Optimal Control of Capital Injections by Reinsurance and Investments

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Julia Eisenberg

aus Ašhabad

1. Berichterstatter: Prof. Dr. Hanspeter Schmidli, Universität zu Köln

2. Berichterstatter: Prof. Dr. Josef G. Steinebach, Universität zu Köln

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Abstract

An insurance company, having an initial capital x, cashes premiums continuously and pays claims of random sizes at random times. In addition to that, the company can buy reinsurance or/and invest money into a riskless or risky assets. The company holders are confronted with the problem of taking decisions on a business policy of the company. Thus, a measure for the risk connected with an insurance portfolio is sorely needed.

The ruin probability, i.e. the probability that the surplus process becomes negative in finite time, is typically the measure for an insurance company's solvency. However, the ruin probability approach has been criticised among other things for not considering the severity of an insolvency and for ignoring the time value of money.

An alternative to measure the risk of a surplus process is to consider the value of expected discounted capital injections, which are necessary to keep the process above zero. Naturally, it raises the question how to minimise this value. If the company holders prefer (or are indifferent) investing tomorrow to investing today, it is optimal to inject capital only when the surplus becomes negative and only as much as is necessary to keep the process above zero.

In the first part of this work, we solve the problem of minimising the expected discounted capital injections over all dynamic reinsurance strategies for the classical risk model and its diffusion approximation. In the second part, we extend the concept by adding the possibility of investing money, if the surplus remains positive, into a riskless asset. In these two cases we are able to show the existence and uniqueness of the optimal reinsurance strategy and the value function as the minimising value of expected discounted capital injections.

In the third part, we consider the surplus process, where the company holders can invest money into a risky asset modeled as a Black-Scholes model. The forth part extends the setup of the third part by possibility of reinsurance. In the last two cases we solve the problem explicitly only for the case of diffusion approximation. In the classical risk model the concept of viscosity solutions introduced by Crandall and Lions has been used.

All the studies are illustrated by simulations, written in Java.

Zusammenfassung

Eine Versicherungsgesellschaft mit Startkapital x erhält fortlaufend eingehende Prämien und zahlt an zufälligen Zeitpunkten Schadenbeträge von zufälliger Höhe aus. Zusätzlich steht es dem Versicherungsunternehmen frei Rückversicherung zu kaufen, das Geld zu einem risikolosen Zinssatz anzulegen oder in Aktien zu investieren. Die Inhaber müssen also Entscheidungen in Hinsicht auf die Unternehmenspolitik treffen. Deshalb braucht man ein Maß für die mit einem Versicherungsportfolio verbundenen Risiken.

Typischerweise wird die Ruinwahrscheinlichkeit, d.h. die Wahrscheinlichkeit, dass der Überschussprozess in endlicher Zeit negativ wird, als Maß für die Solvenz eines Versicherungsunternehmens gewählt. Allerdings wird das Konzept der Ruinwahrscheinlichkeiten unter anderem dafür kritisiert, dass die Ruinstärke und der Geldzeitwert vollkommen ignoriert werden.

Als alternatives Risikomaß betrachtet man den Wert der erwarteten diskontierten Kapitalzuführungen, welche notwendig sind damit der Überschussprozess nichtnegativ bleibt. Es stellt sich die Frage, wie man diesen Wert minimiert. Wir nehmen an, dass die Inhaber der Versicherungsgesellschaft das Geld heute dem Geld morgen bevorzugen. Dann ist es optimal das Kapital nur dann zuzuführen, wenn der Überschussprozess negativ wird, und nur so viel, dass der Überschussprozess wieder auf 0 verschoben wird.

Im ersten Teil der Arbeit lösen wir das Problem der Minimierung der erwarteten diskontierten Kapitalzuführungen über alle dynamischen Rückversicherungsstrategien für das klassische Modell der Risikotheorie und für eine Diffusionsapproximation. Im zweiten Teil erweitern wir das vorherige Konzept durch die Möglichkeit der Anlage des nichtnegativen Überschusses zu einem festen risikolosen Zinssatz. In diesen Fällen werden wir die Existenz und Eindeutigkeit der optimalen Rückversicherungsstrategie und der Wertefunktion als Minimum der erwarteten diskontierten Kapitalzuführungen zeigen können.

Im dritten Teil betrachten wir den Überschussprozess, bei dem die Versicherungsgesellschaft in die Aktien investiert, deren Preis durch Black-Scholes Modell beschrieben wird. Teil vier erweitert den Aufbau des dritten Teils durch die Möglichkeit der Rückversicherung. In den beiden zuletzt genannten Fällen konnten wir eine explizite Lösung nur im Fall der Diffusionsapproximationen finden. Im klassischen Modell wird das Konzept der Viskositätslösungen, eingeführt von Crandall and Lions, verwendet.

All die Betrachtungen werden durch Simulationen, geschrieben in Java, illustriert.

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Shylock:
If you repay me not on such a day
In such a place, such sum or sums as are
Express'd in the condition, let the forfeit
Be nominated for an equal pound
Of your fair flesh, to be cut off and taken
In what part of your body pleaseth me.
...
Antonio:
Come on, in this there can be no dismay,
My ships come home a month before the day.

"Merchant of Venice", Shakespeare

Preface

Insurance and Reinsurance: A Couple of Historical Facts

Googling the word "insurance" one finds as one of the first references the following definition: "Insurance is a promise of compensation for specific potential future losses in exchange for a periodic payment. Insurance is designed to protect the financial well-being of an individual, company or other entity in the case of unexpected loss.". An expected and somehow familiar explanation. We insure our cars, our lives or our belongings; every day we are overwhelmed with information about "credit for reinsurance", "risk transfer", "enterprise risk management" and the like: insurance is an integral part of modern life, we cannot do without. At the same time the simplest mutual insurances in form of mutual aid societies are well known from the humanity's ancient past. The nomadic people of ancient east, engaged in cattle-breeding and caravan trade, by loss of cattle refund the expenses to the concerned tribal members. Also in the Bible one can find examples of insurance dissemination. In the story of Joseph and pharaoh, Gen. 41, 1-8, Joseph creates an insurance fund storing up great abundance of corn during the seven years of plenty. During the following seven years of famine "over all the face of the earth" there was enough corn in Egypt, so that "all countries came into Egypt to Joseph to buy corn".

In the Middle Ages the Florentine, Genovian and Venetian merchants, who had in XII–XV centuries very active maritime trade relations with the countries of Near East, were the first to organise the mutual maritime transport insurances. At first, in the early 14th century, the “verbal agreement” was the rule in commerce. Ships were very expensive to build and maintain, the cargo, they carried was an accumulation of financial risks and the perils of weather and piracy were beyond of individual control of the merchants, who often could not even accompany their vessels. What was needed was a contract which would protect merchants against the caprices of fortune. In special guarantee bonds the merchants, who had sold their goods before transporting them over sea, committed themselves to buy the goods back at a higher price if the ships arrive safely. The difference in the price was a pay for the risk, which got the name premium. The first records of such bonds are believed to date from the beginning of the XIV century. Policies (derived from the Italian word polizza, meaning a promise or undertaking) were given to those insuring marine risks. Venice became an insurance center.

However, to the end of the XVI century the center of maritime commercial insurance moved to England. In 1559 Sir Nicholas Bacon, advisor to Queen Elizabeth I of England and a member of the parliament, said: “Doth not the wise merchant, in every adventure of danger, give part to have the rest assured?”.

The “Merchant of Venice”, a play by William Shakespeare, is believed to have been written between 1596 and 1598. I.e. insurance, as business, was well known not only in the scene of action, Venice, but also in the homeland of the author. But either for the exciting plot or through ignorance, he was not an actuary, Shakespeare did not give his protagonist, the merchant of Venice Antonio, the possibility to “insure” his ships. The “Merchant of Venice” would have been a very different story if some clever insurance agent had persuaded Antonio to insure, and Shylock would have fit in perfectly as a contract lawyer for disputed claims.

It is beyond all question, that if the first insurer incautiously undertakes large risks, it is better to transfer those risks to the other underwriters. Insurance for insurance companies is called reinsurance. The first reinsurance contract was drawn up in Genua in 1370. Thus the reinsurance business originates also from Northern Italian seaside trading towns, which created a rudiment of profit-oriented insurance. In this initial stage of development the whole risk was often shifted from insurer to reinsurer, whereas the reinsurer’s premium was smaller than the insurer’s premium. One could earn money without any risk! As a consequence any marine policy “without further proof of interest than the policy, or by way of gaming or wagering, or without benefit of salvage to the assurer” was prohibited in England from 1746 to 1864. Naturally the development of reinsurance in England was slowed by this legal prohibition of reinsurance. On April 8, 1846 the first independent professional reinsurance company, Cologne Reinsurance Co, was founded. The great fire in Hamburg made the need of such a company evident. Nowadays insurance and reinsurance help to make the modern life possible. Hurricanes, earthquakes, terrorism and the good old pirats present risks beyond the great fires of the past.

Measuring Risks

Now we make a huge jump not only in the time but also in the dimension of discussion of risks and profits in the (re)insurance business. We leave the historical and macroeconomic field and switch to microeconomic considerations.

We consider a non-life insurance company **ABC**, having an initial capital x. As it is common in insurance business **ABC** receives premia and pays claims. The insurer is also allowed to invest either in a risky asset or in a riskless one and to use dynamic reinsurance. Whenever the company makes a decision that affects the surplus. The company is termed to be solvent as long as the initial capital together with the return of investments and already received premia exceed the claims to be paid.

Examples from the previous section reveal, that modern society relies on the stability and strength of the insurance system. An insurance premium paid currently provides coverage for losses that might arise many years in the future. For that reason, the viability of an insurance company is very important. In recent years, a number of insurance companies have become insolvent, leaving their policyholders with no coverage. To guarantee the smooth functioning of the insurance system a proper risk management is needed. Managing the risk means managing an insurance company so as to maintain a comfortable surplus of premia and/or risky assets beyond liabilities. Thus an actuary is interested in a convenient measure for the risk as a basis for the decisions. The problem with describing the risk characteristics of an insurance portfolio is, that in non-life insurance the number of claims, claim sizes and occurrence times are random. In 1903 the Swedish actuary Filip Lundberg [52] introduced a simple model which is capable of describing the basic dynamics of a homogeneous insurance portfolio. Harald Cramér with his books [15, 16] contributed a lot to the understanding and dissemination of Lundberg's collective model, which is now called the Cramér-Lundberg model or the classical risk model. This model is nowdays also one of the most popular in non-life insurance mathematics. Another very popular approach in modeling the surplus consists in replacing the random part of the surplus process in the classical risk model with a diffusion process. The basic idea is to make the claim sizes in the classical model small and simultaneously to let the number of claims grow in such a way that the risk process converges weakly to a diffusion. The idea of application of weak convergence in risk theory was first introduced by Iglehart [43] in 1969, see also Grandell [36] and Schmidli [70].

The next step after modeling the surplus as a stochastic process is to find an appropriate measure for risk. In order to evaluate risks of an insurance portfolio several risk measures can be constructed. For example in Solvency II risk measures as the Value–at–Risk and Expected Shortfall are used. The classical risk measure used by actuaries is the ruin probability, i.e. the probability, that the portfolio becomes negative in finite time.

One says, that ruin occurs when the surplus process, modeled as a stochastic process, becomes negative for the first time.

Stochastic control has been used in finance since the papers of Merton [53, 54]. An introduction to control theory one finds in the books by Fleming and Soner [25], by Fleming and Rishel [26] or by Karatzas and Shreve [49]. In recent years there has been a rapid development in the field of applications of control theory to different aspects of insurance mathematics. One of the popular optimisation criterions in insurance mathematics is to control the process in such a way, that the probability of ruin is minimised. Numerous papers and books have been written on this topic, among others one finds Hipp and Plum [41], Hipp and Schmidli [42], Schmidli [68, 69, 70, 71]. The probability of ruin serves as a kind of litmus paper. It indicates the soundness of the insurer's combination of the income of an insurance company plus the initial capital on the one hand and claims process on the other. Also we obtain a useful tool for portfolio comparison. But despite these positive points, using ruin probabilities also elicited criticism. The main objection issues are

- the ruin probability does not actually represent the probability, that the insurer will go bankrupt in the near future;
- it might take centuries until the ruin will actually happen;
- risk surplus processes usually tend to infinity, which is not the case for a real surplus of an insurer;
- the time and the severity of ruin are ignored.

Another classical control problem of actuarial mathematics is to find strategies for optimal payment of dividends. In 1957 de Finetti [24] considered the optimal control problem of finding the dividend payment strategy, that maximises the expected discounted value of dividends, which are paid to the shareholders until the company is ruined. He was the first to propose to measure the risk as the maximal value of expected discounted dividends. Karl Borch [12] contributed a lot to the dissemination and development of de Finetti's idea. In the framework of the classical compound Poisson model the problem was considered among others by Bühlmann [14], Gerber [28, 30], Gerber and Shiu [33], Gerber et al. [34, 35], Dickson and Waters [18], Dickson and Drekic [19]. In many later papers the problem was formulated and solved also for a diffusion approximation, see Shreve at al. [72], Jeanblanc-Picqué and Shiryaev [46], Paulsen and Gjessing [57], Højgaard and Taksar [38], Asmussen and Taksar [3], Asmussen et al. [4], Paulsen [58].

Typically, by maximising the dividend income, the optimal dividend strategy would be a band strategy leading to certain ruin; and if the ruin probability is minimised, then obviously no dividend will be paid. Christian Hipp [40] considered the maximal dividend income under a ruin constraint. But in real life a technical ruin does not compulsory mean the bankruptcy. Of course, if the liabilities exceed the company value, it is better to wind up the business. Otherwise the company holders have to raise new money to proceed with the business. Therefore, capital injections could be added to the surplus process, and the value of the discounted cash flow can be considered, see Kulenko and Schmidli [50]. However if injecting additional money is not penalised, the optimal strategy would be to pay the income as dividends and to finance the outflow by capital injections. In other words the optimal strategy would keep the surplus at zero.

The idea of this work is to choose as a risk measure the minimal expected discounted capital injections. Let X be the underlying surplus process with $X_0 = x$. Let Y be an increasing process with $Y_{0-} = 0$. The process with capital injections is denoted by $X_t^Y = X_t + Y_t$. We define the value $V^Y(x) = \mathbb{E}_x[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t]$, where $\delta \ge 0$. The injection process Y has to be chosen such that $X_t^Y \ge 0$ for all t (almost surely). The value function is defined as $V(x) = \inf V^Y(x)$, where the infimum is taken over all processes Y such that $X_t^Y \ge 0$ for all t.
It should be noted that δ is not a financial discounting factor. We assume that the process X is already discounted. Choosing $\delta > 0$ then means that the investor prefers investing tomorrow to investing today. If we choose $\delta = 0$ the investor is indifferent to investing tomorrow or today. We will see below that for a diffusion model $\delta = 0$ corresponds to minimising the probability of ruin. Choosing $\delta < 0$ would complicate the problem. It could then be optimal to inject capital already before the process reaches zero. If δ is too small, the value function will become infinite.

Measuring the risk as proposed above has following advantages. As the value of future capital injections, the measure is economically motivated. It is clear, that it is not optimal to inject capital before it is really necessary which means the sub-additivity of the measure. For two underlying processes X and Z with corresponding value functions $V_X(x)$ and $V_Z(x)$ we have for the value function of the sum process $X + Z$: $V_{X+Z} \le V_X + V_Z$. Indeed, it follows readily that $Y_t^{X+Z} \le Y_t^X + Y_t^Z$, so the sub-additivity follows from $\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t = \delta \int_0^\infty e^{-\delta t} Y_t\,\mathrm{d}t$. Compared with the valuation of dividends, the advantage is, that we do not have to solve an optimisation problem in order to find the value.

The problem we will be dealing with in this work is, how the insurer should control the future capital injections by means of reinsurance and/or investments in the Cramér-Lundberg model and in a diffusion approximation if $\delta \ge 0$.

In the first chapter we just give the basic definitions and a short description of the models we will use later. The following four chapters introduce four different settings for the classical risk model and its diffusion approximation. Every chapter opens with the simplest case of a diffusion approximation. Then the classical risk model is considered for the general case $\delta \geq 0$. The special case $\delta = 0$, where we can always show the existence and uniqueness of the solution, is considered in a final step. All considerations are illustrated with examples.

In Chapter 2 we consider the models, in which the only available control is related to dynamic reinsurance. We solve the problem explicitly in the case of a diffusion approximation and give a closed expression for the value function. The optimal strategy in this case is constant and the value function is an exponential function. In the classical risk model we show the existence and the uniqueness of the value function and calculate the value function and the optimal strategy numerically in the special case of proportional reinsurance with exponentially and Pareto distributed claims. In both models we solve the optimality problem with help of the Hamilton–Jacobi–Bellman (HJB) equation. In the case of a diffusion approximation we will go through the following steps. At first we give the HJB equation corresponding to the problem we consider and show that it has a unique solution satisfying the boundary conditions derived from the optimisation problem. Then using the verification theorem we show, that the solution obtained to the HJB equation is the value function of the optimisation problem. In the classical risk model we use another technique. We show directly that the value function solves the HJB equation and that the solution is unique.

In Chapter 3 we change the setup of the second chapter by implementing interest at a constant rate, i.e. in addition to possibility of reinsurance the insurer is allowed to earn on the surplus, provided it remains positive. In this chapter we consider in the case of a diffusion approximation proportional reinsurance only. Although the change in the model was small, the optimal strategy and the value function change a lot. The optimal strategy is not constant any more and the value function is composed of four different functions. As for the classical risk model, the proof techniques are similar to the techniques used in Chapter 2.

In Chapter 4 the insurer is allowed to invest in a risky asset which follows a geometric Brownian motion. While in the diffusion approximation case we show again, that the optimal strategy is constant and the value function is an exponential function, the case of classical risk model becomes much more complicated as in Chapters 2 and 3. For the general case $\delta \geq 0$ we can not show the existence of a classical solution to the corresponding HJB equation and have to use the concept of viscosity solutions. For introduction to viscosity solutions see for example Bardi and Capuzzo-Dolcetta [6] or Crandall et al. [17]. The proof techniques used in the case $\delta \geq 0$, one finds in Benth et al. [8], Azcue and Muler [5] and Albrecher and Thonhauser [1].
In the special case $\delta = 0$ one can show the existence and uniqueness of the classical solution to the HJB equation using the proof techniques from Schmidli [70]. In this case we can also apply the result of Hipp and Plum [41], concerning the asymptotic behavior of the optimal strategy.

In Chapter 5 we consider a combination of both investment and reinsurance. A risky asset is also here modeled as a geometric Brownian motion. Diffusion approximation yields a constant optimal strategy and an exponential function as value function. In the case of the classical risk model we proceed as in Chapter 4. For simplicity we consider in the special case $\delta = 0$ only the proportional reinsurance. But also here we can show the existence and uniqueness of the value function.

In the derivation and proofs of our results we have used a lot of stochastic analysis and techniques from the martingale theory. Because we assume, that the reader of this work is perfectly acquainted with the terms and definitions of stochastic calculus we just state the theorems we have used in the Appendix. For detailed introduction to martingale theory and stochastic analysis we refer to Protter [60], Bhattacharya and Waymire [9], Karatzas and Shreve [49] or Ethier and Kurtz [23].

List of Principal Notation

$\mathbb{N}$	The natural numbers, $\mathbb{N} = \{0, 1, 2, ..\}$
$\mathbb{R}$	The real numbers
$\mathbb{R}_+$	The non-negative real numbers, $\mathbb{R}_+ = [0, \infty)$
$p \wedge q$	$\min\{p, q\}$ for $p, q \in \mathbb{R}$
$p \vee q$	$\max\{p, q\}$ for $p, q \in \mathbb{R}$
p^+	$\max\{p, 0\}$ for $p \in \mathbb{R}$
p^-	$\min\{p, 0\}$ for $p \in \mathbb{R}$
$\bar{E}$	The closure of the set $E \subset \mathbb{R}$
∂E	The boundary of the set $E \subset \mathbb{R}$
C^1	Space of continuously differentiable functions
$\mathbb{P}$	The basic probability measure
Ω	Event space on which probabilities are defined
$\mathcal{F}$	σ-Algebra of the probability space
$\mathbb{F} = \{\mathcal{F}_t\}$	Filtration
$\mathbb{E}$	Expected value with respect to $\mathbb{P}$
$\mathbb{E}_x[X]$	Expected value given initial value x
Var	Variance with respect to $\mathbb{P}$
$\{Z_i\}_{i \in \mathbb{N}}$	iid random variables, representing claim sizes
Z	Generic random variable with the same distribution as Z_i
$M_Z(y)$	Moment-generating function of the random variable Z
$N = \{N_t\}$	Poisson process
λ	Intensity of N
T_i	i-th occurrence time
G	Claim size distribution function
μ, μ_n	The first and the n-th moment of the claim sizes
c	Premium rate
η	Safety coefficient of the first insurer
b	Retention level $b \in [0, \tilde{b}]$
$c(b)$	Premium rate function, depending on retention level b

$r(Z, b)$	Self-insurance function, depending on claim size Z and retention level b
θ	Safety coefficient of the reinsurer
δ	Discounting factor
$1\!\!1_{[\cdot]}$	Indicator function
$X = \{X_t\}$	Stochastic process
$W = \{W_t\}$	Standard Brownian motion
$\mathscr{A}$	Set of admissible investment strategies
$\mathscr{U}$	Set of admissible reinsurance strategies
$B = \{b_t\}$	Reinsurance strategy
$B^* = \{b_t^*\}$	The optimal reinsurance strategy
$A = \{a_t\}$	Investment strategy
$A^* = \{a_t^*\}$	The optimal investment strategy
$Y = \{Y_t\}$	Capital injection process
X_t^B, X_t^A, $X_t^{A,B}$	Stochastic process under the strategies B, A, or (A, B)
$X_t^{B,Y}$, $X_t^{A,Y}$, $X_t^{A,B,Y}$	X_t^B, X_t^A, $X_t^{A,B}$ with capital injections
$V(x)$	Value function
$V^B(x)$, $V^A(x)$, $V^{A,B}(x)$	Return functions of the strategies B, A, (A, B)

1 Preliminaries

We begin with an overview of the models, which provide the motivation for our investigations. The goal of this chapter is just to give a rough review of the concepts we use in the work. Therefore we will abandon the proofs of the theorems stated below and just refer to the books and papers, where a detailed description can be found. At first we consider the Cramér-Lundberg model, state its most important concepts and give a short review of the premium calculation principles and reinsurance treaties. In the second section we derive a diffusion approximation, which will be used throughout this work. In the third section we make some agreements about terminology and notation.

In the following considerations we assume as given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is large enough to carry all the objects defined below. In addition we are given a filtration $\{\mathcal{F}_t\}$. By a filtration we mean a family of σ-algebras, that is increasing, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ for $t \geq s$. For convenience, we will often write $\mathbb{F}$ for the filtration $\{\mathcal{F}_t\}$.

1.1 The Classical Risk Model

1.1.1 Introduction

The Cramér–Lundberg model, also called the classical risk model, is a risk process (or surplus process) X defined as

$$X_t = x + ct - S_t \;, \tag{1.1}$$

where

$$S_t = \sum_{i=1}^{N_t} Z_i \tag{1.2}$$

is the corresponding loss process. The process $\{N_t\}$ represents the number of claims in the time interval $[0, t]$, Z_i for $i \in \mathbb{N}$ are the claim sizes, positive, independent and identically distributed random variables with finite mean μ and finite second moment μ_2. Claims occur at random instants of time $0 < T_1 < T_2 ...$ and interarrival times $T_2 - T_1$, $T_3 - T_2$, $T_4 - T_3$, ... are independent, exponentially distributed random variables with finite mean $\lambda^{-1} > 0$. The processes $\{Z_i\}$ and $\{T_{i+1} - T_i\}$ are independent of each other. c denotes the premium income in a time interval of length 1.

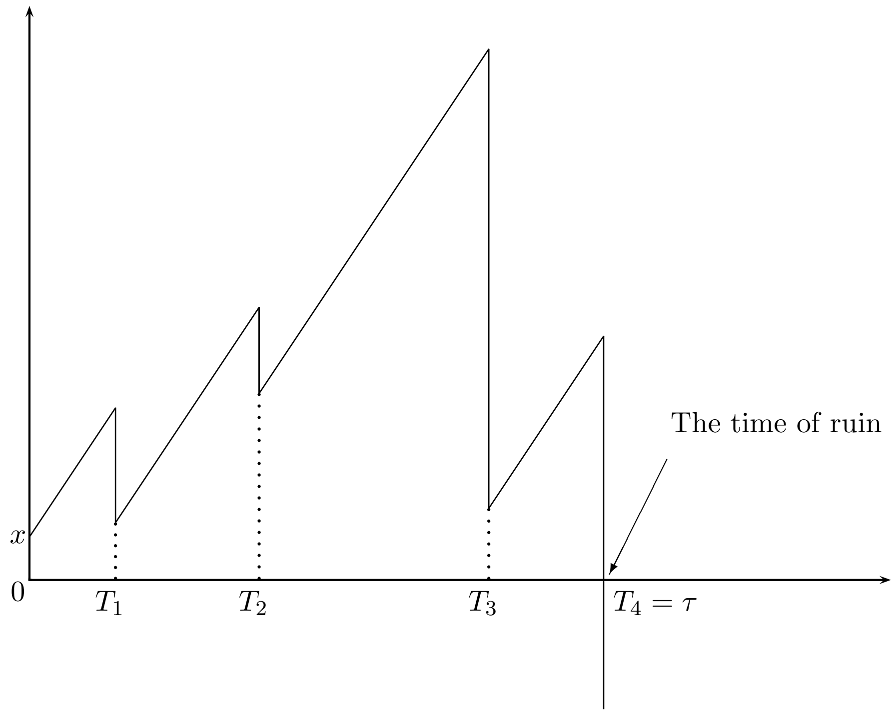

Figure 1.1: The Process $\{X_t\}$.

Given that the interarrival times are exponentially distributed, N_t is a homogeneous Poisson process with intensity $\lambda > 0$, i.e.

$$\mathbb{P}[N_t = k] = e^{-\lambda t}\frac{(\lambda t)^k}{k!}\ , \quad k = 0, 1, 2, ...$$

Typically, the surplus process has the form illustrated in Figure 1.1. X grows linearly and at the random moments of claim arrivals drops by the amounts of the claims.

The concept of ruin is of central importance by modeling the surplus process. Figure 1.1 gives already the first intuitive idea of ruin. Below we give the precise definition.

Definition 1.1.1 (Ruin)
The event, that the process X_t, defined in (1.1)*, ever falls below zero is called* ***ruin****. The time τ, when the process X_t falls below zero for the first time*

$$\tau = \inf\{t > 0 : \ X_t > 0\}$$

is called ***ruin time****. As* ***severity of ruin*** *one denotes the absolute value $|X_\tau|$.*
The ***probability of ruin*** *for the initial capital x is then given by*

$$\psi(x) = \mathbb{P}[\tau < \infty | X_0 = x]\ .$$

Lots of papers on the probability of ruin, on the severity of ruin and on surplus prior to ruin in the different settings for the classical risk model have been written since the pathbreaking paper of Gerber and Shiu [31]. A topic no less important, than the concept of ruin, is the asymptotic behavior of the surplus process. A helpful tool in this regard is the following proposition.

Proposition 1.1.2
The aggregate loss process $\{S_t\}$, defined in (1.2), and the homogeneous Poisson process $\{N_t\}$ satisfy the strong law of large numbers:

$$\frac{S_t}{t} = \frac{S_t}{N_t} \cdot \frac{N_t}{t} \overset{t\to\infty}{\longrightarrow} \lambda\mu \ ,$$

Proof: Confer Mikosch [55, p. 82] □

This implies, $\lim\limits_{t\to\infty} X_t = \infty$ or $\liminf\limits_{t\to\infty} X_t = -\infty$ a.s. according to whether $c > \lambda\mu$ or $c \leq \lambda\mu$. A result from the theory of random walks, see Rolski et al. [61], shows that in the case $c \leq \lambda\mu$ the ruin occurs with probability one.

Definition 1.1.3 (Net profit condition)
*One says, that the **net profit condition** is fulfilled if*

$$c - \lambda\mu > 0 \ .$$

The constant

$$\eta = \frac{c - \lambda\mu}{\lambda\mu}$$

*is called the **safety loading** or **safety coefficient** of the insurer.*

Thus, the net profit condition requires choosing the premium intensity larger than the expected loss in a time interval of length 1.

In connection with the ruin probability one often speaks of adjustment coefficient.

Definition 1.1.4 (Adjustment or Lundberg's coefficient)
*Assume, that the **moment generating function** of Z, $M_Z(s) = \mathbb{E}[e^{sZ}]$, exists in some neighborhood $(-h_0, h_0)$, $h_0 > 0$. If the unique positive solution ρ to the equation*

$$M_Z(s) = \mathbb{E}[e^{sZ}] = 1$$

*exists, it is called the **adjustment** or **Lundberg coefficient**.*

One of the classical results in insurance mathematics is the following proposition.

Proposition 1.1.5 (Lundberg's inequality)
Assume, that the adjustment coefficient ρ exists. Then the following inequality holds for all $x > 0$:

$$\psi(x) = \mathbb{P}[\tau < \infty | X_0 = x] \le e^{-\rho x} \ .$$

Proof: Confer Rolski et al. [61, p. 416]. □

The exponential boundary of Lundberg's inequality ensures that the probability of ruin is very small if one starts with large initial capital x. The result tells us, that the smaller ρ the more risky is the portfolio. For a detailed discussion of Lundberg's coefficient we refer to Rolski et al. [61].

The following operator which is applied to a suitable function f, see Proposition B.2.4 p. 168, is called the infinitesimal generator of the Markov process X,

$$Df(x) = cf'(x) + \lambda \int_0^\infty f(x-z)\, \mathrm{d}G(z) - \lambda f(z) \ ,$$

which will be needed later on.

1.1.2 Premia

Most frequently the insurance premium is described as the "price" in exchange for which the insurer takes over and bears the insured's risk. Below we give three most famous premium calculation principles. We refer to Heilmann [39], to Rolski et al. [61], to Kaas [47] and to Rotar [65] for an explicit description of various premium principles and their properties.

Let Z denote the random variable, which describes the risk of some insurance contract; with c we will denote the premium. We assume, that Z is bounded and non-negative.

1. **Expected value principle**
   The premium is
   $$c = (1+\eta)\mathbb{E}[Z]$$
   for some safety loading of the insurer $\eta > 0$.

2. **Variance principle**:
   The premium is
   $$c = \mathbb{E}[Z] + \alpha \mathrm{Var}[Z]$$
   for some $\alpha > 0$.

3. **Standard deviation principle**: The premium is
   $$c = \mathbb{E}[Z] + \alpha \sqrt{\mathrm{Var}[Z]}$$
   for some $\alpha > 0$.

1.1.3 Reinsurance

In practice an insurer often transfers portions of risk portfolios to other parties in order to reduce large obligations resulting from an insurance claim. An insurance company that transfers a risk to another insurance company is called a **cedent** or a ceding company. An insurance company that assumes all or part of an insurance policy written by a ceding company is called a **reinsurer**. A reinsurance treaty is an agreement between an insurer (cedent) and a reinsurer under which, claims that occur in some fixed period of time are split between the insurer and reinsurer. There are three different types of reinsurance treaties: reinsurance acting on individual claims, reinsurance acting on the aggregate claim over a certain period, and reinsurance acting on the k largest claims occurring during a certain period. For the sake of simplicity of notation we will restrict our considerations only on the first type reinsurance treaties.

Below we discuss some functions of random variables which describe the portion of risk, which the insurer will carry himself. Let Z again describe the loss amount and let r be the function describing that part of the claim, that the insurer intends to pay himself. Because r will depend not only on Z but also on reinsurance treaty-specific variables we write $r(Z,.)$. The function $r(Z,.)$ is called **self-insurance function**. For a general reinsurance treaty it holds $0 \le r(Z,.) \le Z$.

The most famous reinsurance types acting on individual claims are

- $r(Z,b) = b \cdot Z$, **proportional reinsurance** with retention level b;
- $r(Z,b) = \min(Z,b)$, **excess of loss (XL)** reinsurance with retention level b.
- $r(Z,(a,b,\gamma)) = \min(Z,a) + (Z-a-\gamma)^+ + b\min\big((Z-a)^+,\gamma\big)$, **Proportional reinsurance in a layer** for $a,\gamma > 0$ and $b \in (0,1)$.

The proportional reinsurance in a layer has a multidimensional retention level (a,b,γ). For the sake of simplicity we will consider in this work only one-dimensional reinsurance treaties with some retention level b. The case of multidimensional treaties can be treated similarly, if we let the retention level $\mathbf{b}$ be a vector.

The problem by buying reinsurance consists in determining for a given risk the portion, which the insurer should carry himself. The decision is of course influenced by the price, which reinsurer demands for taking over the ceded portion. Using the example of proportional reinsurance, i.e. $r(Z,b) = bZ$, we calculate the premium remaining to the insurer for the three premium calculation principles defined above. Let $c(b)$ denote the premium remaining to the insurer, if the retention level b was chosen.

- Expected value principle. Assume the safety loading of the insurer is given by η and the safety loading of the reinsurer by θ.

$$c(b) = \mathbb{E}[Z](1+\eta) - \mathbb{E}[Z-bZ](1+\theta) = \mathbb{E}[Z](\eta-\theta) + \mathbb{E}[Z](1+\theta)b\,. \tag{1.3}$$

- Variance principle. Assume α is the premium parameter of the insurer and β of the reinsurer.
$$c(b) = b\mathbb{E}[Z] + (\alpha - \beta(1-b)^2)\mathrm{Var}[Z] \; .$$
- Standard deviation principle. Assume α is the premium parameter of the insurer and β of the reinsurer.
$$c(b) = b\mathbb{E}[Z] + (\alpha - \beta(1-b))\mathrm{Var}[Z] \; .$$

For our numerical calculations we will have to choose the distribution function of the claim sizes. Exponential distribution $\mathrm{Exp}(\gamma)$ for some parameter $\gamma > 0$ with density

$$f(x) = \begin{cases} \frac{1}{\gamma} e^{-x/\gamma} & : \; x \ge 0 \; , \\ 0 & : \; x < 0 \; , \end{cases}$$

turns out to be very convenient for the numerical derivations. In insurance mathematics the exponential distribution serves as a benchmark for characterisation of the heaviness of the tails. As a tail of a distribution function F one denotes the function $\bar{F}(x) = 1 - F(x)$. One says F is a **light-tailed** distribution if the tail function $\bar{F}$ fulfils

$$\limsup_{x \to \infty} \frac{\bar{F}(x)}{e^{-x/\gamma}} < \infty$$

for some $\gamma > 0$. If

$$\limsup_{x \to \infty} \frac{\bar{F}(x)}{e^{-x/\gamma}} = \infty$$

for all $\gamma > 0$ we call F **heavy-tailed**.

Thus, $\mathrm{Exp}(\gamma)$ distribution is a light-tailed distribution and the large claims have comparatively small weight. Experience has shown, that the $\mathrm{Pareto}(2, \mu)$ distribution, i.e. the distribution with density

$$f(x) = \frac{2\mu^2}{(x+\mu)^3} \; ,$$

is often appropriate for representing the tail of distributions, where large claims may occur. More about heavy- and light-tailed distributions one finds for example in Mikosch [55]. In our numerical examples we will consider both, the exponential and Pareto distributions.

1.2 Diffusion Approximation

Despite the seeming simplicity of the classical risk model, it is sometimes hard to obtain accurate numerical calculations. A useful method to derive an approximation is the "diffusion approximation". The idea is to substitute the random part of the classical risk model by a diffusion.

First of all we give a definition of a diffusion process.

Definition 1.2.1
A Markov process $X = \{X_t\}$ is said to be a diffusion with infinitesimal drift function $m(x)$ and infinitesimal variance $\sigma^2(x) > 0$, if

$$\begin{aligned}
&\mathbb{E}[(X_{s+t} - X_s)\mathbb{1}_{[|X_t - x| \le \varepsilon]} | X_s = x] = tm(x) + o(t) \ , \\
&\mathbb{E}[(X_{s+t} - X_s)^2\mathbb{1}_{[|X_t - x| \le \varepsilon]} | X_s = x] = t\sigma^2(x) + o(t) \ , \\
&\mathbb{E}[|X_{s+t} - X_s| > \varepsilon | X_s = x] = o(t)
\end{aligned}$$

as $t \downarrow 0$ for every $x \in \mathbb{R}$ and every $\varepsilon > 0$, where $\lim\limits_{t \to 0} \frac{o(t)}{t} = 0$.
We find in Rolski et al. [61, p. 561], that a diffusion X is the unique solution to the stochastic differential equation

$$\mathrm{d}X_t = m(X_t)\ \mathrm{d}t + \sigma(X_t)\ \mathrm{d}W_t \ , \tag{1.4}$$

which is then interpreted as stochastic integral equation

$$X_t = X_0 + \int_0^t m(X_s)\ \mathrm{d}s + \int_0^t \sigma(X_s)\ \mathrm{d}W_s \ .$$

We say, that $\{X_t\}$ is a $(m(x), \sigma^2(x))$-diffusion.

Remark 1.2.2
The most famous examples for diffusion processes are Brownian motion and geometric Brownian motion, see for instance Borodin and Salminen [13, p. 51, 77]. Throughout this work we will denote the standard Brownian motion by $W = \{W_t\}$ and a geometric Brownian motion by $Q = \{Q_t\} = \{Q_0 e^{(m-\sigma^2/2)t + \sigma W_t}\}$ with some m, σ.

Next we give some sufficient conditions, which imply the existence of a unique solution to stochastic differential equation (1.4).

Theorem 1.2.3
Suppose, that the coefficients $m(x)$ and $\sigma(x)$ satisfy the global Lipschitz and linear growth conditions

$$\begin{aligned}
&|m(x) - m(y)| + |\sigma(x) - \sigma(y)| \le K|x - y| \ , \\
&|m(x)|^2 + |\sigma(x)|^2 \le K^2(1 + x^2) \ ,
\end{aligned}$$

for all $x,\ y\ \in \mathbb{R}$, where K is a positive constant. Let further $\{\mathcal{F}_t\}$ be a filtration generated by the standard Brownian motion W. Then there exists a continuous, adapted process $X = \{X_t\}$, which is a strong unique solution to (1.4).

Proof: Confer Karatzas and Shreve [49, p. 287, 289]. □

In the following chapters we will often use the concept of infinitesimal generator of a diffusion in our proofs and derivations. Assume X is a diffusion, then its infinitesimal generator is given by

$$Df(x) = m(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x)$$

for some suitable function f. For proof see Proposition B.2.5 p. 170 in the Appendix.

Because the aim of the present section is to establish diffusion approximations, we will show next how a diffusion approximation to the classical risk model can be constructed. In order to do that we need the concept of weak convergence.

Definition 1.2.4 (Weak Convergence)
A sequence $(X^{(n)})$ of stochastic processes is said to converge weakly to a stochastic process X if for every bounded continuous functional f it follows, that

$$\lim_{n\to\infty} \mathbb{E}[f(X^{(n)})] = \mathbb{E}[f(X)] \,.$$

Then we write $X^{(n)} \Rightarrow X$.

Definition 1.2.5
We say that a risk process X can be approximated by K if there exists a sequence of stochastic processes $(X^{(n)})_{n\geq 1}$, such that $X^{(n)} \Rightarrow K$ and $X_t^{(1)} = X_t$.

If for a classical risk process $\{X_t\}$ there exists a sequence $\{C_t^{(n)}\}$ of classical risk processes such that $C_t^{(1)} = X_t$ and $C^{(n)} \Rightarrow \tilde{W} = \{mt + \sigma W - t\}$ for some m and σ^2, then we call $\tilde{W}$ a **diffusion approximation** to X.

The following basic construction is due to Schmidli. Denote the sequence of corresponding Poisson processes by $N^{(n)} = \{N_t^{(n)}\}$, their intensities by λ_n, the claims by $Z_i^{(n)}$ with distribution function G_n, the initial capitals by x_n, the premium rates by c_n and the loss processes by $\{S_t^{(n)}\}$, i.e.

$$C_t^{(n)} = x_n + c_n t - S_t^{(n)} \,.$$

Let further

$$\mu^{(n)} := \int_0^\infty z \, \mathrm{d}G_n(z) \quad \text{and} \quad \mu_2^{(n)} := \int_0^\infty z^2 \, \mathrm{d}G_n(z) \,,$$

where we assume $\mu_2^{(n)} < \infty$. We write for the safety loadings $\eta_n := \frac{c_n - \lambda_n\mu^{(n)}}{\lambda_n\mu^{(n)}}$.

Note, that all the processes $C^{(n)}$, being classical risk models, have independent and stationary increments. Let $\{W_t\}$ be a standard Brownian motion and consider an (m, σ^2)-Brownian motion $\tilde{W} = \{\tilde{W}_t\}$, i.e. $\tilde{W}_t = \tilde{W}_0 + mt + \sigma W_t$.

Theorem 1.2.6
In order to have $C^{(n)} \Rightarrow \tilde{W}$ the parameters have to satisfy

$$\begin{aligned}
&\lim_{n\to\infty} \lambda_n = \infty \ ; \\
&\lim_{n\to\infty} c_n = \infty \ ; \\
&\lim_{n\to\infty} \eta_n \lambda_n \mu_n = m \ ; \\
&\lim_{n\to\infty} \lambda_n \big((\mu^{(n)})^2 + (\mu_2^{(n)})^2 \big) = \sigma^2 \ ; \\
&\lim_{n\to\infty} \mu^{(n)} = 0 \quad \textit{and} \quad \lim_{n\to\infty} \mu_2^{(n)} = 0 \ .
\end{aligned}$$

Proof: For proof see Schmidli [66, p. 74]. □

A very simple diffusion approximation can be obtained if we choose $\lambda_n = n\lambda$, $Z_i^{(n)} = Z_i/\sqrt{n}$, $\eta_n = \eta/\sqrt{n}$ and $x_n = x$. Note, that $Z_i^{(n)} = Z_i/\sqrt{n}$ implies $\mu^{(n)} = \mu/\sqrt{n}$ and $G_n(z) = G(\sqrt{n}z)$. Thus, it holds

$$C_t^{(n)} = x + \lambda\mu\sqrt{n}(1 + \eta/\sqrt{n})t - \sum_{i=1}^{N_t^{(n)}} Z_i/\sqrt{n} = x + \lambda\mu\eta t + \frac{\lambda\mu n t - \sum_{i=1}^{N_t^{(n)}} Z_i}{\sqrt{n}} \ .$$

Let $W = \{W_t\}$ be the standard Brownian motion again and μ_2 the second moment of claim sizes. With standard propositions from the theory of convergence of probability measures, see for example Billingsley [10], we obtain as a diffusion approximation

$$x + \lambda\mu\eta t + \sqrt{\lambda\mu_2}W_t \ .$$

We have constructed a sequence of risk processes that converges in probability to the risk process, whose random part is the well-known diffusion. The key assumption in our construction was that both the interarrival times as well as the claim severities are in the domain of attraction of stable distributions. This implies, that a diffusion approximation is good in the case, when an individual claim is very small compared to the size of the total reserve.

Assume now, that the insurer models his surplus as the classical risk model and buys proportional reinsurance with retention level $b \in [0,1]$. We assume also, that the retention level can be chosen only at the beginning, i.e. b remains constant over time. Then the surplus process C with initial capital x has the form

$$C_t = x + c(b)t - b\sum_{i=1}^{N_t} Z_i \ ,$$

where $c(b)$ denotes the premium remaining to the insurer. Using the expected value principle, see (1.3), yields

$$C_t = x + \lambda\mu(\eta - \theta + b\theta + b)t - b\sum_{i=1}^{N_t} Z_i \ ,$$

where η is the safety loading of the insurer and θ the safety loading of the reinsurer. Constructing a sequence of classical risk models in the way described above, we obtain as a diffusion approximation

$$X_t = x + \lambda\mu(\eta - \theta + b\theta)t + b\sqrt{\lambda\mu_2}W_t \ ,$$

where W is again the standard Brownian motion. If we assume, that the insurer can change his retention level continuously, i.e. we consider a process $B = \{b_t\}$, then we obtain a diffusion approximation by means of the following proposition.

Proposition 1.2.7
Let $m : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function, $(X^{(n)})_{n\geq 1}$ be a sequence of semimartingales, such that $X_0^{(n)} = 0$, and $Z^{(n)}$ be defined by

$$Z_t^{(n)} = x + X_t^{(n)} + \int_0^t m(Z_s^{(n)}) \, \mathrm{d}s \ .$$

Let further X be a semimartingale with $X_0 = 0$ and Z be defined by

$$Z_t = x + X_t + \int_0^t m(Z_t) \, \mathrm{d}s \ .$$

Then $Z^{(n)}$ converges weakly to Z if and only if $X^{(n)}$ converges weakly to X.

Proof: For proof see Schmidli [67]. □

A diffusion approximation has then the form

$$X_t^B = x + \lambda\mu\int_0^t (\eta - \theta + b_s\theta) \, \mathrm{d}s + \sqrt{\lambda\mu_2}\int_0^t b_s \, \mathrm{d}W_s \ .$$

Remark 1.2.8
In a diffusion approximation the event, that the diffusion approximation process X_t ever falls below zero, is called ruin. The time τ, when the process X_t falls below zero for the first time

$$\tau = \inf\{t > 0 : \ X_t > 0\}$$

is called ruin time. The probability of ruin for the initial capital x is

$$\psi(x) = \mathbb{P}[\tau < \infty | X_0 = x] \ .$$

1.3 On the notation

The following conventions will be applied.

1. In the following we will always use a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all stochastic quantities are defined. We always assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, i.e. $\mathcal{F}$ contains all $\mathbb{P}$-null sets. At the beginning of each section we give a short description of the model we are going to consider. In each model we are given a filtration $\mathbb{F} = \{\mathcal{F}_t\}$. In the following we always assume, that a filtration $\mathbb{F} = \{\mathcal{F}_t\}$ is right continuous, i.e. $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $0 \leq t < \infty$. We do not assume that the given filtration is complete because otherwise we could have problems with change of measure techniques. To simplify the notation we will often omit the expressions almost surely (a.s.) or with probability one.

2. In order to emphasise that the considered process has initial value x we write for the corresponding expected value operator $\mathbb{E}_x[.]$.

3. For the convenience of reading a list of principal notation is given on page 9.

4. An introduction to the optimal control theory goes beyond the scope of this work. Considering the models we simultaneously give an explanation of the optimal control concept concerning each special optimisation problem. To avoid misunderstandings and ambiguity of the notation, we give here, without going into details, some definitions we will use throughout the work.

   The concept of control can be described as the process of influencing the behavior of a dynamical system to achieve a desired goal. Consider some stochastic process $X = \{X_t\}$. Let further $U = \{U_t\}$ be a variable which models the decision process.

   We will say that U is a **feedback control** if $U_t = L(X_t)$ for $t \geq 0$, for a measurable $L : \mathbb{R} \to \mathbb{R}$.

   We call a function, which is associated to some control strategy **return function**. As a **value function** we will denote a function given as an infimum over all return functions.

2 Optimal Control of Capital Injections by Reinsurance

2.1 Diffusion Approximation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, that is large enough to carry all the stochastic objects defined in this section. By $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ we denote the natural filtration of the standard Brownian motion W. The section is organised as follows. In Subsection 2.1.1 we review some results on reflected diffusion processes and calculate the value of an "uncontrolled" surplus process. In Subsection 2.1.2 we introduce the diffusion approximation and add the possibility of reinsurance. We solve the problem of minimising of the expected discounted capital injections. In order to illustrate the results, we give in Subsection 2.1.3 some examples, where we can calculate the value function explicitly.

2.1.1 Reflected Diffusion Processes

In this subsection we consider reflected one-dimensional diffusion processes. These considerations will be used in the analysis of the control problems through out the whole work. Consider the diffusion process X with the state space $\mathbb{R}$ and dynamics

$$\mathrm{d}X_t = m(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\;, \tag{2.1}$$

where the functions m and σ are chosen in such a way that the above stochastic differential equation has a unique strong solution. This is for example the case if m and σ are chosen like in Theorem 1.2.3.

We are interested in the minimal value of the expected discounted capital injections, which are necessary to keep the process X above zero. The data for our problem are the drift rate m, the volatility σ and the discounting rate $\delta \geq 0$. As it has been already explained in Preface, the discounting factor $\delta \geq 0$ expresses the investment preferences of the company holders. $\delta \geq 0$ implies that investing tomorrow is preferred to investing today. It is clear that in this setup for initial value $X_0 < 0$ it will be optimal just to inject $-X_0$. Thus, our attention will be restricted to $x \geq 0$.

It is clear that if X is at zero, we must apply control to stop the process entering $(-\infty, 0)$.

We interpret Y_t as the cumulative capital injections up to time t and associate with $Y = \{Y_t\}$ the controlled process

$$X_t^Y = x + \int_0^t m(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + Y_t\,.$$

Because the discounting factor δ is non-negative, we should inject capital only when the process becomes negative and only as much, that the process is again shifted to zero. It means Y changes only on the set $[X^Y = 0]$ and because $x \ge 0$ it holds $Y_0 = 0$. Thus, the above equation is an equation of Skorokhod type. In Appendix D we find, that the unique solution Y is the **local time** of X at zero and X^Y is a reflected diffusion, which follows the stochastic differential equation

$$\mathrm{d}X_t^Y = m(X_t^Y)\,\mathrm{d}t + \sigma(X_t^Y)\,\mathrm{d}W_t + \mathrm{d}Y_t\,.$$

After we have found the optimal strategy, we are interested in calculating the value function $V(x)$. It is clear that $V(x)$ is a decreasing function. Because of the discounting we also have that $\lim\limits_{x\to\infty} V(x) = 0$. Because the capital injections start at time zero it holds $V(x) = -x + V(0)$ for $x < 0$. We therefore can restrict to positive initial capital.

Consider now the martingale $\{\int_0^t e^{-\delta s}\,\mathrm{d}Y_s + e^{-\delta t}V(X_t^Y)\}$

$$\mathbb{E}_x\Big[\int_0^\infty e^{-\delta s}\,\mathrm{d}Y_s \,\Big|\, \mathcal{F}_u\Big] = \int_0^u e^{-\delta s}\,\mathrm{d}Y_s + e^{-\delta u}V(X_u^Y)\,.$$

Suppose that $V(x)$ is twice continuously differentiable. By Itô's formula, see Proposition C.1.1 p. 171,

$$\begin{aligned} e^{-\delta t}V(X_t^Y) &= \int_0^t e^{-\delta s}\Big(\frac{\sigma^2(X_s^Y)}{2}V''(X_s^Y) + m(X_s^Y)V'(X_s^Y) - \delta V(X_s^Y)\Big)\,\mathrm{d}s \\ &\quad + \int_0^t e^{-\delta s}V'(X_s^Y)\sigma(X_s^Y)\,\mathrm{d}W_s + \int_0^t e^{-\delta s}V'(X_s^Y)\,\mathrm{d}Y_s + V(x)\,. \end{aligned}$$

Rearranging the terms and adding $\int_0^t e^{-\delta s}\,\mathrm{d}Y_s$ on both sides of the equation yields

$$\begin{aligned} & e^{-\delta t}V(X_t^Y) + \int_0^t e^{-\delta s}\,\mathrm{d}Y_s - \int_0^t e^{-\delta s}V'(X_s^Y)\sigma(X_s^Y)\,\mathrm{d}W_s \\ = &\int_0^t e^{-\delta s}\Big(\frac{\sigma^2(X_s^Y)}{2}V''(X_s^Y) + m(X_s^Y)V'(X_s^Y) - \delta V(X_s^Y)\Big)\,\mathrm{d}s \\ &+ \int_0^t e^{-\delta s}\big(1 + V'(X_s^Y)\big)\,\mathrm{d}Y_s \end{aligned}$$

Because we assume that X is the unique strong solution to Equation (2.1), see Chapter D, and $V'(X_t^Y)$ is locally bounded because of the continuity, we obtain that

$\int_0^t e^{-\delta s} V'(X_s^Y)\sigma(X_s^Y)\ \mathrm{d}W_s$ is a local martingale. By means of optional stopping time theorem we obtain that

$$\Big\{\int_0^t e^{-\delta s}\big(1+V'(X_s^Y)\big)\,\mathrm{d}Y_s + \int_0^t e^{-\delta s}\Big(\frac{\sigma^2(X_s^Y)}{2}V''(X_s^Y)+m(X_s^Y)V'(X_s^Y)-\delta V(X_s^Y)\Big)\,\mathrm{d}s\Big\}$$

is a local martingale. Note that since $Y=\{Y_s\}$ is an increasing process, it is of bounded variation. Thus the above local martingale is of bounded variation and therefore constant, say equal to C, see Proposition A.2.5 p. 166. Because for $t=0$ the above local martingale is zero, it follows $C=0$. On the other hand $\{Y_s\}$ only increases on the set $\{X_t^Y=0\}$. Define the stopping time $\tau=\inf\{t\geq 0: X_t<0\}$. Then we obtain

$$\mathbb{E}\Big[\int_0^{\tau\wedge t} e^{-\delta s}\Big(\frac{\sigma^2(X_s^Y)}{2}V''(X_s^Y)+m(X_s^Y)V'(X_s^Y)-\delta V(X_s^Y)\Big)\,\mathrm{d}s\Big\}\Big]=0\,.$$

Dividing by t and letting t go to zero we get by bounded convergence the equation

$$\frac{\sigma^2(x)}{2}V''(x)+m(x)V'(x)-\delta V(x)=0 \tag{2.2}$$

for $x>0$. By continuity the equation also holds for $x=0$, implying $V'(0)=-1$. In particular, $V(x)$ is at 0 continuously differentiable from the right.

Suppose $\lim\limits_{x\to\infty} f(x)=0$, $f(x)$ is twice continuously differentiable and solves (2.2). Let $T_n=\inf\{t: X_t^Y>n\}$. Because as a continuous function, $f'(x)$ is bounded on $[0,n]$ the process $\{\int_0^{T_n\wedge t} f'(X_s^Y)\sigma(X_s^Y)\,\mathrm{d}W_s\}$ is a martingale, see Proposition A.2.6 p. 166. Note further, that $\int_0^t e^{-\delta s}f'(X_s^Y)\,\mathrm{d}Y_s=f'(0)\int_0^t e^{-\delta s}\,\mathrm{d}Y_s$, because Y_t only increases if $X_t^Y=0$. Thus,

$$\Big\{e^{-\delta(T_n\wedge t)}f(X_{T_n\wedge t}^Y)-f'(0)\int_0^{T_n\wedge t}e^{-\delta s}\,\mathrm{d}Y_s\Big\}$$

is a martingale by (2.2). In particular,

$$f(x)=\mathbb{E}_x\Big[e^{-\delta(T_n\wedge t)}f(X_{T_n\wedge t}^Y)-f'(0)\int_0^{T_n\wedge t}e^{-\delta s}\,\mathrm{d}Y_s\Big]\,.$$

By monotone and bounded convergence we can let n and t tend to infinity, giving

$$f(x)=-f'(0)\mathbb{E}_x\Big[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t\Big]\,,$$

This was first shown by Shreve et al. [72].

Shreve et al. [72] also found an explicit solution for a Brownian motion with drift. We review this example, because we will use this result in the following.

Example 2.1.1
Consider the Brownian motion with drift

$$X_t = x + mt + \sigma W_t \; .$$

Equation (2.2) reads

$$\frac{\sigma^2}{2} V''(x) + mV'(x) - \delta V(x) = 0 \; . \tag{2.3}$$

The solution is of the form $V(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$, where $r_1 > 0 > r_2$ are the roots of

$$\sigma^2 r^2 + 2mr - 2\delta = 0 \; ,$$

see Proposition E.1.4. We have to select c_1, c_2 such that

$$V(\infty) = 0, \quad V'(0) = -1 \; .$$

That is, $c_1 = 0$ and $c_2 = \frac{\sigma^2}{m+\sqrt{m^2+2\delta\sigma^2}}$. Thus, the value function has in this case the form

$$V(x) = \begin{cases} \frac{\sigma^2}{m+\sqrt{m^2+2\delta\sigma^2}} \exp\bigl(-\frac{m+\sqrt{m^2+2\delta\sigma^2}}{\sigma^2} x\bigr) & : \; x \ge 0 \; , \\ \frac{\sigma^2}{m+\sqrt{m^2+2\delta\sigma^2}} - x & : \; x < 0 \; . \end{cases}$$

Because this function is twice continuously differentiable on $(0, \infty)$, it is the solution to our problem by the above general considerations. ■

2.1.2 Capital Injections and Reinsurance

We consider a surplus process where the mean number of claims in a time unit is λ and the mean size of a claim is $\mu = \mathbb{E}[Z]$, where Z is a generic claim size. We assume that $\mu_2 = \mathbb{E}[Z^2] < \infty$. The premium is $c = (1+\eta)\lambda\mu$ for some $\eta > 0$. The insurer can now buy reinsurance. That is, a retention level $b \in [0, \tilde{b}]$ has to be chosen, where $\tilde{b} \in (0, \infty]$. The cedent pays then $0 \le r(Z, b) \le Z$. The reinsurance premium is calculated by an expected value principle $(1+\theta)\lambda\bigl(\mu - \mathbb{E}[r(Z, b)]\bigr)$; i.e. the premium rate remaining for the insurer is $c(b) = \lambda(1+\theta)\mathbb{E}[r(Z, b)] - \lambda\mu(\theta - \eta)$. Here $b = 0$ means "full reinsurance" and $b = \tilde{b}$ means "no reinsurance". We assume for simplicity that $r(z, b)$ is increasing and continuous in b. In order that the problem considered below does not have the trivial solution $V(x) = |x| 1\!\!1_{[x<0]}$ we assume that $\theta > \eta$. Here $1\!\!1_A$ denotes the indicator function.

For a constant strategy $B \equiv b$, the diffusion approximation to the surplus process is

$$X_t^b = \bigl\{x + \lambda(\theta\mathbb{E}[r(Z, b)] - (\theta - \eta)\mu)t + \sqrt{\lambda\mathbb{E}[r(Z, b)^2]} W_t\bigr\} \; , \tag{2.4}$$

see Section 1.2. As an extension the insurer can continuously change the reinsurance leading to the process

$$X_t^B = x + \lambda \int_0^t \bigl(\theta\mathbb{E}[r(Z, b_s)] - (\theta - \eta)\mu\bigr) \, \mathrm{d}s + \int_0^t \sqrt{\lambda\mathbb{E}[r(Z, b_s)^2]} \, \mathrm{d}W_s \; . \tag{2.5}$$

Here $B = \{b_t\}$ is an admissible strategy, i.e. adapted reinsurance strategy with $b_t \in [0, \tilde{b}]$. The set of admissible strategies we denote by $\mathscr{U}$. In the following we will denote the time of ruin by $\tau_x^B = \inf\{t \geq 0 : X_t^B < 0,\ X_0^B = x\}$. Adding the capital injections gives

$$X_t^{B,Y} = X_t^B + Y_t \ .$$

The return function associated to a reinsurance strategy B is $V^B(x) = \mathbb{E}_x[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t^B]$, where Y_t^B denotes the local time at 0 of $\{X_t^B\}$. We minimise $V^B(x)$ over all admissible reinsurance strategies B and let the value function be defined as

$$V(x) = \inf_{B \in \mathscr{U}} V^B(x) \ .$$

Because $B = \{\tilde{b} : 0 \leq t\}$ is an admissible strategy, we find from Example 2.1.1 that $0 \leq \lim\limits_{x\to\infty} V(x) \leq \lim\limits_{x\to\infty} V^{\tilde{b}}(x) = 0$. From Subsection 2.1.1 we know, that the capital injection process Y^B for the process (2.5) is given through the local time $Y_t^B = -\min\{\inf\limits_{0\leq s\leq t} X_s^B, 0\}$. Thus, we can write for the return function $V^B(x)$:

$$V^B(x) = \mathbb{E}_x\Big[\int_{\tau_x^B}^\infty e^{-\delta t}\,\mathrm{d}Y_t^B\Big] = V^{B'}(0)\mathbb{E}\big[e^{-\delta\tau_x^B}\big] \ ,$$

where B' is the strategy $B'_s = B_{\tau_x^B + s}$. If we choose an ε-optimal strategy B' for initial capital 0 we have

$$V(0)\mathbb{E}\big[e^{-\delta\tau_x^B}\big] \leq V^B(x) \leq (V(0)+\varepsilon)\mathbb{E}\big[e^{-\delta\tau_x^B}\big] \ .$$

In order to minimise $V^B(x)$ we have to minimise $\mathbb{E}[e^{-\delta\tau_x^B}]$ over all admissible strategies B. Let $L^B(x) = \mathbb{E}[e^{-\delta\tau_x^B}]$ and $L(x) = \inf\limits_{B\in\mathscr{U}} \mathbb{E}[e^{-\delta\tau_x^B}]$. It is clear that $L(x)$ is a decreasing function and $\lim\limits_{x\to\infty} L(x) = 0$.

We assume for the moment, that the optimal reinsurance strategy exists and denote it by B^*, i.e. $L(x) = L^{B^*}(x)$. Let $x, y \in \mathbb{R}_+$. If we start with the initial value $x+y$ we have to cross the level y before $\tau_{x+y}^{B^*}$. This will happen at the time $\tau_x^{B^*}$, i.e. at the ruin time of the process with initial value x. Because the paths are continuous the process with initial capital $x+y$ will have almost surely the value y at the time $\tau_x^{B^*}$. Thus, we can write for the value function:

$$\begin{aligned} L^{B^*}(x+y) &= \mathbb{E}[e^{-\delta\tau_x^{B^*}} e^{-\delta(\tau_{x+y}^{B^*} - \tau_x^{B^*})}] = \mathbb{E}[e^{-\delta\tau_x^{B^*}}]\mathbb{E}[e^{-\delta(\tau_{x+y}^{B^*} - \tau_x^{B^*})}] \\ &= L^{B^*}(x)L^{B^*}(y) \ . \end{aligned}$$

Here we used that for an optimal strategy we can minimise both terms independently, that the underlying process has independent increments and that $\tau_{x+y}^{B^*} - \tau_x^{B^*} = \tau_y^{B''}$, where $B''_t = B^*_{\tau_x^{B^*}+t}$. From the above equation we find that $L(x) = \exp(-\beta x)$ for some

$\beta > 0$. Dividing the interval $[0, x]$ in n intervals of equal length we have to solve the same optimisation problem in each interval $[kx/n, (k+1)x/n]$. This implies that the optimal strategy should be constant. The above argument shows that also for any constant strategy $L^b(x) = \exp(-\beta(b)x)$ for some $\beta(b)$. We therefore need to maximise $\beta(b)$.

We now use the stochastic control approach and motivate the Hamilton–Jacobi–Bellman equation for $x \geq 0$. Let $h > 0$ be very small. Let $b \in [0, \tilde{b}]$ and τ^b be the time of ruin of the process X^b, defined by (2.4), with initial capital x. Suppose further that for each capital x_h at time h there is an admissible strategy $B^\varepsilon = \{b_t^\varepsilon\}$ such that $V(x_h) + \varepsilon > V^{B^\varepsilon}(x_h)$. Construct a strategy $\tilde{B} = \{\tilde{b}_t\}$ as follows. Let $\tilde{b}_t = b$ for $t \leq \tau^b \wedge h$ and $\tilde{b}_{t+h\wedge\tau^b} = b_t^\varepsilon$. Then we have

$$V(x) \leq V^{\tilde{B}}(x) = \mathbb{E}\Big[V^{B^\varepsilon}(X^b_{\tau^b\wedge h})e^{-\delta(\tau^b\wedge h)}\Big] \leq \mathbb{E}\Big[V(X^b_{\tau^b\wedge h})e^{-\delta(\tau^b\wedge h)}\Big] + \varepsilon\;.$$

Because ε was arbitrary we can let it be equal to zero. Suppose now that the value function $V(x)$ is twice continuously differentiable. Then we can apply Ito's formula, see Chapter C, and obtain

$$\begin{aligned} e^{-\delta(\tau^b\wedge h)}V(X^b_{\tau^b\wedge h}) - V(x) \quad &= \int_0^{\tau^b\wedge h} e^{-\delta s}\Big\{\big(\lambda\theta\mathbb{E}[r(Z,b)] - \lambda\mu(\theta-\eta)\big)V'(X_s^b) \\ &\quad + \frac{\lambda\mathbb{E}[r(Z,b)^2]}{2}V''(X_s^b) - \delta V(X_s^b)\Big\}\,\mathrm{d}s \\ &\quad + \int_0^{\tau^b\wedge h}\sqrt{\lambda\mathbb{E}[r(Z,b)^2]}V'(X_s^b)\,\mathrm{d}W_s\;. \end{aligned}$$

Suppose the stochastic integral is a true martingale. Taking expectation yields

$$\begin{aligned} 0 \leq \mathbb{E}\Big[\int_0^{\tau^b\wedge h} e^{-\delta s}\Big\{\big(\lambda\theta\mathbb{E}[r(Z,b)] - \lambda\mu(\theta-\eta)\big)V'(X_s^b) + \frac{\lambda\mathbb{E}[r(Z,b)^2]}{2}V''(X_s^b) \\ -\delta V(X_s^b)\Big\}\,\mathrm{d}s\Big]\;. \end{aligned}$$

Dividing by h and letting h go to zero yields, provided the limit and expectation can be interchanged:

$$\big(\lambda\theta\mathbb{E}[r(Z,b)] - \lambda\mu(\theta-\eta)\big)V'(x) + \frac{\lambda\mathbb{E}[r(Z,b)^2]}{2}V''(x) - \delta V(x) \geq 0\;.$$

This equation must hold for all $b \in [0, \tilde{b}]$. thus

$$\inf_{b\in[0,\tilde{b}]}\big(\lambda\theta\mathbb{E}[r(Z,b)] - \lambda\mu(\theta-\eta)\big)V'(x) + \frac{\lambda\mathbb{E}[r(Z,b)^2]}{2}V''(x) - \delta V(x) \geq 0\;.$$

Suppose further there is an optimal strategy $B^* = \{b_t^*\}$ such that $\lim_{t\to 0} b_t^* = b$, then we can conclude similarly as above

$$0 = \mathbb{E}\Big[\int_0^{\tau^{B^*}\wedge h} e^{-\delta s}\Big\{\big(\lambda\theta\mathbb{E}[r(Z,b_s^*)] - \lambda\mu(\theta-\eta)\big)V'(X_s^{B^*}) + \frac{\lambda\mathbb{E}[r(Z,b_s^*)^2]}{2}V''(X_s^{B^*}) \\ -\delta V(X_s^{B^*})\Big\}\,\mathrm{d}s\Big]\;.$$

Dividing by h and letting h go to zero we obtain

$$\big(\lambda\theta\mathbb{E}[r(Z,b)] - \lambda\mu(\theta-\eta)\big)V'(x) + \frac{\lambda\mathbb{E}[r(Z,b)^2]}{2}V''(x) - \delta V(x) = 0\;.$$

Therefore, the Hamilton–Jacobi–Bellman (HJB) equation for $V(x)$ for $x \ge 0$ is,

$$\inf_{b\in[0,\tilde{b}]} \frac{\lambda\mathbb{E}[r(Z,b)^2]}{2}V''(x) + \lambda\big(\theta\mathbb{E}[r(Z,b)] - (\theta-\eta)\mu\big)V'(x) - \delta V(x) = 0\;. \tag{2.6}$$

Plugging in the solution $V(x) = V(0)e^{-\beta x}$ yields

$$\inf_{b\in[0,\tilde{b}]} \frac{\lambda\mathbb{E}[r(Z,b)^2]}{2}\beta^2 - \lambda\big(\theta\mathbb{E}[r(Z,b)] - (\theta-\eta)\mu\big)\beta - \delta = 0\;. \tag{2.7}$$

The above expression is continuous in b and therefore there is a value b^* where the minimal value is attained.

Similarly, for a constant strategy b, the function $V(x) = (\beta(b))^{-1}e^{-\beta(b)x}$ plugged in Equation (2.3) yields the equation

$$\frac{\lambda\mathbb{E}[r(Z,b)^2]}{2}\beta(b)^2 - \lambda\big(\theta\mathbb{E}[r(Z,b)] - (\theta-\eta)\mu\big)\beta(b) - \delta = 0\;, \tag{2.8}$$

from which we can find $\beta(b)$. We can then maximise $\beta(b)$. One readily can see from the convexity of (2.8) in β that (2.7) and (2.8) yield the same solution and the same minimiser b^* of (2.7) and the same maximiser b^* of $\beta(b)$, respectively.

Define now

$$\alpha(b) = \lambda\big[\theta\mathbb{E}[r(Z,b)] - (\theta-\eta)\mu\big]\;.$$

Then we can express $\beta(b)$ as

$$\beta(b) = \frac{\alpha(b)}{\lambda\mathbb{E}[r(Z,b)^2]} + \sqrt{\Big(\frac{\alpha(b)}{\lambda\mathbb{E}[r(Z,b]^2)}\Big)^2 + \frac{2\delta}{\lambda\mathbb{E}[r(Z,b)^2]}} > 0\;.$$

We see that there are two possibilities to calculate $V(x)$ and b^*: Either, we can solve (2.7) directly or we can maximise $\beta(b)$.

Part of our derivation of $V(x)$ was heuristically. In order to show that our solution is correct, we prove the following verification theorem.

Theorem 2.1.2
The constant strategy $b_t^* = b^*$ is an optimal reinsurance strategy, where b^* is a maximum point of the function $\beta(b)$ on the set $[0, \tilde{b}]$. The value function is given by $V(x) = (\beta(b^*))^{-1} \exp(-\beta(b^*)x)$ for $x \geq 0$ and $V(x) = (\beta(b^*))^{-1} - x$ for $x < 0$.

Proof: We have shown in Example 2.1.1 that the constant strategy $b_t^* = b^*$ yields the value $V^{b^*}(x) = V(x)$. Consider now an arbitrary reinsurance strategy $B = \{b_t\}$. The corresponding surplus process is

$$X_t^B = x + \lambda \int_0^t \big(\theta \mathbb{E}[r(Z, b_s)] - (\theta - \eta)\mu\big)\, \mathrm{d}s + \int_0^t \sqrt{\lambda \mathbb{E}[r(Z, b_s)^2]}\, \mathrm{d}W_s\,.$$

Because $V(x)$ is twice continuously differentiable on $(0, \infty)$ we can apply Ito's formula (note that $X_t^{B,Y} \geq 0$ and therefore we can change $V(x)$ on $(-\infty, 0)$ without changing $V(X_t^{B,Y})$):

$$\begin{aligned} e^{-\delta t} V(X_t^{B,Y}) &= V(x) + \int_0^t e^{-\delta s} V'(X_s^{B,Y})\, \mathrm{d}Y_s^B \\ &\quad + \int_0^t e^{-\delta s} [D_{s,B} V(X_s^{B,Y}) - \delta V(X_s^{B,Y})]\, \mathrm{d}s \\ &\quad + \int_0^t e^{-\delta s} V'(X_s^{B,Y}) \sqrt{\lambda \mathbb{E}[r(Z, b_s)^2]}\, \mathrm{d}W_s \\ &\geq V(x) - \int_0^t e^{-\delta s}\, \mathrm{d}Y_s^B \\ &\quad + \int_0^t e^{-\delta s} V'(X_s^{B,Y}) \sqrt{\lambda \mathbb{E}[r(Z, b_s)^2]}\, \mathrm{d}W_s\,, \end{aligned}$$

where

$$D_{s,B} V(x) = \frac{1}{2} \lambda \mathbb{E}[r(Z, b_s)^2] V''(x) + \lambda \big[\theta \mathbb{E}[r(Z, b_s)] - (\theta - \eta)\mu\big] V'(x)$$

denotes the infinitesimal generator of the process $X_t^{b_s}$, compare Proposition B.2.5, p. 170. We have used (2.6) and the fact that Y_t^B only increases on the set $\{X_t^{B,Y} = 0\}$. Because $|V'(x)| \leq 1$ we obtain that the last term is a martingale with mean value 0. Thus

$$V(x) \leq e^{-\delta t} \mathbb{E}_x\big[V(X_t^{B,Y})\big] + \mathbb{E}_x\Big[\int_0^t e^{-\delta s}\, \mathrm{d}Y_s^B\Big]\,.$$

Because $0 \leq V(X_t^{B,Y}) \leq V(0)$, we get by monotone convergence

$$V(x) \leq \mathbb{E}_x\Big[\int_0^\infty e^{-\delta s}\, \mathrm{d}Y_s^B\Big]\,.$$

This proves the result. $\square$

Remark 2.1.3
Define for a constant b the exponent

$$\tilde{\beta}(b) = \frac{\alpha(b)}{\lambda \mathbb{E}[r(Z,b)^2]} - \sqrt{\Big(\frac{\alpha(b)}{\lambda \mathbb{E}[r(Z,b]^2)}\Big)^2 + \frac{2\delta}{\lambda \mathbb{E}[r(Z,b)^2]}} < 0 \; .$$

Note that $\frac{1}{2}\lambda \mathbb{E}[r(Z,b)^2]\tilde{\beta}^2(b) - \alpha(b)\tilde{\beta}(b) = \delta$. We consider the martingale

$$M_t = \exp(-\tilde{\beta}(b)(X_t^b - x) - \delta t) \; .$$

Note that $\mathbb{E}[M_t] = 1$. Define the new measure on $\mathcal{F}_t$ by $\nu[A] = \int_A M \, \mathrm{d}\mathbb{P}$ for $A \in \mathcal{F}_t$. The measure ν can be extended to a measure on $\mathcal{F}$. For the details see Chapter C, Rolski et al. [61, p. 461] or Schmidli [70, p. 215]. By Girsanov's theorem, see Chapter C, $\{X_t^b\}$ is under ν again a Brownian motion with drift, but the drift to infinity is stronger than under $\mathbb{P}$. By changing the measure we obtain

$$\begin{aligned} L^b(x) &= \mathbb{E}[e^{-\delta \tau_x^b} \mathbb{1}_{[\tau_x^b < \infty]}] = \mathbb{E}_\nu\big[e^{\tilde{\beta}(b) X^b_{\tau_x^b}} \mathbb{1}_{[\tau_x^b < \infty]}\big] e^{-\tilde{\beta}(b)x} \\ &= \mathbb{E}_\nu[\mathbb{1}_{[\tau_x^b < \infty]}] e^{-\tilde{\beta}(b)x} = \nu[\tau_x^b < \infty] e^{-\tilde{\beta}(b)x} \; . \end{aligned}$$

Thus, our problem could be considered as a minimisation of a penalised ruin probability. It is therefore not surprising that the optimal strategy becomes constant as in the diffusion problem in Schmidli [68].

2.1.3 Examples

We now illustrate our result by explicit examples.

Proportional Reinsurance

We consider the case of proportional reinsurance. The self-insurance function has the form $r(Z,b) = bZ$ for some $b \in [0,1]$. In particular, $\mathbb{E}[r(Z,b)] = b\mu$ and $\mathbb{E}[r(Z,b)^2] = b^2\mu_2$. The corresponding Brownian motion becomes

$$X_t^b = x + \lambda\mu\big(b\theta - (\theta - \eta)\big)t + b\sqrt{\lambda\mu_2} W_t \; .$$

Equation (2.7) reads

$$\inf_{b \in [0,1]} \frac{\lambda\mu_2}{2} b^2\beta^2 - \lambda\mu\big(\theta b - (\theta - \eta)\big)\beta - \delta = 0 \; .$$

The minimum is taken at

$$b^* = \frac{\mu\theta}{\mu_2\beta} \wedge 1 \; .$$

Suppose for the moment that $b^* < 1$. In oder to get the optimal β we have to solve the equation

$$\lambda\mu(\theta-\eta)\beta - \frac{\lambda\mu^2\theta^2}{2\mu_2} - \delta = 0\ .$$

The solution is

$$\beta = \frac{\lambda\mu^2\theta^2 + 2\delta\mu_2}{2\mu_2\lambda\mu(\theta-\eta)}\ , \tag{2.9}$$

giving

$$b^* = \frac{2\lambda\mu^2\theta(\theta-\eta)}{\lambda\mu^2\theta^2 + 2\delta\mu_2} \wedge 1\ .$$

This is smaller than 1 if

$$\lambda\mu^2\theta^2 < 2(\delta\mu_2 + \lambda\mu^2\theta\eta)\ .$$

We could write the latter condition as

$$\eta < \theta < \eta + \sqrt{\eta^2 + \frac{2\delta\mu_2}{\lambda\mu^2}}\ .$$

For $b^* = 1$ we have from Example 2.1.1 that

$$\beta = \frac{\lambda\mu\eta + \sqrt{\lambda^2\mu^2\eta^2 + 2\lambda\delta\mu_2}}{\lambda\mu_2}\ .$$

We now consider numerical examples.

Choose for example $\theta = 0.5$, $\eta = 0.3$, $\lambda = 1$, $\delta = 0.04$, $\mu = 1$ and $\mu_2 = 2$. Then we have $b^* = 0.4878$ and $\beta = 0.5125$.

Choosing $\delta = 0$ corresponds to minimising the ruin probability. In this case our findings coincide with the result in Schmidli [70, p. 37]: $b^* = 2\big(1 - \frac{\eta}{\theta}\big) \wedge 1$, i.e. $b^* = 0.8$ and $\beta = 0.3125$. This b^* differs very much from the optimal b with $\delta = 0.04$, and the corresponding value function for $\delta = 0$ lies above the value function for $\delta = 0.04$. We see that discounting has a large influence.

Figure 2.1 compares the value of the optimal strategy with the value in the case without reinsurance, $b = 1$. We can see that buying reinsurance lowers the costs considerably.

In the above example we obtained a strategy b^* such that the drift coefficient of the corresponding Brownian motion was positive. But it may be possible that the drift of the Brownian motion, corresponding to the optimal constant reinsurance strategy, would be negative. It means, that the risk of the portfolio is so large that, because of the discounting, the insurer would be willing to “throw away” money in order to prevent early capital injections. Of course, the insurer should better sell the business in this case. The result is possible because we do not allow the option “no business at all”, which of course has the minimal costs 0.

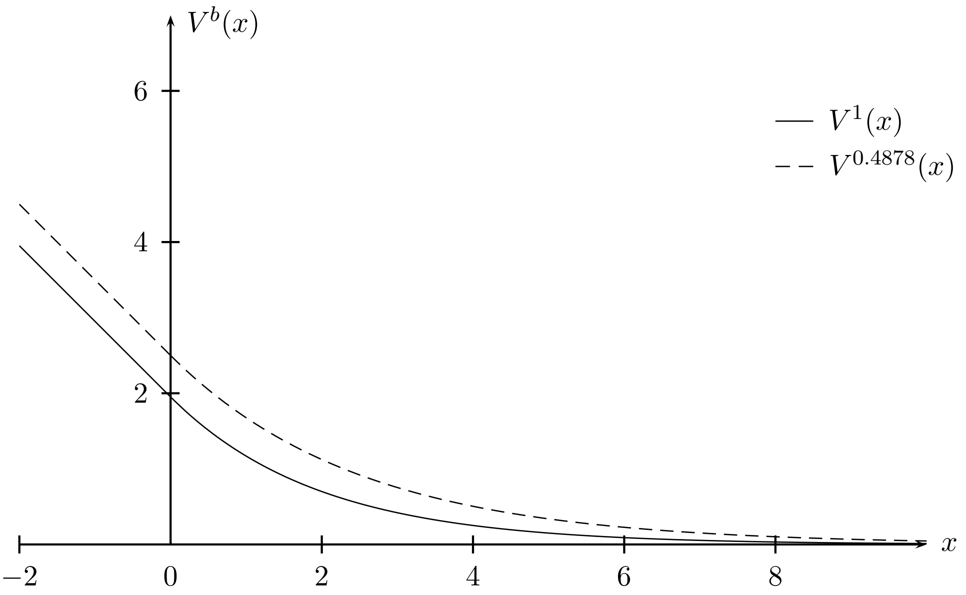

Figure 2.1: Value function with $B^* \equiv 0.4878$ and return function for $B \equiv 1$ for proportional reinsurance.

With parameters $\theta = 0.5$, $\eta = 0.3$, $\lambda = 0.05$, $\delta = 0.04$, $\mu = 20$ and $\mu_2 = 800$ we obtain $b^* = 0.058 < 1$ and $\beta = 0.216$. It is easy to verify, that the drift coefficient is now smaller than zero. Figure 2.2 gives the value functions of the optimal strategy $b^* = 0.058$, for the strategy $b = 1 - \frac{\eta}{\theta} = 0.4$ (the maximal reinsurance such that the drift remains positive) and no reinsurance $b = 1$. We see that reinsurance lowers the costs considerably. Optimal reinsurance moves the costs to the future by lowering the diffusion coefficient. The problem with the non-controlled process is that quite likely there will be early costs. This costs have a larger weight than the future costs in the controlled process.

In the case of proportional reinsurance it was possible to find an optimal strategy b^* without knowing explicitly the claim size distribution. We only needed to know the first two moments. The next example shows, that it will not be always the case.

Excess of Loss Reinsurance

We choose now $r(Z,b) = \min\{Z,b\}$ for $b \in [0,\infty]$. Then we have $\mathbb{E}[r(Z,b)] = \int_0^b (1 - G(x))\,\mathrm{d}x$ and $\mathbb{E}[r(Z,b)^2] = \int_0^b 2x(1-G(x))\,\mathrm{d}x$, where $G(x)$ is the claim size distribution. Equation (2.7) reads

$$\inf_{b\in[0,\infty]} \lambda \int_0^b x(1-G(x))\,\mathrm{d}x\beta^2 - \lambda\Big(\theta \int_0^b (1-G(x))\,\mathrm{d}x - (\theta-\eta)\mu\Big)\beta - \delta = 0\,.$$

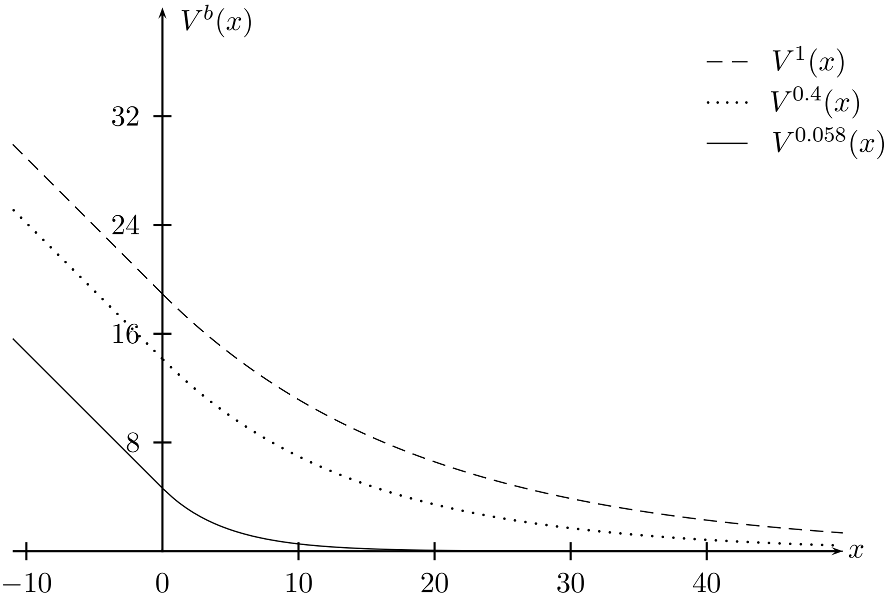

Figure 2.2: Value function with $B^* \equiv 0.058$ and return functions for $B \equiv 0.4$ and $B \equiv 1$ for proportional reinsurance.

The derivative with respect to b is negative for $b < \theta/\beta$ and positive for $b > \theta/\beta$ provided $G(\theta/\beta) < 1$. In any case, the minimum is taken at $b^* = \theta/\beta$. For solving the equation we can let $\beta = \theta/b$ and solve for b:

$$\lambda \int_0^b x(1 - G(x))\,\mathrm{d}x - \lambda b\Big(\int_0^b (1 - G(x))\,\mathrm{d}x - b_0\mu\Big) - \frac{\delta b^2}{\theta^2} = 0\ , \tag{2.10}$$

where $b_0 = 1 - \eta/\theta$. The left hand side of the equation is concave in b with value 0 at zero, derivative $\lambda b_0 \mu$ in zero, and tends to $-\infty$ as $b \to \infty$. Therefore, there is a unique $b^* > 0$ solving the equation. We thus found the solution

$$V(x) = \begin{cases} b^*\theta^{-1} e^{-\theta x/b^*} & : \text{ if } x \geq 0, \\ b^*\theta^{-1} - x & : \text{ if } x < 0. \end{cases}$$

We now ask when it is optimal not to reinsure the portfolio. From the derivations above we get that it is always optimal to buy reinsurance if the support of the claim size distribution is unbounded. Let $\tilde{b} = \sup\{x : \mathbb{P}(Z > x) > 0\}$ and suppose $\tilde{b} < \infty$. No reinsurance is chosen if $b^* \geq \tilde{b}$. This implies by the concavity of the left hand side of (2.10) that

$$\lambda \frac{\mu_2}{2} - \lambda \tilde{b}\mu(1 - b_0) - \frac{\delta \tilde{b}^2}{\theta^2} = \lambda \frac{\mu_2}{2} - \lambda \eta \mu \frac{\tilde{b}}{\theta} - \delta \frac{\tilde{b}^2}{\theta^2} \geq 0\ .$$

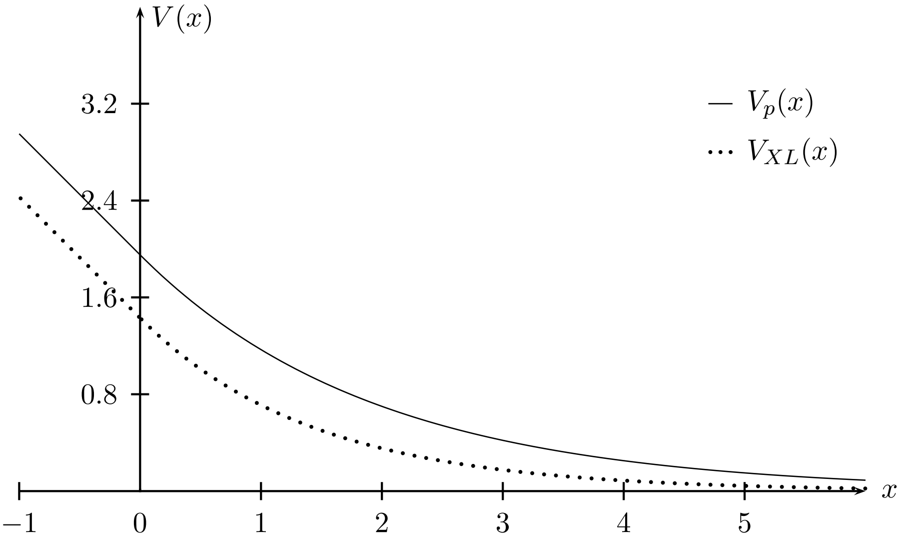

Figure 2.3: Value functions for proportional $V_p(x)$ and Excess of loss V_{XL} reinsurances.

This is equivalent to

$$\frac{\tilde{b}}{\theta} \leq \frac{\sqrt{\lambda^2\mu^2\eta^2 + 2\lambda\mu_2\delta} - \lambda\mu\eta}{2\delta},$$

or

$$\frac{\theta}{\tilde{b}} \geq \frac{\lambda\mu\eta + \sqrt{\lambda^2\mu^2\eta^2 + 2\lambda\mu_2\delta}}{\lambda\mu_2}.$$

We see that we do not reinsure if reinsurance is too expensive or if the maximal claim size is small. Let us now assume, that the claim sizes are exponentially distributed $\mathbf{Z} \sim \mathbf{Exp}(\frac{\mathbf{1}}{\mu})$.
Then the Equation (2.10) reads

$$\lambda \int_0^b x e^{-x/\mu}\,\mathrm{d}x - \lambda b \int_0^b e^{-x/\mu}\,\mathrm{d}x + \lambda b b_0 \mu - \frac{\delta b^2}{\theta^2} = 0\,,$$

which is equivalent to

$$-\lambda\mu^2 e^{-b/\mu} - \lambda\mu b \frac{\eta}{\theta} + \lambda\mu^2 - \frac{\delta b^2}{\theta^2} = 0.$$

Choosing as in Example 2.1.3, $\eta = 0.3$; $\theta = 0.5$; $\delta = 0.04$; $\mu = 1$; $\lambda = 1$, the above equation becomes

$$0 = -\exp(-b) - 0.6b + 1 - 0.16b^2 = 0\,.$$

Numerical solution yields $b^* = 0.715$, i.e. $\beta = 0.6993$. According to this, the value function is

$$V^{0.715}(x) = \begin{cases} \frac{1}{0.6993} \exp(-0.6993 \cdot x) & : \ x \geq 0 \,, \\ \frac{1}{0.6993} - x & : \ x < 0. \end{cases}$$

Gerber [29, p. 130] has shown that excess of loss reinsurance maximises the adjustment coefficient. Because $\beta(b)$ is a quantity similar to an adjustment coefficient it is not surprising that also in our model excess of loss is favourable to proportional reinsurance. Figure 2.3 shows the value functions for optimal proportional and excess of loss reinsurance.

2.2 The Classical Risk Model

From now on we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that should be large enough to carry all the stochastic quantities defined below. We start with introducing the model and calculate the expected discounted value of the capital injections if no reinsurance is taken. In Subsection 2.2.1 we will prove that the value function is Lipschitz continuous and therefore absolutely continuous. We show that the value function satisfies the Hamilton–Jacobi–Bellman equation. In Subsection 2.2.2 we illustrate the theory by some specific examples. Finally, Subsection 2.2.3 treats the special case of no discounting.

Consider the classical risk model. The number of claims is assumed to follow a Poisson process $\{N_t\}$ with intensity λ. Claim sizes $Z_1, Z_2, \ldots$ are independent of $\{N_t\}$, positive and iid with distribution function $G(z)$ and first moment μ. By Z we denote a generic random variable with the same distribution as Z_i. The aggregate claim process is given by

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i \,,$$

where x is the initial capital, $c = \lambda\mu(1+\eta)$ is the premium rate and η is the safety loading. We do not assume that $\eta > 0$ if the discounting parameter δ introduced below is strictly positive. The insurer can buy reinsurance. Choosing the level $b \in [0, \tilde{b}]$, the insurer pays $r(Z_i, b)$ for a claim of size Z_i. Here $b = 0$ means full reinsurance, $b = \tilde{b} \in (0, \infty]$ means no reinsurance. For example, for proportional reinsurance $r(Z, b) = bZ$ and $b \in [0, 1]$. For excess of loss reinsurance we obtain $r(Z, b) = \min\{Z, b\}$ and $b \in [0, \infty]$.

For the reinsurance cover the insurer pays a premium at rate $c - c(b)$. That is, the premium rate left for the cedent is $c(b)$. For simplicity we assume that $r(z, b)$ is continuous and increasing in both z and b and that $c(b)$ is continuous and increasing with $c(0) < 0$ and $c(\tilde{b}) = c$. The assumption $c(0) < 0$ is needed in order that the problem below is not trivial. If the reinsurer uses an expected value principle with safety loading θ, we get

$$c(b) = c - (1 + \theta)\lambda\mathbb{E}[Z - r(Z, b)] = (1 + \theta)\lambda\mathbb{E}[r(Z, b)] - (\theta - \eta)\lambda\mu \,.$$

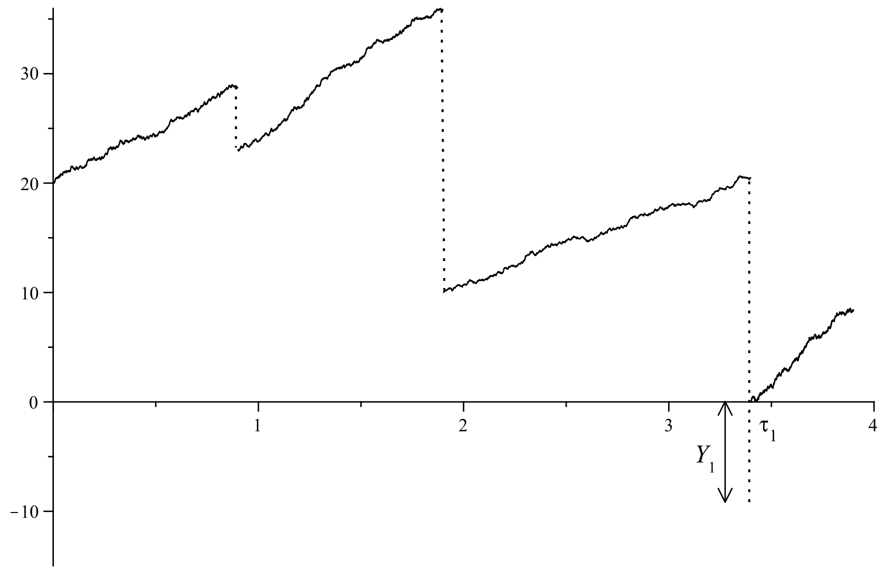

Figure 2.4: The process $\{X_t^{B,Y}\}$.

In order that $c(0) < 0$ we assume $\theta > \eta$. The insurer can choose the level b_t at any time point t. Because no information on the future can be used the process $\{b_t\}$ is assumed to be cadlag and adapted. The surplus of the insurer including reinsurance has then the form

$$X_t^B = x + \int_0^t c(b_s)\,\mathrm{d}s - \sum_{i=1}^{N_t} r(Z_i, b_{T_i-})\,.$$

To prevent, that the surplus process becomes negative, the insurer has to inject additional capital. We denote the accumulated capital injections until time t by $\{Y_t^B\}$. The surplus with capital injections has therefore the form

$$X_t^{B,Y} = x + \int_0^t c(b_s)\,\mathrm{d}s - \sum_{i=1}^{N_t} r(Z_i, b_{T_i-}) + Y_t^B\,.$$

One of the possible structures of the surplus process under some non-constant reinsurance strategy is illustrated in Figure 2.4. We are interested in the expected discounted value

$$V^B(x) = \mathbb{E}_x\Big[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t^B\Big]\,,$$

where $\delta \geq 0$ is a discounting factor. In the case $\delta = 0$ we assume that there is b such that $c(b) > \lambda\mathbb{E}[r(Z, b)$. The discounting expresses the investment preferences of the owners.

$\delta > 0$ means, that the investor prefers capital injections tomorrow to capital injections today. Hence, δ can be seen as a parameter of the risk averseness. $\delta = 0$ means that the investor is indifferent to the time of the injections. Our goal is to find the minimal return function $V(x) = \inf_{B \in \mathscr{U}} V^B(x)$ and an optimal strategy B such that $V(x) = V^B(x)$. Here again $\mathscr{U}$ denotes all cadlag adapted processes $\{b_t\}$ with values in $[0, \tilde{b}]$.

It is clear that capital injections only take place at times where the surplus would be negative, and that after a capital injection the surplus is zero. This follows by the discounting and the possibility that no injections are needed if $x = 0$.

We start by considering the case where no reinsurance is available.

Example 2.2.1
Suppose that there is no possibility to buy reinsurance. The surplus process has then the form

$$X_t^Y = x + ct - \sum_{i=1}^{N_t} Z_i + Y_t \,.$$

We have to inject capital when the process falls below zero. Let

$$\tau_1 = \inf\Big\{t \ge 0 : x + ct - \sum_{i=1}^{N_t} Z_i < 0\Big\}$$

denote the time of the first ruin, compare Figure 1.1, and

$$\tilde{Y}_1 = \sum_{i=1}^{N(\tau_1)} Z_i - x - c\tau_1$$

denote the first injection. At time τ_1 the process starts with initial capital 0. Let for $i \ge 2$,

$$\tau_i := \inf\Big\{t \ge \tau_{i-1} : c(t - \tau_{i-1}) - \sum_{j=N(\tau_{i-1})+1}^{N_t} Z_j < 0\Big\}$$

denote the time of the i-th injection. If $\tau_{i-1} = \infty$ we let $\tau_i = \infty$. The size of the i-th injection becomes, provided $\tau_i < \infty$,

$$\tilde{Y}_i = \sum_{j=N(\tau_{i-1})+1}^{N(\tau_i)} Z_j - c(\tau_i - \tau_{i-1}) \,.$$

The accumulated injections can then be described as

$$Y_t = \sum_{i=1}^{\infty} \tilde{Y}_i 1\!\!1_{[\tau_i \le t]} \,.$$

Denote by

$$\phi(x) = \mathbb{E}_x[\tilde{Y}_1 \exp\{-\delta\tau_1\}; \tau_1 < \infty]$$

the expected discounted penalty at ruin and by $\psi(x) = \mathbb{E}[\exp\{-\delta\tau_1\}; \tau_1 < \infty]$ the discounted ruin probability. Then $V^{\tilde{b}}(x) = \phi(x) + \psi(x)V^{\tilde{b}}(0)$ for $x \geq 0$. It follows recursively that

$$V^{\tilde{b}}(x) = \phi(x) + \psi(x)\phi(0)\sum_{k=0}^{\infty}\psi(0)^k = \phi(x) + \frac{\psi(x)\phi(0)}{1-\psi(0)} \; .$$

Let ρ be the unique positive solution to the generalised Lundberg equation

$$\delta + \lambda - c\rho = \lambda \int_0^\infty e^{-\rho x} \,\mathrm{d}G(x) \; . \tag{2.11}$$

It follows from Gerber and Shiu [31], see also Schmidli [71],

$$\psi(0) = \frac{\lambda}{c}\int_0^\infty e^{-\rho y}(1-G(y))\,\mathrm{d}y = \frac{\lambda}{c\rho}\int_0^\infty (1-e^{-\rho x})\,\mathrm{d}G(x) = 1 - \frac{\delta}{c\rho} \; ,$$

$$\phi(0) = \frac{\lambda}{c\rho}\int_0^\infty (1-e^{-\rho z})(1-G(y))\,\mathrm{d}y = \frac{\delta-(c-\lambda\mu)\rho}{c\rho^2} \; .$$

This gives

$$V^{\tilde{b}}(x) = \phi(x) + \psi(x)\Big(\frac{1}{\rho} - \frac{c-\lambda\mu}{\delta}\Big) \; .$$

In particular, $V^{\tilde{b}}(0) = \phi(0)/(1-\psi(0)) = (\delta-(c-\lambda\mu))/(\delta\rho)$. Expressions for $\phi(x)$ and $\psi(x)$ we find in Gerber and Shiu [31]

$$\phi(x) = \sum_{n=0}^{\infty} g^{*n} * h_2(x) \; , \qquad \psi(x) = \sum_{n=0}^{\infty} g^{*n} * h_1(x) \, ,$$

where

$$\begin{aligned}
\omega_1(u) &= \int_u^\infty \mathrm{d}G(z) = 1 - G(u) \; , \\
\omega_2(u) &= \int_u^\infty (z-u)\,\mathrm{d}G(z) = \int_u^\infty (1-G(z))\,\mathrm{d}z \; , \\
g(x) &= \frac{\lambda}{c}e^{\rho x}\int_x^\infty e^{-\rho z}\,\mathrm{d}G(z) \; , \\
h_1(x) &= \frac{\lambda}{c}e^{\rho x}\int_x^\infty e^{-\rho z}\omega_1(z)\,\mathrm{d}z = \frac{\lambda}{c}e^{\rho x}\int_x^\infty e^{-\rho z}(1-G(z))\,\mathrm{d}z \; , \\
h_2(x) &= \frac{\lambda}{c}e^{\rho x}\int_x^\infty e^{-\rho z}\omega_2(z)\,\mathrm{d}z = \frac{\lambda}{\rho c}\omega_2(x) - \frac{1}{\rho}h_1(x) \, .
\end{aligned}$$

The above functions are all differentiable for $x > 0$. It means, that the function $V^{\tilde{b}}(x)$ is differentiable as well.

More specifically, we consider now exponentially distributed claim sizes with mean value μ. The two solutions to Equation (2.11) are

$$\begin{aligned}\rho &= \frac{[\delta\mu + \lambda\mu - c] + \sqrt{[\delta\mu + \lambda\mu - c]^2 + 4c\mu\delta}}{2c\mu}\,, \\ R &= \frac{[\delta\mu + \lambda\mu - c] - \sqrt{[\delta\mu + \lambda\mu - c]^2 + 4c\mu\delta}}{2c\mu}\,.\end{aligned}$$

Note that for R and ρ the following relations hold:

$$1 + R\mu = \frac{\lambda\mu}{c(1+\rho\mu)} \tag{2.12}$$

$$\rho R = \frac{-\delta}{c\mu}\,. \tag{2.13}$$

From Schmidli [71] it follows readily that

$$\phi(x) = \mu(1+\mu R)e^{Rx}\,, \qquad \psi(x) = (1+\mu R)e^{Rx}\,.$$

This yields the solution

$$V^{\tilde{b}}(x) = \frac{1+R\mu}{-R}e^{Rx}$$

for $x \geq 0$. If we would start with a negative initial capital we get

$$V^{\tilde{b}}(x) = V^{\tilde{b}}(0) - x = \frac{1+R\mu}{-R} - x$$

for $x < 0$.

For the parameters $\mu = 1$, $\lambda = 1$, $\delta = 0.04$, $\theta = 0.5$ and $\eta = 0.3$ we obtain for $x \geq 0$: $V^{\tilde{b}}(x) = 2.3e^{-0.3x}$. This function is plotted in the Figure 2.5.

The analogous calculations can be made for every constant strategy b with $c(b) > 0$. Equation (2.11) reads then

$$\delta + \lambda - c(b)\rho = \lambda \int_0^\infty e^{-\rho r(z,b)}\, \mathrm{d}G(z)\,.$$

Representation of the functions $V^b(x)$ as a sum of two Gerber–Shiu functions has the advantage, that we can approximate $V^b(x)$ corresponding to some complex individual claims distribution by $V^b(x)$ with comparatively simple claim size distribution, for example exponential distribution.

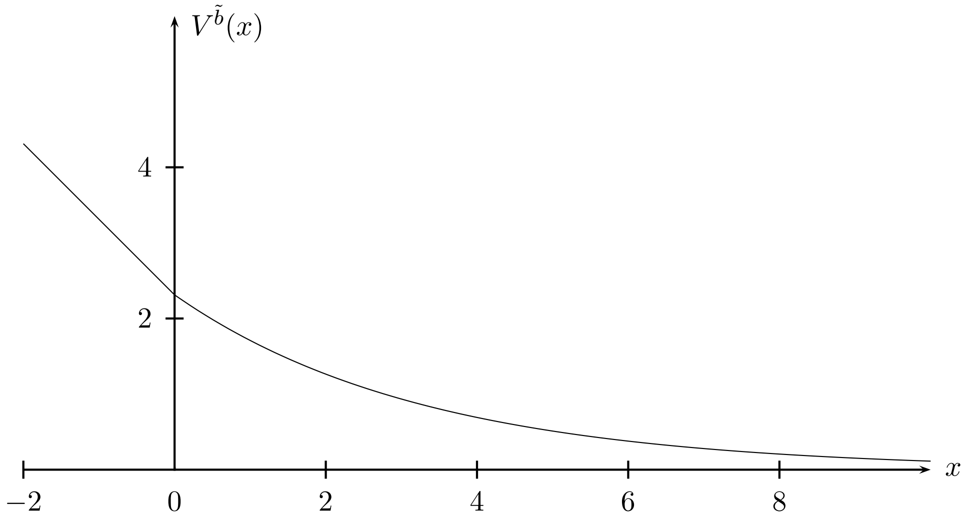

Figure 2.5: Return function $V^{\tilde b}(x)$.

Pitts and Politis [59] suggested to consider a functional Φ, which takes as input the claim size distribution and produces as output the corresponding Gerber–Shiu penalty function. In the case the functional Φ is continuous the desired Gerber–Shiu function will be close to the one with simple claim size distribution if the two distribution functions are close. This type of approximation is called zeroth-order approximations. A better approximation, first-order approximation, can be obtained in the case the functional Φ is Frechet differentiable. ■

2.2.1 Properties of $V(x)$

Lemma 2.2.2
The function $V(x)$ fulfils $\lim\limits_{x\to\infty} V(x) = 0$ and is Lipschitz continuous with $|V(x)-V(y)| \le |x-y|$.

Proof: It is clear that $V(x)$ is decreasing. $\lim\limits_{x\to\infty} V(x) = 0$ follows immediately from Example 2.2.1. Let $z > x$ and $B = \{b_t\}$ be a reinsurance strategy for initial capital z such that $V^B(z) \le V(z) + \varepsilon$. For initial capital x choose the strategy $\tilde B$ (which is not optimal): inject the capital $z - x$ and then follow the strategy B. Thus

$$V(x) - V(z) \le V^{\tilde B}(x) - V^B(z) + \varepsilon = z - x + \varepsilon\ .$$

Because ε is arbitrary we have $|V(x) - V(z)| \le |x - z|$, which proves the Lipschitz-continuity. □

As a consequence, $V(x)$ is absolutely continuous.

We now only look for $V(x)$ for $x \ge 0$ and let $V(x) = V(0) - x$ for $x \le 0$. The Hamilton–Jacobi–Bellman equation is

$$\inf_{b\in[0,\tilde{b}]} \lambda \int_0^\infty V\big(x - r(z,b)\big)\, \mathrm{d}G(z) + c(b)V'(x) - (\delta+\lambda)V(x) = 0\,. \tag{2.14}$$

The derivation of the Hamilton–Jacobi–Bellman equation is described in the proof of Theorem 2.2.3 below. Next we will show that the value function really solves the HJB equation. Let b_0 denote the unique argument at which $c(b_0) = 0$.

Theorem 2.2.3
The function $V(x)$ is continuously differentiable from the right and from the left at all points where $b_t = b_0$ is not optimal. Its derivatives solve Equation (2.14) with the interpretation $c(b_0)V'(x) = 0$, if the derivatives do not exist. Moreover, if there exists a b such that $c(b) \ge \lambda\mathbb{E}[r(Z,b)]$, then any decreasing positive solution to (2.14) coincides with $V(x)$.

Proof: Assume $x > 0$. Let $h > 0$ and $b \in [0,\tilde{b}]$ be fixed. We can assume that $x + c(b)h \ge 0$, i.e. the ruin does not occur because of the premium payments to the reinsurer. Let T_1 be the time of the first claim and choose $\varepsilon > 0$. We further choose $n \in \mathbb{N}$ such that $2(x + c(b)h)/n < \varepsilon$. For each k there is a strategy $B^k = \{b_t^k\}$ such that $V^{B^k}(x_k) \le V(x_k) + \varepsilon/2$. For initial capital $x_k \le x < x_{k+1}$ we choose the strategy B^k. Thus, $V^B(x) \le V^B(x_k) \le V(x_k) + \varepsilon/2 \le V(x) + (x - x_k) + \varepsilon/2 < V(x) + \varepsilon$. This shows that for each $x \in [0, x + c(b)]$ we can find in a measurable way a strategy $\hat{B}(x)$ such that $V^{\hat{B}(x)}(x) < V(x) + \varepsilon$. Consider now the strategy $b_t = b\mathbb{1}_{[t<T_1\wedge h]} + \hat{b}_{t-(T_1\wedge h)}(X_{T_1\wedge h})\mathbb{1}_{[t\ge T_1\wedge h]}$. By conditioning on $\mathcal{F}_{T_1\wedge h}$

$$\begin{aligned}
V(x) &\le V^B(x)\\
&= \mathbb{E}_x\Big(\mathbb{1}_{[T_1\le h]}V^{\hat{B}}(x + c(b)T_1 - r(Z_1,b))e^{-\delta T_1} + \mathbb{1}_{[T_1>h]}V^{\hat{B}}(x + c(b)h)e^{-\delta h}\Big)\\
&= \int_0^h \int_0^\infty \lambda e^{-(\delta+\lambda)t} V^{\hat{B}}\Big(x + c(b)t - r(z,b)\Big)\, \mathrm{d}G(z)\, \mathrm{d}t\\
&\quad + \int_h^\infty e^{-\delta h} V^{\hat{B}}\Big(x + c(b)h\Big)\lambda e^{-\lambda t}\, \mathrm{d}t\\
&\le \lambda \int_0^h \int_0^\infty e^{-(\delta+\lambda)t} V\Big(x + c(b)t - r(z,b)\Big)\, \mathrm{d}G(z)\, \mathrm{d}t\\
&\quad + e^{-(\delta+\lambda)h} V(x + c(b)h) + \varepsilon\,.
\end{aligned}$$

Because ε was arbitrary we can let it be equal to zero. Rearranging the terms and dividing by h yields:

$$\begin{aligned}0 \quad \leq \quad & \frac{V\big(x+c(b)h\big)-V(x)}{h}e^{-(\delta+\lambda)h}+\frac{e^{-(\delta+\lambda)h}-1}{h}V(x) \\ & +\frac{1}{h}\int_0^h\int_0^\infty \lambda e^{-(\delta+\lambda)t}V\big(x+c(b)t-r(z,b)\big)\,\mathrm{d}G(z)\,\mathrm{d}t\,. \end{aligned} \tag{2.15}$$

Now choose a strategy $D(h)=\{d_t(h)\}$, such that $V^D(x)\leq V(x)+h^2$. Let $a(t,h)=\int_0^t c(d_s)\,\mathrm{d}s$. In the same way as above we find, that

$$\begin{aligned}0 \quad \geq \quad & \frac{V(x+a(h,h))-V(x)}{h}e^{-(\delta+\lambda)h}+\frac{e^{-(\delta+\lambda)h}-1}{h}V(x) \\ & +\frac{1}{h}\int_0^h\int_0^\infty \lambda e^{-(\delta+\lambda)t}V\big(x+a(t,h)-r(z,d_t)\big)\,\mathrm{d}G(z)\,\mathrm{d}t-h\,. \end{aligned}$$

All terms with exception of the first one converge. We write

$$\frac{V(x+a(h,h))-V(x)}{h}=\frac{V(x+a(h,h))-V(x)}{a(h,h)}\,\frac{a(h,h)}{h}\,.$$

The first term is in $[-1,0]$, the second term in $[c(0),c]$. Thus, there exists a sequence $h_n\to 0$ such that

$$\lim_{n\to\infty}\frac{V(x+a(h_n,h_n))-V(x)}{h_n}=\limsup_{h\to 0}\frac{V(x+a(h,h))-V(x)}{h}\,.$$

Because $V(x)$ is Lipschitz continuous, the above limit is finite. W.l.o.g. we assume, that $a(h_n,h_n)/h_n$ converges to some $c(\check{b})$. This yields, using that $\{d_t\}$ is cadlag,

$$\begin{aligned}&\lim_{n\to\infty}\frac{V(x+a(h_n,h_n))-V(x)}{h_n}-(\lambda+\delta)V(x) \\ &\qquad +\lambda\int_0^\infty V\big(x-r(z,\check{b})\big)\,\mathrm{d}G(z)\,\mathrm{d}t \quad\geq\quad 0\,.\end{aligned}$$

From (2.15) with $b=\check{b}$ we conclude that equality holds. In particular, the limit

$$\lim_{h\to 0}\frac{V(x+c(\check{b})h)-V(x)}{h}$$

exists because the above limit does not depend on the subsequence chosen. Thus, if $c(\check{b})>0$ then $V(x)$ is differentiable at x from the right. If $c(\check{b})<0$ then $V(x)$ is differentiable at x from the left. In both cases we have shown (2.14) for the corresponding derivative.

Starting with initial capital $x - c(b)h$ yields

$$\begin{aligned} 0 \quad \le \quad & \frac{V\bigl(x-c(b)h\bigr)-V(x)}{-h}e^{-(\delta+\lambda)h} + \frac{e^{-(\delta+\lambda)h}-1}{h}V(x-c(b)h) \\ & + \frac{1}{h}\int_0^h \int_0^\infty \lambda e^{-(\delta+\lambda)t} V\bigl(x-c(b)(h-t)-r(z,b)\bigr)\,\mathrm{d}G(z)\,\mathrm{d}t\,. \end{aligned} \tag{2.16}$$

Suppose now that $V(x)$ is differentiable in x from the right; i.e., $c(\check{b}) > 0$. At each point $z < x$ we have seen that there is $\check{b}(z)$ at which the minimum in (2.14) is taken. Suppose there is a sequence x_n tending to x from the left such that $c(\check{b}_n) > 0$. By taking a subsequence we can assume that $\check{b}_n$ converges to some value b^* with $c(b^*) \ge 0$. We can then find a sequence of h_n such that (2.16) converges and by continuity (2.14) holds. If $c(b^*) > 0$ differentiability from the left follows. If there is a sequence x_n tending to x from below such that $c(\check{b}_n) < 0$, then choose $h_n = -(x - x_n)/c(\check{b}_n)$. If $h_n \to 0$ then differentiability from the left follows. As for the case $c(\check{b}_n) > 0$ Equation (2.14) follows. If h_n does not converge to zero, then $c(\check{b}_n)$ tends to zero, and (2.14) with the minimum taken at b_0. If now $\check{b}_n = b_0$, then in the limit (2.14) holds with the minimum taken at b_0. This shows the result at all points where $\check{b} \ne b_0$, that is, at all points where

$$\lambda \int_0^\infty V(x - r(z, b_0))\,\mathrm{d}G(z) > (\delta + \lambda)V(x)\,.$$

At $x = 0$, we have to consider strategies with $b \le b_0$ and $b > b_0$ separately. For $b > b_0$, Equation (2.15) holds.

If $b \le b_0$, $X_t^B = 0$ and capital is injected at rate $c(b)$ and the claims are paid by capital injections. This yields

$$V(0) = \mathbb{E}\Bigl[\sum_{i=1}^\infty r(Z_i, b)e^{-\delta T_i}\Bigr] - c(b)\int_0^\infty e^{-\delta t}\,\mathrm{d}t = \frac{\lambda}{\delta}\mathbb{E}[r(Z,b)] - \frac{c(b)}{\delta}\,. \tag{2.17}$$

Thus, equality in Equation (2.15) holds. If $V(0)$ is given by (2.17), it follows from (2.15) for $b > b_0$ that the derivative from the right must be -1. If

$$V(0) > \frac{\lambda}{\delta}\mathbb{E}[r(Z,b)] - \frac{c(b)}{\delta}\,,$$

then the arguments above show that (2.14) holds with the infimum taken at some $\check{b} \ge b_0$. Because b_0 is not possible, we get differentiability from the right.

We show now uniqueness of the solution. Let $f(x)$ be a decreasing and positive solution to (2.14). We conclude that $f'(0) \ge -1$. Because otherwise, the right hand side of (2.14) would be strictly negative at $x = 0$ for any b for which $c(b) > \lambda\mathbb{E}[r(Z,b)]$. Consider an arbitrary strategy B. From Proposition A.2.2, Chapter A, we know, that the process, using $f(X_{T_i-} - r(Z_i, b_{T_i-})) = f(X_{T_i}) + \Delta Y_{T_i}$,

$$
\begin{aligned}
M_t &= \sum_{i=1}^{N_t} \Big(f(X_{\tau_i}^{B,Y}) - f(X_{\tau_i-}^{B,Y}) \Big) e^{-\delta\tau_i} + \sum_{s\le t} e^{-\delta s} \Delta Y_s \\
&\quad - \int_0^t e^{-\delta s} \Big(\lambda \int_0^\infty f(X_s^{B,Y} - r(z,b_s)) \,\mathrm{d}G(z) - f(X_s^{B,Y}) \Big) \,\mathrm{d}s
\end{aligned}
$$

is a martingale. Because $f(x)$ is absolutely continuous we also have

$$
\begin{aligned}
& f(X_{\tau_i-}^{B,Y}) e^{-\delta\tau_i} - f(X_{\tau_{i-1}}^{B}) e^{-\delta\tau_{i-1}} \\
&= \int_{\tau_{i-1}}^{\tau_i-} \mathbb{1}_{[X_s^{B,Y}>0]} \big(c(b_s) f'(X_s^{B,Y}) - \delta f(X_s^{B,Y}) \big) e^{-\delta s} \,\mathrm{d}s \\
&\quad - \int_{\tau_{i-1}}^{\tau_i-} \mathbb{1}_{[X_s^{B,Y}=0]} \delta f(0) \,\mathrm{d}s\,.
\end{aligned}
$$

We obtain, that

$$
\begin{aligned}
& f(X_t^{B,Y}) e^{-\delta t} - f(X_0^{B,Y}) - \int_0^t \big(\mathbb{1}_{[X_s^{B,Y}>0]} c(b_s) f'(X_s^{B,Y}) - \delta f(X_s^{B,Y}) \big) e^{-\delta s} \,\mathrm{d}s \\
&- \int_0^t e^{-\delta s} \Big(\lambda \int_0^\infty f(X_s^{B,Y} - r(z,b_s)) \,\mathrm{d}G(z) - f(X_s^{B,Y}) \Big) \,\mathrm{d}s \\
&+ \sum_{s\le t} e^{-\delta s} \Delta Y_s
\end{aligned}
$$

is a martingale. Using the martingale property and taking the expectations yields

$$
\begin{aligned}
& \mathbb{E}_x \Big[f(X_t^{B,Y}) e^{-\delta t} - f(X_0^{B,Y}) + \sum_{s\le t} e^{-\delta s} \Delta Y_s \\
&- \int_0^t e^{-\delta s} \Big(\lambda \int_0^\infty f(X_s^{B,Y} - r(z,b_s)) \,\mathrm{d}G(z) - (\delta+\lambda) f(X_s^{B,Y}) \\
&+ \mathbb{1}_{[X_s^{B,Y}>0]} c(b_s) f'(X_s^{B,Y}) \Big) \,\mathrm{d}s \Big] = 0\,.
\end{aligned}
$$

If $X_s^{B,Y} = 0$ and $c(b_s) < 0$, then $c(b_s) f'(X_s^{B,Y}) \le -c(b_s)$. If $c(b_s) \ge 0$, then $c(b_s) f'(X_s^{B,Y}) \le 0$. Thus

$$
\begin{aligned}
& \mathbb{E}_x \Big[f(X_t^{B,Y}) e^{-\delta t} - f(x) + \int_0^t e^{-\delta s} \,\mathrm{d}Y_s \\
&- \int_0^t e^{-\delta s} \Big(\lambda \int_0^\infty f(X_s^{B,Y} - r(z,b_s)) \,\mathrm{d}G(z) - (\delta+\lambda) f(X_s^{B,Y}) \\
&+ c(b_s) f'(X_s^{B,Y}) \Big) \,\mathrm{d}s \Big] \ge 0\,.
\end{aligned}
$$

Because B was arbitrary and using (2.14) we obtain from the above equation

$$f(x) \le \mathbb{E}_x\Big[f(X_t^{B,Y})e^{-\delta t} + \int_0^t e^{-\delta s}\,\mathrm{d}Y_s\Big]\,.$$

Equality holds if B is the strategy $b_t^* = b(X_t^{B^*,Y})$, where $b(x)$ is an argument at which the minimum in (2.14) is taken. Because $f(X_t^{B,Y}) \le f(0) < \infty$, so that we obtain $\lim\limits_{t\to\infty} \mathbb{E}_x[f(X_t^{B,Y})e^{-\delta t}] = 0$ by bounded convergence. This proves that $f(x) = V(x)$. □

Suppose now that

$$c(b) = \lambda(1+\theta)\mathbb{E}[r(Z,b)] - \lambda\mu(\theta-\eta)\;; \tag{2.18}$$

i.e., the reinsurer uses an expected value principle. Consider the capital $x = 0$. Consider first the case that a strategy with $c(b) \le 0$ is optimal at $x = 0$. Then, compare with (2.17) in the Appendix,

$$\begin{aligned} V(0) &= \frac{\lambda}{\delta}\big[\mathbb{E}[r(Z,b)] - \big\{(1+\theta)\mathbb{E}[r(Z,b)] - (\theta-\eta)\mu\big\}\big] \\ &= \frac{\lambda}{\delta}\big[(\theta-\eta)\mu - \theta\mathbb{E}[r(Z,b)]\big]\,. \end{aligned}$$

This is decreasing in b, hence $b = b_0$ would be optimal.
In particular, $V(0) = \lambda\mathbb{E}[r(Z,b_0)]/\delta$. Let $\kappa, \varepsilon > 0$ such that $\kappa > c(\tilde{b})\varepsilon$. Consider the strategy $b_t = b_0 1\!\!1_{[t \ge T_1 \wedge \varepsilon]}$. This strategy has the return function bounded by

$$\begin{aligned} &e^{-(\lambda+\delta)\varepsilon}\Big(\frac{\lambda}{\delta}\mathbb{E}[r(Z,b_0)] - \frac{\lambda}{\lambda+\delta}c(\tilde{b})\varepsilon(1-G(\kappa))\Big) \\ &\qquad + \int_0^\varepsilon \Big(\lambda\mu + \frac{\lambda}{\delta}\mathbb{E}[r(Z,b_0)] - c(\tilde{b})t(1-G(\kappa))\Big)\lambda e^{-(\lambda+\delta)t}\,\mathrm{d}t\,. \end{aligned}$$

Taking the derivative with respect to ε shows that the function is decreasing in ε, with a derivative bounded away from zero. Thus for κ and ε small enough the return function of the above strategy is smaller than the return function of the strategy $b_t = b_0$. This shows that b_0 cannot be optimal.

Equation (2.14) at $x = 0$ reads

$$\inf_{b \in [0,\tilde{b}]} \lambda\mathbb{E}[r(Z,b)]\big[1 + (1+\theta)V'(0)\big] - \lambda\mu(\theta-\eta)V'(0) - \delta V(0) = 0\,.$$

We see that the minimum is taken either at $b = 0$ or $b = \tilde{b}$. Because $b = 0$ is not optimal, we conclude that $b = \tilde{b}$. Because a level with $b(x) \le b_0$ cannot be crossed, this means that no reinsurance will be taken for capital close to zero. In particular we obtain $V'(0) < -\frac{1}{1+\theta}$ and $V'(x) < -\frac{1}{1+\theta}$ for $x \in [0,y)$ for some $y > 0$.

Remark 2.2.4
Consider now the function

$$g_x(b) := c(b)V'(x) - (\delta+\lambda)V(x) + \lambda\int_0^\infty V\big(x - r(z,b)\big)\,\mathrm{d}G(z)\;.$$

Let further $b_1, b_2 \in [0,\tilde{b}]$ and $b_1 > b_2$ then we obtain with Lipschitz continuity of $V(x)$ for every $x \in [0,\infty)$:

$$\begin{aligned} g_x(b_1) - g_x(b_2) &= \lambda\int_0^\infty V\big(x - r(z,b_1)\big) - V\big(x - r(z,b_2)\big)\,\mathrm{d}G(z) \\ &\quad + (c(b_1) - c(b_2))V'(x) \\ &= \lambda\int_0^\infty V\big(x - r(z,b_1)\big) - V\big(x - r(z,b_2)\big) \\ &\quad + \Big\{r(z,b_1) - r(z,b_2)\Big\}(1+\theta)V'(x)\,\mathrm{d}G(z) \\ &\le \lambda\int_0^\infty \Big\{r(z,b_1) - r(z,b_2)\Big\}\cdot[1 + (1+\theta)V'(x)]\,\mathrm{d}G(z)\;. \end{aligned}$$

$V'(x) \le -\frac{1}{1+\theta}$ implies, that $g_x(b)$ is decreasing, so that the minimum is taken in $b = \tilde{b}$, which is then the optimal strategy if $V(x)$ is differentiable in x. On the other hand, if $b_0 \ne b < \tilde{b}$ is optimal for some $x \in [0,\infty)$, then it must hold $V'(x) > -\frac{1}{1+\theta}$.

Lemma 2.2.5
If the value function $V(x)$ is convex, then it is continuously differentiable.

Proof: From Theorem 2.2.3 we know, that $V(x)$ is continuously differentiable in all $x \in \mathbb{R}_+$, where b_0 is not optimal. Let $V'(x-)$ and $V'(x+)$ denote the derivatives from the right and from the left respectively. Assume further that there is $\tilde{x} \in \mathbb{R}_+$ with $V'(\tilde{x}-) < V'(\tilde{x}+)$. Define

$$f(x) := \lambda\int_0^\infty V(x - r(b_0,z))\,\mathrm{d}G(z) - (\delta+\lambda)V(x)$$

Note that it holds $f(x) \ge 0$. Let further $\tilde{x} := \inf\{x \in \mathbb{R}_+ : f(x) = 0\}$. Because the optimal strategy in 0 is $b^* = \tilde{b}$, it holds $\tilde{x} > 0$. By Theorem 2.2.3 we obtain $f(\tilde{x}) = 0$. W.o.l.g we can assume that $f(x)$ is differentiable on $(0,\tilde{x})$.

Because $f(\tilde{x}) = 0$ there exist sequences $(h_n)_{n\ge 0} \in [0,\tilde{x}]$, $\lim\limits_{n\to\infty} h_n = \tilde{x}$, with $f'(h_n) < 0$ and $(x_n)_{n\ge 0} \in (\tilde{x},\infty)$, $\lim\limits_{n\to\infty} h_n = \tilde{x}$, with $f'(x_n) \ge 0$. Letting n go to infinity we obtain

$$\lambda\int_0^\infty V'(\tilde{x} - r(b_0,z))\,\mathrm{d}G(z) - (\delta+\lambda)V'(\tilde{x}-) \le 0\;,$$
$$\lambda\int_0^\infty V'(\tilde{x} - r(b_0,z))\,\mathrm{d}G(z) - (\delta+\lambda)V'(\tilde{x}+) \ge 0\;.$$

Thus, $V'(\tilde{x}+) \le V'(\tilde{x}-)$, which is a contradiction. $\square$

We will see in the next section that $V(x)$ is convex in the case of proportional reinsurance.

2.2.2 Examples

Example 2.2.6 (Proportional Reinsurance and Z ~Exp$(1/\mu)$.)
For proportional reinsurance $r(Z,b) = bZ$ and the expected value principle, we get that $V(x)$ is a convex function. Indeed, let $x, z \ge 0$, $\alpha \in (0,1)$ and $y = \alpha x + (1-\alpha)z$. Let $\{b_t^x\}$ be the optimal strategy for initial capital x and $\{b_t^z\}$ the optimal strategy for initial capital z. Define the new strategy $b_t^y = \alpha b_t^x + (1-\alpha)b_t^z$. Then for the expected value principle $c(b_t^y) = \alpha c(b_t^x) + (1-\alpha)c(b_t^z)$. Then

$$\begin{aligned}
&X_t^y - Y_t^y + \alpha Y_t^x + (1-\alpha)Y_t^z \\
&= \quad y + \int_0^t c(b_s^y)\,\mathrm{d}s - \sum_{i=1}^{N_t} b_{T_i-}^y Z_i + \alpha Y_t^x + (1-\alpha)Y_t^z \\
&= \quad \alpha X_t^x + (1-\alpha)X_t^z \ge 0\,.
\end{aligned}$$

This implies that $Y_t^y \le \alpha Y_t^x + (1-\alpha)Y_t^z$. Thus

$$V(y) \le V^{B^y}(y) \le \alpha V(x) + (1-\alpha)V(z)\,,$$

which proves the convexity. In particular, all derivatives from the left and from the right exist and therefore solve the HJB equation.

Note, that the convexity in the case of proportional reinsurance can be also shown for all premium calculation principles, where $c(b)$ is concave in b. This is for example the case for the standard deviation and variation principles. For definition see Subsection 1.1.2, p. 14.

Let us assume Z ~Exp$(1/\mu)$. The HJB equation reads now:

$$\inf_{b\in[0,1]} \frac{\lambda}{\mu}\int_0^\infty V(x-bz)\exp(-z/\mu)\,\mathrm{d}z + c(b)V'(x) - (\lambda+\delta)V(x) = 0\,.$$

From Example 2.2.1 we know, that the return function for the constant strategy $b=1$ is given by

$$V^1(x) = \begin{cases} \frac{1+R\mu}{-R}\exp(Rx) & : \ x \ge 0\,, \\ \frac{1+R\mu}{-R} - x & : \ x < 0\,, \end{cases}$$

where

$$R = \frac{\delta\mu + \lambda\mu - c - \sqrt{[\delta\mu+\lambda\mu-c]^2 + 4c\mu\delta}}{2c\mu}\,.$$

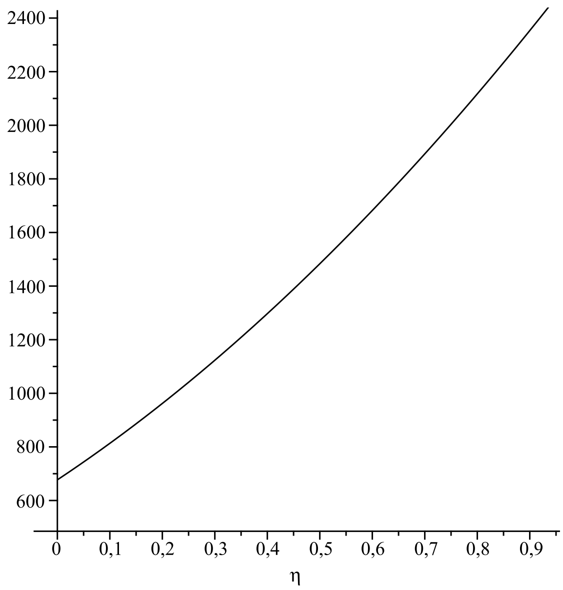

Figure 2.6: Dependence of critical θ on chosen η.

Consider now the function

$$g_x(b) = \frac{\lambda}{\mu}\int_0^\infty V^1(x-bz)e^{-\frac{z}{\mu}}\,\mathrm{d}z + c(b)(V^1)'(x) - (\delta+\lambda)V^1(x)\;.$$

The function $g_x(b)$ is convex in b for each $x \in [0,\infty)$. Hence it holds for the first derivative of $g_x(b)$ with respect to b: $g_x'(b) \le g_x'(1)$. Therefore, $V^1(x)$ is the value function if $g_x'(1) \le 0$ for all $x \in [0,\infty)$, which is equivalent to the inequality

$$\frac{1}{\mu^2}\int_0^\infty \frac{(V^1)'(x-z)}{(V^1)'(x)} \cdot z \cdot e^{-\frac{z}{\mu}}\,\mathrm{d}z \le (1+\theta)\;.$$

The function $\frac{(V^1)'(x-z)}{(V^1)'(x)}$ is increasing in x with $\lim\limits_{x\to\infty}\frac{(V^1)'(x-z)}{(V^1)'(x)} = e^{-Rz}$. Letting x go to infinity in the left hand side of the above inequality we obtain $\theta \ge \frac{1}{(R\mu+1)^2} - 1$.

For the parameters $\delta = 0.04$, $\mu = 1$, $\lambda = 1$ and $\eta = 0.1$ we obtain $\theta \ge 813$. In this case the value function is

$$V(x) = \begin{cases} 3.55\exp(-0.22x) & : \; x \ge 0\;, \\ -x + 3.55 & : \; x < 0\;. \end{cases}$$

In Figure 2.6 the maximal value of θ for which reinsurance should be bouhgt is given as a function of η. The parameters are $\delta = 0.04$, $\mu = 1$ and $\lambda = 1$.

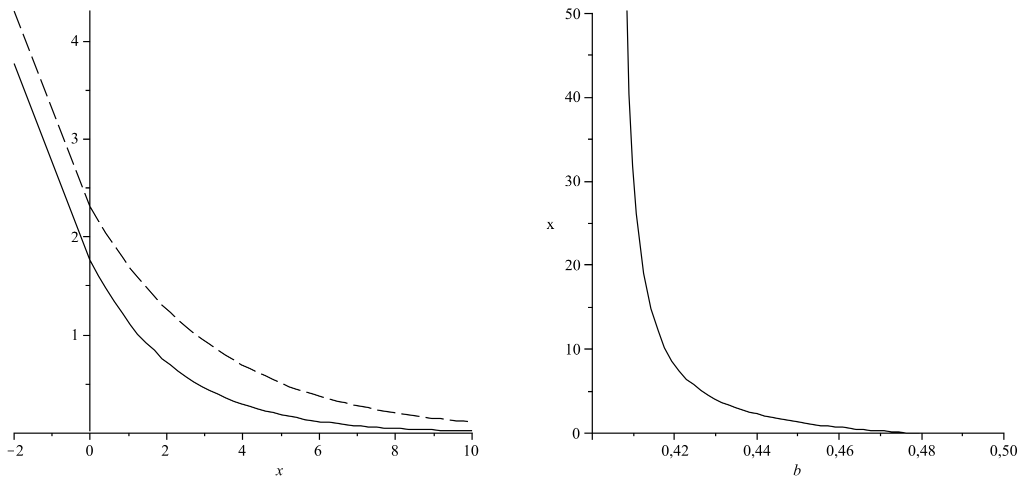

Figure 2.7: Functions $\tilde{V}(x)$ (solid line) and $V^1(x)$ (dashed line) and the corresponding "optimal" level $b(x)$.

The optimal strategy for the case $\theta < \frac{1}{(R\mu+1)^2} - 1$ is more complicated. In this case, the optimal strategy is not constant.

Consider for the moment constant strategies only. It is straight forward to verify, that values corresponding to the constant strategies $b \in (b_0, \tilde{b}]$, $b- \in (0, b_0)$, 0 and b_0, with $c(b_0) = 0$, are given by the functions

$$
\begin{aligned}
V^b(x) &= \begin{cases} -\frac{1+R(b)\mu b}{R(b)} e^{R(b)x} & : \ x \geq 0 \ , \\ -x - \frac{1+R(b)\mu b}{R(b)} & : \ x < 0 \ , \end{cases} \\
V^{b-}(x) &= \begin{cases} \frac{-c(b)+\lambda b\mu}{\delta} e^{\rho(b)x} & : \ x \geq 0 \ , \\ -x + \frac{-c(b)+\lambda b\mu}{\delta} & : \ x < 0 \ , \end{cases} \\
V^0(x) &= \begin{cases} \frac{-c(0)}{\delta} e^{-\frac{\delta}{c(0)}x} & : \ x \geq 0 \ , \\ -x + \frac{-c(0)}{\delta} & : \ x < 0 \ , \end{cases} \\
V^{b_0}(x) &= \begin{cases} \frac{\lambda b_0 \mu}{\delta} e^{-\frac{\delta}{b_0\mu(\delta+\lambda)}x} & : \ x \geq 0 \ , \\ -x + \frac{\lambda b_0 \mu}{\delta} & : \ x < 0 \end{cases}
\end{aligned}
$$

respectively, where

$$
\begin{aligned}
R(b) &= \frac{[\delta\mu b + \lambda\mu b - c(b)] - \sqrt{[\delta\mu b + \lambda\mu b - c(b)]^2 + 4c(b)\mu b\delta}}{2c(b)\mu b} \ , \\
\rho(b) &= \frac{[\delta\mu b + \lambda\mu b - c(b)] + \sqrt{[\delta\mu b + \lambda\mu b - c(b)]^2 + 4c(b)\mu b\delta}}{2c(b)\mu b} \ .
\end{aligned}
$$

The calculations are similar to the calculations in Example 2.2.1. It is easy to see, that $V^b(x) < V^{b-}(x)$ for every choice of $b \in [b_0, 1]$ and $b- \in [0, b_0)$. So we will consider only the constant strategies with positive premium income.

It is easy to see, that the optimal $b \in [b_0, 1]$ minimising $V^b(x)$ depends on x. This shows that the optimal strategy cannot be constant. For parameters given above the function $\tilde{V}(x) = \min\limits_{b\in[b_0,1]} V^b(x)$ is plotted in Figure 2.7

Next we will calculate the value function numerically for exponentially and Pareto distributed claim sizes. Note, that the start value $V(0)$ should be calculated separately. Denote the function, which result from choosing some start value $a > V(0)$ by $f(x)$ and define

$$g(x) := f(x) - V(x) \; .$$

$g(x)$ is continuously differentiable and $g(0) > 0$ by definition. Let further $b^*(x)$ denote the optimal strategy for $V(x)$, then we have

$$c(b^*(x))g'(x) \geq \lambda \int_0^\infty g(x - b^*(x)z) \;\mathrm{d}G(z) - (\delta + \lambda)g(x) \; .$$

Because $g(0) > 0$ and $b^*(0) = 1$ it follows $g'(0) > 0$. Let $\hat{x} = \inf\{x : g'(x) = 0\}$. Since $g'(x) > 0$ on $[0, \hat{x})$ it holds $\lambda \int_0^\infty g(x - b^*(x)z) \;\mathrm{d}G(z) - (\delta + \lambda)g(x) > 0$, which is a contradiction. Therefore, numerically calculated functions with start values bigger than $V(0)$ do not cut the curve $V(x)$, i.e. they do not cut the x axis and begin to increase.

In order to specify $V(0)$ we begin with some start value $V^1(0)$ and calculate the corresponding function. If the function begin to increase, the chosen value V_0 was too big. Proceeding in that way we obtain after a while the right value $V(0)$.

In Figure 2.8 one can see the value function numerically calculated under assumption, that the claims are exponentially distributed. The initial value calculated in the above described way is $V(0) = 1.7$. ■

Example 2.2.7 (Proportional Reinsurance and Z ∼Pareto$(2, \mu)$.)
Consider now Pareto$(2, \mu)$ distributed claim sizes, i.e. $G(x) = 1 - \frac{\mu^2}{(\mu+x)^2}$. It is impossible to give a closed form expression for the function $V^1(x)$ in this case. But we can calculate it numerically using the method of Gerber and Shiu [31]. Consider again the parameters $\delta = 0.04$, $\lambda = \mu = 1$, $\theta = 0.5$ and $\eta = 0.3$. Then it is easy to verify with help of some mathematical programs: $\rho = 0.08331$, $\phi(0) = 1.66343$ and $V^1(0) = 4.50372$. Numerically calculated value function is given in Figure 2.10. The initial value is in this case $V(0) = 2.3847712$. ■

Example 2.2.8 (Excess of Loss Reinsurance and Z ∼Exp$(1/\mu)$.)
For excess of loss reinsurance $r(z, b) = \min(z, b)$ we cannot determine whether the value

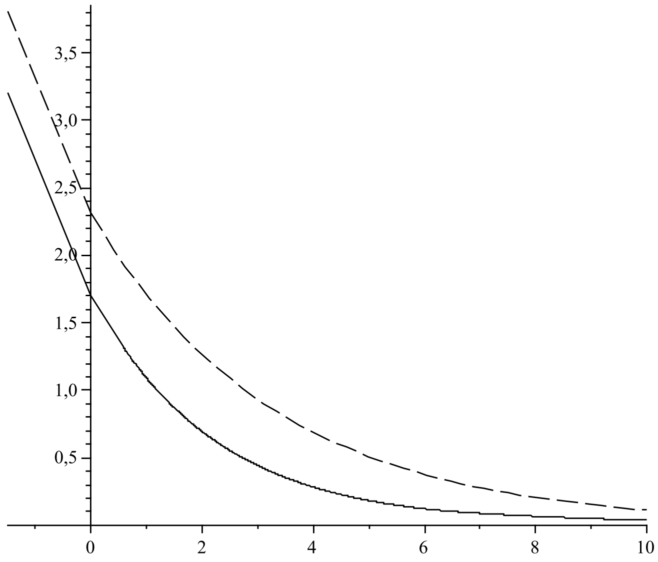

Figure 2.8: Numerically calculated value function $V(x)$ with initial value $V(0) = 1.7$ (solid line) and $V^1(x)$ (dashed line) for exponentially distributed claims.

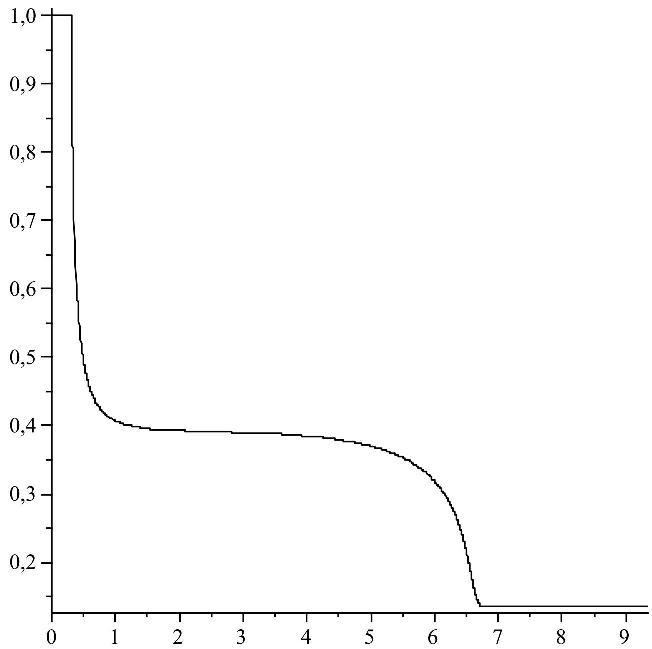

Figure 2.9: Optimal strategy $b(x)$ for exponentially distributed claims.

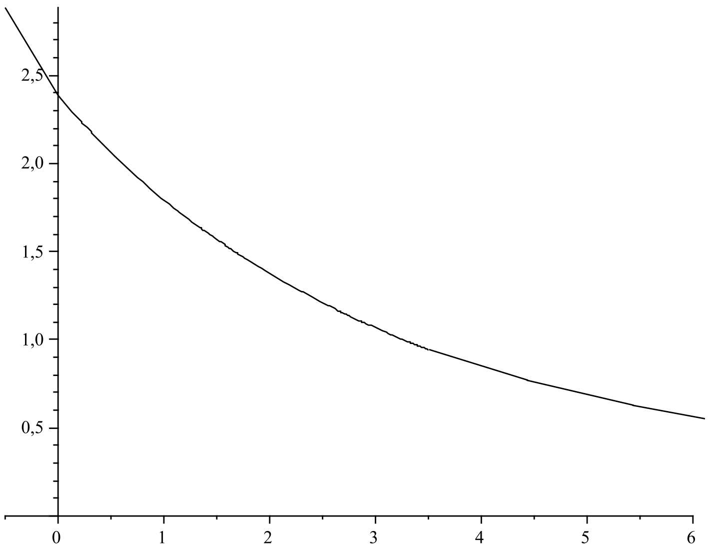

Figure 2.10: Numerically calculated $V(x)$ for Pareto distributed claims with initial value 2.3847.

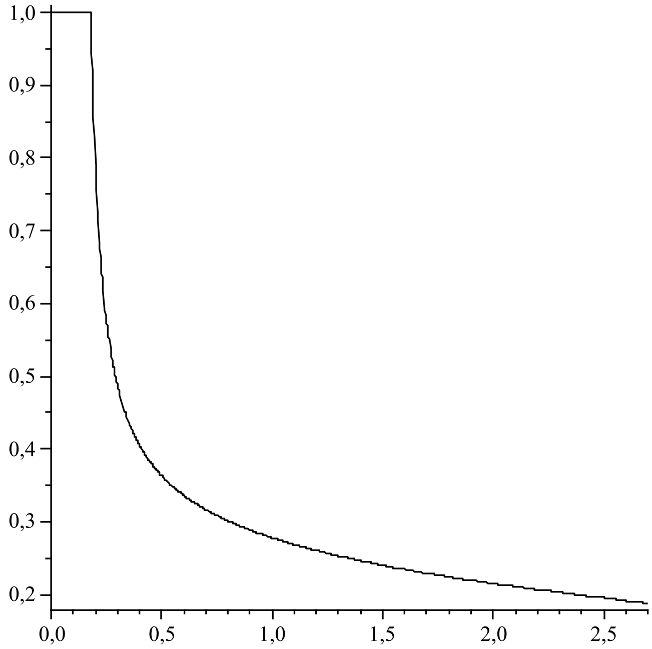

Figure 2.11: Optimal strategy $b(x)$ for Pareto distributed claims.

function is convex. For the HJB equation we get

$$\inf_{b\in[0,\infty]}\Big\{\lambda V(x-b)(1-G(b))+\lambda\int_0^b V(x-z)\,\mathrm{d}G(z)+c(b)V'(x)\\ -(\lambda+\delta)V(x)\Big\}=0\,,$$

where $c(b)=\lambda\mu(\eta-\theta)+\lambda(1+\theta)\big[b(1-G(b))+\int_0^b z\,\mathrm{d}G(z)\big]$.

Consider again

$$\begin{aligned} g_x(b) &= \lambda V(x-b)(1-G(b))+\lambda\int_0^b V(x-z)\,\mathrm{d}G(z)+c(b)V'(x)-(\lambda+\delta)V(x)\\ &= \lambda\int_0^b (1-G(u))[(1+\theta)V'(x)-V'(x-u)]\,\mathrm{d}u-\lambda\mu(\theta-\eta)V'(x)\,. \end{aligned}$$

Then $g_x(b)$ is differentiable at all points where $V(y)$ is differentiable with $y=x-b$. Derivation with respect to b yields

$$g_x'(b)=-\lambda(1-G(b))\big[V'(x-b)-(1+\theta)V'(x)\big]\,.$$

It is clear, that the optimal strategy is $b^*(x)=\infty$ in the case $V'(x)\le-\frac{1}{1+\theta}$. If $V'(x)>-\frac{1}{1+\theta}$ it must hold $0<b^*(x)\le x$. The following relationship holds

$$V'(\{x-b^*(x)\}+)\le(1+\theta)V'(x)\le V'(\{x-b^*(x)\}-)\,.$$

In particular, if we choose an $\varepsilon>0$ very small, we obtain for x large enough

$$h(x)=\lambda\int_0^\infty V(x-z)\,\mathrm{d}G(z)-(\lambda+\delta)V(x)<\varepsilon\,.$$

If $V'(x)\le-\frac{1}{1+\theta}$, we conclude $h(x)+cV'(x)<0$, which is a contradiction to (2.14). Thus, $V'(x)$ tends to zero and $b^*(x)<\infty$ for x large enough. ■

2.2.3 The special case $\delta=0$

We consider now the special case $\delta=0$. We assume the net profit condition $\eta>0$. In this case the Hamilton–Jacobi–Bellman equation has the form

$$\inf_{b\in[0,\tilde b]}\lambda\int_0^\infty V(x-r(z,b))\,\mathrm{d}G(z)+c(b)V'(x)-\lambda V(x)=0\,. \tag{2.19}$$

It is immediately clear, that a strategy b with $c(b)\le0$ can not be optimal for some $x\in\mathbb{R}_+$. Indeed, a level with $c(b)\le0$ cannot be crossed, implying that $V(0)=\infty$. But the strategy $b_t=\tilde b$ yields a finite value.

We transform the HJB equation using the fact, that for the minimiser b we have $c(b) > 0$. Define for this purpose

$$s(x,b) = \inf\{z : r(z,b) > x\}.$$

Then we can write the equation (2.19) as follows:

$$\begin{aligned} \inf_{b\in[0,\tilde b]} \Big\{ \lambda \int_0^{s(x,b)} V\big(x - r(z,b)\big)\, \mathrm{d}G(z) + \lambda\big(1 - G(s(x,b))\big)V(0) - \lambda V(x) \\ + \lambda \int_{s(x,b)}^{\infty} r(z,b)\, \mathrm{d}G(z) + c(b)V'(x) - \lambda\big(1 - G(s(x,b))\big)x \Big\} \quad = \quad 0 \end{aligned}$$

With Fubini's Theorem we obtain:

$$\begin{aligned} \lambda \int_0^{s(x,b)} V\big(x - r(z,b)\big)\, \mathrm{d}G(z) + \lambda\big(1 - G(s(x,b))\big)V(0) - \lambda V(x) \\ = -\lambda \int_0^{x} \big(1 - G(s(y,b))\big)V'(x-y)\, \mathrm{d}y \end{aligned}$$

and

$$\lambda \int_{s(x,b)}^{\infty} r(z,b)\, \mathrm{d}G(z) - \lambda\big(1 - G(s(x,b))\big)x = \lambda \int_x^{\infty} 1 - G(s(y,b))\, \mathrm{d}y.$$

Because for each x there exists a b, we can write for the optimal $b(x)$:

$$\begin{aligned} c(b(x))V'(x) \quad &= \quad \lambda \int_0^{x} \big(1 - G(s(y,b(x)))\big)V'(x-y)\, \mathrm{d}y \\ &\qquad -\lambda \int_x^{\infty} 1 - G(s(y,b))\, \mathrm{d}y\,, \end{aligned} \tag{2.20}$$

and (2.19) transforms to

$$\begin{aligned} f'(x) \quad &= \quad \sup_{b\in(b_0,\tilde b]} \frac{\lambda}{c(b)} \Big[\int_0^{x} \big(1 - G(s(y,b))\big)f'(x-y)\, \mathrm{d}y \\ &\qquad - \int_x^{\infty} 1 - G(s(y,b))\, \mathrm{d}y \Big] \end{aligned} \tag{2.21}$$

From Theorem 2.2.3 we know that the solution to (2.19) and therefore to (2.21) is unique. For $x = 0$ we obtain

$$V'(0) = \sup_{b\in(b_0,\tilde b]} \frac{\lambda \mathbb{E}[r(Z,b)]}{c(b)}\,.$$

In particular, for an expected value principle (2.18) we find $V'(0) = -\frac{1}{1+\eta}$.

Proposition 2.2.9
There is a unique solution $f(x)$ to (2.19), $x \geq 0$, with $f(\infty) = 0$.

Proof: Define an operator F, acting on negative functions $w(x)$ by

$$\begin{aligned} F(w(x)) &= \sup_{b \in (b_0, \tilde{b}]} \frac{\lambda}{c(b)} \Big[\int_0^x \big(1 - G(s(y,b))\big) w(x-y) \, \mathrm{d}y \\ &\quad - \int_x^\infty 1 - G(s(y,b)) \, \mathrm{d}y \Big] \end{aligned} \tag{2.22}$$

We have already seen, that the value function $V(x)$ solves the equation (2.19). Let now $w_0(x) := (V^{\tilde{b}})'(x)$, the solution of Example 2.2.1 with $b_t = \tilde{b}$. Define recursively $w_n(x) = F(w_{n-1}(x))$. We show at first, that the sequence w_n is monotone increasing in n. It is clear, that $w_0(x) \leq w_1(x)$ because $(V^{\tilde{b}})'(x)$ solves the right hand side of (2.20) with $b = \tilde{b}$ instead of the sup. Assume $w_{n-1}(x) \leq w_n(x)$. Because the right side of (2.22) is continuous in b, there is a maximum point $b_n \in [b_0, \tilde{b}]$ for which $w_n(x) = F(w_{n-1}(x))$ attains its maximum. So we have

$$\begin{aligned} & w_{n+1}(x) - w_n(x) = F(w_n(x)) - F(w_{n-1}(x)) \\ &= F(w_n(x)) - \frac{\lambda}{c(b_n)} \Big[\int_0^x \big(1 - G(s(y,b_n))\big) w_{n-1}(x-y) \, \mathrm{d}y \\ &\quad + \int_x^\infty 1 - G(s(y,b_n)) \, \mathrm{d}y \Big] \\ &\geq \frac{\lambda}{c(b_n)} \Big[\int_0^x \big(1 - G(s(y,b_n))\big) [w_n(x-y) - w_{n-1}(x-y)] \, \mathrm{d}y \Big] \\ &\geq 0. \end{aligned}$$

So the sequence $w_n(x)$ is increasing in n and $w_n(x) < 0$. It means we obtain that $w(x) = \lim_{n\to\infty} w_n(x)$ exists pointwise. By the proposition of Lebesgue we have then

$$\lim_{n\to\infty} \int_0^x \big(1 - G(s(y,b))\big) w_n(x-y) dy = \int_0^x \big(1 - G(s(y,b))\big) w(x-y) \, \mathrm{d}y$$

for all x and b.
Let b be a maximal point of $F(w(x))$. Then we have

$$\begin{aligned} w_n(x) &= \frac{\lambda}{c(b_n)} \Big[\int_0^x \big(1 - G(s(y,b_n))\big) w_{n-1}(x-y) \, \mathrm{d}y \\ &\quad - \int_x^\infty 1 - G(s(y,b_n)) dy \Big] \\ &\geq \frac{\lambda}{c(b)} \Big[\int_0^x \big(1 - G(s(y,b))\big) w_{n-1}(x-y) \, \mathrm{d}y \\ &\quad - \int_x^\infty 1 - G(s(y,b)) \, \mathrm{d}y \Big], \end{aligned}$$

which means $w(x) \geq F(g(x))$. On the other hand $w_n(x)$ are increasing in n, i.e. $w_n(x) \leq w(x)$:

$$\begin{aligned}
w_n(x) &= \frac{\lambda}{c(b_n)}\Big[\int_0^x \big(1 - G(s(y,b_n))\big) w_{n-1}(x-y)\,\mathrm{d}y \\
&\quad - \int_x^\infty 1 - G(s(y,b_n))\,\mathrm{d}y\Big] \\
&\leq \frac{\lambda}{c(b_n)}\Big[\int_0^x \big(1 - G(s(y,b_n))\big) w(x-y)\,\mathrm{d}y \\
&\quad - \int_x^\infty 1 - G(s(y,b_n))\,\mathrm{d}y\Big] \\
&\leq \frac{\lambda}{c(b)}\Big[\int_0^x \big(1 - G(s(y,b))\big) w(x-y)\,\mathrm{d}y \\
&\quad - \int_x^\infty 1 - G(s(y,b))\,\mathrm{d}y\Big],
\end{aligned}$$

which means $w(x) \leq F(w(x))$. We have therefore $w(x) = F(w(x))$, and $w(x)$ is continuous.

Because $w_n(x)$ is increasing, we can define

$$f(x) = -\int_x^\infty w(y)\,\mathrm{d}y \leq -\int_x^\infty w_0(y)\,\mathrm{d}y = V^{\tilde{b}}(x)\ .$$

$f(x)$ fulfils (2.22) with $f(\infty) = 0$. $f(x)$ is also decreasing, continuously differentiable and bounded $0 < f(x) \leq V^{\tilde{b}}(x)$.

Suppose now, that $f_1(x)$ and $f_2(x)$ are solutions to (2.19) with $f_1(\infty) = f_2(\infty) = 0$. Denote further by $g_i(x) = f_i'(x)$ the derivatives and by $b_i(x)$ the value, for which the minimum is obtained. Choose now some $x^* > 0$. Because the right-hand side of Equation (2.21) is continuous in b and tends to infinity as $c(b)$ tends to zero, we conclude, that $c(b_i(x))$ is bounded away from zero on $[0, x^*]$. Let $x_1 = \inf\{c(b_1(x)) \wedge c(b_2(x)) : 0 \leq x \leq x^*\}$ and $x_n = nx_1 \wedge x^*$. W.l.o.g. we assume, that $x_1 \leq x^*$. Suppose we have already proved $f_1(x) = f_2(x)$ on the interval $[0, x_n]$. Then for $x \in [x_n, x_{n+1}]$, with $m = \sup\limits_{x_n \leq x \leq x_{n+1}} |g_1(x) - g_2(x)|$ it holds

$$\begin{aligned}
g_1(x) - g_2(x) &= F(g_1(x)) - F(g_2(x)) \\
&\leq \frac{\lambda}{c(b_2(x))}\int_{x_n}^x \Big(g_1(z) - g_2(z)\Big)[1 - G(s(x-z))]\,\mathrm{d}z \\
&\leq \frac{\lambda m x_1}{c(b_2(x))} \leq \frac{m}{2}
\end{aligned}$$

Reversing the roles of $g_1(x)$ and $g_2(x)$, it follows that $|g_1(x) - g_2(x)| \le \frac{m}{2}$. This is only possible for all $x \in [x_n, x_{n+1}]$ if $m = 0$. this shows that $f_1(x) = f_2(x)$ on $[0, x_{n+1}]$. So $f_1(x) = f_2(x)$ on $[0, x^*]$. Because x^* was arbitrary, uniqueness follows. □

Next we illustrate the result by a couple of examples.

Examples

Note, that because we know, that the value function is differentiable we will use the modified HJB equation

$$\inf_{b\in[0,\tilde{b}]} -\lambda \int_0^x V'(z)\big(1 - G((x-z)/b)\big)\,\mathrm{d}z + c(b)V'(x) + \lambda \int_x^\infty 1 - G((x+z)/b)\,\mathrm{d}z = 0\,. \tag{2.23}$$

for numerical calculation of the value function. Furthermore, all the considerations concerning the function $V^{\tilde{b}}(x)$ in the case $\delta > 0$ hold also in the case $\delta = 0$.
In the following examples we will use the expected value principle, i.e. $c(b) = -\lambda\mu(\theta - \eta) + \lambda(1+\theta)\mathbb{E}[r(Z,b)]$.

Example 2.2.10 (Proportional Reinsurance and $Z_i \sim$Exp($\frac{1}{\mu}$).)
For the exponentially distributed claim sizes we obtain as the function $V^1(x)$:

$$V^1(x) = \begin{cases} \frac{\mu}{\eta}\exp\!\big(-\frac{\eta}{\mu(1+\eta)}\cdot x\big) & : \ x \ge 0\,, \\ \frac{\mu}{\eta} - x & : \ x < 0\,. \end{cases}$$

It is easy to verify, that $V^1(x)$ is the value function if $\theta \ge \frac{\eta(c+\lambda\mu)}{\lambda\mu}$. Assume now $\theta < \frac{\eta(c+\lambda\mu)}{\lambda\mu}$. Equation (2.23) becomes

$$\inf_{b\in[0,\tilde{b}]} -\lambda \int_0^x V'(z)e^{-\frac{x-z}{\mu b}}\,\mathrm{d}z + c(b)V'(x) + \lambda\mu b e^{-\frac{x}{\mu b}} = 0\,.$$

The above equation differs very little from the HJB equation for the ruin probability as a function of initial capital, compare Schmidli [70, p. 47]. In the case of ruin probability we have $\lambda\mu e^{-\frac{x}{\mu b}}$ instead of the term $\lambda\mu b e^{-\frac{x}{\mu b}}$ in our case. But this difference is responsible for the behavior of the optimal strategy. While in the ruin probability case the optimal strategy jumps down from the 1-level, in our case it falls down continuously. For the parameters $\mu = \lambda = 1$, $\theta = 0.5$ and $\eta = 0.3$ we can calculate the $\hat{x} := \inf\{x \in \mathbb{R}_+ : b^*(x) < 1\}$ exactly and obtain $\hat{x} = 0.935$. $V(x)$ and the optimal strategy are plotted in Figures 2.12 and 2.13 respectively. Note, that because $V'(0) = -\frac{1}{1+\eta}$ we can calculate the initial value explicitly: $V(0) = 3.11$. ■

Example 2.2.11 (Proportional Reinsurance and $Z_i \sim$Pareto$(2,\mu)$.)
Consider the case, where the claim sizes are Pareto$(2,\mu)$–distributed, i.e. $1 - G(x) = \frac{\mu^2}{(\mu+x)^2}$. In this case we are not able to give a closed expression for the function $V^1(x)$, but we can calculate it numerically with the method of Gerber and Shiu. The value function and the optimal strategy are plotted in Figures 2.14 and 2.15 respectively. The initial value is $V(0) = 18.27$. We see, that the initial value $V(0)$ is considerably larger, than in the case of exponential distribution. ■

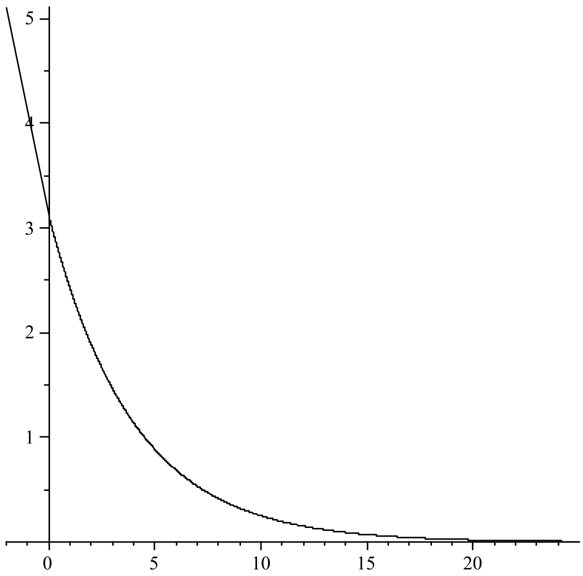

Figure 2.12: Numerically calculated value function for $Z_i \sim \text{Exp}(\frac{1}{\mu})$.

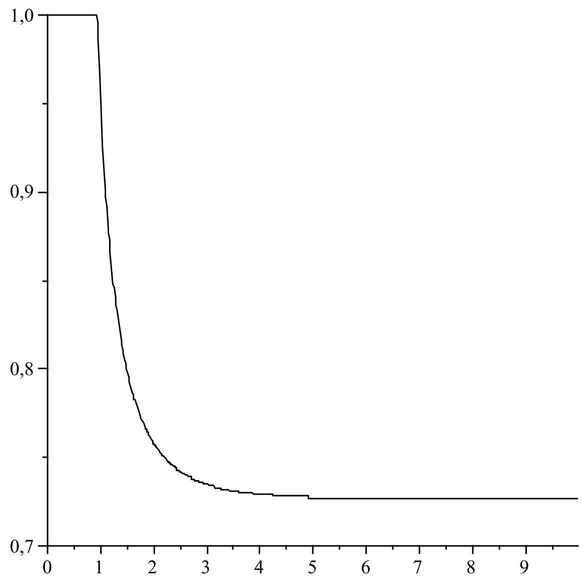

Figure 2.13: Optimal strategy for $Z_i \sim \text{Exp}(\frac{1}{\mu})$.

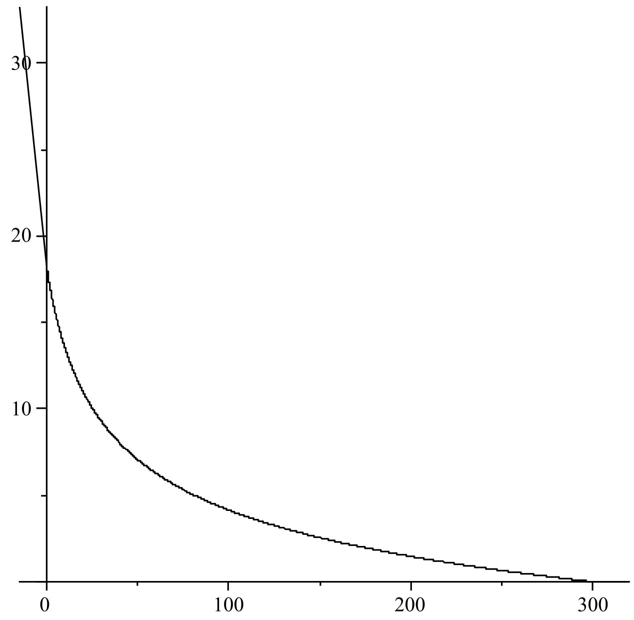

Figure 2.14: Numerically calculated value function for $Z_i \sim$Pareto$(2, \mu)$.

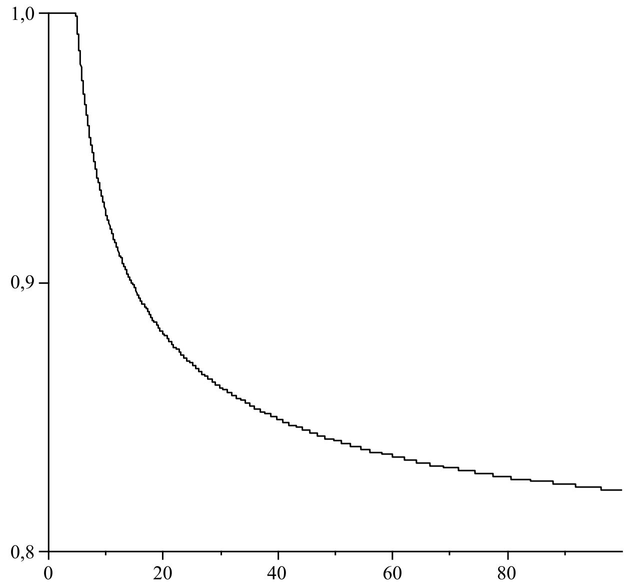

Figure 2.15: Optimal strategy for $Z_i \sim$Pareto$(2, \mu)$.

3 Optimal Control of Capital Injections by Reinsurance with Riskless Rate of Interest

In this chapter we consider the case, where the reinsurer is allowed to invest his positive surplus into a riskless asset with constant interest rate. In the case of diffusion approximation we consider only the case of proportional reinsurance.

3.1 Proportional Reinsurance for a Diffusion Approximation

Consider the surplus process of an insurance company, where the time horizon is infinite

$$C_t = x + ct - \sum_{i=1}^{N_t} Z_i \; .$$

We have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and on this space there is the Poisson process $\{N_t\}$ with intensity $\lambda > 0$ and a sequence of iid random variables $\{Z_i\}_{i\in\mathbb{N}}$. Z_i are assumed to have a distribution G with $\mu = \mathbb{E}[Z_i]$, $\mu_2 = \mathbb{E}[Z_i^2] < \infty$ and be independent of $\{N_t\}$. The premium income of the insurer is $c = (1+\eta)\lambda\mu$ for some $\eta > 0$. Further the insurer can buy proportional reinsurance. That is the insurer has to choose a retention level $b \in [0,1]$ and the reinsurer carries $(1-b)Z_i$ from each claim Z_i. The premium rate remaining to the insurer calculated by an expected value principle is $c(b) = -\lambda\mu(\theta-\eta) + \lambda\mu b(1+\theta)$, where θ is the safety loading of the reinsurer. In order to avoid, that the insurer can make a riskless profit, buying full reinsurance and still receiving a positive premium, we assume $\theta > \eta$. As an extension the insurer can change his retention level continuously.

A diffusion approximation to the above classical risk model fulfils then the differential equation

$$\mathrm{d}X_t^B = \big\{\lambda\mu\big[b_t\theta - (\theta-\eta)\big]\big\}\,\mathrm{d}t + b_t\sqrt{\lambda\mu_2}\,\mathrm{d}W_t \; ,$$

where $B = \{b_t\}$ is some admissible reinsurance strategy with $b_t \in [0,1]$. We call a reinsurance strategy admissible if it is adapted and cadlag; the set of all reinsurance strategies we denote by $\mathscr{U}$.

Usually it is supposed, that because of inflation, the original risk process is discounted, and the riskless interest rate is set to 0. Now we offset this assumption and allow the insurer to earn interest on positive surplus with a constant force of interest. That is, the controlled process with capital injections $Y^B = \{Y_t^B\}$ is then

$$\mathrm{d}X_t^{B,Y,m} = \big\{mX_t^{B,Y,m} + \lambda\mu\big[b_t\theta - (\theta-\eta)\big]\big\}\,\mathrm{d}t + b_t\sqrt{\lambda\mu_2}\,\mathrm{d}W_t + \mathrm{d}Y_t^B\;, \tag{3.1}$$

for a riskless interest rate $m > 0$.

We want to measure the risk, connected to some reinsurance strategy B, by expected discounted capital injections $V^B(x) := \mathbb{E}_x[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t]$. Our goal is to find the value function by minimising $V^B(x)$ over all admissible reinsurance strategies

$$V(x) := \inf_{B\in\mathscr{U}} V^B(x)\;.$$

Here δ expresses just the investing preferences of the insurer and is not a financial discounting factor. In the case $\delta < m$ the insurer prefers to invest all his money into a riskless asset, so that it makes no sense for him to carry on an insurance business. But here we do not restrict to $\delta \ge m$.

It is clear, that because of the discounting, the value function $V(x)$ is decreasing. In particular we obtain for the constant strategy $B \equiv 0$ before the ruin occurs

$$X_t^{0,m} = x - \lambda\mu(\theta-\eta)t + m\int_0^t X_s^{0,m}\,\mathrm{d}s = (x - \lambda\mu(\theta-\eta)m^{-1})e^{mt} + \lambda\mu(\theta-\eta)m^{-1}\;.$$

Since it holds $X_t^{0,m} > 0$ for all t if $x \ge \lambda\mu(\theta-\eta)$, we conclude $\{Y_t^0\} \equiv 0$ and accordingly $V(x) = 0$ for $x \ge \lambda\mu(\theta-\eta)$. Thus we have to consider only $0 \le x < \lambda\mu(\theta-\eta)m^{-1}$.

Remark 3.1.1
Let $\{X_t\}$ be a diffusion process with values in $\mathbb{R}$, fulfilling the stochastic differential equation

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\;,$$

where $\{W_t\}$ is a standard Brownian motion and a, σ are functions, such that the above equation has a unique strong solution. The reflected process fulfils then

$$\mathrm{d}X_t^Y = a(X_t^Y)\,\mathrm{d}t + \sigma(X_t^Y)\,\mathrm{d}W_t + \mathrm{d}Y_t\;,$$

whereas Y is the local time of the process at zero.

We find in Remark 2.1.3, that the corresponding return function $V(x) = \mathbb{E}_x[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t]$ solves the differential equation

$$\frac{\sigma^2(x)}{2}V''(x) + a(x)V'(x) - \delta V(x) = 0 \tag{3.2}$$

for $x \ge 0$ and fulfils $V'(0) = -1$, $\lim\limits_{x\to\infty} V(x) = 0$. From Subsection 2.1.1 we know, that every solution $f(x)$ to the above differential equation, vanishing at infinity, has the form

$$f(x) = f'(0)\mathbb{E}_x\big[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t\big]\;.$$

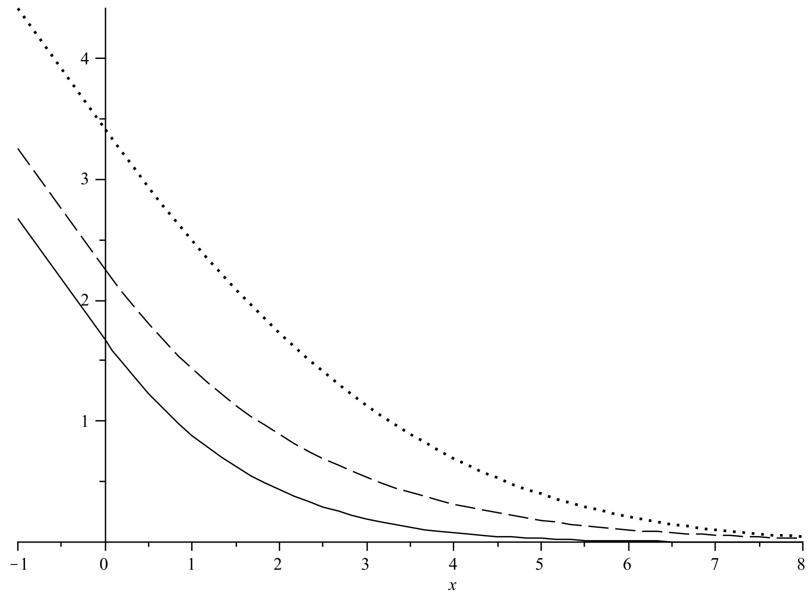

Figure 3.1: Return functions $V_{0.5}^{0.5}(x)$ (solid line), $V_{0.8}^{0.5}(x)$ (dotted line) and $V^1(x)$ dashed line.

Now equipped with the knowledge, how to calculate the return function for a given reinsurance strategy B, we illustrate the method by an example.

Example 3.1.2
Consider now a constant strategy $B \equiv b \in [0,1]$. The process $\{X_t^{b,Y,m}\}$ solves the stochastic differential equation

$$\mathrm{d}X_t^{b,Y,m} = \big\{mX_t^{b,Y,m} + \lambda\mu\big[b\theta - (\theta-\eta)\big]\big\}\,\mathrm{d}t + b\sqrt{\lambda\mu_2}\,\mathrm{d}W_t + \,\mathrm{d}Y_t^b\;.$$

Due to Remark 3.1.1 the corresponding return function $V^b(x)$ solves then the differential equation

$$\frac{b^2\lambda\mu_2}{2}f''(x) + \big(mx + \lambda\mu(b\theta-(\theta-\eta)\big)f'(x) - \delta f(x) = 0\;.$$

With the power series method, see Remark E.1.5 p. 179, we find that solutions to the above differential equation are given by

$$\begin{aligned}
&C_1\Bigg(1+\sum_{n=1}^{\infty}\frac{\prod\limits_{k=1}^{n}(\frac{\delta}{m}+2-2k)}{(2n)!}\Big(\frac{2m}{\lambda\mu_2 b^2}\Big)^n(x+\lambda\mu(b\theta-\theta+\eta)m^{-1})^{2n}\Bigg)\\
&+C_2\Bigg(\sum_{n=1}^{\infty}\frac{\prod\limits_{k=1}^{n}(\frac{\delta}{m}+1-2k)}{(2n+1)!}\Big(\frac{2m}{\lambda\mu_2 b^2}\Big)^n(x+\lambda\mu(b\theta-\theta+\eta)m^{-1})^{2n+1}\\
&+x+\lambda\mu(b\theta-\theta+\eta)m^{-1}\Bigg)\,.
\end{aligned}$$

Using the initial conditions $\lim\limits_{x\to\infty} V^b(x) = 0$ and $(V^b)'(0) = -1$ one can calculate the coefficients C_1 and C_2.

Let for example $b = 0.5$, $\lambda = \mu = 1$, $\mu_2 = 2$, $\theta = 0.5$, $\eta = 0.3$, $\delta = 0.04$ and $m = 0.03$. Then we obtain $C_1 = 4.084921164$ and $C_2 = -1.947322694$. Letting the parameter $\theta = 0.8$ yields $C_1 = 0.9686572638$ and $C_2 = -0.4617685869$. The return functions for the constant strategy $B \equiv 0.5$ $V^{0.5}_{0.5}(x)$ for $\theta = 0.5$ (solid line), $V^{0.5}_{0.8}(x)$ for $\theta = 0.8$ (dotted line) and the return function for $B \equiv 1$ $V^1(x)$ (dashed line) are plotted in Figure 3.1.

We see, that for $\theta = 0.5$ the return function corresponding to $B \equiv 0.5$ lies below $V^1(x)$; and for $\theta = 0.8$ above $V^1(x)$. We will see later, that for some θ it holds $V^1(x) = V(x)$ on some intervals. ■

We have found out, that the value function $V(x)$ fulfils $V(x) = 0$ for $x \ge \lambda\mu(\theta-\eta)m^{-1}$. Consider now $x < \lambda\mu(\theta-\eta)m^{-1}$. For such x the Hamilton–Jacobi–Bellman equation (compare Section 2.1) is given by

$$\inf_{b\in[0,1]} \frac{1}{2}\lambda\mu_2 b^2 V''(x) + \{mx - \lambda\mu(\theta-\eta) + \lambda\mu b\theta\}V'(x) - \delta V(x) = 0\,. \tag{3.3}$$

We abandon the explicit derivation of the HJB equation since it is very similar to the motivation in Subsection 2.1.2. Note that if the value function exists, is twice continuously differentiable and solves the HJB equation above, it must be convex. In fact, if we choose $\hat b = 1 - \frac{\eta}{\theta} - \frac{mx}{\lambda\mu\theta}$ for some $x \in [0, \lambda\mu(\theta-\eta)m^{-1})$ (note that $\hat b \in [0,1]$) we obtain

$$\frac{1}{2}\lambda\mu_2 \hat b^2 V''(x) - \delta V(x) \ge 0\,.$$

We make the ansatz

$$V(x) = C\big(m^{-1}\lambda\mu(\theta-\eta) - x\big)^{\kappa}$$

for some $C, \kappa > 0$. Then (3.3) reads

$$\begin{aligned}0 \;=\; & \inf_{b\in[0,1]} \bigl\{\tfrac{1}{2}\lambda\mu_2 b^2\kappa(\kappa-1)\bigl(m^{-1}\lambda\mu(\theta-\eta)-x\bigr)^{\kappa-2} \\ & - \{mx+\lambda\mu[b\theta-(\theta-\eta)]\}\kappa\bigl(m^{-1}\lambda\mu(\theta-\eta)-x\bigr)^{\kappa-1} \\ & - \delta\bigl(m^{-1}\lambda\mu(\theta-\eta)-x\bigr)^{\kappa}\bigr\}\;.\end{aligned}$$

The optimal b is then given by

$$b(x) = \frac{\theta\mu(m^{-1}\lambda\mu(\theta-\eta)-x)}{\mu_2(\kappa-1)}\;, \tag{3.4}$$

provided $\kappa > 1$ and $b(x) \le 1$; i.e., x is close enough to $m^{-1}\lambda\mu(\theta-\eta)$. For $\kappa < 1$ we should choose $b(x) = 0$. Then we would obtain that V is concave, which is a contradiction. If $\kappa = 1$ then $b(x) = 1$ has to be chosen and our ansatz would not lead to a solution. If $b(x) > 1$ no reinsurance has to be chosen.

Plugging in the optimal $b(x)$ and dividing by $(m^{-1}\lambda\mu(\theta-\eta)-x)^\kappa$ we find

$$m\kappa - \frac{\lambda\kappa\theta^2\mu^2}{2\mu_2(\kappa-1)} - \delta = 0\;. \tag{3.5}$$

Solving for κ yields the solution

$$\kappa = \frac{\delta\mu_2 + m\mu_2 + \frac{1}{2}\lambda\theta^2\mu^2 + \sqrt{(\delta\mu_2 + m\mu_2 + \frac{1}{2}\lambda\theta^2\mu^2)^2 - 4m\mu_2^2\delta}}{2m\mu_2}\;. \tag{3.6}$$

Note that the other solution is smaller than one.

Remark 3.1.3
Let $X^* = X^{b(x),Y,m}$ with initial value $0 \le x < \lambda\mu(\theta-\eta)m^{-1}$, where the reinsurance strategy $b(x)$ is given in (3.4). Consider the process $Z_t = m^{-1}\lambda\mu(\theta-\eta) - X_t^*$ before the process X^* ruins. Then

$$\mathrm{d}Z_t = -\frac{\sqrt{\lambda\mu_2}\theta\mu}{\mu_2(\kappa-1)} Z_t\,\mathrm{d}W_t - \Bigl(\frac{\lambda\mu^2\theta^2}{\mu_2(\kappa-1)} - m\Bigr) Z_t\,\mathrm{d}t\;.$$

That means that $\{Z_t\}$ is a geometric Brownian motion. Taking the logarithm gives

$$\mathrm{d}(\log(Z_t)) = -\frac{\sqrt{\lambda\mu_2}\theta\mu}{\mu_2(\kappa-1)}\,\mathrm{d}W_t + \Bigl(m - \frac{\lambda\mu^2\theta^2(2\kappa-1)}{2\mu_2(\kappa-1)^2}\Bigr)\,\mathrm{d}t\;.$$

In particular, the surplus X^* will never reach the value $m^{-1}\lambda\mu(\theta-\eta)$, where full reinsurance would be bought.

The considerations we used in deriving (3.4) are of heuristic nature. Hence, it remains to prove the verification theorem.

Theorem 3.1.4 (Verification theorem)
Define $\tilde{x} := \Big\{m^{-1}\lambda\mu(\theta-\eta) - \frac{\mu_2(\kappa-1)}{\theta\mu}\Big\} \vee 0$. *Then the strategy*

$$b^*(x) = \begin{cases} 0 & : \ x \geq m^{-1}\lambda\mu(\theta-\eta)\ , \\ b(x) & : \ \tilde{x} < x < m^{-1}\lambda\mu(\theta-\eta)\ , \\ 1 & : \ x \leq \tilde{x}\ , \end{cases}$$

where $b(x)$ *is given in* (3.4), *is an optimal reinsurance strategy. The function* $f(x)$, *given by*

$$f(x) = \begin{cases} 0 & : \ x \geq m^{-1}\lambda\mu(\theta-\eta)\ , \\ f_2(x) & : \ \tilde{x} \leq x < m^{-1}\lambda\mu(\theta-\eta)\ , \\ f_1(x) & : \ 0 < x < \tilde{x}\ , \\ f(0)-x & : \ x \leq 0\,, \end{cases}$$

with

$$\begin{aligned} f_1(x) = C_1\Bigg(1 + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n}(\frac{\delta}{m}+2-2k)}{(2n)!}\Big(\frac{2m}{\lambda\mu_2}\Big)^n (x+\lambda\mu\eta m^{-1})^{2n}\Bigg) \\ + C_2\Bigg(x+\lambda\mu\eta m^{-1} + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n}(\frac{\delta}{m}+1-2k)}{(2n+1)!}\Big(\frac{2m}{\lambda\mu_2}\Big)^n (x+\lambda\mu\eta m^{-1})^{2n+1}\Bigg) \end{aligned}$$

and

$$f_2(x) = C_3\big(r^{-1}\lambda\mu(\theta-\eta)-x\big)^{\kappa}\ ,$$

where κ *is given in* (3.6), *is twice continuously differentiable, solves the HJB equation* (3.3) *and* $f(x) = V(x)$. *If* $\tilde{x} > 0$ *the coefficients* C_1, C_2 *and* C_3 *are uniquely given by the system of equations*

$$\begin{aligned} f_1'(0) &= -1\ , \\ f_1'(\tilde{x}) &= f_2'(\tilde{x})\ , \\ f_1''(\tilde{x}) &= f_2''(\tilde{x})\ ; \end{aligned}$$

if $\tilde{x} \leq 0$, C_3 *is given by* $f_2'(0) = -1$.

Proof: We show at first for each interval, that the strategy $b^*(x)$ yields the function $f(x)$ and that the function $f(x)$ solves the HJB equation. In the second part we prove $f(x) = V(x)$.

1) Consider at first the interval $[\lambda\mu(\theta-\eta)m^{-1},\infty)$. We have already seen, that the strategy $B = 0$ yields $V^0(x) = 0$ for $x \ge \lambda\mu(\theta-\eta)m^{-1}$.

2) On the interval $[\tilde{x},\lambda\mu(\theta-\eta)m^{-1})$ we let $\{X_t^*\}$ be the underlying process and $V^*(x)$ the return function for the strategy $b^*(x)$. Let further $\tau^* = \inf\{t \ge 0 : X_t^* < \tilde{x}\}$, i.e. τ^* is the ruin time of the process $X_t^* - \tilde{x}$. From Remark 3.1.1 we know, that the capital injection process Y^* for the process X^* is given through the local time

$$Y_t^* = -\min\{\inf_{0\le s\le t} X_s^*, 0\} \ .$$

Thus, we can write because the process X_t^* has continuous paths

$$V^*(x) = V^*(\tilde{x})\mathbb{E}_x[e^{-\delta\tau^*}] \ .$$

Note, that $\lambda\mu(\theta-\eta)m^{-1} - \tilde{x} \ge 0$. From Remark 3.1.3 we know, that it holds $X_t^* = \lambda\mu(\theta-\eta)m^{-1} - Z_t$, where Z_t

$$\mathrm{d}Z_t = -\frac{\sqrt{\lambda\mu_2}\theta\mu}{\mu_2(\kappa-1)}Z_t\,\mathrm{d}W_t - \Big(\frac{\lambda\mu^2\theta^2}{\mu_2(\kappa-1)} - m\Big)Z_t\,\mathrm{d}t \ ,$$

i.e. a geometric Brownian motion. This implies

$$\tau^* = \inf\{t \ge 0 : \log\big(\lambda\mu(\theta-\eta)m^{-1} - \tilde{x}\big) - \log(Z_t) < 0\} \ ,$$

i.e. τ^* is the ruin time of a Brownian motion $\tilde{W}_t$

$$\begin{aligned}\tilde{W}_t \;&=\; \log\big(\lambda\mu(\theta-\eta)m^{-1} - \tilde{x}\big) - \log\big(\lambda\mu(\theta-\eta)m^{-1} - x\big) + \frac{\sqrt{\lambda\mu_2}\theta\mu}{\mu_2(\kappa-1)}W_t \\ &\quad - \alpha t\end{aligned}$$

with $\alpha = \Big(m - \frac{\lambda\mu^2\theta^2(2\kappa-1)}{2\mu_2(\kappa-1)^2}\Big)$. Using the change of measure technique, see Chapter C, and Remark 2.1.3 we obtain

$$\begin{aligned}\mathbb{E}_x[e^{-\delta\tau^*}] \;&=\; \exp\Big\{\log\big(\lambda\mu(\theta-\eta)m^{-1} - x\big)^\beta + \log\big(\lambda\mu(\theta-\eta)m^{-1} - \tilde{x}\big)^{-\beta}\Big\} \\ &=\; \big(\lambda\mu(\theta-\eta)m^{-1} - x\big)^\beta \cdot \big(\lambda\mu(\theta-\eta)m^{-1} - \tilde{x}\big)^{-\beta} \ ,\end{aligned}$$

where β is given by

$$\beta = \sqrt{\Big(\frac{\alpha}{\lambda\theta^2\mu^2}\mu_2(\kappa-1)^2\Big)^2 + 2\delta\frac{\mu_2(\kappa-1)^2}{\lambda\mu^2\theta^2}} - \frac{\alpha}{\lambda\theta^2\mu^2}\mu_2(\kappa-1)^2 \ .$$

Define the functions $\chi(u) := \left(m - \frac{\lambda\mu^2\theta^2(2u-1)}{2\mu_2(u-1)^2}\right)$ and

$$\gamma(u) := \sqrt{\left(\frac{\chi(u)}{\lambda\theta^2\mu^2}\mu_2(u-1)^2\right)^2 + 2\delta\frac{\mu_2(u-1)^2}{\lambda\mu^2\theta^2}} - \frac{\chi(u)}{\lambda\theta^2\mu^2}\mu_2(u-1)^2 \,.$$

It is easy to verify, that the function $\gamma(u)$ has a unique fixed point $u_2 = \gamma(u_2)$ and it holds

$$\gamma(u_2) = u_2 = \frac{\delta\mu_2 + m\mu_2 + \frac{1}{2}\lambda\theta^2\mu^2 + \sqrt{(\delta\mu_2 + m\mu_2 + \frac{1}{2}\lambda\theta^2\mu^2)^2 - 4m\mu_2^2\delta}}{2m\mu_2} \,.$$

From (3.6) we then obtain $u_2 = \kappa$ and according to this $\beta = \gamma(\kappa) = \kappa$.

Thus $V^*(x) = C_3\big(\lambda\mu(\theta-\eta)m^{-1} - x\big)^\kappa = f_3(x)$. Plugging in $f_3(x)$ into the HJB equation yields $b(x)$, given in (3.4).

3) Now we assume $\tilde{x} > 0$, otherwise it does not make any sense to consider the interval $[0, \tilde{x}]$. Note, that $\lambda\mu m^{-1}(\theta-\eta) - \frac{\mu_2(\kappa-1)}{\theta\mu} \ge 0$ holds iff $\eta \le \theta\frac{1-\frac{2\delta}{m\kappa}}{2(1-\frac{\delta}{m\kappa})}$.

That the constant strategy $B \equiv 1$ yields the function $f_1(x)$ we have already seen in Example 3.1.2. It remains to show, that plugging in $f_1(x)$ into the HJB equation we obtain $b^*(x) = 1$ for $x \in [0, \tilde{x}]$. Note, that the coefficients C_1, C_2 and C_3 are such, that $b^*(\tilde{x}) = 1$. Consider the HJB Equation (3.3) with the function $f_1(x)$. It is easy to see, that the optimal $b^*(x)$ is given by

$$b^*(x) = \frac{-\mu\theta f_1'(x)}{\mu_2 f_1''(x)} \wedge 1.$$

We need to show, that $g(x) := \frac{-\mu\theta f_1'(x)}{\mu_2 f_1''(x)} \ge 1$ for all $x \in [0, \tilde{x}]$.

Assume for the moment that there exists some $x \in [0, \tilde{x}]$ with $g(x) < 1$. Because $g(\tilde{x}) = 1$ and g is continuous, there exist some interval $[a, b] \subset [0, \tilde{x}]$ and $x^* \in [a, b]$ such, that $g'(x) > 0$ on $[a, b]$ and $g(x^*) < 1$. So we know, that

$$g'(x^*) = -\frac{\theta\mu}{\mu_2} + \frac{\theta\mu}{\mu_2}\frac{f'(x^*)f'''(x^*)}{f''(x^*)^2} > 0.$$

It follows readily

$$1 < \frac{f_1'(x^*)f_1'''(x^*)}{f_1''(x^*)^2} = \frac{-\theta\mu f_1'(x^*)}{\mu_2 f_1''(x^*)} \cdot \frac{-\mu_2 f_1'''(x^*)}{\theta\mu f_1''(x^*)} = g(x^*) \cdot \frac{-\mu_2 f_1'''(x^*)}{\theta\mu f_1''(x^*)} \,.$$

Because $g(x^*)$ was assumed to be smaller than one, $\frac{-\mu_2 f_1'''(x^*)}{\theta\mu f_1''(x^*)}$ has to be larger than one.

The function $f_1(x)$ is smooth. From Example 3.1.2 we know, that $f_1(x)$ fulfils the differential equation

$$\frac{\lambda\mu_2}{2} f_1''(x) + \big(mx + \lambda\mu\eta\big) f_1'(x) - \delta f_1(x) = 0 \, ,$$

from which we obtain the following representation:

$$\frac{\lambda\mu_2}{2} f_1'''(x^*) + \{mx^* + \lambda\mu\eta\} f_1''(x^*) - (\delta - m) f_1'(x^*) = 0 \, .$$

Rearranging the terms, dividing by $f_1''(x^*)$ and using (3.5) yields

$$\begin{aligned}
\frac{-\mu_2 f_1'''(x^*)}{\theta\mu f_1''(x^*)} &= -(\delta - m)\frac{2f_1'(x^*)}{\lambda\theta\mu f_1''(x^*)} + \frac{2}{\lambda\theta\mu}\{mx^* + \lambda\mu\eta\} \\
&< |\delta - m|\frac{2\mu_2}{\lambda\theta^2\mu^2} + \frac{2}{\lambda\theta\mu}\{m\tilde{x} + \lambda\mu\eta\} \\
&= |\delta - m|\frac{2\mu_2}{\lambda\theta^2\mu^2} + \frac{2}{\lambda\theta\mu}\Big\{\lambda\mu\theta - \frac{m\mu_2(\kappa - 1)}{\theta\mu}\Big\} \\
&= |\delta - m|\frac{\kappa}{(\kappa - 1)(m\kappa - \delta)} + 2 - \frac{m\kappa}{m\kappa - \delta} \\
&= \frac{\kappa - 2}{\kappa - 1} < 1 \, ,
\end{aligned}$$

which is a contradiction.

Now we will show $f(x) = V(x)$. Consider an arbitrary admissible reinsurance strategy $B = \{b_t\}$ and denote $\hat{X}_t = X^{B,Y,m}$. Then $\hat{X}$ is given by the differential equation

$$\mathrm{d}\hat{X}_t = \big\{m\hat{X}_t + \lambda\mu\big[b_t\theta - (\theta - \eta)\big]\big\}\,\mathrm{d}t + b_t\sqrt{\lambda\mu_2}\,\mathrm{d}W_t + \,\mathrm{d}Y_t^B \, .$$

Because $f(x)$ is two times continuously differentiable, $\hat{X}_t \geq 0$, $f'(0) = -1$ and using

(3.3) we apply Ito's formula on the function $e^{-\delta t} f(x)$ and obtain:

$$
\begin{aligned}
e^{-\delta t} f(\hat{X}_t) &= f(x) + \int_0^t e^{-\delta s} f'(\hat{X}_s)\, \mathrm{d}Y_s^B \\
&\quad + \int_0^t e^{-\delta s} \Big\{ D_{s,B} f(\hat{X}_s) - \delta f(\hat{X}_s) \Big\}\, \mathrm{d}s \\
&\quad + \int_0^t e^{-\delta s} f'(\hat{X}_s) b_s \sqrt{\lambda \mu_2}\, \mathrm{d}W_s \\
&\geq f(x) + \int_0^t e^{-\delta s} f'(\hat{X}_s)\, \mathrm{d}Y_s^B \\
&\quad + \int_0^t e^{-\delta s} f'(\hat{X}_s) b_s \sqrt{\lambda \mu_2}\, \mathrm{d}W_s \\
&= f(x) - \int_0^t e^{-\delta s}\, \mathrm{d}Y_s^B \\
&\quad + \int_0^t e^{-\delta s} f'(\hat{X}_s) b_s \sqrt{\lambda \mu_2}\, \mathrm{d}W_s \,,
\end{aligned}
$$

where

$$
D_{s,B} f(x) = \frac{\lambda \mu_2 b_s^2}{2} f''(x) + \Big\{ mx + \lambda \mu (b_s \theta - \theta + \eta) \Big\} f'(x)
$$

is the infinitesimal generator of the process $X_t^{b_s,m}$. Because the derivative of the value function is bounded, we can conclude, that the stochastic integral is a martingale with zero-expectation. Thus applying the expectations on the both sides of the above inequality we have

$$
f(x) \leq e^{-\delta t} \mathbb{E}_x\big[f(\hat{X}_t) \big] + \mathbb{E}_x \Big[\int_0^t e^{-\delta s}\, \mathrm{d}Y_s^B \Big] \,.
$$

Because $\hat{X}_t \geq 0$ and f is decreasing, we get $0 \leq f(\hat{X}_t) \leq f(0)$, from which it follows with monotone convergence

$$
f(x) \leq \mathbb{E}_x \Big[\int_0^\infty e^{-\delta s}\, \mathrm{d}Y_s^B \Big] \,.
$$

Since the strategy B was arbitrary, this proves our claim $f(x) = V(x)$. □

Example 3.1.5
Consider now the parameters $\eta = 0.3$, $\lambda = \mu = 1$, $\mu_2 = 2$ and $\delta = 0.04$. In Figure 3.2 we see the curve, which describes the dependence of $\tilde{x}$ on parameters θ and m. The pairs (θ, m) on the curve yield $\tilde{x} = 0$; the pairs to the left from the curve yield $\tilde{x} < 0$ and finally the pairs to the right yield $\tilde{x} > 0$. Note, that for example for $\theta = 0.5$ we would

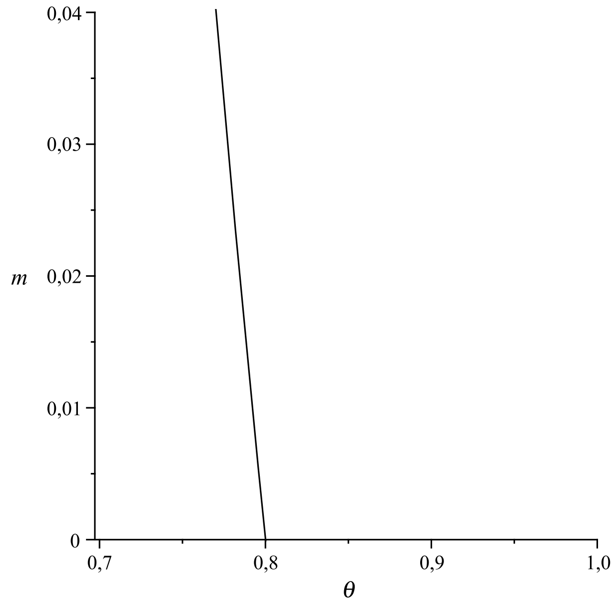

Figure 3.2: $\tilde{x} = m^{-1}\lambda\mu(\theta - \eta) - \frac{\mu_2(\kappa-1)}{\theta\mu} = 0$ in dependence from m and θ.

obtain $\tilde{x} = \left\{m^{-1}\lambda\mu(\theta-\eta) - \frac{\mu_2(\kappa-1)}{\theta\mu}\right\} < 0$ for all $m \le \delta$, so that $V(x) = f_2(x)$ in this case. We also see in Figure 3.1, that for $\theta = 0.5$ even the return function corresponding to the constant strategy $B \equiv 0.5$ (solid line) lies below $f_1(x)$ (dashed line). Choose $\theta = 0.8$ and $m = 0.03$. It is easy to verify, that $\kappa = 7.49$ and the optimal strategy on $[\tilde{x}, \lambda\mu(\theta-\eta)m^{-1}]$ and value function are given by:

$$b(x) = \frac{13.3 - 0.8x}{12.98}$$

$$\begin{aligned}
f_1(x) &= 64.28\left(1 + \sum_{n=1}^{\infty} \frac{1.33\cdots(1.33-2n+2)}{(2n)!} 0.03^n (x+10)^{2n}\right) \\
&\quad - 15.32\left(x + 10 + \sum_{n=1}^{\infty} \frac{(1.33-1)\cdots(1.33-2n+1)}{(2n+1)!} 0.03^n (x+10)^{2n+1}\right) \\
f_2(x) &= 1.575819495 \cdot 10^{-9}(6.7-x)^{7.49}.
\end{aligned}$$

The optimal strategy is mapped in Figures 3.4. In Figure 3.3 we can see the value function, composed of 4 functions. The dashed line corresponds to $f(0) - x$, the dotted line to $f_1(x)$, the solid line to $f_2(x)$ and the line with wide dots to 0. ■

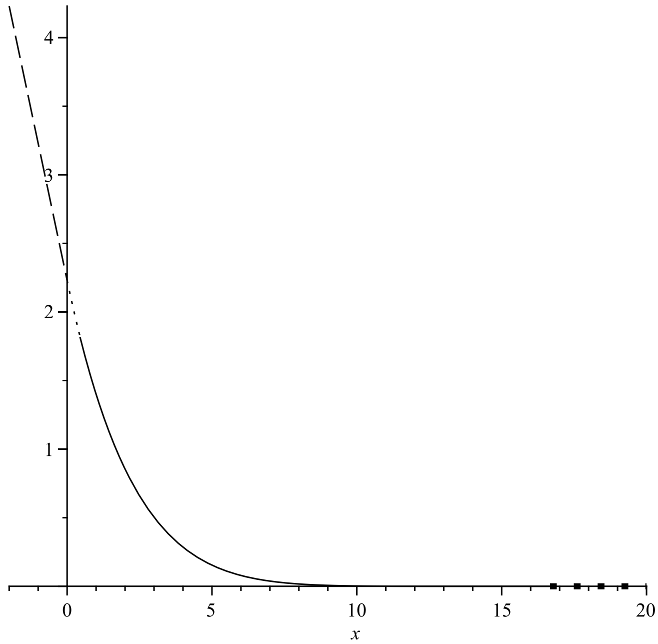

Figure 3.3: Value function for the proportional reinsurance $V(x)$.

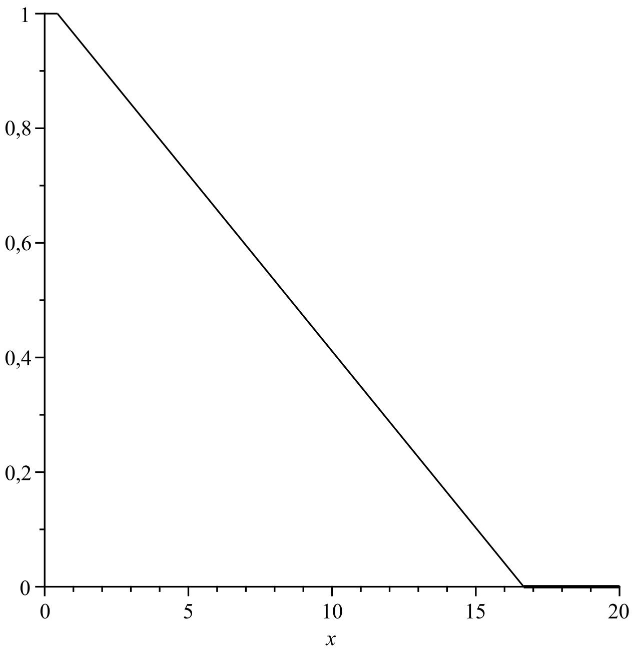

Figure 3.4: Optimal strategy $b^{*}(x)$.

3.2 The Classical Risk Model

In this section we consider the classical risk model. The number of claims is assumed to follow a Poisson process $\{N_t\}$ with intensity λ. Claim sizes $Z_1, Z_2, \dots$ are independent of $\{N_t\}$, positive and iid with distribution function $G(z)$ and first moment μ. By Z we denote a generic random variable with the same distribution as Z_i. The aggregate claim process is given by

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i \,,$$

where x is the initial capital, $c = \lambda\mu(1+\eta)$ is the premium rate and η is the safety loading.

The insurer can buy reinsurance. Choosing the level $b \in [0, \tilde{b}]$, the insurer pays $r(Z_i, b)$ for a claim of size Z_i. Here $b = 0$ means full reinsurance, $b = \tilde{b} \in (0, \infty]$ means no reinsurance. For example, for proportional reinsurance $r(Z, b) = bZ$ and $b \in [0, 1]$. For excess of loss reinsurance we obtain $r(Z, b) = \min\{Z, b\}$ and $b \in [0, \infty]$.

For the reinsurance cover the insurer pays a premium at rate $c - c(b)$. That is, the premium rate left for the cedent is $c(b)$. For simplicity we assume that $r(z, b)$ is continuous and increasing in both z and b and that $c(b)$ is continuous and increasing with $c(0) < 0$ and $c(\tilde{b}) = c$. The assumption $c(0) < 0$ is needed in order that the problem below is not trivial. If the reinsurer uses an expected value principle with safety loading θ, we get

$$c(b) = c - (1+\theta)\lambda\mathbb{E}[Z - r(Z, b)] = (1+\theta)\lambda\mathbb{E}[r(Z, b)] - (\theta - \eta)\lambda\mu \,.$$

In order that $c(0) < 0$ we assume $\theta > \eta$. The insurer can choose the level b_t at any time point t. Because no information on the future can be used the process $\{b_t\}$ is assumed to be cadlag and adapted. The surplus of the insurer including reinsurance has then the form

$$X_t^B = x + \int_0^t c(b_s)\,\mathrm{d}s - \sum_{i=1}^{N_t} r(Z_i, b_{T_i-}) \,.$$

To prevent that the surplus process becomes negative, the insurer has to inject additional capital. We denote the accumulated capital injections until time t by $\{Y_t^B\}$. The surplus with capital injections has therefore the form

$$X_t^{B,Y} = x + \int_0^t c(b_s)\,\mathrm{d}s - \sum_{i=1}^{N_t} r(Z_i, b_{T_i-}) + Y_t^B \,.$$

Assume in addition to this concept, that the insurer can invest his money, if his surplus is positive, with a fixed rate of interest $m > 0$. I.e. his surplus has to fulfil

$$X_t^{B,m,Y} = x + \int_0^t \bigl(c(b_s) + mX_s^{B,m,Y}\bigr)\,\mathrm{d}s - \sum_{i=1}^{N_t} r(Z_i, b_{T_i-}) + Y_t^B \,.$$

We are interested in the minimal value $V(x) = \inf_{B\in\mathscr{U}} \mathbb{E}[\int_0^\infty e^{-\delta t}\, \mathrm{d}Y_t^B]$, where δ is a discounting factor. The discounting expresses the investment preferences of the company holders. If $m \geq \delta$, there would be no sense in holding an insurance company, because investing money in a riskless asset with interest rate m would be more profitable. Here we do not assume $\delta > m$, although $m \geq \delta$ seems to be economically sinless. We also abandon the assumption $\eta > 0$ for the safety loading η.

Note, that our process remains deterministic before the first claim occurs. Let τ denote the ruin time of the process $X^{B,m}$. Then we have for $t < \tau$

$$X_t^{B,m} = x + \int_0^t c(b_s) + m\int_0^t X_s^{B,m}\, \mathrm{d}s\,,$$

so that we can write

$$X_t^{B,m} = \Big(\int_0^t c(b_s)e^{-ms}\, \mathrm{d}s + x\Big)e^{mt}\,. \tag{3.7}$$

Lemma 3.2.1
The function $V(x)$ is decreasing with $V(x) = 0$ for $x \geq \lambda\mu(\theta-\eta)m^{-1}$. $V(x)$ is Lipschitz continuous with $|V(x) - V(y)| \leq |x - y|$.

Proof: It is clear that $V(x)$ is decreasing. Assume now $x \geq c(0) = \lambda\mu(\theta - \eta)m^{-1}$. Consider the process with the constant strategy $B \equiv 0$. Then we obtain from (3.7)

$$X_t^{0,m} = e^{mt}\big\{x + c(0)m^{-1}(1 - e^{-mt})\big\}\,.$$

Because $x \geq c(0) = \lambda\mu(\theta - \eta)m^{-1}$ it holds $X_t^0 \geq 0$ for all $t \geq 0$. Thus for initial capital $x \geq \lambda\mu(\theta-\eta)m^{-1}$ it holds $\{Y_t^0\} \equiv 0$ and accordingly $0 \leq V(x) \leq V^0(x) = 0$.

Let now $x < \lambda\mu(\theta - \eta)m^{-1}$ and choose $z > x$. Let further $B = \{b_t\}$ be a reinsurance strategy for initial capital z such that $V^B(z) \leq V(z)+\varepsilon$. For initial capital x choose the strategy $\tilde{B}$ (which is not optimal): inject the capital $z - x$ and then follow the strategy B. Thus

$$V(x) - V(z) \leq V^{\tilde{B}}(x) - V^B(z) + \varepsilon = z - x + \varepsilon\,.$$

Because ε is arbitrary we have $|V(x) - V(z)| \leq |x - z|$, which proves the Lipschitz-continuity. As a consequence, $V(x)$ is absolutely continuous. □

We conjecture, that the value function $V(x)$ solves the Hamilton–Jacobi–Bellman equation

$$\inf_{b\in[0,\tilde{b}]} \lambda\int_0^\infty V(x - r(z,b))\, \mathrm{d}G(z) + (c(b) + mx)V'(x) - (\delta+\lambda)V(x) = 0\,. \tag{3.8}$$

Theorem 3.2.2
The function $V(x)$ is continuously differentiable from the right and from the left at all points where $b_0(x)$ with $c(b_0(x)) = -mx$ is not optimal. Its derivatives solve Equation (3.8) *with the interpretation $(c(b) + mx)V'(x) = 0$, if the derivatives do not exist. Moreover, if there exists a b such that $c(b) \geq \lambda\mathbb{E}[r(Z, b)]$, then any decreasing positive solution to* (3.8) *coincides with $V(x)$.*

Proof: Assume $x > 0$. Let $h > 0$ and $b \in [0, \tilde{b}]$ be fixed. We can assume that $e^{mt}(x + c(b)m^{-1}) - c(b)m^{-1} \geq 0$, i.e. the ruin does not occur because of the premium payments to the reinsurer. Let T_1 be the time of the first claim and choose $\varepsilon > 0$. We further choose $n \in \mathbb{N}$ such that $2(e^{mh}(x + c(b)m^{-1}) - c(b)m^{-1})/n < \varepsilon$. For each k there is a strategy $B^k = \{b^k_t\}$ such that $V^{B^k}(x_k) \leq V(x_k) + \varepsilon/2$. For initial capital $x_k \leq x < x_{k+1}$ we choose the strategy B^k. Thus $V^B(x) \leq V^B(x_k) \leq V(x_k) + \varepsilon/2 \leq V(x) + (x - x_k) + \varepsilon/2 < V(x) + \varepsilon$ by Lemma 3.2.1. This shows that for each $x \in [0, e^{mh}(x + c(b)m^{-1}) - c(b)m^{-1}]$ we can find in a measurable way a strategy $\hat{B}(x)$ such that $V^{\hat{B}(x)}(x) < V(x) + \varepsilon$.

Consider now the strategy $b_t = b\mathbb{1}_{[t<T_1\wedge h]} + \hat{b}_{t-(T_1\wedge h)}(X_{T_1\wedge h})\mathbb{1}_{[t\geq T_1\wedge h]}$ and define the deterministic process $k_t := -c(b)m^{-1} + (x + c(b)m^{-1})e^{mt}$. Then we obtain by conditioning on $\mathscr{F}_{T_1\wedge h}$ and using (3.7)

$$
\begin{aligned}
V(x) &\leq V^B(x) \\
&= \mathbb{E}_x\Big[\mathbb{1}_{[T_1\leq h]}V^{\hat{B}}\Big(k_{T_1} - r(Z_1, b)\Big)e^{-\delta T_1} \\
&\quad + \mathbb{1}_{[T_1>h]}V^{\hat{B}}\big(k_h\big)e^{-\delta h}\Big] \\
&= \int_0^h \int_0^\infty \lambda e^{-(\delta+\lambda)t}V^{\hat{B}}\big(k_t - r(z, b)\big)\,\mathrm{d}G(z)\,\mathrm{d}t \\
&\quad + \int_h^\infty e^{-\delta h}V^{\hat{B}}(k_h)\lambda e^{-\lambda t}\,\mathrm{d}t \\
&\leq \lambda\int_0^h \int_0^\infty e^{-(\delta+\lambda)t}V\big(k_t - r(z, b)\big)\,\mathrm{d}G(z)\,\mathrm{d}t \\
&\quad + e^{-(\delta+\lambda)h}V(k_h) + \varepsilon\,.
\end{aligned}
$$

Because ε was arbitrary we can let it be equal to zero. Rearranging the terms and dividing by h yields:

$$
\begin{aligned}
0 \;\leq\; &\frac{1}{h}\int_0^h \int_0^\infty \lambda e^{-(\delta+\lambda)t}V\big(k_t - r(z, b)\big)\,\mathrm{d}G(z)\,\mathrm{d}t \\
&+ \frac{V\big(k_h\big) - V(x)}{h}e^{-(\delta+\lambda)h} + \frac{e^{-(\delta+\lambda)h} - 1}{h}V(x)\,.
\end{aligned}
\tag{3.9}
$$

Consider a strategy $D(h) = \{d_t(h)\}$ with $V^D(x) \le V(x) + h^2$. Let further $a(t,h) = \int_0^t c(d_s)e^{-ms}\,\mathrm{d}s$. Then we obtain in the same way as above

$$\begin{aligned} 0 \quad \ge \quad & \frac{V\big(e^{mh}\{x + a(h,h)\}\big) - V(x)}{h} e^{-(\delta+\lambda)h} + \frac{e^{-(\delta+\lambda)} - 1}{h} V(x) \\ & \frac{1}{h}\int_0^h \int_0^\infty \lambda e^{-(\delta+\lambda)t} V\big(e^{mt}\{x + a(t,h)\} - r(z,d_t)\big)\,\mathrm{d}G(z)\,\mathrm{d}t - h\;. \end{aligned}$$

All the terms with exception of the first one converge. And for the first one we obtain

$$\begin{aligned} \frac{V\big(e^{mh}\{x + a(h,h)\}\big) - V(x)}{h} \quad = \quad & \frac{V\big(e^{mh}\{x + a(h,h)\}\big) - V(x)}{e^{mh}\{x + a(h,h)\} - x} \\ & \times \frac{e^{mh}\{x + a(h,h)\} - x}{h}\;. \end{aligned}$$

The first term on the right hand side of the above equation has the values in the interval $[-1, 0]$, because of the Lipschitz continuity; the second one has his values in $\big[c(0) + xm, c + xm\big]$. Thus there exists a sequence $h_n \to 0$ such that

$$\lim_{n\to\infty} \frac{V\big(e^{mh_n}\{x + a(h_n,h_n)\}\big) - V(x)}{h_n} = \limsup_{h\to 0} \frac{V\big(e^{mh}\{x + a(h,h)\}\big) - V(x)}{h}\;.$$

Because of the Lipschitz continuity of the value function $V(x)$ the above limit is finite. Assume now, that $\frac{e^{mh_n}\{x + a(h_n,h_n)\} - x}{h_n}$ goes to $c(\check{b}) + mx$ as n goes to infinity. Using, that the strategy D is cadlag, we obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{V\big(e^{mh}\{x + a(h_n,h_n)\}\big) - V(x)}{h_n} + \lambda \int_0^\infty V\big(x - r(z,\check{b})\big)\,\mathrm{d}G(z) & \\ -(\lambda + \delta)V(x) \le 0\;. & \end{aligned}$$

Together with (3.9) and $b = \check{b}$ equality follows. The above limit does not depend on the chosen subsequence, i.e. the limit

$$\lim_{h\to 0} \frac{V\big(-c(\check{b})m^{-1} + \{x + c(\check{b})m^{-1}\}e^{mh}\big) - V(x)}{h}\;.$$

exists. It means in particular, that in the case $c(\check{b}) + mx > 0$ $V(x)$ is differentiable in x from the right; in the case $c(b) + mx < 0$ from the left. In both cases (3.8) was shown for the corresponding derivatives.

Consider now the initial capital $p_h = -c(\check{b})m^{-1}\big(1 - e^{-mh}\big) + xe^{-mh}$. It holds with the same arguments as by the derivation of (3.9):

$$\begin{aligned} & \frac{1}{h}\int_0^h \int_0^\infty \lambda e^{-(\delta+\lambda)t} V\Big(e^{mt}\{xe^{-mh} + c(\check{b})m^{-1}\big(e^{-mh} - e^{-mt}\big)\}\Big)\,\mathrm{d}G(z)\,\mathrm{d}t \\ & + \frac{V(p_h) - V(x)}{h} + \frac{e^{-(\delta+\lambda)h} - 1}{h} V\big(p_h\big) \ge 0\;. \end{aligned} \tag{3.10}$$

Assume further, that $V(x)$ is differentiable in x from the right, i.e. $c(\check{b}) + mx > 0$. For all $z < x$ there exists some $\check{b}(z)$, where the minimum in (3.8) is taken. Suppose there is a sequence $x_n \to x$ with $x_n \le x$ and $c(\check{b}_n) + mx_n = c(\check{b}(x_n)) + mx_n > 0$. By taking a subsequence we can assume, that $\check{b}_n$ converges to some b^* with $c(b^*) + mx \ge 0$. Then we can find a sequence h_n such that (3.10) converges; (3.8) holds then by continuity. If in addition to the above assumptions $c(b^*) > 0$ we can conclude, that $V(x)$ is also differentiable from the left.

If there is a sequence $x_n \to x$ with $x_n \le x$ and $c(\check{b}_n) + mx_n < 0$, choose $h_n = -\frac{x-x_n}{c(\check{b}_n)+mx_n}$. If $h_n \to 0$, then differentiability from the left follows and Equation (3.8) holds. If h_n does not converge to 0, then $c(\check{b}_n)+mx_n \to 0$ and (3.10) holds with minimum taken at $\check{b}$ with $c(\check{b})+mx = 0$. In the case $c(\check{b}_n) = -mx_n$ (3.10) holds and the minimum is taken in $\check{b}$ with $c(\check{b}) + mx = 0$.

At $x = 0$ we distinguish between $b \le b_0(0)$ and $b > b_0(0)$. For $b > b_0(0)$ (3.9) holds.

For $b \le b_0(0)$ it holds $X_t^{B,m,Y} = 0$ and

$$V(0) = \Big[\sum_{i=1}^{\infty} r(Z_i, b)e^{-\delta T_i}\Big] - \frac{c(b)}{\delta} = \frac{\lambda \mathbb{E}[r(Z, b)] - c(b)}{\delta} .$$

It means that equality in Equation (3.9) holds. However, it follows from (3.9) for $b > b_0(0)$ that the derivative from the right must be -1. If it holds $V(0) > \frac{\lambda \mathbb{E}[r(Z,b)]-c(b)}{\delta}$, then the infimum is taken at some $b \ge b_0(0)$. Because $b_0(0)$ is not possible, it follows the differentiability from the right.

The proof of uniqueness goes perfectly similar to the proof of Theorem 2.2.3. □

The optimal strategy for $x \in [0, \varepsilon)$ for some $\varepsilon > 0$ is given by $b^*(x) = \tilde{b}$. The proof goes perfectly similar to the explanations before and in Remark 2.2.4.

We have seen, that the derivative of the value function may not exist everywhere and if it exists, it is not necessarily continuous. However, there is a condition under which the value function is actually continuously differentiable.

Lemma 3.2.3
If the value function $V(x)$ is convex, then $V(x)$ is continuously differentiable.

Proof: Let $b_0(x)$ denote the root of the equation $c(b) + mx$ for $x \ge 0$ and define

$$f(x) := \lambda \int_0^\infty V(x - r(b_0(x), z))\, \mathrm{d}G(z) - (\delta + \lambda)V(x) .$$

By (3.8), $f(x) \ge 0$. Let $V'(x-)$ and $V'(x+)$ denote the derivatives from the right and from the left, respectively. Assume now, there exists $\tilde{x} \in \mathbb{R}_+$ with $V'(\tilde{x}-) < V'(\tilde{x}+)$. By Theorem 3.2.2, $f(\tilde{x}) = 0$. Note, that because the optimal strategy in 0 is $b^* = 1$, it holds $\tilde{x} > 0$. W.l.o.g. we can assume, that $V(x)$ is continuously differentiable on $(0, \tilde{x})$, which means $f(x)$ is continuously differentiable on $(0, \tilde{x})$. There exist sequences $(h_n)_{n\geq 0} \in (0, \tilde{x})$, $\lim_{n\to\infty} h_n = \tilde{x}$, with $f'(h_n) \leq 0$ and $(x_n)_{n\geq 0} \in (\tilde{x}, \infty)$, $\lim_{n\to\infty} x_n = \tilde{x}$, with $f'(x_n) \geq 0$. Letting n tend to infinity we obtain

$$\lambda \int_0^\infty V'(\tilde{x} - r(b_0(x), z))\, \mathrm{d}G(z) - (\delta + \lambda)V'(\tilde{x}-) \leq 0 \,,$$
$$\lambda \int_0^\infty V'(\tilde{x} - r(b_0(x), z))\, \mathrm{d}G(z) - (\delta + \lambda)V'(\tilde{x}+) \geq 0 \,.$$

Thus $V'(\tilde{x}+) \leq V'(\tilde{x}-)$, which is a contradiction. □

Next we consider some examples in the special case of proportional reinsurance.

3.2.1 Examples

Here we will consider only the proportional reinsurance.

Let us first notice that in the case of proportional reinsurance we obtain as in the case without interest rate, that the value function is convex, if $c(b)$ is concave. Let namely $x, z \geq 0$, $\alpha \in (0, 1)$ and $y = \alpha x + (1 - \alpha)z$. Let $\{b_t^x\}$ be the optimal strategy for initial capital x and $\{b_t^z\}$ the optimal strategy for initial capital z. Define the new strategy $b_t^y = \alpha b_t^x + (1-\alpha)b_t^z$. Then for the expected value principle $c(b_t^y) = \alpha c(b_t^x) + (1-\alpha)c(b_t^z)$. Hence,

$$\begin{aligned} &X_t^y - Y_t^y + \alpha Y_t^x + (1 - \alpha)Y_t^z \\ &= \; y + \int_0^t c(b_s^y)\, \mathrm{d}s - \sum_{i=1}^{N_t} b_{T_i-}^y Z_i + \alpha Y_t^x + (1 - \alpha)Y_t^z \\ &= \; \alpha X_t^x + (1 - \alpha)X_t^z \geq 0 \,. \end{aligned}$$

This implies that $Y_t^y \leq \alpha Y_t^x + (1 - \alpha)Y_t^z$. Thus,

$$V(y) \leq V^{B^y}(y) \leq \alpha V(x) + (1 - \alpha)V(z) \,,$$

which proves the convexity. Like in the case without interest rate, the convexity can be shown for every premium calculation principle, where $c(b)$ is concave in b.

From the convexity we can conclude, that the value function is continuously differentiable, for proof see Lemma 3.2.3.

The next problem in numerical calculation of the value function is, that we have no conditions to calculate the initial value $V(0)$ exactly. But since we know $V(x) = 0$ for $x \geq \lambda\mu(\theta - \eta)m^{-1}$ the calculation of initial value becomes less complicated as in the case without interest rate. We have just to find $V(0)$ with $V(\lambda\mu(\theta - \eta)m^{-1}) = 0$. If the chosen $V(0)$ was too large, we would obtain $V(\lambda\mu(\theta - \eta)m^{-1}) > 0$.

Proof: Define
$$g(x) := f(x) - V(x) \ ,$$
where $f(x)$ is numerically calculated value function with some fixed initial value $f(0)$ and $g(0) > 0$. Let $b^*(x)$ denote the optimal strategy for $V(x)$ and $b_0(x)$ the root of the equation $c(b) + mx = 0$. Replacing the optimal b for $f(x)$ by $b^*(x)$ yields

$$\big(c(b^*(x)) + mx\big)g'(x) + \lambda \int_0^\infty g(x - b^*(x)z)\,\mathrm{d}G(z) - (\delta + \lambda)g(x) \geq 0 \ . \tag{3.11}$$

Note that $g(x) = g(0)$ for $x \leq 0$. Because $g(0) > 0$ and $b^*(0) = 1$ it follows $g'(0) > 0$. Let $\hat{x} = \inf\{x : g'(x) \leq 0\}$. Because $g(x)$ is increasing we conclude that $b^*(x) > b_0(x)$ on $[0, \hat{x})$. From (3.8) and Lemma 3.2.1 we conclude that also $b^*(\hat{x}) > b_0(x)$. Because $g(x)$ is increasing on $[0, \hat{x}]$, it follows from (3.11) that $\big(c(b^*(\hat{x})) + mx\big)g'(\hat{x}) > 0$, which is a contradiction. So the function $g(x)$ is strictly increasing on $\mathbb{R}_+$. Therefore, $f(x)$ will ultimately be increasing. $\square$

First we show, that in the special case of exponentially distributed claim sizes, it is possible to give a closed expression for the return function corresponding to the constant strategy $B \equiv 1$.

Example 3.2.4 (No reinsurance)
In the case, we would not buy reinsurance at all, one can show with usual arguments, that the corresponding return function $V^{\tilde{b}}(x)$ solves the integro-differential equation

$$\lambda \int_0^\infty f(x - z)\,\mathrm{d}G(z) + (c + mx)f'(x) - (\delta + \lambda)f(x) = 0 \ .$$

Consider now the special case, where the claim sizes are exponentially distributed. Then the above integro-differential equation becomes

$$\frac{\lambda}{\mu} \int_0^\infty f(x - y)e^{-\frac{y}{\mu}}\,\mathrm{d}y + (c + mx)f'(x) - (\delta + \lambda)f(x) = 0 \ .$$

Let

$$\begin{aligned} M(a,b,x) &= \sum_{n=0}^\infty \frac{(a)_n}{(b)_n \cdot n!}x^n \ , \\ U(a,b,x) &= \frac{\pi}{\sin(\pi b)}\Big(\frac{M(a,b,x)}{\Gamma(1+a-b)\Gamma(b)} - x^{1-b}\frac{M(1+a-b, 2-b, x)}{\Gamma(a)\Gamma(2-b)}\Big) \end{aligned}$$

denote the Kummer's functions, where $(a)_n = a(a+1)\cdots(a+n-1)$. Then the solutions to the above integro-differential equation are given by a sum of Kummer's functions

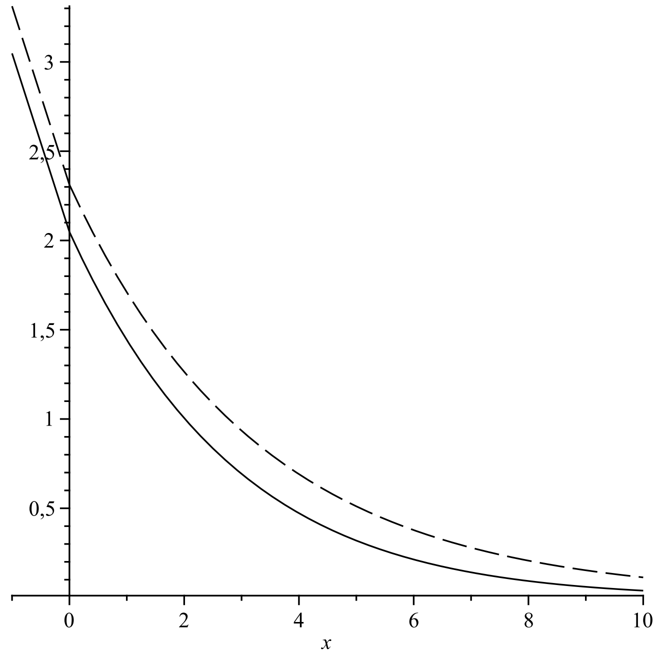

Figure 3.5: $V^{\tilde{b}}(x)$ (solid line) and $V^{\tilde{b}}_{Cl}(x)$ for exponentially distributed claim sizes.

multiplied by an exponential function.

$$
\begin{aligned}
&e^{\frac{-x}{\mu}}\Big\{C_1\cdot M\Big(\frac{m+\delta}{m},\frac{2m+\delta+\lambda}{m},\frac{c+mx}{\mu m}\Big)\cdot(c+mx)^{\frac{\delta+\lambda}{m}}(m+\delta+\lambda)\\
&+C_2\cdot U\Big(\frac{m+\delta}{m},\frac{2m+\delta+\lambda}{m},\frac{c+mx}{\mu m}\Big)\cdot(c+mx)^{\frac{\delta+\lambda}{m}}(m+\delta+\lambda)\\
&+C_1\cdot\frac{m+\delta}{(m+\delta+\lambda)\mu}M\Big(\frac{2m+\delta}{m},\frac{3m+\delta+\lambda}{m},\frac{c+mx}{\mu m}\Big)\cdot(c+mx)^{\frac{m+\delta+\lambda}{m}}\\
&-C_2\cdot\frac{m+\delta}{m\mu}U\Big(\frac{2m+\delta}{m},\frac{3m+\delta+\lambda}{m},\frac{c+mx}{\mu m}\Big)\cdot(c+mx)^{\frac{m+\delta+\lambda}{m}}\Big\}\,.
\end{aligned}
$$

The coefficients C_1 and C_2 can be verified from the equations $\delta V^{\tilde{b}}(0)=\lambda\mu+c(V^{\tilde{b}})'(0)$ and $\lim\limits_{x\to\infty}V^{\tilde{b}}(x)=0$.

For the parameters $c=1.3$, $\mu=\lambda=1$, $\delta=0.04$ and $m=0.03$ we obtain $C_1=0$ and $C_2=0.1076089638$. In particular we find $V^{\tilde{b}}(0)=2.047949073$. $V^{\tilde{b}}(0)$ yields an upper boundary for the value $V(0)$.

The function $V^{\tilde{b}}(x)$ together with the return function $V^{\tilde{b}}_{Cl}$ of the strategy $B\equiv\tilde{b}$ in the classical risk model without interest rate, are plotted in Figure 3.5. One sees, that the curve $V^{\tilde{b}}_{Cl}(x)$ lies considerably above the curve $V^{\tilde{b}}(x)$. Thus one can minimise the cost just investing the excess in some riskless asset. ■

Example 3.2.5 (Exp($\frac{1}{\mu}$) and Pareto distributed claims.)
For the exponentially distributed claim sizes we have to consider

$$\inf_{b\in[0,1]} \frac{\lambda}{\mu}\int_0^\infty V(x-bz)e^{-\frac{z}{\mu}}\,\mathrm{d}z + (\lambda\mu(b\theta-\eta)+mx)V'(x) - (\delta+\lambda)V(x) = 0\,.$$

Numerically calculated optimal strategy and value function are plotted in Figures 3.6 and 3.7 respectively. The initial value is 1.37. For Pareto distributed claims the HJB equation becomes

$$\inf_{b\in[0,1]} \lambda\int_0^\infty V(x-bz)\frac{2\mu^2}{(\mu+z)^3}\,\mathrm{d}z + (\lambda\mu(b\theta-\eta)+mx)V'(x) - (\delta+\lambda)V(x) = 0\,.$$

Numerically calculated optimal strategy and value function are plotted in Figures 3.8 and 3.9 respectively. The initial value is 1.91963. ■

3.2.2 The special case $\delta = 0$

We consider now the special case $\delta = 0$. The methods, we use in this subsection, differ very little from the methods of subsection 2.2.3. But there are some interesting nuances in the case we will consider below. In particular we will see, that investing money into a riskless asset lowers the cost also in the case $\delta = 0$.

Here we have again to assume the net profit condition $\eta > 0$. The Hamilton–Jacobi–Bellman equation has the form

$$\inf_{b\in[0,\tilde{b}]} \lambda\int_0^\infty V(x-r(z,b))\,\mathrm{d}G(z) + (c(b)+mx)V'(x) - \lambda V(x) = 0\,. \tag{3.12}$$

From Lemma 3.2.1 we know, that the value function is Lipschitz continuous with $|V(x)-V(y)| \le |x-y|$ and $V(x)\equiv 0$ for $x \ge \lambda\mu(\theta-\eta)m^{-1}$.

We also obtain due to representation (3.7) of the underlying process, that a constant strategy b with $c(b)+mx \le 0$ for some $x\in\mathbb{R}_+$ cannot be optimal, because in this case $V^b(x)$ will have an infinite value. Because the strategy $b_t = \tilde{b}$ yields a finite value, the value function, provided it exists, will also be finite.

Let in the following $b_0(x)$ denote the root of the equation $c(b)+mx$. Note, that $b \le b_0(x)$ will never be the optimal strategy for x, because for such b it would hold

$$\lambda\int_0^\infty V(x-r(z,b))\,\mathrm{d}G(z) + (c(b)+mx)V'(x) - \lambda V(x) > 0\,.$$

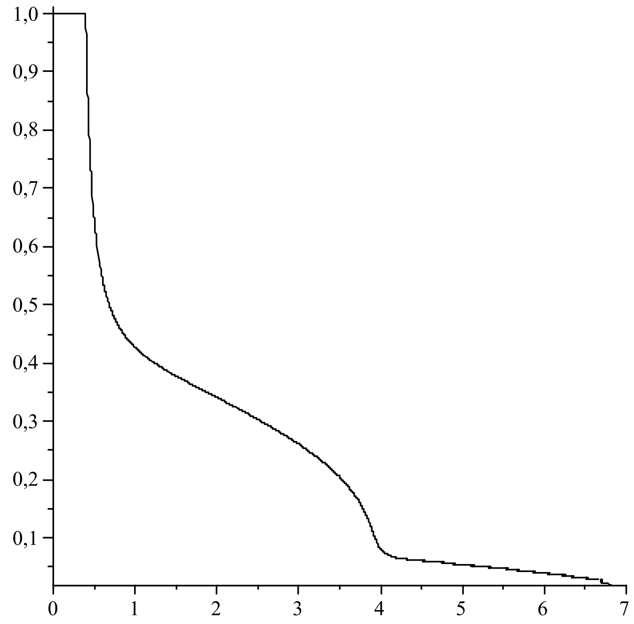

Figure 3.6: Optimal reinsurance strategy for exponentially distributed claim sizes.

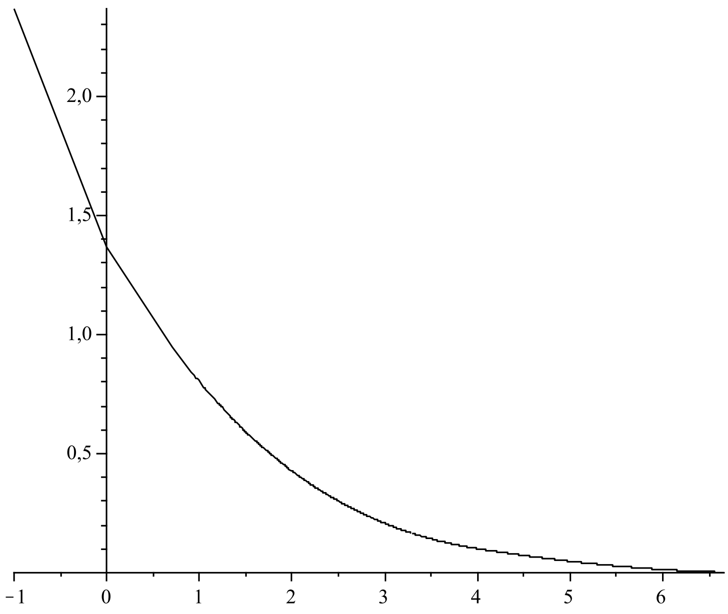

Figure 3.7: Value function for exponentially distributed claim sizes.

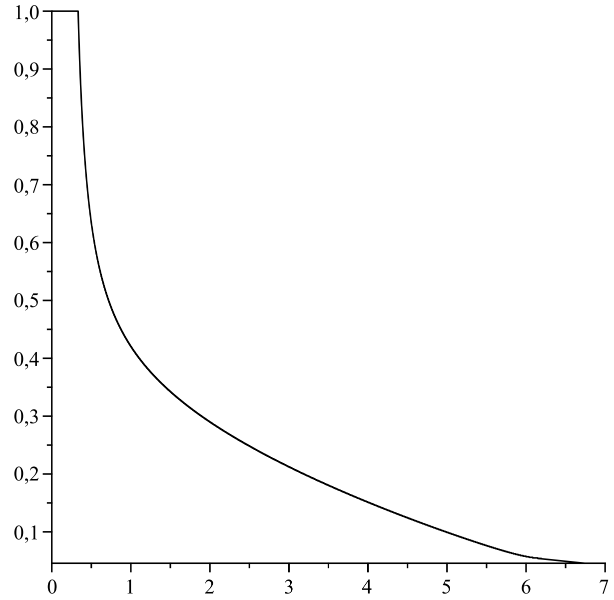

Figure 3.8: Optimal reinsurance strategy for Pareto distributed claim sizes.

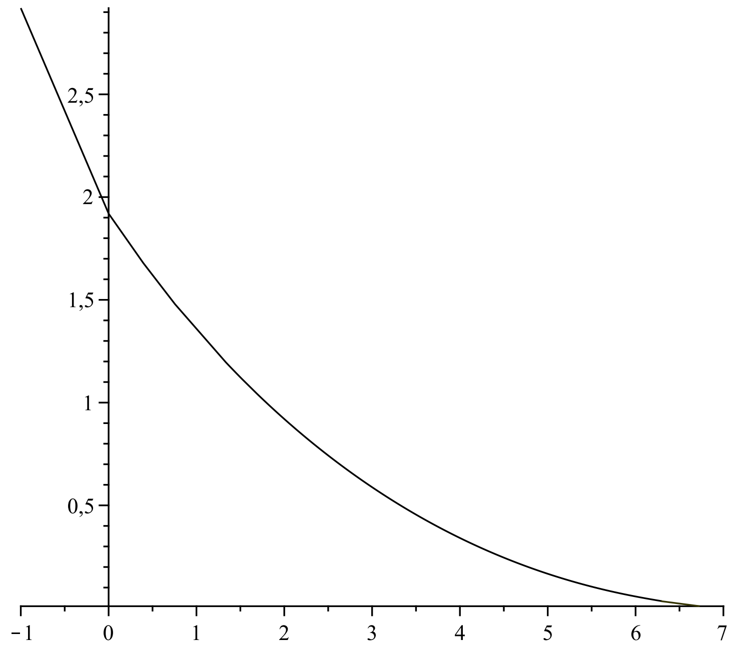

Figure 3.9: Value function for Pareto distributed claim sizes.

Next we transform the HJB equation using the fact, that for the minimiser b we have $c(b) + mx > 0$. Define for this purpose

$$s(x, b) = \inf\{z : r(z, b) > x\}.$$

Then we can rewrite the equation (3.12) as follows:

$$\inf_{b\in[0,\tilde{b}]} \Big\{\lambda \int_0^{s(x,b)} V\big(x - r(z, b)\big)\,\mathrm{d}G(z) + \lambda\big(1 - G(s(x, b))\big)V(0) - \lambda V(x)$$
$$+ \lambda \int_{s(x,b)}^{\infty} r(z, b)\,\mathrm{d}G(z) + (c(b) + mx)V'(x) - \lambda\big(1 - G(s(x, b))\big)x\Big\} = 0\,.$$

With Fubini's Theorem we obtain:

$$\lambda \int_0^{s(x,b)} V\big(x - r(z, b)\big)\,\mathrm{d}G(z) + \lambda\big(1 - G(s(x, b))\big)V(0) - \lambda V(x)$$
$$= -\lambda \int_0^{x} \big(1 - G(s(y, b))\big)V'(x - y)\,\mathrm{d}y$$

and

$$\lambda \int_{s(x,b)}^{\infty} r(z, b)\,\mathrm{d}G(z) - \lambda\big(1 - G(s(x, b))\big)x = \lambda \int_x^{\infty} 1 - G(s(y, b))\,\mathrm{d}y.$$

Because for each x there exists an optimal b, we can write for the optimal $b(x)$:

$$\begin{aligned}(c(b(x)) + mx)V'(x) \quad &= \quad \lambda \int_0^{x} \big(1 - G(s(y, b(x)))\big)V'(x - y)\,\mathrm{d}y \\ &\quad -\lambda \int_x^{\infty} 1 - G(s(y, b(x)))\,\mathrm{d}y\,, \end{aligned} \tag{3.13}$$

and (3.12) transforms to

$$\begin{aligned} f'(x) \quad &= \quad \sup_{b\in(b_0(x),\tilde{b}]} \frac{\lambda}{c(b) + mx}\Big[\int_0^{x} \big(1 - G(s(y, b))\big)f'(x - y)\,\mathrm{d}y \\ &\quad - \int_x^{\infty} 1 - G(s(y, b))\,\mathrm{d}y\Big] \end{aligned} \tag{3.14}$$

From Theorem 3.2.2 we know that the solution to (3.12) and therefore to (3.14) is unique. For $x = 0$ we obtain

$$V'(0) = \sup_{b\in(b_0(0),\tilde{b}]} \frac{\lambda\mathbb{E}[r(Z, b)]}{c(b)}\,.$$

In particular, for an expected value principle (2.18) we find $V'(0) = -\frac{1}{1+\eta}$.

Proposition 3.2.6
There is a unique solution $f(x)$ to (3.12), $x \geq 0$, with $f(\infty) = 0$.

Proof: Define an operator F, acting on negative functions $w(x)$ by

$$
\begin{aligned}
F(w(x)) &= \sup_{b \in (b_0(x), \tilde{b}]} \frac{\lambda}{c(b) + mx} \Big[\int_0^x \big(1 - G(s(y,b))\big) w(x-y) \, \mathrm{d}y \\
&\quad - \int_x^\infty 1 - G(s(y,b)) \, \mathrm{d}y \Big] \qquad (3.15)
\end{aligned}
$$

We have already seen, that the value function $V(x)$ solves the equation (3.12). Let now $w_0(x) := (V^1)'(x)$, the solution of Example 2.2.1 with $b_t = \tilde{b}$. Define recursively $w_n(x) = F(w_{n-1}(x))$. We show at first, that the sequence w_n is monotone increasing in n. It is clear, that $w_0(x) \leq w_1(x)$ because $(V^{\tilde{b}})'(x)$ solves the right hand side of (3.13) with $b = \tilde{b}$ instead of the sup. Assume $w_{n-1}(x) \leq w_n(x)$. Because the right side of (3.15) is continuous in b, there is a maximum point $b_n \in [b_0(x), \tilde{b}]$ for which $w_n(x) = F(w_{n-1}(x))$ attains its maximum. So we have

$$
\begin{aligned}
& w_{n+1}(x) - w_n(x) = F(w_n(x)) - F(w_{n-1}(x)) \\
&= F(w_n(x)) - \frac{\lambda}{c(b_n) + mx} \Big[\int_0^x \big(1 - G(s(y,b_n))\big) w_{n-1}(x-y) \, \mathrm{d}y \\
&\quad + \int_x^\infty 1 - G(s(y,b_n)) \, \mathrm{d}y \Big] \\
&\geq \frac{\lambda}{c(b_n) + mx} \Big[\int_0^x \big(1 - G(s(y,b_n))\big) [w_n(x-y) - w_{n-1}(x-y)] \, \mathrm{d}y \Big] \\
&\geq 0.
\end{aligned}
$$

So the sequence $w_n(x)$ is increasing in n and $w_n(x) < 0$. It means we obtain that $w(x) = \lim\limits_{n \to \infty} w_n(x)$ exists pointwise. By the monotone convergence theorem we have then

$$
\lim_{n \to \infty} \int_0^x \big(1 - G(s(y,b))\big) w_n(x-y) dy = \int_0^x \big(1 - G(s(y,b))\big) w(x-y) \, \mathrm{d}y
$$

for all x and b.
Let b be a maximal point of $F(w(x))$. Then we have

$$
\begin{aligned}
w_n(x) &= \frac{\lambda}{c(b_n) + mx} \Big[\int_0^x \big(1 - G(s(y,b_n))\big) w_{n-1}(x-y) \, \mathrm{d}y \\
&\quad - \int_x^\infty 1 - G(s(y,b_n)) dy \Big] \\
&\geq \frac{\lambda}{c(b) + mx} \Big[\int_0^x \big(1 - G(s(y,b))\big) w_{n-1}(x-y) \, \mathrm{d}y \\
&\quad - \int_x^\infty 1 - G(s(y,b)) \, \mathrm{d}y \Big],
\end{aligned}
$$

which means $w(x) \geq F(g(x))$. On the other hand $w_n(x)$ are increasing in n, i.e. $w_n(x) \leq w(x)$:

$$\begin{aligned} w_n(x) &= \frac{\lambda}{c(b_n)+mx}\Big[\int_0^x \big(1-G(s(y,b_n))\big)w_{n-1}(x-y)\,\mathrm{d}y \\ &\quad -\int_x^\infty 1-G(s(y,b_n))\,\mathrm{d}y\Big] \\ &\leq \frac{\lambda}{c(b_n)+mx}\Big[\int_0^x \big(1-G(s(y,b_n))\big)w(x-y)\,\mathrm{d}y \\ &\quad -\int_x^\infty 1-G(s(y,b_n))\,\mathrm{d}y\Big] \\ &\leq \frac{\lambda}{c(b)+mx}\Big[\int_0^x \big(1-G(s(y,b))\big)w(x-y)\,\mathrm{d}y \\ &\quad -\int_x^\infty 1-G(s(y,b))\,\mathrm{d}y\Big], \end{aligned}$$

which means $w(x) \leq F(w(x))$. We have therefore $w(x) = F(w(x))$, and $w(x)$ is continuous.

Because $w_n(x)$ is increasing, we can define

$$f(x) = -\int_x^\infty w(y)\,\mathrm{d}y \leq -\int_x^\infty w_0(y)\,\mathrm{d}y = V^{\tilde{b}}(x)\;.$$

$f(x)$ fulfils (3.15) with $f(\infty) = 0$. $f(x)$ is also decreasing, continuously differentiable and bounded $0 < f(x) \leq V^1(x)$.

Suppose now, that $f_1(x)$ and $f_2(x)$ are solutions to (3.12) with $f_1(\infty) = f_2(\infty) = 0$. Denote further by $g_i(x) = f_i'(x)$ the derivatives and by $b_i(x)$ the value, for which the minimum is obtained. Choose now some $x^* > 0$. Because the right-hand side of Equation (3.14) is continuous in b and tends to infinity as $c(b)$ tends to zero, we conclude, that $c(b_i(x))$ is bounded away from zero on $[0, x^*]$. Let $x_1 = \inf\{c(b_1(x)) \wedge c(b_2(x)) : 0 \leq x \leq x^*\}$ and $x_n = nx_1 \wedge x^*$. W.l.o.g. we assume, that $x_1 \leq x^*$. Suppose we have already proved $f_1(x) = f_2(x)$ on the interval $[0, x_n]$. Then for $x \in [x_n, x_{n+1}]$, with $l = \sup\limits_{x_n \leq x \leq x_{n+1}} |g_1(x) - g_2(x)|$ it holds

$$\begin{aligned} g_1(x) - g_2(x) &= F(g_1(x)) - F(g_2(x)) \\ &\leq \frac{\lambda}{c(b_2(x))+mx}\int_{x_n}^x \Big(g_1(z)-g_2(z)\Big)[1-G(s(x-z))]\,\mathrm{d}z \\ &\leq \frac{\lambda l x_1}{c(b_2(x))+mx} \leq \frac{l}{2} \end{aligned}$$

Reversing the roles of $g_1(x)$ and $g_2(x)$, it follows that $|g_1(x) - g_2(x)| \le \frac{l}{2}$. This is only possible for all $x \in [x_n, x_{n+1}]$ if $l = 0$. this shows that $f_1(x) = f_2(x)$ on $[0, x_{n+1}]$. So $f_1(x) = f_2(x)$ on $[0, x^*]$. Because x^* was arbitrary, uniqueness follows. □

Example 3.2.7 (Proportional Reinsurance $Z_i \sim$Exp$(\frac{1}{\mu})$ and $Z_i \sim$Pareto$(2,\mu)$.)
We consider here only the proportional reinsurance, i.e. $r(z,b) = zb$.
All the considerations concerning the function $V^1(x)$ in the case $\delta > 0$ hold also in the case $\delta = 0$. But here it is easier to consider the derivative $(V^1)'$. For the exponentially distributed claim sizes, $Z_i \sim \mathrm{Exp}(\frac{1}{\mu})$ we have to solve the integro-differential equation

$$-\lambda \int_0^x (V^1)'(y) e^{-\frac{x-y}{\mu}}\, \mathrm{d}y + c(V^1)'(x) + \lambda\mu e^{-\frac{x}{\mu}} = 0 \, .$$

This equation is easy to solve, and we obtain as the derivative function $(V^1)'(x)$:

$$(V^1)'(x) = -\frac{\lambda\mu}{c}\Big(1 + \frac{mx}{c}\Big)^{\lambda/m-1} e^{-\frac{x}{\mu}}$$

for $x \ge 0$. Choose $\mu = \lambda = 1$, $m = 0.03$ and $\theta = 0.5$ and $\eta = 0.3$. We calculate the value function numerically. The value function and the optimal strategy for $Z_i \sim \mathrm{Exp}(\frac{1}{\mu})$ are plotted in Figures 3.10 and 3.11 respectively. The value function and the optimal strategy for $Z_i \sim$Pareto$(2,\mu)$ are plotted in Figures 3.12 and 3.13 respectively. ■

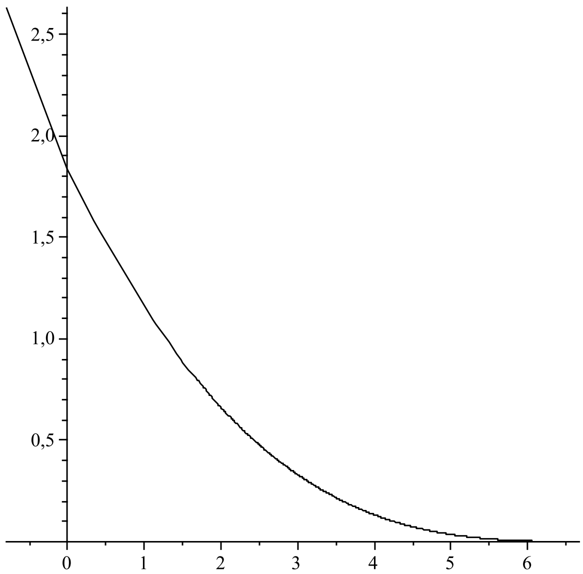

Figure 3.10: Numerically calculated value function for $Z_i \sim \text{Exp}(\frac{1}{\mu})$.

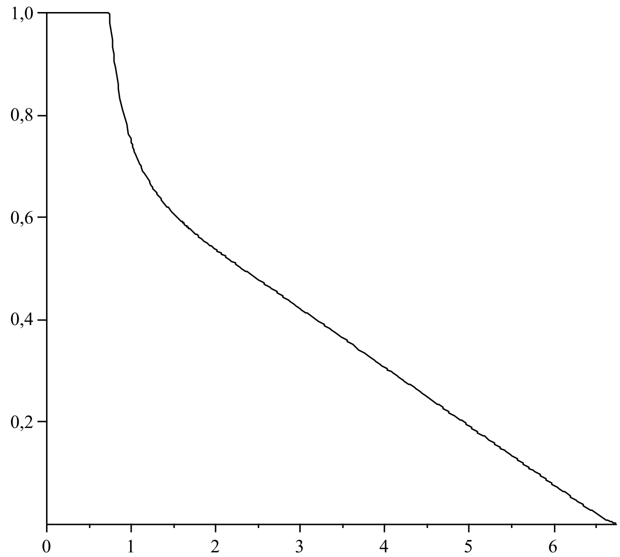

Figure 3.11: The optimal strategy for $Z_i \sim \text{Exp}(\frac{1}{\mu})$.

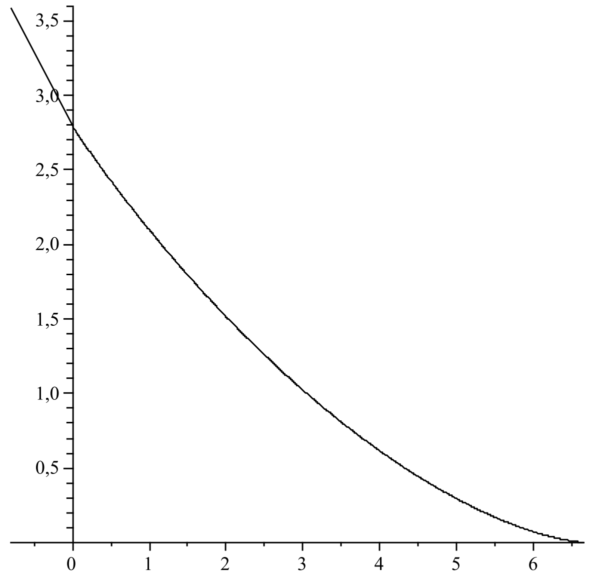

Figure 3.12: Numerically calculated value function for $Z_i \sim$ Pareto$(2, \mu)$.

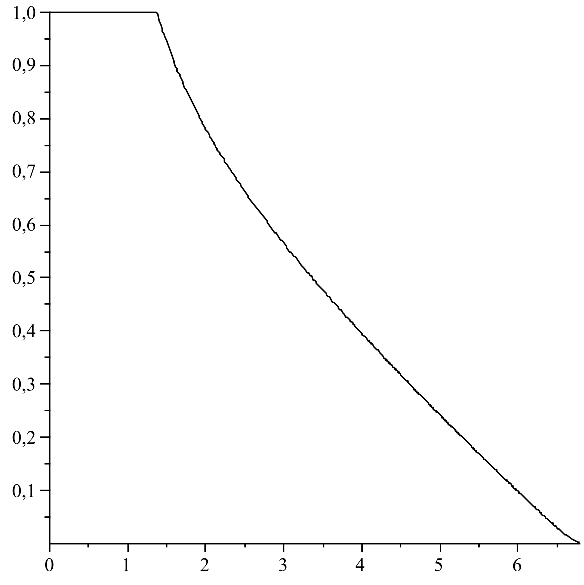

Figure 3.13: The optimal strategy for Z_i ~Pareto$(2, \mu)$.

4 Optimal Control of Capital Injections by Investments, modeled as a Black-Scholes Model

In the following we will consider a classical risk model and a diffusion approximation under the special constraint, which allows the insurer in addition to the reinsurance also to invest money into a risky asset. We also assume, that the scarcity of funds is impossible. That is the money whenever needed can be put up for example by company holders. We start with a diffusion approximation to a classical risk model. Because in this case we will be able to give an explicit expression for the value function and for the optimal strategy.

4.1 Diffusion Approximation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We consider a surplus process, where the mean number of claims in a time unit is λ and the mean size of a claim is $\mu = \mathbb{E}[Z]$, where Z is a generic claim size. We assume that $\mu_2 = \mathbb{E}[Z^2] < \infty$. The premium is $c = (1+\eta)\lambda\mu$ for some $\eta > 0$.

$$X_t = x + \lambda\mu\eta t + \sqrt{\lambda\mu_2}W_s$$

In addition to the above setup assume now, that the insurer can invest money in a risky asset, modeled as a Black-Scholes model

$$Q_t = \exp\left\{(m - \frac{\sigma^2}{2})t + \sigma\tilde{W}_t\right\},$$

where $m, \sigma > 0$. The return of such a process is then the stochastic process $\{Q_t'\}$ given by the stochastic differential equation

$$\mathrm{d}Q_t' = m\ \mathrm{d}t + \sigma\ \mathrm{d}\tilde{W}_t\ .$$

We assume further, that the Brownian motions $\{W_t\}$ and $\{\tilde{W}_t\}$ are independent and denote in following the filtration generated by the couple of Brownian motions $\{(W_t, \tilde{W}_t)\}$ by $\mathbb{F} = \{\mathcal{F}_t\}$. We assume also, that the insurer can change the amount $a_t \in \mathbb{R}$ invested at time t continuously and call every strategy $A = \{a_t\}$, $a_t \in \mathbb{R}$, admissible if it is cadlag and $\mathbb{F}$ adapted. In the following the set of admissible investment strategies will be denoted by $\mathscr{A}$. The surplus process under the investment strategy $A = \{a_t\}$ fulfils the stochastic differential equation:

$$\mathrm{d}X_t^A = (\lambda\mu\eta + a_t m)\,\mathrm{d}t + \sqrt{\lambda\mu_2}\,\mathrm{d}W_t + a_t\sigma\,\mathrm{d}\tilde{W}_t\,. \tag{4.1}$$

If the surplus falls below zero, the company holders have to inject capital to keep the process nonnegative. Our goal is to minimise the expected discounted capital injections over all admissible investment strategies. Denote by $\{Y_t^A\}$ the process of cumulated capital injections corresponding to the investment strategy A. The value of the expected discounted capital injections corresponding to the strategy A is given by $V^A(x) = \mathbb{E}_x\big[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t\big]$, where $\delta \geq 0$. δ expresses the readiness of the company holders to part from money injecting it into the surplus process. Here we assume $\delta \geq 0$, which means, that "injecting money as late as possible" is preferred to "injecting money now", if $\delta > 0$; and "injecting money now" is equally attractive as "injecting money later" if $\delta = 0$. Our goal is to find the value function $V(x) = \inf\limits_{A\in\mathscr{A}} V^A(x)$ and the optimal strategy.

Note, that in the case we would not invest at all, i.e. $A \equiv 0$, the surplus process will have the form

$$X_t^0 = x + \lambda\mu\eta t + \sqrt{\lambda\mu_2}W_t\,,$$

which is a Brownian motion. This case was explicitly treated in Example 2.1.1. The return function for this case is given by $V^0(x) = \frac{1}{\beta}\exp(-\beta x)$ for $x \geq 0$ and $V^0(x) = \frac{1}{\beta} - x$ for $x < 0$. Where $\beta = \frac{\lambda\mu\eta + \sqrt{\lambda^2\mu^2\eta^2 + 2\lambda\mu_2\delta}}{\lambda\mu_2}$. Because $A \equiv 0$ is an admissible strategy we obtain $\lim\limits_{x\to\infty} V(x) = 0$.

Remark 4.1.1
Let $\{X_t\}$ be a diffusion process with values in $\mathbb{R}$, fulfilling the stochastic differential equation

$$\mathrm{d}X_t = m(X_t)\,\mathrm{d}t + \sigma_1(X_t)\,\mathrm{d}W_t^1 + \sigma_2(X_t)\,\mathrm{d}W_t^2\,,$$

where $\{W_t^1\}$ and $\{W_t^2\}$ are standard Brownian motions, m, σ_1 and σ_2 are functions, such that the above equation has a unique strong solution. The reflected process fulfils then

$$\mathrm{d}X_t^Y = m(X_t^Y)\,\mathrm{d}t + \sigma_1(X_t^Y)\,\mathrm{d}W_t^1 + \sigma_2(X_t^Y)\,\mathrm{d}W_t^2 + Y_t\,,$$

whereas $Y = \{Y_t\}$ is the local time of the process at zero.

We find in Subsection 2.1.1, that the corresponding return function $V(x) = \mathbb{E}_x[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t]$ solves the differential equation

$$\frac{\sigma_1^2(x) + \sigma_2^2(x)}{2}V''(x) + m(x)V'(x) - \delta V(x) = 0 \tag{4.2}$$

for $x \geq 0$ and fulfils $V'(0) = -1$, $\lim\limits_{x\to\infty} V(x) = 0$. From Subsection 2.1.1, we know, that every solution $f(x)$ to the above differential equation, vanishing at infinity, has the form

$$f(x) = f'(0)\mathbb{E}_x\big[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t\big]\ .$$

Example 4.1.2
Now we consider a special case of constant strategies, i.e. we let $A \equiv a \in \mathbb{R}$. The underlying process solves then the stochastic differential equation

$$\mathrm{d}X_t^a = (\lambda\mu\eta + am)\,\mathrm{d}t + \sqrt{\lambda\mu_2}\,\mathrm{d}W_t + a\sigma\,\mathrm{d}\tilde{W}_t\ ,$$

and from Remark 4.1.1 it follows, that the corresponding return function $V^a(x)$ solves the differential equation

$$\frac{\lambda\mu_2 + a^2\sigma^2}{2}(V^a)''(x) + (\lambda\mu\eta + am)(V^a)'(x) - \delta V^a(x) = 0\ .$$

The unique solution to the above equation with initial conditions $\lim\limits_{x\to\infty} V^a(x) = 0$ and $(V^a)'(0) = -1$ is given by

$$\frac{\lambda\mu_2 + a^2\sigma^2}{\lambda\mu\eta + am + \sqrt{(\lambda\mu\eta + am)^2 + 2\delta(\lambda\mu_2 + a^2\sigma^2)}}e^{-\frac{\lambda\mu\eta + am + \sqrt{(\lambda\mu\eta+am)^2+2\delta(\lambda\mu_2+a^2\sigma^2)}}{\lambda\mu_2+a^2\sigma^2}x}\ .$$

The natural question is, whether there is a constant value a, which minimises $V^a(x)$ for all x. Later we will give a detailed answer to this question. ∎

Thus, the capital injection Y_t^A for the process (4.1) is given as a local time: $Y_t^A = -\min\{\inf\limits_{0\leq s\leq t} X_s^A, 0\}$. Denoting by τ_x^A the time of ruin of the process $\{X_t^A\}$ with initial capital x we can write for the value $V^A(x)$:

$$V^A(x) = \mathbb{E}\big[\int_{\tau_x^A}^\infty e^{-\delta t}\,\mathrm{d}Y_t\big] = V^{A'}(0)\mathbb{E}\big[e^{-\delta\tau_x^A}\big]\ ,$$

where $A_t' = A_{\tau_x^A + t}$. Choose now a strategy A' in such a way, that $V(x) + \varepsilon > V^A(x)$ for some $\varepsilon > 0$, then we can write for initial capital 0:

$$V(0)\mathbb{E}\big[e^{-\delta\tau_x^A}\big] \leq V^A(x) \leq (V(0) + \varepsilon)\mathbb{E}\big[e^{-\delta\tau_x^A}\big]\ .$$

Thus, in order to minimise $V^A(x)$ we have to minimise $\mathbb{E}\big[e^{-\delta\tau_x^A}\big]$ over all admissible strategies $A \in \mathscr{A}$. For this purpose we define $L^A(x) := \mathbb{E}\big[e^{-\delta\tau_x^A}\big]$ and $L(x) := \inf\limits_{A\in\mathscr{A}} L^A(x)$. Because $V(x)$ is a decreasing function and $\lim\limits_{x\to\infty} V(x) = 0$ we can transfer these characteristics to the function $L(x)$.

Now we assume, that the optimal strategy exists and denote it by $A^* = \{a_t^*\}$, which means $V(x) = V^{A^*}(x)$. Note, that if we start with initial capital $x+y$ for $x, y \geq 0$ we will cross the level y before the ruin at $\tau_{x+y}^{A^*}$. In particular we will cross the level y at the time $\tau_x^{A^*}$, i.e. at the ruin time of the process $\{X_t^{A^*}\}$ with initial value x. Now we take advantage from the fact, that the process $\{X_t\}$ has independent increments and $\tau_{x+y}^{A^*} - \tau_x^{A^*} = \tau_y^{A''}$, where $a_t'' = a^*_{t+\tau_x^{A^*}}$. Using, that for an optimal strategy we can minimise the terms $\mathbb{E}\big[e^{-\delta\tau_x^{A^*}}\big]$ and $\mathbb{E}\big[e^{-\delta(\tau_{x+y}^{A^*} - \tau_x^{A^*})}\big]$ independently we can decompose the function $L(x)$ as follows:

$$\begin{aligned} L(x+y) = L^{A^*}(x+y) &= \mathbb{E}\big[e^{-\delta\tau_{x+y}^{A^*}}\big] = \mathbb{E}\big[e^{-\delta\tau_x^{A^*}} \cdot e^{-\delta(\tau_{x+y}^{A^*} - \tau_x^{A^*})}\big] \\ &= \mathbb{E}\big[e^{-\delta\tau_x^{A^*}}\big] \cdot \mathbb{E}\big[e^{-\delta(\tau_{x+y}^{A^*} - \tau_x^{A^*})}\big] \\ &= L^{A^*}(x) \cdot L^{A^*}(y)\ . \end{aligned}$$

The above equation is the functional equation of the exponential function, i.e. $L(x) = \exp(-\beta x)$ for some $\beta > 0$. In particular we obtain, that for arbitrary constant strategy $A \equiv a$ it holds $L^a(x) = \exp(-\beta(a)x)$ for some $\beta(a) > 0$. Choose now $x > 0$ arbitrary and $n \in \mathbb{N}$. Dividing the interval $[0, x]$ in n equidistant intervals we must solve the same optimisation problem in each interval, which means, that the optimal strategy should be constant. From $L^a(x) = \exp(-\beta(a)x)$ it follows, that in order to minimise $L^a(x)$ (and accordingly $V^a(x)$) over a we have to maximise $\beta(a)$. The other way would be to consider the Hamilton–Jacobi–Bellman (HJB) equation for the function $V(x)$ and to find the optimal a directly. An explicit derivation of the HJB equation, corresponding to the problem we consider, can be found in Subsection 2.1.2.

We obtain for $x \geq 0$:

$$\inf_{a\in\mathbb{R}} \frac{\sigma^2 a^2 + \lambda\mu_2}{2} V''(x) + (\lambda\mu\eta + am)V'(x) - \delta V(x) = 0\ . \tag{4.3}$$

In particular we find in Subsection 2.1.1, that $V'(0) = -1$. The above expression is continuously differentiable in a, so that we obtain by differentiating it with respect to a, that the optimal strategy should be given by $a = -\frac{mV'(x)}{\sigma^2 V''(x)}$. Plugging this ansatz into the HJB equation (4.3) yields the differential equation

$$-\frac{m^2 V'(x)^2}{2\sigma^2 V''(x)} + \frac{\lambda\mu_2}{2} V''(x) + \lambda\mu\eta V'(x) - \delta V(x) = 0\ .$$

Because we conjecture, that the optimal strategy is constant, we make the ansatz $V(x) = V(0)\exp(-\beta x)$ and obtain the equation

$$-\frac{m^2}{2\sigma_2^2} + \frac{\lambda\mu_2}{2}\beta^2 - \lambda\mu\eta\beta - \delta = 0\ .$$

Now we can find β. The above equation is merely a quadratic equation in β, so that the unique positive root is given by

$$\beta = \frac{\lambda\mu\eta + \sqrt{\lambda^2\mu^2\eta^2 + 2\lambda\mu_2(\frac{m^2}{2\sigma_2^2} + \delta))}}{\lambda\mu_2}\,, \tag{4.4}$$

and according to this

$$a = \frac{m}{\sigma^2\beta} = \frac{m\lambda\mu_2/\sigma^2}{\lambda\mu\eta + \sqrt{\lambda^2\mu^2\eta^2 + 2\lambda\mu_2(\frac{m^2}{2\sigma^2} + \delta))}}\,. \tag{4.5}$$

In order to show, that our claim, the optimal strategy is a constant one, is correct, we prove the verification theorem.

Theorem 4.1.3
Let $f(x)$ be a solution to Equation (4.3), then $f(x) = V(x) = \frac{1}{\beta}\exp(\beta x)$ for $x \geq 0$ with β given in (4.4) and $f(x) = V(x) = \frac{1}{\beta} - x$ for $x < 0$. The constant strategy given in (4.5) is optimal.

Proof: We know from Subsection 2.1.1, that the return function corresponding to some constant strategy $A \equiv a \in \mathbb{R}$ is given by

$$V^a(x) = \begin{cases} \frac{1}{\beta(a)}\exp(\beta(a)x) & : \; x \geq 0\,, \\ \frac{1}{\beta(a)} - x & : \; x < 0\,, \end{cases}$$

where the factor $\beta(a)$ depends on a and is given by

$$\beta(a) = \frac{\lambda\mu\eta + am + \sqrt{(\lambda\mu\eta + am)^2 + (a^2\sigma^2 + \lambda\mu_2)\delta}}{a^2\sigma^2 + \lambda\mu_2}\,.$$

Maximising $\beta(a)$ with respect to a yields the values in (4.4) and (4.5). Note, that the function $V^{a^*}(x)$, a^* given in (4.5), is the unique solution to the HJB equation with initial constraint $(V^{a^*})'(0) = -1$. Denote by $A^* = \{a_t^*\}$ the optimal strategy and by X^* the corresponding process. Consider further an arbitrary strategy $A = \{a_t\}$. The corresponding process X_t^A fulfils then

$$X_t^A = x + \lambda\mu\eta t + m\int_0^t a_s \,\mathrm{d}s + \sqrt{\lambda\mu_2}W_t + \sigma\int_0^t a_s \,\mathrm{d}\tilde{W}_s\,.$$

Consider now V^{a^*}, which is a twice continuously differentiable solution to the HJB equation (4.3). We apply Ito's formula and obtain with (4.3):

$$\begin{aligned} e^{-\delta t}V^{a^*}(X_t^{A,Y}) &= f(x) + \int_0^t e^{-\delta s}(V^{a^*})'(X_s^{A,Y})\,\mathrm{d}Y_s^A \\ &\quad + \int_0^t e^{-\delta s}\Big[D_{s,A}V^{a^*}(X_s^{A,Y}) - \delta V^{a^*}(X_s^{A,Y})\Big]\,\mathrm{d}s \\ &\quad + \sigma\int_0^t e^{-\delta t}(V^{a^*})'(X_s^{A,Y})a_s\,\mathrm{d}\tilde{W}_s \\ &\quad + \sqrt{\lambda\mu_2}\int_0^t e^{-\delta t}(V^{a^*})'(X_s^{A,Y})\,\mathrm{d}W_s \\ &\geq V^{a^*}(x) - \int_0^t e^{-\delta s}\,\mathrm{d}Y_s^A \\ &\quad + \sigma\int_0^t e^{-\delta t}(V^{a^*})'(X_s^{A,Y})a_s\,\mathrm{d}\tilde{W}_s \\ &\quad + \sqrt{\lambda\mu_2}\int_0^t e^{-\delta t}(V^{a^*})'(X_s^{A,Y})\,\mathrm{d}W_s \end{aligned}$$

where $D_{s,A}V^{a^*}(x) = \frac{1}{2}(a_s^2\sigma^2 + \lambda\mu_2)(V^{a^*})''(x) + (\lambda\mu\eta + ma_s)(V^{a^*})'(x)$. Note, that for $A = A^*$ the equality holds. Because the derivative $(V^{a^*})'(x)$ is bounded, the processes $\sigma\int_0^t e^{-\delta t}(V^{a^*})'(X_s^{A,Y})a_s\,\mathrm{d}\tilde{W}_s$ and $\sqrt{\lambda\mu_2}\int_0^t e^{-\delta t}(V^{a^*})'(X_s^{A,Y})\,\mathrm{d}W_s$ are martingales with expectation equal to 0. Applying the expectations on the both sides of the above inequality yields

$$\mathbb{E}\big[e^{-\delta t}V^{a^*}(X_t^{A,Y})\big] \geq V^{a^*}(x) - \mathbb{E}\big[\int_0^t e^{-\delta s}\,\mathrm{d}Y_s^A\big]\,.$$

Because it holds $V^{a^*}(X_t^{A,Y}) \leq V^{a^*}(0)$ we obtain by bounded convergence

$$V^{a^*}(x) \leq \mathbb{E}\big[\int_0^\infty e^{-\delta s}\,\mathrm{d}Y_s^A\big]\,.$$

For $X = X^*$ equality holds and thus $V^{a^*}(x) = V(x)$. □

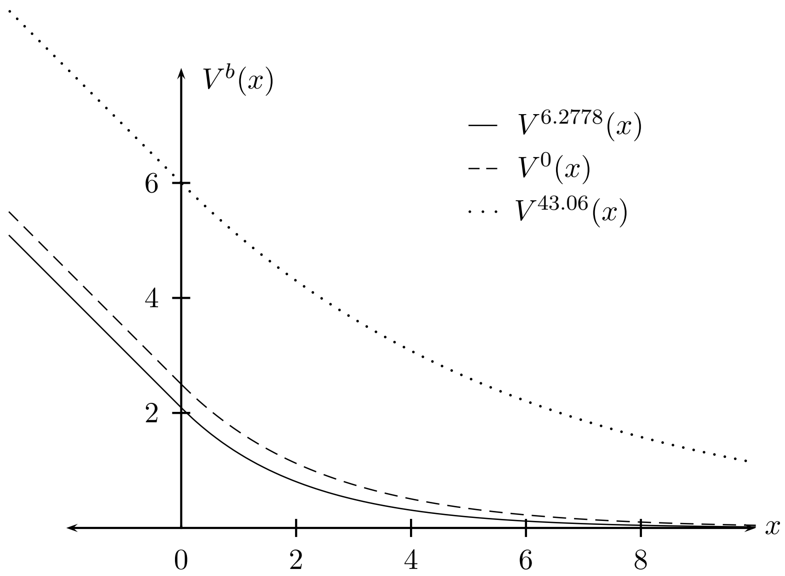

Figure 4.1: Return functions for optimal investment $a^* = 6.2778$, $a = 0$ and $a = 43.06$.

Example 4.1.4
Choose for example $\eta = 0.3$, $\lambda = 1$, $\delta = 0.04$, $\mu = 1$ and $\mu_2 = 2$. Then one can easily calculate, that $\beta = 0.4778719262$ and $a^* = 6.277832691$. Figure 4.1 compares the value of the optimal strategy with the return function in the case without investments, $a = 0$, and return function for the constant strategy $A \equiv 43.06$. We can see that investments can lower the costs. However, the choice of a "wrong" strategy can also cause considerable costs, see the dotted curve in Figure 4.1.

Choosing $\delta = 0$ corresponds to minimising the ruin probability. In this case our findings coincide with the result in Schmidli [70, p. 41]. With the same parameters and $\delta = 0$ we obtain $\beta = 0.4098076211$ and $a^* = 7.320508077$. Also in this case the discounting has a big influence on the optimal strategy and accordingly on the value function. ■

4.2 The Classical Risk Model

Now we switch to the classical risk model. This case is considerably more complicated than the case of diffusion approximations. Here we will not have the comfort of knowing that the value function is two times continuously differentiable, and viscosity solutions come up to take the place of exact solutions. At first we will consider the concept of viscosity solutions for the general case $\delta \geq 0$. Then we show under some special constraints the existence and uniqueness of the exact solution in the special case $\delta = 0$.

Consider now a classical risk model with investments and without the possibility to buy reinsurance. We assume as usual that the aggregate claims amount $\sum\limits_{i=1}^{N_t} Z_i$ obeys compound Poisson distribution, i.e. the distribution where the number of claims is a Poisson process N_t with intensity λ, the individual claims amounts Z_i are iid with mean value μ. By c we denote the premium income in a time unit. I.e. the surplus process has the form

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i \ .$$

In addition to the classical setup we allow the insurer to invest into a risky asset, modeled as a Black-Scholes model

$$Q_t = \exp\{(m - \frac{\sigma^2}{2})t + \sigma W_t\} \ .$$

where $\{W_t\}$ is a standard Brownian motion and $m, \sigma > 0$. The return of such a process is then the stochastic process $\{Q'_t\}$ given by the stochastic differential equation

$$\mathrm{d}Q'_t = m \ \mathrm{d}t + \sigma \ \mathrm{d}W_t \ .$$

We also assume, that the processes $\{\sum\limits_{i=1}^{N_t} Z_i\}$ and $\{W_t\}$ are independent and consider in following the filtration $\{\mathcal{F}_t\}$, generated by the two dimensional process $(\sum\limits_{i=1}^{N_t} Z_i, W_t)$.

The insurer can change the amount $a_t \in \mathbb{R}$, which should be invested at time $t \geq 0$, continuously. If the insurer chooses some admissible ($\{\mathcal{F}_t\}$-adapted and cadlag) strategy $A = \{a_t\}$, then his surplus process fulfils the stochastic differential equation

$$\mathrm{d}X_t^A = c + ma_t \ \mathrm{d}t - \ \mathrm{d}\sum_{i=1}^{N_t} Z_i + \sigma a_t \ \mathrm{d}W_t \ .$$

Cadlag processes are locally bounded, so that the integral $\int_0^t a_s \ \mathrm{d}W_s$ is well defined. To prevent, that the surplus process becomes negative, the insurer has to inject capital. The process of cumulated capital injections corresponding to some admissible investment strategy A we denote by $\{Y_t^A\}$. The process with capital injections has then the form

$$X_t^{A,Y} = x + ct + m\int_0^t a_s \ \mathrm{d}s - \sum_{i=1}^{N_t} Z_i + \sigma \int_0^t a_s \ \mathrm{d}W_t + Y_t^A \ .$$

We are interested in the value function $V(x) = \inf\limits_{A \in \mathscr{A}} \mathbb{E}_x[\int_0^\infty e^{-\delta t} \ \mathrm{d}Y_t^A]$, where $\mathscr{A}$ denotes the set of admissible strategies. We also target to find the optimal strategy A^*, leading to the value function $V(x) = V^{A^*}(x)$. It is clear, that it makes sense to inject capital only if the surplus becomes negative; and directly after the capital injection the surplus is equal to zero. Using this fact we will calculate the return functions for the constant strategies. In general it will be impossible to give a closed expression for the value function. All the more it is important to give as far as possible the closed expressions for return functions corresponding to the constant strategies. Thus, we start by considering constant strategies.

4.2.1 Constant strategies.

In this section we assume, that an investment strategy can be chosen only at the beginning. For some fixed $a \in \mathbb{R}$ the surplus process has then the form

$$X_t^a = x + (c + am)t + \sigma a W_t - \sum_{i=1}^{N_t} Z_i \,.$$

The simplest case, where we can write down the return function without problems is the **Special case $A \equiv a = 0$.**

The surplus process without investments and capital injections has the form

$$X_t^0 = x + ct - \sum_{i=1}^{N_t} Z_i \,,$$

i.e. we have now a classical risk model. Now we distinguish between the case $\delta = 0$ and $\delta > 0$. For $\delta = 0$ we know from Example 2.2.1, that if the net profit condition $c > \lambda\mu$ is fulfilled, the corresponding return function $V^0(x)$ can be expressed as a sum of two Gerber-Shiu penalty functions. For the special case of exponentially distributed claims one can actually give a closed expression for $V^0(x)$:

$$V^0(x) = \frac{\mu}{\eta} e^{-\frac{\eta}{\mu(1+\eta)}x} \,,$$

for $x \geq 0$. For the negative initial capital we have $V^0(x) = V^0(0) - x = \frac{c-\lambda\mu}{\lambda} - x$.

If $\delta > 0$ we do not need to assume $c > \lambda\mu$. We know from Example 2.2.1, that the return function corresponding to the constant strategy $A = 0$ is like in the case $\delta = 0$ just a sum of Gerber-Shiu penalty functions; in the special case of exponentially distributed claim sizes we have

$$V^0(x) = -\frac{1 + R\mu}{R} \exp(Rx)$$

for $x \geq 0$. Where R is the unique negative root of the equation

$$\delta + \lambda + cR = \lambda \int_0^\infty e^{Rx} \, \mathrm{d}G(x) \,. \tag{4.6}$$

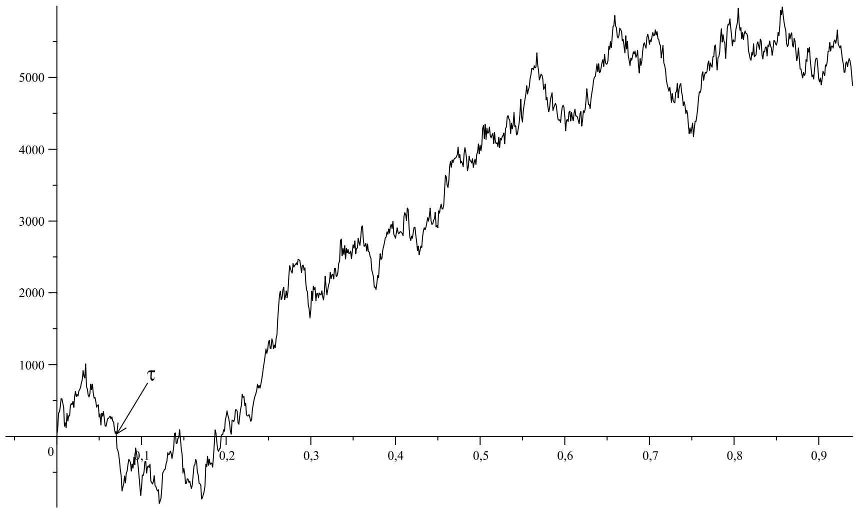

Figure 4.2: Ruin due to the diffusion.

The case $A = 0$ was explicitly treated in Example 2.2.1, thus we skip all further particulars and switch to the
General case $A \equiv a \in \mathbb{R}$.

The underlying process has now the form

$$X^a = x + (c + ma)t + \sigma a W_t - \sum_{i=1}^{N_t} Z_i \ .$$

Note at first, that in this case the ruin can occur in two ways: by a jump, caused by the process $\sum\limits_{i=1}^{N_t} Z_i$; or by oscillation of the Brownian motion. Figure 4.2 illustrates the situation, where the ruin occurs by oscillation. In Figure 4.3 one can see the possible changes in the scenario, if the ruin occurs by a jump: after the ruin, occurring in this example directly at the first claim arrival time T_1, we shift the process to zero by injecting additional money.

In particular we obtain for an arbitrary constant strategy $A \equiv a \in \mathbb{R}$

$$\frac{X^a - x}{t} = c + ma + \sigma a \frac{W_t}{t} - \frac{1}{t}\sum_{i=1}^{N_t} Z_i \overset{t\to\infty}{\longrightarrow} c + ma - \lambda\mu \ .$$

Hence if $\delta = 0$ it holds $V^a(0) = \infty$ for all $a \in \mathbb{R}$ with $a \le \frac{\lambda\mu - c}{m}$, because in the case $a < \frac{\lambda\mu - c}{m}$ the process X_t^a goes to $-\infty$ and in the case $a = \frac{\lambda\mu - c}{m}$ X_t^a oscillates as t goes to ∞. Conversely in the case $a > \frac{\lambda\mu - c}{m}$ X_t^a goes to ∞ as $t \to \infty$. Thus, if the net profit

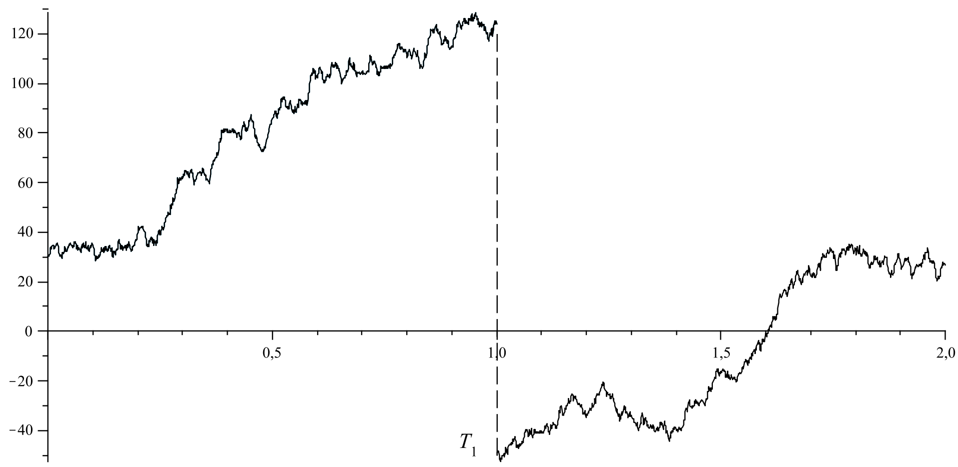

Figure 4.3: Ruin due to a claim.

condition is not fulfilled $c \le \lambda\mu$, choose $a > \frac{\lambda\mu - c}{m}$. For $\delta > 0$ we will have $V^a(0) < \infty$ for all $a \in \mathbb{R}$.
In following we show, how to calculate the return function of a constant strategy for the case $\delta > 0$. All considerations can be repeated also for the case $\delta = 0$ restricted on $a \in \mathbb{R}$ with $V^a(0) < \infty$.

For the constant strategies $A = a$ with $a \neq 0$ arbitrary it is easier to calculate at first $V^a(x)$ in dependence of an unknown initial value $V^a(0)$. In the second step we derive $V^a(0)$ by pointing out some special constraints on it.

Fix $a \in \mathbb{R}$ and let τ define the time of ruin of the process $\{X_t^a\}$. Then we can write for the corresponding return function

$$\begin{aligned} V^a(x) &= V^a(0)\mathbb{E}_x[e^{-\delta\tau}\mathbb{1}_{[X_\tau^a=0]};\tau<\infty] \\ &\quad + \mathbb{E}_x[e^{-\delta\tau}(V^a(0)-X_\tau^a)\mathbb{1}_{[X_\tau^a<0]};\tau<\infty] \ . \end{aligned}$$

It means we distinguish between the case, where ruin occurs due to the oscillation $X_\tau^a = 0$, and the case, when ruin occurs by a jump $X_\tau^a < 0$. Let further ρ_a be the unique positive root of the extended Lundberg equation

$$\lambda + \delta - (c+ma)\rho_a - \frac{\sigma^2 a^2 \rho_a^2}{2} = \lambda \int_0^\infty e^{-\rho_a y}\,\mathrm{d}G(y) \ .$$

Gerber and Landry showed in [32], that the function $V^a(x)$ fulfils the defective renewal equation

$$\begin{aligned} V^a(x) &= \int_0^x V^a(x-y)g_a(y)\,\mathrm{d}y + \int_x^\infty (V^a(0)+(y-x))g_a(y)\,\mathrm{d}y \\ &\quad + V^a(0)H_a(x) - H_a(x)\int_0^\infty (V^a(0)+y)g_a(y)\,\mathrm{d}y\,, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} D_a &= \frac{\sigma^2 a^2}{2}\,, \\ \beta_a &= \frac{c+ma}{D_a} + \rho_a\,, \\ H_a(x) &= e^{-\beta_a x}\,, \\ h_a(s) &= \frac{c+ma}{D_a}\cdot e^{-\beta_a s}\,, \\ \gamma_a(s) &= \frac{\lambda}{c+ma} e^{\rho_a s}\int_s^\infty e^{-\rho_a x}\,\mathrm{d}G(x)\,, \\ g_a(y) &= \int_0^y h_a(y-s)\cdot\gamma_a(s)\,\mathrm{d}s\,. \end{aligned}$$

Using standard methods from renewal theory one can find the unique solution $V^a(x)$ to the above renewal equation, which depends on the still unknown value $V^a(0)$.

Now we are able to specify the initial value $V^a(0)$. We assume $a > 0$, the case $a < 0$ goes similar. $V^a(0)$ can be decomposed into the expected discounted capital injections due to the Brownian motion up to T_1; the expected discounted capital injection at T_1 due to the first jump and finally the expected discounted capital injections from T_1 forward, i.e.

$$V^a(0) = \mathbb{E}\Big[\int_0^{T_1} e^{-\delta t}\,\mathrm{d}Y_t^a\Big] + \mathbb{E}\big[e^{-\delta T_1}V^a(X_{T_1-}^{a,Y} - Z_1)\big]\,. \tag{4.8}$$

Note, that until T_1- the capital injections can be caused only by the oscillation of the Brownian motion, see Figure 4.4. Thus consider the Brownian motion

$$C_t = (c+ma)t + a\sigma W_t\,.$$

The process C_t^Y with capital injections Y_t fulfils then

$$C_t^Y = (c+ma)t + a\sigma W_t + Y_t\,,$$

Consider now the process $x + C_t$ and let $U(x)$ denote the expected discounted capital injections due to the process $x + C_t$ up to the time T_1:

$$U(x) := \mathbb{E}_x\Big[\int_0^{T_1} e^{-\delta t}\,\mathrm{d}Y_t^x\Big]\,.$$

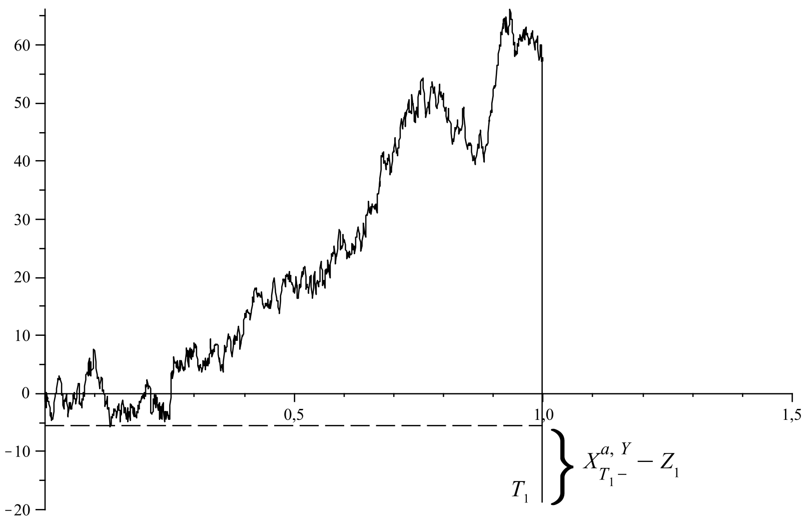

Figure 4.4: Brownian motion up to T_1 and the first claim.

where Y_t^x are the local times and fulfil

$$Y_t^x = -\min\Big\{\inf_{0\le s\le t}(x+(c+ma)t+a\sigma W_t),0\Big\} = \max\Big\{\sup_{0\le s\le t}(-x-(c+ma)t-a\sigma W_t),0\Big\}\,.$$

It is clear, that $U(x)$ is a decreasing function and because of the discounting it follows $\lim_{x\to\infty} U(x) = 0$. For negative capital we let $U(x) = U(0) - x$. Like in Subsection 4.1 the function $U(x)$ satisfies the differential equation

$$\frac{a^2\sigma^2}{2}U''(x) + (c+ma)U'(x) - (\delta+\lambda)U(x) = 0\,.$$

with initial constraints $U'(0) = -1$ and $\lim_{x\to\infty} U(x) = 0$. Every solution to the above equation is given by a sum of exponential functions multiplied with some constants K_1 and K_2, see Proposition E.1.4,

$$\begin{aligned}&K_1 \exp\Big\{-\frac{(c+ma)+\sqrt{(c+ma)^2+2a^2\sigma^2(\delta+\lambda)}}{a^2\sigma^2}\Big\}\\ &+K_2 \exp\Big\{-\frac{(c+ma)-\sqrt{(c+ma)^2+2a^2\sigma^2(\delta+\lambda)}}{a^2\sigma^2}\Big\}\,.\end{aligned}$$

Because $U(x)$ goes to 0 as x goes to infinity we obtain $K_2 = 0$. On the other hand it holds $U'(0) = -1$, which yields $K_1 = \Big\{\frac{(c+ma)+\sqrt{(c+ma)^2+2a^2\sigma^2(\delta+\lambda)}}{a^2\sigma^2}\Big\}^{-1}$. In particular we have $U(0) = K_1$. Thus, it remains to calculate $\mathbb{E}\big[e^{-\delta T_1} V^a(C^Y_{T_1} - Z_1)\big]$. If we denote by $f_t(x)$ the density function of $C^Y_t - Z_1$ we obtain

$$\begin{aligned}\mathbb{E}\big[e^{-\delta T_1} V^a(C^Y_{T_1} - Z_1)\big] &= \int_0^\infty \lambda e^{-(\delta+\lambda)t} \mathbb{E}[V^a(C^Y_t - Z_1)] \,\mathrm{d}t \\ &= \int_0^\infty \lambda e^{-(\delta+\lambda)t} \int_0^\infty V^a(y) f_t(y) \,\mathrm{d}y \,\mathrm{d}t \\ &\quad + \int_0^\infty \lambda e^{-(\delta+\lambda)t} \int_{-\infty}^0 \big(V^a(0) - y\big) f_t(y) \,\mathrm{d}y \,\mathrm{d}t \,.\end{aligned}$$

Thus we have to find $f_t(x)$. We assume $a\sigma > 0$, then we can rewrite the process C^Y_t in the following way

$$C^Y_t = -a\sigma\{-\frac{c+ma}{a\sigma}t - W_t\} + a\sigma \sup_{0\le s\le t}(-\frac{c+ma}{a\sigma}s - W_s) \,,$$

i.e. C^Y_t is just the difference of a Brownian motion $-(c+ma)t - a\sigma W_t$ and its running maximum. For definition of running maximum see Remark A.1.6, p. 165. With change of measure techniques, see Proposition C.2.4 p. 172, it is straight forward to calculate the density function of the process C^Y_t, which is given by

$$\begin{aligned} h_t(x) &= \frac{2}{a\sigma}\frac{1}{\sqrt{2\pi t}} \exp\Big\{-\frac{(x-(c+ma)t)^2}{a^2\sigma^2 t 2}\Big\} \\ &\quad - 2\frac{c+ma}{a^2\sigma^2}\Big(1 - \Phi\big(\frac{x}{a\sigma\sqrt{t}} + \frac{c+ma}{a\sigma}\sqrt{t}\big)\Big) \exp\Big\{2\frac{(c+ma)x}{a^2\sigma^2}\Big\} \,,\end{aligned}$$

where Φ denotes the distribution function of the standard normal distribution. Let $\tilde{G}$ be the distribution function of $-Z_1$, then we obtain for the desired density $f_t(x)$:

$$f_t(x) = \begin{cases} \int_{-\infty}^x h_t(x-y)\,\mathrm{d}\tilde{G}(y) & : \ x \le 0 \,, \\ \int_{-\infty}^0 h_t(x-y)\,\mathrm{d}\tilde{G}(y) & : \ x > 0 \,. \end{cases}$$

Now we are able to calculate $\mathbb{E}\big[e^{-\delta T_1} V^a(C^Y_{T_1} - Z_1)\big]$ using the knowledge of the function $V^a(x)$ for $x > 0$. Finally we can find $V^a(0)$ solving the Equation (4.8). Next we illustrate the method by an example.

Example 4.2.1
Assume, that the claim sizes are exponentially distributed $G(x) = 1 - e^{-\frac{x}{\mu}}$, then we can give a closed expression for the function $g_a(y)$:

$$g_a(y) = \frac{\lambda}{D_a(\rho_a\mu + 1)} \cdot \frac{\mu}{(\beta_a\mu - 1)} \cdot (e^{-\frac{y}{\mu}} - e^{-\beta_a y}) \,.$$

Let $J_a := \frac{\lambda}{D_a(\rho_a\mu+1)}$. The renewal Equation (4.7) becomes

$$\begin{aligned} V^a(x) &= J_a \int_0^x V^a(x-y) \frac{\mu}{\beta_a\mu - 1}(e^{\frac{-y}{\mu}} - e^{-\beta_a y})\, \mathrm{d}y + V^a(0)e^{-\beta_a x} \\ &\quad + \frac{\mu}{\beta_a\mu-1}(e^{-\frac{x}{\mu}} - e^{-\beta_a x})\big(J_a \cdot \mu^2 + J_a \cdot V^a(0)\mu\big)\ . \end{aligned}$$

Note, that $\frac{\mu}{\beta_a\mu-1}(e^{-\frac{x}{\mu}} - e^{-\beta_a x})$ is equal to the convolution of the functions $e^{-\frac{x}{\mu}}$ and $e^{-\beta_a x}$. It is straight forward to calculate, that the solution to the above renewal equation is given by the function

$$\begin{aligned} f(x) &= \mu^2 J_a e^{-\frac{x}{\mu}} \int_0^x e^{-y(\beta_a - \frac{1}{\mu})}\big(\,{}_0F_1(1, J_a xy - J_a y^2) - 1\big)\, \mathrm{d}y \\ &\quad + V^a(0) J_a e^{-\frac{x}{\mu}} \Big\{ \int_0^x e^{-y(\beta_a - \frac{1}{\mu})} y \cdot\ {}_0F_1(2, J_a xy - J_a y^2))\, \mathrm{d}y \\ &\quad + \mu \int_0^x e^{-y(\beta_a - \frac{1}{\mu})}\big(\,{}_0F_1(1, J_a xy - J_a y^2) - 1\big)\, \mathrm{d}y \Big\} \\ &\quad + \frac{J_a\mu(e^{-\frac{x}{\mu}} - e^{-\beta_a x})(\mu^2 V^a(0) + \mu)}{(\beta_a\mu - 1)} + V^a(0)e^{-\beta_a x}\ , \end{aligned}$$

where ${}_0F_1(\alpha, x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+\alpha)n!}$, i.e. a hypergeometric function and $\Gamma(x)$ the Gamma function. For the parameters $m = 0.03$, $\sigma^2 = 0.01$, $\mu = \lambda = 1$, $c = 1.3$ and $a = 12$ we obtain $V^{12}(0) = 1.824317721$. The function $V^{12}(x)$ is plotted as a solid line in Figure 4.5. We see, that the return function corresponding to the strategy $A \equiv 0$, no investment, (dashed line in Figure 4.5) lies above the function $V^{12}(x)$. Thus even a constant strategy, which is not optimal, may reduce the cost. ■

4.2.2 General case $\delta \geq 0$.

For the general case $\delta \geq 0$ we will not be able either to indicate the value function explicitly or to show the existence of the optimal strategy. The following lemma states a couple of useful properties of the value function, which hold for $\delta \geq 0$.

Lemma 4.2.2
The value function $V(x)$ has the following properties

1. *$V(x)$ is decreasing with $\lim_{x\to\infty} V(x) = 0$.*
2. *$V(x)$ is Lipschitz continuous on $[0, \infty)$ with $|V(x) - V(y)| \leq |x - y|$.*
3. *$V(x)$ is convex.*

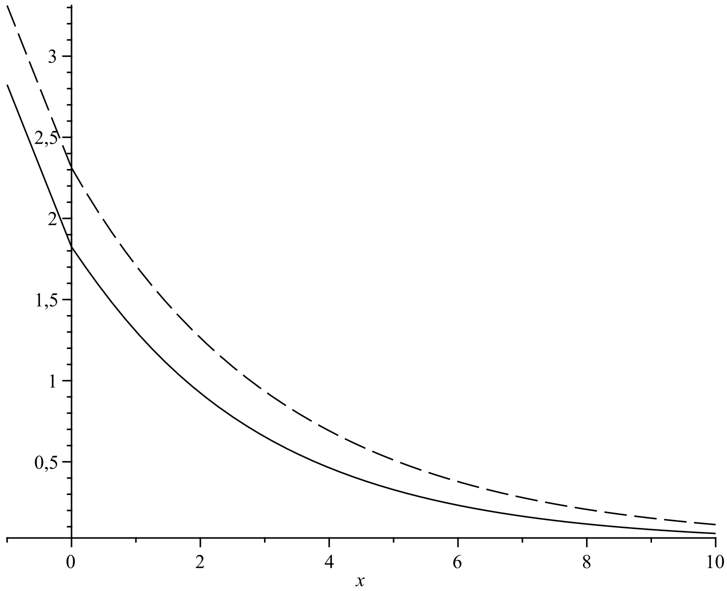

Figure 4.5: Return functions for the constant strategies $A \equiv 0$ (dashed line) and $A \equiv 12$ (solid line) for exponentially distributed claim sizes.

Proof: It is clear, that the value function is decreasing. Consider now $x, z \geq 0$, $\alpha \in (0,1)$ and define $y = \alpha x + (1-\alpha)z$. Let further $A^x = \{a_t^x\}$ be the optimal strategy for the initial capital x and $A^z = \{a_t^z\}$ - for initial capital z. Denote by $A^y = \{a_t^y\}$ the optimal strategy for the initial capital y, then it holds

$$\begin{aligned} & X_t^{A^y,Y} - Y_t^{A^y} + \alpha Y_t^{A^x} + (1-\alpha)Y_t^{A^z} \\ = \;& x + ct + m\int_0^t a_s^y \,\mathrm{d}s + \sigma\int_0^t a_s^y \,\mathrm{d}W_s - \sum_{i=1}^{N_t} Z_i + \alpha Y_t^{A^x} + (1-\alpha)Y_t^{A^z} \\ = \;& \alpha X_t^{A^x,Y} + (1-\alpha)X_t^{A^z,Y} \geq 0\,. \end{aligned}$$

Thus $\alpha Y_t^{A^x} + (1-\alpha)Y_t^{A^z} \geq Y_t^{A^y}$. Since the capital injection processes $\{Y_t\}$ are increasing, they are of bounded variation. With integration by parts we obtain

$$\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t = \delta\int_0^\infty e^{-\delta t} Y_t\,\mathrm{d}t.$$

This yields together with the above result

$$V(\alpha x + (1-\alpha)z) = V(y) \leq \alpha V(x) + (1-\alpha)V(z)\,.$$

Let now $z > x$ and $A = \{a_t\}$ be an investment strategy for initial capital z such that $V^A(z) \leq V(z) + \varepsilon$. For initial capital x choose the strategy $\tilde{A}$ (which is not optimal):

inject the capital $z-x$ and then follow the strategy A. Thus

$$V(x)-V(z)\le V^{\tilde{A}}(x)-V^{A}(z)+\varepsilon=z-x+\varepsilon\;.$$

Because ε is arbitrary we have $|V(x)-V(z)|\le|x-z|$, which proves the Lipschitz-continuity. That is $V(x)$ is absolutely continuous. Because of the Lipschitz-continuity and convexity the value function is differentiable almost everywhere and satisfies $-1\le V'(x)\le 0$. At the points, where $V(x)$ is not differentiable the derivatives from the left and from the right exist.

From Subsection 4.2.1 it follows directly, that $\lim\limits_{x\to\infty}V(x)=0$. □

We conjecture, that the value function $V(x)$ solves the Hamilton–Jacobi–Bellman equation (HJB)

$$\inf_{a\in\mathbb{R}}\frac{\sigma^2a^2}{2}V''(x)+\lambda\int_0^\infty V(x-z)\,\mathrm{d}G(z)+(c+am)V'(x)-(\delta+\lambda)V(x)=0\;. \tag{4.9}$$

For explicit derivation of the HJB equation we refer to the proof of Theorem 4.2.6 or alternatively to Schmidli [70, p. 55]. Because we do not even know, whether the value function is once continuously differentiable, we will be looking for a viscosity solution. Note, that for every twice continuously differentiable function f and continuous function u the minimum of

$$\frac{\sigma^2a^2}{2}f''(x)+\lambda\int_0^\infty u(x-z)\,\mathrm{d}G(z)+(c+am)f'(x)-(\delta+\lambda)u(x)$$

in a is attained at $a=\frac{-f'(x)m}{\sigma^2f''(x)}$. Using this fact we give the precise definition of viscosity solutions.

Definition 4.2.3
We say that a continuous function $\underline{u}:[0,\infty)\to\mathbb{R}_+$ is a **viscosity subsolution** to (4.9) at $x\in(0,\infty)$ if any twice continuously differentiable function $\psi:(0,\infty)\to\mathbb{R}$ with $\psi(x)=\underline{u}(x)$ such that $\underline{u}-\psi$ reaches the maximum at x satisfies

$$-\frac{m^2\psi'(x)^2}{2\sigma^2\psi''(x)}+\lambda\int_0^\infty \underline{u}(x-z)\,\mathrm{d}G(z)+c\psi'(x)-(\delta+\lambda)\underline{u}(x)\ge 0\;, \tag{4.10}$$

and we say that a continuous function $\bar{u}:[0,\infty)\to\mathbb{R}_+$ is a **viscosity supersolution** to (4.9) at $x\in(0,\infty)$ if any twice continuously differentiable function $\phi:(0,\infty)\to\mathbb{R}$ with $\phi(x)=\bar{u}(x)$ such that $\bar{u}-\phi$ reaches the minimum at x satisfies

$$-\frac{m^2\phi'(x)^2}{2\sigma^2\phi''(x)}+\lambda\int_0^\infty \bar{u}(x-z)\,\mathrm{d}G(z)+c\phi'(x)-(\delta+\lambda)\bar{u}(x)\le 0\;.$$

A **viscosity solution** to (4.9) is a continuous function $u:[0,\infty)\to\mathbb{R}_+$ if it is both a viscosity subsolution and a viscosity supersolution at any $x\in(0,\infty)$.

There is an equivalent formulation of viscosity solutions.

Definition 4.2.4
A continuous function $\underline{u} : [0,\infty) \to \mathbb{R}_+$ is a viscosity subsolution to (4.9) at $x \in (0,\infty)$ if any twice continuously differentiable function $\psi : (0,\infty) \to \mathbb{R}$ with $\psi(x) = \underline{u}(x)$, such that $\underline{u} - \psi$ reaches the maximum at x, satisfies

$$-\frac{m^2 \psi'(x)^2}{2\sigma^2 \psi''(x)} + \lambda \int_0^\infty \psi(x-z)\,\mathrm{d}G(z) + c\psi'(x) - (\delta+\lambda)\psi(x) \geq 0\;, \tag{4.11}$$

where $\psi(-x) = \psi(0) - x$ for $x \in (0,\infty)$.
A continuous function $\bar{u} : [0,\infty) \to \mathbb{R}_+$ is a viscosity supersolution to (4.9) at $x \in (0,\infty)$ if any twice continuously differentiable function $\phi : (0,\infty) \to \mathbb{R}$ with $\phi(x) = \bar{u}(x)$, such that $\bar{u} - \phi$ reaches the minimum at x, satisfies

$$-\frac{m^2 \phi'(x)^2}{2\sigma^2 \phi''(x)} + \lambda \int_0^\infty \phi(x-z)\,\mathrm{d}G(z) + c\phi'(x) - (\delta+\lambda)\phi(x) \leq 0\;,$$

where $\phi(-x) = \phi(0) - x$ for $x \in (0,\infty)$.

Given a twice continuously differentiable function f and a continuous function u, we denote in the following

$$\begin{aligned} L(u,f)(x) &= -\frac{m^2 f'(x)^2}{2\sigma^2 f''(x)} + \lambda \int_0^\infty u(x-z)\,\mathrm{d}G(z) + cf'(x) - (\delta+\lambda)u(x)\;, \\ L(f)(x) &= -\frac{m^2 f'(x)^2}{2\sigma^2 f''(x)} + \lambda \int_0^\infty f(x-z)\,\mathrm{d}G(z) + cf'(x) - (\delta+\lambda)f(x)\;. \end{aligned}$$

Proof of equivalence:

The proof technique we use below can be found for example in Benth et al. [8].
We prove the statement for the subsolutions, the supersolution case can be proven similarly.

Let u be a viscosity subsolution and ψ a twice continuously differentiable function, fulfilling the conditions from Definition 4.2.3. Then it holds because $\psi(y) \geq u(y)$ for all $y \in (0,\infty)$ and $\psi(x) = u(x)$:

$$L(\psi)(x) \geq L(u,\psi)(x) \geq 0\;,$$

which yields Definition 4.2.4.

Conversely, let u be a subsolution, satisfying the conditions 1., 2. and 3. from Lemma 4.2.2, and $\tilde{\psi}$ a twice continuously differentiable function in the sense of Definition 4.2.4.

Let further $h \in [0, x]$ be fixed and g_n be a smooth function satisfying $0 \le g_n \le 1$, $g_n(y) = 1$ for $y \in (x - h + \frac{1}{n}, x + h - \frac{1}{n})$ and $g(y) = 0$ for $y \notin [x - h, x + h]$. Define now

$$f(y) = \begin{cases} \exp(-\frac{1}{1-y^2}) & : \ |y| < 1 \ , \\ 0 & : \ |y| \ge 1 \ . \end{cases}$$

Then $f(x)$ is an even, nonnegative, smooth function with support in $(-1, 1)$. Let

$$f_n(y) = \frac{1}{\int_{-1}^{1} f(s)\, \mathrm{d}s} \int_{-1}^{1} u(y - \frac{s+1}{n}) f(s)\, \mathrm{d}s \ .$$

It is obvious, that f_n are smooth functions and f_n converges uniformly to u with $f_n \ge u$. Then define the test functions

$$\psi_n(y) = g_n(y)\tilde{\psi}(y) + (1 - g_n(y)) f_n(y) \ .$$

Apparently ψ_n are twice continuously differentiable, $\psi_n(x) = \tilde{\psi}(x) = u(x)$, $\psi_n(y) \ge u(y)$ for all $y \in (0, \infty)$ and ψ_n converges uniformly to u. Therefore, it holds due to Definition 4.2.4

$$0 \le L(\psi_n)(x) \overset{n\to\infty}{\longrightarrow} L(u, \tilde{\psi})(x) \ .$$

Thus Definition 4.2.3 follows.

□

Remark 4.2.5
Note, that if V is a viscosity solution to (4.9) and is twice continuously differentiable, then V is a classical solution. The optimal investment strategy is then given by $a(x) = -\frac{mV'(x)}{\sigma^2 V''(x)}$.

Note, that for $V''(x) = 0$ the HJB equation would not have any solution, because otherwise there would exist $a > 0$ with

$$\lambda \int_0^\infty V(x - z)\, \mathrm{d}G(z) + (c + am)V'(x) - (\delta + \lambda)V(x) < 0 \ .$$

The function $V(x)$ is in this case strictly convex.

We show at first, that the value function is a viscosity solution. In the proof of the theorem below we used the technique proposed by Benth et al. [8], which was also successfully applied by Albrecher and Thonhauser [1] and Azcue and Muler [5].

Theorem 4.2.6
$V(x)$ is a viscosity solution to (4.9)

Proof: Assume $x > 0$. Let $h > 0$ and $a \in \mathbb{R}$ be fixed. We choose $n \in \mathbb{N}$ such that $|V(x) - V(y)| < \varepsilon/2$ for $|x - y| \le \frac{x}{n}$ for some $\varepsilon > 0$.
Define further $x_k = k\frac{x}{n}$ and $\tau_0 := \inf\{t \ge 0 : |X_t^a - x| > \frac{x}{n}\}$. Then $\tau_1 := \tau_0 \wedge T_1 \wedge h$ is a stopping time. In particular we obtain $X_{\tau_1}^a \in \big(-\infty, \frac{x(n+1)}{n}\big]$. For each k there is a measurable strategy $A^k = \{a_t^k\}$ such that $V^{A^k}(x_k) \le V(x_k) + \varepsilon/2$. For the capital $x_k \le X_{\tau_1}^a < x_{k+1}$, we choose the strategy A^k. Thus $V^{A^k}(X_{\tau_1}^a) \le V^{A^k}(x_k) \le V(x_k) + \varepsilon/2 \le V(X_{\tau_1}^a) + |X_{\tau_1}^a - x_k| + \varepsilon/2 < V(X_{\tau_1}^a) + \varepsilon$ by Lemma 4.2.2. If $X_{\tau_1}^a < 0$ consider $V^{A^k}(0) - X_{\tau_1}^a$. This shows that for each $a \in \mathbb{R}$ we can find in a measurable way a strategy $\hat{A}$ such that $V^{\hat{A}}(X_{\tau_1}^a) < V(X_{\tau_1}^a) + \varepsilon$.

We show at first, that $V(x)$ is a viscosity subsolution. Construct a strategy $\tilde{A} = \{\tilde{a}_t\}$ in following way. Define $\tau^a := \inf\{t \ge 0 : X_t^a < 0\}$ and $T = T_1 \wedge \tau^a \wedge h$ for some $h > 0$ very small. Let further $\tilde{a}_t = a$ for $t \le T$ and $\tilde{a}_t = a_{t-T}$ for $t > T$ and a strategy $A = \{a_t\}$ with $V(x) + \varepsilon \ge V^A(X)$ with $\varepsilon > 0$. Then we obtain

$$\begin{aligned} V(x) \le V^{\tilde{A}}(x) &= \mathbb{E}[e^{-\delta T} V^A(X_T^a)] \\ &= \mathbb{E}\Big[e^{-\delta T_1} V^A(X_{T_1}^a) \mathbb{1}_{[T=T_1]} + e^{-\delta\tau^a} V^A(0) \mathbb{1}_{[T=\tau^a]} \\ &\qquad + e^{-\delta h} V(X_h^a) \mathbb{1}_{[T=h]}\Big] \end{aligned}$$

Rearranging the terms and diving by h yields

$$\begin{aligned} &\mathbb{E}\Big[\frac{1}{h} e^{-\delta T_1} V(X_t^a) \mathbb{1}_{[T=T_1]} + \frac{1}{h} e^{-\delta h} V(X_h^a) \mathbb{1}_{[T=h]} - \frac{1}{h} V(x)\Big] \\ &+ \frac{1}{h} V(0) \mathbb{E}[e^{-\delta\tau^a} \mathbb{1}_{[T=\tau^a]}] \ge 0\,. \end{aligned}$$

Consider at first the term $\frac{1}{h}V(0)\mathbb{E}[e^{-\delta\tau^a}\mathbb{1}_{[T=\tau^a]}]$. Note that $e^{-\delta\tau^a}$ is bounded and

$$\mathbb{P}[T = \tau^a] = \mathbb{P}[\tau^a \le h] - \mathbb{P}[T_1 < \tau^a \le h]\,.$$

It holds $h^{-1}\mathbb{P}[\tau^a \le h] \to 0$ as $h \to 0$, from which it follows, that
$\frac{1}{h}V(0)\mathbb{E}[e^{-\delta\tau^a}\mathbb{1}_{[T=\tau^a]}] \to 0$ as $h \to 0$. So in following we can skip considering this term. Consider further

$$\begin{aligned} &\mathbb{E}\Big[\frac{1}{h} e^{-\delta T_1} V(X_t^a) \mathbb{1}_{[T=T_1]} + \frac{1}{h} V(X_h^a) \mathbb{1}_{[T=h]} - \frac{1}{h} V(x)\Big] \\ = \;& \mathbb{E}\Big[\frac{1}{h} e^{-\delta T_1} V(X_t^a) \mathbb{1}_{[T=T_1]} + \frac{V(X_h^a) - V(x)}{h} e^{-\delta h} \mathbb{1}_{[T=h]} \\ &- V(x) \frac{1 - \mathbb{1}_{[T=h]} e^{-\delta h}}{h}\Big] \\ = \;& \mathbb{E}\Big[\frac{1}{h} e^{-\delta T_1} V(X_t^a) \mathbb{1}_{[T=T_1]} + \frac{V(X_h^a) - V(x)}{h} e^{-\delta h} \mathbb{1}_{[T=h]}\Big] \\ &- V(x) \frac{1 - \mathbb{P}[T = h] e^{-\delta h}}{h}\,. \end{aligned}$$

On the set $\{T = h\}$ it holds $\mathbb{P}[T = h] = \mathbb{P}[\tau^a > h]$. Thus we obtain

$$\lim_{h\to\infty} V(x)\frac{1-\mathbb{P}[T=h]e^{-\delta h}}{h} = (\delta+\lambda)V(x)\ .$$

For the first summand $\frac{1}{h}e^{-\delta T_1}V(X_t^a)1\!\!1_{[T=T_1]}$ we obtain

$$\begin{aligned}\mathbb{E}\Big[\frac{1}{h}e^{-\delta T_1}V(X_t^a)1\!\!1_{[T=T_1]}\Big] &= \mathbb{E}\Big[\frac{1}{h}e^{-\delta T_1}V(X_t^a)1\!\!1_{[T_1\le h]}\Big]\\ &\quad -\mathbb{E}\Big[\frac{1}{h}e^{-\delta T_1}V(X_t^a)1\!\!1_{[\tau^a\le T_1\le h]}\Big]\ .\end{aligned}$$

$V(X_{T_1}^a)1\!\!1_{[T=T_1]}e^{-\delta T_1}$ is bounded and $\mathbb{P}[\tau^a < T_1 \le h]h^{-1} \to 0$ as $h \to 0$. so that we obtain $\mathbb{E}[\frac{1}{h}e^{-\delta T_1}V(X_t^a)1\!\!1_{[\tau^a\le T_1\le h]}] \to 0$ as $h \to 0$. We will skip considering this term in the following proof. For the remaining term we obtain

$$\mathbb{E}\Big[\frac{1}{h}e^{-\delta T_1}V(X_t^a)1\!\!1_{[T_1\le h]}\Big] = \frac{1}{h}\int_0^h\int_0^\infty \mathbb{E}[V(X_t^a)]\ \mathrm{d}G(z)\ \mathrm{d}t\ .$$

Because $V(X_t^a)$ is bounded on the interval $[0,h]$ we have by bounded convergence

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\int_0^\infty \mathbb{E}[V(X_t^a)]\ \mathrm{d}G(z)\ \mathrm{d}t = \int_0^\infty V(x-z)\ \mathrm{d}G(z)\ .$$

Let further ψ be a twice continuously differentiable function such that $V - \psi$ reaches the maximum at x and $\psi(x) = V(x)$. We have then:

$$\begin{aligned}0 &\le \frac{1}{h}\int_0^h\int_0^\infty \mathbb{E}[V(X_t^a)]\ \mathrm{d}G(z)\ \mathrm{d}t - V(x)\frac{1-\mathbb{P}[T=h]e^{-\delta h}}{h}\\ &\quad +\mathbb{E}\Big[\frac{V(X_h^a)-V(x)}{h}1\!\!1_{[T=h]}\Big]\\ &\le \frac{1}{h}\int_0^h\int_0^\infty \mathbb{E}[V(X_t^a)]\ \mathrm{d}G(z)\ \mathrm{d}t - \psi(x)\frac{1-\mathbb{P}[T=h]e^{-\delta h}}{h}\\ &\quad +\mathbb{E}\Big[\frac{\psi(X_h^a)-\psi(x)}{h}1\!\!1_{[T=h]}\Big]\ .\end{aligned}\tag{4.12}$$

Because ψ is two times continuously differentiable we obtain for the generator of the process X_h^a on the set $T = h$:

$$\lim_{h\to 0}\mathbb{E}\Big[\frac{\psi(X_h^a)-\psi(x)}{h}1\!\!1_{[T=h]}\Big] = (c+am)\psi'(x) + \frac{\sigma^2a^2}{2}\psi''(x)\ .$$

So letting h go to 0 in (4.12) we obtain

$$0 \le \lambda\int_0^\infty V(x-z)\ \mathrm{d}G(z) - (\delta+\lambda)V(x) + (c+am)\psi'(x) + \frac{\sigma^2a^2}{2}\psi''(x)$$

Because $a \in \mathbb{R}$ was arbitrary we have

$$\inf_{a\in\mathbb{R}} \frac{\sigma^2 a^2}{2}\psi''(x) + \lambda \int_0^\infty V(x-z)\,\mathrm{d}G(z) + (c+am)\psi'(x) - (\delta+\lambda)V(x) \geq 0 \;,$$

which yields the result.

It remains to show, that $V(x)$ is a viscosity supersolution at any $x > 0$. Arguing by contradiction we assume, that $V(x)$ is not a supersolution at x. Then due to Definition 4.2.4 there exist $\xi > 0$ and a twice continuously differentiable function $\phi_0 : (0,\infty) \to \mathbb{R}$ with $\phi_0(x) = V(x)$ and $V(y) - \phi_0(y) \geq 0$ for all $y > 0$, such that

$$L(\phi_0)(x) > 2\xi \;.$$

Note, that since $-1 < \liminf V'(x) \leq \phi_0'(x) \leq \limsup V'(x)$, we obtain $\phi_0'(x) > -1$. The case $\liminf V'(x) = -1$ is impossible because of Remark 4.2.5.

Consider now the function $\phi_1(y) := \phi_0(y) - \sin^4\left(\frac{(x-y)\pi}{2x}\right)\cdot\frac{\xi}{\lambda}$. It is obvious, that ϕ_1 is twice continuously differentiable, $\phi_1(x) = V(x)$ and $\phi_1(y) \leq V(y) - \sin^4\left(\frac{(x-y)\pi}{2x}\right)\frac{\xi}{\lambda}$. We also have

$$L(\phi_1)(x) > \xi \;,$$

because $\phi_1'(x) = \phi_0'(x)$, $\phi_1''(x) = \phi_0''(x)$ and $\int_0^\infty \sin^4\left(\frac{y\pi}{2x}\right)\,\mathrm{d}G(y) \leq 1$. Since $L(\phi_1)(x)$ is continuous in x there is $\tilde{h} \in (0,\frac{x}{2})$ such that for all $y \in [x-2\tilde{h}, x+2\tilde{h}]$ it holds $L(\phi_1)(y) > \frac{\xi}{2}$.

Like above we conclude $\phi_1'(x) > -1$. Because ϕ_1' is continuous there is $k > 0$ such that $\phi_1'(y) \geq -1$ and $\phi(y) > 0$ for $y \in [x-k, x+k]$. We let in following $[x-2\tilde{h}, x+2\tilde{h}] \cap [x-k, x+k] = [x-2h, x+2h]$.

Let now

$$f(y) = \begin{cases} \exp(-\frac{1}{1-y^2}) & : \; |y| < 1 \;, \\ 0 & : \; |y| \geq 1 \;. \end{cases}$$

Then $f(y)$ is an even, nonnegative, smooth function with support in $(-1,1)$. Define

$$f_n(y) = \frac{1}{\int_{-1}^1 f(s)\,\mathrm{d}s} \int_{-1/n}^{1/n} \Big\{V(y-s) - \sin^4\big(\frac{h\pi}{2x}\big)\frac{\xi}{2\lambda}\Big\} n f(ns)\,\mathrm{d}s \;.$$

It is obvious, that f_n are smooth functions and the sequence $(f_n)_{n\geq1}$ converges uniformly to $V - \sin^4\big(\frac{h\pi}{2x}\big)\frac{\xi}{2\lambda}$. Thus, there is $n_0 \in \mathbb{N}$ such that

$$V(y) - \sin^4\big(\frac{h\pi}{2x}\big)\frac{\xi}{\lambda} \leq f_{n_0}(y) \leq V(y) - \sin^4\big(\frac{h\pi}{2x}\big)\frac{\xi}{4\lambda}$$

for all $y \geq 0$, also we obtain $0 \geq f_{n_0}'(y) \geq -1$ on $[0, x-h]$ because of the construction. In particular it holds $f_{n_0}(y) \geq \phi_1(y) > 0$ on $[x-2h, x-h]$. Then there exists $\kappa > 1$, such that $\phi_1(y)\kappa \geq f_{n_0}(y)$ for all $y \in [x-2h, x-h]$. Let further g be a twice continuously differentiable function satisfying

- $\frac{\pi}{2} \le g(y) \le \pi$
- $g(y) = \pi$ for $y \in [x-h, x+h]$
- $g(y) = \frac{\pi}{2}$ for $y \notin (x-2h, x+2h)$
- $g'(y) \ge 0$ for $y \in [x-2h, x-h]$.

We define

$$\varepsilon = \min\Big\{\sin^4\Big(\frac{h\pi}{2x}\Big)\frac{\xi}{12\lambda}, \frac{\xi}{4\delta}\Big\}$$

and the function ϕ by

$$\phi(y) = \cos^2(g(y))\phi_1(y) + \sin^2\big(g(y)\big)\frac{f_{n_0}(y)}{\kappa}\ .$$

Obviously $\phi(y) = \phi_1(y)$ on $[x-h, x+h]$, from which it follows $L(\phi)(y) > 2\delta\varepsilon$ on $[x-h, x+h]$. Since

$$\begin{aligned}\phi'(y) &= -2g'(y)\cos(g(y))\sin(g(y))\{\phi_1(y) - \frac{f_{n_0}(y)}{\kappa}\}\\ &\quad + \cos^2(g(y))\phi_1'(y) + \sin^2\big(g(y)\big)\frac{f_{n_0}'(y)}{\kappa}\ .\end{aligned}$$

we obtain $\phi'(y) \ge -1$ for $y \in [0, x+h]$ and $\phi(y) \le V(y) - 3\varepsilon$ for $y \in [0, x-h] \cup \{x+h\}$.

Let A be an arbitrary admissible strategy with $\{\hat{X}_t\} = \{X_t^{A,Y}\}$ and τ^* be the exit time from $[x-h, x+h]$. It is clear, that $\hat{X}_{\tau^*} \in [0, x-h] \cup \{x+h\}$, because the paths are continuous between the jumps and the jumps are always downwards. Thus, we obtain

$$V(\hat{X}_{\tau^*}) \ge \phi(\hat{X}_{\tau^*}) + 3\varepsilon\ .$$

Since the function $\phi(x)$ is twice continuously differentiable, we have from Proposition A.2.2, p. 165, that the process

$$\begin{aligned}M_t = -\int_0^{\tau^*\wedge t} e^{-\delta s}\Big(\lambda\int_0^\infty \phi(\hat{X}_s - z)\,\mathrm{d}G(z) - \lambda\phi(\hat{X}_s)\Big)\,\mathrm{d}s&\\ + \sum_{i=1}^{N_{\tau^*\wedge t}} e^{-\delta T_i}\big(\phi(\hat{X}_{T_i}) - \phi(\hat{X}_{T_i-})\big)&\end{aligned}$$

is a martingale with zero-expectation. Rearranging the terms we obtain

$$\begin{aligned}e^{-\delta(\tau^*\wedge t)}\phi(\hat{X}_{\tau^*\wedge t}) &= e^{-\delta(\tau^*\wedge t)}\phi(\hat{X}_{\tau^*\wedge t}) - e^{-\delta T_{N_{\tau^*\wedge t}}}\phi(\hat{X}_{T_{N_{\tau^*\wedge t}}}) + \phi(x)\\ &\quad + \lambda\int_0^{\tau^*\wedge t} e^{-\delta s}\Big(\int_0^\infty \phi(\hat{X}_s - z)\,\mathrm{d}G(z) - \phi(\hat{X}_s)\Big)\,\mathrm{d}s\\ &\quad + \sum_{i=1}^{N_{\tau^*\wedge t}}\big(e^{-\delta T_i}\phi(\hat{X}_{T_i-}) - e^{-\delta T_{i-1}}\phi(\hat{X}_{T_{i-1}})\big) + M_t\ . \qquad (4.13)\end{aligned}$$

On the other hand it holds with Ito's formula

$$
\begin{aligned}
e^{-\delta T_i}\phi(\hat{X}_{T_i-}) - e^{-\delta T_{i-1}}\phi(\hat{X}_{T_{i-1}}) = \int_{T_{i-1}}^{T_i} \sigma a_s e^{-\delta s}\phi'(\hat{X}_s)\,\mathrm{d}W_s \\
+ \int_{T_{i-1}}^{T_i} e^{-\delta s}\phi'(\hat{X}_s)\,\mathrm{d}Y_s^A \\
+ \int_{T_{i-1}}^{T_i} e^{-\delta s}\Big\{\frac{\sigma^2}{2}a_s^2\phi''(\hat{X}_s) + (c+ma_s)\phi'(\hat{X}_s) - \delta\phi(\hat{X}_s)\Big\}\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
e^{-\delta(\tau^*\wedge t)}\phi(\hat{X}_{\tau^*\wedge t}) - e^{-\delta T_{N_{\tau^*\wedge t}}}\phi(\hat{X}_{T_{N_{\tau^*\wedge t}}}) = \int_{T_{N_{\tau^*\wedge t}}}^{\tau^*\wedge t} e^{-\delta s}\phi'(\hat{X}_s)\,\mathrm{d}Y_s^A \\
+ \int_{T_{N_{\tau^*\wedge t}}}^{\tau^*\wedge t} e^{-\delta s}\sigma a_s\phi'(\hat{X}_s)\,\mathrm{d}W_s \\
+ \int_{T_{N_{\tau^*\wedge t}}}^{\tau^*\wedge t} e^{-\delta s}\Big\{\frac{\sigma^2}{2}a_s^2\phi''(\hat{X}_s) + (c+ma_s)\phi'(\hat{X}_s) - \delta\phi(\hat{X}_s)\Big\}\,\mathrm{d}s\,.
\end{aligned}
$$

Plugging in these results into Expression (4.13) we obtain

$$
\begin{aligned}
e^{-\delta(\tau^*\wedge t)}\phi(\hat{X}_{\tau^*\wedge t}) - \phi(x) = \int_0^{\tau^*\wedge t} e^{-\delta s}\Big(\frac{\sigma^2 a_s^2}{2}\phi''(\hat{X}_s) + (c+ma_s)\phi'(\hat{X}_s) \\
+\lambda\int_0^\infty \phi(\hat{X}_s - z)\,\mathrm{d}G(z) - (\delta+\lambda)\phi(\hat{X}_s)\Big)\,\mathrm{d}s \\
+ \int_0^{\tau^*\wedge t} e^{-\delta s}\sigma a_s\phi'(\hat{X}_s)\,\mathrm{d}W_s + M_t + \int_0^{\tau^*\wedge t} e^{-\delta s}\phi'(\hat{X}_s)\,\mathrm{d}Y_s^A\,.
\end{aligned}
$$

Using

$$
\begin{aligned}
L(\phi)(\hat{X}_s) \quad \le \quad & \frac{\sigma^2 a_s^2}{2}\phi''(\hat{X}_s) + (c+ma_s)\phi'(\hat{X}_s) + \lambda\int_0^\infty \phi(\hat{X}_s - z)\,\mathrm{d}G(z) \\
& -(\delta+\lambda)\phi(\hat{X}_s)\,,
\end{aligned}
$$

and $\phi'(\hat{X}_s) \ge -1$ on $[x-h, x+h]$ we obtain

$$
\begin{aligned}
e^{-\delta(\tau^*\wedge t)}\phi(\hat{X}_{\tau^*\wedge t}) - \phi(x) \ge \int_0^{\tau^*\wedge t} e^{-\delta s}L(\phi)(\hat{X}_s)\,\mathrm{d}s + M_t \\
+ \int_0^{\tau^*\wedge t} e^{-\delta s}\sigma a_s\phi'(\hat{X}_s)\,\mathrm{d}W_s - \int_0^{\tau^*\wedge t} e^{-\delta s}\,\mathrm{d}Y_s^A\,.
\end{aligned}
$$

Rearranging the terms and using $\phi(\hat{X}_{\tau^*}) \le V(\hat{X}_{\tau^*}) - 3\varepsilon$ and $L(\phi)(\hat{X}_s) \ge 2\delta\varepsilon$ yields

$$\begin{aligned} e^{-\delta(\tau^*\wedge t)} V(\hat{X}_{\tau^*\wedge t}) + \int_0^{\tau^*\wedge t} e^{-\delta s}\,\mathrm{d}Y_s^A \quad &\ge \quad \phi(x) + 2\delta\varepsilon \int_0^{\tau^*\wedge t} e^{-\delta s}\,\mathrm{d}s \\ &\quad + \int_0^{\tau^*\wedge t} \sigma a_s \phi'(\hat{X}_s)\,\mathrm{d}W_s + M_t \\ &\quad + 3\varepsilon e^{-\delta(\tau^*\wedge t)} . \end{aligned} \tag{4.14}$$

The stochastic integral $\int_0^{\tau^*\wedge t} \sigma a_s \phi'(\hat{X}_s)\,\mathrm{d}W_s$ is a local martingale. Let S_n be a localisation sequence for it and $J_n = S_n \wedge \tau^*$. Then we have from (4.14) applying the expectations

$$\mathbb{E}_x\Big[e^{-\delta J_n} V(\hat{X}_{J_n}) + \int_0^{J_n} e^{-\delta s}\,\mathrm{d}Y_s^A\Big] \ge \phi(x) + 2\varepsilon + \mathbb{E}_x[e^{-\delta J_n}]\big(3\varepsilon - 2\varepsilon\big) .$$

Because the strategy A was arbitrary it follows

$$V(x) = \inf_{A\in\mathscr{A}} \mathbb{E}_x\Big[e^{-\delta J_n} V(\hat{X}_{J_n}) + \int_0^{J_n} e^{-\delta s}\,\mathrm{d}Y_s^A\Big] \ge \phi(x) + 2\varepsilon ,$$

which contradicts the assumption $V(x) = \phi(x)$. □

We have shown, that $V(x)$ is a viscosity solution to (4.9). The uniqueness of the value function follows from the comparison principle, which we show in the next proposition. More about comparison principle one can find for example in Bardi and Capuzzo-Dolcetta [6, p. 82]. The proof technique for $\delta > 0$ used below can be found for example in Azcue and Muler [5].

Proposition 4.2.7
Let now $v(x)$ be a super- and $u(x)$ a subsolution to (4.9), satisfying conditions 1.–3. from Lemma 4.2.2. If it holds $u(0) \le v(0)$, then $u(x) \le v(x)$ on $[0,\infty)$.

Proof: Let u be a sub- and v a supersolution, which fulfil conditions 1.-3. from Lemma 4.2.2. Assume there is $x_0 \in (0,\infty)$ with $u(x_0) - v(x_0) > 0$. Define $v_k(x) = kv(x)$ for $k > 1$. Then $v_k(x)$ is also a supersolution. Choose now $k > 1$ such that $u(x_0) - v_k(x_0) > 0$. It is trivial, that $u(0) \le v_k(0)$. Because u and v_k are decreasing and Lipschitz continuous we obtain the following estimation

$$\begin{aligned} u(x) - v_k(x) \quad &\le \quad u(0) - v_k(x) \le v(0) - v_k(x) \\ &= \quad k(v(0) - v(x)) + (1-k)v(0) \\ &\le \quad kx + (1-k)v(0) . \end{aligned}$$

Thus it holds $u(x) - v_k(x) \leq 0$ for $x \leq \frac{v(0)(k-1)}{k}$. So we will consider only the crucial interval (d, ∞) with $d = \frac{v(0)(k-1)}{k}$.
Define further

$$M := \sup_{x \geq 0}(u(x) - v_k(x)) \; .$$

It follows readily

$$0 < u(x_0) - v_k(x_0) \leq M = \sup_{x \in (d,\infty)} (u(x) - v_k(x)) \; .$$

Let x^* be such, that $M = u(x^*) - v_k(x^*)$.
Define $H := \{(x, y) : d < y < \infty,\ d < x \leq y\}$ and for $\xi > 0$

$$f_\xi(x, y) := u(x) - v_k(y) - \frac{\xi}{2}(x-y)^2 - \frac{2k}{\xi^2(y-x)+\xi} \; .$$

Let $M_\xi := \sup_{(x,y)\in H} f_\xi(x, y)$. Because f_ξ is continuous, there exists $(x_\xi, y_\xi) \in \bar{H}$, where $\bar{H}$ denotes the closure of H, with $M_\xi = f_\xi(x_\xi, y_\xi)$. Because $x^* > d$ it holds $(x^*, x^*) \in H$. Thus we have

$$M_\xi \geq f_\xi(x^*, x^*) = u(x^*) - v_k(x^*) - \frac{2k}{\xi} = M - \frac{2k}{\xi} \; .$$

We can therefore conclude, that $M_\xi > 0$ for $\xi > \frac{4k}{M}$ and $\liminf_{\xi\to\infty} M_\xi \geq M$.
On the other hand we know, that $(x_\xi, y_\xi) \in \bar{H}$, from which it follows $y_\xi \geq x_\xi$.

Now we will show, that there exists ξ_0 such that for all $\xi \geq \xi_0$ it holds $(x_\xi, y_\xi) \notin \partial H$. The boundary ∂H is the union

$$\big\{(d, y) : y \in (d, \infty]\big\} \cup \big\{(x, x) : x \in [d, \infty]\big\} \cup \big\{(x, \infty) : x \in (d, \infty)\big\} \; .$$

It is clear, that for fixed $x \in [d, \infty)$ it holds $\lim_{y\to\infty} f_\xi(x, y) = -\infty$. Consider the set $\big\{(x, x) : x \in [d, \infty]\big\}$. Note, that it holds

$$\lim_{x\to\infty} f_\xi(x, x) = \lim_{x\to\infty} u(x) - v_k(x) - \frac{2k}{\xi} = -\frac{2k}{\xi} < 0$$

and $f_\xi(d, d) = u(d) - v_k(d) - \frac{2k}{\xi} < 0$. For $d < x_1 \leq x_2 < \infty$ we have because of the properties 1. and 2. from Lemma 4.2.2

$$0 \leq \frac{v_k(x_1) - v_k(x_2)}{x_2 - x_1} \leq k \; . \tag{4.15}$$

So it holds

$$\begin{aligned}\frac{f_\xi(x,x)-f_\xi(x,x+h)}{-h} &= \frac{u(x)-v_k(x)-\frac{2k}{\xi}-u(x)+v_k(x+h)+\frac{\xi}{2}h^2+\frac{2k}{\xi^2h+\xi}}{-h}\\ &= -\frac{v_k(x)-v_k(x+h)}{-h}-\frac{\xi}{2}h+\frac{2k}{\xi h+1}\\ &\geq -k-\frac{\xi}{2}h+\frac{2k}{\xi h+1}\ .\end{aligned}$$

Thus $\liminf_{h\to 0}\frac{f_\xi(x,x)-f_\xi(x,x+h)}{-h}\geq k$, which means $f_\xi(x,x)\leq f(x,x+h)$. It remains to consider the set $\big\{(d,y):y\in(d,\infty]\big\}$. For $y>d$ it holds

$$\begin{aligned}\frac{f_\xi(d,y)-f_\xi(d,y-h)}{h} &= \frac{v_k(y-h)-v_k(y)}{h}\\ &\quad+\frac{-\frac{\xi}{2}(d+h-y)^2-\frac{2k}{\xi^2(y-d-h)+\xi}+\frac{\xi}{2}(d-y)^2+\frac{2k}{\xi^2(y-d)+\xi}}{h}\\ &= \frac{v_k(y-h)-v_k(y)}{h}+\frac{\xi}{2}(2y-h-2d)\\ &\quad-\frac{2k}{(\xi(y-d-h)+1)(\xi(y-d)+1)}\\ &\geq \frac{\xi}{2}(2y-h-2d)-2k\ .\end{aligned}$$

Thus, we obtain

$$\liminf_{h\to 0}\frac{f_\xi(d,y)-f_\xi(d,y-h)}{h}\geq \xi(y-d)-2k\ .$$

And on the other hand because $f_\xi(d,d)<0$ there is $\varepsilon>0$ such that for $y\in[d,d+\varepsilon]$ it holds $f_\xi(d,y)<0$. For $\xi\geq\frac{2k}{\varepsilon}$, $y\in[d+\varepsilon,\infty)$ we obtain $f_\xi(d,y)\leq\lim_{x\to\infty}f_\xi(d,x)<0$. Letting $\xi_0:=\max\big\{\frac{2k}{\varepsilon},\frac{4k}{M}\big\}$ we obtain $(x_\xi,y_\xi)\notin\partial H$ for $\xi\geq\xi_0$.

Consider now the functions

$$\begin{aligned}\psi(x) &= v_k(y_\xi)+\frac{\xi}{2}(x-y_\xi)^2+\frac{2k}{\xi^2(y_\xi-x)+\xi}+C\\ \phi(y) &= u(x_\xi)-\frac{\xi}{2}(x_\xi-y)^2-\frac{2k}{\xi^2(y-x_\xi)+\xi}-C\ ,\end{aligned}$$

where

$$C=u(x_\xi)-v_k(y_\xi)-\frac{\xi}{2}(x_\xi-y_\xi)^2-\frac{2k}{\xi^2(y_\xi-x_\xi)+\xi}\ .$$

The functions $\psi(x)$, $\phi(y)$ are twice continuously differentiable. Furthermore, $u(x)-\psi(x)$ attains its maximum at x_ξ; $v_k(y)-\phi(y)$ attains its minimum at y_ξ. Also it holds

$$\psi'(x_\xi) = \phi'(y_\xi) \text{ and } 0 < \psi''(x_\xi) = -\phi''(y_\xi) \ . \tag{4.16}$$

Therefore, it holds per Definition 4.2.4 of viscosity sub- and supersolutions

$$\begin{aligned} -\frac{m^2\psi'(x_\xi)^2}{2\sigma^2\psi''(x_\xi)} + \lambda\int_0^\infty u(x_\xi - z)\,\mathrm{d}G(z) + c\psi'(x_\xi) - (\delta+\lambda)u(x_\xi) \geq 0 \\ -\frac{m^2\phi'(y_\xi)^2}{2\sigma^2\phi''(y_\xi)} + \lambda\int_0^\infty v_k(y_\xi - z)\,\mathrm{d}G(z) + c\phi'(y_\xi) - (\delta+\lambda)v_k(y_\xi) \leq 0 \ . \end{aligned}$$

From above and from (4.16) it follows

$$-\frac{m^2\psi'(x_\xi)^2}{\lambda\sigma^2\psi''(x_\xi)} + \int_0^\infty u(x_\xi - z) - v_k(y_\xi - z)\,\mathrm{d}G(z) \geq \frac{\lambda+\delta}{\lambda}(u(x_\xi) - v_k(y_\xi)) \tag{4.17}$$

It holds (x_ξ, x_ξ), $(y_\xi, y_\xi) \in H$ and

$$f_\xi(x_\xi, x_\xi) + f_\xi(y_\xi, y_\xi) \leq 2f_\xi(x_\xi, y_\xi) \ .$$

Thus, we obtain

$$\xi(x_\xi - y_\xi)^2 \leq u(x_\xi) - u(y_\xi) + v_k(x_\xi) - v_k(y_\xi) + 4k(y_\xi - x_\xi) \ .$$

From (4.15) we conclude, that

$$\xi(x_\xi - y_\xi)^2 \leq 6k|x_\xi - y_\xi| \ . \tag{4.18}$$

Choose a sequence $\xi_n \to \infty$ as $n \to \infty$ such that $(x_{\xi_n}, y_{\xi_n}) \to (\bar{x}, \bar{y}) \in H$. From (4.18) we obtain $|x_{\xi_n} - y_{\xi_n}| \leq \frac{6k}{\xi_n}$, from which it follows $\bar{x} = \bar{y}$; and $\xi_n(x_{\xi_n} - y_{\xi_n})^2 \to 0$. It holds as $\xi_n \to \infty$:

$$\frac{\psi'(x_{\xi_n})^2}{\psi''(x_{\xi_n})} = \frac{\left\{\xi_n(x_{\xi_n} - y_{\xi_n})\cdot\left(\xi_n(y_{\xi_n} - x_{\xi_n}) + 1\right)^2 + 2k\right\}^2}{\xi_n(\xi_n(y_{\xi_n} - x_{\xi_n}) + 1)^3 + 4k\xi_n} \to 0 \ .$$

Thus (4.17) becomes

$$M \geq \int_0^\infty u(\bar{x} - z) - v_k(\bar{y} - z)\,\mathrm{d}G(z) \geq \frac{\lambda+\delta}{\lambda}(u(\bar{x}) - v_k(\bar{y})) \ .$$

Together with $\liminf\limits_{\xi\to\infty} M_\xi \geq M$ we obtain

$$M \leq \liminf_{\xi\to\infty} M_\xi \leq \lim_{\xi_n\to\infty} M_{\xi_n} = u(\bar{x}) - v_k(\bar{y}) \leq \frac{\lambda}{\lambda+\delta}M \ ,$$

which is an obvious contradiction if $\delta > 0$.
Suppose now, that $\delta = 0$. From above it follows, that $\liminf_{\xi\to\infty} M_\xi = M$ and there is some sequence $(\xi_n)_{n\geq 0}$ such that $\xi_n \to \infty$ and $(x_{\xi_n}, y_{\xi_n}) \to (x^*, x^*)$ as $n \to \infty$. Note, that for every $n \in \mathbb{N}$ it holds $f_{\xi_n}(x^*, x^*) < 0$, i.e. there is $\varepsilon > 0$ with $f_{\xi_n}(x, y) \leq 0$ for all $(x, y) \in U_\varepsilon(x^*, x^*)$, where

$$U_\varepsilon(x^*, x^*) = \Big\{(x, y) \in H : ||(x - x^*, y - x^*)||_2 < \varepsilon\Big\}$$

denotes the open ball with center in (x^*, x^*), radius ε and $||.||$ denotes the Euclidean norm. But on the other hand for every $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that

$$||(x_{\xi_n} - x^*, y_{\xi_n} - x^*)||_2 < \varepsilon$$

for all $n \geq n_0$. This contradicts the fact $M_\xi > 0$ for all $\xi > \xi_0$. Thus, we have shown the uniqueness of the value function also in the case $\delta = 0$. □

A direct consequence of the previous propositions is

Corollary 4.2.8
There is a unique viscosity solution to (4.9) *with initial condition* $V(0) = v(0)$.

4.2.3 The special case $\delta = 0$.

Let us now consider the special case $\delta = 0$, i.e. the insurer is indifferent to investing today or tomorrow.

The Hamilton–Jacobi–Bellman equation has the form

$$\inf_{a\in\mathbb{R}} \frac{\sigma^2 a^2}{2} V''(x) + \lambda \int_0^\infty V(x - z)\, \mathrm{d}G(z) + (c + am)V'(x) - \lambda V(x) = 0\ , \tag{4.19}$$

or equivalently applying integration by parts:

$$\inf_{a\in\mathbb{R}} \frac{\sigma^2 a^2}{2} V''(x) - \lambda \int_0^\infty V'(x - z)(1 - G(z))\, \mathrm{d}z + (c + am)V'(x) = 0\ .$$

Under some additional constraints we will be able to show, that the value function $V(x)$ is two times continuously differentiable, solves the HJB equation and is unique. Note, that for $V''(x) = 0$ the HJB equation would not have any solution, because otherwise there would exist $a > 0$ with

$$\lambda \int_0^\infty V(x - z)\, \mathrm{d}G(z) + (c + am)V'(x) - \lambda V(x) < 0\ .$$

Thus we are searching for the function $V(x)$, which is strictly convex.

As for the optimal strategy, it is clear, that the optimal strategy A^* should be given by $a^*(x) = -\frac{mV'(x)}{\sigma^2 V''(x)}$. Because we are searching for $V(x)$, which is decreasing, strictly convex and two times continuously differentiable on $(0,\infty)$, it follows readily $a^*(x) \geq 0$ and $a^*(x)$ is continuous on $(0,\infty)$. Plugging in this ansatz into the HJB equation yields the integro-differential equation

$$-\frac{m^2 V'(x)^2}{2\sigma^2 V''(x)} + \lambda \int_0^\infty V(x-z)\,\mathrm{d}G(z) + cV'(x) - \lambda V(x) = 0\,. \tag{4.20}$$

Next we prove the verification theorem.

Theorem 4.2.9 (Verification theorem)
Let $f(x)$ be a decreasing, vanishing at infinity, twice continuously differentiable solution to (4.19), then $f(x) = V(x)$.

Proof: Let $A = \{a_t\}$ be an arbitrary admissible strategy. Let further X^* denote the surplus process for the optimal strategy and $\hat{X}$ the process $X^{A,Y}$. Choose $n > x$ and define $S_n = \inf\{t \geq 0 : \hat{X}_t \notin [0,n]\}$, then it holds $\hat{X}_{S_n \wedge t} \in [0,n]$, because $\hat{X}_t$ has downward jumps only; in particular we obtain $\hat{X}_{S_n} = n$. We know from Proposition A.2.2, p. 165, that the process

$$M_t = \sum_{i=1}^{N_{S_n\wedge t}} \big(f(\hat{X}_{T_i}) - f(\hat{X}_{T_i-})\big) - \lambda \int_0^{S_n\wedge t} \Big(\int_0^\infty f(\hat{X}_s - z)\,\mathrm{d}G(z) - f(\hat{X}_s)\Big)\,\mathrm{d}s$$

is a martingale. Rearranging the terms we obtain

$$\begin{aligned} f(\hat{X}_{S_n\wedge t}) = f(\hat{X}_{S_n\wedge t}) - f(\hat{X}_{T_{N_{S_n\wedge t}}}) + f(x) + \sum_{i=1}^{N_{S_n\wedge t}} \big(f(\hat{X}_{T_i-}) - f(\hat{X}_{T_{i-1}})\big) \\ + M_t + \lambda \int_0^{S_n\wedge t} \Big(\int_0^\infty f(\hat{X}_s - z)\,\mathrm{d}G(z) - f(\hat{X}_s)\Big)\,\mathrm{d}s\,. \end{aligned} \tag{4.21}$$

On the other hand it holds with Ito's formula

$$\begin{aligned} f(\hat{X}_{T_i-}) - f(\hat{X}_{T_{i-1}}) &= \int_{T_{i-1}}^{T_i} \frac{\sigma^2}{2} a_s^2 f''(\hat{X}_s) + (c + ma_s) f'(\hat{X}_s)\,\mathrm{d}s \\ &\quad + \int_{T_{i-1}}^{T_i} \sigma a_s f'(\hat{X}_s)\,\mathrm{d}W_s + \int_{T_{i-1}}^{T_i} f'(\hat{X}_s)\,\mathrm{d}Y_s^A \\ f(\hat{X}_{S_n\wedge t}) - f(\hat{X}_{T_{N_{S_n\wedge t}}}) &= \int_{T_{N_{S_n\wedge t}}}^{S_n\wedge t} \frac{\sigma^2}{2} a_s^2 f''(\hat{X}_s) + (c + ma_s) f'(\hat{X}_s)\,\mathrm{d}s \\ &\quad + \int_{T_{N_{S_n\wedge t}}}^{S_n\wedge t} \sigma a_s f'(\hat{X}_s)\,\mathrm{d}W_s \\ &\quad + \int_{T_{N_{S_n\wedge t}}}^{S_n\wedge t} f'(\hat{X}_s)\,\mathrm{d}Y_s^A\,. \end{aligned}$$

Plugging in these results into the expression (4.21) we obtain

$$\begin{aligned} f(\hat{X}_{S_n\wedge t}) = f(x) + \int_0^{S_n\wedge t} \Big(\frac{\sigma^2 a_s^2}{2} f''(\hat{X}_s) + (c + m a_s) f'(\hat{X}_s) \\ +\lambda \int_0^\infty f(\hat{X}_s - z)\,\mathrm{d}G(z) - \lambda f(\hat{X}_s)\Big)\,\mathrm{d}s \\ + \int_0^{S_n\wedge t} \sigma a_s f'(\hat{X}_s)\,\mathrm{d}W_s + M_t + \int_0^{S_n\wedge t} f'(\hat{X}_s)\,\mathrm{d}Y_s^A\,. \end{aligned}$$

By HJB Equation (4.19) we obtain

$$f(\hat{X}_{S_n\wedge t}) \ge f(x) + \int_0^{S_n\wedge t} \sigma a_s f'(\hat{X}_s)\,\mathrm{d}W_s + M_t + \int_0^{S_n\wedge t} f'(\hat{X}_s)\,\mathrm{d}Y_s^A\,,$$

and for X^* holds the equality. The stochastic integral $\int_0^{S_n\wedge t} \sigma a_s f'(\hat{X}_s)\,\mathrm{d}W_s$ is a local martingale. Let $\{K_m\}$ be a localisation sequence for it and let $J_n^m = S_n \wedge K_m$. Then we have for the expectations

$$\mathbb{E}_x[f(\hat{X}_{J_n^m\wedge t})] - \mathbb{E}_x\Big[\int_0^{J_n^m\wedge t} f'(\hat{X}_s)\,\mathrm{d}Y_s^A\Big] \ge f(x)\,.$$

Letting m and t go to infinity we obtain by the proposition of Lebesgue

$$\mathbb{E}_x[f(\hat{X}_{S_n})] - \mathbb{E}_x\Big[\int_0^{S_n} f'(\hat{X}_s)\,\mathrm{d}Y_s^A\Big] = f(n) - \mathbb{E}_x\Big[\int_0^{S_n} f'(\hat{X}_s)\,\mathrm{d}Y_s^A\Big] \ge f(x)\,.$$

Because $f(x)$ is Lipschitz continuous with $|f(x) - f(y)| \le |x - y|$, it holds $-f'(x) \le 1$ and we obtain

$$f(n) + \mathbb{E}_x\Big[\int_0^{S_n} \mathrm{d}Y_s^A\Big] \ge f(x)$$

Letting n go to infinity yields $\mathbb{E}_x\big[\int_0^\infty \mathrm{d}Y_s^A\big] \ge f(x)$. For X^* holds the equality, which means $V(x) = f(x)$. □

It remains to show, that there exists a decreasing and twice continuously differentiable solution to (4.19). We will see, that it is possible to show the existence only under the constraint, that the distribution function of the claims amounts is absolutely continuous and the density function is bounded.

But first of all we consider the question of optimal strategy on $(-\infty, 0]$. It is clear, that for $x < 0$ the optimal strategy should be given by $a^*(x) = 0$. In the same model by minimising the ruin probability, compare Schmidli [70, p. 56], it was obvious, that the optimal strategy for $x = 0$ was $a^*(0) = 0$, because otherwise the ruin would occur immediately due to the oscillation of Brownian motion. But in our case it is not self-explanatory, what is the optimal strategy at $x = 0$.

Lemma 4.2.10
Assume the value function $V(x)$ exists, the net profit condition $c > \lambda\mu$ is fulfilled and the claim size distribution $G(x)$ has a bounded density, then the optimal strategy at $x = 0$ is $a^* = 0$.

Proof: Arguing by contradiction we assume $a^* > 0$ and denote by $V(x)$ the value function.

Denote the function, which results from choosing the start strategy $a(0) = 0$ and solving the HJB equation (4.19) by $f(x)$. It is clear, that $f''(0) = \infty$ and $f'(0) = -\frac{\lambda\mu}{c}$. With the same arguments as in Remark 4.2.5 we obtain $f''(x) > 0$ for all $x \in \mathbb{R}_+$. Let further $V^0(x)$ define the return function corresponding to the constant strategy $A \equiv 0$. From Example 2.2.1 we know, that if the claim size distribution $G(x)$ is absolutely continuous and the density function is bounded, $V^0(x)$ is twice continuously differentiable with $(V^0)'(0) = -\frac{\lambda\mu}{c} = f'(0)$, $(V^0)''(0) = \lambda\frac{c-\lambda\mu}{c^2} < \infty$ and solves the integro-differential equation

$$c(V^0)'(x) + \lambda\int_0^\infty V^0(x-z)\,\mathrm{d}G(z) - \lambda V^0(x) = 0\,.$$

Thus we obtain for $h(x) := f(x) - V^0(x)$

$$\begin{aligned} ch'(x) &\geq -\lambda\int_0^\infty h(x-z)\,\mathrm{d}G(z) + \lambda h(x) \\ &= \lambda\big(h(x) - h(0)\big)\big(1 - G(x)\big) + \lambda\int_0^x h(x) - h(x-z)\,\mathrm{d}G(z)\,. \end{aligned}$$

Note, that it holds $h'(0) = 0$ and $h''(0) = \infty$, from which it follows $h'(x) > 0$ for x small enough. Let $\hat{x} = \inf\{x > 0 : h'(x) < 0\}$. Then it holds $h'(x) > 0$ on $(0, \hat{x})$, which implies $ch'(x) \geq \lambda(h(x) - h(0))$ on $(0, \hat{x})$. We can conclude $h(x) > 0$ for all $x > 0$. Thus we obtain $\liminf\limits_{x\to\infty} f(x) - V^0(x) \geq f(0) - V^0(0)$. Because $|f(0) - V^0(0)|$ is finite and $\lim\limits_{x\to\infty} V^0(x) = 0$ we conclude, that $\lim\limits_{x\to\infty} f(x) = -\infty$ is impossible.
Define further

$$g(x) := V(x) - f(x)\,.$$

It is obvious that it holds

$$V'(0) = -\frac{\lambda\mu}{c + a^* m/2} > -\frac{\lambda\mu}{c} = f'(0)$$

as well as $0 < V''(0) < \infty$. Thus $g(x)$ is twice continuously differentiable, $g'(0) > 0$ and $g''(0) = -\infty$ by definition. Let further $a(x)$ denote the optimal strategy for $f(x)$, then we obtain from the HJB equation

$$\frac{a(x)^2\sigma^2}{2}g''(x) + (c + ma(x))g'(x) - \lambda\int_0^x g'(x-z)(1 - G(z))\,\mathrm{d}z \geq 0\,.$$

From $f''(0) = \infty$ and from

$$-\frac{ma(x)}{2} f'(x) = cf'(x) - \lambda \int_0^\infty f'(x-z)(1-G(z))\,\mathrm{d}z$$

we obtain, that $a(x)$ is strictly increasing close to zero, i.e. $a(x) > 0$ for $x \in [0, \varepsilon)$ for some $\varepsilon > 0$. Let $\breve{x} = \inf\{x : g''(x) > 0\} \wedge \inf\{x > \varepsilon : a(x) = 0\}$, then

$$\begin{aligned} 0 > \frac{a(x)^2\sigma^2}{2} g''(x) &\geq \lambda \int_0^x g'(x-z)(1-G(z))\,\mathrm{d}z - (c + ma(x))g'(x) \\ &\geq (\lambda\mu - c - ma(x))g'(x)\,. \end{aligned}$$

on $[0, \breve{x})$. Thus it holds $g'(x) \geq 0$ for all $x \in \mathbb{R}_+$. We conclude, that the function $g(x)$ is increasing, which implies $f(x)$ is decreasing. In particular we have either $\lim\limits_{x\to\infty} f(x) = -\infty$ or $\lim\limits_{x\to\infty} f(x) = d$, with $|d| < \infty$. We have already shown above, that the first case is impossible. In the second case consider the function $f_1(x) = f(x) - d$. $f_1(x)$ solves the HJB equation for the strategy $a(x)$, is twice continuously differentiable, decreasing with $\lim\limits_{x\to\infty} f_1(x) = 0$ and satisfies $f_1'(0) = -\frac{\lambda\mu}{c}$, which is a contradiction to Theorem 4.2.9. $\square$

Thus we can assume $a^*(0) = 0$, i.e. $V'(0) = -\frac{\lambda\mu}{c}$. Using this we show the existence and uniqueness of the value function.

Lemma 4.2.11
If $G(x)$ has a bounded density, then there exists $\xi > 0$, such that there is a solution $f(x)$ to (4.19) on $[0, \xi]$ and $f'(x) = -\frac{\lambda\mu}{c} + \sqrt{x}\frac{\lambda\mu m}{\sigma c^{3/2}} + o(\sqrt{x})$ as $x \downarrow 0$, where $\lim\limits_{x\downarrow 0} \frac{o(\sqrt{x})}{\sqrt{x}} = 0$.

Proof: The proof follows closely the proof in Hipp and Plum [41] see also Schmidli [70, p. 59] and is organised as follows. At first we construct a contraction on the interval $[0, \xi]$ and show the existence of the value function on the half-open interval $[0, \xi)$ by Banach fixed point theorem. Then we extend the solution to the closed interval $[0, \xi]$.

Rewrite at first Equation (4.20) as follows using integration by parts

$$-\frac{m^2 V'(x)^2}{2\sigma^2 V''(x)} - \lambda \int_0^\infty V'(x-z)(1-G(z))\,\mathrm{d}z + cV'(x) = 0\,.$$

Let now $h(x) = V'(x)$ and rewrite the above equation in order to get an expression for $h'(x)$:

$$h'(x) = \frac{m^2 h(x)^2}{2\sigma^2}\Big[-\lambda \int_0^\infty h(x-z)(1-G(z))\,\mathrm{d}z + ch(x)\Big]^{-1}\,.$$

The ansatz $p(x) = h(x^2)$ yields:

$$\begin{aligned} p'(x) &= \frac{m^2 x p(x)^2}{\sigma^2}\Big\{-2\lambda \int_0^x zp(z)\big(1 - G(x^2 - z^2)\big)\,\mathrm{d}z \\ &\quad + \lambda \int_{x^2}^\infty 1 - G(z)\,\mathrm{d}z + cp(x)\Big\}^{-1}\,. \end{aligned} \tag{4.22}$$

Let further $K(f) := \sup\limits_{0<x<\xi} \frac{|f'(x)-f'(0)|}{x}$ and $L := \{f \in C^1[0,\xi] : K(f) < \infty\}$. Define the norm

$$||f|| := \max\{||f||_\infty, |f'(0)|, \xi K(f)\} ,$$

where $||f||_\infty$ is the supremum norm. With this norm the space L is complete. Suppose further $V'(0) = -\frac{\lambda\mu}{c}$ and let

$$D_l := \Big\{f \in L :\ f(0) = -\frac{\lambda\mu}{c}\ ,\ f'(0) = \frac{m\lambda\mu}{\sigma c^{3/2}}\ ,\ \ ||\frac{\lambda\mu}{c} + f||_\infty \le \frac{\lambda\mu}{3c}\ ,\ K(f) \le l\Big\};$$

the set D_l is a closed set. Define an operator F acting on the set D_l:

$$\begin{aligned} F(f(x)) : &= \int_0^x \frac{m^2 y f(y)^2}{\sigma^2}\Big\{ -2\lambda \int_0^y z f(z)\big(1 - G(y^2 - z^2)\big)\,\mathrm{d}z \\ &\quad + \lambda \int_{y^2}^\infty 1 - G(z)\,\mathrm{d}z + c f(y)\Big\}^{-1}\,\mathrm{d}y - \frac{\lambda\mu}{c} . \end{aligned}$$

One can find pairs (l,ξ) (where l is large enough and ξ small enough), such that D_l is mapped into itself and F is a contraction. By Banach fixed point theorem it follows, that there exists a fixed point $\tilde{p}(x)$. Because $\tilde{p} \in D_l$ it follows $\tilde{p}(x) \le -\frac{2\lambda\mu}{3c}$, from which it follows, that the derivative is bounded. The derivative $\tilde{p}'(x)$ can become negative only if $m^2 x\tilde{p}(x)^2 = 0$ for some $x > 0$. But it is impossible because the derivative is bounded. Thus $\tilde{p}'(x) > 0$ on $(0,\xi]$. Moreover the function $h(x) = \tilde{p}(\sqrt{x})$ is then the desired solution and it holds $f'(x) = \tilde{p}(\sqrt{x}) = -\frac{\lambda\mu}{c} + \sqrt{x}\frac{\lambda\mu m}{\sigma c^{3/2}} + o(\sqrt{x})$ as $x \downarrow 0$.

Consider now again Equation (4.20) and let $h(x) = V'(x)$. Rearranging the terms and building the reciprocal yields

$$-\frac{h'(x)}{h(x)^2} = \frac{m^2}{2\sigma^2}\Big\{\lambda \int_0^x h(x-z)(1-G(z))\,\mathrm{d}z - \lambda \int_{x^2}^\infty 1 - G(z)\,\mathrm{d}z - ch(x)\Big\}^{-1} ,$$

from which it follows

$$\begin{aligned} \frac{1}{h(x)} &= -\frac{c}{\lambda\mu} + \int_0^x \frac{m^2}{2\sigma^2}\Big\{\lambda \int_0^y h(y-z)(1-G(z))\,\mathrm{d}z \\ &\qquad -\lambda \int_{y^2}^\infty 1 - G(z)\,\mathrm{d}z - ch(y)\Big\}^{-1}\,\mathrm{d}y . \end{aligned}$$

Thus

$$\begin{aligned} h(x) = \Big\{ -\frac{c}{\lambda\mu} + \int_0^x \frac{m^2}{2\sigma^2}\Big\{\lambda \int_0^y h(y-z)(1-G(z))\,\mathrm{d}z \\ -\lambda \int_{y^2}^\infty 1 - G(z)\,\mathrm{d}z - ch(y)\Big\}^{-1}\,\mathrm{d}y\Big\}^{-1} . \end{aligned}$$

Every solution to Equation (4.22) solves also the above equation. Thus there is a strictly negative, increasing solution to the above equation on some interval $[0,\xi)$ with $\xi > 0$. For this solution, say $p(x)$, it holds $p(x) < 0$ on $[0,\xi)$. If we can show, that

$$\lambda \int_0^\xi p(\xi - z)(1 - G(z))\,\mathrm{d}z - \lambda \int_{\xi^2}^\infty 1 - G(z)\,\mathrm{d}z - cp(\xi) < 0\,,$$

then $p(\xi) < 0$; and the solution can be extended to $[0,\xi]$. Suppose now, that

$$\lambda \int_0^\xi p(\xi - z)(1 - G(z))\,\mathrm{d}z - \lambda \int_{\xi^2}^\infty 1 - G(z)\,\mathrm{d}z - cp(\xi) = 0\,.$$

Then we obtain $p'(\xi) = \infty$. On the other hand it holds for all $x \in [0,\xi)$:

$$\begin{aligned} 0 &> \frac{\lambda \int_0^x p(z)(1 - G(x-z))\,\mathrm{d}z - \lambda \int_{x^2}^\infty 1 - G(z)\,\mathrm{d}z - cp(x)}{\xi - x} \\ &\quad - \frac{\lambda \int_0^\xi p(z)(1 - G(\xi-z))\,\mathrm{d}z - \lambda \int_{\xi^2}^\infty 1 - G(z)\,\mathrm{d}z - cp(\xi)}{\xi - x} \\ &\geq \lambda \int_0^\xi p(z) \frac{(G(\xi - z) - G(x-z))\,\mathrm{d}z}{\xi - x} + c\frac{p(\xi) - p(x)}{\xi - x} \\ &\quad + \lambda \int_{x^2}^{\xi^2} \frac{1 - G(z)}{\xi - x}\,. \end{aligned}$$

Because $G(x)$ has a bounded density and $p(x)$ is bounded, the above expression goes to ∞ as $x \to \xi$. Thus we obtain $0 > \infty$, which is a contradiction and we can extend the solution to $[0,\xi]$. $\square$

Theorem 4.2.12
If $G(x)$ has a bounded density, then there is a twice continuously differentiable solution to (4.19).

Proof: Let ξ be the largest value such that there is a solution $p(x)$ to (4.19) on $[0,\xi]$. From Lemma 4.2.11 we know, that $\xi > 0$. Assume now $\xi < \infty$. Let then

$$d_1 := -\frac{c}{\lambda\mu} + \int_0^\xi \frac{\frac{m^2}{2\sigma^2}}{\lambda \int_0^y p(y-z)(1 - G(z))\,\mathrm{d}z - \lambda \int_{y^2}^\infty 1 - G(z)\,\mathrm{d}z - cp(y)}\,\mathrm{d}y$$

and

$$d_2 := \lambda \int_0^\xi p(\xi - z)(1 - G(z))\,\mathrm{d}z - \lambda \int_{\xi^2}^\infty 1 - G(z)\,\mathrm{d}z - cp(\xi) < 0\,.$$

Define an operator F on the set of continuous, negative and increasing functions $h(x)$ with domain $[0, \xi + \zeta)$, where $h(x) = p(x)$ on $[0, \xi]$ and $h(x) \geq 2\big(d_1 + \frac{m^2}{2\sigma^2}\frac{x-\xi}{d_2}\big)^{-1}$ on $[\xi, \xi + \zeta)$:

$$F(p(x)) = \frac{1}{d_1 + \int_\xi^x \frac{m^2}{2\sigma^2}\Big\{U(y) \vee d_2\Big\}^{-1}\,\mathrm{d}y}\;,$$

where

$$U(y) = \lambda \int_0^y p(y-z)(1-G(z))\,\mathrm{d}z - \lambda \int_{y^2}^\infty 1 - G(z)\,\mathrm{d}z - cp(y)\;.$$

The parameter ζ we will be chosen later.
Let further h_1 and h_2 be two functions in the domain of the operator F with

$$H_i(x) = \Big[\lambda \int_0^x h_i(x-z)(1-G(z))\,\mathrm{d}z - \lambda \int_x^\infty 1 - G(z)\,\mathrm{d}z - ch_i(x)\Big] \vee d_2\;,$$

$i \in \{1, 2\}$. For $x > \xi$ we obtain

$$|F(h_1(x)) - F(h_2(x))| \leq \frac{\frac{m^2}{2\sigma^2}\int_\xi^x |H_1(y)^{-1} - H_2(y)^{-1}|\,\mathrm{d}y}{\prod\limits_{i=1}^{2}\Big(d_1 + \int_\xi^x \frac{m^2}{2\sigma^2} H_i(y)^{-1}\,\mathrm{d}y\Big)}\;.$$

We can estimate the integral in the nominator as follows

$$\begin{aligned}\int_\xi^x |H_1(y)^{-1} - H_2(y)^{-1}|\,\mathrm{d}y &\leq d_2^{-2}\int_\xi^x |H_1(y) - H_2(y)|\,\mathrm{d}y\\ &\leq d_2^{-2}[(x-\xi) + \frac{1}{2}(x-\xi)^2] \sup_{\xi \leq y \leq \xi+\zeta} |h_1(y) - h_2(y)|\;.\end{aligned}$$

Thus for ζ small enough F is a contraction and there is a fixed point $h(x)$. By continuity one can choose ζ so small, that

$$\lambda \int_0^x h(x-z)(1-G(z))\,\mathrm{d}z - \lambda \int_{x^2}^\infty 1 - G(z)\,\mathrm{d}z - ch(x) < d_2$$

for all $x \in [\xi, \xi + \zeta]$. Therefore, there is a solution to (4.19) also on $[\xi, \xi + \zeta)$, which is a contradiction to the assumption, that $[0, \xi]$ is the largest interval. This proves our claim $\xi = \infty$. □

Next we consider the asymptotic behavior of the value function. In order to characterise $V(x)$ for $x \to \infty$ we need the concept of moment generating function, defined as

$$M_Z(r) = \mathbb{E}[e^{rZ}]$$

for $r > 0$. We distinguish between the case, where $M_Z(r) < \infty$ for some $r > 0$, and the case, where $M_Z(r) = \infty$ for all $r > 0$. The characteristics of the value function stated below have been already proven for the ruin probability as a function of initial capital x. Since the proofs will be perfectly similar we will skip them and just refer to Schmildi [70] and Hipp and Plum [41].

Remark 4.2.13
Consider once again the return functions $V^a(x)$, corresponding to the constant strategies $A \equiv a$. In Gerber and Landry [32] it was shown, that in the case there is $R > 0$ such that

$$M_Z(R) = \int_0^\infty e^{Ry}\, \mathrm{d}G(y) = 1 \ ,$$

i.e. for $a = \frac{mR+\sqrt{m^2R^2+2\sigma^2R^2(\delta+cR)}}{\sigma^2R^2}$

$$\lambda + \delta + (c + ma)R - \frac{a^2\sigma^2}{2}R^2 = \lambda \int_0^\infty e^{Ry}\, \mathrm{d}G(y)$$

is fulfilled, and it holds $V^a(x) \sim C_1 e^{-Rx}$ with some constant $C_1 > 0$. Thus, we obtain $V(x) \lesssim C_1 e^{-Rx}$. Due to Example 2.2.1 the return function corresponding to the constant strategy $A \equiv 0$ is given by the function $\frac{1+\tilde{R}}{-\tilde{R}} e^{\tilde{R}x}$, where $\tilde{R}$ is the unique positive solution to the equation

$$\lambda - c\tilde{R} = \lambda \int_0^\infty e^{-\tilde{R}x}\, \mathrm{d}G(x) \ . \tag{4.23}$$

Thus, $V(x) \le \frac{1+\tilde{R}}{-\tilde{R}} e^{-\tilde{R}x}$.

The following lemma deduces the behavior of the value function and the optimal strategy in the case $M_Z(r) = \infty$ for all $r > 0$.

Lemma 4.2.14
Assume $M_Z(r) = \infty$ for all $r > 0$. Then for any $r > 0$ it holds

$$\limsup_{x\to\infty} V(x)e^{rx} = \infty \ .$$

For proof see Schmidli [70, p. 181]. Like in Hipp and Plum [41] we obtain

Proposition 4.2.15
Assume $M_Z(r) = \infty$ for all $r > 0$. Then the optimal strategy $a^*(x)$ is unbounded, i.e. $\limsup_{x\to\infty} a^*(x) = \infty$.

For more special constraints we obtain

Proposition 4.2.16
If the hazard rate $G'(y)/(1-G(y))$ goes to 0 as $y \to \infty$, $\mu = \mathbb{E}[Z] < \infty$ and

$$\lim_{x\to\infty} \int_0^x \frac{(1-G(x-y))(1-G(y))}{1-G(x)} \, \mathrm{d}y = 2\mu \; ,$$

then $\lim\limits_{x\to\infty} a^*(x) = \infty$.

For the proof we also refer to Schmidli [70, p. 183].

4.2.4 Examples

Example 4.2.17 (Exponentially distributed claim sizes)
Consider the special case, where the claims are exponentially distributed, $G(x) = 1 - e^{x/\mu}$. In this case our findings coincide with the considering of ruin probability as a function of initial capital x.
Let now $\mu = \lambda = 1$, $m = 0.03$, $\sigma^2 = 0.01$ and $c = 1.3$. Then we obtain from the Equation (4.20), letting $V'(x) = h(x)$, the following integro-differential equation:

$$-\frac{m^2 h(x)^2}{2\sigma^2 h'(x)} - \lambda e^{-\frac{x}{\mu}} \int_0^x h(y) e^{\frac{y}{\mu}} \, \mathrm{d}y + ch(x) + \lambda\mu e^{-\frac{x}{\mu}} = 0 \; ,$$

which is also the HJB equation for the ruin probability, compare formula (2.19) in Schmidli [70, p. 58]. The numerical solution to the above equation, integrated from x to infinity, is given in Figure 4.6. The optimal investment strategy is given in Figure 4.7. We see, that the optimal investment behaves like stated in Lemma 4.2.11 near zero.

Let

$$C_- = \inf_z \frac{1}{\mathbb{E}[e^{R(Z-z)}|Z > z]} \; , \tag{4.24}$$

where the infimum is taken over the set $\{z : \mathbb{P}[Z > z] > 0\}$, $R = \sup\limits_a R(a)$ with $R(a)$ the unique positive solution to the equation

$$\lambda \int_0^\infty e^{R(a)z} \, \mathrm{d}G(z) - \lambda - (c + am)R(a) + \frac{a^2 m^2 R(a)^2}{2\sigma^2} = 0 \; .$$

It was shown in Schmidli [70, p. 170], that in the case $C_- > 0$, the optimal strategy $A^*(x)$ fulfils $\lim\limits_{x\to\infty} a^*(x) = \hat{a}$ with some $\hat{a} \geq 0$. We see in Figure 4.7, that the optimal strategy converges approximately to 9.5 for $x \to \infty$. Note, that because $V'(0) = -\frac{\lambda\mu}{c}$ we can calculate the initial value $V(0)$ exactly. For the above parameters we have $V(0) = 2.2787$. ■

Example 4.2.18 (Pareto$(2, \mu)$-distributed claim sizes)
Consider now Pareto$(2, \mu)$ distributed claim sizes, i.e. $G(x) = 1 - \frac{\mu^2}{(\mu+x)^2}$. We have to solve the integro-differential equation

$$-\frac{m^2 h(x)^2}{2\sigma^2 h'(x)} - \lambda \int_0^x h(y) \frac{\mu^2}{(\mu + (x-y))^2} \,\mathrm{d}y + c h(x) + \lambda \frac{\mu^2}{\mu + x} = 0 \,,$$

where again $h(x) = V'(x)$. For the parameters $\mu = \lambda = 1$, $m = 0.03$, $\sigma^2 = 0.01$ and $c = 1.3$ numerical calculated $V(x)$ and the optimal strategy are given in Figures 4.8 and 4.9 respectively. Note, that Pareto distribution is in the class of distributions, described in Proposition 4.2.16. It means the optimal investment strategy goes to infinity as $x \to \infty$. In Figure 4.9 we can see, that $a^*(40) \approx 70$. Using $V'(0) = -\frac{\lambda\mu}{c}$ we obtain $V(0) = 4.4271$ for initial value. ∎

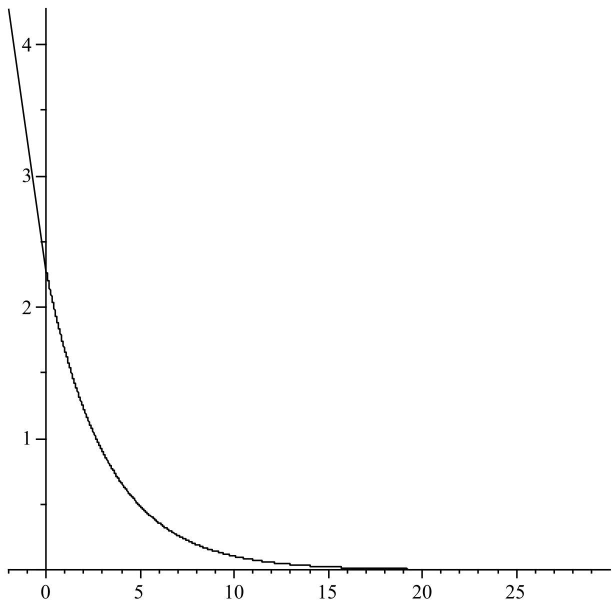

Figure 4.6: $V(x)$ for exponentially distributed claims.

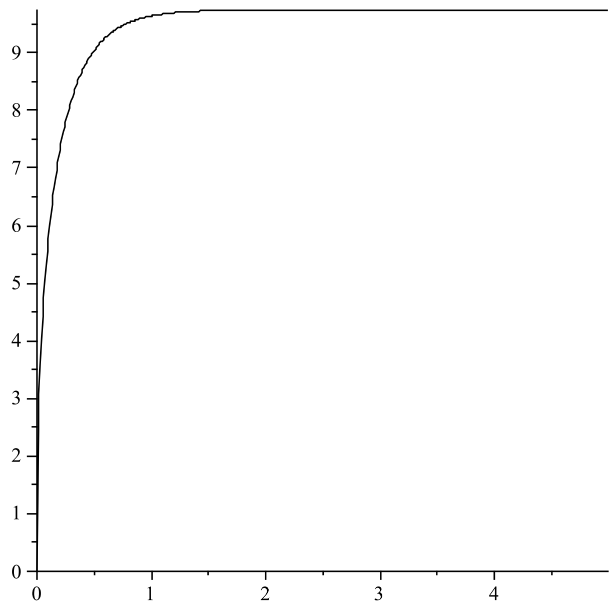

Figure 4.7: Optimal Strategy A^* for exponentially distributed claim sizes.

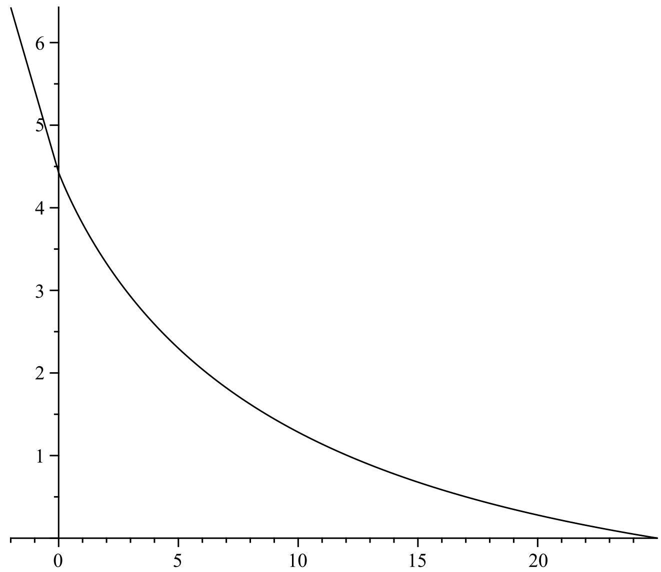

Figure 4.8: $V(x)$ for Pareto$(2, \mu)$-distributed claim sizes.

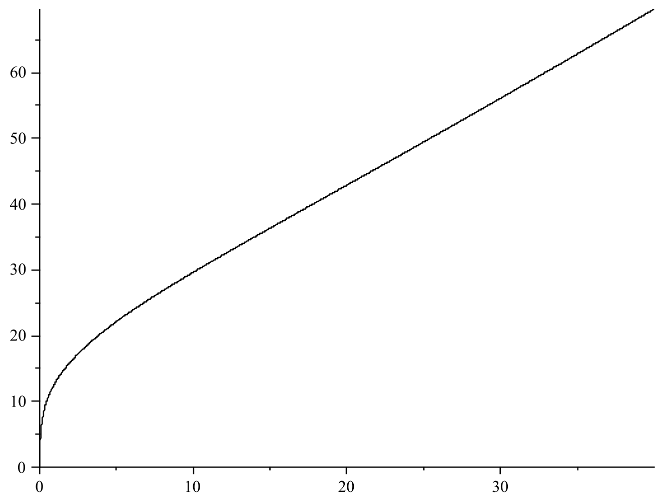

Figure 4.9: The optimal Strategy A for Pareto$(2, \mu)$-distributed claim sizes.

5 Optimal Control of Capital Injections by Reinsurance and Investments, modeled as a Black-Scholes Model

In following we complexify the previous model by adding the possibility of reinsurance. In the examples we will show, that an optimal reinsurance treaty can lower the costs considerably. As in the previous section we start with the simple case of a diffusion approximation to the classical risk model.

5.1 Diffusion Approximation

We consider a surplus process where the mean number of claims in a time unit is λ and the mean size of a claim is $\mu = \mathbb{E}[Z]$, where Z is a generic random variable describing the claim size. We assume that $\mu_2 = \mathbb{E}[Z^2] < \infty$. The premium is $c = (1+\eta)\lambda\mu$ for some $\eta \in \mathbb{R}$. The insurer can now buy reinsurance. That is, a retention level $b \in [0, \tilde{b}]$ has to be chosen, where $\tilde{b} \in (0, \infty]$. The cedent pays then $0 \le r(Z,b) \le Z$. The reinsurance premium is calculated by an expected value principle $(1+\theta)\lambda\big(\mu - \mathbb{E}[r(Z,b)]\big)$; i.e. the premium rate remaining for the insurer is $c(b) = \lambda(1+\theta)\mathbb{E}[r(Z,b)] - \lambda\mu(\theta-\eta)$. Here $b = 0$ means "full reinsurance" and $b = \tilde{b}$ means "no reinsurance". We assume for simplicity that $r(z,b)$ is increasing and continuous in b and z. In order that the problem does not have the trivial solution $V(x) = |x|1\!\!1_{[x<0]}$ we assume that $\theta > \eta$. Here $1\!\!1_A$ denotes the indicator function.

For a constant strategy, the diffusion approximation to the surplus process is

$$\big\{x + \lambda(\theta\mathbb{E}[r(Z,b)] - (\theta-\eta)\mu)t + \sqrt{\lambda\mathbb{E}[r(Z,b)^2]}W_t\big\} ,$$

compare Section 1.2. As an extension, the insurer can continuously change the reinsurance leading to the process

$$X_t^B = x + \lambda\int_0^t \big(\theta\mathbb{E}[r(Z,b_s)] - (\theta-\eta)\mu\big)\,\mathrm{d}s + \int_0^t \sqrt{\lambda\mathbb{E}[r(Z,b_s)^2]}\,\mathrm{d}W_s . \tag{5.1}$$

For the invetsments, the risky asset is again modeled as a Black-Scholes model

$$Q_t = \exp\Big\{\big(m - \frac{\sigma^2}{2}\big)t + \sigma\tilde{W}_t\Big\} ,$$

where $m, \sigma > 0$. We consider the return of such a process, which is then the stochastic process $\{Q'_t\}$ given by the stochastic differential equation

$$\mathrm{d}Q'_t = m\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{W}_t\,.$$

We assume further, that the Brownian motions W_t and $\tilde{W}_t$ are independent and denote the filtration generated by the couple $\{(W_t, \tilde{W}_t)\}$ by $\mathbb{F} = \{\mathcal{F}_t\}$.

The amount $a_t \in \mathbb{R}$ invested at time t can be changed continuously. We call every strategy (A, B), where $A = \{a_t\}$ and $B = \{b_t\}$, admissible if A and B are cadlag and $\mathbb{F}$ adapted. The surplus process under the reinsurance strategy $B = \{b_t\}$ and investment strategy $A = \{a_t\}$ fulfils the stochastic differential equation:

$$\begin{aligned}\mathrm{d}X_t^{A,B} = \Big\{\lambda\mu(\eta-\theta) + \lambda\mathbb{E}[r(Z,b_t)]\theta + m\cdot a_t\Big\}\,\mathrm{d}t + \sqrt{\lambda\mathbb{E}[r(Z,b_t)^2]}\,\mathrm{d}W_t \\ +\sigma a_t\,\mathrm{d}\tilde{W}_t\,.\end{aligned}$$

Note, that here we do not need to assume the net profit condition. We can always find an investment strategy $A = \{a_t\}$, such that the underlying process $X^{A,B}$ has a positive drift.

By $Y^{A,B} = \{Y_t^{A,B}\}$ we denote in following the process of cumulated capital injections corresponding to the strategy (A, B). The underlying process $\{X_t^{A,B}\}$ with capital injections fulfils then

$$\mathrm{d}X_t^{A,B,Y} = \mathrm{d}X_t^{A,B} + \mathrm{d}Y_t\,.$$

Our goal is to minimise the expected discounted capital injections over all admissible strategies (A, B). We define the value associated to the strategy pair (A, B) by $V^{A,B}(x) = \mathbb{E}_x\big[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t^{A,B}\big]$, where $\delta \ge 0$ is the sense of Subsection 4.1, and we are looking for the function

$$V(x) = \inf_{(A,B)} V^{A,B}(x)$$

and for the optimal strategy (A^*, B^*).

Assume for the moment, that the optimal strategy and the value function exist. Then it is clear, that $V(x)$ should be decreasing. Because the constant strategy $(0, \tilde{b})$ is admissible and the corresponding surplus process has the form

$$X_t^{0,\tilde{b}} = x + \lambda\mu\eta t + \sqrt{\lambda\mu_2}W_t$$

we obtain from Example 2.1.1:

$$0 \le \lim_{x\to\infty} V(x) \le \lim_{x\to\infty} V^{0,\tilde{b}}(x) = \lim_{x\to\infty}\frac{1}{\beta}\exp(-\beta x) = 0\,,$$

where

$$\beta = \frac{\lambda\mu\eta + \sqrt{\lambda^2\mu^2\eta^2 + 2\lambda\mu_2\delta}}{\lambda\mu_2}\,.$$

We find in Subsection 2.1.1, that for every constant strategy (a, b) the corresponding return function $V^{a,b}(x)$ solves the differential equation

$$\frac{a^2\sigma^2 + \lambda\mathbb{E}[r(Z,b)^2]}{2}(V^{a,b})''(x) + (\lambda\mu(\eta-\theta) + \lambda\mathbb{E}[r(Z,b)]\theta + ma)(V^{a,b})'(x) - \delta V^{a,b}(x) = 0 \tag{5.2}$$

for $x \geq 0$, fulfils $V'(0) = -1$ and $\lim\limits_{x\to\infty} V^{a,b}(x) = 0$.
It means $V^{a,b}(x) = \frac{1}{\beta(a,b)}\exp(-\beta(a,b)x)$ for some $\beta(a,b) > 0$.

The Hamilton–Jacobi–Bellman equation for $x \geq 0$ is

$$\inf_{\substack{a\in\mathbb{R}\\ b\in[0,\tilde{b}]}} \Big\{\frac{\sigma^2a^2 + \lambda\mathbb{E}[r(Z,b)^2]}{2}V''(x) + (\lambda\mu(\eta-\theta) + \lambda\mathbb{E}[r(Z,b)]\theta + ma)V'(x) - \delta V(x)\Big\} = 0\,.$$

For the explicit derivation of HJB equation we refer to Section 2.1.2. Note, that from above we obtain, that the optimal investment strategy in $x \geq 0$ is given by $a^*(x) = -\frac{mV'(x)}{\sigma^2 V''(x)}$. Using this we can simplify the HJB equation

$$\inf_{b\in[0,\tilde{b}]} \Big\{-\frac{m^2V'(x)^2}{2\sigma^2V''(x)} + \frac{\lambda\mathbb{E}[r(Z,b)^2]}{2}V''(x) + (\lambda\mu(\eta-\theta) + \lambda\mathbb{E}[r(Z,b)]\theta)V'(x) - \delta V(x)\Big\} = 0\,. \tag{5.3}$$

Make now the ansatz $V(x) = \frac{1}{\beta}\exp(-\beta x)$, where β does not depend on x. I.e. we conjecture, that the optimal strategies for both reinsurance and investment are constant. Plugging in the function $\frac{1}{\beta}\exp(-\beta x)$ into the Hamilton–Jacobi–Bellman equation (5.3), we obtain

$$\inf_{b\in[0,\tilde{b}]} -\frac{m^2}{2\sigma^2} + \frac{\lambda\mathbb{E}[r(Z,b)^2]}{2}\beta^2 - (\lambda\mu(\eta-\theta) + \lambda\mathbb{E}[r(Z,b)]\theta)\beta - \delta = 0\,. \tag{5.4}$$

The above expression is a continuous function in b and therefore there is a b^*, where the minimum is attained. Note, that plugging in the return function
$V^{a,b}(x) = \frac{1}{\beta(a,b)}\exp(-\beta(a,b)x)$ corresponding to the constant strategy (a, b) into Equation (5.2) yields

$$\frac{a^2\sigma^2 + \lambda\mathbb{E}[r(Z,b)^2]}{2}\beta(a,b)^2 - (\lambda\mu(\eta-\theta) + \lambda\mathbb{E}[r(Z,b)]\theta + ma)\beta(a,b) - \delta = 0\,.$$

The above equation is just a quadratic equation in $\beta(a,b)$, such that we can easily find $\beta(a,b)$. Define $\alpha(a,b) := \lambda\theta\mathbb{E}[r(Z,b)] - \lambda\mu(\theta-\eta) + ma$, then we obtain for $\beta(a,b)$:

$$\beta(a,b) = \frac{\alpha(a,b) + \sqrt{\alpha(a,b)^2 + 2\delta(\lambda\mathbb{E}[r(Z,b)^2] + a^2\sigma^2)}}{a^2\sigma^2 + \lambda\mathbb{E}[r(Z,b)^2]}\,. \tag{5.5}$$

We can maximise $\beta(a,b)$ with respect to (a,b). Maximising $\beta(a,b)$ with respect to a yields immediatelly $a = \frac{m}{\beta(a,b)\sigma^2}$. Plugging in this result into the above equation yields

$$\frac{\lambda\mathbb{E}[r(Z,b)^2]}{2}\beta(a,b)^2 - (\lambda\mu(\eta-\theta) + \lambda\mathbb{E}[r(Z,b)]\theta)\beta(a,b) - \delta - \frac{m^2}{\sigma^2} = 0\,. \tag{5.6}$$

Because the above equation is convex in β we can conclude, that (5.4) and (5.6) yield the same solution and the same minimiser b^* of (5.4) and the same maximiser b^* of $\beta(a,b)$, respectively.

Next we prove our claim, that the optimal strategy is constant and the value function is an exponential function in the verification theorem.

Theorem 5.1.1 (Verification theorem)
Optimal strategies are given by $B^* \equiv b^*$ and $A^* \equiv a^*$, where (a^*,b^*) is a maximum point of the function $\beta(a,b)$ given in (5.5). The corresponding value function is

$$V(x) = \begin{cases} \frac{1}{\beta(a^*,b^*)} - x & : \ x < 0\,, \\ \frac{1}{\beta(a^*,b^*)}\exp(-\beta(a^*,b^*)x) & : \ x \ge 0\,. \end{cases}$$

Proof: From Subsection 2.1.1, we know, that the return function corresponding to the constant strategy pair (a^*,b^*) $a^* \in \mathbb{R}$, $b^* \in [0,\tilde{b}]$ is given by

$$V^{a^*,b^*}(x) = \begin{cases} \frac{1}{\beta(a^*,b^*)} - x & : \ x < 0\,, \\ \frac{1}{\beta(a^*,b^*)}\exp(\beta(a^*,b^*)x) & : \ x \ge 0\,, \end{cases}$$

where (a^*,b^*) maximises $\beta(a,b)$ given in (5.5). Note, that the function $V^{a^*,b^*}(x)$ is the unique solution to the HJB equation with initial constraints $(V^{a^*,b^*})'(0) = -1$ and $\lim\limits_{x\to\infty} V^{a^*,b^*}(x) = 0$. Consider further an arbitrary strategy (A,B) with $A = \{a_t\}$ and $B = \{b_t\}$ and let $\hat{X}_t = X_t^{A,B,Y}$. The process $\hat{X}_t$ fulfils then

$$\mathrm{d}\hat{X}_t = \Big(\lambda\mu(\eta-\theta) + \lambda\mathbb{E}[r(Z,b_t)]\theta + m\cdot a_t\Big)\,\mathrm{d}t + \sqrt{\lambda\mu_2}\,\mathrm{d}W_t + \sigma a_t\,\mathrm{d}\tilde{W}_t + \,\mathrm{d}Y_t^{A,B}\,.$$

Now we apply the Ito's formula on $V^{a,b}(x)$ and obtain using the HJB equation:

$$\begin{aligned} e^{-\delta t}V^{a^*,b^*}(\hat{X}_t) \ &= \ V^{a^*,b^*}(x) + \int_0^t e^{-\delta s}(V^{a^*,b^*})'(\hat{X}_s)\,\mathrm{d}Y_s^{A,B} \\ &\quad + \int_0^t e^{-\delta s}\Big[D_{s,A,B}V^{a^*,b^*}(\hat{X}_s) - \delta V^{a^*,b^*}(\hat{X}_s)\Big]\,\mathrm{d}s \\ &\quad + \sigma\int_0^t e^{-\delta t}(V^{a^*,b^*})'(\hat{X}_s)a_s\,\mathrm{d}\tilde{W}_s + \sqrt{\lambda\mu_2}\int_0^t e^{-\delta t}(V^{a^*,b^*})'(\hat{X}_s)\,\mathrm{d}W_s \\ &\ge \ V^{a^*,b^*}(x) - \int_0^t e^{-\delta s}\,\mathrm{d}Y_s^{A,B} \\ &\quad + \sigma\int_0^t e^{-\delta t}(V^{a^*,b^*})'(\hat{X}_s)a_s\,\mathrm{d}\tilde{W}_s + \sqrt{\lambda\mu_2}\int_0^t e^{-\delta t}(V^{a^*,b^*})'(\hat{X}_s)\,\mathrm{d}W_s \end{aligned}$$

where $D_{s,A,B}V^{a^*,b^*}(x) = \frac{1}{2}(a_s^2\sigma^2 + \lambda\mu_2)(V^{a^*,b^*})''(x) + (\lambda\mu\eta + ma_s)(V^{a^*,b^*})'(x)$. Note, that for $(A,B) = (a^*,b^*)$ the equality holds. Because the derivative $(V^{a^*,b^*})'(x)$ is bounded, the processes $\sigma\int_0^t e^{-\delta t}(V^{a^*,b^*})'(\hat{X}_s)a_s\,\mathrm{d}\tilde{W}_s$ and $\sqrt{\lambda\mu_2}\int_0^t e^{-\delta t}(V^{a^*,b^*})'(\hat{X}_s)\,\mathrm{d}W_s$ are martingales with zero-expectations. Applying the expectations on the both sides of the above inequality yields

$$\mathbb{E}\big[e^{-\delta t}V^{a^*,b^*}(\hat{X}_t)\big] \ge V^{a^*,b^*}(x) - \mathbb{E}\Big[\int_0^t e^{-\delta s}\,\mathrm{d}Y_s^{A,B}\Big]\ .$$

Because it holds $V^{a^*,b^*}(\hat{X}_t) \le V^{a^*,b^*}(0)$ we obtain by bounded convergence

$$V^{a^*,b^*}(x) \le \mathbb{E}\Big[\int_0^\infty e^{-\delta s}\,\mathrm{d}Y_s^{A,B}\Big]\ .$$

For $X = X^*$ equality holds and thus $V^{a^*,b^*}(x) = V(x)$.

Note, that the uniqueness of the optimal strategy should be shown in every concrete case. □

Next we illustrate the result by two examples.

Example 5.1.2 (Proportional reinsurance)
Consider at first proportional reinsurance, i.e. $r(Z,b) = bZ$. We have to minimise the function

$$\beta(a,b) = \frac{\alpha(a,b) + \sqrt{\alpha(a,b)^2 + 2\delta(\sigma^2a^2 + \lambda\mu_2b^2)}}{\sigma^2a^2 + \lambda\mu_2b^2}\ ,$$

where $\alpha(a,b) = \lambda\theta\mu b + ma - \lambda\mu(\theta-\eta)$.

The optimal investment should be given by $a^* = \frac{m}{\sigma^2\beta}$. Equation (5.6) reads then

$$\frac{\lambda\mu_2b^2}{2}\beta^2 - (\lambda\mu(\eta-\theta) + \lambda\mu b\theta)\beta - \delta - \frac{m^2}{\sigma^2} = 0\ .$$

Minimising with respect to b yields $\beta = \frac{\mu\theta}{\mu_2b^*}$. Plugging in this result into the above equation gives the following:

$$-\frac{\lambda\mu^2\theta^2}{2\mu_2} - \delta - \frac{m^2}{\sigma^2} + \lambda\mu(\theta-\eta)\frac{\mu\theta}{\mu_2b^*} = 0\ .$$

Thus,

$$b^* = \frac{2\theta\sigma^2\lambda\mu^2(\theta-\eta)}{2\sigma^2\delta\mu_2 + \lambda\theta^2\sigma^2\mu^2 + m^2\mu_2} \wedge 1\ .$$

It is straight forward to verify, that the optimal investment strategy is given by

$$a^* = \frac{m\mu_2b^*}{\sigma^2\mu\theta}\ .$$

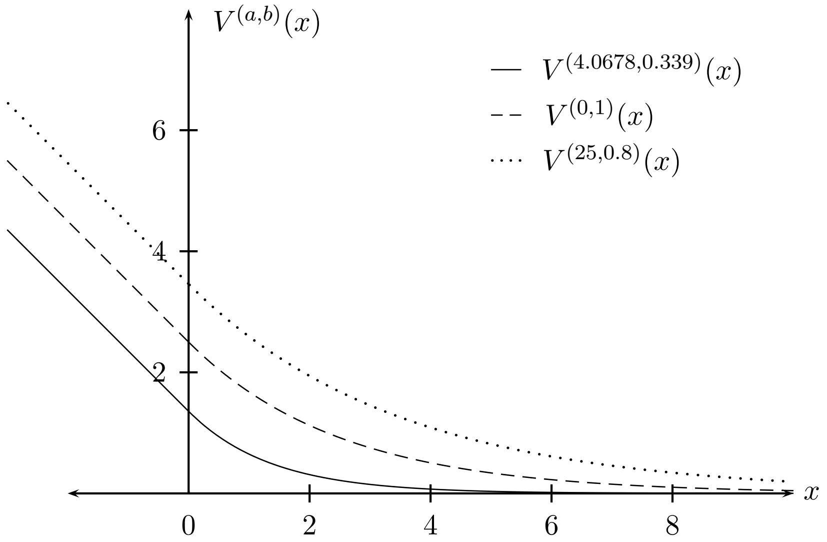

Figure 5.1: Return functions for proportional reinsurance for the optimal strategy $(a^*, b^*) = (4.0678, 0.339)$, $(a, b) = (0, 1)$ and $(a, b) = (25, 0.8)$.

Note, that in the case of proportional reinsurance the function $\beta(a, b)$ has a unique maximum point (a^*, b^*), which means the uniqueness of the optimal strategy.

Let $\lambda = \mu = 1$, $\theta = 0.5$, $\eta = 0.3$, $m = 0.03$, $\sigma^2 = 0.01$ and $\delta = 0.04$. Then we have $\beta = 0.7374999999$ and $(a^*, b^*) = (4.0678, 0.339)$. Figure 5.1 compares the value function, the return function for $(a, b) = (0, 1)$ and return function for $(a, b) = (25, 0.8)$. We can see that investments as well as reinsurance lower the costs considerably. But the choice of a wrong strategy, see the dotted curve in Figure 5.1, can cause higher costs, than in the case without reinsurance and investments. Thus the principle "It is in any case better to reinsure and invest" does not work here. For $\theta \geq \frac{\lambda\eta\mu\sigma + \sqrt{\lambda^2\eta^2\mu^2\sigma^2 + 2\lambda\sigma^2\delta\mu_2 + \lambda m^2\mu_2}}{\lambda\sigma\mu}$ the function $V^{a,1}(x)$ is the value function. The reinsurance is in this case too expensive. ■

Example 5.1.3 (Excess of Loss Reinsurance and $G(x) = 1 - e^{-\frac{x}{\mu}}$.)
For the Excess of Loss reinsurance it holds $r(z, b) = \min\{z, b\}$, where $b \in [0, \infty]$. Then we obtain

$$\mathbb{E}[r(Z, b)] = \int_0^b (1 - G(z))\,\mathrm{d}z \quad \text{and} \quad \mathbb{E}[r(Z, b)^2] = 2\int_0^b z(1 - G(z))\,\mathrm{d}z\,.$$

Equation (5.6) becomes

$$\lambda \int_0^b z(1 - G(z))\,\mathrm{d}z \cdot \beta^2 - (\lambda\mu(\eta - \theta) + \lambda\theta \int_0^b (1 - G(z))\,\mathrm{d}z) \cdot \beta - \delta - \frac{m^2}{\sigma^2} = 0\,.$$

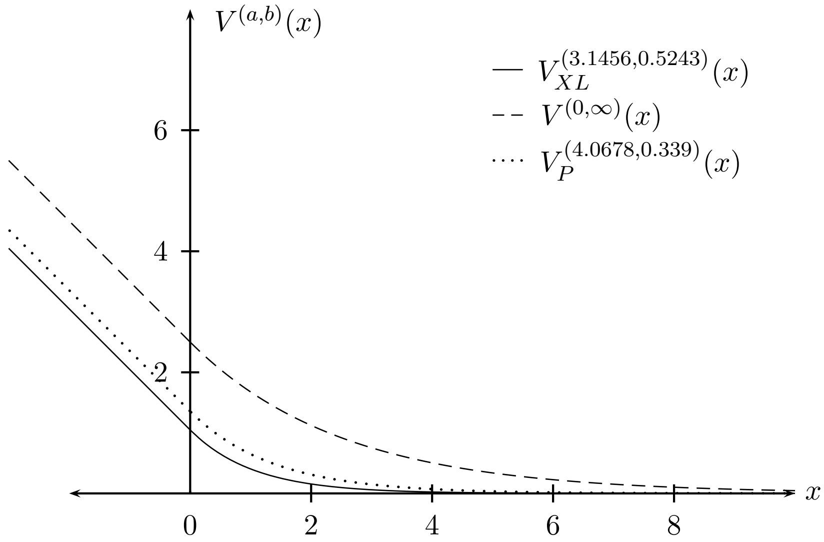

Figure 5.2: Return functions for XL-strategy $(a^*, b^*) = (3.1456, 0.5243)$, proportional strategy $(a, b) = (4.0678, 0.339)$ and $(a, b) = (0, \infty)$.

The derivative with respect to b is equal to zero for $\beta = \frac{\theta}{b}$, provided $1 - G(\theta/\beta) > 0$. Plugging in this result into the above equation gives

$$\lambda \int_0^b z(1 - G(z))\,\mathrm{d}z - (\lambda\mu(\frac{\eta}{\theta} - 1) + \lambda \int_0^b (1 - G(z))\,\mathrm{d}z) \cdot b - (\delta + \frac{m^2}{\sigma^2})\frac{b^2}{\theta^2} = 0\,. \quad (5.7)$$

The left hand side of the above equation is concave in b, is equal to 0 for $b = 0$, tends to $-\infty$ for $b \to \infty$ and the derivative at $b = 0$ is $\lambda\mu(1 - \frac{\eta}{\theta}) > 0$. Thus, there is a unique b^* solving the equation. The optimal investment strategy is then given by $a^* = \frac{mb^*}{\sigma^2\theta}$.

In the case of proportional reinsurance we have found an upper boundary for the safety loading θ of the reinsurer. If θ violates this level, it is optimal not to buy reinsurance, Now we want to analyse, whether we can find such a boundary for the Excess of Loss reinsurance. From the derivations above we get that it is always optimal to buy reinsurance if the support of the claim size distribution is unbounded. Let $\tilde{b} = \sup\{x : \mathbb{P}(Z > x) > 0\}$ and suppose $\tilde{b} < \infty$. No reinsurance is chosen if $b^* \geq \tilde{b}$. This implies by the concavity of the left hand side of (5.7) that

$$\lambda\frac{\mu_2}{2} - \lambda\eta\mu\frac{\tilde{b}}{\theta} - (\delta + \frac{m^2}{\sigma^2})\frac{\tilde{b}^2}{\theta^2} \geq 0\,.$$

This is equivalent to

$$\frac{\tilde{b}}{\theta} \leq \frac{\sqrt{\lambda^2\mu^2\eta^2 + 2\lambda\mu_2(\delta + \frac{m^2}{\sigma^2})} - \lambda\mu\eta}{2(\delta + \frac{m^2}{\sigma^2})}\,,$$

or

$$\frac{\theta}{\tilde{b}} \geq \frac{\lambda\mu\eta + \sqrt{\lambda^2\mu^2\eta^2 + 2\lambda\mu_2(\delta + \frac{m^2}{\sigma^2})}}{\lambda\mu_2} \, .$$

We see that we do not reinsure if reinsurance is too expensive or if the maximal claim size is small.

Assume now, that the claim sizes Z_i are exponentially distributed: $G(x) = 1 - e^{-\frac{x}{\mu}}$. Then we have to maximise the function

$$\beta(a,b) = \frac{\alpha(a,b) + \sqrt{\alpha(a,b)^2 + 2\delta\big(\sigma^2 a^2 + \lambda(2\mu^2 - 2\mu^2 e^{-\frac{b}{\mu}} - 2\mu b e^{-\frac{b}{\mu}})\big)}}{\sigma^2 a^2 + \lambda(2\mu^2 - 2\mu^2 e^{-\frac{b}{\mu}} - 2\mu b e^{-\frac{b}{\mu}})} \, ,$$

where $\alpha(a,b) = \lambda\mu\theta(1 - e^{-\frac{b}{\mu}}) + ma - \lambda\mu(\theta - \eta)$.

Let again $\lambda = \mu = 1$, $\theta = 0.5$, $\eta = 0.3$, $m = 0.03$, $\sigma^2 = 0.01$ and $\delta = 0.04$. With help of mathematical programs one finds easily $(a^*, b^*) = (3.1456, 0.5243)$ and $\beta(3.1456, 0.5243) = 0.9537134001$. Figure 5.2 compares the value function for Excess of Loss (XL) reinsurance, the value function for proportional (P) reinsurance and the return function corresponding to the constant strategy $(0, \infty)$. One can see, that the choice of Excess of Loss reinsurance reduces the costs more, than the choice of proportional reinsurance. ■

5.2 The Classical Risk Model.

Consider again the classical risk model, where the surplus process of the insurer is given by

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i \, .$$

x denotes the initial capital, c - the premium income in a time unit, N_t is a Poisson process with intensity λ and Z_i the i-th claim amount. We let the insurer buy reinsurance and invest money into a risky asset, modeled as a Black-Scholes model, simultaneously. For the investments we use the setup of the Section 4.2.3, i.e. the surplus process under some investment strategy A fulfils

$$\mathrm{d}X_t^A = (c + ma_t)\,\mathrm{d}t - \,\mathrm{d}\sum_{i=1}^{N_t} Z_i + a_t\sigma\,\mathrm{d}W_t \, .$$

As for the reinsurance we use the setup of Section 5.1 with the only difference, that the premium rate remaining for the insurer is $c(b) = c - \lambda(1+\theta)\mathbb{E}[Z - r(Z,b)] =$

$\lambda\mu(\eta-\theta)+\lambda(1+\theta)\mathbb{E}[r(Z,b)]$. We denote the filtration generated by the two dimensional process $(\sum\limits_{i=1}^{N_t} Z_i, W_t)$ by $\{\mathcal{F}_t\}$. Let $B=\{b_t\}_{t\ge 0}$, $b\in[0,\tilde b]$ be a reinsurance strategy and $A=\{a_t\}_{t\ge 0}$, $a\in\mathbb{R}$, an investment strategy. We call a strategy (A,B) admissible if it is cadlag and $\{\mathcal{F}_t\}$ measurable and denote the set of admissible reinsurance strategies by $\mathscr{U}$ and admissible investment strategies by $\mathcal{A}$. The surplus process under reinsurance and investments fulfils

$$\mathrm{d}X_t^{A,B}=(c(b_t)+ma_t)\,\mathrm{d}t-\ \mathrm{d}\sum_{i=1}^{N_t} r(Z_i,b_{T_i-})+a_t\sigma\,\mathrm{d}W_t\,.$$

Our goal is to minimise the expected capital injections over all admissible reinsurance and investment strategies. The surplus process $\{X_t^{A,B}\}$ with corresponding capital injections $\{Y_t^{A,B}\}$ solves then the stochastic differential equation

$$\mathrm{d}X_t^{A,B,Y}=(c(b_s)+a_t m)\,\mathrm{d}t-\ \mathrm{d}\sum_{i=1}^{N_t} r(Z_i,b_{T_i-})+a_t\sigma\,\mathrm{d}W_t+\ \mathrm{d}Y_t^{A,B}\,.$$

We define $V^{A,B}(x)=\mathbb{E}\big[\int_0^\infty e^{-\delta t}\ \mathrm{d}Y_t^{A,B}\big]$ and target to calculate the value function $V(x)=\inf\limits_{\mathscr{U}} V^{A,B}(x)$ and to find the optimal strategy pair (A^*,B^*). Like in the case without reinsurance we state a couple of useful properties of the value function.

Lemma 5.2.1
The function $V(x)$ satisfies

(1) *$V(x)$ is decreasing and fulfils $\lim\limits_{x\to\infty} V(x)=0$;*

(2) *$V(x)$ is Lipschitz continuous with $|V(x)-V(y)|\le|x-y|$.*

$(3)'$ *If we choose the proportional reinsurance, i.e. $r(z,b)=bz$ the corresponding value function is convex, provided the premium calculation principle is chosen such that $c(b)$ is concave in b.*

For the proof see the proof of Lemma 4.2.2. Note, that in general the value function would not be convex.

5.2.1 General case $\delta\ge 0$.

From Lemma 5.2.1 we know, that the value function $V(x)$ is decreasing with $\lim\limits_{x\to\infty} V(x)=0$ and Lipschitz continuous on $[0,\infty)$ with $|V(x)-V(y)|\le|x-y|$. For the same reasons as in the case without reinsurance we will consider here the concept of viscosity solutions. We conjecture, that the value function $V(x)$ solves the Hamilton–Jacobi–Bellman equation

$$\inf_{\substack{a\in\mathbb{R}\\ b\in[0,\tilde{b}]}} \frac{\sigma^2 a^2}{2}V''(x)+\lambda\int_0^\infty V(x-r(z,b))\,\mathrm{d}G(z)+(c(b)+am)V'(x)-(\delta+\lambda)V(x)=0\,. \quad (5.8)$$

For explicit derivation of the HJB equation we refer to Subsections 2.2.1 and 4.2.2, see also Schmidli [70, p. 55]. Note, that for every twice continuously differentiable function f and continuous function u the minimum of

$$\frac{\sigma^2 a^2}{2}f''(x)+\lambda\int_0^\infty u(x-r(z,b))\,\mathrm{d}G(z)+(c(b)+am)f'(x)-(\delta+\lambda)u(x)$$

in a is attained at $a=\frac{-f'(x)m}{\sigma^2 f''(x)}$. Using this fact we define like in Subsection 4.2.2

Definition 5.2.2
We say that a continuous function $\underline{u}:[0,\infty)\to\mathbb{R}_+$ is a **viscosity subsolution** to (5.8) at $x\in(0,\infty)$ if any twice continuously differentiable function $\psi:(0,\infty)\to\mathbb{R}$ with $\psi(x)=\underline{u}(x)$ such that $\underline{u}-\psi$ reaches the maximum at x satisfies

$$-\frac{m^2\psi'(x)^2}{2\sigma^2\psi''(x)}+\inf_{b\in[0,\tilde{b}]}\Big\{\lambda\int_0^\infty \underline{u}(x-r(z,b))\,\mathrm{d}G(z)+c(b)\psi'(x)\Big\}-(\delta+\lambda)\underline{u}(x)\ge 0\,, \quad (5.9)$$

and we say that a continuous function $\bar{u}:[0,\infty)\to\mathbb{R}_+$ is a **viscosity supersolution** to (5.8) at $x\in(0,\infty)$ if any twice continuously differentiable function $\phi:(0,\infty)\to\mathbb{R}$ with $\phi(x)=\bar{u}(x)$ such that $\bar{u}-\phi$ reaches the minimum at x satisfies

$$-\frac{m^2\phi'(x)^2}{2\sigma^2\phi''(x)}+\inf_{b\in[0,\tilde{b}]}\Big\{\lambda\int_0^\infty \bar{u}(x-r(z,b))\,\mathrm{d}G(z)+c(b)\phi'(x)\Big\}-(\delta+\lambda)\bar{u}(x)\le 0\,.$$

A **viscosity solution** to (5.8) is a continuous function $u:[0,\infty)\to\mathbb{R}_+$ if it is both a viscosity subsolution and a viscosity supersolution at any $x\in(0,\infty)$.

And the equivalent formulation

Definition 5.2.3
A continuous function $\underline{u}:[0,\infty)\to\mathbb{R}_+$ is a viscosity subsolution to (5.8) at $x\in(0,\infty)$ if any twice continuously differentiable function $\psi:(0,\infty)\to\mathbb{R}$ with $\psi(x)=\underline{u}(x)$, such that $\underline{u}-\psi$ reaches the maximum at x, satisfies

$$-\frac{m^2\psi'(x)^2}{2\sigma^2\psi''(x)}+\inf_{b\in[0,\tilde{b}]}\Big\{\lambda\int_0^\infty \psi(x-r(z,b))\,\mathrm{d}G(z)+c(b)\psi'(x)\Big\}-(\delta+\lambda)\psi(x)\ge 0\,, \quad (5.10)$$

where $\psi(-x)=\psi(0)-x$ for $x\in(0,\infty)$.

A continuous function $\bar{u} : [0,\infty) \to \mathbb{R}_+$ is a viscosity supersolution to (5.8) at $x \in (0,\infty)$ if any twice continuously differentiable function $\phi : (0,\infty) \to \mathbb{R}$ with $\phi(x) = \bar{u}(x)$, such that $\bar{u} - \phi$ reaches the minimum at x, satisfies

$$-\frac{m^2\phi'(x)^2}{2\sigma^2\phi''(x)} + \inf_{b\in[0,\tilde{b}]} \left\{\lambda \int_0^\infty \phi(x - r(z,b))\, \mathrm{d}G(z) + c(b)\phi'(x)\right\} - (\delta+\lambda)\phi(x) \le 0 \,,$$

where $\phi(-x) = \phi(0) - x$ for $x \in (0,\infty)$.

The proof of equivalence is similar to the proof on page 110 and originates from Benth et al. [8].

Given a twice continuously differentiable function f and a continuous function u, we denote in following

$$\begin{aligned} L_{re}(u,f)(x) &= -\frac{m^2 f'(x)^2}{2\sigma^2 f''(x)} + \inf_{b\in[0,\tilde{b}]} \left\{\lambda \int_0^\infty u(x - r(z,b))\, \mathrm{d}G(z) + c(b)f'(x)\right\} \\ &\quad - (\delta+\lambda)u(x) \,, \\ L_{re}(f)(x) &= -\frac{m^2 f'(x)^2}{2\sigma^2 f''(x)} + \inf_{b\in[0,\tilde{b}]} \left\{\lambda \int_0^\infty f(x - r(z,b))\, \mathrm{d}G(z) + c(b)f'(x)\right\} \\ &\quad - (\delta+\lambda)f(x) \,. \end{aligned}$$

Next we will prove, that the value function is a viscosity solution. We note here again, that the proof originates from Benth et al. [8].

Proposition 5.2.4
V is a viscosity solution to (5.8).

Proof: Here we only prove, that the value function is a supersolution.
For the proof, that the value function is a subsolution see Proposition 4.2.6.

Arguing by contradiction we assume, that $V(x)$ is not a supersolution at x. Then due to Definition 5.2.3 there exist $\xi > 0$ and a twice continuously differentiable function $\phi_0 : (0,\infty) \to \mathbb{R}$ with $\phi_0(x) = V(x)$ and $V(y) - \phi_0(y) \ge 0$ for all $y > 0$, such that

$$L_{re}(\phi_0)(x) > 2\xi \,.$$

Consider now the function $\phi_1(y) := \phi_0(y) - \sin^4\left(\frac{(x-y)\pi}{2x}\right) \cdot \frac{\xi}{\lambda}$. It is obvious, that ϕ_1 is twice continuously differentiable, $\phi_1(x) = V(x)$ and $\phi_1(y) \le V(y) - \sin^4\left(\frac{(x-y)\pi}{2x}\right)\frac{\xi}{\lambda}$. We also have

$$L_{re}(\phi_1)(x) > \xi \,,$$

because $\phi_1'(x) = \phi_0'(x)$, $\phi_1''(x) = \phi_0''(x)$ and $\int_0^\infty \sin^4\left(\frac{y\pi}{2x}\right) \mathrm{d}G(y) \le 1$.

In the proof of Proposition 4.2.6, we have used, that the value function is convex and

according to this $\liminf V'(x) = -1$ was impossible, see Remark 4.2.5. Thus in the case of proportional reinsurance we can just use the proof of Proposition 4.2.6. But if the value function is not convex it may happen for some $x \in \mathbb{R}_+$, that $\liminf V'(x) = -1$. If $\phi_1(x) > -1$ the proof of Proposition 4.2.6 holds.
Assume now $\phi_1(x) = -1$. By construction of ϕ_1 we obtain for $a = -c(0)/m$ and $b = 0$:

$$\frac{a^2\sigma^2}{2}\phi_1''(x) \geq \delta\phi_1(x) + \xi > 0\,.$$

Thus $\phi_1''(x) > 0$, from which it follows $\phi_1'(y) < -1$ for $y < x$ small enough. Since $L_{re}(\phi_1)(y)$ is continuous in y, $c(b)$ and $r(z,b)$ are continuous in b, there are $\tilde{h} \in (0, \frac{x}{2})$ and $j \in \mathbb{N}$, $j > 2$, such that for all $y \in [x - 2\tilde{h}, x + 2\tilde{h}]$ it holds

$$-\frac{m^2\phi_1'(y)^2}{2\sigma^2\phi_1''(y)} + \inf_{b\in[0,\tilde{b}]}\Big\{\frac{c(b)(j-1)}{j}\phi_1'(y) + \lambda\int_0^\infty \phi_1\Big(y - \frac{r(z,b)(j-1)}{j}\Big)\,\mathrm{d}G(z)\Big\} \\ -(\lambda+\delta)\phi_1(y) > \frac{\xi}{2}\,. \quad (5.11)$$

Because ϕ_1 is twice continuously differentiable there is $0 < k < x$ such that $0 \geq \phi_1'(y) \geq -\frac{j}{j-1}$ and $\phi_1(y) > 0$ on $[x-k, x+k]$. We let in following $[x-2\tilde{h}, x+2\tilde{h}] \cap [x-k, x+k] = [x-2h, x+2h]$.

Define like in the proof of Proposition 4.2.6

$$f_n(y) = \frac{1}{\int_{-1}^1 f(s)\,\mathrm{d}s}\int_{-1/n}^{1/n}\Big\{V(y-s) - \sin^4\Big(\frac{h\pi}{2x}\Big)\frac{\xi}{2\lambda}\Big\}nf(ns)\,\mathrm{d}s\,.$$

It is obvious, that f_n are smooth functions and the sequence $(f_n)_{n\geq 1}$ converges uniformly to $V - \sin^4\big(\frac{h\pi}{2x}\big)\frac{\xi}{2\lambda}$. Thus there is $n_0 \in \mathbb{N}$ such that

$$V(y) - \sin^4\Big(\frac{h\pi}{2x}\Big)\frac{\xi}{\lambda} \leq f_{n_0}(y) \leq V(y) - \sin^4\Big(\frac{h\pi}{2x}\Big)\frac{\xi}{4\lambda}$$

for all $y \geq 0$, also we obtain $0 \geq f_{n_0}'(y) \geq -1$ because of the Lipschitz continuity of V and $f_{n_0}(y) \geq 0$ for $y \in [0, x-h]$. In particular it holds $f_{n_0}(y) \geq \phi_1(y) > 0$ on $[x-2h, x-h]$.

Let further g be a twice continuously differentiable function satisfying

- $\frac{\pi}{2} \leq g(y) \leq \pi$
- $g(y) = \pi$ for $y \in [x-h, x+h]$
- $g(y) = \frac{\pi}{2}$ for $y \notin (x-2h, x+2h)$
- $g'(y) \geq 0$ for $y \in [x-2h, x-h]$.

and define $l(y) = \frac{y(j-1)+x}{j}$ for $y \geq 0$. The function $l(y)$ has the following properties

- $l\big([0, x+h]\big) \subset [\frac{x}{j}, x + \frac{h(j-1)}{j}]$ and $l\big([x-h, x+h]\big) = [x - \frac{h(j-1)}{j}, x + \frac{h(j-1)}{j}]$
- l is twice continuously differentiable
- $l(x) = x$
- $l'(y) = \frac{j-1}{j}$.

It means there is $\kappa > 1$ such that $\phi_1(l(y))\kappa \geq f_{n_0}(y)$ on $[x-2h, x-h]$.
We define

$$\varepsilon = \min\Big\{ \sin^4\Big(\frac{h\pi}{2x}\Big)\frac{\xi}{12\lambda}, \frac{\xi}{4\delta}\Big\}$$

and the function ϕ by

$$\phi(y) = \cos^2(g(y))\phi_1\big(l(y)\big) + \sin^2(g(y))\frac{f_{n_0}(y)}{\kappa} \ .$$

Obviously it holds $\phi(x) = \phi_1(x)$ and for each $y \in [x-h, x+h]$ we have $\phi(y) = \phi_1(l(y))$. From (5.11) it follows readily $L_{re}(\phi)(y) > 2\delta\varepsilon$ on $[x-h, x+h]$. Also it holds because l is increasing and ϕ_1 is strictly decreasing on $[x-2h, x]$: $\phi_1(l(y)) \leq \phi_1(y)$ on $[x-2h, x]$. Since

$$\begin{aligned}\phi'(y) \quad &= \quad -2g'(y)\cos(g(y))\sin(g(y))\big\{\phi_1\big(l(y)\big) - \frac{f_{n_0}(y)}{\kappa}\big\} + \cos^2(g(y))l'(y)\phi_1'\big(l(y)\big) \\ &\quad + \sin^2(g(y))\frac{f_{n_0}'(y)}{\kappa}\end{aligned}$$

we obtain $\phi'(y) \geq -1$ for $y \in [0, x+h]$. For $y \in [x-2h, x-h]$ it holds

$$\begin{aligned}\phi(y) \quad &= \quad \cos^2(s(g(y)))\phi_1\big(l(y)\big) + \sin^2(g(y))\frac{f_{n_0}(y)}{\kappa} \\ &\leq \quad \cos^2(g(y))\phi_1(y) + \sin^2(g(y))\frac{f_{n_0}(y)}{\kappa} \\ &\leq \quad V(y) - 3\varepsilon \ .\end{aligned}$$

And for $y \in [0, x-2h]$

$$\phi(y) = \frac{f_{n_0}(y)}{\kappa} \leq V(y) - 3\varepsilon \ .$$

Thus, we obtain

$$\phi(y) \leq V(y) - 3\varepsilon$$

for $y \in [0, x-h] \cup \{x+h\}$.

The remaining part of the proof goes similar to the proof of Proposition 4.2.6. □

We have shown, that $V(x)$ is a viscosity solution to (5.8). Next we show the uniqueness of the value function with the technique one finds in Azcue and Muler [5].

Proposition 5.2.5
Let now $v(x)$ be a super- and $u(x)$ a subsolution to (4.9), *satisfying conditions* (1) *and* (2) *from Lemma 5.2.1. If it holds $u(0) \le v(0)$, then $u(x) \le v(x)$ on $[0,\infty)$.*

Proof: Let u be a sub- and v a supersolution, which fulfil conditions (1) and (2) from Lemma 5.2.1, i.e. u and v are decreasing, Lipschitz continuous and vanish at infinity. Assume there is $x_0 \in (0,\infty)$ with $u(x_0) - v(x_0) > 0$. Define $v_k(x) = kv(x)$ for $k > 1$. Then $v_k(x)$ is also a supersolution. Choose now $k > 1$ such that $u(x_0) - v_k(x_0) > 0$. Clearly $u(0) \le v_k(0)$. Because u and v_k are decreasing and Lipschitz continuous we obtain the following estimation

$$\begin{aligned} u(x) - v_k(x) &\le u(0) - v_k(x) \le v(0) - v_k(x) \\ &= k(v(0) - v(x)) + (1-k)v(0) \\ &\le kx + (1-k)v(0)\ . \end{aligned}$$

Thus it holds $u(x) - v_k(x) \le 0$ for $x \le \frac{v(0)(k-1)}{k}$. So we will consider only the crucial interval (d,∞) with $d = \frac{v(0)(k-1)}{k}$.
Define further

$$M := \sup_{x\ge 0}(u(x) - v_k(x))\ .$$

It follows readily

$$0 < u(x_0) - v_k(x_0) \le M = \sup_{x\in(d,\infty)}(u(x) - v_k(x))\ .$$

Let x^* be such, that $M = u(x^*) - v_k(x^*)$.
Define $H := \{(x,y) : d < y < \infty,\ d < x \le y\}$ and for $\xi > 0$

$$f_\xi(x,y) := u(x) - v_k(y) - \frac{\xi}{2}(x-y)^2 - \frac{2k}{\xi^2(y-x)+\xi}\ .$$

Let $M_\xi := \sup_{(x,y)\in H} f_\xi(x,y)$. Because f_ξ is continuous, there exists (x_ξ, y_ξ) with $M_\xi = f_\xi(x_\xi, y_\xi)$. Because $x^* > d$ it holds $(x^*, x^*) \in H$. Thus, we have

$$M_\xi \ge f_\xi(x^*, x^*) = u(x^*) - v_k(x^*) - \frac{2k}{\xi} = M - \frac{2k}{\xi}\ .$$

We can therefore conclude, that $M_\xi > 0$ for $\xi > \frac{4k}{M}$ and $\liminf_{\xi\to\infty} M_\xi \ge M$.
On the other hand we know, that $(x_\xi, y_\xi) \in \bar{H}$, where $\bar{H}$ denotes the closure of H, from which it follows $y_\xi \ge x_\xi$.

Now we will show, that there exists ξ_0 such that for all $\xi \ge \xi_0$ it holds $(x_\xi, y_\xi) \notin \partial H$. The boundary ∂H is the union

$$\big\{(d,y) : y \in (d,\infty]\big\} \cup \big\{(x,x) : x \in [d,\infty]\big\} \cup \big\{(x,\infty) : x \in (d,\infty)\big\}\ .$$

It is clear, that for fixed $x \in [d, \infty)$ it holds $\lim\limits_{y\to\infty} f_\xi(x,y) = -\infty$. Consider the set $\big\{(x,x) : x \in [d,\infty]\big\}$. Note, that it holds

$$\lim_{x\to\infty} f_\xi(x,x) = \lim_{x\to\infty} u(x) - v_k(x) - \frac{2k}{\xi} = -\frac{2k}{\xi} < 0$$

and $f_\xi(d,d) = u(d) - v_k(d) - \frac{2k}{\xi} < 0$. For $d < x_1 \le x_2 < \infty$ we have because of the properties (1) and (2) in Lemma 5.2.1

$$0 \le \frac{v_k(x_1) - v_k(x_2)}{x_2 - x_1} \le k\ . \tag{5.12}$$

So it holds

$$\begin{aligned}
\frac{f_\xi(x,x) - f_\xi(x,x+h)}{-h} &= \frac{u(x) - v_k(x) - \frac{2k}{\xi} - u(x) + v_k(x+h) + \frac{\xi}{2}h^2 + \frac{2k}{\xi^2 h + \xi}}{-h} \\
&= -\frac{v_k(x) - v_k(x+h)}{-h} - \frac{\xi}{2}h + \frac{2k}{\xi h + 1} \\
&\ge -k - \frac{\xi}{2}h + \frac{2k}{\xi h + 1}\ .
\end{aligned}$$

Thus $\liminf\limits_{h\to 0} \frac{f_\xi(x,x) - f_\xi(x,x-h)}{h} > k$, which means $f_\xi(x,x) \le f_\xi(x,x+h)$ for some small $h > 0$. It remains to consider the set $\big\{(d,y) : y \in (d,\infty]\big\}$. For $y > d$ it holds

$$\begin{aligned}
\frac{f_\xi(d,y) - f_\xi(d,y-h)}{h} &= \frac{v_k(y-h) - v_k(y)}{h} \\
&\quad + \frac{-\frac{\xi}{2}(d+h-y)^2 - \frac{2k}{\xi^2(y-d-h)+\xi} + \frac{\xi}{2}(d-y)^2 + \frac{2k}{\xi^2(y-d)+\xi}}{h} \\
&= \frac{v_k(y-h) - v_k(y)}{h} + \frac{\xi}{2}(2y - h - 2d) \\
&\quad - \frac{2k}{(\xi(y-d-h)+1)(\xi(y-d)+1)} \\
&\ge \frac{\xi}{2}(2y - h - 2d) - 2k\ .
\end{aligned}$$

Thus, we obtain

$$\liminf_{h\to 0} \frac{f_\xi(d,y) - f_\xi(d,y-h)}{h} \ge \xi(y-d) - 2k\ .$$

And on the other hand because $f_\xi(d,d) < 0$ there is $\varepsilon > 0$ such that for $y \in [d, d+\varepsilon]$ it holds $f_\xi(d,y) < 0$. For $\xi \ge \frac{2k}{\varepsilon}$, $y \in [d+\varepsilon, \infty)$ we obtain $f_\xi(d,y) \le \lim\limits_{x\to\infty} f_\xi(d,\infty) < 0$. Letting $\xi_0 := \max\big\{\frac{2k}{\varepsilon}, \frac{4k}{M}\big\}$ we obtain $(x_\xi, y_\xi) \notin \partial H$ for $\xi \ge \xi_0$.

Consider now the functions

$$\begin{aligned}\psi(x) &= v_k(y_\xi) + \frac{\xi}{2}(x-y_\xi)^2 + \frac{2k}{\xi^2(y_\xi - x)+\xi} + C \\ \phi(y) &= u(x_\xi) - \frac{\xi}{2}(x_\xi - y)^2 - \frac{2k}{\xi^2(y-x_\xi)+\xi} - C\ ,\end{aligned}$$

where

$$C = u(x_\xi) - v_k(y_\xi) - \frac{\xi}{2}(x_\xi - y_\xi)^2 - \frac{2k}{\xi^2(y_\xi - x_\xi)+\xi}\ .$$

The functions $\psi(x)$, $\phi(y)$ are twice continuously differentiable. Furthermore, $u(x)-\psi(x)$ attains its maximum at x_ξ; $v_k(y)-\phi(y)$ attains its minimum at y_ξ. Also it holds

$$\psi'(x_\xi) = \phi'(y_\xi) \ \text{ and } \ 0 \le \psi''(x_\xi) = -\phi''(y_\xi)\ . \tag{5.13}$$

Therefore, it holds per Definition 5.2.3 of viscosity sub- and supersolutions

$$-\frac{m^2\psi'(x_\xi)^2}{2\sigma^2\psi''(x_\xi)} + \inf_{b\in[0,\tilde b]}\Big\{\lambda\int_0^\infty u(x_\xi - r(z,b))\,\mathrm{d}G(z) + c(b)\psi'(x_\xi)\Big\} \\ -(\delta+\lambda)u(x_\xi) \ge 0 \tag{5.14}$$

$$-\frac{m^2\phi'(y_\xi)^2}{2\sigma^2\phi''(y_\xi)} + \inf_{b\in[0,\tilde b]}\Big\{\lambda\int_0^\infty v_k(y_\xi - r(z,b))\,\mathrm{d}G(z) + c(b)\phi'(y_\xi)\Big\} \\ -(\delta+\lambda)v_k(y_\xi) \le 0\ . \tag{5.15}$$

Because the functions u, v_k, r and c are continuous, there exist optimal b_1 and b_2, at which the minimum of (5.14) and (5.15) respectively is attained. Plugging in b_1 into Equation (5.15) and using (5.13) we obtain

$$-\frac{m^2\psi'(x_\xi)^2}{\lambda\sigma^2\psi''(x_\xi)} + \int_0^\infty u(x_\xi - r(z,b_1)) - v_k(y_\xi - r(z,b_1))\,\mathrm{d}G(z) \ge \frac{\lambda+\delta}{\lambda}(u(x_\xi)-v_k(y_\xi)) \tag{5.16}$$

It holds (x_ξ,x_ξ), $(y_\xi,y_\xi)\in H$ and

$$f_\xi(x_\xi,x_\xi) + f_\xi(y_\xi,y_\xi) \le 2f_\xi(x_\xi,y_\xi)\ .$$

Thus we obtain

$$\xi(x_\xi - y_\xi)^2 \le u(x_\xi) - u(y_\xi) + v_k(x_\xi) - v_k(y_\xi) + 4k(y_\xi - x_\xi)\ .$$

From above we conclude, that

$$\xi(x_\xi - y_\xi)^2 \le 6k|x_\xi - y_\xi|\ . \tag{5.17}$$

Choose a sequence $\xi_n \to \infty$ as $n \to \infty$ such that $(x_{\xi_n}, y_{\xi_n}) \to (\bar{x}, \bar{y}) \in H$. From (5.17) we obtain $|x_{\xi_n} - y_{\xi_n}| \le \frac{6k}{\xi_n}$, from which it follows $\bar{x} = \bar{y}$; and $\xi_n(x_{\xi_n} - y_{\xi_n})^2 \to 0$. It holds as $\xi_n \to \infty$:

$$\frac{\psi'(x_{\xi_n})^2}{\psi''(x_{\xi_n})} = \frac{\left\{\xi_n(x_{\xi_n} - y_{\xi_n}) \cdot \left(\xi_n(y_{\xi_n} - x_{\xi_n}) + 1\right)^2 + 2k\right\}^2}{\xi_n(\xi_n(y_{\xi_n} - x_{\xi_n}) + 1)^3 + 4k\xi_n} \to 0 \; .$$

Thus (5.16) becomes

$$M \ge \int_0^\infty u(\bar{x} - r(z,b_1)) - v_k(\bar{y} - r(z,b_1)) \,\mathrm{d}G(z) \ge \frac{\lambda + \delta}{\lambda}(u(\bar{x}) - v_k(\bar{y})) \; .$$

Together with $\liminf\limits_{\xi \to \infty} M_\xi \ge M$ we obtain

$$M \le \liminf_{\xi \to \infty} M_\xi \le \lim_{\xi_n \to \infty} M_{\xi_n} = u(\bar{x}) - v_k(\bar{y}) \le \frac{\lambda}{\lambda + \delta} M \; ,$$

which is an obvious contradiction if $\delta > 0$.
Suppose now, that $\delta = 0$. From above it follows, that $\liminf\limits_{\xi \to \infty} M_\xi = M$ and there is some sequence $(\xi_n)_{n \ge 0}$ such that $\xi_n \to \infty$ and $(x_{\xi_n}, y_{\xi_n}) \to (x^*, x^*)$ as $n \to \infty$. Note, that for every $n \in \mathbb{N}$ it holds $f_{\xi_n}(x^*, x^*) < 0$, i.e. there is $\varepsilon > 0$ with $f_{\xi_n}(x,y) \le 0$ for all $(x,y) \in U_\varepsilon(x^*, x^*)$, where

$$U_\varepsilon(x^*, x^*) = \Big\{(x,y) \in H : ||(x - x^*, y - x^*)||_2 < \varepsilon\Big\}$$

denotes the open ball with center in (x^*, x^*), radius ε and $||.||$ denotes the Euclidean norm. But on the other hand for every $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that

$$||(x_{\xi_n} - x^*, y_{\xi_n} - x^*)||_2 < \varepsilon$$

for all $n \ge n_0$. This contradicts the fact $M_\xi > 0$ for all $\xi > \xi_0$. Thus we have shown the uniqueness of the value function also in the case $\delta = 0$. □

A direct consequence of the previous propositions is

Corollary 5.2.6
There is a unique viscosity solution to (5.8) *with initial condition* $V(0) = v(0)$.

5.2.2 The special case $\delta = 0$.

Consider the special case $\delta = 0$. For simplicity we consider only the proportional reinsurance, i.e. the self-insurance function is given by $r(z,b) = zb$ for $b \in [0,1]$. Like in the case without reinsurance we will show the existence and uniqueness of the value function and of the optimal strategy. Finally the method will be illustrated by two examples.

The Hamilton–Jacobi–Bellman equation is

$$\inf_{\substack{a\in\mathbb{R}\\ b\in[0,1]}} \frac{\sigma^2 a^2}{2} V''(x) + \lambda \int_0^\infty V(x-bz)\,\mathrm{d}G(z) + (c(b)+am)V'(x) - \lambda V(x) = 0\,. \tag{5.18}$$

Like in the case without reinsurance it is clear, that we must have $V''(x) > 0$. From Lemma 5.2.1 it follows that the value function is convex in the case of proportional reinsurance, i.e. we have $V''(x) \ge 0$. In the case $V''(x) = 0$ and $V'(x) < 0$ on some interval we would obtain for the strategies $a = \frac{1-c(0)}{m}$, $b = 0$:

$$\frac{\sigma^2 a^2}{2} V''(x) + (c(0)+am)V'(x) < 0\,.$$

In the case $V''(x) = V'(x) = 0$ we would inevitable obtain $V(x) = 0$. Thus, the HJB equation would not have any solution in the case $V''(x) = 0$.

We will see later, that in order to show the existence of the value function we will need the optimal strategy $(a^*(0), b^*(0))$ at $x = 0$. The verification theorem below will help to specify $(a^*(0), b^*(0))$.

Theorem 5.2.7 (Verification theorem)
Let $f(x)$ be a decreasing, twice continuously differentiable solution to (5.18) *with* $\lim\limits_{x\to\infty} f(\infty) = 0$. *Then $f(x) = V(x)$ and the optimal strategy is the strategy of the feedback form $(A^*(X_t), B^*(X_t))$.*

Since the proof is perfectly similar to the proof of verification theorem in the case without reinsurance, see Proposition 4.2.9, we will skip it.

It is clear, that for the minimising strategy A^* it holds $a^*(x) = -\frac{mV'(x)}{\sigma^2 V''(x)}$. Together with $V''(x) > 0$ and $V'(x) \le 0$ we get $a_t^* \ge 0$ for the optimal investment strategy $A^* = \{a_t^*\}$. The knowledge of the minimiser a^* gives us the possibility to simplify the HJB equation:

$$-\frac{m^2 V'(x)^2}{2\sigma^2 V''(x)} + \inf_{b\in[0,1]} \Big\{\lambda \int_0^\infty V(x-bz)\,\mathrm{d}G(z) + c(b)V'(x)\Big\} - \lambda V(x) = 0\,. \tag{5.19}$$

Using integration by parts (5.19) becomes

$$-\frac{m^2 V'(x)^2}{2\sigma^2 V''(x)} + \inf_{b\in[0,1]} \Big\{ -\lambda b \int_0^\infty V'(x-bz)(1-G(z))\,\mathrm{d}z + c(b)V'(x)\Big\} = 0\,.$$

For the initial capital $x = 0$ the Equation (5.19) becomes

$$\inf_{b\in[0,1]} -\frac{m^2V'(0)^2}{2\sigma^2V''(0)} + \lambda\mathbb{E}[r(Z,b)]\big\{1+(1+\theta)V'(0)\big\} + \lambda\mu(\eta-\theta)V'(0) = 0\;.$$

Thus, if it holds $V'(0) < -\frac{1}{1+\theta}$ the optimal reinsurance strategy in 0 will be $b^*(0) = 1$.

In the case $V'(0) = -\frac{1}{1+\theta}$ we have

$$-\frac{m^2V'(0)^2}{2\sigma^2V''(0)} + c(0)V'(0) = 0\;,$$

from which it follows, that the optimal investment strategy in 0 should be given by $a^*(0) = \frac{2\lambda\mu(\theta-\eta)}{m}$ and the reinsurance strategy $b^*(0)$ is arbitrary.

If we assume $0 > V'(0) > -\frac{1}{1+\theta}$, then the optimal strategy in 0 would be $b^*(0) = 0$, from which we would again obtain $a^*(0) = \frac{2\lambda\mu(\theta-\eta)}{m}$.

$V'(0) = 0$ would imply $b = 0$ and $a = 0$ as optimal strategies in 0, from which it would follow because V is decreasing: $V(x) \equiv 0$. But for initial capital 0 and $(a^*(0), b^*(0)) = (0,0)$ the ruin would occur immediately because of the premium payments and accordingly $Y_t > 0$. I.e. the case $(a^*(0), b^*(0)) = (0,0)$ is impossible.

Thus, we have three candidates for the optimal strategy in 0:

$$(a^*, b^*) = \begin{cases} (\frac{2\lambda\mu(\theta-\eta)}{m}, 0) & : \; V'(0) \in (-\frac{1}{1+\theta}, 0)\;, \\ (a^*, 1) & : \; V'(0) \in (-1, -\frac{1}{1+\theta})\;, \\ (\frac{2\lambda\mu(\theta-\eta)}{m}, b) & : \; V'(0) = -\frac{1}{1+\theta}\;, \end{cases}$$

where $b \in [0,1]$. The following remark deduces the behavior of the optimal reinsurance strategy B^* for small initial values $x \geq 0$.

Remark 5.2.8
Consider the function

$$g_x(b) := c(b)V'(x) - \lambda V(x) + \lambda\int_0^\infty V\big(x - r(z,b)\big)\,\mathrm{d}G(z)\;.$$

Let further $b_1, b_2 \in [0,1]$ and $b_1 > b_2$. Then we obtain with the Lipschitz continuity of $V(x)$ for every $x \in [0,\infty)$:

$$\begin{aligned} g_x(b_1) - g_x(b_2) &= \lambda\int_0^\infty V\big(x - r(z,b_1)\big) - V\big(x - r(z,b_2)\big)\,\mathrm{d}G(z) \\ &\quad + (c(b_1) - c(b_2))V'(x) \\ &= \lambda\int_0^\infty V\big(x - r(z,b_1)\big) - V\big(x - r(z,b_2)\big) \\ &\quad + \Big\{r(z,b_1) - r(z,b_2)\Big\}(1+\theta)V'(x)\,\mathrm{d}G(z) \\ &\leq \lambda\int_0^\infty \Big\{r(z,b_1) - r(z,b_2)\Big\}\cdot[1+(1+\theta)V'(x)]\,\mathrm{d}G(z)\;. \end{aligned}$$

$V'(x) \leq -\frac{1}{1+\theta}$ implies, that $g_x(b)$ is decreasing, so that the minimum is taken in $b = 1$.

Lemma 5.2.9
Assume the value function $V(x)$ is the unique, twice continuously differentiable, vanishing at infinity solution to the HJB equation (5.18); and the net profit condition $c > \lambda\mu$ is fulfilled, then the optimal investment strategy at $x = 0$ is $a^* = 0$.

Proof: Arguing by contradiction we assume $a^* > 0$ and denote by $V(x)$ the value function.

Denote the function, which results from choosing the start strategy $(a(0), b(0)) = (0, 1)$ and solving the HJB equation (5.18) by $f(x)$. It is clear, that $f''(0) = \infty$ and $f'(0) = -\frac{\lambda\mu}{c}$. With the same arguments as above we obtain $f''(x) > 0$ for all $x \in \mathbb{R}_+$. Let further $V^{0,1}(x)$ define the return function of the constant strategy $(A, B) \equiv (0, 1)$. From Example 2.2.1 we know, that if the claim size distribution $G(x)$ is absolutely continuous and the density function is bounded, $V^{0,1}(x)$ is twice continuously differentiable with $(V^{0,1})'(0) = -\frac{\lambda\mu}{c} = f'(0)$ and solves the integro-differential equation

$$c(V^{0,1})'(x) + \lambda \int_0^\infty V^{0,1}(x - z)\, \mathrm{d}G(z) - \lambda V^{0,1}(x) = 0\,.$$

Thus, we obtain for $h(x) := f(x) - V^{0,1}(x)$

$$ch'(x) \geq -\lambda \int_0^\infty h(x - z)\, \mathrm{d}G(z) + \lambda h(x)\,.$$

Note, that it holds $h'(0) = 0$ and $h''(0) = \infty$, from which it follows $h'(x) > 0$ for all $x > 0$. Thus, we obtain $\lim\limits_{x\to\infty} f(x) - V^{0,1}(x) \geq f(0) - V^{0,1}(0)$. Because $f(0) - V^{0,1}(0)$ is finite and $\lim\limits_{x\to\infty} V^{0,1}(x) = 0$ we conclude, that $\lim\limits_{x\to\infty} f(x) = -\infty$ is impossible.
Define further

$$g(x) := V(x) - f(x)\,.$$

Because $a^* > 0$ it must hold either

$$V'(0) \geq -\frac{1}{1+\theta} > -\frac{1}{1+\eta} = -\frac{\lambda\mu}{c} = f'(0)$$

or

$$V'(0) = -\frac{\lambda\mu}{c + am/2} > -\frac{\lambda\mu}{c} = f'(0)$$

as well as $0 < V''(0) < \infty$. Thus $g(x)$ is twice continuously differentiable, $g(0) < 0$, $g'(0) > 0$ and $g''(0) = -\infty$ by definition. Let further $(a(x), b(x))$ denote the optimal strategy for $f(x)$, then we obtain from the HJB equation

$$\frac{a^2\sigma^2}{2} g''(x) + (c(b) + ma)g'(x) - \lambda \int_0^x g'(x - bz)(1 - G(z))\, \mathrm{d}z \geq 0\,.$$

Note, that because $b = 1$ and

$$-\frac{ma}{2}f'(x) = cf'(x) - \lambda\int_0^\infty f'(x-z)(1-G(z))\,\mathrm{d}z$$

close to zero, it follows from $f''(0) = \infty$, that $a(x)$ is strictly increasing near zero, i.e. $a(x) > 0$ on $(0,\varepsilon)$ for some $\varepsilon > 0$. Let $\breve{x} = \inf\{x : g''(x) > 0\} \wedge \{x > \varepsilon : a(x) = 0\}$, then

$$\begin{aligned} 0 > \frac{a(x)^2\sigma^2}{2}g''(x) &\geq \lambda\int_0^x g'(x-b(x)z)(1-G(z))\,\mathrm{d}z - (c(b(x)) + ma(x))g'(x) \\ &\geq (\lambda\mu - c(b(x)) - ma(x))g'(x)\;. \end{aligned}$$

on $(0,\breve{x})$. Because $g'(x)$ is continuous, it holds $g'(x) > 0$ and $(\lambda\mu - c(b(x)) - ma(x)) < 0$ on $[0,\breve{x})$. And we can conclude $g'(x) > 0$ for $x \geq 0$. Thus, the function $g(x)$ is increasing, $f(x)$ is decreasing. In particular we have either $\lim\limits_{x\to\infty} f(x) = -\infty$ or $\lim\limits_{x\to\infty} f(x) = d$, with $|d| < \infty$. We have already shown above, that the first case is impossible. In the second case consider the function $f_1(x) = f(x) - d$. $f_1(x)$ solves the HJB equation for the strategy $(a(x), b(x))$, satisfies the conditions from Lemma 5.2.1 and is convex, which contradicts the result of Theorem 5.2.7. □

Thus the optimal strategy in 0 is given by $(0,1)$. Because we are looking for $V(x)$, which is twice continuously differentiable on $(0,\infty)$, it holds $V'(x) < -\frac{1}{1+\theta}$ for x small enough. Due to Lemma 5.2.8 the optimal reinsurance strategy for those x is then also given by 1.

It remains to show, that the strong solution exists. Also in the case with reinsurance the strong solution can exist only under some special constrains, which are specified in the next proposition.

Proposition 5.2.10
There is a unique decreasing, twice continuously differentiable solution to (5.18)*, if the claims distribution function* $G(x)$ *has a bounded density and* $\lim\limits_{b\to 1}\frac{c-c(b)}{1-b} > 0$.

Proof: Note at first, that the optimal reinsurance strategy for x small enough is given by $b^*(x) = 1$, see Remark 5.2.8. Thus, we can conclude from Lemma 4.2.11, that there is a twice continuously differentiable solution to 5.19 on some interval $[0,\xi)$ with $\xi > 0$. Next we show, that the solution can be extended to $[0,\xi]$.
Let $h(x) = V'(x)$ and

$$U(x,b) = -\lambda\int_0^x h(z)\big(1 - G((x-z)/b)\big)\,\mathrm{d}z + c(b)h(x) + \lambda\int_x^\infty 1 - G(y/b)\,\mathrm{d}y\;.$$

Then we obtain from HJB equation (5.18)

$$\frac{m^2h(x)^2}{2\sigma^2h'(x)} = \inf_{b\in[0,1]} U(x,b)\;,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{1}{h(x)}\Big) = -\frac{m^2}{2\sigma^2}\big\{\inf_{b\in[0,1]} U(x,b)\big\}^{-1}\,.$$

Integrating the both sides of the above equation and taking the reciprocal yields

$$h(x) = -\frac{1}{\frac{c}{\lambda\mu} + \frac{m^2}{2\sigma^2}\int_0^x \big\{\inf_{b\in[0,1]} U(y,b)\big\}^{-1}\,\mathrm{d}y}\,. \tag{5.20}$$

It is clear, that $h(\xi)$ is in either case well defined. Next we show, that the solution can be extended to $[0,\xi]$ and $\inf\limits_{b\in[0,1]} U(\xi,b) > 0$. Suppose now $\inf\limits_{b\in[0,1]} U(\xi,b) = 0$. Let x_n be a sequence such that $x_n \uparrow \xi$ as $n\to\infty$ and denote by b_n the optimal strategy at x_n, i.e. $b^*(x_n) = b_n$. We can find a converging subsequence $b_{n_k} \to \hat b$ as $k\to\infty$. By continuity it holds $U(\xi,\hat b) = \inf\limits_{b\in[0,1]} U(\xi,b) = 0$.

Assume $\hat b > 0$. Then it must hold $c(\hat b) > 0$ and $h(\xi) < 0$, because $-\lambda\int_0^x h(z)\big(1 - G((x-z)/b)\big)\,\mathrm{d}z + \lambda\int_x^\infty 1 - G(y/b)\,\mathrm{d}y > 0$. Since $U(\xi,\hat b) = 0$ and (5.20) holds we obtain $h'(\xi) = \infty$. On the other hand it holds for all $x\in[0,\xi)$:

$$\begin{aligned} 0 &> \frac{U(\xi,\hat b) - U(x,\hat b)}{\xi - x} \\ &\ge \lambda\int_0^\xi h(z)\frac{G((\xi-z)/\hat b) - G((x-z)/\hat b)}{\xi - x}\,\mathrm{d}z + c(\hat b)\frac{h(\xi)-h(x)}{\xi - x} \\ &\quad + \lambda\int_x^\xi \frac{1 - G(z/\hat b)}{\xi - x}\,. \end{aligned}$$

Because $G(x)$ has a bounded density and $h(x)$ is bounded, the above expression goes to ∞ as $x\to\xi$. Thus, we obtain $0 > \infty$, which is a contradiction.
Thus we can conclude $\lim\limits_{x\uparrow\xi} b^*(x) = 0$ and $h(\xi) = 0$. Let further $b_0 = \inf\{b : c(b) \ge 0\}$. Because $\lim\limits_{x\uparrow\xi} b^*(x) = 0$ there is $\tilde x$ such that $b^*(\tilde x) < b_0/2$ for all $x > \tilde x$. Then we obtain for $x > \tilde x$: $c(b^*) < 0$ and

$$\begin{aligned} \frac{1}{h(x)} &= -\frac{c}{\lambda\mu} - \int_0^x \frac{m^2}{2\sigma^2}\big\{\inf_{b\in[0,1]} U(x,b)\big\}^{-1} \\ &= -\frac{c}{\lambda\mu} - \int_0^{\tilde x} \frac{m^2}{2\sigma^2}\big\{\inf_{b\in[0,1]} U(y,b)\big\}^{-1}\,\mathrm{d}y - \int_{\tilde x}^x \frac{m^2}{2\sigma^2}\big\{\inf_{b\in[0,1]} U(y,b)\big\}^{-1}\,\mathrm{d}y \\ &> -\frac{c}{\lambda\mu} - \int_0^{\tilde x} \frac{m^2}{2\sigma^2}\big\{\inf_{b\in[0,1]} U(y,b)\big\}^{-1}\,\mathrm{d}y - \int_{\tilde x}^x \frac{m^2}{2\sigma^2 c(b_0/2)h(y)}\,\mathrm{d}y\,. \end{aligned}$$

Let

$$\beta = -\frac{m^2}{2\sigma^2 c(b_0/2)} \ge 0$$

and

$$C = \frac{c}{\lambda\mu} + \int_0^{\tilde{x}} \frac{m^2}{2\sigma^2}\Big\{\inf_{b\in[0,1]} U(y,b)\Big\}^{-1}\,\mathrm{d}y$$

Then we obtain from Gronwall's inequality, see Proposition E.1.6 page 181:

$$-\frac{1}{h(x)} < Ce^{\beta(x-\tilde{x})}\ .$$

This contradicts $h(\xi) = 0$. The remaining part of the proof goes similar to the proof of Theorem 4.2.12. □

5.2.3 Examples

Example 5.2.11 (Proportional Reinsurance and $G(x) = 1 - e^{-\frac{x}{\mu}}$.)
Consider the proportional reinsurance, i.e. $r(z,b) = zb$ and $G = 1 - e^{-\frac{x}{\mu}}$. Like in the case without reinsurance, compare Example 4.2.17, the HJB equation differs from the HJB equation for the ruin probability as a function of initial capital only in the term $\lambda\mu b e^{-\frac{x}{b\mu}}$ Letting $V'(x) = h(x)$ we obtain

$$-\frac{m^2 h(x)^2}{2\sigma^2 h'(x)} + \inf_{b\in[0,1]}\Big\{ -\lambda e^{-\frac{x}{b\mu}} \int_0^x h(y)e^{\frac{y}{b\mu}}\,\mathrm{d}y + c(b)h(x) + \lambda\mu b e^{-\frac{x}{b\mu}}\Big\} = 0\ ,$$

For the parameters $\sigma^2 = 0.01$, $m = 0.03$, $\delta = 0.04$, $\mu = \lambda = 1$, $\eta = 0.3$ and $\theta = 0.5$ we can solve the problem numerically. The value function is plotted in Figure 5.3. The optimal strategies A and B are shown in Figures 5.5 and 5.6 respectively. Also in this case we can specify the value $V(0)$ exactly: $V(0) = 1.5718$. ■

Example 5.2.12 (Proportional reinsurance and $G(x) = 1 - \frac{\mu^2}{(\mu+x)^2}$.)
Consider now the case of Pareto$(\mu,2)$-distributed claims. For the parameters from the Example with exponentially distributed claim sizes the value function is plotted in Figure 5.4. The optimal strategies A and B are shown in Figures 5.7 and 5.8 respectively. We see, that the optimal investment strategy converges approximately to 11.2 and the optimal reinsurance strategy converges to zero. The initial value is given by $V(0) = 2.5$. ■

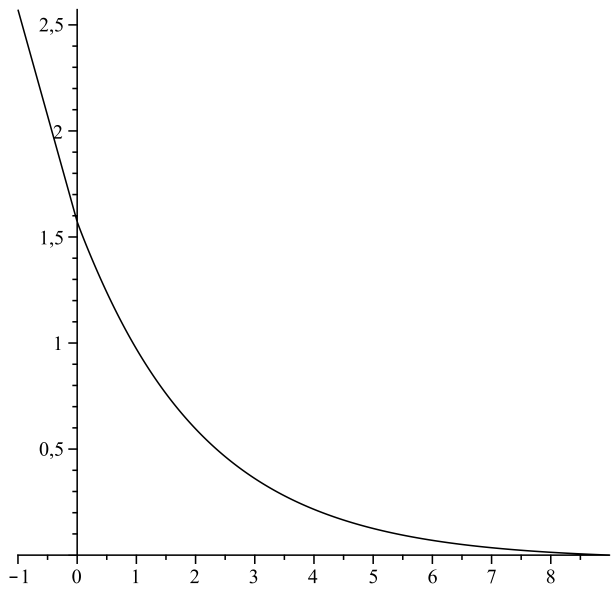

Figure 5.3: Value function $V(x)$ for $G(x) = 1 - e^{-\frac{x}{\mu}}$.

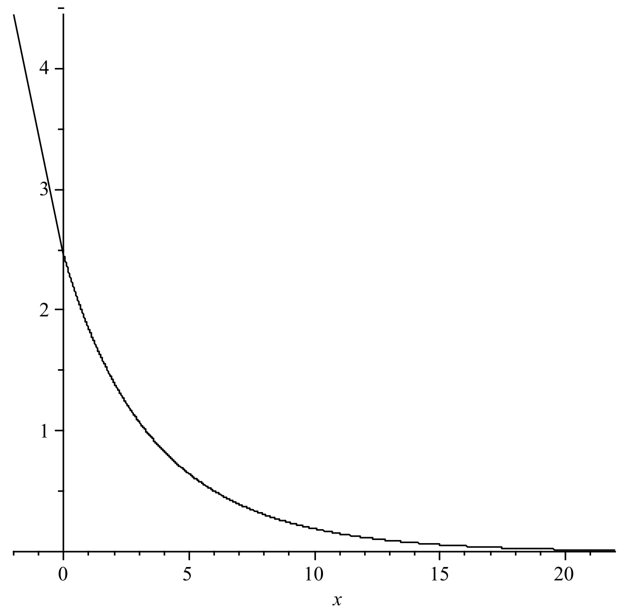

Figure 5.4: Value function $V(x)$ for $G(x) = 1 - \frac{\mu^2}{(\mu+x)^2}$.

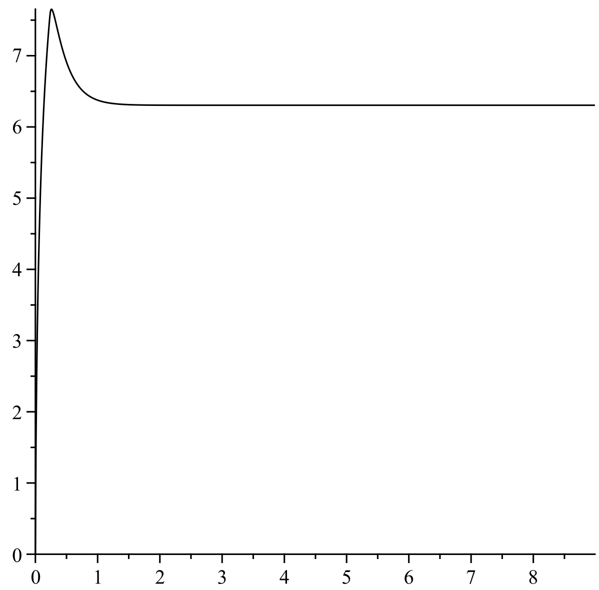

Figure 5.5: Optimal investment strategy A for $G(x) = 1 - e^{-\frac{x}{\mu}}$.

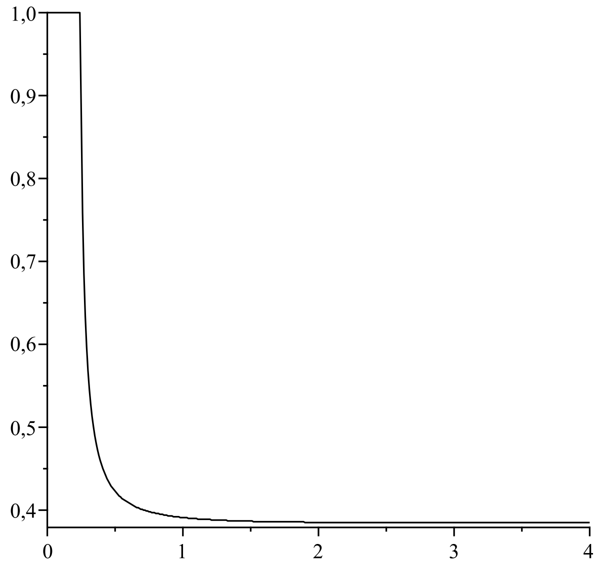

Figure 5.6: Optimal reinsurance strategy B for $G(x) = 1 - e^{-\frac{x}{\mu}}$.

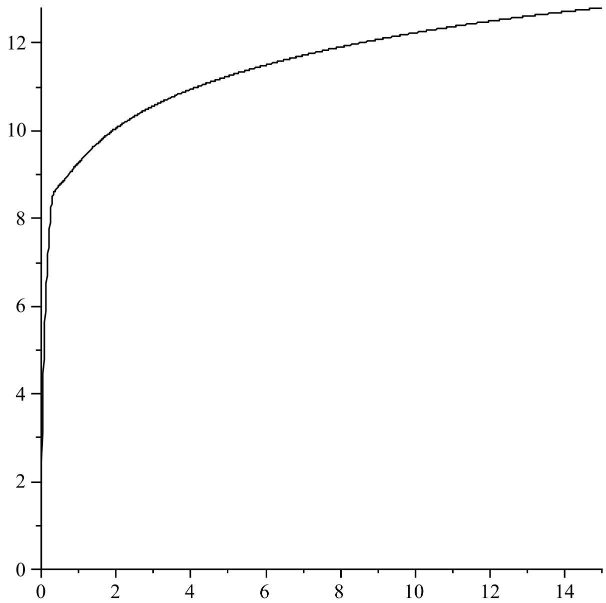

Figure 5.7: Optimal investment strategy A for $G(x) = 1 - \frac{\mu^2}{(\mu+x)^2}$.

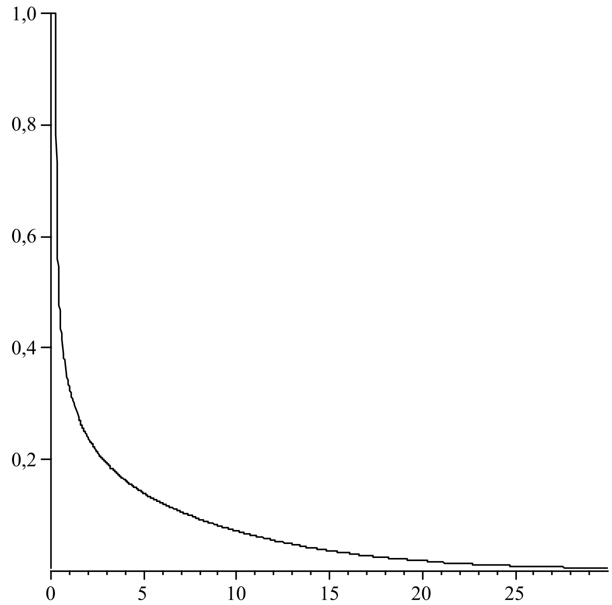

Figure 5.8: Optimal reinsurance strategy B for $G(x) = 1 - \frac{\mu^2}{(\mu+x)^2}$.

Appendix

A Stochastic Processes and Martingales

Martingales have been an important tool in the proofs of existence of the value function. In this appendix we consider some well known results, which state the processes to be martingales under some special constrains. For detailed introduction to martingales we refer for example to Protter [60], to Bhattacharya and Waymire [9] or to Rogers and Williams [63, 64].

A.1 Stochastic processes

We assume as given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $I = \mathbb{N}$ or $I = \mathbb{R}$, then the stochastic process on I with state space $\mathbb{R}$, is a family of $\mathbb{R}$-valued random variables $X = \{X_t : t \in I\}$, or in the short notation $X = \{X_t\}$. X is said to be **cadlag**, if its sample paths are right continuous with left limits. We define the **variation** of X over $(0, t]$ by

$$F_X(t) = \sup \left\{ \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}| \right\} ,$$

where the supremum is taken over all $n \geq 1$ and all partitions $0 = t_0 < t_1 < \cdots < t_n = t$. The process X is said to be of **bounded variation** if $F_X(t) < \infty$ for all $t > 0$.

Definition A.1.1
A **filtration** $\mathbb{F} = \{\mathcal{F}_t\}$ is a non-decreasing family of sub-σ-algebras. For any process $Y = \{Y_t\}$, the natural filtration $\mathbb{F}^Y$ is given by $\mathcal{F}_t^Y = \sigma\{Y_s; s \leq t\}$. Thus $\mathcal{F}_t^Y$ is the σ-algebra generated by Y up t and represents the history of Y up to time t.

We say Y is **adapted** to $\mathbb{F}$ if Y_t is $\mathcal{F}_t$ measurable for all $t \geq 0$.

In the following we always assume, that a filtration $\mathbb{F} = \{\mathcal{F}_t\}$ is right continuous, i.e. $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $0 \leq t < \infty$.

A σ-algebra $\mathcal{P}$ on $\Omega \times [0, \infty)$ is called **predictable** if it is generated by all processes $\{X_{t-}\}$, where $\{X_t\}$ is $\{\mathcal{F}_t\}$ is adapted. A process X is called predictable if it is $\mathcal{P}$ measurable.

Definition A.1.2
A random variable $T : \Omega \to [0, \infty]$ is a **stopping time**, if the event $\{T \leq t\} \in \mathcal{F}_t$ for every $0 \leq t \leq \infty$.

In the classical risk model and its diffusion approximation the concepts of Poisson process and Brownian motion are of central importance.

Definition A.1.3 (Poisson Process)
An $\{\mathcal{F}_t\}$- adapted process $N = \{N_t\}$ defined by

$$N_t = \sum_{n \geq 1} \mathbb{1}_{[t \geq T_n]} ,$$

*where T_n are stopping times, is a **Poisson process** if*

1. *for any s, t with $0 \leq s < t < \infty$, $N_t - N_s$ is independent of $\mathcal{F}_s$;*
2. *for any s, t, u, v with $0 \leq s < t < \infty$, $0 \leq u < v < \infty$ and $t - s = v - u$ the distribution of $N_t - N_s$ is the same as that of $N_v - N_u$.*
3. *It holds*
$$\mathbb{P}[N_t = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!} .$$

*$\{T_n\}$ are called the **arrival times**.*

Definition A.1.4 (Brownian Motion)
*An $\{\mathcal{F}_t\}$-adapted continuous process $W = \{W_t\}$ taking values in $\mathbb{R}$ is called **standard Brownian motion** if*

1. *for $0 \leq s < t < \infty$, $W_t - W_s$ is independent of $\mathcal{F}_s$;*
2. *for $0 < s < t$, $W_t - W_s$ is a Gaussian random variable with mean zero and variance $(t - s)$.*

A standard Brownian motion starts at 0, $\mathbb{P}[W_0 = 0] = 1$.
A process $\{mt + \sigma W_t\}$ is called an (m, σ^2)-Brownian motion.

Proposition A.1.5 (Strong law of large numbers)
Let $W = \{W_t\}$ be a standard Brownian motion. Then it holds

$$\lim_{t \to \infty} \frac{W_t}{t} = 0 .$$

Proof: See Rolski et al. [61, p. 431]. □

Among other processes the running maximum of the standard Brownian motion played an important role in our considerations.

Remark A.1.6 (Brownian Motion and Its Running Maximum)
Let W be a standard Brownian motion and define the running maximum

$$M_t = \max_{0 \le s \le t} W_t \, . \tag{A.1}$$

The density function of the pair (W_t, M_t) is given by

$$f(a,b) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\Big\{ - \frac{(2b-a)^2}{2t} \Big\} \mathbb{1}_{[a \le b]} \mathbb{1}_{[b \ge 0]} \, .$$

Let further X be an (m, σ^2)-Brownian motion with $m > 0$. Define $\tau^x = \inf\{t \ge 0 : X_t = -x\}$, $x > 0$. Then τ^x is a stopping time and

$$\mathbb{P}[\tau^x \le t] = \Phi\Big(- \frac{mt + x}{\sigma\sqrt{t}} \Big) + e^{-2mx/\sigma^2} \Phi\Big(\frac{mt - x}{\sigma\sqrt{t}} \Big) \, ,$$

where Φ denotes the distribution function of the standard normal distribution.
For proofs see Karatzas and Shreve [49, p. 95].

A.2 Martingales

In this section we give the concepts of martingales and local martingales and the propositions, we have used in the work. In some of the propositions below stochastic integrals play a central role. The concept of stochastic integral goes beyond the scope of this appendix. We abandon any definitions and explanations and just refer for example to Roger and Williams [64].

Definition A.2.1
A real-valued, adapted process $X = \{X_t\}$ is called a **martingale** with respect to the filtration $\mathbb{F} = \{\mathcal{F}_t\}$ if

1. $\mathbb{E}[|X_t|] < \infty$ for all $t \ge 0$;
2. $\mathbb{E}[X_t | \mathcal{F}_s] = X_s$ for all $s \le t$.

The next two propositions state very significant for this work properties of the Brownian motion and Poisson process.

Proposition A.2.2
Let $N = \{N_t\}$ be a Poisson process with intensity λ and arrival times $\{T_n\}$. Then $M_t = N_t - \lambda t$, the compensated Poisson process, is a martingale. For an $\{\mathcal{F}_t\}$ predictable process $H = \{H_t\}$ with $\mathbb{E}[\int_0^t |H_s| \; \mathrm{d}s] < \infty$, the process $I_t = \int_0^t H_s \; \mathrm{d}M_s$ is an $\{\mathcal{F}_t\}$ martingale.

Proposition A.2.3
Let $W = \{W_t\}$ be a standard Brownian motion and $l > 0$. Then the following processes are martingales

$$\begin{aligned}&\{W_t\} \ ,\\ &\{W_t^2 - t\} \ ,\\ &\{e^{lW_t - l^2 t/2}\} \ .\end{aligned}$$

Definition A.2.4
An adapted process X is a **local martingale**, if there exists a sequence of increasing stopping times, T_n with $\lim_{n\to\infty} T_n = \infty$ a.s., such that $X_{t\wedge T_n} \mathbb{1}_{[T_n>0]}$ is a martingale for each n.

Proposition A.2.5
A continuous local martingale of bounded variation is constant.
A predictable local martingale of bounded variation is constant.

For proof see Rolski et al. [61, p. 566].

Proposition A.2.6
Let $X = \{X_t\}$ be an arbitrary stochastic process and let $W = \{W_t\}$ be a standard Brownian motion.

1. *If $\int_0^t X_s^2 \, \mathrm{d}s < \infty$, then the stochastic integral $\int_0^t X_s \, \mathrm{d}W_s$ is a local martingale.*
2. *If $\int_0^t \mathbb{E}[X_s^2] \, \mathrm{d}s < \infty$, then the stochastic integral $\int_0^t X_s \, \mathrm{d}W_s$ is a true martingale with zero expectation.*

In particular, if X is a bounded process, then the stochastic integral is a martingale.

Definition A.2.7
*A **semimartingale** is an adapted real-valued process $X = \{X_t\}$ of the form*

$$X_t = M_t + A_t \ ,$$

where $M = \{M_t\}$ is a local martingale and $A = \{A_t\}$ is a process of bounded variation.

B Markov Processes and Infinitesimal Generators

In this work we have often dealt with Markov processes, and used their properties in several proofs. Thus, we start with the definition of a Markov process.

B.1 Markov processes

Definition B.1.1
Let ν be a probability measure on the space $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, where $\mathscr{B}(\mathbb{R})$ is the Borel σ-Algebra on $\mathbb{R}$. An $\mathbb{F}$ adapted real-valued one-dimensional process $X = (X_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Markov process with initial distribution ν if

1. $\mathbb{P}[X_0 \in B] = \nu[B]$ *for all* $B \in \mathscr{B}(\mathbb{R})$,
2. *for* $s, t \geq 0$ *and* $B \in \mathscr{B}(\mathbb{R})$

$$\mathbb{P}[X_{t+s} \in B | \mathcal{F}_s] = \mathbb{P}[X_{t+s} \in B | X_s] \ , \quad \mathbb{P}\text{-a.s.}$$

We define the ***transition function*** *of the process X by*

$$P_t(s, x, B) = \mathbb{P}[X_{t+s} \in B | X_t = x] \ .$$

If the transition function does not depend on t, we call the Markov process homogeneous, and we will omit the index t.

In the following we let $C_b(E)$, $E \subset \mathbb{R}$ denote the set of all measurable bounded real functions on E. We endow $C_b(E)$ with the supremum norm $||f||_\infty = \sup_{x \in E} |f(x)|$.

Definition B.1.2
Let $\{T(h) : h \geq 0\}$ be a family of bounded linear operators from $C_b(E)$ to $C_b(E)$. Then $\{T(h)\}$ is called a contraction semigroup on $C_b(E)$ if

1. $T(0) = I$, *where I is the identity function $I : C_b(E) \to C_b(E)$, $f \mapsto f$;*
2. $T(s+t) = T(s)T(t)$ *for all* $s, t \geq 0$*;*
3. $||T(h)f||_\infty \leq ||f||_\infty$.

Proposition B.1.3
Let $\{X_t\}$ be a Markov process with transition functions $P(t,x,B)$. The mapping family

$$T(t)f(x) = \int f(y)P(t,x,dy) = \mathbb{E}_x[f(X_t)] \,, \tag{B.1}$$

where $f \in C_b(E)$, is a contraction semigroup.

For proof see Rolski et al. [61, p. 440].

B.2 Generators

Definition B.2.1
The infinitesimal generator of a Markov process $\{X_t\}$ or of its corresponding semigroup $\{T(t)\}$, defined in (B.1), *is the linear operator D defined by*

$$Df(x) = \lim_{t \downarrow 0} \frac{T(t)f(x) - f(x)}{t} \,,$$

*provided the limit exists. We call the class of functions, where the right side converges to some function uniformly in x, the **domain** $\mathfrak{D}(D)$ of generator D.*

Theorem B.2.2
Assume that $X = \{X_t\}$ is an E-valued Markov process, $E \subset \mathbb{R}$ with transition functions $P(h,x,B)$. Let further $\{T(h)\}$ denote the corresponding semigroup, defined by (B.1), *and let D be its generator. Then for each $f \in \mathfrak{D}(D)$ the stochastic process $\{M_t\}$ is an $\{\mathcal{F}_t^X\}$-martingale, where*

$$M_t = f(X_t) - f(X_0) - \int_0^t Df(X_s)\,\mathrm{d}s \,.$$

For proof see Rolski et al. [61, p. 442].

B.2.1 The classical risk model

Lemma B.2.3
Every stochastic process X with stationary and independent increments is a Markov process.

The process defined in (1.1), i.e. the classical risk model, is a Markov process since it has stationary and independent increments.

Proposition B.2.4
The process X defined in (1.1) *has the infinitesimal generator*

$$Df(x) = cf'(x) + \lambda \int_0^\infty f(x-z)\,\mathrm{d}G(z) - \lambda f(z) \,,$$

where f is a bounded real-valued, differentiable function and G is the distribution function of the claim sizes Z_i.

Proof: Consider the semigroup $T(h)$ associated to $X_t = x + ct - S_t$. It is given by

$$T(h)f(x) = \sum_{k=0}^{\infty} \frac{(\lambda h)^k}{k!} e^{-\lambda h} \mathbb{E}[f(x + ch - S_h)|N_h = k]$$

for $x \in \mathbb{R}$ and $f \in C_b(\mathbb{R})$. Thus, we obtain

$$\begin{aligned} T(h)f(x) &= e^{-\lambda h} f(x + ch) + \lambda h e^{-\lambda h} \mathbb{E}[f(x + ch - Z_1)|N_h = 1] \\ &\quad + h^2 \sum_{k=0}^{\infty} \frac{\lambda^{k+2} h^k}{(k+2)!} e^{-\lambda h} \mathbb{E}[f(x + ch - S_h)|N_h = k + 2];. \end{aligned}$$

Because f is bounded and differentiable we have

$$\begin{aligned} \lim_{h \to 0} \frac{T(h)f(x) - f(x)}{h} &= \lim_{h \to 0} \Big\{ e^{-\lambda h} \frac{f(x + ch) - f(x)}{h} \\ &\quad + f(x) \frac{e^{-\lambda h} - 1}{h} \\ &\quad + \lambda e^{-\lambda h} \mathbb{E}[f(x + ch - Z_1)|N_h = 1] \\ &\quad + h \sum_{k=0}^{\infty} \frac{\lambda^{k+2} h^k}{(k+2)!} e^{-\lambda h} \mathbb{E}[f(x + ch - S_h)|N_h = k + 2] \Big\} \\ &= cf'(x) - \lambda f(x) + \lambda \int_0^{\infty} f(x - z)\, \mathrm{d}G(z)\,. \end{aligned}$$

□

B.2.2 Diffusion approximation

Next we will give the infinitesimal generator of a diffusion process. The concept of a diffusion we have already explained in Definition 1.2.1 p. 17. In Bhattacharya and Waymire [9, p. 367] one finds that a diffusion X, solving the stochastic differential equation

$$\mathrm{d}X_t = m(x)\, \mathrm{d}t + \sigma(x)\, \mathrm{d}W_t\,,$$

fulfils

$$\begin{aligned} &\mathbb{E}[(X_{s+t} - X_s)\mathbb{1}_{[|X_t - x| \le \varepsilon]}|X_s = x] = tm(x) + o(t)\,, \\ &\mathbb{E}[(X_{s+t} - X_s)^2\mathbb{1}_{[|X_t - x| \le \varepsilon]}|X_s = x] = t\sigma^2(x) + o(t)\,, \qquad \text{(B.2)} \\ &\mathbb{E}[|X_{s+t} - X_s| > \varepsilon|X_s = x] = o(t) \end{aligned}$$

as $t \downarrow 0$ for every $x \in \mathbb{R}$ and every $\varepsilon > 0$, where $\lim\limits_{t \to 0} \frac{o(t)}{t} = 0$.

Proposition B.2.5
Let $\{X_t\}$ be a diffusion defined above. Then for all bounded twice continuously differentiable real valued f the infinitesimal generator of $\{X_t\}$ is given by

$$Df(x) = m(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) \ .$$

Proof: Let $x \in \mathbb{R}$ be fixed and $\delta > 0$. Because f is twice continuously differentiable, there is $\varepsilon > 0$ such that $|f''(x) - f''(y)| < \delta$ for all $|x - y| \leq \varepsilon$. Further it holds

$$T(t)f(x) = \mathbb{E}_x[f(X_t)\mathbb{1}_{[|X_t - x| \leq \varepsilon]}] + \mathbb{E}_x[f(X_t)\mathbb{1}_{[|X_t - x| > \varepsilon]}] \ .$$

Taking a Taylor expansion on $f(X_t)$ in a neighborhood of x

$$\begin{aligned} \mathbb{E}_x[f(X_t)\mathbb{1}_{[|X_t - x| \leq \varepsilon]}] &= \mathbb{E}\Big[\Big\{f(x) + (X_t - x)f'(x) + \frac{(X_t - x)^2}{2}f''(x) \\ &\quad + \frac{(X_t - x)^2}{2}(f''(\xi_t) - f''(x))\Big\}\mathbb{1}_{[|X_t - x| \leq \varepsilon]}\Big] \ . \end{aligned}$$

where ξ_t is a number between X_t and x. Using (B.2) the expectations of the first three terms add up to

$$f(x) + tm(x)f'(x) + \frac{t\sigma^2(x)}{2}f''(x) + o(t) \ ,$$

$t \downarrow 0$. Thus, we obtain

$$\mathbb{E}_x\Big[\frac{(X_t - x)^2}{2}(f''(\xi_t) - f''(x))\mathbb{1}_{[|X_t - x| \leq \varepsilon]}\Big] \leq \frac{\delta\sigma^2(x)}{2} + \limsup \frac{o(t)}{t}$$

for $t \downarrow 0$. Because f is bounded and using (B.2) we obtain

$$\mathbb{E}_x[|f(X_t)|\mathbb{1}_{[|X_t - x| > \varepsilon]}] \leq ||f||\mathbb{E}[\mathbb{1}_{[|X_t - x| > \varepsilon]}] = o(t)$$

for $t \downarrow 0$. It means $\limsup\limits_{t \downarrow 0} \mathbb{E}_x[f(X_t)\mathbb{1}_{[|X_t - x| > \varepsilon]}] = 0$. Thus we have

$$\limsup_{t \downarrow 0} \Big|\frac{T(t)f(x) - f(x)}{t} - m(x)f'(x) - \frac{\sigma^2(x)}{2}f''(x)\Big| \leq \frac{\delta\sigma^2(x)}{2} \ .$$

Because $\delta > 0$ was arbitrary it follows

$$\lim_{t \downarrow 0} \frac{T(t)f(x) - f(x)}{t} = m(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) \ .$$

□

C Change of Measure and Change of Variables Formulae

In this appendix we give the important change of measure and change of variables formulae.

C.1 Ito's Lemma

We start with the change of variables formula, also called Ito's lemma. Here we give just a one-dimensional version for the processes of the form

$$X_t = x + \int_0^t K_s^{(1)} \, \mathrm{d}s + \int_0^t K_s^{(2)} \, \mathrm{d}W_s + \int_0^t K_{s-}^{(3)} \, \mathrm{d}Y_s \,, \tag{C.1}$$

where W is the standard Brownian motion and Y is a process of bounded variation. Such process, if it is well defined, is called Ito process. A general form of Ito's result can be found for example in Protter [60].

Proposition C.1.1 (Ito's Lemma)
Let X be an Ito process defined in (C.1) *and $f(t,x)$ be a function which is continuously differentiable with respect to t and twice continuously differentiable with respect to x. Then $\tilde{X}_t = f(t, X_t)$ fulfils*

$$\begin{aligned}\tilde{X}_t \;=\; & \int_0^t \left\{ \frac{\partial f}{\partial s}(s, X_s) + K_s^{(1)} \frac{\partial f}{\partial x}(s, X_s) + \frac{(K_s^{(2)})^2}{2} \frac{\partial^2 f}{\partial x^2}(s, X_s) \right\} \mathrm{d}s \\ & + \int_0^t K_s^{(2)} \frac{\partial f}{\partial x}(s, X_s) \, \mathrm{d}W_s + \int_0^t \frac{\partial f}{\partial x}(s, X_s) K_{s-}^{(3)} \, \mathrm{d}Y_s \,.\end{aligned}$$

Proof: For proof see Rolski et al. [61, p. 559]. □

C.2 Change of Measure

In order to introduce the change of measure formula we need some preliminary remarks. Let $W = \{W_t\}$ be an $\mathbb{F} = \{\mathcal{F}_t\}$ adapted standard Brownian motion living on $(\Omega, \mathcal{F}, \mathbb{P})$. Let further $\{l_s\}$ be an adapted process such that the stochastic integral $\int_0^t l_s \, \mathrm{d}W_s$ is well defined. Define further

$$K_t = \exp\Big\{ \int_0^t l_s \, \mathrm{d}W_s - \frac{1}{2} \int_0^t l_s^2 \, \mathrm{d}s \Big\} .$$

Theorem C.2.1 (Novikov)
If for each $t \geq 0$

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2} \int_0^t l_s^2 \, \mathrm{d}s \Big) \Big] < \infty ,$$

then it holds $\mathbb{E}[K_t] = 1$ for all $t \geq 0$. In this case the process $\{K_t\}$ is a positive martingale.

For the change of measure formula we do not really need Novikov's theorem. But it gives us a useful tool for constructing a strictly positive martingale with expected value 1.

Definition C.2.2
*Let ν and $\varkappa$ be two measures ν and $\varkappa$ on the same measurable space, $\varkappa$ is said to be **absolutely continuous** with respect to ν, or dominated by ν if $\varkappa(A) = 0$ for every set A for which $\nu(A) = 0$. This is written as $\varkappa \ll \nu$.*

Remark C.2.3
Let M be a strictly positive martingale with respect to the filtration $\{\mathcal{F}_t\}$ and expected value 1. Then

$$\tilde{P}_t[A] = \int_A M_t \, \mathrm{d}\mathbb{P} = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A M_t]$$

is a probability measure on $\mathcal{F}_t$. If T is a given time, then for any $A \in \mathcal{F}_t \subset \mathcal{F}_T$ it holds

$$\tilde{P}[A] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A M_T] = \tilde{P}_t[A] ,$$

and $\tilde{P}$ is a probability measure on $(\Omega, \mathcal{F}_T)$. Moreover $\tilde{P}$ is absolutely continuous with respect to $\mathbb{P}$ and M_t is a version of its Radon-Nikodym derivative when the measures are restricted to $\mathcal{F}_t$, $0 \leq t \leq T$

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = M_t \quad \mathbb{P} - \text{a.s.}$$

Proposition C.2.4 (Girsanov's theorem)
Let $J = \{J_t\}$ be an adapted process satisfying $\int_0^t J_s^2 \, \mathrm{d}s < \infty$ a.s. and such that the process $M = \{M_t\}$ defined by

$$M_t = \exp\Big(- \int_0^t J_s \, \mathrm{d}W_s - \frac{1}{2} \int_0^t J_s^2 \, \mathrm{d}s \Big)$$

is an $\{\mathcal{F}_t\}$ martingale. Then, under the measure $\tilde{P}$, defined as in Remark C.2.3, the process $\{W_t + \int_0^t J_s^2 \, \mathrm{d}s : \ 0 \le t \le T\}$ is an $\{\mathcal{F}_t\}$-standard Brownian motion.

Proof: For the proof see Rogers and Williams [63, p. 82], where uncompleted filtrations are used and the result is formulated in $\mathbb{R}$. □

Remark C.2.5
Consider an (m, σ^2)-Brownian motion $X_t = x + mt + \sigma W_t$ on $(\Omega, \mathcal{F}, \mathbb{P})$, where $W = \{W_t\}$ is again the standard Brownian motion.

(i) Let J be a constant process $J = \{J_t\} = \{\frac{m}{\sigma^2}\}$. Then, due to Proposition A.2.3

$$M_t = \exp\Big(- \frac{m}{\sigma^2} W_t - \frac{m^2}{2\sigma^4} t \Big)$$

is a martingale with expectation 1. Under the measure $\tilde{P}$, defined in Remark C.2.3, the process X_t is a $(0, \sigma^2)$-Brownian motion with initial value x. Note, that because the $(0, \sigma^2)$-Brownian motion exists, the measure $\tilde{P}$ can be extended to $\mathcal{F}$.

(ii) From change of measure formula we also obtain for every positive β and stopping time $\tau = \inf\{t \ge 0 : X_t < 0\}$:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[e^{-\beta\tau} \mathbb{I}_{[\tau<\infty]}] &= \mathbb{E}_{\tilde{P}}[e^{X_\tau (m+\sqrt{m^2+2\beta\sigma^2})/\sigma^2}] \cdot e^{-x(m+\sqrt{m^2+2\beta\sigma^2})/\sigma^2} \\ &= e^{-x(m+\sqrt{m^2+2\beta\sigma^2})/\sigma^2} \ . \end{aligned}$$

D Stochastic Differential Equations and Local Times

In this appendix we make a short digression into the theory of diffusions. Our considerations are restricted to the one-dimensional case. For a detailed introduction to stochastic differential equations we refer to Karatzas and Shreve [49], to Protter [60] or to Revuz and Yor [62].

D.1 Stochastic Differential Equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a one-dimensional standard Brownian motion $W = \{W_t\}$ is given. Let further $\{\mathcal{F}_t\}$ be the natural filtration of the Brownian motion W. In Preliminaries we have already given a definition of a diffusion and in Section 2.1 we consider a diffusion, which is a strong solution to some stochastic differential equation (SDE). Next we give a definition, what a strong solution is.

Definition D.1.1
Let $m : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ be Borel-measurable functions. A process X, $X_0 = x$ taking values in $\mathbb{R}$ is said to be a ***strong solution*** *to the stochastic differential equation*

$$\mathrm{d}X_t = m(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t \tag{D.1}$$

if

1. *X is $\{\mathcal{F}_t\}$-adapted;*
2. *$\mathbb{P}[X_0 = x] = 1$;*
3. *$\int_0^t \left(|\sigma(X_t)|^2 + |m(X_t)|\right)\,\mathrm{d}s < \infty$ for all $t \geq 0$ a.s.;*
4. *$X_t = x + \int_0^t m(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s$ for all $t \geq 0$ a.s.*

If X and $\tilde{X}$ are two strong solutions to (D.1) *with $\mathbb{P}[X_t = \tilde{X}_t\ , 0 \leq t < \infty] = 1$, then we say that Equation* (D.1) *has a* ***unique strong solution****.*

Note that due to *3.* the integrals in *4.* are well defined. It is also clear from *4.* that X is a continuous semimartingale.

D.2 Local Times

We start with the most simple diffusion, namely with the standard Brownian motion. Let $W = \{W_t\}$ be the standard Brownian motion.

Definition D.2.1
*By the **local time** of W we denote a family of non-negative random variables $L^x = \{L_t^x,\ x \in \mathbb{R}\}$ such that with probability one, the following holds*

1. *$(t,x) \mapsto L_t^x$ is continuous;*
2. *for every Borel subset $A \subset \mathbb{R}$ and $t \in [0,\infty)$*

$$\int_0^t \mathbb{1}_A(W_s)\ \mathrm{d}s = \int_A L_t^x\ \mathrm{d}x\ .$$

If such a family exists, then it is unique and is given by

$$L_t^x = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{[(x-\varepsilon,x+\varepsilon)]}(W_s)\ \mathrm{d}s\ .$$

Theorem D.2.2
The local time $\{L_t^x\}$ of W exists.

Proof: Confer Ikeda and Watanabe [44, p. 113] □

The local time at zero $\{L_t^0\}$ has been of particular interest for our considerations. From now on, speaking of local time, we will mean the local time at zero. Itô's formula for twice continuously differentiable functions f can be extended to functions, satisfying less restrictive assumptions. The first result in this direction is known as Tanaka's formula. Let

$$\mathrm{sgn}(x) = \begin{cases} 1 & , \quad : x \geq 0\ , \\ -1 & , \quad : x < 0\ . \end{cases}$$

Then it holds

$$|W_t| = \int_0^t \mathrm{sgn}(W_s)\ \mathrm{d}W_s + L_t^0\ .$$

The process $|W| = \{|W_t|\}$ we call the **reflected standard Brownian motion**.
Proof: See for example Protter [60, p. 217]. □

Next theorem states that the reflected Brownian motion can be characterised by the Skorokhod equation.

Theorem D.2.3 (Skorokhod)
Let $x : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function with $x(0) \geq 0$. Then there exists a unique pair (a, y) of continuous functions such that

1. *$a(t) = x(t) + y(t)$,*
2. *$a(t) \geq 0$, y is increasing, $y(0) = 0$,*
3. *$\int_0^t \mathbb{1}_{[a(s) \neq 0]} \, \mathrm{d}y(s) = 0$, $\forall t$. Moreover, the function $y(t)$ is explicitly given by*

$$y(t) = -\min\{\inf_{0 \leq s \leq t} x(s), 0\} \ .$$

Proof: See Rogers and Williams [64, p. 117]. □

In particular, we obtain that the Brownian local time at zero L^0 starts at zero, is continuous, non-decreasing and $\mathrm{d}L$ is almost surely supported by $\{t \geq 0 : |W_t| = 0\}$. Moreover, the local time L^0 is given by $L_t^0 = -\min\{\inf_{0 \leq s \leq t} W_s, 0\}$. Also it holds

$$|W_t| = W_t + L_t^0 \ .$$

Note that $|W|$ is a diffusion process solving the stochastic differential equation $\mathrm{d}W_t + \mathrm{d}L_t^0$. With change of measure and time-change formulae the results for the standard Brownian motion can be transfered, under some assumptions, to strong solutions to SDEs (see Rogers and Williams [64, p. 289]).

E Differential equations

In this appendix we review some results from the theory of differential equations, which we have used in the work. We start with an equation, which we have had to solve in every diffusion approximation model with the exception of the model with constant interest rate, Section 3.1. We find in Kamke [48] for linear homogeneous second-order differential equations with constant coefficients:

Proposition E.1.4
Every solution to the differential equation of the form

$$y'' + by' + cy = 0$$

is given by

$$y = C_1 \cdot \exp\Big(\frac{-b + \sqrt{b^2 - 4c}}{2} \cdot x\Big) + C_2 \cdot \exp(-\frac{b + \sqrt{b^2 - 4c}}{2} \cdot x) \ ,$$

provided $b^2 - 4c \geq 0$.

In Section 3.1 we have had to solve the differential equations of the form

$$y''(x) + p(x)y'(x) + qy(x) = 0 \ ,$$

where $p(x)$ is a linear function and q is a constant. This is a linear second-order homogeneous differential equation. In remark below we explain, how to solve this differential equation with the so-called power series method.

Remark E.1.5
Consider then

$$y''(x) + p(x)y'(x) + qy(x) = 0 \ ,$$

where $p(x) = J_1 x + J_2$ and q is a constant. Let $\phi(p(x)) = y(x)$, then it holds $y'(x) = J_1\phi'(p(x))$ and $y''(x) = J_1^2\phi''(p(x))$. Thus, the above differential equation can be transformed to

$$J_1^2\phi''(p) + pJ_1\phi'(p) + q\phi(p) = 0 \ .$$

The power series method calls for the construction of a power series solution

$$\phi(p) \quad = \quad \sum_{n=0}^{\infty} a_n p^n$$

with $\phi'(p) = \sum_{n=1}^{\infty} n a_n p^{n-1}$ and $\phi''(p) = \sum_{n=2}^{\infty} n(n-1) a_n p^{n-2}$. Subsetting these into the differential equation yields

$$\begin{aligned} 0 &= J_1^2 \sum_{n=2}^{\infty} n(n-1) a_n p^{n-2} + p J_1 \sum_{n=1}^{\infty} n a_n p^{n-1} + q \sum_{n=0}^{\infty} a_n p^n \\ &= \sum_{n=0}^{\infty} p^n \Big\{ J_1^2 (n+2)(n+1) a_{n+2} + (J_1 n + q) a_n \Big\} . \end{aligned}$$

Now, if this series is a solution, all the coefficients must be zero, so:

$$a_{n+2} = \frac{a_n(-J_1 n - q)}{J_1^2 (n+1)(n+2)} .$$

Solving this relation recursively, we obtain an expression for every a_n, $n \geq 2$ in terms of a_0 and a_1:

$$\begin{aligned} a_2 &= \frac{-a_0 q}{J_1^2 2} , \\ a_3 &= \frac{-J_1 - q}{J_1^2 2 \cdot 3} a_1 , \\ &\vdots \\ a_{2n} &= \frac{\prod_{k=0}^{n-1} (-2k - q/J_1)}{J_1^n (2n)!} a_0 , \\ a_{2n+1} &= \frac{\prod_{k=1}^{n} (-(2k-1) - q/J_1)}{J_1^n (2n+1)!} a_1 . \end{aligned}$$

Thus, it holds

$$\begin{aligned} \phi(p) &= \sum_{n=0}^{\infty} a_n p^n = a_0 \Big(1 + \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n-1} (-2k - q/J_1)}{J_1^n (2n)!} p^{2n} \Big) \\ &\quad + a_1 \Big(p + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n} (-(2k-1) - q/J_1)}{J_1^n (2n+1)!} p^{2n+1} \Big) \end{aligned}$$

Plugging in $p(x) = J_1 x + J_2$ yields

$$
\begin{aligned}
y(x) &= a_0\Big(1 + \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n-1}(-2k - q/J_1)}{(2n)!} J_1^n (x + J_2/J_1)^{2n}\Big) \\
&\quad + a_1 J_1\Big(x + J_2/J_1 + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n}(-(2k-1) - q/J_1)}{(2n+1)!} J_1^n (x + J_2/J_1)^{2n+1}\Big) .
\end{aligned}
$$

For example for $J_1 = \frac{2m}{\lambda\mu_2}$, $J_2 = \frac{2\lambda\mu\eta}{\lambda\mu_2}$ and $q = -\frac{2\delta}{\lambda\mu_2}$ we obtain

$$
\begin{aligned}
y(x) &= a_0\Big(1 + \sum_{n=1}^{\infty} \frac{\frac{\delta}{m}\cdots(\frac{\delta}{m} - 2n + 2)}{(2n)!} \Big(\frac{2m}{\lambda\mu_2}\Big)^n (x + \frac{\lambda\mu\eta}{m})^{2n}\Big) \\
&\quad + a_1 \frac{2m}{\lambda\mu_2}\Big(x + \frac{\lambda\mu\eta}{m} + \sum_{n=1}^{\infty} \frac{(\frac{\delta}{m} - 1)\cdots(\frac{\delta}{m} - 2n + 1)}{(2n+1)!} \Big(\frac{2m}{\lambda\mu_2}\Big)^n (x + \frac{\lambda\mu\eta}{m})^{2n+1}\Big) .
\end{aligned}
$$

Gronwall's lemma, which we will consider below, has been an important tool to obtain the existence of a solution in Section 5.2. Gronwall's lemma allows one to bound a function, that is known to satisfy a certain differential or integral inequality, by the solution of the corresponding differential or integral equation. There are two forms of the lemma, a differential form and an integral form. We will consider here the integral form.

Proposition E.1.6
Let $I = [a, b] \subset \mathbb{R}$ and u, $C : I \to \mathbb{R}$ and $\beta : I \to [0, \infty)$ be continuous functions. Suppose

$$u(x) \leq C(x) + \int_a^b \beta(y) u(y) \, \mathrm{d}y$$

for all $x \in I$. Then it holds

$$u(x) \leq C(x) + \int_a^b C(y)\beta(y) e^{\int_y^x \beta(s)\, \mathrm{d}s} \, \mathrm{d}y .$$

for all $x \in I$.
If C and β are constants with $\beta \geq 0$, then we have

$$u(x) \leq C e^{\beta(x-a)} .$$

Proof: For proof see Amann and Escher [2, p. 131]. □

F The Black-Scholes Model

The Black Scholes model is the most widely used model for pricing options. The model and associated call and put option formulas remained the industry standard in equity and currency markets for the last thirty years. That is why, giving the insurer the possibility to invest his money, we have used the Black-Scholes setup for modeling of a risky asset.
In this section we give a short introduction of the Black-Scholes model. For detailed introduction we refer to Müller-Möhl [56], Lamberton and Lapeyre [51] and Rolski et al. [61].

We have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is large enough to carry all the stochastic quantities defined below. With $\{\mathcal{F}_t\}$ we denote the smallest right-continuous filtration such that the standard Brownian motion $W = \{W_t\}$ is adapted with respect to $\{\mathcal{F}_t\}$.

The Black-Scholes model is a continuous-time model, which describes the behavior of the price of a risky asset with price Q_t at time t and a riskless asset with price Q_t^0 at time t. We suppose that the behavior of Q^0 follows the differential equation

$$\mathrm{d}Q_t^0 = rQ_t^0 \,\mathrm{d}t \,,$$

where r is a non-negative constant, which is called **risk-free rate**. Further we assume that the behavior of the stock price is determined by the stochastic differential equation

$$\mathrm{d}Q_t = Q_t(m \,\mathrm{d}t + \sigma \,\mathrm{d}W_t) \quad Q_0 > 0 \,,$$

where m and σ are constants.
We see, that the process $Q = \{Q_t\}$ is a solution to the above differential equation if and only if the process $\{\log(Q_t)\}$ is an (m, σ)- Brownian motion; and it holds

$$Q_t = Q_0 \exp\Big((m - \sigma^2/2)t + \sigma W_t\Big) \,.$$

The return of the process Q is given by

$$Q_t' = \frac{\mathrm{d}Q_t}{Q_t} = m \,\mathrm{d}t + \sigma \,\mathrm{d}W_t \,.$$

m is called the **expected rate of return** and $\sigma > 0$ the **volatility**.

An European Option assumes that it can be exercised only at expiration. The Black-Scholes model requires that the risk-free rate r, the expected rate of return m and the volatility of the underlying stock price σ^2 remain constant over the period of analysis. The model also assumes that the underlying stock does not pay dividends. The Black-Scholes formula calculates the price of an European call option to be:

$$C = S\Phi(d_1) - Xe^{-rT}\Phi(d_2) ,$$

where

$$\begin{aligned}
C &= \text{price of the call option} \\
X &= \text{option exercise price} \\
T &= \text{current time until expiration} \\
\Phi &= \text{standard Normal distribution} \\
d_1 &= \tfrac{\ln(Q/X)+(r+\sigma^2/2)T}{\sigma T^{1/2}} \\
d_2 &= d_1 - \sigma T^{1/2}
\end{aligned}$$

Put-call parity requires that:

$$P = C - Q + Xe^{-rT}$$

Then the price of an European put option is:

$$P = Xe^{-rT}\Phi(-d_2) - Q\Phi(-d_1) .$$

Bibliography

[1] Albrecher, H. and Thonhauser, S. (2007). Dividend maximization under consideration of the time value of ruin. *Insurance: Mathematics and Economics* **41**, 163–184.

[2] Amann, H. and Escher, J. (1999). *Analysis II.* Birkhäuser Verlag, Basel.

[3] Asmussen, S. and M. Taksar (1997). Controlled diffusion models for dividend payout. *Insurance: Mathematics and Economics* **20**, 1–15.

[4] Asmussen, S., Højgaard B. and Taksar M. (2000). Optimal risk control and dividend distribution policies. Example of excess-of-loss reinsurance for an insurance corporation. *Finance Stoch.* **4**, 299–324.

[5] Azcue, P. and Muler, N. (2005). Optimal reinsurance and dividend distribution policies in the Cramér–Lundberg model. *Math. Finance* **15**, 261–308.

[6] Bardi, M. and Capuzzo-Dolcetta, I. (1997). *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations.* Birkhäuser, Boston.

[7] Beard, R.E., Pentikäinen, T. and Pesonen, E. (1984). *Risk Theory.* Chapman and Hall, London.

[8] Benth, F.E., Karlsen, K.H. and Reikvam, K. (2001). Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: A viscosity solution approach. *Finance Stoch.* **5(3)**, 275–303.

[9] Bhattacharya, R. N. and Waymire, E. C. (1990). *Stochastic Processes with Applications.* Wiley, New York.

[10] Billingsley, P. (1999) *Convergence of Probability Measures.* Wiley, New York.

[11] Bremaud, P. (1981). *Point Processes and Queues, Martingale Dynamics.* Springer-Verlag, New York.

[12] Borch, K.H. (1974). *The Mathematical Theory of Insurance.* Cambridge, MA: Lexington Books, D.C. Heath.

[13] Borodin, A.N. and Salminen P. (2002). *Handbook of Brownian Motion – Facts and Formulae.* Birkhäuser Verlag, Basel.

[14] Bühlmann, H. (1970). *Mathematical methods in risk theory.* Springer-Verlag, Berlin.

[15] Cramér, H. (1930) *On the Mathematical Theory of Risk.* Scandia Jubilee Volume, Stockholm.

[16] Cramér, H. (1955) *Collective risk theory.* Scandia Jubilee Volume, Stockholm.

[17] Crandall, M.G., Ishii, H. and Lions, P.L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc* **27**, 1–67.

[18] Dickson, D.C.M and Waters, H.R. (2004). Some optimal dividend problems. *ASTIN Bulletin* **34**, 49–74.

[19] Dickson, D.C.M. and Drekic, S. (2006). Optimal dividends under a ruin probability constraint. *Annals of Actuarial Science* **1 (2)**, 291–306.

[20] Durrett, R. (1996). *Stochastic Calculus: A Practical Introduction.* CRC Press, Boca Raton.

[21] Eisenberg, J. and Schmidli, H. (2008). Optimal control of capital injections by reinsurance in a diffusion approximation. *Blätter der DGVFM* **30**, 1–13

[22] Eisenberg, J. and Schmidli, H. (2009). Minimising Expected Discounted Capital Injections by Reinsurance in a Classical Risk Model. To appear in *Scandinavian Actuarial Journal.*

[23] Ethier, S.N. and Kurtz, T.G. (1986). *Markov Processes. Characterization and Convergence.* Wiley, New York.

[24] de Finetti, B. (1957). Su un'impostazione alternativa della teoria collettiva del rischio. *Transactions of the XVth International Congress of Actuaries* **2**, 433–443.

[25] Fleming, W.H. and Soner, H.M. (1993). *Controlled Markov Processes and Viscosity Solutions.* Springer-Verlag, New York.

[26] Fleming, W.H. and Rishel, R.W. (1976). *Deterministic and Stochastic Optimal Control.* Springer-Verlag, New York.

[27] Gerber, H.U. (1969). Entscheidungskriterien für den zusammengesetzten Poisson-Prozess. *Schweiz. Verein. Versicherungsmath. Mitt.* **69**, 185–228.

[28] Gerber, H.U. (1974). The dilemma between dividends and safety and a generalisation of the Lundberg-Cramer formulas. *Scandinavian Actuarial Journal* 46–57.

[29] Gerber, H.U. (1979). *An Introduction to Mathematical Risk Theory.* Huebner Foundation Monographs, Philadelphia.

[30] Gerber, H.U. (1981). On the probability of ruin in the presence of a linear dividend barrier. *Scandinavian Actuarial Journal* 105–115.

[31] Gerber, H.U. and Shiu E.S.W. (1997). On the time value of ruin. *Insurance: Mathematics and Economics* **69**, 145–199.

[32] Gerber, H.U. and Landry B. (1998). On the discounted penalty at ruin in a jump-diffusion and the perpetual put option. *Insurance: Mathematics and Economics* **22**, 263–276.

[33] Gerber, H.U. and Shiu, E.S.W. (2006) On the optimal dividend stategies in the compound Poisson model. *North American Actuarial Journal* **10(2)**, 76–93.

[34] Gerber, H.U., Lin, X.S. and Yang, H. (2006) A note on the dividends-penalty identity and the optimal dividend barrier. *Astin Bulletin* **36(2)**, 489–503.

[35] Gerber, H.U., Shiu, E.S.W. and Smith, N. (2008) Methods for estimating the optimal dividend barrier and the probability of ruin. *Insurance: Mathematics and Economics* **42**, 243–254

[36] Grandell, J. (1977) A class of approximations of ruin probabilities. *Scandinavian Actuarial Journal*, 37–52.

[37] Grandell, J. (1991). *Aspects of Risk Theory.* Springer-Verlag, New York.

[38] Højgaard, B. and M. Taksar, (1999). Controlling risk exposure and dividends payout schemes: Insurance company example. *Mathematical Finance* **9**, 153–182.

[39] Heilmann, W. (1987). *Grundbegriffe der Risikotheorie.* VVW, Karlsruhe.

[40] Hipp, C. (2003). Optimal dividend payment under a ruin constraint: Discrete time and state space. *Blätter der DGVFM* **26**, 255–264.

[41] Hipp, C. and Plum, M. (2000). Optimal investment for insurers. *Insurance: Mathematics and Economics* **27**, 215–228.

[42] Hipp, C. and Schmidli, H. (2004). Asymptotics of ruin probabilities for controlled risk processes in the small claims case. *Scandinavian Actuarial Journal*, 321–335.

[43] Iglehart, D.L. (1969). Diffusion approximations in collective risk theory. *Journal of Applied Probability* **6**, 285–292.

[44] Ikeda, N. and Watanabe, S. (1989) *Stochastic Differential Equations and Diffusion Processes.* Elsevier Science Publishers B.V., Amsterdam.

[45] Itô, K. and McKean, H.P.,Jr. (1974) *Diffusion Processes and Their Sample Paths.* Springer-Verlag, Berlin.

[46] Jeanblanc-Picqué, M. and Shiryaev, A.N. (1995). Optimisation of the flow of dividends. *Russian Math. Surveys* **50**, 257–277.

[47] Kaas, R., Goovaerts, M., Dhaene, J. and Denuit M. (2008) *Modern Actuarial Risk Theory.* Springer-Verlag, Berlin.

[48] Kamke, E. (1944) *Differentialgleichungen Lösungsmethoden und Lösungen.* Akademische Verlagsgesellschaft Becker and Erler Kom.-Ges., Leipzig.

[49] Karatzas, I. and Shreve, S.E. (2000). *Brownian Motion and Stochastic Calculus.* Springer-Verlag, New York.

[50] Kulenko, N. and Schmidli, H. (2008). Optimal dividend strategies in a Cramér–Lundberg model with capital injections. *Insurance: Mathematics and Economics* **43**, 270–278

[51] Lamberton, D. and Lapeyre, B. (2008). *Introduction to Stochastic Calculus Applied to Finance.* Chapman and Hall, London.

[52] Lundberg, F. (1909). On the theory of Risk. *Transactions of the Sixth International Congress of Actuaries* **1**, p. 877

[53] Merton, R.C. (1969) Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and Statistics* **51**, 247–257.

[54] Merton, R.C. (1971) Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* **3**, 373–413.

[55] Mikosch, T. (2004). *Non-Life Insurance Mathematics.* Springer-Verlag, Berlin.

[56] Müller-Möhl, E. (1999). *Optionen und Futures.* Schäffer-Poeschel Verlag, Stuttgart.

[57] Paulsen, J. and Gjessing H.K., (1997). Optimal choice of dividend barriers for a risk process with stochastic return on investments. *Insurance: Mathematics and Economics* **20**, 215–223.

[58] Paulsen, J. (2003). Optimal dividend payouts for diffusions with solvency constraints. *Finance Stochast.* **7**, 457–473.

[59] Pitts, S.M. and Politis, K. (2007). Approximations for the Gerber-Shiu expected discounted penalty function in the compound Poisson risk Model. *Advances in Applied Probability* **39**, 385–406

[60] Protter, P.E. (2004). *Stochastic Integration and Differential Equations.* Springer-Verlag, Berlin.

[61] Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. (1999). *Stochastic Processes for Insurance and Finance.* Wiley, Chichester.

[62] Revuz, D. and Yor, M. (1991). *Continuous Martingales and Brownian Motion.* Springer, Berlin.

[63] Rogers, L.C.G. and Williams, D. (2000). *Diffusions, Markov Processes and Martingales 1. Foundations.* Cambridge University Press, Cambridge.

[64] Rogers, L.C.G. and Williams, D. (2000). *Diffusions, Markov Processes and Martingales 2. Itô Calculus.* Cambridge University Press, Cambridge.

[65] Rotar, V. (2000). *Risk Processes.* Cambridge University Press, Cambridge.

[66] Schmidli, H. (1992). *A General Insurance Risk Model.* Diss. ETH Nr. 9881, ETH Zürich.

[67] Schmidli, H. (1994). Diffusion approximation for a risk process with the possibility of borrowing and investment. *Communications in statistics. Stochastic models* **10(2)**, 365–388.

[68] Schmidli, H. (2001). Optimal proportional reinsurance policies in a dynamic setting. *Scand. Actuarial J.*, 55–68.

[69] Schmidli, H. (2002). On minimising the ruin probability by investment and reinsurance. *Annals of Applied Probability* **12**, 890–907.

[70] Schmidli, H. (2008). *Stochastic Control in Insurance.* Springer-Verlag, London.

[71] Schmidli, H. (2008). On the Gerber–Shiu Function and Change of Measure. Preprint, University of Cologne.

[72] Shreve, S.E., Lehoczky, J.P. and Gaver, D.P. (1984). Optimal consumption for general diffusions with absorbing and reflecting barriers. *SIAM J. Control and Optimization* **22**, 55–75.

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