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Testing for Rational Bubbles in Financial Markets

A Comparison of Different Methods and Real World Applications

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The codes and data used are available on my GitHub page at `https://github.com/NicolasRoever/TestingForRationalBubbles`.
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List of Abbreviations and Acronyms

AGTHX Ticker symbol of *The Growth Fund of America, Class A*

CLT Central Limit Theorem

CMT Continuous Mapping Theorem

iid Independent and identically distributed

OLS Ordinary least squares

supB Refers to the statistic given by Equation (35)

supBT Refers to the statistic given by Equation (37)

supDF Refers to the statistic given by Equation (40)

supDFC Refers to the statistic given by Equation (49)

supK Refers to the statistic given by Equation (38)

1 Introduction

The financial crisis of 2008 was one of the most severe financial crises in history by almost any measure (Reinhart and Rogoff, 2009). Countries worldwide experienced severe recessions, banks around the world failed and many companies went bankrupt (Erdem, 2020). Barnichon et al. (2018) estimate a lifetime present-value income loss of 70,000 $ for every US-American as a result of the crisis.
Current research points to the real estate market of the United States as one of the major causes of the financial crisis in 2008 (Shiller, 2020 among others): Real estate prices in the United States surged exponentially due to a rapid increase in the supply of credit in the early 2000s. House prices grew significantly faster than household incomes, investors began to buy real estate in anticipation of selling at a higher price and even households with impaired credit records were able to get mortgage loans (Shiller, 2020). However, in 2006 the boom slowed down and house prices started to decline significantly (Reinhart and Rogoff, 2009).[1] This sharp decline in prices eventually lead to the bankruptcy of Lehmann Brothers in September 2008, one of the largest US investment banks at the time - Lehman Brothers was largely exposed to the real estate market. Their bankruptcy triggered a domino effect within the financial ecosystem resulting in a worldwide financial crisis.
Financial historians have used the term *financial bubble* to characterize the US housing market leading up to the financial crisis in 2008 (Cooper, 2008; Kindleberger and Aliber, 2011; Shiller, 2015). A financial bubble is defined as a surge in prices that is not in line with the intrinsic value of the underlying asset (Brunnermeier, 2008; Stiglitz, 1990). When studying banking crises in general, bubbles in stocks and real estate prices emerge as a common pattern (Ferguson, 2008; Kindleberger and Aliber, 2011). This is due to the fact that macroeconomic research attributes the excessive supply of capital through the supply of credits by banks or cross border capital flows to be the underlying cause of many financial crises (Cooper, 2008; Ferguson, 2008; Kindleberger and Aliber, 2011). Such increase in money supply eventually leads to prices surging to an unsustainably high level not in line with intrinsic values - or in other words to a *financial bubble*.[2]

[1] For a detailed discussion of why the US housing market collapsed beginning in 2006, we refer to Kindleberger and Aliber (2011).
[2] Recent research in the field of behavioural finance also stresses the impact of herd behaviour and prevalent economic narratives on financial bubbles (Shiller, 2020).

Detecting bubbles proves to be difficult: We cannot observe intrinsic asset values as they usually depend on *future* events. The value of a stock for instance depends on future dividend payments. However, Homm and Breitung (2012) present a model framework allowing to test empirically for the existence of bubbles in stock markets by only considering historical time series data of stock prices and dividends. Based on this framework, they suggest various testing procedures. Their approach can be generalized to other asset classes with similar return structures and is therefore highly relevant in two ways:
(1) As an ex-post confirmation it provides evidence that a bubble was indeed present during a specific period in financial history. (2) Valid testing procedures can serve as an ex-ante warning mechanism alerting regulators to dysfunctional market behaviour. When having detected a bubble during its inflationary phase, regulators may prevent financial crises by intervening.
The purpose of this thesis is to elaborate on Homm and Breitung (2012), as their influential paper presents recent econometric theory concerning testing for bubbles in financial markets:
We explain statistical concepts used by Homm and Breitung (2012) and rigorously cover their model framework presenting the concept of a *rational* bubble in stock prices, which is consistent with *rational* market participants. In line with Homm and Breitung (2012) we reason that explosive behaviour in stock prices together with non-explosive behaviour in stock dividends implies the existence of a rational bubble. We motivate the specific testing procedures considered by Homm and Breitung (2012) and discuss their limiting distributions. In our applications, we implement the test statistics using the statistical software package R and provide additional critical values by conducting our own Monte Carlo study. Hereby we find evidence pointing to a rational bubble in the US stock market of the 1990s and the US housing market leading up to the financial crisis in 2008.
We first cover general statistical theory in Chapters 2 and 3. In Chapter 4 we develop the mathematical framework which allows us to test empirically for the existence of bubbles. We review the different testing strategies in Chapter 5 and apply them to the US stock market of the 1990s and the US housing market leading up to the financial crisis in 2008 in Chapter 6. Chapter 7 concludes with discussing the strengths and weaknesses of the presented statistics.

2 Central Limit and Continuous Mapping Theorem

We explain and apply statistical testing procedures to detect rational bubbles in financial markets in the forthcoming chapters. In order to understand the behaviour of those statistics, their limiting distributions are put forward. This requires utilizing two theorems in particular, the Central Limit Theorem and the Continuous Mapping Theorem. Under its conditions, the Central Limit Theorem (CLT) provides the distribution of a sum of individual random variables. As the price of a stock can be considered to be the result of a sum of many individual events, the CLT can be used to derive distributions for the price process of stocks. Since the statistics in Chapter 5 do not consider the price process itself, but transformed versions of the price process (the statistics in Sections 5.3 and 5.4 for instance resemble a variance), we also need the Continuos Mapping Theorem (CMT) for deriving the distributions of the test statistics. It helps to get the distribution for transformed random variables.

2.1 Central Limit Theorem

In order to understand the CLT, we first define *convergence in distribution*. Suppose, a stock pays a dividend which is uncertain and follows a continuous uniform distribution with bounds 0 \$ and 3 \$. The density function is given by

$$f(x) = \begin{cases} \frac{1}{3} & \text{for } 0 \leq x \leq 3, \\ 0 & \text{for } x < 0 \text{ or } x > 3, \end{cases} \quad x \in \mathbb{R}. \tag{1}$$

Let a second stock have an identical dividend distribution which is independent of the first stock. The mean dividend of the two stocks can be described by the following distribution:

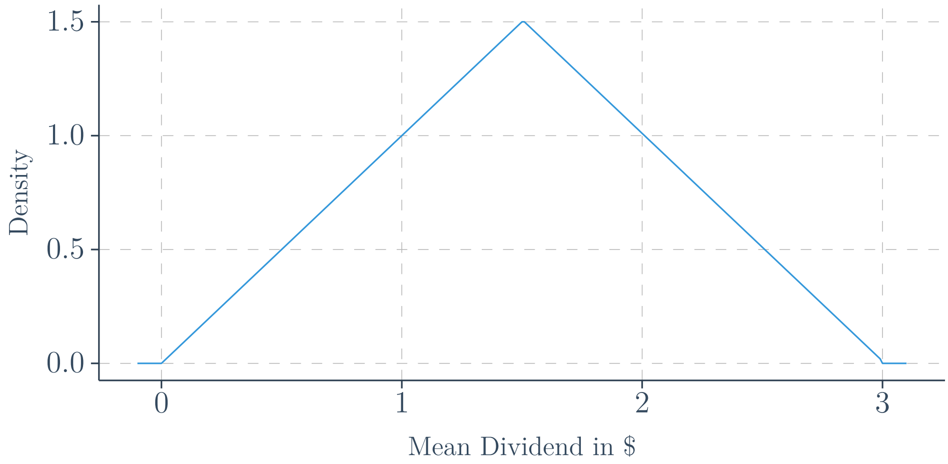

Figure 1: Density Function of the Mean Dividend of Two Independent Stocks. Each stock follows the dividend distribution given by (1).

The density plot illustrates that given density function (1) it is highly unlikely for *both* stocks to pay dividends close to e.g. 3 \$. Values closer to the intervall bounds are thus less likely.
But how does the distribution of the mean dividend develop if independent dividends of more and more stocks are averaged? If the mean dividend approaches a certain distribution, it is said to converge in distribution to a *limiting* distribution for every $x \in \mathbb{R}$ at which the limiting distribution function is continuous. The definition is stated in formal terms following Fuller (2009):

Definition 1. *Let* $\{X_n\}_{n\in\mathbb{N}}$ *be a set of random variables with distribution functions* $\{F_{X_n}(x)\}_{n\in\mathbb{N}}$, $x \in \mathbb{R}$, *with* $F_{X_n} : \mathbb{R} \to [0, 1]$. *If*

$$\lim_{n\to\infty} F_{X_n}(x) \ = F_X(x),$$

at all x *for which* $F_X : \mathbb{R} \to [0, 1]$ *is continuous.* $\{X_n\}_{n\in\mathbb{N}}$ *is said to converge in distribution to the random variable* X*. This is denoted by* $X_n \xrightarrow{d} X$*.*

An example of such convergence is given by the CLT. It states that a sum of independent and identically distributed (iid) random variables does not converge to *any* distribution, but to a normally distributed random variable.
To illustrate this, consider a portfolio of 1,000 stocks each paying a dividend independent of the other stocks and following the disribution given by (1). The mean dividend of this portfolio is obtained by summing up all individual dividends and dividing the result by the number of stocks. As the dividends are modeled to be iid random variables, all conditions of the CLT are met in this

example and the distribution of the mean dividend will converge to a normal distribution - despite every individual dividend being uniformly distributed. The following graph shows a Monte Carlo simulation with 10,000 replications for the mean dividend of this portfolio of 1,000 stocks. It can be seen, that the histogram and the density plot approach the bell shaped curve characteristic for the normal distribution:

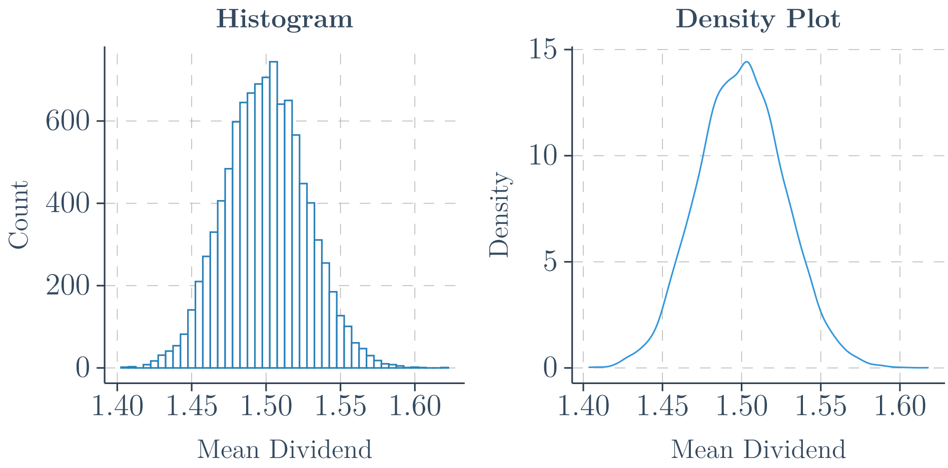

Figure 2: Monte Carlo Simulation for the Mean Dividend of 1,000 Stocks with 10,000 Replications.
In each replication, 1,000 draws are simulated according to (1) and the mean is taken.

But the CLT is more powerful than this example may convey: Even if the distribution of a single random variable is unknown, the distribution of the sum of this variable can be obtained - under the conditions of the following theorem the sum will converge to a normal distribution. The theorem stated here is a weaker version following Bomsdorf et al. (2003):[3]

Theorem 1 (Central Limit Theorem)**.** *Let* $S_n \in \mathbb{R}, n \in \mathbb{N}$ *be a random variable,* $\mu, \sigma \in \mathbb{R}$ *and* $\{X_n\}_{n\in\mathbb{N}} \in \mathbb{R}$ *be a set of iid random variables with mean* μ *and standard deviation* σ*. Define* $S_n := \sum_{i=1}^{n} X_i$*. Then for* $x \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right) = \Phi(x),$$

[3] For convergence to a normally distributed variable under weaker assumptions, see e.g. Lindeberg (1922).

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx.$$

$\Phi : \mathbb{R} \rightarrow [0,1]$ *is called the cumulative distribution function of a normal distribution.*

2.2 Continuous Mapping Theorem

Let us reconsider a portfolio of 1,000 stocks with independent dividends generated by the uniform distribution given in (1). Applying the CLT, the mean dividend of this portfolio can be well approximated to follow a normal distribution. Let a financial derivative have a payoff equal to the *squared* mean dividend of this portfolio. The Continuous Mapping Theorem (CMT), which is only presented for convergence in distribution here, helps to get the approximate distribution for the payoff of such a derivative:

Theorem 2 (Continuous Mapping Theorem)**.** *Let* $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ *be a continuous function,* $X_n, X \in \mathbb{R}^m$*. If* $X_n \xrightarrow{d} X$*, then*

$$f(X_n) \xrightarrow{d} f(X).$$

As x^2 is a continuous function, the CMT can be applied to get the distribution of the squared mean dividend of the portfolio - it approximately follows a squared normal distribution, which is also referred to as a $\chi^2(1)$ distribution in statistical literature.[4] Thus if the distribution of a random variable is known, the distribution for a transformation of this variable can be easily obtained - as long as the transformation is continuous.

3 Time Series Analysis

Building on the theory of Chapter 2, we now introduce mathematical models used to analyze the dynamics of asset prices and dividends. By applying those concepts in Chapter 4, we develop a specific framework which allows to detect patterns indicative of rational bubbles.

We use theory from the field of statistical *time series analysis*, which is concerned with studying data points recorded in time order. The models suggested in this

[4] The distribution is an approximation as convergence to the normal distribution according to the CLT is only exact in the limit.

paper to describe adaquately the time series of prices and dividends are called *autoregressive* processes or short AR-processes. They are very popular with applications ranging from the foraging of wild animals (Bovet and Benhamou, 1988) to financial theory. The AR-process has been controversially used for instance to falsify the efficient market hypothesis advanced especially by Fama (1970), who argues that asset prices generally reflect all relevant information. Section 3.1 introduces AR-processes formally, Sections 3.2 and 3.3 discuss the behaviour of different AR-processes and Sections 3.4 and 3.5 are concerned with the limiting distribution of a particular autoregressive process.

3.1 AR-Processes

An **autoregressive process** specifies that the output variable depends only on a linear combination of past realisations of the variable and a stochastic error term. The specific model is abbreviated as an AR(p)-process. p refers to the order of the process and corresponds to the number of past realizations of the variable included in the model. The error term is **white-noise** e_t, a process of iid random variables with mean zero. An AR(p)-process is therefore defined as follows:

Definition 2. *Let $\sigma_e^2, \phi_i \in \mathbb{R}, i = 1, ..., p,\ p \in \mathbb{N}$ and $e_t \in \mathbb{R}$ be a set of random variables. Then an AR(p)-process $\{Y_t\}_{t\in\mathbb{N}}$, where $Y_t \in \mathbb{R}$ is a set of random variables, satisfies the equation*

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} + e_t,\ \textit{where } e_t \sim (0, \sigma_e^2)\ \forall\ t,\ Cov[e_i, e_j] = 0\ \forall\ i, j \in \mathbb{N}, i \neq j.$$

The specifcation of an AR(1)-model is therefore

$$Y_t = \phi_1 Y_{t-1} + e_t,\ t \in \mathbb{N}. \tag{2}$$

The following plot shows simulations for an AR(1)-model with different parameter values for ϕ_1 and the initial condition $Y_0 = 0$:

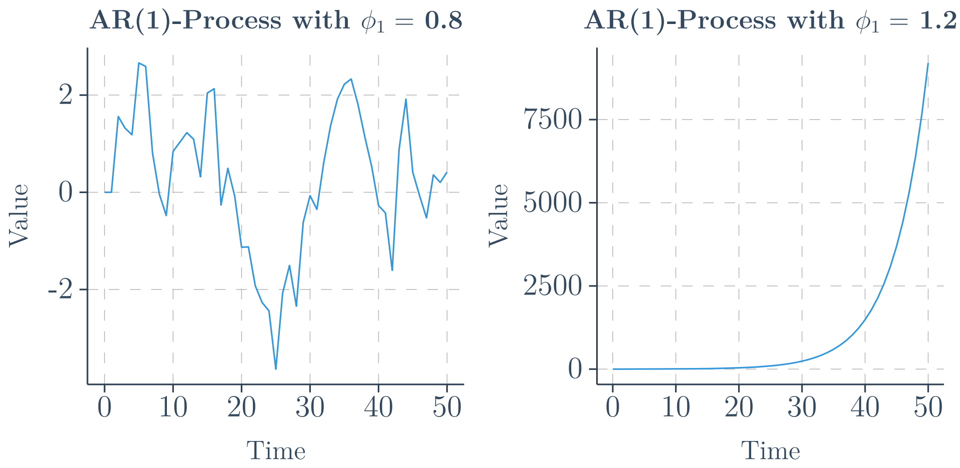

Figure 3: Simulated AR(1)-Processes.
We define the autoregressive parameter as ϕ_1.

Both processes appear to be very different. While the time series on the left maeanders around zero, the time series with $\phi_1 = 1.2$ diverges as t gets larger.

3.2 Stationarity of AR-Processes

In order to understand this varying behaviour of AR(p)-processes the concept of weak stationarity is introduced. It helps to distinguish the two processes from Figure 3. Weak stationarity is defined by Verbeek (2008) as follows:

Definition 3. *Let $\mu, \gamma_i \in \mathbb{R}, i \in \mathbb{N}$ and $Y_t \in \mathbb{R}$ be a set of random variables. A process $\{Y_t\}_{t\in\mathbb{N}}$ is defined to be weakly stationary if for all $t \in \mathbb{N}$ it holds that*

$$\text{(I)} \quad E[Y_t] = \mu,$$

$$\text{(II)} \quad Var[Y_t] = \gamma_0,$$

$$\text{(III)} \quad Cov[Y_t, Y_{t-i}] = \gamma_i.$$

In other words, a stationary process has similar statistical properties no matter which moment in time is considered: The time series has a constant mean and a constant variance; the covariance only depends on the time distance between two observations.
Whether an AR-process is stationary critically depends on the parameter values of ϕ_i - they determine the dynamics of the process. The stationarity restrictions for AR(p)-processes are shown exemplary for an AR(1)-process:

The equation describing an AR(1)-model can be solved through forward iteration using an initial condition $Y_0 \in \mathbb{R}$.[5] This means by substituting $Y_1 = Y_0 + e_1$ and so forth we obtain

$$Y_2 = \phi_1 Y_1 + e_2 = \phi_1(\phi_1 Y_0 + e_1) + e_2 = \phi_1^2 Y_0 + \phi_1 e_1 + e_2. \tag{3}$$

This way, we get the general solution

$$Y_t = \phi_1^t Y_0 + \sum_{i=0}^{t-1} \phi_1^i e_{t-i}. \tag{4}$$

As e_t has mean zero, taking the expected value $E[\cdot]$ of (4) we get

$$E[Y_t] = \phi_1^t Y_0. \tag{5}$$

It follows that Condition (I) of Definition 3 is not met and the process is therefore non-stationary. The mean is dependent on the moment in time as Equation (5) is dependent on t.
However if we consider the limiting value of t and if $|\phi_1| < 1$, the expected value of $\phi_1^t Y_0$ converges to zero. This means that for sufficiently large t, the mean of the time series does not depend on t anymore, formally

$$\lim_{t\to\infty} E[Y_t] = \lim_{t\to\infty} \ \phi_1^t Y_0 = 0, \text{ for } |\phi_1| < 1. \tag{6}$$

Hence there are two conditions that need to be met for the mean of an AR(1)-process to be time-independent: (1) The data-generating process needs to be infinite and (2) the parameter ϕ_1 needs to be less than 1 in absolute value. It can be shown analogously that the variance of the process is constant and the covariance is time-independent if these conditions are met. The resulting process is weakly stationary according to Definition 3.[6]
We summarize this in the following lemma:

[5] This works analogously, if there is no initial condition given.
[6] This theory can be easily extended to AR(p)-processes without an initial condition. This would also yield requirements for the absolute value of the parameters ϕ_1, ϕ_2.... and necessitate an infinite data generating process.

Lemma 1. *An AR(1)-process $\{Y_t\}_{t\in\mathbb{N}}$ satisfiying Definition 2 fulfills the stationarity conditions given in Definition 3, if*

1. $\lim_{t\to\infty} Y_t$ *is considered and*
2. $|\phi_1| < 1$.

The implications of Lemma 1 can be clearly seen in Figure 3. As $|\phi_1|$ is greater than unity, the left time series does not have a constant mean and the simulation grows exponentially, which is why an AR(1)-series with $|\phi_1| > 1$ is referred to as *explosive*. On the contrary, the time series on the left shows a constant variance and maeanders around its constant mean 0.

3.3 Random Walk Model and Order of Integration

A random walk is a particular AR(1)-process. Within the model framework of Chapter 4, identifying a bubble critically depends on rejecting a random walk model for the respective asset price time series. The random walk is defined as an AR(1)-process with the autoregressive parameter set to 1:

Definition 4. *Let σ_e^2, $\phi_1 \in \mathbb{R}$ and $e_t \in \mathbb{R}$ be a set of random variables. A process $\{Y_t\}_{t\in\mathbb{N}}$, where $Y_t \in \mathbb{R}$ is a set of random variables, is defined as a random walk if and only if it satisfies the equation*

$$Y_t = Y_{t-1} + e_t,\ \ where\ e_t\ \sim\ (0, \sigma_e^2)\ \forall\ t,\ Cov[e_i, e_j] = 0\ \forall\ i, j \in \mathbb{N}, i \neq j.$$

The process is non-stationary, as $|\phi_1| \not< 1$. This can be intuitively understood by considering the variance. Because of $\phi_1 = 1$, the impact of every Y_t does not converge to zero over time. Rather, going forward in time, every single observation Y_t adds up. This results in the variance of the process increasing over time, which contradicts the second stationarity condition (II) given by Theorem 3.

However, the process can be rewritten by subtracting Y_{t-1} on both sides of the equation for all $t \in \mathbb{N}$.

$$\begin{aligned} Y_t - Y_{t-1} &= Y_{t-1} - Y_{t-1} + e_t, \\ \Delta Y_t &= e_t. \end{aligned} \tag{7}$$

ΔY_t is called the first difference of Y_t and is defined as $\Delta Y_t := Y_t - Y_{t-1}$. The resulting process is obviously stationary, as e_t is defined as a stationary process. A time series which is stationary after first-differencing is called **integrated of order 1** or short I(1). If a time series is already stationary in levels, it is called I(0). This concept of stationarity through differencing can be extended to an arbitrary number of differences. We could also consider the difference between the first differences of a time series, in other words considering $\Delta(\Delta Y_t) = \Delta(Y_t - Y_{t-1}) = (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2})$. If the resulting series is stationary, the process is *integrated of order 2*.

3.4 The Wiener Process

In order to express the limiting distributions of the statistics in Chapter 5 we need to find the limiting distribution for a random walk process - the null hypothesis to be introduced in Chapter 4 asserts that asset prices follow a random walk. Obtaining those limiting distributions allows us to better understand the behaviour of the test statistics under the null hypothesis.
For expressing the limiting distribution of a random walk, a second stochastic process is introduced called *Wiener Process*. In this section we show that the random walk converges in distribution to the Wiener Process:
Suppose, we buy a stock for 100 \$ in $t = 0$ and we are interested in the price change of the stock after buying it. The development of the price during the following year could be modeled by a *simple symmetric random walk*:

Definition 5. *Let $\{S_t\}_{t\in\mathbb{N}} \in \mathbb{R}$ be a set of random variables and $\{X_t\}_{t\in\mathbb{N}} \in \mathbb{R}$ be a set of iid random variables following the distribution*

$$X_t = \begin{cases} +1 & \text{with } p = 0.5, \\ -1 & \text{with } p = 0.5. \end{cases}$$

S_t is defined as $S_t := \sum_{i=1}^{t} X_i,\ S_0 = 0$. S_t is then called a ***simple symmetric random walk****.*

This model assumes that a new price change is recorded at the end of every month. This change can either be $+1$ \$ or -1 \$. The random variable X_t follows a binomial distribution, which is why the model is called *simple*. As every outcome is equally likely, the random walk is called *symmetric*. Within the model, the price change after twelve months S_{12} is equal to the sum of all price changes during the year, i.e.

$$S_{12} = \sum_{i=1}^{12} X_i. \tag{8}$$

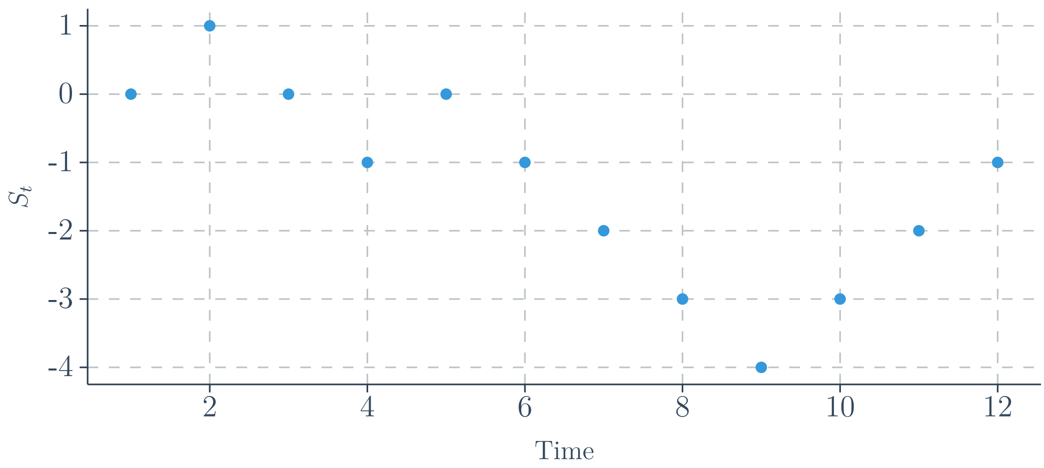

Figure 4: Sample Path of a Simple Symmetric Random Walk for $t = 1, 2, ..., 12$.

This stock price model is *time-discrete*, modeling the stock price change only at the end of every month, i.e. during the first year t is only defined for $t = 1, 2, 3, ..., 12$. We could consider measuring price changes more often such as every week, day or hour. If we model those time intervals to be infinitesimally small, we obtain the limiting distribution of a simple symmetric random walk: First the inital model from (8) is rescaled and standardized to be distributed with mean zero and variance 1 at $t = 12$. As every price change is independent, the variance of the process from $t = 1$ to $t = 12$ is given by

$$Var[S_{12}] = \sum_{i=1}^{12} Var[X_i] = 12 \cdot 1 = 12 \ . \tag{9}$$

We denote the largest integer smaller than x by $\lfloor x \rfloor$. This way, $W^{12}(t)$ is defined for all $t \in [0, 1]$ as

$$W^{12}(t) := \frac{S_{\lfloor 12t \rfloor}}{\sqrt{12}} \ . \tag{10}$$

We can write this model dependent on $N \in \mathbb{N}^+$,

$$W^N(t) := \frac{S_{\lfloor Nt \rfloor}}{\sqrt{N}} \text{ , with } t \in [0, 1]. \tag{11}$$

In the context of our example N corresponds to the number of price changes recorded in the year after the stock acquisition. The following plot visualizes the model given by (11):

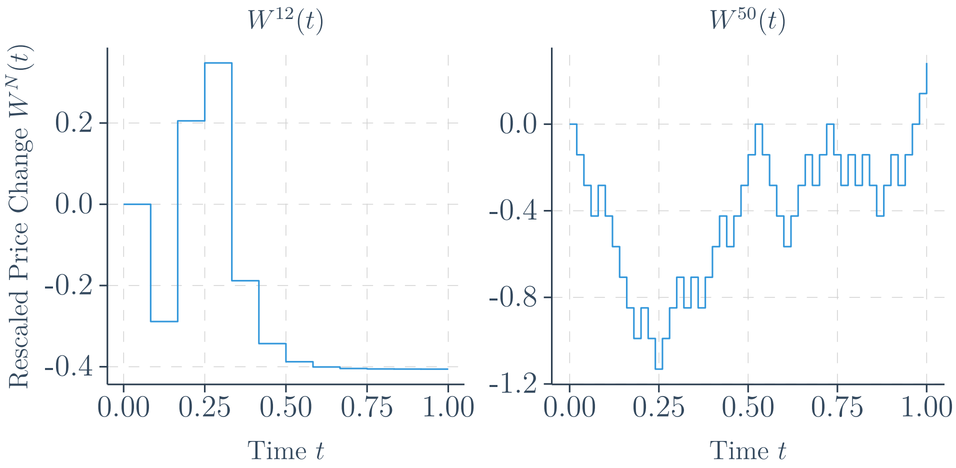

Figure 5: A Simulation of Two Price Change Models $W^N(t)$ with $N = 12$ and $N = 50$.

As the number of time-increments N approaches infinity, the process $W^N(t)$ meets all conditions of the CLT (Theorem 1). This means at $t = 1$ it converges to a standard normal distribution, i.e.

$$\lim_{N\to\infty} W^N(1) \xrightarrow{d} N(0,1). \tag{12}$$

This result can be extended to any point in time $t \in [0,1]$. With N approaching infinity, the distribution at time t converges to a normal distribution with mean 0 and variance t, i.e.

$$W^N(t) = \frac{S_{\lfloor Nt \rfloor}}{\sqrt{N}} = \underbrace{\frac{S_{\lfloor Nt \rfloor}}{\sqrt{\lfloor Nt \rfloor}}}_{\xrightarrow{d} N(0,1)} \underbrace{\frac{\sqrt{\lfloor Nt \rfloor}}{\sqrt{N}}}_{\sqrt{t}} \xrightarrow{d} N(0,t). \tag{13}$$

We now show that model (11) converges in distribution to a specifically defined stochastic process called *Wiener Process*. This way we can express the limiting distributions of a simple symmetric random walk and therefore of the test statistics in Chapter 5 with the help of this thoroughly defined Wiener Process. It is formally introduced by Durrett (2019) as follows:

Definition 6. *A one-dimensional standard Wiener Process is a process* $W : \mathbb{R}_0^+ \to \mathbb{R}$*, meeting the following properties:*

1. $W(0) = 0$*, i.e. the process starts at zero with probability one.*
2. $W(t)$ *has independent increments, i.e.* $W(t_1) - W(t_0), ..., W(t_n) - W(t_{n-1})$ *are independent for* $0 \leq t_0 < t_1 < ... < t_n$*.*
3. *Normal Distribution, i.e. if* $s, t \geq 0$*, then* $W(s+t) - W(t) \sim N(0, s)$*.*
4. $W(t)$ *has continuous paths, i.e.* $W : t \mapsto W(t)$ *is continuous with probability one.*

So at time $t \in \mathbb{R}_0^+$ the standard Wiener Process is normally distributed with mean 0 and variance t, i.e. $W(t) \sim N(0, t)$. Equation (13) shows that the rescaled symmetric random walk model given by (11) converges to exactly this distribution as $\lim_{N\to\infty}$ for $t \in [0, 1]$. Hence for all $t \in [0, 1]$, the rescaled symmetric random walk from Definition 5 converges in distribution to the just introduced standard Wiener Process.
Up until now, convergence of $W^N(t)$ is limited to a specific value of t. So if we are interested in the probability distribution of the price change of the stock after 6 months for instance, this limiting distribution could be obtained by using the Wiener Process from Definition 6.
Donsker's Theorem first introduced in Donsker (1951) extends this convergence in distribution even further. It states that the constructed process $W^N(t)$ does not only converge in distribution at time t, but that the entire function $W^N(t)$ converges in distribution to a standard Wiener Process for $t \in [0, 1]$. Therefore Donsker's theorem is often refered to as the *functional* Central Limit Theorem. This means, also limiting joint distributions of model (11) can be obtained using the standard Wiener Process. For instance we can express the limiting probability distribution that the stock has lost in value 6 months after the acquisition and has gained in value 10 months after the acquisition by using the Wiener Process.

Theorem 3 (Donsker's Theorem)**.** *Let* $\{X_i\}_{i\in\mathbb{N}} \in \mathbb{R}$ *be a set of iid random variables with* $E[X_i] = 0$ *and* $E[X_i^2] = 1$*. Let* $S_j \in \mathbb{R}$ *be a random variable with* $j \in \mathbb{N}^+$*,* $S_j := \sum_{i=1}^{j} X_i$*,* $S_0 = 0$*. Then* $W^N(t) := \frac{S_{\lfloor Nt \rfloor}}{\sqrt{N}}$*,* $N \in \mathbb{N}^+$*, converges in distribution to the standard Wiener Process from Definition 6 for* $t \in [0, 1]$*, i.e.*

$$W^N(t) := \frac{S_{\lfloor Nt \rfloor}}{\sqrt{N}} \xrightarrow{d} W(t) \text{ for } t \in [0, 1]. \tag{14}$$

During this chapter, the very restrictive simple symmetric random walk was shown to converge in distribution to the standard Wiener Process. It should be noted that Donsker's Theorem gives less restrictive conditions for the process $\{X_i\}_{i\in\mathbb{N}}$. Therefore, convergence in distribution to the Wiener Process can be used to express limiting distributions in Chapter 5, although the underlying random walk model may be neither symmetric nor simple.

3.5 Stochastic Calculus

In order to obtain the limiting distributions of the test statistics in Chapter 5, we need to consider integrals of the Wiener Process. Due to its characteristic properties, integrals of the Wiener Process cannot be understood as the limit of a Riemann sum. Differentials involving the Wiener Process also require distinct rules, which is why we need to introduce *stochastic calculus.*
Let us consider a function f dependent on the standard Wiener Process $W(t)$,

$$\begin{aligned} f : [0,1] &\to \mathbb{R}, \\ t &\mapsto (W(t))^2. \end{aligned} \tag{15}$$

We could think of $W(t)$ being a model for the development of a stock price and f describing the payoff of a financial derivative dependent on this stock.

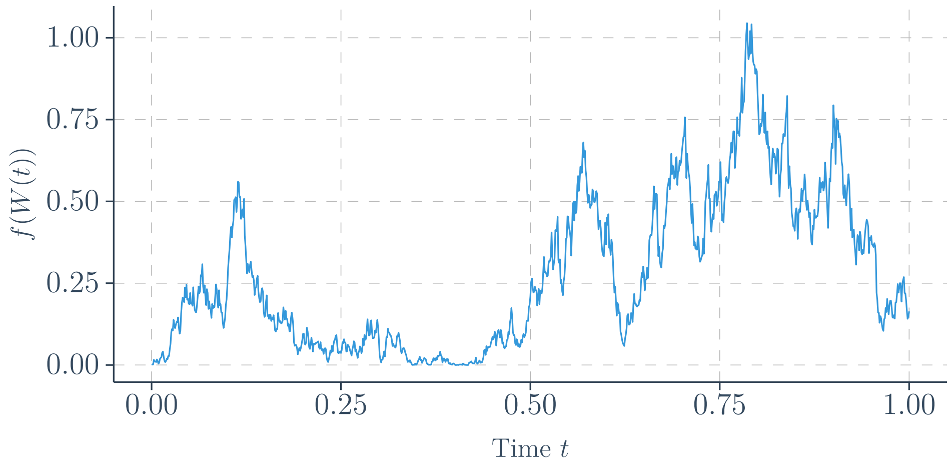

Figure 6: A Simulation of Equation (15).

In order to derive the dynamics of equation (15), we could think of using the chain rule from calculus and heuristically rearrange to write (15) as differential form

$$\frac{df}{dt} = \left(f'(W(t)) \ \frac{dW(t)}{dt} \right),$$
$$df = \left(f'(W(t)) \ \frac{dW(t)}{dt} \right) dt. \tag{16}$$

However, (16) does not make any sense due to the following property of the Wiener Process $W(t)$:

Proposition 1. *A sample path of the standard Wiener Process according to Definition 6 is nowhere differentiable with probability 1, i.e.*

$$\frac{dW(t)}{dt} \text{ does not exist.}$$

This can be intuitively understood by looking at Figure 6. The behaviour of $(W(t))^2$ is very erratic, meaning that it is not possible to find tangents fitting the graph. Hence equation (16) does not have a mathematical meaning within the framework of ordinary calculus. $dW(t)$ cannot be understood as a traditional differential as the standard rules of calculus do not apply.
Thus $dW(t)$ is considered a *stochastic* differential. The Japanese mathematician Kiyoshi Itô introduces a rigorous calculus to deal with stochastic differentials. Itô (1951) provides the following formula for explicitly stating differential equations involving the Wiener Process:

Theorem 4 (Itô's Lemma)**.** *Let $f(t, W(t))$ be a smooth function of two variables, with $f : \mathbb{R}^2 \to \mathbb{R}$, where $W(t)$ denotes a one-dimensional standard Wiener Process according to Definition 6 and $t \in \mathbb{R}_0^+$. Then*

$$df(t, W(t)) = \left(\frac{df}{dt} + \frac{1}{2} \ \frac{d^2 f}{d(W(t))^2} \right) dt + \frac{df}{dW(t)} \ dW(t).$$

This formula allows to quantify the dynamics of the stochastic system in (15) without using undefined expressions. For (15) Itô's Lemma yields

$$d(W(t))^2 = dt + 2W(t)dW(t). \tag{17}$$

Equation (17) is called a *stochastic* differential equation, as it contains the stochastic element $W(t)$.

Let us write (17) by using integrals. Itô defined the integral as being the inverse of differentiation:[7]

Definition 7. *For stochastic differential equations, integration is defined as the inverse of differentiation, i.e.*

$$F(t, W(t)) = \int h(t, W(t))dW(t) + \int g(t, W(t))dt$$

if and only if

$$dF(t, W(t)) = h(t, W(t))dW(t) + g(t, W(t))dt.$$

Rearranging and writing (17) with the use of integrals yields

$$\begin{aligned} \frac{1}{2}\ d(W(t))^2 = W(t)\ dW(t) + \frac{1}{2}\ dt \Leftrightarrow \\ \frac{1}{2}\ (W(T))^2 = \int_0^T W(t)\ dW(t) + \int_0^T \frac{1}{2}\ dt = \int_0^T W(t)\ dW(t) + \frac{T}{2}. \end{aligned} \tag{18}$$

Rearranging (18) gives

$$\int_0^T W(t)\ dW(t) = \frac{1}{2}\ (W(T))^2 - \frac{T}{2}. \tag{19}$$

Result (19) is in contrast to the Rieman integral

$$\int_0^T x\ dx = \frac{1}{2}\ T^2. \tag{20}$$

This illustrates that calculus involving the Wiener Process cannot be done in the traditional way due to its characteristic properties, i.e. the erratic behaviour of $W(t)$. Stochastic calculus provides a framework which allows to use differential equations and integrals nonetheless. When deriving distributions of the test statistics in Chapter 5, calculations involving the Wiener Process need to be done following the distinct rules of stochastic calculus.

[7] See Kempthorne et al. (2013). Itô also defined the integral to have a meaningful interpretation by adapting the concept of Riemann sums to fit the distinct properties of the Wiener Process.

4 The Time Series of Stock Prices and Dividends

As Chapters 2 and 3 covered general statistical methods for anaylzing time series data, we now motivate a specific model for the dynamics of stock prices and dividends and formally introduce the concept of a rational bubble. This theory allows to test empirically for the existence of rational bubbles. While we present the theory by considering stock prices, it can be easily extended to other asset classes with a similar return structure.

4.1 The Price of a Stock

We model the price of a stock following Homm and Breitung (2012) and Campbell et al. (1997):
If a stock is held for one period and sold after the dividend payment, the return R_{t+1} is given by

$$R_{t+1} := \frac{P_{t+1} + D_{t+1}}{P_t} - 1, \tag{21}$$

where t is a time index, P_t is the stock price and D_t is the dividend.

4.1.1 Assumptions

In order to get the price of the stock P_t, we need to rearrange (21). We make three assumptions:

Assumption 1. *An arbitrage-free market:*
Arbitrage describes a situation where a risk-free profit can be made. For instance this is the case when the same stock is priced differently at two stock exchanges. Then we could buy the stock at the cheaper price at one stock exchange and sell it at the other exchange profiting from the price difference.

Assumption 1 is a standard assumption in many economic pricing models (Albrecher et al., 2011) such as Black and Scholes (1973), who develop an influential model used for pricing particular financial derivatives. It seems plausible that opportunities for making risk-free profits in modern competitive financial market are negligible - especially since the technologies of the information age fosters transparency and flow of information. A consequence of the no-arbitrage assumption is often referred to as *the law of one price* - the same asset can only have **one price** at any time t. If Assumption 1 did not hold, we could not specify a price for an asset, as every asset could have multiple prices.

Assumption 2. *All investors are risk-neutral:*
Prices and dividends in future periods are volatile and therefore associated with risk. It is assumed that investors do not need extra compensation for taking on this risk.

Empirical findings within the field of economics show investors to be risk averse (Chiappori and Paiella, 2011; Paravisini et al., 2017 among others). Therefore the return of an asset must depend on its volatility. However our testing strategy is not based on estimating rates of return correctly, but on comparing the dynamics of dividends and prices. The specific rate of return itself is not needed. Hence we can assume risk premia to be zero and simplify our notation. This way we do not need to distinguish rates of return between different assets.

Assumption 3. *A constant expected return:*
We assume $E[R_t] = R \; \forall \; t$, *where* $E[\cdot]$ *is the expected value.*

Assumption 3 allows us to develop a model which we can test empirically. In the short run, returns can of course vary by large. However, we consider a long time horizon as stocks are expected to pay dividends for *infinity*. Assuming a constant return in this long run can be considered plausible: Jordà et al. (2019) examine historical rates of return from 1870 to 2015 and find stable returns for risky assets. When valuating a company, analysts usually estimate the future long term rate of return to be constant and in line with the general growth rate of the economy (Ballwieser and Hachmeister, 2016).

4.1.2 The Stock Price Model

The three assumptions lead to the price of a stock at time t

$$P_t = \frac{E_t[P_{t+1} + D_{t+1}]}{1 + R}. \tag{22}$$

The expected value at time t is denoted by $E_t[\cdot]$. Equation (22) can be solved by employing the law of iterated expectations and forward iteration substituting $P_{t+1} = \frac{E_{t+1}[P_{t+2}+D_{t+2}]}{1+R}$ and so forth, yielding what we refer to as the *fundamental* *stock price*

$$P_t^f := \sum_{i=1}^{\infty} \frac{1}{(1+R)^i} \; E_t[D_{t+i}]. \tag{23}$$

According to (23), P_t^f only depends on the development of the dividends - or in other words only on economic *fundamentals*.
However, we can find alternating solutions for (22). We consider a process $\{B_t\}_{t=1}^{\infty}$ with

$$E_t[B_{t+1}] = (1+R)B_t. \tag{24}$$

If we add B_t to the solution (23), we also solve (22). This can be shown by substituting $P_t = P_t^f + B_t$ in Equation (22) and using property (24), which leads to a true statement.

$$P_t^f + B_t = \frac{E_t[(P_{t+1}^f + B_{t+1}) + D_{t+1}]}{1+R},$$

$$P_t^f + B_t = \frac{E_t[P_{t+1}^f + D_{t+1}]}{1+R} + \frac{(1+R)B_t}{1+R},$$

$$P_t^f + B_t = \underbrace{P_t^f}_{\text{fundamental stock price}} + \underbrace{B_t}_{\text{bubble component}} = P_t. \tag{25}$$

There is an infinite number of solutions to (22) all taking the form of (25), which decomposes the stock price into a fundamental component and a bubble component. As long as $B_t \neq 0$, a rational bubble is present - which is defined here as a deviation from the *fundamental stock price* P_t^f.
Considering (24), the bubble component is expected to grow at rate $1+R$. Therefore even a rational investor is willing to spend more than the fundamental value on a stock, since he is compensated sufficiently well in two ways: (1) The fundamental stock price P_t^f pays off in the form of future dividends. (2) The investment in the bubble component also gives an expected return of R in the form of capital gains, as it is expected to grow at rate $1+R$ (cf. (24)). If enough market participants expect (24), they will buy shares as they expect sufficient return and the price will indeed grow - even if it cannot be justified by market fundamentals.
There is no irrational behaviour within this model. This is why we call a price development including $B_t \neq 0$ in line with equation (24) and (25) a *rational* bubble.

4.2 A Time Series Model for Dividends and Prices

Within our testing strategy, we try to identify any $B_t \neq 0$ present in the price process. However we can not observe the fundamental stock price P_t^f as it depends on *future* dividend payments. By making assumptions about the time series of P_t^f, we can test for the existence of a bubble component B_t by considering only the historical time series of dividends D_t and prices P_t - which we can both observe.

Following Homm and Breitung (2012), dividends are assumed to follow a random walk with drift, i.e. there is a constant μ added to the random walk model in Definition 4.

$$D_t = \mu + D_{t-1} + e_t, \tag{26}$$

where D_t is the dividend, t indexes the time and e_t describes a white noise process, i.e. an iid random variable with mean zero.

This model assumes dividends to be stationary in first differences, which is consistent with the literature on rational bubbles (Diba and Grossman, 1988; Li et al., 2019; Phillips et al., 2011 among others) and in line with the more general literature on testing for the presence of an autoregressive parameter with value 1 in other macroeconomic time series (cf. Perron, 1988). The intuitive motivation behind this is that macroeconomic variables such as aggregate dividends of a stock index tend not to change rapidly in absence of a significant macroeconomic shock and can therefore be modelled well by a random walk with drift.

If we substitute (26) in Equation (23) and simplify, we get an equation describing the time series of P_t^f,

$$P_t^f = \sum_{i=1}^{\infty} \frac{1}{(1+R)^i} \; E_t[D_{t+i}] = \sum_{i=1}^{\infty} \frac{1}{(1+R)^i} E_t[\mu + D_{t+i-1} + e_t]. \tag{27}$$

In order to express D_{t+i-1}, we use iteration (see Chapter 3, Equation (3)) and obtain

$$P_t^f = \sum_{i=1}^{\infty} \frac{1}{(1+R)^i}(i\mu + D_t) = \sum_{i=1}^{\infty} \frac{i\mu}{(1+R)^i} + \sum_{i=1}^{\infty} \frac{D_t}{(1+R)^i} \; . \tag{28}$$

As $\frac{i\mu}{(1+R)^i} = 0$ for $i = 0$, we can start this sum at 0 and use formulas for simplifying a geometric series.

$$\begin{aligned} P_t^f &= \mu \sum_{i=0}^{\infty} \frac{i}{(1+R)^i} + \sum_{i=1}^{\infty} \frac{D_t}{(1+R)^i} \\ &= \mu \frac{\frac{1}{1+R}}{(\frac{1}{1+R}-1)^2} + \frac{1}{R} D_t \\ &= \mu \frac{1+R}{R^2} + \frac{1}{R} D_t \ . \end{aligned} \tag{29}$$

As D_t is by assumption a random walk with drift, (29) shows that P_t^f also follows a random walk with drift, since it is a linear transformation of D_t for all t.

This is in contrast to the *bubble component* of the general solution (25). The bubble component is characterised by an explosive process, as the coefficient for B_t is given by $(1 + R)$ which is greater than unity (see Equation (24)).

The following plot shows simulations for the time series of P_t^f (according to (29)) and B_t (according to (24)):

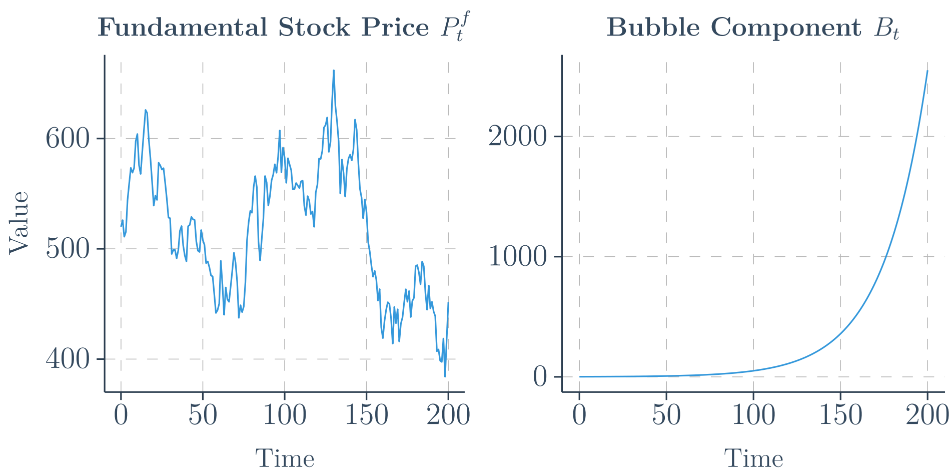

Figure 7: Simulation of the Time Series Model for the Fundamental Stock Price and the Bubble Component.
$D_0 = 5, \ R = 0.05, \ \mu = 0.01, \ B_0 = 1, \ e_t \sim N(0, 1)$.

As (25) states, the stock price P_t is given by the sum $P_t^f + B_t$. If there is no bubble present, the stock price can be modeled adequately by a random walk with drift as it only includes the fundamental stock price P_t^f. However if a bubble component is present in the stock price, the price process P_t will show explosive behaviour:

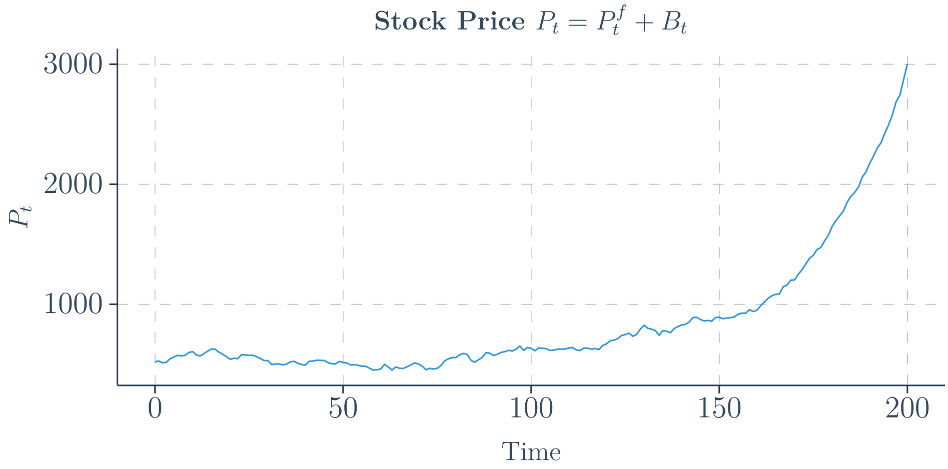

Figure 8: Plot of P_t According to (25).
P_t^f and B_t are simulated as in Figure 7, i.e. $D_0 = 5,\ R = 0.05,\ \mu = 0.01,\ B_0 = 1,\ e_t \sim N(0, 1)$. We clearly see explosive behaviour in the time series of P_t as the bubble component starts to singinificantly influence the price process.

Figure 8 sums up our identification strategy: We try to detect a bubble component $B_t \neq 0$ by testing for explosive behaviour in the time series of stock prices and dividends. If dividends follow a random walk with drift and our model for P_t^f is correct, explosive behaviour in P_t has to be caused by an existing bubble component.

5 Testing for Rational Bubbles

Based on the model of Chapter 4, we now introduce test statistcs to detect rational bubbles. We test for explosive behaviour in the time series of stock prices and dividends and identify a bubble if we can reject a random walk model for prices and cannot reject a random walk model for dividends.
Section 5.1 covers a method for detrending a time series, which we use to simplify the test statistics. Section 5.2 formalizes the test hypotheses and sections 5.3 to 5.7 put forward five different test statistics.

5.1 Ordinary Least Squares Detrending

Before calculating test statistics we detrend the time series. This way, we remove a possible constant (the detrended time series has mean zero) and a

possible linear trend in the time series of prices and dividends. After detrending we can test a simpler model and use less complicated test statistics.
We detrend by using ordinary least squares (OLS) and regress on a constant and a linear time trend. The regression model is therefore given by

$$X_t = \beta_0 + \beta_1 t + u_t, \tag{30}$$

where X_t is the respective value of the time series, t is a time index, β_0 and β_1 are the coefficients of the regression and u_t is the residual. The tests are then computed by using the residuals $\hat{u}_t$ from an OLS-estimation of (30). We denote those residuals by Y_t in the following chapters.

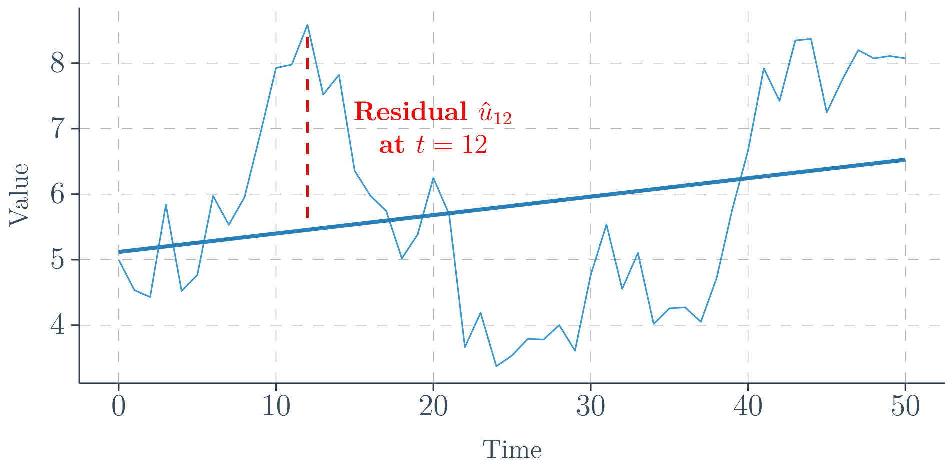

Figure 9: Detrending Using OLS.
An AR(1)-process is simulated and the OLS-model is illustrated by the straight blue line.

5.2 Test Hypotheses

Detrending removes a possible linear trend in stock prices and dividends (which we would model by including the drift parameter μ) and a possible constant. Therefore we can test the simple AR(1)-model

$$Y_t = \rho_t Y_{t-1} + e_t. \tag{31}$$

t is a time index, Y_t is the estimated residual $\hat{u}_t$ of the regression from Section 5.1, ρ_t is the autoregressive parameter and e_t is a stochastic white noise error term, i.e. an iid random variable with mean zero. The null hypothesis of the tests asserts a random walk for the time series of prices and dividends for all t. If this holds true for the time series of stock prices, no significant bubble component is present in the price process:

$$H_0 : \rho_t = 1 \text{ for } t = 1, 2, 3, ..., T \ , \tag{32}$$

where T is the last time index of the time series.
Under the alternative hypothesis there is a structural change in the autoregressive parameter: If a bubble component starts to significantly influence the price process, the time series of prices changes will change from from I(1) to explosive. We denote this *moment of change* by t^*.

$$H_1 : \rho_t = 1 \text{ for } t = 1, 2, 3, ..., t^* \text{ and } \rho_t > 1 \text{ for } t = t^* + 1, ..., T \ . \tag{33}$$

The time series under the alternative hypothesis is visualized in the following plot:

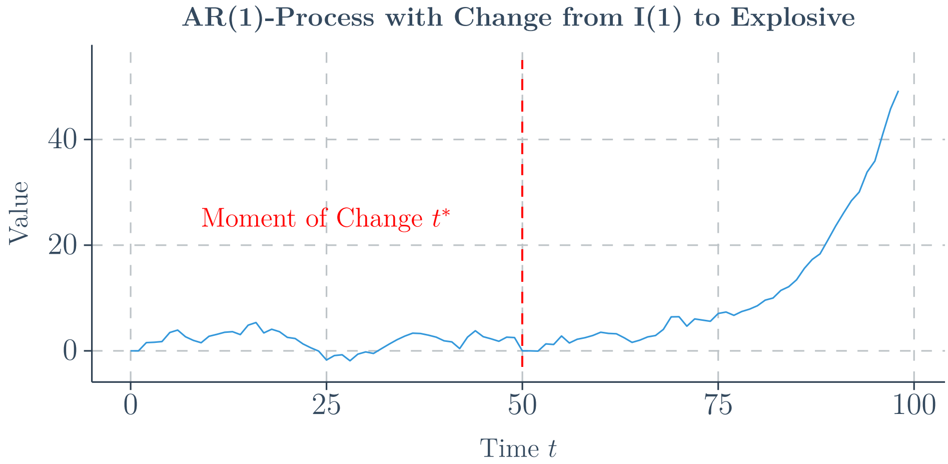

Figure 10: Simulation of an AR(1)-Process with a Change in the Autoregressive Parameter from 1 to 1.1 at $t = 50$.
The value at $t = 0$ is set to 0, $e_t \sim N(0, 1)$.

If we can reject H_0 for the time series of prices and there is no explosive behaviour in dividends (in Chapter 4 we assumed that dividends follow a random walk with drift), we conclude that a rational bubble is present in the price process.

5.3 Bhargava Statistic

Bhargava (1986) proposes a test statistic, which is adapted by Homm and Breitung (2012) to fit the hypotheses from section 5.2. Let

$$s_\tau^2 := \frac{1}{T - \lfloor \tau T \rfloor} \sum_{t=\lfloor \tau T \rfloor + 1}^{T} (Y_t - Y_{t-1})^2, \tag{34}$$

with $\tau \in [\tau_0, 1 - \tau_0]$ and $\tau_0 \in (0, 0.5)$. As the moment of change from a random walk to explosive t^* is not known beforehand, the statistic is recursively calculated for subsamples of the time series and the supremum is taken. The size of the smallest subsample is determined by the choice of τ_0. The statistic is given by

$$supB(\tau_0) := \sup_{\tau \in [\tau_0, 1-\tau_0]} B\tau, \ B_\tau := \frac{1}{s_\tau^2 (T - \lfloor \tau T \rfloor)^2} \sum_{t=\lfloor \tau T \rfloor + 1}^{T} \underbrace{(Y_t - Y_{\lfloor \tau T \rfloor})^2}_{\text{forecast error for period t}} . \tag{35}$$

Equation (35) can be motivated by the following considerations: Recall that a random walk has no clear tendency to move in any direction. For a random walk, the expected value in period t for all further periods is Y_t. Hence the present value is a reasonable forecast. The statistic sums up all squared errors of this forecast Y_t and divides the result by the variance of the process. If the process of Y_t can be modeled by random walk, the forecast errors are small and so is the value of the test statistic. However if the process is explosive, Y_t will grow exponentially and the forecast error will become very large.

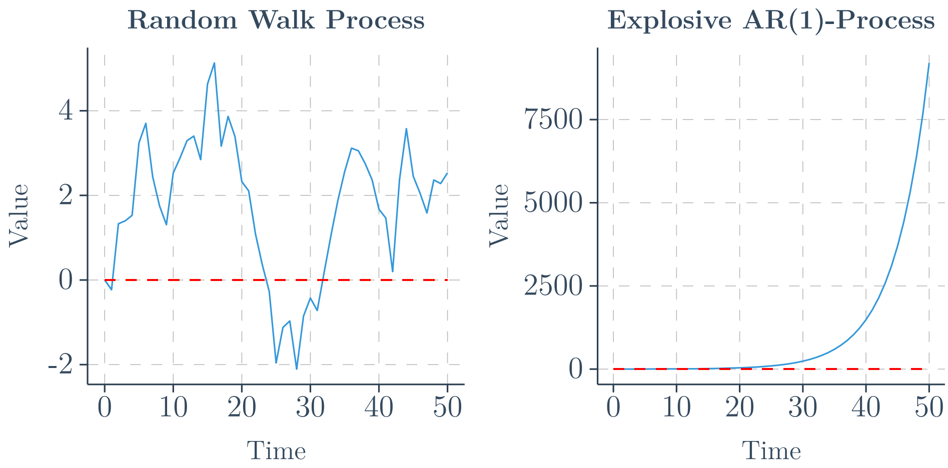

Figure 11: Intuition Behind the Bhargava Statistic.
In the left plot, forecasting the present value in $t = 0$ (red line) proves to be reasonable for a random walk process. In the right plot however, the forecast is very much off. Therefore the test statistic will have a large value and reject the null hypothesis.

The limiting distribution of the test statistic can be expressed using a Wiener Process:

Proposition 2. *If Y_t is generated by the random walk model from* (31)*, the limiting distribution of the test statistic in* (35) *under the null hypothesis* (32) *is given by*

$$supB(\tau_0) \xrightarrow{d} \sup_{\tau \in [\tau_0, 1-\tau_0]} \left\{ \frac{1}{(1-\tau)^2} \int_0^{1-\tau} (W(r))^2 dr \right\}. \tag{36}$$

Proof. First the statistic is simplified using the binomial formula:

$$B_\tau = \frac{1}{s_\tau^2 (T - \lfloor \tau T \rfloor)^2} \sum_{t=\lfloor \tau T \rfloor + 1}^{T} \left(Y_t^2 - 2Y_t Y_{\lfloor \tau T \rfloor} + Y_{\lfloor \tau T \rfloor}^2 \right).$$

Secondly, we rearrange $\frac{1}{s_\tau^2 (T - \lfloor \tau T \rfloor)^2}$ and factorize $(1-\tau)^{-2}$.

$$\begin{aligned} B_\tau &= \sum_{t=\lfloor \tau T \rfloor + 1}^{T} \left[\left(\frac{Y_t}{s_\tau \sqrt{T - \lfloor \tau T \rfloor}} \right)^2 (T - \lfloor \tau T \rfloor)^{-1} - \frac{2 Y_t Y_{\lfloor \tau T \rfloor}}{s_\tau^2 (T - \lfloor \tau T \rfloor)^2} + \frac{Y_{\lfloor \tau T \rfloor}^2}{s_\tau^2 (T - \lfloor \tau T \rfloor)^2} \right] \\ &= (1-\tau)^{-2} \sum_{t=\lfloor \tau T \rfloor + 1}^{T} \left[\left(\frac{Y_t}{s_\tau \sqrt{T}} \right)^2 T^{-1} - 2 Y_{\lfloor \tau T \rfloor} s_\tau T^{-0.5} \frac{Y_t}{s_\tau \sqrt{T}} T^{-1} + \frac{Y_{\lfloor \tau T \rfloor}^2}{s_\tau^2 T^2} \right]. \end{aligned}$$

Under the null hypothesis Y_t follows a random walk, i.e. $Y_t = Y_{t-1} + e_t$, which can be solved by iteration (see Chapter 3): Y_t solves to be the sum of all previous error terms e_t. We define S_t as the sum of all error terms up until time t, i.e. $S_t := \sum_{i=1}^{t} e_i$ with $S_0 = 0$. Hence we can substitute S_t for Y_t.

$$B_\tau = (1-\tau)^{-2} \sum_{t=\lfloor \tau T \rfloor+1}^{T} \Big[\Big(\frac{S_t}{s_\tau \sqrt{T}}\Big)^2 T^{-1} - 2Y_{\lfloor \tau T \rfloor} s_\tau T^{-0.5} \frac{S_t}{s_\tau^2 \sqrt{T}} T^{-1} + \frac{Y_{\lfloor \tau T \rfloor}^2}{s_\tau^2 T^2}\Big].$$

The next step involves using an integral to simplify $\frac{S_t}{\sigma_\tau \sqrt{T}} T^{-1}$. The intuition behind this can be easily visualized:

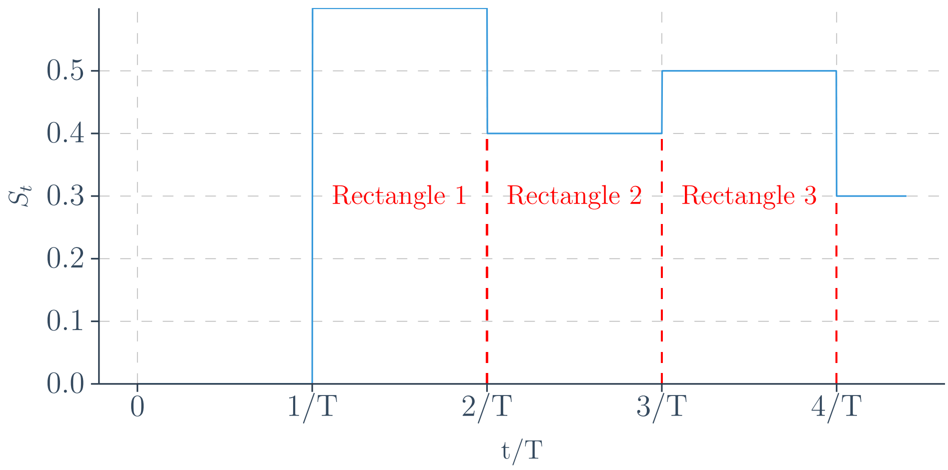

Figure 12: Intuition for Simplifiying $\frac{S_t}{\sigma_\tau \sqrt{T}} T^{-1}$ with the Help of Integration.

In Figure 12 a segment of a time-discrete random walk S_t with $S_0 = 0$ starting at $t = 0$ is plotted. Every $\frac{1}{T}$ a new error term is added to the sum of the previous error terms. The Riemann-integral operator calculates the area under the graph. This means, for every interval $[(t-1/T), t/T]$, the area is calculated using the value of the function and multiplying it by $\frac{1}{T}$ - as width times length is the formula for the area of a rectangle. This is why we can write $\frac{S_t}{\sigma_\tau \sqrt{T}} T^{-1}$ as $\int_{(t-1)/T}^{t/T} \frac{S_{\lfloor tr \rfloor}}{\sigma_\tau \sqrt{T}}\, dr$. Both expressions multiply the value of $\frac{S_t}{\sigma_\tau \sqrt{T}}$ with $\frac{1}{T}$.

$$B_\tau = (1-\tau)^{-2} \sum_{t=\lfloor \tau T \rfloor+1}^{T} \Big[\int_{(t-1)/T}^{t/\tau} \Big(\frac{S_{\lfloor tr \rfloor}}{\sigma_\tau \sqrt{T}}\Big)^2 dr - 2Y_{\lfloor \tau T \rfloor} s_\tau T^{-0.5} \int_{(t-1)/T}^{t/\tau} \frac{2S_{\lfloor tr \rfloor}}{s_\tau^2 \sqrt{T}}\, dr + \frac{Y_{\lfloor \tau T \rfloor}^2}{s_\tau^2 (T - \lfloor \tau T \rfloor)^2}\Big].$$

The next step involves simplyfying the sum of the integrals $\Sigma_{t=1}^{T} \int_{(t-1)/T}^{t/T} (\frac{S_{\lfloor tr \rfloor}}{\sigma_\tau \sqrt{T}})\, dr$. As an integral is additive, instead of summing all individual "rectangles", the entire integral from $[\tau T]/T$ to 1 can be taken.

$$B_\tau = (1-\tau)^{-2}\Big[\int_{[\tau T]/T}^{1} \Big(\frac{S_{\lfloor tr \rfloor}}{\sigma_\tau \sqrt{T}}\Big)^2 dr - 2Y_{\lfloor \tau T \rfloor} s_\tau T^{-0.5} \int_{[\tau T]/T}^{1} \Big(\frac{S_{\lfloor tr \rfloor}}{\sigma_\tau \sqrt{T}}\Big) dr + \sum_{t=\lfloor \tau T \rfloor+1}^{T} \frac{Y_{[\tau T]}^2}{s_\tau^2 T^2}\Big].$$

The last step involves considering the limit of T, i.e. $T \to \infty$. In this case the last two summands converge to zero. As established by Donsker's Theorem (Theorem 3), the standardized and rescaled random walk sequence $\frac{S_{\lfloor tr \rfloor}}{\sigma_\tau \sqrt{T}}$ converges in distribution to a standard Wiener Process. Since the integral itself is continuous and x^2 is a continuous function, the CMT states that the transformed random walk converges to the similiarly transformed Wiener process. As the statistic is not calculated for the entire sample, but only for the respective subsample, the length of the Wiener Process to be integrated is not given by 1, but by $1-\tau$. So taking the limit gives the distribution of Proposition 2:

$$B_\tau = (1-\tau)^{-2}\Big[\underbrace{\int_{[\tau T]/T}^{1} \Big(\frac{S_{\lfloor tr \rfloor}}{\sigma_\tau \sqrt{T}}\Big)^2 dr}_{\xrightarrow{d} \int_0^{1-\tau} (W(r))^2 dr} - \underbrace{2Y_{\lfloor \tau T \rfloor} s_\tau T^{-0.5} \int_{[\tau T]/T}^{1} \Big(\frac{S_{\lfloor tr \rfloor}}{\sigma_\tau \sqrt{T}}\Big) dr}_{\to 0} + \underbrace{\sum_{t=\lfloor \tau T \rfloor+1}^{T} \frac{Y_{[\tau T]}^2}{s_\tau^2 T^2}\Big]}_{\to 0}.$$

□

5.4 Busetti-Taylor Statistic

Homm and Breitung (2012) also adapt a statistic proposed by Busetti and Taylor (2004) yielding

$$supBT(\tau_0) := \sup_{\tau \in [\tau_0, 1-\tau_0]} BT_\tau, \; BT_\tau := \frac{1}{s_0^2 (T - \lfloor \tau T \rfloor)^2} \sum_{t=\lfloor \tau T \rfloor+1}^{T} (Y_T - Y_{t-1})^2, \tag{37}$$

where s_0^2 is the variance estimator given by (34) based on the entire sample and $\tau_0 \in (0, 0.5)$. There are two differences compared to statistic (35). First, the entire sample is used to estimate the variance of the process. Second, only the final value is forecasted based on the previous periods under the random walk hypothesis. Again, large values indicate explosive behaviour. In the explosive case, the last value of the time series Y_T differs significantly from the previous values. The following plot visualizes this:

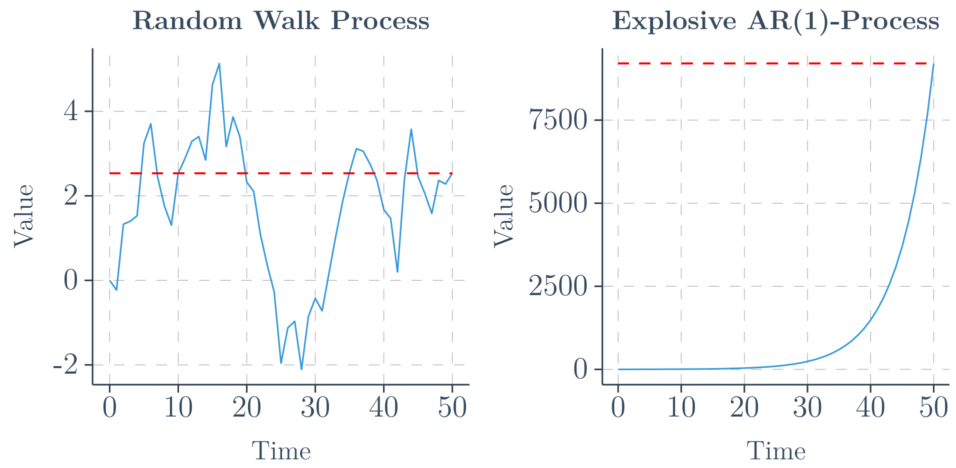

Figure 13: Intuition Behind the Busetti-Taylor Statistic. The red line gives the value of Y_T.
While the Bhargava statistic sums the squared difference to the **first** value of the resepective intervall, the Busetti-Taylor statistic sums the squared difference to the **last** value.

The limiting distribution of the statistic is obtained in a similar way to the Bhargava statistic and is presented in the following proposition:

Proposition 3. *If Y_t is generated by the random walk model from* (31)*, the limiting distribution of the test statistic in* (37) *under the null hypothesis* (32) *is given by*

$$supBT(\tau_0) \xrightarrow{d} \sup_{\tau \in [\tau_0, 1-\tau_0]} \left\{ \frac{1}{(1-\tau)^2} \int_\tau^1 (W(1-r))^2 dr \right\}.$$

5.5 Kim Statistic

Kim (2000) is once again slightly adapted by Homm and Breitung (2012). The statistic is given by

$$supK(\tau_0) = \sup_{\tau \in [\tau_0, 1-\tau_0]} K_\tau, \; K_\tau = \frac{(T - \lfloor \tau T \rfloor)^{-2} \sum_{t=\lfloor \tau T \rfloor + 1}^{T} (Y_t - Y_{\lfloor \tau T \rfloor})^2}{\lfloor \tau T \rfloor^{-2} \sum_{t=1}^{\lfloor \tau T \rfloor} (Y_t - Y_1)^2}, \quad (38)$$

with $\tau_0 \in (0, 0.5)$. The statistic splits the sample in two. Under the null hypothesis, Y_t follows a random walk in both parts of the sample. This is why the first value of the respective subsample is a reasonable forecasts for the subsample if H_0 is true. If the second part of the sample is an explosive time series, the forecast $Y_{\lfloor \tau T \rfloor}$ is not accurate. In the explosive case, the nominator will become large and therefore large values of the statistic indicate rejection. The following proposition states the limiting distribution of the Kim statistic:

Proposition 4. *If Y_t is generated by the random walk model from Equation* (31)*, the limiting distribution of the test statistic in* (38) *under the null hypothesis* (32) *is given by*

$$\underset{\tau \in [\tau_0, 1-\tau_0]}{sup} K_\tau \xrightarrow{d} \underset{\tau \in [\tau_0, 1-\tau_0]}{sup} \left\{ \left(\frac{\tau}{1-\tau} \right)^2 \frac{\int_\tau^1 (W(r-\tau))^2 dr}{\int_0^\tau (W(r))^2 dr} \right\}. \tag{39}$$

This result can be made clear by recalling the limiting distribution of the Bhargava statistic in Proposition 2. The Kim statistic could be interpreted as taking the quotient of a Bhargava statistic for the second subsample and a Bhargava statistic for the first subsample. So the limiting distribution is obtained by utilizing the Continuous Mapping Theorem and dividing the limiting distributions of the two subsamples.

5.6 Philipps et al. Statistic

Phillips et al. (2011) suggest to use Dickey-Fuller tests[8] for testing the hypotheses. This involves estimating the AR(1)-process given by (31) using ordinary least squares (OLS).

[8] Dickey and Fuller (1979) introduce very popular tests for the existence of autoregressive parameters with the value of 1 in a time series.

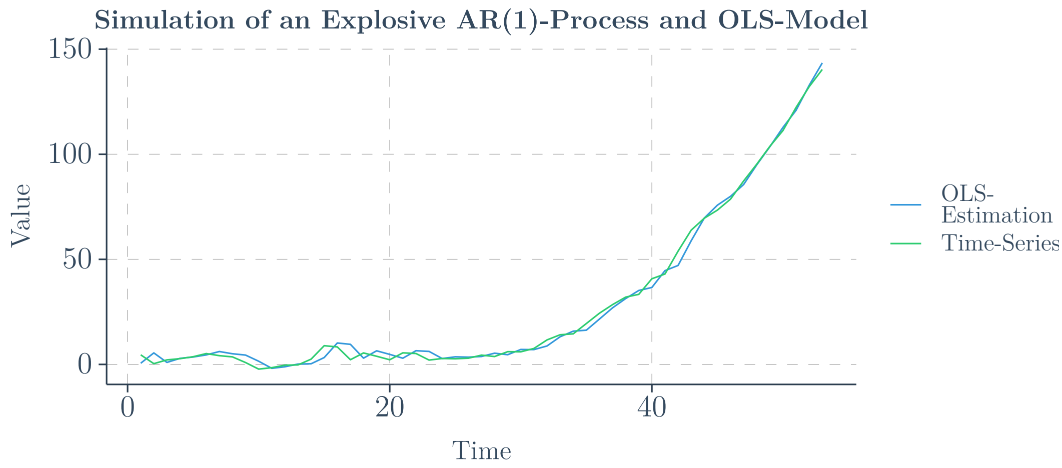

Figure 14: Simulated AR(1)-Model According to (31) and the Respective OLS-Estimation.
The autoregressive parameter is set to 1.1, $Y_0 = 0$, $e_t \sim N(0,1)$.

Their test statistic resembles a standard t-test for a parameter of an OLS-regression. Contrary to Phillips et al. (2011), we do not use a constant in our regression, as we have detrended the time series (cf. (31)). As the moment of change is once again unknown, the statistic is recursively calculated for subsamples of the time series and the supremum is taken. The statistic is thus given by

$$supDF(\tau_0) := \sup_{\tau_0 \leq \tau \leq 1} DF_\tau = \frac{\hat{\rho}_\tau - 1}{\hat{\sigma}_{\hat{\rho}_\tau}}, \tag{40}$$

where $\hat{\rho}_\tau$ is the autoregressive parameter obtained by an OLS-estimation of (31) when only considering the subsample $\{Y_1, Y_2, ..., Y_{\lfloor \tau T \rfloor}\}$, $\hat{\sigma}_{\hat{\rho}_\tau}$ is the usual unbiased estimator for the variance of $\hat{\rho}_\tau$ and $\tau \in [\tau_0, 1]$ with $\tau_0 \in (0, 0.5)$. The size of the smallest subsample is therefore determined by the choice of τ_0.
The intuition behind this statistic is straightforward: If the true value of the parameter ρ_τ exceeds 1 in the respective subsample, the OLS-estimation should fit a model with the parameter $\hat{\rho}_\tau$ also greater than one. This means, large values of statistic (40) indicate explosive behaviour in the time series. The limiting distribution of this statistic is presented following Homm and Breitung (2012) as:

Proposition 5. *If Y_t is generated by the random walk model from (31), the limiting distribution of the test statistic in (40) under the null hypothesis (32) is given by*

$$\sup_{\tau_0 \leq \tau \leq 1} DF_\tau \xrightarrow{d} \sup_{\tau_0 \leq \tau \leq 1} \frac{\int_0^\tau W(r) dW(r)}{\int_0^\tau (W(r))^2 dr}. \tag{41}$$

The following considerations do not prove Proposition 5, but illustrate how it is obtained:
The OLS-method computes the estimator $\hat{\rho}_\tau$ for a given subsample as a fraction of covariance and variance,

$$\hat{\rho}_\tau = \frac{Cov[Y_t, Y_{t-1}]}{Var[Y_{t-1}]}. \tag{42}$$

As we have detrended the time series, the mean of Y_t is equal to zero. This helps to simplify the calculation of (42):

$$\begin{aligned}\hat{\rho}_\tau &= \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} Y_t Y_{t-1}}{\sum_{t=2}^{T} Y_t^2} = \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} (Y_{t-1} + \Delta Y_t) Y_{t-1}}{\sum_{t=2}^{T} Y_t^2} \\ &= \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} (Y_{t-1}^2 + e_t Y_{t-1})}{\sum_{t=2}^{\lfloor \tau T \rfloor} Y_t^2} = 1 + \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} e_t Y_{t-1}}{\sum_{t=2}^{\lfloor \tau T \rfloor} Y_t^2}.\end{aligned} \tag{43}$$

This simplified solution is used to express $\lfloor \tau T \rfloor (\hat{\rho}_\tau - 1)$,

$$\lfloor \tau T \rfloor (\hat{\rho}_\tau - 1) = \frac{\lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} e_t Y_{t-1}}{\lfloor \tau T \rfloor^{-2} \sum_{t=2}^{\lfloor \tau T \rfloor} Y_t^2}. \tag{44}$$

The limiting distribution of the nominator and denominator of (44) can be derived in a similar way as in Section 5.3. Utilizing the Continuous Mapping Theorem, the distribution of a quotient is given by the the quotient of the respective limiting distributions of nominator and denominator. After a few extra steps of simplification and using Itô-Calculus, we get the distribution of Proposition 5.

5.7 A Chow-Type Statistic

The idea behind the last test introduced by Homm and Breitung (2012) is to incorporate the information that both under H_0 and H_1 the time series follows a random walk at first. In order to use this information, the model from (31) is rewritten by using $\mathbb{1}_{\{.\}}$, which is an indicator function that equals one if the statement in braces is true and equals zero if the statement is false.

$$\begin{aligned} Y_t &= \rho_\tau (Y_{t-1} \mathbb{1}_{\{t > \lfloor \tau T \rfloor\}}) + e_t, \\ Y_t - Y_{t-1} &= (1 - \rho_\tau)(Y_{t-1} \mathbb{1}_{\{t > \lfloor \tau T \rfloor\}}) + e_t, \\ \Delta Y_t &= \delta_\tau (Y_{t-1} \mathbb{1}_{\{t > \lfloor \tau T \rfloor\}}) + e_t, \end{aligned} \tag{45}$$

where ρ_τ is the autoregressive parameter for a given τ, $\delta_\tau = (1 - \rho_\tau)$, $\tau \in [\tau_0, 1 - \tau_0]$ with $\tau_0 \in (0, 0.5)$.
This way the sample is split in two. In the first part of the sample $\{Y_1, ..., Y_{\lfloor \tau T \rfloor}\}$, the indicator function $\mathbb{1}_{\{.\}}$ equals zero and (45) simplifies to

$$\Delta Y_t = e_t, \tag{46}$$

which corresponds to the first difference of a random walk model. In the second part of the sample $\{Y_{\lfloor \tau T \rfloor + 1}, ..., Y_T\}$, the indicator function equals one and the model is given by

$$\Delta Y_t = \delta_\tau Y_{t-1} + e_t. \tag{47}$$

If the time series is explosive in the second part of the sample, i.e. $\rho_\tau > 1$, the correct specification for (47) includes a $\delta_\tau > 0$. If the time series follows a random walk in the second part of the sample, the correct specification of (47) corresponds to $\delta_\tau = 0$, as the first difference of a random walk is given by $\Delta Y_t = e_t$.
In order to test the null hypothesis of a random walk against a change from I(1) to explosive, we therefore test $H_0 : \delta_\tau = 0$ against $H_1 : \delta_\tau > 0$.
We estimate (45) with OLS and obtain the t-statistic by dividing the OLS-estimator $\hat{\delta}_\tau$ by its variance $\hat{\sigma}_{\hat{\delta}_\tau}$.

$$DFC_\tau := \frac{\hat{\delta}_\tau}{\hat{\sigma}_{\hat{\delta}_\tau}} = \frac{\sum_{t=\lfloor \tau T \rfloor + 1}^{T} \Delta Y_t Y_{t-1}}{\hat{\sigma}_\tau \sqrt{\sum_{t=\lfloor \tau T \rfloor + 1}^{T} Y_{t-1}^2}}, \tag{48}$$

with

$$\hat{\sigma}_\tau := \frac{1}{T-2} \sum_{t=2}^{T} (\Delta Y_t - \hat{\delta}_\tau (Y_{t-1} \mathbb{1}_{\{t > \lfloor \tau T \rfloor\}})^2.$$

As the moment of change is once again not known, the statistic is recursively calculated for different splits of the sample and the supremum is taken:

$$supDFC := \sup_{\tau \in [\tau_0, 1 - \tau_0]} DFC_\tau. \tag{49}$$

The limiting distribution is obtained by Homm and Breitung (2012) and stated in the following proposition:

Proposition 6. *Let Y_t be generated by the random walk model from (31). Under the null hypothesis (45), the limiting distribution of (49) is given by*

$$supDFC \xrightarrow{d} \sup_{\tau \in [\tau_0, 1-\tau_0]} \frac{\frac{1}{2}[\{(W(1))^2 - (W(\tau))^2 - (1-\tau)\}]}{\sqrt{\int_\tau^1 (W(r))^2 dr}}.$$

6 Applications

In this section we apply the testing procedures from Chapter 5 to two important periods in financial history. Thus, Section 6.1 examines the US stock market of the 1990s and Section 6.2 studies the US housing market leading up to the financial crisis in 2008.

6.1 The Dot-com Bubble

Our first application concerns the US stock market of the late 1990s. Its price dynamics were characterized as *irrational exuberance* (Shiller, 2015) and the episode is widely considered to be a classical example of a bubble (Kindleberger and Aliber, 2011).

During the 1990s optimism was widespread among investors in the United States (Shiller, 2015). Economic indicators such as the unemployment rate or the US Treasury's annual fiscal balance signaled that the economy was in good shape and US stock prices surged - in the year 1999 alone, the NASDAQ Composite Index grew by 85.6% (Kindleberger and Aliber, 2011). This surge in prices was mainly driven by a focus on companies involved in the rapidly developing field of information technology. The Initial Public Offering of VA Linux, an American company involved in server technology, on December 9, 1999 provides ancedotal evidence for remarkable price dynamics: Its stock value jumped by 698% on the first day of the stocks' listing (Gimein, 1999). In March 2000, the period of growing stock prices ended abruptly and prices dropped sharply as the economic context started to change: The Federal Reserve Bank of America withdrew liquidity and raised interest rates. The United States eventually experienced a recession starting in the beginning of 2002 (Kindleberger and Aliber, 2011).

In order to examine the price development of US stocks in the 1990s, we turn to an investment fund called *The Growth Fund of America* with ticker symbol AGTHX. Its strategy is based on holding so-called *growth stocks*, which are expected to exhibit growth significantly higher than the market average in the

future. The fund therefore held and still holds stocks of many companies in the information sector. As of July 30, 2020, 26.3% of its net assets are invested in the stocks of Netflix, Facebook, Amazon, Microsoft and Alphabet (The Capital Group Companies, Inc., 2020).
A visual inspection of dividends and prices of the AGTHX seems to confirm the presence of a bubble in stockprices which burst in the year 2000:

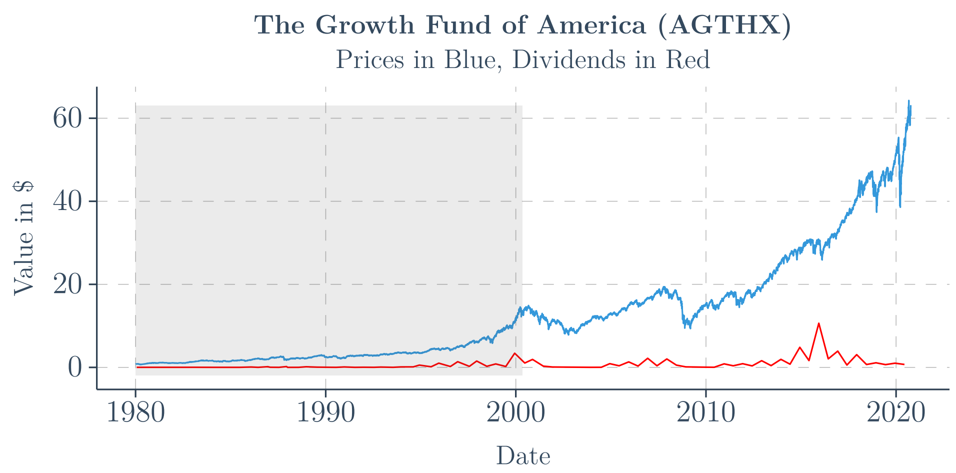

Figure 15: The Growth Fund of America (AGTHX), Prices (Blue) and Dividends (Red).
The time period of interest is shaded.

6.1.1 Data

We obtain prices and dividends of the fund from Yahoo! Finance. Our observation period ranges from January 2, 1980 to October 9, 2020 and includes 10,283 daily observations of prices and 81 dividend payments. We adjust our data for inflation by using the Consumer Price Index for the United States from the Organization for Economic Co-operation and Development.
As Homm and Breitung (2012) show, the power of the tests is very low if the observations after the peak are included in the sample. A visual inspection of the graph suggests that the funds' value peaks in the year 2000. Evaluating the data set only in the year 2000 we find the fund to have its highest value in March. Therefore we limit the time series of prices to March 1, 2000 and test a total of $T = 5{,}097$ observations. As asset prices reflect expectations about future dividends, we present the results for the restricted and unrestricted dividend time series.

6.1.2 Dividends

Following Chapter 5, we test the detrended time series of dividends. As our unrestricted sample has $T = 81$ and our restricted sample has $T = 60$ observations, we take the critical values obtained from Homm and Breitung (2012) for $T = 100$. We choose $\tau_0 = 0.3$, as our sample is rather small and we do not want to consider very small subsamples. The following table reports our results:

Table 1: Testing for Explosive Behaviour in the Dividends of The Growth Fund of America.

	Test Statistics				
	supB(0.3)	supBT(0.3)	K(0.3)	supDF(0.3)	supDFC(0.3)
(a) Restricted Sample January 2, 1980 to March 1, 2000					
Value	0.0564	1.2180	14.9782	-0.9142	-6.3282
(b) Unrestricted Sample January 2, 1980 to October 9, 2020					
Value	0.0745	0.0752	36.8726**	-1.3886	-6.7781
Upper tail critical values for T = 100					
1 %	2.3514	1.8448	24.294	0.0599	0.9071
5 %	2.7720	2.3928	33.729	0.3402	1.2864
10%	2.3514	3.7456	55.935	0.7871	2.0408

* p-value $<$ 0.1, ** p-value $<$ 0.05, *** p-value $<$ 0.01.
The test statistics from Chapter 5 are applied to the restricted and unrestricted sample of the detrended time series of dividends of The Growth Fund of America. Critical values are obtained from Homm and Breitung (2012).

If we restrict the sample to end in March 2000, no statistic rejects the null hypothesis on a 10 % significance level.
Only the Kim Statistic is significant on a 5 % level if we test the entire sample from January 1980 to October 2020. The reason for this is the construction of the Kim statistic. It opposes the first part of the sample $\{Y_1, ..., Y_{\lfloor \tau T \rfloor}\}$ to the second part of the sample $\{Y_{\lfloor \tau T \rfloor +1}, ..., Y_T\}$. As the funds dividends were very close to zero from 1980 to about 1995, this half of the sample exhibits very low variance. The fund starts to pay higher dividends from 1995 onwards and consequently shows a variance significantly larger than zero when considering only the subsample starting in 1995. Since the statistic divides

variance measures of the first and second part of the sample, it yields large values for splits of the sample around the year of 1995 - the nominator value is divided by a very small number.
But all other tests do not classify the behaviour of the time-series as explosive even on a 10% level. Concluding, we do not find convincing statistical evidence for explosive behaviour in the time series of dividends of the AGTHX.

6.1.3 Prices

For testing the detrended time series of real prices of the AGTHX we choose $\tau_0 = 0.05$ as our sample includes $T = 5{,}097$ observations and compute critical values by Monte Carlo simulation: We generate data according to (31) with $\rho_t = 1 \; \forall \; t$, the initial value $Y_0 = 0$, a normally distributed error term, i.e. $e_t \sim N(0,1) \; \forall \; t$ and sample size $T = 5{,}000$. In line with Section 5.1 we apply the test statistics to the detrended series during the Monte Carlo study. For every test statistic, $\tau_0 = 0.05$ as this is the value chosen in the application and 1,000 replications are performed.
We report the following results:

Table 2: Testing for Explosive Behaviour in the Real Prices of The Growth Fund of America.

	Test Statistics				
	supB(0.05)	supBT(0.05)	supK(0.05)	supDF(0.05)	supDFC(0.05)
Value	2.5391	21.1612***	146.4415***	4.3660***	6.9398***
Upper tail critical values for T = 5000					
10 %	3.1534	1.9440	38.6562	1.1894	1.1282
5 %	3.8479	2.3308	53.1084	1.5759	1.5160
1%	5.0723	3.4897	90.0695	2.2057	2.1308

* p-value < 0.1, ** p-value < 0.05, *** p-value < 0.01.
The test statistics from Chapter 5 are applied to the time series of prices of The Growth Fund of America from January 2, 1980 to March 1, 2000. Critical values are obtained by Monte Carlo simulation.

All tests but the Bhargava statistic clearly reject the null hypothesis of a random walk.
The Bhargava statistic does not detect explosive behaviour. This is in line

with the results of Homm and Breitung (2012), who find that the statistic has poor power properties. In their Monte Carlo simulations the Busetti-Taylor statistic shows superior power properties to the Bhargava statistic. It seems more effective to compare the values of the time series with the **last** value of the series, rather than with the **first**.
Summing up, we find that all statistics apart from the Bhargava statistic detect explosive behaviour on a 1 % significance level. Since we did not find convincing evidence for explosive behaviour in dividends, we conclude a rational bubble was present in the AGTHX.

6.2 The Housing Market of Philadelphia, USA

As shortly discussed in Chapter 1, a dysfunctional housing market can have severe consequences. We now turn to the Housing Market of Philadelphia, USA to analyze the price developments which eventually resulted in the financial crisis of 2008. We use the identification strategy to detect a rational bubble introduced in Chapters 4 and 5. As we analyze real estate, we do not consider dividends, but rents.
A visual inspection of the plot of rents and house prices indeed suggests the presence of a bubble bursting in January 2008. However, we might also suspect a minor bubble from about 1982 to 1989:

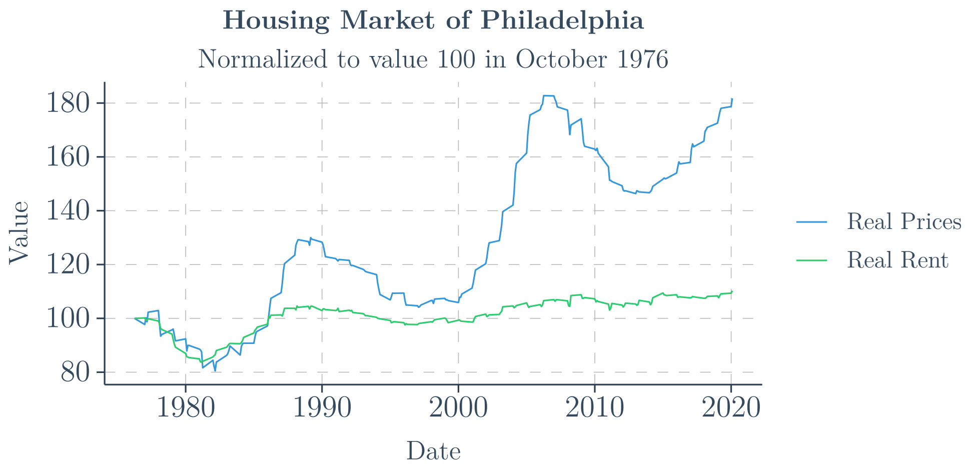

Figure 16: Housing Market of Philadelphia, Real Prices and Real Rents. Both time series are normalized to value 100 in October 1976.

6.2.1 Data

We use the House Price Index (HPI) from the U.S. Federal Housing Finance Agency for Philadelphia, USA to measure prices. The quarterly time series starts in the fourth quarter of 1976 and ends in the second quarter of 2020. It includes $T = 175$ observations. Rents are examined by considering the Consumer Price Index for All Urban Consumers: Rent of Primary Residence in Philadelphia-Camden-Wilmington from the U.S. Bureau of Labor Statistics. In order to match the time series of prices we examine the monthly time series of the rent index starting in October 1976. As the Bureau of Labour Statistics reports the index only every other month from October 1976 to December 1977, we impute missing values by averaging the neighboring values. For example, if an index value of 10 was reported for October 1977 and an index value of 20 for December 1977, we would impute an index value of 15 for November 1977. This is a common procedure when dealing with missing values in a time series (Le et al., 2018).

6.2.2 Rents

We test the detrended time series of real rents. As a visual inspection of Figure 16 suggests that the housing bubble burst at the beginning of 2008, we only consider the time series of prices up until the first quarter of 2008 in Section 6.2.3. But as prices reflect expectations about future rents, we present the results for the restricted and unrestricted time series of rents. The unrestricted sample has $T = 526$ observations and the restricted sample has $T = 376$ observations. We use the critical values obtained from Homm and Breitung (2012) with τ_0 set to 0.2.

Table 3: Testing for Explosive Behaviour in the Real Rents of the Housing Market of Philadelphia, USA.

	Test Statistics				
	supB(0.2)	supBT(0.2)	supK(0.2)	supDF(0.2)	supDFC(0.2)
Unrestricted Sample October 1976 to July 2020					
Value	1.0262	0.0679	1.3098	-0.4539	-1.6224
Restricted Sample October 1976 to January 2008					
Value	1.3146	0.3125	0.9388	-0.5037	-0.6898
Upper tail critical values for $T = 400$					
10 %	2.6240	1.7661	26.692	0.1192	0.9542
5 %	2.9635	2.28	36.402	0.3562	1.3451
1%	3.9984	3.6151	62.72	0.8264	2.0168

The test statistics from Chapter 5 are applied to the detrended time series of real rents in Philadelphia, USA. Critical values are obtained from Homm and Breitung (2012).

Neither the unrestricted, nor the restricted sample shows statistically significant explosive behaviour. If our theoretical framework from Chapter 4 is correct, explosive behaviour in prices can only be caused by a bubble component.

6.2.3 Prices

As the plot suggests a burst of the bubble in January 2008, we restrict the time series of prices from the fourth quarter of 1976 to the first quarter of 2008. This yields $T = 126$ observations. We use critical values from Homm and Breitung (2012) for $T = 100$ and set $\tau_0 = 0.3$ in order to keep subsamples reasonably large. We obtain the following results:

Table 4: Testing for Explosive Behaviour in the Real Prices of the Housing Market of Philadelphia, USA.

	Test Statistics				
	supB(0.3)	supBT(0.3)	supK(0.3)	supDF(0.3)	supDFC(0.3)
Prices (restricted)	4.8631***	6.6193***	5.5021	0.4291**	0.6392
Upper tail critical values for $T = 100$					
1 %	2.3514	1.8448	24.294	0.0599	0.9071
5 %	2.7720	2.3928	33.729	0.3402	1.2864
10%	2.3514	3.7456	55.935	0.7871	2.0408

* p-value < 0.1, ** p-value < 0.05, *** p-value < 0.01.
The test statistics from Chapter 5 are applied to the quarterly time series of the house price index ranging from the fourth quarter of 1976 to the first quarter of 2008 and including $T = 126$ observations. Critical values are obtained from Homm and Breitung (2012).

The results are not unambiguous and reveal the strengths and weaknesses of the testing procedures. They illustrate that the tests are highly dependent on the specific trajectory of the time series data. Plotting the real detrended house price data yields the following:

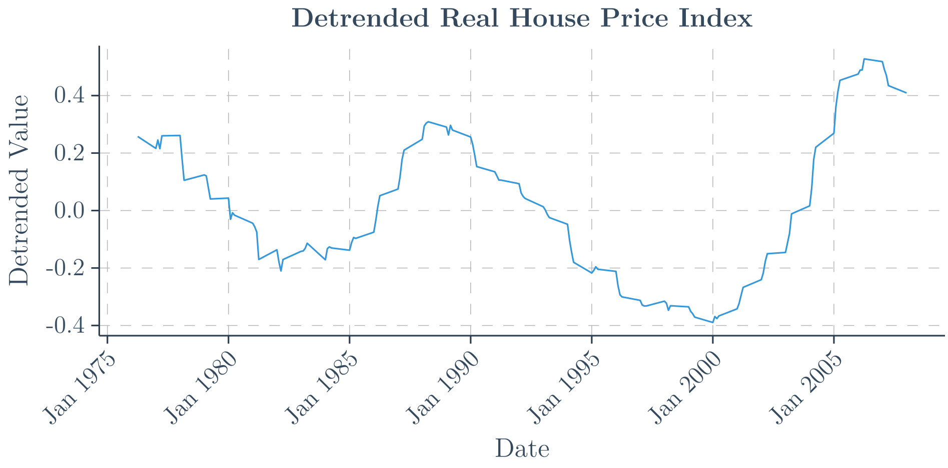

Figure 17: Real Detrended House Price Data for Philadelphia, USA.

The US housing market boomed in the 1980s and dropped significantly in the early 1990s as the result of a market correction.[9] The first part of the sample does not appear to exhibit the pathway of a random walk. This is not in line with our test hypotheses: H_0 and H_1 both assert that prices follow a random walk in the first part of the sample. However, the test statistics handle this differently well:

The Bhargava statistic and the Busetti-Taylor statistic clearly reject H_0. As they only build on information of the last part of the sample, they are not affected by the behaviour of the time series in the first part of the sample.

The Kim statistic compares the variance of the first sample part and the second sample part. Since the time series also exhibits a high variance due to rising and then falling prices during the 1980s and early 1990s, the Kim statistic does not detect explosiveness in the second part of the sample. This is in line with the results from the Monte Carlo study of Homm and Breitung (2012). They find the Kim statistic to have poor power properties.

The supDF statistic builds on information of the subsample $\{Y_1, Y_2, ..., Y_{\lfloor \tau T \rfloor}\}$ and fits a regression model. Thus, the statistic is always calculated involving the first part of the sample. This is why the statistic only detects explosiveness in the second part of the sample at a 5 % significance level: It can not fit a regression model clearly indicating explosive behaviour as it always takes the first part of the sample into account:

[9] For a rigorous analysis of this period, we refer to Wheelock (2006).

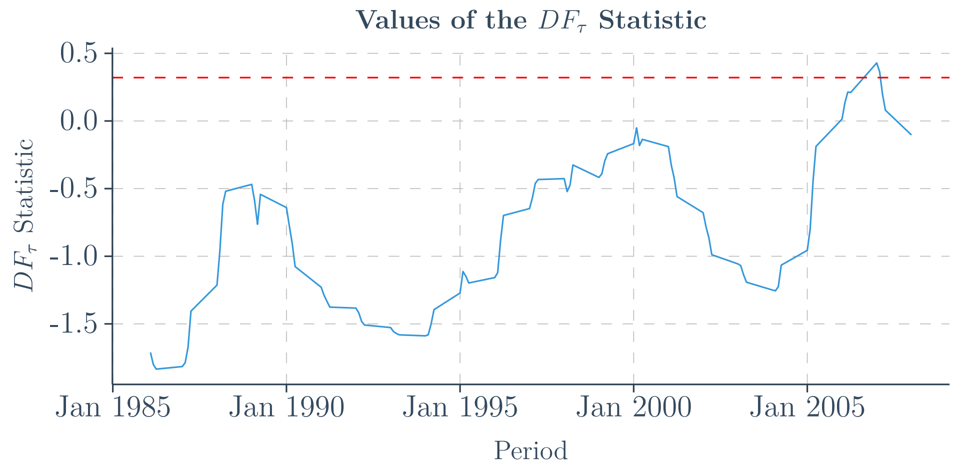

Figure 18: Values of the DF_τ Statistic for the Detrended Series of Real House Prices in Philadelphia, USA.
The red line gives the 5 % critical value obtained by Homm and Breitung (2012) and the x-axis marks the last period of the respective subsample.

The supDFC statistic does not detect explosiveness at a 10 % significance level. As the statistic fits a regression model asserting a random walk for the first part of the sample, it seems to be misspecified in this application. Thus, the variance of the regression estimators $\hat{\delta}_\tau$ is rather high and the values of the test statistic remain smaller than the 10% critical value.
Summing up, supB, supBT and supDF detect explosive behaviour the US housing market from October 1976 to January 2008. As we did not find evidence for explosive behaviour in the time series of rents, they support the view of a rational bubble in the housing market. supK and supDFC seem to have low power properties due to the distinct pathway of the time series and can not reject the null hypothesis of a random walk.

7 Conclusion

In this thesis we elaborate on Homm and Breitung (2012), who suggest statistical testing procedures to identify rational bubbles in stock markets. Chapter 2 explains the Central Limit and Continuous Mapping Theorem, which we use to derive the limiting distributions of the test statistics. In Chapter 3 we cover relevant theory from the field of statistical time series analysis, mainly focusing on the AR(1)-process and its distribution. By utilizing this theory in Chapter 4 we suggest specific models for the time series of asset prices and dividends. Within this theoretical framework, non-explosive behaviour in dividends together with explosive behaviour in prices implies a rational bubble. We discuss the five different testing strategies suggested by Homm and Breitung (2012) in Chapter 5. They each test the null hypothesis of a random walk against the alternative of a change from I(1) to explosive and can therefore be used to detect rational bubbles.

The supB statistic and the supBT statistic only consider the second part of the sample period. While the supB statistic is based on the idea of comparing the *first* value of the respective subsample to the other data points, the supBT statistic is based on comparing the *last* value of the subsample to the other data points. The supK statistic opposes the first part of the sample to the second part of the sample and the supDF and supDFC statistic are based on estimating a regression model with OLS and testing for specific values of the autoregressive parameter: The supDF statistic is concerned with estimating an AR(1)-model for recursive subsamples of the time series. In contrast, the supDFC statistic incorporates the information that both under H_0 and H_1 the time series follows a random walk at first by considering an adjusted regression model.

We apply those statistics to the US stock market of the 1990s and the US housing market leading up to the financial crisis in 2008 in Chapter 6. We hereby find strong evidence for a rational bubble in the 1990s US stock market and also find indications of a bubble in the US housing market.

The discussed testing strategies show significant weaknesses in detecting explosive behaviour:

Homm and Breitung (2012) show in their Monte Carlo study that the supB and supK statistic have weak power properties and generally exhibit the lowest power among the presented tests. This becomes apparent in our applications, since the supB statistic does not detect explosive bahviour in the time series of real prices of the AGTHX and the supK statistic does not reject the null hypothesis of a random walk for the time series of real house prices of Philadelphia, USA.

Furthermore, the power of the tests critically depends on the specific pathway of the time series. If the time series does not exhibit a random walk in the first part of the sample, supK, supDF and supDFC statistic appear to have weak power (see Chapter 6.2.3). On this matter Homm and Breitung (2012) show that all statistics have difficulties detecting explosive behaviour if the sample includes multiple collapsing bubbles. We refer to Phillips et al. (2015a) and Phillips et al. (2015b) who address this issue by advancing the supDF statistic. Additionally it should be noted that we tried to identify bubbles in retrospect in Chapter 6. This allowed us to examine a time series including its peak. When testing in real time, this is usually not possible and the test statistics could yield less conclusive results.

Despite those weaknesses, every testing procedure presented in this thesis is a valid tool for detecting rational bubbles in financial markets. The theoretical framework presented in Homm and Breitung (2012), which those testing procedures are based on, is a significant contribution to the study of financial markets as it allows to test empirically for the existence of rational bubbles.

The presented theory enabled us to identify a rational bubble in *The Growth Fund of America* in our applications. Homm and Breitung (2012) and Phillips et al. (2011) also detect a bubble in the NASDAQ composite index for the same period using the presented statistics. Those findings provide strong evidence that a rational bubble was indeed present in the US stock market of the 1990s. Additionally, supDF, supB and supBT statistic identify a bubble in the US housing market leading up to the financial crisis in 2008 in our application.

Those results show that the presented statistics are helpful tools for analyzing financial markets in retrospect. They also encourage employing the testing strategies to monitor financial markets. The tests can provide important hints to dysfunctional market behaviour and prompt regulators to action. We therefore support the efforts of Martínez-García et al. (2020), who are working on a computationally efficient implementation of the presented statistics in the programming language R - in order to facilitate the widespread usage of the testing procedures.

When reviewing recent research on bubbles in financial markets we find the presented methods used in a wider context: Süssmuth (2019) for instance examines bubbles in cryptocurrencies by testing the time series of internet search queries. Future research could look into other applications of the test statistics.

In the future we also expect further development of the framework presented in Homm and Breitung (2012). We see potential benefits in incorporating additional variables. Considering measures of indebtness for instance could yield more conclusive results when studying bubbles, as rising indebtness of individuals is regarded as one of the major causes of bubbles (see Chapter 1).

8 References

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