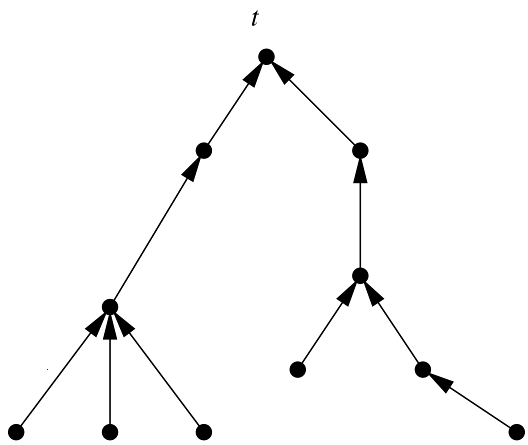

Figure 1.3: T_t, a subnetwork of paths to t. Each radius must dominate the outgoing arc.

for some cheaper edge in T, making MST even cheaper, a contradiction.) So we have

$$r(t) = d(t, s) \geq \max_{e \in T} d(e) \geq \max_{e \in MST} d(e),$$

proving our claim. □

We do not have such a nice lower bound for BROADCAST. E.g., a star with $n-1$ nodes around source s, all edges of weight 1, has a feasible BROADCAST range assignment of constant cost 1. This already partly explains the lack of good (constant factor) approximations for BROADCAST, as good lower bounds are a key ingredient for approximation algorithms. In the next chapter, we will see that BROADCAST is indeed *provably* harder to approximate than the other two range assignment problems.

1.4 Outline

We roughly sketch the outline of this thesis. More detailed summaries and discussions can be found at the end of each chapter.

In chapter 2, we analyze the computational complexity of range assignment problems in very abstract settings. We examine how much (or how little) and what kind of structure the distance function may have such that the addressed range assignment problems remain efficiently solvable, and at what point they become hard to solve or approximate. This is the first thorough treatment of this kind for these problems. This may be due to the fact that the study of wireless network design problems is relatively young and very

Proof. We use a rather generic reduction from the SET COVER problem. Given an instance of SET COVER with elements $\{1,\ldots,n\}$ and subsets $\{S_1,\ldots,S_m\}$, we construct a graph as follows. Let $V = \{u, w_1, \ldots, w_m, v_1, \ldots, v_n\}$ be the set of vertices. As the notation may suggest, node v_i corresponds to element i and node w_j to subset S_j. Accordingly, we call the v_is *element nodes* and the w_js *set nodes*. The set of edges E contains all edges $\{u, w_j\}$ for $j \in \{1,\ldots,m\}$. Furthermore, an edge $\{w_j, v_i\}$ is contained in E iff $i \in S_j$. For an edge $e = \{u, w_j\}$, we define $d(e) = 1$. For edges $e = \{w_j, v_i\}$, we set $d(e) = 2$. See Figure 2.1 for an illustration of this reduction.

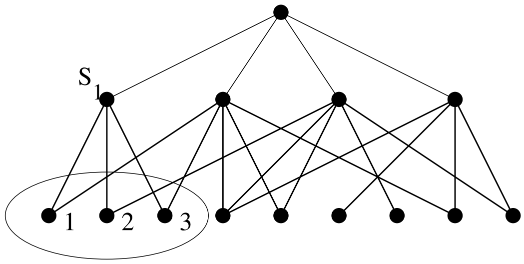

Figure 2.1: An illustration of a reduced SET COVER instance, with $S_1 = \{1,2,3\}$, etc.

In the minimal configuration r_{min}, we have that $r_{min}(u) = r_{min}(w_j) = 1$ for all j, and $r_{min}(v_i) = 2$ for all i.

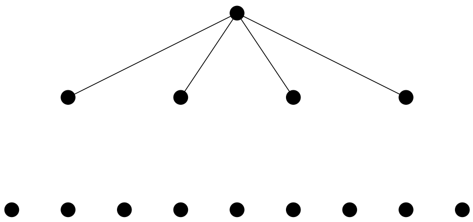

Figure 2.2: The undirected communication graph $G_{r_{min}}$...

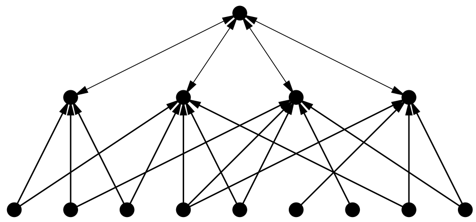

Figure 2.3: ...and the directed communication graph $\vec{G}_{r_{min}}$.

The only way to add more edges to the minimal configuration is to increase $r_{min}(w_j)$, for some of the js, from 1 to 2. (Increasing some radius above 2 would only add to the total costs, but no new edges.) Let r be a feasible range assignment, and $C_r = \{j \mid r(w_j) = 2\}$. An edge from a w_j with $j \in C_r$ to v_i is obviously established iff $i \in S_j$. Note that as the communication graph of r is (strongly) connected, C_r has to constitute a set cover

connected to the rest of the graph. (Recall here that we are in the case that an optimal range assignment includes a range 3 node.)

But now we know the structure of an optimal solution quite well: We have 3-supernodes connected in a clique-wise fashion, and each 3-supernode may have one 2-supernode attached to it. Figure 2.9 shows an example of such an optimal solution.

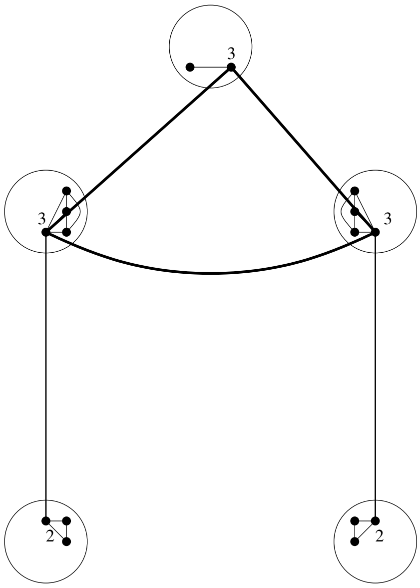

Figure 2.9: An optimal connected range assignment for the above example.

How can we optimize this kind of structure? This question is easy to answer: We want to use as many 2-supernodes as possible, so that we need the least number of 3-supernodes. In other words, we want to maximize the number of distinct pairs of 2- and 3-supernodes, connected by distance 2 edges, the edges of G. This means nothing else than constructing a maximum matching M in G, an efficiently solvable task.[Edm65, KV02] So here, we need additional cost $2|V| - |M|$ to establish connectivity.

Concluding our quite lengthy proof for our short remark, we give an algorithm to construct an optimal range assignment for $(1,2,3)$-CONNECTIVITY in $\hat{G}$.

- **Set** $\hat{r} = \mathbb{1}$.
- Construct a maximum matching M in G.
  Let $U \subseteq V$ be the set of nodes which are not covered by M.
- **If** $(|M| = 1)$,

which is easy:

$$2\sqrt{2}^{\alpha} + \left(\frac{3}{4}\right)^{\alpha} \;>\; 2\sqrt{2}^{\alpha} + \frac{1}{\sqrt{2}^{\alpha}} \;>\; \sqrt{2}^{\alpha} + 2 \;>\; 3.$$

We have now argued that in a minimal solution to CONNECTIVITY, only radii of stations corresponding to nodes of D' are increased from the minimal configuration, and if so, they are increased from $\frac{3}{4}$ to 1. Note that for nodes where this is the case, all incident edges are thereby connected to the backbone and its connected component (cf. fig. 3.8). So in order for the range assignment to be connected, these nodes have to form a vertex cover of D'. On the other hand, by virtue of the backbone, increasing each node in a vertex cover for D' already makes a minimal configuration connected.

Finally, we notice that by Lemma 3.3, a minimal vertex cover for D' is of size $m + k$, which means that a minimal range assignment with CONNECTIVITY has cost $M + \gamma(m + k)$, which concludes this proof.

□

3.3.2 NP-hardness for well-spread instances

Our reduction produces CONNECTIVITY instances where all stations lie on lines of a graph, not really spread-out in the available 2-d space, with many "holes" with no stations at all. It is nevertheless possible to make this construction well-spread, with the following simple trick.

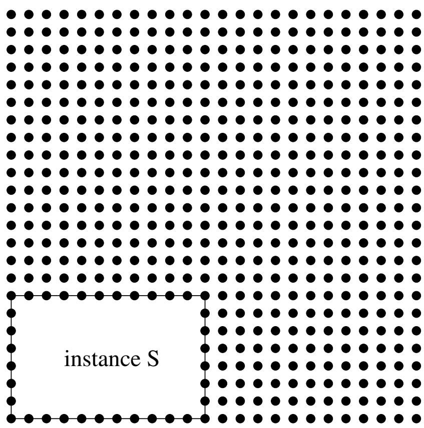

Figure 3.9: Making S well-spread by adding a well-spread grid.

The set of stations $S \subseteq \mathbb{R}^2$ is contained in its 'bounding box' R, the smallest axis-parallel rectangle such that all stations in S lie inside R. Let a be the longer side-length of R, and put a square Q of side-length $2a$ around

- Construct a polynomial orthogonal drawing of $\bar{D}$ in $\mathbb{R}^3$.
- Scale the drawing by factor 3.
- For all polylines representing original edges $e \in E$, erase the first and last open unit interval of the polyline (i.e., do not erase any integer points).
- For all polylines representing backbone edges $e \in \bar{E}$, erase the first (i.e., the interval starting at the tail of e) open interval of length σ/s, where $\sigma \in \mathbb{N}$ is chosen such that $(\sigma - 1)/s < \beta \leq \sigma/s$. Again, erase neither the first nor the last point of this interval.
- Let $\varepsilon < 1 - \beta^\alpha$, and replace all remaining line segments of length 1 (resp. $1 - \sigma/s$) with $s + 1$ (resp. $s - \sigma + 1$) stations along this line at distance $1/s$.

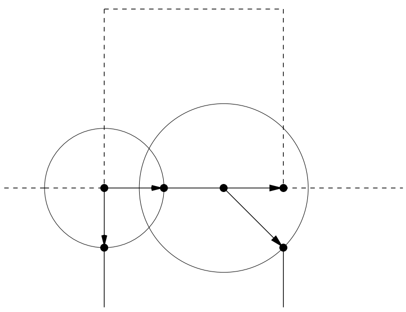

Figure 3.12: A cheaper way to connect edges when $\alpha < 2$.

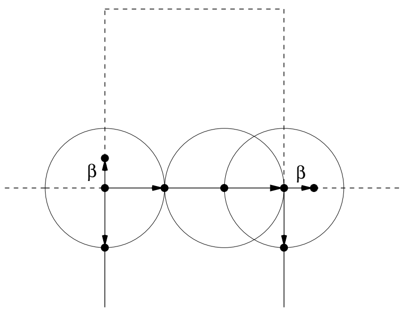

Figure 3.13: By leaving spaces of length β in the backbone, this problem is fixed.

So this time, the minimal configuration consists of one strongly connected component for each original or backbone edge. Note that $0 < \beta < 2 - \sqrt{2} < 0.6$. In order for S to be strongly connected, at least one of the end stations of each backbone edge must have radius at least β; setting all vertex radii to β suffices to make the backbone strongly connected. As it has no advantage not to do so, we assume this is the case.

This means now every vertex station wlog. already has radius β. If we would now choose to transmit data with an edge station, we would pay at least additional cost $\sqrt{2}^\alpha - 1$ instead of paying $1 - \beta^\alpha$ for an incident vertex station, which due to the choice of β is not cheaper (see also fig. 3.13). Hence, we can assume that only vertex stations are used for connectivity, and the vertex stations with radius 1 in a minimal range assignment with Strong