations in mind, based on the **representation** of the simulation, the users decide their next action leading to a new *input* for the simulation.

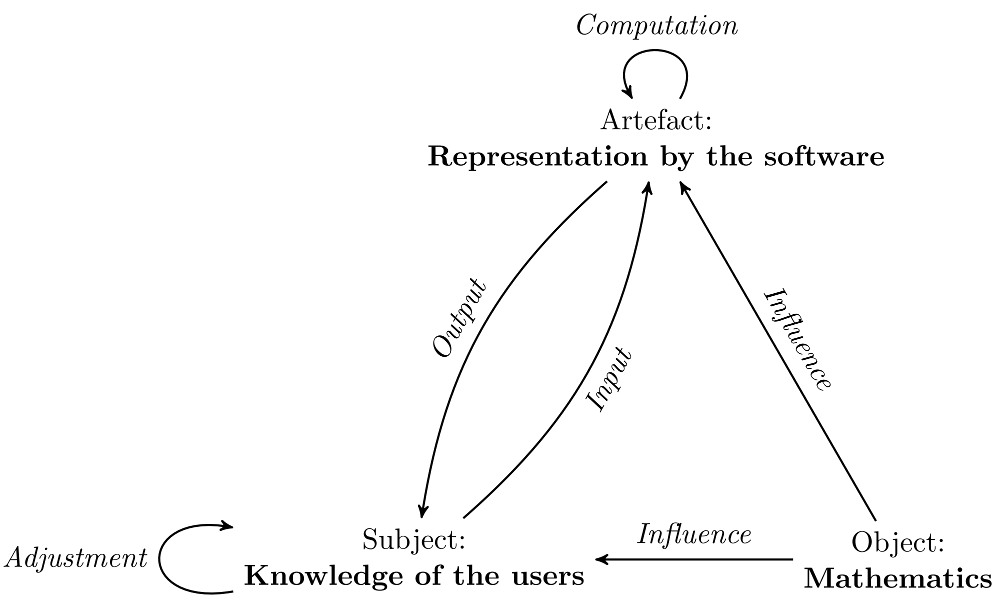


Figure 1: An interaction process with the mathematical simulation

2.4 Errors in the interaction process with the MS

For a productive use of mathematical simulations, the process described in Sec. 2.3 must function properly. Different types of errors can disrupt this process, which are characterized in this section. We distinguish two categories: errors by the user and errors by the software. An error by the user means an incorrect interpretation of the representation of mathematics of the mathematical simulation, and an error by the simulation an incorrect representation of mathematics by the mathematical simulation.

It is, of course, impossible to determine what, in general, an incorrect representation of mathematics is. But the representation of mathematics in a mathematical simulation constitutes an agreement between user and software to the "language" used to carry information. So even if a representation generated by a simulation – like any representation – cannot be false by itself, it can be judged by its adherence to the representation mode agreed upon and thus implicit to the simulation. In the following, the words "correct" or "error" will be used in this sense. In the triangle from Fig. 1, this error would be situated in "Representation by the software".

Nevertheless, a mathematical error originating from interaction with the software can only be made by the user if the simulation's dissenting representation leads to false assumptions about the represented mathematics. Notably, this implies that the error was observed by the user; if an error of the mathematical simulation has no consequences on the representation given as output or is overlooked, then it cannot have implications on the expectations and on the mathematical knowledge of the user.

In addition, the user cannot only hold correct or incorrect assumptions on mathematics, but also make errors in the interpretation of the simulation's representation. The chance of misinterpretation may increase if the representation mode is not defined explicitly. In Fig. 1, this would correspond to errors in the arrows between artefact and subject.

This leads to the distinction shown in Tab. 1. The user can correctly or incorrectly interpret the representation by the mathematical simulation. In the same way, the mathematical simulation can render the representation agreed upon correctly or incorrectly. Even if all possible interactions may be situated in this grid, not all errors can be identified in this way; the user can also draw the right or wrong conclusions, corresponding to errors in the subject corner in Fig. 1, or the software can go beyond its representation capabilities.

Table 1: The different types of errors along two dimensions, accounting for representation-related errors.

MS representation / User interpretation	Correct	Incorrect
Correct	Case (a)	Case (b)
Incorrect	Case (c)	Case (d)

To illustrate the types of errors given in Tab. 1 with examples, suppose a fictional software was given. These simple examples do not represent proving situations, but suffice to showcase the different possible errors of user and software on the mathematical content.

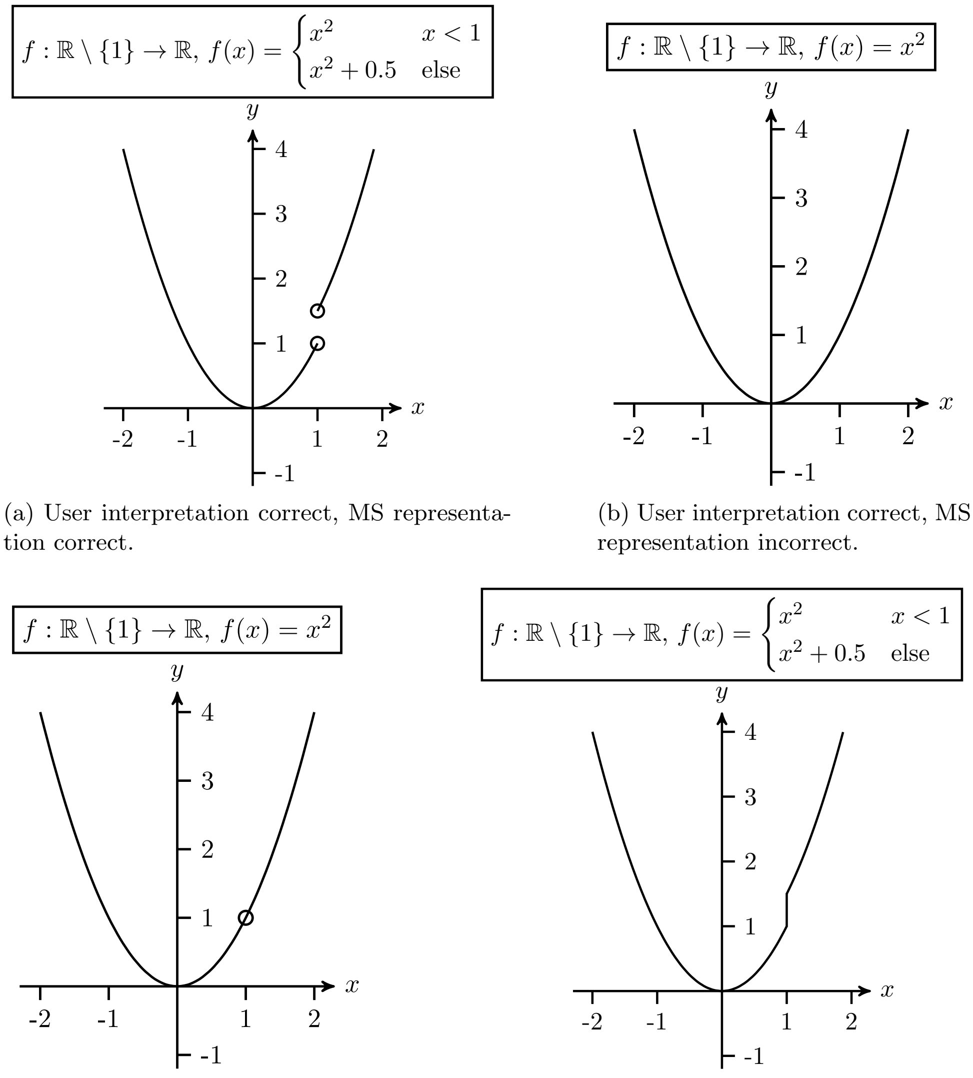
The software is a function plotter, which can represent a given polynomial function by drawing it in a coordinate system. It does so by randomly sampling points and connecting them by linear splines. It leaves a gap where the function is not defined. For all presented functions, the user wants to investigate singularities of the function using the software, disregarding the fact that a thoughtful user might not trust the software (compare Sec. 4.3.1), or double-check the result by other means. The error types in Fig. 2 can then be described as follows.

(a) The representation is correctly understood by the users. They see the singularity pointed out by the software.

(b) The users have the correct assumptions about the representation used, but the software depicts an incorrect representation. It might be that the software did not include 1 in the approximation of the curve, and thus did not observe the singularity. The users "correctly" assume that the function does not have a singularity.

(c) The users have incorrect assumptions about the representation used, but the software has no fault in the display. The user does not understand that the circled point is a singularity, even if removable, and believes the function to be free of singularities.

(d) The users have incorrect assumptions about mathematics and the simulation has a fault in the display. The software displays a continuous line along the jump discontinuity, which is not the agreed representation for this type of singularity (correct representation in Fig. 2a). The users think such an abrupt and non-differentiable change of slope is a sign of a discontinuity, and interpret the function to have singularities at the points $(0.99, 1)$ and $(1.01, 1.5)$.



(a) User interpretation correct, MS representation correct.

(b) User interpretation correct, MS representation incorrect.

(c) User interpretation incorrect, MS representation correct.

(d) User interpretation incorrect, MS representation incorrect.

Figure 2: The error types, as seen in a software for function plotting, according to Tab. 1.

3 Dimensions of proofs

Mathematical proofs can be classified along several dimensions, for example along the dimension of formality (Lakatos, 1978). Besides this dimension, we concentrate on "interactivity" as a second dimension, to account for the focus on more dynamic proof forms, which are the focus of this article. The classification helps to distinguish the various currently existing environments for basing proofs upon, and to subsequently point out the place simulation-based proofs take in relation to these other forms of proof. We will also discuss the concept of "transferability" of proofs, which certainly plays an important role in this context. All these dimensions are not necessarily orthogonal, but represent a way of distinguishing some proofs or mathematical environments.

3.1 Formality

Formality describes the adherence to standardized mathematical notation in the representation of mathematics, which is distinct from "formal," meaning written in a formalized language with formalized derivation rules (Krabbe, 2008). Here, formal means adhering to a strict, structured symbolic representation.

Proofs, as all mathematical content, can be distinguished with regard to their representation. Different systems have been proposed to categorize representation modes, for example enactive, iconic, and symbolic by Bruner (1966), or somatic, mythic, romantic, philosophical, and ironical by Egan (1997).

We use the much coarser distinction into formal–informal, as this suffices to classify existing environments allowing proofs for the purposes of this article.

3.2 Interactivity

Interactivity is defined by the dictionary Merriam-Webster as "mutually or reciprocally active." That means, not only has the medium to be active, such as in a video, but the user's actions have consequences on the activity of the medium.

Proofs can present themselves at different levels of interactivity. One end of the spectrum are *static* proofs. These can range, in different levels of structuredness, from Proofs Without Words (Nelsen, 1993), over "traditional" formal proofs, such as proofs in textbooks, to proofs that can be checked by a computer, the "platonic ideal" (Lamport, 2012).

More interactive proofs are given by videos or even animations, which may even be altered by tools such as sliders. These give the users some kind of control over the way the proof is presented to them, but does not constitute a simulation proof in the sense described in Sec. 2.2.

At the other end of the spectrum are fully interactive proofs in mathematical simulations, such as the ones presented in ARIADNE. These are proofs in which the user has total control to alter the proof, albeit limited by the allowances of the simulation used.

3.3 Overview on representation modes of proofs

Several different types of environments in which proofs can be shown or done, such as digital environments or simply textbooks, are depicted in Fig. 3. The aim of this overview is to give a sense of the place mathematical simulations and with them simulation-based proofs take in comparison to other forms of proof-supporting environments. We will now give some more information about the objects referenced in the figure.

The standard type of proof is a text with symbols, sometimes accompanied by images for clarification or illustration purposes, such as in a standard textbook. Standard non-interactive proofs can, however, range from informal to formal. A very informal category are Proofs Without

Words, where the image itself is the proof, only sometimes accompanied by text or symbols for clarification. Their status as to being a proof or representing an idea of a proof is not clear[5].

A relatively new type of argumentations are so-called "Proofs Without Words 2.0" (Doyle, Kutler, Miller, & Schueller, 2014), which are animated versions of Proofs Without Words, sometimes giving the user some control over the animation. This places them at the same level of formality as traditional Proofs Without Words, but more interactive, together with digital textbooks incorporating them such as Mathigon Legner (n.d.). Further up in the figure are environments that allow the user to manipulate the objects more freely and to come up with own proofs, such as DGEs or Ariadne. These programs are mostly informal in their representation, which is, however, no requirement for a mathematical simulation.

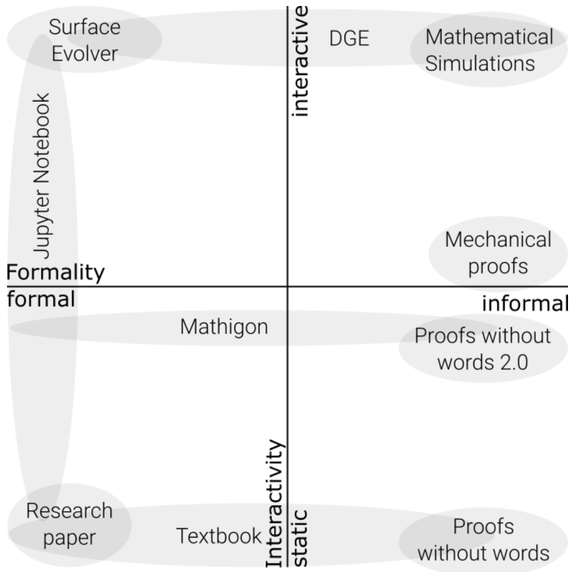


Figure 3: Some proofs and environments for doing proofs sorted along the dimensions interactivity and formality, ranging from static to interactive and formal to informal, respectively. For examples of mechanical proofs, see Richard et al. (2019).

On the more formal side of mode of representation are software such as Surface Evolver (Brakke, 1992) or Jupyter Notebook (Kluyver et al., 2016). Surface Evolver is a program to do operations, such as the simulation of geometric flows, on surfaces, working with a text-based interface. Jupyter Notebook is a web-based interactive computational environment, making it possible to mix text with programming language outputs of, for example, Python, making "seamless the communication between human and machine" (Barba, 2015).

An environment, which is both highly interactive and formal, is also possible, for example in the form of proof assistants such as Isabelle (Paulson, n.d.) or Lurch (Carter & Monks, 2013), or even general mathematical text editors, possibly giving another mode of distinction, which is, however, not the focus of this paper.

3.4 Transferability

While the idea behind a proof is in theory independent of its representation, it is nevertheless hard to do comparisons between proofs in different representation modes (Giaquinto, 2020). It is even hard to compare proofs in the same representation: given two proofs of the same theorem, there is

[5] In his famous collections of Proofs Without Words (Nelsen, 1993, 2000, 2015), Roger Nelsen writes in the introduction to Volume I: "Of course, 'proofs without words' are not really proofs", while the introduction to Volume II, quite ironically, contains: "Of course, some argue that 'proofs without words' are not really proofs" concluding with a quote from Brown (1999) ending with: "In short, pictures can prove theorems".

no formal notion of which proof is "simpler" (Cain, 2019) (also compare Inglis and Aberdein's (2014) discussion on the meaning of "simple" in mathematics). Also, not every proof can be represented in all modes of representation with the same ease, as many Proofs Without Words demonstrate; many would be challenging to formalize, and probably even lose their elegance.

There are, however, cases where the representation of a proof can be changed without changing the proof's idea. We describe the degree to which this is possible and the amount of "mental work" necessary to do so by *transferability*. This concept is certainly related to the concept of cognitive unity (Boero, Garuti, Lemut, & Mariotti, 1996; Pedemonte, 2007), but does not consider the transition from a person's argumentation to a formal proof, but the comparison of the products of proving processes in general. It stands out of question that, based on this vague definition, transferability cannot easily be quantified. It is nevertheless an important concept for understanding the role of proofs in different modes of representation. The dimension of transferability is relevant in mathematics as well as mathematics education, as the current *de facto* representation of choice for proofs is formal-symbolic and all other representation modes are measured as for their alignment to this representation (Brunner & Reusser, 2019).

4 Functions of simulation proofs

We follow De Villiers' (1990) extension of a categorization by Bell (1976) to analyze the functions of simulation proofs. De Villiers highlights the functions of explanation, systematization, discovery, verification, and communication. We split up the function of *verification* in relative and absolute conviction, to emphasize the difference between believing the statement to have a high probability to be true, which is relative conviction, and believing the statement to be surely true, which is absolute conviction (Weber & Mejia Ramos, 2015).

We identify several criteria particular to simulation-based proofs that influence the relative conviction function: *trust in the technology*, *level of detail*, and *limits of the representation*. These criteria add to the established ones for proofs in general as well as perhaps also to the criteria surrounding computer-based proofs.

4.1 Explanation

Explaining *why* a result holds is one of the main motivations for a formal proof and goes beyond the relative conviction *that* it holds. That relative conviction can be achieved by, for example, a lack of counterexamples after some searching, is also one of the criticisms of the use of dynamic geometry in educational settings; by being able to provide a large, seemingly infinite, number of examples, students are so convinced of the result they should prove that they no longer feel the need to prove it (Bolite Frant & Rabello de Castro, 2000). In the framework of Harel (2013), the students draw certainty from an "inductive proof scheme" instead of the educationally desired "transformational proof scheme".

On the other hand, numerous Proofs Without Words give examples of visual arguments explaining why a result holds. This shows that the capacity of a proof for explanation does not necessarily depend on the representation. "Proofs Without Words 2.0" denote animated versions of such proofs (Doyle et al., 2014), which can certainly be seen as being something between static visual proofs and simulation-based proofs, and retain the explanatory capacity of Proofs Without Words while being more interactive.

A mathematical simulation goes even further, giving the user more freedom to explore the phenomenon being reviewed and thus more opportunities to find an explanation. There are many examples of explanations using dynamic geometry software, which show that the problem addressed by the criticism above should be directed more to the overemphasis of the relative conviction aspect of proofs in education (Hanna, 1998), which leads to the question of "why" being overshadowed and thus neglected by students.

4.2 Relative conviction

While the notion of "relative conviction" was only put forward in (Weber & Mejia Ramos, 2015), the connection between exploration of mathematical situations using technology and conviction through quasi-empirical testing of the truth of a result is a well-explored concept (see the examples and references in the introduction). It is the main reason for the use of simulations such as MATHEMATICA to check statements by computation in research, and the employment of dynamic geometry software in education. In a such a setting, verification is often, especially in lower grades, understood as *quasi-empirical* verification providing relative conviction (Hanna, 1998), as opposed to a deductive proof, which would provide absolute conviction. We believe that simulation proofs can go beyond this kind of relative conviction and also provide absolute conviction by being more than just a collection of examples, however compelling, but providing insights and explanations as to why a result holds.

4.3 Absolute conviction

This aspect not only denotes the absolute conviction of the truth of the result, but also the conviction of the validity of the proof (which certainly implies the former). As there is no consensus on a definition of a proof, there are no generally accepted criteria for its validity (Hanna & Jahnke, 1996). Proofs have evolved through history and range over a wide variety of type, and while there are some techniques of proof accepted by most, such as mathematical induction or *reductio ad absurdum*, there is no one proof type that fulfills all needs and demands of every mathematician (McAllister, 2005). Thus, the question of the validity of a proof is a deeply subjective one, influenced by the ever-changing norms of the mathematical community (Sommerhoff & Ufer, 2019). This sort of change of norms is not unique to mathematics; quantum theory offering no visualizations led to some physicists not accepting it, until overwhelming evidence forced physicists to reshape their criteria for theory acceptance, abandoning the need for visual representations of phenomena (McAllister, 2005).

However, different criteria can be identified that may influence this absolute conviction to a varying extent. We propose that these be grouped into the following categories:

1. Mathematical-logical reasons, the personal understanding of the theorem, and an *a priori* judgment of its validity; the logical consistency of the proof.

2. Socio-cultural reasons, such as the trust in the author or in the mathematical community that examines the proof.

3. Technological reasons.

The first two categories contain established criteria for proof acceptance (Hilbert, 1931; Sommerhoff & Ufer, 2019; Yackel & Cobb, 1996) and are not the focus of this paper.

The third category of technological reasons is of particular importance for proofs in the context of computers, including mathematical simulations. Several criteria of this category can be identified.

4.3.1 Criterium 1: Trust in the technology

If a proof involves the work of a computer, then trust in the computer may be a factor in accepting the proof. This is a multi-faceted aspect, which encompasses several subcategories.

It can mean the trust in the operations done by a computer, such as in computer-assisted proofs. This can further be elaborated, as, for example, the original Appel-Haken proof of the Four-Color Theorem not only needed the trust in the operations executed, but also in the validity of the computer program generating the proof itself. This was eliminated only decades later with a formal proof in Coq (The Coq Development Team, 2019), reducing the trust to the Coq system and the computer operations (Gonthier, 2008).

The main problem with computer-based proofs, such as computer-assisted proofs, is often their length, which makes them *non-surveyable*, that is inaccessible for verification to the mathematical community (Tymoczko, 1979). But even if a proof seems surveyable, its acceptance by the recipient can still need trust in the technology, as the following example demonstrates.

If users drag a triangle in a DGE, the users most probably have the *expectation* that only the position of the triangle should change, not its area or other intrinsic features. Each time they perform the dragging operation, they can perform a visual inspection and check if the triangle has changed, strengthening their trust that dragging is a faithful representation of this translation they have in mind. They could further strengthen this trust by looking at the source code of the application, if accessible.

In some points, the users will check against their expectations and mathematical knowledge if the simulation is indeed one, that is if the mathematics are accurately represented. At other times, they will trust the software to be honest with them, and will adjust their expectations and mathematical knowledge according to their interpretation of the visualization given by the software.

4.3.2 Criterium 2: Level of detail

In a traditional paper-and-pencil proof, every argument can be broken down further, until the steps performed are small enough to be understood by the recipient or the axioms are reached. This strengthens the trust of the users in a proof, as they have the certainty that they could, in theory, probe every part of it.

In a digital environment, such as a mathematical simulation, the detail of a proof is limited by the resources the software provides. In addition to the interference of the trust, and with it, the conviction of the users in the proof, this massively inhibits the users in their own creativity, as they, opposed to the software designer, are not able to construct their own objects independently of the software's functionality, a limitation that can hopefully be overcome by future simulations.

A prime example is the "winding number tool" in Ariadne, which provides a way of computing the winding number of a path around a point. By doing this in a certain way, it not only inhibits the users to think of their own way to compute the winding number of a path, but also which invariant of a path to choose in the first place. Therefore, proving in such an environment is always only proving given these constraints.

4.3.3 Criterium 3: Limits of the representation

Every representation mode has its limits, which are independent of the technology used. This is a problem in some types of proof such as visual Proofs Without Words that can hamper proof acceptance (Bardelle, 2010). This factor is magnified in dynamic proofs in digital environments, as the representations of mathematical objects here are not directly generated by the prover, i.e. the mathematician. They are thus even more susceptible to error, as the possibility of a simulation going beyond the capabilities of its representation always exists (compare Sec. 2.4). The users know this, which can impede their relative conviction.

Instances of these limits are found readily, for example given a mathematical simulation plotting a function with a removable singularity such as $f : \mathbb{R}\backslash\{2\} \to \mathbb{R}$, $x \mapsto x^2$. A programmer would have had to think of a way to highlight this singularity, for example by drawing a small circle at $(2, 4)$, as no resolution would be sufficiently fine to show its existence. In such a way, the programmer must have thought about the representation of *every possible* function that might be plotted by the software.

The more general and powerful the simulation, the more cases have to be covered in beforehand. In the light of the research-level proving process, one is certainly in the realm of new mathematical objects and connections between them, which makes it all but impossible to construct the environment to account for all possible cases. Herein lies the special challenge for the designers of such mathematical simulations, but also for the user interacting with the software, who has to take this type of limitation into account.

4.4 Systematization

Systematization describes the organization of several results into a (deductive) system. This was certainly made easier with the standardized formal Bourbaki-style notation ubiquitous in mathematics today, allowing the description of results from different fields in the same language.

For a specialized piece of software such as a simulation designed to simulate a certain part of mathematics, the systematization of results from different fields is certainly hard to achieve. If the simulation uses visualization as its mode of representation, this may add to this problem, as at least today, generality and power of a simulation stand opposed to its intuitiveness and informality. This means that a visual simulation often tries to incorporate an informal and intuitive interface, which then limits its generality. Hopefully, future simulations will be able to overcome this limitation.

One could say that systematization is thus a weak point of a mathematical simulation.

Local systematization on the other hand is, however, very much possible; the organization of several definitions, lemmas, and theorems in one (sub-)field being more homogeneous in their representation. An example is a function plotter, which can very well classify different functions such as polynomials using their coefficients, or Ariadne systematizing paths on the plane by their homotopy classes, showing the relation between theorems on these objects.

4.5 Discovery

Discovery of new results is certainly not limited to proofs, but is an aspect relevant to proofs. The historical example of sphere eversions illustrates a discovery process in mathematics.

The eversion of the sphere is a regular homotopy turning $\mathbb{S}^2$ in $\mathbb{R}^3$ inside out. Constructions of such eversions were contrived by, for example, Shapiro (Levy & Thurston, 1995), Morin (turned into a video by Max (Max, 1977)), and Thurston (featured in a movie (Levy, Maxwell, & Munzner, 1994)[6]).

Visualization in itself can bring new ideas into mathematical research (Bartzos et al., 2018). Now imagine a software had been available to these mathematicians, making possible the deformation of manifolds shown in the videos by directly controlling their manipulation. It is quite possible that the construction of such an eversion would have been more accessible. This is even more plausible as the discovery of the later eversions correlates with the expanded use of visualization software.

In education, one of the main uses of dynamic geometry software is to facilitate the forming of conjectures in students by analyzing specific mathematical settings given by the teacher (Mogetta et al., 1999). This is certainly a part of the discovery aspect of proofs; by investigating arguments for the validity of statements, new statements are conjectured. Furthermore, there is no reason to believe this feature is only of use in education, given a software capable of representing objects of interest to research-level mathematicians, it would certainly be used to discover new results.

4.6 Communication

It is a central function of a proof to communicate mathematical ideas. In a simulation, this function encompasses both the communication with the mathematical community and the communication with the software itself. In both situations, the software provides a medium to express thought, which is strongly dependent of the representation form of the proof.

[6]There is even a more interactive animation available at `http://profs.etsmtl.ca/mmcguffin/eversion/`

The most used form to communicate proofs is by large a static and formal representation. To quote Mazur (2014) "As with almost all advances, something was lost," talking about losing "the public" (i.e. non-mathematicians) by this "code, unintelligible for the uninitiated." However, Victor (2014) argues that even more was lost: The use of formal notation restricts many human capabilities, forcing humans to use only a part of their cognitive abilities as they are confined to sitting in front of a small screen or piece of paper, manipulating it indirectly with a pen or a mouse creating static objects.

Simulation-based proofs may give the possibility to externalize thoughts in a way closer to the way we think. Touch- or gesture-based interfaces can express argumentation in a more *embodied* fashion, reacting dynamically to human input (Abrahamson & Bakker, 2016). While this may also be seen as an economical argument by allowing "more" work to be done, the idea is to achieve more *humane* means of communication.

Software proofs also allow, using the internet, a collaborative form of proving, by being able to manipulate objects simultaneously (Borba et al., 2017). While this has been a standard way of working in mathematics for a long time, this collaboration can be now made independent of physical restraints.

A mode of representation often chosen by simulations is a visual one, which has several influencing factors. Visualizations have a long history in mathematics, and in the acceptance of theories in general, having been seen a prerequisite or at least necessary component of a "proof" (von Fritz, 1955).[7] Also, visual representation modes of mathematics may be "closer" to the way we think, as mathematical thinking more often deals with images than with formulas, at least in some areas of mathematics (Hadamard, 1954). Hopefully, future simulations can unite the advantages of visual embodied communication with the power, precision and universality of formal-symbolic representation modes.

5 Conclusion

This article points out the role mathematical simulations can play in the context of proving. As this connection is not yet well explored and as proving is a core activity in mathematics, this has far-reaching implications for mathematics educational practice and research, as well as for mathematics research itself.

These implications require, in a way, that the presented simulation-based proofs can indeed be regarded as proofs. This issue can surely not be resolved in this general formulation, but will depend on the same criteria other proofs need to fulfill as well, such as their exact form and implementation, and the context of their use.

5.1 Implications for mathematics education practice and research

The development of new technologies and the programming of new software is changing the educational landscape. Decades ago, calculators replaced slide rules and logarithm tables in classrooms. Years ago, DGE and CAS have revolutionized teaching. Their impact on proof has been to give a preliminary exploratory and quasi-empirical step to increase relative conviction regarding a conjecture, without influencing the *actual* proof, the content and form of which has not changed. Now, with more mathematical simulations such as Ariadne appearing, the form of the actual proof is challenged. This makes it all the more important to do research in this area.

[7] See Giaquinto (2020), citing Gauss: "anybody who is acquainted with the essence of geometry knows that [the logical principles of identity and contradiction] are able to accomplish nothing by themselves, and that they put forth sterile blossoms unless the fertile living intuition of the object itself prevails everywhere."

As simulation-based proofs can fulfill all functions of proof, and can do so without the at times inhibiting factor of formulaic representation, they are of particular interest for educators in undergraduate courses. Here, the restrictions of creativity through design choices of the software (compare Sec. 4.3.2) can even be thought of as a feature, as such restriction can constitute an aid in doing proofs by narrowing the number of choices available. Note that it is not the purpose of simulation-based proofs, nor is it the question addressed in this paper, if simulation-based proofs are to be considered as "real" proofs, in the sense of being a perfect substitute for traditional symbolic proofs. Furthermore, such a discussion cannot be settled by theoretical debate, as the status of proofs depend on the norms of the community (compare Sec. 2.1). Such proofs can rather be a gateway into proving, giving an alternative *access* to proofs in a non-formal highly interactive setting. They may guide learners on their transition from mental argumentations in the sense of Mamona-Downs and Downs (2010) to formal proofs by giving them an appropriate environment to project their thoughts upon, maybe with the guiding rails of the affordances of the environment.

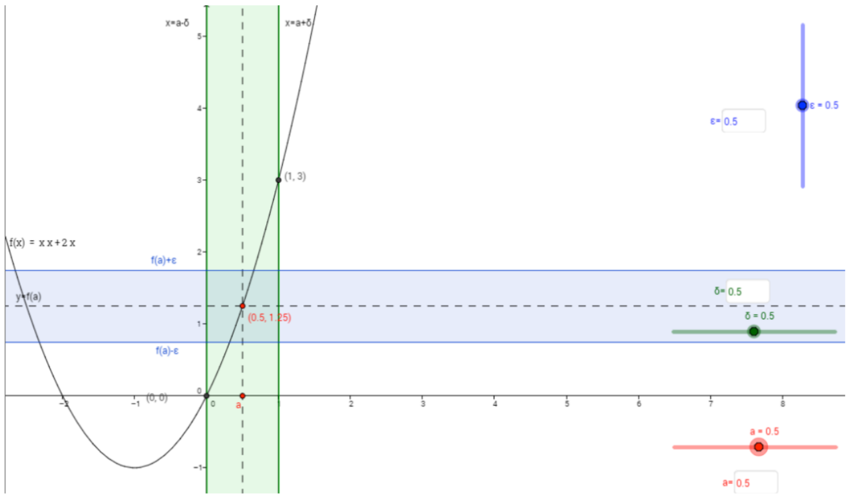


Figure 4: A GeoGebra-Applet letting the user explore the uniform continuity of some functions Dikovic (2017).

Examples may be the introduction of the concept of continuity through a suitable mathematical simulation software, for example giving a visual representation of the ϵ-δ definition of continuity, maybe along the lines of Fig. 4. This software might then be used to prove the continuity, or non-continuity of some functions. However, as described in Sec. 2.2 and Sec. 4.5, the software must be powerful enough to allow real discovery by being more than just an animation, but a mathematical simulation.

There may be an obstruction to the implementation of simulation-based proofs in educational settings; the absence, at times, of transferability or "parallelism" (Miller, 2012) to formal representation also implies a deficiency of connectivity with further study or even other fields of mathematics. In an educational setting, visual proofs or arguments are mostly made to support a formal proof, which currently is the gold standard in mathematics, and aid in its understanding. A visual proof thus has to be transferable into a formal proof to a certain degree, which may not always be possible (compare Sec. 3.4). It is however possible, with future development in mathematics visualization, that this can be made more balanced, or even reversed; a proof will first be provided visually, and translated into formal proof as an addition, for example for automated verification purposes.

Following the distinction into "learning to argue" and "arguing to learn" (Baker, Andriessen, & Schwarz, 2019), simulation-based proofs may not only be used to learn mathematics by proving in a mathematical simulation, but also for inciting a discussion with students on the nature of proof.

The debate on whether, or to which extent, the arguments made in a simulation constitute a proof, or how visual arguments relate to proofs in general in the sense of the above discussion, can be used to foster understanding on the concept of proofs.

The consideration of simulation proofs also has implications for mathematics education research. Approaching the subject from the students point of view, researchers investigating ways of teaching proofs or the learning of proofs may consider the use of simulations for either purpose. The implications of their use might then be assessed for their influence on the beliefs on the nature of proof. From the technological side, researchers working on the assessment or development of learning software might be driven to consider existing software for use in learning and teaching of proofs. For educational designers, requisites for the design of such environments must also be established.

5.2 Implications for mathematics research

Visual proofs are playing an ever larger role in research (Bartzos et al., 2018). As the possibilities of representing mathematics in computer environments as well as the possibilities to interact with computer-generated content continue to expand, this progress will surely sustain. We believe that simulation-based proofs may be a bridge between traditional proofs and experimental mathematics, combining the deductiveness of the former and the explorational capacities of the latter.

As this is highly dependent on available software, which is arguably harder to develop for research-level mathematics than for undergraduate or school mathematics in general, the long-term developments in this area are hard to predict.

5.3 Outlook

While simulation-based proofs can already fulfill all functions of proofs, we are still at the beginning of the digital age, so many more changes are to be expected. Further advances both in technology as well as representations will hopefully lead to simulations vastly more powerful while still being intuitive to use, realizing many of the features outlined in this article. This will undoubtedly lead to a shift in the use of technology away from purely exploratory capacities to other areas of mathematics practice. Research is and will be needed to understand which areas, such as proofs or problem posing and solving, these are, and how they are affected.

As remarked in Sec. 4.3, proof is not a static concept, but shaped by the community. It would be interesting to examine the opinions of research mathematicians as well as educators on the status of simulation proofs as acceptable proofs, in mathematics as a scientific discipline and as an item in the curriculum, and maybe identify further factors influencing their opinion.

Acknowledgments

A Examples of proofs in Ariadne

Theorem A.1. *Given the plane with one puncture $\mathbb{R}^2 \setminus \{0\}$, there exists a path γ with $W_\gamma(0) = 1$.*

Proof. See Sümmermann (2019b):

Table 2: The proof of Theorem A.1

	Step of proof	In Ariadne
1	Designate a point.	Touch the canvas
2	Construct a path. γ	Drag the point around somehow, such that it does not stay in one quadrant relative to the puncture, and then back to itself.
3	Compute the winding number of γ.	Click on winding number, then on γ, then on the puncture.

□

Theorem A.2. *Given the plane with one puncture* $\mathbb{R}^2 \setminus \{0\}$*, the path* γ *with* $W_\gamma(0) = 1$ *is not null-homotopic.*

This is a theorem asking for a proof of non-existence, as it postulates there cannot exist a homotopy between the path with winding number 1 and the constant path.

Proof. See Sümmermann (2019c):

Table 3: The proof of Theorem A.2

	Step of proof	In Ariadne
1	Construct a path δ.	Drag the point around somehow, such that it stays in one quadrant relative to the puncture, and then back to itself.
2	Compute the winding number of δ.	Click on winding number, then on δ, then on the puncture.
3	Compute the winding number of homotopic paths constructed by dragging δ.	Drag δ around and compute the winding number of the dragged path. If necessary, do this first for small dragging steps, and then for larger ones; see that larger drags are “made up” of smaller ones.
4	Conclude that the winding number is invariant under homotopy.	By understanding the construction of the winding number, and seeing how the dragging of paths influences this computation.
5	Show that δ is null-homotopic.	Drag the path to the puncture.
6	Construct another path γ around the puncture.	Drag the point around the obstacle.
7	Compute the winding number of γ.	Click on winding number, then on γ, then on the puncture.
8	As the two winding numbers differ, conclude that γ is not homotopic to δ.	By invariance of the winding number under homotopy.
9	Conclude that γ is not null-homotopic.	As a null-homotopic path has winding number zero.

□

The arguably most tricky step is Step 3, where the user has to convince herself/himself of the invariance of the winding number under homotopies. This seems to be a situation as described in Hanna (1998), but is decidedly different.

Hanna describes a student who convinces himself of the truth of a theorem, that the perpendicular bisectors of a triangle intersect at a point, by dragging the corners of the triangle, thereby deforming it, while the bisectors always intersect (technically speaking a homotopy (Lehrer & Chazan, 2012)). Here, Hanna argues that the student does not learn *why* this theorem holds.

We argue that the student has no chance to see why the theorem holds as the *construction* of the perpendicular bisectors does not give the same insight as their *defining property*: that all points on the bisector have equal distance to the endpoints of the bisected line segment.

In Ariadne, the winding number $W_\gamma(p)$ of a path γ around a point p is not given as the result of a calculation invisible to the user, but as the result of a construction following the definition. So, when users drag a path and recompute its winding number, they can see *how* the dragging of the path changed the computation of the winding number. This enables the generalization to the fact that the winding number is null-homotopic. This is an example of a use of a mathematical simulation in contrast to the construction above, as the users gain insight into the mechanism of action behind the winding number, and thus understanding and not just conviction.

Theorem A.3. *Given the plane with one puncture* $\mathbb{R}^2 \setminus \{0\}$*. If* γ *is the path with* $W_\gamma(0) = 1$*, then* $W_{\gamma^n}(0) = n$ *for all* $n \geq 1$*.*

Proof. See Sümmermann (2019d):

Table 4: The proof of Theorem A.3

	Step of proof	In Ariadne
1	Designate a point.	Touch the canvas
2	Construct a path with winding number 1, possible by Theorem A.1. γ	As detailed in the proof for Theorem A.1.
3	Redraw the same path to obtain a new path δ.	Drag the starting point of γ along γ.
4	Concatenate the two paths, obtaining a new path ϵ.	First touch γ, then δ, touching both simultaneously.
5	Compute the winding number of ϵ. If necessary, repeat from Step 3, concatenating a new path to ϵ.	Click on winding number, then on ϵ, then on the puncture.
6	Conclude that concatenating γ with a path increases its winding number by one.	Observe the construction of the winding number.

□

This proof relies on the user seeing how the winding number is computed. As the winding number is computed iteratively by analyzing how the path changes locally, the concatenation of paths results in the addition of winding numbers. This naturally and inductively generalizes to any number of paths n.

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2.5 Knotted Portals in Virtual Reality

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This paper presents KnotPortal, a software for the visualization of branched coverings of knots in virtual reality. The article not only describes the inner workings of the software, but also provides a short introduction into the mathematical theory of branched coverings and offers a rich collection of examples of branched coverings.

Knotted Portals in Virtual Reality

Moritz L. Sümmermann

Abstract

KnotPortal is a software for the visualization of branched covers of knots based on an idea by Bill Thurston (Thurston, 2012). It imagines knots made of a magical material which "rips the universe apart", leading to the creation of portals to other worlds. This makes possible the visualization of three-manifolds constructed through gluing of different sheets along the knot as a branching curve. To recreate the experience of "stepping through the knot" described by Thurston, our implementation allows users to explore these knotted portals in virtual reality using a head-mounted device with room-tracking. Users not in possession of such a device can alternatively use the software on a normal computer screen and with keyboard controls.

This article gives a short introduction into branched coverings and the history of branched covers of knots as well as the mathematical background to the ideas described by Thurston and used in the software. It also provides examples of branched coverings and the associated deck transformation groups, which are required as input for KnotPortal.

KnotPortal can be used to enable students to learn about knots, gluing, (branched) covers, or just to have a fun looking at portals and knots. It is open-source and available for free download at the website of the imaginary foundation at `https://imaginary.org/program/knotportal`.

1 Introduction

In a video titled "Knots to Narnia" (Thurston, 2012), Bill Thurston presents an approach to "visualize" the cyclic branched cover of a knot by interpreting the knot as a portal to other universes.[1] He demonstrates this using a wire to create different life-sized knotted portals. The wire is "magical" and, when its ends are joined, creates a "rip in the fabric of the universe," creating a portal from our world to a parallel world called "Narnia" in reminiscence of the novels by C.S. Lewis. The only rule governing the portal is that by circling around the boundary curve twice, one returns to the original world one started in. He then proceeds to explain the phenomena arising in the context of such portals by walking through this wire portal (see Fig. 1).

[1] The video was recorded by Tony Phillips as he asked topologists to do "demos" with knots. To his knowledge, Thurston was the first to illustrate this phenomenon of a branched world in this way.

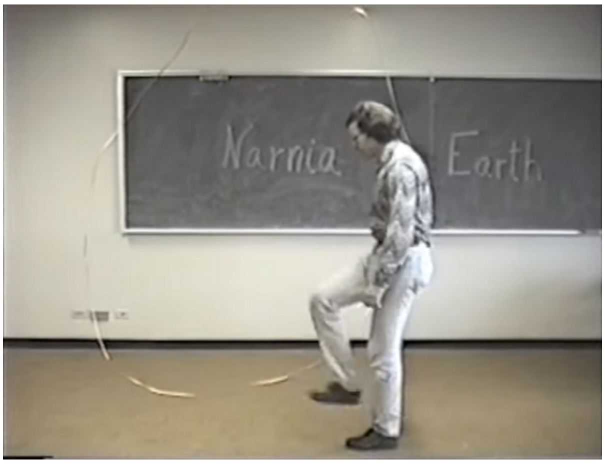


Figure 1: Thurston stepping through a portal generated by the unknot from Earth to Narnia.

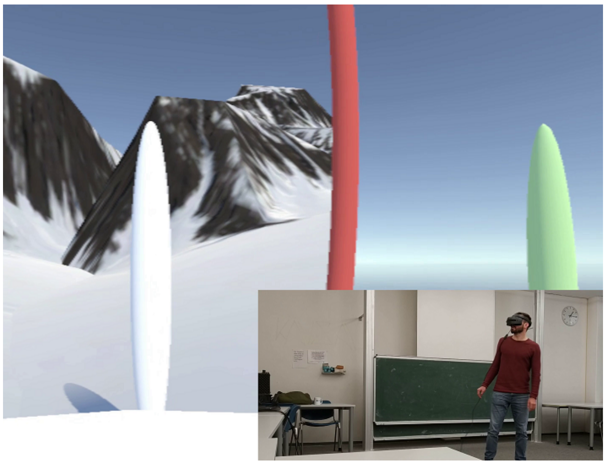


Figure 2: The author stepping through a portal given by the unknot. Screenshot from `https://youtu.be/Pgmfsl1e_vA`

This notion of a portal being generated by a ring-shaped object is a quite common theme in movies and videogames, and is mathematically quite simple. Thurston then proceeds to ask a question: What if the wire generating the portal was to be knotted? This leads to different regions in the knot, generating multiple portals. But how many different portals would be generated, and in how many worlds would they lead?

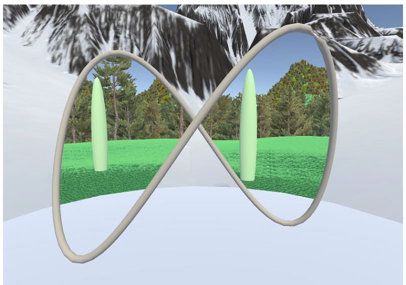


Figure 3: A twisted unknot in KnotPortal, showing two portals into a different world, as seen from the first (ice) world.

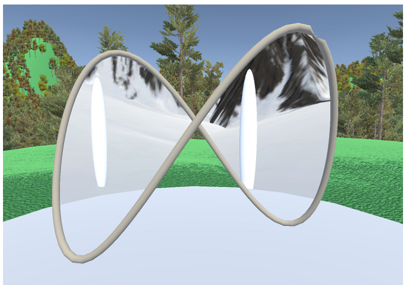


Figure 4: The same twisted unknot, now seen from the second (forest) world.

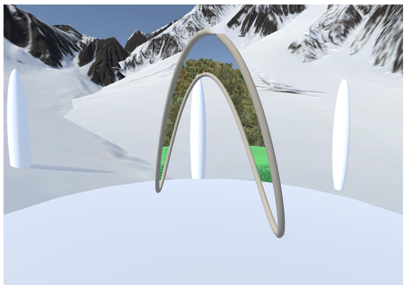


Figure 5: A sideways view of the twisted unknot, revealing why "both" portals must lead into the same world; there is in fact only one portal.

The object being studied is a cyclic branched cover of order 2. This means that a knot defines a gluing of several sheets of $\mathbb{R}^3$, by regarding it as a branching *curve*. Each world is cut along *surfaces* generated by the knot in a way specified in Sec. 5, and then glued together according to permutations subject to certain rules. This is analogous to the two-dimensional case, where one has branching *points* and cut *lines* in the construction of, for example, the complex logarithm (see Fig. 6).[2]

[2]This is also an explanation for the common cartoon trope "behind a stick", where a character vanishes by running around a tree. It is also what a "portal" in two-dimensional "Flatland" would look like.

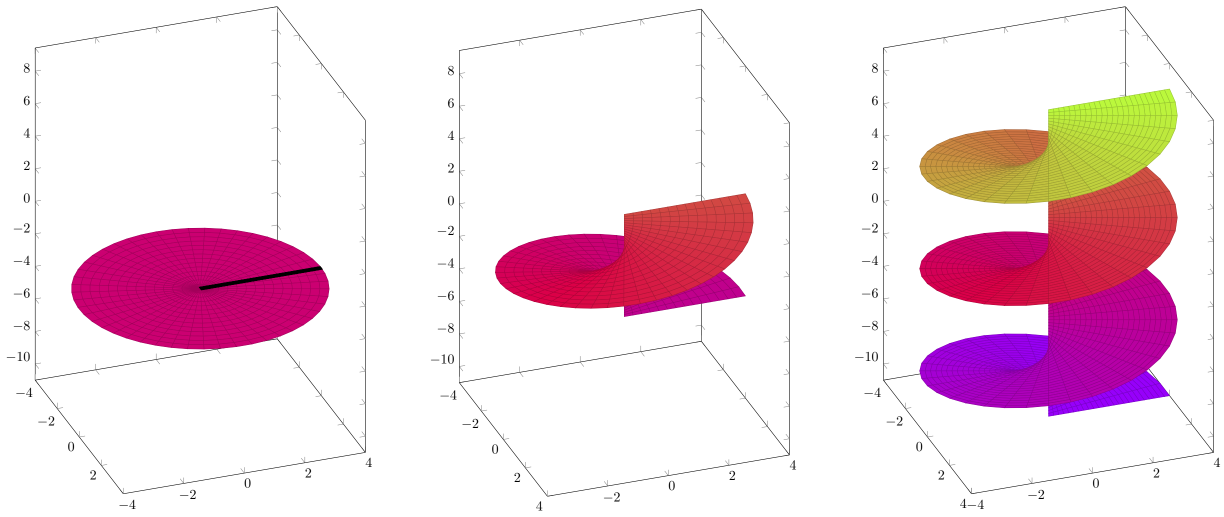


Figure 6: Creating a branched cover of the complex plane by first cutting from the branch point to infinity, and then gluing together copies of the cut surface along the cutting line.

This representation of branched covers of knots is fascinating, and for the unknot, it is easy enough to imagine.[3] If, however, the branching curve is knotted, it requires quite a lot of imagination to be able to picture these portals, even for simple cases. This gave the motivation to implement this vision as a computer program, to further recreate Thurston's experience of being able to step through portals as a virtual reality software, giving users the possibility to not only *see* these portals but actually be able to *walk through* them as Thurston did.

In this paper, we describe the implementation of this software and a description of the mathematics involved in the construction of the portals as well as the group structures given by them.

2 How to read this article

Sec. 3 gives details of previous work in recreating Thurston's idea. Sec. 4 contains a short introduction into branched coverings with some interesting examples. In Sec. 5, the software KnotPortal is described in detail. Finally, Sec. 6 provides examples of branched coverings and the corresponding deck transformation groups.

Readers only interested in the mathematical background of branched coverings of knots need only read Sec. 4 and maybe 6 for some examples. For understanding the project, all sections should be read in order, jumping to the examples in Sec. 6 on occasion. This last section is of particular interest to those wanting to add their own knots to KnotPortal, as it gives an algorithm for doing so.

Regardless the motivation, the reader is strongly advised to try out the software, or at least watch videos of its use, at `https://imaginary.org/program/knotportal`.

3 Project history

Previous projects also concerned with the modeling of branched covers of knots include the software "Polycut" by Ken Brakke (Brakke, n.d.). This software was designed "for visualizing multiple universes connected by a certain kind of wormhole," with the purpose of illustrating "the author's contention that soap films are best viewed as minimal cuts in covering spaces." In the software,

[3] Although it is not completely trivial: If you step through the portal defined by the unknot, and turn around, what do you see?

the user can view different knots and links and some of their branched covers as differently colored regions, as well as soap films, which are the minimal surfaces separating the sheets.

We wanted to achieve something different, as our goal was to give a real "world" instead of just colors, as well as to realize a virtual reality experience.

There was an attempt to achieve this by porting Ken Brakke's code to CAVE virtual reality technology by George Francis, Alison Ortony, Elizabeth Denne, Stuart Levy and John Sullivan during the illiMath2001 research program, however, this attempt remained unfruitful: "Though a complete solution to this visualization problem still eludes us, extensive geometrical documentation and evaluation of extant software was undertaken this summer and presented as a PME talk at MathFest, Madison, WI.", as reported at `http://new.math.uiuc.edu/oldnew/im2001/`.

In this project, we achieved our goal through a new software called KnotPortal, by using the combination of a game engine and a head-mounted virtual reality device capable of room-scale tracking (see Fig. 2). In our software, the user can move around in a fully immersive experience featuring different real worlds. It is adaptable as new knots can easily be added, and a non-VR version for use with a normal desktop computer can be used if a VR-headset is not available.

4 Mathematics background

4.1 Branched coverings

While this section gives a short overview on branched coverings, interested readers in this topic might want to consult a more comprehensive resource. Most standard textbooks on algebraic topology will do.

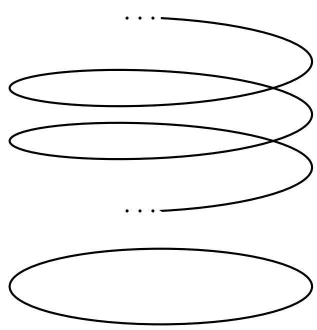


Figure 7: The real line $\mathbb{R}$ coiled over the sphere $\mathbb{S}^1$. Every neighborhood of a point on the sphere has countably infinite many copies above it

.

A *covering map* is a map p from a "covering space" E to a "base space" X, such that for any $x \in X$, the pre-image $p^{-1}(U_x)$ of any neighborhood U_x of x is a disjoint union of open sets $\tilde{U}_{i \in I}$, with $\tilde{U}_i$ homeomorphic to U_x for every $i \in I$. The cardinality of the index set I is also called the degree of the cover. In words, this means that every part of the base space has copies of itself "above" it. Besides the trivial covering of the disjoint union of copies of a space covering the space itself, the classical example is the "exponential spiral". It is defined by the covering map $p : \mathbb{R} \to \mathbb{S}^1$ from the covering space $\mathbb{R}$ to the base space $\mathbb{S}^1$, $p(t) = \exp^{2\pi i t}$, see Fig. 7.

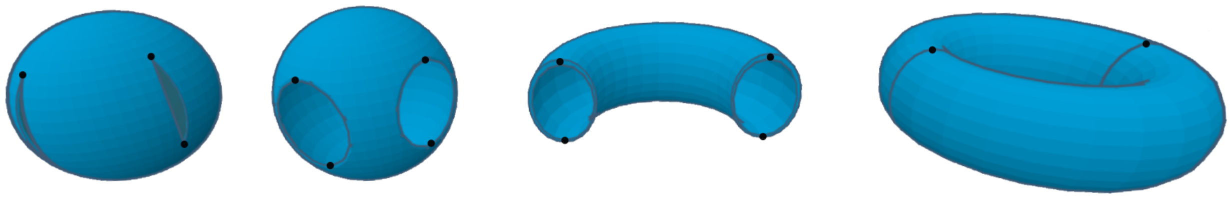


Figure 8: From left to right, the construction of a double cover of the sphere by the torus, by cutting up a sphere and gluing it to a copy of itself. Marked in black are the four branching points. (Screenshot from TinkerCAD)

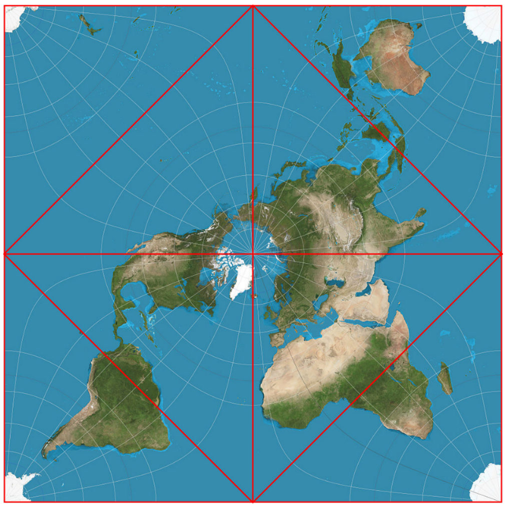
If the assumption of *every* point being covered as described above is relaxed to *most* points, one obtains *branched covering maps*. To be precise, a map p is a branched covering map if it is a covering map for all points but those in a nowhere dense set $S \subseteq B$, the set of *branch points*. Here, a classical example is the complex logarithm used as a countably infinite cover of the complex plane, giving rise to the "logarithmic spiral" in Fig. 6.

Another example is depicted in Fig. 8 and describes the construction of a branched double cover of the sphere by a torus with four branching points. A sphere is cut twice, which is homeomorphic to an open cylinder or a half-torus. Two half-tori are glued together to yield a torus, so that every point on the sphere except the points on the cuts has a corresponding point on each half-torus.

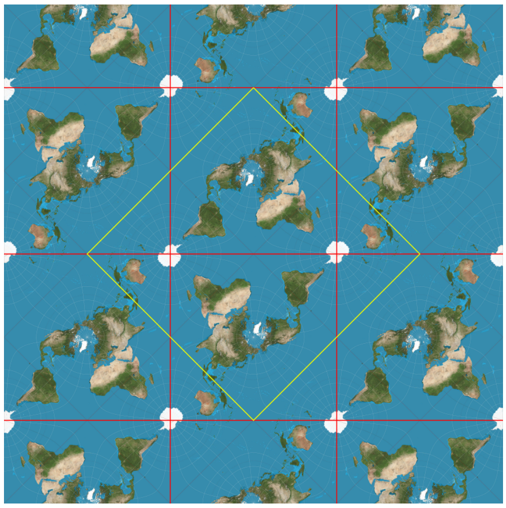
A second somewhat more complex example of a branched cover of the sphere by a torus with four branching points is given by the Peirce quincuncial projection (see Fig. 9, or consult Baez (2006) for a complete explanation). In this case the two disjoint arcs that we cut along in the previous example are chosen to cross each other. This can be visualized by projecting the sphere to an octahedron, and then unfolding the octahedron by cutting all edges adjacent to a vertex on the square equator. The flattened version gives a square with the south pole at all corners (See Fig. 9a). This square can tile the plane by point reflection on the midpoint of the sides, as shown in Fig. 9b. This then defines a branched double covering of the sphere by the torus, depicted in Fig. 9d, with covering space the torus and base space the sphere.

Every point on the globe is present on the torus twice, except for the branch points, which are only present once. Going around one of the branching points in the covering space also means going around the point on the globe twice. This is not apparent on the map, as Peirce placed the branch points in oceans, making them less visible.

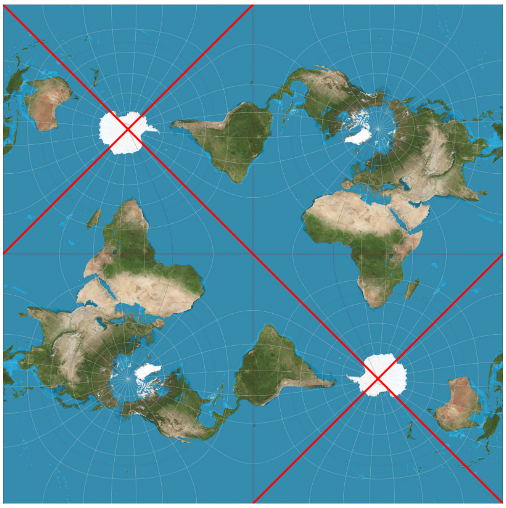
Branched covers of the sphere, such as the ones presented in these examples, are ubiquitous, as made precise by the Riemann existence theorem: every Riemann surface is the branched cover of the sphere Harbater (2015).



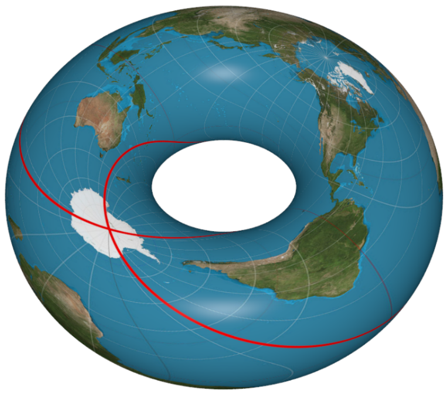
(a) The map projection into a square. The red lines indicate the edges of the octahedron, which is obtained by folding the triangles at the corners backwards. Image from Strebe (2012).



(b) Tiling of the plane by the maps, which are marked in red.



(c) The yellow fundamental region of the torus from Fig. 9b, with identical corners and identical opposing sides. The four branch points are now at the corner, the center, and the two midpoints of the sides.



(d) The fundamental region wrapped on a torus. The red lines both connect the Antarcticas. Image generated with Persistence of Vision Raytracer Pty. Ltd. (n.d.).

Figure 9: Different views of the Pierce quincuncial map projection.

4.2 History of the relationship between knots and branched coverings

Knots are everywhere in our world, and applications of knot theory range from understanding why headphones get tangled spontaneously (Raymer & Smith, 2007) to phenomena in quantum physics (Planat, Aschheim, Amaral, & Irwin, 2018). Although knots are found throughout human history, such as the famous Gordian Knot, their modern mathematical study first began in the 18$^{\text{th}}$ century by Vandermonde (Vandermonde, 1771) and rised together with topology (Przytycki, 2007). The first applications of known mathematical methods to knots came with Poincarï¿½'s *Analysis Situs* (Poincaré, 1895). Heegaard used topological methods to compute the 2-fold branch cover of the trefoil knot (Heegaard, 1898), but did not use the result to discriminate the trefoil from the unknot, as this now central problem of knot theory was not of interest to him and was only proved by Tietze in 1908 using the fundamental group (Stillwell, 1980, p. 226). He used the cover to construct "Riemann spaces," analog to the construction of Riemann surfaces in one dimension higher (Stillwell, 2012).

Alexander then proved in Alexander (1920) that "*Every* closed orientable triangulable n-manifold M is a branched covering of the n-dimensional sphere", an extension of branched coverings of spheres of the Riemann existence theorem. The theory was even further developed when Hilden, Lozano, and Montesinos (1983) provided a *universal knot*, a knot such that every 3-manifold is a branched cover of the sphere with the knot as a branching set.[4]

The knot itself came into the center of attention when Wirtinger extended Heegaard's results and, together with his student Tietze, used the construction to compute a presentation of the fundamental group of the knot complement for every knot (Epple, 1999). The knot group is thus a result of considerations of branched coverings of knots.

5 Software

The software was created with Unity3D (Unity Technologies, 2017), the virtual reality gear is HP Mixed Reality[5]. Scripts are in C# or, for the shaders, in DirectX 9-style HLSL. The deck transformation groups determining the gluing of the worlds as quotients of the respective knot group, as well as the associated multiplication tables were computed with the help of GAP (GAP, 2019).

5.1 Input

As input, the software is given a knot through some parametrization, as well as a group multiplication table which can be generated with GAP. Examples for knot parametrizations together with group multiplication tables are given in Sec. 6. The software further needs a map defining which "cone segment" (see below) gets assigned to which group element, the generator-to-cone map.

5.2 The setting up of the cut surface

At the start of the program, the following steps are carried out.

1. Build all needed worlds

2. Set up a camera in each world, moving and rotating as the player camera moves and rotates.

3. Let each camera render to a full-screen sized texture, and assign the textures to the post-processing shader.

[4]For a more complete history, consult Artal, Costa, and Izquierdo (2017).

[5]This is not to be confused with augmented reality; Mixed Reality is just the brand name Microsoft has given its virtual reality technology.

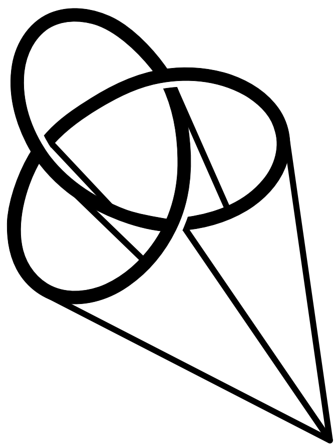


Figure 10: The Heegaard (Heegaard, 1898)/Reidemeister (Reidemeister, 1932) cone construction for the trefoil knot. The cone has three self-intersecting lines, resulting in three cone segments (as depicted in Stillwell (1980)).

Then, in the first world, we apply the cone construction from Heegaard (1898) to the knot, see Fig. 10. The goal is to provide a cut surface for the gluing of the worlds. This is analogous to the cut *line* given in the construction of the domain of the complex logarithm in Fig. 6. In our case, we cut from the branch *curve* to a point at "infinity" (in the implementation a point sufficiently far away) so that the knot is in general position from its point of view. This defines a cone or cylinder[6] and glue together the different worlds along the cutting *surface*.

1. The knot is placed in the world as a tubular mesh around a Catmull-Rom non self intersecting closed spline, given the control points from the discretized parametrization.
2. A point p is chosen, from which a normal knot projection is obtained.
3. A cone is built from this point by building a mesh formed by the triangles obtained through filling all line segments from p to every start and end of the line segments of the knot. This results in a sort of cone, possibly self-intersecting.
4. The cone is cut along the intersections, leading to a number of mesh pieces. These are duplicated and the duplicated has its normals flipped to give a backside.
5. Each "cone segment" is assigned a generator of the group according to the provided generator-to-cone map. Its backside gets assigned the inverse of the generator.

Now, in each frame, if the knot is visible, perform the following steps on the CPU:

1. Transform the knot's anchor points from world space into screen space.
2. Using the line segments, divide the screen space into polygonal regions by an algorithm of de Berg, Cheong, van Kreveld, and Overmars (2008).
3. Find a central point in each region using a C# port of the "polylabel" algorithm from `https://github.com/mapbox/polylabel` to find the pole of inaccessibility of the region.
4. Raycast each point from the camera, multiplying the current world generator with every generator from a cone segment encountered along the way. In this way, build a map assigning a generator to each polygonal screen region.

[6] also called Reidemeister's cylinder (Epple, 1999)

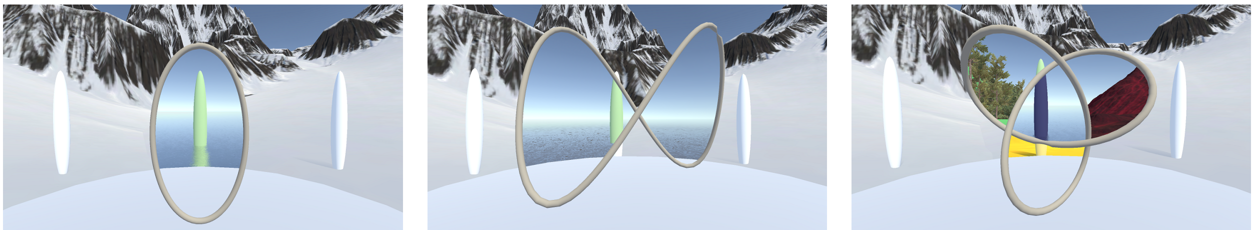


Figure 11: Portals through the unknot, the twisted unknot, and the trefoil knot

Then run the following steps in the post-processing shader:

1. For each pixel, perform an optimized[7] point-in-polygon test.
2. Assign the pixel the pixel from the camera texture of the world corresponding to the polygon's generator.

5.3 Player teleportation

In each frame, perform a raycast from the players old position to his new one. Multiply the current world generator with every cone segment's generator encountered by the raycast, giving the new world. Teleport the player to the point in the same place, but the new world.

This implies that in contrast to expectation, teleportation occurs much later (or earlier, depending on the direction of approach to the knot) as one might think. It does not happen as one "passes through the portal," but as one passes through the cut surfaces, i.e. the cone segments, which are the "real" portal.

5.4 World design

The software comes with a two different sets of worlds, *simple* and *real* ones. The simple worlds are featureless colored places to enable low-end hardware to run the program, and for a more minimalist experience.

The other kind are the real worlds (such as in Fig. 2), which give the more rich experience. They were designed with several goals in mind. Firstly, they should be interesting enough to give the user a real motivation to step through the portal and look into other worlds. Secondly, they should not be too interesting, as to keep the focus of the experience on the knot and the portals, and not the world. The worlds are also color-coded, to enable the user to speak about "the white world" or "the blue world," which is also helpful in keeping the worlds apart, as well as easing the transition between simple and real worlds. The color codes where taken mainly from naturally occurring colors, with the addition of some colors not present on this planet but possible on other ones (Kiang et al., 2007).

6 Example cases

These cases all describe branched covers of order 2, i.e. the knot as the branching curve has order 2. So a path going around a knot segment twice is back in the same world (sheet) it started in.

In general, the construction of the deck transformation groups is well-known. Given a (based) cyclic branched covering $p : (E, e_0) \to (X, x_0)$ the deck transformation groups can be computed through the Wirtinger presentation together with the fundamental theorem of covering spaces.

The Wirtinger presentation gives the generators of the knot group as loops around the knot strands, together with relations between them for every crossing of the strands.

[7] Optimized by first checking if the pixel lies in a bounding box around the polygon, or in a circle of small enough radius around the pole of inaccessibility of the region.

The fundamental theorem then states that the deck transformation group is isomorphic to

$$\pi_1(X, x_0)/p_\star(\pi_1(E, e_0)) \tag{1}$$

. Given a presentation

$$\langle g_1, \ldots, g_n \mid R_1, \ldots, R_m \rangle \tag{2}$$

of the knot group, as the covering is cyclic, we have

$$p_\star(\pi_1(E, e_0)) \cong \langle g_1^{k_1}, \ldots, g_n^{k_n} \mid R_1, \ldots, R_m \rangle \tag{3}$$

for some coefficients $k_1, \ldots, k_n$. As we restrict ourselves to branched covers of order 2, the coefficients are all 2.

6.1 Unknot

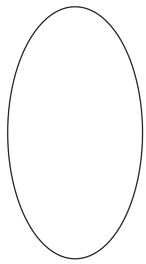


Figure 12: The unknot under the z-projection

For the unknot K, the knot group is $\pi_1(\mathbb{S}^3 \setminus K)$ which is $\pi_1(\mathbb{S}^1 \times D^2) \cong \mathbb{Z}$ with presentation $\langle a \rangle$. Taking the quotient of this group and the subgroup $\langle a^2 \rangle$, which is the induced by the fundamental group of the covering space, as the simple generating loop has to go around the unknot twice before returning to the basepoint. This results in the presentation $\langle a \mid a^2 \rangle$. This is thus a two-fold covering with deck transformation group $\mathbb{Z}_2$, or equivalently the (Coxeter) group A_1.

The unknot is represented in the software through the parametric equations

$$\begin{pmatrix} 0.8 \sin t \\ 1.5 \cos t \\ 0 \end{pmatrix}$$

, generates $|A_1| = 2$ worlds, and has 1 portal. The group multiplication matrix of A_1 is $\begin{pmatrix} e & a \\ a & e \end{pmatrix}$. As the cone associated to this knot has no self-intersections, the generator-to-cone map is trivial, assigning every cone segment the group element a.

6.2 Twisted Unknot

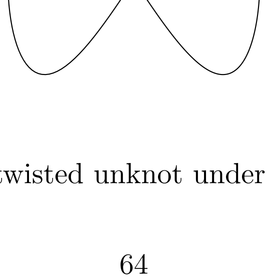


Figure 13: The twisted unknot under the z-projection

This case is of course the same as the unknot from a knot theoretical standpoint.

As for the implementation, the knot is given by

$$\begin{pmatrix} 2\sin(t+1) \\ 3\sin(t+1)\cos(t+1) \\ \sin t \end{pmatrix}$$

, but as there are two portals leading to the same world, the generator-to-cone map assigns a to both cone segments.

6.3 Trefoil knot

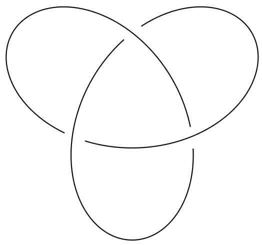


Figure 14: The trefoil knot under the z-projection

For the trefoil knot K, the knot group is $\langle a, b \mid a^3 = b^2 \rangle$ as the trefoil knot is the $(2, 3)$ torus knot (Stillwell, 1980). Alternatively, it can be given by $\langle x, y \mid xyx = yxy \rangle$ (Rolfsen, 2003, p. 61). By using $xyx = yxy \cong xyxxyx = xyxyxy \cong yxyxyx = (xy)^3 \cong yxxyxx = (xy)^3 \cong (yxx)^2 = (xy)^3$ we can see the isomorphism between the two presentations. Adding the relations x^2 and y^2, we obtain the presentation $\langle a, b \mid (xy)^3, x^2, y^2 \rangle$. This is the dihedral group of the triangle, and a Coxeter group with Coxeter matrix $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. The group order 6 implies the construction of 6 worlds from this knot. In general, the r-fold branched covering of the torus knots of type (p, q) is a Brieskorn manifold $M(p, q, r)$, the intersection of the 5-sphere $\mathbb{S}^5$ in $\mathbb{C}^3$ with the equation given through $z_1^p + z_2^q + z_3^r = 1$ (Planat et al., 2018).

In KnotPortal, the trefoil knot is represented through the parametric equations

$$\begin{pmatrix} \sin t + 2 * \sin 2t \\ \cos t - 2 * \cos 2t \\ -\sin 3t \end{pmatrix}$$

. The group multiplication matrix of D_3 is

$$\begin{pmatrix} e & a & b & c & d & f \\ a & e & d & f & b & c \\ b & f & e & d & c & a \\ c & d & f & e & a & b \\ d & c & a & b & f & e \\ f & b & c & e & a & d \end{pmatrix}$$

. The generator-to-cone map assigns the elements a, b, and c to the three cone segments, respectively.

The relationship between the group and the portals of the trefoil knot is detailed in Fig. 15

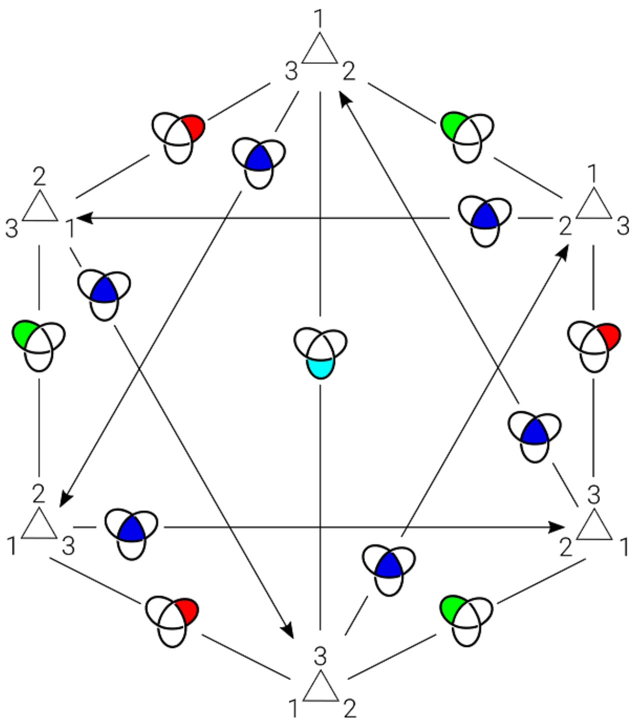


Figure 15: The relationship between the elements of the symmetry group of the triangle D_3 and the portals generated by the trefoil knot, after the drawing in Thurston (2012). The outer portals correspond to reflections, the inner portal to a rotation.

6.4 Figure eight knot

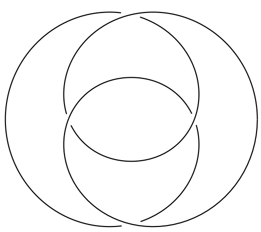


Figure 16: The figure eight knot under the z-projection

The presentation of the figure eight knot is $\langle x, y \mid x^{-1}yxy^{-1} = yx^{-1}yx \rangle$ (Rolfsen, 2003, p. 58). Again adding the relations x^2 and y^2, one obtains $\langle x, y \mid (xy)^5, x^2, y^2 \rangle$, which is again a Coxeter group, namely H_2, which is of order 10. This knot thus generates 10 worlds.

In the software, it is represented through

$$\begin{pmatrix} (2+\cos 2t)\cos 3t \\ (2+\cos 2t)\sin 3t \\ \sin 4t \end{pmatrix}$$

The group multiplication table is

$$\begin{pmatrix} a & b & c & d & e & f & g & h & i & j \\ b & a & d & c & f & e & h & g & j & i \\ c & j & e & b & g & d & i & f & a & h \\ d & i & f & a & h & c & j & e & b & g \\ e & h & g & j & i & b & a & d & c & f \\ f & g & h & i & j & a & b & c & d & e \\ g & f & i & h & a & j & c & b & e & d \\ h & e & j & g & b & i & d & a & f & c \\ i & d & a & f & c & h & e & j & g & b \\ j & c & b & e & d & g & f & i & h & a \end{pmatrix}$$

6.5 Solomon's Seal knot

This is the $(5,2)$-torus knot. Its parametric equation is thus given by Von Seggern (2016):

$$\begin{pmatrix} (3+\cos 5t)\cos 2t \\ (3+\cos 5t)\sin 2t \\ \sin 5t \end{pmatrix}$$

and the presentation of its group is $\langle x, y \mid xyxyxy^{-1}x^{-1}y^{-1}x^{-1}y^{-1} \rangle$ (Livingston, 1993). After adding the relations for the generators, the order two covering group of this knot is thus the same as for the Figure eight knot.

6.6 Hopf Link

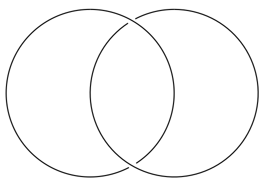


Figure 17: The Hopf link under the z-projection

Each of the branching curves gives a generator, and the two commute, so the deck transformation group is $\langle a, b \mid a^2, b^2, (ab)^2 \rangle$. This group is a Coxeter group with matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, which is $\mathbb{Z}_2^2$, or equivalently, A_1^2. This results in 4 worlds and 3 portals.[8]

Acknowledgments

Thanks to Marc Sauerwein, John Sullivan, and Roice Nelson for their valuable advice.

[8] At the present time, the support of links is not implemented in the software, but could certainly be achieved without much change to the methods.

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2.6 Embodied Mathematics: Forming Concepts in Topology by Moving Through Virtual Reality

Authors: Moritz L. Sümmermann, Benjamin Rott
In review, submitted to a journal on October 13, 2020.

This article exhibits opportunities of embodied learning enabled through KnotPortal, by giving a theoretical analysis coupled to a case study. As a secondary objective, this article aims to provide a template for such a study into embodied learning in mathematics, going beyond general educational frameworks.

Forming Concepts through Embodied Mathematics in Virtual Reality. A Digital Environment Analysis and Case Study

Moritz L. Sümmermann, Benjamin Rott

October 14, 2020

Abstract

KnotPortal is a virtual-reality-based software for experiencing embodied mathematics based on an idea by mathematician and Fields medalist Bill Thurston. In the software, the user can travel between worlds using knotted portals, describing the mathematical idea of branched covers in three dimensions. This article presents a description of the software along with a case study with a mathematician and a student, delineating how users interact with this special topic in mathematics in such a highly immersive environment using full-body locomotion. By analyzing the role of embodied cognition in elucidating the presented structures, we demonstrate that the study of embodiment in a mathematics learning environment must be specific to the mathematical content and its representation. These claims are backed by examples of the relationship between the users' actions and the mathematical interpretation. We further discuss possible detrimental effects of immersion, leading to a loss of overview, and thus hindering the learning process; it seems that distance from a problem may help solving it.

1 Introduction

> Children are not taught to read or write music at all – they sing, listen, and move their bodies to the sound of music. [. . .] So why are children still taught mathematics as a paper and pencil exercise [. . .]? For most of us, mathematics, like music, needs to be expressed in physical actions and human interactions before its symbols can evoke the silent patterns of mathematical ideas [. . .].
>
> Skemp (1971/2012)

Mathematics is often seen as being the *product* of an *abstract mind*. But the era of this Cartesian duality, the mind being disconnected from the body, is over, as provocatively stated by Macedonia (2019). Instead, learning is now often considered in a more holistic way, with embodied learning receiving considerable attention. This theory focuses on the relationship between the learner's actions and cognition and is gaining traction in the educational sciences, including mathematics education (Kosmas, Ioannou, & Zaphiris, 2018). In education, the main ideas of embodiment are not new and known at least since Piaget placed bodily movement at the center of his theory of cognitive development, a theory which, however, propagated the pursuit of "abstract" thinking and saw action-based cognition only as a stepping stone for the former (Ionescu & Vasc, 2014; Marshall, 2016).

This poses a challenge to us educational researchers and especially educational designers to develop environments supporting embodied learning of mathematics (Abrahamson & Bakker, 2016), especially for use outside of traditional classroom settings (Albirini, 2007). The development is being facilitated by technological advancements, widening the range of options to provide such environments (Georgiou & Ioannou, 2019; Lindgren & Johnson-Glenberg, 2013; Tran, Smith, & Buschkuehl, 2017). This starts with touch screen devices, enabling the interaction of the user with gestures, and goes to fully immersive virtual reality environments, where users can use the full range of their bodily movements to interact with the content (Johnson-Glenberg, Birchfield, Tolentino, & Koziupa, 2014; Lindgren & Johnson-Glenberg, 2013; Malinverni, Pares, Malinverni, & Pares, 2014; Price, Yiannoutsou, & Vezzoli, 2020).

Of course, not all movement, even if situated in a mathematics-rich environment, results in embodied learning. The special challenge is not to construct environments where movement is possible or even encouraged, but where it *makes sense* in a content-related way (compare Lindgren and Johnson-Glenberg (2013)).

In this paper, we present a learning environment which presents an opportunity for embodied learning of mathematical concepts. We argue why this environment is capable of providing embodied learning by analyzing the way the user can interact with the presented mathematics using *paths* as well as the new perspectives on knots given by the software. We complement the theoretical analysis by presenting a case study of user interaction with the environment, giving proof of "embodied heuristics" playing a role in exploration in this environment.

The digital environment is KnotPortal[1], a virtual reality software where users can experience a special discipline of mathematics in an embodied fashion as an example of a learning environment as described above. It consists of a virtual reality headset which fully immerses users in a world centering on *knotted portals*, letting them explore this concept from the mathematical topic of "knot theory" using movement of head and body. Knot theory is well suited for investigating embodied learning (compare McCallum (2019)), as most users do not have previous knowledge of the subject from their school education, but it its fundamentals are easy enough to be grasped in a short time. Many operations in knot theory also have a "bodily" component given the physical nature of the objects of study (Freitas & McCarthy, 2014).

The aim of this article is to provide a substantial example of the use of technology to enable embodied learning in mathematics, as to contribute to research not lacking in theories but in actual implementations. This environment is then also to function as a counterweight to those which do not provide deep-rooted content-related affordances for embodied learning, such as projects which use virtual or augmented reality without need, simply duplicating reality (Lindgren & Johnson-Glenberg, 2013). The offered case study presents a prototypical investigation into a mathematical learning environment designed for embodied learning.

2 Theory

2.1 Theory of Embodiment

Embodiment is a relatively new school of thinking in cognitive sciences and especially mathematics education (Gerofsky, 2015), but drawing on a long history of debate on the subject of the relation between mind and body (Johnson-Glenberg, 2014). It provides a contrast to the Descartian dichotomy of body and mind by rejecting this division and replacing it with a more holistic view: "Physical movement [...] is not the executive arm of an abstracted intelligence. Rather, moving is situated in dynamical cognition" (Abrahamson & Bakker, 2016). The theory of embodied cognition postulates that bodily actions are not the consequence of mental processes, but that this is in fact reversed, as "mental concepts" are shaped by our actions and experiences (Johnson-Glenberg et al., 2014). This can demonstrate itself in metaphors, such as the temperature "warm" indicating emotional proximity, which is shaped by our experience of warmth of human bodies (Lakoff & Johnson, 2008), or the "grasping" of a concept. Another example would be studies showing brain activity in regions associated to the physical actions when having participants listening to words

[1] Available at `https://imaginary.org/program/knotportal`

such as “lick,” “pick,” or “kick” (Hauk, Johnsrude, & Pulvermüller, 2004), indicating the embodied nature of the understanding of these words.

Mathematics learning also offers many examples of such influence of actions on “mental concepts.” Some research is specific to the mathematical content, such as the importance of fingers in counting (Fischer & Brugger, 2011; Soylu, Lester, & Newman, 2018). More general research shows that some actions such as gestures play a central role in doing and understanding all mathematics (Alibali & Nathan, 2012; Goldin-Meadow, Kim, & Singer, 1999).

There exist several theories proposing a taxonomy to classify research or projects in embodiment. Examples are the framework of Johnson-Glenberg et al. (2014), stating the dimensions of *amount of motoric engagement*, *gestural congruency*, and *perception of immersion*. The amount of motoric engagement simply describes how much users use their body while engaging in the task, ranging from seated activities where only the finger or the hand of the user moves, to activities promoting full-body locomotion. Gestural congruency describes “how well-mapped the evoked gesture is to the content to be learned” (Johnson-Glenberg, 2014, p. 282). There is no reason this category cannot be extended to account for non-gesture movements, so we will talk about *congruency* instead, referring to how well-mapped bodily movement in general is to content. Lastly, *perception of immersion* is concerned with how well the user is immersed in the environment providing embodied learning, and is specific to technology. It ranges from small screens to virtual or mixed reality environments.

Another taxonomy is proposed by Skulmowski and Rey (2018) along the dimensions of *bodily engagement* and *task integration*, where the former roughly corresponds to *amount of motoric engagement* and the latter, *task integration*, describes describes “whether bodily activities are related to a learning task in a meaningful way or not,” which is to be seen as distinct to congruency. Skulmowski and Rey (2018) define task integration to be more general, as it does not distinguish at the level of concrete examples but more of study design.

Regardless of the details of the taxonomy, high relatedness of task to activity and high body engagement have been associated with more positive learning outcomes and are thus to be seen favorably (Brooks & Goldin-Meadow, 2015; Goldin-Meadow et al., 1999; Johnson-Glenberg et al., 2014; Lindgren, Tscholl, Wang, & Johnson, 2016; Malinverni et al., 2014; Ruiter, Loyens, & Paas, 2015).

2.2 Description of the software KnotPortal

The notion of having portals to other worlds is not new, and a common theme in movies and computer games. A portal is defined by its boundary, which is a ring-shaped curve in space. We will call this the “boundary curve.”

In a video titled “Knots to Narnia,” the mathematician and Fields medalist Bill Thurston gives an example of such a portal (Thurston, 2012) (see Fig. 1). This portal is subject to a rule: passing “through the portal,” i.e. going around the curve defined by a knot, changes the world, but by circling around the boundary curve twice, one returns to the original world one started in. He then raises a question: What happens if this curve is not a simple loop, but is knotted?

In this way, a knot gives rise to different “universes” or “worlds,” being created through the “rip in the fabric of the universe” defined by the knot as a boundary curve of the portal(s). He exemplifies this construction by discussing the portal structure of the trefoil knot (for diagrams of this knot, see Fig. 4 or 9).

Figure 1: Thurston stepping through a portal generated by the "unknot" (see Fig. 3) from Earth to Narnia. Screenshot from a video available at `https://youtu.be/IKSrBt2kFD4`.

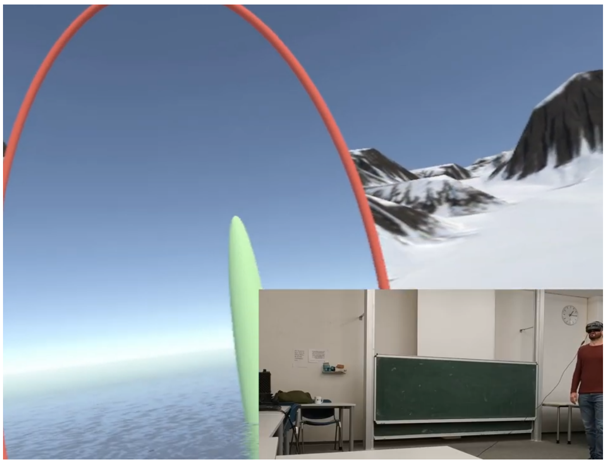

Figure 2: The first author stepping through a portal given by the "unknot". He goes from the "mountain world" into the "water world", also distinguishable by the green cone color. The red arc is a section of the knot.[3]

Mathematically speaking, Thurston is talking about phenomena from knot theory, a subfield of topology. Topology in general is the study of spaces without measuring, so objects are seen as the same if they can be deformed into one another. Knot theory is the part of topology concerned with knots and links, as in the sort of objects arising when one or more pieces of rope are entangled (see Fig. 3, 4, 5, and 6 for some examples). The only "special" thing about mathematical knots, compared to seaman's knots or shoelaces, is that the ends of a knot may not be loose, but are connected. Otherwise, all knots could be unknotted easily by pulling the rope through the entanglement, and thus would all be the same.

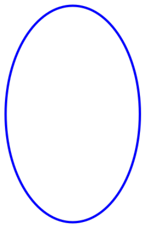

Figure 3: A diagram of the unknot, the simplest "knot", without any crossings.

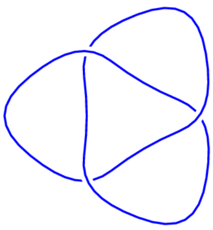

Figure 4: A diagram of the trefoil knot, the simplest non-trivial knot, with 3 crossings.

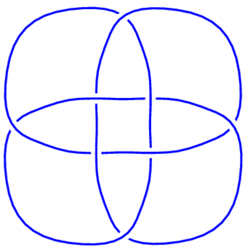

Figure 5: A diagram of a knot with 8 crossings.

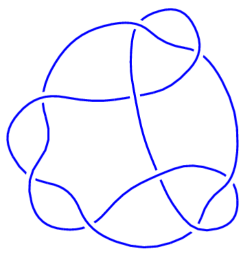

Figure 6: A diagram of a knot with 9 crossings.

Thurston talks about cyclic "branched covers" of knots of order 2. This means that by going around a strand of the knot, he goes into another world, but if he goes around the same strand twice (as the order is 2), he returns to the original world, see Fig. 7, 8 and 9. This can also be thought of as a staircase, or as the spiral ramp in a car park, but going up the spiral ramp two stories would return to the original floor. Some paths are of course more interesting than others, going back and

[3]Screenshot from a video available at `https://www.youtube.com/watch?v=Pgmfsl1e_vA`

forth through a portal does not change the world, as it corresponds to going a staircase up and then down again. The order of the cover can be chosen to be any natural number; a cyclic covering of order four would return to the original world after four trips around the strand. "Cyclic" means that no shuffling of the worlds is permitted; going around a strand starting in World 1, for a cover of order n the ordering is always World 1 $\rightarrow$ World 2 $\rightarrow \ldots \rightarrow$ World n $\rightarrow$ World 1 $\rightarrow \ldots$.

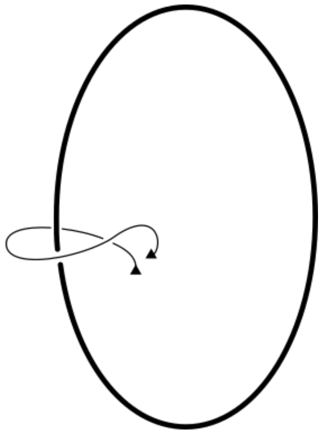

Figure 7: This path goes around a strand of the knot once, thus leading into another world.

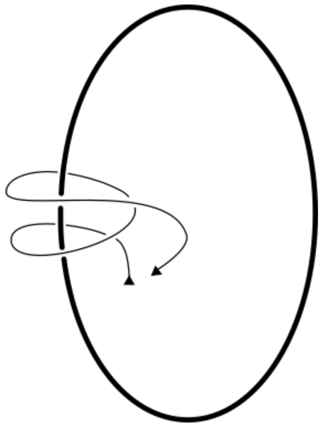

Figure 8: If the order of the cover is two, this path returns to the same world it started in.

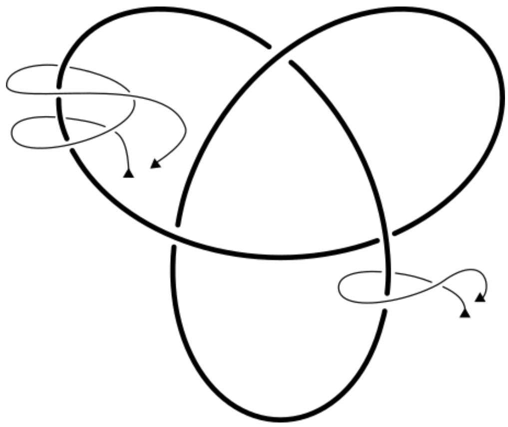

Figure 9: Given a cover of the trefoil knot of order two, the left path would return to the starting world, while the right path would lead into a new world.

He examines these covers using the "deck transformation group," which means nothing else than examining where different paths lead, and which paths are actually the same, i.e. if one can be deformed into the other. It is a very common method in mathematics, and especially in topology, to understand a space by looking at paths in it. For more information on the mathematical background of these spaces, consult Sümmermann (2020).

KnotPortal is an adaptation of this idea to a software. It combines the visuals of the different worlds with an immersion through virtual reality. This enables users to explore this mathematical concept by moving around and through the knotted portals into different worlds (see Fig. 2).

2.3 Embodiment in KnotPortal

While the placement of KnotPortal as presenting high body engagement, high task relatedness, and full immersion in the aforementioned categories might be seen as evidence to the ability of this software to be "better" for learning in some sense, such a quantitative statement should not be taken as the main message of this article. What we focus on is the ability of embodied actions to *elucidate mathematical concepts* and contribute to learners *construction of meaning* (Alibali & Nathan, 2012; Marshall, 2016).

This follows Papert's (1980) definition of embodiment. As he states in the foreword to his seminal book *Mindstorms,* as a child, much of his mathematical knowledge was was composed from gears: "I became adept at turning wheels in my head and at making chains of cause and effect. [...] I saw multiplication tables as gears, and my first brush with equations in two variables (e.g., $3x + 4y = 10$) immediately evoked the differential. By the time I had made a mental gear model of the relation between x and y, figuring how many teeth each gear needed, the equation had become a comfortable friend."[4] Having this in mind, we can formulate a rationale for the research presented here:

[4]Note also the emotional relation, which Papert himself considers vital to have a productive relationship with an environment.

Our goal is to provide an environment which gives learners an *object-to-think-with*[5] which gives rise to embodied concepts via actions carried out in it.

There are several aspects of KnotPortal relating to an embodied building of concepts.

Embodied cognition explains the use of expressions in language, as described in Sec. 2.1. This also holds for mathematics, where some definitions have non-technical meanings. A prominent example is the notion of a "path." Being able to move in this world, while remembering previous positions, gives humans the concept of a *path*. Obstacles such as trees give humans a choice, to pass left or right of the obstacle. This gives the notion of *different paths*, with "different" not in a metric meaning of distance between them, but in the topological sense of homotopicness.

In this project, we use this embodied provenance of paths and homotopies by letting users explore the given structure through their own footpaths. This results in full *congruency* (Johnson-Glenberg et al., 2014; Segal, 2011) between the embodied action and the concept, as paths in the embodied sense of the world here coincide completely with the mathematical notion of paths. Furthermore, paths are mathematically the tool of choice to study the branched covers mentioned in Sec. 2.2, as the structure to be revealed is the deck transformation group. This is the group of permutations of the fiber of the base point, which is just to say that one looks at the effect of going all possible paths through the knot.

Users can explore the concept of branched worlds by starting from a point in the real world room they are in, going around the room, and then returning back to their starting point. In the virtual world, this can also correspond to a closed loop if the path does not go through the knot. If it does, this leads to them not returning to the point they started at, but at another point in another world, existing parallel to ours.

Another aspect is the movement of the body and head, enabling the user to see the knot and the portals through it from different sides and perspectives. The usual viewpoint is that of a small knot which is manipulated and turned around by the user (Atiyah, Dijkgraaf, & Hitchin, 2010; Hotz, 2008; Strohecker, 1996), leaving the user in a static position with a top-down view of the knot. This image is probably heavily influenced by the usual representation of knot diagrams as drawn on paper as in Figs. 3 to 6, which is the usual way knots are represented. In KnotPortal, the user can "go around the knot," changing not the position of the knot but moving himself, which offers a new perspective on knots. Thurston (1994) himself noted: "An interesting phenomenon in spatial thinking is that scale makes a big difference. We can think about little objects in our hands, or we can think of bigger human-sized structures that we scan, or we can think of spatial structures that encompass us and that we move around in. We tend to think more effectively with spatial imagery on a larger scale: it's as if our brains take larger things more seriously and can devote more resources to them." The presentation of a larger knot to the user may thus have consequences on the learning about it.

3 Methods

To gain insights on the use of KnotPortal and its effects on grounding concepts in action, we performed a case study with two subjects. This type of study permits us to probe into the effects of embodiment experienced in KnotPortal in an open and explorative way. This study consists of an analysis of the participants usage of the software in a virtual reality setting and their responses to questions asked during their examination of the structures underlying the portals. The questions were asked by the first author in a unstructured interview, in order to maximize adaptability to the subjects. The usage was recorded on a video camera with audio and simultaneously by screen

[5] In this case probably about knot theory and topology in general. But who knows? Gears would not have been associated with the kind of mathematics Papert uses them for.

capture. The analysis was then performed following Hartley (2004).

The first participant will be denoted by “N”. He is an undergraduate university student visiting a seminar on topology, with no prior knowledge of neither knot theory in general nor this software and its aims in particular. The second participant will be named “E”. He is a researcher in differential and geometric topology, but with little experience in knot theory. He was aware of the goal of the software, and stated to have watched Thurston’s video several times a few days before the interview, until he had understood the structures explained there.

These participants were chosen for several reasons. Firstly, they needed to have a mathematical education on a level high enough to be able to grasp the concepts involved such as groups. The second was to expose people of different mathematical skill levels to the software, so a novice and an expert were chosen. Both participants had never tried virtual reality headsets before, so prior to the interview, they walked around in the Windows virtual reality home screen to get accustomed to this new and unusual experience.

The duration of the usage was 49 minutes for N and 42 minutes for E. In this time, both participants were presented several scenarios or “levels” in ascending order of complexity, as judged by the first author. The next level was presented to them when they claimed to have understood the shown structure, with the option to revisit previous levels if asked for by the participant.

For each level, there are two design options available, a realistic world and a color-coded simple world. The latter is to reduce the load on the computer. The participants were shown the realistic worlds until the computer could not handle the necessary computing power, which is when the level was switched to the color-coded world. The levels presented to the participants were:

Unknot of order two This is the universe generated by the unknot, a simple non-knotted circle, gluing together two worlds in a cyclic fashion. Here, going around the knot is the same as going back through the knot, as there are only two worlds.

Figure 10: A front view of an unknot, defining a portal into a second (water) world.

Figure 11: The same view from the second world, showing a portal back into the first world.

Unknot of order three Similar to the first case, but with a variation of the rule stated above. These are *three* worlds glued together cyclically along the portal defined by the unknot, so one has to go through and around the knot three times to return to the starting point.

Figure 12: A front view of an unknot, defining a portal into a second (desert) world.

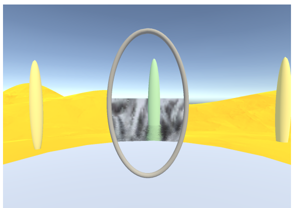

Figure 13: The same view from the second world, showing a portal into the third (water) world.

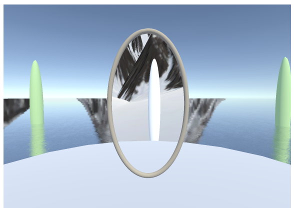

Figure 14: The same view from the third world, showing a portal back into the first world.

Twisted unknot of order two Mathematically equivalent to the first case, as the unknot is just twisted to give the appearance of a different knot, but could still be deformed to be the unknot again. This is, however, not directly apparent and must be discovered.

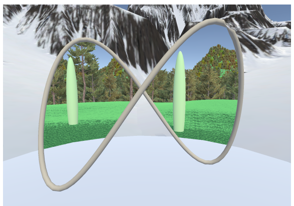

Figure 15: A front view of an a twisted unknot, defining a portal into a second (forest) world.

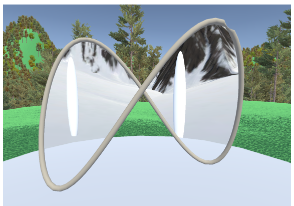

Figure 16: The same view from the second world, showing a portal back into the first world.

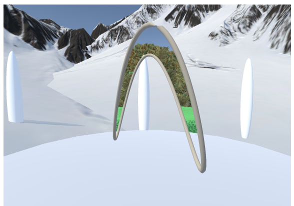

Figure 17: The same twisted unknot from Fig. 15 from a different viewpoint.

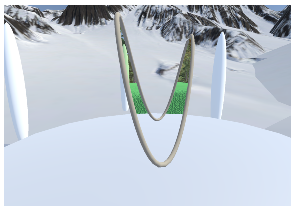

Figure 18: The same twisted unknot from Fig. 15 from yet another viewpoint.

Trefoil of order two Certainly the most challenging of the presented cases. This is the most simple non-trivial knot, but it still generates 6 worlds glued together along the 4 portals defined by the knot (see Fig. 19 to 24). It is still of order two, i.e. any path going around a segment of the knot twice returns to its starting point.

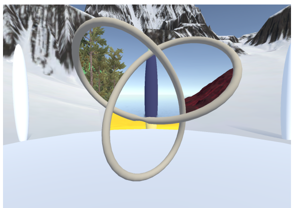

Figure 19: View into 4 other worlds through the portals generated by the trefoil.

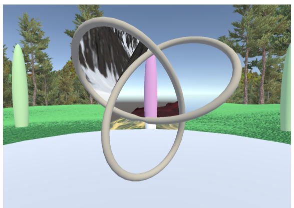

Figure 20: The front view of the knot after passing through the upper left portal in Fig. 19.

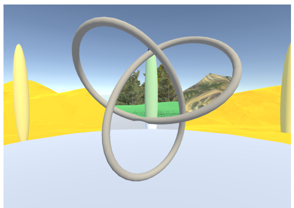

Figure 21: The front view of the knot after passing through the bottom portal in Fig. 19.

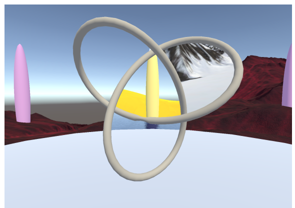

Figure 22: The front view of the knot after passing through the upper right portal in Fig. 19.

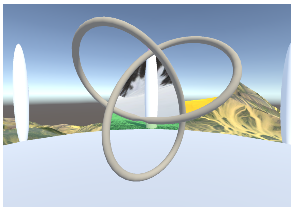

Figure 23: The front view of the knot after passing through the center portal in Fig. 19.

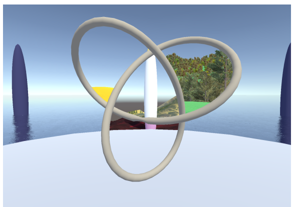

Figure 24: The front view of the knot in the sixth (hill) world, not visible in Fig. 19.

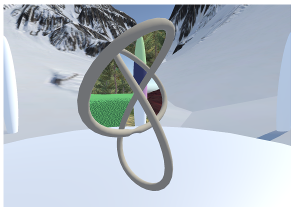

Figure 25: A sideways view of the knot in Fig. 19.

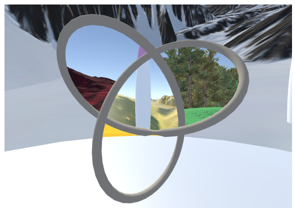

Figure 26: The view of the back side of the knot in Fig. 19. Note the center portal leading into a different world.

4 Results and discussion

We present three observed interactions in the software environment, particular to the embodied nature of the environment, which could be identified in the analysis of the use cases. The presented effects concentrate on the embodiment aspect of the software use; many of the behaviors and statements that could be observed in the case studies could be interesting from a mathematics education standpoint, such as shown misconceptions on different mathematical concepts and their implications, which are beyond the scope of this article.

4.1 Understanding through paths

As predicted, both participants took full advantage of the possibility of exploration through movement in virtual reality, trying out different *paths* to understand the structure of the knotted portals. These paths were mostly *closed* paths, meaning the users started at some real-world point, went around the room, and returned to their starting point, while observing the world changes effected by that path in virtual reality. An example is a scenario from E in the world(s) of the trefoil knot depicted in Fig. 27 as a schematic top view, with his taken path broken up into two paths for a better overview. Starting at a point p_R in the real world and the corresponding point p_Y in the yellow world, he traveled along a path (the left path in the figure) through the middle portal into the purple world, turned around, and went to the right portal into the white world, returning to p_R, but now at the corresponding point p_W in the white world. This is a closed path in the real world, but not closed in the portal world, as he has not yet returned to p_Y. He then continued (along the right path in the figure) by walking again through the middle portal into the pink world, before returning back to the yellow world and the point p_R corresponding to p_Y again, now closing the virtual path as well.

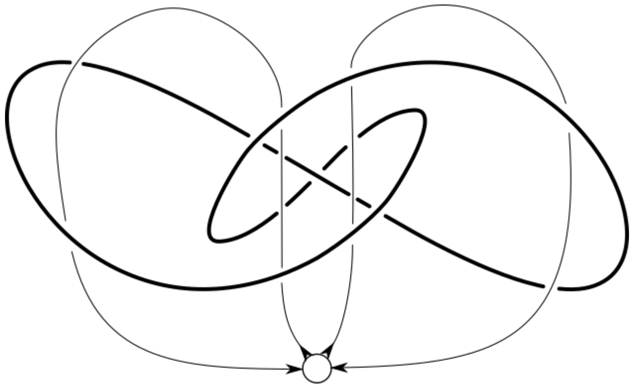

Figure 27: An exemplary path in the trefoil knot-level of E from a top view. He first traveled along the left path, then along the right. The circle represents the starting point, either p_R, p_W, or p_Y depending on the world the participant is currently in.

These paths also shaped the language used; when talking about world changes during the session, the participants both used language along the lines of "you go around this loop..." or "if you look from the other side." It must be said, however, that other language patterns would have been surprising, given the lack of previous knowledge on knot theory of both participants.

4.2 Patterns of movement

An unexpected behavior was revealed in the analysis of the usage cases. In the process of understanding the structures presented by the software, the participants used *embodied* problem-solving strategies. After passing to the next level, both participants used the movements which had helped them understand a feature of the previous level in trying to understand the current one.

We illustrate this with an excerpt from the use case scenario of participant E. In the level "Unknot of order three," he successfully understood the structure of the relationship between the worlds by passing through the portal in a circular motion (see Fig. 7). After subsequently being confronted with the new level "Twisted unknot of order two," he immediately tried to *reuse* this movement pattern. But in contrast to the setting before, the portal is no longer a circle, where it is clear what "passing through" means, leading to the failure of this pattern. This puzzles him, leading him to rethink on what his aims were in the first place.

The movement was re-initiated without closer inspection of the setting, which could have lead to the conclusion that this movement is no longer viable. *Bodily movement* is thus used as a *heuristic strategy* in the same way as in non-embodied problem solving, which identifies *patterns of movement* as an embodied version of the "patterns of reasoning" described in Pólya (1957/2004).

4.3 Loss of overview through immersion

Immersion describes how much the user is "in the virtual world" and perceives it as real, and is typical of virtual reality (Price et al., 2020; Winkler, Roethke, Siegfried, & Benlian, 2020). It is composed of three features: lack of awareness of time, loss of awareness of the real world, and involvement and a sense of being in the task environment (Jennett et al., 2008). Immersion is generally seen as a desirable trait of learning software (Dede, Salzman, & Loftin, 1996; Gutiérrez et al., 2007; Huang, Rauch, & Liaw, 2010; Johnson-Glenberg et al., 2014; Winkler et al., 2020).

The experience given through KnotPortal was certainly immersive, as both users could be observed weaving their head around virtual obstacles, or instances of E raising his feet as if he were really stepping through a portal, forgetting that only the movement of the head is tracked.

But, as the case study reveals, immersion has not only the potential to give the user greater access to a phenomenon, but also to hinder understanding by losing the ability to "step back" from the problem at hand. On a sheet of paper, a user can tackle a problem by aligning several copies of a scene next to each other. The lack of distance to the problem in a virtual reality environment as provided by KnotPortal makes it difficult to gain overview even of one scene, not to mention several scenes in parallel, which is simply impossible. The users literally "lose themselves in VR," as well put in the title of Winkler et al. (2020); but in this context this is a disadvantageous and not desirable effect. This was demonstrated in the behavior of E as well as N. While trying to understand the structure of the worlds, both showed indications of loss of overview, forgetting which worlds they had been to, and how they had gone there. The paths involved were quite complex, as seen in Fig. 27 and could not be easily remembered.

On the other hand, this immersive experience enabled the insight into the lack of understanding in the first place. The user is forced to implement actions through whole-body movement, experiencing all the effects happening "along the way." An example is an excerpt of E. At the beginning, E is sure of his understanding of the structure, given his pre-usage knowledge. Even the mistakes in identifying the structure of earlier scenarios, which were subsequently resolved, did not shake this confidence. But in the last scenario with the unknot, he tries a strategy which worked before, returning to the starting world by going in a sort of "circular" path. Contrary to his expectations, this does not work, leading him into a new world. His reaction, "Ah! That's funny…" marks the onset on him rethinking the structure he had before, leading to him switching to other strategies and trying to understand the problem from a new perspective. This leads to a complete overload: "That's amazing how overwhelmed I am.", standing in stark contrast to his initial assessment of having understood the problem completely.

Same as with the "patterns of movement" from the previous section, this effect went contrary to the expected ease in understanding through the software.

5 Conclusion

As the theoretical and case analysis demonstrates, "abstract" mathematics can be made accessible in an embodied way with software such as KnotPortal. By letting users explore mathematical structures using movement of their body, resulting in different effects being observable, namely understanding of the structure through movement in paths, use of "embodied heuristics" such as patterns of movement, and loss of overview through immersion. The latter two effects have implications for the research of embodiment in all educational settings and the design of educational environments, both of which must take them into account. This can mean the interpretation of certain behaviors as embodied heuristics, or the explicit design of an environment making use of this type of problem-solving behavior.

Detecting and pinpointing the nature of this embodied learning is done by a content-specific analysis of the software and the users' interactions. This can serve as a blueprint for further studies of embodied learning, be it in mathematics education or other fields. This study also revealed that an embodied representation of knowledge can have detrimental effects, which once again shows that also in embodied design, "there is no such thing as a free lunch."

6 Outlook

Gestures did not play a role in this study. The participants did, however, show gestures in real life of different natures during the use of the software. Implementing hands or hand-held controllers in VR would increase the ability of users to use gestures, for example to point out objects or symbolize paths.

To overcome some problems with orientation (situations such as "where did I come from again?"), a so-called "trail renderer" could be implemented, showing the path the user has taken through the world(s).

Methodically, the study could be expanded, including participants with even more (or less) knowledge in the topic. Pre- and post-interviews would further the understanding of the impact on language and gestures participants use to knot theory, be it recounting their experiences in this environment or when talking about knot theory or covering spaces in general.

Another highly interesting path to be explored is multi-user support (Dillenbourg, Järvelä, & Fischer, 2009; Malinverni et al., 2014). Seeing the actions of other users, how they disappear into and reappear from other worlds, enabling a sophisticated hide and seek. This is an ambitious project, and a first step could be the implementation of non-player characters. The NPC and the other player characters could be implemented as birds flying between the worlds, extending the Flock VR experience of Lobser, Perlin, Fang, and Romero (2017).

The diversity of effects at play in the design and use of such embodied learning environments calls for more study on the subject, especially in providing paradigmatic examples showcasing different affordances and different goals.

Acknowledgments

We would like to thank Elizabeth de Freitas for bringing the topic of embodied cognition to our attention, John Sullivan for clarifying questions on the topic of branched coverings of knots, and Roice Nelson for providing ideas on an optimized implementation of the software.

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2.7 On the Future of Design in Mathematics Education Research

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The message of this article/opinion piece is simple: design is important for mathematics education research and practice, but is not considered to be research in itself; this should change. We advocate for taking this view by giving arguments on what can be gained by choosing to accept design as research.

On the Future of Design in Mathematics Education Research

Moritz L. Sümmermann, Benjamin Rott

Mathematics education research is still a relatively young field, with a variety of different research topics, objectives and methods. As Mogens Niss critically discussed in FLM issue 40(1) publications of empirical, theory-guided research are playing an ever-larger role, which does not reflect the true diversity of the field. He asserts that theoretical research is accepted into publication only to the extent it refers to and is corroborated by empirical studies.

We want to further Niss' discussion, focusing on an issue mentioned in his essay: the role of the design of objects of study in mathematics education research. Our case is that the design of such objects is not considered to be research [1]. We consider 'design' to encompass a wide variety of different activities and associated products. The products of design can be a mathematics textbook, a questionnaire for an empirical study, a collection of problems, a model of a learner trajectory through a certain topic, a software for mathematics learning, or even simply a picture visualizing a certain aspect of mathematics. As Bakker (2019) puts it, Niss talks about how form is valued more than content; we join Niss in claiming that there is much content being disregarded.

Design and science

The relationship between design and science has been a lasting debate, as is the debate if design constitutes a science (Cross, 2001; Galle & Kroes, 2014). The main arguments are that while science describes or explains, design, such as engineering, constructs: "the natural sciences are concerned with how things are. [...] Design, on the other hand, is concerned with how things ought to be" (Simon, 1996 [1969], p. 5). Another argument is on the fundamentally different object of study: the natural vs. the artificial. While this has been theoretically debated, in reality design has been a part of many sciences, and this 'scientific design' is "not a controversial concept, but merely a reflection of the reality of modern design practice"(Cross, 2001, p. 52).

Let us consider physics, for instance. There, artifacts are needed, for example complex measuring instruments for use in large-scale experiments, such as detectors for use in particle colliders. This requires the design of such instruments, which is accepted as research without 'scientific' experiments being done, nor reflection on the design choices being carried out. Nobel prizes have even been given for the design of a construction, as in the case of the scanning tunnel microscope (Binnig, Rohrer, Gerber, & Weibel, 1982)[2].

This is an example of the incorporation of elements of an artificial designing science, in the sense of engineering as a science, in this otherwise empirical natural science. Interestingly enough, even though the design was considered research, the invention of the scanning tunnel microscope was neglected by the physics community until valuable insights were gained by its use (Mody, 2004).

The state of design in mathematics education research

Returning to mathematics education, we also construct objects to enhance our understanding of learning. In addition, mathematics education is not a natural science, so the phenomena we investigate do not, in most cases, arise naturally and without our intervention; design in the educational sciences is 'doubly artificial' (compare Cole and Packer (2016)) through artificial design of artificial settings. In most research models, some sort of artifact is placed at the center of the study, with research questions addressing the interaction of learners with it, either to understand the learning of mathematics, or to allow this learning in the first place. This artifact can be as simple as a problem, which learners will solve, or as complex as a curriculum implemented over years.

An example of such an artifact are dynamic geometry softwares. They have had an enormous impact on the mathematics teaching and the mathematics education research, engendering conferences and journals. Reports of research using dynamic geometry software have appeared in practically all high ranking journals in mathematics education. But would a presentation of the software itself, not framed by any empirical study or educational theory building, have found a way into one of these journals [3]?

This is the way that such artifacts are presented, as by-products of a theoretical or, more probably, an empirical study. Oftentimes, the artifact is constructed with the sole purpose of serving as a mediating object in the study, to exhibit some properties with which the learner will interact in a certain manner. The object may play a pivotal role; it is still presented only as a tool bringing insights on the learner, not having a purpose in itself. Even if the purpose of the study is the comparison of two or more such artifacts, the design of the artifacts themselves seems to play a subordinate role. An example could be the PISA studies (OECD, 2003), for which tasks were created that enable researchers to differentiate competency levels of students. This could have been a publication on its own. Instead, the tasks were constructed solely to serve the purpose of the study to measure student performance.

The prevalence of theory, methods and empiricism in such studies is not a problem per se, as their goal is to further understanding on the learning of mathematics by investigating learners, not the design of some object.

Approaches incorporating design in mathematics education research

There is an approach to mathematics education research which has, at first glance, design at its core. In the early 1990s, researchers argued for a refocusing of research, identifying the central role of design in mathematics education, portraying it as a 'design science' (Collins, 1992; Wittmann, 1995). This has then found its way into mathematics research through the 'design(-based) research' approach (van den Akker, Bannan, Kelly, Nieveen, & Plomp, 2013). This method consists of many design cycles, where an artifact and a theory are refined by empirical analysis. While the design research process includes the development of the artifact, the main purpose of this process is however not a refined designed object, but a theory, and the application of theory to practice (Gravemeijer & Cobb, 2013). Researchers engaging in this type of research aim to further the application of their improved theory, it being field-tested in many cycles. This is a valuable aim, but again it does not focus on developing an artifact, but, as in the example of PISA, only to the extent that theory building requires it.

There is another variant of design-based research, named 'research-based design', in which the design of the object seems to be the main objective. But design in the sense of research-based design means a design to solve an empirical problem with its validation and optimization in several cycles, not design stemming from the perspective of mathematics education itself (Plomp, 2013). Even this type of design-based research sets the focus on and requires a cycle of empirical validation of an initial design. Returning to the example of the scanning tunnel microscope: Following the line of thought of research-based design, one would have started with the design of an earlier type of microscope such as an electron microscope. Then, to address a problem encountered in practice, in this case a resolution too low to picture single atoms, empirical refinement would be carried out. Would this process somehow end up with the scanning tunnel microscope? That is highly doubtful. The design of such a complex artifact requires mental leaps to be made and problems to be addressed which are not answered or even raised by empirical testing cycles. The ideas for its design were not gained through iterative improvement of an existing design, but by implementing a fundamentally new concept, using ideas derived from other fields of physics. Of course, every object is refined and empirical evidence can help in this process, but regarding its nascence, not every complex object is the result of a gradually improving design of a simple starting construction.

There is a strand of mathematics education focusing on design, named 'educational design' as opposed to 'educational research'. The community is small in comparison with 'educational research', but nevertheless they are organized in a society, the International Society for Design and Development in Education, and publish an online journal, the Educational Designer. It is devoted to the research into the design process of educational tools, with the hope of identifying best practices and in this way helping researchers to improve the design process, as well as "increasing the impact of professional design on educational practice" [4]. In Cross' (2001) characterization of design and science, this would correspond to the 'science of design'.

Allowing researchers to demonstrate their design process and in this way further the 'theory of design' will certainly help improving the design of future educational tools, and is a unique and important undertaking. But even here, the design, as in the construction of an artifact itself or 'scientific design' (Cross, 2001), is not considered to be research and cannot be published in mathematics education journals.

In any case, summing up the discussion of design-based research, research-based design, and educational design, despite the recognition of its central role in mathematics education research, the design of objects for the learning of mathematics has not been recognized as research in itself. Designers can only publish their designed object by attaching it to an empirical study, embedding it in a design-based research process or by reflecting and documenting the design process.

As the obligatory exception to the rule, some designs and constructions can be published in practitioner journals; these are however not considered to be reporting on mathematics education research. These journals are intended for teachers and provide ideas which relate to classroom practice, bridging the gap between research and school.

'Stoffdidaktik'

There is of course a school of research devoted to the analysis and construction of certain objects, called in German Stoffdidaktik, or subject-matter didactics (H. N. Jahnke, Hefendehl-Hebeker, & Leuders, 2019). Its focus, however, is on purely mathematical analysis and construction. The design of a syllabus based on content analysis for teaching differentiation would be a prime example. If the design of a series of questions and tasks on a certain topic would be considered research in this sense could be debated. Finally, the design of an app engaging students in activities around fractions would certainly not fall into this category.

Subject-matter didactics, the dominant type of research in the early 20th century, faced much criticism for not including empirical evidence. Curricula were designed at the drawing board, without considering how students actually reacted to them in this process. This led to the inclusion of empirical studies in subject-matter didactics, to the point where Schubring (2015) states, "In contrast to the traditional subject-matter didactics free of empiricism, nowadays an empirical component should be self-evident in every teaching proposal" (p. 36). This quote also showcases the shift of German-speaking didactics from content analysis to teaching proposals and their evaluation; traditional subject-matter didactics has faded from the journal-published research landscape (T. Jahnke, 2010). There is still research being carried out in subject-matter didactics, but its decline certainly falls into the category of "conceptually or theoretically oriented reflective research without an empirical component" which Niss (2019, p. 6) identifies as the kind of research suffering the most from the trend to empiricism in modern mathematics education.

In the case of empirical studies of the kind Niss describes, the view of the object as a tool limits the amount of work put into it. By its very nature, this kind of 'small-scale research' also produces small, strictly bounded objects; every step beyond these bounds would not only mean unnecessary work, but also a blurring of the exactly defined research parameters. Again, the PISA studies are a prototypical example. While this may be useful to showcase behavior or performance of learners, it is the opposite of a rich learning environment. It is designed to understand and sometimes measure learning, but not to promote it or to extend our understanding of it.

The role of technology

Technological advances amplify this problem by expanding the field of learning software design. In the past, mathematics was learned only in classrooms with textbooks, a context within the reach of mathematics education research. Nowadays, there are thousands of mathematics learning apps and programs on the market, which are used at home more than in the controlled and supervised school environment. They stem from companies, with only a small percentage having any relationship with mathematics education research. The 'edtech' market is growing, resulting in disrupting technology such as apps that can solve textbook equations instantly, and also investments in education by some of the world's largest companies, seeking to shape education through their own policies.

The development of artifacts, especially larger projects, is happening in the education industry and through independent developers outside of academic education research, leaving mathematics education in the passive role of an observer (see Abrahamson (2015)). To manage a large project in mathematics education would mean to have to divide the projects into several parts, such as artifact design, theory, and empirical validation. The latter two roles count as mathematics education research, but design in itself does not, and thus is done by researchers as a side-line. This is a limiting factor in the project size and scope, restricting research to projects in which the design part is small enough to be handled by researchers not using their full resources for it. Technology influences this trend by enhancing our ability to collect and generate data as well as providing a larger variety of comprehensive learning environments. This leads to the number of large projects growing, aggravating the need for designers, which need their place in the mathematics education community.

Furthermore, dismissing the constructions would also mean the inability to construct such artifacts, and with it not having knowledge on their functioning. It would not only make mathematics education research passive, but blind, forced to treat artifacts as black boxes, only capable of measuring outcomes. It cannot be in the interests of the research community to exclude design knowledge from mathematics education.

Examples and future directions

An example of research, which should be considered as such in mathematics education, is a well-thought out curriculum proposal together with the motivation and reasons why the author might consider it especially well suited for implementation. Another kind could be a set of interesting problems, which are capable of enlightening aspects of a certain theory, for example able to provoke moments of intuitive reasoning.

The foundation of these examples is a collection of mathematics, which has undergone didactical exposition of some kind. Given this vital role of mathematics exposition, this line of research in the tradition of German subject-matter didactics should certainly find its place in modern mathematics education. T. Jahnke (2010) already pointed out this "gradual disappearance of the subject" (p. 22) from mathematics education research, relying on publication records of the Journal für Mathematik-Didaktik [5]. While this kind of research is strictly speaking not design, the two are strongly tied, and it represents another aspect of not well-represented mathematics education research.

Of course, not every design constitutes research. As Niss (2019) puts it,

> as most important designs and constructions are required to have certain properties and meet certain specifications before the resulting constructions are installed, design disciplines are scientific only to the extent they can provide well-founded evidence and

reasons to believe that their designs possess certain such properties to a satisfactory degree. (p. 10)

In the case of physics, the test which a construction has to pass is subject to engineering and physical standards, requiring for example a detector to achieve a certain fidelity with regard to measurements. In mathematics education, such criteria would have to be established in a dialectic process. A starting point could be given through the criteria for intervention in design-based research given in Plomp (2013): relevance, consistency, practicality, and effectiveness. As an example, consistency might describe the degree to which a learning software demonstrates its fidelity to mathematics, by explaining the mathematics behind the software and its representation for the user, together with the manipulation choices of the user on this software. Another criterion could be practicality, i.e. if the design has a potential use in mathematics education, justified by a comprehensible argumentation. These criteria do have to be applied carefully, as not to confine designs into strict boundaries, thereby restricting creativity and innovation.

Our aim is not to diminish and not even to criticize the ways researchers work or do design, but quite the opposite. We believe that mathematics education research encompasses a variety of different approaches that all make valuable contributions, and that designing artifacts for mathematics education is part of these valuable approaches.

Mathematics education as a scientific discipline has over the time acquired autonomy from mathematics, as its importance was acknowledged as being too great to be just research being done as side projects of mathematicians such as Pólya, Hadamard, or Klein [6]. Given the importance of design in mathematics education research, it is reasonable to give it space in the communication channels of our community. Designs also play a crucial role in the link of mathematics education research and practice, a link which would be strengthened by legitimizing design as research.

In the same way that physicists chose to consider the construction of experimental tools to be a part of physics and not to outsource it into engineering, to accept design as research in mathematics education is a choice that the field can make. This would not change the nature of research in mathematics education in any way. It represents a division of labor in the current type of research. There is no need for every researcher to do every part of a research project; some researchers may design an artifact, others collect data using it [7], and again others build theories explaining observed phenomena. The acceptance of this would be a choice that can benefit both sides, designers and empirical researchers, and avoid the pitfalls portrayed by Niss of research which lacks clarity and purpose. This would of course require journals and conferences to accept contributions presenting artifacts of interest to mathematics education, following certain standards.

With this in mind, it is pleasing to hear the editor-in-chief of one of the most influential journals in mathematics education emphasizes the importance of "non-empirical articles with important messages" (Bakker, 2019, p. 44). This message cannot only be a theoretical perspective on a topic as asked for by Niss, but also the presentation of a certain way of seeing, doing or interacting with mathematics through a well-thought-out design.

Notes

[1] This is disregarding some exceptions that prove this rule. See, for example Hewitt (2016) or Jankvist and Niss (2015).

[2] The Nobel prize was in fact "for their design of the scanning tunneling microscope"; see `https://www.nobelprize.org/prizes/physics/1986/binnig/facts/`

[3] Again, there are rare exceptions (e.g., ZDM issue 43(3)), but their rarity supports our point.

[4] From the inaugural editorial in Educational Designer, by Burkhardt, McKenney & Pead; see `https://www.educationaldesigner.org/ed/volume1/issue1/article0/`

[5] Publishing design in mathematics seems to follow a similar decline, noting the publication of Papert's LOGO and the Labordes' Cabri-Géomètre in the sixties and eighties together with the absence of similar, more recent publications.

[6] Of course, mathematics education has its roots not only in mathematics but also in other fields such as psychology, sociology, and in the profession of mathematics teaching which should not be forgotten.

[7] There even exist journals for publishing data, such as 'Scientific Data', `https://www.nature.com/sdata/`.

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2.8 Conference and non-peer reviewed contributions

1. "Touchbasierte Lernumgebung für Homotopien"
   Talk at the "3. Gemeinsame Jahrestagung der Deutschen Mathematiker-Vereinigung und der Gesellschaft für Didaktik der Mathematik" in 2018 as well as the associated non-peer reviewed publication in the conference proceedings (Sümmermann, 2018), see appendix A.1.

2. "Drawing Topology"
   Talk and exhibition of Ariadne as touch and virtual reality versions at the "Imaginary Conference on Open and Collaborative Communication of Mathematical Research" in 2018.

3. "Drawing topology using Ariadne"
   Talk and poster presentation at the "Eleventh Congress of the European Society for Research in Mathematics Education" in 2019, as well as the associated peer reviewed publication in the conference proceedings (Sümmermann, 2019c), see appendix A.2.

4. "Drawing Topology"
   Poster presentation at the "53. Jahrestagung der Gesellschaft für Didaktik der Mathematik" in 2019, as well as the non-peer reviewed publication of the poster in the conference proceedings (Sümmermann, 2019b).

5. "Exploring Topology by Touch and in Virtual Reality"
   Poster presentation at the workshop "Illustrating Geometry and Topology" in 2019, see appendix A.3.

6. Contribution to *Illustrating Mathematics*
   This is a short two-page presentation on the project about knotted portals in virtual reality, published in the book *Illustrating Mathematics* (Davis, 2020). This is a book named after the semester program at The Institute for Computational and Experimental Research in Mathematics, collecting work done by researchers attending the program, see appendix A.5.

7. "Entwicklung von mathematischen Lernumgebungen als mathematikdidaktische Forschung"
   Talk at the mini-symposium "MS 9: Digitalisierung und mathematisches Lernen und Lehren in den Sekundarstufe" in the "54. Jahrestagung der Gesellschaft für Didaktik der Mathematik" in 2020 as well as the associated peer reviewed publication in the conference proceedings (Sümmermann, 2020a), see appendix A.6.

3 Further unpublished research

3.1 Concept images in topology

This was a research project intended as a follow-up to the first article on Ariadne. Its aim was to elucidate the concept images (also "basic notions" or, in German, "Grundvorstellungen") involved in topology, especially in the topic surrounding homotopies and paths. This was meant to pave the way for an empirical study on the "effectiveness" of Ariadne in enabling students to learn about these concepts. Concept images would provide a way of measuring or capturing *understanding* of concepts depicted in the software.

While it is indeed interesting to analyze the different metaphors involved in the expression of thoughts on notions such as paths, this project was nevertheless abandoned. In researching the use of such concept images, in many instances they seemed to be used in a *normative* rather than *descriptive* fashion. This raises the question of the motives involved in such a use, which are informed by a certain view on the role of mathematics education. This view states that in "traditional" mathematics lessons (in Germany), conceptual understanding is neglected in favor of procedures and rote learning Vohns (2005). There is thus a need to build "Grundvorstellungen" in the students, which enable such conceptual understanding. This then raises the issue of procuring such concept images capturing understanding, which is done by a mixture of empirical methods and subject analysis.

So far, I do not object to such research. The problem lies in the view, not uncommon in mathematics education, on the application of these concept images. The following quote from Malle (2004) captures this view quite accurately (translation by the author): "Concept images are a central topic of mathematical didactics today. For more than 10 years, the Institute of Mathematics at the University of Vienna has been conducting a research program dealing with concept images. Lists of concept images have been compiled for all subject areas of school mathematics and control tasks have been developed with which the existence of these basic ideas can be checked. Empirical studies with more than 2.500 students were carried out to find out to what extent these concept images are present in our current students. The results can be summarized in two words: a catastrophe! Concept images are largely absent or insufficiently present."[3]

This quote summarizes the problematic view of concept images as a complete set of notions both necessary and sufficient for understanding. It rejects the possibility of some images being irrelevant for some students, and more importantly, condemns deviation from these prescribed images that are then used as the definition of understanding.

The termination of this project on concept images was thus morally induced, to not support this kind of standardization of thinking. This decision then also entailed the discarding of the empirical study originally based on concept images.

Nevertheless, I present an excerpt from the notes of this work are below, where I collected some basic notions associated with the terms of "path" and "winding number." I also collected some definitions that might be associated with basic notions in the sense of Thurston (1994), who gives a list of definitions of the derivative: "This is a list of different ways of thinking about or conceiving of the derivative, rather than a list of different logical definitions. Unless great efforts are made to

[3] In the original: "Grundvorstellungen bilden heute ein zentrales Thema der Mathematikdidaktik. Am Institut für Mathematik der Universität Wien wird seit über 10 Jahren ein Forschungsprogramm durchgeführt, das sich mit Grundvorstellungen beschäftigt. Es wurden Listen von Grundvorstellungen zu allen Stoffgebieten der Schulmathematik erstellt und Kontrollaufgaben entwickelt, mit denen man das Vorhandensein dieser Grundvorstellungen überprüfen kann. In empirischen Untersuchungen an über 2500 Schülern wurde eruiert, inwiefern diese Grundvorstellungen bei unseren derzeitigen Schülerinnen und Schülern vorhanden sind. Die Ergebnisse können in zwei Worten zusammengefasst werden: eine Katastrophe! Grundvorstellungen sind weitgehend nicht oder nicht ausreichend vorhanden."

maintain the tone and flavor of the original human insights, the differences start to evaporate as soon."

1.2 Path

Definition 1. *A* path γ *is a continuous function* $\gamma : [0,1] \to Y$*.*

Basic notions:

- Function: Each point of the interval is mapped to a point in the space
- Continuous or even differentiable: If we wiggle a point, its image wiggles accordingly
- Time parameter: $\gamma(t)$ is the point after t time units have passed. This is connected to the notion of directedness of a path, as time has a natural orientation.
- Locus of a movement in the space: Each point in the image of γ represents the position of the point at a certain time

Basic experiences or properties: paths

- connect points: this may cause problems with closed paths
- have endpoints: paths are images of closed intervals, and as such have a definite beginning and end
- can self-intersect: paths are not necessarily injective/embeddings
- have an orientation: the interval has a canonical orientation
- cannot traverse obstacles: paths are continuous
- can be subdivided/connected/paths with appropriate start- and endpoints can be concatenated – there is a map $I \to I \vee I$
- are homotopies

Paths as branches of a tree in stochastics in school.

[...]

1.6 Winding Number

The winding number is at the intersection of topology and geometry. In being a function, a winding number brings all *functional* basic notions. Same for being a number. There are several definitions, all related to different basic notions:

$$\begin{aligned} \gamma : [0,1] &\to \mathbb{R}^2 \setminus \{p = (a,b)\} \\ t &\mapsto \gamma(t) \end{aligned}$$

Covering Space

Definition 2.

$$\frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi} \tag{1}$$

for a path γ lifted to $\mathbb{R}$.

Angle of Endpoints

Definition 3.

$$\frac{\theta(1) - \theta(0)}{2\pi} \tag{2}$$

for a path γ given as $\gamma(t) = p + r(t)e^{i\theta(t)}$.

Alexander Numbering

Definition 4. *Let a "well-behaved" curve γ on the plane be given, which does not retrace itself. This curve then divides the plane into regions, one of which is unbounded. The winding number of a point on the plane is then determined as follows. The winding numbers of γ around any two points in the same region are equal; the winding number around any point in the unbounded region is zero; and winding numbers around points in adjacent regions differ by 1, with the larger winding number appearing on the left side of the curve (Chang & Erickson, 2015).*

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Definition 5.

$$\frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t) - p}dt \tag{3}$$

with $\mathbb{R}^2 \cong \mathbb{C}$.

3.2 Ariadne3D

As mentioned in the outlook section of the paper on Ariadne (see (Sümmermann, 2019a) in Sec. 2.3), a next step in its development was the integration of virtual reality gear, making the environment immersive and aiding in the establishment of intuitive controls for drawing paths on two-dimensional surfaces in 3D, such as a sphere or a torus (see Fig. 11).

This turned out to be a helpful extension, as was indicated during a mathematics exhibition during the IC18 conference with Uruguayan mathematics undergraduates unfamiliar with topology. They used the software to draw all kinds of paths on the torus, quickly turning to drawing "the most interesting" ones, namely the meridian, the equator, and combinations of them, yielding curves twisting around the torus.

3.3 Constructing manifolds in 3D

Orientable closed surfaces are classified as being topologically equivalent to the sphere with handles attached, or equivalently to the sphere or the sum of tori. Given the ability to draw paths on a sphere or a torus from Ariadne3D, this project added a cutting and gluing as well as a duplication

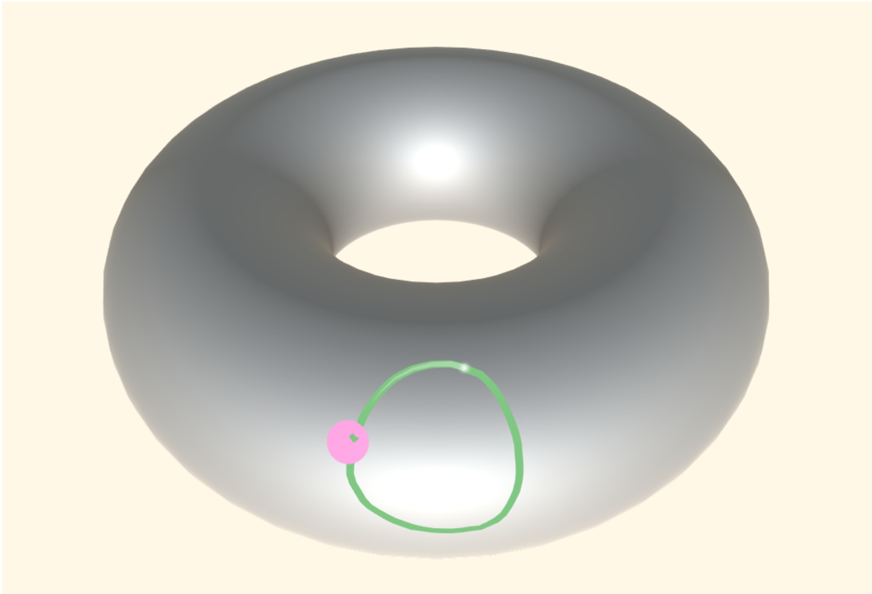


Figure 11: A torus with a path drawn on it.

functionality to the software. The long-range goal of the project was to let the user freely construct surfaces, starting with the most simple ones. This could mean starting with a flat sheet, gluing opposite edges, thus first making a cylinder and then a torus.

An intermediate target was the construction of orientable surfaces starting from a torus. As an example, a user can draw a loop on a torus and cut out along the path, resulting in a separation of the torus in a disc and a punctured torus. The punctured torus can be duplicated and turned, so that the two holes are facing each other. The two tori are then glued along the boundary curves defined by the drawn paths, resulting in the construction of a double torus. Screenshots of this process can be seen in Fig. 9 in the introduction.

In this way, the objects have a *genesis* and are not presented to the user without any motivation, which was the approach carried out in the Ariadne3D project.

To cut the surface along a path, a mesh-cutting algorithm needed to be implemented. It was based on a so-called snapping-and-refinement method (Wu, Westermann, & Dick, 2015) described in the following. The surface is given as a mesh of triangles, and the path as a sequence of line segments. Every triangle intersected by a line segment is then cut between the two intersection points, and the resulting pieces are then triangulated. If the start or endpoint of a line segment are close to the vertex of a triangle, it is snapped to its position, to avoid numerical problems with small triangles. The gluing of two surfaces along their boundary curves is completed by gluing the ends of a cylinder to the two curves.

A problem arising in the construction of such objects is their appearance. Following the classical joke on topology on the homeomorphism between a doughnut and a coffee cup, a user might not be able to visualize why the object she or he built is the same as some "standard" one. The surfaces resulting from the processes described above do not immediately give nice-looking results, but it might also be an illuminating experience to see a transformation into a "nice" shape.

To this end, a project to implement a feature for the transformation of topological surfaces into "standard" shapes was started. This involves a continuous deformation into some desired state and is also called "surface fairing." This process can be realized by minimizing some energy through the implementation of a discretized volume-preserving energy flow, such as curvature flow, Ricci flow, or Willmore flow (Bobenko & Schröder, 2005). Several algorithms implementing different flows were tested, such as a simple Laplacian smoothing (Botsch et al., 2007), mean curvature flow (Desbrun, Meyer, Schröder, & Barr, 1999), and Gaussian curvature flow (Zhao & Xu, 2006).

The results were adequate for small deformations such as gluing of a cylinder to a boundary curve as in Fig. 9f, but not for the envisioned ultimate goal of deforming whole objects into pleasing shapes. While Willmore flow was the most promising candidate for this, as per Sullivan (2012) and conversations with Keenan Crane, there exists no implementation efficient enough to allow real-time deformation. This put a stop to this project, shifting the focus to create an implementation of KnotPortal.

3.4 Surface Walker

This project was started to let users draw paths on surfaces, while materializing the relationship between surfaces and their universal covering surface and the associated fundamental polygon. It features a keyboard-controlled ball that the user can roll around on a plane. The plane is tesselated by polygons, which each represent a fundamental polygon of a surface. The user-controlled ball is replicated on each fundamental polygon, giving the illusion of many mirror images of the ball being moved around simultaneously. The surface and the plane are connected via a map, translating the movement of the ball on the plane to the movement of a dot on the surface.

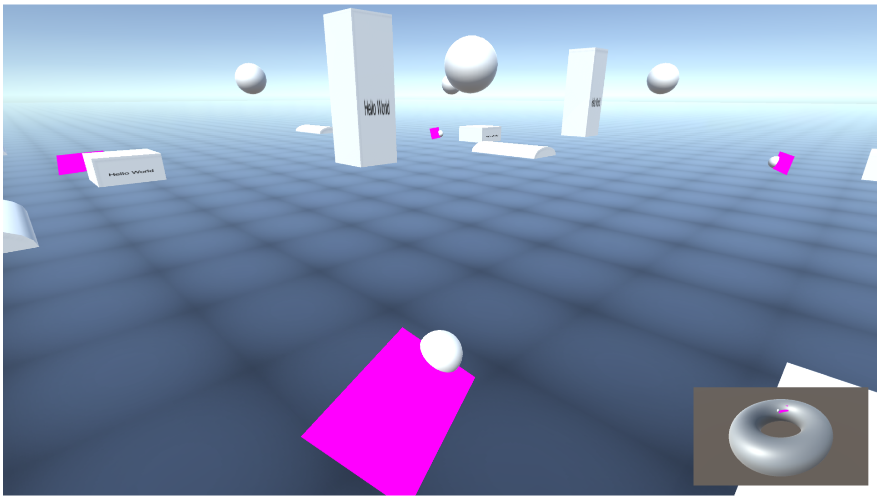


Figure 12: A screenshot of a ball rolling on the plane tessellated by the flat torus, closeup of Fig. 8a. Notice the orientation of the "Hello World" on the pillar, as well as the parallel movement of the "other" balls.

Several features where added to aid the users understanding of this construction. As an example, the plane has some objects with a non-symmetrical marking placed on it, so the user has some orientation to her/his position and orientation on the plane. Another feature is the ability of the user to press the space bar to fire a projectile from her/his ball. As the other balls are duplicates of the user-controlled one, *all* balls fire projectiles. This leads to interesting phenomena, such as the inability to dodge the projectile while trying to hit the "others," reinforcing the notion of the covering map involved.

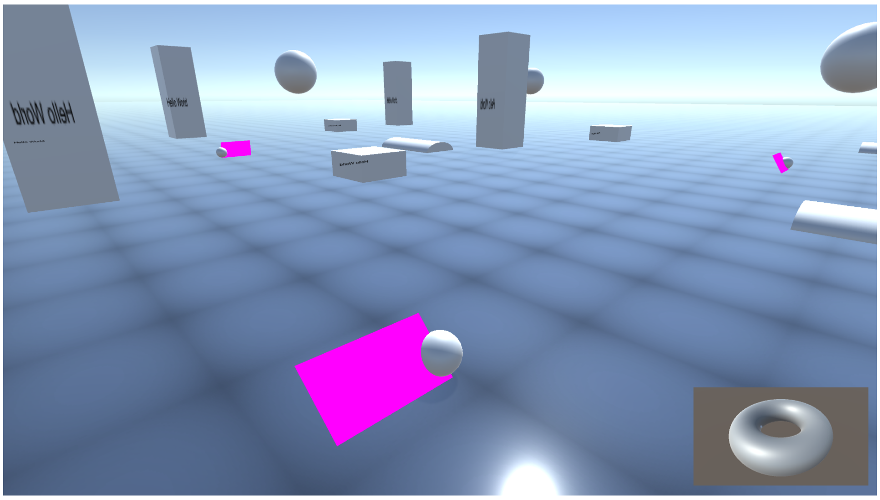


Figure 13: A screenshot of a ball rolling on the plane tessellated by the flat Klein bottle, closeup of Fig. 8b. Notice the inverted "Hello World" on the pillar, as well as the parallel movement of the balls in one direction, and the counter-directional movement of the balls in the other.

The torus (Fig. 12) and Klein bottle (Fig. 13) examples were quite easy to implement. For higher-genus surfaces, things can get quite complicated. There are two main issues in the implementation of a software with the aforementioned capabilities. Firstly, one needs a "parametrization" of the surface, i.e. a nice, maybe even conformal map between the surface X and its corresponding fundamental polygon P_X. Such maps would have to be constructed and discretized for every possible genus. Then, the plane has to be tesselated with P_X, in most cases resulting in a hyperbolic geometry. This tessalation may lead to a word problem, as already "occupied" parts on the plane need not be covered twice. While the word problem was solved by Dehn in 1922 (see (Dehn, 1987)), an algorithm respecting or circumventing this word problem must be carefully chosen.

The deviation of the topic from topology to geometry, together with the prospect of this amount of work necessary, led to the abandonment of this project.

3.5 Dirichlet Diagrams

Figure 14: A screenshot from the software generating Dirichlet diagrams (see `https://youtu.be/1CEDMdXEQGk`). Note the curvature of the region boundaries, depending on the time gap between activation of the spreading from the points.

This was inspired by work of my research group colleague Anne Möller on a problem-based exploratory learning approach in the context of perpendicular bisectors. A starting point is the so-called well-problem, where the student is asked to aid a sheep herder in finding the areas with the shortest distance to certain marked wells on a map. This generates a Voronoi diagram, with the boundaries of the regions given through the perpendicular bisectors of the wells.

Another representation of Voronoi diagrams is given by dropping paint onto predefined points, letting it spread radially until it meets already painted areas. This raises an interesting question: What if some paint drops are placed earlier than others? The result are so-called (additively) weighted Voronoi diagrams, also called Dirichlet diagrams. The edges of the regions in this diagram are then not given by perpendicular bisectors anymore, but by hyperbolic arc segments (see Fig. 14).

To make this phenomenon accessible, I wrote a software visualizing Dirichlet diagrams. Users can set points and start (or stop) the radial spread of paint from a point by a click or touch, or start (or stop) all points at once. The former gives hyperbolic edges, the latter the "usual" Voronoi diagrams. The software is available at the Imaginary website, see Sec. 2.2.

4 Discussion

Programs work, even if they don't run.
Visualizations work, even when they aren't seen.

-Stephenson (2019)

The aim of this thesis was to explore the potential of technology to make mathematics more accessible. The examples and their analyses from a small range of mathematical topics provided in the published and unpublished work give a glimpse into the large space of possibilities for mathematics to be made interactive in a visual representation. In this way, the presented work complements attempts from other parts of mathematics as well as other modes of representation.

The most work in this particular direction has arguably been done in classical geometry, where visual manipulatives have a long history. Examples are of course dynamic geometry systems such as Cabri géomètre, Geometers Sketchpad or the more recent Cinderella (CindyJS) and GeoGebra. One could even go so far as to give geometry a special place in this regard, as visual manipulatives (used to) be the defining objects and the axiomatic foundation for mathematics being carried out. This has since been replaced by a Hilbert-style abstract axiomatic, defining objects implicitly through relations given in axioms. This sort of development has ostensibly led to the abandonment of visual manipulatives in favor of formal-symbolic representations.

As many examples show, this trend of mathematics is becoming more formal and abstract is an illusion more than anything else; mathematicians in every field use visual representations and their manipulation to devise definitions, proofs, methods, and in all other mathematical work.

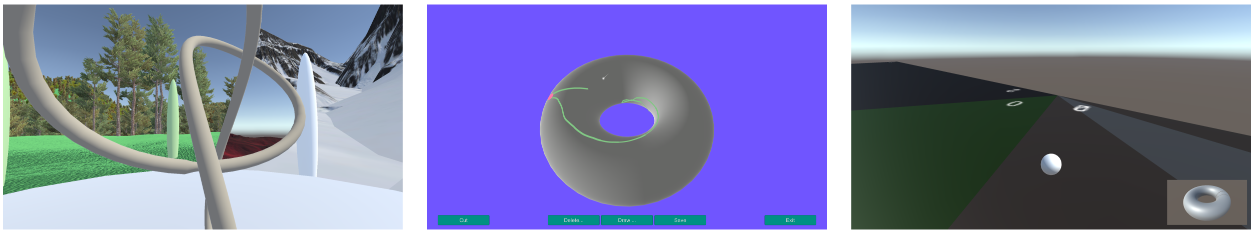
Modern technology undoubtedly gives an advantage in this realm of visualization, providing new ways of depicting and changing representations, be it through more powerful graphics engines or hardware devices such as virtual or augmented reality. It has also the capability to make possible the visualization of things which before were confined to the mind, giving users the ability to express their ideas through the software. This line of thought follows the view of technology for mathematics from Papert (1980). He talks about giving users *objects-to-think-with*, constructing microworlds able to incorporate some mathematical principles in a way that is accessible to the user. The work in this thesis was done in this spirit.

Ariadne represents such a microworld. In the environment, users can freely explore the relationships between dots and paths given different constraints through obstacles. By exploring, conjecturing and reasoning, concepts for the meaning of these terms can be formed, giving the visual representation of the environment as a complement to formal representations, not substituting for them.

As delineated in the paper in Sec. 2.4 on Mathematics in the Digital Age: The Case of Simulation-Based Proofs, Ariadne also provides an opportunity for research into the role of technology in proofs in mathematics education. The combination of the topic of topology and the affordance of a tool visually computing an invariant, making possible argumentations that feel like *real* proofs, inspire a new viewpoint on the role technology can play in teaching and doing proofs, not only in the classroom. This article starts the discussion on functions of technology beyond exploration, conjecturing and the gathering of pseudo-empirical evidence, which is seen as their main use in education as well as in research.

Another insight of relevance to current mathematics education research was the use of embodied cognition by users of KnotPortal, described in the article in Sec. 2.6 Embodied Mathematics: Forming Concepts in Topology by Moving Through Virtual Reality. The article provides a template for analyzing embodied cognition in mathematical learning environments through content-related analysis of the environment and the user interactions. By investigating KnotPortal in this way, several interesting instances of embodied cognition were found, such as "embodied paths" or "patterns of movement" as embodied heuristics, furthering the discussion on embodied learning in mathematics.

While this thesis is concerned with mathematics education, and not computer science, the issue of “bugs” or non-functioning software in general must nevertheless be addressed, given the relative amount and importance of software presented here. Fig. 15 allows a glance into the many malfunctions that can be found in the different software; some are due to unimplemented features or algorithms, other represent errors in their implementation, even if the algorithm is technically correct.[4]



(a) An error in the division of the screen into segments leads to the wrong worlds being displayed.

(b) The normal vector of the path's mesh is not normal to the surface, i.e. the path is twisted, making it partly invisible to the user.

(c) Several bugs can be seen here: the tesselation ends, the piece of the plane where the ball is located is tesselated twice, the lighter grey shines flickers through, and the image in the corner should not be a genus one surface.

Figure 15: Some different examples of bugs in the software presented in this thesis.

Such bugs are inevitable in all software which is complex enough, and become more frequent and are harder to detect the more freedom the user has in the use of the software. Ariadne as a mathematical simulation, but also KnotPortal, were both designed to give users a maximal amount of freedom of use, which aggravates the problem. Bugs in a mathematical software are not only annoying, but may actually impede their use by destroying the *trust in the technology*, as pointed out in Sec. 2.4 on Mathematics in the Digital Age: The Case of Simulation-Based Proofs. On the other hand, a bug may also present an opportunity, as it may present a safeguard against the user trusting a software to the point of not retracing the arguments brought forward anymore.

The overall reaction to the work presented in this thesis often followed the words of the tiresome young man in Bishop's poem at the start of this thesis: “It's pretty, but what is it for?” These impressions, together with an impression in mind of how mathematics education research might look like, led to the article On the Future of Design in Mathematics Education Research.

In the paper, we argue that there are many kinds of design, and most researchers in mathematics education practice one kind of design or another, as designed objects are often at the center of studies in mathematics education as an “artificial” field of study. Examples are of a mathematical or mathematics educational nature, such as sets of interesting problems or specific questionnaires, respectively.

Disregarding their central role in mathematics education research, design is, however, not accepted as research in itself, but only in conjunction with an empirical study or theory building. This has not always been the case, but is a trend, which is indicated by the demise of German traditional “Stoffdidaktik” and the simultaneous rise of design-based research in mathematics education.

The article concludes with the open question of criteria for regarding design as research, which is left open as to not presumptuously and externally impose criteria on a branch of research before it is even established.

[4]This brings to mind Donald Knuth's famous quote “Beware of bugs in the above code; I have only proved it correct, not tried it” to end “Notes on the van Emde Boas construction of priority deques: An instructive use of recursion (1977).”

While the article on the role of design is most certainly heavily influenced by negative personal experiences, it must be said that there are many wonderful design projects with amazing researchers behind them "at the edges" of mathematics education, one must only look at the work by the Imaginary group.

Ariadne, KnotPortal and the other projects presented in this thesis are not mentioned as an example of design-as-research in the paper, but I certainly consider them as such, and the paper was written with them in the back of my mind. If one compares the presented software projects with the suggestions of criteria given near the end of the article, at least the criteria of "consistency" and "practicality" must be regarded as met by Ariadne and KnotPortal, given the already presented uses and their design rationales. Whether the other software, especially the more bug-ridden ones, are design, is certainly more controversial. In my opinion, this nevertheless holds true, in the spirit of the quote by Stephenson (2019) above; the construction and the reflection on obstacles that would have to be cleared, as well as the process of imagining such a visualization, all help in learning about mathematics and mathematics education. So why shouldn't their design be research?

5 Summary and outlook

> Mathematicians usually have fewer and poorer figures in their papers and books than in their heads.
>
> -Thurston (1994)

This thesis contains some examples of visualizations in topology, together with analyses of interactions with them, as well as a meta-comment on their role in mathematics education research.

The analyses are mainly to be seen as a probing into the nature of such visualizations and examples for studies that could be conducted. Much more could be discussed in their context, in particular concerning their use in practice. An example could be the role of the winding number tool in Ariadne, as it provides an opportunity for introduction of users to negative numbers. While users might, at first, consider the winding number to be simply a natural number, users then learn that concatenation of paths leads to addition of winding numbers. The users may then consider the example of a path γ with winding number 1 around a point p, and the concatenations $\gamma \star \gamma$ and $\gamma \star \gamma^{-1}$, with winding numbers 2 and 0, respectively. This forces the distinction into a sort of handedness of paths around a point; as γ and γ^{-1} seemed to both have winding number 1. This finally concludes with the establishment of the concepts of *left-handed* and *right-handed* winding numbers, corresponding to positive and negative integers.

Another interesting subject could be the examination of the learning of concepts using the software. An example could be group theory. With Ariadne, this could mean discussing the concepts of commutativity with the example of the fundamental group of the twice punctured plane versus the fundamental group of the torus. With KnotPortal, one could investigate the role of generators and relations in the presentation of a group, as this seemed to be an issue in the interviews conducted for Embodied Mathematics: Forming Concepts in Topology by Moving Through Virtual Reality. Another investigation could be into the learning of continuity, as discussed in Ariadne – A Digital Topology Environment, in terms of time and motion, using paths.

The software may also be used in learning situations, such as with children in schools or in university-level courses on topology, and assessed as to whether their use is beneficial to the students' understanding.

The software projects themselves could be extended in many ways. This includes first and foremost bug-fixing and other quality assurance measures, as the projects were merely built as prototypes to show a proof of concept. On the other hand, the software being prototypes also means that many features with regard to content could be introduced.

In Ariadne, a feature could be added showing the genesis of punctures. In the current version, holes are generated in the level builder in a similar way as dots, by a touch of the screen at the desired location. This does not show the fundamental difference of a puncture to a defined point.[5] This could be alleviated by changing this construction method to the cutting out of a drawn path, which would correspond more closely to the notion of a "hole" in the surface. Furthermore, the algebra tool, which is included in Ariadne in a very rudimentary version, could be extended to ease the transfer from a visual to a formal-symbolic representation of paths and homotopies. There are also smaller changes, which might be worth considering, such as the speed of the moving dot on a concatenated path. At the moment, the result of a concatenation is only up to reparametrization, i.e. the speed is uniform over the length.[6] The usual definition of concatenation defines the concatenated path of two paths γ_1 and γ_2 as being the path γ_1 on the first half of the interval, and γ_2 on the second half. This would imply a different speed on the different parts of the concatenated path, if their lengths differ.

[5] Interestingly, these are the same words in German: *Punktierung*, i.e. puncturing, can mean a puncture in a topological setting, or a defined point in a geometrical one.

[6] Compare the notions of associahedra and the associahedron operad, e.g. (Stasheff, 1963).

The three-dimensional version of Ariadne could be extended to accommodate non-orientable surfaces; it would certainly be fun to draw paths on a Klein bottle.

For KnotPortal, there are also many possible ways the software could be further developed. A smaller change would be the ability of users to leave a trace behind, easing orientation by showing the traveled path. The most exciting addition would, however, be the incorporation of a multiplayer environment. Allowing users to interact with each other, watching other users disappear and reappear through the different portals and worlds would possibly greatly influence understanding. From a technological standpoint, the software could benefit from the integration of "infinite walking" technologies such as presented in Sun et al. (2018), which allows movement without restraint from the room size.

The embodiment aspect would greatly benefit from the implementation of finger- or hand-tracking, which could be analyzed using the large body of research on gestures in mathematics.

While this thesis is mathematically only concerned with topology, other areas of mathematics, besides the obvious choice of elementary geometry with its rich array of dynamic geometry software, offer an at least as large potential for simulations. Examples could be graph theory, analysis or probability theory, but also algebra or number theory. For all of these fields, great animations exist already and it would be great to see a shift from animations to simulations.

I certainly hope that the discussion started with the article on On the Future of Design in Mathematics Education Research will be continued, for example through further debate on criteria for regarding designs as research, and will lead to the acceptance of more articles on design in mathematics education journals. Papert stated 40 years ago: "[i]n current professional definitions physicists think about how to do physics, educators think about how to teach it. There is no recognized place for people whose research is really physics, but physics oriented in directions that will be educationally meaningful" (Papert, 1980, p. 188); maybe this debate will even help create job positions for these people, with design for mathematics education as their focus.

The feeling I have after writing this thesis is that the presented articles and projects only scratch the surface of the addressed issues and topics. Embodiment and proofs are only two aspects of the multi-faceted nature of visualization software, and the software examples themselves are far from showing the full potential of technology in mathematics learning and visualization.

I also believe that this potential is not to be fulfilled by improvements in technology, but by improvements in the use of technology, as indicated by the actuality of the propositions put forward in Papert's *Mindstorms*. His book focuses on researching the way that technology changes (mathematics) education, not on the effects of a specific product or technology; it talks about general issues but in terms of examples, as "[y]ou can't think seriously about thinking without thinking about thinking about something." (Papert, 1980, p. 10). This is what I tried to achieve in this thesis: present general thoughts on mathematics education based on concrete mathematical examples.

It is certainly my hope that this thesis as an example as well as the initiation of the discussion through On the Future of Design in Mathematics Education Research will help in advancing the goal of showing the intertwined relationship of mathematics and mathematics education, and the dependence of mathematics education on good design of mathematical content. In this way, instead of drifting apart, as is observable at least in parts, mathematics education and its related disciplines such as mathematics and mathematics communication may be brought together.

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A Conference contributions

A.1 Touchbasierte Lernumgebung für Homotopien

Talk at the "3. Gemeinsame Jahrestagung der Deutschen Mathematiker-Vereinigung und der Gesellschaft für Didaktik der Mathematik" in 2018 as well as the associated non-peer reviewed publication in the conference proceedings (Sümmermann, 2018).

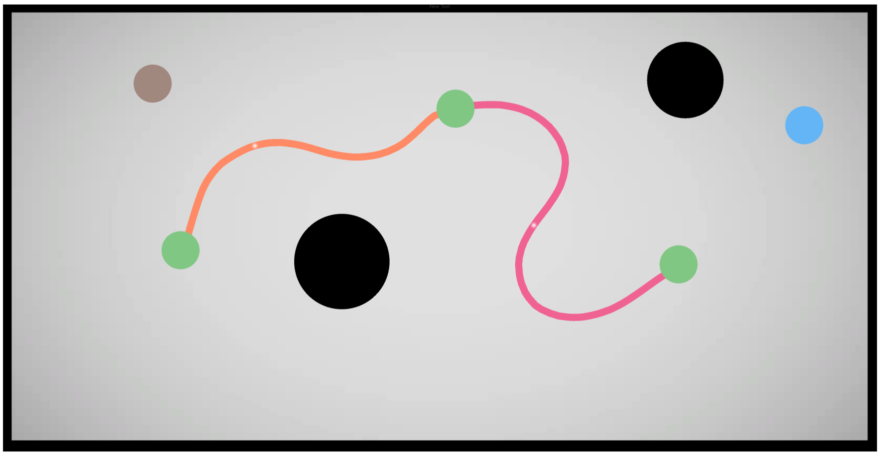
Touchbasierte Lernumgebung für Homotopien

Moritz L. Sümmermann

In diesem Beitrag stelle ich kurz und informell eine Lernumgebung namens Ariadne in Form eines Tablet-PC-Programms vor. Es ist schwierig, solch ein dynamisches Programm in dieses statische Format zu fassen, der geneigte Leser ist daher eingeladen sich die Videos über Ariadne auf meiner Homepage anzuschauen oder es auf einem Gerät mit Touchfunktion selber auszuprobieren (`www.mathedidaktik.unikoeln.de/11924.html`).

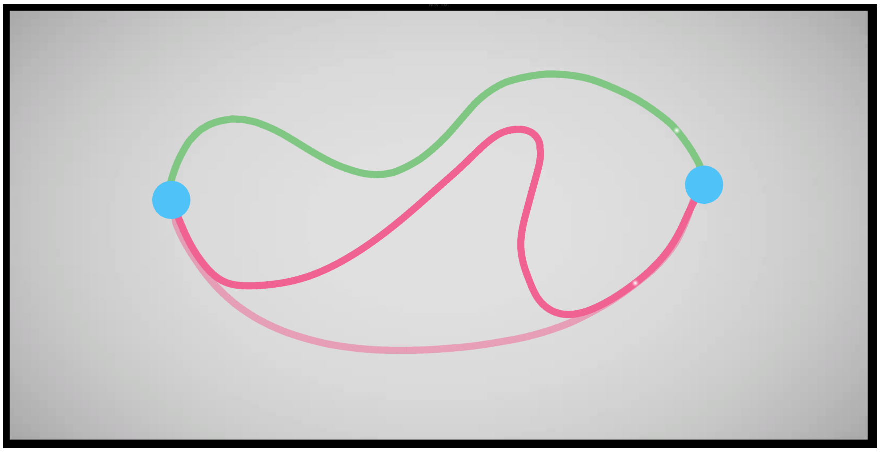
Die Forschungsfrage meiner Dissertation lautet, ob man mit dieser Lernumgebung in der Lage ist, das Konzept von Homotopien zu verstehen. Meine Zielgruppe ist die Primarstufe, um ein möglichst unvoreingenommenen Blick auf Mathematik zu ermöglichen.

Zunächst ist im Programm nur ein graues Feld mit schwarzen Bereichen zu sehen. Durch Berührung des Bildschirms kann man nun Punkte auf dem Feld setzen, aber nicht auf den schwarzen Bereichen. Die schwarzen Bereiche spielen mathematisch die Rolle von „Aussparungen“ bzw. „Hindernisse“ der Ebene. Durch ziehen der Punkte mit dem Finger entstehen Wege zwischen Punkten, wobei auch diese nicht die Hindernisse queren können. Dabei haben durch Wege verbundene Punkte dieselbe Farbe, um diese Beziehung der Verbundenheit zu verdeutlichen.



Screenshot 1: Punkte, Wege und Hindernisse

Zwei aneinander anknüpfbare Wege, das heißt bei denen der Endpunkt des einen Weges Startpunkt des anderen ist, können durch gleichzeitige Berührung von je einem Finger verbunden werden. Nun gibt es auch „Wege zwischen Wegen“, sogenannte Homotopien. Diese werden durch das Ziehen eines Weges mit einem Finger realisiert. Sobald man einen Weg vollständig auf einen anderen Weg gezogen habe, bekommen auch diese Wege dieselbe Farbe. Diese Wege heißen „homotop“ zueinander. Auch hier stellen die schwarzen Bausteine wieder Hindernisse dar. Ein Sonderfall stellt hier das Ziehen eines Weges mit dem gleichen Start- und Endpunkt auf ebendiesen dar, solch einen Weg nennt man „nullhomotop“.



Screenshot 2: Eine Homotopie zwischen zwei Wegen

Eine weitere Funktion von Ariadne ist die Anzeige der sog. „Windungszahl“ eines Weges um ein Hindernis. Nicht dargestellt, aber ebenfalls im Programm implementiert, sind dreidimensionale Objekte wie der Torus oder die Sphäre, auf denen wie auf dem Feld in den Screenshots Punkte gesetzt und Wege gezogen werden können.

Diese Lernumgebung beruht auf Mathematik aus dem Teilgebiet der algebraischen Topologie. Für Informationen über diese verweise ich auf Lehrbücher wie Hatcher (2000) oder Bredon (1993). Nur mithilfe dieser einfachen Funktionen können sowohl einfache als auch sehr komplexe Aufgaben aufgeworfen und bearbeitet werden. Das Vokabular gilt es natürlich dem Entwicklungsstand des Kindes anzupassen, möglicherweise auch von ihm selber wählen zu lassen. Verschiedene Fragen, in verschiedener Schwierigkeit, sind beispielsweise:

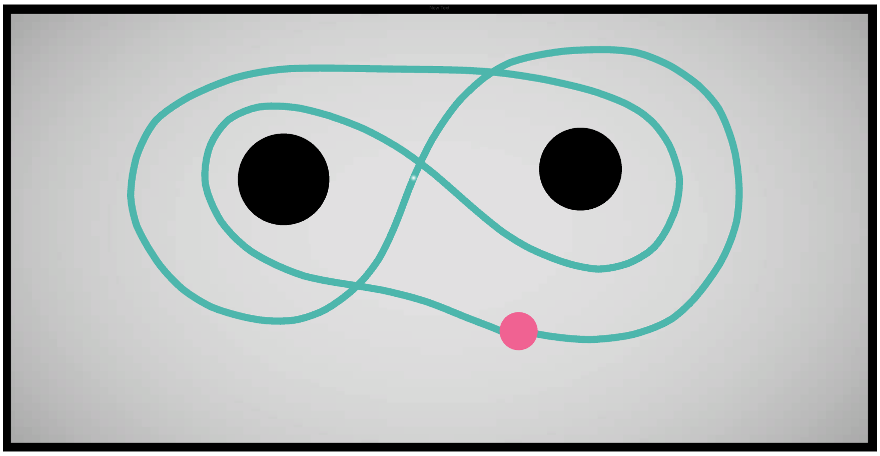
1. Welche Punkte kann man verbinden?
2. Wie viele verschiedene Farben der Punkte gibt es minimal in diesem Bild?
3. Kann man diesen Weg zu diesem anderen umformen?
4. Sind diese Wege nullhomotop? Sind sie zueinander homotop? Was ist ihre Windungszahl?
5. Wie viele verschiedene, das heißt nicht homotope, Wege gibt es hier?
6. Kann man Hindernisse setzen, so dass dieser Weg nicht nullhomotop ist?
7. Kann man einen Weg mit Windungszahl 3 malen?
8. Wie viele verschiedene Wege gibt es hier?
9. ...

Dies sind alles genuine mathematische Fragen aus der algebraischen Topologie. Manche dieser Fragen sind auch für Grundschulkinder schnell zu beantworten, andere brauchen viel Arbeit in Form von Beispielen und Gegenbeispielen. Dabei kann auch die zugrundeliegende mathematische Struktur genutzt werden. Beispielsweise gibt es einen Weg von Punkt A zu Punkt C, falls es einen Weg von A nach B und von B nach C gibt. Dem ist so, da „homotop zu“ eine Äquivalenzrelation ist. Dies kann genutzt werden, ohne den Begriff der Äquivalenzrelation explizit einzuführen, und somit das Denken in mathematischen Strukturen fördern.

Das Thema Homotopien und Wege bietet sich für Computerumgebungen an, da es keine physikalischen Materialien mit den gewünschten Eigenschaften eines Weges gibt. Ein solches Material müsste beliebig verformbar und dehnbar sein. Kandidaten wie Lehm (Nobel Committee for Physics, 2016) oder Gummibänder (Szpiro, 2008) erfüllen dies zwar teilweise, haben aber natürliche Grenzen.

Mithilfe des Computers können diese Grenzen zulasten der dann fehlenden Haptik überwunden werden.

Ein weiteres durch ARIADNE angesprochenes Thema ist das Machen von Mathematik im Gegensatz zum Wissen darüber (Papert, 1972). Das bedeutet in diesem Fall die Konstruktion von Objekten durch den Benutzer, um Fragen aufzuwerfen oder zu beantworten. Aufgestellte Hypothesen über diese Objekte können durch Ausprobieren bestätigt oder widerlegt werden, ohne eine externe Autorität, wie beispielsweise einen Lehrer, zurate ziehen zu müssen.



Screenshot 3: Der „Pochhammer-Weg“, ein nicht nullhomotoper Weg mit Windungszahl 0 um beide Hindernisse (Siehe, z.B., Wang und Guo (1989, S. 105)

Nicht zuletzt ist es ein interessanter Aspekt, ein zentrales Thema der modernen Mathematik elementar vermitteln zu können.

Die Lernumgebung wurde bisher nur mit Erwachsenen Nicht-Mathematikern getestet, der nächste Schritt ist die Analyse vermöge qualitativer Untersuchungen mit Grundschulkindern.

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A.2 Drawing topology using Ariadne

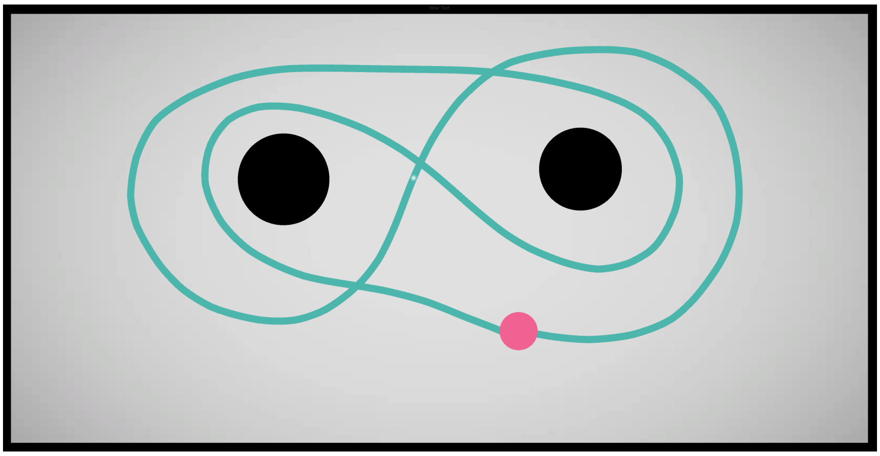
Conference proceedings of the "Eleventh Congress of the European Society for Research in Mathematics Education", as well as the poster presented there and at the "53. Jahrestagung der Gesellschaft für Didaktik der Mathematik". It should be noted that as in the poster in Sec. A.4, a tablet-PC was attached to the middle of the screen. The tablet-PC showed a pre-recorded video of the software.

Drawing topology using Ariadne

Moritz L. Sümmermann

Research overview

The focus of this work is to build a learning environment, making it possible to learn about paths and homotopies without the use of formalism. Here, learning environment means a microworld given through a software in the sense of Papert (1987). To help achieve this goal I have developed Ariadne, a software tool for the visualization of and interaction with paths. These paths can be constructed on a wide variety of surfaces, from the plane to manifolds of arbitrary genus, and punctured versions thereof. A more detailed account of Ariadne's capabilities is given in Sümmermann (2019).



Screenshot 1: The Pochhammer Contour, a non-nullhomotopic green path starting and ending at the magenta dot with winding number zero around both black punctures, constructed in Ariadne.

Topology in general is not present in the school curriculum, which limits the extent of research in the field of topology education. It is, however, a very important part of modern mathematics, so there have been some attempts to visualize topology, either without (Strohecker, 1996; Sugarman, 2014) or with software (Culler, Dunfield, Goerner, & Weeks, n.d.; Scharein, 1998). There has been no attempt to implement interactive continuous deformations as represented by homotopies, which is the focus of Ariadne. It also follows a different approach didactically, as its purpose is not only to visualize concepts already known to the user, but to teach the user these concepts by letting him interact with the visualization. The theoretical framework behind the design of Ariadne is based on the design principles of Devlin (2013) and the Artefact Centric Activity Theory from Ladel and Kortenkamp (2013). Ariadne is split into a two- and a three-dimensional mode. Both are usable on any touchscreen device, such as tablet-PCs or smartphones. In 2D, the user can construct points, paths and homotopies of paths on the plane with an arbitrary number of punctures, as well as compute the winding number around these punctures. This allows the user to tackle questions on the existence and equivalence of paths, and thus the treatment of the fundamental group. The same can be done for closed orientable surfaces of genus g in three-dimensional mode.

For the 3D-mode, a mixed reality environment is implemented. This mode facilitates the interaction with two-dimensional surfaces in three-dimensional space, such as the sphere or the torus, and thus alleviates handling issues inherent to the two-dimensional touchscreen. The three-dimensional mode also allows the construction of paths on the universal cover of the chosen surface, which is for most surfaces the hyperbolic plane. Ariadne is being evaluated through individual interviews with students from all age groups, in which they are being posed questions to assess their understanding of the used concepts. Further research directions are a didactical analysis of the topological notions involved in Ariadne to ensure that the answer quality is representative for the understanding of the content, planned to be implemented as a qualitative empirical study with mathematicians. The questions can then be refined based on this analysis. Another direction of research is the development of course material for Ariadne for the use in schools and universities.

Poster contents

The poster contains a short summary of the mathematical objects involved using some formulas and pictures, so it is clear what mathematics are conveyed with Ariadne. This is by no means exhaustive, but intends to sensitize the audience to the subtleties of the concepts involved. In the center of the poster is a tablet-PC, which the conference participants can use to test Ariadne for themselves. Another part of the poster is a list of sample questions which can be answered with the help of Ariadne, as a demonstration of Ariadne's capabilities. The last part is a short overview on the technicalities of the program for those interested in the mechanisms of action behind Ariadne.

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Drawing Topology

Ariadne

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Motivation

- Topology is an important field in modern mathematics
- Topology and in particular homotopies are inherently visual, but are taught with formulas
- There is no real-world material able to emulate homotopies
- Ariadne makes it possible to learn about topology without formalism

Sample questions

- Which points can be connected by paths?
- Are these paths homotopic?
- How many different paths can be drawn on a surface from a point to itself?
- Are all paths with the same winding number homotopic?
- Why is the winding number of a closed path an integer?

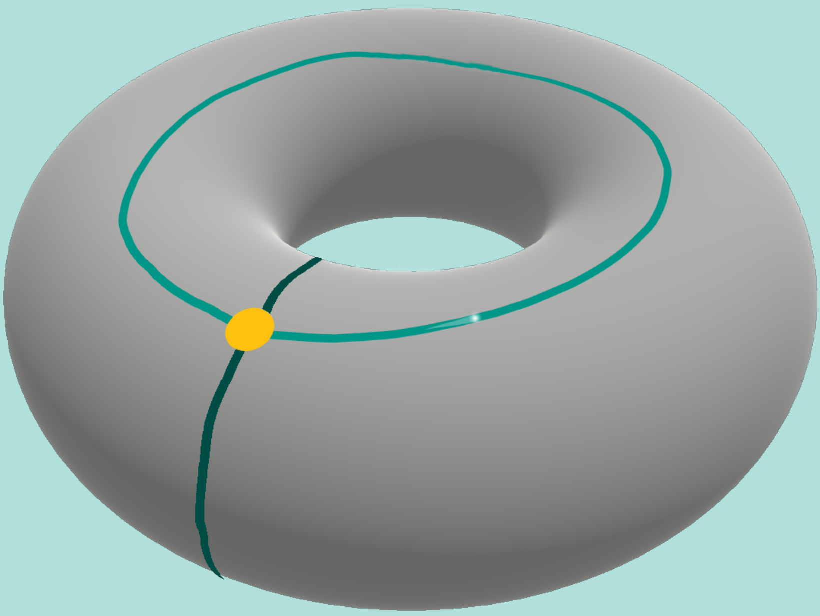
Mathematics

- Fundamental group $\pi_1(X, x_0)$
- Winding number of a path
- Homotopies from a path f to a path g

$$H : [0,1] \times [0,1] \to Y$$
$$H(x,0) = f(x)$$
$$H(x,1) = g(x)$$

- Cutting and gluing along paths
- Realization of all orientable surfaces with boundary curves and arbitrary genus

Try it



Virtual reality

- Manipulation of paths and surfaces using Windows Mixed Reality controllers
- Overcomes the limitations of a 2D interface
- Experience self-made surfaces with the sense of place

Implementation

- Programmed in C# using Unity3D
- Multitouch and virtual reality roomscale interface
- Built-in computation of homotopy classes in 2D

Download from Google Play or from imaginary.org

A.3 Exploring Topology by Touch and in Virtual Reality

Poster presented at the workshop “Illustrating Geometry and Topology” at the Institute for Computational and Experimental Research in Mathematics in 2019.

Knotted portals

This is the realization of an idea by Bill Thurston ("Knots to Narnia"), to think of cyclic branched coverings of knots (of order two) as knotted portals to other worlds.

The realization allows the exploration of these portals in virtual reality, making it possible to explore the group structure (Fig. 4) by moving around and in between these worlds in virtual reality.

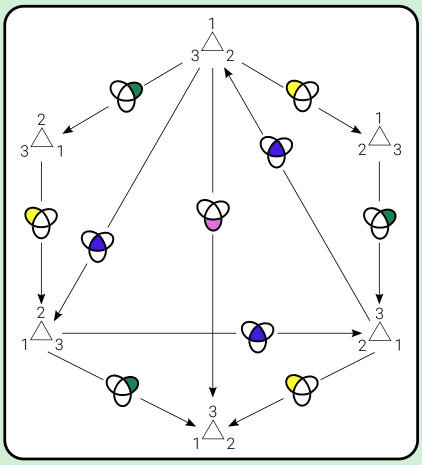


Figure 4: The group structure for the trefoil knot and its correspondence to the symmetry group of a triangle.

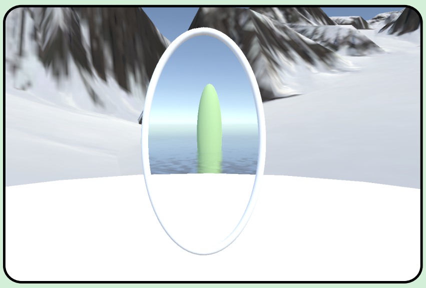


Figure 1: The unknot as a portal between two worlds.

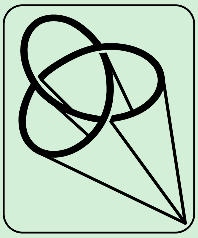


Figure 2: The model for the trefoil knot, exhibiting the cone sections.

The software is made with Unity3D and C#. The worlds are glued together along cone sections (Fig. 2) obtained by tracing the knot from a point. To correctly compute the world transitions, the user has to specify the group multiplication table for the knot.

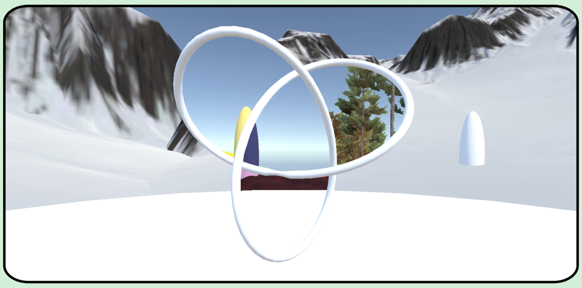


Figure 3: The trefoil knot with 4 portals.

Ariadne

Drawing homotopies by touch

This software enables the user to do basic topology on the (punctured) plane. He can create dots by touch, paths by dragging dots and homotopies by dragging paths. Paths can be cut out to define punctures, which leads to the distinction of homotopy classes of paths via the winding tool feature (Fig. 4), giving the ability to make first contact with the concept of invariants.

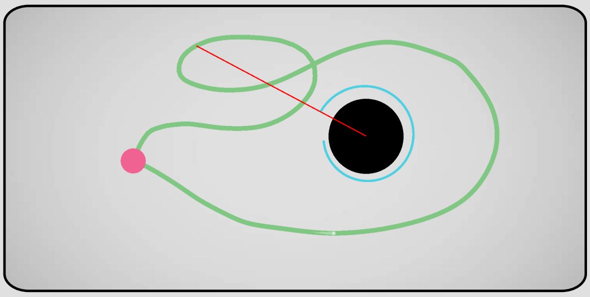


Figure 4: Winding number of a path is being computed as a spiral indicating the number of revolutions around a point

This makes it possible to learn some basic concepts from topology without having to use formulas, ranging from the number of connected components to the fundamental group, as paths can be concatenated to give the group structure. The software is designed to be used by children, or students in topology courses.

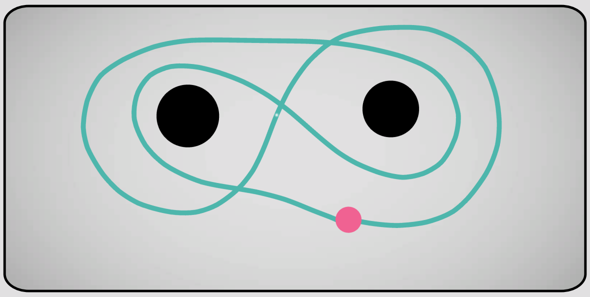


Figure 5: The Pochhammer contour, a non null-homotopic path with winding number zero around both punctures.

Creating surfaces in virtual reality

This software is an extension of Ariadne to 3D. The user can draw paths on surfaces such as a sphere or a torus and drag the paths to construct homotopies between paths.

In addition manifolds can be moved around and rotated. The simple closed paths drawn by the user can be cut, resulting in a manifold with boundary. These boundary curves can be filled with a cone (a cap), or surfaces can be glued together along their boundaries.
This enables the construction of all orientable closed surfaces, as any surface can also be duplicated, resulting in the workflow from Fig. 6.

These surfaces can then again be explored using paths and homotopies of paths.

These software projects are part of my PhD thesis in mathematics education.

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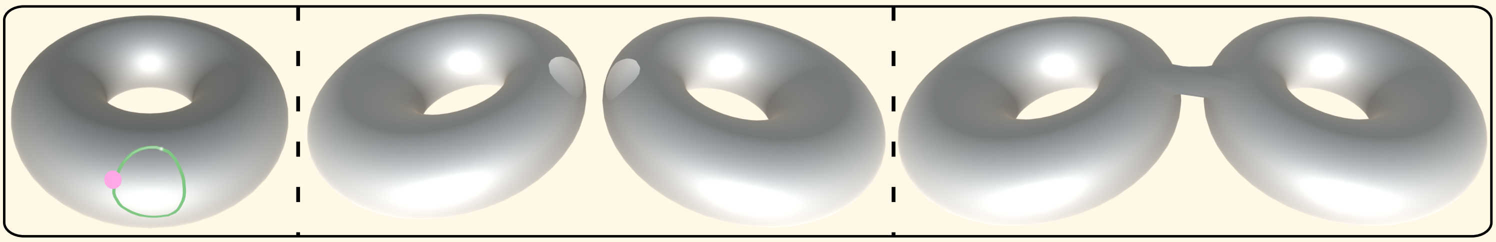

Figure 6: Construction of a double torus from a single torus in three steps. First, a dot and a closed path are drawn on the torus. Then the torus is cut along the path, duplicated and rotated. In the last step, the tori are glued together along the boundary curves.

A.4 Verknotete Portale

This is not a conference contribution, but is nevertheless listed as it is both informative and visually appealing. This poster was presented at the poster session of the summer festival of the Department of Didactics of Mathematics and Natural Sciences in 2019. It should be noted that as in the poster for CERME in Sec. A.2, a tablet-PC was attached to the middle of the screen. The tablet-PC showed a pre-recorded video of the software.

Verknotete Portale

Überlagerungen

Stellen Sie sich einen Draht aus einem magischen Material vor. Wenn Sie die beiden Enden des Drahts verbinden, öffnet sich ein Portal in eine andere Welt. Aber was geschieht, wenn man erst einen Knoten in den Draht macht und danach erst die Enden verbindet? Wie viele Portale öffnen sich, und in wie viele Welten? Diese Fragen behandelt man mathematisch unter der Überlagerungstheorie, in diesem Fall entlang dem Draht verzweigte Überlagerungen.

Was können Studenten über diese Theorie mithilfe der Software erlernen?

Embodiment & Virtual Reality

Embodiment bezeichnet die Theorie, dass Kognition mehr als nur das Gehirn umfasst, sondern mittels unserers Körpers und unserer Umgebung geschieht. Virtual Reality unterstützt diese verkörperte Art der Wahrnehmung, indem es den Nutzer vollständig in eine andere Welt versetzt, in der er die Mathematik mit seinem Körper erkunden kann.

Wie setzen Studenten ihren Körper in einer VR-Umgebung zur Wahrnehmung ein?

Umsetzung

Microworlds

Eine Microworld nach Seymour Papert ist eine konstruierte Realität, die so strukturiert ist, dass ein Lernender bestimmte mächtige Ideen erkunden oder intellektuelle Fähigkeiten ausüben kann. Der Stoff muss hinreichend eingegrenzt sein, um eine konstruktive Erforschung zu ermöglichen, aber hinreichend ergiebig, um bedeutende Entdeckungen machen zu können.

Entspricht die Software den Anforderungen an Microworlds?

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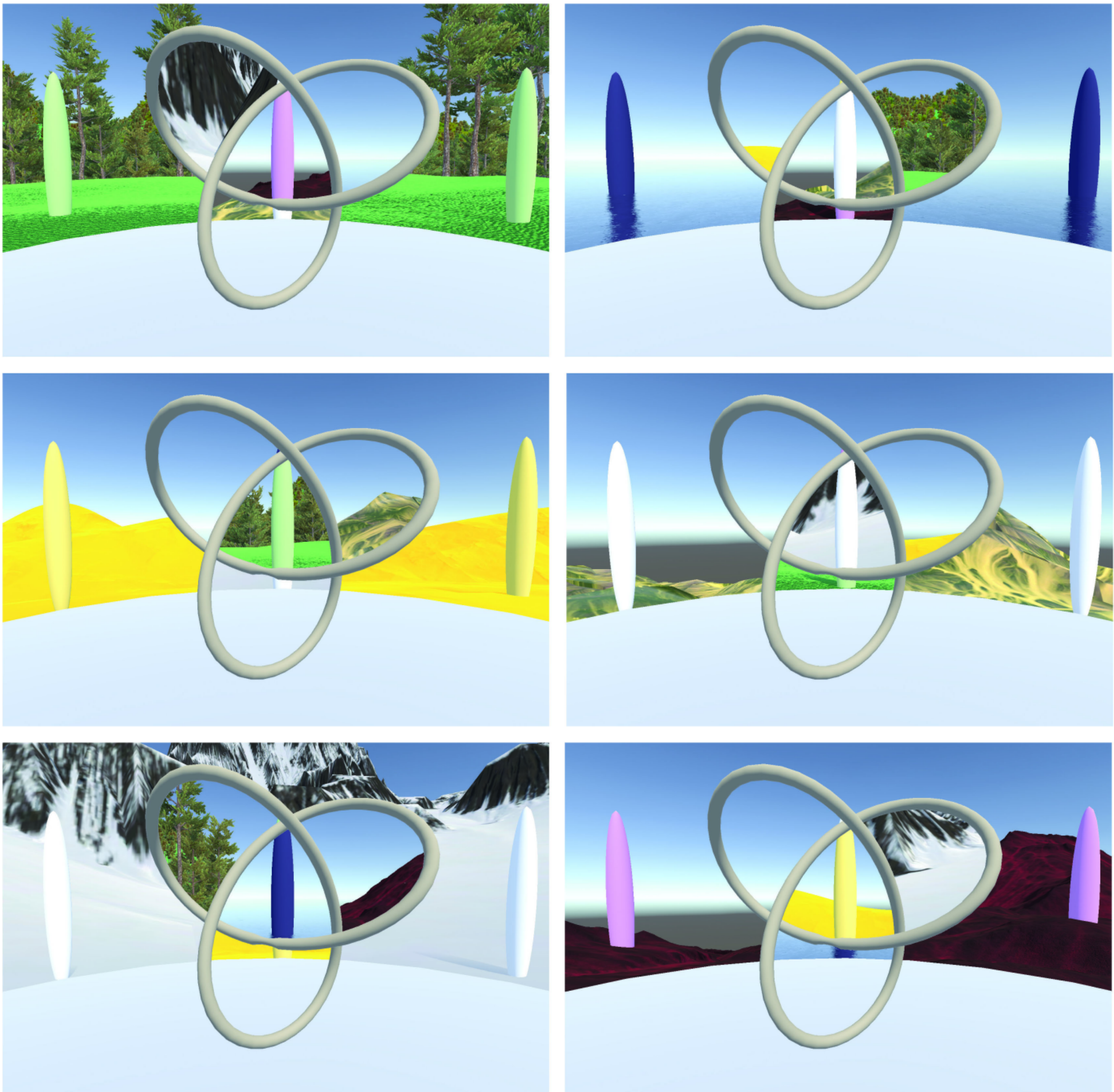
A.5 Contribution to *Illustrating Mathematics*

This is a short two-page presentation on the project about knotted portals in virtual reality, published in the book *Illustrating Mathematics* (Davis, 2020). This is a book named after the semester program at The Institute for Computational and Experimental Research in Mathematics, collecting work done by researchers attending the program.



In his video “Knots to Narnia,” William Thurston uses large wire knots to demonstrate his concept of knots as portals, where he actually steps through the knot to move back and forth between, in his conception, Earth and Narnia (see previous page). Virtual reality gives us the ability to bring to life this experience of actually stepping into other worlds and seeing, not just imagining, what it looks like on the other side. These pictures show the six-fold branched covering of order two of the trefoil knot, generating the dihedral group of the triangle. The three outer loops of the knot correspond to reflections, and the inner region to the rotation by 120° of the triangle.

To create this virtual reality experience, I had to learn a lot about cyclic branched covers of knots, as I had to construct them in the software implementation. This led me to discover a construction from Poul Heegaard’s dissertation from 1898, which could be implemented to simulate these portals in virtual reality. It consists of gluing a cone to the knot, which serves as the branch cut, along which the different worlds are glued together.

At first, I thought about implementing portals as surfaces somehow spanned by the knot. The most obvious choice was to try out Seifert surfaces, which turned out to be a dead end. The next attempt was to not construct the branched covering, but only simulate it through the knot projection on the screen. This approach used a variant of Reidemeister moves to keep track of the worlds the regions lead to, but it turned out to be quite complicated and unstable at the crossings. Luckily, I then found the reference to Heegaard’s construction in John Stillwell’s “Classical Topology and Combinatorial Group Theory” mentioned above, and I was able to implement the virtual reality world.

It was a challenge to reduce the computational load of the software, which has to compute which world to show for each pixel of both screens in the head-mounted display. This could be achieved by using shaders to offload much of the calculations on the graphics processing unit, thanks to some tips from Roice Nelson at the “Illustrating Topology and Geometry” workshop. The main computational load is now the rendering of the worlds. They could have been just color-coded, but I wanted to create worlds which are interesting enough to look at, but not so interesting as to take away the focus from the knot itself.

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virtual reality

Software download and information:
https://imaginary.org/program/knotportal
Thurston’s original video, “Knots to Narnia”:
https://www.youtube.com/watch?v=IKSrBt2kFD4

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A.6 Entwicklung von mathematischen Lernumgebungen als mathematikdidaktische Forschung

Peer reviewed publication in the conference proceedings of the “54. Jahrestagung der Gesellschaft für Didaktik der Mathematik” (Sümmermann, 2020a).

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Entwicklung von mathematischen Lernumgebungen als mathematikdidaktische Forschung

Die Digitalisierung verändert alle Aspekte unseres Lebens und damit auch die Art, wie wir Mathematik betreiben können. Beispiele dafür gibt es zur Genüge, von digital erschaffenen und von 3D-Druckern realisierten Modellen (z. B. die Skulptur in Abb. 1, links), über Visualisierungen (z. B. Hyperbolic VR in Abb. 1, rechts, KnotPortal in Abb. 3), bis hin zu interaktiver Software (z. B. Ariadne in Abb. 2).

Abb. 1: 3D-Druck des Menger-Schwamms (Bill Owens, CC-BY-CC0), Hyperbolic VR (http://h3.hypernom.com, unter MIT Licence Expat, https://www.tldrlegal.com/l/mit)

Diese Umgebungen müssen nicht notwendigerweise einen didaktischen Hintergrund haben, teils handelt es sich dabei nicht um ein Werkzeug zum Lernen, sondern es dient der Darstellung von Mathematik und folgt dabei einem Selbstzweck (wie die Skulptur in Abb. 1). Bei anderen Entwicklungen ist es durchaus das Ziel, einem Lernenden Einblicke zu ermöglichen; über einen neuen und ungewohnten Blickwinkel wie bei GeoGebra, welches es radikal einfacher macht, geometrische Objekte zu erzeugen und zu verändern als in traditionellen Papier-und-Bleistift-Umgebungen, oder sogar aus einer zuvor nicht möglichen Perspektive, wie in Projekten wie Hyperbolic VR (Hart, Hawksley, Matsumoto & Segerman, 2017) oder KnotPortal.

Die Frage, welche nach Erfahrung des Autors dabei oft gestellt wird, lautet, ob diese Entwicklung solcher letzterer didaktisch orientierten Umgebungen *an sich* ein Teil mathematikdidaktischer Forschung sein sollte.

Dabei geht es nicht um eine „ingenieurwissenschaftliche Sicht der Fachdidaktik als design science“ (Wittmann, 1998), bei der die Entwicklung von Lernumgebungen in Einheit mit der Evaluierung durch empirische Studien steht. Stattdessen geht es um das Ansehen der Entwicklung selbst als eigenständige wissenschaftliche Arbeit, welches zurzeit in der durch empirische Bildungsforschung geprägten Mathematikdidaktik keinen Platz findet. Dies

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geht hinaus über den stoffdidaktischen Mehrwert, den solche Umgebungen haben können.

Diese Frage haben andere Disziplinen wie Physik oder Informatik schon entschieden. Dort ist die Entwicklung z. B. eines Chips für einen Detektor (Germic, 2019) ein klarer Teil der Forschung, nicht nur eine Vorarbeit zu ebenjener.

Um diese Frage auch für die Mathematikdidaktik entschieden zu bejahen, werden im Folgendem zwei Projekte des Autors vorgestellt, bei denen die Entwicklung einer Software zu didaktischen Erkenntnissen geführt hat. Dadurch wird aufgezeigt, wie Entwicklung und Untersuchung solcher Umgebungen miteinander verwoben sind. Dabei stellen die Untersuchungen lediglich Beispiele von Erkenntnisprozessen dar, welche durch das Anfertigen von Umgebungen ausgelöst werden können.

1. Ariadne

Ariadne ist ein Projekt zum Konstruieren von und Interagieren mit Homotopien von Wegen auf der Ebene (Sümmermann, 2019). Der Nutzer kann durch Berührungsgesten auf einem geeigneten Gerät wie einem Tablet-PC oder einem Smartphone Punkte, Wege zwischen Punkten und Homotopien zwischen Wegen konstruieren.

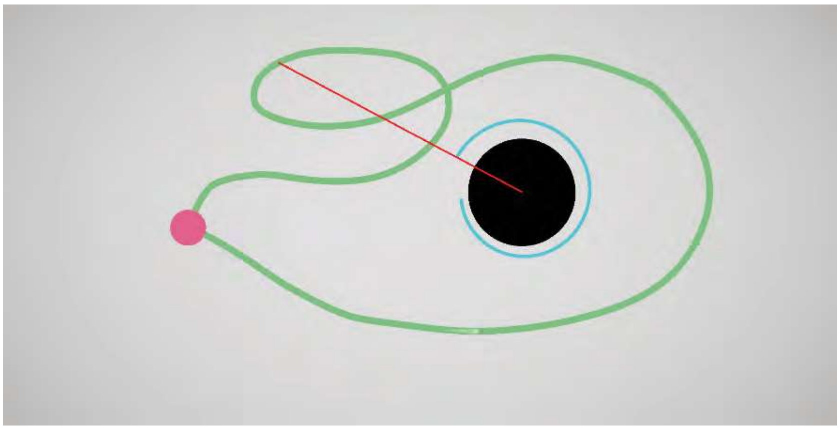

Abb. 2: Ein vom Nutzer gemalter Weg um eine Punktierung auf der Ebene in Ariadne. Es wird gerade die Windungszahl des Weges um die Punktierung berechnet.

Diese Konstruktionen, zusammen mit der Darstellung von Invarianten wie der Windungszahl, ermöglicht die mathematische Untersuchung von Phänomenen wie der Fundamentalgruppe der mehrfach punktierten Ebene.

1.1 Untersuchungen zu Ariadne

Bei der Entwicklung und Erprobung von Ariadne ist aufgefallen, dass die Lernumgebung zusammen mit dem Lerngegenstand der Topologie neue Arten der Beweisführung möglich macht (Sümmermann, Sommerhoff & Rott, in review). Dabei wurden Erkenntnisse über die Bedeutung von Beweisen in solchen Umgebungen an sich gewonnen.

Bisher ist die gängige Meinung, dass Computerumgebungen einen Beitrag zum Beweisen nur über das Sammeln von Beispielen und der damit verbundenen Erkundung eines Problems dienen können (Hanna, 1998). Mit Ariadne konnte beispielhaft gezeigt werden, dass man Argumentationen auf Software in einer solchen Art und Weise stützen kann, dass durchaus von Beweisen die Rede sein kann.

2. KnotPortal

KnotPortal ist eine Software zum Erzeugen von begehbaren dreidimensionalen Überlagerungen. In KnotPortal wird eine Frage angegangen, welche von Bill Thurston aufgeworfen wurde: Angenommen, man hätte einen Draht aus einem magischen Material. Sobald man die Enden des Drahtes verbindet, reißt man ein Portal in ein anderes Universum auf. Aber was geschieht, wenn man, bevor man die Enden verbindet, einen Knoten in den Draht bindet? Wie viele Portale entstehen und in wie viele (und welche) Welten führen sie?

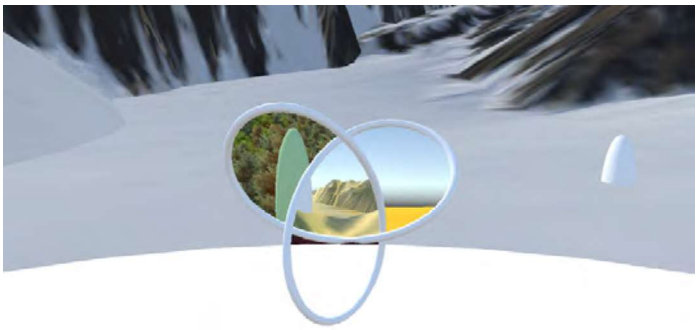

Abb. 3: Blick auf den Kleeblattknoten in KnotPortal. Man kann durch die gebildeten Portale in 4 andere Welten sehen.

In KnotPortal werden diese Portale in Virtual Reality dargestellt und sind frei begehbar. Damit kann der Nutzer durch Bewegen seines Körpers und seines Kopfes neue Perspektiven gewinnen und so besser verstehen, wie die Portale aufgebaut sind.

Mathematisch „verbirgt" sich hinter dieser Darstellung die Theorie der Überlagerungen, in diesem Fall entlang eines Knotens verzweigte Überlagerungen des Komplements des Knotens. Wie im Fall von Ariadne kann die interaktive Visualisierung ein Thema zugänglich machen, welches für die formale Aufarbeitung einiges an Vorarbeit verlangen würde.

2.1 Untersuchungen zu KnotPortal

KnotPortal stellt eine Umgebung bereit, in der Embodiment (Gerofsky, 2015) in einem mathematikdidaktischen Kontext untersucht werden kann. Dabei geht es tatsächlich um Nutzung des Körpers zur Erkenntnisgewinnung, nicht nur, wie in der Literatur oft beschrieben, Nutzung der Finger bei Einsatz von touchbasierter Software. KnotPortal gibt damit die Möglichkeit, den Begriff des Embodiments mit Inhalt zu füllen.

3. Implikationen

Ohne die Software wären die Erkenntnisse nicht möglich gewesen. Digitalisierung wird in der Mathematikdidaktik eine Worthülse sein, wenn nicht die Forschung aktiv an der Erstellung von Materialien und Untersuchung von dabei auftretenden neuen Phänomenen teilhat. Daher muss der Erstellung solcher Umgebungen der notwendige Platz eingeräumt werden.

Dies ist zu unterscheiden von der empirischen Untersuchung existierender Umgebungen mit dem Ziel einen bestimmten Effekt, wie zum Beispiel die Wirksamkeit auf den Lernerfolg, zu untersuchen. Es geht vielmehr um das Erkennen der Möglichkeiten und Grenzen, welche diese neuen Umgebungen bieten. Der Prozess der Digitalisierung darf dabei nicht als „Black Box“ behandelt werden, welcher aufs Geratewohl Objekte wie Apps o. ä. produziert, die dann durch die Forschung auf ihre Eigenarten untersucht werden. Vielmehr muss die Forschung es als ihren ureigenen Auftrag wahrnehmen, die Eigenarten eben dieses Prozesses zu untersuchen.

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B Eidesstattliche Erklärung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne die Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Werken dem Wortlaut oder dem Sinn nach entnommen wurden, sind als solche kenntlich gemacht. Ich versichere an Eides statt, dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen und eingebundenen Artikeln und Manuskripten – noch nicht veröffentlicht worden ist sowie, dass ich eine Veröffentlichung der Dissertation vor Abschluss der Promotion nicht ohne Genehmigung des Promotionsausschusses vornehmen werde. Die Bestimmungen dieser Ordnung sind mir bekannt. Darüber hinaus erkläre ich hiermit, dass ich die Ordnung zur Sicherung guter wissenschaftlicher Praxis und zum Umgang mit wissenschaftlichem Fehlverhalten der Universität zu Köln gelesen und sie bei der Durchführung der Dissertation zugrundeliegenden Arbeiten und der schriftlich verfassten Dissertation beachtet habe und verpflichte mich hiermit, die dort genannten Vorgaben bei allen wissenschaftlichen Tätigkeiten zu beachten und umzusetzen. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.

Teilpublikationen:

1. Ariadne – A Digital Topology Environment
2. Mathematics in the Digital Age: The Case of Simulation-Based Proofs
3. Knotted Portals in Virtual Reality
4. Embodied Mathematics: Forming Concepts in Topology by Moving Through Virtual Reality
5. On the Future of Design in Mathematics Education Research
6. Contribution to *Illustrating Mathematics*
7. Exploring Topology by Touch and in Virtual Reality
8. Drawing topology using Ariadne
9. Touchbasierte Lernumgebung für Homotopien
10. Verknotete Portale
11. Entwicklung von mathematischen Lernumgebungen als mathematikdidaktische Forschung