Some results on finite dimensional modules of current and loop algebras

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Kurzzusammenfassung

Auf dem Gebiet der endlichdimensionalen Moduln von Strom- und Schleifenalgebren wurden in den letzten zwei Jahrzehnten erhebliche Fortschritte gemacht. Darauf aufbauend haben wir in der vorliegenden Arbeit bewiesen, dass Demazuremoduln als Moduln für die Stromalgebra isomorph sind zu einem Fusionsprodukt “kleinerer“ Demazuremoduln, ausserdem berechneten wir die Zerlegungen der fundamentalen Demazuremoduln als $\mathfrak{g}$-Moduln. Durch die Kombination dieser beiden Resultate ergeben sich neue Dimensions- und Charakterformeln für Demazuremoduln. In einer Anwendung konstruierten wir affine Höchstgewichtsmoduln als direkte Limes von Fusionsprodukten von Demazuremoduln.
Darüber hinaus bewiesen wir, dass die fundamentalen Demazuremoduln als Moduln für die Stromalgebra isomorph sind zu Kirillov-Reshetikhin-Moduln, analog zeigten wir in der kombinatorischen Darstellungstheorie, dass die Kirillov-Reshetikhin-Kristalle jeweils die Demazure-Kristalle enthalten.
Wir fanden einen neuen, elementareren Beweis der Dimensionsformel für Weylmoduln der Schleifenalgebra im einfach verbundenen Fall. Für die getwisteten Schleifenalgebren haben wir eine Klassifikation der Weylmoduln angegeben und die bekannten Theoreme aus dem ungetwisteten Fall, d.h. Tensorproduktstruktur, Dimensions- und Charakterformeln, übertragen und bewiesen.

Abstract

In the field of finite dimensional modules of current and loop algebras a lot of research was done and progress was made in the last two decades. Resuming the discussions we showed in the present thesis that Demazure modules are fusion products of “smaller“ Demazure modules and calculated a decomposition of the fundamental Demazure modules as $\mathfrak{g}$-modules. Combining both results we obtained new dimension and character formulas for Demazure modules. As an application we constructed certain affine highest weight modules as the direct limit of fusion products of Demazure modules.
We proved that the fundamental Demazure modules are isomorphic to Kirillov-Reshetikhin modules for the current algebra. Furthermore we proved as an analogon in the combinatorial representation theory that the Kirillov-Reshetikhin crystal contains the Demazure crystal.
We give a new and elementary proof of the dimension formula of Weyl modules for the loop algebra in the simply laced case. For twisted loop algebras we provide a classification of Weyl modules and proved the analog theorems from the untwisted case, e.g. tensor product structure, dimension and character formulas.

Introduction

In the study of algebraic objetcs, e.g. Lie algebras, the representation theory becomes more and more important. To simplify the analysis of these objects by representing them as "easier" obejcts is discovered to be a powerful tool. Namely finding maps that preserve the structure into endomorphisms of vector spaces. The idea is to find symmetries and to decude from them properties of these obejcts. A natural question is to ask how many of these representations (resp. modules) can be found and how do they look like.
Lie algebras and their representations were initially defined in the 19th century and became very important in quantum mechanics in the second half of the last century. The classification of simple Lie algebras and their irreducible representations, the basic tools in the theory, where subjects to a lot of research in the last hundred and more years. In the last 50 years the research on variations of these simple Lie algebras, e.g. affine Lie algebras, quantum algebras, loop algebras etc., was growing explosively and became a huge and intensively studied field. The theory of finite dimensional representations of these algebras has been subject to a lot of research and articles in the recent years. See for example [1], [2], [3], [4], [5], [6], [8], [9], [10], [12], [14], [16], [17], [24], [28], [34] for different approaches and different aspects of this subject.

There are a lot of classification and structure problems still unsolved for these Lie algebras and their modules. For instance, Chari and Pressley gave a classification of the irreducible representations of loop algebras. Since the category of finite dimensional representations is not semisimple, this is far from being a classification of all finite dimensional representations. They provide a class of these representations, called Weyl modules, which are in some sense the maximal finite dimensional representations. The Weyl modules are far from being well understood. One of the main interests of this thesis is to provide progress in the analysis of the structure of these representations.

This thesis consists of four articles, written with various coauthors which are published yet or will be published soon

I G. Fourier, P. Littelmann
Tensor product structure of affine Demazure modules and limit constructions
Nagoya Math. J. Vol. 182(2006), 171-198

II G. Fourier, P. Littelmann,
Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions,
Advances in Mathematics, Volume 211, Issue 2, 1 June 2007, Pages 566-593

III G. Fourier, A. Schilling, M.Shimozono,

Demazure structure inside Kirillov-Reshetikhin crystals,
J. of Algebra 309 (2007), no. 1, 386–404

IV V. Chari, G. Fourier, P. Senesi
Weyl modules for the twisted loop algebras,
submitted to J. of Algebra

A special kind of finite dimensional modules are the Demazure modules. One of the most interesting modules for an affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ is the so called basic representation, that is the irreducible highest weight module of highest weight Λ_0. Let us denote by $V(m\Lambda_0)$ the irreducible highest weight module of highest weight $m\Lambda_0$, a multiple of the fundamental weight of the basic representation. For each element $\lambda^\vee$ of the coweight lattice $P^\vee$ one can associate an element of the (extended) affine Weyl group $t_{-\lambda^\vee}$ and so a Demazure module $D(m, \lambda^\vee)$. The Demazure module is, by definiton, a finite dimensional, cyclic module for the affine Borel subalgebra of $\widehat{\mathfrak{g}}$. $D(m, \lambda^\vee)$ is a module for the current algebra $\mathcal{C}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t]$ as well and so especially for the underlying simple complex Lie algebra $\mathfrak{g}$. So a natural question is to ask for the dimension and the $\mathfrak{g}$-character. In [I] we provide the following: Let $\lambda^\vee = \lambda_1^\vee + \cdots + \lambda_r^\vee$ be a sum of dominant coweights then

$$D(m, \lambda^\vee) \simeq D(m, \lambda_1^\vee) \otimes \ldots \otimes D(m, \lambda_r^\vee)$$

The proof deals with Demazure operators and the Demazure character formula ([13], [29]), so it is quite elementary.
To compute the dimension and give a character formula we provide a list of almost all of the “smallest“ Demazure modules, corresponding to fundamental coweights. The “bigger“ Demazure modules are just tensor products of these fundamental Demazure modules.

The Demazure modules are modules for the current algebra. Since the tensor product ist not cyclic, the tensor product factorization doesn't respect the current structure. One can modifiy the product structure to obtain a cyclic module. Let ϕ_a be the automorphism of the current algebra induced by $x \otimes t \mapsto x \otimes t - a$ for some complex number a. Let $a_1, \ldots, a_r$ be pairwise distinct complex numbers, and twist the action of $\mathcal{C}\mathfrak{g}$ on the i-th factor by ϕ_{a_i}. Then the tensor product of these twisted modules is cyclic. But this module is not graded, while the Demazure module has a grading induced by the operation of t. So in [15] the *fusion product* (denoted with $*$ instead of $\otimes$) is introduced and defined to be the graded associated module of the twisted tensor product, where the filtration is induced from the grading of the current algebra. In [II] we proved that

$$D(m, \lambda^\vee) \simeq D(m, \lambda_1^\vee)_{a_1} * \ldots * D(m, \lambda_r^\vee)_{a_r}.$$

This shows that the Demazure modules are fusion products of “smaller“ Demazure modules (this was proved before for $\mathfrak{sl}_n$ in [5] and [14]).

The tensor and fusion product structure can be used to construct the irreducible highest weight $\widehat{\mathfrak{g}}$-module $V(m\Lambda_0)$ as a $\mathcal{C}\mathfrak{g}$-module. It is known that this module is isomorphic as a module for the affine borel subalgebra to the direct limit of Demazure modules. For example $V(m\Lambda_0) \simeq \lim_{n\to\infty} D(m, n\lambda^\vee)$ for any nonzero dominant coweight $\lambda^\vee$. In [I] and [II] we provided a construction of $V(m\Lambda_0)$ as a $\mathcal{C}\mathfrak{g}$-module as a direct limit of fusion products of Demazure modules. A kind of this has been done in the framework of perfect crystals for classical Lie algebras and for a few exceptional types, see for example [19], [26] or by Littelmann paths in [31].

Another intensively studied class of modules are the Kirillov-Reshetikhin modules, a name that originally refers to evaluation modules of the Yangian. To each node of the Dynkin diagram of simple Lie algebra $\mathfrak{g}$ and each positive integer r, one associates a certain cyclic finite dimensional module $KR(i, r)$ with highest classical weight $r\omega_i$. In [4] a definition of these modules for the current algebra in terms of generators and relations is given.
In [I] and [II] we proved a connection to Demazure modules. Demazure modules are defined by Weyl group elements, and the extended affine Weyl group is isomorphic to a semi direct product of the classical Weyl group and the translations at the coweight lattice. By identifying the Cartan subalgebra with its dual via the invariant bilinear form, one can identify the coweight lattice with a sublattice of the weight lattice. The fundamental coweights $\omega_i^\vee$ are mapped to multiples of the fundamental weights, so $\nu(\omega_i^\vee) = d_i\omega_i$. The d_i are determined by the kernel of the Cartan matrix (and its transpose) and $\in \{1, 2, 3\}$. For example in the simply laced case, all the $d_i = 1$.
In [I] we have calculated an almost complete list of the $\mathfrak{g}$ decompositions of the 'smallest" Demazure modules $D(m, d_i\omega_i)$, all the classical Lie algebras are treated, a few nodes of exceptional types are missing. By comparing our list with a list of the $\mathfrak{g}$ decompositions of the Kirillov-Reshetikhin modules, recently calculated in [4], we showed, that the decomposition of $D(m, d_i\omega_i)$ coincides with the one of $KR(d_im, i)$ in the calculated cases. In [II] we proved in general ($\mathfrak{g}$ of arbitrary type, i an arbitrary node of $\mathfrak{g}$ and $m \in \mathbb{N}$)

$$D(m, d_i\omega_i) \simeq KR(d_im, \omega_i) \text{ as modules for } \mathcal{C}\mathfrak{g}$$

This was done before for the classical types, for example in [7]. The proof given in [II] compares the generators and relations of the Kirillov-Reshetikhin modules ([4]) with the generators and relations of the Demazure modules ([20], [32] and [36]). So especially the $\mathfrak{g}$ decompositions are the same, although they are still unknown in a few cases.

The notion of a Weyl module in this context was introduced in [9] for the affine Kac-Moody algebra and its quantized version. These modules can be described in terms of generators and relations, and they are characterized by the following universal property: any finite dimensional highest weight module which is generated by a one dimensional highest weight space, is a quotient of a Weyl module. This notion can be extended to the

category of finite dimensional graded representations of the current algebra ([5], [15]). So for each dominant weight λ of $\mathfrak{g}$ one associates a graded $\mathcal{C}\mathfrak{g}$ module $W(\lambda)$.
A natural question is again the dimension and character formula of this $W(\lambda)$. There had been conjectures for these formulas and they were proved in [34]. In [II] we gave a more elementary proof for these conjectures in the case where $\mathfrak{g}$ is of simply-laced type. The idea is to show that the Weyl module $W(\lambda)$ is nothing else than the Demazure module $D(1, \lambda)$. This was proved for the $\mathfrak{sl}_2$ case in [9] and for $\mathfrak{sl}_n$ in [5]. We proved the general simply-laced case by comparing the generators and relations for both modules and reducing the assertion to the $\mathfrak{sl}_2$ case. By identifying the Weyl module with a Demazure module we obtain the following formula (for $\lambda = \sum_{i=1}^n m_i\omega_i$ as a linear combination of the fundamental weights)

$$W(\lambda) \simeq W(\omega_1)^{*m_1} * \ldots * W(\omega_n)^{*m_n}$$

The Weyl modules for the loop algebra are parametrized by n-tuple (where n ist the rank of $\mathfrak{g}$) of polynomials (π_i) with constant term 1. In [9] it is proved, that the Weyl modules statisfy a tensor product property, i.e. the decomposition of the n-tuple into polynomials with distinct roots corresponds to a decomposition of the Weyl module into a tensor product of "smaller" Weyl module. The Weyl modules with a unique fixed root can be parametrized by a dominant weight and a nonzero complex number $W(\lambda, a)$.

The Weyl modules were also defined and analyzed for twisted quantum affine algebra in [12]. The non-quantum counterpart, namely the twisted loop modules, was missing. The main difficulty is, that the twisted affine algebras are not only built up from $A_1^{(1)}$'s but also from $A_2^{(2)}$.
So let $\mathcal{L}\mathfrak{g}^\sigma$ be the twisted loop algebra, let $\mathfrak{g}_0$ be the simple complex Lie fixpoint algebra. In [IV] we defined for each r-tuple (where r is the rank of $\mathfrak{g}_0$) (π_i^σ) of polynomials with constant term 1 a module of the twisted loop algebra with the suitable properties. We showed that for each (π_i^σ) there exists an untwisted Weyl module $W((\pi_i))$, an ideal I of $U(\mathcal{L}\mathfrak{g})$ and an ideal I^σ of $U(\mathcal{L}\mathfrak{g}^\sigma)$ such that the following holds

1. $W((\pi_i))$ is cyclic for $U(\mathcal{L}\mathfrak{g})/I$
2. $W((\pi_i^\sigma))$ is cyclic for $U(\mathcal{L}\mathfrak{g}^\sigma)/I^\sigma$
3. The natural embedding $U(\mathcal{L}\mathfrak{g}^\sigma) \hookrightarrow U(\mathcal{L}\mathfrak{g})$ becomes an isomorphism $U(\mathcal{L}\mathfrak{g}^\sigma)/I^\sigma \simeq U(\mathcal{L}\mathfrak{g})/I$

With these, one can deduce all the desired properties of twisted Weyl modules from their untwisted counterpart. So the twisted Weyl modules admit an analog tensor product structure, analog dimension and character formulas and analog universal properties. Some of the results in [IV] were proven in [9] and [12] but at least we simplify some of the proofs.

In [23] Kashiwara introduced the notion of a good $U_q'(\widehat{\mathfrak{g}})$–module, which, roughly speaking, is an irreducible finite dimensional $U_q'(\widehat{\mathfrak{g}})$–module with a crystal basis and a global basis. It is conjectured that the quantum Kirillov-Reshetikhin modules are good and that every good module is a tensor product of KR modules. The existence of their crystal basis is proved for nonexceptional types ([35]). The KR cyrstals corresponding to KR modules which are isomorphic to Demazure modules are those whose crystal basis is conjectured to be perfect. A few paper in the last decade made progress in defining these KR crystals also in a combinatorial way (for example [18]). In [IV] we proved, that these combinatorial crystals are, under some natural assumptions, unique. We showed, that the Demazure crystal [22] is a subcrystal of the Kirillov-Reshetikhin crystal, in the sense, that one has to add certain arrows labelled with 0 to obtain the KR-crystal. In [26] and [37] the combinatorial KR-crystal is given for a few types.

In [33] the crystals of the quantum Weyl modules are given in terms of Littelmann paths. In [II] we showed that, in the same sense as above, the Demazure path model is as submodel of the Weyl path model. I.e. the Weyl path model can be obtained from the Demazure path model by adding certain arrows labelled with 0. So the path model for the Weyl module is strongly determind by the Demazure path model.

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Tensor product structure of affine Demazure modules and limit constructions

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Abstract

Let $\mathfrak{g}$ be a simple complex Lie algebra, we denote by $\widehat{\mathfrak{g}}$ the affine Kac–Moody algebra associated to the extended Dynkin diagram of $\mathfrak{g}$. Let Λ_0 be the fundamental weight of $\widehat{\mathfrak{g}}$ corresponding to the additional node of the extended Dynkin diagram. For a dominant integral $\mathfrak{g}$–coweight $\lambda^\vee$, the Demazure submodule $V_{-\lambda^\vee}(m\Lambda_0)$ is a $\mathfrak{g}$–module. We provide a description of the $\mathfrak{g}$–module structure as a tensor product of "smaller" Demazure modules. More precisely, for any partition of $\lambda^\vee = \sum_j \lambda_j^\vee$ as a sum of dominant integral $\mathfrak{g}$–coweights, the Demazure module is (as $\mathfrak{g}$–module) isomorphic to $\bigotimes_j V_{-\lambda_j^\vee}(m\Lambda_0)$. For the "smallest" case, $\lambda^\vee = \omega^\vee$ a fundamental coweight, we provide for $\mathfrak{g}$ of classical type a decomposition of $V_{-\omega^\vee}(m\Lambda_0)$ into irreducible $\mathfrak{g}$–modules, so this can be viewed as a natural generalization of the decomposition formulas in [14] and [17]. A comparison with the $U_q(\mathfrak{g})$–characters of certain finite dimensional $U_q'(\widehat{\mathfrak{g}})$–modules (Kirillov–Reshetikhin–modules) suggests furthermore that all quantized Demazure modules $V_{-\lambda^\vee,q}(m\Lambda_0)$ can be naturally endowed with the structure of a $U_q'(\widehat{\mathfrak{g}})$–module. We prove, in the classical case (and for a lot of non-classical cases), a conjecture by Kashiwara [11], that the "smallest" Demazure modules are, when viewed as $\mathfrak{g}$-modules, isomorphic to some KR-modules. For an integral dominant $\widehat{\mathfrak{g}}$–weight Λ let $V(\Lambda)$ be the corresponding irreducible $\widehat{\mathfrak{g}}$–representation. Using the tensor product decomposition for Demazure modules, we give a description of the $\mathfrak{g}$–module structure of $V(\Lambda)$ as a semi-infinite tensor product of finite dimensional $\mathfrak{g}$–modules. The case of twisted affine Kac-Moody algebras can be treated in the same way, some details are worked out in the last section.

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Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra, we denote by $\widehat{\mathfrak{g}}$ the affine Kac–Moody algebra associated to the extended Dynkin diagram of $\mathfrak{g}$. (The twisted case is considered separately in the last section). Let Λ_0 be the fundamental weight of $\widehat{\mathfrak{g}}$ corresponding to the additional node of the extended Dynkin diagram. The basic representation $V(\Lambda_0)$ is one of the most important representations of $\widehat{\mathfrak{g}}$ because its structure determines strongly the structure of all other highest weight representations $V(\Lambda)$, Λ an arbitrary dominant integral weight for $\widehat{\mathfrak{g}}$.

Let $P^\vee$ be the coweight lattice of $\mathfrak{g}$. An element $\lambda^\vee$ in the coroot lattice, can be viewed as an element of the affine Weyl group W^{aff} (see section 1), and one can associate to $\lambda^\vee$ the Demazure submodule $V_{\lambda^\vee}(\Lambda)$ of $V(\Lambda)$ (see section 2).

Actually, this construction generalizes to arbitrary $\lambda^\vee \in P^\vee$ in the following way: one can write $\lambda^\vee$ as $w\sigma \in \widetilde{W}^{\mathrm{aff}}$ in the extended affine Weyl group, where $w \in W^{\mathrm{aff}}$ and σ corresponds to an automorphism of the Dynkin diagram of $\widehat{\mathfrak{g}}$. Denote by $V_{\lambda^\vee}(\Lambda)$ the Demazure submodule $V_w(\Lambda')$ of the highest weight module $V(\Lambda')$, where $\Lambda' = \sigma(\Lambda)$.

If $\lambda^\vee$ is a dominant coweight, then the Demazure module $V_{-\lambda^\vee}(m\Lambda_0)$ is in fact a $\mathfrak{g}$–module, and it is interesting to study its structure as $\mathfrak{g}$–module. So one would like to get a restriction formula expressing $V_{-\lambda^\vee}(m\Lambda_0)$ as a direct sum of simple $\mathfrak{g}$–representations. We write $\overline{V}_{-\lambda^\vee}(m\Lambda_0)$ for the Demazure module viewed as a $\mathfrak{g}$–module.

A first reduction step is the following theorem describing the Demazure module as a tensor product. Such a decomposition formula for Demazure modules was first observed by Sanderson [21] in the affine rank two case, and was later studied in the case of classical groups in the framework of perfect crystals for example in [15], see [7] for a more complete account. We provide in this article a description of the Demazure module as a tensor product of modules of the same type, but for "smaller coweights". More precisely, let $\lambda^\vee$ be a dominant coweight and suppose we are given a decomposition

$$\lambda^\vee = \lambda_1^\vee + \lambda_2^\vee + \ldots + \lambda_r^\vee$$

of $\lambda^\vee$ as a sum of dominant coweights. The following theorem is a generalization of a result in [17], where the statement has been proved in the case $m = 1$ and under the additional assumption that all the $\lambda_i^\vee$ are minuscule fundamental weights, and the decomposition formulas in [5], [6], [14] and [15], where in the framework of perfect crystals for classical groups many cases have been discussed. (The corresponding version for a twisted Kac-Moody algebra can be found in section 4.)

Theorem 1. *For all $m \geq 1$, we have an isomorphism of $\mathfrak{g}$–representations between the Demazure module $\overline{V}_{-\lambda^\vee}(m\Lambda_0)$ and the tensor product of Demazure modules:*

$$\overline{V}_{-\lambda^\vee}(m\Lambda_0) \simeq \overline{V}_{-\lambda_1^\vee}(m\Lambda_0) \otimes \overline{V}_{-\lambda_2^\vee}(m\Lambda_0) \otimes \cdots \otimes \overline{V}_{-\lambda_r^\vee}(m\Lambda_0).$$

Of course, to analyse the structure of $\overline{V}_{-\lambda^\vee}(m\Lambda_0)$ as a $\mathfrak{g}$–module, the simplest way is to take a decomposition of $\lambda^\vee$ as a sum of fundamental coweights $\lambda^\vee = \sum a_i\omega_i^\vee$. So by Theorem 1, it remains to describe the structure of the $\overline{V}_{-\omega_i^\vee}(m\Lambda_0)$ as a $\mathfrak{g}$–module. We give such a description below for all fundamental coweights for the classical groups. For the exceptional groups we give the decomposition in the cases interesting for the limit constructions considered later. The enumeration of the fundamental weights is as in [1], we write ω_0 for the trivial weight. For more details on the notation see section 2, we only recall here that we use the abbreviations $V_{-\omega^\vee}(m\Lambda_0)$ and $V_{-\nu(\omega^\vee)}(m\Lambda_0)$ for the Demazure submodule associated to the translation $t_{-\nu(\omega^\vee)}$, viewed as an element in the extended affine Weyl group. We write V^* for the contragradient dual of a representation V. Many special cases of the list below have been calculated before, for example by the Kyoto school ([11], [14], [15], [22]). (The corresponding version for a twisted Kac-Moody algebra can be found in section 4.)

Theorem 2. *Let $\omega^\vee$ be a fundamental coweight and let $V_{-\omega^\vee}(m\Lambda_0)$ be the associated Demazure module. Viewed as a $\mathfrak{g}$–module, $\overline{V}_{-\omega^\vee}(m\Lambda_0)$ decomposes into the direct sum of irreducible $\mathfrak{g}$–modules as follows:*

- *Type $\mathtt{A}_n$: $\overline{V}_{-\omega_i^\vee}(m\Lambda_0) = \overline{V}_{-\omega_i}(m\Lambda_0) \simeq V(m\omega_i)^*$ as $\mathfrak{sl}_n$–module for all $i = 1, \dots, n$.*
- *Type $\mathtt{B}_n$: Set $\theta = 0$ for i even and $\theta = 1$ for i odd, then we have for $1 \le i < n$:*

$$\overline{V}_{-\omega_i^\vee}(m\Lambda_0) = \overline{V}_{-\omega_i}(m\Lambda_0) \simeq \bigoplus_{a_i+a_{i-2}+\ldots+a_\theta=m} V(a_i\omega_i + a_{i-2}\omega_{i-2} + \ldots + a_\theta\omega_\theta)$$

  and for $i = n$:

$$\overline{V}_{-\omega_n^\vee}(m\Lambda_0) = \overline{V}_{-2\omega_n}(m\Lambda_0) \simeq \bigoplus_{a_n+a_{n-2}+\ldots+a_\theta=m} V(2a_n\omega_n + a_{n-2}\omega_{n-2} + \ldots + a_\theta\omega_\theta)$$

  as $\mathfrak{so}_{2n+1}$–module.

- *Type $\mathtt{C}_n$: for $j < n$ we have*

$$\overline{V}_{-\omega_j^\vee}(m\Lambda_0) = \overline{V}_{-2\omega_j}(m\Lambda_0) \simeq \bigoplus_{a_1+\ldots+a_j\le m} V(2a_1\omega_1 + \ldots + 2a_j\omega_j)$$

  and for $j = n$: $\overline{V}_{-\omega_n^\vee}(m\Lambda_0) = \overline{V}_{-\omega_n}(m\Lambda_0) \simeq V(m\omega_n)$ as $\mathfrak{sp}_n$–module.

- *Type $\mathtt{D}_n$: Set $\theta = 0$ for i even and $\theta = 1$ for i odd, then we have for $2 \le i \le n-2$:*

$$\overline{V}_{-\omega_i^\vee}(m\Lambda_0) = \overline{V}_{-\omega_i}(m\Lambda_0) \simeq \bigoplus_{a_i+a_{i-2}+\ldots+a_\theta=m} V(a_i\omega_i + a_{i-2}\omega_{i-2} + \ldots + a_\theta\omega_\theta)$$

  and for $i = 1, n-1, n$: $\overline{V}_{-\omega_i^\vee}(m\Lambda_0) = \overline{V}_{-\omega_i}(m\Lambda_0) \simeq V(m\omega_i)^*$ as $\mathfrak{so}_{2n}$–module.

- *Type* $\mathtt{E}_6$*:*

$$\overline{V}_{-\omega_i^\vee}(m\Lambda_0) = \overline{V}_{-\omega_i}(m\Lambda_0) \simeq V(m\omega_i)^* \textit{ for } i = 1, 6$$

$$\overline{V}_{-\omega_2^\vee}(m\Lambda_0) = \overline{V}_{-\omega_2}(m\Lambda_0) \simeq \bigoplus_{r=0}^{m} V(r\omega_2)$$

as $\mathtt{E}_6$*–module.*

- *Type* $\mathtt{E}_7$*:*

$$\overline{V}_{-\omega_7^\vee}(m\Lambda_0) = \overline{V}_{-\omega_7}(m\Lambda_0) \simeq V(m\omega_7)$$

$$\overline{V}_{-\omega_1^\vee}(m\Lambda_0) = \overline{V}_{-\omega_1}(m\Lambda_0) \simeq \bigoplus_{r=0}^{m} V(r\omega_1)$$

as $\mathtt{E}_7$*–module.*

- *Type* $\mathtt{E}_8$*:* $\overline{V}_{-\omega_8^\vee}(m\Lambda_0) = \overline{V}_{-\omega_8}(m\Lambda_0) \simeq \bigoplus_{r=0}^{m} V(r\omega_8)$ *as* $\mathtt{E}_8$*–module.*

- *Type* $\mathtt{F}_4$*:*

$$\overline{V}_{-\omega_1^\vee}(m\Lambda_0) = \overline{V}_{-\omega_1}(m\Lambda_0) \simeq \bigoplus_{r=0}^{m} V(r\omega_1)$$

and

$$\overline{V}_{-\omega_4^\vee}(m\Lambda_0) = \overline{V}_{-2\omega_4}(m\Lambda_0) \simeq \bigoplus_{r+s\leq m} V(r\omega_1 + s2\omega_4)$$

as $\mathtt{F}_4$*–module.*

- *Type* $\mathtt{G}_2$*:* $\overline{V}_{-\omega_2^\vee}(m\Lambda_0) = \overline{V}_{-\omega_2}(m\Lambda_0) \simeq \bigoplus_{r=0}^{m} V(r\omega_2)$ *as* $\mathtt{G}_2$*–module.*

There is a very interesting conjectural connection with certain $U_q'(\widehat{\mathfrak{g}})$–modules. Here $U_q'(\widehat{\mathfrak{g}})$ denotes the quantized affine algebra without derivation.
Let $KR(m\omega_i)$ be the Kirillov–Reshetikhin–module for a multiple of a fundamental weight of $\mathfrak{g}$, for the precise definition see [5], it is irreducible as $U_q'(\widehat{\mathfrak{g}})$ and the highest weight, when viewed as a $U_q(\mathfrak{g})$-module, is $m\omega_i$. In [10] Kashiwara introduced the notion of a good $U_q'(\widehat{\mathfrak{g}})$–module, which, roughly speaking, is an irreducible finite dimensional $U_q'(\widehat{\mathfrak{g}})$–module with a crystal basis and a global basis, and he proved that the tensor product of good modules is a good module. It is conjectured that the KR-modules are good. For all fundamental $\mathfrak{g}$–weights ω_i Kashiwara constructed this irreducible finite-dimensional integrable $U_q'(\widehat{\mathfrak{g}})$–module $KR(\omega_i)$ and showed that it is good and even more that the crystal is isomorphic to a certain generalized Demazure crystal as a $\mathfrak{g}$–crystal.

Let $c_k^\vee = \frac{a_k}{a_k^\vee}$ (for the definition of the a_k see section 1.1) and $l \in \mathbb{N}$. Let $KR(lc_k^\vee\omega_k)$ be the Kirillov–Reshetikhin–module for $U_q'(\widehat{\mathfrak{g}})$ associated to the weight $lc_k^\vee\omega_k$. It is more generally conjectured that the $KR(lc_k^\vee\omega_k)$ the crystal is isomorphic to the crystal of a Demazure module, after omitting the 0–arrows in both crystals [11]. Chari and the Kyoto school have calculated for classical Lie-algebras and some fundamental weights for non-classical Lie-algebras the decomposition of the Kirillov–Reshetikhin module $KR(lc_k^\vee\omega_k))$ into irreducible $U_q(\mathfrak{g})$–modules [2]. By comparing the $U_q(\mathfrak{g})$–structure of the Kirillov–Reshetikhin module $KR(lc_k^\vee\omega_k))$ with the list in Theorem 2 we conclude:

Corollary 1. *In all cases stated in Theorem 2, the Demazure module* $(\overline{V}_{-\omega^\vee,q}(l\Lambda_0))$ *and the Kirillov–Reshetikhin module* $KR(lc_k^\vee{\omega_k}^*)$ *are, as* $U_q(\mathfrak{g})$*–modules, isomorphic.*

In particular, if $KR(lc_k^\vee{\omega_k}^*)$ has a crystal basis, then the crystal is isomomorphic to the crystal of $(V_{-\omega_k^\vee}(l\Lambda_0))$ after omitting the arrows with label zero. By using the $U_q(\mathfrak{g})$–module isomorphism, we see that (in the cases above) the quantized Demazure modules $V_{-\omega_k^\vee,q}(l\Lambda_0)$ can be equipped with the structure of an irreducible $U_q'(\widehat{\mathfrak{g}})$–module. In fact, using the Theorem 1, we see that for classical groups all quantized Demazure modules $V_{-\lambda^\vee,q}(l\Lambda_0)$, $\lambda^\vee$ a dominant coweight, can be equipped with the structure of an $U_q'(\widehat{\mathfrak{g}})$–module. Of course, in the exceptional case the same argument shows that when $\lambda^\vee$ can be written as linear combinations of the fundamental weights listed in Theorem 2, then again $V_{-\lambda^\vee,q}(l\Lambda_0)$ can be equipped with the structure of an $U_q'(\widehat{\mathfrak{g}})$–module. This leads to the following:

Conjecture 1. Let $\mathfrak{g}$ be a semisimple Lie algebra, let $U_q(\widehat{\mathfrak{g}})$ be the associated untwisted quantum affine algebra and let $U_q'(\widehat{\mathfrak{g}})$ be its subalgebra without derivation. For all dominant coweights $\lambda^\vee$ and for all $l > 0$, the Demazure module $V_{-\lambda^\vee,q}(l\Lambda_0)$ can be endowed with the structure of a $U_q'(\widehat{\mathfrak{g}})$–module admitting a crystal basis. Its crystal graph is isomorphic to the crystal of the Demazure module, after omitting the arrows labelled with zero.

The tensor decomposition structure in Theorem 1 holds in the following more general situation. Let Λ_i, $1 \leq i \leq n$, be a fundamental weight of $\widehat{\mathfrak{g}}$ such that the corresponding coweight $\omega_i^\vee$ is minuscule. Let $\lambda^\vee$ be a dominant coweight and suppose we are given a decomposition
$$\lambda^\vee = \omega_i^\vee + \lambda_2^\vee + \ldots + \lambda_r^\vee$$
of $\lambda^\vee$ as a sum of dominant coweights and denote ω_i^* the highest weight of the irreducible $\mathfrak{g}$–module $V(\omega_i)^*$.

Theorem 1 A. *For all* $m \geq 0$ *and* $s \geq 1$*, we have an isomorphism of* $\mathfrak{g}$*–representations between the Demazure module* $\overline{V}_{-\lambda^\vee}(m\Lambda_0 + s\Lambda_i)$ *and the tensor product of Demazure modules:*

$$\overline{V}_{-\lambda^\vee}(m\Lambda_0 + s\Lambda_i) \simeq V(s\omega_i^*) \otimes \overline{V}_{-\lambda_2^\vee}((m+s)\Lambda_0) \otimes \cdots \otimes \overline{V}_{-\lambda_r^\vee}((m+s)\Lambda_0).$$

Let Λ be an arbitrary dominant integral weight for $\widehat{\mathfrak{g}}$. The $\widehat{\mathfrak{g}}$–module $V(\Lambda)$ is the direct limit of the Demazure-modules $V_{-N\lambda^\vee}(\Lambda)$ for some dominant, integral, nonzero coweight of $\mathfrak{g}$. We give a construction of the $\mathfrak{g}$–module $\overline{V}(\Lambda)$ as a direct limit of tensor products of Demazure modules. This has been done before in the case of classical Lie-algebras for $\Lambda = r\Lambda_0$ (and corresponding weights obtained by automorphisms as in the statement of Theorem 2) by Kang, Kashiwara, Kuniba, Misra et al. [7], [14] via the theory of perfect crystals. In addition they have also considered some special weights in the case of non-classical groups. For $\mathtt{G}_2$, such a construction has been given by Yamane [22]. For the Lie algebras of type $\mathtt{E}_6$ and $\mathtt{E}_7$ a construction (only for the case $\Lambda = \Lambda_0$) was given by Peter Magyar [17] using the path model.

We provide in this article such a direct limit construction for arbitrary simple Lie algebras $\mathfrak{g}$. Let Λ be a dominant, integral weight for $\widehat{\mathfrak{g}}$, then we can write $\Lambda = r\Lambda_0 + \lambda$ with λ dominant, integral for $\mathfrak{g}$.

Let W be the $\mathfrak{g}$-module $W := \overline{V}_{-\theta^\vee}(r\Lambda_0)$, where θ is the highest root of $\mathfrak{g}$, we show that W contains a unique one-dimensional submodule. Fix $w \neq 0$ a $\mathfrak{g}$-invariant vector in W. Let $V(\lambda)$ be the irreducible $\mathfrak{g}$-module with highest weight λ and define the $\mathfrak{g}$-module $V^\infty_{\lambda,r}$ to be the direct limit of:

$$V^\infty_{\lambda,r}: \quad V(\lambda) \hookrightarrow W \otimes V(\lambda) \hookrightarrow W \otimes W \otimes V(\lambda) \hookrightarrow W \otimes W \otimes W \otimes V(\lambda) \hookrightarrow \dots$$

where the inclusions are always given by taking a vector u to its tensor product $u \mapsto w \otimes u$ with the fixed $\mathfrak{g}$-invariant vector in W.

Recall the notation $\overline{V}(\Lambda)$ for $V(\Lambda)$ viewed as a $\mathfrak{g}$-module. (The corresponding version for a twisted Kac-Moody algebra can be found in section 4.)

Theorem 3. *For any integral dominant weight Λ of $\widehat{\mathfrak{g}}$, $\Lambda = r\Lambda_0 + \lambda$, the $\mathfrak{g}$-modules $V^\infty_{\lambda,r}$ and $\overline{V}(\Lambda)$ are isomorphic.*

Remark 1. The choice of W is convenient because it avoids case by case considerations. But, in fact, one could choose any other module $W = V_{-\mu^\vee}(r\Lambda_0)^{\otimes m}$, where $V_{-\mu^\vee}(r\Lambda_0)$ is the Demazure module for a dominant, integral, nonzero coweight $\mu^\vee$ and m is such that $V_{-\mu^\vee}(r\Lambda_0)^{\otimes m}$ contains a one-dimensional submodule.

1 The affine Kac–Moody algebra

1.1 Notations and basics

In this section we fix the notation and the usual technical padding. Let $\mathfrak{g}$ be a simple complex Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b} \supseteq \mathfrak{h}$. Denote $\Phi \subseteq \mathfrak{h}^*$ the root system of $\mathfrak{g}$, and, corresponding to the choice of $\mathfrak{b}$, let Φ^+ be the set of positive roots and let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the corresponding basis of Φ.

For a root $\beta \in \Phi$ let $\beta^\vee \in \mathfrak{h}$ be its coroot. The basis of the dual root system (also called the coroot system) $\Phi^\vee \subset \mathfrak{h}$ is denoted $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$.

We denote throughout the paper by $\Theta = \sum_{i=1}^n a_i\alpha_i$ the highest root of Φ, by $\Theta^\times = \sum_{i=1}^n a_i^\times \alpha_i^\vee$ the highest root of $\Phi^\vee$ and by $\Theta^\vee = \sum_{i=1}^n a_i^\vee \alpha_i^\vee$ the coroot of Θ. Note that $\Theta^\vee \neq \Theta^\times$ in general. The Weyl group W of Φ is generated by the simple reflections $s_i = s_{\alpha_i}$ associated to the simple roots.

Let P be the weight lattice of Φ and let $P^\vee$ be the weight lattice of the dual root system $P^\vee$. Denote $P^+ \subset P$ the subset of dominant weights and let $\mathbb{Z}[P]$ be the group algebra of P. For a simple root α_i let ω_i be the corresponding fundamental weight, we use the same notation for simple coroots and coweights. Recall that ω_i is called *minuscule* if $a_i^\times = 1$, and the coweight $\omega_i^\vee$ is called *minuscule* if $a_i = 1$.

Denote by $\mathfrak{h}_\mathbb{R} \subset \mathfrak{h}$ the real span of the coroots and let $\mathfrak{h}_\mathbb{R}^* \subset \mathfrak{h}^*$ be the real span of the fundamental weights. We fix a W–invariant scalar product $(\cdot,\cdot)$ on $\mathfrak{h}$ and normalize it such that the induced isomorphism $\nu : \mathfrak{h}_\mathbb{R} \longrightarrow \mathfrak{h}_\mathbb{R}^*$ maps $\Theta^\vee$ to Θ. With the notation as above it follows for the weight lattice $P^\vee$ of the dual root system $\Phi^\vee$ that

$$\nu(\alpha_i^\vee) = \frac{a_i}{a_i^\vee}\alpha_i \quad \text{and} \quad \nu(\omega_i^\vee) = \frac{a_i}{a_i^\vee}\omega_i, \quad \forall\, i = 1, \ldots, n.$$

Let $\widehat{\mathfrak{g}}$ be the affine Kac–Moody algebra corresponding to the extended Dynkin diagram of $\mathfrak{g}$ (see [8], Chapter 7):

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_\mathbb{C} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

Here d denotes the derivation $d = t\frac{d}{dt}$ and K is the canonical central element. The Lie algebra $\mathfrak{g}$ is naturally a subalgebra of $\widehat{\mathfrak{g}}$. In the same way, $\mathfrak{h}$ and $\mathfrak{b}$ are subalgebras of the Cartan subalgebra $\widehat{\mathfrak{h}}$ respectively the Borel subalgebra $\widehat{\mathfrak{b}}$ of $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \widehat{\mathfrak{b}} = \mathfrak{b} \oplus \mathbb{C}K \oplus \mathbb{C}d \oplus \mathfrak{g} \otimes_\mathbb{C} t\mathbb{C}[t] \tag{1}$$

Denote by $\widehat{\Phi}$ the root system of $\widehat{\mathfrak{g}}$ and let $\widehat{\Phi}^+$ be the subset of positive roots. The positive non-divisible imaginary root in $\widehat{\Phi}^+$ is denoted δ. The simple roots are $\widehat{\Delta} = \{\alpha_0\} \cup \Delta$ where $\alpha_0 = \delta - \Theta$. Let $\Lambda_0, \ldots, \Lambda_n$ be the corresponding fundamental weights, then for $i = 1, \ldots, n$ we have

$$\Lambda_i = \omega_i + a_i^\vee \Lambda_0. \tag{2}$$

The decomposition of $\widehat{\mathfrak{h}}$ in (1) has its corresponding version for the dual space $\widehat{\mathfrak{h}}^*$:

$$\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta, \tag{3}$$

here the elements of $\mathfrak{h}^*$ are extended trivially, $\langle \Lambda_0, \mathfrak{h}\rangle = \langle \Lambda_0, d\rangle = 0$ and $\langle \Lambda_0, K\rangle = 1$, and $\langle \delta, \mathfrak{h}\rangle = \langle \delta, K\rangle = 0$ and $\langle \delta, d\rangle = 1$. Let $\widehat{\Delta}^\vee = \{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \widehat{\mathfrak{h}}$ be the corresponding basis of the coroot system, then $\alpha_0^\vee = K - \Theta^\vee$.

Set $\widehat{\mathfrak{h}}^*_{\mathbb{R}} = \mathbb{R}\delta + \sum_{i=0}^n \mathbb{R}\Lambda_i$, by (3) and (2) we have $\mathfrak{h}^*_{\mathbb{R}} \subseteq \widehat{\mathfrak{h}}^*_{\mathbb{R}}$. The affine Weyl group W^{aff} is generated by the reflections $s_0, s_1, ..., s_n$, where again $s_i = s_{\alpha_i}$ for a simple root. The cone $\widehat{C} = \{\Lambda \in \widehat{\mathfrak{h}}^*_{\mathbb{R}} | \langle \Lambda, \alpha_i^\vee \rangle \geq 0, i = 0, ..., n\}$ is the fundamental Weyl chamber for $\widehat{\mathfrak{g}}$.

We put a $\widehat{\ }$ on (almost) everything related to $\widehat{\mathfrak{g}}$. Let $\widehat{P}$ be the weight lattice of $\widehat{\mathfrak{g}}$, let $\widehat{P}^+$ be the subset of dominant weights and let $\mathbb{Z}[\widehat{P}]$ be the group algebra of $\widehat{P}$. Recall the following properties of δ (see for example [8], Chapter 6):

$$\langle \delta, \alpha_i^\vee \rangle = 0 \, \forall \, i = 0, \ldots, n \quad w(\delta) = \delta \, \forall \, w \in W^{\mathrm{aff}}, \quad \langle \alpha_0, \alpha_i^\vee \rangle = -\langle \Theta, \alpha_i^\vee \rangle \text{ for } i \geq 1 \tag{4}$$

Put $a_0 = a_0^\vee = 1$ and let $A = (a_{i,j})_{0 \leq i,j \leq n}$ be the (generalized) Cartan matrix of $\widehat{\mathfrak{g}}$. We have a non–degenerate symmetric bilinear form $(\cdot, \cdot)$ on $\widehat{\mathfrak{h}}$ defined by ([8], Chapter 6)

$$\begin{cases} (\alpha_i^\vee, \alpha_j^\vee) = \frac{a_j}{a_j^\vee} a_{i,j} & i, j = 0, \ldots, n \\ (\alpha_i^\vee, d) = 0 & i = 1, \ldots, n \\ (\alpha_0^\vee, d) = 1 & (d, d) = 0. \end{cases} \tag{5}$$

The corresponding isomorphism $\nu : \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^*$ maps

$$\nu(\alpha_i^\vee) = \frac{a_i}{a_i^\vee} \alpha_i, \quad \nu(K) = \delta, \quad \nu(d) = \Lambda_0.$$

Denote by $\mathfrak{g}_{\mathrm{sc}}$ the subalgebra of $\widehat{\mathfrak{g}}$ generated by $\mathfrak{g}$ and $\alpha_0^\vee = K - \Theta^\vee$, then $\mathfrak{h}_{\mathrm{sc}} = \mathfrak{h} \oplus \mathbb{C}K$ is a Cartan subalgebra of $\mathfrak{g}_{\mathrm{sc}}$. The inclusion $\mathfrak{h}_{\mathrm{sc}} \to \widehat{\mathfrak{h}}$ induces an epimorphism $\widehat{\mathfrak{h}}^* \to \mathfrak{h}^*_{\mathrm{sc}}$ with one dimensional kernel. Now (4) implies that we have in fact an isomorphism

$$\widehat{\mathfrak{h}}^* / \mathbb{C}\delta \to \mathfrak{h}^*_{\mathrm{sc}}$$

and we set $\mathfrak{h}^*_{\mathrm{sc},\mathbb{R}} = \widehat{\mathfrak{h}}^*_{\mathbb{R}} / \mathbb{R}\delta$. Since $\mathbb{R}\delta \subset \widehat{C}$, we use the same notation $\widehat{C}$ for the image in $\mathfrak{h}^*_{\mathrm{sc},\mathbb{R}}$. In the following we are mostly interested in characters of $\mathfrak{g}$–modules respectively $\mathfrak{g}_{\mathrm{sc}}$–modules obtained by restricion from $\widehat{\mathfrak{g}}$–modules, so we consider also the ring

$$\mathbb{Z}[P_{\mathrm{sc}}] := \mathbb{Z}[\widehat{P}] / I_\delta,$$

where $I_\delta = (1 - e^\delta)$ is the ideal in $\mathbb{Z}[\widehat{P}]$ generated by $(1 - e^\delta)$.

1.2 The extended affine Weyl group

Since W^{aff} fixes δ, the group can be defined as the subgroup of $GL(\mathfrak{h}^*_{\mathrm{sc},\mathbb{R}})$ generated by the induced reflections $s_0, \ldots, s_n$.

Let $M \subset \mathfrak{h}^*_{\mathbb{R}}$ be the lattice $M = \nu(\bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee)$. If $\mathfrak{g}$ is simply laced, then M is the root lattice in $\mathfrak{h}^*_{\mathbb{R}}$, otherwise M is the lattice in $\mathfrak{h}^*_{\mathbb{R}}$ generated by the long roots. An element

$\Lambda \in \mathfrak{h}^*_{\mathrm{sc},\mathbb{R}}$ can be uniquely decomposed into $\Lambda = \lambda + b\Lambda_0$ such that $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$. For an element $\mu \in M$ let $t_\mu \in GL(\mathfrak{h}^*_{\mathrm{sc},\mathbb{R}})$ be the map defined by

$$\Lambda = \lambda + b\Lambda_0 \mapsto t_\mu(\Lambda) = \lambda + b\Lambda_0 + b\mu = \Lambda + \langle \Lambda, K\rangle \mu. \tag{6}$$

Obviously we have $t_\mu \circ t_{\mu'} = t_{\mu+\mu'}$, denote t_M the abelian subgroup of $GL(\mathfrak{h}^*_{\mathrm{sc},\mathbb{R}})$ consisting of the elements t_μ, $\mu \in M$. Then W^{aff} is the semidirect product $W^{\mathrm{aff}} = W \ltimes t_M$.

The *extended affine Weyl group* $\widetilde{W}^{\mathrm{aff}}$ is the semidirect product $\widetilde{W}^{\mathrm{aff}} = W \ltimes t_L$, where $L = \nu(\bigoplus_{i=1}^n \mathbb{Z}\omega_i^\vee)$ is the image of the coweight lattice. The action of an element t_μ, $\mu \in L$, is defined as above in (6).

Let Σ be the subgroup of $\widetilde{W}^{\mathrm{aff}}$ stabilizing the dominant Weyl chamber $\widehat{C}$:

$$\Sigma = \{\sigma \in \widetilde{W}^{\mathrm{aff}} \mid \sigma(\widehat{C}) = \widehat{C}\}.$$

Then Σ provides a complete system of coset representatives of $\widetilde{W}^{\mathrm{aff}}/W^{\mathrm{aff}}$ and $\widetilde{W}^{\mathrm{aff}} = \Sigma \ltimes W^{\mathrm{aff}}$. The elements $\sigma \in \Sigma$ are all of the form (one can verify this easily or see [1])

$$\sigma = \tau_i t_{-\nu(\omega_i^\vee)} = \tau_i t_{-\omega_i},$$

where $\omega_i^\vee$ is a minuscule coweight and $\tau_i = w_0 w_{0,i}$, where w_0 is the longest word in W and $w_{0,i}$ is the longest word in W_{ω_i}, the stabilizer of ω_i in W.

We extend the length function $\ell : W^{\mathrm{aff}} \to \mathbb{N}$ to a length function $\ell : \widetilde{W}^{\mathrm{aff}} \to \mathbb{N}$ by setting $\ell(\sigma w) = \ell(w)$ for $w \in W^{\mathrm{aff}}$ and $\sigma \in \Sigma$.

2 Demazure modules

2.1 Definitions

For a dominant weight $\Lambda \in \widehat{P}^+$ let $V(\Lambda)$ be the (up to isomorphism) unique irreducible $\widehat{\mathfrak{g}}$–highest weight module of highest weight Λ.

Let $U(\widehat{\mathfrak{b}})$ be the enveloping algebra of the Borel subalgebra $\widehat{\mathfrak{b}} \subset \widehat{\mathfrak{g}}$. Given an element $w \in W^{\mathrm{aff}}/W_\Lambda$, fix a generator $v_{w(\Lambda)}$ of the line $V(\Lambda)_{w(\Lambda)} = \mathbb{C}v_{w(\Lambda)}$ of $\widehat{\mathfrak{h}}$–eigenvectors in $V(\Lambda)$ of weight $w(\Lambda)$.

Definition 1. The $U(\widehat{\mathfrak{b}})$–submodule $V_w(\Lambda) = U(\widehat{\mathfrak{b}}) \cdot v_{w(\Lambda)}$ generated by $v_{w(\Lambda)}$ is called the *Demazure submodule of* $V(\Lambda)$ associated to w.

To associate more generally to every element $\sigma w \in \widetilde{W}^{\mathrm{aff}} = \Sigma \ltimes W^{\mathrm{aff}}$ a Demazure module, recall that elements in Σ correspond to automorphisms of the Dynkin diagram of $\widehat{\mathfrak{g}}$, and thus define an associated automorphism of $\widehat{\mathfrak{g}}$, also denoted σ. For a module V of $\widehat{\mathfrak{g}}$ let V^σ be the module with the twisted action $g \circ v = \sigma^{-1}(g)v$. Then for the irreducible module of highest weight $\Lambda \in \widehat{P}^+$ we get $V(\Lambda)^\sigma = V(\sigma(\Lambda))$.

So for $\sigma w \in \widetilde{W}^{\mathrm{aff}} = \Sigma \ltimes W^{\mathrm{aff}}$ we set

$$V_{w\sigma}(\Lambda) = V_w(\sigma(\Lambda)) \quad \text{respectively} \quad V_{\sigma w}(\Lambda) = V_{\sigma w \sigma^{-1}}(\sigma(\Lambda)). \tag{7}$$

Recall that for a simple root α the Demazure module $V_{w\sigma}(\Lambda)$ is stable for the associated subalgebra $\mathfrak{sl}_2(\alpha)$ if and only if $s_\alpha w\sigma \leq w\sigma \mod W_\Lambda$ in the (extended) Bruhat order. In particular, $V_{w\sigma}(\Lambda)$ is a $\mathfrak{g}$–module if and only if $s_i w\sigma \leq w\sigma \mod W_\Lambda^{\mathrm{aff}}$ for all $i = 1, \ldots, n$.

The example which will interest us are the Demazure modules associated to the weight $r\Lambda_0$ for $r \geq 1$, in this case $W_\Lambda^{\mathrm{aff}} = W$, so $\widetilde{W}^{\mathrm{aff}}/W = L$. The Demazure module $V_{t_{\nu(\mu^\vee)}}(\Lambda_0)$ is a $\mathfrak{g}$–module if and only if $\mu^\vee$ is an anti-dominant coweight, or, in other words, $\mu^\vee = -\lambda^\vee$ for some dominant coweight.

To simplify the notation, we write in the following

$$V_{-\lambda^\vee}(m\Lambda_0) \quad \text{for} \quad V_{t_{-\nu(\lambda^\vee)}}(m\Lambda_0), \tag{8}$$

and we write

$$\overline{V}_{-\lambda^\vee}(m\Lambda_0), \tag{9}$$

for $V_{-\lambda^\vee}(m\Lambda_0)$ viewed as a $\mathfrak{g}$–module. So we view $\mathrm{Char}\,\overline{V}_{-\lambda^\vee}(m\Lambda_0)$ as an element in $\mathbb{Z}[P]$ obtained from the $\widehat{\mathfrak{h}}$–character by projection.

2.2 Demazure operators

Let β be a real root of the root system $\widehat{\Phi}$. We define the *Demazure operator:*

$$D_\beta : \mathbb{Z}[\widehat{P}] \to \mathbb{Z}[\widehat{P}], \quad D_\beta(e^\lambda) = \frac{e^\lambda - e^{s_\beta(\lambda)-\beta}}{1 - e^{-\beta}}$$

Lemma 1. *1. For $\lambda, \mu \in \widehat{P}$ we have:*

$$D_\beta(e^\lambda) = \begin{cases} e^\lambda + e^{\lambda-\beta} + \cdots + e^{s_\beta(\lambda)} & \text{if } \langle \lambda, \beta^\vee \rangle \geq 0 \\ 0 & \text{if } \langle \lambda, \beta^\vee \rangle = -1 \\ -e^{\lambda+\beta} - e^{\lambda+2\beta} - \cdots - e^{s_\beta(\lambda)-\beta} & \text{if } \langle \lambda, \beta^\vee \rangle \leq -2 \end{cases} \tag{10}$$

2. $D_\beta(e^{\lambda+\mu}) = e^\lambda D_\beta(e^\mu) + e^{s_\beta(\mu)} D_\beta(e^\lambda) - e^{\lambda+s_\beta(\mu)}$

3. Let $\chi \in \mathbb{Z}[\widehat{P}]$ be such that $s_\beta(\chi) = \chi$, then $D_\beta(\chi) = \chi$.

4. Let $\chi \in \mathbb{Z}[\widehat{P}]$, then $D_\beta(\chi)$ is stable under s_β. In particular, if $D_\beta(\chi) = \chi$, then $s_\beta(\chi) = \chi$.

5. D_β is idempotent, i.e., $D_\beta(D_\beta(e^\mu)) = D_\beta(e^\mu)$ for all μ

Proof. For 1., 3., 4., and 5. see [3], (1.5)–(1.8). The proof of part 2. is a simple calculation. •

Lemma 1 implies:

Corollary 2. *If $\langle \mu, \beta^\vee \rangle = 0$, then $D_\beta(e^{\lambda+\mu}) = e^\mu D_\beta(e^\lambda)$.*

The corollary is in fact a special case of the following more general exchange rule, which follows easily from Lemma 1:

Lemma 2. *Let $\chi, \eta \in \mathbb{Z}[\widehat{P}]$. If $D_\beta(\eta) = \eta$, then*

$$D_\beta(\chi \cdot \eta) = \eta \cdot (D_\beta(\chi)).$$

Since $D_{\alpha_i}(1 - e^\delta) = (1 - e^\delta)$ for all $i = 0, \ldots, n$, Lemma 2 shows that the ideal I_δ is stable under all Demazure operators D_β. Thus we obtain induced operators (we still use the same notation D_β)

$$D_\beta : \mathbb{Z}[P_{\mathrm{sc}}] \longrightarrow \mathbb{Z}[P_{\mathrm{sc}}], \quad e^\lambda + I_\delta \mapsto D_\beta(e^\lambda) + I_\delta,$$

Recall further that $\langle \delta, \beta^\vee \rangle = 0$ (see (4)), so it makes sense to define on $\mathbb{Z}[P_{\mathrm{sc}}]$ the function $e^\lambda \mapsto \langle \lambda, \beta^\vee \rangle$.

Lemma 3. *If $\lambda \in \widehat{P} \cap \mathfrak{h}^*$, then $D_{\alpha_0}(e^\lambda) = D_{-\Theta}(e^\lambda)$ in $\mathbb{Z}[P_{\mathrm{sc}}]$.*

Proof. Since $\lambda \in \mathfrak{h}^*$ we have $\langle \lambda, \alpha_0^\vee \rangle = \langle \lambda, c - \Theta^\vee \rangle = -\langle \lambda, \Theta^\vee \rangle$. Further, $\alpha_0 = \delta - \Theta$, so equation (10) can be read in $\mathbb{Z}[P_{\mathrm{sc}}]$ as

$$\begin{aligned} D_{\alpha_0}(e^\lambda) &= \begin{cases} e^\lambda + e^{\lambda+\Theta} + \cdots + e^{\lambda+n\Theta} & \text{if } n = \langle \lambda, \alpha_0^\vee \rangle = \langle \lambda, -\Theta^\vee \rangle \geq 0 \\ 0 & \text{if } \langle \lambda, \alpha_0^\vee \rangle = \langle \lambda, -\Theta^\vee \rangle = -1 \\ -e^{\lambda-\Theta} - \cdots - e^{\lambda-(|n|-1)\Theta} & \text{if } n = \langle \lambda, \alpha_0^\vee \rangle = \langle \lambda, -\Theta^\vee \rangle \leq -2 \end{cases} \qquad (11) \\ &= D_{-\Theta}(e^\lambda) \end{aligned}$$

•

2.3 Demazure character formula

We want to extend the notion of a Demazure operator also to elements of Σ. We define for $\sigma \in \Sigma$:

$$D_\sigma : \mathbb{Z}[\widehat{P}] \to \mathbb{Z}[\widehat{P}], \quad D_\sigma(e^\Lambda) = e^{\sigma(\Lambda)}.$$

Since $\sigma(\delta) = \delta$, we get an induced operator D_σ on $\mathbb{Z}[P_{\mathrm{sc}}]$.

Lemma 4. *$D_\sigma D_\beta = D_{\sigma(\beta)} D_\sigma$.*

Proof. Let $\Lambda \in \widehat{P}$, then $\langle \Lambda, \beta^\vee \rangle = \langle \sigma(\Lambda), \sigma(\beta^\vee) \rangle$, which implies the claim by equation (10). $\bullet$

In the following we denote by D_i, $i = 0, \ldots, n$ the Demazure operator D_{α_i} corresponding to the simple root α_i. Recall that for any reduced decomposition $w = s_{i_1} \cdots s_{i_r}$ of $w \in W^{\mathrm{aff}}$ the operator $D_w = D_{i_1} \cdots D_{i_r}$ is independent of the choice of the decomposition (see [12], Corollary 8.2.10).

We associate an operator to any element $w\sigma \in \widetilde{W}^{\mathrm{aff}}$ by setting

$$\begin{array}{rcl} D_{w\sigma} : \mathbb{Z}[P_{\mathrm{sc}}] & \to & \mathbb{Z}[P_{\mathrm{sc}}] \\ e^\Lambda & \mapsto & D_w(e^{\sigma(\Lambda)}) \end{array}$$

By Lemma 4 we have for $\sigma w \in \widetilde{W}^{\mathrm{aff}} = \Sigma \ltimes W^{\mathrm{aff}}$:

$$\begin{array}{rcl} D_{\sigma w} : \mathbb{Z}[P_{\mathrm{sc}}] & \to & \mathbb{Z}[P_{\mathrm{sc}}] \\ e^\Lambda & \mapsto & \sigma\Big(D_w(e^\Lambda)\Big) = D_{\sigma w \sigma^{-1}}(e^{\sigma(\Lambda)}) \end{array}$$

Let $w\sigma \in \widetilde{W}^{\mathrm{aff}}$ and let $\Lambda \in \widehat{P}^+$ be a dominant weight.

Theorem 1 ([12] Chapter VIII, [13, 18])**.**

$$\mathrm{Char}\, V_w(\sigma(\Lambda)) = D_{w\sigma}(e^\Lambda).$$

Let $\lambda^\vee$ be a dominant coweight. Associated to $t_{-\nu(\lambda)^\vee} \in \widetilde{W}^{\mathrm{aff}}$ we have a Demazure operator $D_{t_{-\nu(\lambda^\vee)}}$, we write for simplicity just $D_{-\lambda^\vee}$.

Lemma 5. *Let $\lambda_1^\vee, \lambda_2^\vee$ be two dominant coweights, and set $\lambda^\vee = \lambda_1^\vee + \lambda_2^\vee$. Then*

$$D_{-\lambda_1^\vee} D_{-\lambda_2^\vee} = D_{-\lambda^\vee}$$

Lemma 6. *Let V be a finite dimensional $\mathfrak{g}_{\mathrm{sc}}$–module such that* $\mathrm{Char}\, V \in \mathbb{Z}[P]$*, then*

$$D_i(\mathrm{Char}\, V) = \mathrm{Char}\, V \quad \forall i = 0, \ldots, n; \text{ and } D_\sigma(\mathrm{Char}\, V) = \mathrm{Char}\, V. \tag{12}$$

Proof. The character of a finite dimensional $\mathfrak{g}$–module is stable under the Weyl group W and hence stable under D_i for all $i = 1, \ldots, n$ by Lemma 1. It remains to consider the case $i = 0$. Now all weights lie in $\mathfrak{h}^*$, so by Lemma 3 we have:

$$D_0(\mathrm{Char}\, V) = D_{-\Theta}(\mathrm{Char}\, V) = \mathrm{Char}\, V$$

where the right hand side is again a consequence of Lemma 1.

Now $\sigma = y t_{-\nu(\omega_j^\vee)}$ for some minuscule fundamental coweight $\omega_j^\vee$ and some $y \in W$. Since $t_{-\nu(\omega_j^\vee)}$ operates trivially on $\mathbb{Z}[P]$ and $D_y(\mathrm{Char}\, V) = \mathrm{Char}\, V$, the claim follows. $\bullet$

3 The proofs

3.1 Proof of Theorem 1

Let $\lambda^\vee$ be a dominant coweight and suppose we are given a decomposition

$$\lambda^\vee = \lambda_1^\vee + \lambda_2^\vee + \ldots + \lambda_r^\vee$$

of $\lambda^\vee$ as a sum of dominant coweights. For the notation see (8) and (9).

Theorem 1 *As $\mathfrak{g}$–representations, the modules*

$$\overline{V}_{-\lambda^\vee}(m\Lambda_0) \quad \text{and} \quad \overline{V}_{-\lambda_1^\vee}(m\Lambda_0) \otimes \overline{V}_{-\lambda_2^\vee}(m\Lambda_0) \otimes \cdots \otimes \overline{V}_{-\lambda_r^\vee}(m\Lambda_0)$$

are isomorphic.

More precisely, we will show that, on the level of characters of $\mathfrak{g}_{\mathrm{sc}}$–modules:

Theorem 1'.

$$\mathrm{Char}\, V_{-\lambda^\vee}(m\Lambda_0) = e^{m\Lambda_0}\mathrm{Char}\,\overline{V}_{-\lambda_1^\vee}(m\Lambda_0)\mathrm{Char}\,\overline{V}_{-\lambda_2^\vee}(m\Lambda_0)\cdots\mathrm{Char}\,\overline{V}_{-\lambda_r^\vee}(m\Lambda_0).$$

Theorem 1' obviously implies Theorem 1, so it suffices to prove Theorem 1'.
A first step is the following lemma:

Lemma 7. *Let $\chi \in \mathbb{Z}[P_{\mathrm{sc}}]$ be a character of the form $e^{m\Lambda_0}\mathrm{Char}\,\overline{V}$, where $\overline{V}$ is a finite dimensional $\mathfrak{g}$–module. Suppose $\lambda^\vee \in P^\vee$ is a dominant coweight and let $t_{-\nu(\lambda^\vee)} = s_{i_1}\ldots s_{i_t}\sigma$ be a reduced decomposition in $\widetilde{W}^{\mathrm{aff}}$. Then*

$$D_{i_1}\ldots D_{i_t}D_\sigma(e^{m\Lambda_0}\mathrm{Char}\,\overline{V}) = D_{i_1}\ldots D_{i_t}D_\sigma(e^{m\Lambda_0})\mathrm{Char}\,\overline{V}.$$

Proof. The lemma is proven exactly in the same way as Lemma 2, only using now in addition Lemma 6 for the operators D_0 and D_σ. •

Proof of Theorem 3.1'. The proof is by induction on r. Suppose $r = 1$ and $\lambda^\vee = w\sigma$ where $\sigma \in \Sigma$ and $w \in W^{\mathrm{aff}}$. The character of $V_{-\lambda^\vee}(m\Lambda_0)$ is the character of the Demazure submodule $V_w(\sigma(m\Lambda_0)) = V_w(m\Lambda_0 + m\omega_i^*)$ for some appropriate minuscule fundamental weight of $\mathfrak{g}$. So all $\mathfrak{g}_{\mathrm{sc}}$–weights occuring in the module are of the form $m\Lambda_0 + m\omega_i^* +$ a sum of roots in Φ (possibly positive and negative, see Lemma 3), and hence the character is of the desired form $e^{m\Lambda_0}\mathrm{Char}\,\overline{V}_{-\lambda^\vee}(m\Lambda_0)$.

Suppose now $r \geq 2$ and the claim holds already for $r-1$. By the definition in equation (7) we have for $t_{\nu(-\lambda^\vee)} = w\sigma \in \widetilde{W}^{\mathrm{aff}}$:

$$\mathrm{Char}\, V_{-\lambda^\vee}(m\Lambda_0) = \mathrm{Char}\, V_w(m\sigma(\Lambda_0)),$$

by the Demazure character formula (Theorem 1) the latter is equal to $D_{-\lambda^\vee}(e^{m\Lambda_0})$, so

$$\mathrm{Char}\, V_{-\lambda^\vee}(m\Lambda_0) = D_{-\lambda^\vee}(e^{m\Lambda_0}),$$

by Lemma 5 the right hand side can be rewritten as

$$\mathrm{Char}\, V_{-\lambda^\vee}(m\Lambda_0) = D_{-\lambda_1^\vee}\Big(D_{-\lambda_2^\vee}\cdots D_{-\lambda_r^\vee}(e^{m\Lambda_0})\Big),$$

by induction the right hand side can be reformulated as

$$\mathrm{Char}\, V_{-\lambda^\vee}(m\Lambda_0) = D_{-\lambda_1^\vee}\Big(e^{m\Lambda_0}\mathrm{Char}\,\overline{V}_{-\lambda_2^\vee}(m\Lambda_0)\cdots\mathrm{Char}\,\overline{V}_{-\lambda_r^\vee}(m\Lambda_0)\Big),$$

by Lemma 7 this is equivalent to

$$\mathrm{Char}\, V_{-\lambda^\vee}(m\Lambda_0) = \Big(D_{-\lambda_1^\vee}(e^{m\Lambda_0})\Big)\mathrm{Char}\,\overline{V}_{-\lambda_2^\vee}(m\Lambda_0)\cdots\mathrm{Char}\,\overline{V}_{-\lambda_r^\vee}(m\Lambda_0).$$

Now the arguments for the proof of the case $r=1$ show that this implies

$$\mathrm{Char}\, V_{-\lambda^\vee}(m\Lambda_0) = e^{m\Lambda_0}\mathrm{Char}\,\overline{V}_{-\lambda_1^\vee}(m\Lambda_0)\mathrm{Char}\,\overline{V}_{-\lambda_2^\vee}(m\Lambda_0)\cdots\mathrm{Char}\,\overline{V}_{-\lambda_r^\vee}(m\Lambda_0),$$

which finishes the proof. •

3.2 Proof of Theorem 1 A

The proof is similar to the proof above, so we give just a short sketch. As above, we have

$$\mathrm{Char}\, V_{-\lambda^\vee}(m\Lambda_0 + r\Lambda_i) = D_{-\lambda^\vee}\Big(e^{m\Lambda_0+r\Lambda_i}\Big) = D_{-\lambda_2^\vee}D_{-\lambda_3^\vee}\cdots D_{-\lambda_r^\vee}D_{-\omega_i^\vee}\Big(e^{m\Lambda_0+r\Lambda_i}\Big).$$

Now $t_{-\nu(\omega_i)} = t_{-\omega_i} = \tau_i\sigma_i$. Here $\tau_i = w_{0,i}w_0$, where w_0 is the longest element in W and $w_{0,i}$ is the longest word in the stabilizer W_{ω_i} of ω_i, and σ_i is a diagram automorphism. Note that $\sigma_i(\Lambda_i) = \tau_i^{-1}t_{-\omega_i}(\Lambda_0+\omega_i) = \tau_i^{-1}(\Lambda_0) = \Lambda_0$ and

$$\sigma_i(\Lambda_0) = \tau_i^{-1}t_{-\omega_i}(\Lambda_0) = \tau_i^{-1}(\Lambda_0-\omega_i) = \Lambda_0 + \tau_i^{-1}(-\omega_i) = \Lambda_0 + w_0w_{0,i}(-\omega_i) = \Lambda_0+\omega_i^*,$$

where ω^* denotes the highest weight of the irreducible $\mathfrak{g}$-representation $V(\omega_i)^*$. Note that $\Lambda_0+\omega_i^*$ is again a fundamental weight (for the Kac–Moody algebra $\widehat{\mathfrak{g}}$), and recall that

$$\tau_i = w_{0,i}w_0 = w_0(w_0^{-1}w_{0,i}w_0) = w_0w_{0,i}^*,$$

where $w_{0,i}^*$ is the longest word in the stabilizer $W_{\omega_i^*}$ of ω_i^*. So

$$\begin{aligned}
D_{-\omega_i^\vee}\Big(e^{m\Lambda_0+r\Lambda_i}\Big) &= D_{\tau_i}D_{\sigma_i}\Big(e^{m\Lambda_0+r\Lambda_i}\Big)\\
&= D_{\tau_i}\Big(e^{m\Lambda_0+m\omega_i^*+r\Lambda_0}\Big)\\
&= e^{(m+r)\Lambda_0}D_{\tau_i}\Big(e^{m\omega_i^*}\Big)\\
&= e^{(m+r)\Lambda_0}D_{w_0w_{0,i}^*}\Big(e^{m\omega_i^*}\Big)\\
&= e^{(m+r)\Lambda_0}\mathrm{Char}\, V(m\omega_i^*).
\end{aligned}$$

Now the same induction procedure as above applies to finish the proof. •

3.3 Proof of Theorem 2

The proof is divided into several case by case considerations. Suppose first that $\omega^\vee$ is a minuscule coweight. In this case (for $m = 1$ this has already been proved in [17]) $t_{-\omega_i} = w_{i,0}w_0\sigma_i$ and hence

$$\operatorname{Char}\overline{V}_{-\omega^\vee}(m\Lambda_0) = D_{-\omega^\vee}e^{m\Lambda_0} = D_{w_{i,0}w_0}e^{m\Lambda_0+m\omega^*} = D_{w_0w^*_{i,0}}e^{m\Lambda_0+m\omega^*} = e^{m\Lambda_0}\operatorname{Char}V(m\omega)^*.$$

In particular, this finishes the proof for the Lie algebras of type $\mathtt{A}_n$. For the next few cases we need the following:

Lemma 8. *Let w_0 be the longest element in the Weyl group of $\mathfrak{g}$, let z be an arbitrary element of $Stab_W(\Theta)$, where Θ is the highest root of $\mathfrak{g}$, let $r \in \mathbb{N}$. Then*

$$\overline{V}_{w_0zs_0}(r\Lambda_0) \simeq \bigoplus_{m=0}^{r} V(m\Theta)$$

as $\mathfrak{g}$-representations.

Proof.

$$\begin{aligned} D_{(w_0z)s_0}(e^{r\Lambda_0}) &= D_{w_0z}D_{-\Theta}(e^{r\Lambda_0}) \\ &= D_{w_0z}(e^{r\Lambda_0} + e^{r\Lambda_0+\Theta} + \ldots + e^{r\Lambda_0+r\Theta}) \\ &= D_{w_0}(e^{r\Lambda_0} + e^{r\Lambda_0+\Theta} + \ldots + e^{r\Lambda_0+r\Theta}) \\ &= e^{r\Lambda_0}(D_{w_0}(e^0 + D_{w_0}(e^{\Theta}) + \ldots + D_{w_0}(e^{r\Theta}))) \end{aligned}$$

which finishes the proof. •

In the cases E_6, E_7, E_8, F_4, G_2 the highest root Θ is also a fundamental weight, say ω_i. Let $p_i := \frac{a_i}{a_i^\vee}$. Then $\nu(\omega_i^\vee) = \frac{a_i}{a_i^\vee}\omega_i = p_i\Theta$. In fact, for the adjoint representations considered here one sees that $p_i = 1$ in all cases. Since $t_{-\omega_i} = s_\theta s_0$, it follows by Lemma 8:

$$\overline{V}_{-\omega_i^\vee}(r\Lambda_0) \simeq \bigoplus_{m=0}^{r} V(m\omega_i)$$

Next we consider the types $\mathtt{B}_n$ and $\mathtt{D}_n$ with the Bourbaki indexing of the simple roots, i.e., we consider the root system as embedded in $\mathbb{R}^n$ with the canonical basis $\{\epsilon_1, \ldots, \epsilon_n\}$ and the standard scalar product. The basis of the root system is given by the simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $i = 1, \ldots, n-1$ and $\alpha_n = \epsilon_n$ (type $\mathtt{B}_n$, $n \geq 3$) respectively $\alpha_n = \epsilon_{n-1} + \epsilon_n$ (type $\mathtt{D}_n$, $n \geq 4$), the highest root is $\epsilon_1 + \epsilon_2$ in both cases. We have

$$\begin{aligned} t_{-\omega_2} &= s_{\epsilon_1+\epsilon_2}s_0 \\ &= (s_2 \cdots s_n \cdots s_2)s_1(s_2 \cdots s_n \cdots s_2)s_0 \end{aligned}$$

In the following we consider only the non-minuscule fundamental coweights. We get for $2i \le n$ (case $\mathtt{B}_n$) respectively $2i \le n-2$ (case $\mathtt{D}_n$):

$$\begin{aligned}
t_{-\nu(\omega_{2i}^\vee)} &= t_{-\epsilon_1-\epsilon_2} t_{-\epsilon_3-\epsilon_4} \cdots t_{-\epsilon_{2i-1}-\epsilon_{2i}} \\
&= t_{-\omega_2} \Big((s_2 s_1 s_3 s_2) t_{-\omega_2} (s_2 s_1 s_3 s_2) \Big) \cdots \Big((s_{2i-2} \cdots s_2 s_1) \\
&\quad (s_{2i-1} \cdots s_3 s_2) t_{-\omega_2} (s_2 s_3 \cdots s_{2i-1})(s_1 s_2 \cdots s_{2i-2}) \Big) \\
&= s_{\epsilon_1+\epsilon_2} s_0 s_{\epsilon_3+\epsilon_4} \Big((s_2 s_1 s_3 s_2) s_0 (s_2 s_1 s_3 s_2) \Big) s_{\epsilon_5+\epsilon_6} \cdots s_{\epsilon_{2i-1}+\epsilon_{2i}} \\
&\quad \Big((s_{2i-2} \cdots s_2 s_1)(s_{2i-1} \cdots s_3 s_2) s_0 (s_2 s_3 \cdots s_{2i-1})(s_1 s_2 \cdots s_{2i-2}) \Big) \\
&= \Big[s_{\epsilon_1+\epsilon_2} s_{\epsilon_3+\epsilon_4} \cdots s_{\epsilon_{2i-1}+\epsilon_{2i}} \Big] \Big[s_0 \Big((s_2 s_1 s_3 s_2) s_0 (s_2 s_1 s_3 s_2) \Big) \cdots \\
&\quad \Big((s_{2i-2} \cdots s_2 s_1)(s_{2i-1} \cdots s_3 s_2) s_0 (s_2 s_3 \cdots s_{2i-1})(s_1 s_2 \cdots s_{2i-2}) \Big) \Big]
\end{aligned} \tag{13}$$

We see that we can write the word as a product $w_1 w_2$ of two words, the first being an element of the Weyl group W and the second being a word in the subgroup of W^{aff} generated by the simple reflections $s_0, s_1, \ldots, s_{2i-1}$, this is (in the $\mathtt{B}_n$ as well as in the $\mathtt{D}_n$ case) a group of type $\mathtt{D}_{2i}$.

Since we look for a character of a $\mathfrak{g}$–module, we know the character is stable under the operators D_i, $1 \le i \le n$. So to determine the character of $V_{-\omega_{2i}^\vee}(m\Lambda_0)$, it suffices to get a reduced decomposition of the word w_2 above modulo the right and left action of W, the character of $V_{-\omega_{2i}^\vee}(m\Lambda_0)$ can be reconstructed by applying the Demazure operators D_i, $1 \le i \le n$.

The strategy is the following. We show that the decomposition above of w_2 is a reduced decompostion. Further, we show that $\tau = s_1 s_3 \ldots s_{2i-1} w_2$ is the longest word of the Weyl group of the subgroup of W^{aff} of type $\mathtt{D}_{2i}$.

Before we give a more detailed account on how to prove this, let us show how this solves the problem. Let $\mathfrak{d} \subset \widehat{\mathfrak{g}}$ be the semisimple Lie algebra of type $\mathtt{D}_{2i}$ associated to the simple roots $\alpha_0, \ldots, \alpha_{2i-1}$, then $V_\tau(m\Lambda_0)$ is an irreducible $\mathfrak{d}$–module. More precisely, it is the irreducible m–th spin representation (associated to the node of α_0). Let $\mathfrak{d}'$ be the semisimple subalgebra of $\mathfrak{d}$ corresponding to the simple roots $\alpha_1, \ldots, \alpha_{2i-1}$, then $\mathfrak{d}'$ is also the semisimple part of a Levi subalgebra of $\mathfrak{g}$. Since $V_\tau(m\Lambda_0)$ is a $\widehat{\mathfrak{b}}$–module, it is hence a $\mathfrak{b}$ and a $\mathfrak{d}'$–module. By the Borel-Weil-Bott theorem we know that the induced $\mathfrak{g}$–module (which is the module $V_{-\omega_{2i}^\vee}(m\Lambda_0)$) has the same direct sum decomposition as $V_\tau(m\Lambda_0)$ has as $\mathfrak{d}'$–module. Since the latter has been already given in [16], this finishes the proof.

We come now back to the proof of the first claim. We make the calculations in the following modulo δ, so the set of positive roots for the type $\mathtt{D}_{2i}$–subdiagram (modulo δ) is the set

$$\{\epsilon_s - \epsilon_t \mid 1 \le s < t \le 2i\} \cup \{-\epsilon_s - \epsilon_t \mid 1 \le s < t \le 2i\}.$$

In these terms the decomposition of w_2 as the second part in the square brackets in (13), reads as

$$w_2 = s_{-\epsilon_1-\epsilon_2} s_{-\epsilon_3-\epsilon_4} \cdots s_{-\epsilon_{2i-1}-\epsilon_{2i}}.$$

and all positive roots above are sent to negative by w_2 roots except $\alpha_1, \alpha_3, \ldots, \alpha_{2i-1}$. This implies that the decomposition above of length $4i^2 - 3i$ is reduced, and

$$\begin{aligned}\tau = s_1 s_3 \ldots s_{2i-1} s_0 \Big((s_2 s_1 s_3 s_2) s_0 (s_2 s_1 s_3 s_2) \Big) \cdots \Big(&(s_{2i-2} \cdots s_2 s_1)(s_{2i-1} \cdots s_3 s_2) \\ &s_0 (s_2 s_3 \cdots s_{2i-1})(s_1 s_2 \cdots s_{2i-2}) \Big)\end{aligned} \tag{14}$$

is a reduced decomposition of the longest word of the Weyl group of the subgroup of type $\mathtt{D}_{2i}$. This shows that τ a subword of $t_{-\omega_{2i}}$, and $V_{-\omega_{2i}^\vee}(m\Lambda_0)$ is the $\mathfrak{g}$–module generated by the $\mathfrak{b}$–$\mathfrak{d}'$–submodule $V_\tau(m\Lambda_0)$.

It has been already pointed out above that the decomposition of $V_{-\omega_{2i}^\vee}(m\Lambda_0)$ as $\mathfrak{g}$–module is completely determined by the $\mathfrak{h}$–module structure of $V_\tau(m\Lambda_0)$ and the decomposition of $V_\tau(m\Lambda_0)$ as $\mathfrak{d}'$–module. So it remains to describe the decompositon of the m-th spin–representation $V_\tau(m\Lambda_0)$ with respect to the subalgebra $\mathfrak{d}'$, and to describe the highest weights as weights for the Cartan subalgebra $\mathfrak{h}$.

The decomposition of the m-th spin–representation $V_\tau(m\Lambda_0)$ with respect to the subalgebra $\mathfrak{d}'$ can be found in [16] (see section 1.4). The description of the possible highest weights occuring ([16], Proposition 3.2) in the decomposition implies for $2i < n$ (case $\mathtt{B}_n$) respectively $2i \leq n-2$ (case $\mathtt{D}_n$):

$$\operatorname{Char} \overline{V}_{-\omega_{2i}^\vee}(m\Lambda_0) = \sum_{a_1+\ldots+a_i=m} \operatorname{Char} V(a_1\omega_2 + \ldots + a_i\omega_{2i}),$$

and for $2i = n$ in the case $\mathtt{B}_n$:

$$\operatorname{Char} \overline{V}_{-\omega_n^\vee}(m\Lambda_0) = \sum_{a_1+\ldots+a_{n/2}=m} \operatorname{Char} V(a_1\omega_2 + \ldots + a_{(n-2)/2}\omega_{n-2} + 2a_n\omega_n).$$

The calculation for the odd case is similar. We assume $2i+1 \leq n$ in the case $\mathtt{B}_n$ and $2i+1 \leq n-2$ in the case $\mathtt{D}_n$

$$\begin{aligned} t_{-\omega_{2i+1}^\vee} &= t_{-\epsilon_1-\epsilon_2} \cdots t_{-\epsilon_{2i-1}-\epsilon_{2i}} t_{-\epsilon_{2i+1}} \\ &= t_{-\epsilon_1-\epsilon_2}(s_2 s_1 s_3 s_2 t_{-\epsilon_1-\epsilon_2} s_2 s_3 s_1 s_2) \cdots \\ &\quad (s_{2i-2} \ldots s_1 s_{2i-1} \ldots s_2 t_{-\epsilon_1-\epsilon_2} s_2 \ldots s_{2i-1} s_1 \ldots s_{2i-2}) \\ &\quad\ (s_{2i} \ldots s_1 t_{-\epsilon_1} s_1 \ldots s_{2i}) \\ &= s_{\epsilon_1+\epsilon_2} s_0 s_{\epsilon_3+\epsilon_4}(s_2 s_1 s_3 s_2 s_0 s_2 s_3 s_1 s_2) \cdots \\ &\quad s_{\epsilon_{2i-1}+\epsilon_{2i}}(s_{2i-2} \ldots s_1 s_{2i-1} \ldots s_2 s_0 s_2 \ldots s_{2i-1} s_1 \ldots s_{2i-2}) \\ &\quad\ s_{\epsilon_{2i+1}}(s_{2i} \ldots s_1 \sigma_1 s_1 \ldots s_{2i}) \\ &= [s_{\epsilon_1+\epsilon_2} s_{\epsilon_3+\epsilon_4} \cdots s_{\epsilon_{2i-1}+\epsilon_{2i}} s_{\epsilon_{2i+1}}][s_0(s_2 s_1 s_3 s_2 s_0 s_2 s_3 s_1 s_2) \cdots \\ &\quad (s_{2i-2} \ldots s_1 s_{2i-1} \ldots s_2 s_0 s_2 \ldots s_{2i-1} s_1 \ldots s_{2i-2})(s_{2i} \ldots s_1 s_0 s_2 \ldots s_{2i})\sigma_1] \end{aligned}$$

It follows as above that the second part of the word is reduced. In fact, after multiplying the word with $s_1 s_3 \ldots s_{2i-1}$, we obtain a reduced decomposition of the longest word

$$\begin{aligned}\tau = s_1 s_3 \ldots s_{2i-1} s_0 (s_2 s_1 s_3 s_2 s_0 s_2 s_3 s_1 s_2) \cdots &(s_{2i-2} \ldots s_1 s_{2i-1} \ldots s_2 s_0 s_2 \ldots s_{2i-1} s_1 \ldots s_{2i-2}) \\ &(s_{2i} \ldots s_1 s_0 s_2 \ldots s_{2i})\end{aligned}$$

in the Weyl group of the semisimple Lie algebra $\mathfrak{d} \subset \widehat{\mathfrak{g}}$ of type $\mathtt{D}_{2i+1}$ associated to the simple roots $\alpha_0, \ldots, \alpha_{2i}$. The Demazure module $V_{\tau\sigma_1}(m\Lambda_0)$ is an irreducible $\mathfrak{d}$–module, it is the m–th spin representation, associated to the node corresponding to α_1. Consider the decomposition of $V_{\tau\sigma_1}(m\Lambda_0)$ as an $\mathfrak{h}$– and a $\mathfrak{d}'$–module, where $\mathfrak{d}' \subset \mathfrak{d}$ is the semisimple Lie subalgebra associated to the simple roots $\alpha_1, \ldots, \alpha_{2i}$. By [16], we get as $\mathfrak{d}'$–$\mathfrak{h}$–module the decomposition ($2i + 1 < n$ in the $\mathtt{B}_n$ case):

$$\overline{V}_{\tau\sigma_1}(m\Lambda_0) = \overline{V}_{\tau}(m\Lambda_1) = \bigoplus_{a_1+\ldots+a_i=m} V(a_1\omega_1 + a_1\omega_3 + \ldots + a_i\omega_{2i+1})$$

and, again by the Borel–Weil–Bott theorem, the same decomposition holds for the Demazure module $V_{-\omega^\vee_{2i+1}}(m\Lambda_0)$ as $\mathfrak{g}$–module. The case $n = 2i + 1$ is treated similarly.

Next we consider the Lie algebra of type $\mathtt{C}_n$. We have for $j = 1, \ldots, n-1$ ($\omega^\vee_n$ is minuscule)

$$t_{-\omega^\vee_j} = t_{-2\omega_j} = t_{-2\epsilon_1}t_{-2\epsilon_2}\cdots t_{-2\epsilon_j} = t_{-2\epsilon_1}(s_1t_{-2\epsilon_1}s_1)\cdots(s_{j-1}\cdots s_1t_{-2\epsilon_1}s_1\cdots s_{j-1}).$$

Replacing $t_{-2\epsilon_1}$ by $s_{2\epsilon_1}s_0$ we get

$$\begin{aligned} t_{-\omega^\vee_j} &= s_{2\epsilon_1}s_0(s_1s_{2\epsilon_1}s_0s_1)(s_2s_1s_{2\epsilon_1}s_0s_1s_2)\cdots(s_{j-1}\cdots s_1s_{2\epsilon_1}s_0s_1\cdots s_{j-1}) \\ &= s_{2\epsilon_1}s_0s_{2\epsilon_2}(s_1s_0s_1)s_{2\epsilon_3}(s_2s_1s_0s_1s_2)\cdots s_{2\epsilon_j}(s_{j-1}\cdots s_1s_0s_1\cdots s_{j-1}) \\ &= [s_{2\epsilon_1}s_{2\epsilon_2}s_{2\epsilon_3}\cdots s_{2\epsilon_j}][s_0(s_1s_0s_1)(s_2s_1s_0s_1s_2)\cdots(s_{j-1}\cdots s_1s_0s_1\cdots s_{j-1})] \end{aligned}$$

We proceed now with the same strategy as before. For the moment we omit the reflections $s_{2\epsilon_1}s_{2\epsilon_2}s_{2\epsilon_3}\cdots s_{2\epsilon_j}$. The second part, the word

$$\tau = s_0(s_1s_0s_1)(s_2s_1s_0s_1s_2)\cdots(s_{j-1}\cdots s_1s_0s_1\cdots s_{j-1}),$$

is a reduced decomposition of the longest word of the semisimple subalgebra $\mathfrak{d} \subset \widehat{\mathfrak{g}}$ of type $\mathtt{C}_j$ associated to the simple roots $\alpha_0, \ldots, \alpha_{j-1}$.

The Demazure module $V_\tau(m\Lambda_0)$ is, as $\mathfrak{d}$–module, irreducible. Let $\mathfrak{d}' \subset \mathfrak{d}$ be the semisimple Lie algebra associated to the simple roots $\alpha_1, \ldots, \alpha_{j-1}$, it follows again from [16] that the restriction of $V_\tau(m\Lambda_0)$ decomposes as $\mathfrak{d}'$– and $\mathfrak{h}$–module

$$\overline{V}_\tau(m\Lambda_0) \simeq \bigoplus_{a_1+\ldots+a_j\leq m} \overline{V}(2a_1\omega_1 + \ldots + 2a_j\omega_j),$$

which, as above, implies the corresponding decomposition as $\mathfrak{g}$-module.

Fot $\mathfrak{g}$ of type $\mathtt{F}_4$ and $\omega^\vee_4$ we use the same strategy as above. Using the same notation as in [1], one sees $2\omega_4 = 2\epsilon_1 = (\epsilon_1 + \epsilon_2) + (\epsilon_1 - \epsilon_2) = \Theta + s_1s_2s_3s_2s_1(\Theta)$, so

$$\begin{aligned} t_{-\nu(\omega^\vee_4)} = t_{-2\omega_4} = t_{-\epsilon_1-\epsilon_2}t_{-\epsilon_1+\epsilon_2} &= (s_\Theta s_0)(s_1s_2s_3s_2s_1s_\Theta s_1s_2s_3s_2s_1) \\ &= (s_\Theta s_{\epsilon_1-\epsilon_2})(s_0s_1s_2s_3s_2s_1s_0s_1s_2s_3s_2s_1) \end{aligned}$$

Again we decompose the translation into a product of two words $w\tau$ such that $w \in W$ and τ is a subword of a reduced decomposition of the longest word of the Weyl group of type $\mathtt{B}_4$ corresponding to the roots $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$. For the corresponding Levi subalgebra of $\widehat{\mathfrak{g}}$ we have $V_\tau(m\Lambda_0)$ is the Cartan component in the m-th symmetric power of the action of the orthogonal Lie algebra on $\mathbb{C}^n$. Now by looking at the decomposition of this space with respect to the Levi subalgebra of $\mathfrak{g}$ corresponding to the simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$ (using again the tables in [16]), we obtain the desired formula.

3.4 Proof of Theorem 3

We will need the following simple:

Lemma 9. *Let $\lambda^\vee$ be a dominant, integral coweight of $\mathfrak{g}$, let w_0 be the longest element of the Weyl group of $\mathfrak{g}$, then:*

$$l(t_{-\lambda^\vee} w_0) = l(t_{-\lambda^\vee}) + l(w_0)$$

So reduced decompositions of $t_{-\lambda^\vee}$ and w_0 give a reduced decomposition of $t_{-\lambda^\vee} w_0$.

Lemma 10. *Let W be the $\mathfrak{g}$-module $W := \overline{V}_{-\theta^\vee}(r\Lambda_0)$, then there exists a unique one-dimensional submodule in W.*

Proof. The proof is by case by case consideration.

- For type $\mathtt{A}_n$ we have $\theta^\vee = \omega_1^\vee + \omega_n^\vee$, so by Theorem 2

  $$\overline{V}_{-\theta^\vee}(r\Lambda_0) \simeq V(r\omega_1^*) \otimes V(r\omega_n^*)$$

  contains an unique one-dimensional submodule.

- For type $\mathtt{B}_n$ and $\mathtt{D}_n$, $\theta^\vee = \omega_2^\vee$. By Theorem 2 $\overline{V}_{-w_2^\vee}(r\Lambda_0)$ contains a unique one-dimensional submodule.

- For type $\mathtt{C}_n$, $\theta^\vee = \omega_1^\vee$ and $\theta = 2\omega_1$, so again by Theorem 2 $\overline{V}_{-\theta^\vee}(r\Lambda_0)$ contains a unique one-dimensional submodule.

- If $\mathfrak{g}$ is of type $\mathtt{E}_6$, $\mathtt{E}_7$, $\mathtt{E}_8$, $\mathtt{F}_4$, $\mathtt{G}_2$, then $\theta^\vee = \omega_2^\vee, \omega_1^\vee, \omega_8^\vee, \omega_1^\vee, \omega_2^\vee$ respectively and the claim follows again by Theorem 2. •

We come to the proof of the theorem:
Proof. Let W be the $\mathfrak{g}$-module $\overline{V}_{s_\theta s_0}(r\Lambda_0)$. Consider the following sequence of Weyl group elements:

$$w_0 < s_\theta s_0 w_0 < (s_\theta s_0)^2 w_0 < (s_\theta s_0)^3 w_0 < \dots .$$

Note that the length is additive (recall $t_{-\theta} = s_\theta s_0$ and Lemma 9), and in a reduced decomposition of s_θ every simple reflection $s_i, i = 1,\ldots,n$, has to occur. So given an arbitrary element $\kappa \in W^{\text{aff}}$, there exists an $N \in \mathbb{N}$ such that $w \leq (s_\theta s_0)^N w_0$. Hence:

$$V(\Lambda) = \lim_{N\to\infty} V_{(s_\theta s_0)^N w_0}(\Lambda)$$

Write $\Lambda = r\Lambda_0 + \lambda$, then we obtain (using the Demazure operator)

$$\begin{aligned} D_{(s_\theta s_0)^N w_0}(e^{(r\Lambda_0+\lambda)}) &= D_{(s_\theta s_0)^N} D_{w_0}(e^{(r\Lambda_0+\lambda)}) \\ &= D_{(s_\theta s_0)^N}(e^{r\Lambda_0} \text{ Char } V(\lambda)) \\ &= e^{r\Lambda_0}(\text{ Char } W)^N \text{ Char } V(\lambda) \end{aligned}$$

This shows that in the sequence of inclusions

$$V(\lambda) \hookrightarrow W \otimes V(\lambda) \hookrightarrow W \otimes W \otimes V(\lambda) \hookrightarrow \ldots$$

the submodules $W^{\otimes N} \otimes V(\lambda)$ are, as $\mathfrak{g}$–modules, isomorphic to $\overline{V}_{(s_\theta s_0)^N w_0}(\Lambda)$. Now the same arguments as in [17], chapter 3, prove the theorem. •

4 The twisted case

In this section we would like to extend the results to twisted affine Kac-Moody algebras and by the way to so called special vertices. Let $\mathtt{X}_\mathtt{n}^{(\mathtt{r})}$ be Dynkin diagram of affine type, r the order of the automorphism, in this section we consider only $r > 1$. A vertex k of the Dynkin diagram is called special if $\delta - a_k\alpha_k$ is a positive root, here δ, a_k, α_k and so on are defined in the same way as in chapter 2. For example, 0 is always special vertex, one has $a_0 = 2$ for $\mathtt{A}_{21}^{(2)}$ and $a_0 = 1$ for the other case.

Suppose k is a special vertex. Set $\theta_k = \delta - a_k\alpha_k$, we have the finite Weyl group $W_k = \langle s_i \,|\, i \neq k\rangle$ and let M_k be the $\mathbb{Z}$-lattice spanned by $\nu(W_k(\theta_k^\vee))$ (see [8] for more details). One knows ([8]) that the affine Weyl group of $\mathtt{X}_\mathtt{n}^{(\mathtt{r})}$ is isomorphic to $W_k \ltimes t_{M_k}$, the semi-direct product of W_k with the translations (modulo δ) by M_k. The following Lemma holds:

Lemma 11. *Let k be a special vertex, then $s_k s_{\theta_k} = t_{\nu(\beta^\vee)}$ modulo δ. For λ with $\langle\lambda, K\rangle = 0$ it follows:*

$$s_k s_{\theta_k}(\lambda) = \lambda$$

Proof.

$$\begin{aligned} s_k s_{\theta_k}(\lambda) &= s_k(\lambda - \lambda(\theta_k^\vee)\theta_k) \\ &= s_k(\lambda - \lambda(\theta_k^\vee)(\delta - a_k\alpha_k)) \\ &= \lambda - (\lambda(\alpha_k^\vee) + a_k\lambda(\theta_k^\vee))\alpha_k - \lambda(\theta_k^\vee)\delta \end{aligned}$$

So the lemma follows, because $\lambda(\alpha_k^\vee + a_k\theta_k^\vee) = 0$, since $\lambda(K) = 0$. •

In section 2 we have defined the Demazure operator D_β for every real root β, with Lemma 1 and Lemma 11 it follows:

Lemma 12. *Let $\chi \in \mathbb{Z}[\widehat{P} \cap \mathfrak{h}^*]$. If $s_{\theta_k}(\chi) = \chi$, then $D_{\alpha_k}(\chi) = \chi$.*

If one deletes in the Dynkin diagram of $\mathtt{X}_\mathtt{n}^{(\mathtt{r})}$ the zero node, then one gets the diagram (let us call it $\mathtt{Y}_\mathtt{n}$) of a simple Lie Algebra. The following list shows which diagram one gets after removing the zero node, and further, it shows that the positive root $\delta - a_0\alpha_0$ is a root of $\mathtt{Y}_\mathtt{n}$.

- for $\mathtt{A}_2^{(2)}$: $\mathtt{A}_1$ and $\delta - a_0\alpha_0 = \alpha_1 = \theta$, the highest root of $\mathtt{A}_1$
- for $\mathtt{A}_{21}^{(2)}$: $\mathtt{C}_1$ and $\delta - a_0\alpha_0 = \theta^l$, the highest long root of $\mathtt{C}_1$
- for $\mathtt{A}_{21-1}^{(2)}$: $\mathtt{C}_1$ and $\delta - a_0\alpha_0 = \theta^s$, the highest short root of $\mathtt{C}_1$
- for $\mathtt{D}_{1+1}^{(2)}$: $\mathtt{B}_1$ and $\delta - a_0\alpha_0 = \theta^s$, the highest short root of $\mathtt{B}_1$
- for $\mathtt{E}_6^{(2)}$: $\mathtt{F}_4$ and $\delta - a_0\alpha_0 = \theta^s$, the highest short root of $\mathtt{F}_4$
- for $\mathtt{D}_4^{(3)}$: $\mathtt{G}_2$ and $\delta - a_0\alpha_0 = \theta^s$, the highest short root of $\mathtt{G}_2$

More generally, a vertex k is special if and only if there exists an automorphism σ of the Dynkin diagram, such that $\sigma(k) = 0$. In the untwisted case special is the same as minuscule. In the twisted case, there are only for $\mathtt{A}_{21-1}^{(2)}$ and $\mathtt{D}_{1+1}^{(2)}$ nontrivial automorphisms. We make a new list now for the twisted case, we delete a special vertex $k \neq 0$.

- for $\mathtt{A}_{21-1}^{(2)}$ deleting 1: $\mathtt{C}_1$ and $\delta - a_1\alpha_1 = \theta_1^s$, the highest short root of $\mathtt{C}_1$
- for $\mathtt{D}_{1+1}^{(2)}$ deleting l: $\mathtt{B}_1$ and $\delta - a_l\alpha_l = \theta_l^s$, the highest short root of $\mathtt{B}_1$

We get an analog of Lemma 6. Let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra associated to $\mathtt{X}_\mathtt{n}^{(\mathtt{r})}$, let $\mathfrak{a}$ be the simple Lie algebra associated to $\mathtt{Y}_\mathtt{n}$ and denote P the weight lattice of $\mathfrak{a}$.

Lemma 13. *Let V be a finite dimensional $\mathfrak{a}$ module such that* $\operatorname{Char} V \in \mathbb{Z}[P]$, *then*

$$D_i(\operatorname{Char} V) = \operatorname{Char} V \;\forall\, i = 0, \dots, n \tag{15}$$

Proof. $\operatorname{Char} V$ is stable under D_i , $i \geq 1$. In fact, $\operatorname{Char} V$ is stable under D_β for all roots of the Lie algebra $\mathfrak{a}$. So only the case $i = 0$ has to be considered. Now all weights in V are of level 0, so $D_0 = D_{-a_0\theta_0}$ $\theta_0 = \delta - a_0\alpha_0$, on these weights, which finishes the proof. This suffices to prove this, because if χ is stable under D_β, then it is stable under $D_{n\beta}$, even if it is not a root. •

Recall P is the $\mathbb{Z}$-lattice spanned by the fundamental weights of $\mathfrak{a}$. One can now formulate a statement analogous to Theorem 1. Let $\lambda^\vee$ be a dominant element of $M_k \subset P_k$, where P_k are the integral, dominant weights of $\mathfrak{a}$. Let $\lambda^\vee = \lambda_1^\vee + \lambda_2^\vee + \ldots + \lambda_r^\vee$ be a decomposition of $\lambda^\vee$ as a sum of dominant elements of M_k.

Theorem 2. *Let k be a special vertex of a twisted affine Kac–Moody algebra of type $\mathtt{X}_{\mathtt{n}}^{(\mathtt{r})}$, and let $\mathfrak{a}, \ldots$ be as above. For all $m \geq 1$, we have an isomorphism of $\mathfrak{a}$-modules between the Demazure module $\overline{V}_{-\lambda^\vee}(m\Lambda_k)$ and the tensor product of Demazure modules:*

$$\overline{V}_{-\lambda^\vee}(m\Lambda_k) \simeq \overline{V}_{-\lambda_1^\vee}(m\Lambda_k) \otimes \overline{V}_{-\lambda_2^\vee}(m\Lambda_k) \otimes \cdots \otimes \overline{V}_{-\lambda_r^\vee}(m\Lambda_k).$$

With the Lemma above, the proof is the same as in the untwisted case.
As in the untwisted case we can now look in more detail at the smallest Demazure modules $V_{-\omega_i}(l\Lambda_0)$, where ω_i is a fundamental weight for $\mathfrak{a}$. The decompositions listed below have been partially calculated (or conjectured) in [5], the remaining cases (and the proofs of the conjectured decompositions) have been calculated by Naito and Sagaki (unpublished result) as in [20] With a bar we denote again the $\mathfrak{a}$-module, where $\mathfrak{a}$ denotes the simple Lie algebra associated to diagram obtained after removing the zero node. Let $\epsilon = 1$ for i odd and 0 for i even.

- $\mathtt{A}_{\mathtt{2n}}^{(2)}$, $\mathfrak{a}$ is of type $\mathtt{C}_{\mathtt{n}}$

$$\overline{V_{-\omega_i}(l\Lambda_0)} \simeq \bigoplus_{s_1+\ldots+s_i \leq l} V(s_1\omega_1 + \ldots + s_i\omega_i)$$

- $\mathtt{A}_{\mathtt{2n-1}}^{(2)}$,$\mathfrak{a}$ is of type $\mathtt{C}_{\mathtt{n}}$

$$\overline{V_{-\omega_i}(l\Lambda_0)} \simeq \bigoplus_{s_{p_i}+s_{p_i+2}+\ldots+s_i = l} V(s_{p_i}\omega_{p_i} + s_{p_i+2}\omega_{p_i+2} + \ldots + s_i\omega_i)$$

- $\mathtt{D}_{\mathtt{n+1}}^{(2)}$, $\mathfrak{a}$ is of type $\mathtt{B}_{\mathtt{n}}$

$$i = n : \overline{V_{-\omega_i}(l\Lambda_0)} \simeq V(l\omega_n)$$
$$i \neq n : \overline{V_{-\omega_i}(l\Lambda_0)} \simeq \bigoplus_{s_1+\ldots+s_i \leq l} V(s_1\omega_1 + \ldots + s_i\omega_i)$$

- $\mathtt{E}_{\mathtt{6}}^{(2)}$, $\mathfrak{a}$ is of type $\mathtt{F}_{4}$

$$i = 1 : \overline{V_{-\omega_i}(l\Lambda_0)} \simeq \bigoplus_{0 \leq s \leq l} V(s\omega_1)$$
$$i = 4 : \overline{V_{-\omega_i}(l\Lambda_0)} \simeq \bigoplus_{0 \leq s_1+s_4 \leq l} V(s_1\omega_1 + s_4\omega_4)$$

- $\mathtt{D}_{\mathtt{4}}^{(3)}$, $\mathfrak{a}$ is of type $\mathtt{G}_{2}$

$$i = 1 : \overline{V_{-\omega_i}(l\Lambda_0)} \simeq \bigoplus_{0 \leq s \leq l} V(s\omega_1)$$

For the other special vertices the decompositions can be computed by taking automorphisms.
Theorem 3 holds in the same way, for the basic module W of the direct limit one choose $\overline{V}_{-\theta_k^\vee}(r\Lambda_k)$. Then the direct sum decomposition of W contains obviously an one dimensional module, namely the one who corresponds in the Demazure module $V_{-\theta_k^\vee}(r\Lambda_k)$ to the weight $r\Lambda_k$. Again let $V^\infty_{\lambda,r}$ be the direct limit constructed above. Then it follows

Theorem 3. *For any integral dominant weight Λ of $\widehat{\mathfrak{g}}$, $\Lambda = r\Lambda_k + \lambda$, the $\mathfrak{a}$-modules $V^\infty_{\lambda,r}$ and $\overline{V}(r\Lambda_k)$ are isomorphic.*

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Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions

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Abstract

We study finite dimensional representations of current algebras, loop algebras and their quantized versions. For the current algebra of a simple Lie algebra of type ADE, we show that Kirillov-Reshetikhin modules and Weyl modules are in fact all Demazure modules. As a consequence one obtains an elementary proof of the dimension formula for Weyl modules for the current algebra, the loop algebra and its quantized version (see also [3, 10]), and the fact that the Weyl modules of the loop algebra are specializations of irreducible modules of its quantized analogue (see also [3, 27]). Further, we show that the crystals of the Weyl and the Demazure module are the same up to some additional label zero arrows for the Weyl module.

For the current algebra $\mathcal{C}\mathfrak{g}$ of an arbitrary simple Lie algebra, the fusion product of Demazure modules of the same level turns out to be again a Demazure module. As an application we construct the $\mathcal{C}\mathfrak{g}$-module structure of the Kac-Moody algebra $\widehat{\mathfrak{g}}$-module $V(\ell\Lambda_0)$ as a semi-infinite fusion product of finite dimensional $\mathcal{C}\mathfrak{g}$–modules.

Contents

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1 Introduction

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. The theory of finite dimensional representations of its loop algebra $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, its quantized loop algebra $U_q(\mathcal{L}\mathfrak{g})$ and its current algebra $\mathcal{C}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t]$ have been the subject of many articles in the recent years. See for example [2], [3], [5], [7], [8], [10], [11], [12], [13], [16], [20], [21], [26], [28], [34] for different approaches and different aspects of this subject.

The notion of a Weyl module in this context was introduced in [11] for the affine Kac-Moody algebra and its quantized version. These modules can be described in terms of generators and relations, and they are characterized by the following universal property: any finite dimensional highest weight module which is generated by a one dimensional highest weight space, is a quotient of a Weyl module. This notion can be naturally extended to the category of finite dimensional representations of the current algebra ([7], [18]). Another intensively studied class of modules are the Kirillov-Reshetikhin modules, a name that originally refers to evaluation modules of the Yangian. In [6] Chari gave a definition of these modules for the current algebra in terms of generators and relations.

The current algebra is a subalgebra of a maximal parabolic subalgebra of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$. Let Λ be a dominant weight for $\widehat{\mathfrak{g}}$ and denote by $V(\Lambda)$ the associated (infinite dimensional) irreducible $\widehat{\mathfrak{g}}$-representation. Another natural class of finite dimensional representations of the current algebra are provided by certain Demazure submodules of $V(\Lambda)$. Of particular interest for this paper are the twisted (see section 2.2) $\mathcal{C}\mathfrak{g}$-stable Demazure submodules $D(m, \lambda)$ of $V(m\Lambda_0)$, where Λ_0 is the fundamental weight associated to the additional node of the extended Dynkin diagram of $\mathfrak{g}$.

If $\mathfrak{g}$ is simply laced, then we can identify the weight and the coweight lattice, so the Weyl modules as well as the twisted $\mathcal{C}\mathfrak{g}$-stable Demazure submodules of $V(m\Lambda_0)$ are classified by dominant weights $\lambda \in P^+$.

Theorem A *For a simple complex Lie algebra of simply laced type, the Weyl module $W(\lambda)$ and the Demazure module $D(1,\lambda)$ are isomorphic as $\mathcal{C}\mathfrak{g}$-modules.*

Also the Demazure modules of higher level are related to an interesting class of finite dimensional modules for $\mathcal{C}\mathfrak{g}$. Let $\mathfrak{g}$ be an arbitrary simple complex Lie algebra, the $\mathcal{C}\mathfrak{g}$-stable Demazure modules $D(m,\lambda^\vee)$ are classified by dominant coweights $\lambda \in \check{P}^+$.

Theorem B *For a fundamental coweight $\omega_i^\vee$ let $d_i = 1, 2$ or 3 be such that $d_i\omega_i = \nu(\omega_i^\vee)$. The Kirillov-Reshetikhin module $KR(d_im\omega_i)$ is, as $\mathcal{C}\mathfrak{g}$-module, isomorphic to the Demazure module $D(m,\omega_i^\vee)$. In particular, in the simply laced case all Kirillov-Reshetikhin modules are Demazure modules.*

Remark 1. The fact that $D(m,\omega_i^\vee)$ is a quotient of a Kirillov-Reshetikhin module has been already pointed out in [9]. In the same paper Chari and Moura have shown that $D(m,\omega_i^\vee)$ is isomorphic to $KR(d_im\omega_i)$ for all classical groups using character calculations. Our proof is independent of the type of the algebra.

To stay inside the class of cyclic highest weight modules, the tensor product of cyclic $\mathcal{C}\mathfrak{g}$-modules is often replaced by the fusion product of modules [17].

Theorem C *Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\lambda^\vee = \lambda_1^\vee + \ldots + \lambda_r^\vee$ be a decomposition of a dominant coweight as a sum of dominant coweights. Then $D(m,\lambda^\vee)$ and the fusion product $D(m,\lambda_1^\vee) * \cdots * D(m,\lambda_r^\vee)$ are isomorphic as $\mathcal{C}\mathfrak{g}$-modules.*

Remark 2. The theorem shows in particular that the fusion product of Demazure modules of the same level is associative and independent of the parameters used in the fusion construction. In [2] it is shown that the fusion product of Kirillov-Reshetikhin modules of arbitrary levels is independent of the parameters.

As a consequence we obtain for the Weyl module $W(\lambda)$ in the simply laced case:

Corollary A *Suppose $\mathfrak{g}$ is of simply laced type. Let $\lambda = a_1\omega_1 + \ldots + a_n\omega_n$ be a decomposition of a dominant weight $\lambda \in P^+$ as a sum of fundamental weights. Then the Weyl module $W(\lambda)$ for the current algebra is the fusion product of the fundamental Weyl modules:*

$$W(\lambda) \simeq \underbrace{W(\omega_1) * \cdots * W(\omega_1)}_{a_1} * \cdots * \underbrace{W(\omega_n) * \cdots * W(\omega_n)}_{a_n}.$$

The Weyl modules for the loop algebra are classified by n-tuples $\pi = (\pi_1,\ldots,\pi_n)$ of polynomials $\pi_j \in \mathbb{C}[u]$ with constant term 1 [11]. The associated dominant weight is $\lambda_\pi = \sum_i \deg \pi_i\omega_i$. Similarly, the Weyl modules for the quantized loop algebra are classified by n-tuples $\pi_q = (\pi_{q,1},\ldots,\pi_{q,n})$ of polynomials $\pi_{q,j} \in \mathbb{C}(q)[u]$ with constant term 1, the associated weight λ_{π_q} is defined as above.

It follows from the existence of a global basis for the extremal weight modules [27] and the special structure of the crystal basis [3] that the dimension of the Weyl modules depend only on λ_{π_q} respectively λ_π, and that (whenever one has an appropriate lattice) the specialization of the quantum Weyl module is the classical Weyl module. It was shown in [10] that, as a consequence, the conjectured dimension formula [11, 12] for the Weyl modules for the quantum loop algebra.

A different approach was suggested in [11, 12]. In fact, using the specialization and dimension arguments outlined there, Theorem A and Theorem C imply in the simply laced case the dimension formula and the specialization result:

Corollary B *Let $\mathfrak{g}$ be a simple Lie algebra of simply laced type, let $\lambda = \sum m_i\omega_i$ be a dominant weight (for $\mathfrak{g}$), let π (resp. π_q) be an n-tuple of polynomials in $\mathbb{C}[u]$ (resp. in $\mathbb{C}(q)[u]$) with constant term 1 such that $\lambda = \lambda_\pi = \lambda_{\pi_q}$.*

1. $\dim W(\lambda) = \dim W(\pi) = \dim W_q(\pi_q) = \dim D(1, \lambda^\vee) = \prod_i (\dim W(\omega_i))^{m_i}$
2. *If π_q is integral, then the specialization $\overline{W_q(\pi_q)}$ at $q = 1$ of the quantum Weyl module is isomorphic to the Weyl module $W(\overline{\pi_q})$ for the loop algebra associated to the specialized n-tuple of polynomials.*

Remark 3. For $\mathfrak{g} = \mathfrak{sl}_n$, the connection between Demazure modules in $V(\Lambda_0)$ and Weyl modules had been already obtained by Chari and Loktev in [7]. The isomorphism between the Weyl module $W(\lambda)$ and the Demazure module $D(1, \lambda)$ has been conjectured in [16].

For a dominant weight $\lambda = \sum m_i\omega_i$ let $\pi_{\lambda,a}$ be the tuple having $(1 - au)^{m_i}$ as i-th entry. The quantum Demazure module $D_q(m, \lambda)$ has an associated crystal graph which is a subgraph of the crystal graph of the corresponding irreducible $U_q(\widehat{\mathfrak{g}})$-representation. We conjecture that by adding appropriate label zero arrows, one gets the graph of an irreducible $U_q(\widehat{\mathfrak{g}})$-representation. In the simply laced case and level one we have:

Proposition *The crystal graph of $D_q(1, \lambda)$ is obtained from the crystal graph of $W_q(\pi_{q,\lambda,1})$ by omitting certain label zero arrows. More precisely, let $B(\lambda)_{cl}$ be the path model for $W_q(\pi_{q,\lambda,1})$ described in [37], then the crystal graph of the Demazure module is isomorphic to the graph of the concatenation $\pi_{\Lambda_0} * B(\lambda)_{cl}$.*

In the simply laced case, the restriction of the loop Weyl module $W(\pi_{\lambda,a})$ to $\mathcal{C}\mathfrak{g}$ is (up to a twist by an automorphism) the Weyl module $W(\lambda)$. It follows:

Corollary C *The Demazure module $D(m, \lambda)$ of level m can be equipped with the structure of a cyclic $U(\mathcal{L}\mathfrak{g})$-module such that the $\mathfrak{g}$-module structure coincides with the natural $\mathfrak{g}$-structure coming from the Demazure module construction.*

Let $V(m\Lambda_0)$, $m \in \mathbb{N}$, be the irreducible highest weight module of highest weight $m\Lambda_0$ for the affine Kac-Moody $\widehat{\mathfrak{g}}$. In [19] we gave a description of the $\mathfrak{g}$-module structure of this representation in terms of a semi-infinite tensor product. Using Theorem C, we are able to lift this result to the level of modules for the current algebra. The theorem holds in a much more general setting (see Remark 18), but for the convenience of a uniform presentation, let Θ be the highest root of the root system of $\mathfrak{g}$.

Theorem D *Let $D(m, n\Theta) \subset V(m\Lambda_0)$ be the Demazure module of level m corresponding to the translation at $-n\Theta$. Let $w \neq 0$ be a $\mathcal{C}\mathfrak{g}$-invariant vector of $D(m, \Theta)$. Let V_m^∞ be the direct limit*

$$D(m, \Theta) \hookrightarrow D(m, \Theta) * D(m, \Theta) \hookrightarrow D(m, \Theta) * D(m, \Theta) * D(m, \Theta) \hookrightarrow \dots$$

where the inclusions are given by $v \mapsto w \otimes v$.
Then $V(m\Lambda_0)$ and V_m^∞ are isomorphic as $U(\mathcal{C}\mathfrak{g})$-modules.

The semi-infinite fusion construction can be seen as an extension of the construction of Feigin and Feigin [15] ($\mathfrak{g} = \mathfrak{sl}_2$) and Kedem [28] ($\mathfrak{g} = \mathfrak{sl}_n$) to arbitrary simple Lie algebras. We conjecture (see Conjecture 2) that, as in [15] and [28], the semi-infinite fusion construction works for arbitrary dominant weights and not only for multiples of Λ_0.

Remark 4. Naito and Sagaki [35], [36], [37] gave a path model for the Weyl modules $W(\omega)$ for all fundamental weights and $\mathfrak{g}$ of arbitrary type. Since the Weyl modules coincide with the level-one Demazure modules provided $\mathfrak{g}$ is simply-laced, the semi-infinite limit construction above gives on the combinatorial side a combinatorial limit path model for the representation $V(\Lambda_0)$ as a semi-infinite concatenation of a finite path model, extending in this sense the approach of Magyar in [32].

After introducing some notation, we will recall in more detail the definition of Demazure and Weyl modules and fusions products. The proof of the Theorems A – C, their corollaries and the proposition is given in section 3 (see Theorem 4, 7 and 8). The proof of Theorem D is given in section 4, see Theorem 9.

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2 Notation and basics

2.1 Affine Kac-Moody algebras

In this section we fix the notation and the usual technical padding. Let $\mathfrak{g}$ be simple complex Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b} \supseteq \mathfrak{h}$. Denote $\Phi \subseteq \mathfrak{h}^*$ the root system of $\mathfrak{g}$, and, corresponding to the choice of $\mathfrak{b}$, let Φ^+ be the set of positive roots and let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the corresponding basis of Φ.

For a root $\beta \in \Phi$ let $\beta^\vee \in \mathfrak{h}$ be its coroot. The basis of the dual root system (also called the coroot system) $\Phi^\vee \subset \mathfrak{h}$ is denoted $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$.

We denote throughout the paper by $\Theta = \sum_{i=1}^{n} a_i\alpha_i$ the highest root of Φ and by $\Theta^\vee = \sum_{i=1}^{n} a_i^\vee \alpha_i^\vee$ its coroot. Note that $\Theta^\vee$ is in general not the highest root of $\Phi^\vee$. (For more details concerning the connection with the dual root system of the affine root system $\widehat{\Phi}$ see [24], Chapter 6.)

The Weyl group W of Φ is generated by the simple reflections $s_i = s_{\alpha_i}$ associated to the simple roots.

Let P be the weight lattice of $\mathfrak{g}$ and let P^+ be the subset of dominant weights. The group algebra of P is denoted $\mathbb{Z}[P]$, we write $\chi = \sum m_\mu e^\mu$ (finite sum, $\mu \in P$, $m_\mu \in \mathbb{Z}$) for an element in $\mathbb{Z}[P]$, where the embedding $P \hookrightarrow \mathbb{Z}[P]$ is defined by $\mu \mapsto e^\mu$.

We denote the coweight lattice by $\check{P}$, i.e., this is the lattice of integral weights for the dual root root system. The dominant coweights are denoted $\check{P}^+$.

Corresponding to the enumeration of the simple roots let $\omega_1, \dots, \omega_n$ be the fundamental weights. Let $\mathfrak{h}_\mathbb{R}$ be the "real part" of $\mathfrak{h}$, i.e., $\mathfrak{h}_\mathbb{R}$ is the real span in $\mathfrak{h}$ of the coroots $\alpha_1^\vee, \dots, \alpha_n^\vee$, and let $\mathfrak{h}_\mathbb{R}^*$ be the real span of the fundamental weights $\omega_1, \dots, \omega_n$. Let $(\cdot, \cdot)$ be the unique invariant symmetric non-degenerate bilinear form on $\mathfrak{g}$ normalized such that the restriction to $\mathfrak{h}$ induces an isomorphism

$$\nu : \mathfrak{h}_\mathbb{R} \longrightarrow \mathfrak{h}_\mathbb{R}^*, \quad \nu(h) : \left\{ \begin{array}{ccc} \mathfrak{h} & \to & \mathbb{R} \\ h' & \mapsto & (h, h') \end{array} \right.$$

mapping $\Theta^\vee$ to Θ. With the notation as above it follows for the weight lattice $P^\vee$ of the dual root system $\Phi^\vee$ that

$$\nu(\alpha_i^\vee) = \frac{a_i}{a_i^\vee}\alpha_i \quad \text{and} \quad \nu(\omega_i^\vee) = \frac{a_i}{a_i^\vee}\omega_i, \quad \forall\, i = 1, \dots, n.$$

Let $\widehat{\mathfrak{g}}$ be the affine Kac–Moody algebra corresponding to the extended Dynkin diagram of $\mathfrak{g}$ (see [24], Chapter 7):

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_\mathbb{C} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

Here d denotes the derivation $d = t\frac{d}{dt}$, K is the canonical central element, and the Lie bracket is given by

$$[t^m \otimes x + \lambda K + \mu d, t^n \otimes y + \nu K + \eta d] = t^{m+n} \otimes [x, y] + \mu n t^n \otimes y + \eta m t^m \otimes x + m\delta_{m,-n}(x, y)K. \quad (1)$$

The Lie algebra $\mathfrak{g}$ is naturally a subalgebra of $\widehat{\mathfrak{g}}$. In the same way, the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ are subalgebras of the Cartan subalgebra $\widehat{\mathfrak{h}}$ respectively the Borel subalgebra $\widehat{\mathfrak{b}}$ of $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \widehat{\mathfrak{b}} = \mathfrak{b} \oplus \mathbb{C}K \oplus \mathbb{C}d \oplus \mathfrak{g} \otimes_\mathbb{C} t\mathbb{C}[t] \quad (2)$$

Denote by $\widehat{\Phi}$ the root system of $\widehat{\mathfrak{g}}$ and let $\widehat{\Phi}^+$ be the subset of positive roots. The positive non-divisible imaginary root in $\widehat{\Phi}^+$ is denoted δ. The simple roots are $\widehat{\Delta} = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$

where $\alpha_0 = \delta - \Theta$. We identify the root system Φ of $\mathfrak{g}$ with the root subsystem of $\widehat{\Phi}$ generated by the simple roots $\alpha_1, \ldots, \alpha_n$.

Let $\Lambda_0, \ldots, \Lambda_n$ be the corresponding fundamental weights, then for $i = 1, \ldots, n$ we have

$$\Lambda_i = \omega_i + a_i^\vee \Lambda_0. \tag{3}$$

The decomposition of $\widehat{\mathfrak{h}}$ in (2) has its corresponding version for the dual space $\widehat{\mathfrak{h}}^*$:

$$\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta \tag{4}$$

Here the elements of $\mathfrak{h}^*$ are extended trivially, $\langle \Lambda_0, \mathfrak{h} \rangle = \langle \Lambda_0, d \rangle = 0$ and $\langle \Lambda_0, K \rangle = 1$, and $\langle \delta, \mathfrak{h} \rangle = \langle \delta, K \rangle = 0$ and $\langle \delta, d \rangle = 1$. Let $\widehat{\Delta}^\vee = \{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \widehat{\mathfrak{h}}$ be the corresponding basis of the coroot system, then $\alpha_0^\vee = K - \Theta^\vee$. Recall that the positive affine roots are precisely the roots of the form

$$\widehat{\Phi}^+ = \{\beta + s\delta \mid \beta \in \Phi^+,\, s \geq 0\} \cup \{-\beta + s\delta \mid \beta \in \Phi^+,\, s > 0\} \cup \{s\delta \mid s > 0\}$$

For a real positive root $\beta + s\delta$ respectively $-\beta + s\delta$ the corresponding coroot is

$$(\beta + s\delta)^\vee = \beta^\vee + s\frac{(\beta^\vee, \beta^\vee)}{2}K \quad \text{respectively} \quad (-\beta + s\delta)^\vee = -\beta^\vee + s\frac{(\beta^\vee, \beta^\vee)}{2}K \tag{5}$$

Let $\widehat{\mathfrak{h}}^*_{\mathbb{R}}$ be the real span $\mathbb{R}\delta + \sum_{i=0}^n \mathbb{R}\Lambda_i$, note that by the decomposition (4) and by (3) we have $\mathfrak{h}^*_{\mathbb{R}} \subseteq \widehat{\mathfrak{h}}^*_{\mathbb{R}}$. The affine Weyl group W^{aff} is generated by the reflections $s_0, s_1, ..., s_n$ acting on $\widehat{\mathfrak{h}}^*_{\mathbb{R}}$. (We use again the abbreviation $s_i = s_{\alpha_i}$ for a simple root α_i.) The cone $\widehat{C} = \{\Lambda \in \widehat{\mathfrak{h}}^*_{\mathbb{R}} | \langle \Lambda, \alpha_i^\vee \rangle \geq 0, i = 0, ..., n\}$ is the fundamental Weyl chamber for $\widehat{\mathfrak{g}}$.

We keep the convention and put a $\widehat{}$ on (almost) everything related to $\widehat{\mathfrak{g}}$. We denote by $\widehat{P}$ the weight lattice of $\widehat{\mathfrak{g}}$ and by $\widehat{P}^+$ the subset of dominant weights. As before, let $\mathbb{Z}[\widehat{P}]$ be the group algebra of $\widehat{P}$, so an element in the algebra is a finite sum of the form $\sum m_\mu e^\mu$, $\mu \in \widehat{P}$ and $m_\mu \in \mathbb{Z}$. Recall the following special properties of the imaginary root δ (see for example [24], Chapter 6):

$$\langle \delta, \alpha_i^\vee \rangle = 0 \,\forall\, i = 0, \ldots, n \quad w(\delta) = \delta \,\forall\, w \in W^{\mathrm{aff}}, \quad \langle \alpha_0, \alpha_i^\vee \rangle = -\langle \Theta, \alpha_i^\vee \rangle \text{ for } i \geq 1 \tag{6}$$

Put $a_0 = a_0^\vee = 1$ and let $A = (a_{i,j})_{0 \leq i,j \leq n}$ be the (generalized) Cartan matrix of $\widehat{\mathfrak{g}}$. We have a non–degenerate symmetric bilinear form $(\cdot, \cdot)$ on $\widehat{\mathfrak{h}}$ defined by ([24], Chapter 6)

$$\begin{cases} (\alpha_i^\vee, \alpha_j^\vee) = \frac{a_j}{a_j^\vee} a_{i,j} & i, j = 0, \ldots, \ell \\ (\alpha_i^\vee, d) = 0 & i = 1, \ldots, \ell \\ (\alpha_0^\vee, d) = 1 & (d, d) = 0. \end{cases} \tag{7}$$

The corresponding isomorphism $\nu : \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^*$ maps

$$\nu(\alpha_i^\vee) = \frac{a_i}{a_i^\vee}\alpha_i, \quad \nu(K) = \delta, \quad \nu(d) = \Lambda_0.$$

Since W^{aff} fixes δ, the affine Weyl W^{aff} can be defined as the subgroup of $GL(\mathfrak{h}^*_{\mathrm{sc},\mathbb{R}})$ generated by the induced reflections $s_0, \dots, s_n$. Another well–known description of the affine Weyl group is the following. Let $M \subset \mathfrak{h}^*_{\mathbb{R}}$ be the lattice $M = \nu(\bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee)$. If $\mathfrak{g}$ is simply laced, then M is the root lattice in $\mathfrak{h}^*_{\mathbb{R}}$, otherwise M is the lattice in $\mathfrak{h}^*_{\mathbb{R}}$ generated by the long roots.

An element $\Lambda \in \mathfrak{h}^*_{\mathrm{sc},\mathbb{R}}$ can be uniquely decomposed into $\Lambda = \lambda + b\Lambda_0$ such that $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$. For an element $\mu \in M$ let $t_\mu \in GL(\mathfrak{h}^*_{\mathrm{sc},\mathbb{R}})$ be the map defined by

$$\Lambda = \lambda + b\Lambda_0 \mapsto t_\mu(\Lambda) = \lambda + b\Lambda_0 + b\mu = \Lambda + \langle \Lambda, K\rangle \mu. \tag{8}$$

Obviously we have $t_\mu \circ t_{\mu'} = t_{\mu+\mu'}$, denote t_M the abelian subgroup of $GL(\mathfrak{h}^*_{\mathrm{sc},\mathbb{R}})$consisting of the elements t_μ, $\mu \in M$. Then W^{aff} is the semi-direct product $W^{\mathrm{aff}} = W \ltimes t_M$.

The *extended affine Weyl group* $\widetilde{W}^{\mathrm{aff}}$ is the semi-direct product $\widetilde{W}^{\mathrm{aff}} = W \ltimes t_L$, where $L = \nu(\bigoplus_{i=1}^n \mathbb{Z}\omega_i^\vee)$ is the image of the coweight lattice. The action of an element t_μ, $\mu \in L$, is defined as above in (8).

Let Σ be the subgroup of $\widetilde{W}^{\mathrm{aff}}$ stabilizing the dominant Weyl chamber $\widehat{C}$:

$$\Sigma = \{\sigma \in \widetilde{W}^{\mathrm{aff}} \mid \sigma(\widehat{C}) = \widehat{C}\}.$$

Then Σ provides a complete system of coset representatives of $\widetilde{W}^{\mathrm{aff}}/W^{\mathrm{aff}}$, so we can write in fact $\widetilde{W}^{\mathrm{aff}} = \Sigma \ltimes W^{\mathrm{aff}}$.

The elements $\sigma \in \Sigma$ are all of the form

$$\sigma = \tau_i t_{-\nu(\omega_i^\vee)} = \tau_i t_{-\omega_i},$$

where $\omega_i^\vee$ is a minuscule fundamental coweight. Further, set $\tau_i = w_0 w_{0,i}$, where w_0 is the longest word W and $w_{0,i}$ is the longest word in W_{ω_i}, the stabilizer of ω_i in W.

We extend the length function $\ell : W^{\mathrm{aff}} \to \mathbb{N}$ to a length function $\ell : \widetilde{W}^{\mathrm{aff}} \to \mathbb{N}$ by setting $\ell(\sigma w) = \ell(w)$ for $w \in W^{\mathrm{aff}}$ and $\sigma \in \Sigma$.

2.2 Definition of Demazure modules

For a dominant weight $\Lambda \in \widehat{P}^+$ let $V(\Lambda)$ be the (up to isomorphism) unique irreducible highest weight module of highest weight Λ.

Let $U(\mathfrak{I}_\mathfrak{b})$ be the enveloping algebra of the Iwahori subalgebra $\mathfrak{I}_\mathfrak{b} = \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathfrak{b} \otimes 1$, and let $U(\widehat{\mathfrak{n}})$ be the enveloping algebra of $\widehat{\mathfrak{n}} = \mathfrak{n}^+ \otimes \mathbb{C}[t] \oplus \mathfrak{h} \otimes t\mathbb{C}[t] \oplus \mathfrak{n}^- \otimes t\mathbb{C}[t]$.

Given an element $w \in W^{\mathrm{aff}}/W_\Lambda$, fix a generator $v_{w(\Lambda)}$ of the line $V(\Lambda)_{w(\Lambda)} = \mathbb{C}v_{w(\Lambda)}$ of $\widehat{\mathfrak{h}}$–eigenvectors in $V(\Lambda)$ of weight $w(\Lambda)$.

Definition 1. *The $U(\widehat{\mathfrak{b}})$–submodule $V_w(\Lambda) = U(\widehat{\mathfrak{b}}) \cdot v_{w(\Lambda)}$ generated by $v_{w(\Lambda)}$ is called the Demazure submodule of $V(\Lambda)$ associated to w.*

Remark 5. Since v is an $\widehat{\mathfrak{h}}$–eigenvector, we can also view the Demazure module $V_w(\Lambda)$ as a cyclic $U(\mathfrak{I}_{\mathfrak{b}})$–module or a cyclic $U(\widehat{\mathfrak{n}})$–module generated by $v_{w(\Lambda)}$:

$$V_w(\Lambda) = U(\mathfrak{I}_{\mathfrak{b}}) \cdot v_{w(\Lambda)} = U(\widehat{\mathfrak{n}}) \cdot v_{w(\Lambda)}.$$

To associate more generally to every element $\sigma w \in \widetilde{W}^{\mathrm{aff}} = \Sigma \ltimes W^{\mathrm{aff}}$ a Demazure module, recall that elements in Σ correspond to automorphisms of the Dynkin diagram of $\widehat{\mathfrak{g}}$, and thus define an associated automorphism of $\widehat{\mathfrak{g}}$, also denoted σ. For a module V of $\widehat{\mathfrak{g}}$ let V^σ be the module with the twisted action $g \circ v = \sigma^{-1}(g)v$. Then for the irreducible module of highest weight $\Lambda \in \widehat{P}^+$ we get $V(\Lambda)^\sigma = V(\sigma(\Lambda))$.

So for $\sigma w \in \widetilde{W}^{\mathrm{aff}} = \Sigma \ltimes W^{\mathrm{aff}}$ we set

$$V_{w\sigma}(\Lambda) = V_w(\sigma(\Lambda)) \quad \text{respectively} \quad V_{\sigma w}(\Lambda) = V_{\sigma w \sigma^{-1}}(\sigma(\Lambda)). \tag{9}$$

Recall that for a simple root α the Demazure module $V_{w\sigma}(\Lambda)$ is stable for the associated subalgebra $\mathfrak{sl}_2(\alpha)$ if and only if $s_\alpha w\sigma \le w\sigma \mod W_\Lambda$ in the (extended) Bruhat order. In particular, $V_{w\sigma}(\Lambda)$ is a $\mathfrak{g}$–module if and only if $s_i w\sigma \le w\sigma \mod W_\Lambda^{\mathrm{aff}}$ for all $i = 1, \ldots, n$.

We are mainly interested in Demazure modules associated to the weight $\ell\Lambda_0$ for $\ell \ge 1$. In this case $W_\Lambda^{\mathrm{aff}} = W$, so $\widetilde{W}^{\mathrm{aff}}/W = L$. The Demazure module $V_{t_{\nu(\mu^\vee)}}(\Lambda_0)$ is a $\mathfrak{g}$–module if and only if $\mu^\vee$ is an anti-dominant coweight, or, in other words, $\mu^\vee = -\lambda^\vee$ for some dominant coweight. Since we will mainly work with these $\mathfrak{g}$–stable Demazure modules, to simplify the notation, we write in the following

$$D(\ell, \lambda^\vee) \text{ for } \quad V_{t_{-\nu(\lambda^\vee_*)}}(\ell\Lambda_0) \tag{10}$$

where $\lambda^\vee_* = -w_0(\lambda^\vee)$, the dual coweight of $\lambda^\vee$. This notation is justified by the fact that $D(\ell, \lambda^\vee)$ is, considered as $\mathfrak{g}$-module, far from being irreducible, but this $\mathfrak{g}$-module still has a unique maximal highest weight: $\ell\nu(\lambda^\vee)$, i.e., if $V(\mu)$ is an irreducible $\mathfrak{g}$-module of highest weight μ and $\mathrm{Hom}\,(V(\mu), D(\ell, \lambda)) \neq 0$, then necessarily we have $\ell\nu(\lambda^\vee) - \mu$ is a non-negative sum of positive roots. For more details on the $\mathfrak{g}$-module structure of $D(\ell, \lambda^\vee)$ see also Theorem 2 respectively [19].

2.3 Properties of Demazure modules

A description of Demazure modules in terms of generators and relations has been given by Joseph [23] (semisimple Lie algebras, characteristic zero) and Polo [38] (semisimple Lie algebras, characteristic free), and Mathieu [33] (symmetrizable Kac–Moody algebras). We give here a reformulation for the affine case.

Theorem 1 ([33]). *Let $\Lambda \in \widehat{P}^+$ and let w be an element of the affine Weyl group of $\widehat{\mathfrak{g}}$. The Demazure module $V_w(\Lambda)$ is as a $U(\widehat{\mathfrak{b}})$-module isomorphic to the following cyclic* *module, generated by $v \neq 0$ with the following relations: for all positive roots β of $\mathfrak{g}$ we have*

$$(X_\beta \otimes t^s)^{k_\beta+1}.v = 0 \quad \textit{where } s \geq 0, \quad k_\beta = max\{0, -\langle w(\Lambda), (\beta + s\delta)^\vee\rangle\}$$

$$(X_\beta^- \otimes t^s)^{k_\beta+1}.v = 0 \quad \textit{where } s > 0, \quad k_\beta = max\{0, -\langle w(\Lambda), (-\beta + s\delta)^\vee\rangle\}$$

$$(h \otimes t^s).v = 0 \ \ \forall\, h \in \mathfrak{h},\ s > 0,$$

$$(h \otimes 1).v = w(\Lambda)(h)v \qquad \forall\, h \in \mathfrak{h}, \qquad d.v = w(\Lambda)(d).v,\ K.v = level(\Lambda)v$$

Let $\lambda^\vee \in \check{P}^+$ be a dominant coweight. We reformulate now the description of the Demazure modules above for the Demazure modules $D(\ell, \lambda^\vee)$ we are interested in.

Corollary 1. *As a module for the current algebra $\mathcal{C}\mathfrak{g}$, $D(\ell, \lambda^\vee)$ is isomorphic to the cyclic $\mathcal{C}\mathfrak{g}$–module generated by a vector v subject to the following relations:*

$$\mathfrak{n}^+ \otimes \mathbb{C}[t].v = 0\ ,\ \mathfrak{h} \otimes t\mathbb{C}[t].v = 0\ ,\ h.v = \ell\nu(\lambda^\vee)(h)v \textit{ for all } h \in \mathfrak{h},$$

and for all positive roots $\beta \in \Phi^+$ one has

$$(X_\beta^- \otimes t^s)^{k_\beta+1}.v = 0 \textit{ where } s \geq 0 \textit{ and } k_\beta = \ell \max\{0, -\langle \Lambda_0 + \nu(\lambda^\vee), (-\beta + s\delta)^\vee\rangle\} \quad (11)$$

Proof. Denote by M the cyclic $U(\mathcal{C}\mathfrak{g})$–module obtained by the relations above. Recall (see (8)) that $t_{\nu(\lambda^\vee)}(\ell\Lambda_0) = \ell\Lambda_0 + \ell\nu(\lambda^\vee)$ and set $\mu = t_{\nu(\lambda^\vee)}(\ell\Lambda_0)$. Write $t_{\nu(\lambda^\vee)} = w\sigma$ where $w \in W^{\text{aff}}$ and $\sigma \in \Sigma$. Set $\Lambda = \sigma(\Lambda_0)$, then the highest weight $\widehat{\mathfrak{g}}$-module $V(\ell\Lambda)$ has a unique line of $\widehat{\mathfrak{h}}$–eigenvectors of weight μ, let v_μ be a generator. Fix also a generator $v_{w_0(\mu)}$ of weight $w_0(\mu)$. Restricted to the current algebra, we have $\mu|_{\mathfrak{h}} = \ell\nu(\lambda^\vee)$. The submodule $U(\mathcal{C}\mathfrak{g}).v_\mu$ of $V(\ell\Lambda)$ is the Demazure module $D(\ell, \lambda^\vee)$ because:

$$D(\ell, \lambda^\vee) = V_{t_{-\nu(\lambda^\vee_*)}}(\ell\Lambda_0) = U(\widehat{\mathfrak{n}}) \cdot v_{w_0(\mu)} = U(\mathcal{C}\mathfrak{g}).v_\mu.$$

Since v_μ is an extremal weight vector, using $\mathfrak{sl}_2$-representation theory one verifies easily that v_μ satisfies the relations above. For example, if the root is of the form $\beta + s\delta$, where $s \geq 0$ and $\beta \in \Phi^+$ is a positive root, then the corresponding coroot is of the form $\beta^\vee + s'K$, $s' \geq 0$. It follows that

$$\langle \ell\Lambda_0 + \ell\nu(\lambda^\vee), \beta^\vee + s'K\rangle = \ell s' + \langle \ell\nu(\lambda^\vee), \beta^\vee\rangle \geq 0,$$

and hence $(\mathfrak{n}^+ \otimes \mathbb{C}[t])v_\mu = 0$. So we have an obvious surjective $\mathcal{C}\mathfrak{g}$–equivariant morphism $M \longrightarrow D(\ell, \lambda^\vee)$, which maps the cyclic generator v to the cyclic generator v_μ.

To prove that this map is an isomorphism it suffices to prove: $\dim M \leq \dim D(\ell, \lambda^\vee)$. The module M is not trivial by the above, and the generator $v \in M$ is a highest weight vector for the Lie subalgebra $\mathfrak{g} \subset \mathcal{C}\mathfrak{g}$. In fact, the relations imply that the $\mathfrak{g}$-submodule $U(\mathfrak{g}).v \subseteq M$ is an irreducible, finite dimensional highest weight $\mathfrak{g}$-module $V(\ell\nu(\lambda^\vee)) \subseteq M$. So we may replace for convenience the generator v by a generator $v' \in V(\ell\nu(\lambda^\vee))$ of weight $w_0(\ell\nu(\lambda^\vee))$, i.e., we replace a $\mathfrak{g}$-highest weight vector by a $\mathfrak{g}$-lowest weight vector. By construction, the following relations hold:

$$\begin{array}{lrcl} 1) & (X_\beta^+ \otimes t^s)^{k_\beta+1}.v' & = & 0 \text{ where } s \geq 0;\ k_\beta = \ell \max\{0, -\langle \Lambda_0 + w_0(\nu(\lambda^\vee)), (\beta + s\delta)^\vee\rangle\} \\ 2) & (h \otimes 1).v' & = & \ell\nu(w_0(\lambda^\vee))(h)v' \text{ where } h \in \mathfrak{h}. \\ 3) & \mathfrak{h} \otimes t\mathbb{C}[t].v' & = & 0 \quad 4) \quad \mathfrak{n}^- \otimes \mathbb{C}[t].v' = 0 \end{array}$$

Now in 4) we have roots of the form $-\beta + s\delta$, where β is a positive root and $s \geq 0$. It follows

$$\langle \Lambda_0 + w_0(\nu(\lambda^\vee)), -\beta^\vee + s'K\rangle = s' + \langle -w_0(\nu(\lambda^\vee)), \beta^\vee\rangle \geq 0,$$

so we can reformulate 4) in the following way:

$$4') \quad (X_\beta^- \otimes t^s)^{k_\beta+1}.v' = 0 \text{ where } s \geq 0\ ;\ k_\beta = \ell \max\{0, -\langle \Lambda_0 + w_0(\nu(\lambda^\vee)), (-\beta + s\delta)^\vee\rangle\}$$

Now 4) implies $M = U(\mathfrak{g} \otimes \mathbb{C}[t]).v' = U(\mathfrak{I}_\mathfrak{b}).v'$, and $1), 2), 3), 4')$ show that the cyclic generator v' for M as $U(\mathfrak{I}_\mathfrak{b})$–module satisfies the same relations as the generator for the Demazure module $D(\ell, \lambda^\vee)$. Hence we have a surjective $U(\mathfrak{I}_\mathfrak{b})$–module homomorphism $D(\ell, \lambda^\vee) \to M$, which finishes the proof. •

Remark 6. We can easily extend the defining relations in Corollary 1 to an action of $\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ by letting K act by ℓ, the level of $\ell\Lambda_0$. This follows immediately from (1) since in the current algebra there are no elements of the form $x \otimes t^{-s}$, $s > 0$.

The $\mathfrak{g}$–module structure of these special Demazure modules has been investigated in [19]: let $\lambda^\vee = \lambda_1^\vee + \ldots + \lambda_r^\vee$ be a sum of dominant integral coweights for $\mathfrak{g}$ and let $\ell \in \mathbb{N}$.

Theorem 2 ([19])**.** *As $\mathfrak{g}$-modules the following are isomorphic*

$$D(\ell, \lambda^\vee) \simeq D(\ell, \lambda_1^\vee) \otimes \ldots \otimes D(\ell, \lambda_r^\vee)$$

In this paper we will extend this isomorphism to an isomorphism of $\mathcal{C}\mathfrak{g}$-modules by replacing the tensor product by the fusion product.

2.4 Weyl modules for the loop algebra

The Weyl modules for the loop algebra $\mathcal{L}\mathfrak{g}$ have been introduced in [11]. These modules are classified by n-tuples of polynomials $\pi = (\pi_1, \ldots, \pi_n)$ with constant term 1, and they have the following universal property: every finite dimensional cyclic $\mathcal{L}\mathfrak{g}$ highest weight module generated by a one-dimensional highest weight space is a quotient of some $W(\pi)$ (for a more precise formulation see [11]). So these can be considered as maximal finite dimensional cyclic representations in this class. A special class of tuples of polynomials is defined as follows: fix $\lambda = \sum_{j=1}^{n} m_j\omega_j$ a dominant integral weight for $\mathfrak{g}$ and a nonzero complex number $a \in \mathbb{C}^*$, and set

$$\pi_{\lambda,a} = ((1-au)^{m_1}, \ldots, (1-au)^{m_n}). \tag{12}$$

The Weyl modules $W(\pi_{\lambda,a})$ are of special interest because

1. it has been shown in [11] that a Weyl module $W(\pi)$ is isomorphic to a tensor product $\bigotimes_j W(\pi_{\lambda_j,a_j})$ of Weyl modules corresponding to this special class of polynomials.
2. the defining relations for the Weyl module $W(\pi_{\lambda,a})$ reduce to (see [7]): $W(\pi_{\lambda,a})$ is the cyclic module generated by an element $w_{\lambda,a}$, subject to the relations

   $$(\mathfrak{n}^+ \otimes \mathbb{C}[t,t^{-1}])w_{\lambda,a} = 0,\ (h \otimes t^s)w_{\lambda,a} = a^s\lambda(h)w_{\lambda,a},\ (x_{\alpha_i}^- \otimes 1)^{m_i+1}w_{\lambda,a} = 0$$

   for all $h \in \mathfrak{h}$, $1 \le i \le n$, $s \in \mathbb{Z}$.

In the following we denote by $\lambda_\pi = \sum_i \deg \pi_i\omega_i$ the weight associated to a n-tuple of polynomials π.

2.5 Weyl modules for the current algebra

Let $\lambda = \sum m_i\omega_i$ be a dominant weight for $\mathfrak{g}$. A class of Weyl modules $W(\lambda)$ has also been introduced for the current algebra. In terms of generators and relations one has:

Definition 2. *Let λ be a dominant integral weight of $\mathfrak{g}$, $\lambda = \sum m_i\omega_i$. Denote by $W(\lambda)$ the $\mathcal{C}\mathfrak{g}$-module generated by an element v with the relations:*

$$\mathfrak{n}^+ \otimes \mathbb{C}[t].v = 0\ ,\ \mathfrak{h} \otimes t\mathbb{C}[t].v = 0\ ,\ h.v = \lambda(h).v\ ,\ (x_{\alpha_i}^- \otimes 1)^{m_i+1}.v = 0$$

for all $h \in \mathfrak{h}$ and all simple roots α_i. This module is called the Weyl module for $\mathcal{C}\mathfrak{g}$ associated to $\lambda \in P^+$.

The same proofs as those in [11] show that $W(\lambda)$ exists, is finite dimensional and has the same universal property (see also [8] and [7]).

Remark 7. It follows easily that for all positive roots β the following relation holds in $W(\lambda)$:

$$(X_\beta^- \otimes 1)^{k_\beta+1}.v = 0 \ \text{ for } \ k_\beta = \lambda(\beta^\vee).$$

For $a \in \mathbb{C}^*$ consider the Lie algebra homomorphism φ_a defined as follows:

$$\varphi_a : \mathcal{C}\mathfrak{g} \longrightarrow \mathcal{C}\mathfrak{g}, \quad x \otimes t^m \mapsto x \otimes (t+a)^m.$$

Now $W(\pi_{\lambda,a})$ is module for the loop algebra and hence by restriction also a module for the current algebra. It has been shown in [11, 8] that the twisted $\mathcal{C}\mathfrak{g}$-module

$$\varphi_a{}^*(W(\pi_{\lambda,a})),\ \text{where the action is defined by } (x \otimes t^m) \circ_{\varphi_a} w = (x \otimes (t-a)^m)w$$

is a cyclic $\mathcal{C}\mathfrak{g}$-module satisfying the relations in Definition 2, so:

Lemma 1. *As a $\mathcal{C}\mathfrak{g}$-module, ${\varphi_a}^*W(\pi_{\lambda,a})$ is a quotient of $W(\lambda)$.*

In [16] the so called higher level Weyl modules were introduced:

Definition 3. *Let W be a cyclic $\mathcal{C}\mathfrak{g}$-module, with fixed generator w. We denote by $W^{[k]}$ the $\mathcal{C}\mathfrak{g}$ submodule of $W^{\otimes k}$ generated by $w^{\otimes k}$.*

For a dominant integral weight λ let $W(\lambda)$ be the Weyl module for the current algebra. The Weyl module of level k corresponding to λ is defined as

$$W(\lambda)^{[k]}$$

Remark 8. Let $V_w(\Lambda)$ denote the Demazure submodule in the irreducible highest weight $\widehat{\mathfrak{g}}$-module $V(\Lambda)$ corresponding to the Weyl group element w. Then

$$V_w(\Lambda)^{[k]} = V_w(k\Lambda)$$

Remark 9. [16] Let V, W be cyclic $\mathcal{C}\mathfrak{g}$-modules, and suppose that V is a quotient of W. Then $V^{[k]}$ is a quotient of $W^{[k]}$.

2.6 Fusion products for the current algebra

In this section we recall some facts on tensor products and fusion products of cyclic $\mathcal{C}\mathfrak{g}$-modules. Let W be $\mathcal{C}\mathfrak{g}$-module and let a be a complex number. Let W_a be the $\mathcal{C}\mathfrak{g}$-module defined by the pullback $\varphi_a^* W$, so $x \otimes t^s$ acts as $x \otimes (t-a)^s$. The following is well known:

Lemma 2 ([17])**.** *Let $W^1, \ldots, W^r$ be cyclic graded, finite-dimensional $\mathcal{C}\mathfrak{g}$-modules with cyclic vectors $w_1, \ldots, w_r$ and let $C = \{c_1, \ldots, c_r\}$ be pairwise distinct complex numbers. Then $w_1 \otimes \ldots \otimes w_r$ generates $W^1_{c_1} \otimes \ldots \otimes W^r_{c_r}$.*

The Lie algebra $\mathcal{C}\mathfrak{g}$ has a natural grading and an associated natural filtration $F^\bullet(\mathcal{C}\mathfrak{g})$, where $F^s(\mathfrak{g} \otimes \mathbb{C}[t])$ is defined to be the subspace of $\mathfrak{g}$-valued polynomials with degree smaller or equal s. One has an induced filtration also on the enveloping algebra $U(\mathcal{C}\mathfrak{g})$. Let now W be a cyclic module and let w be a cyclic vector for W. Denote by W_s the subspace spanned by the vectors of the form $g.w$, where $g \in F^s(U(\mathcal{C}\mathfrak{g}))$, and denote the associated graded $\mathcal{C}\mathfrak{g}$–module by $\mathrm{gr}(W)$

$$\mathrm{gr}(W) = \bigoplus_{i \geq 0} W_s / W_{s-1} \text{ where } W_{-1} = 0.$$

As $\mathfrak{g}$-modules, W and $\mathrm{gr}(W)$ are naturally isomorphic, but in general not as $\mathcal{C}\mathfrak{g}$-modules.

Definition 4 ([17])**.** *Let W^i and c_i as above in Lemma 2 . The $\mathcal{C}\mathfrak{g}$-module*

$$W^1 * \ldots * W^r := \mathrm{gr}_C(W^1_{c_1} \otimes \ldots \otimes W^r_{c_r})$$

is called the fusion product.

Remark 10. It would be more appropriate to write $W^1_{c_1} * \ldots * W^r_{c_r}$ for the fusion product, since a priori the structure of the fusion product depends on the choice of C. It has been conjectured in fact in [17] that the fusion product a) does not depend on the choice of the pairwise distinct complex numbers $C \in \mathbb{C}^r$, and b) is associative. This has been proved in the case $\mathfrak{g} = \mathfrak{sl}_n$ for various fusion products. In this paper we will prove the independence and the associativity property for the fusion product of the Demazure modules $D(\ell, \lambda)$, which justifies the fact that we omit almost always the pairwise distinct complex numbers in the notation for the fusion product. In [2] it is shown that the fusion product of Kirillov-Reshetikhin modules of arbitrary levels is independent of the parameters.

Remark 11. The case $r = 1$ is of course not excluded. For example, let W be a graded cyclic $\mathcal{C}\mathfrak{g}$-module. Let $C = \{c\}$, where $c \in \mathbb{C}$, then $\mathrm{gr}_C(W) \simeq W$ as $\mathcal{C}\mathfrak{g}$-modules.

2.7 Kirillov-Reshetikhin modules

In [6] (see also [9]) for each multiple of a fundamental weight $m\omega_i$ a $\mathcal{C}\mathfrak{g}$-module $KR(m\omega_i)$ has been defined. These modules are called Kirillov-Reshetikhin module because in many cases (Lie algebras of simply laced type or of classical type, see [6]) they can also be obtained from the quantum Kirillov-Reshetikhin module by specialization and restriction to the current algebra.

Definition 5. *Let $KR(m\omega_i)$ be the $\mathcal{C}\mathfrak{g}$-module generated by a vector $v \neq 0$ with relations*

$$(\mathfrak{n}^+ \otimes \mathbb{C}[t]).v = 0\ ,\ (\mathfrak{h} \otimes t\mathbb{C}[t]).v = 0\ ,\ hv = m\omega_i(h)\ ,\ h \in \mathfrak{h} \tag{13}$$

$$(X^-_{\alpha_i})^{m+1}v = (X^-_{\alpha_i} \otimes t)v = 0 \quad \text{and } (X^-_{\alpha_j})v = 0 \textit{ for } j \neq i. \tag{14}$$

2.8 Quantum Weyl modules

Let $U_q(\mathcal{L}\mathfrak{g})$ be the quantum loop algebra over $\mathbb{C}(q)$, q an indeterminate, associated to $\mathfrak{g}$ (see [14]). As in the classical case, one can associate finite-dimensional modules of $U_q(\mathcal{L}\mathfrak{g})$ to n-tuples of polynomials π_q with constant term 1 and coefficients in $\mathbb{C}(q)$ (see [11]). These modules are called quantum Weyl modules. Again, the following universal property holds: every highest weight module generated by a one-dimensional highest weight space is a quotient of $W_q(\pi_q)$ for some n-tuple π_q (see [11]). Each such module has a unique irreducible quotient which we denote by $V_q(\pi_q)$.

For such a n-tuple $\pi_q = (\pi_{q,1}, \ldots, \pi_{q,n})$ set $\lambda_{\pi_q} = \sum_i \deg \pi_{q,i}\omega_i$, and let $\pi_{q,\omega_i,1}$ be defined as in the classical case.

The connection with Demazure modules is given by a theorem due to Kashiwara. We state the theorem only for the simply laced type, but it holds in much more generality.

Theorem 3 ([25]). *Let $\mathfrak{g}$ be a simple Lie algebra of simply laced type, then*

$$\dim W_q(\pi_{q,\omega_i,1}) = \dim D_q(1,\omega_i^\vee)$$

and $W_q(\pi_{q,\omega_i,1})$ is irreducible.

Remark 12. The classical Demazure module $V_w(\Lambda)$ (resp. $D(m,\lambda^\vee)$) is the $q \to 1$ limit of the quantized Demazure module $V_{q,w}(\Lambda)$ (resp. $D_q(m,\lambda^\vee)$).

Definition 6. *The n-tuple π_q is called integral if all coefficients are in $\mathcal{A}$, and if the coefficient of the highest degree term is in $\mathbb{C}^* q^{\mathbb{Z}}$.*

Let $U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})$ be the $\mathcal{A}$ subalgebra defined in [11]. It has been shown in [11] that for an integral n-tuple π_q the corresponding quantum Weyl module $W_q(\pi_q)$ admits a $U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})$-stable $\mathcal{A}$-lattice $W_{\mathcal{A}}(\pi_q) \subset W_q(\pi_q)$.

Further, let $\mathbb{C}_1$ be the $\mathcal{A}$-module with q acting by 1, then $U(\mathcal{L}\mathfrak{g})$ is a quotient of $U_{\mathcal{A}}(\mathcal{L}\mathfrak{g}) \otimes \mathbb{C}_1$, and $\overline{W_q(\pi_q)} := W_{\mathcal{A}}(\pi_q) \otimes \mathbb{C}_1$ becomes in a natural way a $U(\mathcal{L}\mathfrak{g})$-module.

Let $\overline{\pi_q}$ be the n-tuple of polynomials obtained by setting $q = 1$, so the coefficients are in $\mathbb{C}$. The universality property of the Weyl modules implies [11]:

Lemma 3. *If π_q is integral, then the $U(\mathcal{L}\mathfrak{g})$-module $\overline{W_q(\pi_q)}$ is a quotient of the classical Weyl module $W(\overline{\pi_q})$.*

It was already pointed out in [12] that the cyclicity result of Kashiwara ([25], Theorem 9.1, see [39] for the simply laced case) for twisted tensor products of cyclic modules implies a lower bound for the dimension:

$$\dim W_q(\pi_q) \geq \prod_i (\dim W_q(\pi_{q,\omega_i,1}))^{\deg \pi_{q,i}} \tag{15}$$

The existence of a global basis for the extremal weight modules [27] and the special structure of the crystal basis [3] implies that the dimension of the Weyl modules depend only on λ_{π_q}, as conjectured in [11, 12]. Further (whenever one has an appropriate lattice), the specialization of the quantum Weyl module is the classical Weyl module. As a consequence (see [10]), one gets that the inequality in (15) is an equality, but we will not need this in the following.

3 Connections between the modules

3.1 Quotients

We have some obvious maps between the Weyl modules for the current algebra and certain Demazure modules.

Lemma 4. *Let $\lambda^\vee$ be a dominant integral coweight of $\mathfrak{g}$. For all $m \geq 1$, the Demazure module $D(m, \lambda^\vee)$ is a quotient of the Weyl module $W(m\nu(\lambda^\vee))$.*

Proof. This follows immediately by comparing the relations for the Weyl module in Definition 2 and the relations for the Demazure module in Corollary 1 •

Lemma 5. *Let $\lambda_i^\vee$, $i = 1, \dots, r$ be dominant integral coweights, let $\lambda^\vee = \lambda_1^\vee + \dots + \lambda_r^\vee$, and let $a_1, \dots, a_n$ be pairwise distinct complex numbers. Then*

$$D(1, \lambda_1^\vee)_{a_1} * \dots * D(1, \lambda_r^\vee)_{a_r}$$

is a quotient of $W(\nu(\lambda^\vee))$.

Proof. Let $v_i \in D(1, \lambda_j^\vee)$ be the cyclic generator as in Definition 2 and let $\nu(\lambda_i^\vee) = \sum_j m_j^i \omega_j$, then the following relations hold:

$$\mathfrak{n}^+ \otimes \mathbb{C}[t].v_i = 0\ ,\ (\mathfrak{h} \otimes t\mathbb{C}[t])v_i = 0;\ h \otimes 1.v_i = \nu(\lambda_i^\vee)(h)v_i\ ,\ (x_{\alpha_j}^- \otimes 1)^{m_j^i+1}.v_i = 0$$

Let $\nu(\lambda^\vee) = \sum_j m_j \omega_j$, then the following relations for the fusion product follow from the relations above:

$$\mathfrak{n}^+ \otimes \mathbb{C}[t].(v_i^{\otimes_{i=1}^r}) = 0\ ,\ h \otimes 1.(v_i^{\otimes_{i=1}^r}) = \nu(\lambda^\vee)(h)(v_i^{\otimes_{i=1}^r})\ ,\ (x_{\alpha_j}^- \otimes 1)^{m_j^i+1}.(v_i^{\otimes_{i=1}^r}) = 0.$$

To see that all the relations of the Weyl module are satisfied in the fusion product, it remains to show that $\mathfrak{h} \otimes t\mathbb{C}[t]$ annihilates $v_1 \otimes \dots \otimes v_r$. So (recall that $(h \otimes t^k).v_i = 0$ for $k > 0$) we have for $n \geq 1$:

$$\begin{aligned}
(h \otimes t^n).v_1 \otimes \dots \otimes v_r &= \sum_i v_1 \otimes \dots \otimes (h \otimes (t + c_j)^n)v_i \otimes \dots \otimes v_r \\
&= \sum_j c_j^n \nu(\lambda_j^\vee)(h) v_1 \otimes \dots \otimes v_r \\
&= (\sum_j c_j^n \nu(\lambda_j^\vee)(h)) v_1 \otimes \dots \otimes v_r
\end{aligned}$$

By definition, this an element in the n-th part of the filtration, but obviously the vector $v_1 \otimes \dots \otimes v_r$ in also 0-th part of the filtration. Hence in the fusion product we have $(h \otimes t^n).v_1 \otimes \dots \otimes v_r = 0$ for $n \geq 1$. It follows: $(\mathfrak{h} \otimes t\mathbb{C}[t]).v_1 \otimes \dots \otimes v_r = 0$, which finishes the proof. •

3.2 KR-modules

Theorem 4. *For a fundamental coweight $\omega_i^\vee$ let d_i such that $d_i\omega_i = \nu(\omega_i^\vee)$. The Kirillov-Reshetikhin module $KR(d_i m \omega_i)$ is, as $\mathcal{C}\mathfrak{g}$-module isomorphic to the Demazure module $D(m, \omega_i^\vee)$. In particular, in the simply laced case (i.e., the root system is of type $\mathtt{A}_n, \mathtt{D}_n, \mathtt{E}_n$) all Kirillov-Reshetikhin modules are Demazure modules.*

Remark 13. The fact that $D(m, \omega_i^\vee)$ is a quotient of a Kirillov-Reshetikhin module has been already pointed out in [9]. In the same paper Chari and Moura have shown that $D(m, \omega_i^\vee)$ isomorphic to $KR(d_i m\omega_i)$ for all classical groups using character calculations. Our proof is independent of these results, and holds for all types, and gives an alternative proof of the fact that these modules are finite dimensional.

Proof. The fact that $D(m, \omega_i^\vee)$ is a quotient of $KR(d_i m\omega_i)$ is obvious by comparing the relations of the KR modules with the relations of the Demazure module from Corollary 1.

To show that the KR-modules above are quotients of Demazure modules, it remains to verify that the relations (14) above imply the relations (11) in Corollary 1. So let β be a positive root, set $\alpha = \alpha_i$, $d = d_i$ and $\omega = \omega_i$. Note that $[h \otimes t^k, X_\beta^- \otimes t^\ell] = \beta(h) X_\beta^- \otimes t^{\ell+k}$ implies:

$$(X_\beta^- \otimes t^s).v = 0 \quad \Rightarrow (X_\beta^- \otimes t^r).v = 0 \; \forall r \geq s. \tag{16}$$

The fact that (11) holds for the elements X_β^- follows by $\mathfrak{sl}_2$-theory. If $\langle \omega, \beta^\vee \rangle = 0$, then (11) holds for all elements $X_\beta^- \otimes t^s$, $s \geq 0$, by (16).

Assume now $\langle \omega, \beta^\vee \rangle > 0$ and consider an element of the form $X_\beta^- \otimes t^s$ for some $s \geq 1$. Let $\gamma \neq \alpha$ be a simple root, to verify the relation for $X_\beta^- \otimes t^s$ is equivalent to verify it for $X_{s_\gamma(\beta)}^- \otimes t^s$. By replacing β by $s_\gamma(\beta)$ if $\langle \beta, \gamma^\vee \rangle > 0$, without loss of generality we may assume that either $\beta = \alpha$, in which case the relations are satisfied, or $\beta \neq \alpha$ and α is the only simple root such that $\langle \beta, \alpha^\vee \rangle > 0$.

We have $\langle \beta, \alpha^\vee \rangle = j$, $j = 1, 2, 3$. So $\beta' = s_\alpha(\beta) = \beta - j\alpha$ and, if $t \geq j$, then, up to a scalar,

$$X_\beta^- \otimes t^s = [X_\alpha^- \otimes t, [..., [X_\alpha^- \otimes t, X_{\beta'}^- \otimes t^{s-j}]...].$$

Except for the case where α, β' are two short roots in a root system of type $\mathtt{G}_2$, the elements $X_\alpha^- \otimes t, X_{\beta'}^- \otimes t^{s-j}$ generate the nilpotent part of a Lie algebra of type $\mathtt{A}_2$, $\mathtt{B}_2$ or $\mathtt{G}_2$.

We consider first the case α, β' are short roots in a root system of type $\mathtt{G}_2$. Let γ be the long simple root, then $\beta' = \gamma + \alpha$ and $\beta = \gamma + 2\alpha = \omega$. We have $X_\gamma^- v = 0$, $(X_\alpha^- \otimes t)v = 0$ and hence $(X_{\beta'}^- \otimes t)v = [X_\gamma^-, X_\alpha^- \otimes t]v = 0$. In the same way one concludes $(X_\beta^- \otimes t^2)v = 0$. Now using the commutation relations, one sees that $X_\beta.((X_\beta^- \otimes t)^k.v) = 0$ for all $k \geq 0$. So if $(X_\beta^- \otimes t)^k.v \neq 0$, then this a highest weight vector for the Lie algebra generated by X_β and X_β^-, and hence $(X_\beta^- \otimes t)^{3m+1}.v = 0$. It follows that the elements $X_{\beta'}^- \otimes t^s, X_\beta^- \otimes t^s$, $s \geq 0$, satisfy in this case the relations for the Demazure module $D(m, \omega^\vee)$.

Suppose now α, β' form a basis of a root system of type $\mathtt{X}_2$, $X = \mathtt{A}, \mathtt{B}, \mathtt{G}$. Using the higher order Serre relations (see for example [29], Corollary 7.1.7), one sees by induction that $(X_\alpha^- \otimes t).v = 0$ implies for some constant $c \in \mathbb{C}$:

$$\begin{aligned}(X_\beta^- \otimes t^s)^m.v &= ([X_\alpha^- \otimes t, [..., [X_\alpha^- \otimes t, X_{\beta'}^- \otimes t^{s-j}]...])^m.v \\ &= c \cdot (X_\alpha^- \otimes t)^{jm} (X_{\beta'}^- \otimes t^{s-j})^m.v\end{aligned}$$

Now if $X^-_{\beta'} \otimes t^{s-j}$ satisfies the relations for the Demazure module in (11), then so does $X^-_\beta \otimes t^s$.

In the simply laced case this finishes the proof since the arguments above provide an inductive method reducing the verification of the relations to the case either $\beta = \alpha$ or $s = 0$, and in both cases we know already that the relations hold. In the case $\mathfrak{g}$ is of type $\mathtt{B}_n, \mathtt{C}_n$ or $\mathtt{F}_4$, the procedure reduces the proof to the cases 1) $\beta = \alpha$, 2) $s = 0$ (now in these two cases the proof is finished), or 3) β is a long root, α is a simple short root and $\langle \beta, \alpha^\vee \rangle = 2$. In this case the relations have to be verified for the root vector $X^-_\beta \otimes t$.

Now except for one case (in type $\mathtt{F}_4$) the pair $(\alpha, \beta' = \beta - 2\alpha)$ is such that $\omega(\beta'^\vee) = 0$, so $X^-_{\beta'}.v = 0$ and hence $(X^-_\beta \otimes t^2).v = 0$. Now as above, using the commutation relations one sees that $X_\beta (X^-_\beta \otimes t)^k.v = 0$ for all k, so if $(X^-_\beta \otimes t)^k.v \neq 0$, then this is a highest weight vector for the Lie subalgebra generated by $X^\pm_\beta$. It follows $(X^-_\beta \otimes t)^{m+1}.v = 0$, and hence $X^-_\beta \otimes t$ satisfies the relations for the Demazure module.

In the remaining case in type $\mathtt{F}_4$ (using reflections by simple roots $\gamma \neq \alpha$) it suffices to consider the pair where $\alpha = \alpha_3$ and $\beta = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$ (notation as in [4]). Now β' is a positive root for which we already know that the relations for the Demazure module hold. (Using simple reflections s_γ, $\gamma \neq \alpha$, to verify the relations for β' is equivalent to verify them for $\alpha_2 + 2\alpha_3$, this is a positive root of the type discussed above). One has $\omega(\beta'^\vee) = 2$, so $(X^-_{\beta'} \otimes t^2).v = 0$, $(X^-_{\beta'} \otimes t)^{m+1}.v = 0$ and $(X^-_{\beta'})^{2m+1}.v = 0$, and the induction procedure shows $(X^-_\beta \otimes t^4).v = 0$, $(X^-_\beta \otimes t^3)^{m+1}.v = 0$ and $(X^-_\beta \otimes t^2)^{2m+1}.v = 0$. It remains to show that $(X^-_\beta \otimes t)^{3m+1}.v = 0$. Suppose $u = (X^-_\beta \otimes t)^{3m+1}.v \neq 0$, then, by $\mathfrak{sl}_2$-theory, $X^{2m+2}_\beta u \neq 0$. Now using the commutation relations, a simple induction procedure shows that $X^n_\beta (X^-_\beta \otimes t)^{3m+1}.v$ is a linear combination of terms of the form

$$(X^-_\beta \otimes t)^{3m+1-2n+j} (X^-_\beta \otimes t^2)^{n-2j} (X^-_\beta \otimes t^3)^j v.$$

For $n = 2m+2$ one has $j \geq m+2$, and hence all the terms vanish. It follows $u = 0$.

Now for $\mathfrak{g}$ of type $\mathtt{G}_2$ the induction procedure (respectively the arguments above for the two short roots) reduce the proof of the relations to the cases of the root vectors $X^-_\beta \otimes t$ and $X^-_\beta \otimes t^2$. Here α is the short simple root, β' is the long simple root and $\beta = s_\alpha(\beta')$. Since $(X^-_\beta)^{3m+1} v = 0$ and $(X^-_\beta \otimes t^3).v = 0$, the commutation relations as above show that $X^{m+2}_\beta (X^-_\beta \otimes t)^{2m+1} v = 0$ and hence, by $\mathfrak{sl}_2$-theory, $(X^-_\beta \otimes t)^{2m+1} v = 0$.

Note that the root vectors X^-_β and $X^-_{\beta-\alpha}$ commute. Since $(X^-_{\beta-\alpha} \otimes t^2) v = 0$, we see that $X_\alpha (X^-_\beta \otimes t^2)^k.v = 0$ for all $k \geq 0$. So if $(X^-_\beta \otimes t^2)^k.v$ is nonzero, then this is a highest weight vector for the Lie subalgebra generated by $X^\pm_\alpha$. Since $\langle 3m\omega - k\beta, \alpha^\vee \rangle = 3m - 3k$, it follows that $(X^-_\beta \otimes t^2)^{m+1} v = 0$. •

3.3 The $\mathfrak{sl}_2$-case

Before we come to the proof of the main results, let us recall the case $\mathfrak{g} = \mathfrak{sl}_2$. Note that for $\mathfrak{sl}_2$ we have $\nu(\lambda^\vee) = \lambda = \lambda_*$. Recall:

Theorem 5 ([11])**.** *Let $\lambda = m\omega$, then dim $W(\lambda) = 2^m$.*

As immediate consequence one obtains (already proved in [11], see also [7]):

Theorem 6. *For $\mathfrak{g} = \mathfrak{sl}_2$ one has $W(\lambda) \simeq D(1, \lambda)$ as $\mathcal{C}\mathfrak{g}$-modules.*

Proof. The Demazure module is a quotient of the Weyl module, and by [19] and the theorem above one knows that $\dim D(1, \lambda) = (\dim\ D(1, \omega))^m = 2^m = \dim W(\lambda)$. •

3.4 The simply-laced case

In this section let $\mathfrak{g}$ be a simple simply laced Lie algebra, so $\mathfrak{g}$ is of type $\mathtt{A}_n$, $\mathtt{D}_n$ or $\mathtt{E}_n$. Note that in this case $\nu(\lambda^\vee) = \lambda$. We are now ready to prove

Theorem 7. *Let $\mathfrak{g}$ be simply laced. Let $\lambda^\vee \in \check{P}^+$ be a dominant integral coweight for $\mathfrak{g}$. The $\mathcal{C}\mathfrak{g}$-Weyl module $W(\lambda)$ is isomorphic to the Demazure module $D(1, \lambda^\vee)$.*

Remark 14. In [7] the result has been proved for $\mathfrak{sl}_n$ by showing that the dimension conjecture of [11] is true for the classical Weyl module for $\mathfrak{g} = \mathfrak{sl}_n$. Our approach is different and uses the relations defining a Demazure module. On the other hand, we obtain a proof of the dimension conjecture of [11] for the simply laced case by combining the result above with Theorem 2, see Proposition 1.

Proof. We know already that the Demazure module is a quotient of the Weyl module. By comparing the defining relations in Corollary 1 and in Definition 2, we see that to prove that this map is an isomorphism, it is sufficient to show for the Weyl modules that the following set of relations hold: for all positive roots $\beta \in \Phi^+$ and all $s \geq 0$ one has

$$(X_\beta^- \otimes t^s)^{k_\beta+1}.v = 0 \text{ where } s \geq 0 \text{ and } k_\beta = \max\{0, -\langle \Lambda_0 + \nu(\lambda^\vee), (-\beta + s\delta)^\vee \rangle\} \quad (17)$$

Let β be a positive root of $\mathfrak{g}$, let $s \in \mathbb{N}$ be a nonnegative integer and set

$$k = \max\ \{0, \lambda(\beta^\vee) - s\}.$$

Let $w_\lambda \in W(\lambda)$ be a generator of weight λ. To prove (17), we have to show

$$(X_\beta^- \otimes t^s)^{k+1}.w_\lambda = 0.$$

Let $\mathfrak{sl}_\beta$ be the Lie subalgebra generated by $X_\beta^-, X_\beta, \beta^\vee$. Let V be the $\mathfrak{sl}_\beta \otimes \mathbb{C}[t]$-submodule of $W(\lambda)$ generated by w_λ, i.e., $V = U(\mathfrak{sl}_\beta \otimes \mathbb{C}[t]).w_\lambda$. Then V satisfies obviously the defining relations for the $\mathfrak{sl}_\beta \otimes \mathbb{C}[t]$-Weyl module $W_\beta(\lambda(\beta^\vee))$ (see Remark 7), so V is a quotient of this Weyl module $W_\beta(\lambda(\beta^\vee))$. By Theorem 6 we know for the current algebra $\mathfrak{sl}_\beta \otimes \mathbb{C}[t]$ that the Weyl module $W_\beta(\lambda(\beta^\vee))$ is the same as the Demazure module $D_\beta(1, \lambda(\beta^\vee))$. In particular, the defining relations of $D_\beta(1, \lambda(\beta^\vee))$ hold for the corresponding generator of $W_\beta(\lambda(\beta^\vee))$, and hence also for the corresponding generator of V. It follows: $(X_\beta^- \otimes t^s)^{k+1}.w_\lambda = 0$. •

The following proposition is an immediate consequence of Theorem 2 and 7.

Proposition 1. *Let $\mathfrak{g}$ be a simple, simply laced Lie algebra and let $\lambda^\vee = \sum m_i \omega_i^\vee$ be a dominant integral coweight. The dimension of $W(\lambda)$ is*

$$\dim W(\lambda) = \prod (\dim W(\omega_i))^{m_i} = \prod (\dim D(1, \omega_i^\vee))^{m_i}$$

We can now describe the current algebra module $\varphi_a^*(W(\pi_{\lambda,a}))$ obtained as a pull back from the Weyl module for the loop algebra. Here $\lambda = \sum m_i \omega_i$ and $\pi_{\lambda,a}$ is the n-tuple of polynomials as in section 2.4.

Proposition 2. *Let λ be a dominant, integral weight for $\mathfrak{g}$ of simply laced type. Then*

$$\varphi_a^*(W(\pi_{\lambda,a})) \simeq W(\lambda)$$

Proof. We know that $\varphi_a^*(W(\pi_{\lambda,a}))$ is a quotient of $W(\lambda)$, so it suffices to show that $\dim W(\pi_{\lambda,a}) \geq \dim W(\lambda)$. We have already seen that the specialization $\overline{W_q(\pi_{q,\lambda,a})}$ at $q = 1$ of a quantum Weyl module is a quotient of the Weyl module $W(\pi_{\lambda,a})$ (see Lemma 3). By Theorem 3, the inequality (15) and Proposition 1 it follows hence:

$$\begin{aligned}\dim W(\pi_{\lambda,a}) \geq \dim W_q(\pi_{q,\lambda,a}) &\geq \prod (\dim W_q(\pi_{q,\omega_i,1}))^{m_i} \\ &= \prod (\dim D(1, \omega_i^\vee))^{m_i} = \prod (\dim W(\omega_i))^{m_i} = \dim W(\lambda).\end{aligned}$$

•

As an immediate consequence we see (note, except for the connection with the current algebras, these results follow also from the results on global basis theory in [3, 27]):

Corollary 2. *Let $\mathfrak{g}$ be a simple Lie algebra of simply laced type, let λ be a dominant weight (for $\mathfrak{g}$), let π (resp. π_q) be an n-tuple of polynomials in $\mathbb{C}[u]$ (resp. in $\mathbb{C}(q)[u]$) with constant term 1 such that $\lambda = \lambda_\pi = \lambda_{\pi_q}$.*

1. $\dim W(\lambda) = \dim W(\pi) = \dim W_q(\pi_q) = \dim D(1, \lambda^\vee) = \prod_i (\dim W(\omega_i))^{m_i}$
2. *If π_q is integral, then $\overline{W_q(\pi_q)} \simeq W(\overline{\pi_q})$ as $U(\mathcal{L}\mathfrak{g})$-modules.*
3. *The quantum Weyl module $W_q(\pi)$ is irreducible (note, the π_i have complex coefficients), and its specialization at $q = 1$ is the Weyl module $W(\pi)$ for the classical loop algebra.*

Proof. The first claim follows from Theorem 7, Proposition 1, Proposition 2, the tensor product decomposition property (see section 2.4) and the specialization arguments outlined in [12] (see section 2.8).

Now 2) is an immediate consequence of 1). To prove 3), let m_{a^j} be the multiplicity of the root $a^j \in \mathbb{C}^*$ of the polynomial $\pi_j(u)$. The tensor product $W_q = \bigotimes_{j,a^j} W_q(\omega_j, a^j)^{\otimes m_{a^j}}$ over all j and all roots a^j of $\pi_j(u)$ is irreducible by Theorem 9.2, [26], it is again a highest weight module associated to the right n-tuple of polynomials, and has the right dimension by 1), so it follows $W_q = W_q(\pi)$. The rest of the claim follows from 2). •

We can now also prove the first step of Conjecture 1 in [19] for $\mathfrak{g}$ of simply laced type. In the case of a multiple of a fundamental weight, this provides a method to reconstruct the KR-module structure for $U(\mathcal{L}\mathfrak{g})$ from the $U(\mathcal{C}\mathfrak{g})$-structure on the Demazure module.

Corollary 3. *Let $D(m, \lambda^\vee)$ be a Demazure module of level m, corresponding to $\lambda^\vee$. Then $D(m, \lambda^\vee)$ can be equipped with the structure of a $U(\mathcal{L}\mathfrak{g} \oplus \mathbb{C}K)$-module such that the $\mathfrak{g}$-module structure of $D(m, \lambda^\vee)$ coming from the construction of the Demazure module and the $\mathfrak{g}$-module structure of $D(m, \lambda^\vee)$ obtained by the restriction of the $U(\mathcal{L}\mathfrak{g} \oplus \mathbb{C}K)$-module structure coincide.*

Proof. As a $\mathcal{C}\mathfrak{g}$-module, $D(m, \lambda^\vee)$ is a quotient of $W(m\lambda)$ (Lemma 4). Let $N(m\lambda)$ be the kernel of the map, so $D(m, \lambda^\vee) \simeq W(m\lambda)/N(m\lambda)$ as $\mathcal{C}\mathfrak{g}$-modules. By Corollary 2, we know that $W(m\lambda)$ is isomorphic to $\varphi_1^* W(\pi_{m\lambda,1})$ as module for the current algebra.

Let $N_1(m\lambda) = \varphi_{-1}^* N(m\lambda)$ be the submodule of $W(\pi_{m\lambda,1})$ corresponding to $N(m\lambda)$. Using [8], Proposition 3.3 (see also [11]), one can show that $x \otimes t^{-s}$ operates as a linear combination of elements of $U(\mathcal{C}\mathfrak{g})$ on $W(\pi_{m\lambda,1})$. So a $U(\mathcal{C}\mathfrak{g})$-submodule of $W(\pi_{m\lambda,1})$ is actually a $U(\mathcal{L}\mathfrak{g})$-submodule. Since K is central (and operates trivially), we conclude that $N_1(m\lambda)$ is a $U(\mathcal{L}\mathfrak{g} \oplus \mathbb{C}K)$-submodule of $W(\pi_{m\lambda,1})$.

So the quotient $W(\pi_{m\lambda,1})/N_1(m\lambda)$ is a $U(\mathcal{L}\mathfrak{g} \oplus \mathbb{C}K)$-module, isomorphic to the Demazure module $D(m, \lambda^\vee)$ as vector space. Further, since φ^* does not change the $\mathfrak{g}$-structure of a $\mathcal{C}\mathfrak{g}$-module, we see that the $\mathfrak{g}$-module structure on $D(m, \lambda^\vee)$ and on the quotient $W(\pi_{m\lambda,1})/N_1(m\lambda)$ are identical. •

We conjecture that the corresponding statement also holds for the quantum algebras and that the module admits a crystal basis as $U_q(\mathcal{L}\mathfrak{g})$-module. Its crystal graph should be obtained from the crystal graph of the quantum Demazure module just by adding certain arrows with label zero.

In the level 1 case we know that we can identify $W_q(\pi_{q,\lambda,1})$ with $D_q(1, \lambda)$. To compare the crystals, let $P_{cl} = P/\mathbb{Z}\delta$ be the quotient of the weight lattice by the imaginary root and let $\psi : P \otimes_{\mathbb{Z}} \mathbb{R} \to P_{cl} \otimes_{\mathbb{Z}} \mathbb{R}$ be the projection of the associated real spaces. For a weight ν let $\pi_\nu : [0, 1] \to P \otimes_{\mathbb{Z}} \mathbb{R}$, $t \mapsto t\nu$, be the straight line path joining the origin with ν, and let $\psi(\pi_\nu)$ be the image of the path in $P_{cl} \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition 3. *The crystal graph of $D_q(1, \lambda)$ is obtained from the crystal graph of $W_q(\pi_{q,\lambda,1})$ by omitting certain arrows with label zero. More precisely, let $B(\lambda)_{cl}$ be the path model for $W_q(\pi_{q,\lambda,1})$ described in [37], then the crystal graph of the Demazure module is isomorphic to the graph of the concatenation $\psi(\pi_{\Lambda_0}) * B(\lambda)_{cl}$.*

Proof. Write $t_{-\lambda_*}$ as $w\sigma$, so $D_q(1, \lambda)$ is the Demazure submodule $V_{q,w}(\sigma(\Lambda_0))$. The path model theory (see [31]) is independent of the choice of an initial path, we are going to choose an appropriate path. Instead of the straight line $\pi_{\sigma(\Lambda_0)}$ joining 0 and $\sigma(\Lambda_0)$, consider the two straight line paths π_{Λ_0} and $\pi_{-\lambda_*}$ joining the origin with Λ_0 respectively $-\lambda_*$ in $P \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\eta = \pi_{\Lambda_0} * \pi_{-\lambda_*}$ be the concatenation of these two and denote by $B(\eta)$ the set of paths generated by applying the root operators to η. By [31], section 6, η is linked for arbitrary L to the straight line path $\pi_{\Lambda_0-\lambda_*}$, which is an LS-path of shape $\sigma(\Lambda_0)$. It follows that the two path models are isomorphic, and hence: a) the crystal associated to the set of paths $B(\eta)$ is isomorphic to the crystal of $V_q(\sigma(\Lambda_0))$, and b) in $B(\eta)$ there exists a unique path π_0 contained in the dominant Weyl chamber and ending in $\sigma(\Lambda_0)$. Denote by $B(\eta)_{cl}$ the image of this set of paths under the projection ψ. The root operators e_α, f_α, α a simple root for $\widehat{\mathfrak{g}}$, are still well defined on paths in $P_{cl} \otimes_{\mathbb{Z}} \mathbb{R}$ since δ vanishes on all coroots. In fact, the operators commutes with the map ψ. So the uniqueness of η (as path contained in the dominant Weyl chamber) implies that ψ induces a bijection between the crystals $B(\eta)$ and $B(\eta)_{cl}$.

Let $B(\lambda)$ be the set of all LS-paths of shape λ and denote by $B(\lambda)_{cl}$ the image of this set under the projection ψ. Combining part 3 of the theorem above with the result of Naito and Sagaki in [37], we see that $B(\lambda)_{cl}$ is a combinatorial model for the crystal of the Weyl module $W_q(\pi_{q,\lambda,1})$. The concatenation $\psi(\pi_{\Lambda_0}) * B(\lambda)_{cl}$ in $P_{cl} \otimes_{\mathbb{Z}} \mathbb{R}$ provides a set of paths stable under all root operators e_α, α a simple root, and f_{α_i}, $i = 1, \ldots, n$.

To describe in $B(\eta)_{cl}$ the set of path corresponding to the Demazure module $D(1, \lambda)$, recall that the latter is the union of all paths of the form $f_{\alpha_{j_1}}^{n_1} \cdots f_{\alpha_{j_t}}^{n_t} \pi_0$, where $t_{-\lambda_*} = w\sigma$. $w = s_{\alpha_{j_1}} \cdots s_{\alpha_{j_t}}$ is a reduced decomposition and $n_i \in \mathbb{N}$. Recall that π_0 is of the form $\pi_{\Lambda_0} * \pi'$, where $\pi' \in B(\lambda)$. Since we work modulo δ and λ is a level zero weight, $e_\alpha \pi_0 = 0$ for all simple roots of $\widehat{\mathfrak{g}}$ means that π' is a path completely contained in the fundamental alcove of the root system of $\mathfrak{g}$. This path is obtained from the straight line path π_λ by folding it successively back into the alcove (in the same way as in [30], proof of the PRV-conjecture). Next consider the sequence of turning points.

If λ is regular and generic (i.e., $\lambda \neq m\mu$ for some $m \geq 2$, $\mu \in P^+$), then this are exactly the points where π_0 meets the codimension one faces of the fundamental alcove Δ_f, and the corresponding product of the simple reflections is exactly a reduced decomposition of w', where $t_\lambda = w'\sigma$. We get a reduced decomposition of w (recall, $t_{-\lambda_*} = w\sigma$) by multiplying the given reduced decompositions with appropriate simple reflection s_{α_i}, $i \geq 1$. By the choice of the reduced decomposition above, the paths $f_{\alpha_{j_1}}^{n_1} \cdots f_{\alpha_{j_t}}^{n_t} \pi_0$ are all of the form $\psi(\pi_{\Lambda_0}) * \pi'$, where $\pi' \in B(\lambda)$.

The same holds also in the general case, only that the turning points are not anymore associated to just one simple reflection an element of maximal length in a coset W'/W'' of subgroups of W^{aff}. Here W', W'' are associated to the turning point and the path π_0, for details in terms of galleries see for example [22], Example 4.

So the set of paths in the path crystal of $V_q(\sigma(\Lambda_0))$ corresponding to the subcrystal of $D(1, \lambda)$ is a subset of $\psi(\pi_{\Lambda_0}) * B(\lambda)_{cl}$. By the equality of the number of elements, the two sets have in fact to be equal. •

3.5 Demazure modules as fusion modules

In this section let $\mathfrak{g}$ be a simple Lie algebra of arbitrary type. So, unless it is explicitly mentioned, in this section we do not assume that $\mathfrak{g}$ is necessarily simply laced.

Theorem 8. *Let $\lambda^\vee = \sum_{i=1}^s \lambda_i^\vee$ be a sum of dominant integral coweights and let $c_1, \dots, c_s$ be pairwise distinct complex numbers, then*

$$D(1, \lambda^\vee) \simeq D(1, \lambda_1^\vee) * \dots * D(1, \lambda_s^\vee)$$

as modules for the current algebras $\mathfrak{g} \otimes \mathbb{C}[t]$.

In the simply laced case we have of course equivalently:

Corollary 4. *Let $\mathfrak{g}$ be a simple simply laced Lie algebra. For $\lambda = \sum\limits_{i=1}^{s} \lambda_i$, λ_i dominant integral coweights, and $c_1, \dots, c_s$ pairwise distinct complex numbers:*

$$W(\lambda) \simeq W(\lambda_1) * \dots * W(\lambda_s)$$

as $\mathcal{C}\mathfrak{g}$-modules

Remark 15. For $\mathfrak{g} = \mathfrak{sl}_n$ and the λ_i, $i = 1, \dots, s$, all fundamental weights, the theorem above (and its corollary) was proved by Chari and Loktev in [7].

Corollary 5. *Let $\mathfrak{g}$ be again a simple Lie algebra of arbitrary type and let $\lambda^\vee = \sum_{i=1}^s \lambda_i^\vee$ be a sum of dominant integral coweights and let $c_1, \dots, c_s$ be pairwise distinct complex numbers, then for all $k \geq 1$*

$$D(k, \lambda^\vee) \simeq D(k, \lambda_1^\vee) * \dots * D(k, \lambda_s^\vee)$$

as modules for the current algebras $\mathfrak{g} \otimes \mathbb{C}[t]$.

As obvious consequences we have:

Corollary 6.
1. *The fusion product of the Demazure modules $D(k, \lambda_j)$ is associative and independent of the choice of the pairwise distinct complex numbers $\{c_1, \dots, c_s\}$.*
2. *Let d_i be as in Theorem 4, then $KR(md_i\omega_i)$ is the m-fold fusion product $KR(d\omega_i)^{*m}$.*

Proof of Corollary 5. It follows from Remark 8 and Theorem 8 that

$$D(k, \lambda^\vee) = D(1, \lambda^\vee)^{[k]} \simeq (D(1, \lambda_1^\vee) * \dots * D(1, \lambda_s^\vee))^{[k]}.$$

By Proposition 2.10 in [16] the latter is a quotient of $D(1, \lambda_1^\vee)^{[k]} * \dots * D(k, \lambda_s^\vee)^{[k]} = D(k, \lambda_1^\vee) * \dots * D(k, \lambda_s^\vee)$. The dimension formula (Theorem 2) implies again that the map is an isomorphism. •

Proof of Theorem 8. In the simply laced case the result follows immediately from the equality of Demazure and Weyl modules: the right hand side is a Weyl module by Theorem 7, and the left hand side is a quotient of this Weyl module by Lemma 5. Now by Theorem 2 the dimension of both modules is equal, which finishes the proof.

In the general case we need to use the defining equations for Demazure module (see Corollary 1). In the proof of Lemma 5 we have already seen that the fusion module:

$$D(1,\lambda_1^\vee) * \ldots * D(1,\lambda_r^\vee)$$

is a quotient of the Weyl module and hence satisfies the relations:

$$\mathfrak{n}^+ \otimes \mathbb{C}[t].(v_i^{\otimes_{i=1}^r}) = 0\ ,\ h \otimes 1.(v_i^{\otimes_{i=1}^r}) = \nu(\lambda^\vee)(h)(v_i^{\otimes_{i=1}^r}) \text{ and } \mathfrak{h} \otimes t\mathbb{C}[t](v_i^{\otimes_{i=1}^r}) = 0.$$

Let now $\beta \in \Phi^+$ be a positive root. The following lemma implies that the fusion product is a quotient of the Demazure module. Since both have the same dimension by Theorem 2, they are isomorphic, which finishes the proof. •

Lemma 6.

$$(X_\beta^- \otimes t^s)^{k_\beta+1}(v_i^{\otimes_{i=1}^r}) = 0 \textit{ for } k_\beta = \max\{0, \langle \Lambda_0 + \nu(\lambda^\vee), (-\beta + s\delta)^\vee \rangle\}, \tag{18}$$

The proof of Lemma 6 is by reduction to the $\widehat{\mathfrak{sl}}_2$–case. Note that in this case we know already that Theorem 8 and Corollary 5 hold.

We fix first some notation. For a positive root $\beta \in \Phi^+$ let $Z_\beta \subset \widehat{\mathfrak{g}}$ be the Lie subalgebra generated by the root spaces $\widehat{\mathfrak{g}}_{\pm\beta+s\delta}$, $s \in \mathbb{Z}$, the elements in the Cartan subalgebra $(\pm\beta \pm s\delta)^\vee$, and the derivation d. Then Z_β is an affine Kac-Moody algebra isomorphic to $\widehat{\mathfrak{sl}}_2$ with Cartan subalgebra $\widehat{\mathfrak{h}}_\beta = \langle \beta^\vee, \epsilon K, d \rangle_{\mathbb{C}}$, where $\epsilon = (\beta^\vee, \beta^\vee)/2$ (see equation (5)) is 1 if β and Θ have the same length, and $\epsilon = 2$ or 3 if β is a short root. Set

$$\widehat{\mathfrak{n}}_\beta^\pm = \widehat{\mathfrak{n}}^\pm \cap Z_\beta \quad \text{and} \quad \mathfrak{sl}_2(\beta) \otimes \mathbb{C}[t] = Z_\beta \cap \mathfrak{g} \otimes \mathbb{C}[t]$$

Write $t_{\nu(\lambda^\vee)} = w\sigma$ (see equations (9) and (10)), so $w \in W^{\text{aff}}$. Set $\mu = w(\sigma(\Lambda_0))$ and let $v_\mu \in V(\sigma(\Lambda_0))$ be an extremal weight vector of weight μ. The submodule $M = U(Z_\beta)v_\mu \subset V(\sigma(\Lambda_0))$ is an irreducible (since v_μ is an extremal weight vector) Z_β-submodule, say $M = V^\beta(\Omega)$ is the Z_β-representation of highest weight Ω. The subspace

$$M(\nu(\lambda^\vee)) := U(\mathfrak{sl}_2(\beta) \otimes \mathbb{C}[t]).v_\mu = U(\widehat{\mathfrak{n}}_\beta^+)s_\beta(v_\mu) \tag{19}$$

is then a Demazure module, stable under $U(\mathfrak{sl}_2(\beta) \otimes \mathbb{C}[t])$. Now $V(\sigma(\Lambda_0))$ is a level one module for $\widehat{\mathfrak{g}}$, but the irreducible Z_β-submodule V_β is a level ϵ-module for the affine Kac-Moody algebra $Z_\beta \simeq \widehat{\mathfrak{sl}}_2$ (recall, the canonical central element of Z_β is ϵK). We need the following more precise statement:

Lemma 7. *As $\mathfrak{sl}_2(\beta)\otimes\mathbb{C}[t]$-module, the submodule $M(\nu(\lambda^\vee))$ is isomorphic to $D(\epsilon, m\omega^\beta)$, where $m = \nu(\lambda^\vee)(\beta^\vee)/\epsilon$ and ω^β denotes the fundamental weight for the Lie algebra $\mathfrak{sl}_2(\beta)$.*

Proof of Lemma 7. The first step is to show that the highest weight Ω is a multiple of a fundamental weight for Z_β. The only non-trivial case is when Θ and β have different lengths. We show first that in this case:

$$\nu(\lambda^\vee)(\beta^\vee) \equiv 0 \bmod \epsilon\mathbb{Z}. \tag{20}$$

To prove this, recall that for $\lambda^\vee = \sum_i m_i\omega_i^\vee$ one has $\nu(\lambda^\vee) = \sum_i m_i \frac{a_i}{a_i^\vee}\omega_i$. If α_i is a short root, then $\frac{a_i}{a_i^\vee} = \epsilon$, so $\frac{a_i}{a_i^\vee}\omega_i(\beta^\vee) \equiv 0 \bmod \epsilon\mathbb{Z}$. Now a case by case consideration shows that if α_i is a long simple root and β is a short positive root, then again $\frac{a_i}{a_i^\vee}\omega_i(\beta^\vee) \equiv 0 \bmod \epsilon\mathbb{Z}$.

Now $s_\beta(\nu(\lambda^\vee)|_{\widehat{\mathfrak{h}}_\beta}) \equiv t_\eta(\Omega) \bmod \mathbb{Z}\Lambda_0$ (respectively $t_\eta(\sigma(\Omega)) \bmod \mathbb{Z}\Lambda_0$) for some $\mathfrak{sl}_2(\beta)$-weight η. Since $t_\eta(\Omega) = \Omega + \epsilon\eta$ respectively $t_\eta(\sigma(\Omega)) = \sigma(\Omega) + \epsilon\eta$, it follows

$$\Omega(\beta^\vee) \equiv 0 \bmod \epsilon\mathbb{Z} \quad \text{respectively} \quad \sigma(\Omega)(\beta^\vee) \equiv 0 \bmod \epsilon\mathbb{Z}. \tag{21}$$

But this is only possible if $\Omega = \epsilon\Lambda_0^\beta$ or $\Omega = \epsilon\Lambda_1^\beta$ as highest weight for the irreducible $Z_\beta \simeq \widehat{\mathfrak{sl}}_2$–representation M, and hence $M(\nu(\lambda^\vee)) \simeq D(\epsilon, m\omega^\beta)$ for some m. Since $\nu(\lambda^\vee)(\beta^\vee) = (\epsilon\Lambda_0^\beta + \epsilon m\omega^\beta)(\beta^\vee)$, it follows that $m = \nu(\lambda^\vee)(\beta^\vee)$. •

Proof of Lemma 6. For each of the Demazure modules $D(1, \lambda_i^\vee)$ denote by $M(\nu(\lambda_i^\vee))$ the Z_β–Demazure submodule generated by v_i, as in (19). By the lemma above we have $M(\nu(\lambda_i^\vee)) \simeq D(\epsilon, m_i\omega^\beta)$, where $m_i = \nu(\lambda_i^\vee)(\beta^\vee)/\epsilon$. Taking the tensor product, we get an embedding

$$M(\nu(\lambda_1^\vee)) \otimes \cdots \otimes M(\nu(\lambda_s^\vee)) \hookrightarrow D(1, \lambda_1^\vee) \otimes \cdots \otimes D(1, \lambda_s^\vee)$$

Now the filtration on $M(\nu(\lambda_1^\vee)) \otimes \cdots \otimes M(\nu(\lambda_s^\vee))$ as $\mathfrak{sl}_2(\beta)\otimes\mathbb{C}[t]$–module is compatible with the filtration of $D(1, \lambda_1^\vee) \otimes \cdots \otimes D(1, \lambda_s^\vee)$ as $\mathfrak{g}\otimes\mathbb{C}[t]$–module, so we get an induced map

$$M(\nu(\lambda_1^\vee)) * \cdots * M(\nu(\lambda_s^\vee)) \longrightarrow D(1, \lambda_1^\vee) * \cdots * D(1, \lambda_s^\vee)$$

Since we are in the simply laced case, we know by Corollary 5 that the left $\mathfrak{sl}_2(\beta)\otimes\mathbb{C}[t])$-module is isomorphic to $M(\nu(\lambda^\vee))$. By construction, the generator of this module satisfies the equation (18), and hence also the image $v_i^{\otimes_{i=1}^s}$ satisfies equation (18).

Remark 16. It is not anymore true that $W(\nu(\lambda^\vee)) \simeq D(1, \lambda^\vee)$. As a counter example consider $\mathfrak{g}$ of type $\mathtt{C}_2$ and take $\lambda^\vee = \omega_1^\vee$. Note that $\nu(\omega_1^\vee) = 2\omega_1$. By [19], $D(1, \omega_1^\vee)$ has dimension 11, and by [6], $KR(\omega_1)$ has dimension 4. The fusion product $KR(\omega_1)*KR(\omega_1)$ is a quotient of $W(2\omega_1)$, so $\dim W(2\omega_1) \geq 16 > \dim D(1, \omega_1^\vee)$.

We conjecture that Corollary 4 also holds in the non-simply laced case:

Conjecture 1. *Let $\lambda = \sum \lambda_i$ be a sum of dominant integral weights, $c_1, \ldots, c_n$ be pairwise distinct complex numbers, then*

$$W(\lambda) \simeq W(\lambda_1) * \ldots * W(\lambda_n)$$

4 Limit constructions

In this section we start with a simple Lie algebra $\mathfrak{g}$ of arbitrary type. We want to reconstruct the $\mathcal{C}\mathfrak{g}$-module structure of the irreducible highest weight $U(\widehat{\mathfrak{g}})$-module $V(l\Lambda_0)$ as a direct limit of fusion products of Demazure modules.

In [19] we have given such a construction of the $\mathfrak{g}$-module structure of $V(l\Lambda_0)$ as a semi-infinite tensor product of finite dimensional $\mathfrak{g}$-module. In this section we want to extend this construction to the $U(\mathcal{C}\mathfrak{g})$-module structure by replacing the tensor product by the fusion product.

We need first a few facts about inclusions of Demazure modules. Set $\tilde{\mathfrak{b}} = \mathfrak{h} \oplus \widehat{\mathfrak{n}}^+ \oplus \mathbb{C}K$, and, as before, we denote by W^{aff} the affine Weyl group. Let Λ be an integral dominant weight for $\widehat{\mathfrak{g}}$. We fix for all $w \in W^{\mathrm{aff}}/{W^{\mathrm{aff}}}_{\Lambda}$ a generator v_w of the line of weight $w(\Lambda) \subset V(\Lambda)$. Denote $V_w(\Lambda) = U(\widehat{\mathfrak{b}}).v_w$ the Demazure module and let $\iota_w : V_w(\Lambda) \hookrightarrow V(\Lambda)$ the inclusion.

Lemma 8. *Let Λ be an integral dominant weight for $\widehat{\mathfrak{g}}$. Given $w \in W^{\mathrm{aff}}/{W^{\mathrm{aff}}}_{\Lambda}$, there is a unique (up to scalar multiplication) nontrivial morphism of $U(\tilde{\mathfrak{b}})$-modules*

$$V_w(\Lambda) \longrightarrow V(\Lambda).$$

In fact, this morphism is, up to scalar multiples, the canonical embedding of the Demazure module.

Proof. We want to prove that, up to scalar multiples, $\iota_w : V_w(\Lambda) \longrightarrow V(\Lambda)$ is the only nontrivial morphism of $U(\tilde{\mathfrak{b}})$-modules. The proof is by induction on the length of w.

For $w = id$, the Demazure module is one-dimensional. The generator v is killed by $U(\widehat{\mathfrak{n}}^+)$, so its image in $V(\Lambda)$ is a highest weight vector. But such a vector is unique (up to scalar multiple) in $V(\Lambda)$, and hence there exists, up to a scalar multiples, only one such morphism.

Suppose now $\ell(w) \geq 1$, and let $\tau = s_\alpha w$, α a simple root, be such that $\tau < w$, and let $\varphi : V_w(\Lambda) \longrightarrow V(\Lambda)$ be a non trivial $U(\tilde{\mathfrak{b}})$-equivariant morphism. Let v_w be a generator of the weight space in $V_w(\Lambda)$ corresponding to the weight $w(\Lambda)$, and set $m_\alpha = w(\Lambda)(\alpha^\vee)$. Then $(x_\alpha)^{m_\alpha}.v_w \neq 0$, but $(x_\alpha)^{m_\alpha+1}.v_w = 0$.

Now φ is an $U(\tilde{\mathfrak{b}})$-morphism, so the image $\varphi(v_w) \in V(\Lambda)$ is again an eigenvector for $\mathfrak{h} \oplus \mathbb{C}K$ of weight $w(\Lambda)|_{\mathfrak{h}\oplus\mathbb{C}K}$. Since $V(\Lambda)$ is a $\widehat{\mathfrak{g}}$-module, $\mathfrak{sl}_2$-representation theory implies $(x_\alpha)^{m_\alpha}.\varphi(v_w) \neq 0$, and since φ is an $U(\tilde{\mathfrak{b}})$-morphism, we have $(x_\alpha)^{m_\alpha+1}.\varphi(v_w) = 0$.

Now $(x_\alpha)^{m_\alpha}.v_w$ is a generator of the Demazure module $V_\tau(\Lambda) \subset V_w(\Lambda)$, so $\varphi|_{V_\tau(\Lambda)}$ provides a non-trivial $U(\tilde{\mathfrak{b}})$-morphism, which by induction can only be a non-zero scalar multiple of the standard inclusion. Hence $(x_\alpha)^{m_\alpha}.\varphi(v_w)$ is a non-zero multiple of v_τ. Further, by weight reasoning and $\mathfrak{sl}_2$-representation theory, it follows that $x_{-\alpha}\varphi(v_w) = 0$. By the usual exchange relation we get

$$x_{-\alpha}^{m_\alpha}(x_\alpha)^{m_\alpha}\varphi(v_w) = c\varphi(v_w),$$

for some nonzero complex number c, and hence $\varphi(v_w)$ is an extremal weight vector of weight $w(\Lambda)$, which finishes the proof. •

Corollary 7. *Let $\tau < w$, then there exists (up to scalar multiples) a unique morphism of $U(\tilde{\mathfrak{b}})$-modules $V_\tau(m\Lambda_0) \longrightarrow V_w(m\Lambda_0)$.*

Consider the Demazure module $D(m, n\Theta) = V_{-n\Theta}(m\Lambda_0)$. We fix a generator $w \neq 0$ of the unique $U(\mathcal{C}\mathfrak{g})$-fixed line in $D(m, \Theta)$. Note (see [19]) that w spans the line of the highest weight vectors for $\widehat{\mathfrak{g}}$ in $V(m\Lambda_0)$. By Theorem 8 we have for $c_1 \neq c_2$ an isomorphism

$$D(m, (n+1)\Theta) \simeq D(m, \Theta)_{c_2} * D(m, n\Theta)_{c_1}.$$

We extend this to an isomorphism of $U(\mathcal{C}\mathfrak{g} \oplus \mathbb{C}K)$-modules by letting K operate on $D(m, (n+1)\Theta)$ by the level m, and letting K act on the second module by 0 on the first factor and on the second factor by the level m. Define the map

$$\tilde{\varphi} : D(m, n\Theta)_{c_1} \longrightarrow D(m, \Theta)_{c_2} \otimes D(m, n\Theta)_{c_1}$$

by $\tilde{\varphi}(v) = w \otimes v$. This map is an $U(\mathcal{C}\mathfrak{g})$-module morphism because w is $U(\mathcal{C}\mathfrak{g})$-invariant, which extends, as above, to a $U(\mathcal{C}\mathfrak{g} \oplus \mathbb{C}K)$-module morphism.

The map respects the filtrations up to a shift: let $v_2 \in D(m, \Theta)$ be a generator and let q be minimal such that $w \otimes v_2 \in F^q(D(m, \Theta)_{c_2} \otimes D(m, n\Theta)_{c_1})$. By the $U(\mathcal{C}\mathfrak{g})$-equivariance it follows that

$$\tilde{\varphi}(F^j(D(m, n\Theta)_{c_1}) \subseteq F^{j+q}(D(m, \Theta)_{c_2} \otimes D(m, n\Theta)_{c_1})$$

So we get an induced $U(\mathcal{C}\mathfrak{g} \oplus \mathbb{C}K)$-morphism φ between the associated graded modules by $\varphi(\overline{v}) = \overline{w \otimes v}$. φ is nontrivial and so by Corollary 7 it is (up to multiplication by a scalar) the embedding of Demazure modules ι. We proved:

Lemma 9. *The map $\varphi : D(m, n\Theta) \longrightarrow D(m, \Theta)_{c_2} * D(m, n\Theta)_{c_1} \simeq D(m, (n+1)\Theta)$ induced by $\varphi(v) = \overline{w \otimes v}$ is an embedding of $U(\tilde{\mathfrak{b}})$-modules.*

One knows that $V(m\Lambda_0) = \lim\limits_{n\to\infty} D(m, n\Theta)$ as $U(\mathcal{C}\mathfrak{g})$-modules, and also as $U(\tilde{\mathfrak{b}})$-modules. It follows by the above:

Lemma 10. *Let $\mathfrak{g}$ be a simple Lie algebra.*

The following is a commutative diagram of $U(\mathcal{C}\mathfrak{g})$-modules

$$\begin{tikzcd}
D(m,n\Theta) \arrow[r, hook, "\iota"] \arrow[d, "\wr"] & D(m,(n+1)\Theta) \arrow[d, "\wr"] \\
D(m,n\Theta) \arrow[r, "\varphi"] \arrow[d, "\wr"] & D(m,\Theta)_{c_{n+1}} * D(m,n\Theta)_c \arrow[d, "\wr"] \\
D(m,\Theta)_{c_n} * \ldots * D(m,\Theta)_{c_1} \arrow[r, "\varphi"] & D(m,\Theta)_{c_{n+1}} * D(m,\Theta)_{c_n} * \ldots * D(m,\Theta)_{c_1}
\end{tikzcd}$$

where the down arrows are the isomorphism of Corollary 5

Theorem 9. *Let $\mathfrak{g}$ be a simple Lie algebra. As $U(\mathcal{C}\mathfrak{g})$-module, $V(m\Lambda_0)$ is isomorphic to the semi-infinite fusion product*

$$V(m\Lambda_0) \simeq \lim_{n\to\infty} D(m,\Theta) * \ldots * D(m,\Theta)$$

We expect the following to hold:

Conjecture 2. *Let $\Lambda = m\Lambda_0 + \lambda$ be a dominant integral weight for $\widehat{\mathfrak{g}}$, then $V(\Lambda)$ and*

$$\lim_{n\to\infty} D(m,\Theta) * \ldots * D(m,\Theta) * V(\lambda)$$

are isomorphic as $\mathcal{C}\mathfrak{g}$-modules.

Remark 17. This isomorphism holds for the $\mathfrak{g}$-module structure, see [19].

Remark 18. As in [19], the limit construction above works in a much more general setting. Let $D(m,\mu^\vee)$ be a Demazure module with the property that for some k the fusion product $W = D(m,\mu^\vee) * \cdots * D(m,\mu^\vee) \simeq D(m,k\mu^\vee)$ contains a highest weight vector of weight $m\Lambda_0$. Instead of $D(m,\Theta)$ one can then use the module W in the direct limit construction above.

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DEMAZURE STRUCTURE INSIDE KIRILLOV–RESHETIKHIN CRYSTALS

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Abstract. The conjecturally perfect Kirillov-Reshetikhin (KR) crystals are known to be isomorphic as classical crystals to certain Demazure subcrystals of crystal graphs of irreducible highest weight modules over affine algebras. Under some assumptions we show that the classical isomorphism from the Demazure crystal to the KR crystal, sends zero arrows to zero arrows. This implies that the affine crystal structure on these KR crystals is unique.

1. Introduction

The irreducible finite-dimensional modules over a quantized affine algebra $U_q'(\mathfrak{g})$ were classified by Chari and Pressley [3, 4] in terms of Drinfeld polynomials. We are interested in the subfamily of such modules which possess a global crystal basis. Kirillov–Reshetikhin (KR) modules are finite-dimensional $U_q'(\mathfrak{g})$-modules $W^{r,s}$ that were introduced in [7, 8]. It is expected that each KR module has a crystal basis $B^{r,s}$, and that every irreducible finite-dimensional $U_q'(\mathfrak{g})$-module with crystal basis, is a tensor product of the crystal bases of KR modules.

The KR modules $W^{r,s}$ are indexed by a Dynkin node r of the classical subalgebra (that is, the distinguished simple Lie subalgebra) $\mathfrak{g}_0$ of $\mathfrak{g}$ and a positive integer s. In general the existence of $B^{r,s}$ remains an open question. For type $A_n^{(1)}$ the crystal $B^{r,s}$ is known to exist [18] and its combinatorial structure has been studied [24]. In many cases, the crystals $B^{1,s}$ and $B^{r,1}$ for nonexceptional types are also known to exist and their combinatorics has been worked out in [16, 18] and [9, 14], respectively.

Viewed as a $U_q(\mathfrak{g}_0)$-module by restriction, $W^{r,s}$ is generally reducible; its decomposition into $U_q(\mathfrak{g}_0)$-irreducibles was conjectured in [7, 8]. This was verified by Chari [1] for the nontwisted cases.

Kashiwara [13] conjectured that as classical crystals, many of the KR crystals (the ones conjectured to be perfect in [7, 8]) are isomorphic to certain Demazure subcrystals of affine highest weight crystals. Kashiwara's conjecture was confirmed by Fourier and Littelmann [5] in the untwisted cases and Naito and Sagaki [22] in the twisted cases.

In this paper we prove that the classical isomorphism from the Demazure crystals to KR crystals sends zero arrows to zero arrows (see Theorem 4.4). It is not an affine crystal isomorphism but becomes an isomorphism after tensoring with an appropriate affine highest weight crystal. This recovers some of the isomorphisms given by the Kyoto path model. We emphasize this is accomplished without the

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assumption of perfectness of the KR crystals. The automorphisms on the crystals that are used in the definition of the ground state path in the Kyoto path model, come from affine Dynkin diagram automorphisms which can be calculated using the factorization of a translation element in the extended affine Weyl group in our setting. For the proof of our results we require the assumptions of regularity of KR crystals, the existence and uniqueness of a certain special element u in a KR crystal, and the existence of automorphisms on KR crystals coming from certain Dynkin automorphisms (see Assumption 1). We show that under these assumptions, the KR crystals admit a unique affine crystal structure (see Corollary 4.6), and we give an algorithm which shows that twofold tensor products of KR crystals are connected (see Corollary 6.1). We expect that Assumption 1 holds, that is, if the existence of the KR crystals were established these hypotheses could be removed.

In Section 2 we establish notation and review some results about the extended affine Weyl group. The definition of Demazure crystals and KR crystals is given in Section 3. Section 4 contains our main result stated in Theorem 4.4 showing that all zero arrows of the Demazure crystal are present in the KR crystal. In Section 5 we provide explicit sequences of lowering operators leading from the special element u of a KR crystal to all classical highest weight elements of the KR crystal. The connectedness of tensor products of KR crystals and an application regarding the algorithmic calculation of the combinatorial R-matrix can be found in Section 6.

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2. Notation and basics

2.1. **Affine Kac-Moody algebras.** Let $\mathfrak{g}$ be an affine Kac-Moody algebra with Cartan subalgebra $\mathfrak{h}$, Dynkin node set $I = \{0, 1, \dots, n\}$, Cartan matrix $A = (a_{ij})_{i,j\in I}$, realized by the set of linearly independent simple roots $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and simple coroots $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$, such that $\langle \alpha_i^\vee , \alpha_j\rangle = a_{ij}$ [10]. Let $d \in \mathfrak{h}$ be the scaling element, which is any element such that $\langle d , \alpha_i\rangle = 0$ for $i \in I\backslash\{0\}$ and $\langle d , \alpha_0\rangle = 1$. Let $(a_i \mid i \in I)$ be the unique tuple of relatively prime positive integers that give a linear dependence relation among the columns of A, and let $(a_i^\vee \mid i \in I)$ be the tuple for the rows of A. Let $\delta = \sum_{i\in I} a_i\alpha_i$ be the null root, $\theta = \sum_{i\in I\backslash\{0\}} a_i\alpha_i$, and $c = \sum_{i\in I} a_i^\vee\alpha_i^\vee$ the canonical central element. We have $\langle d , \delta\rangle = a_0$. Let $\{\Lambda_i \mid i \in I\} \subset \mathfrak{h}^*$ be the fundamental weights, which, together with δ/a_0, are defined to the dual basis to the basis $\{\alpha_i^\vee \mid i \in I\} \cup \{d\}$ of $\mathfrak{h}$. In particular $\langle \alpha_i^\vee , \Lambda_j\rangle = \delta_{ij}$. Let $P = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}(\delta/a_0) \subset \mathfrak{h}^*$ be the weight lattice, $P^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\Lambda_i \oplus \mathbb{Z}(\delta/a_0) = \{\lambda \in P \mid \langle \alpha_i^\vee , \lambda\rangle \geq 0 \text{ for all } i \in I\}$ the set of dominant weights and $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ the root lattice. The level of a weight $\lambda \in P$ is defined by $\langle c , \lambda\rangle$. Let W be the affine Weyl group, generated by the simple reflections $\{s_i \mid i \in I\}$. W acts on P by $s_i\lambda = \lambda - \langle \alpha_i^\vee , \lambda\rangle\alpha_i$.

Let $(\cdot \mid \cdot)$ be the nondegenerate W-invariant symmetric form on $\mathfrak{h}^*$; it is defined by $(\alpha_i \mid \alpha_j) = a_i^\vee a_i^{-1} a_{ij}$ for $i, j \in I$, $(\alpha_i \mid \Lambda_0) = 0$ for $i \in I\backslash\{0\}$, $(\alpha_0 \mid \Lambda_0) = a_0^{-1}$, and $(\Lambda_0 \mid \Lambda_0) = 0$. One may check that [10, (6.4.1)]

$$(\theta \mid \theta) = 2a_0 = \begin{cases} 4 & \text{for } A_{2n}^{(2)} \\ 2 & \text{otherwise.} \end{cases} \tag{2.1}$$

The pairing $(\cdot \mid \cdot)$ induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ given by $\langle \nu(h)\,,\, h'\rangle = (h \mid h')$ for all $h, h' \in \mathfrak{h}$. So $\nu(\alpha_i^\vee) = a_i (a_i^\vee)^{-1}\alpha_i$ for $i \in I$, $\nu(d) = a_0\Lambda_0$, and $\nu(c) = \delta$. Define $\theta^\vee \in \mathfrak{h}$ by $\nu(\theta^\vee) = 2\theta/(\theta \mid \theta) = \theta/a_0$.

Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the simple Lie subalgebra whose Dynkin node set is $I\backslash\{0\}$, with Weyl group $W_0 \subset W$, root lattice Q_0, weight lattice P_0, and fundamental weights $\{\omega_i \mid i \in I\backslash\{0\}\} \subset P_0$.

Let $P' = P/\mathbb{Z}(\delta/a_0)$. The natural projection $P' \to P_0$ has a section $P_0 \to P'$ defined by $\omega_i \mapsto \Lambda_i - a_i^\vee \Lambda_0$ for $i \in I\backslash\{0\}$. The image of this section is the set of elements in P' of level zero.

2.2. **Dynkin automorphisms.** Let X denote the affine Dynkin diagram and $\mathrm{Aut}(X)$ denote the group of automorphisms of X. By definition an element of $\mathrm{Aut}(X)$ is a permutation of the Dynkin node set I which preserves the kind of bonds between nodes. Observe that

$$\begin{aligned} a_{\tau(i)} &= a_i \\ a_{\tau(i)}^\vee &= a_i^\vee \end{aligned} \qquad \text{for all } i \in I \text{ and } \tau \in \mathrm{Aut}(X). \tag{2.2}$$

There is an action of $\mathrm{Aut}(X)$ on P given by

$$\begin{aligned} \sigma(\Lambda_i) &= \Lambda_{\sigma(i)} \qquad \text{for } i \in I \\ \sigma(\delta) &= \delta \end{aligned}$$

for $\sigma \in \mathrm{Aut}(X)$. By (2.2) this action restricts to an action of $\mathrm{Aut}(X)$ on P_0 called the level zero action.

2.3. **Translations.** For $\alpha \in P_0$, define the element $t_\alpha \in \mathrm{Aut}(P)$ by [10, (6.5.2)]

$$t_\alpha(\lambda) = \lambda + \langle c\,,\, \lambda\rangle\alpha - \big((\lambda \mid \alpha) + \frac{1}{2}(\alpha \mid \alpha)\langle c\,,\, \lambda\rangle\big)\delta. \tag{2.3}$$

The map $\alpha \mapsto t_\alpha$ defines an injective group homomorphism $P_0 \to \mathrm{Aut}(P)$ whose image shall be denoted $T(P_0)$. For any $w \in W_0$,

$$w t_\alpha w^{-1} = t_{w(\alpha)}. \tag{2.4}$$

Therefore $W_0 \ltimes T(P_0)$ acts on P. There is an induced action of $W_0 \ltimes T(P_0)$ on P' that preserves the level of a weight. For every $m \in \mathbb{Z}$ there is an action of $W_0 \ltimes T(P_0)$ on P_0 called the level m action, given by $w *_m \mu = w(m\Lambda_0 + \mu) - m\Lambda_0$ for $\mu \in P_0$. Under the level one action, the element t_α is precisely translation by α.

2.4. **Extended affine Weyl group.** For each $i \in I\backslash\{0\}$, define $c_i = \max(1, a_i/a_i^\vee)$; these constants were introduced in [7]. Using the Kac indexing of the affine Dynkin diagrams [10, Table Fin, Aff1 and Aff2], we have $c_i = 1$ except for $c_i = 2$ for $\mathfrak{g} = B_n^{(1)}$ and $i = n$, $\mathfrak{g} = C_n^{(1)}$ and $1 \le i \le n-1$, $\mathfrak{g} = F_4^{(1)}$ and $i = 3, 4$, and $c_2 = 3$ for $\mathfrak{g} = G_2^{(1)}$. Consider the sublattices of P_0 given by

$$\begin{aligned} M &= \bigoplus_{i \in I\backslash\{0\}} \mathbb{Z} c_i \alpha_i = \mathbb{Z} W_0 \cdot \theta/a_0 \\ \widetilde{M} &= \bigoplus_{i \in I\backslash\{0\}} \mathbb{Z} c_i \omega_i. \end{aligned}$$

It is easy to check that $M \subset \widetilde{M}$ and that the action of W_0 on P_0 restricts to actions on M and $\widetilde{M}$. Let $T(\widetilde{M})$ (resp. $T(M)$) be the subgroup of $T(P_0)$ generated by t_λ for $\lambda \in \widetilde{M}$ (resp. $\lambda \in M$).

There is an isomorphism [10, Prop. 6.5]

$$W \cong W_0 \ltimes T(M) \tag{2.5}$$

as subgroups of $\mathrm{Aut}(P)$. Under this isomorphism we have

$$s_0 = t_{\theta/a_0} s_\theta. \tag{2.6}$$

Define the extended affine Weyl group to be the subgroup of $\mathrm{Aut}(P)$ given by

$$\widetilde{W} = W_0 \ltimes T(\widetilde{M}). \tag{2.7}$$

When $\mathfrak{g}$ is of untwisted type, $M \cong Q^\vee$, $\widetilde{M} \cong P^\vee$, $c_i\omega_i = \nu(\omega_i^\vee)$, and $c_i\alpha_i = \nu(\alpha_i^\vee)$ for $i \in I\backslash\{0\}$.

Let $C \subset P \otimes_{\mathbb{Z}} \mathbb{R}$ be the fundamental chamber, the set of elements λ such that $\langle \alpha_i^\vee, \lambda\rangle \geq 0$ for all $i \in I$. Define the subgroup $\Sigma \subset \widetilde{W}$ to be the set of elements that send C into itself.

It follows from (2.4) and (2.5) that W is a normal subgroup of $\widetilde{W}$. Thus Σ acts on W by conjugation. Since the Weyl chambers adjacent to C are precisely those of the form $s_i(C)$ for $i \in I$, the element $\tau \in \Sigma$ induces a permutation (also denoted τ) of the set I given by

$$\tau s_i \tau^{-1} = s_{\tau(i)} \qquad \text{for } i \in I. \tag{2.8}$$

Since the braid relations in W are preserved, Σ is a subgroup of $\mathrm{Aut}(X)$.

2.5. **Special automorphisms.** We identify the subgroup Σ explicitly. Say that an affine Dynkin node $i \in I$ is *special* if there is an automorphism $\tau \in \mathrm{Aut}(X)$ of the affine Dynkin diagram such that $\tau(i) = 0$. In the untwisted case, i is special if and only if $\omega_i^\vee$ is a minuscule coweight. Let $I^0 \subset I$ denote the set of special vertices. Explicitly, using the Kac labeling [10]:

$$I^0 = \begin{cases} \{0,1,\dots,n\} & \text{for } A_n^{(1)} \\ \{0,1\} & \text{for } B_n^{(1)},\, A_{2n-1}^{(2)} \\ \{0,n\} & \text{for } C_n^{(1)},\, D_{n+1}^{(2)} \\ \{0,1,n-1,n\} & \text{for } D_n^{(1)} \\ \{0,1,5\} & \text{for } E_6^{(1)} \\ \{0,6\} & \text{for } E_7^{(1)} \\ \{0\} & \text{otherwise.} \end{cases}$$

Proposition 2.1. *For each $i \in I^0$ there is a unique element $\tau_i \in \Sigma$ such that $\tau_i(i) = 0$. Moreover $\Sigma = \{\tau_i \mid i \in I^0\}$.*

We call τ_i the special automorphism associated with $i \in I^0$.

Note that every Dynkin automorphism is determined by its action on I^0. We describe the special automorphisms explicitly. τ_0 is the identity automorphism. If $\mathfrak{g}$ is of untwisted affine type and $i \in I^0$ then for all $j \in I^0$, $\tau_i(j) = k \in I^0$ where $-\omega_i + \omega_j \cong \omega_k \mod Q_0$ and $\omega_0 = 0$ by convention. For $\mathfrak{g}$ of twisted type the only nonidentity (special) automorphisms are the elements of $\mathrm{Aut}(X)$ which on I^0 are given by $\tau_1 = (0,1)$ in type $A_{2n-1}^{(2)}$ and $\tau_n = (0,n)$ in type $D_{n+1}^{(2)}$.

We now specify Σ explicitly as a subgroup of permutations of I^0. In all cases but $D_n^{(1)}$ and n even, Σ is a cyclic group. This determines τ_i and Σ completely except for types $A_n^{(1)}$ and $D_n^{(1)}$. For $A_n^{(1)}$, $\Sigma \cong \mathbb{Z}/(n+1)\mathbb{Z}$ where $\tau_i(j) = j - i \mod (n+1)$

for all $i, j \in I^0$. For $D_n^{(1)}$ and n odd, Σ is cyclic with $\tau_{n-1} = (0, n, 1, n-1)$, $\tau_1 = (0,1)(n-1,n)$ and $\tau_n = (0, n-1, 1, n)$ in cycle notation acting on I^0. For n even, $\Sigma \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $\tau_1 = (0,1)(n-1,n)$, $\tau_{n-1} = (0, n-1)(1,n)$ and $\tau_n = (0,n)(1,n-1)$.

Proposition 2.2. *$\Sigma \cong \widetilde{M}/M$ via $\tau_i \mapsto \omega_i + M$ for $i \in I^0$ and*

$$\widetilde{W} \cong W \rtimes \Sigma. \tag{2.9}$$

as subgroups of $\mathrm{Aut}(P_0)$.

If $i \in I^0$ then $c_i = 1$ and we have

$$\tau_i = w_0^{\omega_i} t_{-\omega_i} \tag{2.10}$$

where, for $\lambda \in P_0^+$,

$$w_0^\lambda \in W_0 \text{ is the shortest element such that } w_0^\lambda \lambda \text{ is antidominant.} \tag{2.11}$$

2.6. **Dynkin automorphisms revisited.** Let X_0 be the Dynkin diagram for the classical subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$.

Lemma 2.3. *There is a group homomorphism*

$$\begin{aligned} \mathrm{Aut}(X) &\to \mathrm{Aut}(X_0) \\ \sigma &\mapsto \sigma' \end{aligned} \tag{2.12}$$

where $\sigma'(i) = j$ if and only if $\sigma(\omega_i) \in W_0\omega_j$.

Proof. We first claim that there is a group action of $\mathrm{Aut}(X)$ on $W_0 \backslash P_0$ defined by $\sigma(W_0\lambda) = W_0\sigma\lambda$ where $\mathrm{Aut}(X)$ acts on P_0 via the level zero action. The level zero action of s_0 on P_0 is the same as that of $s_\theta \in W_0$, by (2.6) and (2.3). Thus for the level zero action, $W\lambda = W_0\lambda$ for $\lambda \in P_0$. By (2.8), $\sigma W_0 \sigma^{-1} \subset W$ as it is generated by $s_{\sigma(i)}$ for $i \in I\backslash\{0\}$. Thus we have $W_0\sigma W_0 \tau\lambda = W_0(\sigma W_0 \sigma^{-1})\sigma\tau\lambda = W_0\sigma\tau\lambda$. Therefore $\mathrm{Aut}(X)$ acts on $W_0\backslash P_0$.

Next we show that this action restricts to an action on $F \subset W_0 \backslash P_0$ where F is the set of W_0-orbits of fundamental weights ω_i for $i \in I\backslash\{0\}$. Due to the above group action we need only that $\sigma F \subset F$ for generators σ of $\mathrm{Aut}(X)$. By (2.2) we have $\sigma(\omega_r) = \omega_{\sigma(r)} - a_r^\vee \omega_{\sigma(0)}$ where we write $\omega_i = \Lambda_i - a_i^\vee \Lambda_0$ for all $i \in I$. Using this one may straightforwardly check the lemma for each affine root system. $\square$

$\mathrm{Aut}(X_0)$ is trivial except in the following cases, where the homomorphism is described explicitly. The elements of $\mathrm{Aut}(X)$ and $\mathrm{Aut}(X_0)$ are given by their action as permutations of I^0 and $I^0 \setminus \{0\}$ respectively.

1. $\mathrm{Aut}(A_n)$ is generated by the involution $i \mapsto n+1-i$ for $i \in I\backslash\{0\}$. In this case $\mathrm{Aut}(A_n^{(1)})$ is the dihedral group $D_{2(n+1)}$. For $\sigma \in \mathrm{Aut}(A_n^{(1)})$, σ' is the nontrivial element in $\mathrm{Aut}(A_n)$ if and only if σ reverses orientation.
2. $\mathrm{Aut}(D_n)$ is generated by $(n-1,n)$ when $n > 4$. In this case $\mathrm{Aut}(D_n^{(1)})$ is generated by $(0,1)$, $(n-1,n)$ and $(0,n)(1,n-1)$. All these map to the nontrivial element of $\mathrm{Aut}(D_n)$ except in the case that n is even, when $(0,n)(1,n-1)$ maps to the identity.
3. $\mathrm{Aut}(D_4)$ is the symmetric group on the three "satellite" vertices $\{1,3,4\}$. $\mathrm{Aut}(D_4^{(1)})$ is the symmetric group on the vertices $\{0,1,3,4\}$ and is generated by $(0,i)$ for $i \in \{1,3,4\}$. The generator $(0,i)$ is sent to the element (j,k) in $\mathrm{Aut}(D_4)$ where $\{0,i,j,k\} = \{0,1,3,4\}$ as sets.

(4) $\mathrm{Aut}(E_6)$ is generated by $(1,5)$. $\mathrm{Aut}(E_6^{(1)})$ is isomorphic to the S_3 that permutes the special vertices $\{0,1,5\}$. Then each of the elements of order two in $\mathrm{Aut}(E_6^{(1)})$ is sent to the nontrivial element of $\mathrm{Aut}(E_6)$.

Remark 1. In all cases, for all $\tau \in \Sigma$, τ' is the identity in $\mathrm{Aut}(X_0)$. However for $\sigma = (0,1) \in \mathrm{Aut}(D_n^{(1)})$ we have $\sigma' = (n-1,n) \in \mathrm{Aut}(D_n)$.

3. Crystals

3.1. **Definition of crystals.** A P-weighted I-crystal is a set B, equipped with Kashiwara operators $e_i, f_i : B \to B \sqcup \{\emptyset\}$, and weight function $\mathrm{wt} : B \to P$ such that $e_i(f_i(b)) = b$ if $f_i(b) \neq \emptyset$, $f_i(e_i(b)) = b$ if $e_i(b) \neq \emptyset$, $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$ if $f_i(b) \neq \emptyset$, $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$ if $e_i(b) \neq \emptyset$, and $\langle \alpha_i^\vee, \mathrm{wt}(b)\rangle = \varphi_i(b) - \varepsilon_i(b)$ where $\varphi_i(b) = \min\{m \mid f_i^m(b) \neq \emptyset\}$ and $\varepsilon_i(b) = \min\{m \mid e_i^m(b) \neq \emptyset\}$ are assumed to be finite for all $b \in B$ and $i \in I$. If $f_i(b) \neq \emptyset$ we draw an arrow colored i from b to $f_i(b)$. The connected components of the graph obtained by removing all arrows from B except the arrows colored i, are called the i-strings of B. We write $\varepsilon(b) = \sum_{i\in I} \varepsilon_i(b)\Lambda_i$ and $\varphi(b) = \sum_{i\in I} \varphi_i(b)\Lambda_i$.

An I-crystal B is *regular* if, for each subset $K \subset I$ with $|K| = 2$, each K-component of B is isomorphic to the crystal basis of an irreducible integrable highest weight $U_q'(\mathfrak{g}_K)$-module where $\mathfrak{g}_K$ is the subalgebra of $\mathfrak{g}$ with simple roots α_i for $i \in K$.

The crystal reflection operator $S_i : B \to B$ is defined by the property that $S_i(b)$ is the unique element in the i-string of b such that $\varepsilon_i(S_i(b)) = \varphi_i(b)$ or equivalently $\varphi_i(S_i(b)) = \varepsilon_i(b)$. This defines an action of the Weyl group W on B if B is regular [12].

If B and B' are P-weighted I-crystals, their tensor product $B \otimes B'$ is a P-weighted I-crystal as follows (we use the opposite of Kashiwara's convention). As a set $B \otimes B'$ is just the Cartesian product $B \times B'$ where traditionally one writes $b \otimes b'$ instead of (b, b'). The Kashiwara operators are given by

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes f_i(b') & \text{if } \varepsilon_i(b) < \varphi_i(b') \end{cases}$$

$$e_i(b \otimes b') = \begin{cases} e_i(b) \otimes b' & \text{if } \varepsilon_i(b) > \varphi_i(b') \\ b \otimes e_i(b') & \text{if } \varepsilon_i(b) \leq \varphi_i(b'). \end{cases}$$

Given any P-weighted I-crystal B and Dynkin automorphism σ, there is a P-weighted I-crystal B^σ whose vertex set is written $\{b^\sigma \mid b \in B\}$ and whose edges are given by $f_i(b) = b'$ in B if and only if $f_{\sigma(i)}(b^\sigma) = (b')^\sigma$. The weight function satisfies $\mathrm{wt}(b^\sigma) = \sigma(\mathrm{wt}(b))$ where the second σ is the automorphism of P defined by σ. A similar statement holds for P_0-weighted I-crystals, using the level zero action of σ on P_0 defined in Subsection 2.2.

Given any P-weighted I-crystal B, define the *contragredient dual* crystal $B^\vee = \{b^\vee \mid b \in B\}$ with $\mathrm{wt}(b^\vee) = -\mathrm{wt}(b)$ and $f_i(b) = b'$ if and only if $e_i(b^\vee) = {b'}^\vee$.

3.2. **Branching.** The following ideas have been applied extensively (in [18] and [25], for example) to identify the 0-arrows in KR crystals. We shall use them here for the same purpose.

Let B be the crystal graph of a $U_q'(\mathfrak{g})$-module and $K \subset I$. A *K-component* of B is a connected component of the graph obtained from B by removing all i-edges for

$i \notin K$. A *K-highest weight vector* is an element $b \in B$ such that $\varepsilon_i(b) = 0$ for all $i \in K$. Suppose K is a proper subset of I. Since the subalgebra of $\mathfrak{g}$ with simple roots $\{\alpha_i \mid i \in K\}$ is semisimple, each K-component of B has a unique K-highest weight vector. When $K = I \setminus \{0\}$ we call the K-components and K-highest weight vectors *classical* components and highest weight vectors.

Suppose σ is a Dynkin automorphism that fixes K and induces an automorphism (also denoted σ) on B that sends i-arrows to $\sigma(i)$-arrows for all $i \in I$. Then by definition σ preserves i-arrows for all $i \in K$. There is a projection from the classical weight lattice to that of the subalgebra with simple roots α_i for $i \in K$; we refer to the latter as the *K-weight lattice*. In particular σ permutes the collection of K-components, sending K-highest weight vectors to those with the same K-weight (that is, $\varphi_i \circ \sigma = \varphi_i$ for $i \in K$).

3.3. **Demazure modules and crystals.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra and $U_q(\mathfrak{g})$ its quantized universal enveloping algebra. For a dominant weight Λ denote by $V(\Lambda)$ the irreducible integrable highest weight $U_q(\mathfrak{g})$-module with highest weight Λ. Write $B(\Lambda)$ for its crystal basis. Let $\mathfrak{b}$ be a Borel Lie subalgebra of $\mathfrak{g}$. For $\mu \in W \cdot \Lambda$ let u_μ be a generator of the line of weight μ in $V(\Lambda)$. Write $\mu = w\Lambda$ where w is shortest in its coset wW^Λ and $W^\Lambda = \{w \in W \mid w\Lambda = \Lambda\}$. When writing an element $w\Lambda \in W \cdot \Lambda$ we shall always assume w is of minimum length. Define the Demazure module

$$V_w(\Lambda) := U_q(\mathfrak{b}) \cdot u_{w(\Lambda)}.$$

It is known that $V_w(\Lambda)$ has a crystal base $B_w(\Lambda)$ [11]; it is the full subgraph of $B(\Lambda)$ whose vertex set consists of the elements in $B(\Lambda)$ that are reachable by raising operators, from the unique element $u_{w\Lambda} \in B(\Lambda)$ of weight $w\Lambda$. We shall make use of the following result. By abuse of notation let

$$f_w(b) = \{\, f_{i_N}^{m_N} \cdots f_{i_1}^{m_1}(b) \mid m_k \in \mathbb{Z}_{\geq 0}\} \tag{3.1}$$

where $w = s_{i_N} \cdots s_{i_1}$ is any fixed reduced decomposition of w. It is known [15, 20, 21] that as sets,

$$B_w(\Lambda) = f_w(u_\Lambda). \tag{3.2}$$

For $\mathfrak{g}$ affine, let $w \in \widetilde{W}$. By (2.9) we may express it uniquely as $w = z\tau$ where $z \in W$ and $\tau \in \Sigma$. We define the Demazure module to be

$$V_w(\Lambda) := V_z(\tau(\Lambda)).$$

Its crystal graph is denoted $B_w(\Lambda) = B_z(\tau\Lambda)$. For a dominant $\lambda \in \widetilde{M}$, let $\lambda^* = -w_0(\lambda)$, where w_0 is the longest element in W_0. Define $D(\lambda, s) = V_{t_{-\lambda^*}}(s\Lambda_0)$ and by abuse of notation, $D(\lambda, s) = B_{t_{-\lambda^*}}(s\Lambda_0)$. For any $\sigma \in \mathrm{Aut}(X)$ let $D^\sigma(\lambda, s) = B_{t_{-\sigma(\lambda)^*}}(s\Lambda_{\sigma(0)})$; it is obtained from $D(\lambda, s)$ by changing every i arrow into a $\sigma(i)$ arrow.

3.4. **KR crystals.** Kirillov–Reshetikhin (KR) modules $W^{r,s}$, labeled by $(r, s) \in I\backslash\{0\} \times \mathbb{Z}_{>0}$, are finite-dimensional $U_q'(\mathfrak{g})$-modules. See [7] for the precise definition. It is conjectured that $W^{r,s}$ has a global crystal basis $B^{r,s}$.

In [7] a conjecture is given for the decomposition of each Kirillov–Reshetikhin (KR) module $W^{r,c_r s}$ into its $\mathfrak{g}_0$-components. Chari [1] proved this conjecture for the nonexceptional untwisted algebras and for the exceptional cases for the nodes r such that either $r \in I^0$ or ω_r is the highest root. Recently the G_2 case was treated in full [2]. In [5], the $\mathfrak{g}_0$-structure of the Demazure modules was calculated for the same cases as in [1], and it was verified that the Demazure and KR modules agree as $\mathfrak{g}_0$-modules. In addition, it was shown in [6] that no matter what the precise $\mathfrak{g}_0$-structure is, the Demazure and the KR modules agree as $\mathfrak{g}_0$-modules for all untwisted algebras. Naito and Sagaki [22] proved the conjectures of [7] on the level of crystals for the twisted cases under the assumption that the KR crystals for the untwisted algebras exist. In unpublished work, Naito and Sagaki did the same construction for the twisted cases on the Demazure modules.

Remark 2. Assuming that B^{r,c_rs} exists, the Demazure crystal $D(c_r\omega_r, s)$ and the KR crystal B^{r,c_rs} have the same classical crystal structure.

In this paper we assume that the KR crystal B^{r,c_rs} has the properties of Assumption 1, which we expect to hold if the KR crystals exist. In the next section we will see that with these assumptions the Demazure crystal sits inside the KR crystal (see Theorem 4.4) and that the KR crystal is unique (see Corollary 4.6). For types $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ let σ be the Dynkin automorphism exchanging the Dynkin nodes 0 and 1 and fixing all others. For types $C_n^{(1)}$ and $D_{n+1}^{(2)}$ let σ be the Dynkin automorphism defined by $i \mapsto n-i$ for all $i \in I$. We also write σ for the induced automorphism of P.

Assumption 1. The KR crystal B^{r,c_rs} has the following properties:

(1) B^{r,c_rs} is regular.
(2) There is a unique element $u \in B^{r,c_rs}$ such that

$$\varepsilon(u) = s\Lambda_0 \quad \text{and} \quad \varphi(u) = s\Lambda_{\tau(0)},$$

where $t_{-c_r\omega_r} = w\tau$ with $w \in W$ and $\tau \in \Sigma$.
(3) For all types different from $A_{2n}^{(2)}$, B^{r,c_rs} admits the automorphism corresponding to σ (also denoted σ) such that

$$\varepsilon \circ \sigma = \sigma \circ \varepsilon \qquad \varphi \circ \sigma = \sigma \circ \varphi. \tag{3.3}$$

For type $A_{2n}^{(2)}$ we assume that B^{r,c_rs} is given explicitly by the virtual crystal construction in [23].

4. Relation between Demazure and KR crystals

In this section we show that the Demazure crystal sits inside the KR crystals in Theorem 4.4 and, assuming their existence, that the KR crystals are unique in Corollary 4.6.

The main technique that we use in the proof is a decomposition of the translation elements $t_{-c_r\omega_r}$ that ends in a word for the subalgebra associated to the nodes $\{0, 1, \dots, r-1\}$ of the Dynkin diagram in analogy to the results of [5].

Proposition 4.1. *Let $\mathfrak{g}$ be of nonexceptional affine type, $r \in I \setminus I^0$ and $t_{-c_r\omega_r} = w\tau$ for $w \in W$ and $\tau \in \Sigma$. Then a reduced word for the minimum length coset* *representative w_2 in $W_0 w$ is given by*

$$w_2 = \begin{cases} \prod_{k=i}^{1} s_0(s_2 s_3 \cdots s_{2k-1})(s_1 s_2 \cdots s_{2k-2}) & \text{for } r = 2i \text{ and } B_n^{(1)},\ D_n^{(1)},\ A_{2n-1}^{(2)} \\ \prod_{k=i}^{1} s_0(s_2 s_3 \cdots s_{2k})(s_1 s_2 \cdots s_{2k-1}) & \text{for } r = 2i+1 \text{ and } B_n^{(1)},\ D_n^{(1)},\ A_{2n-1}^{(2)} \\ \prod_{k=i}^{1} s_0(s_1 s_2 \cdots s_{k-1}) & \text{for } r = i \text{ and } C_n^{(1)},\ A_{2n}^{(2)},\ D_{n+1}^{(2)} \end{cases} \tag{4.1}$$

where the index k decreases as the product is formed from left to right.

Proof. All nodes for $A_n^{(1)}$ are special so we may assume $\mathfrak{g}$ is not of this type.

Applying the sequence of reflections in (4.1) to $\Lambda_{\tau(0)}$, we see that each reflection s_j changes the weight by a positive multiple of α_j, and the final weight is $\Lambda_0 + c_r\omega_r - i\delta$. It follows that (4.1) yields a reduced decomposition of some element $w_2 \in W$.

Using (2.3), in all cases we have

$$w\Lambda_{\tau(0)} = t_{-c_r\omega_r}\tau^{-1}\Lambda_{\tau(0)} = \Lambda_0 - c_r\omega_r - i\delta/a_0.$$

Since $r \notin I^0$ we have $w_0^{\omega_r}\omega_r = -\omega_r$ where $w_0^{\omega_r}$ is defined in (2.11). Moreover $w_0^{\omega_r}$ is also the shortest element of W_0 sending $\Lambda_0 + c_r\omega_r - i\delta/a_0$ to $\Lambda_0 - c_r\omega_r - i\delta/a_0$. It follows that $w = w_0^{\omega_r}w_2$ is a length-additive factorization and that w_2 is the minimum length coset representative in $W_0 w$. $\square$

Remark 3. Let $K = \{0, 1, \ldots, r-1\} \subset I$, $\mathfrak{g}_K \subset \mathfrak{g}$ the simple subalgebra with Dynkin nodes K, $\{\widetilde{\omega}_j \mid j \in K\}$ the fundamental weights for $\mathfrak{g}_K$, and $W_K = \langle s_j \mid j \in K\rangle \subset W$ the Weyl group of $\mathfrak{g}_K$. This given, we have $w_2 = w_0^{\widetilde{\omega}_{\tau(0)}}$ where $w_0^{\widetilde{\omega}_j} \in W_K$ is defined with respect to $\mathfrak{g}_K$.

Lemma 4.2. *All of the weights of $B^{r,c_r s}$ are in the convex hull of the W_0-orbit $W_0 \cdot c_r s\omega_r$. Moreover for every $\mu \in W_0 \cdot c_r s\omega_r$, there is a unique element $u_\mu \in B(c_r s\omega_r) \subset B^{r,c_r s}$ of the extremal weight μ.*

Proof. By [5, 22] the classical decomposition of $D(c_r\omega_r, s)$ agrees with that specified in [7]. In every case the above condition holds. $\square$

Lemma 4.3. *Let $\mathfrak{g}$ be of nonexceptional affine type, $r \in I \setminus I^0$, $s \in \mathbb{Z}_{>0}$, $k < r$ where $B(c_r s\omega_k)$ occurs in $B^{r,c_r s}$, and $b = u_{c_r s\omega_k} \in B(c_r s\omega_k) \subset B^{r,c_r s}$. Define*

$$y = \begin{cases} S_2 \cdots S_{k+1} S_1 \cdots S_k(b) & \text{for } B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, \\ S_1 \cdots S_k(b) & \text{for } C_n^{(1)}, D_{n+1}^{(2)}, A_{2n}^{(2)}. \end{cases}$$

Then

$$f_0^s(y) = \begin{cases} u_{c_r s\omega_{k+2}} & \text{for } B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, \\ u_{c_r s\omega_{k+1}} & \text{for } C_n^{(1)}, D_{n+1}^{(2)}, A_{2n}^{(2)}. \end{cases} \tag{4.2}$$

Proof. By definition the element y is an extremal weight vector within the classical crystal $B(c_r s\omega_k)$. By weight considerations one may check that

$$y = \begin{cases} f_2^s \cdots f_k^s f_{k+1}^s f_1^s f_2^s \cdots f_{k-1}^s f_k^s(b) & \text{for } B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, \\ f_1^{c_r s} f_2^{c_r s} \cdots f_k^{c_r s}(b) & \text{for } C_n^{(1)},\ D_{n+1}^{(2)},\ A_{2n}^{(2)}. \end{cases}$$

We claim that

$$\begin{aligned}
&\varepsilon(y) = s(\Lambda_0 + \Lambda_2) && \varphi(y) = s(\Lambda_0 + \Lambda_{k+2}) && \text{for } B_n^{(1)},\ D_n^{(1)}, A_{2n-1}^{(2)},\ k > 0\\
&\varepsilon(y) = s(\Lambda_0 + c_r\Lambda_1) && \varphi(y) = s(\Lambda_0 + c_r\Lambda_{k+1}) && \text{for } C_n^{(1)},\ A_{2n}^{(2)},\ D_{n+1}^{(2)},\ k > 0\\
&\varepsilon(y) = s\Lambda_0 && \varphi(y) = s\Lambda_0 && \text{for } k = 0.
\end{aligned}$$

By extremality and Lemma 4.2, y is in the indicated position within its i-strings for $i \in I\backslash\{0\}$. It remains to show that $\varepsilon_0(y) = \varphi_0(y) = s$ and (4.2) holds. In each case we shall use Assumption 1 (3) either for the existence of a crystal automorphism σ on $B^{r,c_r s}$ or, in type $A_{2n}^{(2)}$, for the virtual crystal construction of $B^{r,c_r s}$.

We begin with type $D_n^{(1)}$. We have $c_r = 1$ and $\mu := \mathrm{wt}(y) = (0^2, s^k, 0^{n-k-2})$. Here we realize $P_0 \subset ((1/2)\mathbb{Z})^n$ with i-th standard basis element ϵ_i, with $\omega_i = (1^i, 0^{n-i})$ for $1 \le i \le n-2$ (we do not need the spin weights) and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n-1$. Let $b' = u_{s\omega_{k+2}} \in B(s\omega_{k+2}) \subset B^{r,s}$. We have $\varphi_0(b') = 0$, for otherwise $f_0(b') \in B^{r,s}$ has weight contradicting Lemma 4.2. Since $\langle \alpha_0^\vee, \mathrm{wt}(b')\rangle = 2s$, we have $\varepsilon_0(b') = 2s$.

For type $D_n^{(1)}$, the automorphism σ of $B^{r,c_r s}$ satisfies $e_0 = \sigma \circ e_1 \circ \sigma$. Define $z = e_1^s(\sigma(b'))$. It suffices to show that

$$y = \sigma(z).$$

Let $K = \{2, 3, \dots, n\} \subset I$. The subalgebra of $\mathfrak{g}$ with simple roots α_i for $i \in K$, is of type D_{n-1}. For this reason we shall refer to D_{n-1}-components and D_{n-1}-highest weight vectors instead of K-components and K-highest weight vectors. Our proof rests on the following fact:

> $B^{r,s}$ contains a unique element of weight μ that satisfies $\varepsilon_1 = 0$ and whose associated D_{n-1}-highest weight vector has D_n-weight $\lambda := (0, s^k, 0^{n-k-1})$.

For the classical components of $B^{r,s}$ that contain D_{n-1}-components of weight λ, are precisely those of the form $B((s-t)\omega_k + t\omega_{k+2})$ for $0 \le t \le s$, and only for $t = 0$ does the classical component contain an element of weight μ with $\varepsilon_1 = 0$ (and by extremality $B(s\omega_k)$ contains a unique element of weight μ).

y clearly satisfies the above property. It suffices to show that $\sigma(z)$ does also.

$\sigma(b')$ is a D_{n-1}-highest weight vector with $\mathrm{wt}(\sigma(b')) = (-s, s^{k+1}, 0^{n-k-2})$. So $\mathrm{wt}(z) = \mu$. By weight considerations and Lemma 4.2, $z' = S_{k+1} \cdots S_2(z)$ is a D_{n-1}-highest weight vector of weight λ. Therefore $\sigma(z)$ has weight $\sigma(\mu) = \mu$ and has associated D_{n-1}-highest weight vector $\sigma(z')$, which has weight $\sigma(\lambda) = \lambda$. Since the Dynkin nodes 0 and 1 are nonadjacent we have $\varepsilon_1(\sigma(z)) = \varepsilon_1(e_0^s(b')) = \varepsilon_1(b') = 0$. Thus $\sigma(z)$ fulfills the above criteria and so must be equal to y.

The proof is analogous for types $B_n^{(1)}$ and $A_{2n-1}^{(2)}$ using the same set K, which defines subalgebras of types B_{n-1} and C_{n-1} respectively.

For type $C_n^{(1)}$ we have $c_r = 2$ for all $1 \le r \le n-1$. Let $K = \{1, 2, \dots, n-1\}$; the associated subalgebra is of type A_{n-1}. Here we realize $P_0 \cong \mathbb{Z}^n$ with $\omega_i = (1^i, 0^{n-i})$ for $1 \le i \le n$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = 2\epsilon_n$. Our argument uses the fact that

> $B^{r,2s}$ contains a unique element of weight $\mu := (0, (2s)^k, 0^{n-k-1})$ such that $\varepsilon_n = 0$ and whose associated A_{n-1}-highest weight vector has C_n-weight $2s\omega_k$.

For the classical components in $B^{r,2s}$ that contain such an A_{n-1}-component, are precisely those of the form $B(2(s-t)\omega_k + 2t\omega_{k+1})$ for $0 \le t \le s$, and among these, only for $t = 0$ does the classical component contain an element of weight μ for which $\varepsilon_n = 0$ (and by extremality $B(2s\omega_k)$ contains a unique element of weight μ).

By construction y satisfies this property. It suffices to show that $\sigma(z)$ does also, where $z = e_n^s \circ \sigma(b')$ and $b' = u_{2s\omega_{k+1}} \in B(2s\omega_{k+1}) \subset B^{r,s}$.

We have $\varphi_0(b') = 0$ for otherwise $f_0(b') \in B^{r,2s}$ would have weight contradicting Lemma 4.2. Since $\langle \alpha_0^\vee, \mathrm{wt}(b') \rangle = 2s$ we have $\varepsilon_0(b') = 2s$.

$\sigma(b')$ is an A_{n-1}-highest weight vector of weight $\sigma(2s\omega_{k+1}) = (0^{n-k-1}, (-2s)^{k+1})$. Therefore z has weight $(0^{n-k-1}, (-2s)^k, 0)$ and associated A_{n-1}-highest weight vector $z' = S_{n-k} \cdots S_{n-1}(z)$, which has weight $(0^{n-k}, (-2s)^k)$. It follows that $\sigma(z)$ has weight μ and its associated A_{n-1}-highest weight vector has weight $2s\omega_k$. Now $\varepsilon_n(\sigma(z)) = \varepsilon_n(e_0^s(b')) = \varepsilon_n(b') = 0$ since the Dynkin nodes 0 and n are nonadjacent. We have shown that $\sigma(z)$ satisfies the above criteria and so must be equal to y.

Type $D_{n+1}^{(2)}$ is similar to type $C_n^{(1)}$.

For type $A_{2n}^{(2)}$, the above kind of argument is not available since $A_{2n}^{(2)}$ admits no nontrivial Dynkin automorphism. Instead we apply virtual crystals. Under Assumption 1 (3), by [23] the crystal $B^{r,s}$ is realized as the subset of $V^{r,s} = B_A^{2n-r,s} \otimes B_A^{r,s}$ of type $A_{2n-1}^{(1)}$ generated from $u_{s\omega_{2n-r}} \otimes u_{s\omega_r}$ by the virtual crystal operators $\hat{f}_i = f_i f_{2n-i}$ for $1 \le i \le n$ and $\hat{f}_0 = f_0^2$ where f_i are the crystal operators of the $A_{2n-1}^{(1)}$-crystal $V^{r,s}$. Denote the virtualization by $v : B^{r,s} \hookrightarrow V^{r,s}$. We perform explicit computations using the tableau realization of $U_q(A_{2n-1})$-crystals in [19] and 0-arrows given by [24]. We have

$$\begin{aligned}
v(b) &= (2n-k)^s \cdots (r+2)^s (r+1)^s k^s \cdots 2^s 1^s \otimes r^s \cdots 2^s 1^s \\
v(y) &= (2n)^s (2n-k-1)^s \cdots (r+2)^s (r+1)^s (k+1)^s \cdots 3^s 2^s \otimes r^s \cdots 2^s 1^s \\
v(f_0^s y) &= (2n-k-1)^s \cdots (r+2)^s (r+1)^s (k+1)^s \cdots 2^s 1^s \otimes r^s \cdots 2^s 1^s \\
&= v(u_{s\omega_{k+1}}).
\end{aligned}$$

□

The next theorem is the main result of this paper. It shows that under the isomorphism between the Demazure and the KR crystals as classical crystals zero arrows map to zero arrows. In addition it yields the isomorphism (4.3) without the assumption that the KR crystal $B^{r,c_r s}$ is perfect.

Theorem 4.4. *Let $(r,s) \in I \backslash \{0\} \times \mathbb{Z}_{>0}$. Suppose that $r \in I^0$, or $c_r \omega_r = \theta$, or $\mathfrak{g}$ is of nonexceptional affine type. Write $t_{-c_r\omega_r^*} = w\tau$ with $w \in W$ and $\tau \in \Sigma$. Then there is an affine crystal isomorphism*

$$\begin{aligned}
B(s\Lambda_{\tau(0)}) &\cong B^{r,c_r s} \otimes B(s\Lambda_0) \\
u_{s\Lambda_{\tau(0)}} &\mapsto u' := u \otimes u_{s\Lambda_0}
\end{aligned} \tag{4.3}$$

where u is the element specified by Assumption 1 (2). It restricts to an isomorphism

$$D(c_r\omega_r, s) \cong B^{r,c_r s} \otimes u_{s\Lambda_0} \tag{4.4}$$

where both sides of (4.4) *are regarded as full subcrystals of their respective sides in* (4.3).

Proof. Let w_2 be the minimum length coset representative in $W_0 w$. Then $w = w_1 w_2$ is a length-additive factorization with $w_1 = w w_2^{-1} \in W_0$. We choose a reduced word of w by concatenating reduced words of w_1 and w_2. We claim that it suffices to establish the following assertions.

(A1) There is a bijection

$$\begin{aligned} B_{w_2}(s\Lambda_{\tau(0)}) &\to B' := f_{w_2}(u') \\ u_{s\Lambda_{\tau(0)}} &\mapsto u' \end{aligned} \tag{4.5}$$

that preserves all arrows in f_{w_2}.

(A2) $B' \subset B^{r,c_r s} \otimes u_{s\Lambda_0}$.

Suppose (A1) and (A2) hold. Since $w_1 \in W_0$, $B_{w_2}(s\Lambda_{\tau(0)})$ contains all the classical highest weight vectors of $D(c_r\omega_r, s)$. By (A1) these classical highest weight vectors correspond to the classical highest weight vectors in B'. Let $B'' \subset B^{r,c_r s} \otimes B(s\Lambda_0)$ be the classical subcrystal generated by B'; by (A2) $B'' \subset B^{r,c_r s} \otimes u_{s\Lambda_0}$. By Demazure theory for highest weight modules over simple Lie algebras, the bijection (4.5) extends uniquely to a classical crystal isomorphism $D(c_r\omega_r, s) \cong B''$. By Assumption 1 and Remark 2 we have $B'' = B^{r,c_r s} \otimes u_{s\Lambda_0}$. So we have a bijection

$$D(c_r\omega_r, s) \cong B^{r,c_r s} \otimes u_{s\Lambda_0} \tag{4.6}$$

which is an isomorphism of classical crystals that extends the bijection (4.5). It follows that $B^{r,c_r s} \otimes u_{s\Lambda_0}$ and therefore $B^{r,c_r s} \otimes B(s\Lambda_0)$, have a unique affine highest weight vector, namely, u'. By [17, Prop. 2.4.4] there is an affine crystal isomorphism (4.3). It must extend the bijection (4.6), and the Theorem follows.

We prove (A1) and (A2) by cases.

If $r \in I^0$ then by (2.10) w_2 is the identity, $B_{w_2}(s\Lambda_{\tau(0)}) = \{u_{s\Lambda_{\tau(0)}}\}$, $B' = \{u'\}$, $c_r = 1$, and $B^{r,s} \cong B(s\omega_r)$ as a classical crystal with classical highest weight vector u. In this case (A1) and (A2) are immediate. This is the only case where $\omega_r^* \neq \omega_r$.

If $c_r\omega_r = \theta$ then τ is the identity, $w_1 = s_\theta$ and $w_2 = s_0$. By Assumption 1 (2), $B_{w_2}(s\Lambda_0)$ and B' are the 0-strings of $u_{s\Lambda_0}$ and u' respectively. The elements are at the dominant ends of their respective 0-strings, which both have length s. This gives (A1). (A2) follows by the signature rule and Assumption 1 (2).

Otherwise we assume that $\mathfrak{g}$ is of nonexceptional affine type and $r \in I \setminus I^0$. Then w_2 is given in Proposition 4.1. We use the notation of Remark 3 throughout the rest of the proof. Since $K \subsetneq I$, $\mathfrak{g}_K$ is a simple Lie algebra and Assumption 1 (1) implies that $B^{r,c_r s}$ decomposes into a direct sum of K-components, each of which is isomorphic to the crystal graph of an irreducible highest weight module for $U_q(\mathfrak{g}_K)$. We have the K-crystal isomorphisms

$$B_{w_2}(s\Lambda_{\tau(0)}) \cong B_{w_2}(s\widetilde{\omega}_{\tau(0)}) = B(s\widetilde{\omega}_{\tau(0)}) \cong B'. \tag{4.7}$$

The first isomorphism holds by restriction from an I-crystal to a K-crystal. The equality holds by Remark 3 and Demazure theory for the simple Lie algebra $\mathfrak{g}_K$. We have $B_{w_2}(s\widetilde{\omega}_{\tau(0)}) \cong B'$, since both sides are generated by f_{w_2} (with $w_2 \in W_K$) applied to K-highest weight vectors of K-weight $s\widetilde{\omega}_{\tau(0)}$; see Assumption 1 (2). This establishes (A1).

For types $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$ we have $c_r = 1$ for all r and $\tau = \tau_0$ or $\tau = \tau_1$ (and $\tau(0) = 0$ or $\tau(0) = 1$) according as r is even or odd. Here $u = u_{s\omega_{\tau(0)}} \in B(s\omega_{\tau(0)}) \subset B^{r,c_r s}$, where $\omega_0 = 0$ by convention.

We consider the decomposition of $B^{r,c_r s}$ into K-components, which we call D_r-components. Note that 0 and 1 are the spinor nodes in D_r. Now $u_{c_r s\omega_r} \in B^{r,c_r s}$

is a D_r-lowest weight vector of D_r-weight $-2s\widetilde{\omega}_0$. Therefore there is a D_r-crystal embedding

$$\begin{aligned} B(2s\widetilde{\omega}_0)\otimes u_{s\Lambda_0} &\to B(s\widetilde{\omega}_{\tau(0)})^{\otimes 2}\otimes B(s\Lambda_0) \\ u_{c_r s\omega_r}\otimes u_{s\Lambda_0} &\mapsto u_{-s\widetilde{\omega}_0}^{\otimes 2}\otimes u_{s\Lambda_0}. \end{aligned}$$

But by Lemma 4.3 there is a D_r-path from u' to $u_{c_r s\omega_r}\otimes u_{s\Lambda_0}$ that never changes the right hand tensor factor. Therefore there is a D_r-embedding

$$\begin{aligned} B' &\to B(s\widetilde{\omega}_{\tau(0)})^{\otimes 2}\otimes B(s\Lambda_0) \\ u' = u\otimes u_{s\Lambda_0} &\mapsto u_{s\widetilde{\omega}_{\tau(0)}}\otimes u_{-s\widetilde{\omega}_0}\otimes u_{s\Lambda_0}. \end{aligned}$$

The image of u' is uniquely determined by Assumption 1 (2) since $u_{s\widetilde{\omega}_{\tau(0)}}\otimes u_{-s\widetilde{\omega}_0}$ is the unique element of $B(s\widetilde{\omega}_{\tau(0)})^{\otimes 2}$ with $\varepsilon = s\Lambda_0$ and $\varphi = s\Lambda_{\tau(0)}$.

The form of the image of u' now clearly shows that when f_{w_2} is applied to u' it only acts on the left hand tensor factor. This implies (A2).

Next let us consider type $C_n^{(1)}$ for $r\notin I^0$; for such r, $c_r = 2$ and τ is the identity. Here u is the unique element in the one-dimensional C_n-crystal in $B^{r,2s}$. We decompose $B^{r,2s}$ as a K-crystal, which is a C_r-crystal in this case. All other arguments go through as for type $D_n^{(1)}$.

Types $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$ follow in the same fashion. In this case the decomposition of $B^{r,c_r s}$ as a K-crystal is a B_r crystal. □

Remark 4. We expect Theorem 4.4 to hold for any affine algebra $\mathfrak{g}$ and any Dynkin node $r\in I\backslash\{0\}$. Our proof requires a special property, that the minimum length coset representative w_2 of Proposition 4.1 has a certain form, namely, in the notation of (2.11), $w_2 = w_0^{\lambda}$ where λ is a fundamental weight for some subalgebra $\mathfrak{g}_K$ where $K\subsetneq I$. This property of w_2 does not hold for the trivalent node in type $E_6^{(1)}$. For such nodes a different strategy is required.

Remark 5. In the notation of Lemma 2.3 we expect that for any affine algebra $\mathfrak{g}$ with affine Dynkin diagram X and any $\sigma\in\mathrm{Aut}(X)$, there is a bijection $\sigma : B^{r,c_r s}\to B^{\sigma'(r),c_r s}$ such that (3.3) holds. In particular, for any $\sigma\in\mathrm{Aut}(X)$, we expect that there is an automorphism σ on $B^{r,c_r s}$ satisfying (3.3) if and only if $\sigma'(r) = r$. By Remark 1 this means that every special Dynkin automorphism $\sigma\in\Sigma$ should induce an automorphism of each $B^{r,c_r s}$. In contrast, for the nonspecial automorphism $\sigma = (0,1)$ of $D_n^{(1)}$, $\sigma' = (n-1,n)$ is not the identity and σ induces a bijection $B^{n-1,s}\to B^{n,s}$ satisfying (3.3).

Remark 5 comes into play in Section 6 and the following Theorem.

Theorem 4.5. *For the cases in Assumption 1 (3) where σ is defined, there exist unique maps*

$$\Psi : D(\omega_r, s)\hookrightarrow B^{r,c_r s} \text{ and } \Psi^{\sigma} : D^{\sigma}(\omega_r, s)\hookrightarrow B^{r,c_r s}.$$

The maps are induced by $\Psi(u_{s\Lambda_0}) = u$ and $\Psi^{\sigma}(u_{s\Lambda_{\sigma(0)}}) = \sigma(u)$.

Proof. The map Ψ^{σ} is obtained by applying σ to everything in sight. □

Corollary 4.6. *The affine structure of $B^{r,c_r s}$ is uniquely determined.*

Theorem 4.7. *Suppose that* $\lambda = \sum_{r \in I \setminus \{0\}} m_r c_r \omega_r$ *with* $m_r \in \mathbb{Z}_{\geq 0}$ *and* $m_r > 0$ *only when* r *is as in Theorem 4.4. Write* $t_{-\lambda^*} = w\tau$ *for* $w \in W$ *and* $\tau \in \Sigma$. *Assume that for each* $k \in I^0$ *and every* $r \in I \setminus \{0\}$ *with* $m_r > 0$, *the special Dynkin automorphism* $\tau_k \in \Sigma$ *induces an automorphism of* $B^{r,c_r s}$ *that sends* i*-arrows to* $\tau_k(i)$*-arrows. Then for every* $r' \in I^0$ *there is an isomorphism*

$$B(s\Lambda_{\tau(r')}) \cong \Big(\bigotimes_{r \in I \setminus \{0\}} (B^{r,c_r s})^{\otimes m_r} \Big) \otimes B(s\Lambda_{r'})$$

which restricts to an isomorphism of full subcrystals

$$B_{\tau_{r'}^{-1} w \tau_{r'}}(s\Lambda_{\tau(r')}) \cong \Big(\bigotimes_{r \in I \setminus \{0\}} (B^{r,c_r s})^{\otimes m_r} \Big) \otimes u_{s\Lambda_{r'}}.$$

Proof. Induction allows a straightforward reduction to the case of one KR tensor factor. Applying a special Dynkin automorphism allows the reduction to the case $r' = 0$, which is Theorem 4.4. $\square$

Corollary 4.8. *Let* λ *be as in Theorem 4.7. Then the Demazure crystal* $D(\lambda, s)$ *can be extended to a full affine crystal by adding 0-arrows.*

Remark 6. This proves Conjecture 1 in [5] on the level of crystals. However it is not yet clear whether there exists a global basis of the Demazure module, whose corresponding crystal basis is the one given in Theorem 4.7. For level $s = 1$, Theorem 4.7 was proved using the Littelmann path model in [6, Proposition 3].

5. Reaching the classical highest weight vectors of a KR crystal

In the proof of Lemma 4.3, explicit paths in the KR crystal were given, from the element u to certain classical highest weight vectors in the KR crystal. For $\mathfrak{g}$ of nonexceptional affine type and for each KR crystal $B^{r,c_r s}$, we shall give (without proof) an explicit way to reach each classical highest weight vector in $B^{r,c_r s}$ from the element u of Assumption 1.

If $r \in I^0$ then the KR crystal $B^{r,c_r s}$ is connected as a classical crystal and the problem is trivial. This includes all $r \in I \setminus \{0\}$ for $A_n^{(1)}$.

So we now assume $r \notin I^0$.

We shall use the standard realizations of the weight lattices of B_n, C_n, D_n by sublattices of $((1/2)\mathbb{Z})^n$. We let $\omega_i = (1^i, 0^{n-i})$ for $i \in I \setminus \{0\}$ nonspin. Since $r \notin I^0$ the only spin weight we need is $\omega_n = (1/2)(1^n)$ in type B_n, and in that case $c_n = 2$. Thus all the weights we must consider, correspond to partitions, elements in $\mathbb{Z}_{\geq 0}^n$ consisting of weakly decreasing sequences. Moreover, for the nonexceptional affine algebras the KR crystals are multiplicity-free as classical crystals.

For $\mathfrak{g}$ of type $B_n^{(1)}$, $D_n^{(1)}$, or $A_{2n-1}^{(2)}$, $B(\lambda)$ occurs in $B^{r,c_r s}$ if and only if the diagram of the partition corresponding to λ, is obtained from the $r \times s$ rectangular partition by removing vertical dominoes. Let $t = 0$ or $t = 1$ according as r is even or odd. We have

$$u_\lambda = \left(\prod_{i=(r-t)/2}^{1} f_0^{\lambda_{2i}} (f_2^{\lambda_{2i}} f_3^{\lambda_{2i}} \cdots f_{2i-1+t}^{\lambda_{2i}})(f_1^{\lambda_{2i}} f_2^{\lambda_{2i}} \cdots f_{2i-2+t}^{\lambda_{2i}}) \right) u$$

where the product is formed from left to right using decreasing indices i.

Example 1. Let $\mathfrak{g}$ be of type $D_7^{(1)}$, $(r,s)=(5,4)$ and λ be the weight $\omega_5+\omega_3+2\omega_1$. Then $t=1$, λ is the partition $(4,2,2,1,1)$, and the sequence of lowering operators is $(f_0f_2f_3f_4f_1f_2f_3)(f_0^2f_2^2f_1^2)$. This is applied to the classical highest weight vector of weight given by the partition (4), and the parenthesized subexpressions successively yield classical highest weight vectors corresponding to the partitions $(4,2,2)$, and $(4,2,2,1,1)$ respectively.

For $\mathfrak{g}$ of type $C_n^{(1)}$, $A_{2n}^{(2)}$ or $D_{n+1}^{(2)}$, the partitions corresponding to classical highest weights in B^{r,c_rs} are precisely those of the form $c_r\lambda=(c_r\lambda_1,c_r\lambda_2,\dots)$ where λ runs over the partitions contained in the $r\times s$ rectangle. We have

$$u_{c_r\lambda}=\left(\prod_{i=r}^{1} f_0^{c_r\lambda_i}f_1^{c_r\lambda_i}\cdots f_{i-1}^{c_r\lambda_i}\right)u$$

where the product of operators is formed from left to right as i decreases.

Example 2. Let $\mathfrak{g}$ be of type $C_3^{(1)}$, $(r,s)=(2,3)$, and $\lambda=\omega_2+2\omega_1$. Then we have $c_r=2$, the partition $\lambda=(3,1)$, and the sequence of lowering operators $(f_0^2f_1^2)(f_0^6)$. This is applied to the classical highest weight vector of weight 0 (corresponding to the empty partition). After f_0^6 the classical weight is given by the partition (6) and after $f_0^2f_1^2$ one has the partition $(6,2)=2\lambda$.

6. Connectedness

Theorem 4.4 shows that the KR crystals B^{r,c_rs} are connected. In this section we show that the tensor product of two KR crystals is also connected by providing an algorithm which for any given element in the crystal yields a string of operators e_i (or f_i) to reach a given special element. This algorithm is also useful in defining crystal morphisms such as the combinatorial R-matrix. Since KR crystals and their tensor products are not highest weight crystals, it is not completely obvious which sequence of raising operators e_i will yield a given special element.

Here we give a construction on how to reach $u_1\otimes u_2\in B^{r_1,c_{r_1}s_1}\otimes B^{r_2,c_{r_2}s_2}$ where u_1 is the unique elements of $B^{r_1,c_{r_1}s_1}$ with $\varepsilon(u_1)=s_1\Lambda_0$ and $\varphi(u_1)=s_1\Lambda_{\tau_1(0)}$ as required in Assumption 1 (2), and u_2 is the unique element in $B^{r_2,c_{r_2}s_2}$ with $\varepsilon(u_2)=s_2\Lambda_{\tau_2^{-1}(0)}$ and $\varphi(u_2)=s_2\Lambda_0$ as required in Assumption 1 (2) and Remark 5.

By Theorems 4.4 and 4.5 we have the following isomorphism of affine crystals

$$\begin{aligned} B^{r_1,c_{r_1}s_1}\otimes B^{r_2,c_{r_2}s_2}\otimes B(s_2\Lambda_{\tau_2^{-1}(0)}) &\cong B^{r_1,c_{r_1}s_1}\otimes B(s_2\Lambda_0)\\ u_1\otimes u_2\otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}} &\mapsto u_1\otimes u_{s_2\Lambda_0}. \end{aligned}$$

Assume that $s_1\geq s_2$. Acting with raising operators e_i with $i\in I$ one can bring any element $b_1\otimes b_2\otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}}$ into the form $c_1\otimes u_2\otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}}$ since by the tensor product rule the e_i will eventually act on the right tensor factors and by Theorem 4.4 $b_2\otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}}$ is connected to $u_2\otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}}$. Once such an element is reached, tensor from the right by $u_{(s_1-s_2)\Lambda_0}\in B((s_1-s_2)\Lambda_0)$ to obtain

$$\begin{aligned} B^{r_1,c_{r_1}s_1}\otimes B^{r_2,c_{r_2}s_2}\otimes B(s_2\Lambda_{\tau_2^{-1}(0)})\otimes B((s_1-s_2)\Lambda_0)&\\ \cong B^{r_1,c_{r_1}s_1}\otimes B(s_2\Lambda_0)\otimes B((s_1-s_2)\Lambda_0)& \end{aligned}$$

under which $c_1 \otimes u_2 \otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}} \otimes u_{(s_1-s_2)\Lambda_0}$ maps to $c_1 \otimes u_{s_2\Lambda_0} \otimes u_{(s_1-s_2)\Lambda_0}$. The latter element is the image of the vector $c_1 \otimes u_{s_1\Lambda_0}$ under the embedding of affine crystals $B^{r_1,c_{r_1}s_1} \otimes B(s_1\Lambda_0) \to B^{r_1,c_{r_1}s_1} \otimes B((s_1-s_2)\Lambda_0) \otimes B(s_2\Lambda_0)$.

Now from $c_1 \otimes u_{s_1\Lambda_0} \in B^{r_1,c_{r_1}s_1} \otimes B(s_1\Lambda_0)$ one can reach $u_1 \otimes u_{s_1\Lambda_0}$ using e_i with $i \in I$.

If $s_1 < s_2$ we tensor from the left with the dual crystals. Explicitly,

$$B^\vee(s_1\Lambda_{\tau_1(0)}) \otimes B^{r_1,c_{r_1}s_1} \otimes B^{r_2,c_{r_2}s_2} \cong B^\vee(s_1\Lambda_0) \otimes B^{r_2,c_{r_2}s_2}.$$

The lowest weight element $u^\vee_{s_1\Lambda_0} \in B^\vee(s_1\Lambda_0)$ corresponds to $u^\vee_{s_1\Lambda_{\tau_1(0)}} \otimes u_1 \in B^\vee(s_1\Lambda_{\tau_1(0)}) \otimes B^{r_1,c_{r_1}s_1}$. Acting with lowering operators f_i with $i \in I$ one can bring any element $u^\vee_{s_1\Lambda_{\tau_1(0)}} \otimes b_1 \otimes b_2$ into the form $u^\vee_{s_1\Lambda_{\tau_1(0)}} \otimes u_1 \otimes c_2$. Once this element is reached, tensor on the left by $u^\vee_{(s_2-s_1)\Lambda_0} \in B^\vee((s_2-s_1)\Lambda_0)$, obtaining the element $u^\vee_{(s_2-s_1)\Lambda_0} \otimes u^\vee_{s_1\Lambda_{\tau_1(0)}} \otimes u_1 \otimes c_2$, which can be identified with $u^\vee_{s_2\Lambda_0} \otimes c_2 \in B^\vee(s_2\Lambda_0) \otimes B^{r_2,c_{r_2}s_2}$. Now move down to the lowest weight vector $u^\vee_{s_2\Lambda_0} \otimes u_2$ using f_i with $i \in I$.

As a result of the above construction we obtain the following corollary:

Corollary 6.1. *The tensor product $B^{r_1,c_{r_1}s_1} \otimes B^{r_2,c_{r_2}s_2}$ of KR crystals is connected.*

The combinatorial R-matrix is a crystal morphism. More precisely

$$R : B^{r_1,c_{r_1}s_1} \otimes B^{r_2,c_{r_2}s_2} \to B^{r_2,c_{r_2}s_2} \otimes B^{r_1,c_{r_1}s_1}$$

satisfies $R \circ e_i = e_i \circ R$ and $R \circ f_i = f_i \circ R$ for all $i \in I$. There exists a unique element $u_{c_{r_k}s_k\omega_{r_k}} \in B^{r_k,c_{r_k}s_k}$ and by weight considerations R must map $R(u_{c_{r_1}s_1\omega_{r_1}} \otimes u_{c_{r_2}s_2\omega_{r_2}}) = u_{c_{r_2}s_2\omega_{r_2}} \otimes u_{c_{r_1}s_1\omega_{r_1}}$. Assume that $s_1 \geq s_2$. Then for any element $b_1 \otimes b_2 \in B^{r_1,c_{r_1}s_1} \otimes B^{r_2,c_{r_2}r_2}$ the above algorithm provides a sequence $e_{\{i\}} := e_{i_1}e_{i_2}\cdots e_{i_\ell}$ such that $e_{\{i\}}(b_1 \otimes b_2) = u_1 \otimes u_2$. In particular, $e_{\{j\}}(u_{c_{r_1}s_1\omega_{r_1}} \otimes u_{c_{r_2}s_2\omega_{r_2}}) = u_1 \otimes u_2$. Set $f_{\{\leftarrow i\}} := f_{i_\ell}\cdots f_{i_1}$. Then

$$R(b_1 \otimes b_2) = f_{\{\leftarrow i\}} e_{\{j\}}(u_{c_{r_2}s_2\omega_{r_2}} \otimes u_{c_{r_1}s_1\omega_{r_1}}).$$

For the case $s_1 < s_2$ a similar construction works where f_i and e_i are interchanged.

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WEYL MODULES FOR THE TWISTED LOOP ALGEBRAS

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ABSTRACT. The notion of aWeyl module, previously defined for the untwisted affine algebras, is extended here to the twisted affine algebras. We describe an identification of the Weyl modules for the twisted affine algebras with suitably chosen Weyl modules for the untwisted affine algebras. This identification allows us to use known results in the untwisted case to compute the dimensions and characters of the Weyl modules for the twisted algebras.

1. INTRODUCTION

The notion of Weyl modules for the untwisted affine Lie algebras was introduced in [6] and was motivated by an attempt to understand the category of finite dimensional representations of the untwisted quantum affine algebra. Specifically, the Weyl modules were conjectured to be the $q = 1$ limit of certain irreducible representations of the quantum affine algebras. It was proved that the conjecture was true for $\mathfrak{sl}_2$ and that this conjecture would follow if the dimensions of the Weyl modules were known. H. Nakajima has pointed out recently that the dimension formula follows by using results of [2] and [12].

Another approach to proving the dimension formula for the Weyl modules can be found in [4] for $\mathfrak{sl}_n$ and in [10] for the general simply laced case. These papers also make the connection between Weyl modules and the Demazure modules for affine Lie algebras and also with the fusion product defined by [7]. The approach in these papers is rather simple and show that one can study the Weyl modules from a purely classical viewpoint. Other points of interest and generalizations of these can be found in [8].

We now turn our attention to the case of the twisted affine algebras. None of the quantum machinery is available and in fact there are rather few results on the category of finite dimensional representations of the twisted quantum affine algebras [1], [5]. These results do show however that one can make a similar conjecture; i.e that one can define a notion of the Weyl module for the twisted affine Lie algebras such that they are the specializations of irreducible modules in the quantum case. To do this, one requires the Weyl modules to be universal in a suitable sense. One of the difficulties is in the case of the algebras of type $A_{2n}^{(2)}$, which are not built up entirely of algebras isomorphic to $A_1^{(1)}$; and indeed one needs to understand $A_2^{(2)}$ on its own. Thus, we use results of [9], [13] to arrive at the correct definition of the Weyl modules.

The next question clearly is to determine the dimensions of the Weyl modules and also their decomposition as modules for the underlying finite–dimensional simple Lie algebra. In the untwisted case these questions can be answered either by using the fusion product of [7] or

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the fact that the modules are specializations of modules for the quantum affine algebra. Both these techniques are unavailable to us in the twisted case, as far as we know the notion of fusion product does not admit a generalization to the twisted algebras. We get around these difficulties by identifying the Weyl modules for the twisted algebras $X_n^{(m)}$, $m > 1$ with suitably chosen Weyl modules for the untwisted algebra $X_n^{(1)}$. We then use all the known results in the untwisted case to complete our analysis of the twisted algebras. In conclusion, we note that some of the methods we use in this paper give simpler proofs of some of the results in [6].

2. The untwisted loop algebras and the modules $W(\boldsymbol{\pi})$.

2.1. Throughout the paper $\mathbf{C}$ (resp. $\mathbf{C}^\times$) denotes the set of complex (resp. non–zero complex) numbers, and $\mathbf{Z}$ (resp. $\mathbf{Z}_+$) the set of integers (resp. non–negative) integers. Given a Lie algebra $\mathfrak{a}$ we denote by $\mathbf{U}(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$ and by $L(\mathfrak{a})$ denotes the loop algebra of $\mathfrak{a}$. Specifically, we have

$$L(\mathfrak{a}) = \mathfrak{a} \otimes \mathbf{C}[t, t^{-1}],$$

with commutator given by

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}$$

for $x, y \in \mathfrak{a}$, $r, s \in \mathbf{Z}$. We identify $\mathfrak{a}$ with the subalgebra $\mathfrak{a} \otimes 1$ of $L(\mathfrak{a})$. Given $a \in \mathbf{C}^\times$, we let $\tau_a : L(\mathfrak{a}) \to L(\mathfrak{a})$ be the automorphism defined by extending $\tau_a(x \otimes t^k) = a^k(x \otimes t^k)$ for all $x \in \mathfrak{g}$, $k \in \mathbf{Z}$.

Given $\ell, N \in \mathbf{Z}_+$ and $\mathbf{a} = (a_1, \cdots, a_\ell) \in (\mathbf{C}^\times)^\ell$ let $\mathfrak{a}_{\mathbf{a},N}$ be the quotient of $L(\mathfrak{a})$ by the ideal $\mathfrak{a} \otimes \prod_{k=1}^{\ell}(t - a_k)^N \mathbf{C}[t, t^{-1}]$.

Lemma. *Let $\mathbf{a} = (a_1, \cdots, a_\ell) \in (\mathbf{C}^\times)^\ell$ be such that $\mathbf{a}$ has distinct coordinates. For all $N \in \mathbf{Z}_+$, we have*

$$\mathfrak{a}_{\mathbf{a},N} \cong \oplus_{r=1}^{N} \mathfrak{a}_{a_r,N}.$$

Proof. Since $a_r \neq a_s$ if $1 \leq r \neq s \leq \ell$, it is standard that

$$\mathbf{C}[t, t^{-1}]/\prod_{r=1}^{\ell}(t - a_r)^N \mathbf{C}[t, t^{-1}] \cong \oplus_{r=1}^{\ell} \mathbf{C}[t, t^{-1}]/(t - a_r)^N \mathbf{C}[t, t^{-1}]$$

and the lemma now follows trivially. □

2.2. The simple Lie algebras and their representations. Let $\mathfrak{g}$ be any finite-dimensional complex simple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $W_\mathfrak{g}$ the corresponding Weyl group. Let $R_\mathfrak{g}$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, $I_\mathfrak{g}$ an index set for a set of simple roots (and hence also for the fundamental weights), $R_\mathfrak{g}^+$ the set of positive roots, $Q_\mathfrak{g}^+$ (resp. $P_\mathfrak{g}^+$) the $\mathbf{Z}_+$ span of the simple roots (resp. fundamental weights) and $\theta_\mathfrak{g}$ be the highest root in $R_\mathfrak{g}^+$. Given $\alpha \in R_\mathfrak{g}$ let $\mathfrak{g}_\alpha$ be the corresponding root space, we have

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}.$$

Fix a Chevalley basis $x_\alpha^\pm, h_\alpha$, $\alpha \in R^+$ for $\mathfrak{g}$ and set

$$x_{\alpha_i}^\pm = x_i^\pm, \quad h_{\alpha_i} = h_i, \quad i \in I.$$

In particular for $i \in I$,

$$[x_i^+, x_i^-] = h_i, \quad [h_i, x_i^\pm] = \pm 2x_i^\pm.$$

Given a finite–dimensional representation of $\mathfrak{g}$ on a complex vector space V, we can write

$$V = \oplus_{\mu \in \mathfrak{h}^*} V_\mu, \quad V_\mu = \{v \in V : hv = \mu(h)v \ \ \forall\ h \in \mathfrak{h}\}.$$

Set $\mathrm{wt}(V) = \{\mu \in \mathfrak{h}^* : V_\mu \neq 0\}$. It is well–known that

$$V_\mu \neq 0 \quad \Longrightarrow \mu \in P \ \text{ and } \ w\mu \in \mathrm{wt}(V) \ \ \forall \ \ w \in W,$$

and that V is isomorphic to a direct sum of irreducible representations. The set of isomorphism classes of irreducible finite–dimensional $\mathfrak{g}$–modules is in bijective correspondence with P^+ and for any $\lambda \in P^+$ let $V_{\mathfrak{g}}(\lambda)$ be an element of the corresponding isomorphism class. Then $V_{\mathfrak{g}}(\lambda)$ is generated by an element v_λ satisfying the relations:

$$\mathfrak{n}^+.v_\lambda = 0, \quad hv_\lambda = \lambda(h)v_\lambda, \quad (x_i^-)^{\lambda(h_i)+1}v_\lambda = 0. \tag{2.1}$$

2.3. Identities in $\mathbf{U}(L(\mathfrak{g}))$. For $i \in I$ it is easy to see that the elements $\{x_i^\pm \otimes t^k, h_i \otimes t^k : k \in \mathbf{Z}_+\}$ span a subalgebra of $L(\mathfrak{g})$ which is isomorphic to $L(\mathfrak{sl}_2)$. We shall need the following formal power series in u with coefficients in $\mathbf{U}(L(\mathfrak{g}))$. For $i \in I$, set

$$\mathbf{p}_i^\pm(u) = \exp\left(-\sum_{k=1}^{\infty} \frac{h_i \otimes t^k}{k} u^k\right),$$

$$\mathbf{x}_i^\pm(u) = \sum_{k=0}^{\infty}(x_i^\pm \otimes t^k)u^{k+1}, \quad \widetilde{\mathbf{x}}_i^\pm(u) = \sum_{k=-\infty}^{\infty}(x_i^\pm \otimes t^k)u^{k+1}$$

Given a power series $\mathbf{f}$ in u with coefficients in an algebra A, let $(\mathbf{f})_m$ be the coefficient of u^m ($m \in \mathbf{Z}$). The following result was proved in [11, Lemma 7.5], (see [6, Lemma 1.3] for the formulation in this notation).

Lemma. *Let $r \in \mathbf{Z}_+$.*

$$(x_i^+ \otimes t)^{(r)}(x_i^- \otimes 1)^{(r+1)} = (-1)^r \left(\mathbf{x}_i^-(u)\mathbf{p}_i^+(u)\right)_{r+1} \quad \bmod \mathbf{U}(L(\mathfrak{g}))\widetilde{\mathbf{x}}_i^+(u).$$

□

2.4. The monoid $\mathcal{P}^+$. Let $\mathcal{P}^+$ be the monoid of I–tuples of polynomials $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_n)$ in an indeterminate u with constant term one, with multiplication being defined component wise. For $i \in I$ and $a \in \mathbf{C}^\times$, set

$$\boldsymbol{\pi}_{i,a} = ((1-au)^{\delta_{ij}} : j \in I) \in \mathcal{P}^+, \tag{2.2}$$

and for $\lambda \in P^+$, set

$$\boldsymbol{\pi}_{\lambda,a} = \prod_{i \in I}(\boldsymbol{\pi}_{i,a})^{\lambda(h_i)}, \quad \lambda \neq 0.$$

Clearly any $\boldsymbol{\pi}^+ \in \mathcal{P}^+$ can be written uniquely as a product

$$\boldsymbol{\pi}^+ = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_i,a_i},$$

for some $\lambda_1, \cdots, \lambda_\ell \in P^+$ and distinct elements $a_1, \cdots, a_\ell \in \mathbf{C}^\times$ and in this case we set $\boldsymbol{\pi}^- = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_i,\, a_i^{-1}}$. Define a map $\mathcal{P}^+ \to P^+$ by $\boldsymbol{\pi} \to \lambda_{\boldsymbol{\pi}} = \sum_{i\in I} \deg(\pi_i)\omega_i$.

2.5. The modules $W(\boldsymbol{\pi})$, $V(\boldsymbol{\pi})$. Given $\boldsymbol{\pi} = (\pi_i)_{i\in I} \in \mathcal{P}^+$, let $W(\boldsymbol{\pi})$ be the $L(\mathfrak{g})$–module generated by an element $w_{\boldsymbol{\pi}}$ with relations:

$$L(\mathfrak{n}^+)w_{\boldsymbol{\pi}} = 0, \quad hw_{\boldsymbol{\pi}} = \lambda_{\boldsymbol{\pi}}(h)w_{\boldsymbol{\pi}}, \quad (x_i^-)^{\lambda_{\boldsymbol{\pi}}(h_i)+1}w_{\boldsymbol{\pi}} = 0,$$
$$\left(\mathbf{p}_i^{\pm}(u) - \pi_i^{\pm}(u)\right) w_{\boldsymbol{\pi}} = 0,$$

where $\lambda_{\boldsymbol{\pi}} = \sum_{i\in I}(\deg \pi_i)\omega_i$, $\boldsymbol{\pi}^+ = \boldsymbol{\pi}$, $i \in I$ and $h \in \mathfrak{h}$. It is not hard to see that if we write $\boldsymbol{\pi} = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_\ell, a_\ell}$ where $a_1, \cdots, a_\ell$ are all distinct, then for $i \in I$

$$\left(\mathbf{p}_i^{\pm}(u) - \pi_i^{\pm}(u)\right) w_{\boldsymbol{\pi}} = 0 \iff (h_i \otimes t^r)w_{\boldsymbol{\pi}} = \left(\sum_{j=1}^{\ell} \lambda_j(h_i)a_j^r\right) w_{\boldsymbol{\pi}}.$$

Let $b \in \mathbf{C}^\times$ and let $\tau_b W(\boldsymbol{\pi})$ be the $L(\mathfrak{g})$–module obtained by pulling back $W(\boldsymbol{\pi})$ through the automorphism τ_b of $L(\mathfrak{g})$. The next result is standard.

Lemma. (i) *Let $\boldsymbol{\pi} \in \mathcal{P}^+$. Then $W(\boldsymbol{\pi}) = \mathbf{U}(L(\mathfrak{n}^-))w_{\boldsymbol{\pi}}$, and hence we have,*

$$\mathrm{wt}(W(\boldsymbol{\pi}) \subset \lambda_{\boldsymbol{\pi}} - Q^+, \quad \dim W(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}} = 1.$$

In particular, the module $W(\boldsymbol{\pi})$ has a unique irreducible quotient $V(\boldsymbol{\pi})$.

(ii) *For $b \in \mathbf{C}^\times$, we have $\tau_b W(\boldsymbol{\pi}) \cong W(\boldsymbol{\pi}_b)$, where $\boldsymbol{\pi} = (\pi_i(u))_{i\in I}$ and $\boldsymbol{\pi}_b = (\pi_i(b^{-1}u))_{i\in I}$. In particular we have*

$$W(\boldsymbol{\pi}_{\lambda,a}) \cong_{\mathfrak{g}} W(\boldsymbol{\pi}_{\lambda,ab}).$$

2.6. The modules $W(\boldsymbol{\pi})$ were initially defined and studied in [6] and a formula was conjectured for their dimension. Parts (i) and (ii) of the next theorem were proved in [6]. Part (iii) was proved in [6] in the case of $\mathfrak{sl}_2$, for $\mathfrak{sl}_{\mathfrak{n}}$ it was proved in [4] and for the general simply laced case in [10]. Part (iii) can be deduced for the general case by using results of [2],[12],[14] for quantum affine algebras.

Theorem 1. (i) Given $\boldsymbol{\pi} = (\pi_i)_{i\in I}$ with unique decomposition $\boldsymbol{\pi} = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_\ell, a_\ell}$, we have an isomorphism of $L(\mathfrak{g})$–modules

$$W(\boldsymbol{\pi}) \cong \otimes_{k=1}^{\ell} W(\boldsymbol{\pi}_{\lambda_k, a_k}).$$

(ii) Let V be any finite–dimensional $L(\mathfrak{g})$–module generated by an element $v \in V$ such that

$$L(\mathfrak{n}^+)v = 0, \quad L(\mathfrak{h})v = \mathbf{C}v.$$

Then there exists $\boldsymbol{\pi} \in \mathcal{P}^+$ such that the assignment $w_{\boldsymbol{\pi}} \to v$ extends to a surjective homomorphism $W(\boldsymbol{\pi}) \to V$ of $L(\mathfrak{g})$–modules.

(iii) Let $\lambda \in P^+$ and $a \in \mathbf{C}^\times$. Suppose that $\lambda = \sum_{i\in I} m_i\omega_i$. Then

$$W(\boldsymbol{\pi}_{\lambda,a}) \cong_{\mathfrak{g}} \bigotimes_{i\in I} W(\boldsymbol{\pi}_{\omega_i,1})^{\otimes m_i}.$$

$\square$

2.7. Annihilating ideals for $W(\boldsymbol{\pi})$. The next proposition is implicit in [6] but since it plays a big role in this paper we make it explicit and give a proof.

Proposition. *Let $\boldsymbol{\pi} = \prod_{r=1}^{\ell} \boldsymbol{\pi}_{\lambda_r, a_r} \in \mathcal{P}^+$. There exists an integer $N = N(\boldsymbol{\pi})$ such that*

$$\left(\mathfrak{g} \otimes \prod_{r=1}^{\ell} (t - a_r)^N \mathbf{C}[t, t^{-1}] \right) W(\boldsymbol{\pi}) = 0.$$

Proof. We begin by proving that for all $i \in I$

$$x_i^- \otimes \prod_{r=1}^{\ell} (t - a_r)^{\lambda_r(h_i)} w_{\boldsymbol{\pi}} = 0. \tag{2.3}$$

Set $N_i = \lambda_{\boldsymbol{\pi}}(h_i)$. Using the defining relations of $W(\boldsymbol{\pi})$ and Lemma 2.3,

$$0 = (x_i^+ \otimes t)^{N_i} (x_i^- \otimes 1)^{N_i + 1} w_{\boldsymbol{\pi}} = (-1)^{N_i} \left(\mathbf{x}_i^-(u) \mathbf{p}_i^+(u) \right)_{N_i} w_{\boldsymbol{\pi}}.$$

We also have

$$\mathbf{p}_i(u).w_{\boldsymbol{\pi}} = \prod_{r=1}^{\ell} (1 - a_r u)^{\lambda_r(h_i)}.w_{\boldsymbol{\pi}} \equiv \left(\sum_{j=0}^{N_i} p_{i,j} u^j \right) .w_{\boldsymbol{\pi}}.$$

Combining these we get

$$\left(\mathbf{x}_i^-(u) \mathbf{p}_i^+(u) \right)_{N_i} w_{\boldsymbol{\pi}} = \left(\sum_{j=0}^{N_i} x_i^- \otimes p_{i, N_i - j} t^j \right) w_{\boldsymbol{\pi}} = x_i^- \otimes \left(\sum_{j=0}^{N_i} t^j p_{i, N_i - j} \right) w_{\boldsymbol{\pi}} = 0.$$

But it is elementary to see that

$$\sum_{j=0}^{N_i} t^j p_{i, N_i - j} = \prod_{r=1}^{\ell} (t - a_r)^{\lambda_r(h_i)},$$

which proves (2.3). Since $\mathfrak{n}^-$ is generated by the elements x_i^-, $i \in I$, it is immediate that there exists $N \gg 0$ such that

$$\left(x_\theta^- \otimes \prod_{r=1}^{\ell} (t - a_r)^N \right) w_{\boldsymbol{\pi}} = 0. \tag{2.4}$$

Since $[\mathfrak{n}^-, x_\theta^-] = 0$ and $W(\boldsymbol{\pi}) \cong \mathbf{U}(L(\mathfrak{n}^-)) w_{\boldsymbol{\pi}}$ as vector spaces, we get

$$\left(x_\theta^- \otimes \prod_{r=1}^{\ell} (t - a_r)^N \right) W(\boldsymbol{\pi}) = 0.$$

Since any element in $\mathfrak{g}$ is in the span of elements of the form $\{[x_{i_1}^+ [x_{i_2}^+ [\cdots [x_{i_k}^+, x_\theta^-], \cdots]] : i_1, \cdots, i_k \in I\}$, we now get

$$\left(\mathfrak{g} \otimes \prod_{r=1}^{\ell} (t - a_r)^N \mathbf{C}[t, t^- 1] \right) W(\boldsymbol{\pi}) = 0.$$

$\square$

Corollary. *Given $\boldsymbol{\pi} \in \mathcal{P}^+$ with unique decomposition $\boldsymbol{\pi} = \prod_{r=1}^{\ell} \boldsymbol{\pi}_{\lambda_r,a_r} \in \mathcal{P}^+$, there exists $N \in \mathbf{Z}_+$ such that the action of $L(\mathfrak{g})$ on $W(\boldsymbol{\pi})$ factors through to an action of $\mathfrak{g}_{\mathbf{a},N}$ on $W(\boldsymbol{\pi})$ and $W(\boldsymbol{\pi}) = \mathbf{U}(L(\mathfrak{n}^-_{\mathbf{a},N}))w_{\boldsymbol{\pi}}$.*

3. The twisted algebras $L^\sigma(\mathfrak{g})$ and the modules $W(\boldsymbol{\pi}^\sigma)$

3.1. Assume from now on that $\mathfrak{g}$ is simply–laced and that $\sigma : \mathfrak{g} \to \mathfrak{g}$ is a non–trivial diagram automorphism of $\mathfrak{g}$ of order m. In particular σ induces a permutation of I and R^+ and we have

$$\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma(\alpha)}, \quad \sigma(\mathfrak{h}) = \mathfrak{h}, \ \sigma(\mathfrak{n}^\pm) = \mathfrak{n}^\pm.$$

Let ζ be a primitive m^{th} root of unity, we have

$$\mathfrak{g} = \bigoplus_{\epsilon=0}^{m-1} \mathfrak{g}_\epsilon, \quad \mathfrak{g}_\epsilon = \{x \in \mathfrak{g} : \sigma(x) = \zeta^\epsilon x\}.$$

Given any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which is preserved by σ, set $\mathfrak{a}_\epsilon = \mathfrak{g}_\epsilon \cap \mathfrak{a}$. It is known that $\mathfrak{g}_0$ is a simple Lie algebra, $\mathfrak{h}_0$ is a Cartan subalgebra and that $\mathfrak{g}_\epsilon$ is an irreducible representation of $\mathfrak{g}_0$ for all $0 \ leq \epsilon \le m-1$. Moreover,

$$\mathfrak{n}^\pm \cap \mathfrak{g}_0 = \mathfrak{n}_0^\pm = \bigoplus_{\alpha \in R^+_{\mathfrak{g}_0}} (\mathfrak{g}_0)_{\pm\alpha}.$$

The following table describes the various possibilities for $\mathfrak{g}$, $\mathfrak{g}_0$ and the structure of $\mathfrak{g}_k$ as a $\mathfrak{g}_0$–module, here θ_0^s is the highest short root of $\mathfrak{g}_0$ and $B_1 = A_1$.

m	$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_k$
2	A_{2n},	B_n	$V_{\mathfrak{g}_0}(2\theta_0^s)$
2	$A_{2n-1},\ n \ge 2$	C_n	$V_{\mathfrak{g}_0}(\theta_0^s)$
2	$D_{n+1},\ n \ge 3$	B_n	$V_{\mathfrak{g}_0}(\theta_0^s)$
2	E_6	F_4	$V_{\mathfrak{g}_0}(\theta_0^s)$
3	D_4	G_2	$V_{\mathfrak{g}_0}(\theta_0^s)$

From now we set $R_{\mathfrak{g}} = R^+$, $R_{\mathfrak{g}_0} = R_0$, the sets I, P^+ etc. are defined similarly. The set of σ– orbits of I has the same cardinality as I_0 and we identify I_0 with a subset of I. In the case when $\mathfrak{g}$ is of type A_{2n} we assume that $n \in I_0$ corresponds to the unique short simple root of $\mathfrak{g}_0$. We shall also fix ζ a primitive m^{th} root of unity.

Suppose that $\{y_i : i \in I\}$ is one of the sets $\{h_i : i \in I\}$, $\{x_i^+ : i \in I\}$ or $\{x_i^- : i \in I\}$ and assume that $m = 2$ and that $i \ne n$ if $\mathfrak{g}$ is of type A_{2n}. Define subsets $\{y_{i,\epsilon} : i \in I_0,\ 0 \le \epsilon \le 1\}$ of $\mathfrak{g}_\epsilon$ by

$$y_{i,0} = y_i \quad \text{if } i = \sigma(i), \qquad y_{i,0} = y_i + y_{\sigma(i)} \quad \text{if } i \ne \sigma(i),$$
$$y_{i,1} = y_i - y_{\sigma(i)} \quad \text{if } i \ne \sigma(i) \qquad y_{i,1} = 0 \quad \text{if } i = \sigma(i),$$

If $\mathfrak{g}$ is of type A_{2n}, then we set,

$$h_{n,0} = 2(h_n + h_{n+1}), \quad x^{\pm}_{n,0} = \sqrt{2}(x^{\pm}_n + x^{\pm}_{n+1}),$$
$$x^{\pm}_{n,1} = -\sqrt{2}(x^{\pm}_n - x^{\pm}_{n+1}), \quad h_{n,1} = h_n - h_{n+1},$$
$$y^{\pm}_{n,1} = \mp\frac{1}{4}\left[x^{\pm}_{n,0}, x^{\pm}_{n,1}\right].$$

Finally if $\mathfrak{g}$ is of type D_4 and $m = 3$, set,

$$y_{i,0} = y_i \quad \text{if } i = \sigma(i), \qquad y_{i,0} = \sum_{j=0}^{m-1} y_{\sigma^j(i)} \quad \text{if } i \neq \sigma(i),$$
$$y_{i,1} = y_{i,2} = 0 \quad \text{if } i = \sigma(i),$$
$$y_{i,1} = y_i + \zeta^2 y_{\sigma(i)} + \zeta y_{\sigma^2(i)}, \qquad y_{i,2} = y_i + \zeta y_{\sigma(i)} + \zeta^2 y_{\sigma^2(i)} \quad \text{if } i \neq \sigma(i),$$

In the rest of this paper in the case when $\mathfrak{g}$ is of type A_{2n}, we shall only be interested in elements $\lambda \in P_0^+$ such that $\lambda(h_{n,0}) \in 2\mathbf{Z}_+$ and we let P_σ^+ denote this subset of P_0^+. Moreover we regard $\lambda \in P_\sigma^+$ as an element of P^+ as follows:

$$\lambda(h_i) = \begin{cases} \lambda(h_{i,0}), \quad i \in I_0, & \text{if } \mathfrak{g} \text{ is not of type } A_{2n} \\ 0 \quad i \notin I_0, \\ (1 - \delta_{i,n}/2)\lambda(h_{i,0}), & \text{if } \mathfrak{g} \text{ is of type } A_{2n}. \end{cases}$$

3.2. Let $\widetilde{\sigma} : L(\mathfrak{g}) \to L(\mathfrak{g})$ be the automorphism defined by extending,

$$\widetilde{\sigma}(x \otimes t^k) = \zeta^k \sigma(x) \otimes t^k,$$

for $x \in \mathfrak{g}$, $k \in \mathbb{Z}$. Then $\tilde{\sigma}$ is or order m and we let $L^\sigma(\mathfrak{g})$ be the subalgebra of fixed points of $\tilde{\sigma}$. Clearly,

$$L^\sigma(\mathfrak{g}) \cong \bigoplus_{\epsilon=0}^{m-1} \mathfrak{g}_\epsilon \otimes t^{m-\epsilon}\mathbf{C}[t^m, t^{-m}].$$

Lemma. *Let $i \in I_0$ and assume that $i \neq n$ if $\mathfrak{g}$ is of type A_{2n}. The subalgebra of $L^\sigma(\mathfrak{g}))$ spanned by the elements $\{x^{\pm}_{i,\epsilon} \otimes t^{mk-\epsilon}, h_{i,\epsilon} \otimes t^{mk-\epsilon} : k \in \mathbf{Z},\ 0 \leq \epsilon \leq m-1\}$ is canonically isomorphic to $L(\mathfrak{sl}_2)$. If $\mathfrak{g}$ is of type A_{2n} the subalgebra of $L^\sigma(\mathfrak{g}))$ spanned by the elements $\{x^{\pm}_{n,\epsilon} \otimes t^{2k+\epsilon}, h_{n,\epsilon} \otimes t^{2k+\epsilon}, \mp\frac{1}{4}[x^{\pm}_{n,0}, x^{\pm}_{n,1}] \otimes t^{2k+1} : k \in \mathbf{Z},\ 0 \leq \epsilon \leq m-1\}$ is canonically isomorphic to $L^\sigma(\mathfrak{sl}_3)$.* □

3.3. Identities in $\mathbf{U}(L^\sigma(\mathfrak{g}))$. Suppose that either $\mathfrak{g}$ is not of type A_{2n} and $\alpha_i \in (R_0)^+_s$ or that $\mathfrak{g}$ is of type A_{2n} and $i \neq n$. Define power series with coefficients in $\mathbf{U}(L(^\sigma(\mathfrak{g}))$ by,

$$\mathbf{p}^{\pm}_{i,\sigma}(u) = \exp\left(-\sum_{k=1}^{\infty}\sum_{\epsilon=0}^{m-1} \frac{h_{i,\epsilon} \otimes t^{mk-\epsilon}}{mk-\epsilon} u^{mk-\epsilon}\right),$$

$$\mathbf{x}^-_i(u) = \sum_{k=0}^{\infty}\sum_{\epsilon=0}^{m-1}(x^-_{i,m-\epsilon} \otimes t^{mk+\epsilon})u^{mk+\epsilon+1}, \qquad \tilde{\mathbf{x}}^+_i(u) = \sum_{k=-\infty}^{\infty}\sum_{\epsilon=0}^{m-1}(x^+_{i,m-\epsilon} \otimes t^{mk+\epsilon})u^{mk+\epsilon+1}.$$

If $\mathfrak{g}$ is not of type A_{2n} and $\alpha_i \in (R_0)_l^+$, then we set

$$\mathbf{p}^{\pm}_{i,\sigma}(u) = \exp\left(-\sum_{k=1}^{\infty} \frac{h_{i,0}\otimes t^{mk}}{k} u^k\right),$$

$$\mathbf{x}_i^-(u) = \sum_{k=0}^{\infty} (x_{i,0}^- \otimes t^{mk} u^{k+1}) \qquad \tilde{\mathbf{x}}_i^+(u) = \sum_{k=0}^{\infty} (x_{i,0}^+ \otimes t^{mk}) u^{k+1}.$$

Finally, if $\mathfrak{g}$ is of type A_{2n} and $i = n$ we have,

$$\mathbf{p}^{\pm}_{n,\sigma}(u) = \exp\left(-\sum_{k=1}^{\infty} \frac{h_{n,0}/2\otimes t^{2k}}{2k} u^{2k} + \sum_{k=1}^{\infty} \frac{h_{n,1}\otimes t^{2k-1}}{2k-1} u^{2k-1}\right),$$

$$\mathbf{x}_n^-(u) = \sum_{k=0}^{\infty}\sum_{\epsilon=0}^{m-1} (x_{n,\epsilon}^- \otimes t^{mk+\epsilon}) u^{mk+\epsilon+1}, \qquad \tilde{\mathbf{x}}_n^+(u) = \sum_{k=-\infty}^{\infty}\sum_{\epsilon=0}^{m-1} (x_{n,\epsilon}^+ \otimes t^{mk+\epsilon}) u^{mk+\epsilon+1}.$$

Lemma. *Let $r \in \mathbf{Z}_+$.*

(i) *If $\mathfrak{g}$ is not of type A_{2n} and $\alpha_i \in (R_0)_s^+$ or $\mathfrak{g}$ is of type A_{2n} and $\alpha_i \in (R_0)_l^+$, we have*

$$(x_{i,1}^+ \otimes t)^{(r)} (x_{i,0}^- \otimes 1)^{(r+1)} = (-1)^r \left(\mathbf{x}_i^-(u)\mathbf{p}^+_{i,\sigma}(u)\right)_{r+1} \mod \mathbf{U}(L^\sigma(\mathfrak{g}))\widetilde{\mathbf{x}}_i^{\,+}(u).$$

(ii) *If $\mathfrak{g}$ is not of type A_{2n} and $\alpha_i \in (R_0)_l^+$,*

$$(x_{i,0}^+ \otimes t^2)^{(r)} (x_{i,0}^- \otimes 1)^{(r+1)} = (-1)^r \left(\mathbf{x}_i^-(u)\mathbf{p}^+_{i,\sigma}(u)\right)_{r+1} \mod \mathbf{U}(L^\sigma(\mathfrak{g}))\widetilde{\mathbf{x}}_i^{\,+}(u).$$

(iii) *If $\mathfrak{g}$ is of type A_{2n}, we have*

(a) $(x_{n,0}^+ \otimes 1)^{(2r-1)} (y_{n,1}^- \otimes t)^{(r)} = -\left(\mathbf{x}_n^-(u)\mathbf{p}^+_{n,\sigma}(u)\right)_r \mod \mathbf{U}(L^\sigma(\mathfrak{g}))\widetilde{\mathbf{x}}_n^+(u)$,

(b) $(x_{n,0}^+ \otimes 1)^{(2r)} (y_{n,1}^- \otimes t)^{(r)} = -\left(\mathbf{p}^+_{n,\sigma}(u)\right)_r \mod \mathbf{U}(L^\sigma(\mathfrak{g}))\widetilde{\mathbf{x}}_n^+(u)$,
where

$$y_{n,1}^- = \frac{1}{4}\left[x_{n,0}^-, x_{n,1}^-\right].$$

Proof. Parts (i) and (ii) are immediate consequences of Lemma 2.3 and Lemma 3.2. Part (iii) is deduced from [13], [9, Lemma 5.36], exactly as (i) and (ii) were deduced from Garland in [6]. □

3.4. The monoid $\mathcal{P}_\sigma^+$. Let $(\ ,\ )$ be the form on $\mathfrak{h}_0^*$ induced by the Killing form of $\mathfrak{g}_0$ normalized so that $(\theta_0, \theta_0) = 2$. For $i \in I_0$ and $a \in \mathbf{C}^\times$, $\lambda \in P_0^+$ and $\mathfrak{g}$ not of type A_{2n} let

$$\boldsymbol{\pi}^\sigma_{i,a} = ((1 - a^{(\alpha_i,\alpha_i)}u)^{\delta_{ij}} : j \in I_0), \qquad \boldsymbol{\pi}^\sigma_{\lambda,a} = \prod_{i\in I_0} \left(\boldsymbol{\pi}^\sigma_{i,a}\right)^{\lambda(h_i)},$$

while if $\mathfrak{g}$ is of type A_{2n} we set for $i \in I_0$, $a \in \mathbf{C}^\times$, $\lambda \in P_\sigma^+$,

$$\boldsymbol{\pi}^\sigma_{i,a} = ((1 - au)^{\delta_{ij}} : j \in I_0), \qquad \boldsymbol{\pi}^\sigma_{\lambda,a} = \prod_{i\in I_0} \left(\boldsymbol{\pi}^\sigma_{i,a}\right)^{(1-\frac{1}{2}\delta_{i,n})\lambda(h_i)}.$$

Let $\mathcal{P}_\sigma^+$ be the monoid generated by the elements $\boldsymbol{\pi}_{\lambda,a}^\sigma$. Define a map $\mathcal{P}_\sigma^+ \to P_\sigma^+$ by

$$\lambda_{\boldsymbol{\pi}^\sigma} = \sum_{i\in I_0} (\deg \pi_i)\omega_i,$$

if $\mathfrak{g}$ is not of type A_{2n} and

$$\lambda_{\boldsymbol{\pi}^\sigma} = \sum_{i\in I_0} (1+\delta_{i,n})(\deg \pi_i)\omega_i,$$

if $\mathfrak{g}$ is of type A_{2n}. It is clear that any $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$ can be written (non–uniquely) as product

$$\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell} \prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k},$$

where $\mathbf{a} = (a_1, \cdots, a_\ell)$ and $\mathbf{a}^m$ have distinct coordinates. We call any such expression a standard decomposition of $\boldsymbol{\pi}^\sigma$.

3.5. The set $\mathbf{i}(\boldsymbol{\pi}^\sigma)$. Given $\lambda = \sum_{i\in I} m_i\omega_i \in P^+$ and $0 \le \epsilon \le m-1$, define elements $\lambda(\epsilon) \in P_\sigma^+$ by,

$$\lambda(0) = \sum_{i\in I_0} m_i\omega_i, \quad \lambda(1) = \sum_{i\in I_0:\sigma(i)\neq i} m_{\sigma(i)}\omega_i, \quad \text{if } m=2 \text{ and } \mathfrak{g} \text{ not of type } A_{2n}$$

$$\lambda(0) = \sum_{i\in I_0} (1+\delta_{i,n}) m_i\omega_i, \quad \lambda(1) = \sum_{i\in I_0:\sigma(i)\neq i} (1+\delta_{\sigma(i),n}) m_{\sigma(i)}\omega_i, \quad \text{if } m=2 \text{ and } \mathfrak{g} \text{ of type } A_{2n}$$

$$\lambda(0) = m_1\omega_1 + m_2\omega_2, \quad \lambda(1) = m_3\omega_1, \quad \lambda(2) = m_4\omega_1, \quad \text{if } m=3.$$

Define a map $\mathbf{r} : \mathcal{P}^+ \to \mathcal{P}_\sigma^+$ as follows. Given $\boldsymbol{\pi} \in \mathcal{P}^+$ write

$$\boldsymbol{\pi} = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_k,a_k}, \quad a_k \neq a_p, \quad 1 \le k \neq p \le \ell,$$

and set

$$\mathbf{r}(\boldsymbol{\pi}) = \prod_{k=1}^{\ell} \prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_k(\epsilon),\zeta^\epsilon a_k}.$$

Note that $\mathbf{r}$ is well defined since the choice of (λ_k, a_k) is unique and set

$$\mathbf{i}(\boldsymbol{\pi}^\sigma) = \{\boldsymbol{\pi} \in \mathcal{P}^+ : \mathbf{r}(\boldsymbol{\pi}) = \boldsymbol{\pi}^\sigma\}.$$

We now give an explicit description of the set $\mathbf{i}(\boldsymbol{\pi}^\sigma)$. Recall that given $\lambda \in P_\sigma^+$, we also regard $\lambda \in P^+$ as in Section 3.1. In addition, define $\sigma(\omega_i) = \omega_{\sigma(i)}$ for $i \in I$.

Lemma. (i) *Let $i \in I_0$ and $a \in \mathbf{C}^\times$. We have,*

$$\mathbf{i}(\boldsymbol{\pi}^\sigma_{\omega_i,a}) = \{\boldsymbol{\pi}_{\sigma^r(\omega_i),\zeta^{m-r}a} \,|\, 0 \le r < m\},$$

and for A_{2n}^2 and $i = n$,

$$\mathbf{i}(\boldsymbol{\pi}^\sigma_{2\omega_n,a}) = \{\boldsymbol{\pi}_{\omega_n,a} \,, \boldsymbol{\pi}_{\omega_{n+1},-a}\}$$

(ii) *Let* $\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\prod_{i\in I_0}(\boldsymbol{\pi}^\sigma_{\omega_i,\zeta^\epsilon a_k})^{m_{k,\epsilon,i}}$ *be a decomposition of* $\boldsymbol{\pi}^\sigma$ *into linear factors for* $\mathfrak{g}$ *not of type* A_{2n}. *Then*

$$\mathbf{i}(\boldsymbol{\pi}^\sigma) = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\prod_{i\in I_0}\{\boldsymbol{\pi}_{\sigma^r(\omega_i),\zeta^{m-r+\epsilon}a_k}\,|\,0\le r<m\}^{m_{k,\epsilon,i}}$$

where the product of the sets is understood to be the set of products of elements of the sets.
In the case of $A_{2n}^{(2)}$, *let* $\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{1}\prod_{i\in I_0}(\boldsymbol{\pi}^\sigma_{(1+\delta_{i,n})\omega_i,\zeta^\epsilon a_k})^{m_{k,\epsilon,i}}$ *be a decomposition of* $\boldsymbol{\pi}^\sigma$ *into linear factors. Then*

$$\mathbf{i}(\boldsymbol{\pi}^\sigma) = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{2}\prod_{i\in I_0}\{\boldsymbol{\pi}_{\sigma^r(\omega_i),\zeta^{2-r+\epsilon}a_k}\,|\,0\le r<2\}^{m_{k,\epsilon,i}}$$

(iii) *In particular, we have*

$$\prod_{k=1}^{\ell}\boldsymbol{\pi}_{\mu_k,a_k} = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\prod_{i\in I_0}\boldsymbol{\pi}^{m_{k,\epsilon,i}}_{\sigma^\epsilon(\omega_i),a_k}\in\mathbf{i}(\boldsymbol{\pi}^\sigma),$$

where $\mu_k = \sum_{\epsilon=0}^{m-1}\sum_{i\in I_0} m_{k,\epsilon,i}\sigma^\epsilon(\omega_i)$ *and* $a_i^m\neq a_j^m$.

Proof. The first statement is trivially checked, noting that if i is a fixed point of σ, then $\boldsymbol{\pi}^\sigma_{\omega_i,a} = \boldsymbol{\pi}^\sigma_{\omega_i,\zeta^r a}$ for $0\le r<m$. The other statements follow immediately from the first one. $\square$

From here on we shall assume that, unless otherwise noted, the element $\boldsymbol{\pi}\in\mathbf{i}(\boldsymbol{\pi}^\sigma)$ chosen is of the form given in (iii) of the lemma.

3.6. The modules $W(\boldsymbol{\pi}^\sigma)$, $V(\boldsymbol{\pi}^\sigma)$. Given $\boldsymbol{\pi}^\sigma = (\pi_{i,\sigma})_{i\in I_0}\in\mathcal{P}_\sigma^+$, the Weyl module $W(\boldsymbol{\pi}^\sigma)$ is the $\mathbf{U}(L^\sigma(\mathfrak{g}))$-module generated by an element $w_{\boldsymbol{\pi}^\sigma}$ with relations:

$$L^\sigma(\mathfrak{n}^+)w_{\boldsymbol{\pi}^\sigma}=0,\quad hw_{\boldsymbol{\pi}} = \lambda_{\boldsymbol{\pi}}(h)w_{\boldsymbol{\pi}^\sigma},\quad (x_{i,0}^-)^{\lambda_{\boldsymbol{\pi}}(h_i)+1}w_{\boldsymbol{\pi}^\sigma}=0,$$

$$\left(\mathbf{p}^\pm_{i,\sigma}(u)-\pi^\pm_{i,\sigma}(u)\right)w_{\boldsymbol{\pi}^\sigma}=0,$$

for all $i\in I_0$ and $h\in\mathfrak{h}_0$. If $\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell}\boldsymbol{\pi}^\sigma_{\lambda_k,a_k}\in\mathcal{P}_\sigma^+$, it is not hard to see that for $i\in I_0$, we have if $\mathfrak{g}$ not of type A_{2n},

$$\left(\mathbf{p}^\pm_{i,\sigma}(u)-\pi^\pm_{i,\sigma}(u)\right)w_{\boldsymbol{\pi}^\sigma}=0 \iff (h_{i,\epsilon}\otimes t^{mk-\epsilon})w_{\boldsymbol{\pi}^\sigma} = \sum_{j=1}^{\ell}\lambda_j(h_{i,0})a_j^{mk-\epsilon}w_{\boldsymbol{\pi}^\sigma}, \tag{3.1}$$

and for $\mathfrak{g}$ of type A_{2n},

$$\left(\mathbf{p}^\pm_{i,\sigma}(u)-\pi^\pm_{i,\sigma}(u)\right)w^\sigma_{\lambda,a}=0 \iff (h_{i,\epsilon}\otimes t^{mk-\epsilon})w_{\boldsymbol{\pi}^\sigma} = \sum_{j=1}^{\ell}(1-\frac{1}{2}\delta_{i,n})\lambda_j(h_{i,\epsilon})a_j^{mk-\epsilon}w_{\boldsymbol{\pi}^\sigma}. \tag{3.2}$$

3.7. For $b \in \mathbf{C}^\times$ we have $\tau_b(L^\sigma(\mathfrak{g})) \subset L^\sigma(\mathfrak{g})$ and we let $\tau_b W(\boldsymbol{\pi}^\sigma)$ be the $L^\sigma(\mathfrak{g})$–module obtained by pulling back $W(\boldsymbol{\pi}^\sigma)$ through τ_b. The next result is proved by standard methods.

Lemma. (i) *Let $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$. Then $W(\boldsymbol{\pi}^\sigma) = \mathbf{U}(L^\sigma(\mathfrak{n}^-))w^\sigma_{\boldsymbol{\pi}}$, and hence we have,*

$$\mathrm{wt}(W(\boldsymbol{\pi}^\sigma) \subset \lambda_{\boldsymbol{\pi}^\sigma} - Q_0^+, \qquad \dim W(\mathbf{P}^\sigma)_{\lambda_{\boldsymbol{\pi}^\sigma}} = 1.$$

In particular, the module $W(\boldsymbol{\pi}^\sigma)$ has a unique irreducible quotient $V(\boldsymbol{\pi}^\sigma)$.

(ii) *For $b \in \mathbf{C}^\times$, we have $\tau_b W(\boldsymbol{\pi}^\sigma) \cong W(\boldsymbol{\pi}^\sigma_b)$, where $\boldsymbol{\pi}^\sigma = (\pi_i(u))_{i\in I}$ and $\boldsymbol{\pi}^\sigma_b = (\pi_i(b^{-1}u))_{i\in I}$. In particular we have*

$$W(\boldsymbol{\pi}^\sigma_{\lambda,a}) \cong_{\mathfrak{g}_0} W(\boldsymbol{\pi}^\sigma_{\lambda,ba}).$$

□

3.8. The main theorem. In the rest of this paper we shall prove the following result.

Theorem 2. (i) Let $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$. For all $\boldsymbol{\pi} \in \mathbf{i}(\boldsymbol{\pi}^\sigma)$, we have

$$W(\boldsymbol{\pi}^\sigma) \cong_{L^\sigma(\mathfrak{g})} W(\boldsymbol{\pi}), \qquad V(\boldsymbol{\pi}^\sigma) \cong_{L^\sigma(\mathfrak{g})} V(\boldsymbol{\pi}).$$

(ii) Let $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$ and assume that $\prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k} \in \mathcal{P}_\sigma^+$ is a standard decomposition of $\boldsymbol{\pi}$. As $L^\sigma(\mathfrak{g})$–modules, we have

$$W(\boldsymbol{\pi}^\sigma) \cong \bigotimes_{k=1}^{\ell} W(\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k}).$$

(iii) Suppose that $\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a} \in \mathcal{P}_\sigma^+$. Then

$$W(\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}) \cong_{\mathfrak{g}_0} \bigotimes_{\epsilon=0}^{m-1} W(\boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}).$$

(iv) Let $\lambda = \sum_{i\in I_0} m_i\omega_i \in P_\sigma^+$ and $a \in \mathbf{C}^\times$. We have for $\mathfrak{g}$ not of type A_{2n}

$$W(\boldsymbol{\pi}^\sigma_{\lambda,a}) \cong_{\mathfrak{g}_0} \bigotimes_{i=1}^{n} W(\boldsymbol{\pi}^\sigma_{\omega,1})^{\otimes m_i}$$

and for $\mathfrak{g}$ of type A_{2n}

$$W(\boldsymbol{\pi}^\sigma_{\lambda,a}) \cong_{\mathfrak{g}_0} W(\boldsymbol{\pi}^\sigma_{2\omega_n,1})^{\otimes \frac{m_n}{2}} \otimes \bigotimes_{i=1}^{n-1} W(\boldsymbol{\pi}^\sigma_{\omega_i,1})^{\otimes m_i}.$$

(v) Let V be any finite–dimensional $L^\sigma(\mathfrak{g})$–module generated by an element $v \in V$ such that

$$L^\sigma(\mathfrak{n}^+)v = 0, \quad L^\sigma(\mathfrak{h})v = \mathbf{C}v.$$

Then there exists $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$ such that the assignment $w_{\boldsymbol{\pi}^\sigma} \to v$ extends to a surjective homomorphism $W(\boldsymbol{\pi}^\sigma) \to V$ of $L^\sigma(\mathfrak{g})$–modules.

□

4. Proof of Theorem 2

4.1. Annihilating ideals for $W(\boldsymbol{\pi}^\sigma)$.

Proposition. *Let $\boldsymbol{\pi}^\sigma = \prod_{r=1}^{\ell} \boldsymbol{\pi}^\sigma_{\lambda_r,a_r} \in \mathcal{P}^+_\sigma$. There exists an integer $N = N(\boldsymbol{\pi})$ such that*

$$\left(\bigoplus_{\epsilon=0}^{m-1}(\mathfrak{g}_\epsilon \otimes t^{m-\epsilon}\prod_{r=1}^{\ell}(t^m - a_r^m)^N \mathbf{C}[t^m, t^{-m}])\right) W(\boldsymbol{\pi}^\sigma) = 0.$$

Proof. The subalgebra $L^m(\mathfrak{g}_0) = \mathfrak{g}_0 \otimes \mathbf{C}[t^m, t^{-m}]$ is canonically isomorphic to $L(\mathfrak{g}_0)$. It follows from the defining relations that

$$L^m(\mathfrak{n}_0^+)w_{\boldsymbol{\pi}^\sigma} = 0, \qquad (h_0 \otimes t^{mk})w_{\boldsymbol{\pi}^\sigma} = \left(\sum_{r=1}^{\ell}\lambda_r(h_0)a_r^{mk}\right) w_{\boldsymbol{\pi}^\sigma},$$

and hence, $\mathbf{U}(L^m(\mathfrak{g}_0))w_{\boldsymbol{\pi}^\sigma}$ is a quotient of the $L(\mathfrak{g}_0)$–module $W_{\mathfrak{g}_0}(\boldsymbol{\pi}_m)$ where

$$\boldsymbol{\pi}_m = \prod_{r=1}^{\ell}\boldsymbol{\pi}_{\lambda_r, a_r^m}.$$

It follows from (2.4) that

$$(x^-_{\theta_0} \otimes \prod_{r=1}^{m}(t^m - a_r^m))w^\sigma_{\boldsymbol{\pi}} = 0, \tag{4.1}$$

for some $N \in \mathbf{Z}_+$, where $\theta_0 \in R_0^+$ is the highest root in R_0^+.

Assume first that $\mathfrak{g}$ is not of type A_{2n}, then

$$[x^-_{\theta_0}, L^\sigma(\mathfrak{n}^-)] = 0, \qquad [\mathfrak{h}_\epsilon, \mathfrak{g}_0] = \mathfrak{g}_\epsilon, \quad 0 \le \epsilon \le m-1. \tag{4.2}$$

The first equality in (4.2) gives $\left(x^-_{\theta_0} \otimes \prod_{r=1}^{\ell}(t^m - a_r^m)^N\right) W(\boldsymbol{\pi}^\sigma) = 0,$. One deduces now as in the untwisted case that

$$\left(\mathfrak{g}_0 \otimes \left(\prod_{r=1}^{\ell}(t^m - a_r^m)^N\right)\mathbf{C}[t^m, t^{-m}]\right) W(\boldsymbol{\pi}^\sigma) = 0.$$

Applying $\mathfrak{h}_\epsilon \otimes t^{m-\epsilon}$ to the preceding equation and using the second equality in (4.2) gives

$$\left(\mathfrak{g}_\epsilon \otimes t^{m-\epsilon}\left(\prod_{r=1}^{\ell}(t^m - a_r^m)\right)\mathbf{C}[t^m, t^{-m}]\right) W(\boldsymbol{\pi}^\sigma) = 0,$$

for all $0 \le \epsilon \le m-1$ and the result is proved.

Assume now that $\mathfrak{g}$ is of type A_{2n}. This time, we use the fact that

$$(x^-_{n,\epsilon} \otimes t^\epsilon \mathbf{C}[t^2, t^{-2}])w_{\boldsymbol{\pi}} \in \mathbf{U}(L^\sigma(\mathfrak{h} \oplus \mathfrak{n}^+))(x^-_{\theta_0} \otimes \mathbf{C}[t^2, t^{-2}])w_{\boldsymbol{\pi}}$$

together with (4.1) to conclude that

$$(x^-_{n,\epsilon} \otimes \prod_{r=1}^{\ell} t^\epsilon(t^2 - a_r^2)^N)w_{\boldsymbol{\pi}^\sigma} = 0.$$

Hence

$$\left([x_{\theta_0}^-, x_{n,1}^-] \otimes \prod_{r=1}^{\ell} t(t^2 - a_r^2)^N)\right) w_{\boldsymbol{\pi}}^{\sigma} = 0,$$

for some $N \gg 0$. Since the element $[x_{\theta_0}^-, x_{n,1}^-] \in \mathfrak{g}_1$ generates $\mathfrak{g}_1$ as a $\mathfrak{g}_0$–module and $[\mathfrak{n}^-, [x_{\theta_0}^-, x_{n,1}^-]] = 0$, we can now prove by similar arguments that for some $N \gg 0$,

$$(\mathfrak{g}_1 \otimes \prod_{r=1}^{\ell} t(t^2 - a_r^2)^N) W(\boldsymbol{\pi}^\sigma) = 0.$$

Next, using the fact that $[x_{\theta_0}^-, \mathfrak{n}_1^-] = \mathbf{C}[x_{\theta_0}^-, x_{n,1}^-]$, we get

$$(x_{\theta_0}^- \otimes \prod_{r=1}^{\ell} (t^2 - a_r^2)^N) W(\boldsymbol{\pi}^\sigma) = 0,$$

which finally gives

$$\left(\mathfrak{g}_0 \otimes \prod_{r=1}^{\ell} (t^2 - a_r^2)^N \mathbf{C}[t^2, t^{-2}]\right) W(\boldsymbol{\pi}^\sigma) = 0,$$

and completes the proof.

□

Given positive integers $\ell, N \in \mathbf{Z}_+$, $\mathbf{a} = (a_1, \cdots, a_\ell) \in (\mathbf{C}^\times)^\ell$ and a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that $\sigma(\mathfrak{a}) \subset \mathfrak{a}$, let

$$\mathfrak{a}_{\mathbf{a},N}^\sigma = L^\sigma(\mathfrak{g}) / \oplus_{\epsilon=0}^{m-1} (\mathfrak{a}_\epsilon \otimes t^\epsilon \prod_{k=1}^{\ell} (t^m - a_k)^N \mathbf{C}[t^m, t^{-m}]). \tag{4.3}$$

Corollary. *Let* $\boldsymbol{\pi}^\sigma = \prod_{r=1}^{\ell} \boldsymbol{\pi}_{\lambda_r, a_r}^\sigma \in \mathcal{P}_\sigma^+$ *be a standard decomposition of* $\boldsymbol{\pi}^\sigma$ *and set* $\mathbf{a} = (a_1, \cdots, a_\ell)$ *There exists* $N >> 0$ *such that*

$$W(\boldsymbol{\pi}^\sigma) = \mathbf{U}((\mathfrak{n}_{\mathbf{a}^m,N}^-)^\sigma) w_{\boldsymbol{\pi}^\sigma}$$

4.2.

Proposition. *For all* $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$*, the* $L^\sigma(\mathfrak{g})$*–module* $W(\boldsymbol{\pi}^\sigma)$ *is finite–dimensional.*

Proof. Let $u \in W(\boldsymbol{\pi}^\sigma)$ and write $u = y w_{\boldsymbol{\pi}^\sigma}$ for some $y \in \mathbf{U}(L^\sigma(\mathfrak{n}^-))$. The adjoint action of the subalgebras $\mathfrak{n}_0^\pm$ on $L^\sigma(\mathfrak{g})$ and hence on $\mathbf{U}(L^\sigma(\mathfrak{g}))$ is nilpotent. Using the defining relations we get immediately that for some $r = r(u) > 0$, we have

$$(x_\alpha^\pm \otimes 1)^r u = 0, \quad \forall \;\; \alpha \in R_0^+.$$

This implies that $\mathbf{U}(\mathfrak{g}_0)u$ is a finite–dimensional $\mathfrak{g}_0$–submodule of $W(\boldsymbol{\pi}^\sigma)$, and hence $W(\boldsymbol{\pi}^\sigma)$ is isomorphic to a direct sum of $\mathfrak{g}_0$–modules. Write,

$$W(\boldsymbol{\pi}^\sigma) = \bigoplus_{\eta \in Q_0^+} W(\boldsymbol{\pi}^\sigma)_\mu,$$

where $W(\boldsymbol{\pi}^\sigma)_\mu = \{u \in W(\boldsymbol{\pi}^\sigma) : hu = \mu(h)u, \ \ \forall \ \ h \in \mathfrak{h}_0\}$. The representation theory of $\mathfrak{g}_0$ now implies that

$$W(\boldsymbol{\pi}^\sigma)_\mu \neq 0 \iff W(\boldsymbol{\pi}^\sigma)_{w(\mu)} \neq 0, \ \ \forall \ w \in W_0.$$

Since $W(\boldsymbol{\pi}^\sigma)_\mu = 0$ unless $\mu \in \lambda - Q_0^+$ and the number of elements in P_0^+ with this property is finite we get that $W(\boldsymbol{\pi}^\sigma)_\nu = 0$, for all but finitely many $\nu \in P_0^+$. The proposition follows if we prove that $\dim(W(\boldsymbol{\pi}^\sigma)_\nu) < \infty$ for all $\nu \in P_0^+$.

Choose $\mathbf{a}$ and N as in Corollary 4.1. Then

$$W(\boldsymbol{\pi}^\sigma)_\nu = \mathbf{U}((\mathfrak{n}^-_{\mathbf{a},N})^\sigma)_{\lambda_{\boldsymbol{\pi}}-\nu} w_{\boldsymbol{\pi}^\sigma}$$

where

$$\mathbf{U}((\mathfrak{n}^-_{\mathbf{a},N})^\sigma)_{\lambda_{\boldsymbol{\pi}}-\nu} = \{y \in \mathbf{U}((\mathfrak{n}^-_{\mathbf{a},N})^\sigma)_{\lambda_{\boldsymbol{\pi}}-\nu} : [h,y] = (\lambda_{\boldsymbol{\pi}} - \nu)(h)y, \ \ \forall \ h \in \mathfrak{h}_0\}.$$

Since this subspace is finite–dimensional it follows that $\dim(W(\boldsymbol{\pi}^\sigma)_\nu) < \infty$ as required. $\square$

4.3. Let $N \in \mathbf{Z}_+$ and $\mathbf{a} \in (\mathbf{C}^\times)^\ell$. The inclusion $\iota : L^\sigma(\mathfrak{g}) \to L(\mathfrak{g})$ obviously induces a Lie algebra map $\iota_{\mathbf{a},N} : \mathfrak{g}^\sigma_{\mathbf{a}^m,N} \to \mathfrak{g}_{\mathbf{a},N}$, where $\mathbf{a}^m = (a_1^m, \cdots, a_\ell^m)$. The following proposition will play a crucial role in the proof of Theorem 2.

Proposition. *Let $\mathbf{a} \in (\mathbf{C}^\times)^\ell$ be such that $\mathbf{a}$ and $\mathbf{a}^m$ have distinct coordinates. For all $N \in \mathbf{Z}_+$ we have an isomorphism of Lie algebras,*

$$\mathfrak{g}_{\mathbf{a},N} \cong \bigoplus \mathfrak{g}^\sigma_{a_i^m,N} \cong \mathfrak{g}^\sigma_{\mathbf{a}^m,N}$$

for all $N \in \mathbf{Z}_+$. In particular, the composite map $L^\sigma(\mathfrak{g}) \to L(\mathfrak{g}) \to \mathfrak{g}_{\mathbf{a},N}$ is surjective.

Proof. The proof that

$$\bigoplus \mathfrak{g}^\sigma_{a_i^m,N} \cong \mathfrak{g}^\sigma_{\mathbf{a}^m,N}$$

is an obvious modification of the one given in Lemma 2.1 which also shows now that it is sufficient to prove the proposition when $\ell = 1$. For this, let $a \in \mathbf{C}^\times$ and $f = t^\epsilon g$ where $g \in \mathbf{C}[t^m, t^{-m}]$. Then,

$$f \in (t-a)^N \mathbf{C}[t,t^{-1}] \quad \iff \quad f \in t^\epsilon (t^m - a^m)^N \mathbf{C}[t^m, t^{-m}],$$

which proves that $\iota_{a,N}$ is injective. The proposition follows by noting that

$$\dim \mathfrak{g}^\sigma_{a^m,N} = \dim \mathfrak{g}_{a,N} = N \dim \mathfrak{g}.$$

$\square$

4.4. We note some elementary observations which we use without further comment. Any $\mathfrak{g}_{\mathbf{a},N}$–module (resp. $\mathfrak{g}^\sigma_{\mathbf{a},N}$) is obviously a $L(\mathfrak{g})$–module (resp. $L^\sigma(\mathfrak{g})$–module). Moreover if $\mathbf{a} \in (\mathbf{C}^\times)^\ell$ is such that $\mathbf{a}$ and $\mathbf{a}^m$ have distinct coordinates then for all $N \in \mathbf{Z}_+$, any $\mathfrak{g}_{\mathbf{a},N}$–module V is also a $\mathfrak{g}_{\mathbf{a}^m,N}$–module and we write it as $V_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}}$. Similarly if we start with a $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$–module V we get a $\mathfrak{g}_{\mathbf{a},N}$–module which we write as $V_{\mathfrak{g}_{\mathbf{a},N}}$. Note also that if V is an $\mathfrak{g}_{\mathbf{a},N}$–module, then

$$(V_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}})_{\mathfrak{g}_{\mathbf{a},N}} \cong_{\mathfrak{g}_{\mathbf{a},N}} V, \qquad (V_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}})_{\mathfrak{g}_{\mathbf{a},N}} \cong_{L(\mathfrak{g})} V. \tag{4.4}$$

4.5.

Lemma. *Let $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$, and assume that $\boldsymbol{\pi} \in \mathbf{i}(\boldsymbol{\pi}^\sigma)$.*

(i) *There exists $\ell, N \in \mathbf{Z}_+$ and $\mathbf{a} \in (\mathbf{C}^\times)^\ell$ with $\mathbf{a}$ and $\mathbf{a}^m$ having distinct coordinates such that $W(\boldsymbol{\pi})$ and $W(\boldsymbol{\pi}^\sigma)$ are modules for both $\mathfrak{g}_{\mathbf{a},N}$ and $\mathfrak{g}_{\mathbf{a}^m,N}$.*

(ii) *In particular,*

$$W(\boldsymbol{\pi})_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}} = \mathbf{U}(\mathfrak{g}^\sigma_{\mathbf{a}^m,N})w_{\boldsymbol{\pi}},$$

and $V(\boldsymbol{\pi})_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}}$ is an irreducible $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$–module.

Proof. Let $\boldsymbol{\pi} = \prod_{k=1}^\ell \boldsymbol{\pi}_{\lambda_k,a_k}$, where $\mathbf{a} = (a_1, \cdots, a_k)$ and $\mathbf{a}^m$ have distinct coordinates. Proposition 2.7 implies that $W(\boldsymbol{\pi}) = \mathbf{U}(\mathfrak{g}_{\mathbf{a},N})w_{\boldsymbol{\pi}}$. Using proposition 4.3 we see that $W(\boldsymbol{\pi})$ is also module for $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$ and so we get

$$W(\boldsymbol{\pi})_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}} = \mathbf{U}(\mathfrak{g}^\sigma_{\mathbf{a}^m,N})w_{\boldsymbol{\pi}}.$$

Similarly Proposition 4.1 implies that $W(\boldsymbol{\pi}^\sigma)$ is a module for $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$ and hence for $\mathfrak{g}_{\mathbf{a},N}$. Since $V(\boldsymbol{\pi})$ is an irreducible module for $\mathfrak{g}_{\mathbf{a},N}$, it follows that it is also irreducible as a $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$–module and the proposition is proved. $\square$

The following proposition proves (i) of Theorem 2.

Proposition. *Let $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$, $\boldsymbol{\pi} \in \mathbf{i}(\boldsymbol{\pi}^\sigma)$.*

(i) *Regarded as $L^\sigma(\mathfrak{g})$–module $W(\boldsymbol{\pi})$ is a quotient of $W(\boldsymbol{\pi}^\sigma)$ and hence*

$$V(\boldsymbol{\pi}) \cong_{L^\sigma(\mathfrak{g})} V(\boldsymbol{\pi}^\sigma).$$

(ii) *For $N \gg 0$, the $L^\sigma(\mathfrak{g})$–module structure on $W(\boldsymbol{\pi}^\sigma)$ (resp. $V(\boldsymbol{\pi}^\sigma)$) extends to an $L(\mathfrak{g})$–module action on $W(\boldsymbol{\pi}^\sigma)$ (resp. $V(\boldsymbol{\pi}^\sigma)$).*

(iii) *The module $W(\boldsymbol{\pi}^\sigma)_{\mathfrak{g}_{\mathbf{a},N}}$ is a $L(\mathfrak{g})$–module quotient of $W(\boldsymbol{\pi})$.*

Proof. Using (3.1), (3.2) and the fact that $\mathbf{r}(\boldsymbol{\pi}) = \boldsymbol{\pi}^\sigma$, we see that $w_{\boldsymbol{\pi}}$ satisfies the defining relations of $W(\boldsymbol{\pi}^\sigma)$. Part (i) follows if we prove that $W(\boldsymbol{\pi}) = \mathbf{U}(L^\sigma(\mathfrak{g}))w_{\boldsymbol{\pi}}$. But this is true because proposition 4.3 and proposition 4.5 prove that there exists $\mathbf{a} \in (\mathbf{C}^\times)^\ell$ such that

$$W(\boldsymbol{\pi})_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}} = \mathbf{U}(\mathfrak{g}^\sigma_{\mathbf{a}^m,N})w_{\boldsymbol{\pi}} = \mathbf{U}(L^\sigma(\mathfrak{g}))w_{\boldsymbol{\pi}}.$$

It now follows that $V(\boldsymbol{\pi})_{\mathfrak{g}_{\mathbf{a}^m,N}}$ is the irreducible quotient of $W(\boldsymbol{\pi}^\sigma)$ and hence is isomorphic to $V(\boldsymbol{\pi}^\sigma)$ as $L^\sigma(\mathfrak{g})$–modules.

To prove (ii), note that that we have a surjective homomorphism of Lie algebras

$$\mathbf{p} : L(\mathfrak{g}) \to \mathfrak{g}_{\mathbf{a},N} \to \mathfrak{g}^\sigma_{\mathbf{a}^m,N},$$

such that the restriction of $\mathbf{p}$ to $L^\sigma(\mathfrak{g})$ is just the canonical surjection. Moreover

$$\mathbf{p}(L(\mathfrak{n}^\pm)) \subset (\mathfrak{n}^\pm)^\sigma_{\mathbf{a}^m,N}, \quad \mathbf{p}(L(\mathfrak{h})) \subset \mathfrak{h}^\sigma_{\mathbf{a}^m,N},$$

and hence

$$L(\mathfrak{n}^+)w_{\boldsymbol{\pi}^\sigma} = 0, \quad L(\mathfrak{h})w_{\boldsymbol{\pi}^\sigma} = \mathbf{C}w_{\boldsymbol{\pi}^\sigma}.$$

Since $\dim(W(\boldsymbol{\pi}^\sigma)) < \infty$, it follows from Theorem 1(i) that $W(\boldsymbol{\pi}^\sigma)_{\mathfrak{g}_{\mathbf{a},N}}$ is a quotient of $W(\tilde{\boldsymbol{\pi}})$ for some $\tilde{\boldsymbol{\pi}} \in \mathcal{P}^+$. Since the module $W(\tilde{\boldsymbol{\pi}})$ has an unique irreducible quotient $V(\tilde{\boldsymbol{\pi}})$, part (iii) follows if we prove that

$$V(\boldsymbol{\pi}) \cong_{L(\mathfrak{g})} V(\boldsymbol{\pi}^\sigma)_{\mathfrak{g}_{\mathbf{a},N}}.$$

But this follows from part (i) and (4.4) and part (iii) is now proved. □

4.6. The next proposition proves part (ii) of Theorem 2.

Proposition. *Let $\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k} \in \mathcal{P}^+_\sigma$ and assume that $\mathbf{a}$ and $\mathbf{a}^m$ have distinct coordinates. As $L^\sigma(\mathfrak{g})$–modules, we have*

$$W(\boldsymbol{\pi}^\sigma) \cong \bigotimes_{k=1}^{\ell} W\left(\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k}\right).$$

Proof. For $1 \le k \le \ell$, set

$$\boldsymbol{\pi}^\sigma_k = \prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k}.$$

It is checked easily that the element $\otimes_{k=1}^{\ell} w_{\boldsymbol{\pi}^\sigma_k}$ satisfies the defining relations of $w_{\boldsymbol{\pi}^\sigma}$ and hence we have an $L^\sigma(\mathfrak{g})$–module map,

$$\eta : W(\boldsymbol{\pi}^\sigma) \to \bigotimes_{k=1}^{\ell} W(\boldsymbol{\pi}^\sigma_k).$$

The proposition follows if we prove that this map is surjective. For then, taking $\boldsymbol{\pi} \in \mathbf{i}(\boldsymbol{\pi}^\sigma)$ and $\boldsymbol{\pi}_k \in \mathbf{i}(\boldsymbol{\pi}^\sigma_k)$, we have

$$\dim W(\boldsymbol{\pi}) = \dim W(\boldsymbol{\pi}^\sigma) \ge \prod_{k=1}^{\ell}\dim W(\boldsymbol{\pi}^\sigma_k) = \prod_{k=1}^{\ell}\dim W(\boldsymbol{\pi}_k) = \dim W(\boldsymbol{\pi}),$$

where the first equality uses part (i) of Theorem 2 and the last equality follows from Theorem 1(iii). To prove that η is surjective choose $N >> 0$ so that $W(\boldsymbol{\pi}^\sigma)$ is a module for $\mathfrak{g}_{\mathbf{a}^m,N}$ and also so that for all $1 \le k \le \ell$ the algebra $\mathfrak{g}^\sigma_{\mathbf{a}^m_k,N}$ acts on $W(\boldsymbol{\pi}^\sigma_k)$ where $\mathbf{a}^m_k = \{a^m_k\}$ and we have

$$W(\boldsymbol{\pi}^\sigma_k) = \mathbf{U}(\mathfrak{g}^\sigma_{\mathbf{a}^m_k,N})w_{\boldsymbol{\pi}^\sigma_k}.$$

On the other hand by Proposition 4.3 we have

$$\mathfrak{g}^\sigma_{\mathbf{a}^m,N} \cong \bigoplus_{k=1}^{\ell}\mathfrak{g}^\sigma_{\mathbf{a}^m_k,N},$$

and hence $\otimes_{k=1}^{\ell} W(\boldsymbol{\pi}^\sigma_k)$ is cyclic for $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$, i.e

$$\mathbf{U}(\mathfrak{g}^\sigma_{\mathbf{a}^m,N})(\otimes_{k=1}^{\ell} w_{\boldsymbol{\pi}^\sigma_k}) = \bigotimes_{k=1}^{\ell} W(\boldsymbol{\pi}^\sigma_k).$$

This proves that η is a surjective map of $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$–modules and the proof of the proposition is complete.

□

4.7. We now prove (iii) of Theorem 2. Recall that in Section 2.1, we have identified elements of P_σ^+ with elements of P^+ and hence for each $a \in \mathbf{C}^\times$ and $\lambda \in P_\sigma^+$ we have elements $\boldsymbol{\pi}_{\lambda,a} \in \mathcal{P}^+$ and $\boldsymbol{\pi}^\sigma_{\lambda,a} \in \mathcal{P}_\sigma^+$. Moreover, $\boldsymbol{\pi}_{\lambda,a} \in \mathbf{i}(\boldsymbol{\pi}^\sigma_{\lambda,a})$.

Proof. Choose $b_\epsilon \in \mathbf{C}^\times$, $0 \le \epsilon \le m-1$ such that

$$b_r \neq b_s, \quad b_r^m \neq b_s^m, \quad r \neq s.$$

Using Lemma 3.7, Theorem 2(ii) and Theorem 2(i) in that order gives,

$$\bigotimes_{\epsilon=0}^{m-1} W(\boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}) \cong_{\mathfrak{g}_0} \bigotimes_{\epsilon=0}^{m-1} W(\boldsymbol{\pi}^\sigma_{\lambda_\epsilon,b_\epsilon}) \cong_{L^\sigma(\mathfrak{g})} W(\prod \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,b_\epsilon}).$$

Since $\lambda_\epsilon \in \mathcal{P}_\sigma^+$, we have $\prod \boldsymbol{\pi}_{\sigma^\epsilon(\lambda_\epsilon),\zeta^\epsilon b_\epsilon} \in \mathbf{i}(\prod \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,b_\epsilon})$ and so by Theorem 2(i) we get,

$$W(\prod \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,b_\epsilon}) \cong_{L^\sigma(\mathfrak{g})} W(\prod \boldsymbol{\pi}_{\sigma^\epsilon(\lambda_\epsilon),\zeta^\epsilon b_\epsilon}).$$

Theorem 1 gives,

$$W(\prod \boldsymbol{\pi}_{\sigma^\epsilon(\lambda_\epsilon),\zeta^\epsilon b_\epsilon}) \cong_{\mathfrak{g}} W(\prod \boldsymbol{\pi}_{\sigma^\epsilon(\lambda_\epsilon),1}) \cong_{\mathfrak{g}} W(\boldsymbol{\pi}_{\sum_{\epsilon=0}^{m-1}\sigma^\epsilon(\lambda_\epsilon),a}).$$

And since $\boldsymbol{\pi}_{\sum_{\epsilon=0}^{m-1}\sigma^\epsilon(\lambda_\epsilon),a} \in \mathbf{i}(\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a})$, we get

$$W(\prod \boldsymbol{\pi}_{\sigma^\epsilon(\lambda_\epsilon),\zeta^\epsilon b_\epsilon}) \cong_{\mathfrak{g}_0} W(\boldsymbol{\pi}_{\lambda,a}) \cong_{L^\sigma(\mathfrak{g})} W(\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}),$$

which completes the proof.

□

4.8. We now prove Theorem 2 (iv). By Theorem 2(i), we have

$$W^\sigma(\boldsymbol{\pi}^\sigma_{\lambda,a}) \cong_{L^\sigma(\mathfrak{g})} W(\boldsymbol{\pi}_{\lambda,a}).$$

Theorem 1 gives if $\mathfrak{g}$ not of type A_{2n} that

$$W(\boldsymbol{\pi}_{\lambda,a}) \cong_{\mathfrak{g}} \bigotimes_{i=1}^{n} W(\boldsymbol{\pi}_{\omega_i,1})^{\otimes m_i}$$

and for $\mathfrak{g}$ of type gives A_{2n}

$$W(\boldsymbol{\pi}_{\lambda,a}) \cong_{\mathfrak{g}} W(\boldsymbol{\pi}_{2\omega_n,1})^{\otimes \frac{m_n}{2}} \otimes \bigotimes_{i=1}^{n-1} W(\boldsymbol{\pi}_{\omega_i,1})^{\otimes m_i},$$

which completes the proof.

4.9. We now prove Theorem 2(v). This part of the proof is very similar to the one given in [6] in the untwisted case and we shall only give a sketch of the proof. Thus, let V be an $L^\sigma(\mathfrak{g})$–module, assume that V is finite–dimensional and that it is generated by an element $v \in V$ such that

$$L^\sigma(\mathfrak{n}^+)v = 0, \quad \mathbf{U}(L^\sigma(\mathfrak{h}))v = \mathbf{C}v.$$

Let $\lambda \in P_\sigma^+$ be such that $hv = \lambda(h)v$ for all $h \in \mathfrak{h}_0$. Since V is finite–dimensional it follows from the representation theory of the subalgebras $\{x_{i,0}^\pm, h_{i,0}\}$, $i \in I_0$ that $\lambda \in P_\sigma^+$ and also that

$$(x_{i,0}^-)^s = 0, \;\; i \in I_0, \;\; s \in \mathbf{Z}_+, \;\; s \geq \lambda(h_i) + 1 \tag{4.5}$$

Moreover if $\mathfrak{g}$ is of type A_{2n}, we find by working with the subalgebra $\left\{\frac{1}{2}h_{n,0}, y_{n,1}^\pm \otimes t^{\mp 1}\right\}$ that

$$(y_{n,1}^- \otimes t)^s v = 0, \quad s \in \mathbf{Z}_+, \quad s \geq \frac{1}{2}\lambda(h_{n,0}) + 1. \tag{4.6}$$

Applying $(x_{i,0}^+ \otimes t)^s$ to both sides of (4.5), ($i \neq n$ if $\mathfrak{g}$ of type A_{2n}) we find by using Lemma 3.3 (i), (ii), that

$$\left(\mathbf{p}_{i,\sigma}^+(u)\right)_s = 0, \quad s > \lambda(h_{i,0}),$$

while if $\mathfrak{g}$ is of type A_{2n}, we apply $(x_{n,0}^+)^{2s}$ to both sides of (4.6) and using Lemma 3.3(iii), we find

$$\left(\mathbf{p}_{n,\sigma}^+(u)\right)_k = 0, \quad k > \lambda(\frac{1}{2}h_{n,0}).$$

Set

$$\pi_i^\sigma(u) = \sum_{k=0}^\infty (\mathbf{p}_i^\sigma(u))_k u^k,$$

and let $\boldsymbol{\pi}^\sigma = (\pi_i^\sigma)_{i \in I_0}$. The preceding arguments show that $\boldsymbol{\pi}^\sigma$ is an I_0–tuple of polynomials. We claim that

$$\lambda = \lambda_{\boldsymbol{\pi}}, \qquad \mathbf{p}_{i,\sigma}^-(u)v = (\pi_i^\sigma(u))^- v, \tag{4.7}$$

which now shows that V is a quotient of $W(\boldsymbol{\pi}^\sigma)$. To prove that $\lambda = \lambda_{\boldsymbol{\pi}}$ is equivalent to proving that

$$\left(\mathbf{p}_{i,\sigma}^+\right)_{\lambda(h_i)} v \neq 0, \tag{4.8}$$

for all $i \in I$, if if $\mathfrak{g}$ is not of type A_{2n} and for all $i \neq n$ if $\mathfrak{g}$ is of type A_{2n} and if $\mathfrak{g}$ is of type A_{2n}

$$\left(\mathbf{p}_{n,\sigma}^+(u)\right)_{\frac{1}{2}\lambda(h_{n,0})} .v \neq 0. \tag{4.9}$$

It is now easy to see (keeping in mind that $(\mathbf{p}_{i\sigma}(u))_0 = 1$) that the following Lemma implies (4.7).

Lemma. *Let V be a finite–dimensional $L^\sigma(\mathfrak{g})$–module and let $v \in V_\lambda)$ be such that $L^\sigma(\mathfrak{n}^+)v = 0$. For all $i \in I_0$ ($i \neq n$ for $\mathfrak{g}$ of type A_{2n}), we have*

$$(\mathbf{p}_{i,\sigma}^+(u))_{\lambda(h_{i,0})}(\mathbf{p}_{i,\sigma}^-(u))_k.v = (\mathbf{p}_{i,\sigma}^+(u))_{\lambda(h_{i,0})-k}.v, \quad 0 \leq k \leq \lambda(h_{i,0}),$$

and for $\mathfrak{g}$ of type A_{2n}, we have

$$(\mathbf{p}_{n,\sigma}^+(u))_{\frac{1}{2}\lambda(h_{n,0})}(\mathbf{p}_{n,\sigma}^-(u))_k.v = (\mathbf{p}_{n,\sigma}^+(u))_{\frac{1}{2}\lambda(h_{n,0})-k}.v, \quad 0 \leq k \leq \frac{1}{2}\lambda(h_{n,0}).$$

Proof. The proof of the first statement is given in [6, Proposition 1.1] and the key ingredient in that proof is Lemma 3.4 (i). The proof when $i = n$ and $\mathfrak{g}$ of type A_{2n} is entirely similar and one uses Lemma 3.4 (iii)(a) with $r = \frac{1}{2}\lambda(h_{n,0}) + 1$.

□

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Veröffentlichungen

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