Construction of MV-polytopes via LS-galleries

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Abstract

For an algebraic group G, Anderson defined the notion of MV-polytopes in [And03], images of the MV-cycles, defined in [MV04], under the moment map of the corresponding affine Grassmannian. It was shown by Kamnitzer in [Kam05a] and [Kam05b] that these polytopes can be described via tropical relations involving the heights of their edges, and that they give rise to a crystal structure on the set of MV-cycles. Another crystal structure can be introduced using LS-galleries which were defined by Gaussent and Littelmann in [GL05], a more discrete version of Littelmann's path model. In [BG06], it was shown by Baumann and Gaussent that these two crystal structures coincide.

The aim of this dissertation is to obtain a direct combinatorial construction of the MV-polytopes using LS-galleries, especially without using the moment map. Whose image is hard to calculate, as the MV-cycles are not very explicitly described in such a way that one can obtain the set of torus fixed points contained in them. In addition we want to link this construction to the retractions of the affine building corresponding to G. This leads to a definition of MV-polytopes not involving the tropical Plücker relations. Herefore it provides a description of the polytopes independent of the type of the algebraic group via the gallery model and affine buildings.

This reproves two theorems due to Kamnitzer about the relations between the MV-polytopes and the Lusztig as well as the Kashiwara datum.

The main result of this thesis is, starting from a fixed LS-gallery, to give a construction of a set of galleries in such a way that the convex hull of their weights defines the MV-polytope. This is done by using the Gelfand-Goresky-MacPherson-Serganova stratum and retractions in the affine building.

Zusammenfassung

Das Ergebnis dieser Arbeit ist eine Verbindung zwischen dem LS-Galerien Modell von [GL05] für endlich-dimensionale Darstellungen einer zusammenhängenden komplexen halbeinfachen algebraischen Gruppe G und den MV-Polytopen (kurz für Mirković und Vilonen), via Retraktionen im affinen Tits Gebäude herzustellen. Die MV-Polytope wurden in [And03] und [Kam05a] untersucht, es handelt sich dabei um die Bilder der zugehörigen MV-Zykel unter der Moment- oder Impulsabbildung. Es ist wohlbekannt, dass beide, sowohl das LS-Galerien Modell als auch die MV-Polytope, eine kombinatorische Realisierung der endlich-dimensionalen Darstellungen von G sind. Wir konstruieren in dieser Arbeit eine kombinatorische Methode, die die beiden Modelle miteinander in Verbindung bringt. Dies geschieht, indem wir, ausgehend von einer LS-Galerie δ, für jeden Eckpunkt des zugehörigen MV-Polytopes eine neue Galerie konstruieren, die diesen Eckpunkt genau als Endpunkt besitzt. Wir zeigen, dass diese Galerien genau die Bilder einer offenen dichten Teilmenge der zu δ assoziierten Bialynicki-Birula Zelle unter geeigneten Retraktionen im affinen Tits Gebäude sind.

Dies führt zu einer alternativen Definition der MV-Polytope und neue Beweise für einige von Kamnitzers Resultaten über MV-Polytope. Dies beinhaltet vor allem die Zusammenhänge zwischen MV-Polytopen und dem Lusztig sowie Kashiwara Datum. Der Beweis ist hierbei unabhängig vom Typ und benutzt daher nicht die tropischen Plücker Relationen.

Wie bereits angemerkt, treten MV-Polytope auf natürliche Weise als Bild der MV-Zykel unter der Momentabbildung auf. Nach [MV04] sind MV-Zykel eine Klasse von algebraischen Zykeln der affinen Grassmann-Varietät. Diese wird rein mengentheoretisch definiert als $\mathcal{G} = G(\mathcal{K})/G(\mathcal{O})$, wobei $\mathcal{K} = \mathbb{C}((t))$ und $\mathcal{O} = \mathbb{C}[[t]]$ sind. Wir fixieren nun einen maximalen Torus $T \subset G$ sowie eine Borel-Untergruppe und bezeichnen mit $X = \mathrm{Mor}(T, \mathbb{C}^*)$ die Charaktergruppe und mit $X^\vee = \mathrm{Mor}(\mathbb{C}^*, T)$ die Kocharaktergruppe des Torus T. Letzteres ist ebenfalls die Charaktergruppe des dualen maximalen Torus $T^\vee$ in $G^\vee$, der Langlands dualen Gruppe zu G. Diese Gruppe indiziert zwei verschiedene Arten von Objekten, geometrische und darstellungstheoretische. Erstens können wir $X^\vee$ als eine Menge von Punkten in $\mathcal{G}$ sehen und $\mathcal{G}$ dann in disjunkte $G(\mathcal{O})$-Orbiten zerlegen, $\mathcal{G}_\lambda = G(\mathcal{O}).\lambda$ für $\lambda \in X^\vee_+$. Zweitens können wir jeder endlich-dimensionalen irreduziblen Darstellung von $G^\vee$ auf eindeutige Weise ihr Höchstgewicht in $X^\vee_+$ zuordnen.

Diese beiden Arten von Objekten führen genau zu den beiden kombina-

torischen Objekten mit denen wir uns in dieser Arbeit auseinander setzen wollen, den LS-Galerien und den MV-Polytopen. Die LS-Galerien wurden in [GL05] als eine gebäudetheoretische Alternative zum Pfadmodell eingeführt (siehe [Lit95], [Lit03] und [Lit97]). Die Autoren haben außerdem bewiesen, dass man mit Hilfe der LS-Galerien eine dichte Teilmenge der MV-Zykel beschreiben kann. Diese sind definiert als die irreduziblen Komponenten von $\overline{U^-(\mathcal{K}).\mu \cap \mathcal{G}_\lambda}$ für $\lambda \in X_+^\vee$ und $\mu \in X^\vee$, falls dieser Schnitt nicht leer ist. Hierbei bezeichnet $U^-(\mathcal{K})$ die $\mathcal{K}$-wertigen Punkte der unipotenten Radikals innerhalb der zur gewählten Borel entgegengesetzten Borel Untergruppe.

Man erhält MV-Polytope indem man $\mathcal{G}$ in einen unendlich-dimensionalen projektiven Raum $\mathcal{P}$ einbettet und benutzt die klassische Momentabbildung von $\mathcal{P}$ um Teilmengen von $X^\vee \otimes \mathbb{R}$ zu erhalten. Anderson hat in [And03] gezeigt, dass es sich bei diesen Teilmengen wirklich um Polytope handelt. Eine andere Möglichkeit, diese Polytope zu erhalten, wurde von Kamnitzer entdeckt: Er bewies, dass es sich bei den MV-Polytopen, um eine Klasse von Polytopen handelt, die eine endliche Anzahl von tropischen Gleichungen erfüllen, die so genannten *tropischen Plücker Relationen*.

Wir beginnen diese Arbeit mit der Wiederholung einiger Definitionen und Eigenschaften der Gruppe G und der zugehörigen affinen Kac-Moody-Gruppe $\hat{\mathcal{L}}(G)$ in Abschnitt 2, der affinen Grassmann Varietät, MV-Zykeln und MV-Polytopen in Abschnitt 3 und LS-Galerien und Retraktionen im affinen Tits-Gebäude in Abschnitt 4.

In Abschnitt 5 zeigen wir auf, wie man die Menge der Alkoven, aus der eine gegebene fest gewählte Galerie δ besteht, auf sinnvolle Art unterteilen kann. Dies geschieht, indem man Alkoven nach verschiedenen Typen unterscheidet und die Galerie dann entsprechend dieser Typen aufteilt. Wir untersuchen ebenfalls die Art und Weise, wie sich die Wurzeloperatoren des LS-Galerien Modells mit der definierten Unterteilung vertragen. Mit Hilfe dieser Unterteilung können wir dann für jedes Element w der Weyl-Gruppe von G, eine Galerie $\Xi_w(\delta)$ definieren.

In Abschnitt 6 werden wir dann unser Hauptresultat beweisen.

Theorem 0.1. *Sei M_δ ein MV-Zykel zu einer LS-Galerie δ und $P = \boldsymbol{\mu}(M_\delta)$ das zugehörige MV-Polytop, dann gilt*

$$P = P_\delta := \mathrm{conv}(\{wt(\Xi_w(\delta)) \mid w \in W\}).$$

Um dies ohne die tropischen Plücker-Relationen zu beweisen, berechnen wir für einen, in geeigneter Art, generischen Punkt der zu δ gehörigen

Bialynicki-Birula Zelle das Bild der Retraktionen an Unendlich. Wir zeigen, dass für eine dichte Teilmenge das Bild der Retraktion an Unendlich zu w immer genau $\Xi_w(\delta)$ ist. Dies gibt einen Beweis für die obige Aussage unter Benutzung der GGMS-Strata, kurz für Gelfand-Goresky-MacPherson-Serganova.

Im letzten Abschnitt 7 zeigen wir, dass viele der Eigenschaften von MV-Polytopen die bereits von Kamnitzer bewiesen wurden auch aus dieser Definition von MV-Polytopen folgen, meist auf schnellere und leichtere Art. In diesem Abschnitt wird auch ein Vorteil dieser Konstruktion der MV-Polytope deutlich, nämlich die Tatsache, dass hier die Eckpunkte auf kombinatorische Art beschrieben werden. Wir geben außerdem eine kleine Anzahl von Beispielen für die konstruierten Galerien und die zugehörigen MV-Polytope.

Contents

1 Introduction

The aim of this dissertation is to provide a connection between the LS-gallery model [GL05] for finite-dimensional representations of a connected complex semisimple algebraic group G and the Mirković-Vilonen polytopes (MV-polytopes for short). These appear in [And03] and [Kam05a] as images of MV-cycles [MV04] under the moment map, via retractions in the affine Tits building associated with G. It is well known that both, the LS-gallery model and the MV-polytopes are combinatorial realisations of finite-dimensional representations of G. We provide a combinatorial link between these two objects and explicitly construct MV-polytopes by starting with an LS-gallery δ and then defining a new gallery for each vertex of the MV-polytope. In addition we show that the so constructed galleries are the images of the retractions of the affine Tits building, when applied to an open subset of the Bialynicki-Birula cell corresponding to δ.

This gives an alternative definition of MV-polytopes as well as a new proof for Kamnitzer's result that the MV-polytopes are a set of polytopes whose edge lengths are the Lusztig datum of the canonical basis element corresponding to the polytope. This new proof is independent of the type and does not involve the tropical Plücker relations.

As mentioned above MV-polytopes occur naturally as images of MV-cycles under the moment map. According to [MV04], these algebraic cycles are cycles of the affine Grassmannian, which, is defined set theoretically as $\mathcal{G} = G(\mathcal{K})/G(\mathcal{O})$, where $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Now, one fixes a maximal torus $T \subset G$ and denotes by $X = \mathrm{Mor}(T, \mathbb{C}^*)$ its character group and by $X^\vee = \mathrm{Mor}(\mathbb{C}^*, T)$ its co-character group. This is also the character group of the dual maximal torus $T^\vee$ in $G^\vee$, the Langlands dual group of G, and is an indexing set for two different classes of objects, one geometric the other representation theoretic. First, we can view $X^\vee$ as a set of points in $\mathcal{G}$, and $\mathcal{G}$ decomposes into the disjoint union of $G(\mathcal{O})$-orbits $\mathcal{G}_\lambda = G(\mathcal{O}).\lambda$, for $\lambda \in X^\vee_+$. Second, every finite dimensional irreducible representation of $G^\vee$ can be characterised by its highest weight $\lambda \in X^\vee_+$.

This leads to the two types of objects that we want to study in this article: LS-galleries and MV-polytopes. The LS-galleries were introduced in [GL05] as a building-theoretic alternative to the path-model (see [Lit95], [Lit03], and [Lit97]).The authors also proved that the LS-galleries can be used to index and describe dense subsets in the MV-cycles, which are the irreducible components of $\overline{U^-(\mathcal{K}).\mu \cap \mathcal{G}_\lambda}$ for $\lambda \in X^\vee_+$ and $\mu \in X^\vee$, such that the intersection is non-empty. Here $U^-(\mathcal{K})$ denotes the $\mathcal{K}$ valued points of the unipotent radical inside the Borel opposite to the chosen one, with respect to which the dominant weights are defined.

To obtain the MV-polytopes, one embeds $\mathcal{G}$ into an infinite dimensional projective space $\mathcal{P}$ and uses the classical moment map for $\mathcal{P}$ to obtain a subset of $X^\vee \otimes \mathbb{R}$. Anderson showed that this subset is indeed a polytope [And03]. Another way to obtain these polytopes is the approach used by Kamnitzer: He proves that the MV-polytopes are a certain class of polytopes satisfying a finite set of tropical relations, called tropical Plücker relations.

We start by recalling the necessary definitions for the group G and the corresponding affine Kac-Moody group $\hat{\mathcal{L}}(G)$ (in Section 2), the affine Grassmannian, MV-cycles and MV-polytopes (in Section 3), LS-galleries and retractions in the affine Tits building (in Section 4).

Afterwards, in Section 5, we analyse the alcoves appearing in a fixed gallery δ and how one can partition a gallery, according to the type of its alcoves, into parts consisting of alcoves of the same type each. We also discuss the interaction of the root operators for LS-galleries and our defined partition. In addition we show that by using these partitions one can construct a specific gallery for each element w of the Weyl group of G, named $\Xi_w(\delta)$.

In Section 6 we prove our main theorem.

Theorem 1.1. *Let M_δ be an MV-cycle corresponding to an LS-gallery δ and $P = \boldsymbol{\mu}(M_\delta)$ the corresponding MV-polytope, then*

$$P = P_\delta := \mathrm{conv}(\{wt(\Xi_w(\delta)) \mid w \in W\}).$$

To obtain a proof that this convex hull is equal to the MV-polytope without utilising the tropical Plücker relations, we calculate the images of the retraction for each chamber of the building at infinity in Section 6. We show that for a dense subset of the MV-cycle the image of the retraction at a chamber at infinity, corresponding to a Weyl group element w, is equal to our constructed gallery $\Xi_w(\delta)$. This gives the proof for the above mentioned theorem by using the GGMS-stratum (short for Gelfand-Goresky-MacPherson-Serganova stratum).

In the final Section 7 we show that some of the properties of MV-polytopes that were proven by Kamnitzer can also be obtained when using our approach. We also give a small number of examples for the constructed galleries and the corresponding MV-polytopes. Finally we define a bijection between different sets of galleries to obtain LS-galleries whose endpoints define the

MV-polytope, by using the Weyl group action on the set of LS-galleries defined by Kashiwara, [Kas94].

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2 Algebraic and Kac-Moody groups

We want to begin with fixing the notations for our group G and the associated affine Kac-Moody group $\hat{\mathcal{L}}(G)$. We also want to recall a number of technical rules for computations and calculations in these groups from [Tit87] and [Ste68, §6].

2.1 Notations for the group G

Let G be a complex, simply-connected, semisimple algebraic group. We also fix a *Borel subgroup* $B \subset G$ and a *maximal torus* $T \subset B$ and denote by B^- the opposite Borel subgroup of B relative to T. We denote the *unipotent radicals* of B and B^- by U and U^-. We denote by $N_G(T)$ the normaliser of T in G and by $W = N_G(T)/T$ the Weyl group of G and T.

For our maximal torus T we denote by $X = X^*(T) := \mathrm{Mor}(T, \mathbb{C}^*)$, respectively $X^\vee = X_*(T) := \mathrm{Mor}(\mathbb{C}^*, T)$, its *character*, respectively *cocharacter group*. For a point $\mu \in X^\vee$, we write $\mu : \mathbb{C}^* \to T$, $s \mapsto s^\mu$ to simplify the notations in the calculations. Furthermore we write Φ and $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ for the *root* and *coroot system*. Corresponding to our choice of B we denote by Φ^+ and Φ^- the positive and negative roots of G and use the notation $\Phi^\vee_+$ and $\Phi^\vee_-$ for the corresponding subsets of the coroots. By $X_+ = \{\lambda \in X \mid \forall \alpha^\vee \in \Phi^\vee_+, \langle \lambda, \alpha^\vee \rangle \geq 0\}$ and $X^\vee_+ = \{\lambda^\vee \in X^\vee \mid \forall \alpha \in \Phi_+, \langle \alpha, \lambda^\vee \rangle \geq 0\}$ we denote the sets of *dominant weights* and *coweights*.

We also want to fix a numbering of the simple roots $(\alpha_i)_{i \in I}$, with I the indexing set for the corresponding Dynkin diagram. Inside X^+ we denote by Λ_i the *fundamental weight* corresponding to α_i for each $i \in I$. We also fix the following numbers $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$. Via the Coxeter representation, we view the Weyl group as the group of real reflections, generated by the reflections along the simple roots, on the real vector spaces spanned by X. We denote by s_i the element of the Weyl group inducing the reflection along α_i. By $l(w)$ for an element $w \in W$ we denote the length of the element w, i.e., the minimum of the lengths of all expressions that write w as the product of the reflections s_i. For elements in X, respectively $X^\vee$ we have the dominance order with respect to the cone generated by the positive roots: $\nu \geq \mu \Leftrightarrow \nu - \mu \in \mathbb{N}\Phi_+$ for $\nu, \mu \in X$, respectively $\nu^\vee \geq \mu^\vee \Leftrightarrow \nu^\vee - \mu^\vee \in \mathbb{N}\Phi^\vee_+$ for $\nu, \mu \in X^\vee$.

To be able to calculate with elements of the affine Grassmannian, we want to fix certain elements inside G and write down their commutator relations (see also [Tit87]). For every simple root α, we fix a non-trivial additive 1-parameter subgroup $U_\alpha = \{x_\alpha(t) \mid t \in \mathbb{C}\}$ of U such that $s^{\lambda^\vee} x_\alpha(t) s^{-\lambda^\vee} = x_\alpha(s^{\langle \alpha, \lambda^\vee \rangle} t)$ holds for all $\lambda^\vee \in X^\vee$, $s \in \mathbb{C}^*$, and $t \in \mathbb{C}$. By general theory we know that, for each $i \in I$ there exists a unique morphism $\phi_i : \mathrm{SL}_2 \to G$ such that

$$\phi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_{\alpha_i}(t) \text{ and } \phi_i \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} = s^{\alpha_i^\vee}$$

for all $s \in \mathbb{C}^*$ and $t \in \mathbb{C}$. Furthermore we fix the elements

$$x_{-\alpha_i}(t) = \phi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \text{ and } \overline{s_i} = \phi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For an arbitrary element $w \in W$, we will write $\overline{w}$ for the lift in $N_G(T)$ by using the elements $\overline{s_i}$ instead of the s_i's in any reduced expression for w, which is well defined as the $\overline{s_i}$ satisfy the braid relations.

For an arbitrary positive root α, we choose a simple root α_i and an element $w \in W$ such that $\alpha = w\alpha_i$. We define analogous one-parameter subgroups U_α and $U_{-\alpha}$ for α by

$$x_\alpha(t) := \overline{w} x_{\alpha_i}(t) \overline{w}^{-1} \text{ and } x_{-\alpha}(t) := \overline{w} x_{-\alpha_i}(t) \overline{w}^{-1}$$

and write $\overline{s_\alpha}$ for the element $\overline{w s_i w}^{-1}$. With these elements we have the following computation rules that will be used later on (see also [Tit87, §3.6] or [Ste68, §6]):

(i) For all $\lambda^\vee \in X^\vee$, a root α, $s \in \mathbb{C}^*$, and $t \in \mathbb{C}$,

$$s^\lambda x_\alpha(t) = x_\alpha(s^{\langle \alpha, \lambda^\vee \rangle} t) s^\lambda.$$

(ii) For $\alpha \in \Phi$ and $t, t' \in \mathbb{C}$ such that $1 + tt' \neq 0$,

$$x_\alpha(t) x_{-\alpha}(t') = x_{-\alpha}(t'/(1+tt'))(1+tt')^{\alpha^\vee} x_\alpha(t/(1+tt')).$$

(iii) For a positive root α and $t \in \mathbb{C}^*$,

$$x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t) = x_{-\alpha}(-t^{-1}) x_\alpha(t) x_{-\alpha}(-t^{-1}) = t^{\alpha^\vee} \overline{s}_\alpha = \overline{s}_\alpha t^{-\alpha^\vee}.$$

(iv) (Chevalley's commutator formula) If α and β are two linearly independent roots, then there are numbers $c_{i,j,\alpha,\beta} \in \{\pm 1, \pm 2, \pm 3\}$ such that

$$x_\beta(s)^{-1} x_\alpha(t)^{-1} x_\beta(s) x_\alpha(t) = \overrightarrow{\prod_{i,j>0}} x_{i\alpha + j\beta}(c_{i,j,\alpha,\beta} t^i s^j)$$

for all $s, t \in \mathbb{C}$. The product is taken over all pairs $i, j \in \mathbb{Z}^+$ such that $i\alpha + j\beta$ is a root and in order of increasing height of the occurring roots, i.e., with increasing products $\langle i\alpha + j\beta, \rho^\vee \rangle$, with $\rho^\vee$ being the half-sum over the positive coroots.

Especially the third relation is very important in the later calculations as it will pretty much correspond to the folding of a gallery at a certain face. While the second relation will be important in some independence arguments in the proof of the main result in Section 6.

Remark 2.1. By abuse of notations we often just write w, respectively s_i, instead of $\overline{w}$, respectively $\overline{s_i}$, for representatives of the Weyl group elements in $N_G(T)$. In all situations where this happens it will be clear which element is meant.

2.2 The affine Kac-Moody group $\hat{\mathcal{L}}(G)$

We write $\mathcal{O}$ for the formal power series ring $\mathbb{C}[[t]]$ and $\mathcal{K}$ for its field of fractions $\mathbb{C}((t))$. We denote by $G(\mathcal{O})$ and $G(\mathcal{K})$, the sets of $\mathcal{O}$-valued and $\mathcal{K}$-valued points of G. Unless stated otherwise all the definitions and statements of this section are from [Kum02, §13] and follow the notation from [GL05]. The field $\mathcal{K}$ is naturally equipped with a map, the rotation operation, $\gamma : \mathbb{C}^* \to \mathrm{Aut}(\mathcal{K})$ that acts by "rotating" the indeterminate, $\gamma(z)(f(t)) = f(zt)$. This operation can easily be lifted to an operation on the group $G(\mathcal{K})$, $\gamma_G : \mathbb{C}^* \to \mathrm{Aut}(G(\mathcal{K}))$ and we will denote the semidirect product $\mathbb{C}^* \ltimes G(\mathcal{K})$ by $\mathcal{L}(G)$, the loop group corresponding to G. Since the operation obviously restricts to $\mathcal{O}$ we also obtain $\mathcal{L}(G(\mathcal{O})) := \mathbb{C}^* \ltimes G(\mathcal{O})$ as a natural subgroup.

The *affine Kac-Moody group* $\hat{\mathcal{L}}(G)$ is defined as a central extension of the loop group

$$1 \to \mathbb{C}^* \to \hat{\mathcal{L}}(G) \xrightarrow{\pi} \mathcal{L}(G(\mathcal{K})) \to 1,$$

see or [Kum02, 13.2.11] We denote by $\mathcal{P}_\mathcal{O} \subset \hat{\mathcal{L}}(G)$ the *parabolic subgroup* $\pi^{-1}(\mathcal{L}(G(\mathcal{O})))$, this group is important for the definitions of the affine Grassmannian as well as the Bott-Samelson resolution later on.

Of course we also have the notion of a *Weyl group* for the affine Kac-Moody group. It can be constructed in different ways. Let us denote by $N_\mathcal{K}$ the subgroup of $G(\mathcal{K})$ generated by $N_G(T)$ and $T(\mathcal{K})$, write $\overline{T}$ for the standard maximal torus of $\mathcal{L}(G(\mathcal{K}))$ and $\overline{N}$ for the extension of $N_\mathcal{K}$. The affine Weyl group is then defined as

$$W^\mathfrak{a} \cong \mathcal{N}/\mathcal{T} \cong N_\mathcal{K}/T \cong \overline{N}/\overline{T},$$

for the first isomorphism see [Kum02, 6.2.9(b)], for the second [Kum02, 13.2.E(5)], and the last is a property of the semidirect product. where $\mathcal{T}$ and $\mathcal{N}$ denote the maximal torus of $\hat{\mathcal{L}}(G)$ and its normalizer therein.

Remark 2.2. The affine Weyl group can also be realized as the semidirect product of the classical Weyl group of G with the coroot lattice on which the Weyl group acts naturally.

We denote by $ev : G(\mathcal{O}) \to G$ and $ev_{\mathcal{L}} : \mathcal{L}(G(\mathcal{O})) \to \mathbb{C}^* \times G$ the evaluation maps at $t = 0$. We can then define the corresponding *Iwahori subgroups* as the preimages of the Borel subgroup $\mathcal{I} = ev^{-1}(B)$ respectively $\mathcal{I}_{\mathcal{L}} = ev_{\mathcal{L}}^{-1}(\mathbb{C}^* \times B)$ and denote by $\hat{\mathcal{B}} = \pi^{-1}(\mathcal{B}_{\mathcal{L}})$ the *Borel subgroup* of $\hat{\mathcal{L}}(G)$.

Remark 2.3. **([GL05, §2])** The definitions of the Iwahori and Borel subgroups lead to a number of Bruhat decompositions of $G(\mathcal{K})$, $\mathcal{L}(G(\mathcal{K}))$, and $\hat{\mathcal{L}}(G)$, but we will not explicitly use them and just want to remark that all of them are indexed by the affine Weyl group. This means that the orbits under the Iwahori and Borel subgroups in these groups are in one-one correspondence to each other. For the affine Grassmannian it means that there is a one-one correspondence of $G(\mathcal{O})$-orbits in $G(\mathcal{K})$ with orbits of $\mathcal{L}(G(\mathcal{O}))$ in $\mathcal{L}(G(\mathcal{K}))$ and $\mathcal{P}_{\mathcal{O}}$-orbits in $\hat{\mathcal{L}}(G)$.

It is known that by this construction we obtain a new simple root α_0 and a corresponding reflection $s_0 \in W^{\mathfrak{a}}$. All notations and computation rules in the previous part apply to this root as well. Although we will not define any operators for these affine roots, they still occur in the description of elements in the Bott-Samelson variety as we will see later.

As in [BG06, §5.1] we will identify the real affine roots with $\Phi \times \mathbb{Z}$, where we identify $\alpha \in \Phi$ with $(\alpha, 0)$ and α_0 with $(-\theta, -1)$, where θ is the highest root in the classical root system. To each pair (α, n) we associate a reflection of $X^\vee \otimes \mathbb{R}$ as $s_{\alpha,n}(x) := x - (\langle \alpha, x \rangle - n)\alpha^\vee$. The corresponding (affine) reflection hyperplane $H_{\alpha,n}$ will become more important later in Section 4 in the theory of affine buildings.

Via these reflections we can define an action of the affine Weyl group on $X^\vee \otimes \mathbb{R}$. Recall that the affine Weyl group can also be seen as the semi-direct product of the coroot lattice with the classical Weyl group. This allows us to define an action of it on the set of real affine roots, via $(\tau_\lambda, w)(\alpha, n) := (w\alpha, n + \langle \alpha, \lambda \rangle)$, for τ_λ the translation with respect to an element $\lambda \in \mathbb{Z}\Phi^\vee$ and $w \in W$. These two actions are compatible with each other, meaning that if we apply an element w of the affine Weyl group to a reflection hyperplane $H_{\alpha,n}$ we obtain the reflection hyperplane corresponding to $w.(\alpha, n)$.

We denote the element $\alpha_0 - \theta$ by $\underline{\delta}$. In the above notations this means that $\underline{\delta}$ corresponds to $(0, -1)$ and thus any real affine root can be written, in a unique way, in the form $\alpha + n\underline{\delta}$, for $\alpha \in \Phi$ and $n \in \mathbb{Z}$.

Looking at the action of the affine Weyl group on the set of real affine roots it is also obvious that for the root (α, n) (or $\alpha - n\underline{\delta}$) we associate the one-parameter subgroup

$$U_{\alpha,n} = \{x_\alpha(at^n) \mid a \in \mathbb{C}\}.$$

3 Grassmannians and MV-cycles

We now want to introduce the main geometric objects that we will work with, the affine Grassmannian associated to our algebraic group, which is closely related to the affine Kac-Moody group, and the MV-cycles which are subvarieties in the affine Grassmannian, both of these follow [Kum02] and [MV04]. In addition we will also recall the definition of MV-polytopes like it can be found in [And03], [Kam05a], and [Kam05b].

3.1 The affine Grassmannian

We want to recall the notations and basic definitions of the affine Grassmannian for our group G, which is a variety quite similar to the generalized flag variety $\hat{\mathcal{L}}(G)/\hat{\mathcal{B}}$.

Definition 3.1. ([Kum02, 13.2.12]) For the group G, we denote by $\mathcal{G}_G := G(\mathcal{K})/G(\mathcal{O})$ the *affine Grassmannian* associated to G, following [MV04]. We want this set to be equipped with a reduced projective ind-scheme structure as in [Kum02, 13.2.12]. If no confusion can arise we will usually denote $\mathcal{G}_G$ by $\mathcal{G}$.

Remark 3.2. For sake of completeness we also want to give some alternative definitions for the affine Grassmannian, which involve the affine Kac-Moody group and the loop group

$$\mathcal{G} = \mathcal{L}(G(\mathcal{K}))/\mathcal{L}(G(\mathcal{O})) = \hat{\mathcal{L}}(G)/\mathcal{P}_{\mathcal{O}} = G(\mathbb{C}[t,t^{-1}])/G(\mathbb{C}[t]),$$

see [GL05, §2, (1)]. There are two variations that we want to mention, see also [MV04]. First, if instead of a semisimple group G we take $G = T$, a torus, then, as a reduced ind-scheme, $\mathcal{G}_T = X_*(T)$, the cocharacters of G. Second, if we take a reductive group G, we denote by $Z(G)$ the centre of G and by $Z = Z(G)^0$ its connected component of the neutral element. Furthermore let $\overline{G} = G/Z$, then $\mathcal{G}_G$ is a trivial covering of $\mathcal{G}_{\overline{G}}$ with covering group $X_*(Z)$. In addition, the connected components of $\mathcal{G}$ are indexed by the component group of $G(\mathcal{K})$, which is also isomorphic to $\pi_1(G)$, the topological fundamental group of G.

Inside the affine Grassmannian we are interested in two different types of orbits. We first remark that $G(\mathcal{O})$, which is a group scheme itself, acts on $\mathcal{G}$ by left multiplication with finite dimensional orbits and one can easily describe a way to index these orbits. Let $\lambda \in X_+^\vee$, by definition we can view λ as an element in $G(\mathcal{K})$ and denote its image in $\mathcal{G}$ by L_λ. We denote by $\mathcal{G}_\lambda = G(\mathcal{O}).L_\lambda$ the corresponding $G(\mathcal{O})$-orbit. It suffices to use $\lambda \in X_+^\vee$ as the orbit is closed under the Weyl group action. We can also describe the closure relations between these orbits as

$$\overline{\mathcal{G}_\lambda} = \bigcup_{\nu \in X_+^\vee,\ \nu \leq \lambda} \mathcal{G}_\nu,$$

see [MV04, §2, (2.2)] The closure $\overline{\mathcal{G}_\lambda}$ is called a *generalized Schubert variety.* This union is of course only a finite union as we only have to take into account dominant cocharacters and not arbitrary ones.

The second type of orbit we are interested in are the semi-infinite orbits S_ν^w for $\nu \in X^\vee$ and $w \in W$. These are defined as $S_\nu^w = wU^-(\mathcal{K})w^{-1}L_\nu$ and we usually denote S_ν^{id} by S_ν.

As before there are well known closure relations for these types of orbits, for this we introduce the following partial order $\geq_w$ on $X^\vee$ for any $w \in W$ as $\mu \geq_w \nu$ if and only if $w^{-1}\mu \geq w^{-1}\nu$. The closure of a *semi-infinite orbit* can then be decomposed as

$$\overline{S_\mu^w} = \bigcup_{\mu \geq_w \nu} S_\nu^w, \tag{1}$$

see [MV04, 2.1] or [Kam05b, 2.2, (4)]

3.2 MV-cycles and MV-polytopes

As described in the previous section, we have the orbits for two different sub groups and we want to look at the intersection of them or to be more precise at the closure of their intersection. The next definition is based on [MV04] and [MV99].

Definition 3.3. ([And03, §5.3, Def. 2] and [Kam05b, 2.2]) Let $\lambda \in X_+^\vee$ and $\mu \in X^\vee$. If the intersection $\mathcal{G}_\lambda \cap S_\mu$ is not empty we call the irreducible components of $\overline{\mathcal{G}_\lambda \cap S_\mu}$ the *MV-cycles of coweight* (λ, μ).

It was shown by Mirković and Vilonen, [MV04], based on earlier work by Ginzburg, [G.95], that the collection of all MV-cycles of coweights (λ, ν) for $\nu \in X^\vee$, form a natural basis of the irreducible representation of highest weight λ, $V(\lambda)$, for $G^\vee$, the Langlands dual group of G. The action of the Langlands dual group on the basis of MV-cycles was described more precisely by Vasserot, [Vas02], using first Chern classes of different canonical sheaves. Another way to define MV-cycles which is better suited for the viewpoint of Lusztig's canonical basis is to take irreducible components of $\overline{S_\mu \cap S_\lambda^{w_0}}$ with $\lambda \leq \mu$, those irreducible components that lie inside $\mathcal{G}_\lambda$ are then the ones described above. This also leads to the notion of stable MV-cycles as presented in [Kam05a], but we will omit this here, as it is not needed in our situation. This is due to the fact that stable MV-cycles are cosets of MV-cycles and there is always at most one representative in such a coset that is contained in $\mathcal{G}_\lambda$.

Definition 3.4. ([And03, §6, Prop 4] and [Kam05b, 2.2]) Let A be an MV-cycle of coweight (λ, μ) then we define its corresponding *MV-polytope* $P(A)$ as the convex hull of the set $\{\mu \in X^\vee \mid L_\mu \in A\}$ inside $X^\vee \otimes \mathbb{R}$. We call polytopes in $X^\vee \otimes \mathbb{R}$ arising in this way *MV-polytopes*.

As each MV-cycle is a T-invariant closed subvariety of the affine Grassmannian, it was shown by Anderson ([And03]) that this convex hull is the image of A under the moment map $\boldsymbol{\mu}$ of the affine Grassmannian with respect to the T-action. This is done by embedding the affine Grassmannian into an infinite dimensional projective space $\mathbb{P}(V)$ with V being the highest weight representation of the affine Kac Moody group $\hat{\mathcal{L}}(G)$ with highest weight Λ_0, the fundamental weight corresponding to α_0. This representation decomposed into weight spaces and after decomposing $v = \sum_{\nu \in X} v_\nu$ we define

$$\boldsymbol{\mu}([v]) = \sum_{\nu \in X} \frac{|v_\nu|^2}{|v|^2} \nu,$$

see [And03, §6] In addition to the images of the MV-cycles under the moment map it is also nice to know what the images of the semi-infinite cells are. Due to their closure relations we recalled in (1), it follows that

$$\boldsymbol{\mu}(\overline{S_\mu^w}) = C_\mu^w := \{p \in X^\vee \otimes \mathbb{R} \mid \langle p, w \cdot \Lambda_i \rangle \geq_w \langle \mu, w \cdot \Lambda_i \rangle \text{ for all } i\},$$

where the Λ_i's are the fundamental coweights, [Kam05b, 2.2]. Kamnitzer showed that the MV-polytopes are polytopes of a very special form, he called these pseudo-Weyl polytopes, an idea that is due to Anderson.

For $\lambda \in X_+^\vee$ we define the λ-*Weyl polytope* W_λ as $\mathrm{conv}(W.\lambda)$, the convex hull of the points in $W.\lambda$ in $\mathbb{R} \otimes X^\vee$. This image can also be described as an intersection of half spaces

$$W_\lambda = \bigcap_w C^w_{w\cdot\lambda},$$

see and [Kam05b, 2.3]. As done by Berenstein and Zelevinsky [BZ01], we call a weight of the form $w \cdot \Lambda_i$ a *chamber weight of level* i and denote the *set of all chamber weights by* Γ. Given a collection of coweights $\mu_\bullet = (\mu_w)_{w\in W}$ such that $\mu_v \geq_w \mu_w$ for all v, $w \in W$, we can define the polytope

$$P(\mu_\bullet) := \bigcap_w C^w_{\mu_w},$$

see [Kam05b, 2.3].

Definition 3.5. ([Kam05b, §2.3]) A polytope P in $X^\vee \otimes \mathbb{R}$ is called a *pseudo-Weyl polytope* if there exists a set of coweights $\mu_\bullet = (\mu_w)_{w\in W}$ satisfying $\mu_v \geq_w \mu_w$ for all v, $w \in W$, such that $P = P(\mu_\bullet)$.

Remark 3.6. A Weyl polytope is by definition also a pseudo-Weyl polytope.

As mentioned above, it was shown by Kamnitzer, [Kam05a], that the MV-polytopes are pseudo-Weyl polytopes. Since the MV-polytopes have the MV-cycle as a geometric counterpart it is non surprising that one can already associate to each pseudo-Weyl polytope a geometric object, such that when applying the moment map to this object, the original pseudo-Weyl polytope should occur as the image. These are the GGMS-strata.

Definition 3.7. ([Kam05b, §2.4]) Let $\mu_\bullet = (\mu_w)_{w\in W}$ be a set of coweights, such that $\mu_v \geq_w \mu_w$ for all v, $w \in W$. The corresponding *Gelfand-Goresky-MacPherson-Serganova* (or short GGMS) strata on the affine Grassmannian is

$$A(\mu_\bullet) := \bigcap_{w\in W} S^w_{\mu_w}.$$

The assumed inequalities for the coweights are exactly those that one needs to demand for the intersection to be non-empty. We will later go into more detail, why the GGMS-strata is of such great importance for the gallery model. It is a quite easy calculation to show that indeed this definition produces the right variety and that one has

$$\boldsymbol{\mu}(\overline{A(\mu_\bullet)}) = P(\mu_\bullet),$$

for a set of coweights satisifying the needed inequalities, see [Kam05b, 2.5].

In order to describe which pseudo-Weyl polytopes are indeed already MV-polytopes, Kamnitzer used the generalized minors of Berenstein and Zelevinsky [BZ01] and gave a set of equations for each case, that the heights of the walls of the MV-polytope have to satisfy. We first want to recall the needed functions on the affine Grassmannian and the group $G(\mathcal{K})$.

The first type of function we want to introduce is the valuation function for a chamber weight γ. To do this we want to define a valuation function on a $\mathcal{K}$-vector space that arises from a $\mathbb{C}$-vector space by extension of scalars. Let V be a $\mathbb{C}$ vector space. For an element $u \in V \otimes \mathcal{K}$ we define $\mathrm{val}(u) = k$ if $u \in V \otimes t^k\mathcal{O}$ and $u \notin V \otimes t^{k+1}\mathcal{O}$.

We now fix a highest weight vector v_{Λ_i} in each fundamental representation $V(\Lambda_i)$ of G. In addition we fix a vector $v_\gamma = \overline{w} v_{\Lambda_i}$ for each chamber weight $\gamma = w\Lambda_i$. Following [Kam05b, 2.5], we obtain an action of $G(\mathcal{K})$ on $V(\Lambda_i)\otimes\mathcal{K}$ in the usual way and define

$$\begin{aligned} D_\gamma : \mathcal{G} &\longrightarrow \mathbb{Z} \\ [g] &\mapsto \mathrm{val}(g v_\gamma). \end{aligned}$$

This is well-defined as $G(\mathcal{O})$ acts trivially with respect to the valuation.

To connect these functions with the GGMS-strata, it was shown by Kamnitzer that the following holds

$$S_\mu^w = \{L \in \mathcal{G} \mid D_{w\Lambda_i}(L) = \langle \mu, w\Lambda_i \rangle \text{ for all } i\}$$

for $w \in W$ and $\mu \in X^\vee$.

The second type of function that we will define are the generalized minors. Let γ be a chamber weight of level i, then we define

$$\begin{aligned} \Delta_\gamma : G &\longrightarrow \mathbb{C} \\ g &\mapsto \langle g v_\gamma, v_{-\Lambda_i} \rangle, \end{aligned}$$

where $v_{-\Lambda_i}$ is the fixed vector of weight $-\Lambda_i$ in $V(-w_0\Lambda_i) \cong V(\Lambda_i)^*$, the dual representation of $V(\Lambda_i)$.

These two types of functions are related in the following way.

Proposition 3.8. ***([Kam05b, §4.5])*** *Let $L \in \mathcal{G}$. Then there exists a $g \in G(\mathcal{K})$ such that $[g] = L$ and $D_\gamma(L) = val(\Delta_\gamma(g))$ for all chamber weights γ.*

As Berenstein and Zelevinsky already established a set of three-term Plücker relations for the generalized minors, it is only natural to try to obtain a similar set of relations for the valuation function. This was done by Kamnitzer via tropicalization. We want to recall these relations for sake of completeness.

Definition 3.9. ([Kam05b, §3.2]) Let $w \in W$, $i, j \in I$ be such that $ws_i > w$, $ws_j > w$, and $i \neq j$. Let $M_\bullet = (M_\gamma)_{\gamma\in\Gamma}$ be a set of integers indexed by chamber weights, we say that $M_\bullet$ satisfies the *tropical Plücker relations at* (w, i, j) if

(i) $a_{ij} = 0$,

(ii) if $a_{ij} = a_{ji} = -1$, and

$$M_{ws_i\Lambda_i} + M_{ws_j\Lambda_j} = \min(M_{w\Lambda_i} + M_{ws_is_j\Lambda_j}, M_{ws_js_i\Lambda_i} + M_{w\Lambda_j});$$

(iii) if $a_{ij} = -1$, $a_{ji} = -2$, and

$$\begin{aligned} M_{ws_j\Lambda_j} + M_{ws_is_j\Lambda_j} + M_{ws_i\Lambda_i} = \min\ &(2M_{ws_is_j\Lambda_j} + M_{w\Lambda_i},\\ &2M_{w\Lambda_j} + M_{ws_is_js_i\Lambda_i},\\ &M_{w\Lambda_j} + M_{ws_js_is_j\Lambda_j} + M_{ws_i\Lambda_i}),\end{aligned}$$

$$\begin{aligned} M_{ws_js_i\Lambda_i} + 2M_{ws_is_j\Lambda_j} + M_{ws_i\Lambda_i} = \min\ &(2M_{w\Lambda_j} + 2M_{ws_is_js_i\Lambda_i},\\ &2M_{ws_js_is_j\Lambda_j} + 2M_{ws_i\Lambda_i},\\ &M_{ws_is_js_i\Lambda_i} + 2M_{ws_is_j\Lambda_j} + M_{w\Lambda_i});\end{aligned}$$

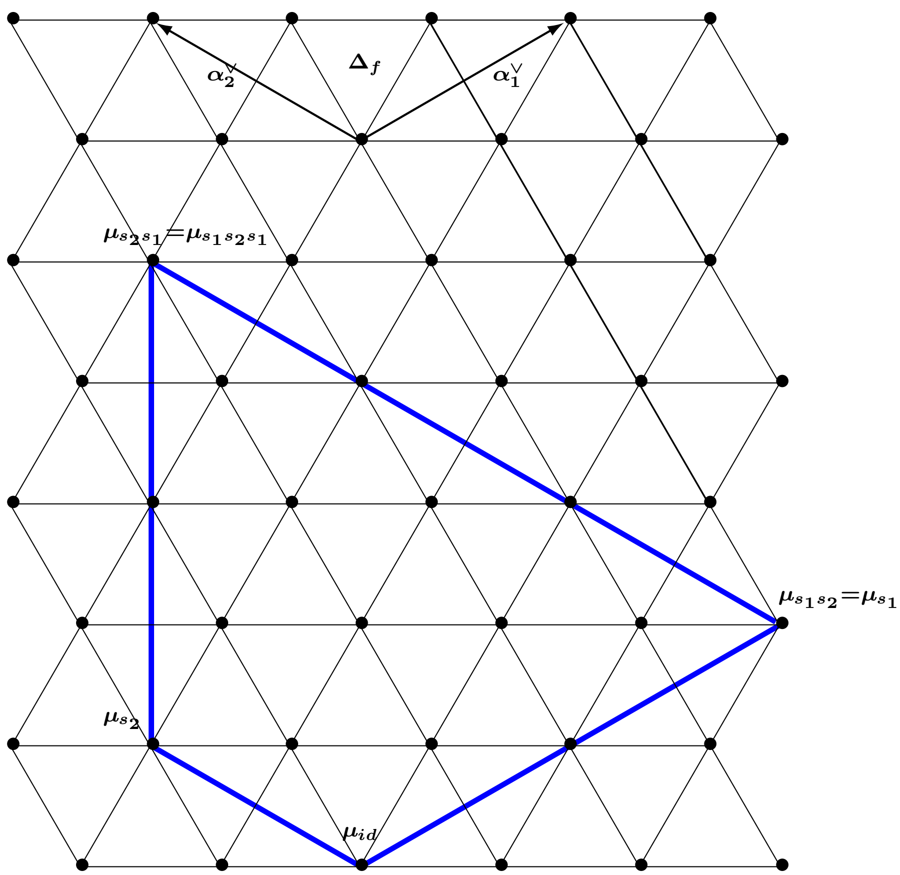
(iv) if $a_{ij} = -2$, $a_{ji} = -$, and

$$\begin{aligned} M_{ws_js_i\Lambda_i} + M_{ws_i\Lambda_i} + M_{ws_is_j\Lambda_j} = \min\ &(2M_{ws_i\Lambda_i} + M_{ws_js_is_j\Lambda_j},\\ &2M_{ws_is_js_i\Lambda_i} + M_{w\Lambda_j},\\ &M_{ws_is_js_i\Lambda_i} + M_{w\Lambda_i} + M_{ws_is_j\Lambda_j}),\end{aligned}$$

$$\begin{aligned} M_{ws_j\Lambda_j} + 2M_{ws_i\Lambda_i} + M_{ws_is_j\Lambda_j} = \min\ &(2M_{ws_is_js_i\Lambda_i} + 2M_{w\Lambda_j},\\ &2M_{w\Lambda_i} + 2M_{ws_is_j\Lambda_j},\\ &M_{w\Lambda_j} + 2M_{ws_i\Lambda_i} + M_{ws_js_is_j\Lambda_j}).\end{aligned}$$

We say that such a set satisfies the tropical Plücker relations if it satisfies the relations at each (w, i, j) that satisfies the conditions above.

Example 3.10. We want to give an example for an MV-polytope whose corresponding cycle forms a basis element for the representation $V(3\alpha_1^\vee + 3\alpha_2^\vee)$. Hence we have to start with a pseudo-Weyl polytope.



We have $\mu_{id} = -3\alpha_1^\vee - 3\alpha_2^\vee$, $\mu_{s_1} = \mu_{s_1 s_2} = -\alpha_1^\vee - 3\alpha_2^\vee$, $\mu_{s_2} = -3\alpha_1^\vee - 2\alpha_2^\vee$, and $\mu_{s_2 s_1} = \mu_{s_1 s_2 s_1} = -\alpha_1^\vee$.

Hence one calculates that

$$\begin{array}{rcl} M_{\Lambda_1} = \langle \mu_{id}, \Lambda_1 \rangle &=& -3 = M_{s_2\Lambda_1}, \\ M_{\Lambda_2} = \langle \mu_{id}, \Lambda_2 \rangle &=& -3 = M_{s_1\Lambda_2}, \\ M_{s_1\Lambda_1} = \langle \mu_{s_1}, s_1\Lambda_1 \rangle &=& -2 = M_{s_1s_2\Lambda_1}, \\ M_{s_2\Lambda_2} = \langle \mu_{s_2}, s_2\Lambda_2 \rangle &=& -1 = M_{s_2s_1\Lambda_2}, \\ M_{s_1s_2\Lambda_2} = \langle \mu_{s_1s_2}, s_1s_2\Lambda_2 \rangle &=& 1 = M_{s_1s_2s_1\Lambda_2}, \text{ and} \\ M_{s_2s_1\Lambda_1} = \langle \mu_{s_2s_1}, s_2s_1\Lambda_1 \rangle &=& 0 = M_{s_2s_1s_2\Lambda_1}. \end{array}$$

We now need to check the tropical Plücker relations, the only Weyl group element w that satisfies both $ws_1 > w$ and $ws_2 > w$ is of course $w = id$, hence we only have the equation

$$M_{s_1\Lambda_1} + M_{s_2\Lambda_2} = \min(M_{\Lambda_1} + M_{s_1s_2\Lambda_2}; M_{s_2s_1\Lambda_1} + M_{\Lambda_2}).$$

Inserting the above values, we obtain that both sides are equal to -3, thus the polytope is an MV-polytope of coweight $(3\alpha_1^\vee + 3\alpha_2^\vee, -\alpha_1^\vee)$.

For a pseudo-Weyl polytope $P(\mu_\bullet)$ we can now define such a set of integers in the following way. Kamnitzer's main result about the MV-polytope can then be summed up as follows.

Theorem 3.11. ***([Kam05b, Theorem 3.1])*** *Let $P(\mu_\bullet)$ be a pseudo-Weyl polytope. Let $M_{w\Lambda_i} := \langle \mu_w, w\Lambda_i \rangle$ and $M_\bullet = (M_\gamma)_{\gamma\in\Gamma}$. $P(\mu_\bullet)$ is an MV-polytope if and only if $M_\bullet$ satisfies the tropical Plücker relations.*

Corollary 3.12. ***([Kam05b, Theorem 3.1])*** *Let $\mu_\bullet$ be a collection of coweights satisfying the same assumptions as in the theorem above. Then $\overline{A(\mu_\bullet)}$ is an MV-cycle if and only if $M_\bullet$ satisfies the tropical Plücker relations.*

Remark 3.13. Let M be an MV-cycle and P the corresponding MV-polytope, then for any $L \in M$ it holds $D_\gamma(L) = M_\gamma$ for any $\gamma \in \Gamma$.

In addition to the description given above, there is a second possibility to define MV-polytopes by using Lusztig's canonical basis and its parametrisation. Let us write $\mathbb{B}$ for *Lusztig's canonical basis* for $U_-^\vee$, the lower triangular part of the quantized enveloping algebra of $G^\vee$, it was shown in [Lus90] that for any choice of a reduced expression $w_0^{\underline{i}}$ of w_0 there exists a bijection $\phi_{\underline{i}} : \mathbb{B} \to \mathbb{N}^{|\Phi^+|}$. We will call $\phi_{\underline{i}}(b)$ the $\underline{i}$-Lusztig datum of b, see [BZ01].

Following [Kam05a], we can also define an $\underline{i}$-Lusztig datum for each pseudo-Weyl polytope as follows. Let $w_0^{\underline{i}} = s_{i_1} \cdot \ldots \cdot s_{i_m}$ be a reduced expression of w_0 with $m = |\Phi^+|$ and P a fixed pseudo-Weyl polytope, then there is a unique path through the 1-skeleton of P going through the vertices μ_e, $\mu_{s_{i_1}}$, $\mu_{s_{i_1}s_{i_2}}$, ..., μ_{w_0}. Let n_1, ..., n_m be the length of the edges of this path, the sequence $(n_j)_{1 \leq j \leq m}$ is then called the $\underline{i}$-Lusztig datum $n_{\underline{i}}(P)$. A result by Kamnitzer is the following theorem, which essentially says that for the set of MV-polytopes this gives a bijection.

Theorem 3.14. ***([Kam05a, Theorem 3.2])*** *Fix $\lambda \in X_+^\vee$ and let $\mathbb{B}^\lambda$ be the set of canonical bases elements that do not act as zero on the highest weight vector of $V(\lambda)$. Then there exists a crystal bijection Ψ between the set of MV-polytopes inside $W.\lambda$ and $\mathbb{B}^\lambda$ with the property that the $\underline{i}$-Lusztig datum of an MV-polytope P equals the $\underline{i}$-Lusztig datum of $\Psi(P)$.*

This was proved by Kamnitzer by showing that both Lusztig datums satisfy the same tropical relations, derived from the Plücker relations.

We will give an alternative proof of this fact in the last section that only deals with the crystal combinatorics of the LS-gallery model as well as some results of [BZ01] and [MG03].

4 Galleries

In this part we want to recall the definitions of different buildings and their properties that will be needed later on. Most of this follows [GL05] with a small amount of modifications at some places and simplifications at others. In addition we also want to recall the definition of the Bott-Samelson variety as it is also used in both [Kum02] and [GL05]. For some of the original statements, we refer to [Dem74] and [Han73].

4.1 Buildings and galleries

As mentioned above we start by recalling some of the definitions and results about buildings and continue with the associated gallery model for representations of G. We will mostly follow [GL05] and also use the same notations in most cases. We will also recall some basic facts about buildings from [Tit74], [Ron89], and [Bro98].

The first building that appears in this set-up is the *spherical building*, which as a set is just the set of parabolic subgroups of G with the opposite of the inclusion making it into a simplical complex. To each torus one associated an apartment consisting of all those parabolic subgroups that contain the given torus. This gives the simplical complex the structure of a building, denoted by $\mathcal{J}^s$, more details can be found in [Ron89, §4 - §6]. Each of its apartments is isomorphic to the Coxeter complex $C(W, S)$ of the Weyl group G, that is defined via the Coxeter presentation.

However for our purposes we need the affine version of this building, see [Ron89, §9 and §10]. As a set one can identify the affine building $\mathcal{J}^{\mathfrak{a}}$ with the set of all parahoric subgroups of $G(\mathcal{K})$. But the explicit construction is a bit more involved as we will define it in a way that makes the definition of galleries simpler.

Let $\mathcal{A} := X^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, then the affine Weyl group $W^{\mathfrak{a}}$ acts on $\mathcal{A}$ as an affine reflection group. We denote by $H^{\mathfrak{a}} \subset \mathcal{A}$ the set of affine reflection hyperplanes for this action. All hyperplanes in $H^{\mathfrak{a}}$ are of the form $H_{\alpha,m} = \{x \in \mathcal{A} \mid \langle x, \alpha\rangle = m\}$ for some positive root α and some integer m. The reflection to such a hyperplane will be denoted by $s_{\alpha,m}$ and the two corresponding *half-spaces* by $H^+_{\alpha,m} = \{x \in \mathcal{A} \mid \langle x, \alpha\rangle \geq m\}$ and $H^-_{\alpha,m} = \{x \in \mathcal{A} \mid \langle x, \alpha\rangle \leq m\}$.

The connected components of the $\mathcal{A} \setminus H^{\mathfrak{a}}$ are called the *open alcoves* and hence the closure of such an open alcove is called a *closed alcove* or *alcove* for short. In addition to alcoves, there is also the slightly more general notion of faces, with alcoves being the faces of maximal dimension.

Definition 4.1. ([Bro98, I.4]) A face F is a subset of $\mathcal{A}$ of the form

$$\bigcap_{(\beta,m)\in\Phi^+\times\mathbb{Z}} H_{\beta,m}^{\epsilon_{\beta,m}},$$

with $\epsilon_{\beta,m} \in \{+,-,\emptyset\}$ and $H^{\emptyset}_{\beta,m} = H_{\beta,m}$. In an analogous way we define open faces, by using open half-spaces instead of closed ones.

The *fundamental alcove* is the subset $\Delta_f := \{x \in \mathcal{A} \mid 0 \leq \langle x, \beta\rangle \leq 1, \forall \beta \in \Phi^+\}$ and we define $S^{\mathfrak{a}} = \{s_{\beta,m} \mid H_{\beta,m} \text{ is a wall for } \Delta_f\}$ the set of affine reflections that generate the affine Weyl group $W^{\mathfrak{a}}$. Furthermore, for a face F of Δ_f we define

$$S^{\mathfrak{a}}(F) = \{s_{\beta,m} \in S^{\mathfrak{a}} \mid F \subset H_{\beta,m}\}$$

and call this the *type* of F. This means that

$$S^{\mathfrak{a}}(0) = S = \{s_{\beta,0} \mid H_{\beta,0} \text{ is a wall for } \Delta_f\}$$

and $S^{\mathfrak{a}}(\Delta_f) = \emptyset$. One defines the type of an arbitrary face F' as the type of the unique face of Δ_f that lies in the $W^{\mathfrak{a}}$-orbit of F'. Hence the type of an alcove if always trivial, while the type of a codimension 1 face always consists of exactly one simple reflection.

In addition to the faces, one also defines subsets that are larger than alcoves and resemble the spherical building, this is needed for the definition of retractions. An connected component of $\mathcal{A} \setminus \bigcup_{\beta\in\Phi^+} H_{\beta,0}$ is called an *open chamber* and its closure is called a *chamber*. Corresponding to our choice of the Borel subgroup B, we have the *dominant chamber* $\mathfrak{C}_f$ as the chamber that contains Δ_f and the *anti-dominant chamber* $\mathfrak{C}_{-f} = w_0\mathfrak{C}_f$. Of course all chambers are in the W-orbit of $\mathfrak{C}_f$. This leads to the definition of sectors, which are $W^{\mathfrak{a}}$-translates of chambers.

Definition 4.2. ([Ron89, 9.1]) A *sector* $\mathfrak{s}$ in $\mathcal{A}$ is a $W^{\mathfrak{a}}$-translate of a chamber. Two sectors $\mathfrak{s}$, $\mathfrak{s}'$ are called equivalent if there exists a third sector $\mathfrak{s}''$ in the intersection $\mathfrak{s} \cap \mathfrak{s}'$.

To define the building itself we also need a number of unipotent subgroups of $U_\beta(\mathcal{K})$, with $U_\beta = \{x_\beta(t) \mid t \in \mathbb{C}\}$:

$$U_{\beta,r} := \{1\} \cup \{exp(X_\beta \otimes f) \mid f \in \mathcal{K}^*, v(f) \geq r\},$$

where X_β is the corresponding root subspace of the Lie algebra associated to G and r is an arbitrary integer. For a non-empty subset $\Omega \subset \mathcal{A}$ let $\ell_\beta(\Omega) = -\inf_{x\in\Omega}\beta(x)$ and we define

$$U_\Omega := \left\langle U_{\beta,\ell_\beta(\Omega)} \mid \beta \in \Phi \right\rangle .$$

Then we can define the affine building in the following way.

Definition 4.3. The *affine building* $\mathcal{J}^{\mathfrak{a}} := G(\mathcal{K}) \times \mathcal{A}/\sim$ associated to G is the quotient of $G(\mathcal{K}) \times \mathcal{A}$ via the equivalence relation

$$(g,x) \sim (h,y) \quad \text{if } \exists n \in N(\mathcal{K}) \text{ such that } nx = y \text{ and } g^{-1}hn \in U_x.$$

The affine building is naturally equipped with a $G(\mathcal{K})$ action $g \cdot (h,y) := (gh,y)$ for $g \in G(\mathcal{K})$ and $(h,y) \in \mathcal{J}^{\mathfrak{a}}$ and the obvious map that embeds $\mathcal{A}$ into $\mathcal{J}^{\mathfrak{a}}$, via the 1 in $G(\mathcal{K})$, is injective and $N(\mathcal{K})$ equivariant, hence we can identify $\mathcal{A}$ with its image in the affine building. Together with the above mentioned action of $G(\mathcal{K})$ we can look at the orbit of the set $\mathcal{A}$.

Definition 4.4. A subset $g\mathcal{A}$ of $\mathcal{J}^{\mathfrak{a}}$ is called an *apartment*.

We can now introduce the last of the three occurring buildings, the *spherical building at infinity* $\mathcal{J}^\infty$, see [Ron89, 9.3]. This building has the same apartments as $\mathcal{J}^{\mathfrak{a}}$, but the structure of the simplical complex for each apartment is different. The chambers of the complex are the equivalence classes of sectors, hence the structure is similar to the spherical building as the equivalence classes of sectors are in one-to-one correspondence with the spherical chambers. To define retractions for all Weyl group elements, we make two following definitions.

Definition 4.5. ([Ron89, 9.3]) For a $w \in W$, the equivalence class of sectors of the Weyl chamber $w\mathfrak{C}_f$ is called the *chamber of w at infinity* and is denoted by $\mathfrak{C}_w^\infty$.

Definition 4.6. ([Bro98, I.4]) A *gallery* in the affine building is a sequence of faces γ in $\mathcal{J}^{\mathfrak{a}}$

$$\gamma = (\Gamma_0' \subset \Gamma_0 \supset \Gamma_1' \subset \ldots \supset \Gamma_p' \subset \Gamma_p \supset \Gamma_{p+1}'),$$

such that

- Γ_0' and Γ_{p+1}' are vertices of $\mathcal{J}^{\mathfrak{a}}$,

- the Γ_j's are alcoves,
- each Γ'_j, for $j \in \{1, \ldots, p\}$, is a face of Γ_{j-1} and Γ_j, of relative dimension one.

Remark 4.7. Is is not exactly the definition of galleries as it was given in [GL05], as their definition was much more general and also allowed galleries where all the "large" faces Γ_i were not alcoves but arbitrary faces of the same dimension and the "small" faces Γ'_i were faces of codimension one.

A gallery that is contained in our fixed apartment $\mathcal{A}$ will be called a *combinatorial gallery*.

For any two alcoves Δ and Δ' we define $d(\Delta, \Delta')$ to be the length of a minimal gallery of alcoves connecting these two, i.e., a gallery such that $\Gamma_0 = \Delta$ and $\Gamma_p = \Delta'$. We call $d(\Delta, \Delta')$ the *distance* between these two alcoves.

Then for any alcove Δ in the fixed apartment $\mathcal{A}$ one can define a map $r_{\Delta,\mathcal{A}} : \mathcal{J}^{\mathfrak{a}} \to \mathcal{A}$ of chamber complexes, called the retraction onto $\mathcal{A}$ with centre Δ. This map has a number of important properties:

(i) For any face F of Δ, including Δ itself, $r^{-1}_{\Delta,\mathcal{A}}(F) = \{F\}$.

(ii) For any alcove Δ', $d(\Delta, \Delta') = d(\Delta, r_{\Delta,\mathcal{A}}(\Delta'))$.

(iii) The map is type-preserving.

For the second property, $d(\Delta, \Delta')$ denotes the distance between the two alcoves, which is equal to a minimal length of a gallery of alcoves connecting those Δ and Δ'. By a gallery we denote a sequence of adjacent alcoves, i.e., where successive alcoves have a common codimension one face. This map restricts to an isomorphism of chamber complexes for any apartment $g\mathcal{A}$ and $\mathcal{A}$.

To define the retractions at infinity, we have to choose alcoves that are sufficiently far away from each other. To make this more precise, if we take an alcove $\Delta' \subset \mathcal{J}^{\mathfrak{a}}$, then there exists an element $\mathfrak{s}$ in $\mathfrak{C}^{\infty}_w$, a sector in $\mathcal{A}$, such that Δ' and $\mathfrak{s}$ lie in the same apartment $g\mathcal{A}$. For an arbitrary alcove $\Delta \subset \mathfrak{s}$, the retraction $r_{\Delta,\mathcal{A}}$ restricts to an isomorphism between $g\mathcal{A}$ and $\mathcal{A}$, which fixes the common sector $\mathfrak{s}$. The map is indeed independent of the chosen $\Delta \subset \mathfrak{s}$.

Definition 4.8. ([Ron89, §3.3], [Tit74, ??], and [Bro98, VI.8]) For $w \in W$, the map $r_w : \mathcal{J}^{\mathfrak{a}} \to \mathcal{A}$, defined by $r_w(\Delta') = r_{\Delta,\mathcal{A}}(\Delta')$ for some alcove $\Delta \subset \mathfrak{s}$, with $\mathfrak{s} \in \mathfrak{C}_w^\infty$, contained in a common apartment with Δ', is called the *retraction of centre w at infinity.*

The following important property of the retractions at infinity, was already stated in [GL05] and can be found in more details in [Ron89].

Proposition 4.9. ***([GL05, §3.4, Prop. 1])*** *The fibres of the map $r_w : \mathcal{J}^{\mathfrak{a}} \to \mathcal{J}$ are the $wU(\mathcal{K})w^{-1}$-orbits on $\mathcal{J}^{\mathfrak{a}}$.*

This is already a good indication that the retractions are related to the semi-infinite cells of the affine Grassmannian. To make this more precise we need the notation of the Bott-Samelson resolution of the generalized Schubert varieties.

For any subset Ω of and an alcove F in the same apartment A, we say that a wall H separates them if Ω and F are not contained in the same closed half-space corresponding to H. If E is a face and E and F are contained in an apartment A, we denote by $\mathcal{M}(E, F)$ the set of walls that separate them in A. This leads to the definition of maximal distance. Let Δ be an alcove in A, such that $E \subset \Delta$, then there exists a unique alcove in A, see [GL05, § 4], denoted by $proj_F(\Delta)$, such that any face of the convex hull of Δ and F containing F is contained in $proj_F(\Delta)$.

Definition 4.10. ([GL05, §4, Def. 9]) The alcove $\Delta \supset E$ is said to be at *maximal distance* to F if the length of a minimal gallery between Δ and $proj_F(\Delta)$ is $\#\mathcal{M}(E, F)$.

Remark 4.11. If a gallery

$$\gamma = (F_f = \Gamma_0' \subset \Gamma_0 \supset \Gamma_1' \subset \ldots \supset \Gamma_p' \subset \Gamma_p \supset F'),$$

is minimal, then the set $\mathcal{M}(F_f, F)$ is the disjoint union of the sets

$$\mathcal{H}_j := \{H \in H^A \mid \Gamma_j' \in H, \Gamma_j \notin H\},\ j = 0, \ldots, p.$$

In addition, any apartment that contains Γ_0 and $proj_F(\Delta)$ also contains all minimal galleries between the two alcoves.

For the LS-gallery model we only need to look at galleries that join the origin with a certain coweight λ.

Definition 4.12. ([GL05, §4, Def. 11]) A *combinatorial gallery joining* 0 *with* λ is a gallery γ in $\mathcal{A}$ that starts at F_f (the face of type S in $\mathcal{A}$) and ends in F_λ (the face corresponding to λ, via the embedding $X^\vee \to \mathcal{A}$):

$$\gamma = (F_f = \Gamma_0' \subset \Gamma_0 \supset \Gamma_1' \subset \ldots \supset \Gamma_p' \subset \Gamma_p \supset F_\lambda).$$

If one chooses a regular dominant coweight, i.e., a dominant coweight that does not lie on the boundary of the fundamental chamber, and demands the gallery to be minimal, it is clear that the whole gallery is contained in the dominant chamber $\mathfrak{C}_f$ and Γ_0 is the fundamental alcove Δ_f. As we did for an alcove or face, we can also define the notion of type for a whole gallery.

Definition 4.13. ([GL05, §4]) Let $\lambda \in X_+^\vee$ be a dominant coweight and let γ_λ be a minimal combinatorial gallery joining F_f and F_λ

$$\gamma_\lambda = (F_f = \Gamma_0' \subset \Gamma_0 \supset \Gamma_1' \subset \ldots \supset \Gamma_p' \subset \Gamma_p \supset F_\lambda).$$

Then the *gallery of types* associated to γ_λ is the list of the types of all the faces of the gallery γ_λ:

$$t_{\gamma_\lambda} = type(\gamma_\lambda) = (S = t_0' \supset t_0 \subset t_1' \supset \ldots \subset t_p' \supset t_p \subset t_\lambda),$$

where t_λ is the type of the face F_λ and t_j, respectively t_j', is the type of the face of Γ_j, respectively Γ_j' for all $0 \leq j, j' \leq p$.

As we have restricted ourselves to galleries of alcoves and not arbitrary ones, the types of the large faces will always be trivial by definition, hence we can omit them and only have the type $t_{\gamma_\lambda} = (S = t_0', t_1', \ldots, t_p', t_\lambda)$.

We will denote by $\Gamma(\gamma_\lambda)$ the set of all combinatorial galleries of type t_{γ_λ} starting at the origin.

Using the type we can associate to each element in $\Gamma(\gamma_\lambda)$ a sequence of reflections. Therefore we consider the following group

$$W \times W_1' \times W_2' \times \ldots \times W_p',$$

where W_j' is the Coxeter subgroup of $W^{\mathfrak{a}}$ generated by the reflections in t_j'. The elements of this group are in bijection to the set of all combinatorial galleries of type t_{γ_λ} starting at the origin.

We will write $[\delta_0, \delta_1, \ldots, \delta_p]$ for the image of a gallery under this bijection and s_{i_j} for the generator of W_j'.

We will say that a gallery $\delta = [\delta_0, \delta_1, \ldots, \delta_p] = (F_f \subset \Gamma_0 \supset \Gamma'_1 \subset \ldots \supset \Gamma'_p \subset \Gamma_p \supset F_\lambda)$ is folded around the small face Γ'_j if $\delta_j \neq s_{i_j}$. By applying affine reflections at a wall containing the face Γ'_j to the part of the gallery that comes thereafter we inductively obtain a sequence of galleries in $\Gamma(\gamma_\lambda)$:

$$\gamma_0 = [\delta_0, s_{i_1}, s_{i_2}, \ldots, s_{i_p}],$$

$$\gamma_1 = [\delta_0, \delta_1, s_{i_2}, \ldots, s_{i_p}],$$

$$\gamma_2 = [\delta_0, \delta_1, \delta_2, s_{i_3}, \ldots, s_{i_p}],$$

$$\vdots$$

$$\gamma_{p-1} = [\delta_0, \delta_1, \delta_2, \ldots, \delta_{p-1}, s_{i_p}],$$

$$\gamma_{p-1} = \delta.$$

For each $i > 0$, the gallery γ_i is obtained from its predecessor γ_{i-1} by either folding at the hyperplane containing Γ'_j or doing nothing.

Definition 4.14. ([GL05, §4, Def. 13]) The gallery $\delta \in \Gamma(\gamma_\lambda)$ is called *positively folded* at Γ'_j if $\delta_j = id$ and the half-space, which contains Γ_j and corresponds to the reflection hyperplane (that contains Γ'_j), can be separated from the anti-dominant chamber (i.e., there exists a sector $\mathfrak{s}$ of $\mathfrak{C}^\infty_{w_0}$ such that the half-space and $\mathfrak{s}$ are separated by the reflection hyperplane). We say that the gallery is positively folded if all foldings are positive.

The set of positively folded galleries of type t_{γ_λ} will be denoted by $\Gamma^+(\gamma_\lambda)$. This set is still a bit to big to be a combinatorial model for the representation $V(\lambda)$. To reduce the number of galleries, we have to introduce the notion of LS-galleries and dimension of a gallery.

Let $\gamma = (F_f \subset \Gamma_0 \supset \Gamma'_1 \subset \ldots \supset \Gamma'_p \subset \Gamma_p \supset F_\nu)$ be in $\Gamma^+(\gamma_\lambda)$. For $j = 0, \ldots, p$ let $\mathcal{H}_j$ be the set of all affine reflection hyperplanes H in $H^\mathfrak{a}$ such that $\Gamma'_j \subset H$. In all cases except $j = 0$ this set will only consist of a single hyperplane as we have restricted ourselves to galleries of alcoves and for $j = 0$ it will consist of all classical reflection hyperplanes. We say that an affine hyperplane H is a *load-bearing* wall for γ at Γ_j if $H \in \mathcal{H}_j$ and H separates Γ_j from $\mathfrak{C}^\infty_{w_0}$, see [GL05, §5].

Definition 4.15. ([GL05, §10]) The set of all load-bearing walls of a combinatorial gallery δ, is denoted by $J_{-\infty}(\delta)$. This set can be divided into two subsets $J^+_{-\infty}(\delta)$ and $J^-_{-\infty}(\delta)$. An index $j \in J_{-\infty}(\delta)$ is in $J^+_{-\infty}(\delta)$ if Γ_{j-1} is not separated from $\mathfrak{C}^\infty_{w_0}$, otherwise $j \in J^-_{-\infty}(\delta)$

This partition into non load-bearing walls and the two different types of load-bearing walls will be useful for describing a dense open subset of the MV-cycles using our galleries.

Remark 4.16. By definition all folding hyperplanes of a positively folded gallery are load-bearing walls and the corresponding indices are in $J^-_{-\infty}(\delta)$.

This leads to the definition of the dimension of a gallery.

Definition 4.17. ([GL05, §5, Def. 14]) The *dimension of a gallery* $\gamma \in \Gamma^+(\gamma_\lambda)$ is defined as:

$$\dim\gamma = \#\{(H, \Gamma_j) \mid H \text{ is a load-bearing wall for } \gamma \text{ at } \Gamma_j\}.$$

Definition 4.18. ([BG06, §5.2] and [GL05, §5, Def. 15]) A positively folded gallery γ of type t_{γ_λ} joining the origin with ν is called an *LS-gallery of type* t_{γ_λ} if $\dim\gamma = \langle \lambda + \nu, \rho \rangle + \dim(P_\lambda/B)$, where P_λ is the standard parabolic subgroup P_J for $J = \{j \in I \mid \langle \alpha_j, \lambda \rangle = 0\}$. The set of all positively folded LS galleries joining the origin and ν of type t_{γ_λ} is denoted by $\Gamma^+_{LS}(\gamma_\lambda)$.

The set of LS-galleries are the set of galleries with maximal dimension as it was shown by [GL05] that a gallery in $\Gamma^+(\gamma_\lambda)$ joining the origin and ν is of dimension smaller or equal to $\langle \lambda + \nu, \rho \rangle$.

As we will need them in the following we want to introduce a few other sets of galleries that are, more or less, only slight modifications of the existing ones and simplify the notations. Basically one has to substitute $\mathfrak{C}^\infty_{w_0 w}$ for $\mathfrak{C}^\infty_{w_0}$ in the last 4 definitions to obtain the following sets.

- $\Gamma^w(\gamma_\lambda)$, the set of combinatorial galleries of type t_{γ_λ} that are positively folded with respect to the chamber at infinity $\mathfrak{C}^\infty_{w_0 w}$.
- $\Gamma^w_{LS}(\gamma_\lambda)$, the set of combinatorial galleries of type t_{γ_λ} that are positively folded and LS with respect to the chamber at infinity $\mathfrak{C}^\infty_{w_0 w}$.
- $\Gamma^w(\gamma_\lambda, \nu)$, the set of combinatorial galleries of type t_{γ_λ} that are positively folded with respect to the chamber at infinity $\mathfrak{C}^\infty_{w_0 w}$ that join the origin and ν.
- $\Gamma^w_{LS}(\gamma_\lambda, \nu)$, the set of combinatorial galleries of type t_{γ_λ} that are positively folded and LS with respect to the chamber at infinity $\mathfrak{C}^\infty_{w_0 w}$ that join the origin and ν.

4.2 Root operators

We now want to define the folding operators which will be used in the rest of this paper that endow the set $\Gamma^+_{LS}(\gamma_\lambda)$ with the structure of a crystal. The following definitions also make sense for arbitrary combinatorial galleries, not necessarily LS or positively folded, but we will not need them in that generality.

Most of these definitions are also available for the path model and much of the combinatorics that we will deal with in Section 5 works with path as well. For more details about the path-model we refer to [Lit98b], [Lit94], [Lit95], [Lit03], and [Lit97].

To define the crystal structure on the set of LS-galleries we first want to recall what a crystal is.

Definition 4.19. ([Kas95, §7.2]) A finite set $\mathbb{B}$ endowed with the following map

$$wt : \mathbb{B} \longrightarrow X^\vee$$

and for each simple root α with the maps

$$\begin{aligned} \varepsilon_\alpha : \mathbb{B} &\longrightarrow \mathbb{Z} \\ \varphi_\alpha : \mathbb{B} &\longrightarrow \mathbb{Z} \\ e_\alpha : \mathbb{B} &\longrightarrow \mathbb{B}\dot\cup\{0\} \\ f_\alpha : \mathbb{B} &\longrightarrow \mathbb{B}\dot\cup\{0\}. \end{aligned}$$

is called a *finite dimensional crystal* of the group $G^\vee$, if the following axioms are fulfilled.

(i) $\varphi_\alpha(b) = \varepsilon_\alpha(b) + \langle \alpha, wt(b)\rangle$ for each simple root α

(ii) If $b \in \mathbb{B}$ and satisifies $e_\alpha(b) \neq 0$, then
$\varepsilon_\alpha(e_\alpha(b)) = \varepsilon_\alpha(b) - 1$
$\varphi_\alpha(e_\alpha(b)) = \varphi_\alpha(b) + 1$
$wt(e_\alpha(b)) = wt(b) + \alpha^\vee$.

(iii) If $b \in \mathbb{B}$ and satisifies $f_\alpha(b) \neq 0$, then
$\varepsilon_\alpha(f_\alpha(b)) = \varepsilon_\alpha(b) + 1$

$$\varphi_\alpha(f_\alpha(b)) = \varphi_\alpha(b) - 1$$
$$wt(f_\alpha(b)) = wt(b) - \alpha^\vee.$$

(iv) For $b_1, b_2 \in \mathbb{B}$, $b_2 = f_\alpha b_1$ if and only if $b_1 = e_\alpha b_2$.

In the definition of e_α and f_α, we understand 0 as an element that is not contained in $\mathbb{B}$.

For this, let α be a simple root, λ a dominant coweight, and $\nu \prec \lambda$ an arbitrary coweight. We fix a combinatorial gallery $\gamma \in \Gamma^+(\gamma_\lambda)$ joining the origin and ν for a given type γ_λ:

$$\gamma = (F_f = \Gamma'_0 \subset \Gamma_0 \supset \Gamma'_1 \subset \ldots \supset \Gamma'_p \subset \Gamma_p \supset F_\nu).$$

Let $m \in \mathbb{Z}$ be minimal with the following property: A face Γ'_k is contained in $H_{\alpha,m}$. Note that this automatically implies $m \leq 0$.

Definition 4.20. ([GL05, §6, Def. 16])
Let $\gamma \in \Gamma^+_{LS}(\gamma_\lambda)$ and m as above.

I) If $m \leq -1$. Let k be minimal such that $\Gamma'_k \subset H_{\alpha,m}$ and fix $0 \leq j \leq k$ maximal such that $\Gamma'_j \subset H_{\alpha,m+1}$.

Then $e_\alpha\gamma$ is defined by

$$e_\alpha\gamma = (F_f = \Delta'_0 \subset \Delta_0 \supset \Delta'_1 \subset \ldots \supset \Delta'_p \subset \Delta_p \supset F_{\tilde{\nu}}),$$

where $\Delta_i = \begin{cases} \Gamma_i & \text{for } i < j \\ s_{\alpha,m+1}(\Gamma_i) & \text{for } j \leq i < k \\ t_{\alpha^\vee}(\Gamma_i) & \text{for } i \geq k. \end{cases}$
and $t_{\alpha^\vee}$ is the translation by $\alpha^\vee$.

II) If $\langle \nu, \alpha \rangle - m \geq 1$. Let j be maximal such that $\Gamma'_j \subset H_{\alpha,m}$ and fix $j \leq k \leq p+1$ minimal such that $\Gamma'_k \subset H_{\alpha,m+1}$.

Then $f_\alpha\gamma$ is defined by

$$f_\alpha\gamma = (F_f = \Delta'_0 \subset \Delta_0 \supset \Delta'_1 \subset \ldots \supset \Delta'_p \subset \Delta_p \supset F_{\tilde{\nu}}),$$

$$\text{where } \Delta_i = \begin{cases} \Gamma_i & \text{for } i < j \\ s_{\alpha,m}(\Gamma_i) & \text{for } j \leq j \leq k-1 \\ t_{-\alpha^\vee}(\Gamma_i) & \text{for } i \geq k. \end{cases}.$$

Remark 4.21. The first remark about this definition is of course the fact, that if the operators e_α or f_α are defined, their images are again LS-galleries, [GL05].

In addition to the two types of operators we also want to have to define the maps ε_α, φ_α and wt.

Definition 4.22. Let γ be a combinatorial gallery, then we define

(i) $wt(\gamma) = \nu$,

(ii) $\epsilon_\alpha(\gamma) = \max_m\{e_\alpha^m(\gamma) \text{ is defined}\}$,

(iii) $\varphi_\alpha(\gamma) = \max_m\{f_\alpha^m(\gamma) \text{ is defined}\}$.

We want to recall a few of the main properties of the root operators that can be found in [GL05] and that will be used later in the part about sections and root operators.

Proposition 4.23. ***([GL05, §6, Lemma 5])*** *Let $\gamma \in \Gamma(\gamma_\lambda, \nu)$, then the following properties hold:*

(i) The gallery $e_\alpha\gamma$ is not defined if and only if $m = 0$, and if $e_\alpha\gamma$ is defined, then $e_\alpha\gamma \in \Gamma(\gamma_\lambda, \nu - \alpha^\vee)$.

(ii) The gallery $f_\alpha\gamma$ is not defined if and only if $m = \langle\nu, \alpha^\vee\rangle$, and if $f_\alpha\gamma$ is defined, then $f_\alpha\gamma \in \Gamma(\gamma_\lambda, \nu + \alpha^\vee)$.

(iii) If $e_\alpha\gamma$ is defined, then $f_\alpha(e_\alpha\gamma)$ is defined and equal to γ. In addition $m + 1$ is minimal such that a face of the gallery $e_\alpha\gamma$ is contained in a hyperplane corresponding to α.

(iv) If $f_\alpha\gamma$ is defined, then $e_\alpha(f_\alpha\gamma)$ is defined and equal to γ. In addition $m - 1$ is minimal such that a face of the gallery $f_\alpha\gamma$ is contained in a hyperplane corresponding to α.

(v) $\epsilon_\alpha(\gamma) - \varphi_\alpha(\gamma) = \langle\nu, \alpha^\vee\rangle$.

The following Proposition is a summary of results found in [GL05, § 6] and describes the LS-gallery model.

Proposition 4.24. ***([GL05, §6, Cor. 1 and Theorem 2])*** *Let $\lambda \in X_+$ be regular and γ_λ a minimal gallery joining the origin and λ. Then the following hold:*

(i) $e_\alpha \gamma_\lambda$ *is not defined for any simple root* α.

(ii) *The set of LS-galleries* $\Gamma^+_{LS}(\gamma_\lambda)$ *is generated from* γ_λ *by successively applying the operators* f_α, *whenever they are defined.*

(iii) *The set of LS-galleries* $\Gamma^+_{LS}(\gamma_\lambda)$ *together with the maps wt and for each simple root* α *the maps* ε_α, φ_α, e_α, *and* f_α *form a finite dimensional crystal.*

(iv) *In addition, the character of* $V(\lambda)$, *the highest weight representation with highest weight* λ *of* $G^\vee$ *is equal to* $\sum_{\gamma \in \Gamma^+_{LS}(\gamma_\lambda)} e^{wt(\gamma)}$.

As with the Lusztig datum for the canonical basis, we can also attach a vector of integers to each gallery, the Kashiwara datum, sometimes also called String datum.

Let $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$ and $w_0 = s_{i_1} \cdots s_{i_r}$ a reduced expression of the longest element of the Weyl group, then we define the following set of galleries

$$\delta_0 = \delta,$$

$$\delta_j = e_{\alpha_{i_j}}^{\epsilon_{\alpha_{i_j}}(\delta_{j-1})}(\delta_{j-1}) \text{ for } 1 \leq j \leq r.$$

The vector consisting of $c^{\underline{i}}_j = \epsilon_{\alpha_{i_j}}(\delta_{j-1})$ for $1 \leq j \leq r$ is then called the $\underline{i}$-Kashiwara datum of δ.

4.3 Bott-Samelson resolution

Our first definition of the Bott-Samelson variety is quite general and works for any gallery, not only alcove galleries and has similar versions for non-affine Kac-Moody groups as well. We fix the type t_{γ_λ} of a minimal combinatorial gallery. To define the Bott-Samelson variety $\hat{\Sigma}(\gamma_\lambda)$, we denote by $\hat{\mathcal{P}}_j$, respectively $\hat{\mathcal{Q}}_{j_j}$ the parabolic subgroup of type t'_j, respectively t_j, containing $\hat{\mathcal{B}}$. Most of the following definitions have their origin in [Dem74] and [Han73], but we will quote them from more recent works to have notations that are closer to ours.

Definition 4.25. ([Gau01, §3.1, Def. 4], [GL05, §7, Def. 21], or [Kum02, 7.1.3]) The *Bott-Samelson variety* $\hat{\Sigma}(\gamma_\lambda)$ is defined as

$$\hat{\Sigma}(\gamma_\lambda) = \mathcal{P}_0 \times_{\hat{\mathcal{Q}}_0} \hat{\mathcal{P}}_1 \times_{\hat{\mathcal{Q}}_1} \ldots \times_{\hat{\mathcal{Q}}_{p-1}} \hat{\mathcal{P}}_p / \hat{\mathcal{Q}}_p,$$

i.e., the algebraic variety defined as the quotient of the group $\mathcal{P}_0 \times \hat{\mathcal{P}}_1 \times \ldots \times \hat{\mathcal{P}}_p$ by the subgroup $\hat{\mathcal{Q}}_0 \times \hat{\mathcal{Q}}_1 \times \ldots \times \hat{\mathcal{Q}}_p$ under the right action given by $g.q = (g_0 q_0, q_0^{-1} q_1 q_1, \ldots, q_{p-1}^{-1} g_p q_p)$ where $q = (q_0, \ldots, q_p) \in \hat{\mathcal{Q}}_0 \times \hat{\mathcal{Q}}_1 \times \ldots \times \hat{\mathcal{Q}}_p$ and $g = (g_1, \ldots, g_p) \in \mathcal{P}_0 \times \hat{\mathcal{P}}_1 \times \ldots \times \hat{\mathcal{P}}_p$.

The two facts that on the one hand we have restricted ourselves to galleries of alcoves and on the other hand that the Kac-Moody group is affine simplifies this definition, see [BG06, § 5.2].

Definition 4.26. ([GL05, §3.3, Example 3]) Let F be a face of the fundamental alcove and let us denote by $W^{\mathfrak{a}}(F)$ the subgroup of $W^{\mathfrak{a}}$ generated by its type $S^{\mathfrak{a}}(F)$, then we define the *standard parahoric subgroup of type F* as

$$P_F := \bigcup_{w \in W^{\mathfrak{a}}(F)} \mathcal{I} w \mathcal{I}.$$

Definition 4.27. ([BG06, §5.2]) For $0 \leq j \leq p$, let us denote by $\mathcal{P}_j$ the parahoric subgroup of $G(\mathcal{K})$ of type t'_j, containing $\mathcal{I}$. The *Bott-Samelson variety* $\hat{\Sigma}(\gamma_\lambda)$ is defined as

$$\hat{\Sigma}(\gamma_\lambda) = G(\mathcal{O}) \times_{\mathcal{I}} \mathcal{P}_1 \times_{\mathcal{I}} \ldots \times_{\mathcal{I}} \mathcal{P}_p / \mathcal{I}.$$

In other words it is defined as the quotient of the group $G(\mathcal{O}) \times \mathcal{P}_1 \times \ldots \times \mathcal{P}_p$ by the subgroup $\mathcal{I} \times \mathcal{I} \times \ldots \times \mathcal{I}$, the $p+1$st power of $\mathcal{I}$ under the right action given by $g.q = (g_0 q_0, q_0^{-1} q_1 q_1, \ldots, q_{p-1}^{-1} g_p q_p)$ where $q = (q_0, \ldots, q_p) \in G(\mathcal{O}) \times \mathcal{P}_1 \times \ldots \times \mathcal{P}_p$ and $g = (g_1, \ldots, g_p) \in \mathcal{I} \times \mathcal{I} \times \ldots \times \mathcal{I}$.

As one can see the simplified version of the Bott-Samelson variety is basically defined by substituting parabolic for parahoric subgroups and specialising to the case of galleries of alcoves, which means that we have a product of minimal parahorics and divide by the action of a product of Iwahori groups.

This will be one of the two realisations of the Bott-Samelson variety that we will use, the other one is the following embedding.

Proposition 4.28. *Let $\hat{\Sigma}(\gamma_\lambda) = G(\mathcal{O}) \times_{\mathcal{I}} \mathcal{P}_1 \times_{\mathcal{I}} \ldots \times_{\mathcal{I}} \mathcal{P}_p / \mathcal{I}$, be the Bott-Samelson variety, then the map*

$$\phi : \hat{\Sigma}(\gamma_\lambda) \longrightarrow (G(\mathcal{K})/\mathcal{I})^{p+1},$$

with $\phi([g_0, \ldots, g_p]) := (\overline{g_0}, \overline{g_0 g_1}, \ldots, \overline{g_0} \cdots \overline{g_p})$, is injective and T-equivariant and its image is a closed subvariety $\widetilde{\Sigma}(\gamma_\lambda)$ consisting of those points $(x_0, \ldots, x_p)$ satisfying

(i) $x_0 \in G(\mathcal{O})$,

(ii) $x_{j-1}^{-1} x_j \in \mathcal{P}_j$ *for* $1 \leq j \leq$.

Proof. It is obvious that by definition the image of the map is contained in the above mentioned subvariety and that the map is indeed a map of varieties. As the torus acts diagonally on the right side, the T-equivariance is also obvious by definition.

Let us assume that $\Phi([g_0, \ldots, g_p]) = \Phi([h_0, \ldots, h_p])$, then it follows that there exists an $a_0 \in \mathcal{I}$ such that $h_0 a_0 = g_0$, on the other hand using the definition of the Bott-Samelson this implies

$$[h_0, h_1, \ldots, h_p] = [h_0 a_0, a_0^{-1} h_1, \ldots, h_p].$$

this can of course be done at every position and we define a_j for $1 \leq j \leq p$ by the property $h_0 \cdots h_j a_j = g_0 \cdots g_j$. Then it is evident that $a_{j-1}^{-1} h_j a_j = g_j$ for $1 \leq j \leq p$ and hence we have

$$[h_0, h_1, \ldots, h_p] = [h_0 a_0, a_0^{-1} h_1 a_1, a_1^{-1} h_2 a_2, \ldots, a_{p-1}^{-1} h_p a_p] = [g_0, \ldots, g_p].$$

Hence the map is injective. As already said, it is obvious that the image lies in the above mentioned subvariety, so we only need to check surjectivity. Let $(\overline{x_0}, \ldots, \overline{x_p}) \in \widetilde{\Sigma}(\gamma_\lambda)$, then it is easy to see that $[x_0, x_0^{-1} x_1, x_1^{-1} x_2, \ldots, x_{p-1}^{-1} x_p]$ is a preimage. □

Following a result of Contou-Carrère [CC83] and a generalization of it in [GL05], we can view the Bott-Samelson variety in another way.

Proposition 4.29. ***([GL05, §7, Def.-Prop. 1] or [CC83, I, §6])*** *For* $0 \leq j \leq p$, *let us denote by* $\mathcal{P}_j$, *respectively* $\mathcal{Q}_j$, *the parahoric subgroup of* $G(\mathcal{K})$ *of type* t'_j, *respectively* t_j, *containing* $\mathcal{I}$. $\hat{\Sigma}(\gamma_\lambda)$ *is the closed subvariety of the product*

$$G(\mathcal{K})/\mathcal{Q}_0 \times \left(\prod_{1 \leq j \leq p} G(\mathcal{K})/\mathcal{P}_j \times G(\mathcal{K})/\mathcal{Q}_j \right) \times G(\mathcal{K})/\mathcal{Q}_\lambda$$

given by all the sequences of parahoric subgroups of the shape

$$G(\mathcal{O}) \supset \mathcal{Q}'_0 \subset \mathcal{P}'_1 \supset \mathcal{Q}'_1 \subset \ldots \subset \mathcal{P}'_p \supset \mathcal{Q}'_p \subset \mathcal{Q}'_\lambda,$$

where type$(\mathcal{P}'_i) = t'_i$, *type*$(\mathcal{Q}'_i) = t_i$, *and* $\mathcal{Q}'_\lambda$ *is a subgroup associated to a vertex of type* t_λ.

Remark 4.30. In particular we can realize the Bott-Samelson variety as the the closed subvariety consisting of all the galleries in the affine building $\mathscr{J}^{\mathfrak{a}}$ of type t_{γ_λ} starting at the origin, where the type of each face corresponds to the type of the corresponding parahoric subgroup.

As mentioned in the title of this part, the Bott-Samelson is a resolution for the Schubert cells in the affine Grassmannian. It is a smooth projective variety of dimension $\dim(\hat{\Sigma}(\gamma_\lambda)) = 2\langle\lambda,\rho\rangle$. We will write a point in this variety as $g = [g_0, \ldots, g_p]$ and will call it a gallery. By the above Remark 4.30 this naming is reasonable. The natural multiplication map then provides us with a map

$$\begin{array}{rcl} \pi : \hat{\Sigma}(\gamma_\lambda) & \rightarrow & X_\lambda \\ g & \mapsto & g_0 g_1 \cdots g_p \end{array}$$

that is birational and proper.

The set of combinatorial galleries of type t_{γ_λ} starting at the origin is a special subset, when viewing the Bott-Samelson variety in this way, they are the T-fixed points under the natural torus action. In addition we also have the action of $G(\mathcal{O})$ on the variety, this comes from the action on the building, as $G(\mathcal{O})$ acts by simplical maps. This action preserves the type of a gallery, and if a gallery is minimal, then so are all the galleries in the $G(\mathcal{O})$-orbit.

If we view the Bott-Samelson variety as the set of galleries, the map $\pi : \hat{\Sigma}(\gamma_\lambda) \rightarrow X_\lambda$ is the restriction of the projection on the last factor. This means that the minimal gallery γ_λ is mapped to L_λ, hence the map π induces a morphism between the $G(\mathcal{O})$-orbit of γ_λ and $\mathcal{G}_\lambda$. As all minimal galleries lie in the same $G(\mathcal{O})$-orbit, see [GL05, § 7], this induces a bijection between them and the open orbit $\mathcal{G}_\lambda$ in X_λ via π.

As the retractions were defined for alcoves, they are naturally extended to galleries by applying them to every alcove of the gallery. Since the retractions preserve the type of a face and hence of a gallery, the image of an arbitrary gallery of type t_{γ_λ} is a combinatorial gallery of the same type. By using this fact one has the following property.

Proposition 4.31. ***([GL05, §7, Prop. 5])*** *The retraction r_w with centre w at infinity induces a map $\hat{r}_w : \hat{\Sigma}(\gamma_\lambda) \rightarrow \Gamma(\gamma_\lambda)$. The fibres $C_w(\delta) = \hat{r}_w^{-1}(\delta)$, $\delta \in \Gamma(\gamma_\lambda)$, are endowed with the structure of a locally closed subvariety, which are isomorphic to some affine spaces.*

The $C_w(\delta)$ are the Bialynicki-Birula cells $\{x \in \hat{\Sigma}(\gamma_\lambda) \mid \lim_{s\to 0} s^\eta.x = \delta\}$ of centre δ in $\hat{\Sigma}(\gamma_\lambda)$, associated to a generic one-parameter subgroup η of T in the Weyl chamber $w\mathfrak{C}_f$.

This is a slight modification of the proposition in [GL05] as it uses retractions for every element of the Weyl group and not for w_0 alone. This does not change anything for the proof of the proposition as any other Weyl group element besides w_0 just corresponds to a different choice of a Borel and a different set of positive roots.

As mentioned above we can identify the open orbit $\mathcal{G}_\lambda$ with the set of minimal galleries in $\hat{\Sigma}(\gamma_\lambda)$. One of the main results in [GL05] is the following theorem, which relates the intersection with the semi-infinite orbit with the retractions at infinity.

Theorem 4.32. ***([GL05, §7, Theorem 3])*** *The restriction of $\hat{r}_w$ induces a map $r_w : \mathcal{G}_\lambda \to \Gamma^w(\gamma_\lambda)$. Furthermore the union $\bigcup_\delta r_w^{-1}(\delta)$ of the fibres over all galleries in $\Gamma^w(\gamma_\lambda)$ with target ν is the intersection $S^w_\nu \cap \mathcal{G}_\lambda$.*

To give a better description the fibres, we introduce an open covering of the Bott-Samelson variety.

For a combinatorial gallery $\delta = [\delta_0, \ldots, \delta_p] \in \Gamma(\gamma_\lambda)$ and a $w \in W$ we define

$$\mathcal{U}_0^w = \prod_{\substack{\beta \in w(\Phi^+),\\ \delta_0^{-1}\beta < 0}} U_\beta \cdot \prod_{\substack{\alpha \in w(\Phi^-),\\ \delta_0^{-1}\alpha < 0}} U_\alpha \cdot \delta_0.$$

Furthermore for $1 \leq j \leq p$ we define

$$\mathcal{U}_j = \begin{cases} U_{\alpha_{i_j}} s_{i_j} & \text{if } \delta_j = s_{i_j} \\ U_{-\alpha_{i_j}} & \textit{otherwise.} \end{cases}$$

The following definition is a modification of the one in [GL05, § 8], it is just the translation of the definition given there to the case of the simplified definition of the Bott-Samelson we introduced earlier.

Definition 4.33. ([GL05, §8, Def. 23]) Let $\delta = [\delta_0, \ldots, \delta_p] \in \Gamma(\gamma_\lambda)$ be a combinatorial gallery and $w \in W$. Then we define the subvariety $\mathcal{U}_\delta^w$ of $\hat{\Sigma}(\gamma_\lambda) = G(\mathcal{O}) \times_{\mathcal{I}} \mathcal{P}_1 \times_{\mathcal{I}} \ldots \times_{\mathcal{I}} \mathcal{P}_p/\mathcal{I}$ as follows

$$\mathcal{U}_\delta^w = \mathcal{U}_0^w \times \mathcal{U}_1 \times \ldots \times \mathcal{U}_p.$$

The collection of these subvarieties for a given $w \in W$ form an affine open covering of $\hat{\Sigma}(\gamma_\lambda)$.

Remark 4.34. By Proposition 4.31, it is evident that for a combinatorial gallery δ, the cell $C_w(\delta)$ lies inside $\mathcal{U}_\delta^w$.

Remark 4.35. If we talk about galleries in such an open piece $\mathcal{U}_\delta^w$ of the Bott-Samelson variety corresponding to a combinatorial gallery, we always mean that the gallery is written in the corresponding chart as given above, i.e., we have the form

$$g_0 = \prod_{\substack{\beta \in w(\Phi^+), \\ \delta_0^{-1}\beta < 0}} x_\beta(a_\beta) \cdot \prod_{\substack{\alpha \in w(\Phi^-), \\ \delta_0^{-1}\alpha < 0}} x_\alpha(a_\alpha) \cdot \delta_0,$$

for some arbitrary complex parameters and for $1 \leq j \leq p$ we have

$$g_j = \begin{cases} x_{\alpha_{i_j}}(a_j)s_{i_j} & \text{if } \delta_j = s_{i_j} \\ x_{-\alpha_{i_j}}(a_j) & \textit{otherwise}, \end{cases}$$

for some $a_j \in \mathbb{C}$.

The structure of the fibres of the retraction at infinity can then be described more precise in the following way, stated in [GL05].

Theorem 4.36. ***([GL05, §11, Theorem 4])*** *Let $\lambda \in X_+^\vee$ and let $\delta = [\delta_0, \delta_1, \dots, \delta_p] \in \Gamma^+(\gamma_\lambda)$. Then $\hat{r}_{w_0}^{-1}(\delta)$ is a subvariety of $\mathcal{U}_\delta$ isomorphic to a product of $\mathbb{C}$'s and $\mathbb{C}^*$'s. More precisely, the fibre consists of all galleries $[g_0, g_1, \dots, g_p]$ such that*

$$g_0 \in \prod_{\beta<0, \delta_0^{-1}(\beta)<0} U_\beta \cdot \delta_0 \text{ and } g_j = \begin{cases} \delta_j & \text{if } j \notin J_{-\infty}(\delta) \\ x_{-\alpha_{i_j}}(a_j), a_j \neq 0 & \text{if } j \in J_{-\infty}^-(\delta) \\ x_{\alpha_{i_j}}(a_j)s_{i_j} & \text{if } j \in J_{-\infty}^+(\delta) \end{cases}$$

By definition $\overline{S_\nu^w \cap \mathcal{G}_\lambda}$ is the union of $Z(\delta) = \overline{r_w^{-1}(\delta)}$ for $\delta \in \Gamma^w(\gamma_\lambda, \nu)$. Since the intersection $S_\nu^w \cap \mathcal{G}_\lambda$ is equidimensional, by [MV99], we can reduce this to the union of over those galleries whose fibre has the maximal dimension. Hence, using a statement from [GL05] that a gallery and its fibre have the same dimension, we reduce to

$$\overline{S_\nu^w \cap \mathcal{G}_\lambda} = \bigcup_{\delta \in \Gamma_{LS}^w(\gamma_\lambda, \nu)} Z(\delta).$$

As seen in the theorem above an open subset of the MV-cycle can be given in a quite explicit way.

This can be used to describe the MV-cycle and MV-polytope by using the retractions in the directions for all Weyl group elements and using the GGMS-strata. As a GGMS-stratum that lies inside $\mathcal{G}_\lambda$ and is the intersection of S_ν with other semi-infinite cells defines an open subset of a MV-cycle of $\overline{S_\nu \cap \mathcal{G}_\lambda}$, we obtain the following proposition by combining the above mentioned results.

Proposition 4.37. *Let $A(\mu_\bullet) \subset \overline{S_\nu \cap \mathcal{G}_\lambda}$ be an MV-cycle of coweight (λ, ν).*

Then there exist an open subset $O \subset A(\mu_\bullet) \cap \mathcal{G}_\lambda$ and galleries $\delta_w \in \Gamma^w_{LS}(\gamma_\lambda, \mu_w)$ for each element of the Weyl group, such that $r_w(x) = \delta_w$ for all $x \in O$.

Proof. For each $w \in W$ there exists $\delta_w \in \Gamma^w_{LS}(\gamma_\lambda, \mu_w)$ such that $r_w^{-1}(\delta_w) \subset A(\mu_\bullet) \cap \mathcal{G}^\lambda$ and it is an open subset. Hence

$$\bigcap_{w \in W} r_w^{-1}(\delta_w)$$

is dense in $A(\mu_\bullet) \cap \mathcal{G}^\lambda$. □

For a fixed LS-gallery δ, we thus have to construct galleries δ_w for $w \in W$ with properties as in the proposition. By using the coweights of these galleries $d_\bullet = (wt(\delta_w))_{w \in W}$ we then obtain the MV-cycle as the closure of the GGMS-stratum $A(d_\bullet)$. In the next part we will construct these galleries.

5 Sections

In this part we will divide a positively folded LS-gallery into parts that will be called sections and study the behaviour of these sections with respect to the root operators that were defined in the last part. Everything stated about sections would, in a very similar matter, also work for the path model described in [Lit95], [Lit03], and [Lit97].

5.1 Stable and directed sections

We start by analysing certain alcoves of an LS-gallery that, when applying the operators e_i or f_i, for a chosen simple root α_i, successively to the gallery, are not reflected at any hyperplane, but only translated.

In the following let $\delta = (\Delta'_0 \subset \Delta_0 \supset \Delta'_1 \subset \ldots \supset \Delta'_p \subset \Delta_p \supset \Delta'_{p+1})$ be a gallery and unless otherwise stated $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$.

Definition 5.1. Let α be a simple root. An alcove Δ_i of δ is called *α-stable at m* if the following condition holds:

> m is minimal such that there exists a pair $r(\Delta_i)$ and $l(\Delta_i)$ with the following property:
>
> (i) $r(\Delta_i) > i$ and $\Delta'_{r(\Delta_i)} \subset H_{\alpha,m}$
>
> (ii) $l(\Delta_i) \leq i$ and $\Delta'_{l(\Delta_i)} \subset H_{\alpha,m}$
>
> (iii) for all $s \in \mathbb{N}$ with $l(\Delta_i) \leq s < r(\Delta_i) : \Delta'_s \not\subset H_{\alpha,m-1}$

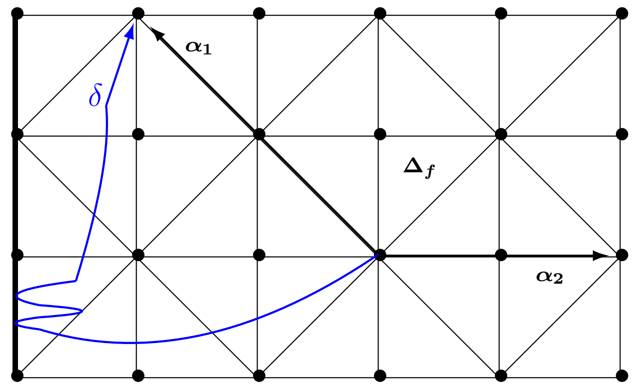
Let $R_{\alpha,m}(\delta)$ be the set of all indices j, such that Δ_j is α-stable at m and

$$R_\alpha(\delta) := \bigcup_{m\in\mathbb{Z}} R_{\alpha,m}(\delta),$$

the set of all indices corresponding to α-stable alcoves for arbitrary m.

The pair $r_\alpha(\Delta_i)$ and $l_\alpha(\Delta_i)$ is the pair of indices such that the three conditions above are satisfied for minimal m and $r_\alpha(\Delta_i)$ is maximal with properties (i) and (iii), while $l_\alpha(\Delta_i)$ is minimal with properties (ii) and (iii).

Example 5.2. We want to give a short example on how a stable section looks like.



One can see that on the wall corresponding to α_2 of height -3 (the bolded hyperplane), the gallery has two foldings and the part of the gallery between these two foldings lies to the right of the gallery, hence it does not cross a hyperplane corresponding to α_2 of lower height than -3. Hence Δ_5 is α_2 stable at -3 as well as Δ_6. The number $l_{\alpha_2}(\Delta_5) = 5$ and $r_{\alpha_2}(\Delta_5) = 7$ as those are the two small faces of the galleries that lie inside the wall of height -3.

Remark 5.3. If for Δ_i there exists an m as in the definition such that all three conditions are satisfied, but m is not minimal with this property, then we still have $i \in R_\alpha(\delta)$. This is due to the fact that $i \in R_{\alpha,n}(\delta)$ for some $n < m$.

We first look at some basic properties of these indices. For example that all alcoves between $r_\alpha(\Delta_i)$ and $l_\alpha(\Delta_i)$ are in fact stable for the same α and the same m.

Lemma 5.4. *If $i \in R_{\alpha,m}(\delta)$, then $j \in R_{\alpha,m}(\delta)$ for all $l_\alpha(\Delta_i) \leq j < r_\alpha(\Delta_i)$.*

Proof. We first assume that $j \notin R_{\alpha,m}(\delta)$. As $l_\alpha(\Delta_i) \leq j < r_\alpha(\Delta_i)$ and condition (i), (ii), and (iii) of Definition 5.1 are already satisfied, this implies that for Δ_j, m is in fact not minimal. Hence there exist $l(\Delta_j) \leq j < r(\Delta_j)$ such that properties (i) - (iii) of Definition 5.1 are already satisfied for $n < m$. But as $\Delta'_s \not\subset H_{\alpha,m-1}$ for $l_\alpha(\Delta_i) \leq s < r_\alpha(\Delta_i)$ it means that $l(\Delta_j) < l_\alpha(\Delta_i)$ and $r_\alpha(\Delta_i) < r(\Delta_j)$, which would show that i is in fact α-stable at n, which is a contradiction to m being minimal. □

Example 5.5. (Example 5.2 continued) In the example this can be seen as Δ_5 is α_2-stable at -3 and $l_{\alpha_2}(\Delta_5) = 5$ and $r_{\alpha_2}(\Delta_5) = 7$. Hence by the above lemma Δ_6 has to be α_2-stable as well, which it obviously is.

In addition we also want to see how the neighbourhood of indices not α-stable at any m looks like. It turns out that all of the prior alcoves down to the last crossing of a hyperplane corresponding to α and all of the subsequent alcoves up to the next crossing of a hyperplane corresponding to α are not α-stable as well. In addition the two mentioned hyperplane crossings occur at different, but adjacent hyperplanes.

Lemma 5.6. *If $i \notin R_\alpha(\delta)$ and $j \leq i$ is maximal such that $\Delta'_j \subset H_{\alpha,m}$ for some $m \in \mathbb{Z}$ and $k > i$ is minimal such that $\Delta'_k \subset H_{\alpha,m'}$ for some $m' \in \mathbb{Z}$, then $\{j, \dots, k-1\} \cap R_\alpha(\delta) = \emptyset$ and furthermore $m - m' = \pm 1$.*

Proof. If there exists $s \in j, \dots, k-1$ and an $n \in \mathbb{Z}$ such that $s \in R_{\alpha,n}(\delta)$, then $l_\alpha(\Delta_s) \leq j$ and $r_\alpha(\Delta_s) \geq k$ and thus $l_\alpha(\Delta_s) \leq i < r_\alpha(\Delta_s)$. By Lemma 5.4 this implies $i \in R_{\alpha,n}(\delta)$ which is a contradiction.
It is obvious that $m - m' \in \{-1, 0, 1\}$. Suppose $m - m' = 0$, then $i \in R_{\alpha,m}(\delta)$ unless m is not minimal, as the pair (j, k) satisfies condition (i) - (iii) of Definition 5.1. But by Remark 5.3 this shows that i is α-stable, which is a contradiction. □

Example 5.7. (Example 5.2 continued) In the example it is easy to see that if we look at any alcove Δ_i and take the last index smaller than i and the next one greater than i that lies in an α_2 hyperplane that the differences between the heights of these hyperplanes is either ± 1 or 0.

Definition 5.8. For a gallery $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$, an interval $[i, j] \subset [0, p]$ is called

- *α-directed section* if there exists an $m \in \mathbb{Z}$, such that for all $k \in [i, j-1]$

$$\left\{ \begin{array}{c} \Delta'_k \subset H_{\alpha,m} \Rightarrow k = i \\ \Delta'_k \subset H_{\alpha,m+1} \Rightarrow k = j \end{array} \right\} \wedge k \notin R_\alpha(\delta)$$

  holds.

- *$(-\alpha)$-directed section* if there exists an $m \in \mathbb{Z}$, such that for all $k \in [i, j-1]$

$$\left\{ \begin{array}{c} \Delta'_k \subset H_{\alpha,m} \Rightarrow k = i \\ \Delta'_k \subset H_{\alpha,m-1} \Rightarrow k = j \end{array} \right\} \wedge k \notin R_\alpha(\delta)$$

  holds.

- *α-stable section* at m if there exists a $k \in [i, j-1]$, such that $k \in R_{\alpha,m}(\delta)$ and $r_\alpha(\Delta_k) = j$ and $l_\alpha(\Delta_k) = i$.

Remark 5.9. For the directed section the inclusion conditions also imply that no Δ'_k lies in any hyperplane as long as k is not equal to i or j.

This definition allows us to divide our gallery into parts, each being one of the three sections above.

Lemma 5.10. *For a gallery $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$, there exists a unique partition $i_1 < i_2 < \ldots < i_t$, such that for each k, $[i_k, i_{k+1}]$ is an α-directed, $(-\alpha)$-directed, or α-stable.*

Proof. The existence of a partition follows from Lemma 5.4 and Lemma 5.6 and the uniqueness is obvious for the directed sections and for the stable sections is implicated by the following remark. □

Remark 5.11. It is obvious that if $[i_k, i_{k+1}]$ is an α-stable section, that neither $[i_{k-1}, i_k]$ nor $[i_{k+1}, i_{k+2}]$ are α-stable. If one of them would be α-stable as well, it would have to be for the same m. This would imply that for any $s \in [i_k, i_{k+1}]$, $r_\alpha(\Delta_s)$, respectively $l_\alpha(\Delta_s)$, would not be maximal, respectively minimal, with respect to m.

Remark 5.12. We can also see which kind of sections appear in which order, namely the following two situations can not occur. Let $i < j < k < l$, such that $[i, j]$ is an α-directed section. In the first case we suppose that $[j, k]$ is a $(-\alpha)$-directed section, but this implies that $[i, k]$ would already be a α-stable section, hence this order cannot appear. In the second case we assume that $[j, k]$ is a stable section and $[k, l]$ is a $(-\alpha)$-directed section. Again this implies $[i, l]$ is already an α-stable sections and hence can also not occur.

For our partition this means that all α-directed section appear after the $(-\alpha)$-directed ones, with α-stable sections in between.

Example 5.13. (Example 5.2 continued) For the example one can see that the partition is equal to $(0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11)$. For these sections of the gallery one has the following types, $[0, 1], \ldots [4, 5]$ are $(-\alpha_2)$-directed, $[5, 7]$ is α_2-stable, and $[7, 8], \ldots, [10, 11]$ are α_2-directed. Which is exactly the ordering of these types of sections as it is mentioned in the last remark.

5.2 Sections and root operators

In this section we want to see how the defined partition of our gallery interacts with the operators f_α and e_α. For this we need the following technical lemma. It tells us that the part of the gallery that will be reflected at an α-hyperplane, when applying f_α, consists only of non-α-stable alcoves.

Lemma 5.14. *Let m be minimal such that $\exists i$ with $\Delta'_i \subset H_{\alpha,m}$ and $\langle wt(\delta), \alpha\rangle - m \geq 1$ and j max such that $\Delta'_j \subset H_{\alpha,m}$ and $k > j$ minimal such that $\Delta'_k \subset H_{\alpha,m+1}$. Then $\Delta_s \notin R_\alpha(\delta)$ for all $j \leq s < k$.*

Proof.

- Suppose Δ_s is α-stable at $n < m$ then it follows that there exists an r, such that $\Delta'_r \subset H_{\alpha,n}$, but this is a contradiction to m being minimal.
- Suppose Δ_s is α-stable at m then it follows that there exists an $r > s$, such that $\Delta'_r \subset H_{\alpha,m}$, but this is a contradiction to j being maximal with the property that $\Delta'_j \subset H_{\alpha,m}$ as $j \leq s < r$.
- Suppose Δ_s is α-stable at $n > m$ then there exists $l \leq s$, such that $\Delta'_l \subset H_{\alpha,n}$ and $\Delta'_t \not\subset H_{\alpha,n-1}$ for $l \leq t \leq s$, in particular it implies that $\Delta'_t \not\subset H_{\alpha,m}$ for $l \leq t \leq s$. Hence $k \leq l$ as both l and k are strictly bigger than j, but k is minimal with the property that $\Delta'_k \subset H_{\alpha,m+1}$. But this would be a contradiction to $l \leq s$ as $j \leq s < k \leq l$.

□

In other words, we have shown a bit more than we stated in the lemma. The part that will be reflected at an α-hyperplane by the operator f_α must be an α-directed section for δ.

Example 5.15. (Example 5.2 continues) In our example, if we look at hyperplanes with respect to α_2, $j = 7$ and $k = 11$. None of the alcoves $\Delta_7, ldots, \Delta_{10}$ are α_2 stable at any height.

We now want to prove, that if we can apply the operator f_α to a gallery, an α-stable alcove remains α-stable and an alcove that is α-stable afterwards was obtained from one.

Proposition 5.16. *Let m, j, k be as in Lemma 5.14, then $f_\alpha\delta$ is defined. Let*

$$f_\alpha\delta = ([t^0] \subset \widetilde{\Delta_0} \supset \widetilde{\Delta'_1} \subset \ldots \supset \widetilde{\Delta'_p} \subset \widetilde{\Delta_p} \supset \widetilde{\Delta'_{p+1}})$$

Then for all i, it holds

$$i \in R_\alpha(\delta) \Longleftrightarrow i \in R_\alpha(f_\alpha\delta).$$

Proof. We start with $i < j$.

(i) Let $i \in R_{\alpha,n}(\delta)$. It is clear that $n \geq m$ and as $l_\alpha(\Delta_i) \leq i < j$ there are two possibilities. Either $n = m$, then $r_\alpha(\Delta_i) \leq j$ as j is maximal with the property that $\Delta'_j \subset H_{\alpha,m}$. Or on the other hand $n > m$, then $r_\alpha(\Delta_i) < j$ because of the property that $\Delta'_t \not\subset H_{\alpha,m-1}$ for all $l_\alpha(\Delta_i) \leq t < r_\alpha(\Delta_i)$.

Thus $\Delta_{l_\alpha(\Delta_i)}, \ldots, \Delta_{r_\alpha(\Delta_i)-1}$ are not changed by f_α. As $\widetilde{\Delta'_k}$ is the only face with the property $\widetilde{\Delta'_k} \subset H_{\alpha,m-1}$ and as no face between $\widetilde{\Delta'_j}$ and $\widetilde{\Delta'_k}$ lies in any α wall, we have that $\{\Delta_{l_\alpha(\Delta_i)}, \ldots, \Delta_{r_\alpha(\Delta_i)-1}\} \subset R_{\alpha,n}(f_\alpha\delta)$ and especially $\Delta_i \in R_{\alpha,n}(f_\alpha\delta)$.

(ii) If on the other hand $i \in R_{\alpha,n}(f_\alpha\delta)$ then $n \geq m$ and like in the previous case there exists $r_\alpha(\widetilde{\Delta_i})$ and $l_\alpha(\widetilde{\Delta_i})$ both lower or equal to j as $\widetilde{\Delta'_k}$ is the only face that lies in an α-wall with a height lower than m. But this means of course that $\widetilde{\Delta_s} = \Delta_s$ for all $l_\alpha(\widetilde{\Delta_i}) \leq s < r_\alpha(\widetilde{\Delta_i})$ and thus $i \in R_{\alpha,n}(\delta)$.

Next we look at $j \leq i < k$. After Lemma 5.14 it is already obvious that $i \notin R_\alpha(\delta)$, thus it remains to show that $i \notin R_\alpha(f_\alpha\delta)$. This is essentially the same argument, we suppose that $i \in R_{\alpha,n}(f_\alpha\delta)$ for some $n \in \mathbb{Z}$. As the next face after $\widetilde{\Delta'_i}$ that lies in an α-wall is $\widetilde{\Delta'_k}$ it is clear that by condition (iii) of Definition 5.1 $n = m - 1$. But by definition of f_α, $\widetilde{\Delta'_k}$ is the only face that lies in the α-wall with height $m-1$ which is a contradiction to condition (ii) of Definition 5.1 as the index $l_\alpha(\widetilde{\Delta_i})$ would just not exist.

At last we want to look at the case $i \geq k$.

(i) Let $i \in R_{\alpha,n}(\delta)$. Since Δ'_j was the last face inside the hyperplane of height m, it is clear that $n > m$. Also as Δ'_k was the first face afterwards that lied in any α-wall, it follows that $r_\alpha(\Delta_i)$ and $l_\alpha(\Delta_i)$ are both greater or equal to k.

This implies that $\widetilde{\Delta_s} = t_{-\alpha^\vee}(\Delta_s)$ for all $l_\alpha(\Delta_i) \leq s < r_\alpha(\Delta_i)$, which means that $i \in R_{\alpha,n-2}(f_\alpha\delta)$ unless $n-2$ is not minimal with the stability property, i.e. $i \in R_{\alpha,n'}(f_\alpha\delta)$ for $n' < n-2$. But this can only occur if $l_\alpha(\widetilde{\Delta_i}) \leq j$. In the case where $n' > m-1$ this is not possible because of condition (iii) in Definition 5.1 and in the case $n' = m-1$ there are no faces before Δ'_j that lie in $H_{\alpha,m-1}$ by the definition of f_α.

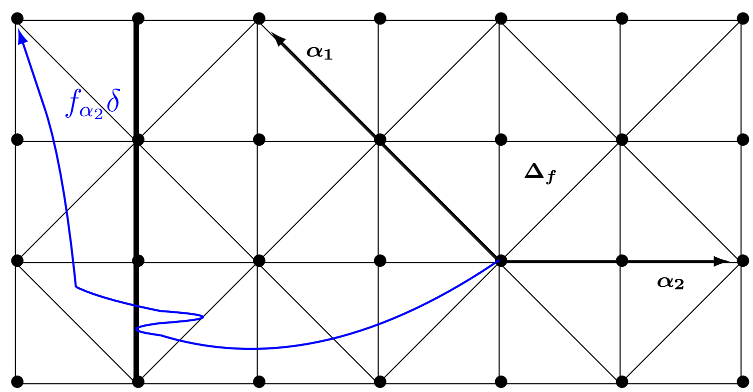
(ii) Let $i \in R_{\alpha,n}(f_\alpha\delta)$, as $H_{\alpha,m-1}$ is the minimal wall containing any faces, it is clear that $n \geq m-1$ and as above $r_\alpha(\widetilde{\Delta_i})$ and $l_\alpha(\widetilde{\Delta_i})$ are greater or equal to k as $\widetilde{\Delta'_k}$ is the first face contained in $H_{\alpha,m-1}$. This means that

$i \in R_{\alpha,n+2}(\delta)$, unless $n+2$ is not minimal with the stability property, but as above this would mean that $l_\alpha(\Delta_i) \leq j$ which as above is impossible as there is no face after Δ'_j contained in $H_{\alpha,m}$, thus i would have to be stable with respect to $n' > m$, but then $l_\alpha(\Delta_i) \leq j$ would be a contradiction to condition (iii) of Definition 5.1.

□

Remark 5.17. This proposition of course also holds for e_α, as the two operators are partially inverse.

Example 5.18. (Example 5.2 continues) In our example we will look at $f_{\alpha_2}(\delta)$



As one can see in both galleries, δ and $f_{\alpha_2}\delta$, the same indices correspond to α_2-stable alcoves.

It is also easy to see that the condition for f_α to be defined, is related to the number of α-directed sections.

Proposition 5.19. *It holds:*

$$f_\alpha\delta \text{ is defined } \iff \text{ there exists an } \alpha\text{-directed section.}$$

Proof. If $f_\alpha\delta$ is defined, let m, j, and k be as in Lemma 5.14. Then by Lemma5.14, the interval $[j,k]$ is an α-directed section.

If on the other hand let $[s,t]$ be the first α-directed section and $\Delta'_s \subset H_{\alpha,n}$, then n is the minimum of δ with respect to α. This follows from the fact that all previous sections of our gallery where either α-stable or $(-\alpha)$-directed and all subsequent sections will be either α-stable or α-directed. Since n is the minimum and $[s,t]$ is α-directed, we have $\langle wt(\delta),\alpha\rangle - n \geq 1$. Now suppose there exists a $q > s$ maximal such that $\Delta'_q \subset H_{\alpha,n}$ and let $p \leq s$ be minimal such that $\Delta'_p \subset H_{\alpha,n}$. Then by definition $\Delta'_l \not\subset H_{\alpha,n-1}$ for $p \leq l < q$ and thus Δ'_s is α-stable at n, which is a contradiction. Thus Δ'_s is the last face contained in the minimal wall with respect to α and Δ'_t is contained in $H_{\alpha,n+1}$ and by definition of α-directed also minimal. Hence f_α is defined. □

Remark 5.20. As one can also see in the proof of Proposition 5.19, if f_α is defined it will reflect the first α-directed section at the wall orthogonal to α of minimal height and thus produce a $(-\alpha)$-directed section at that part of the gallery and translate or leave invariant the rest.

Proposition 5.21. *It holds:*

$$e_\alpha\delta \text{ is defined} \iff \text{there exists an } (-\alpha)\text{-directed section.}$$

Proof. If $e_\alpha\delta$ is defined then $f_\alpha e_\alpha\delta$ is defined and equal to δ but by Proposition 5.19 this means that $e_\alpha\delta$ possesses an α-directed section that transforms into a $(-\alpha)$-directed section in δ.

If on the other hand $[s,t]$ is the last $(-\alpha)$-directed section of δ with $\Delta'_t \subset H_{\alpha,m}$, then all faces before Δ'_s that are contained in α-walls are contained in ones with height strictly bigger than m, as $[s,t]$ was the last $(-\alpha)$-directed section and all faces after Δ'_t that are contained in α-walls are contained in ones with height greater than or equal to m, thus Δ'_t is the first face contained in the minimal wall with respect to α and $m \leq -1$ (Remark 5.12) and thus e_α is defined. □

The two propositions and their proofs can be combined to give the following result about the relation between the partitions and the operators e_α and f_α.

Theorem 5.22. *Let $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$ and $i_1 < i_2 < \ldots i_k$ the corresponding partition into α-directed, $(-\alpha)$-directed, and α-stable sections. Then the following holds:*

(i) If $e_\alpha\delta$ is defined and $[i_s, i_{s+1}]$ is the last $(-\alpha)$-directed section of δ, then $e_\alpha\delta$ also has the partition $i_1 < i_2 < \ldots < i_k$ and all sections are of the same type as before, except that $[i_s, i_{s+1}]$ is now an α-directed section. Furthermore, if $\Delta'_{i_{s+1}+1} \subset H_{\alpha,m}$ and $e_\alpha\delta = (\widetilde{\Delta'_0} \subset \widetilde{\Delta_0} \supset \widetilde{\Delta'_1} \subset \ldots \supset$

$\widetilde{\Delta'_p} \subset \widetilde{\Delta_p} \supset \widetilde{\Delta'_{p+1}})$, *then*

$$\widetilde{\Delta_j} = \begin{cases} \Delta_j & \text{if } j < i_s, \\ s_{\alpha,m}\Delta_j & \text{if } i_s \leq j \leq i_{s+1}, \\ t_{\alpha^\vee}\Delta_j & j > i_{s+1}. \end{cases}$$

(ii) If $f_\alpha\delta$ is defined and $[i_s, i_{s+1}]$ is the first α-directed section of δ, then $f_\alpha\delta$ also has the partition $i_1 < i_2 < \ldots < i_k$ and all sections are of the same type as before, except that $[i_s, i_{s+1}]$ is now a $(-\alpha)$-directed section. Furthermore, if $\Delta'_{i_s+1} \subset H_{\alpha,m}$ and $f_\alpha\delta = (\widetilde{\Delta'_0} \subset \widetilde{\Delta_0} \supset \widetilde{\Delta'_1} \subset \ldots \supset \widetilde{\Delta'_p} \subset \widetilde{\Delta_p} \supset \widetilde{\Delta'_{p+1}})$, then

$$\widetilde{\Delta_j} = \begin{cases} \Delta_j & \text{if } j < i_s, \\ s_{\alpha,m}\Delta_j & \text{if } i_s \leq j \leq i_{s+1}, \\ t_{-\alpha^\vee}\Delta_j & j > i_{s+1}. \end{cases}$$

Corollary 5.23. *$\varphi_\alpha(\delta) = \#\{\alpha -$ directed sections$\}$ and $\varepsilon_\alpha(\delta) = \#\{(-\alpha) -$ directed sections$\}$*

Proof. This follows immediately from Proposition 5.19 and Proposition 5.21. □

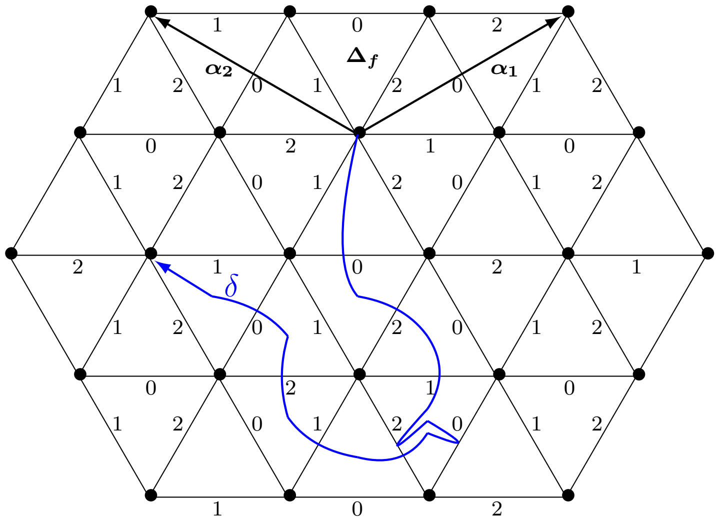
Definition 5.24. The *flipping* with respect to α of δ, $(\delta)_{-\alpha}$ is defined as the gallery $(\widetilde{\Delta'_0} \subset \widetilde{\Delta_0} \supset \widetilde{\Delta'_1} \subset \ldots \supset \widetilde{\Delta'_p} \subset \widetilde{\Delta_p} \supset \widetilde{\Delta'_{p+1}})$:

$$\widetilde{\Delta_r} = \begin{cases} s_{\alpha,m}\Delta_r & \text{if } \Delta_r \in R_{\alpha,m}(\delta), \\ \Delta_r & \text{otherwise.} \end{cases}$$

This produces a well defined gallery by the definition of stableness in 5.1 and Lemma 5.4.

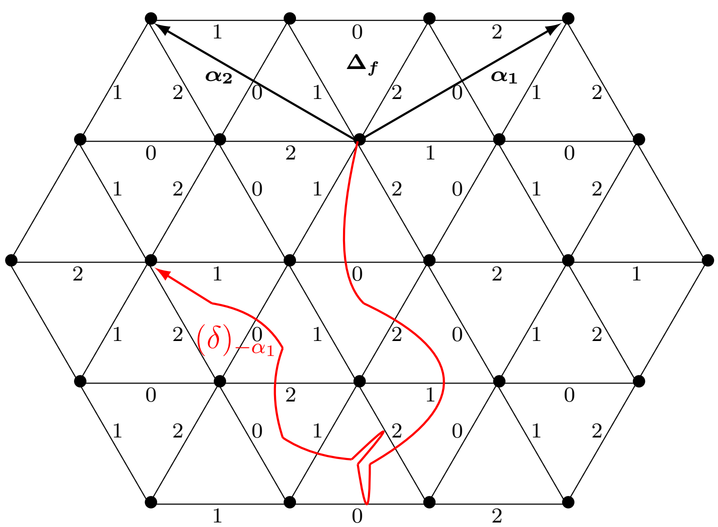
It should be noted that, in general, when applying the flipping operator $(\cdot)_{-\alpha}$ to a gallery in $\Gamma^+_{LS}(\gamma_\lambda)$, the new gallery is not an LS-gallery, neither for $\mathfrak{C}^\infty_{w_0}$ nor for $\mathfrak{C}^\infty_{w_0 s_\alpha}$

Example 5.25. Let us illustrate this with a small example. We start with a combinatorial gallery δ of the following form.



Combinatorial gallery δ.

If we now apply our flipping operator $(\cdot)_{-\alpha_1}$, most of the gallery is not changed at all, but the small part at the bottom, where the gallery crosses the same hyperplane twice is reflected to the other side.



Combinatorial gallery $(\delta)_{-\alpha_1}$.

This provides us with a means to compare the two galleries lying at the end of an α-chain in the corresponding crystal of LS-galleries, as we see in the following theorem.

Theorem 5.26. $(f_\alpha^{\varphi_\alpha(\delta)}(\delta))_{-\alpha} = s_\alpha.e_\alpha^{\varepsilon_\alpha(\delta)}(\delta)$, *where* s_α *operates on the gallery by using the identification* $\mathcal{A} = X^\vee \otimes \mathbb{R}$.

Proof. This follows immediately from Theorem 5.22. □

Remark 5.27. This also implies that $s_\alpha.(f_\alpha^{\varphi_\alpha(\delta)}(\delta))_{-\alpha} \in \Gamma^+_{LS}(\gamma_\lambda)$ or equivalently $(f_\alpha^{\varphi_\alpha(\delta)}(\delta))_{-\alpha} \in \Gamma^{s_\alpha}_{LS}(\gamma_\lambda)$.

Definition 5.28. Let $w \in W$ and $w^{\underline{i}} = s_{i_1} \dots s_{i_n}$ a reduced decomposition of w and $w^{\underline{i}}_k = s_{i_1} \dots s_{i_k}$ for $0 \leq k \leq n$, we define the following series of galleries for $1 \leq k \leq n$

$$\delta_0^{\underline{i}} := \delta$$

$$\begin{aligned} \delta_k^{\underline{i}} &:= w_k^{\underline{i}} \left(e_{\alpha_{i_k}}^{\varepsilon_{\alpha_{i_k}}((w^{\underline{i}}_{k-1})^{-1}\delta^{\underline{i}}_{k-1})}((w^{\underline{i}}_{k-1})^{-1}\delta^{\underline{i}}_{k-1}) \right) \\ &= w_{k-1}^{\underline{i}} \left(f_{\alpha_{i_k}}^{\varphi_{\alpha_{i_k}}((w^{\underline{i}}_{k-1})^{-1}\delta^{\underline{i}}_{k-1})}((w^{\underline{i}}_{k-1})^{-1}\delta^{\underline{i}}_{k-1}) \right)_{-\alpha_{i_k}}. \end{aligned}$$

We define $\Xi^{\underline{i}}_w(\delta) := \delta^{\underline{i}}_n$.

Example 5.29. We want to look at a small example for this last definition. For this let us choose a gallery δ for the representation $V(3\alpha_1^\vee + 3\alpha_2^\vee)$ of type B_2 and a Weyl group element w. We take

$$\delta = [s_2s_1s_2; s_0, s_1, s_2, s_1, \mathrm{id}, \mathrm{id}, \mathrm{id}, s_2, s_1, s_0] \text{ and } w^{\underline{i}} = s_2s_1s_2.$$

By the above definition $\delta_0^{\underline{i}} = \delta$, for $\delta_1^{\underline{i}}$ we have to calculate $\varepsilon_{\alpha_2}(\delta_0^{\underline{i}})$, this is equal to 3. Thus we apply our operator e_{α_2} three times and obtain the gallery

$$e^3_{\alpha_2}(\delta_0^{\underline{i}}) = [s_1s_2; s_0, s_1, s_2, s_1, s_0, \mathrm{id}, \mathrm{id}, s_2, s_1, s_0].$$

Last we have to apply the reflection s_2 and obtain

$$\delta_1^{\underline{i}} = [s_2s_1s_2; s_0, s_1, s_2, s_1, s_0, \mathrm{id}, \mathrm{id}, s_2, s_1, s_0].$$

Next we do the same process with α_1, hence we first calculate $\varepsilon_{\alpha_1}(s_2\delta_1^{\underline{i}})$, which is 2 and apply e_{α_1} twice and apply the Weyl group element s_2s_1 to the result. We obtain

$$\delta_2^{\underline{i}} = [s_2s_1s_2; s_0, s_1, s_2, s_1, s_0, s_1, \mathrm{id}, s_2, s_1, s_0].$$

If we do the process one last time we obtain

$$\delta_3^{\underline{i}} = [s_2s_1s_2; s_0, s_1, s_2, s_1, s_0, s_1, s_0, s_2, s_1, s_0].$$

These galleries are the galleries δ, $\Xi_{s_2}(\delta)$, $\Xi_{s_2s_1}(\delta)$, and $\Xi_{s_2s_1s_2}(\delta)$ in 7.4.

Of course we would like this operator to be independent of the chosen reduced decomposition of w, hence the following proposition.

Proposition 5.30. *Let w, $w_k^{\underline{i}}$, and $\delta_k^{\underline{i}}$ be as in Definition 5.28. We also set $\widetilde{\delta}_0^{\underline{i}} := \delta$ and $\widetilde{\delta}_k^{\underline{i}} := e_{\alpha_{i_k}}^{\varepsilon_\alpha(\widetilde{\delta}_{k-1}^{\underline{i}})}(\widetilde{\delta}_{k-1}^{\underline{i}})$ for $1 \leq k \leq n$. Then it holds:*

(i) $\delta_l^{\underline{i}} = w_l^{\underline{i}}.\widetilde{\delta}_l^{\underline{i}}$ for $0 \leq l \leq n$,

(ii) $\widetilde{\delta}_n^{\underline{i}}$ is independent of the choice of the reduced decomposition.

Proof. (i) We prove this by induction on l, for $l = 0$ this is true by definition. For $l > 0$ we have $\delta_l = w_l^{\underline{i}}\left(e_{\alpha_{i_l}}^{\varepsilon_\alpha((w_{l-1}^{\underline{i}})^{-1}\delta_{l-1}^{\underline{i}})}((w_{l-1}^{\underline{i}})^{-1}\delta_{l-1}^{\underline{i}})\right)$, but by induction $(w_{l-1}^{\underline{i}})^{-1}\delta_{l-1}^{\underline{i}} = \widetilde{\delta}_{l-1}^{\underline{i}}$ and hence $\delta_l^{\underline{i}} = w_l^{\underline{i}} e_{\alpha_{i_l}}^{\varepsilon_\alpha(\widetilde{\delta}_{l-1}^{\underline{i}})(\widetilde{\delta}_{l-1}^{\underline{i}})} = w_l^{\underline{i}}\widetilde{\delta}_l^{\underline{i}}$.

(ii) If we have two reduced decompositions that are obtained one from the other by using a single braid relation, we only have to consider the rank 2 case for two different decompositions of the corresponding longest element of the Weyl group, but then the independence is true after [Lit98a]. For two arbitrary decompositions we use a series of successive decompositions each obtained from the former by using a single braid relation.

□

Thus the operators $\Xi_w^{\underline{i}}$ is independent of the reduced decomposition, hence we can define Ξ_w.

Definition 5.31. For $w \in W$, we define the *vertex gallery* $\Xi_w(\delta)$ of δ with respect to w to be $\Xi_w^{\underline{i}}(\delta)$ for an arbitrary reduced decomposition $\underline{i}$ of w.

Examples for this definition can be found in 7.3 and 7.4 of Section 7. Proposition 5.30 also implies the following.

Corollary 5.32. *The gallery $\Xi_w(\delta) \in \Gamma_{LS}^w(\gamma_\lambda)$ for all $w \in W$.*

We give two examples of this construction in Section 7 together with a few more remarks.

Remark 5.33. We also want to introduce a small simplification of the notation as follows. For $\delta \in \Gamma_{LS}^+(\gamma_\lambda)$ we define $e_\alpha^{\max}(\delta) := e_\alpha^{\epsilon_\alpha(\delta)}(\delta)$, i.e., we apply e_α as often as it is defined. Furthermore for a reduced expression $w = s_{i_1} \ldots s_{i_r}$ we define $e_w^{\max}(\delta) := e_{\alpha_{i_r}}^{\max} \cdots e_{\alpha_{i_1}}^{\max}(\delta)$, hence we first apply $e_{\alpha_{i_1}}$ as often as it is defined, then $e_{\alpha_{i_2}}$ and so on, up to $e_{\alpha_{i_r}}$. As in the proof of the proposition, it follows from [Lit98a], that this is independent of the reduced expression. This allows us to write the first part of the proposition less technical as

$$\Xi_w(\delta) = we_w^{\max}(\delta).$$

From this it easily follows that the definition of the Ξ_w's is recursive in a certain way. Let $w \in W$ and $\alpha \in \Phi^+$, such that $l(ws_\alpha) > l(w)$, then

$$\Xi_{ws_\alpha}(\delta) = \Xi_{s_{w\alpha}}(\Xi_w(\delta)).$$

This is essentially the reason why we can hope to use an inductive argument in Section 6.

6 Retractions

We introduced the notion of flipping to obtain a link between the positively folded LS-galleries and the retractions at infinity for all Weyl group elements. In this section we prove the main theorem about the retractions in the affine building when applied to a dense subset in a given cell $C(\delta)$, for a given combinatorial gallery $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$. This relates the retractions with the combinatorial galleries $\Xi_w(\delta)$, defined in the last section, and will give a proof for the fact that these galleries can be used to define the MV-polytope, as well as the MV-cycle via the GGMS-strata.

6.1 Special case: w_0

We first want to deal with the case $\Xi_{w_0}(\delta)$, this is not essential for the general proof but highlights the type of transformations and calculations that have to be done.

Proposition 6.1. *Let $g = [g_0, g_1, \ldots, g_p] \in C(\delta)$, such that if $g_j = x_{-\alpha_{i_j}}(a_j)$ then $a_j \neq 0$, i.e., the gallery is minimal. Then $r_{id}(g) = \Xi_{w_0}(\delta)$.*

Proof. For some $t \in \mathbb{Z}_{\geq 0}$, let $r_1, \ldots, r_t$ be the set of indices where the gallery is folded, i.e., where $g_{r_s} = x_{-\alpha_{i_{r_s}}}(a_{r_s})$. We then start by looking at the last folding:

$$\begin{aligned} g &= [g_0, \ldots, g_{r_t-1}, x_{-\alpha_{i_{r_t}}}(a_{r_t}), g_{r_t+1}, \ldots, g_p] \\ &= [g_0, \ldots, g_{r_t-1}, x_{\alpha_{i_{r_t}}}(a_{r_t}^{-1}) s_{i_{r_t}} x_{\alpha_{i_{r_t}}}(a_{r_t})(a_{r_t})^{\alpha^\vee_{i_{r_t}}}, g_{r_t+1}, \ldots, g_p]. \end{aligned}$$

If we use the commutator formula to move $(a_{r_t})^{\alpha^\vee_{i_{r_t}}}$ to the right, we only change the elements of the form $x_{\alpha_{i_j}}(a_{i_j})$ for $j > r_t$ by multiplying a_{i_j} with a non-zero complex number, which is of no concern for the retraction later on. The term $x_{\alpha_{i_{r_t}}}(a_{r_t})$ is in $G(\mathcal{O})$ hence it leaves the set of minimal galleries invariant, see [GL05, § 7]. As the partial gallery $[g_{r_t+1}, \ldots, g_p]$ is minimal, the multiplication with $x_{\alpha_{i_{r_t}}}(a_{r_t})$ will change this part of the gallery, but it will be left minimal. Hence we might change all the coefficients after j, but the general structure of the gallery remains the same. Thus after using the relations to move both terms to the right we have a gallery of the form

$$g = [g_0, \ldots, g_{r_t-1}, x_{\alpha_{i_{r_t}}}(a_{r_t}^{-1}) s_{i_{r_t}}, \widetilde{g_{r_t+1}}, \ldots, \widetilde{g_p}],$$

with $\widetilde{g_j} = x_{\alpha_{i_j}}(\widetilde{a_j})s_{i_j}$ for $j > r_t$. Now the gallery is minimal after r_{t-1}.

Hence we iterate this process and obtain the following form for our gallery

$$g = [g_0, \overline{g_1}, \dots, \overline{g_p}],$$

with $\overline{g_j} = x_{\alpha_{i_j}}(\overline{a_j})s_{i_j}$ for $j > 0$ and $g_0 = \delta_0 \prod_{\beta<0,\delta_0(\beta)<0} x_\beta(a_\beta)$.

We still have to deal with g_0, for this we fix an ordering $\beta_1, \dots, \beta_s$ for some $s \in \mathbb{Z}_{\geq 0}$, such that $\beta_i < 0$ and $\delta_0(\beta_i) < 0$ and such that these are the only negative roots with this property. All the following computations will change the terms after g_0, but they are changed as above, only effecting the parameter of the one-parameter subgroups. Then we start with β_1 and obtain

$$\delta_0 \prod_{i=2}^{s} x_{\beta_i}(a_{\beta_i}) \cdot x_{-\beta_1}(a_{\beta_1}^{-1})s_{\beta_1}x_{-\beta_1}(a_{\beta_1})(a_{\beta_1})^{\beta_1^\vee}$$

as a new leading term.

As before we have to use the commutator formula to move the last two terms to the right in the gallery, this changes the gallery in the same way as described for the previous calculation. Finally we also use the relations to move s_{β_1} to the front and obtain

$$g_0 = \delta_0 s_{\beta_1} \prod_{i=s}^{2} x_{s_{\beta_1}\beta_i}(a_{\beta_i}) \cdot x_{\beta_1}(a_{\beta_1}^{-1})$$

as the new form for the first term. Of course $l(\delta_0 s_{\beta_1}) > l(\delta_0)$ and the set $\{s_{\beta_1}\beta_i \mid 2 \leq i \leq s\}$ is exactly the set of negative roots that is left negative by $\delta_0 s_{\beta_1}$. Hence we can do the same again, but we have to be a bit more careful as we will see. At the second step we arrive at

$$\begin{aligned}
& \delta_0 s_{\beta_1} \prod_{i=s}^{2} x_{s_{\beta_1}\beta_i}(a_{\beta_i}) \cdot x_{\beta_1}(a_{\beta_1}^{-1}) \\
= \quad & \delta_0 s_{\beta_1} \prod_{i=s}^{3} x_{s_{\beta_1}\beta_i}(a_{\beta_i}) \cdot x_{-\beta_2}(a_{\beta_2}^{-1})s_{\beta_2}x_{-\beta_2}(a_{\beta_2}) \cdot x_{\beta_1}(a_{\beta_1}^{-1}) \\
= \quad & \delta_0 s_{\beta_1} s_{\beta_2} \prod_{i=s}^{3} x_{s_{\beta_2}s_{\beta_1}\beta_i}(a_{\beta_i}) x_{\beta_2}(a_{\beta_2}^{-1}) x_{-\beta_2}(a_{\beta_2}) \cdot x_{\beta_1}(a_{\beta_2}^{\langle \beta_1, \beta_2^\vee \rangle} a_{\beta_1}^{-1}).
\end{aligned}$$

We now see that we have elements from three different one-parameter subgroups at the end and the subgroups correspond to negative as well as positive roots. Hence we need the commutator formula to order these elements such that we have elements from subgroups corresponding to negative roots to the left of those corresponding to positive roots. For these positive terms we use the same argument as above and multiply the rest of the gallery with them. We leave the negative ones as they are and repeat this process until we arrive at the following expression

$$\delta_0 s_{\beta_s} \dots s_{\beta_1} \prod_{\beta<0} x_\beta(b_\beta),$$

for some new elements $x_\beta(b_\beta)$ for negative roots only. Thus our gallery is now of the form

$$g = [\delta_0 s_{\beta_s} \dots s_{\beta_1} \prod_{\beta<0} x_\beta(b_\beta), x_{\alpha_{i_1}}(\overline{c_1})s_{i_1}, x_{\alpha_{i_2}}(\overline{c_2})s_{i_2}, \dots, x_{\alpha_{i_p}}(\overline{c_p})s_{i_p}],$$

for some complex numbers c_j for $1 \le j \le p$ and b_β for $\beta \in \Phi^-$.

Hence if we multiply the whole gallery with the toral element s^ρ for $s \in \mathbb{C}^*$ we obtain only positive powers of s in the one-parameter subgroups occurring in g_0 and also in all others as this is the exact same situation as if we would apply $s^{-\rho}$ to a non-folded gallery with only positive crossings instead of only negative ones. Thus if we take the limit s to 0, we obtain $\Xi_{w_0}(\delta)$ which is the unique minimal combinatorial gallery of the same type as g starting in the anti-dominant alcove.

□

6.2 Special case: s_α

We now want to proceed and make the fundamental calculations for the general case.

We will start with a gallery $g \in C(\delta)$, let $g = [g_0(\underline{a_\beta}), g_1(a_1) \dots, g_p(a_p)]$ with

$$g_0(\underline{a_\beta}) = \prod_{\beta<0, \delta_0^{-1}(\beta)<0} x_\beta(a_\beta) \cdot \delta_0$$

and

$$g_j(a_j) = \begin{cases} \delta_j & \text{if } j \notin J_{-\infty}(\delta) \\ x_{-\alpha_{i_j}}(a_j), a_j \neq 0 & \text{if } j \in J^-_{-\infty}(\delta) \\ x_{\alpha_{i_j}}(a_j)s_{i_j} & \text{if } j \in J^+_{-\infty}(\delta) \end{cases}$$

Hence our gallery has coordinates a_j for $1 \leq j \leq p$ as above and a_β for negative roots β. In the latter case the coordinates for those roots $\beta < 0$ such that $\delta_0^{-1}(\beta) > 0$ are just zero.

The first thing to do is to rewrite the gallery in the coordinates for $\mathcal{U}^{s_\alpha}_{\Xi_{s_\alpha}(\delta)}$. This will of course not work for an arbitrary gallery g, we will need to impose some assumptions on the coordinates to make it work.

To rewrite the coordinates, we recall that by Theorem 5.26 and Definition 5.31, the sequences of simple reflections of the galleries δ and $\Xi_{s_\alpha}(\delta)$ only differ at the index where the gallery attains its last minimum with respect to α and at the first and last index of stable sections that occur before this point. Hence we have to proceed in two steps:

(i) First we have to eliminate the folding at the minimum of the gallery or change the leading term of the gallery if the minimum is at 0.

(ii) Second for every stable section occurring before the minimum we have to change the folding at the first index to a crossing and the crossing at the last index to a folding.

Changing a folding to a crossing, which corresponds to eliminating the occurring simple reflection at that position, basically only means inverting the coefficient and moving some newly created terms to the right, similar to what we have already seen in 6.1, only this time we have to be a lot more careful as the remainder of the gallery is not minimal anymore.

For the calculations at the stable sections we define a special subset of indices that play an important role, the critical indices.

Definition 6.2. Let $\delta \in \Gamma^w_{LS}(\gamma_\lambda)$ and $[a, b]$ an α-stable section of δ, stable at n. An index $i \in [a, b]$ is called *critical* (or more precise α-critical), if $\Delta'_i \subset H_{\alpha,n}$.

Remark 6.3. By definition the first index of every α-stable section is always critical.

Another property of galleries that will be important is summarized in the next proposition, it deals with the way the roots corresponding to the types of the various faces of the gallery behave when they are "pulled back to the origin".

Proposition 6.4. *Let $\delta = [\delta_0, \delta_1, \ldots, \delta_p] \in \Gamma^+(\gamma_\lambda)$ and let $1 \leq j \leq p$ and $\Delta'_j \subset H_{\beta,n}$ for some positive root β and some integer n. Then it holds*

$$\delta_0 \ldots \delta_{j-1}(-\delta_j \alpha_{i_j}) = \begin{cases} -\beta + n\delta & \text{if } j \in J_{-\infty}(\delta) \\ \beta - n\delta & \text{if } j \notin J_{-\infty}(\delta) \end{cases}$$

Proof. We will only make the calculations for the load bearing case, the other one is exactly the same with opposite signs. We will use the embedding of the affine roots into $\Phi \times \mathbb{Z}$ and the corresponding notations of [BG06, §5.2]. Let H' be the wall, that contains the face of Δ_f of type t'_j. Then $H = \delta_0 \ldots \delta_j H'$ contains Δ'_j by definition of the type. By Remark 2.2 we can decompose the element $\delta_0 \ldots \delta_j$ into $s^\mu w$ with $\mu \in \mathbb{Z}\Phi^+$ and $w \in W$. As we are assuming that j is load-bearing it holds that $\Delta_j = \delta_0 \ldots \delta_j \Delta_f \subset H^+$, the positive closed half-space corresponding to H. This implies that $w\Delta_f \subset (t^{-\mu}H)^+$. We will now distinguish two cases.

(i) $\alpha_{i_j} \neq \alpha_0$**:** In this case Δ'_j and thus also H contains the unique vertex of Δ_j of type S. This implies that $t^{-\mu}H$ contains 0 as $w0 = 0$. Thus $s^{-\mu}H$ is a hyperplane containing the origin and hence $\Delta_f \subset (s^{-\mu}H)^+$. As we now have Δ_f and $w\Delta_f$ contained in $(s^{-\mu}H)^+$ we obtain $\beta := w\alpha_{i_j}$ must be positive and we set $n = \langle \beta, \mu \rangle$, hence $H = H_{\beta,n}$. We can conclude

$$\begin{aligned} (\delta_0 \ldots \delta_{j-1}) x_{\delta_j \alpha_{i_j}}(a) (\delta_0 \ldots \delta_{j-1})^{-1} &= s^\mu x_{w\alpha_{i_j}}(\pm a) s^{-\mu} \\ &= x_\beta(\pm a s^{\langle \beta, \mu \rangle}) \\ &= x_\beta(\pm a s^n). \end{aligned}$$

If the gallery is positively folded at j, we conclude that

$$\delta_0 \ldots \delta_{j-1}(-\alpha_{i_j}) = -\beta + n\delta$$

as $\delta_j \alpha_{i_j} = \alpha_{i_j}$. While for the case of a positive crossing we have $\delta_0 \ldots \delta_{j-1} \alpha_{i_j} = -\beta + n\delta$.

(ii) $\alpha_{i_j} = \alpha_0$**:** In this case we have that the hyperplane H' must be $H_{\theta,1}$ and thus $t^{-\mu}H = wH' = H_{w\theta,1}$ and $w\Delta_f \subset (H_{w\theta,1})^+$. Thus we have that $w\theta$ is a negative root and set $\beta = w(-\theta)$ the corresponding positive root and $n = \langle \beta, \mu \rangle - 1$, hence $H = H_{-\beta,-n} = H_{\beta,n}$. Again we calculate

$$\begin{aligned} (\delta_0 \ldots \delta_{j-1}) x_{\delta_j \alpha_{i_j}}(a) (\delta_0 \ldots \delta_{j-1})^{-1} &= s^\mu x_{w\alpha_{i_j}}(\pm a t^{-1}) s^{-\mu} \\ &= x_\beta(\pm a s^{\langle \beta, \mu \rangle - 1}) \\ &= x_\beta(\pm a s^n). \end{aligned}$$

Hence we again obtain for a positive folding $\delta_0 \ldots \delta_{j-1}(-\alpha_{i_j}) = -\beta + n\delta$ and for a positive crossing $\delta_0 \ldots \delta_{j-1}\alpha_{i_j} = -\beta + n\delta$.

□

Remark 6.5. In the case of a folding we apply $\delta_0 \ldots \delta_{j-1}$ to $-\alpha_{i_j}$ in the statement of the above proposition and in the case of a crossing to α_{i_j}. Thus one has to be careful as in a few cases during the calculations we will apply $\delta_0 \ldots \delta_{j-1}$ to the additive inverse of the roots we mention in the Proposition 6.4.

For the following propositions and calculations we want to fix a number of notations.

Notation 6.6. We fix a simple root α, an LS-gallery $\delta = [\delta_0, \delta_1, \ldots, \delta_p]$, and $g \in C(\delta)$ with the coordinates written as above. Let m be minimal such that there exists an index j with $\Delta'_j \subset H_{\alpha,m}$ and $k = \max\{j \mid \Delta'_j \subset H_{\alpha,m}\}$ the last of these indices. By $[u_1, v_1], \ldots, [u_r, v_r]$ we denote the α-stable sections of δ such that $v_1 \leq k$ and $u_i > v_{i+1}$, hence the α-stable sections that occur before k in reverse order, thus the ones that are relevant for the calculations as they will change. For $[u_i, v_i]$ we denote by $C([u_i, v_i])$ the set of critical indices of this section.

We will now have to deal with the different positions where we have to change the coordinates of our gallery. As we do not want to change the different positions in an arbitrary order, we will fix a sequence of galleries κ^i, $0 \leq i \leq r$, that basically lie between δ and $\Xi_{s_\alpha}(\delta)$. The galleries $\kappa^i = [\kappa^i_0, \kappa^i_1, \ldots, \kappa^i_p]$ are defined as follows:

$$\kappa^0_j = \begin{cases} s_{\alpha_{i_k}}\delta_k & j = k \text{ and } k < p \\ \delta_j & \text{otherwise} \end{cases},$$

$$\kappa^i_j = \begin{cases} s_{\alpha_{i_j}}\kappa^{i-1}_j & j = u_i \text{ or } j = v_i \\ \kappa^{i-1}_j & \text{otherwise} \end{cases} \quad \text{for } 1 \leq i \leq r.$$

Remark 6.7. Obviously $\kappa^r = \Xi_{s_\alpha}(\delta)$.

If we change the coordinates of g for the section $[u_i, v_i]$ we will always assume that our gallery is already written in the coordinates for the combinatorial gallery κ^{i-1}. This means that we will first do the changes at the position k and then proceed with the stable sections $[u_1, v_1]$ up to $[u_r, v_r]$, in exactly this order. By the above remark we will finish with a gallery written in the coordinates for the combinatorial gallery $\Xi_{s_\alpha}(\delta)$.

We will have to prove four propositions that deal with the possible cases, two for dealing with the coordinate changes from δ to κ^0, one for $k = 0$ and one for $k \neq 0$, and two for the coordinate changes from κ^i to κ^{i+1}, again we have to differentiate between the cases that $u_{i+1} = 0$ or not.

The different proofs of the following propositions are always quite elongate, but they always follow the same pattern. We begin by changing some position in the gallery by using the relations from part 2 and afterwards make sure that a number of unwanted terms can be moved to the end of the gallery without posing any problems.

We commence with the change of coordinates from δ to κ^0 and $k \neq 0$, the calculations are very similar to the ones in 6.1.

Proposition 6.8. *We assume that $k \neq 0$. If $g_k = x_{-\alpha_{i_k}}(a_k)$ with $a_k \neq 0$, then $g \in \mathcal{U}_{\kappa^0}$.*

Proof. In the case that $k = p$ we have nothing to show as δ and κ^0 coincide.

Hence we assume that $0 < k < p$ and only have to change δ_k from id to s_{i_k}. Thus we only look at the following part of our gallery

$$[x_{-\alpha_{i_k}}(a_k), g_{k+1}(a_{k+1}), \dots, g_p(a_p)].$$

We start by using the Chevalley relations to change the gallery at the position k, to obtain

$$[x_{\alpha_k}(a_k^{-1}) s_{\alpha_{i_k}} x_{\alpha_{i_k}}(a_k) a_k^{\alpha_{i_k}^\vee}, g_{k+1}(a_{k+1}), \dots, g_p(a_p)].$$

Next we move the two terms $x_{\alpha_{i_k}}(a_k) a_k^{\alpha_{i_k}^\vee}$ to the right using the commutator formula. The term $a_k^{\alpha_{i_k}^\vee}$ poses no problem in this regard, as it will multiply all parameters a_j occurring afterwards by a non-zero complex number. In contrast to that $x_{\alpha_{i_k}}(a_k)$ may produce new terms when moving to the right, because of the Chevalley commutator formula.

The first thing that has to be verified is, if $x_{\alpha_{i_k}}(a_k)$ can move through the whole gallery up to the end and then vanish in the right coset by $\mathcal{I}$. Let us assume that this is not true, hence there exists $j > k$ such that

$$\delta_{j-1} \dots \delta_{k+1} \alpha_{i_k} = \alpha_{i_j},$$

with the property that $\delta_j = s_{i_j}$, otherwise we would not create an element corresponding to a negative root anyway. We then apply $\delta_0 \cdots \delta_{j-1}$ to this equation and obtain

$$\alpha - m\underline{\delta} = \begin{cases} -\gamma + n\underline{\delta} & \text{for a positive crossing} \\ \gamma - n\underline{\delta} & \text{for a negative crossing} \end{cases},$$

for some positive root γ and some integer n. This is a contradiction in the first case as α and γ are positive and in the second as m was the minimum and k the last index where this minimum occurs and hence $m < n$.

Hence we can move the term $x_{\alpha_{i_k}}(a_k)$ to the right until we arrive at the end and it vanishes in $\mathcal{I}$. But we still need to deal with the new terms that are created during this process.

We have to make sure that these new terms do not eliminate any existing coefficients or create new coefficients in front of former negative crossings.

Let us first assume that a newly created term cannot move all the way to the right, hence it arrives at a position $j > k$ where it is made negative. Then there are $k < j_1 < \ldots < j_l < j$ and positive integers $p_1, q_1, \ldots, p_l, q_l$ that describe the root of the new term, i.e., the indices $j_1, \ldots, j_l$ are the positions where the new term was created using the Chevalley commutator formula and let $p_1, q_1, \ldots, p_l, q_l$ be the occurring coefficients. Then our new term arrives at the position j as $x_\beta(c)$ for some c and

$$\beta = p_l \cdots p_1 \delta_{j-1} \cdots \delta_{k+1} \alpha_{i_k} + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h \delta_{j-1} \cdots \delta_{j_h} \beta_{j_h},$$

with $\beta_{j_h} = -\delta_{j_h}(\alpha_{i_{j_h}})$. We will first deal with the case that the gallery has a negative crossing at the position j. As before we assume $\beta = \alpha_{i_j}$ and apply $\delta_0 \cdots \delta_{j-1}$ to this equality to obtain

$$\gamma - m'\underline{\delta} = p_l \cdots p_1(\alpha - m\underline{\delta}) + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h(-\gamma_h + m_h\underline{\delta}),$$

for some positive roots γ_h and γ and some integers m_h and m' if the gallery has a negative crossing at j. On the right side we have a multiple of the simple root α and substract positive roots γ_h, while on the right hand side we have a simple root γ as a real part of this equation. This can only be true if all the roots γ_h and γ are equal to α, hence for a negative crossing we arrive at

$$\alpha - m'\underline{\delta} = p_l \cdots p_1(\alpha - m\underline{\delta}) + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h(-\alpha + m_h\underline{\delta}).$$

This can be divided into an equality for the real part and one for the imaginary part

$$1 + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h = p_l \cdots p_1$$

and

$$m' + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h m_h = p_l \cdots p_1 m.$$

But m was the minimum of the gallery with respect to α and the index k was the last time this minimum occurred, hence $m_h > m$ for all h and $m' > m$, thus the two equalities contradict each other. Thus we have seen that the new terms will never arrive at a negative crossing with the same root, hence they can always move past these positions and hence there are no coefficients created at such a position.

In the case that the new term arrives with the same root at a positive crossing we just add the new term to the existing coefficient.

Finally we need to look at what happens if new terms arrive at a folding position j. This can only be problematic if the root β is equal to α_{i_j}, with the same notations as above. This of course leads to the same equation as for the negative crossing and we obtain the same result, that the new terms never change the coefficients in front of any foldings.

Hence we have seen that the new terms either move all the way to the right or add themselves to coefficients in front of positive crossings. This completes the proof. □

Of course we need the same result for the case that $k = 0$, in this case the coordinates change in a more complicated way as the first part of the gallery is not only a simple reflection but possibly an arbitrary Weyl group element. To look at the case $k = 0$ one has to remember that this means that the alcove $\Delta_0 = \delta_0\Delta_f$ lies in the positive half-space $H^+_{\alpha,0}$, hence $\delta_0^{-1}(-\alpha) < 0$, as $\langle \delta_0\rho, \alpha^\vee \rangle > 0$. Thus $-\alpha$ is one of the roots whose root group may occur in the term $g_0(\underline{a_\beta})$ with a non-zero paramater.

Proposition 6.9. *We assume that $k = 0$. If*

$$g_0(\underline{a_\beta}) = \prod_{\beta<0,\delta_0^{-1}(\beta)<0} x_\beta(a_\beta) \cdot \delta_0,$$

with $a_{-\alpha} \neq 0$, then $g \in \mathcal{U}_{\kappa^0}^{s_\alpha}$.

Proof. As mentioned in Section 2, we can assume that the product occurring in g_0 is already ordered corresponding to the height of the occurring roots, with those of lowest height (as we have negative roots) to the right. For this we will write

$$g_0(\underline{a_\beta}) = \overleftarrow{\prod_{\beta<0,\delta_0^{-1}(\beta)<0}} x_\beta(a_\beta) \cdot \delta_0,$$

with the arrow pointing in the direction of the highest root in the product.

The first thing we want to do with the product is to move the coefficient $x_{-\alpha}(a_{-\alpha})$ to the right such that it is located in front of δ_0. While we move it to the right it will produce new terms by the Chevalley commutator formula when it moves past some $x_\beta(a_\beta)$, but the set of roots for the product is closed under summation and the new terms that are created to the right of our terms $x_{-\alpha}(a_{-\alpha})$ and $x_\beta(a_\beta)$ have a lower height than both of them, thus the corresponding roots already occur further right in the product. Hence we can not only move $x_{-\alpha}(a_{-\alpha})$ to the right, but also know that the newly created terms just add themselves to already existing terms by just moving all of them to the right as well. Thus we arrive at the following form

$$g_0(\underline{a_\beta}) = \overleftarrow{\prod_{\beta<0,\beta\neq-\alpha,\delta_0^{-1}(\beta)<0}} x_\beta(a'_\beta) x_{-\alpha}(a_{-\alpha}) \cdot \delta_0,$$

where

$$a'_\beta = a_\beta + f(a_\gamma \mid ht(-\alpha) \geq ht(\gamma) > ht(\beta))$$

and f is a polynomial. It is obvious that if we view the parameters in our coefficients as algebraically independent transcendent elements over $\mathbb{C}$ that these new parameters are algebraically independent as well, as they transform in a triangular pattern.

Next we proceed with our usual transformation of $x_{-\alpha}(a_{-\alpha})$. With the assumptions we can write $g_0(\underline{a_\beta})$ as

$$g_0(\underline{a_\beta}) = \overleftarrow{\prod_{\beta<0,\beta\neq-\alpha,\delta_0^{-1}(\beta)<0}} x_\beta(a'_\beta) x_\alpha(a_{-\alpha}^{-1}) s_\alpha x_\alpha(a_{-\alpha}) a_{-\alpha}^{\alpha^\vee} \cdot \delta_0.$$

By our assumptions we know that $\delta_0^{-1}\alpha$ is positive, hence we can move $x_\alpha(a_{-\alpha})a_{-\alpha}^{\alpha^\vee}$ past δ_0. The changes that occur afterwards when this term moves through the gallery are exactly the same as before in Proposition 6.8.

Hence we are left with

$$g_0(\underline{a_\beta}) = \overleftarrow{\prod_{\beta<0,\beta\neq-\alpha,\delta_0^{-1}(\beta)<0}} x_\beta(a'_\beta)x_\alpha(a_{-\alpha}^{-1}) \cdot s_\alpha\delta_0.$$

Of course we have to make sure that the coefficients that we now have are the right ones for the open chart $\mathcal{U}^{s_\alpha}_{\Xi_{s_\alpha}(\delta)}$. To check this we have the following two conditions that need to be satisfied for a root γ, whose one-parameter subgroup may appear in front of $s_\alpha\delta_0$:

$$\gamma <_{s_\alpha} 0 :\Leftrightarrow s_\alpha\gamma < 0$$

and

$$(s_\alpha\delta_0)^{-1}\gamma < 0.$$

The first of these conditions is satisfied by all roots that we have, for α it is obvious and for the others it just follows from the fact that they are negative roots different from $-\alpha$. In contrast to that the second condition does not have to be satisfied by all appearing roots. For α it is true by assumption, but for the others it might not be.

Thus we start by dividing our set of roots that appear into two disjoint subsets

$$R_\alpha = \{\beta < 0 \mid \beta \neq -\alpha, \delta_0^{-1}\beta < 0, (s_\alpha\delta_0)^{-1}\beta < 0\},$$

which are the ones that satisfy the second condition and

$$R'_\alpha = \{\beta < 0 \mid \delta_0^{-1}\beta < 0, (s_\alpha\delta_0)^{-1}\beta > 0\},$$

the ones that do not.

Later in 6.3 we will take a closer look at the roots which we have to substitute for the ones in R'_α. But for this proof we will just eliminate the coefficients that we don't need. Hence we will just move all elements corresponding to roots in R'_α past $x_\alpha(a''_\alpha)$ and $s_\alpha\delta_0$, starting with the one furthest to the left. This will of course produce new terms, but as all of these correspond to negative roots, different from $-\alpha$, they satisfy the first condition above. Again we just observe if the new terms don't satisfy the second condition above, move them past $s_\alpha\delta_0$ as well, and leave them where they are if they do satisfy the condition.

Finally we have to take a look at what happens when some of the created terms move through the rest of the gallery. Let $x_{\delta_0^{-1}s_\alpha\gamma}(c)$ be such a term that moves through the gallery with a negative root γ such that $(s_\alpha\delta_0)^{-1}\gamma > 0$ and $c \in \mathbb{C}$. We have seen that these are the only ones that are created and move past $s_\alpha\delta_0$. We denote $\delta_0^{-1}s_\alpha\gamma$ by γ' for short. We now have to do a similar calculation as in Proposition 6.8 and assume that the term or any terms it could create cannot move through the whole gallery or arrive at a folding with the same root as the coefficient at the folding. Let us assume that $j > 0$ is such a position where the new term cannot move past. Then there are $0 < j_1 < \ldots < j_l < j$ and positive integers $p_1, q_1, \ldots, p_l, q_l$ that describe the root of the new term, i.e., the indices $j_1, \ldots, j_l$ are the positions where the new term was created using the Chevalley commutator formula. Then our new term arrives at the position j with the root

$$p_l \cdots p_1 \delta_{j-1} \cdots \delta_1 \delta_0^{-1} s_\alpha \gamma + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h \delta_{j-1} \cdots \delta_{j_h} \beta_{j_h},$$

where $\beta_{j_h} = -\delta_{j_h}(\alpha_{i_{j_h}})$.

As above we assume that this root is equal to α_{i_j} and apply $\delta_0 \cdots \delta_{j-1}$ to this equality to obtain

$$\beta - m'\underline{\delta} = p_l \cdots p_1 s_\alpha \gamma + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h (-\beta_h + m_h \underline{\delta}),$$

for some positive roots β_h and β and some integers m_h and m' if the gallery has a negative crossing at j. We can again divide this into two equalities to obtain

$$\beta + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h \beta_h = p_l \cdots p_1 s_\alpha \gamma$$

and

$$-m' = \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h m_h.$$

But in this case the first equality taken alone is already a contradiction as the left side is a positive linear combination of positive roots, while the right side is a positive multiple of a negative root, as $\gamma \neq \alpha$. Hence the new terms can also only contribute to the coefficients at positive crossings and nowhere else. This completes the proof. □

Next we will have to deal with the stable sections $[u, v]$ that occur before the minimum with respect to α. If, in the following, we deal with a stable section $[u_i, v_i]$ we will always assume that we have already dealt with the part of the gallery after v_i. More precisely we assume that we can already write the gallery in the coordinates for the gallery κ^{i-1}.

In other words we have already applied the change of coordinates from Proposition 6.8 to the gallery as well as the result of the coming Proposition 6.10 for all stable section $[u_l, v_l]$ for $l < i$. This especially means that all crossings of α-walls that occur after v_i are at walls whose height is strictly smaller than the height of the wall that the v_i-th face of κ^i lies in.

Proposition 6.10. *We fix one of the section $[u_i, v_i]$ and assume $u_i \neq 0$. Let $g = [g_0(\underline{a_\beta}), g_1(a_1) \ldots, g_p(a_p)] \in \mathcal{U}_{\kappa^{i-1}}$ be the coordinates of g with respect to the gallery κ^{i-1}. If for any $I \subset C([u_i, v_i])$ the inequality*

$$\sum_{n \in I} a_n \neq 0,$$

holds, then $g \in \mathcal{U}_{\kappa^i}$.

Proof. For simplicity we will write $[u, v]$ instead of $[u_i, v_i]$ and assume that $[u, v]$ is α-stable at n. Since we look at an α-stable section we need to change two elements in our gallery κ^{i-1}. We have to change κ^i_u to id_W and κ^{i-1}_v to $s_{\alpha_{i_v}}$. Hence we look at

$$[x_{-\alpha_{i_u}}(a_u), g_{u+1}(a_{u+1}), \ldots, g_p(a_p)].$$

We start by using the Chevalley relations to change the gallery at the position u, to obtain

$$[x_{\alpha_{i_u}}(a_u^{-1}) s_{\alpha_{i_u}} x_{\alpha_{i_u}}(a_u) a_u^{\alpha_{i_u}^\vee}, g_{u+1}(a_{u+1}), \ldots, g_p(a_p)].$$

As in Proposition 6.8, we want to move the two terms $x_{\alpha_{i_u}}(a_u) a_u^{\alpha_{i_u}^\vee}$ to the right using the commutator formula. As before, the term $a_u^{\alpha_{i_u}^\vee}$ poses no problem in this regard, as it will multiply all parameters a_j occurring afterwards by a non-zero complex number, but the term $x_{\alpha_{i_u}}(a_u)$ may produce new terms when moving to the right, because of the commutator formula. In addition to the possibilities in Proposition 6.8 we now also have to be careful about the critical indices of this gallery, as they correspond to small faces in the same wall as the small face at the position u, hence we do not always have strict inequalities for the coefficients of $\underline{\delta}$ in the equations that also appeared in Proposition 6.8.

Let $u = k_1 < \ldots < k_s$ be the critical indices in $[u, v]$. We first move $x_{\alpha_{i_u}}(a_u)a_u^{\alpha_{i_u}^\vee}$ to the right until it arrives in front of $g_{k_2}(a_{k_2})$. On the way, $x_{\alpha_{i_u}}(a_u)$ may create new terms through the Chevalley commutator formula. As in Proposition 6.8, these new terms appear to the right of $x_{\alpha_{i_u}}(a_u)$ when it moves through the gallery. Hence we will move them to the right as well. If they create new terms themselves we move these to the right as well. There are now basically four possibilities for new terms that may occur when we move them to the right. The first and second case basically correspond to the possible behaviour in Proposition 6.8, while the other two are new as they depend on the existence of critical indices.

(i) The new term can move to the right and ends up in front of $g_{k_2}(a_{k_2})$ with a root different from $\pm\alpha_{i_{k_2}}$.

(ii) The new term cannot move to the right up to $g_{k_2}(a_{k_2})$.

(iii) The new term can move to the right and ends up in front of $g_{k_2}(a_{k_2})$ with a root equal to $-\alpha_{i_{k_2}}$.

(iv) The new term can move to the right and ends up in front of $g_{k_2}(a_{k_2})$ with a root equal to $\alpha_{i_{k_2}}$.

Case (i): This is the usual case, as the term can then simple move past the position k_2 and move further to the right.

Case (ii): If the new term moves to the right and has to move past a reflection $s_{\alpha_{i_j}}$, for some $k_1 < j < k_2$ such that its root is made negative, then there are two possibilities. δ can either have a negative or a positive crossing at Δ'_j. If it crosses in a positive direction, we just add our new term to the one that is already in front of $s_{\alpha_{i_j}}$. If it is a negative crossing we make a calculation similar to the one in Proposition 6.8 to see that this cannot happen. Hence we look at positions $u = k_1 < j_1 < \ldots j_l < j$ and positive integers $p_1, q_1, \ldots, p_l, q_l$ such that the new term has

$$\gamma = p_l \cdots p_1 \delta_{j-1} \cdots \delta_{u+1} \alpha_{i_u} + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h \delta_{j-1} \cdots \delta_{j_h} \beta_{j_h}$$

as its root.

In this case we assume that $\gamma = \alpha_{i_j}$ and again apply $\delta_0 \cdots \delta_{j-1}$ to the whole equation and obtain

$$\gamma' - n'\underline{\delta} = p_l \cdots p_1(\alpha - n\underline{\delta}) + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h(-\gamma_h + m_h\underline{\delta}),$$

for some positive roots γ' and γ_h for all $1 \leq h \leq l$ and integers m_h for all h and n'. The left side is of such a form, with γ' positive, because we assumed that the gallery has a negative crossing at j. Again we can split this into two equalities

$$\gamma' = p_l \cdots p_1\alpha - \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h \gamma_h$$

and

$$n' = p_l \cdots p_1 n_r - \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h m_h.$$

The first equality means that $\gamma' \in p_l \cdots p_1\alpha - \mathbb{Z}_{>0}\Phi^+$, which means that $\gamma' = \alpha$ as α is simple. Hence with the same argument as in Proposition 6.8, the first equation simplifies to

$$1 = p_l \cdots p_1 - \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h.$$

As $n_r < n'$ and $n_R \leq m_h$, the two equalities would imply

$$n' + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h m_h = \left(1 + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h\right) n_r < n' + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h m_h,$$

which is a contradiction. Hence if the case occurs that a new term cannot move to the right, it will simply merge with an existing coefficient in front of a positive crossing. The same calculations as for the negative crossing also shows that the new terms can never change the coefficients at a non-critical folding, see for this the arguments in the proof of Proposition 6.8.

Case (iii): This case cannot appear at all as the newly created terms always correspond to positive roots. The cases were these may be transformed into negative roots, were already dealt with in case (ii). We have seen there that it either does not happen at all, or the new term just adds itself to the already existing one at a positive crossing. Thus this case is already covered by the previous one.

Case (iv): This is the most problematic case as a term with these properties will change the parameter at the position k_2 in a significant way when moving past it and will make it impossible to control the change of coordinates at that position. Hence let us take a closer look at one of these new terms. Again, there have to be positions $u = k_1 < j_1 < \ldots j_l < k_2$ where the new term was created using the Chevalley commutator formula and positive integers $p_1, q_1, \ldots, p_l, q_l$ that occurred as coefficients of the new root in the formula. Then the new term will appear in front of $g_{k_2}(a_{k_2})$ as $x_\gamma(c)$ for some c and

$$\gamma = p_l \cdots p_1 \delta_{k_2-1} \cdots \delta_{u+1} \alpha_{i_u} + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h \delta_{k_2-1} \cdots \delta_{j_h} \beta_{j_h},$$

with $\beta_{j_h} = -\delta_{j_h}(\alpha_{i_{j_h}})$.

In this case we assume that $\gamma = \alpha_{k_2}$, as above we apply $\delta_0 \cdots \delta_{k_2-1}$ to the whole equation and obtain

$$\alpha - n\underline{\delta} = p_l \cdots p_1(\alpha - n\underline{\delta}) + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h(-\gamma_h + m_h\underline{\delta}),$$

for some positive roots γ_h and integers m_h, corresponding to the hyperplane containing Δ'_{j_h}. The terms in the sum are all of the form $-\gamma_j + m_j\delta$ as there are no coefficients at places where the gallery crosses a hyperplane in the negative direction and as we have seen in case (ii) this also does not change via the new terms. If we look at the equality above, we can take its classical and its imaginary part as usual and obtain two equalities

$$(p_l \cdots p_1 - 1)\alpha = \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h \gamma_h$$

and

$$(p_l \cdots p_1 - 1)n = \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h m_h.$$

In the first equality we have a non-negative multiple of a simple root on the left and a sum of positive roots on the right, this can only be equal if all occurring γ_h are equal to α. Hence the new terms have to be created at positions that lie inside a hyperplane corresponding to α. As there are no critical indices between k_1 and k_2 all the m_j have to be greater than n. Hence we can simplify the first equations to

$$(p_l \cdots p_1 - 1) = \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h$$

and the second equation to

$$(p_l \cdots p_1 - 1) n_r = \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h m_h,$$

it holds that $m_h \leq n$ for all h and $m_h > n$ for at least one h. But these two equalities can not hold at the same time. Hence there are no new terms that arrive at $g_{k_2}(a_{k_2})$ with root α_{k_2}. We only assume that at least one $m_h > n$ as this allows us to use the same argument when we look at critical indices occurring after k_2.

Summary: We have now shown that the new terms that are created through the Chevalley commutator formula can always be moved to the right and they either add themselves to a coefficient at a positive crossing or they move past the critical index k_2 and past all other foldings as well as negative crossing without changing the coefficients.

As, in the final case, we have only made the assumption that there exists at least one $m_h > n$ and that this does not need to hold for all m_h we can also conclude that any new terms will also move past all other critical indices occurring after k_2. This is true, as there can be no terms that were created by the commutator formula that only involve the critical indices, because the term $x_{\alpha_{i_u}}(a_u)$ will always arrive at a critical index k_h as $x_{\alpha_{i_{k_h}}}(a_u)$ and will move past the critical index without creating any new terms at that position, as the commutator formula does not apply in this situation.

How the coefficients at the critical indices change exactly will be calculated in Lemma 6.11 afterwards, but just by the Chevalley relations it is obvious that they remain algebraically independent and non-zero, while non of the newly created terms affect them.

By Lemma 6.11 the terms $x_{\alpha_{i_u}}(a_u) a_u^{\alpha_{i_u}^\vee}$ will move to the right and arrive at the position v as $x_{\alpha_{i_v}}(a_{k_1} + \ldots + a_{k_s})(-(a_{k_1} + \ldots + a_{k_s}))^{\alpha_{i_v}^\vee}$. The gallery κ^{i-1} always has a negative crossing at the position v and hence no coefficient, except in the special case where $v = k$, the minimum with respect to α, there is a coefficient $a_v \neq 0$, see Proposition 6.8. Hence we have to do the following calculation with this special case in mind

$$\begin{aligned}
& x_{\alpha_{i_v}}(A)(-A)^{\alpha_{i_v}^\vee} x_{\alpha_{i_v}}(a_v) s_{i_v} \\
= \ & x_{\alpha_{i_v}}(A + A^2 a_v) s_{v_i} (-A)^{-\alpha_{i_v}^\vee} \\
= \ & x_{-\alpha_{i_v}}\left((A + A^2 a_v)^{-1}\right) \left(A + A^2 a_v\right)^{\alpha_{i_v}^\vee} x_{\alpha_{i_v}}\left(-(A + A^2 a_v)^{-1}\right) (-A)^{-\alpha_{i_v}^\vee} \\
= \ & x_{-\alpha_{i_v}}\left((A + A^2 a_v)^{-1}\right) x_{\alpha_{i_v}}\left(-(A + A^2 a_v)\right) \left(A + A^2 a_v\right)^{\alpha_{i_v}^\vee} (-A)^{-\alpha_{i_v}^\vee} \\
= \ & x_{-\alpha_{i_v}}\left((A + A^2 a_v)^{-1}\right) x_{\alpha_{i_v}}\left(-(A + A^2 a_v)\right) (1 + A a_v)^{\alpha_{i_v}^\vee} (-1)^{-\alpha_{i_v}^\vee},
\end{aligned}$$

where $A = \sum_{j=1}^{s} a_{k_j}$. Again we have some terms that will move through the gallery after v:

$$x_{\alpha_{i_v}}\left(-(A + A^2 a_v)\right) (1 + A a_v)^{\alpha_{i_v}^\vee} (-1)^{-\alpha_{i_v}^\vee}.$$

For the rest of the gallery, the terms $(1 + A a_v)^{\alpha_{i_v}^\vee} (-1)^{-\alpha_{i_v}^\vee}$ are of no real concern as they again only multiply some coefficients with non-zero complex numbers. The term $x_{\alpha_{i_v}}(-(A + A^2 a_v))$ on itself is also of no real concern as it can only change terms that lie at small faces in the hyperplane $H_{\alpha,n}$, but as mentioned above, all hyperplanes for α occurring after v have a height strictly smaller than n. Hence we only have to take care of the new terms that are created by $x_{\alpha_{i_v}}(-(A + A^2 a_v))$ and herein lies the problem.

For the newly created terms there are two different possibilities how they originated, either before or after v_i. The difference is the way they behave if they are pulled back to the origin. One other thing that has to be kept in mind is, that the gallery after v_i is now negatively folded with regards to α and positively folded with regards to the other positive roots or in other words it is positively folded with respect to the fundamental alcove $s_\alpha \Delta_f$.

Terms created after v:

In this case let $t > v$ and $v < j_1 < \ldots < j_s \leq t$ be the indices were the new term is created using the Chevalley commutator formula and as before the corresponding positive integers $p_1, q_1, \ldots, p_l, q_l$. Hence our term has the form $x_\gamma(c)$, for some paramter and

$$\gamma = p_l \cdots p_1 \delta_t \cdots \delta_{v+1} \alpha_{i_v} + \sum_{h=1}^{l} p_l \cdots p_{h+1} q_h \delta_t \cdots \delta_{j_h+1} \beta_{h_j},$$

with $\beta_{j_h} = -\delta_{j_h}(\alpha_{i_{j_h}})$. We have to be a bit careful as the gallery is not positively folded after v_i for all positive roots, hence we have to differentiate between those roots β_{j_h} that correspond to positions of the gallery where it crosses or is folded at a hyperplane corresponding to α or another positive root. For this let $J_\alpha \subset \{j_1, \ldots, j_l\}$ be the subset of indices corresponding to small faces of the gallery that lie in a hyperplane corresponding to α.

We now assume $\gamma = \alpha_{i_{t+1}}$.

(i) **$\Delta'_{i_{t+1}}$ lies in a hyperplane corresponding to α:** In this case we have to differentiate between two possibilities, either the gallery crosses in a negative direction or it has a negative folding or crosses in a positive direction. In the case that we have a positive crossing or a negative folding we have to make sure that this situation cannot arise. Let us apply $\delta_0 \cdots \delta_t$ to the equality $\gamma = \alpha_{i_{t+1}}$ to obtain

$$\begin{aligned} -\alpha + n'\underline{\delta} = p_l \cdots p_1 \left(-\alpha + n\underline{\delta}\right) \quad &+ \sum_{h=1, j_h \notin J_\alpha}^{l} p_l \cdots p_{h+1} q_h \left(-\gamma_{j_h} + n_{j_h}\underline{\delta}\right) \\ &+ \sum_{h=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h \left(-\alpha + \widetilde{n_{j_h}}\underline{\delta}\right), \end{aligned}$$

with integers n', n_{j_h}, and $\widetilde{n_{j_h}}$, positive roots γ_{j_h} different from α. Of course this equality can only hold if the first sum vanishes as this is composed of roots whose classical part is different from $\pm\alpha$, hence this sum cannot be equal to a multiple of α. If this sum vanishes we are only left with

$$-\alpha + n'\underline{\delta} = p_l \cdots p_1 \left(-\alpha + n\underline{\delta}\right) + \sum_{h=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h \left(-\alpha + \widetilde{n_{j_h}}\underline{\delta}\right),$$

with $n > n'$ and $n > \widetilde{n_{j_h}}$ for all $j_h \in J_\alpha$. This was the main reason why we assumed the way g was written in the beginning. As in this first case, we now divide the equality into two equalities in the classical and the imaginary component and obtain

$$p_l \cdots p_1 = 1 + \sum_{h=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h$$

and

$$p_l \cdots p_1 n = n' + \sum_{h=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h n_{j_h}.$$

This is a contradiction as we have the inequalities about the coefficients n, n', and n_{j_h} above.

In the case of a negative crossing we just add the new term to the existing coefficient. This means that we have a similar condition as above, that our newly created terms cannot change the coefficients in front of the position where our gallery is folded or has a positive crossing corresponding to α. Again one should remember that the gallery is negatively folded after v with respect to α.

(ii) **$\Delta'_{i_{t+1}}$ lies in a hyperplane not corresponding to α:** In this case we add the new term to the existing coefficient if our gallery has a positive crossing. In addition we have to make sure that we do not run into trouble if it is positive folding or a negative crossing. If this occurs, we again apply $\delta_0 \cdots \delta_t$ to the equality $\gamma = \alpha_{i_{t+1}}$ and obtain

$$\begin{aligned} \beta + n'\underline{\delta} = p_l \cdots p_1 (-\alpha + n\underline{\delta}) \quad &+ \sum_{h=1, j_h \notin J_\alpha}^{l} p_l \cdots p_{h+1} q_h (-\gamma_{j_h} + n_{j_h}\underline{\delta}) \\ &+ \sum_{h=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h (-\alpha + \widetilde{n_{j_h}\underline{\delta}}), \end{aligned}$$

with the same conditions as above and a positive root $\beta \neq \alpha$. Looking at the classical part of this equality we see that it cannot hold as we can move the first sum to the left side of the equality and obtain a sum of classical positive roots (all different from α) on the left and a multiple of α on the right. Hence we have again that the newly created terms cannot change the coefficients in front of folding positions or create new coefficients at positions with negative crossings.

Terms created before v:

If we are in this this case let $t > v$ and $u < j'_1 < \ldots < j'_{l'} < v < j_1 < \ldots < l_j \leq t$ be the indices were the new term is created using the Chevalley commutator formula and as before $p'_1, q'_1, \ldots, p'_{l'}, q'_{l'}$ and $p_1, q_1, \ldots, p_l, q_l$.

Hence our term has the form $x_\gamma(c)$, for some parameter c and

$$\begin{aligned}\gamma &= p_l\cdots p_1\cdot p'_{l'}\cdots p'_1\delta_t\cdots\delta_{u+1}\alpha_{i_u}\\ &+ \sum_{h=1}^{l'} p_l\cdots p_1\cdot p'_{l'}\cdots p'_{h+1}q'_h\delta_t\cdots\delta_{j'_h+1}\beta_{j'_h}\\ &+ \sum_{h=1}^{l} p_l\cdots p_{h+1}q_h\delta_t\cdots\delta_{j_h+1}\beta_{j_h},\end{aligned}$$

with $\beta_N = -\delta_N(\alpha_{i_N})$.

As above we have to differentiate between the same two cases. We now assume that $\gamma = \alpha_{i_{t+1}}$.

(i) $\Delta'_{i_{t+1}}$ **lies in a hyperplane corresponding to** α**:** Again we have the two possibilities, either we have a negative crossing at $\Delta'_{i_{t+1}}$, then we just add the new term or we have a negative folding or positive crossing. If we are in the latter situation we apply $\delta_0\cdots\delta_t$ to the equality $\gamma = \alpha_{i_{t+1}}$ and obtain

$$\begin{aligned}-\alpha + n'\underline{\delta} &= p_l\cdots p_1\cdot p'_{l'}\cdots p'_1(\alpha - n\underline{\delta})\\ &+ \sum_{h=1}^{l'} p_l\cdots p_1\cdot p'_{l'}\cdots p'_{h+1}q'_h(-\gamma'_{j'_h} + n_{j'_h}\underline{\delta})\\ &+ \sum_{h=1, j_h\notin J_\alpha}^{l} p_l\cdots p_{h+1}q_h(-\gamma_{j_h} + n_{j_h}\underline{\delta})\\ &+ \sum_{h=1, j_h\in J_\alpha}^{l} p_l\cdots p_{h+1}q_h(\alpha - \widetilde{n}_{j_h}\underline{\delta}),\end{aligned}$$

with J_α as above, integers n', n_{j_h}, $n'_{j'_h}$, and $\widetilde{n}_{j_h}$, positive roots γ_{j_h} different from α and positive roots $\gamma'_{j'_h}$. For the same reason as above we can see that the second sum cannot occur and also that all $\gamma'_{j'_h}$ are equal to

α. Hence we arrive at

$$\begin{aligned}
-\alpha + n'\underline{\delta} &= p_l \cdots p_1 \cdot p'_{l'} \cdots p'_1(\alpha - n\underline{\delta}) \\
&+ \sum_{h=1}^{l'} p_l \cdots p_1 \cdot p'_{l'} \cdots p'_{h+1} q'_h(-\alpha + n_{j'_h}\underline{\delta}) \\
&+ \sum_{h=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h(\alpha - \widetilde{n}_{j_h}\underline{\delta}),
\end{aligned}$$

with $n > n'$ and $n > \widetilde{n}_{j_h}$ for all $j_h \in J_\alpha$ and $n_i \leq n_{j'_h}$, as those were positions inside the stable section and hence in the positive halfspace corresponding to the hyperplane H_{α,n_i}. Again we split the equality into its classical and imaginary part to obtain

$$\begin{aligned}
-1 = p_l \cdots p_1 \cdot p'_{l'} \cdots p'_1 &- \sum_{h=1}^{l'} p_l \cdots p_1 \cdot p'_{l'} \cdots p'_{h+1} q'_h \\
&+ \sum_{h=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h
\end{aligned}$$

and

$$\begin{aligned}
n' = -p_l \cdots p_1 \cdot p'_{l'} \cdots p'_1 n &+ \sum_{h=1}^{l'} p_l \cdots p_1 \cdot p'_{l'} \cdots p'_{h+1} q'_h n_{j'_h} \\
&- \sum_{j=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h \widetilde{n}_{j_h}.
\end{aligned}$$

While the second one can be rewritten as

$$\begin{aligned}
\sum_{h=1}^{l'} p_l \cdots p_1 \cdot p'_{l'} \cdots p'_{h+1} q'_h n_{j'_h} = n' &+ p_l \cdots p_1 \cdot p'_{l'} \cdots p'_1 n \\
&+ \sum_{h=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h \widetilde{n}_{j_h},
\end{aligned}$$

the first equality together with the inequalities about the various coefficients of δ, yields

$$\begin{aligned}
&\sum_{h=1}^{l'} p_l \cdots p_1 \cdot p'_{l'} \cdots p'_{h+1} q'_h n_{j'_h} \\
\geq\ & \left(\sum_{h=1}^{l'} p_l \cdots p_1 \cdot p'_{l'} \cdots p'_{h+1} q'_h \right) n_i \\
=\ & p_l \cdots p_1 \cdot p'_{l'} \cdots p'_1 n + n + \sum_{h=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h n \\
>\ & p_l \cdots p_1 \cdot p'_{l'} \cdots p'_1 n + n' + \sum_{h=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h \widetilde{n}_{j_h},
\end{aligned}$$

which is a contradiction. Thus we have again the situation that these terms will not change any coefficients at foldings or negative crossings corresponding to α.

(ii) **$\Delta'_{i_{t+1}}$ lies in a hyperplane not corresponding to α:** This part is much easier than the last one, we add the new term to the existing coefficient if our gallery has a positive crossing. As usual we have to make sure that we do not run into trouble if it is positive folding or a negative crossing. If this occurs we again apply $\delta_0 \cdots \delta_t$ to the equality $\gamma = \alpha_{i_{t+1}}$ and obtain

$$\begin{aligned}
\beta - n'\underline{\delta} &= p_l \cdots p_1 \cdot p'_{l'} \cdots p'_1 (\alpha - n\underline{\delta}) \\
&+ \sum_{h=1}^{l'} p_l \cdots p_1 \cdot p'_{l'} \cdots p'_{h+1} q'_h (-\gamma'_{j'_h} + n_{j'_h} \underline{\delta}) \\
&+ \sum_{h=1, j_h \notin J_\alpha}^{l} p_l \cdots p_{h+1} q_h (-\gamma_{j_h} + n_{j_h} \underline{\delta}) \\
&+ \sum_{h=1, j_h \in J_\alpha}^{l} p_l \cdots p_{h+1} q_h (\alpha - \widetilde{n_{j_h} \underline{\delta}}),
\end{aligned}$$

with the same conditions as above and a positive root β different from α. In contrast to the previous case, we already have a contradiction in this equality as we can move the first and second sum to the left and obtain a linear combination of classical positive roots on the left that is not a multiple of α and a multiple of α on the right. Hence this case is already dealt with and we again have the situation that the newly created terms do not change the coefficients at folding positions or negative crossings.

Hence we have obtained the needed result, the term $x_{\alpha_{i_u}}(a_u)$ can be moved through the whole gallery and will only change the coefficients at positive crossings for roots different from α, positive crossing with respect to an α-hyperplane occurring before v, or negative crossings with respect to an α-hyperplane occurring after v. How these coefficient change will be dealt with in 6.3.

It should also be noted that the negative crossings between v and u_{i+1} (or k if we look at the last stable section) are also not changed. If we look at a term that was created after v we have the same calculation as before with no difference. If we look at a term created before v we have a long equality with four sums on the right side, but as the set J_α is empty in that case we see that the last sum vanishes and all roots in the second one are equal to α and one can immediately deduce a contradiction after splitting the equality into its imaginary and real part. □

We want to take a closer look at the coordinate changes at the critical indices.

Lemma 6.11. *We use the same notations as in the proof of Proposition 6.10, with $g = [g_0'(\underline{b_\beta}), g_1'(b_1) \ldots, g_p'(b_p)]$ being the coordinates with respect to κ^i, then the following holds*

$$b_u = a_u^{-1} \text{ and}$$

$$b_{k_j} = \frac{a_{k_j}}{\left(\sum_{l=1}^{j-1} a_{k_l}\right)\left(\sum_{l=1}^{j} a_{k_l}\right)}.$$

Proof. The equality $b_u = a_u^{-1}$ is obvious by the proof of Proposition 6.10. When moving $x_{\alpha_{i_u}}(a_u)a_u^{\alpha_{i_u}^\vee}$ to the right the non-critical indices will be changed as described above, but as proved in Proposition 6.10 the terms that are created through the Chevalley commutator formula will play no role when looking at the critical indices. When we arrive with our two terms at the index k_2, one has to remember that $\delta_{k_2-1}\cdots\delta_{k_1+1}\alpha_{k_1} = \alpha_{k_2}$, by Proposition 6.4, we do the following transformation

$$\begin{aligned} & x_{\alpha_{i_{k_2}}}(a_{k_1})(-a_{k_1})^{\alpha^\vee_{i_{k_2}}} x_{-\alpha_{i_{k_2}}}(a_{k_2}) \\ = \; & x_{\alpha_{i_{k_2}}}(a_{k_1}) x_{-\alpha_{i_{k_2}}}(a_{k_1}^{-2})(-a_{k_1})^{\alpha^\vee_{i_{k_2}}} \\ = \; & x_{-\alpha_{i_{k_2}}}\left(\frac{a_{k_1}^{-2}a_{k_2}}{1+a_{k_1}^{-1}a_{k_2}}\right) x_{\alpha_{i_{k_2}}}(a_{k_1}(1+a_{k_1}^{-1}a_{k_2}))(1+a_{k_1}^{-1}a_{k_2})^{\alpha^\vee_{i_{k_2}}}(-a_{k_1})^{\alpha^\vee_{i_{k_2}}} \\ = \; & x_{-\alpha_{i_{k_2}}}\left(\frac{a_{k_1}^{-1}a_{k_2}}{a_{k_1}+a_{k_2}}\right) x_{\alpha_{i_{k_2}}}(a_{k_1}+a_{k_2})(-(a_{k_1}+a_{k_2}))^{\alpha^\vee_{i_{k_2}}}. \end{aligned}$$

Again the last two terms will be moved to the right and we only have to look at this term to see how the critical indices change. Hence we have the same calculation with $a_{k_1}+a_{k_2}$ instead of a_{k_1}. Thus, inductively, we can see that after we do this transformation at all places k_j for $2 \leq j \leq s$ we have

$$b_{k_j} = \frac{a_{k_j}}{\left(\sum_{l=1}^{j-1} a_{k_l}\right)\left(\sum_{l=1}^{j} a_{k_l}\right)}.$$

□

Finally we have to deal with the version of Proposition 6.10 under assumption $u_i = 0$. This will only be done very briefly as the arguments in this case are a combination of the arguments in the proofs of the Propositions 6.9 and 6.10.

Proposition 6.12. *Assume $u_r = 0$. Let $g = [g_0(\underline{a_\beta}), g_1(a_1)\ldots, g_p(a_p)] \in \mathcal{U}_{\kappa^{r-1}}$ be the coordinates of g with respect to the gallery κ^{r-1}. If for any $I \subset C([u_r, v_r])$ the inequality*

$$\sum_{n\in I} a_n \neq 0$$

holds, then $g \in \mathcal{U}^{s_\alpha}_{\kappa^r}$.

Proof. As mentioned above, the proof is a combination of the proofs for Proposition 6.9 and Proposition 6.10. The way κ_0^{r-1} is changed is exactly the same as in Proposition 6.9, while the rest of the calculations that the occurring terms can be moved to the right in the gallery and do not change the structure of the gallery is exactly as in Proposition 6.10. □

Combining these propositions we arrive at the following statement.

Theorem 6.13. *Let $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$. There exists a dense open subset $O_{id} \subset C(\delta)$ such that $r_{w_0 s_\alpha}(g) = \Xi_{s_\alpha}(\delta)$ for all simple roots α and $g \in O_{id}$.*

Proof. This follows from the propositions 6.8, 6.9, 6.10, and 6.12, as one can see in their proofs that when we change the coordinates for the minimum k or one of the stable section $[u_i, v_i]$ by using the transformation in the corresponding proposition, we only change the parameters after k, respectively u_i.

Hence we can start with the transformation at k and then proceed with the transformations at the stable sections $[u_i, v_i]$ in increasing order. This yields a finite set of inequalities for our parameters, with each proposition giving us inequalities for different, pairwise non-intersecting, sets of parameters.

In addition we have also seen that none of the transformations will change the coefficients at positions were the gallery δ had negative crossings. Hence there are no difficulties with applying the retraction, as we have not only written the gallery in the coordinates with respect to $\Xi_{s_\alpha}(\delta)$ but also know that the gallery is contained in $C_{s_\alpha}(\Xi_{s_\alpha}(\delta))$.

Hence there is a dense subset O_{id} of $C(\delta)$ such that any $g \in O_{id}$ satisfies all the inequalities for each simple root α and hence $r_{w_0 s_\alpha}(g) = \Xi_{s_\alpha}(\delta)$. □

6.3 Algebraic independence of parameters

To be in a position to use Theorem 6.13 as the basis for an inductive proof we have to make sure that we have a better understanding of how the parameters change, if we move from the coordinates with respect to δ to the ones with respect to $\Xi_{s_\alpha}(\delta)$ for an arbitrary simple root α.

As before let $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$, α a simple root and let k, u_i, and v_i, for $1 \leq i \leq r$ be the indices as in Notation 6.6.

We have seen that for a suitable $g \in C(\delta)$ we can apply the coordinate changes of the different propositions 6.8, 6.9, 6.10, and 6.12 and how they transform the coordinates in different ways.

As mentioned in Theorem 6.13, the transformations always only change the coefficients after the stable section they are applied to, or after k if we apply Proposition 6.9 or Proposition 6.12. Hence we can first change the gallery at the position k and proceed with the stable sections $[u_i, v_i]$ in increasing order.

What still needs to be dealt with, is to take a closer look at how the parameters change if we apply these transformations. After changing the coordinates at the minimum k and afterwards at all stable sections $[u_i, v_i]$ we denote by b_β, respectively b_i, the new coordinates while we denote the old ones by a_β, respectively a_i. Let $k_i^1, \ldots, k_i^{t_i} < k$ be the set of critical indices for the stable section $[u_i, v_i]$. To exclude the index k at this point is just convenient as we want to view the index k as his own stable section of length 0. We set $k_r^1 = \alpha$ if $u_r = 0$ for this purpose. In addition we define $t_0 = 1$ and set $k_0^1 = k$. We denote by

$$I_c = \{k_i^j \mid 0 \leq i \leq r, 0 \leq j \leq t_i\}$$

the set of critical indices together with the starting indices of the stable sections. For an index j we denote by

$$I_c^{<j} = \{i \in I_c \mid i < j\}$$

the set of indices in I_c that are strictly smaller than j. In addition we define $I_v = \{v_i \mid 1 \leq i \leq r, v_i < k\}$, the set of final indices for the stable sections and again we exclude k. For the change of coordinates we then have a number of different behaviours depending on the type of the index $j > 0$.

(i) j **is a non-critical positive folding and** $j \neq k$: In this case we have seen that the parameter will be changed by the elements of the torus that move through the gallery but nothing else, hence the new parameter has the simple form

$$b_j = F(a_l \mid l \in I_c^{<j})a_j,$$

for some non-zero rational function F.

(ii) j **is a positive crossing**: In this case we have the same changes as in the previous case, but we have also seen that the parameter can be changed by adding terms that were created by using the Chevalley relations. But these new terms always consist of parameters belonging to smaller indices, hence the form is as follows

$$b_j = F(a_l \mid l \in I_c^{<j})a_j + G(\underline{a_\beta}, a_l \mid l < j),$$

for a non-zero rational function F and a rational function G, which might be zero.

(iii) j **is a negative crossing,** $j \neq v_i$ **for all** i: At all of these positions we have seen that nothing happens at all. Hence $b_j = 0$ as before.

(iv) $j = k$ **and** $k \neq v_1$: In this case we know by Proposition 6.8 that the parameter gets changed to its inverse and afterwards we have the same behaviour as for a positive crossing by propositions 6.10 and 6.12, hence we obtain
$$b_j = F(a_l \mid l \in I_c^{<k})a_k^{-1} + G(\underline{a_\beta}, a_l \mid l < j),$$
for a non-zero rational function F and a rational function G, which might be zero.

(v) $j = k$ **and** $k = v_1$: As above the parameter is first changed to its inverse, but we have seen in the proof of Proposition 6.10 that the next change was a bit more complicated. But as the gallery is then folded at this position, the following changes of coordinates will only affect this index through their toral elements that move through the gallery. Finally we obtain
$$b_j = F(a_l \mid l \in I_c^{<k})(A + A^2 a_k^{-1})^{-1},$$
for a non-zero rational function F and $A = a_{k_1^1} + \ldots + a_{k_1^{t_1}}$.

(vi) $j = k_i^1 \in I_c$ **for some** i: In this case the transformation in Proposition 6.10 changes the parameter to its inverse and it will afterwards only be changed by toral elements. Hence we obtain
$$b_j = F(a_l \mid l \in I_c^{<j})a_j^{-1},$$
for a non-zero rational function F.

(vii) $j = k_i^m \in I_c$ **for some** i **and** $m \geq 2$: In this case we have the explicit change of coordinates seen in Lemma 6.11 followed by changes through toral elements from stable section with smaller indices, thus we obtain
$$b_j = F(a_l \mid l \in I_c^{<u_i}) \frac{a_j}{\left(\sum_{l=1}^{m-1} a_{k_i^l}\right)\left(\sum_{l=1}^{m} a_{k_i^l}\right)},$$
for a non-zero rational function F.

(viii) $j = v_i \in I_v$ **for some** i: This is the only case were we start with a negative crossing and obtain a non-zero parameter in the end. We have seen in the proof of Proposition 6.10 that the first change is just the inverse of the sum of all the parameters of the critical indices of the current stable section. Afterwards this can of course be changed by toral elements, hence we obtain

$$b_j = F(a_l \mid l \in I_c^{<u_i})(a_{k_i^1} + \ldots + a_{k_i^{t_i}})^{-1},$$

for a non-zero rational function F.

We also need to look at the parameters at the first position of our gallery. This is a special situation, as we usually have more than one parameter here and have seen in propositions 6.9 and 6.12 that the changes are more complicated.

For this we return to the notations and assumptions on α used in the proof of Proposition 6.9. We have already seen that the coefficients stay algebraically independent in the first steps of the proof until we arrive at

$$g_0(\underline{a_\beta}) = \overleftarrow{\prod_{\beta<0,\beta\neq-\alpha,\delta_0^{-1}(\beta)<0}} x_\beta(a'_\beta)x_\alpha(a_{-\alpha}^{-1}) \cdot s_\alpha\delta_0.$$

This was due to the fact that the parameters change in a triangular pattern.

We want to make a short observation on the question for which roots we have to exchange the ones in R'_α.

Lemma 6.14. *For the set of roots* $R' = \{\beta \in \Phi \mid \beta <_{s_\alpha} 0, (s_\alpha\delta_0)^{-1}\beta < 0\}$ *it holds*

$$R' = R_\alpha \cup \{\alpha\} \cup s_\alpha R'_\alpha.$$

Proof. The first observation is the fact that the set R has the same cardinality as the set of roots that satisfy our original conditions, $\beta < 0$ and $\delta_0^{-1}\beta < 0$. This is just due to the fact that $s_\alpha(\Phi^- \setminus \{-\alpha\}) = \Phi^- \setminus \{-\alpha\}$. It is also obvious that the set R_α is only permuted by s_α as a root in this set stays negative and unequal to $-\alpha$ and the last two conditions for the root are just interchanged. While the set R'_α are exactly the negative roots that after applying s_α are still negative but when applying δ_0^{-1} afterwards they become positive.

The obvious candidate for the roots that we need are the ones in $s_\alpha R'_\alpha$, by definition these are roots γ such that $\gamma <_{s_\alpha} 0$ and $\delta_0^{-1}\gamma > 0$. But they satisfy $(s_\alpha\delta_0)^{-1}\gamma < 0$ by definition of R'_α. Thus we have a set of roots

$$\{\alpha\} \cup R_\alpha \cup s_\alpha R'_\alpha$$

that satisfy our two conditions and the set has the right cardinality. $\square$

While the set $R_\alpha \cup \{\alpha\}$ is already present in g_0 we need to obtain the ones in $s_\alpha R'_\alpha$ and get rid of the ones in R'_α. For this we make a small calculation.

Lemma 6.15. *Let $\gamma \in s_\alpha R'_\alpha$, then $\langle \gamma, \alpha^\vee \rangle > 0$ holds.*

Proof. Let $\gamma \in s_\alpha R'_\alpha$ then by definition we have the following

$$\langle s_\alpha \gamma, \delta_0 \rho^\vee \rangle < 0,$$
$$\langle \gamma, \delta_0 \rho^\vee \rangle > 0, \text{ and}$$
$$\langle \alpha, \delta_0 \rho^\vee \rangle > 0.$$

From the equality $s_\alpha \gamma = \gamma - \langle \gamma, \alpha^\vee \rangle \alpha$, thus follows $\langle \gamma, \alpha^\vee \rangle > 0$. □

This lemma means that for $\gamma \in s_\alpha R'_\alpha$, we can write

$$\gamma = s_\alpha \gamma + p\alpha,$$

with $s_\alpha \gamma \in R'_\alpha$ and $p > 0$. Thus for any element $\gamma \in s_\alpha R'_\alpha$ we find at least one root $s_\alpha \gamma$ in R'_α such that an element in the one-parameter subgroup corresponding to γ can be found in the commutator of $x_{s_\alpha \gamma}(a'_{s_\alpha \gamma})$ and $x_\alpha(a_{-\alpha})$.

We start by moving the coefficient corresponding to roots in R'_α to the right, starting again with the highest one. As the set R'_α is closed under taking sums, we can again order the product in the same way as the one for R_α. This will again change the parameters in a triangular pattern and we will write a''_β for the new ones that are still algebraically independent. We obtain

$$g_0(\underline{a_\beta}) = \overleftarrow{\prod_{\beta \in R_\alpha}} x_\beta(a''_\beta) \overleftarrow{\prod_{\beta \in R'_\alpha}} x_\beta(a''_\beta) \cdot x_\alpha(a''_\alpha) \cdot s_\alpha \delta_0.$$

To be able to make statements about the way the parameters change when we move the second product past $x_\alpha(a''_\alpha)$, we first need to take a closer look at its structure.

Lemma 6.16. *Let $\beta \in R'_\alpha$ such that $\beta - \alpha \in \Phi^-$, then $\beta - \alpha \in R'_\alpha$.*

Proof. We just do the calculations

$$\delta_0^{-1}(\beta - \alpha) = \delta_0^{-1}\beta + \delta_0^{-1}(-\alpha) < 0 \text{ and}$$
$$(s_\alpha \delta_0)^{-1}(\beta - \alpha) = (s_\alpha \delta_0)^{-1}\beta + \delta_0^{-1}\alpha > 0,$$

by definition of R'_α and by the assumptions on α. Thus $\beta - \alpha \in R'_\alpha$. □

Hence we denote by $\beta_1, \ldots, \beta_l$ for some $l \geq 1$ those roots in R'_α such that $\beta_i - \alpha \notin \Phi$ and let $k_i \in \mathbb{Z}^{\geq 0}$ such that k_i is maximal with the property that $\beta_i + k_i\alpha \in R'_\alpha$.

Definition 6.17. The set of elements $\{\beta_i, \ldots, \beta_i + k_i\alpha\}$ is called an *α-chain* of R'_α starting at β_i.

Thus we can divide our set R'_α as follows

$$R'_\alpha = \coprod_{i=1}^{l} \{\beta_i, \ldots, \beta_i + k_i\alpha\}.$$

It is obvious that all elements in a fixed α-chain will generate the same elements in $s_\alpha R'_\alpha$ when they move past $\alpha(a''_\alpha)$. Hence we need the following.

Lemma 6.18. *Let $\beta_i \in R'_\alpha$ be the start of an α-chain. Then $k_i = 0$.*

Proof. As β_i and α are linearly independent and we already know that $\langle \beta_i, \alpha^\vee \rangle < 0$ there are only two possible cases (this follows for example from [Spr98, 7.5.1]).

(i) Case $\langle \beta_i, \alpha^\vee \rangle = -1$: Let us assume that $\beta_i + \alpha \in R'_\alpha$. Then we conclude that

$$\beta_i = s_\alpha(\beta_i + \alpha) \in s_\alpha R'_\alpha,$$

which would imply $\beta_i \in R'_\alpha \cap s_\alpha R'_\alpha = \emptyset$. Which is a contradiction.

(ii) Case $\langle \beta_i, \alpha^\vee \rangle = -2$: Let us again assume that $\beta_i + \alpha \in R'_\alpha$. Then we conclude that

$$\beta_i + \alpha = s_\alpha(\beta_i + \alpha) \in s_\alpha R'_\alpha,$$

which would imply $\beta_i + \alpha \in R'_\alpha \cap s_\alpha R'_\alpha = \emptyset$. Which is a contradiction as well.

Thus every α-chain consists of only one element. □

Hence we can divide our set $R'_\alpha = \{\beta_1, \ldots, \beta_l\}$ with $\beta_i - \alpha \notin \Phi$ and $\beta_i + \alpha \in \Phi \setminus R'_\alpha$. The fact that $\beta_i + \alpha$ is a root for all i follows from [Car72, 5.2]. Precisely [Car72] states that if β_i and α are linearly independent roots and $\beta_i + \alpha$ is not a root, then their corresponding one-parameter subgroups commute. As this is not the case here, it follows inductively that $\beta_i + \alpha, \ldots \beta_i + (\langle \beta_i, \alpha^\vee \rangle - 1)\alpha$ are roots.

Definition 6.19. For $\alpha \in \Phi$ we denote by $ht_w(\beta) = \langle \beta, w\rho^\vee \rangle$, the *w-height* of β.

As stated above we know that when we move $x_{\beta_i}(a''_{\beta_i})$ past $x_\alpha(a''_\alpha)$ we will obtain an element of the one-parameter subgroup corresponding to $s_\alpha\beta_i$. The question that remains is, if we also obtain other elements from one-parameter subgroups corresponding to $s_\alpha\beta_j$ for $j \neq i$.

Lemma 6.20. *Let $\beta \in R'_\alpha$. Let $\beta' \in R'_\alpha$ be a root such that an element of the one-parameter subgroup $U_{s_\alpha\beta}$ is contained in the commutator of $x_{\beta'}(a''_{\beta'})$ and $x_\alpha(a''_\alpha)$, then it holds that*

$$ht(\beta) < ht(\beta').$$

Proof. Let $\gamma = s_\alpha\beta = \beta + p\alpha$, where $p = \langle \beta, \alpha^\vee \rangle \in \{1, 2\}$ (again [Spr98, 7.5.1]). Let us assume that there exist $k, l > 0$ such that $\gamma = k\beta' + l\alpha$, i.e., its one-parameter subgroup can appear when $x_{\beta'}(a''_{\beta'})$ moves past $x_\alpha(a''_\alpha)$. Then there are four cases that one needs to deal with.

(i) Case $k = 1$ and $l \geq 1$: In this case β and β' differ by a multiple of α, meaning that they would lie in the same α-chain, which means that they must be equal.

(ii) Case $k \geq 1$ and $l = 1$: In this case we have $k\beta' = \beta + (p-1)\alpha$, which by the same argument as above using [Car72] is in Φ. This implies that $k = 1$ and thus β and β' lie again in the same α-chain and are equal.

(iii) Case $k = 3$ and $l = 2$: This implies $3\beta' = \beta + (p-2)\alpha$. If $p = 2$ this equality cannot hold as one root is never a rational multiple of another if this muplte is not ± 1. In the case that $p = 1$ we obtain $\beta = 3\beta' + \alpha$. But as $ht(\beta') < 0$ this implies

$$ht(\beta) < ht(\beta').$$

(iv) Case $k = 2$ and $l = 3$: This implies $2\beta' = \beta + (p-3)\alpha$. If $p = 1$ this yields $2(\beta' + \alpha) = \beta$, but $\beta' + \alpha \in \Phi$ as $\beta' \in R'_\alpha$ thus this equality cannot hold. In the case that $p = 2$ we obtain $2\beta' + \alpha = \beta$ and as $ht(\beta') < -1$ (as $\beta' \in \Phi^-$ and not simple) we obtain again

$$ht(\beta) < ht(\beta').$$

∎

We can now start to move the elements corresponding to roots in R'_α past $x_\alpha(a''_\alpha)$. We start with one $x_{\beta_l}(a''_{\beta_l})$ furthest to the right in our product, thus it is one of minimal height. If we move it past $x_\alpha(a''_\alpha)$ it will produce a number of new terms including one corresponding to $s_\alpha\beta_l$. As we have seen above in the third and fourth case, the only other terms in the commutator corresponding to elements in $s_\alpha R'_\alpha$ correspond to roots of lower height, hence none are present in this case as we took an element of minimal height. Thus we can move all the new terms except for the one corresponding to $s_\alpha\beta_l$ to the right and past $s_\alpha\delta_0$. We obtain

$$g_0(\underline{a_\beta}) = \overleftarrow{\prod_{\beta\in R_\alpha}} x_\beta(a''_\beta)\prod_{i=1}^{l-1} x_{\beta_i}(a''_{\beta_i})\cdot x_\alpha(a''_\alpha)x_{s_\alpha\beta_l}(c{a''_\alpha}^{\langle\beta_l,\alpha^\vee\rangle}a_{\beta_l})\cdot s_\alpha\delta_0.$$

Let us now assume we have already moved all terms up to an index $1\leq t<l$ past $x_\alpha(a''_\alpha)$ and we start with

$$\begin{aligned} g_0(\underline{a_\beta}) &= \overleftarrow{\prod_{\beta\in R_\alpha}} x_\beta(a''_\beta)\prod_{i=1}^{t} x_{\beta_i}(a''_{\beta_i})\cdot x_\alpha(a''_\alpha)\cdot \\ &\quad \prod_{i=t+1}^{l} x_{s_\alpha\beta_i}(c_i{a''_\alpha}^{\langle\beta_i,\alpha^\vee\rangle}a_{\beta_i}+f_i(a_\beta\mid\beta\in R'_\alpha, ht(\beta)>ht(\beta_i)))\cdot s_\alpha\delta_0. \end{aligned}$$

We now want to move $x_{\beta_t}(a''_{\beta_t})$ past $x_\alpha(a''_\alpha)$, we already know that doing so produces a number of new terms, one of them is the one corresponding to $s_\alpha\beta_t$ and at most two others that correspond to different roots of lower height, i.e., roots whose corresponding one-parameter subgroups were already moved past $x_\alpha(a''_\alpha)$, and a number of other terms whose corresponding roots do not lie in R'_α. All the terms that do not correspond to $s_\alpha\beta_t$ are now moved to the right. For ease of notation let us just write $x_\gamma(c)$ for one of the new terms and see what can happen when it moves past the term corresponding to $s_\alpha\beta_i$, for $i\geq t+1$.

(i) Case $\gamma=s_\alpha\beta_i$: In this case we just add the two terms and the coefficient of $x_{s_\alpha\beta_{t+1}}$ is still of the form $c_i{a''_\alpha}^{\langle\beta_i,\alpha^\vee\rangle}a_{\beta_i}+f_i(a_\beta\mid ht(\beta)>ht(\beta_i)))$.

(ii) Case $\gamma\neq s_\alpha\beta_i$: In this case we interchange the two terms. But any terms that are created as a positive linear combination of γ and $s_\alpha\beta_i$ have the property that their s_α-height is strictly smaller than the one of both γ and $s_\alpha\beta_i$. Thus if they correspond to roots in $s_\alpha R'_\alpha$ these roots roots can be found even further to the right in the product.

Using these two cases we can move all the new terms that do not correspond to $s_\alpha\beta_t$ past the third product or they add themselves to some factor in that product, but they do not alter the structure of the coefficient of any factor. We arrive at

$$\begin{aligned} g_0(\underline{a_\beta}) &= \overleftarrow{\prod_{\beta\in R_\alpha}} x_\beta(a''_\beta)\cdot x_\alpha(a''_\alpha)\cdot \\ &\quad \prod_{i=1}^{l} x_{s_\alpha\beta_i}(c_i a_\alpha''^{\langle\beta_i,\alpha^\vee\rangle} a_{\beta_i} + f_i(a_\beta \mid \beta\in R'_\alpha, ht(\beta)>ht(\beta_i)))\cdot s_\alpha\delta_0. \end{aligned}$$

Thus all parameters of g_0 stay algebraically independent of each other and as we have seen above, also independent of all other parameters. In addition we have also seen that we arrive at a generic element of the form

$$\prod_{\beta<_{s_\alpha}0,(s_\alpha\delta_0)^{-1}\beta} x_\beta(c_\beta)s_\alpha\delta_0,$$

when we start with generic parameters a_β.

6.4 General case

We now want to use the result for the simple roots to obtain the general result by an inductive argument. What we have seen so far is the following, if we start with $\delta\in\Gamma^+_{LS}(\gamma_\lambda)$, we know that there exists a dense subset $O_{id}\subset C(\delta)$ such that $r_{w_0s_\alpha}(g)=\Xi_{s_\alpha}(\delta)$ for all simple roots α and $g\in O_{id}$. If we now fix a simple root β, the same is also true for $\Xi_{s_\beta}(\delta)\in\Gamma^{s_\beta}_{LS}(\gamma_\lambda)$, hence there exists a dense subset $O_{s_\beta}\subset C_{s_\beta}(\Xi_{s_\beta}(\delta))$ such that $r_{w_0s_\beta s_\alpha}(g)=\Xi_{s_\beta s_\alpha}(\delta)$ for all simple roots α and $g\in O_{s_\beta}$.

Thus to obtain the result for the general case we have to show that $O_{id}\cap O_{s_\beta}$ is dense in O_{id} and also more generally if we take any $w\in W\setminus\{w_0\}$ that $O_{id}\cap O_w$ is dense in O_{id}, where O_w is defined in an analogous way as O_{id}.

Theorem 6.21. *For $\delta\in\Gamma^+_{LS}(\gamma_\lambda)$, there exists a dense subset $O_\delta\subset C(\delta)$ such that for every $g\in O_\delta$ it holds*

$$r_{w_0w}(g)=\Xi_w(\delta) \text{ for all } w\in W.$$

Proof. We want to define the subset O_δ inductively. For this let

$$W_1\subset W_2\subset\ldots\subset W_l=W,$$

be the subsets of W such that $W_i = \{w \in W \mid l(w) \leq i\}$ and $l = l(w_0)$. We want to define a decreasing series of dense subsets O_δ^i of $C(\delta)$ such that for any $g \in O_\delta^i$ and $w \in W_i$ it holds that

$$r_{w_0 w}(g) = \Xi_w(\delta).$$

Of course the subset O_δ^l is the one needed for the proof of the statement.

Theorem 6.13 is already the proof for W_1, but to use an inductive argument we have to be a bit more careful about the parameters to make it work. As one can see in part 6.3 we may not know explicitly how the parameters change, but the structure can be seen quite well. One thing that one can see is that the parameters for all indices of positive foldings stay algebraically independent if we view them as rational functions with our original parameters as indeterminantes.

Hence let us take $w \in W_i$ and assume that the theorem already holds for W_{i-1}. Let us now take $w' \in W_{i-1}$ and $w'\alpha \in w'\Phi^+$ a simple root such that $w = w's_\alpha = w's_\alpha w'^{-1}w'$. For g in O_δ^{i-1} we know that we can successively change the coordinates until we have written g with respect to $\Xi_{w'}(\delta)$.

If we now want to apply the coordinate change to obtain g with respect to the combinatorial gallery $\Xi_w(\delta)$, we have to make sure that the parameters of g fulfil all the needed inequalities. But by part 6.3 we know that if we start with generic coefficients, we will always have generic coefficients at all foldings, the only dependencies occur with some positive crossings that for the calculations are of no concern. The only position where we have to make sure that we are not missing any coefficients, is g_0.

For this let us take $w = s_{i_1} \cdots s_{i_m}$ a reduced decomposition of w, such that $s_{i_m} = s_\alpha$. Hence we need to successively transform the gallery to $\Xi_{w^l}(\delta)$ for $0 \leq l \leq m$. For ease of notation we write $w^l = s_{i_1} \cdots s_{i_l}$ for $1 \leq l \leq m$, and define $\beta_l = w^{l-1}\alpha_{i_l}$. We assume that we could already write our gallery with respect to w^{m-1}, hence we now need to apply the changes with respect to β_m. If propositions 6.9 and 6.12 do not apply in this situation it just means that we do not need to change anything at the first position of our gallery, thus no problem can arise. Hence let us assume that one of the propositions applies.

We need to fix some integers $0 = k_1 < \ldots < k_r = m$ in $\{0, \ldots, m\}$. These shall be all the positions where we needed to change the first position of our gallery when moving from the coordinates with respect to $\Xi_{w^{k_l}}(\delta)$ to the ones with respect to $\Xi_{w^{k_l+1}}(\delta)$. We also fix the following elements

$$t_l = w^{k_l-1} s_{i_{k_l}} (w^{k_l-1})^{-1},$$

these are exactly the reflections that are added to our element of the Weyl group at the first position in each of the steps where we apply one of the two proposition. Hence our current element at the first position is of the form.

$$t_{r-1} \cdots t_1 \delta_0.$$

In addition we know that

$$(t_{r-1} \cdots t_1 \delta_0)^{-1}(-\beta_{k_r}) < 0,$$

as we otherwise would not need to apply one of the propositions. The main question is if the one-parameter subgroup corresponding to $-\beta_{k_r}$, does exist with a non-zero parameter in front of our current element of the Weyl group. For this let us define $\gamma_r = \beta_{k_r}$. Then we know

$$(t_{r-1} \cdots t_1 \delta_0)^{-1}(-\gamma_r) < 0,$$

hence

$$(t_{r-2} \cdots t_1 \delta_0)^{-1}(-t_{r-1}\gamma_r) < 0.$$

In addition its a straight forward calculation to see that

$$(w^{k_{r-1}-1})^{-1}(-t_{r-1}\gamma_r) < 0 \text{ and } (w^{k_{r-1}-1})^{-1}(-\gamma_r) < 0,$$

just by definition and the fact that $w = s_{i_1} \cdots s_{i_m}$ is a reduced expression.

If we now look at $(t_{r-2} \cdots t_1 \delta_0)^{-1}(-\gamma_r)$, there are two possibilities. Either this is negative, then by definition

$$-\gamma_r \in R_{\beta_{k_{r-1}}}$$

and we define $\gamma_{r-1} = \gamma_r$. On the other hand, if $(t_{r-2} \cdots t_1 \delta_0)^{-1}(-\gamma_r)$ is positive, we know

$$-t_{r-1}\gamma_r \in R'_{\beta_{k_{r-1}}}$$

by definition and we define $\gamma_{r-1} = t_{r-1}\gamma_r$.

No matter how we defined γ_{r-1} in each case, the one parameter subgroup corresponding to $-\beta_{k_r}$ exists in our situation if the one for $-\gamma_{r-1}$ existed in the previous transformation step k_{r-1}. Thus we iterate this process.

If we inductively defined γ_{r-j} such that

$$(t_{r-j-1}\cdots t_1\delta_0)^{-1}(-\gamma_{r-j})<0,\ (w^{k_{r-j}-1})^{-1}(-\gamma_{r-j})<0, \text{ and}$$

$$(w^{k_{r-j}-1})^{-1}(-t_{r-j}\gamma_{r-j})<0,$$

i.e., γ_{r-j} is one of the roots whose one-parameter subgroup is allowed in the transformation step k_{r-j}. Then we can again look at

$$(t_{r-j-2}\cdots t_1\delta_0)^{-1}(-\gamma_{r-j})$$

and if it is negative define γ_{r-j-1} as γ_{r-j} and otherwise as $t_{r-j-1}\gamma_{r-j}$. The properties of γ_{r-j} above imply

$$(w^{k_{r-j-1}-1})^{-1}(-\gamma_{r-j-1})<0 \text{ and } (w^{k_{r-j-1}-1})^{-1}(-t_{r-j-1}\gamma_{r-j-1})<0.$$

Thus again we have an element that is either in $R_{\beta_{k_{r-j-1}}}$ or $R'_{\beta_{k_{r-j-1}}}$.

After we have inductively defined all these elements we look at $-\gamma_0$, which is an element that satisfies $-\gamma_0<0$ and $\delta_0^{-1}(-\gamma_0)<0$. Thus the one-parameter subgroup corresponding to $-\gamma_0$ must exist by assumption on the genericness of the initial parameters. By definition it will be transformed to a element in the one-parameter subgroup corresponding to $-\beta_{k_r}$, after the transformations to $\Xi_{w^{m-1}}(\delta)$.

Thus we can use the proposition and have a generic non-zero parameter to work with. This means that we can perform the transformation for w and we can thus define O_δ^i by looking at all elements w of length i. This completes the proof. □

Remark 6.22. As one can see in the proof of the above theorem, during the transformations of g_0, we produce a positive root at each step where we apply one of the two propositions 6.9 or 6.12. It can be seen in example calculations for types $A_2,\dots,A_6$, $B_2,\dots,B_5$, D_4, E_6, F_4, and G_2 using MAGMA that the created positive roots always lie in the sets R_β for later transformations at a root $-\beta$, why this is the case we cannot say at the moment. But as one can see above this is also not needed for the proof itself.

6.5 Results

This leads to the following results about MV-cycles and MV-polytopes.

Corollary 6.23. *Let M_δ be an MV-cycle corresponding to $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$, then*

$$M_\delta := \overline{\bigcap_{w \in W} S^w_{\nu_w}}, \text{ with } \nu_w := wt(\Xi_w(\delta)).$$

Corollary 6.24. *Let M_δ be as above and $P = \Phi(M_\delta)$ the corresponding MV-polytope, then*

$$P = P_\delta := \mathrm{conv}(\{wt(\Xi_w(\delta)) \mid w \in W\}).$$

Corollary 6.25. *Let $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$, then*

$$\bigcap_{w \in W} C_w(\Xi_w(\delta))$$

is dense in $C(\delta)$.

7 Remarks

We want to state and prove a few remarks about the constructed galleries $\Xi_w(\delta)$ and give proofs for some of the properties of MV-polytopes appearing in [Kam05a] and [Kam05b] using our definition of MV-polytopes.

7.1 Lusztig and Kashiwara Datums

As the constructed galleries $\Xi_w(\delta)$ define the MV-polytope corresponding to δ, they especially define a pseudo-Weyl polytope. Hence it is reasonable to take a closer look at the edges of this polytope to obtain the same results about them as Kamnitzer in [Kam05a] and [Kam05b].

Proposition 7.1. *Let w, $w_k^{\mathbf{i}}$, $\delta_k^{\mathbf{i}}$, and $\widetilde{\delta}_k^{\mathbf{i}}$ be as in Proposition 5.30 and let $\beta_k := w_{k-1}^{\mathbf{i}}.\alpha_{i_k}$ for $1 \leq k \leq n$. Then*

$$wt(\Xi_{ws_{i_n}}(\delta)) - wt(\Xi_w(\delta)) = \varphi_{\alpha_{i_n}}(\widetilde{\delta}_{n-1}^{\mathbf{i}})\beta_n.$$

Proof. Let $\lambda := wt(\Xi_{ws_{i_n}}(\delta)) - wt(\Xi_w(\delta))$, then it holds

$$(w_{k-1}^{\mathbf{i}})^{-1}\lambda = wt((w_{k-1}^{\mathbf{i}})^{-1}\Xi_{ws_{i_n}}(\delta)) - wt((w_{k-1}^{\mathbf{i}})^{-1}\Xi_w(\delta)).$$

By definition it is clear that $(w_{k-1}^{\mathbf{i}})^{-1}\Xi_{ws_{i_n}}(\delta) = \widetilde{\delta}_{n-1}^{\mathbf{i}}$, while

$$(w_{k-1}^{\mathbf{i}})^{-1}\Xi_w(\delta) = s_{i_n}\widetilde{\delta}_n^{\mathbf{i}}.$$

Substituting these yields

$$(w_{k-1}^{\mathbf{i}})^{-1}\lambda = wt(\widetilde{\delta}_{n-1}^{\mathbf{i}}) - wt(s_{i_n}\widetilde{\delta}_n^{\mathbf{i}}).$$

We now apply the definition of $\widetilde{\delta}_n^{\mathbf{i}}$ and obtain

$$(w_{k-1}^{\mathbf{i}})^{-1}\lambda = wt(\widetilde{\delta}_{n-1}^{\mathbf{i}}) - wt(s_{i_n}e_{\alpha_{i_n}}^{\varepsilon_\alpha(\widetilde{\delta}_{n-1}^{\mathbf{i}})}(\widetilde{\delta}_{n-1}^{\mathbf{i}})).$$

By Theorem 5.26 we have $s_{i_n}e_{\alpha_{i_n}}^{\varepsilon_\alpha(\widetilde{\delta}_{n-1}^{\mathbf{i}})}(\widetilde{\delta}_{n-1}^{\mathbf{i}}) = (f_{\alpha_{i_n}}^{\varphi_\alpha(\widetilde{\delta}_{n-1}^{\mathbf{i}})}(\widetilde{\delta}_{n-1}^{\mathbf{i}}))_{-\alpha}$, but of course

$$wt((f_{\alpha_{i_n}}^{\varphi_\alpha(\widetilde{\delta}_{n-1}^{\mathbf{i}})}(\widetilde{\delta}_{n-1}^{\mathbf{i}}))_{-\alpha}) = wt(f_{\alpha_{i_n}}^{\varphi_\alpha(\widetilde{\delta}_{n-1}^{\mathbf{i}})}(\widetilde{\delta}_{n-1}^{\mathbf{i}})) = wt(\widetilde{\delta}_{n-1}^{\mathbf{i}}) - \varphi_\alpha(\widetilde{\delta}_{n-1}^{\mathbf{i}})\alpha_{i_n}.$$

Thus we have

$$(w_{k-1}^{\mathbf{i}})^{-1}\lambda = \varphi_{\alpha_{i_n}}(\widetilde{\delta}_{n-1}^{\mathbf{i}})\alpha_{i_n}$$

which implies the statement. ∎

By the definition of pseudo-Weyl polytope as given in Section 3, this already implies the following.

Theorem 7.2. $P_\delta = \mathrm{conv}(\{wt(\Xi_w(\delta)) \mid w \in W\})$ *is a pseudo-Weyl polytope.*

Proof. This follows from Proposition 7.1, the fact that obviously $P_\delta \subset \mathrm{conv}(W.\lambda)$, and [Kam05a]. □

We also want to take a closer look at those numbers $\varphi_{\alpha_{i_k}}(\widetilde{\delta}^{\underline{i}}_{k-1})$ occurring as the face length of this polytope to give a new proof for the fact that the edge length of the 1-skeleton of the constructed polytopes form exactly the different Lusztig datums of the corresponding gallery.

Lemma 7.3. *Let* $w := w_0$ *the longest element in the Weyl group,* $r := l(w_0)$ *and* $w^{\underline{i}}_k$, $\delta^{\underline{i}}_k$, *and* $\widetilde{\delta}^{\underline{i}}_k$ *be as in Proposition 5.30 and let* $\beta_k := w^{\underline{i}}_{k-1}.\alpha_{i_k}$ *for* $1 \le k \le r$. *Then we have*

$$\varphi_{\alpha_{i_k}}(\widetilde{\delta}^{\underline{i}}_{k-1}) = b^{\underline{i}}_k(\delta),$$

where $b^{\underline{i}}_k(\delta)$ *is the k's component of the* $\underline{i}$*-Lusztig datum of* δ*.*

Proof. We have

$$\varphi_{\alpha_{i_k}}(\widetilde{\delta}^{\underline{i}}_{k-1}) = \varepsilon_{\alpha_{i_k}}(\widetilde{\delta}^{\underline{i}}_{k-1}) + \left\langle \alpha^\vee_{i_k}, wt(\widetilde{\delta}^{\underline{i}}_{k-1}) \right\rangle.$$

Using the definition of the string datum $c^{\underline{i}}_j(\delta)$, we have

$$\varphi_{\alpha_{i_k}}(\widetilde{\delta}^{\underline{i}}_{k-1}) = c^{\underline{i}}_k(\delta) + \left\langle \alpha^\vee_{i_k}, \lambda \right\rangle - \sum_{j=k}^{r} c^{\underline{i}}_j(\delta) \left\langle \alpha^\vee_{i_k}, \alpha_{i_j} \right\rangle.$$

Which is of course equal to

$$\varphi_{\alpha_{i_k}}(\widetilde{\delta}^{\underline{i}}_{k-1}) = -c^{\underline{i}}_k(\delta) + \left\langle \alpha^\vee_{i_k}, \lambda \right\rangle - \sum_{j=k+1}^{r} c^{\underline{i}}_j(\delta) \left\langle \alpha^\vee_{i_k}, \alpha_{i_j} \right\rangle.$$

This is, by [BZ01] or [MG03], equal to $b^{\underline{i}}_k(\delta)$. □

By using Kamnitzers analysis of the faces of MV-polytopes being related to the corresponding Lusztig datum, this would have already led to a proof of our main result. But this would still need the tropical Plücker relations and would not yield any information about the images of the retractions and the situation in the Bott-Samelson variety.

Corollary 7.4. *P_δ is the MV-polytope associated to δ.*

Proof. This follows from Kamnitzers work together with Lemma 7.3. □

We can also obtain a new proof for a result of Kamnitzer concerning the relations between MV-polytopes and the Kashiwara datum, appearing in [Kam05a].

Proposition 7.5. *Let $c_j^{\underline{i}}(\delta)$ be the j's component of the $\underline{i}$-Kashiwara datum of δ and let $M_{w_j^{\underline{i}}\Lambda_k} := \left\langle wt(\Xi_{w_j^{\underline{i}}}(\delta)), w_j^{\underline{i}}\Lambda_k \right\rangle$. Then we have*

$$M_{w_{k+1}^{\underline{i}}\Lambda_{i_{k+1}}} - M_{w_k^{\underline{i}}\Lambda_{i_{k+1}}} = c_{k+1}^{\underline{i}}(\delta).$$

Proof. Because the pairing is invariant under the Weyl group, we have

$$M_{w_k^{\underline{i}}\Lambda_j} := \left\langle (w_k^{\underline{i}})^{-1} wt(\Xi_{w_k^{\underline{i}}}(\delta), \Lambda_j \right\rangle$$

and thus

$$\begin{aligned} M_{w_{k+1}^{\underline{i}}\Lambda_{i_{k+1}}} - M_{w_k^{\underline{i}}\Lambda_{i_{k+1}}} &= \left\langle wt(\widetilde{\delta}_{k+1}^{\underline{i}}) - wt(\widetilde{\delta}_k^{\underline{i}}), \Lambda_{i_{k+1}} \right\rangle \\ &= \left\langle \varepsilon_{\alpha_{i_{k+1}}}(\widetilde{\delta}_k^{\underline{i}})\alpha_{i_{k+1}}^\vee, \Lambda_{i_{k+1}} \right\rangle \\ &= \varepsilon_{\alpha_{i_{k+1}}}(\widetilde{\delta}_k^{\underline{i}}) \\ &= c_{k+1}^{\underline{i}}(\delta) \end{aligned}$$

□

In all of this we basically only used the definition of Ξ_w that involved the operators e_α. We did not need the relation between the operator e_α, f_α, and the flipping operator as seen in Theorem 5.26. Thus this whole calculation is independent of the gallery model itself and can be done with any model for the crystal and its Kashiwara operators, as one only needs to construct the elements $e_w x$, for an element x in the crystal, which can be done with any model, and afterwards apply the Weyl group element w to the weights of these elements. Through this, one obtains the exact same set of weights as $\{wt(\Xi_w(\delta)) \mid w \in W\}$.

7.2 Crystal bijections

As we have constructed positively folded galleries for different systems of positive roots, we would like to have weight respecting bijections between all of these sets of galleries to obtain positively folded LS-galleries for a single system of positive roots. This would mean that instead of our galleries $\Xi_w(\delta)$, we would construct a set of galleries that defines the same polytope and is positively folded. In special cases we see that if we look at the MV-polytopes corresponding to these galleries that we obtain polytopes which are contained in the polytope P_δ.

In the following let $\Phi^+ = \{\alpha_1, \ldots, \alpha_r\}$ be the simple roots and $\Phi^+_w = \{w.\alpha_1, \ldots, w.\alpha_r\}$ the system of simple roots for the Weyl group element w, with $\Phi^+_{id} = \Phi^+$. Furthermore let $s_i^w := ws_iw^{-1}$ the reflection with respect to $w.\alpha_i$ and $\Gamma^w_{LS}(\gamma_\lambda)$ the positively folded LS-galleries with respect to Φ^+_w as before.

Definition 7.6. For $1 \leq i \leq r$, we define the following map between two sets of galleries:

$$\begin{aligned}\Psi_i : (\Gamma^+_{LS}(\gamma_\lambda))^w &\longrightarrow (\Gamma^+_{LS}(\gamma_\lambda))^{ws_i} \\ \delta &\mapsto f_{ws_i.\alpha_i}^{\varphi_{w.\alpha_i}(\delta)}(\Xi_{s_i^w}(\delta)) = \\ &\quad f_{ws_i.\alpha_i}^{\varphi_{w.\alpha_i}(\delta)}(ws_iw^{-1}(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta))) = \\ &\quad ws_i.f_{\alpha_i}^{\varphi_{\alpha_i}(w^{-1}.\delta)}(e_{\alpha_i}^{\varepsilon_{\alpha_i}(w^{-1}.\delta)}(w^{-1}.\delta))\end{aligned}$$

This map is not a map of crystals, but it is a weight respecting map. In the language of Kashiwara this can be read as follows, where S_i is the map defined in [Kas94, 7] that produces the Weyl group action on the crystal.

Proposition 7.7. *Let $\delta \in (\Gamma^+_{LS}(\gamma_\lambda))^w$ and $1 \leq i \leq r$, then we have $\Psi_i(\delta) = ws_i.S_i(w^{-1}.\delta)$.*

Proof. This is just the calculation that $f_{\alpha_i}^{\varphi_{\alpha_i}(\delta)}(e_{\alpha_i}^{\varepsilon_{\alpha_i}(\delta)}(\delta)) = S_i(\delta)$. □

These maps also commute with the reflections corresponding to the roots with the same index.

Proposition 7.8. *Let $1 \leq i \leq r$ and $w \in W$, then $s_i^{ws_i} \circ \Psi_i = \Psi_i \circ s_i^w$ and furthermore*

$$(\Psi_i \circ s_i^w)(\delta) = f_{w.\alpha_i}^{\varphi_{w.\alpha_i}(\delta)}(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta)).$$

Proof. Let $\gamma := s_i^w.\delta = ws_iw^{-1}.\delta$, then

$$\begin{aligned}
\Psi_i(\gamma) &= f_{w.\alpha_i}^{\varphi_{ws_i.\alpha_i}(\gamma)}(ws_iw^{-1}.(e_{ws_i.\alpha_i}^{\varepsilon_{ws_i.\alpha_i}(\gamma)}(ws_iw^{-1}.\gamma))) \\
&= f_{w.\alpha_i}^{\varphi_{w.\alpha_i}(\delta)}(w.(e_{\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(w^{-1}.\delta))) \\
&= f_{w.\alpha_i}^{\varphi_{w.\alpha_i}(\delta)}(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta))) \\
&= (ws_iw^{-1})^2.f_{w.\alpha_i}^{\varphi_{w.\alpha_i}(\delta)}(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta))) \\
&= ws_iw^{-1}.f_{ws_i.\alpha_i}^{\varphi_{w.\alpha_i}(\delta)}(ws_iw^{-1}.(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta))) \\
&= ws_iw^{-1}.\Psi_i(\delta) = s_i^{ws_i}.\Psi_i(\delta) \ .
\end{aligned}$$

□

The following proposition either holds by using the corresponding result for S_i or by the following straightforward calculation.

Proposition 7.9. *Let $\delta \in (\Gamma_{LS}^+(\gamma_\lambda))^w$ and $1 \leq i \leq r$, then it holds that $wt(\delta) = wt(\Psi_i(\delta))$.*

Proof. We will just calculate this directly.

$$\begin{aligned}
wt(\Psi_i(\delta)) &= ws_sw^{-1}.(wt(\delta) + \varepsilon_{w.\alpha_i}(\delta)w.\alpha_i) - \varphi_{w.\alpha_i}(\delta)ws_i.\alpha_i \\
&= ws_sw^{-1}.(wt(\delta)) + \varepsilon_{w.\alpha_i}(\delta)ws_i.\alpha_i - \varphi_{w.\alpha_i}(\delta)ws_i.\alpha_i \\
&= wt(\delta) - \langle wt(\delta), w.\alpha_i\rangle\, w.\alpha_i + \underbrace{(\varepsilon_{w.\alpha_i}(\delta) - \varphi_{w.\alpha_i}(\delta))}_{=-\langle wt(\delta), w.\alpha_i\rangle} ws_i.\alpha_i \\
&= wt(\delta)
\end{aligned}$$

□

In addition to the fact that the maps preserve the weights of all galleries, they are also bijections.

Proposition 7.10. *Let $\delta \in (\Gamma_{LS}^+(\gamma_\lambda))^w$, then $\Psi_i^2(\delta) = \delta$.*

Proof. We set $\gamma := \Psi_i(\delta)$, then we have

$$\begin{aligned}
\Psi_i^2(\delta) &= f_{w.\alpha_i}^{\varphi_{ws_i.\alpha_i}(\gamma)}(ws_iw^{-1}.(e_{ws_i.\alpha_i}^{\varepsilon_{ws_i.\alpha_i}(\gamma)}(f_{ws_i.\alpha_i}^{\varphi_{w.\alpha_i}(\delta)}(ws_iw^{-1}.(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta)))))) \\
&= f_{w.\alpha_i}^{\varphi_{ws_i.\alpha_i}(\gamma)}(ws_iw^{-1}.(e_{ws_i.\alpha_i}^{\varepsilon_{ws_i.\alpha_i}(ws_iw^{-1}.(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta)))}(ws_iw^{-1}.(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta))))) \\
&\overset{(1)}{=} f_{w.\alpha_i}^{\varphi_{ws_i.\alpha_i}(\gamma)}(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta)) \\
&\overset{(2)}{=} \delta
\end{aligned}$$

In this calculations we use

$$
\begin{aligned}
(1)\quad \varepsilon_{ws_i.\alpha_i}(ws_iw^{-1}.(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta))) &= \varepsilon_{\alpha_i}(w^{-1}.(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta))) \\
&= \varepsilon_{\alpha_i}(e_{\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(w^{-1}.\delta)) \\
&= \varepsilon_{\alpha_i}(e_{\alpha_i}^{\varepsilon_{\alpha_i}(w^{-1}.\delta)}(w^{-1}.\delta)) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
(2)\quad \varphi_{ws_i.\alpha_i}(\gamma) &= \varphi_{ws_i.\alpha_i}(ws_i.(f_{\alpha_i}^{\varphi_{w.\alpha_i}(\delta)}(w^{-1}.(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta))))) \\
&= \varphi_{\alpha_i}(w.(f_{w.\alpha_i}^{\varphi_{w.\alpha_i}(\delta)}(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta)))) \\
&= \varphi_{w.\alpha_i}(f_{w.\alpha_i}^{\varphi_{w.\alpha_i}(\delta)}(e_{w.\alpha_i}^{\varepsilon_{w.\alpha_i}(\delta)}(\delta))) \\
&= \varepsilon_{w.\alpha_i}(\delta)
\end{aligned}
$$

□

This is of course basically just the calculation that the relation $S_i^2 = id$ mentioned in [Kas94, 7] holds.

Corollary 7.11. *The map Ψ_i is a bijection of sets between the two crystals $(\Gamma^+_{LS}(\gamma_\lambda))^w$ and $(\Gamma^+_{LS}(\gamma_\lambda))^{ws_i}$*

This provides us with a possibility to construct maps between sets of positively folded LS-galleries of the same type for any two different Weyl group elements, although we must check that this choice is independent of the used reduced decompositions for both elements. This can of course be reduced to showing this for only one element w and an arbitrary reduced expression of this element and the identity.

Theorem 7.12. *Let $w^{\underline{i}} = s_{i_1} \dots s_{i_m}$ be a reduced expression of $w \in W$ and $w_k^{\underline{i}} = s_{i_1} \dots s_{i_k}$ for $0 \le k \le m$, then $\Psi_w : (\Gamma^+_{LS}(\gamma_\lambda))^w \longrightarrow \Gamma^+_{LS}(\gamma_\lambda)$ with $\Psi_w := \Psi_{i_1} \circ \dots \circ \Psi_{i_m}$ is independent of the choice of the reduced expression.*

Proof. We know that $\Psi_j : (\Gamma^+_{LS}(\gamma_\lambda))^w \longrightarrow (\Gamma^+_{LS}(\gamma_\lambda))^{ws_j}$ is equal to $ws_jS_jw^{-1}$ and thus for $\delta \in (\Gamma^+_{LS}(\gamma_\lambda))^w$ we have

$$
\begin{aligned}
\Psi_w(\delta) &= (w_0^{\underline{i}} S_{i_1} (w_1^{\underline{i}})^{-1}) \circ (w_1^{\underline{i}} S_{i_2} (w_2^{\underline{i}})^{-1}) \circ \dots \circ (w_{m-1}^{\underline{i}} S_{i_m} (w_m^{\underline{i}})^{-1})(\delta) \\
&= (S_{i_1} \circ \dots \circ S_{i_m})((w_m^{\underline{i}})^{-1}.\delta)
\end{aligned}
$$

Which is independent of the choice of the reduced expression after [Kas94, 7.2.1]. □

By using one of these maps and the inverse of another we can define maps from any $(\Gamma^+_{LS}(\gamma_\lambda))^w$ to any other $(\Gamma^+_{LS}(\gamma_\lambda))^v$ for $v, w \in W$.

We want to apply these maps to the set of galleries $\{\Xi_w(\delta) \mid w \in W\}$ to obtain the following corollary.

Corollary 7.13. *Let $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$, then $P_\delta = \operatorname{conv}(\{wt(\Psi_w(\Xi_w(\delta))) \mid w \in W\})$ and each of the $\Psi_w(\Xi_w(\delta))$ is a positively folded LS gallery for our original root system.*

In addition we can look at the corresponding MV-polytopes $P_{\Psi_w(\Xi_w(\delta))}$ for arbitrary $w \in W$ and see how they relate to P_δ.

In a special case we can easily describe these new galleries and see that their corresponding MV-polytope lies inside the original one, which follows immediately from a result of Baumann and Gaussent.

Proposition 7.14. *Let $\delta \in \Gamma^+_{LS}(\gamma_\lambda)$, then $\Psi_i(\Xi_{s_i}(\delta)) = f_{\alpha_i}^{\varphi_{\alpha_i}(\delta)}(\delta)$.*

Proof. Let $\gamma := \Xi_{s_i}(\delta)$, then

$$\begin{aligned}
\Psi_i(\Xi_{s_i}(\delta)) &= f_{\alpha_i}^{\varphi_{-\alpha_i}(\gamma)}(s_i.(e_{-\alpha_i}^{\varepsilon_{-\alpha_i}(\gamma)}(s_i.e_{\alpha_i}^{\varepsilon_{\alpha_i}(\delta)}(\delta)))) \\
&= f_{\alpha_i}^{\varphi_{\alpha_i}(e_{\alpha_i}^{\varepsilon_{\alpha_i}(\delta)}(\delta))}(e_{\alpha_i}^{\varepsilon_{\alpha_i}(\delta)}(\delta)) \\
&= f_{\alpha_i}^{\varphi_{\alpha_i}(\delta)}(\delta)
\end{aligned}$$

□

With Proposition 7.14 and [BG06, Proposition 12], we have the following corollary about the inclusion of polytopes.

Corollary 7.15. *Let $1 \leq i \leq r$, then $P_{\Psi_i(\Xi_{s_i}(\delta))} \subset P_\delta$.*

We hope that this holds for all elements in our set of galleries.

Conjecture 7.16. *Let $w \in W$, then $P_{\Psi_w(\Xi_w(\delta))} \subset P_\delta$.*

What one can do is to better describe the gallery $\Psi_w(\Xi_w(\delta))$.

Lemma 7.17. *For $w \in W$ and $w^{\underline{i}} := s_{i_1} \dots s_{i_m}$ a reduced decomposition and $\widetilde{\delta}^{\underline{i}}_m$ as in Proposition 5.30, it holds*

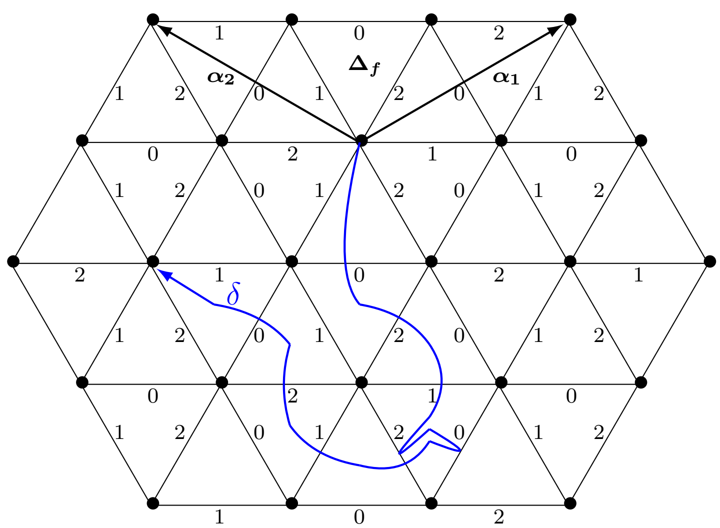
$$\Psi_w(\Xi_w(\delta)) = (S_{i_1} \circ \ldots \circ S_{i_m})(\widetilde{\delta}^{\underline{i}}_m)$$

Proof. This follows from the proofs of Theorem 7.12 and Proposition 5.30 □

7.3 Example: A_2

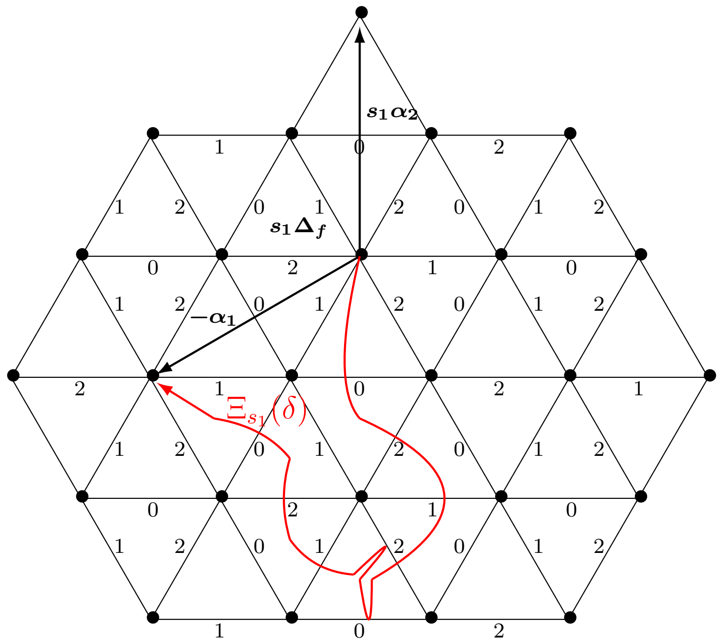
We want to illustrate the construction of the galleries $\Xi_w(\delta)$ with a small example, where we construct all galleries from a given positively folded LS-gallery δ, of type $[s_2s_1s_2, s_0, s_2, s_1, s_2, s_0, s_2, s_1, s_2, s_0]$, for the irreducible representation of highest weight $3\Lambda_1 + 3\Lambda_2$.

The following figures show all the galleries $\Xi_w(\delta)$, for all elements of the Weyl group of type A_2. The galleries are pictured as path through the apartment, this is not related to the paths model of Littelmann and others, it should only make it easier to see which sequence of alcoves form the gallery. If the path only touches a face of an alcove and does not cross it, it means that the gallery is folded around that face. In each step we also marked the two simple roots and the fundamental alcove for the given gallery.



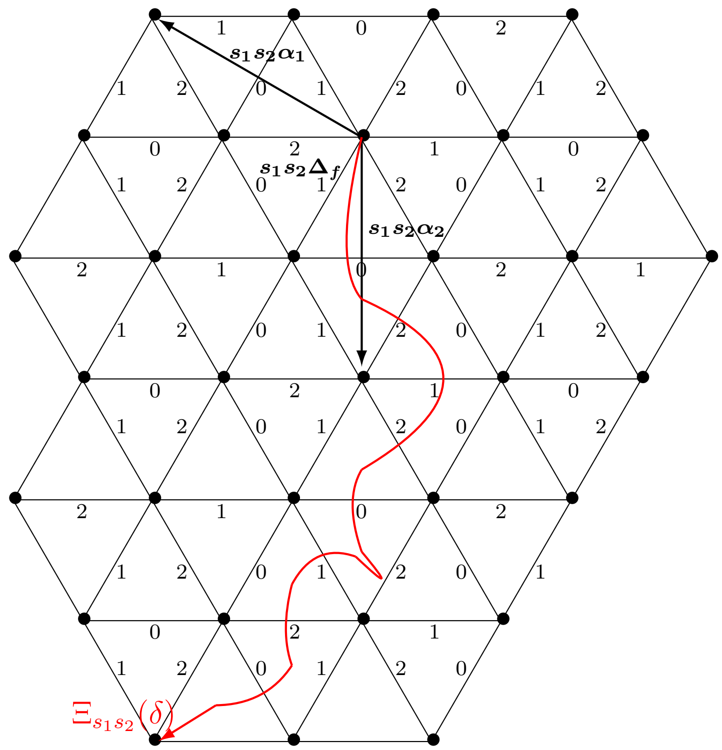
Combinatorial gallery δ

As the endpoint of the gallery is already minimal with respect to α_1, we only apply the flipping operator to the gallery and not f_{α_1}. Thus the new gallery $\Xi_{s_1}(\delta)$ has the same weight as δ, but just a small part in the middle was folded around a wall.



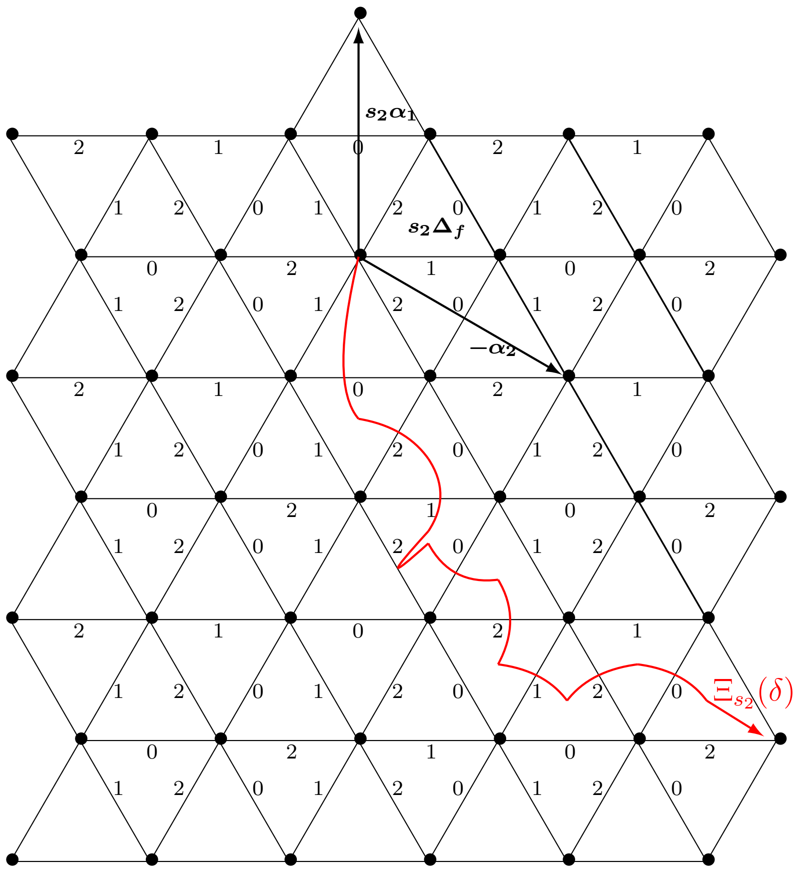
Combinatorial gallery $\Xi_{s_1}(\delta)$

This time we can see that the minimum of the gallery with respect to $s_1\alpha_2$ is not the endpoint of the gallery, but it only possesses a single folding in that direction, i.e., we only have to apply $f_{s_1\alpha_2}$ as often as it is defined, which is two times.



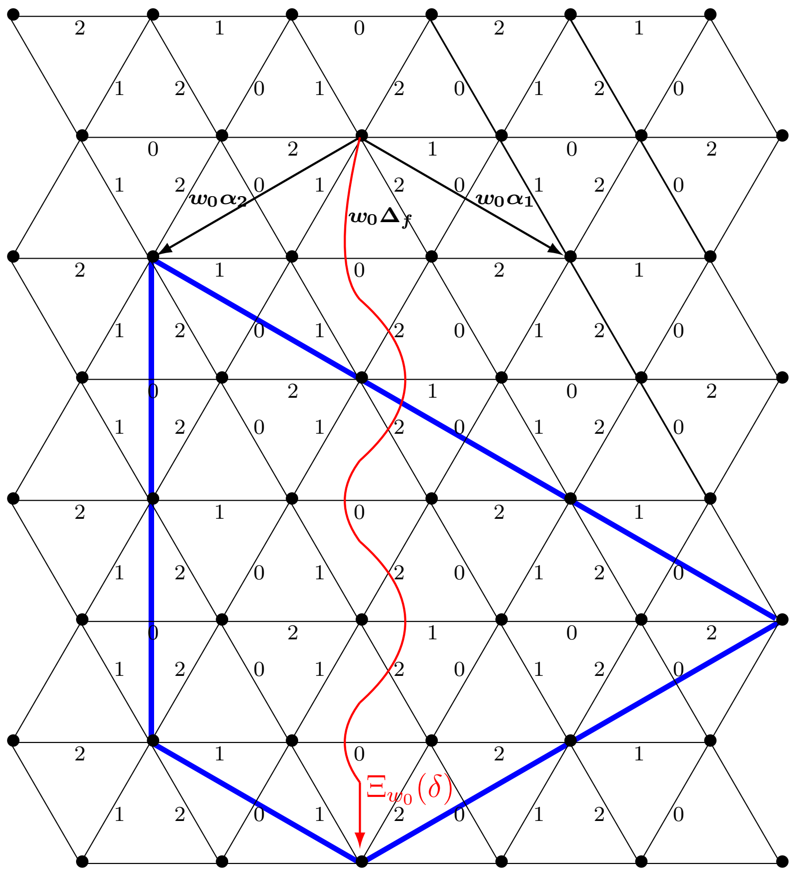
Combinatorial gallery $\Xi_{s_1s_2}(\delta)$

Again the minimum with respect to $s_1s_2\alpha_1$ is not the endpoint and thus we again only apply $f_{s_1s_2\alpha_1}$. Of course this time we will obtain the gallery corresponding to the lowest weight vector of the representation. We will first look at the other possible reduced expression of our longest Weyl group element, in this case we only need to apply f_{α_2} three times.



Combinatorial gallery $\Xi_{s_2}(\delta) = \Xi_{s_2 s_1}(\delta)$

In this case, there is no folding in the direction of the now simple root $s_2\alpha_1$ and the endpoint of the gallery is already minimal for that direction, hence the galleries $\Xi_{s_2}(\delta)$ and $\Xi_{s_2 s_1}(\delta)$ coincide. As with the gallery $\Xi_{s_1 s_2}(\delta)$ we have one folding remaining for the root $s_2 s_1 \alpha_2$ and applying the operator $f_{s_2 s_1 \alpha_2}$ as often as defined will result in the gallery corresponding to the lowest weight.

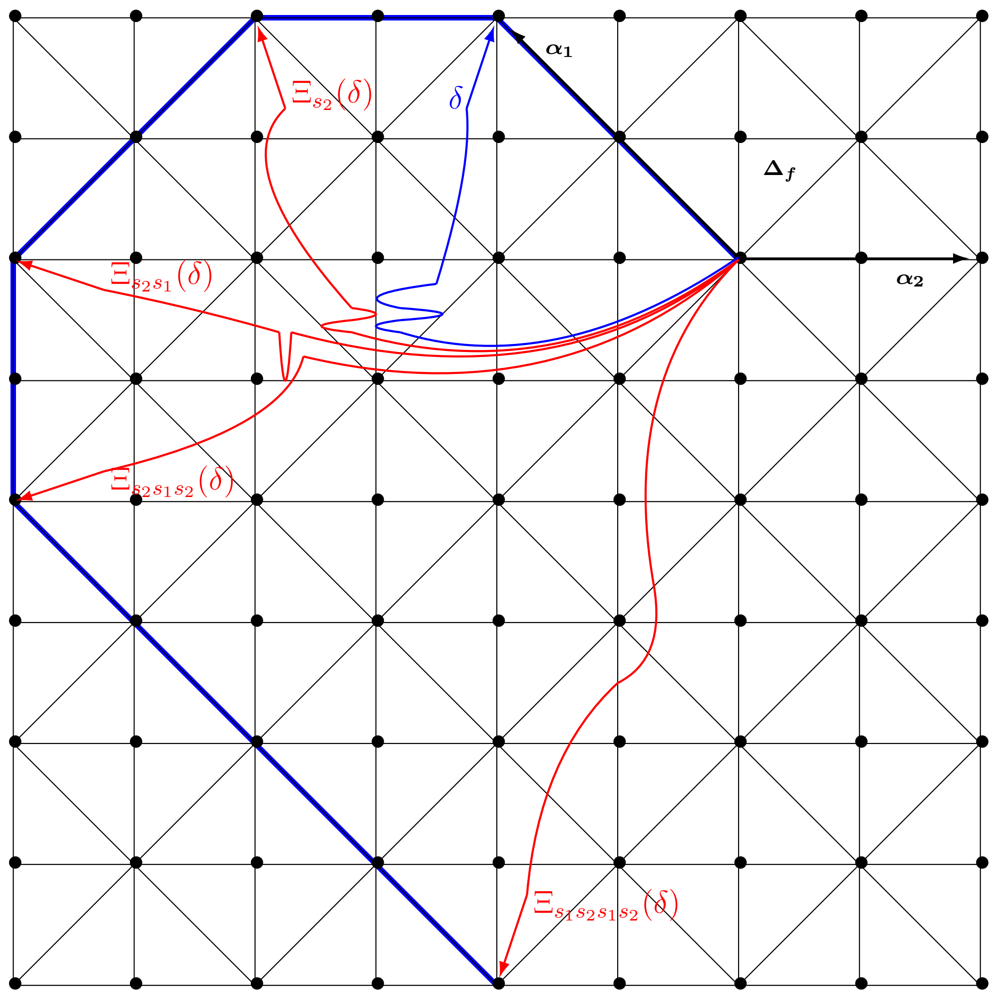


Combinatorial gallery $\Xi_{w_0}(\delta)$ and corresponding MV-polytope P_δ

In addition to the gallery for the lowest weight vector of the representation, the figure also shows the MV-polytope corresponding to the gallery δ that one obtains by taking the convex hull of all the endpoints of the constructed galleries.

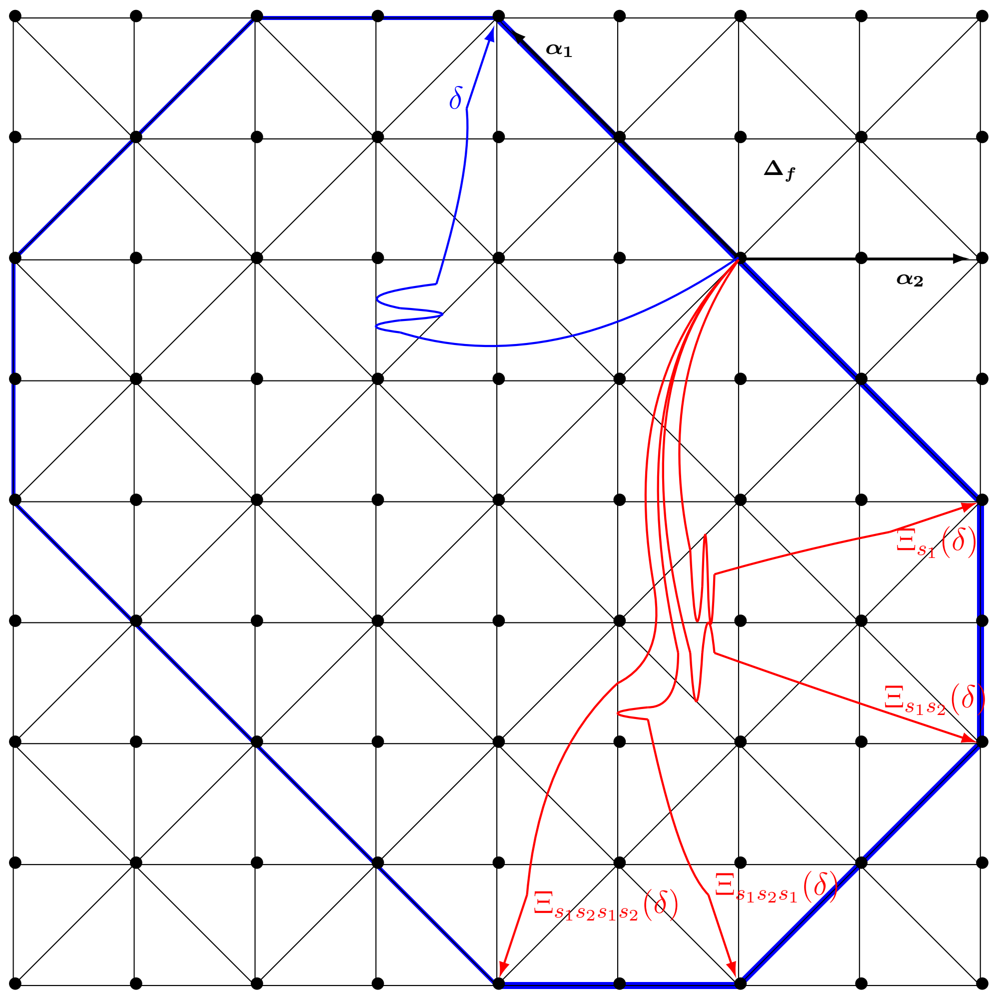
7.4 Example: B_2

We want to look at a second example for the constructed galleries, this time for the root system of type B_2. This time for the representation of highest weight $3\Lambda_1 + 3\Lambda_2$. In the first figure one can see the galleries for the reduced decomposition $s_2 s_1 s_2 s_1$ of w_0 and the corresponding part of the 1-skeleton of the MV-polytope.



Combinatorial galleries for $w_0 = s_2 s_1 s_2 s_1$

This figure shows the remaining galleries for the reduced expression $s_1s_2s_1s_2$ as well as the corresponding 1-skeleton of the MV-polytope (very thick lines) as well as the part corresponding to the previous reduced expression (semi-thick lines) to show the whole MV-polytope.



Combinatorial galleries for $w_0 = s_1s_2s_1s_2$

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Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Peter Littelmann betreut worden.

Michael Ehrig