Positivity and regularity of solutions to higher order Dirichlet problems on smooth domains

Inaugural-Dissertation

zur

Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Fakultät

der Universität zu Köln

vorgelegt von

Inka Schnieders

aus Leverkusen

Köln, 2021

Berichterstatter:	Prof. Dr. Guido Sweers
	apl. Prof. Dr. Dirk Horstmann
Tag der mündlichen Prüfung:	12.01.2021

Abstract

For second order elliptic boundary value problems a maximum principle holds true and as a consequence one finds a priori estimates for the solutions or a useful comparison principle. For higher order elliptic boundary value problems no direct generalization of the maximum principle is valid and therefore, questions that can be answered for second order problems remain open for higher order problems. In this thesis we investigate whether results such as a comparison principle or the existence of classical solutions to nonlinear problems hold for some elliptic Dirichlet problems of order $2m$.

We consider a weighted polyharmonic problem $(-\Delta)^m u - \lambda w u = f$ in a bounded domain Ω with smooth boundary and $(\frac{\partial}{\partial\nu})^k u = 0$ on $\partial\Omega$ for $k \in \{0, 1, \dots, m-1\}$. One of the main results is the following: One assumes that there is a function u_0 that can be estimated from below by $d(\cdot)^m$ and which fulfills $(-\Delta)^m u_0 > 0$ in classical sense. Here $d(\cdot)$ is the distance to the boundary. Then one finds a strictly positive weight function w and an interval $I \subset \mathbb{R}$, such that for $\lambda \in I$ the following holds for the Dirichlet problem described above: f positive implies that u is positive. Such a result is called a *positivity preserving property.*

The proof is based on the construction of an appropriate weight function w and a corresponding strongly positive eigenfunction for the weighted polyharmonic eigenvalue problem. Then, applying a converse of the Krein-Rutman theorem for the weighted polyharmonic Dirichlet problem, one obtains the main result concerning positivity of solutions. As a special case it is shown that one finds for all smooth domains an appropriate weight function, such that the weighted bilaplace problem is positivity preserving for λ in some small interval. Also some examples and special cases for higher order problems ($m > 2$) are described.

Moreover, further consequences of known estimates for the polyharmonic Green function are presented. Using these estimates and regularity results, we investigate the classical solvability of a higher order semilinear Dirichlet problem. We consider the differential equation $(-\Delta)^m u + g(\cdot, u) = f$ with zero Dirichlet boundary conditions, where g fulfills a sign condition $g(x,t)t \geq 0$ for all $(x,t) \in \Omega \times \mathbb{R}$ and satisfies a growth condition. One may improve known results about classical solvability and prove that there exists a solution $u \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$.

Zusammenfassung

Für elliptische Randwertprobleme zweiter Ordnung gilt ein Maximumprinzip, woraus Abschätzungen für die Lösungen oder ein nützliches Vergleichsprinzip folgen. Für elliptische Probleme höherer Ordnung existiert keine direkte Verallgemeinerung des Maximumprinzips, weshalb einige Fragen noch offen sind, die im Falle zweiter Ordnung beantwortet wurden. In dieser Dissertation untersuchen wir, ob Ergebnisse wie ein Vergleichsprinzip oder die Existenz klassischer Lösungen von semilinearen Problemen für einige elliptische Randwertprobleme der Ordnung $2m$ erfüllt sind.

Wir betrachten das gewichtete polyharmonische Problem $(-\Delta)^m u - \lambda w u = f$ in einem beschränkten Gebiet Ω mit glattem Rand $\partial\Omega$ und $(\frac{\partial}{\partial\nu})^k u = 0$ auf $\partial\Omega$ für $k \in \{0, 1, \dots, m-1\}$. Eines der Hauptergebnisse ist das Folgende: Es wird die Existenz einer genügend glatten Funktion u_0 angenommen, die von unten durch $d(\cdot)^m$ abzuschätzen ist und $(-\Delta)^m u_0 > 0$ im klassischen Sinne erfüllt. Hier ist $d(\cdot)$ die Distanz zum Rand des Gebietes. Dann existiert eine strikt positive Gewichtsfunktion w und ein Intervall $I \subset \mathbb{R}$, sodass für alle $\lambda \in I$ folgt: f positiv impliziert u positiv. Dies nennt man eine *positivitätserhaltende Eigenschaft*.

Der Beweis basiert auf der Konstruktion einer geeigneten Gewichtsfunktion und einer zugehörigen positiven Eigenfunktion für das gewichtete Eigenwertproblem. Wendet man anschließend eine Umkehrung des Theorems von Krein-Rutman für das gewichtete polyharmonische Dirichlet Problem auf glatten Gebieten an, findet man das genannte Ergebnis über die Positivität von Lösungen. Als Spezialfall erhält man für alle glatten Gebiete die Existenz einer Gewichtsfunktion, sodass das gewichtete biharmonische Problem für ein kleines Intervall für λ positivitätserhaltend ist. Es werden zudem Beispiele und Spezialfälle für Probleme höherer Ordnung ($m > 2$) dargestellt.

Darüber hinaus werden weitere Folgerungen aus einer bekannten Abschätzung für die Greensche Funktion des polyharmonische Problems erläutert. Es werden dieses Resultat und Regularitätsergebnisse verwendet, um die klassische Lösbarkeit eines semilinearen Dirichlet Problems höherer Ordnung zu untersuchen. Dabei wird die Differentialgleichung $(-\Delta)^m u + g(\cdot, u) = f$ mit Dirichlet Randdaten betrachtet, wobei g die Bedingung $g(x,t)t \geq 0$ für alle $(x,t) \in \Omega \times \mathbb{R}$ und zusätzliche Wachstumsbedingungen erfüllt. Man kann bekannte Ergebnisse über die klassische Lösbarkeit des semilinearen Problems verbessern, indem die Existenz einer Lösung $u \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$ bewiesen wird.

Contents

Chapter 1

Introduction

1.1 Overview of the problem

For second order elliptic boundary value problems, a maximum principle or results concerning classical solvability of semilinear boundary value problems are well known. It is surprising that these are important features of second order problems that distinguish them from higher order problems or systems of differential equations. So, the following question may arise:

> *Do similar results regarding the positivity of solution operators or existence of classical solutions for nonlinear Dirichlet problems also hold in the case of higher order elliptic differential operators?*

One can consider the problem $Lu = f$ on some bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. For $L = -\Delta$ and Dirichlet boundary conditions $u = 0$ on $\partial\Omega$ one finds that a nonnegative right-hand side f implies a nonnegative solution u. This result is often called maximum principle, but when we refer to this property we call it a *positivity preserving property* (PPP) to make a distinction between the positivity result and the local maximum principle. Moreover, using the maximum principle, one may find a priori estimates and with Hopf's boundary point lemma one obtains informations about the behavior of the solution near the boundary. In addition, since the solution operator of $L = -\Delta$ with Dirichlet boundary conditions is positive, the Krein-Rutman theorem provides results concerning simplicity of the first eigenvalue or positivity of the corresponding eigenfunction. In this thesis, we will investigate the validity of similar results for higher order problems.

For fourth or higher order elliptic Dirichlet problems a positivity preserving property does not hold in general. There are a lot of counterexamples for the clamped plate problem, that is

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain and ν is the outer normal unit vector on $\partial\Omega$. The model in (1.1) describes the deviation of a thin plate due to a force density

f. The plate is clamped at its boundary. One might suppose that if one considers sufficiently smooth and simply connected domains, then positivity preserving holds true, since this is the case in one dimension or if Ω is a ball.

Hadamard also conjectured in 1908 after a discussion with Boggio, see [36], that for convex domains a positivity preserving property holds for (1.1). The first well known counterexample was proven by Duffin in 1949, see [17]. He considered the biharmonic Dirichlet problem on an infinitely long strip and found a nonnegative right-hand side f with sign-changing solution u for problem (1.1). Only two years later, Garabedian constructed a counterexample in the case where the underlying domain is a sufficiently eccentric ellipse, see [20]. An elementary proof that the biharmonic Green function of an eccentric ellipse changes sign can be found in [64]. In [69, 70], a counterexample for the bi- and trilaplacian in an ellipse is shown. More examples can be found in Section 1.3. So, no positivity preserving property holds even if one considers bounded, smooth and convex domains. Therefore, it is not obvious under which assumptions a positivity result is fulfilled.

There are only a few domains where positivity preserving for (1.1) can be proven. Boggio constructed in [5] an explicit Green function for the ball in every dimension and since this function is positive, there is a positivity preserving property on balls. Moreover, in [28] a positivity result was shown for small perturbations of balls in two dimensions.

It is frustrating that for a lot of results in the case of second order problems one presupposes the maximum principle, so that there are no obvious extensions for higher order problems. In general, there is no replacement for a comparison principle. For higher order elliptic operators that are not a product of second order operators, the fundamental solution does not even have to be positive, see [27]. However, if we consider the special case of the polyharmonic operator with Dirichlet boundary conditions, then we will find a replacement for a comparison principle. So we get a better understanding of the behavior of solutions to some higher order problems. Instead of a maximum or comparison principle our estimates and proofs of a positivity preserving property, and the existence of classical solutions to some semilinear problems are based on sharp two-sided estimates for the polyharmonic Green function:

For every bounded and sufficiently smooth domain $\Omega \subset \mathbb{R}^n$, that is $\partial\Omega \in C^{2m,\gamma}$, there is a constant $c_{\Omega,m} > 0$ such that

$$G_{\Omega,m}(x,y) + c_{\Omega,m} d(x,\partial\Omega)^m d(y,\partial\Omega)^m \geq 0 \text{ for all } (x,y) \in \Omega^2 \text{ with } x \neq y.$$

The function $G_{\Omega,m}$ is the Green function for the polyharmonic Dirichlet problem

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu} u = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

that is $u(x) = \int_\Omega G_{\Omega,m}(x,y) f(y) dy$ solves (1.2) and $d(\cdot,\partial\Omega)$ is the distance to the boundary $\partial\Omega$. More precisely, there exists a positive function

$H_{\Omega,m}(\cdot,\cdot)$ that contains the singularity of $G_{\Omega,m}(\cdot,\cdot)$ in the sense that we find two constants $\tilde{c}_{\Omega,m}, \hat{c}_{\Omega,m} > 0$ such that

$$\tilde{c}_{\Omega,m} H_{\Omega,m}(x,y) \leq G_{\Omega,m}(x,y) + c_{\Omega,m} d(x,\partial\Omega)^m d(y,\partial\Omega)^m \leq \hat{c}_{\Omega,m} H_{\Omega,m}(x,y) \tag{1.3}$$

for all $(x,y) \in \Omega^2$ with $x \neq y$.

For the fourth order problem the two-sided estimate can be found in [26] and for $m > 2$ Pulst proved the estimate in his dissertation [53, Theorem 0.1]. Pulst even included lower order derivatives in the differential equation, but the leading order term has to be $(-\Delta)^m$.

Using results from Krein-Rutman, one finds that if the higher order elliptic boundary value problem

$$\begin{cases} (-\Delta)^m u - \lambda u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu} u = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

for $\lambda = 0$ is positivity preserving, then the first eigenvalue λ_1 of the polyharmonic eigenvalue problem

$$\begin{cases} (-\Delta)^m \varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = \frac{\partial}{\partial\nu} \varphi = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

is positive, simple and the eigenfunction φ_1 is positive in Ω. Moreover, by a Neumann series expansion it was shown in [31, Proposition 4.1] that if problem (1.4) satisfies a positivity preserving property for $\lambda = 0$, then this property holds true for all $\lambda \in [0, \lambda_1)$. Furthermore, it was proven that φ_1 is positive in the sense that there is a constant $c > 0$ such that $\varphi_1(x) \geq c\, d(x,\partial\Omega)^m$, where $d(x,\partial\Omega) := \inf_{y\in\partial\Omega} |x-y|$ is the distance to the boundary.

This raises the question whether the positivity of the solution operator is related to the existence of a positive eigenfunction. Indeed, one may show that the existence of a positive eigenfunction for (1.5) with simple eigenvalue leads to the positivity of the solution operator to (1.4) for a small interval for λ. This result can be understood as a reverse of the Krein-Rutman theorem and is published in [57] for a fourth order problem. Obviously, it would be a stronger result if we could apply this to problem (1.4) for all domains. However, the existence of a positive eigenfunction with simple eigenvalue cannot be guaranteed and is also difficult to examine for general domains. But for some smooth and bounded domains a weight function can be added so that the inverse to Krein-Rutman can be applied, see [58] for $m = 2$.

Therefore, we will search for a weighted eigenvalue problem such that one gets a simple eigenvalue with corresponding positive eigenfunction. Then, using an extended version of the two-sided Green function estimate in (1.3) for the weighted differential operator $(-\Delta)^m - \lambda w$ with weight function w and parameter λ, we derive a positivity preserving property for λ in some interval. This is possible since the singularity of the Green function, respectively the function $H_{\Omega,m}$ in (1.3), is positive and the negative part can be canceled out.

So, we add a positive weight function w to problem (1.4) and consider the following Dirichlet problem for $2 \leq m \in \mathbb{N}$ in some smooth and bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} (-\Delta)^m u - \lambda w u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial \nu} u = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

As a special case we will show that for $m = 2$ and every bounded and sufficiently smooth domain Ω there is a Hölder continuous and positive weight function w such that positivity preserving holds for λ in some interval. This result can be found in [58] and is accepted for publication in *Pure and Applied Analysis*.

***Remark* 1.1.1** *The following biharmonic problem for the deviation of a thin plate is known, see for example [73, Chapters 3, 4]:*

$$\begin{cases} \Delta(s\Delta u) = f & \textit{in } \Omega, \\ u = \frac{\partial}{\partial \nu} u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^2$, s could be seen as the varying thickness assuming the thickness and the stiffness have a linear relation, u is the deviation of the plate and f a force density. We assume that the stiffness may depend on x but neglect the second and third order terms in (1.7) *and consider $s\Delta^2 u = f$ instead. If we set $s = w^{-1}$, we find*

$$\begin{cases} w^{-1}\Delta^2 u = f & \textit{in } \Omega, \\ u = \frac{\partial}{\partial \nu} u = 0 & \textit{on } \partial\Omega. \end{cases}$$

So, when asked about a physical meaning of the weighted problem, we would understand w as a measure of stiffness or thickness of the plate, even if we assume that the weight function w may depend on x.

If problem (1.1) *is not positivity preserving in Ω, one expects some negativity close to the boundary since this is the same phenomenon that appears for limaçons which are close to the cardioid or some ellipses, see Section 1.3. In order to maintain a positivity preserving property for* (1.6)*, one suspects that one has to consider plates which are stiff in a neighborhood of the boundary and rather flexible away from the boundary of Ω. Accordingly, we expect to find a weight function w that takes on larger values near the boundary compared to the interior.*

In the second part of this thesis, we will consider another $2m$-order Dirichlet problem. We show existence of classical solutions to some nonlinear problems. This is a longstanding problem that has already been considered by Tomi in 1976, von Wahl in 1978, Luckhaus in 1979 or Grunau in his dissertation in 1990. Instead of adding a term with a weight function to problem (1.4), we add a semilinear term and investigate

$$\begin{cases} (-\Delta)^m u + g(\cdot, u) = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu} u = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where f is Hölder continuous in $\overline{\Omega}$ and g is Hölder continuous in $\overline{\Omega} \times \mathbb{R}$ and satisfies the sign condition

$$g(x,t) \cdot t \geq 0 \text{ for all } x \in \Omega, t \in \mathbb{R}. \tag{1.9}$$

To this end, we will present the results of [59], accepted for publication in *Nonlinear Analysis*. The following summary of known results about classical solvability of (1.8) can also be found in [59, Introduction]:

If g is some monotone nonlinearity, then it is well known that there exists a distributional solution to (1.8), see [7], [9], [37]. We are interested in the following question: *Under which additional conditions on function g does problem* (1.8) *have a classical solution $u \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$?*

For $m = 1$ one finds a classical solution independently of the growth of the nonlinear term. Indeed, one just needs the maximum principle for second order linear elliptic problems, or more precisely a comparison principle, and the property that one may split problem (1.8) into two Dirichlet problems on $\Omega^+ := \{x \in \Omega, u(x) > 0\}$ and $\Omega^- := \{x \in \Omega, u(x) < 0\}$ to get a priori estimates for $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$. With $\|u\|_\infty$ bounded, one uses some iteration steps: first, known regularity results imply $u \in W^{2,p}(\Omega)$ for all $p \in (1, \infty)$ and then, using Sobolev imbeddings and regularity results again, $u \in C^{2,\gamma}(\overline{\Omega}) \cap C_0(\overline{\Omega})$ follows. For $m \geq 2$ there is no direct generalization of these properties, so some additional assumptions seem necessary.

In the literature there are results that include classical solvability of higher order problems. Tomi in [74] proved that with some additional monotonicity assumptions for g and $m = 2$, one finds a solution $u \in C^{4,\gamma}(\Omega) \cap W_0^{2,2}(\Omega)$ to (1.8). Using the growth condition $|g(\cdot, u)| \leq C(1 + |u|^q)$ with $1 \leq q \leq \frac{n+2m}{n-2m}$ for $n > 2m$, von Wahl [76] and Luckhaus [44] proved that there is a classical solution $u \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$. In [29] and [23] the growth condition for g was weakened. Applying [29, Theorem 1], one finds that with some growth condition for $g(x,t)$ with $t \leq 0$ and arbitrarily strong growth of $g(x,t)$ with $t \geq 0$, or vice versa, there is a solution $u \in C^{2m,\gamma}(\Omega) \cap W_0^{m,2}(\Omega)$. It is well known and can be proven using the Sobolev imbedding $W_0^{m,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ that for $n < 2m$ the sufficiently monotone function g may have an arbitrary power type growth, and one still finds a classical solution $u \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$. So, in the last part of this thesis, we assume that $n \geq 2m$ and using the same assumptions for the semilinear term g as in [29], we want to improve the result in [29] and [24] to find solutions which take on the boundary values in classical sense. To prove this result we use the Green function estimates described in (1.3). So, surprisingly the replacement of the maximum principle by the Green function estimates leads to an improved result for a nonlinear problem as well.

In the whole thesis, the known estimate from below and above for the polyharmonic Green function in (1.3), see also [26] and [53], will be an important argument. We use it not only for the proof of the existence of classical solutions to (1.8) but also for the proof of positivity preserving of the weighted problem in (1.6). So in this thesis, the Green function estimates will appear in many proofs and therefore represent a link between the individual topics for higher order problems.

The main theorems of this thesis are generalizations of the results in [58] or contained in [59] and can be found in the next section. After that, some examples and known positivity results for the clamped plate problem are presented. In Chapter 2 the preliminaries like the maximum principle, Krein-Rutman's theorem and Sobolev imbeddings, and the notation that will be used are presented. In Chapter 3 we prove the converse to the Krein-Rutman theorem that we mentioned above. This theorem provides sufficient conditions for the eigenfunctions and eigenvalues of the weighted boundary value problem so that (1.6) is positivity preserving for λ in some interval. In joint work with Guido Sweers, this result was first proven for $m = 2$ without a weight function and can be found in [57]. We will then construct a problem that satisfies these conditions, i.e. we find a weight function such that an eigenvalue of the weighted eigenvalue problem becomes simple and the corresponding eigenfunction is positive. In Chapter 4 we construct an appropriate weight function such that we gain a strongly positive eigenfunction. Since we want to apply the results of Chapter 3, we have to find a small perturbation of this weight function to obtain simplicity of the eigenvalue which corresponds to the positive eigenfunction. After that, in Chapter 5 we consider special cases like the weighted biharmonic Dirichlet problem or the polyharmonic problem on an ellipsoid. In Chapter 6 we investigate a semilinear Dirichlet problem of higher order. Using estimates for the Green function of the polyharmonic Dirichlet problem, regularity results and an approximation with bounded functions for the semilinear term g, we may find uniform bounds for weak solutions to the changed problem. Then, we can prove classical solvability of the original semilinear problem and expand known results proven by Grunau and Sweers in [29], where the authors apply local maximum principles instead of global estimates.

1.2 Main results

1.2.1 Positivity preserving property of a weighted Dirichlet problem

We will use the existence of a positive eigenfunction to prove positivity preserving of a weighted problem. More specifically, in Chapter 3 we consider problem (1.6) and the corresponding weighted eigenvalue problem

$$\begin{cases} (-\Delta)^m \varphi = \lambda w \varphi & \text{in } \Omega, \\ \varphi = \frac{\partial}{\partial\nu}\varphi = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

with $m \geq 2$ and show a converse to Krein-Rutman's result:

If there is a simple eigenvalue with strongly positive eigenfunction, that is $\varphi(x) \geq c\, d(x, \partial\Omega)^m$, to the weighted problem (1.10)*, then there exists an interval, such that for all λ in that interval* (1.6) *is positivity preserving.*

In [57] we have shown the result for $m = 2$ without a weight function, and in [58] we have presented an alternative proof that includes a weight function.

***Remark* 1.2.1** *We will note the eigenvalues and eigenfunctions of* (1.10) *as $\lambda_{i,m,w}$ and $\varphi_{i,m,w}$ for $i \in \mathbb{N}^+$, where the eigenvalues are counted with their multiplicity, that is $0 < \lambda_{1,m,w} \leq \lambda_{2,m,w} \leq \dots$. If we write $\lambda_{i,m,1}$ respectively $\varphi_{i,m,1}$, then we refer to the eigenvalues and eigenfunctions to* (1.5)*, that is the polyharmonic eigenvalue problem without a weight function.*

There are domains where there is no simple eigenvalue with positive eigenfunction to problem (1.5), or where it is difficult to prove that this holds true. Therefore, we derive conditions for Ω and a suitable positive weight function $w \in C^{0,\gamma}(\overline{\Omega})$ with $\gamma \in (0, 1)$, such that we find a simple eigenvalue with positive eigenfunction to the weighted eigenvalue problem.

Let $m \in \mathbb{N}$ with $m \geq 2$. Before we introduce sufficient conditions for a positivity preserving property, we give the following three definitions:

Definition 1.2.2 *We call a function $u \in C^{2m,\gamma}(\overline{\Omega})$ with $\gamma \in (0, 1)$ a m-polyharmonic Dirichlet supersolution if*

$$\begin{cases} (-\Delta)^m u \geq 0 & \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu} u = \dots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} u = 0 & \text{on } \partial\Omega. \end{cases}$$

Definition 1.2.3 *1. We call a function $u \in C^{k,\gamma}(\overline{\Omega})$ with $m \leq k \in \mathbb{N}$ and $\gamma \in (0, 1)$ strongly positive if there exists a constant $C_{SP} > 0$ such that*

$$u(x) \geq C_{SP}\, d(x, \partial\Omega)^m \text{ for all } x \in \Omega. \tag{1.11}$$

2. We call a function $u \in C^{0,\gamma}(\overline{\Omega})$ with $\gamma \in (0, 1)$ strictly positive if

$$\min_{x \in \overline{\Omega}} u(x) > 0.$$

***Remark* 1.2.4** *We say that a function $u \in W^{m,2}(\Omega)$ is strongly positive if there is a constant $C_{SP} > 0$ for which* (1.11) *is satisfied for almost every $x \in \Omega$.*

In the following we consider weak solutions in $W_0^{m,2}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{m,2}(\Omega)}}$ to problem (1.6).

Definition 1.2.5 *A function $u \in W_0^{m,2}(\Omega)$ is a weak solution to* (1.6) *if for all $v \in W_0^{m,2}(\Omega)$*

$$\begin{cases} \displaystyle\int_\Omega \left(\Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}} v - \lambda w u v - f v\right) dx = 0 & \text{for even } m \in \mathbb{N}^+, \\ \displaystyle\int_\Omega \left(\nabla \Delta^{\frac{m-1}{2}} u \cdot \nabla \Delta^{\frac{m-1}{2}} v - \lambda w u v - f v\right) dx = 0 & \text{for odd } m \in \mathbb{N}^+. \end{cases} \tag{1.12}$$

In the following chapters we assume throughout that Condition A and Condition B are satisfied:

Condition A *Suppose that* $\Omega \subset \mathbb{R}^n$ *(*$n \geq 2$*) is a bounded domain and such that* $\partial\Omega \in C^{2m,\gamma}$ *for some* $\gamma \in (0,1)$.

Condition B *Suppose that there is a function* $u_0 \in C^{2m,\gamma}(\overline{\Omega})$ *which is a strongly positive, m-polyharmonic Dirichlet supersolution and such that there is* $m_0 \in \mathbb{N}$ *with* $0 \leq m_0 \leq m$ *and a strictly positive function* $f_0 \in C^{0,\gamma}(\overline{\Omega})$ *such that*

$$(-\Delta)^m u_0(x) = d(x, \partial\Omega)^{m_0} f_0(x) \text{ for all } x \in \Omega. \tag{1.13}$$

We use Condition A to be able to apply standard results such as regularity results from Agmon, Douglis and Nirenberg. However, Condition B is a restriction since it is not known whether it is satisfied for all smooth domains and all $m \in \mathbb{N}$ with $m > 2$.

***Remark* 1.2.6** *Since the distance function* $d(\cdot, \partial\Omega)$ *is Lipschitz-continuous on* $\overline{\Omega}$, *we find that* $d(\cdot, \partial\Omega)^{m_0} f_0 \in C^{0,\gamma}(\overline{\Omega})$.

***Remark* 1.2.7** *For smooth domains and* $m = 2$ *one possibility to find a function* u_0 *that satisfies Condition B is to consider a suitable Dirichlet problem for the Poisson equation. Indeed, Condition B is satisfied if we find a function* $\mathbf{e} \in C^{4,\gamma}(\overline{\Omega})\backslash\{0\}$ *with*

$$\begin{cases} -\Delta \mathbf{e} \geq 0 & \text{in } \Omega, \\ \mathbf{e} = 0 & \text{on } \Omega, \end{cases} \tag{1.14}$$

such that $\mathbf{e}^2$ *is a positive biharmonic Dirichlet supersolution in* $C^{4,\gamma}(\overline{\Omega})$ *with* $(-\Delta)^2\mathbf{e}^2$ *strictly positive. Then using the maximum principle for the Laplacian, it follows that* $\mathbf{e} > 0$ *in* Ω, *and with Hopf's boundary point lemma [22, Section 3.2] and the mean value theorem we obtain constants* $c_1, c_2 > 0$ *such that*

$$c_1 d(x, \partial\Omega) \leq \mathbf{e}(x) \leq c_2 d(x, \partial\Omega) \text{ for all } x \in \Omega, \tag{1.15}$$

so $\mathbf{e}^2(x) \geq c_1^2 d(x, \partial\Omega)^2$. *We will use this result in Section 5.1 to show a positivity preserving property for a weighted Dirichlet-bilaplace problem. For* $m = 2$ *one may use* $-\Delta\mathbf{e} = 1$ *in* (1.14) *and one finds the desired result. For* $m > 2$ *we do not necessarily get* $(-\Delta)^m \mathbf{e}^m \geq 0$ *for all smooth domains, see Remark 5.1.1.*

The following theorems can be found in [58, Theorem 2, Corollary 4] for $m = 2$. In the case $m = 2$ there is always a function on smooth and bounded domains that satisfies (1.13), as we will see in Chapter 5. Therefore, the following version differs from [58] by the additional assumption in Condition B. The proof can be found in Chapter 4.

Theorem 1.2.8 *Suppose that* Ω *satisfies Condition A. Moreover, let Condition B be fulfilled. Then, there exists a strictly positive weight function* $w \in C^{0,\gamma}(\overline{\Omega})$ *such that the eigenvalue problem (1.10) has the simple eigenvalue* $\lambda_{\mathrm{p},m,w} = 1$ *with a strongly positive eigenfunction* $\varphi_{\mathrm{p},m,w} \in C^{2m,\gamma}\left(\overline{\Omega}\right) \cap C_0^{m-1}(\overline{\Omega})$.

***Remark* 1.2.9** *The simple eigenvalue with strongly positive eigenfunctions does not have to be the first one. In [14], Duffin and coauthors showed that for an annulus with small inner radius the simple eigenvalue with positive eigenfunction will be the third one. Therefore, we assume that it is the* p*-th eigenvalue.*

Using Theorem 1.2.8 and the results in Chapter 3 for a converse of the Krein-Rutman theorem, we find a positivity preserving property:

Theorem 1.2.10 (PPP) *Suppose that Conditions A and B are fulfilled. Let* w *and* $\lambda_{\mathrm{p},m,w} = 1$ *be as in Theorem 1.2.8. Then there is* $\lambda_c < \lambda_{\mathrm{p},m,w}$ *such that for* $0 \leq f \in L^2(\Omega)$ *with* f *nontrivial and* u *the weak solution to (1.6):*

1. *If* $\lambda \in [\lambda_c, \lambda_{\mathrm{p},m,w})$*, then* $u > 0$ *in* Ω.
2. *If* $\lambda \in (\lambda_c, \lambda_{\mathrm{p},m,w})$*, then a Hopf type result holds: There exists* $c_{f,\lambda} > 0$ *such that*
$$u(x) \geq c_{f,\lambda}\, d(x, \partial\Omega)^m \text{ for almost every } x \in \Omega.$$

Moreover, if $\lambda_{\mathrm{p},m,w}$ *is not the first eigenvalue of* (1.10)*, then it holds*

$$\lambda_c \geq \lambda_{\mathrm{p}-1,m,w} + \frac{\lambda_{\mathrm{p},m,w} - \lambda_{\mathrm{p}-1,m,w}}{2}. \tag{1.16}$$

***Remark* 1.2.11** *For the unweighted second-order problem, i.e.* $m = 1$ *and* $w \equiv 1$*, one gets positivity preserving for all* $\lambda \in (-\infty, \lambda_{1,1,1})$*. For higher order problems* ($m \geq 2$) *there is a lower bound for* λ_c*, since for* $\lambda \ll 0$ *problem* (1.4) *and* (1.6) *are not positivity preserving, see [30, Theorem 6.1, Lemma 6.3].*

In one dimension with $\Omega = (0, 1)$*, it is known that the fourth order problem*

$$\begin{cases} u'''' - \lambda u = f \text{ in } (0,1), \\ u(0) = u'(0) = 0, \\ u(1) = u'(1) = 0 \end{cases}$$

is positivity preserving if $\lambda \in [\lambda_c, \lambda_{1,2,1})$ *with* $\lambda_{1,2,1}$ *the principle eigenvalue to the biharmonic Dirichlet problem and*

- $\lambda_{1,2,1} = (2\mu_1)^4$ *with* μ_1 *the first positive solution of* $\tan(\mu) + \tanh(\mu) = 0$*;*
- $\lambda_c = -4\mu_c^4$ *with* μ_c *the first positive solution of* $\tan(\mu) = \tanh(\mu)$.

This result can be found in [68, Lemma 2.3] and [75, Theorem 1.2].

One notices in Chapter 3 that using similar arguments as in the proof of Theorem 1.2.10, one finds a result for λ in a right neighborhood of a simple eigenvalue with strongly positive eigenfunction. For sufficiently smooth right-hand side f one can show a reverse result for the sign of the solution to problem (1.6). The result that a right-hand side $f \gneqq 0$ implies $u < 0$ is called *anti-maximum principle* (AMP) and can be found in the next theorem.

There are some results in the literature about anti-maximum principles for problem (1.4), see [10], [11], [12] and [33]. Indeed, if we apply [10] to the second order problem (1.4), i.e. $m = 1$, one finds for right-hand sides $0 \leq f \in L^q(\Omega)$ with $q > n$ a value $\delta_f > 0$ such that the solution u to the boundary value problem is negative for $\lambda \in (\lambda_{1,1,1}, \lambda_{1,1,1} + \delta_f)$. The results in [12] and [33] imply for $m = 1$ and Ω arbitrary but smooth or $m \geq 2$ with Ω a ball: For $0 \leq f \in L^q(\Omega)$ with $q > \max\left\{1, \frac{n}{m}\right\}$ there exists a small right neighborhood of the first eigenvalue, such that for λ in this neighborhood, the solution to (1.4) is negative. In [12] and [33] only $\Omega = B_R(0)$ is investigated for $m \geq 2$ since the existence of a simple first eigenvalue with corresponding positive eigenfunction is used. As in this thesis, the authors in [33] also make use of estimates for the Green function. So, if we assume that the weighted problem in (1.6) has a simple eigenvalue with positive eigenfunction, then we find a supplement of the known results using similar arguments.

Theorem 1.2.12 (AMP) *Suppose that Conditions A and B are fulfilled. Let w and $\lambda_{\mathrm{p},m,w} = 1$ be as in Theorem 1.2.8. Moreover, let $0 \leq f \in L^q(\Omega)$ with f nontrivial and $q > \max\{1, \frac{n}{m}\}$. Then, there exists $\delta_f > 0$ such that for all $\lambda \in (\lambda_{\mathrm{p},m,w}, \lambda_{\mathrm{p},m,w} + \delta_f)$ the following holds: There is a constant $c_{f,\lambda,q} > 0$ such that the solution $u_{m,\lambda,w} \in W^{2m,q}(\Omega) \cap W_0^{m,q}(\Omega)$ of (1.6) satisfies*

$$u_{m,\lambda,w}(x) \leq -c_{f,\lambda,q} d(x, \partial\Omega)^m \quad \textit{for all } x \in \Omega.$$

***Remark* 1.2.13** *We notice that for $f \in L^q(\Omega)$ with $q > \max\{1, \frac{n}{m}\}$ and $\lambda \in (\lambda_{\mathrm{p},m,w}, \lambda_{\mathrm{p},m,w} + \delta_f)$, the weak solution of (1.6) is an element of $C^m(\overline{\Omega})$. So, the solution takes on the boundary conditions in classical sense. Moreover we note that δ_f depends on the right-hand side f, and we do not get a uniform result as in Theorem 1.2.10.*

***Remark* 1.2.14** *For $n > m$ it is shown in [66] for $m = 1$ and in [33] for $m \geq 1$ that the condition $q > \frac{n}{m}$ in Theorem 1.2.12 is sharp. It is proven that for $q = \frac{n}{m}$ with $\Omega = B_R(0)$ or $m = 1$ one finds a function $0 < f \in L^q(\Omega)$ such that for all $\lambda > \lambda_{1,m,1}$ the solution to problem (1.4) changes sign.*

Fortunately, using the results in Theorem 1.2.10 and 1.2.12, we may find informations about positivity of solutions to some higher order problems. However, an open problem is whether Condition B is satisfied for all smooth domains and all $m \in \mathbb{N}$ with $m > 2$.

1.2.2 Classical solutions to some semilinear Dirichlet problems

For second order problems, such as the Poisson-Dirichlet problem, the positivity preserving property follows directly from the maximum principle. Moreover, the existence of a classical solution to (1.8) with $m = 1$ can be shown using the maximum principle and other properties that are important features of second order problems. For higher order problems these results cannot be used.

An interesting and open problem is whether there is a classical solution to problem (1.8) for any Hölder continuous function g that satisfies the sign condition (1.9). Unforunately, we cannot answer this general question, but we can improve some results proven by Grunau respectively Grunau and Sweers, see [24], [23] and [29]. They showed that with additional growth conditions for g there exists a solution $u \in C^{2m,\gamma}(\Omega) \cap W_0^{m,2}(\Omega)$ for problem (1.8).

The following theorem and remark about classical solvability of (1.8) are contained in [59].

Theorem 1.2.15 *Let $n \geq 2m$, Condition A be satisfied, $f \in C^{0,\gamma}(\overline{\Omega})$ and $g \in C^{0,\gamma}(\overline{\Omega} \times \mathbb{R})$ satisfies (1.9) and one of the following growth conditions:*

- *$n \in [2m, 6m)$ and $\sigma \in [0, \infty)$ exists with $(n-2m)\sigma < 4m$ such that for some $c_1 \in \mathbb{R}^+$, it holds that*

$$g(x,t) \leq c_1(1+t^\sigma) \text{ for all } x \in \Omega, t > 0; \tag{1.17}$$

- *or $n \geq 6m$ and for some constant $c_1 \in \mathbb{R}^+$ it holds that*

$$g(x,t) \leq c_1(1+t) \text{ for all } x \in \Omega, t > 0; \tag{1.18}$$

- *or $n > 2m$ and $\sigma, \tau \in [0, \infty)$ exist with $\tau \geq \frac{n+2m}{n-2m}$ and $\sigma < \frac{4m}{n-2m} + \frac{1}{\tau}\frac{n+2m}{n-2m}$ such that for some $c_1, c_2 \in \mathbb{R}^+$ it holds*

$$-c_2(1+|t|^\tau) \leq g(x,t) \leq c_1(1+|t|^\sigma), \text{ for all } x \in \Omega, t \in \mathbb{R}.$$

Then, the semilinear Dirichlet problem in (1.8) has a classical solution $u \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$.

In Theorem 1.2.15, the value of σ determines the growth condition from above and τ determines the growth from below. However, they are interchangeable. So, instead of a growth condition from above in (1.17) and (1.18), we could have restricted the growth of g from below. The permissible growth conditions are displayed in Figure 1.1.

***Remark* 1.2.16** *To prove the main result, we use regularity estimates that follow from known estimates for the Green operator of the polyharmonic Dirichlet problem, that is problem (1.4) with $\lambda = 0$, approximation of the nonlinear term g with bounded functions and Sobolev imbeddings. However, these results may also be applied to a more general differential operator than $(-\Delta)^m$ with additional lower order terms, see also [53, Theorem 0.1]. For the Green function estimates the leading term has to be $(-\Delta)^m$. So, we may consider the following problem and find a similar result:*

$$\begin{cases} (-\Delta)^m u(x) + \sum\limits_{\ell=0}^{m-1} \sum\limits_{|\alpha|=|\beta|=\ell} D^\beta \left(a^\ell_{\alpha,\beta}(x) D^\alpha u(x)\right) + g(x,u(x)) = f(x) & \text{for } x \in \Omega, \\ u(x) = \frac{\partial}{\partial\nu} u(x) = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{1.19}$$

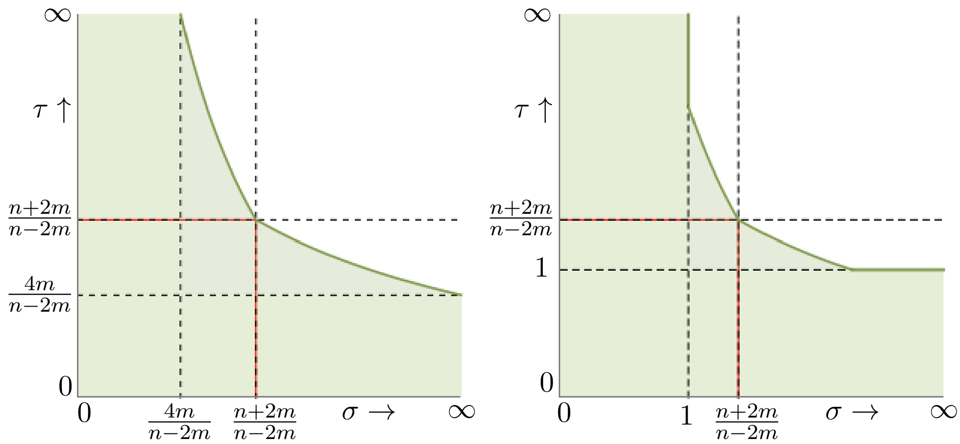


Figure 1.1: Range of admissable growth rates for some $n \in (2m, 6m]$ (left) and some $n > 6m$ (right); a similar picture, that was created by Guido Sweers, appears in [59].

where $a^\ell_{\alpha,\beta}$ *are sufficiently smooth, for example* $a^\ell_{\alpha,\beta} \in C^{m-1,\gamma}(\overline{\Omega})$ *and symmetric, that is* $a^\ell_{\alpha,\beta} = a^\ell_{\beta,\alpha}$. *Moreover, we assume that there is a constant* $K > 0$ *such that*

$$\|a^\ell_{\alpha,\beta}\|_{C^{m-1,\gamma}(\overline{\Omega})} \leq K$$

and that the differential operator is coercive, that is

$$\int_\Omega \left((-\Delta)^m u(x) + \sum_{\ell=0}^{m-1} \sum_{|\alpha|=|\beta|=\ell} D^\beta \left(a^\ell_{\alpha,\beta}(x) D^\alpha u(x) \right) \right) u(x) dx \geq C \|u\|^2_{W^{m,2}(\Omega)}$$

for all $u \in C^{2m}(\overline{\Omega}) \cap W^{m,2}_0(\Omega)$.

The proof of this theorem can be found in Chapter 6 and in Sections 4 and 5 of [59].

1.3 Some examples for the clamped plate problem

In this section we recall some known counterexamples for positivity preserving of the clamped plate problem, that is problem (1.1). Some paragraphs of this summary can also be found in [57, Section 3], and the content serves as a motivation why we have to look at a changed fourth-order problem or more explicitly, why we have to add a weight function to problem (1.4) to get a positivity preserving property.

There are several ways to prove that positivity preserving does not hold. One can show that the first eigenfunction is sign-changing, see also [67] for some examples. Then it follows by the Krein-Rutman theorem that the problem is not positivity preserving. Another way is to show that the Green function is not positive, or to construct an explicit positive right-hand side with sign-changing solution.

Annulus The annulus with inner radius $\varepsilon > 0$ is defined as follows:

$$A_\varepsilon := \{x \in \mathbb{R}^2; \varepsilon < |x| < 1\}.$$

Around 1907, Hadamard considered problem (1.1) on $\Omega = A_\varepsilon$, see [36]. He mentioned that positivity preserving cannot be true for the annulus, but he did not provide a detailed proof. Nakai and Sario proved in [45] that the Green function for the clamped plate problem is sign-changing for small inner radii. Moreover,

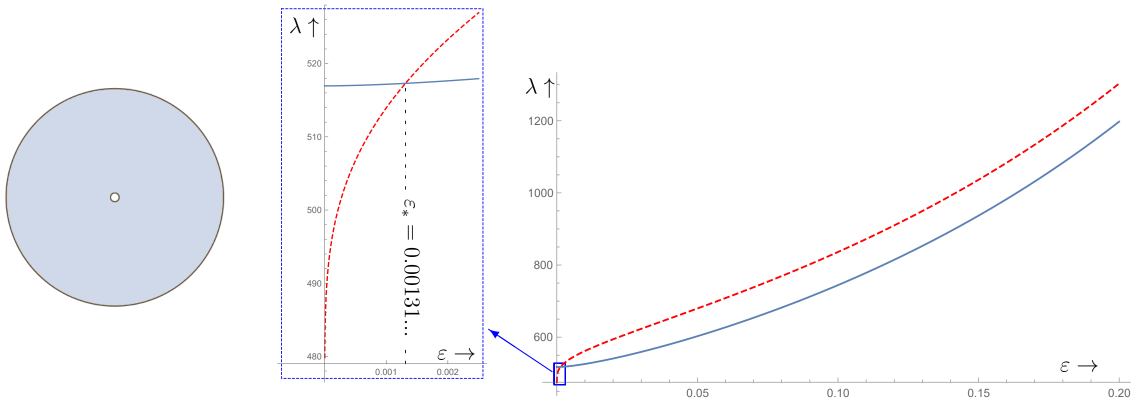


Figure 1.2: Graphs of the first eigenvalues for the annulus $A_\varepsilon = \left\{x \in \mathbb{R}^2; \varepsilon < |x| < 1\right\}$ as a function of ε. In blue $\varepsilon \mapsto \lambda_{\mathrm{p},2,1}$ and in dashed red $\varepsilon \mapsto \min\{\lambda \text{ is eigenvalue of (1.10)}; \lambda \neq \lambda_{\mathrm{p},2,1}\}$. At ε_* the eigenvalues cross. The section around the crossing is enlarged in the rectangle (first published in [57] and created by Guido Sweers).

Coffman, Duffin and Shaffer [14, 18] showed that for small inner radii the positive eigenfunction is not the first one. Numerically, they found the value $\varepsilon^* = 0.00131...$ such that for $\varepsilon > \varepsilon^*$ the positive eigenfunction corresponds to the first eigenvalue. For $\varepsilon < \varepsilon^*$ it corresponds to the third one. An explicit computation of the eigenvalues can be found in [71]. Moreover, Engliš and Peetre [19] proved in 1996 that the Green function for A_ε is sign-changing, even if for large inner radii the first eigenfunction is positive and the corresponding eigenvalue is simple. Hence, a positivity preserving property does not hold true for problem (1.1). Using the information that one finds an eigenvalue with positive eigenfunction, the question arises whether one may prove a positivity preserving property for a weighted problem.

Let $\lambda_{1,2,1}$ be the smallest eigenvalue to the eigenvalue problem (1.5) for $\Omega = A_\varepsilon$, $\lambda_{2,2,1}$ the second and $\lambda_{3,2,1}$ the third one. We will see that there exists $\lambda_\varepsilon > 0$ with

- $\lambda_\varepsilon \in (0, \lambda_{1,2,1})$ such that (1.4) is positivity preserving for $\lambda \in [\lambda_\varepsilon, \lambda_{1,2,1})$ and $\varepsilon \in (\varepsilon^*, 1)$,
- $\lambda_\varepsilon \in (\lambda_{1,2,1}, \lambda_{3,2,1}) = (\lambda_{2,2,1}, \lambda_{3,2,1})$ such that (1.4) is positivity preserving for $\lambda \in [\lambda_\varepsilon, \lambda_{3,2,1})$ and $\varepsilon \in (0, \varepsilon^*)$,
- $\lambda_\varepsilon \in (0, 1)$ such that (1.6) is positivity preserving for $\lambda \in [\lambda_\varepsilon, 1)$, $\varepsilon = \varepsilon^*$ and a sufficiently chosen positive weight function.

For the three dimensional variant of the annulus, the spherical shell, numerical approximations of the first eigenfunctions and the assumption that the second one can be written as $x \mapsto x_1\, \varphi(|x|)$ show that the first one is positive for all inner radii. So in that case, an additional weight function is not needed, see [57, Appendix].

Limaçon of Pascal One may identify $\mathbb{R}^2$ with $\mathbb{C}$ and define the set

$$\Omega_a = f_a(B_1(0)) \text{ with } f_a : \mathbb{C} \to \mathbb{C}, f_a(z) = z + az^2$$

for $a \in \left[0, \frac{1}{2}\right]$, where Ω_0 is the unit ball and $\Omega_{\frac{1}{2}}$ is a cardioid, see Figure 1.3. Hadamard constructed an explicit Green function for the Limaçon of Pascal G_a [35, Supplement]. However, he conjectured that it is positive for all limaçons. Dall'Acqua and Sweers proved in [15] that this conjecture is not true. They showed that $G_a(x, y) \geq 0$ for all $(x, y) \in \Omega_a \times \Omega_a$ if and only if $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$, that is if G_a is not far from a ball. In addition, no eigenvalues or eigenfunctions are known. But we will still find that Condition B is satisfied, and thus a suitable fourth order Dirichlet problem with a positivity preserving property in Ω_a can be found for all $a \in \left[0, \frac{1}{2}\right)$. The case where $a = \frac{1}{2}$ has to be excluded since $\Omega_{\frac{1}{2}}$ does not fulfill Condition A.

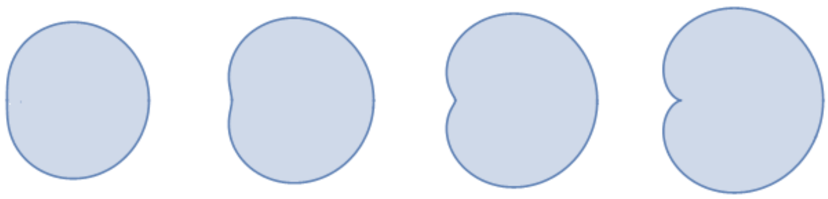


Figure 1.3: Limaçons for $a = \frac{1}{4}, \frac{1}{3}, \frac{\sqrt{6}}{6}, \frac{1}{2}$.

Ellipse For the bilaplace and trilaplace Dirichlet problem one can consider some eccentric ellipse, see [69, 70]

$$\Omega = \{(x, y) \in \Omega; x^2 + 144y^2 < 1\} \tag{1.20}$$

and find with

$$u(x, y) = (1 - x^2 - 144y^2)^m (1 - x + 200(1 - x)^2 - 21y^2 - \varepsilon) \tag{1.21}$$

and small $\varepsilon > 0$ a sign-changing solution to (1.4) with $\lambda = 0$ and $(-\Delta)^m u \geq 0$ for $m \in \{2, 3\}$, see Figure 1.3 for $m = 3$ and $\varepsilon = 0.0001$. So even if we investigate convex and smooth domains, we do not obtain a positivity preserving property. In Chapter 5 we prove that we can find a weight function $w \in C^{0,\gamma}(\overline{\Omega})$ for any ellipsoid, even in higher dimensions, and any $m \in \mathbb{N}^+$ such that positivity is preserved for problem (1.6).

There are more examples for problem (1.1) such that positivity is not preserved. It is also surprising that it is not known which conditions the domain or the differential operator have to fulfill such that a positivity preserving property for (1.1) on

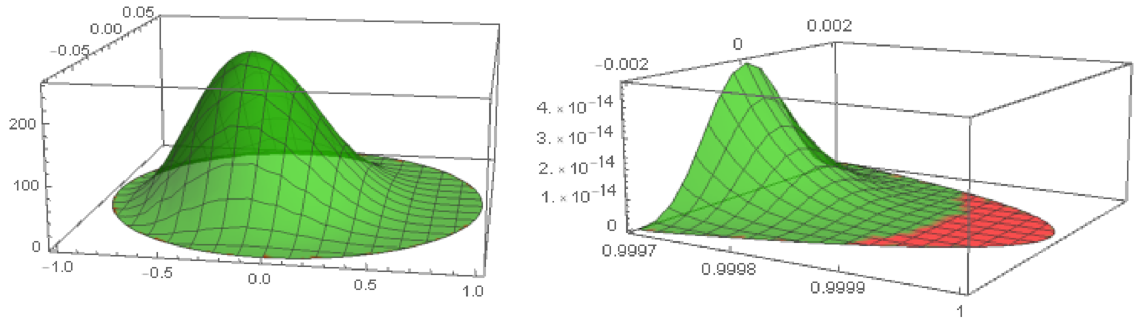


Figure 1.4: Left: u as defined in (1.21) with $m = 3$ and $\varepsilon = 0.0001$; right: enlarged graph of u for (x, y) in a neighborhood of $(1, 0)$ with positive values of u in green and negative values in red. A similar picture can be found in [69].

Ω is valid. For this reason, it is interesting to investigate how we can change the problem to be able to make more precise statements.

Chapter 2

Preliminaries

In this chapter we present some notations and recall important results such as the existence of a Green function for problem (1.6), the maximum principle, Hopf's boundary point lemma or Sobolev imbeddings. We rely on these results throughout the following chapters. In addition, the content serves for a better understanding of the thesis and as background information.

2.1 Basic notations

In this section we list some notations that are used throughout the thesis. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers including 0 and $\mathbb{N}^+ = \mathbb{N}\backslash\{0\}$.

By $\Omega \subset \mathbb{R}^n$ we denote a bounded domain as mentioned in Condition A. A set in $\mathbb{R}^n$ is a domain whenever it is open and connected. For short notation we use

$$d(x) := d(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x - y|$$

for the distance to the boundary and

$$\operatorname{diam}(\Omega) := \sup_{x,y\in\Omega} |x - y|$$

for the diameter of Ω.

The space $C^{k,\gamma}(\overline{\Omega})$ is the space of all k-th times continuously differentiable functions such that all k-th partial derivatives are Hölder continuous with Hölder exponent $\gamma \in (0, 1)$. The space $(C^{k,\gamma}(\overline{\Omega}), \|\cdot\|_{C^{k,\gamma}(\overline{\Omega})})$ is a Banach space, where

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} = \sum_{|\alpha|\le k} \sup_{x\in\overline{\Omega}} |D^\alpha u(x)| + \sum_{|\alpha|=k} \sup_{x\neq y\in\overline{\Omega}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\gamma}.$$

When we write $C_0^k(\overline{\Omega})$, we mean all functions $u \in C^k(\overline{\Omega})$ such that $D^\alpha u = 0$ on $\partial\Omega$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \in \{0, \dots, k\}$. One finds that $(C_0^k(\overline{\Omega}), \|\cdot\|_{C^k(\overline{\Omega})})$ is a Banach space since it is a closed subspace of $(C^k(\overline{\Omega}), \|\cdot\|_{C^k(\overline{\Omega})})$.

The space $L^p(\Omega)$ for $p \ge 1$ denotes the space of measurable functions such that the p-th power of the absolute value is Lebesgue integrable. Functions which agree almost everywhere are identified. By $W^{k,p}(\Omega)$ we denote the Sobolev space of functions in $L^p(\Omega)$ such that all weak derivatives up to order k exist and are elements of $L^p(\Omega)$. The space $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$ is a Banach space with norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

Also, we will use the Sobolev space $W_0^{k,p}(\Omega)$ which is defined as the closure of $C_c^\infty(\Omega)$, that is the space of all smooth functions with compact support, in $W^{k,p}(\Omega)$. For functions $u \in C^k(\overline{\Omega})$ or $u \in W^{k,p}(\Omega)$ with $k \in \mathbb{N}^+$ we write for (weak) derivatives of u

$$D^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u \quad \text{for } \alpha \in \mathbb{N}^n \text{ with } |\alpha| \le k$$

and

$$D^l u = \left\{ \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_l}} u \right\}_{i_1,i_2,\dots,i_l \in \{1,\dots,n\}} \quad \text{for } l \in \mathbb{N}^+ \text{ with } l \le k.$$

A special case is the gradient $\nabla u := (\frac{\partial}{\partial x_1}u, \dots, \frac{\partial}{\partial x_n}u)^\top = D^1 u$ and the Laplace operator $\Delta u := \left(\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_n}\right)^2 \right) u = \sum_{i=1}^n D^{2e_i} u$, where e_i are the standard unit vectors in $\mathbb{R}^n$.

Moreover, for normed vector spaces X, Y we write $BL(X, Y)$ for the set of all linear and bounded operators from X into Y. The space $(BL(X, Y), \|\cdot\|_{BL(X,Y)})$ is a normed space where $\|\cdot\|_{BL(X,Y)}$ is defined by

$$\|T\|_{BL(X,Y)} = \sup \{ \|Tv\|_Y ; v \in X \text{ with } \|v\|_X \le 1 \}.$$

Also, we use $BL(X) := BL(X, X)$ for short notation, and $X^* := BL(X, (\mathbb{R}, |\cdot|))$ is the dual space of X.

In the following chapters we will use estimates for operators and for kernels of integral operators. Therefore, we use the following notation, see also [58, Notation 12]: If an operator is defined through a kernel function, we use capital letters for the kernel function and script letters for the integral operator, unless otherwise stated. For example, let $\mathcal{A} : L^2(\Omega) \to L^2(\Omega)$ be an integral operator defined through a kernel function, that is

$$(\mathcal{A}f)(x) = \int_\Omega A(x, y) f(y) dy. \tag{2.1}$$

For $\mathcal{A}, \mathcal{B} : L^2(\Omega) \to L^2(\Omega)$ we define $\mathcal{A} \ge \mathcal{B}$ whenever for all $f \in L^2(\Omega)$ with $f \ge 0$

almost everywhere it holds that

$$(\mathcal{A}f)(x) \geq (\mathcal{B}f)(x) \text{ for almost every } x \in \Omega.$$

Obviously, if $\mathcal{A}, \mathcal{B}$ are defined through kernels $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ as in (2.1), and $A(x,y) \geq B(x,y)$ holds for all $x, y \in \overline{\Omega}$, then one also gets $\mathcal{A} \geq \mathcal{B}$. In this thesis, we only consider kernels that are continuous or continuous except for singularities on the diagonal $\{(x,x); x \in \overline{\Omega}\}$. Therefore, we may also use that $\mathcal{A} \geq \mathcal{B}$ implies $A(x,y) \geq B(x,y)$ for all $x, y \in \overline{\Omega}$ with $x \neq y$.

2.2 Green function for the polyharmonic Dirichlet problem

In this section we recall some definitions of the polyharmonic Green function and the Green function for the weighted problem.

One uses the fundamental solution for the polyharmonic operator $(-\Delta)^m$, [21, p. 48]:

$$F_{n,m}(x) = \begin{cases} \frac{2\Gamma(n/2-m)}{nb_n 4^m \Gamma(n/2)(m-1)!}|x|^{2m-n} & \text{if } n > 2m \text{ or } n \text{ is odd}, \\ \frac{(-1)^{m-n/2}}{nb_n 4^{m-1}\Gamma(n/2)(m-n/2)!(m-1)!}|x|^{2m-n}(-\log(|x|)) & \text{if } n \leq 2m \text{ is even}, \end{cases}$$

where

$$b_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \tag{2.2}$$

is the volume of the n-dimensional unit ball. For bounded and smooth domains $\Omega \subset \mathbb{R}^n$ and f in a suitable functional space one finds the solution to (1.4) with $\lambda = 0$ through a Green function. Therefore, we recall the following definition [21, Definition 2.26]:

Definition 2.2.1 *A Green function for the polyharmonic Dirichlet problem in* (1.4) *with* $\lambda = 0$ *is a function* $(x,y) \mapsto G_{m,0,1}(x,y) : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R} \cup \{\infty\}$ *such that*

1. $x \mapsto G_{m,0,1}(x,y) - F_{m,n}(x-y) \in C^{2m}(\Omega) \cap C^{m-1}(\overline{\Omega})$ *for all* $y \in \Omega$ *if defined suitably for* $x = y$,
2. $(-\Delta_x)^m(G_{m,0,1}(x,y) - F_{m,n}(x-y)) = 0$ *for all* $(x,y) \in \Omega^2$ *if defined suitably for* $x = y$,
3. $D_x^\alpha G_{m,0,1}(x,y) = 0$ *for all* $(x,y) \in \partial\Omega \times \Omega$ *and* $|\alpha| \leq m-1$.

The weak solution to (1.4) with $\lambda = 0$ and $f \in L^2(\Omega)$ can then be written as

$$u(x) = \int_\Omega G_{m,0,1}(x,y)f(y)dy.$$

***Remark* 2.2.2** *As mentioned in the introduction, the polyharmonic Green function for the ball $\Omega = B_1(0)$ is known, see [21, Lemma 2.27]:*

$$G_{m,0,1}(x,y) = \ell_{n,m}|x-y|^{2m-n} \int_1^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} (s^2-1)^{m-1} s^{1-n} ds,$$

where the positive constant $\ell_{n,m}$ is defined by

$$\ell_{m,n} = \frac{1}{nb_n 4^{m-1}((m-1)!)^2}.$$

This function is positive, so the polyharmonic problem in (1.4) *is positivity preserving.*

***Remark* 2.2.3** *The positivity of the fundamental solution plays an important role in the entire thesis. If the leading order part of the differential operator is not a product of second order operators, then it is possible that the associated fundamental solution changes sign, see [27]. Therefore, there exists no direct generalization of the results in this thesis to any 2m-order Dirichlet problem.*

Let $\lambda_{1,m,w}$ be the first eigenvalue of (1.10). In his PhD Thesis, Pulst showed the existence and described the construction of a Green function for (1.6) with $\lambda < \lambda_{1,m,w}$ and strictly positive Hölder continuous weight function w which has the following properties [53, Proposition 2.1]:

1. $x \mapsto G_{m,\lambda,w}(x,y) \in L^1(\Omega) \cap C^{2m,\gamma}(\overline{\Omega}\backslash\{y\})$;
2. $\left(\frac{\partial}{\partial\nu}\right)_x^j G_{m,\lambda,w}(x,y)_{|\partial\Omega} = 0$ for $j = 0,\dots,m-1$;
3. $G_{m,\lambda,w}(x,y) = G_{m,\lambda,w}(y,x)$ for $x \neq y$;
4. For all $\varphi \in C^{2m}(\overline{\Omega})$ with $\left(\frac{\partial}{\partial\nu}\right)^j \varphi_{|\partial\Omega} = 0$ for $j = 0,\dots,m-1$ one has the representation formula

$$\begin{aligned}\varphi(x) &= \int_\Omega ((-\Delta)^m\varphi(y) - \lambda w(y)\varphi(y))\, G_{m,\lambda,w}(x,y)dy \\ &=: \left(\tilde{\mathcal{G}}_{m,\lambda,w}\left((-\Delta)^m\varphi - \lambda w\varphi\right)\right)(x). \end{aligned} \tag{2.3}$$

Hence, for $\lambda < \lambda_{1,m,w}$ we find that for $f \in C^{0,\gamma}(\overline{\Omega})$ there is a pointwise defined kernel function, and the solution $u_{m,\lambda,w}$ to (1.6) is well-defined through

$$u_{m,\lambda,w}(x) = \left(\tilde{\mathcal{G}}_{m,\lambda,w} f\right)(x) = \int_\Omega G_{m,\lambda,w}(x,y)f(y)dy. \tag{2.4}$$

The operator $\tilde{\mathcal{G}}_{m,\lambda,w}$ can be extended on $f \in L^2(\Omega)$, and one notices that the integral operator in (2.4) is well-defined for all $f \in L^2(\Omega)$. For $\lambda > \lambda_{1,m,w}$ not a weighted eigenvalue, similar arguments as in [21, Section 4.4] and [53, Section 2.2] yield a pointwise defined Green function for problem (1.6).

In this thesis we will use the following slightly modified definition of the Green operator:

Definition 2.2.4 *For $w \in C^{0,\gamma}(\overline{\Omega})$ and strictly positive, we use the notation*

$$f_w = \tfrac{f}{w}. \tag{2.5}$$

Let $G_{m,\lambda,w}$ denote the Green function and $\mathcal{G}_{m,\lambda,w}$ the Green operator for

$$\begin{cases} ((-\Delta)^m - \lambda w)u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu}u = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

in the sense that

$$u_{m,\lambda,w}(x) = \int_\Omega G_{m,\lambda,w}(x,y)f(y)dy = (\mathcal{G}_{m,\lambda,w}f_w)(x) \tag{2.7}$$

solves (2.6) *if defined. By $G_{m,\lambda,1}$ we mean the Green function for* (2.6) *without a weight function, i.e. $w \equiv 1$.*

***Remark* 2.2.5** *We notice that for $\lambda = 0$ we find $\mathcal{G}_{m,0,w} = \mathcal{G}_{m,0,1}(w\cdot)$, where $\mathcal{G}_{m,0,1}$ is the polyharmonic Green operator. Also, this definition corresponds to the definition of Pulst in* (2.3) *if $w \equiv 1$. If $w \not\equiv 1$, then $\mathcal{G}_{m,\lambda,w}f = \tilde{\mathcal{G}}_{m,\lambda,w}(wf)$ for all $f \in L^2(\Omega)$. In the following we will derive estimates for the Green function $G_{m,\lambda,w}$ using estimates for the corresponding Green operator $\mathcal{G}_{m,\lambda,w}$. Since the functions f and f_w differ only by the additional weight function, which is positive and bounded, the estimates for $\mathcal{G}_{m,\lambda,w}$ that we prove in the next chapter can be transferred to the Green function $G_{m,\lambda,w}$.*

***Remark* 2.2.6** *The reason why we apply $\mathcal{G}_{m,\lambda,w}$ to f_w instead of f is that in Section 2.6, we draw conclusions about the eigenvalues and eigenfunctions of the operator $\frac{1}{w}(-\Delta)^m$ and derive these results with standard arguments. Actually, the operator $\mathcal{G}_{m,\lambda,w}$ is the solution operator to problem*

$$\begin{cases} \frac{1}{w}\left(-\Delta\right)^m u - \lambda u = f_w & \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu}u = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} u = 0 & \text{on } \partial\Omega. \end{cases}$$

Problem (1.6) is positivity preserving if and only if the corresponding Green function is nonnegative. Hence, estimates for the Green function and Green operator play a major role in the proof of Theorem 1.2.10 and Theorem 1.2.12. Using some estimates for the polyharmonic Green function which can be found in [21, Theorem 4.6] for the ball and in [53, Theorem 4.1] for general smooth domains, we obtain estimates for $\mathcal{G}_{m,\lambda,w}$ in Chapter 3.

2.3 The maximum principle and Hopf's boundary point lemma

When working with elliptic partial differential equations of second order, for example the Poisson-Dirichlet problem, one finds a maximum principle. For

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad\; u = \psi & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

that is [22, Theorems 2.2, 2.3]:

Theorem 2.3.1 (Strong maximum principle) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is the solution to (2.8). Then one finds:*

- *If $f \leq 0$ and there exists a point $y \in \Omega$ with $u(y) = \sup_{x\in\Omega} u(x)$, then u is constant.*
- *If $f \geq 0$ and there exists a point $y \in \Omega$ with $u(y) = \inf_{x\in\Omega} u(x)$, then u is constant.*

Theorem 2.3.2 (Weak maximum principle) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is the solution to (2.8). Then one finds:*

- *If $f \leq 0$, then $\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)$.*
- *If $f \geq 0$, then $\min_{x\in\overline{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x)$.*

So one finds for $\psi \equiv 0$ that a nonnegative and nontrivial right-hand side provides a positive solution. Furthermore, we recall Hopf's boundary point lemma which was proven by Hopf in 1952 and can be found in [52, Theorem 2.7]:

Lemma 2.3.3 (Hopf's boundary point lemma) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $0 \leq u \in C^2(\Omega)$ satisfy $(-\Delta)u \geq 0$ in Ω. Moreover, let $u(x_0) = 0$ for some $x_0 \in \partial\Omega$. Assume that x_0 lies on the boundary of a ball $B \subset \Omega$. If u is continuous on $\Omega \cup \{x_0\}$ and if the outward directional derivative $\frac{\partial}{\partial\nu}u$ exists in x_0, then $u \equiv 0$ or $\frac{\partial}{\partial\nu}u(x_0) < 0$.*

One implication is that for a nonnegative and nontrivial right-hand side f and zero Dirichlet boundary conditions the solution u to (2.8) is strongly positive, so there exists a constant $c_f > 0$, dependent on f, such that

$$u(x) \geq c_f d(x) \text{ for all } x \in \Omega. \tag{2.9}$$

The inequality (2.9) can be made more precisely. Zhao proved in 1986 estimates for the Green function for (2.8) with $\psi \equiv 0$, see [79] and [80]. These results imply

$$G_{1,0,1}(x, y) \geq c\, d(x)d(y),$$

where c depends only on the domain. Therefore, we find for the solution to (2.8) with $\psi \equiv 0$

$$u(x) = \int_\Omega G_{1,0,1}(x,y)f(y)dy \geq c\left(\int_\Omega f(y)d(y)dy\right)d(x).$$

This estimate can also be found in [8, Lemma 3.2] for smooth domains and $f \in L^\infty(\Omega)$. We already mentioned that the property about sign-preserving is often called the maximum principle. Since we want to distinguish between this property and Theorem 2.3.2, we introduce the following formal definition:

Definition 2.3.4 *If problem* (1.6) *for* $m \in \mathbb{N}^+$, $w \in C^{0,\gamma}(\overline{\Omega})$ *strictly positive and* $\lambda \in \mathbb{R}$ *not a weighted eigenvalue fulfills the property that* $f \geq 0$ *implies* $u \geq 0$*, then one says that it has the positivity preserving property.*

Using the maximum principle for the Dirichlet-Poisson problem, one finds that (1.6) has the positivity preserving property for $m = 1$, $\lambda = 0$ and $w \equiv 1$. For fourth or higher order problems there is no maximum principle or positivity preserving property for most domains.

2.4 The Krein-Rutman theorem

In Jentzsch's article [38] one finds one of the first results that link positivity preserving in one dimension of some integral operator with the simplicity of the first eigenvalue and positivity of the corresponding eigenfunction. A generalized version is the result from Krein and Rutman, see [42]. If one can prove a positivity preserving property for (1.6) with $\lambda = 0$, one can use the Krein-Rutman theorem to obtain the existence of a simple first eigenvalue with positive eigenfunction. In order to recall the result, we need the following two definitions, see [21, p. 63]:

Definition 2.4.1 *Let* $(X, \|\cdot\|, \geq)$ *be an ordered Banach space. Then the set* $\mathcal{K} = \{u \in X; u \geq 0\}$ *is called the positive cone in* X*.*

Definition 2.4.2 *Let* $(X, \|\cdot\|, \geq)$ *be an ordered Banach space and set*

$$|f| := \inf\{h \in X; h \geq f \text{ and } h \geq -f\}.$$

- $(X, \|\cdot\|, \geq)$ *is called a Banach lattice if*

$$f, g \in X \text{ implies } \inf\{h \in X; h \geq f \text{ and } h \geq g\} \in X \tag{2.10}$$

  and

$$f, g \in X \text{ with } |f| \leq |g| \text{ implies } \|f\| \leq \|g\|.$$

- *A linear subspace* $A \subset X$ *is called lattice ideal if*

$$|f| \leq |g| \text{ and } g \in A \text{ implies } f \in A.$$

***Remark* 2.4.3** *For the Banach spaces* $(C(\overline{\Omega}), \|\cdot\|_{C(\overline{\Omega})})$ *and* $(L^2(\Omega), \|\cdot\|_{L^2(\Omega)})$ *with the pointwise order* $f \geq g$ *iff* $f(x) \geq g(x)$ *for (almost) every* $x \in \Omega$*, we may read* (2.10) *as*

$$f, g \in X \text{ implies } \max\{f, g\} \in X$$

and $|f| = \max\{f, 0\} + \max\{-f, 0\}$.

There are many different versions of the Krein-Rutman theorem. Here, we want to recall a generalized version which is a combination of the Krein-Rutman theorem and a result of De Pagter, see [21, p. 63].

Theorem 2.4.4 (Krein-Rutman) *Let* X *be a Banach lattice with* $\dim(X) > 1$ *and let* $T : X \to X$ *be a linear operator satisfying the following three properties:*

1. T *is compact,*

2. T *is positive, which means that for the positive cone* $\mathcal{K} \subset X$ *we find* $T(\mathcal{K}) \subset \mathcal{K}$*,*

3. T *is irreducible, which means that* $\{0\}$ *and* X *are the only closed lattice ideals invariant under* T.

Then, the spectral radius $r(T)$ *of* T *is strictly positive and there exists an element* $u \in \mathcal{K}\backslash\{0\}$ *with* $Tu = r(T)u$*. Furthermore, the algebraic multiplicity of* $r(T)$ *is one, all other eigenvalues* λ *satisfy* $|\lambda| < r(T)$ *and no other eigenfunction is positive.*

Example 2.4.5 *We can apply this theorem to problem* (1.6) *if the corresponding Green function is positive in* $\Omega \times \Omega\backslash\{(x, x); x \in \Omega\}$*. For* $X = L^2(\Omega)$ *or* $X = C_0(\overline{\Omega})$ *we find that the solution operator, which can be expressed by the Green function as the kernel function, is compact and irreducible, see [21, p. 61]. The irreducibility follows from the assumption that the Green function is positive. For the Poisson-Dirichlet problem we find a positive Green function on smooth domains. Therefore, we get a positive first eigenvalue with corresponding positive eigenfunction in the case of the second-order problem.*

2.5 Sobolev imbedding

In the next chapter, we consider operators defined on Sobolev spaces and therefore weak solutions of the weighted $2m$-th order problem (1.6). However, we also want to apply results for continuously differentiable functions like the mean value theorem. So we have to be able to infer results in Sobolev spaces from results in Hölder spaces. To this end, we will use the following Sobolev imbeddings [1, Theorem 4.12]:

Theorem 2.5.1 *Let $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}^+$ be a bounded domain and $\partial\Omega \in C^{2m,\gamma}$. Then there exist the following imbeddings for $p \in [1,\infty)$ and $m \in \mathbb{N}^+$:*

$$\begin{array}{rlll} i. & \text{for } (m-1)p < n < mp: & W^{2m,p}(\Omega) \hookrightarrow C^{m,\mu}(\overline{\Omega}) & \text{with } 0 < \mu \le m - \frac{n}{p}, \\ ii. & \text{for } n \le (m-1)p: & W^{2m,p}(\Omega) \hookrightarrow C^{m,\mu}(\overline{\Omega}) & \text{with } 0 < \mu < 1, \\ iii. & \text{for } n < 2mp: & W^{2m,p}(\Omega) \hookrightarrow L^q(\Omega) & \text{with } p \le q \le p_n^* := \infty, \\ iv. & \text{for } n = 2mp: & W^{2m,p}(\Omega) \hookrightarrow L^q(\Omega) & \text{with } p \le q < p_n^* := \infty, \\ v. & \text{for } n > 2mp: & W^{2m,p}(\Omega) \hookrightarrow L^q(\Omega) & \text{with } p \le q \le p_n^* := \frac{np}{n-2mp}. \end{array} \tag{2.11}$$

The imbeddings in (2.11) are implications of the well known Morrey and Gagliardo-Sobolev-Nirenberg inequalities.

If we replace the inequality $0 < \mu \le m - \frac{n}{p}$ in *i.* by $0 < \mu < m - \frac{n}{p}$ and the inequalities $p \le q \le p_n^*$ in *iii.* and *v.* by $p \le q < p_n^*$, we get that the imbeddings in (2.11) are even compact, see [1, Theorem 6.3].

When we use the imbedding $X \hookrightarrow Y$ with X, Y Hölder- or Sobolev spaces, then we write it as

$$\mathcal{I} : X \hookrightarrow Y.$$

Which spaces X and Y are meant, when only $\mathcal{I}$ is written, is mentioned in each case or is clear from the context.

2.6 The weighted setting

In this section we will present the weighted setting for problem (1.6). Therefore, we describe some standard arguments for the existence of weak solutions and some properties of the eigenfunctions and corresponding eigenvalues. To this end, we follow the steps presented in [58] for $m = 2$ and adapt the setting to the general case. We use the Hilbert space $L^2_w(\Omega) := (L^2(\Omega), \langle\cdot,\cdot\rangle_{L^2_w(\Omega)})$, where the scalar product is defined by

$$\langle u, v\rangle_{L^2_w(\Omega)} := \int_\Omega u(x)v(x)w(x)dx \text{ for } u, v \in L^2(\Omega).$$

This is equivalent to the standard inner product because $w \in C^{0,\gamma}(\overline{\Omega})$ is bounded from below and from above by positive constants. Since for all $u \in C_c^\infty(\Omega)$ and $m \in \mathbb{N}^+$ we find with partial integration

$$\int_\Omega \sum_{j_1,j_2,\dots,j_m=1}^{n} \left(\frac{\partial}{\partial x_{j_1}}\frac{\partial}{\partial x_{j_2}}\cdots\frac{\partial}{\partial x_{j_m}}u(x)\right)^2 dx = \begin{cases} \int_\Omega (\Delta^{\frac{m}{2}}u(x))^2dx & \text{for even } m, \\ \int_\Omega |\nabla\Delta^{\frac{m-1}{2}}u(x)|^2dx & \text{for odd } m, \end{cases}$$

and $C_c^\infty(\Omega)$ is dense in $W_0^{m,2}(\Omega)$, it holds true for all $u \in W_0^{m,2}(\Omega)$. Using this calculation and the Poincaré-Friedrichs inequality, we obtain that the standard norm on $W_0^{m,2}(\Omega)$, that is

$$\|u\|_{W^{m,2}(\Omega)} := \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}},$$

is equivalent to the norm

$$\|u\|_{m,\lambda} := \begin{cases} \sqrt{\|\Delta^{\frac{m}{2}} u\|^2_{L^2(\Omega)} - \lambda \langle u, u\rangle_{L^2_w(\Omega)}} & \text{for even } m \in \mathbb{N}^+, \\ \sqrt{\||\nabla \Delta^{\frac{m-1}{2}} u|\|^2_{L^2(\Omega)} - \lambda \langle u, u\rangle_{L^2_w(\Omega)}} & \text{for odd } m \in \mathbb{N}^+, \end{cases}$$

for all $\lambda \leq 0$. So, for $\lambda \leq 0$ one gets that $W_0^{m,2}(\Omega)$ is a Hilbert space with scalar product

$$\langle u, v\rangle_{\lambda, W_0^{m,2}(\Omega)} := \begin{cases} \int_\Omega \left(\Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}} v - \lambda w u v \right) dx & \text{for even } m \in \mathbb{N}^+, \\ \int_\Omega \left(\nabla \Delta^{\frac{m-1}{2}} u \cdot \nabla \Delta^{\frac{m-1}{2}} v - \lambda w u v \right) dx & \text{for odd } m \in \mathbb{N}^+. \end{cases}$$

Using Riesz' Representation Theorem, we find for every $f \in L^2(\Omega)$ a weak solution $u_{m,\lambda,w}$ to (1.6). Applying results by Agmon-Douglis-Nirenberg, see [21, Theorems 2.19, 2.10], we find that $u_{m,\lambda,w} \in W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega)$.

The Green operator $\mathcal{G}_{m,0,w} : L^2_w(\Omega) \to W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega)$ is a linear operator, since we investigate a linear boundary value problem. Using the compact Sobolev imbedding $\mathcal{I} : W^{2m,2}(\Omega) \hookrightarrow L^2_w(\Omega)$, one finds that $\mathcal{I} \circ \mathcal{G}_{m,0,w} : L^2_w(\Omega) \to L^2_w(\Omega)$ is compact and since $\mathcal{G}_{m,0,w}$ is an isomorphism, we obtain the inverse operator

$$A_{m,w} : D(A_{m,w}) \subset L^2_w(\Omega) \to L^2_w(\Omega)$$

defined by

$$D(A_{m,w}) = W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega) \text{ with } A_{m,w} = \frac{1}{w}(-\Delta)^m.$$

Since $\mathcal{I} \circ \mathcal{G}_{m,0,w}$ is compact, the spectrum of $A_{m,w}$ is discrete, see [4, Theorem 9.9]. We also find that $A_{m,w}$ is selfadjoint since for $m \in \mathbb{N}^+$ even and $u, v \in W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega)$ we obtain

$$\begin{aligned} \langle A_{m,w} u, v\rangle_{L^2_w(\Omega)} &= \langle A_{m,1} u, v\rangle_{L^2(\Omega)} = (-1)^m \int_\Omega (\Delta^m u(x))\, v(x) dx \\ &= \int_\Omega (\Delta^{\frac{m}{2}} u(x))(\Delta^{\frac{m}{2}} v(x)) dx = \langle u, A_{m,w} v\rangle_{L^2_w(\Omega)} \end{aligned}$$

and analogously for $m \in \mathbb{N}^+$ odd. Since $A_{m,w}$ is also positive in $L^2_w(\Omega)$, that is

$$\langle A_{m,w} u, u\rangle_{L^2_w(\Omega)} = \langle A_{m,1} u, u\rangle_{L^2(\Omega)} > 0 \text{ for } u \not\equiv 0,$$

and using the spectral theorem for self-adjoint operators with compact resolvent, see [4, Theorem 10.12], one finds that the spectrum consists of countably many positive real eigenvalues $\{\lambda_{i,m,w}\}_{i\in\mathbb{N}^+}$ with

$$0 < \lambda_{1,m,w} \le \lambda_{2,m,w} \le \cdots \to \infty$$

and corresponding eigenfunctions $\{\varphi_{i,m,w}\}_{i\in\mathbb{N}^+} \subset W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega)$.

***Remark* 2.6.1** *Using the Rayleigh quotient and applying the Poincaré-Friedrichs inequality several times, one also sees that the first eigenvalue is positive.*

***Remark* 2.6.2** *Still assuming $\lambda \le 0$ and using Agmon-Douglis-Nirenberg results, see [21, Theorems 2.19, 2.20], one finds that the restriction of $\mathcal{G}_{m,\lambda,w}$ to $C^{0,\gamma}(\overline{\Omega})$ with $\gamma \in (0,1)$ or $L^q(\Omega)$ with $q \in (1,\infty)$ are isomorphisms in the following way:*

$$\begin{aligned} \mathcal{G}_{m,\lambda,w} &: C^{0,\gamma}(\overline{\Omega}) \to C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega}), \\ \mathcal{G}_{m,\lambda,w} &: L^q(\Omega) \to W^{2m,q}(\Omega) \cap W_0^{m,q}(\Omega). \end{aligned}$$

Then one obtains with a bootstrapping argument and Sobolev imbeddings (2.11) *that for $\partial\Omega \in C^{2m,\gamma}$, the $W_0^{m,2}(\Omega)$-eigenfunctions are in $C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$.*

The eigenfunctions can be chosen such that they are normalised by

$$\langle \varphi_{i,m,w}, \varphi_{j,m,w} \rangle_{L^2_w(\Omega)} = \delta_{ij}, \tag{2.12}$$

where δ_{ij} is the Kronecker delta. By the Riesz-Schauder theorem, see [4, pp. 395, 409–410], we get that $\{\varphi_{i,m,w}\}_{i\in\mathbb{N}^+}$ is a complete orthonormal system of eigenfunctions in $L^2_w(\Omega)$, such that for $f \in L^2_w(\Omega)$ it holds that

$$f = \sum_{i=1}^{\infty} \varphi_{i,m,w} \langle \varphi_{i,m,w}, f \rangle_{L^2_w(\Omega)}, \tag{2.13}$$

and for $\lambda \notin \{\lambda_{i,m,w}\}_{i\in\mathbb{N}^+}$ we find

$$\mathcal{G}_{m,\lambda,w} f = \sum_{i=1}^{\infty} \frac{1}{\lambda_{i,m,w} - \lambda} \varphi_{i,m,w} \langle \varphi_{i,m,w}, f \rangle_{L^2_w(\Omega)}. \tag{2.14}$$

This series converges when applied to some $f \in L^2(\Omega)$ since (2.13) converges by Bessel's inequality, see [6, p. 87], and $|\lambda_{i,m,w} - \lambda|^{-1}$ is bounded from above.

In the following chapter we will use the integral operators with kernel functions $d(x)^m d(y)^m$ and $\varphi_{i,m,w}(x)\varphi_{i,m,w}(y)w(y)$ to find estimates for $\mathcal{G}_{m,\lambda,w}$.

Definition 2.6.3 *1. The orthogonal projections $\mathcal{P}_{i,m,w}, \mathcal{P}_{j*,m,w} : L^2(\Omega) \to L^2(\Omega)$*

onto the eigenspaces in $L^2_w(\Omega)$ *are defined by*

$$(\mathcal{P}_{i,m,w}v)(x) := \varphi_{i,m,w}(x)\int_\Omega \varphi_{i,m,w}(y)\, v(y)w(y)dy \text{ for } i \in \mathbb{N}^+, x \in \Omega, \quad (2.15)$$

$$\mathcal{P}_{j*,m,w} := \mathcal{I} - \sum_{i=1}^{j} \mathcal{P}_{i,m,w} \qquad \text{for } j \in \mathbb{N}^+. \quad (2.16)$$

2. *The operator* $\mathcal{D}_m : L^2(\Omega) \to L^2(\Omega)$ *is defined by*

$$(\mathcal{D}_m v)(x) := d(x)^m \int_\Omega d(y)^m v(y)dy \text{ for } x \in \Omega. \quad (2.17)$$

***Remark* 2.6.4** *We note that for all* $i \in \mathbb{N}^+$

$$\|\mathcal{P}_{i,m,w}\|_{BL(L^2_w(\Omega))} = \sup\left\{\left|\int_\Omega \varphi_{i,m,w}(y)w(y)v(y)dy\right| ; \|v\|_{L^2_w(\Omega)} \le 1\right\} = 1.$$

Using this definition, we may also write instead of (2.13) and (2.14) the following representation formulas:

$$f = \sum_{i=1}^{\infty} \mathcal{P}_{i,m,w} f \quad \text{and} \quad \mathcal{G}_{m,\lambda,w} f = \sum_{i=1}^{\infty} \frac{1}{\lambda_{i,m,w} - \lambda} \mathcal{P}_{i,m,w} f.$$

Chapter 3

A converse to Krein-Rutman

For the proof of Theorem 1.2.8 we need an estimate for the polyharmonic Green function $G_{m,\lambda,w}$ and Green operator $\mathcal{G}_{m,\lambda,w}$ defined in (2.7). In [57] we have proven such estimates for the special case $m = 2$ and $w \equiv 1$ and in [58] for $m = 2$ and some strictly positive and Hölder continuous weight function w. We follow similar steps with the only difference that we consider the Dirichlet problem of order $2m$ instead of the special case $m = 2$. First, we show the converse of Krein-Rutman's theorem using regularity results as in [58]. This result is stated in Section 3.1 and is proven in Sections 3.2, 3.3 and 3.4. In Section 3.5 we use arguments from [57] and show an asymptotic behavior of the eigenvalues and eigenfunctions and thus derive an alternative proof of the main theorem. In Section 3.6 we note that similar arguments can also be used to prove an anti-maximum principle.

***Remark* 3.0.1** *The results are consequences of estimates for the polyharmonic Green function and since for a strictly positive weight function $w \in C^{0,\gamma}(\overline{\Omega})$ there exist two constants $c_{w,1}, c_{w,2} > 0$ such that*

$$c_{w,1} \leq w(x) \leq c_{w,2} \text{ for all } x \in \overline{\Omega}, \tag{3.1}$$

we can follow analogous steps as in [57] with adjusted constants and replace 2 with m or we follow the steps in [58] with small changes.

By extending the results for fourth order problems, Pulst proved in his dissertation [26] the following inequality for the Green function of (1.6) with $\lambda = 0$ in bounded $C^{2m,\gamma}$-smooth domains, see [53, Theorem 3.1]:

$$c_2^{-1} H_{n,m}(x,y) \leq G_{m,0,1}(x,y) + c_1 d(x)^m d(y)^m \leq c_2 H_{n,m}(x,y) \tag{3.2}$$

for all $(x,y) \in \overline{\Omega} \times \overline{\Omega} \backslash \{(x,x); x \in \overline{\Omega}\}$, where $c_1, c_2 > 0$ are dependent on the domain and m, and $H_{n,m} : \overline{\Omega} \times \overline{\Omega} \backslash \{(x,x); x \in \overline{\Omega}\} \to \mathbb{R}$ is defined by

$$H_{n,m}(x,y) := \begin{cases} |x-y|^{2m-n} \min\left\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\} & \text{if } n > 2m, \\ \log\left(1 + \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) & \text{if } n = 2m, \\ d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\} & \text{if } n < 2m. \end{cases} \tag{3.3}$$

We will use the calligraphic $\mathcal{H}_{n,m}$ for the integral operator with kernel function $H_{n,m}$:

$$\mathcal{H}_{n,m} : L^2(\Omega) \to L^2(\Omega), \qquad (\mathcal{H}_{n,m} f)(x) = \int_\Omega H_{n,m}(x,y) f(y) dy. \tag{3.4}$$

***Remark* 3.0.2** *Some useful result, Pulst proved in his doctoral thesis is, that there is a constant $c > 0$ such that*

$$c\, d(x)^m d(y)^m \leq H_{n,m}(x,y) \text{ for all } (x,y) \in \overline{\Omega} \times \overline{\Omega} \backslash \{(x,x); x \in \overline{\Omega}\}.$$

One finds this result using (3.3), $|x-y| \leq \operatorname{diam}(\Omega)$ and estimates in [21, Lemma 4.5]. Moreover, it is included in Corollary 3.4.2 in Section 3.4.

***Remark* 3.0.3** *If Ω is a ball, we find the estimate in (3.2) with $c_1 = 0$, see [30]. Two-sided estimates for the second order problem ($m = 1$) were proven by Zhao, see [79] and [80].*

We will extend this result to the Green function of (1.6) with λ in some bounded interval and with some Hölder continuous, strictly positive weight function. First, we present the extension of (3.2) to the Green function $G_{m,\lambda,w}$ and then, the asymptotic behavior of the constants for $\lambda \uparrow \lambda_{p,m,w}$ is shown.

3.1 Pointwise estimates for the Green function and idea of the proof

Using (2.14), we note that formally the Green function $G_{m,\lambda,w}$ can be written as

$$G_{m,\lambda,w}(x,y) = \sum_{i=1}^{\infty} \frac{1}{\lambda_{i,m,w} - \lambda} \varphi_{i,m,w}(x) \varphi_{i,m,w}(y). \tag{3.5}$$

But even if (2.14) converges in $L^2_w(\Omega)$ for λ not an eigenvalue, the series in (3.5) does not have to converge as a function in $\Omega \times \Omega$, especially for higher dimensions. For $n < 4m$ we can show a convergence in $L^2_w(\Omega \times \Omega) = \left(L^2(\Omega \times \Omega), \langle \cdot, \cdot \rangle_{L^2_w(\Omega \times \Omega)}\right)$, where we define

$$\langle u, v \rangle_{L^2_w(\Omega \times \Omega)} := \int_\Omega \int_\Omega u(x,y) v(x,y) w(x) w(y) dx\, dy \quad \text{for } u, v \in L^2(\Omega \times \Omega).$$

Indeed, we find for a fixed λ, which is not an eigenvalue, that $\frac{\lambda_{i,m,w}^2}{(\lambda_{i,m,w}-\lambda)^2} \leq C$ for some constant $C > 0$, independent of $i \in \mathbb{N}^+$ and

$$\langle G_{m,\lambda,w}, G_{m,\lambda,w}\rangle_{L^2_w(\Omega\times\Omega)} = \sum_{i=1}^{\infty} \frac{1}{(\lambda_{i,m,w}-\lambda)^2} \leq C \sum_{i=1}^{\infty} \lambda_{i,m,w}^{-2}.$$

We will see in Chapter 3.5 that $\lambda_{i,m,w} \geq c\, i^{\frac{2m}{n}}$ for all $i \in \mathbb{N}^+$ with $c > 0$ independent of i, so the series on the right-hand side converges for $n < 4m$. Even if we cannot write the Green function as in (3.5) for large dimensions, one gets the intuition that it becomes positive for λ in a small left neighborhood of a simple eigenvalue with corresponding positive eigenfunction. In the same way, one expects that it becomes negative for λ in a small right neighborhood of this eigenvalue.

If we do not choose λ close to an eigenvalue, we obtain a result similar to (3.2):

Theorem 3.1.1 *Suppose that Condition A is fulfilled. Moreover, let* $0 < w \in C^{0,\gamma}(\overline{\Omega})$ *and* $\{\lambda_{i,m,w}\}_{i\in\mathbb{N}^+} \subset (0,\infty)$ *denote the eigenvalues for (1.10) and take* $M, \delta_1 \in \mathbb{R}^+$. *Set*

$$I_{M,\delta_1} = [-M, M] \setminus \bigcup\nolimits_{i=1}^{\infty} (\lambda_{i,m,w} - \delta_1, \lambda_{i,m,w} + \delta_1)\,. \tag{3.6}$$

Let $G_{m,\lambda,w}$ *be the Green function for* (2.6). *Then there are* $c_1, c_2, c_3 > 0$, *depending on the domain,* M, δ_1, m *and* w, *such that for all* $\lambda \in I_{M,\delta_1}$ *it holds:*

$$c_1\ H_{n,m}(x,y) \leq G_{m,\lambda,w}(x,y) + c_2\ d(x)^m\ d(y)^m \leq c_3\ H_{n,m}(x,y) \text{ for all } x,y \in \Omega. \tag{3.7}$$

***Remark* 3.1.2** *For* $m = 2$ *this result can be found in [58, Theorem 14]. For* $w \equiv 1$, $\lambda = 0$ *and* $m = 2$ *it is proven in [26, Theorem 1].*

We want to find positivity of the Green function. So, as mentioned above, we will choose λ in a left neighborhood of the simple eigenvalue with corresponding positive eigenfunction. More precisely, we find the following estimate, see also [58, Theorem 16] or [57, Theorem 2] for $m = 2$:

Theorem 3.1.3 *Suppose that Condition A is satisfied and let* $\delta_2 > 0$. *Suppose* $0 < w \in C^{0,\gamma}(\overline{\Omega})$ *and that* $\lambda_{\mathrm{p},m,w}$ *is a simple eigenvalue of* (1.10) *with the corresponding eigenfunction* $\varphi_{\mathrm{p},m,w}$ *strongly positive as in (1.11). Moreover, suppose the interval*

$$I_{\delta_2} = [\lambda_{\mathrm{p},m,w} - \delta_2, \lambda_{\mathrm{p},m,w}) \tag{3.8}$$

contains no eigenvalue. Let $G_{m,\lambda,w}$ *be the Green function for* (2.6). *Then there exist constants* $C_1, C_2, C_3 > 0$, *depending on the domain,* m, δ_2 *and* w, *such that for all* $\lambda \in I_{\delta_2}$ *and* $x, y \in \Omega$:

$$G_{m,\lambda,w}(x,y) \geq C_1\, H_{n,m}(x,y) + \left(\frac{C_2}{\lambda_{\mathrm{p},m,w} - \lambda} - C_3\right) \varphi_{\mathrm{p},m,w}(x)\ \varphi_{\mathrm{p},m,w}(y). \tag{3.9}$$

These theorems can be proven using Lemma 3.2.2, Corollary 3.3.3 and Proposition 3.4.5 in the following sections.

Corollary 3.1.4 *If there exists a strictly positive weight function* $w \in C^{0,\gamma}(\overline{\Omega})$ *such that there is a simple eigenvalue* $\lambda_{\mathrm{p},m,w}$ *of* (1.10) *with corresponding strongly positive eigenfunction* $\varphi_{\mathrm{p},m,w}$ *in the sense of* (1.11)*, then problem* (1.6) *is positivity preserving for* λ *in a small left neighborhood of* $\lambda_{\mathrm{p},m,w}$.

***Remark* 3.1.5** *We will see in Chapter 4 that Condition B is sufficient for our construction and the existence of a weight function that satisfies the requirements of Corollary 3.1.4.*

***Remark* 3.1.6** *If we assume that there exists a strongly positive eigenfunction* $\varphi_{\mathrm{p},m,w}$ *with corresponding eigenvalue* $\lambda_{\mathrm{p},m,w}$ *which has multiplicity* $M \geq 2$*, then there are* $M-1$ *sign-changing orthogonal eigenfunctions* $\varphi_{\mathrm{p}+1,m,w}, \ldots, \varphi_{\mathrm{p}+M-1,m,w}$ *in the sense of* (2.12) *with eigenvalues* $\lambda_{\mathrm{p},m,w} = \lambda_{\mathrm{p}+1,m,w} = \cdots = \lambda_{\mathrm{p}+M-1,m,w}$*. So, if we restrict ourselves to the following space for the right-hand side* f *in* (1.6)*, we obtain a positive solution for* λ *in a small left neighborhood of* $\lambda_{\mathrm{p},m,w}$ *even if* $\lambda_{\mathrm{p},m,w}$ *is not simple:*

$$\left\{ f \in L^2(\Omega); f \geq 0 \text{ and } \langle f, \varphi_{\mathrm{p}+k,m,w} \rangle_{L^2(\Omega)} = 0 \text{ for all } k \in \{1, \ldots, M-1\} \right\}.$$

If the assumptions in Corollary 3.1.4 are fulfilled, (1.6) is positivity preserving for some $\lambda < \lambda_{\mathrm{p},m,w}$ and $\lambda_{\mathrm{p},m,w}$ is not the first eigenvalue of (1.10), then we can calculate a lower bound for λ. This also proves inequality (1.16).

Lemma 3.1.7 *Suppose that Conditions A is satisfied. Moreover, assume that there exists a strictly positive weight function* $w \in C^{0,\gamma}(\overline{\Omega})$ *such that there is a simple eigenvalue* $\lambda_{\mathrm{p},m,w}$ *with* $\mathrm{p} > 1$ *and corresponding strongly positive eigenfunction* $\varphi_{\mathrm{p},m,w}$ *for* (1.10)*. Let* $\lambda \in (\lambda_{\mathrm{p}-1,m,w}, \lambda_{\mathrm{p},m,w})$ *be such that* (1.6) *is positivity preserving. Then, we find*

$$\lambda \geq \lambda_{\mathrm{p}-1,m,w} + \frac{\lambda_{\mathrm{p},m,w} - \lambda_{\mathrm{p}-1,m,w}}{2}. \tag{3.10}$$

Proof. Let $\lambda \in (\lambda_{\mathrm{p}-1,m,w}, \lambda_{\mathrm{p},m,w})$ be such that (1.6) is positivity preserving and define

$$c_1 := \left(\sup_{x \in \Omega} \frac{\varphi_{\mathrm{p}-1,m,w}(x)}{\varphi_{\mathrm{p},m,w}(x)} \right)^{-1} > 0 \quad \text{and} \quad c_2 := \left(\inf_{x \in \Omega} \frac{\varphi_{\mathrm{p}-1,m,w}(x)}{\varphi_{\mathrm{p},m,w}(x)} \right)^{-1} < 0.$$

It holds that $c = c_1$ is the largest and $c = c_2$ the smallest value such that $\varphi_{\mathrm{p},m,w} - c\,\varphi_{\mathrm{p}-1,m,w}$ is nonnegative in Ω. We choose

$$f = w\varphi_{\mathrm{p},m,w} - c\, w\varphi_{\mathrm{p}-1,m,w} \quad \text{with} \quad c = \begin{cases} c_1 & \text{if } c_1 \geq -c_2, \\ c_2 & \text{if } c_1 < -c_2. \end{cases}$$

Then we find $f \geq 0$. Considering problem (1.6) with right-hand side f, one obtains the solution

$$u = \frac{1}{\lambda_{\mathrm{p},m,w} - \lambda}\left(\varphi_{\mathrm{p},m,w} - c\,\frac{\lambda_{\mathrm{p},m,w} - \lambda}{\lambda_{\mathrm{p}-1,m,w} - \lambda}\varphi_{\mathrm{p}-1,m,w}\right).$$

Using the assumption that (1.6) is positivity preserving, we find $u \geq 0$. Hence

$$\left|c\,\frac{\lambda_{\mathrm{p},m,w} - \lambda}{\lambda_{\mathrm{p}-1,m,w} - \lambda}\right| \leq \max\{c_1, |c_2|\}$$

which is equivalent to

$$\left|\frac{\lambda_{\mathrm{p},m,w} - \lambda}{\lambda_{\mathrm{p}-1,m,w} - \lambda}\right| \leq 1.$$

So, we get (3.10). ■

Before we provide the technical details for the proof of Theorem 3.1.1 and 3.1.3 and the necessary lemmata in the next sections, a first idea of the proof is given, see also [58, Section 3]. We note that instead of (3.7) we may show the inequalities for the corresponding integral operators, i.e. that there are three constants $c_1, c_2, c_3 > 0$ such that

$$c_1\ \mathcal{H}_{n,m} \leq \mathcal{G}_{m,\lambda,w} + c_2\ \mathcal{D}_m \leq c_3\ \mathcal{H}_{n,m} \text{ for all } \lambda \in I_{M,\delta_1}$$

and instead of (3.9) we prove that there are constants $C_1, C_2, C_3 > 0$ such that

$$\mathcal{G}_{m,\lambda,w} \geq C_1\ \mathcal{H}_{n,m} + \left(\frac{C_2}{\lambda_{\mathrm{p},m,w} - \lambda} - C_3\right)\mathcal{P}_{\mathrm{p},m,w} \text{ for all } \lambda \in I_{\delta_2}$$

with $\mathcal{H}_{n,m}$, $\mathcal{D}_m$ and $\mathcal{P}_{\mathrm{p},m,w}$ as defined in (3.4), (2.17) and (2.15). First, we recall the asymptotic formula for $\mathcal{G}_{m,\lambda,w}$ using Neumann series which contains $\mathcal{G}_{m,0,w}$, respectively $\mathcal{G}_{m,0,1}(w\cdot)$, and powers of this operator. The idea is similar to the steps in [58, Section 3]:

Suppose that $|\lambda| < \lambda_{1,m,w}$ and $u_{m,\lambda,w} = \mathcal{G}_{m,\lambda,w} f_w$, where f_w is defined as in (2.5). We can also write

$$u_{m,\lambda,w} = \mathcal{G}_{m,0,w}\left(\lambda u_{m,\lambda,w} + f_w\right).$$

This is equivalent to

$$\left(\mathcal{I} - \lambda\mathcal{G}_{m,0,1}(w\cdot)\right) u_{m,\lambda,w} = \mathcal{G}_{m,0,1}(w f_w).$$

The spectral radius of $\mathcal{G}_{m,0,w}$ is $\lambda_{1,m,w}^{-1}$, so we can invert the operator $(\mathcal{I} - \lambda\mathcal{G}_{m,0,1}(w\cdot))$

and using Neumann series, we find

$$u_{m,\lambda,w} = \sum_{k=0}^{\infty} \lambda^k \left(\mathcal{G}_{m,0,1}(w\cdot)\right)^{k+1} f_w.$$

We also want to get a representation formula for $u_{m,\lambda,w}$ if $|\lambda| > \lambda_{1,m,w}$ and $\lambda \neq \lambda_{i,m,w}$ for all $i \in \mathbb{N}^+$. Therefore, let $M \in \mathbb{R}^+$ be as in Theorem 3.1.1 and set

$$\lambda_{j,m,w} = \min\{\lambda \in \{\lambda_{i,m,w}\}_{i\in\mathbb{N}^+};\ \lambda > M\}. \tag{3.11}$$

Without restriction we assume in the whole chapter that

$$\lambda_{j,m,w} \geq \lambda_{\mathrm{p},m,w}. \tag{3.12}$$

Then we find for

$$\lambda \in (-\lambda_{j+1,m,w}, \lambda_{j+1,m,w})\backslash\{\lambda_{i,m,w}\}_{i\leq j} \tag{3.13}$$

the solution for (1.6)

$$\begin{aligned} u_{m,\lambda,w} = \mathcal{G}_{m,\lambda,w} f_w &= \sum_{i=1}^{j} \mathcal{G}_{m,\lambda,w} \mathcal{P}_{i,m,w} f_w + \mathcal{G}_{m,\lambda,w} \mathcal{P}_{j*,m,w} f_w \\ &= \underbrace{\sum_{i=1}^{j} \frac{1}{\lambda_{i,m,w} - \lambda} \mathcal{P}_{i,m,w} f_w}_{I} + \sum_{k=0}^{\infty} \lambda^k \left(\mathcal{G}_{m,0,1}(w\cdot)\right)^{k+1} \mathcal{P}_{j*,m,w} f_w. \end{aligned} \tag{3.14}$$

We split the series on the right in a finite part and an infinite remainder. For λ as in (3.13) we then find

$$\begin{aligned} &\sum_{k=0}^{\infty} \lambda^k \left(\mathcal{G}_{m,0,1}(w\cdot)\right)^{k+1} \mathcal{P}_{j*,m,w} f_w \\ = &\underbrace{\sum_{k=2k_{n,m}}^{\infty} \lambda^k \left(\mathcal{G}_{m,0,1}(w\cdot)\right)^{k+1} \mathcal{P}_{j*,m,w} f_w}_{II} + \underbrace{\sum_{k=0}^{2k_{n,m}-1} \lambda^k \left(\mathcal{G}_{m,0,1}(w\cdot)\right)^{k+1} \mathcal{P}_{j*,m,w} f_w}_{III}, \end{aligned} \tag{3.15}$$

where $k_{n,m} \in \mathbb{N}^+$ is defined by

$$k_{n,m} = \left[\frac{n+2m}{4m}\right] + 1. \tag{3.16}$$

We will describe in the next section how to derive the value in (3.16) and we show that I and II can be estimated by $\tilde{c}_1 \mathcal{D}_m f_w$ and III can be estimated by $\tilde{c}_2 \mathcal{H}_{n,m} f_w - \tilde{c}_3 \mathcal{D}_m f_w$ for some constants $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 > 0$ and all $0 \leq f_w \in L^2(\Omega)$.

The regularity results and pointwise estimates in the following three sections imply these three estimates.

***Remark* 3.1.8** *The restriction for $k_{n,m}$ in* (3.16) *is sufficient to apply some regularity properties for $\mathcal{G}_{m,0,w}^{k_{n,m}}$. We need this regularity result to estimate* II. *For* III, *the value of $k_{n,m}$ does not matter. It is only important that we consider a finite sum.*

3.2 Estimates for the orthogonal projections

First, we present some elementary results to make sure that all series converge which appear in the following arguments. Therefore, we have to consider powers of the operator $\mathcal{G}_{m,0,w}$ and compositions of this operator with orthogonal projections onto the eigenspaces. We obtain the following result which can be found for $m = 2$ in [57, Lemma 7]:

Lemma 3.2.1 *Let $\mathcal{G}_{m,\lambda,w}$, $\mathcal{P}_{i,m,w}$ and $\mathcal{P}_{j*,m,w}$ be as defined in* (2.7), (2.15) *and* (2.16) *with $j \in \mathbb{N}^+$ as in* (3.11). *Moreover, let I_{M,δ_1} and I_{δ_2} be as in* (3.6) *and* (3.8). *Then, we find for $\lambda \in I_{M,\delta_1} \cup I_{\delta_2}$*

$$\mathcal{G}_{m,\lambda,w}\mathcal{P}_{i,m,w} = \mathcal{P}_{i,m,w}\mathcal{G}_{m,\lambda,w} = \frac{1}{\lambda_{i,m,w} - \lambda}\mathcal{P}_{i,m,w} \text{ for all } i \in \mathbb{N}^+, \tag{3.17}$$

$$\mathcal{G}_{m,0,w}^k = \sum_{i=1}^{\infty} \frac{1}{\lambda_{i,m,w}^k}\mathcal{P}_{i,m,w} \text{ for all } k \in \mathbb{N}^+, \tag{3.18}$$

$$\mathcal{G}_{m,0,w}^k\mathcal{P}_{j*,m,w} = \sum_{i=j+1}^{\infty} \frac{1}{\lambda_{i,m,w}^k}\mathcal{P}_{i,m,w} \text{ for all } k \in \mathbb{N}^+. \tag{3.19}$$

The series in (3.18) *and* (3.19) *converge when applied to some $f \in L_w^2(\Omega)$.*

Proof. It holds that $\{\varphi_{i,m,w}\}_{i\in\mathbb{N}^+}$ is a complete orthonormal system in $L_w^2(\Omega)$, so we recall that the series $\sum_{i=1}^{\infty}\langle\varphi_{i,m,w}, f\rangle_{L_w^2(\Omega)}\varphi_{i,m,w}(\cdot)$ converges in $L_w^2(\Omega)$ to f. Also we find

$$\|f\|_{L_w^2(\Omega)}^2 = \sum_{i=1}^{\infty}\langle\varphi_{i,m,w}, f\rangle_{L_w^2(\Omega)}^2. \tag{3.20}$$

Since $\varphi_{i,m,w}$ are eigenfunctions of $\mathcal{G}_{m,0,w}$, we obtain that

$$\mathcal{G}_{m,\lambda,w}\,\varphi_{i,m,w} = \frac{\varphi_{i,m,w}}{\lambda_{i,m,w} - \lambda}.$$

This implies (3.17). Since $\mathcal{G}_{m,0,w}$ is linear and continuous on $L^2_w(\Omega)$, we find that

$$\begin{aligned}\mathcal{G}^k_{m,0,w}f &= \sum_{i=1}^{\infty}\langle\varphi_{i,m,w}, f\rangle_{L^2_w(\Omega)}\mathcal{G}^k_{m,0,w}\varphi_{i,m,w}\\ &= \sum_{i=1}^{\infty}\frac{1}{\lambda^k_{i,m,w}}\langle\varphi_{i,m,w}, f\rangle_{L^2_w(\Omega)}\varphi_{i,m,w} = \sum_{i=1}^{\infty}\frac{1}{\lambda^k_{i,m,w}}\mathcal{P}_{i,m,w}f\end{aligned}$$

for all $f \in L^2_w(\Omega)$. The series converges since (3.20) converges and $\lambda_{i,m,w} \to \infty$ for $i \to \infty$. This implies (3.18). Since the eigenfunctions are orthonormal in $L^2_w(\Omega)$, we find that

$$\mathcal{G}^k_{m,0,w}\mathcal{P}_{i,m,w} = \frac{1}{\lambda^k_{i,m,w}}\mathcal{P}_{i,m,w}. \tag{3.21}$$

Hence, we get

$$\mathcal{G}^k_{m,0,w}\mathcal{P}_{j*,m,w} = \mathcal{G}^k_{m,0,w} - \sum_{i=1}^{j}\mathcal{G}^k_{m,0,w}\mathcal{P}_{i,m,w}. \tag{3.22}$$

and using (3.18), (3.21) and (3.22), we obtain (3.19). ■

To find an estimate for I in (3.14), we only make use of the regularity of the eigenfunctions and the mean value theorem:

Lemma 3.2.2 *Suppose that Condition A is satisfied. Let $\mathcal{D}_m$, $\mathcal{P}_{i,m,w}$ and $\mathcal{P}_{j*,m,w}$ be as defined in* (2.17), (2.15) *and* (2.16) *with $j \in \mathbb{N}^+$ as in* (3.11) *and* (3.12). *Moreover, let I_{M,δ_1} and I_{δ_2} be as in* (3.6) *and* (3.8). *Then, we find three constants $c_j, \tilde{c}_j, \hat{c}_j > 0$, depending also on the domain, m, w, M, δ_1 and δ_2, such that*

1. $-c_j\mathcal{D}_m \le \sum_{i=1}^{j}\frac{1}{\lambda_{i,m,w}-\lambda}\mathcal{P}_{i,m,w} \le c_j\mathcal{D}_m$ *for all $\lambda \in I_{M,\delta_1}$ and*
2. $\sum_{i=1}^{j}\frac{1}{\lambda_{i,m,w}-\lambda}\mathcal{P}_{i,m,w} \ge \frac{1}{\lambda_{\mathrm{p},m,w}-\lambda}\mathcal{P}_{\mathrm{p},m,w} - \tilde{c}_j\mathcal{D}_m \ge \left(\frac{1}{\lambda_{\mathrm{p},m,w}-\lambda} - \hat{c}_j\right)\mathcal{P}_{\mathrm{p},m,w}$ *for all $\lambda \in I_{\delta_2}$.*

Proof.

1. Using the mean value theorem, we find for $\varphi \in C^m(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$ and all $x \in \Omega$

$$|\varphi(x)| \le \|\varphi\|_{C^m(\overline{\Omega})}d(x)^m. \tag{3.23}$$

In Remark 2.6.2 we mentioned that the eigenfunctions are elements of $C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$, so (3.23) holds for every eigenfunction $\varphi_{i,m,w}$. With

Definition 2.6.3 and (3.1) we obtain that for $0 \leq f \in L^2(\Omega)$ and $1 \leq i \leq j$

$$\begin{aligned} |(\mathcal{P}_{i,m,w}f)(x)| &= \left|\varphi_{i,m,w}(x) \int_\Omega \varphi_{i,m,w}(y) f(y) w(y) dy\right| \\ &\leq c_{w,2} \|\varphi_{i,m,w}\|^2_{C^m(\overline{\Omega})} d(x)^m \int_\Omega d(y)^m f(y) dy \\ &= c_i (\mathcal{D}_m f)(x), \end{aligned} \tag{3.24}$$

where $c_i = c_{w,2} \|\varphi_{i,m,w}\|^2_{C^m(\overline{\Omega})}$. Using the constant

$$c_j^* = \max_{i \leq j} \sup_{\lambda \in I_{M,\delta_1}} |\lambda_{i,m,w} - \lambda|^{-1},$$

we find

$$\left| \sum_{i=1}^{j} \frac{1}{\lambda_{i,m,w} - \lambda} \mathcal{P}_{i,m,w} \right| \leq \left(j c_j^* \max_{i \leq j} c_i \right) \mathcal{D}_m$$

for all $\lambda \in I_{M,\delta_1}$.

2. For $\lambda \in I_{\delta_2}$ we single out $\mathcal{P}_{\mathrm{p},m,w}$, use the estimates in (3.24), the assumption that $\varphi_{\mathrm{p},m,w}$ is strongly positive and that the corresponding eigenvalue is simple.

■

3.3 Regularity results and dual estimates

In this section we derive estimates for *II* in (3.15). Therefore, we use regularity results of the Green operator and make use of dual spaces and maps. To this end, we adapt the proof of Theorem 14 and 16 for $m = 2$ in [58] to the general case and divide it into partial results. Some steps and paragraphs of the proofs are identical to the proof in [58] except that we replace 2 by m. Some arguments are described and proven in more detail.

In the proof of the following lemma it becomes clear why we chose $k_{n,m}$ as in (3.16).

Lemma 3.3.1 *Suppose that Condition A is satisfied. Let $k_{n,m}$ be defined as in (3.16). Then $(\mathcal{G}_{m,0,1}(w\cdot))^{k_{n,m}} f \in C^m(\overline{\Omega})$ for $f \in L^2(\Omega)$ and there exists a constant $C > 0$, depending on the domain, m and w, such that*

$$\left| \left((\mathcal{G}_{m,0,1}(w\cdot))^{k_{n,m}} f \right)(x) \right| \leq C \|f\|_{L^2_w(\Omega)} d(x)^m \text{ for all } f \in L^2(\Omega) \text{ and every } x \in \Omega.$$

Proof. For $\partial\Omega \in C^{2m,\gamma}$ we find by Agmon-Douglis-Nirenberg results that for all $p \in (1, \infty)$

$$\mathcal{G}_{m,0,w} : L^p(\Omega) \to W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$$

is a bounded operator. Let $\mathcal{I} : W^{2m,p}(\Omega) \hookrightarrow L^q(\Omega)$ be the Sobolev imbedding in (2.11) with sufficiently chosen q. Then we find that

$$\mathcal{I} \circ \mathcal{G}_{m,0,w} : L^p(\Omega) \to L^q(\Omega)$$

is a bounded operator. Hence, applying $\mathcal{G}_{m,0,1}(w\cdot)$ $k_{n,m}$-times and using Sobolev imbeddings after each step, we obtain that

$$(\mathcal{G}_{m,0,1}(w\cdot))^{k_{n,m}} : L^2_w(\Omega) \to W^{2m,q}(\Omega) \cap W^{m,q}_0(\Omega)$$

is a bounded operator for some $q > \frac{n}{m}$. Indeed, we find with analogous arguments as in [57, Lemma 13]:

- For $n \in \{2, \ldots, 2m-1\}$ it holds that $2 > \frac{n}{m}$ and Agmon-Douglis-Nirenberg results imply that $\mathcal{G}_{m,0,1}(w\cdot)$ is a bounded operator from $L^2_w(\Omega)$ to $W^{2m,2}(\Omega) \cap W^{m,2}_0(\Omega)$.
- For $n \in \{2m, \ldots, 6m-1\}$ we may use Agmon-Douglis-Nirenberg results and Sobolev imbeddings as in (2.11). We find with

$$\begin{cases} 2^*_n = \infty & \text{for } 2m \leq n \leq 4m, \\ 2^*_n = \frac{2n}{n-4m} \geq \frac{2n}{2m-1} & \text{for } 4m+1 \leq n \leq 6m-1, \end{cases}$$

  and $q = \frac{4n}{4m-1} \in (\frac{n}{m}, 2^*_n)$ that

$$\begin{aligned} (\mathcal{G}_{m,0,1}(w\cdot))^2 : \quad & L^2_w(\Omega) \xrightarrow{\mathcal{G}_{m,0,w}} W^{2m,2}(\Omega) \cap W^{m,2}_0(\Omega) \\ & \hookrightarrow L^q(\Omega) \xrightarrow{\mathcal{G}_{m,0,w}} W^{2m,q}(\Omega) \cap W^{m,q}_0(\Omega) \end{aligned}$$

  is a bounded operator.
- For $n \geq 6m$ we set $\ell := \left[\frac{n-2m}{4m}\right]$ and for $k \leq \ell$

$$p_0 = 2 \text{ and } p_{k+1} := (p_k)^*_n,$$

  with $(p_k)^*_n$ as in (2.11). Then we find

$$p_k = \frac{2n}{n-4mk} \text{ for } k \leq \ell$$

  and

$$2 = p_0 < p_1 < \cdots < p_\ell \leq \frac{n}{m} < p_{\ell+1} \leq \infty. \tag{3.25}$$

  Moreover, it holds that

$$p_{\ell+1} = \begin{cases} (p_\ell)^*_n = \infty & \text{for } 6m \leq n \leq 4m(\ell+1), \\ (p_\ell)^*_n = \frac{2n}{n-4m(\ell+1)} \geq \frac{2n}{2m-1} & \text{for } 4m(\ell+1)+1 \leq n. \end{cases} \tag{3.26}$$

The inequality in (3.26) follows from the fact that $\ell \geq \frac{n-2m}{4m} - 1 + \frac{1}{4m}$. Hence, setting

$$q = \frac{4n}{4m-1} \in \left(\frac{n}{m}, (p_\ell)_n^*\right)$$

and using the imbedding $\mathcal{I} : W^{2m,p_k}(\Omega) \cap W_0^{m,p_k}(\Omega) \hookrightarrow L^{p_{k+1}}(\Omega)$ for $k \in \{0, \ldots, \ell-1\}$ and $\mathcal{I} : W^{2m,p_\ell}(\Omega) \cap W_0^{m,p_\ell}(\Omega) \hookrightarrow L^q(\Omega)$, we obtain for

$$k_{n,m} = \left[\frac{n+2m}{4m}\right] + 1 = \ell + 2$$

that the operator

$$\begin{aligned} (\mathcal{G}_{m,0,1}(w\cdot))^{k_{n,m}} : \quad & L^2_w(\Omega) \xrightarrow{\mathcal{G}_{m,0,w}} W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega) \\ & \hookrightarrow L^{p_1}(\Omega) \xrightarrow{\mathcal{G}_{m,0,w}} \ldots \xrightarrow{\mathcal{G}_{m,0,w}} W^{2m,p_\ell}(\Omega) \cap W_0^{m,p_\ell}(\Omega) \\ & \hookrightarrow L^q(\Omega) \xrightarrow{\mathcal{G}_{m,0,w}} W^{2m,q}(\Omega) \cap W_0^{m,q}(\Omega) \end{aligned}$$

is bounded.

So, there exists a constant $c > 0$ such that

$$\|(\mathcal{G}_{m,0,1}(w\cdot))^{k_{n,m}} f\|_{W^{2m,q}(\Omega)\cap W_0^{m,q}(\Omega)} \leq c\|f\|_{L^2_w(\Omega)} \text{ for all } f \in L^2(\Omega). \tag{3.27}$$

Applying (2.11), one obtains that $W^{2m,q}(\Omega) \cap W_0^{m,q}(\Omega)$ imbeds in $C^m(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$, so there exists a constant $\tilde{c} > 0$ such that

$$\begin{aligned} \left| \left((\mathcal{G}_{m,0,1}(w\cdot))^{k_{n,m}} f \right)(x) \right| &\leq d(x)^m \| (\mathcal{G}_{m,0,1}(w\cdot))^{k_{n,m}} f\|_{C^m(\overline{\Omega})} \\ &\leq \tilde{c}\, d(x)^m \| (\mathcal{G}_{m,0,1}(w\cdot))^{k_{n,m}} f\|_{W^{2m,q}(\Omega)\cap W_0^{m,q}(\Omega)}. \end{aligned}$$

With (3.27), we find

$$\left| \left((\mathcal{G}_{m,0,1}(w\cdot))^{k_{n,m}} f \right)(x) \right| \leq c\,\tilde{c}\, d(x)^m \|f\|_{L^2_w(\Omega)}$$

for all $f \in L^2(\Omega)$. ■

If we use estimates for the norms in the dual spaces of $L^2_w(\Omega)$ or $W^{2m,q}(\Omega) \cap W_0^{m,q}(\Omega)$, we can also estimate $\mathcal{G}_{m,0,1}(w\cdot)^{k_{n,m}} f$ in $L^2_w(\Omega)$-norm, as we will see in the proof of the following lemma.

Lemma 3.3.2 *Suppose that Condition A is satisfied. Let $k_{n,m}$ be defined as in (3.16). Then there exists a constant $C > 0$, depending on the domain, w and m, such that*

$$\|(\mathcal{G}_{m,0,1}(w\cdot))^{k_{n,m}} f\|_{L^2_w(\Omega)} \leq C \int_\Omega |f(y)| d(y)^m dy \text{ for all } f \in L^2(\Omega).$$

Proof. In the proof of the previous lemma we showed that $(\mathcal{G}_{m,0,1}(w\cdot))^{k_{n,m}}$ is a linear and bounded operator from $L^2_w(\Omega)$ to $W^{2m,q}(\Omega)\cap W^{m,q}_0(\Omega)$ for some $q>\frac{n}{m}$ and $q\geq 2$. So, we find that its adjoint operator

$$(\mathcal{G}_{m,0,1}(w\cdot)^*)^{k_{n,m}}:\left(W^{2m,q}(\Omega)\cap W^{m,q}_0(\Omega)\right)^*\to\left(L^2_w(\Omega)\right)^*$$

is linear and bounded for $k_{n,m}$ as above. Hence, there is a constant $c>0$ such that for all $g\in(W^{2m,q}(\Omega)\cap W^{m,q}_0(\Omega))^*$

$$\|(\mathcal{G}_{m,0,1}(w\cdot)^*)^{k_{n,m}}g\|_{L^2_w(\Omega)^*}\leq c\,\|g\|_{\left(W^{2m,q}(\Omega)\cap W^{m,q}_0(\Omega)\right)^*}. \tag{3.28}$$

Moreover, we obtain that $L^2(\Omega)\subset(W^{2m,q}(\Omega)\cap W^{m,q}_0(\Omega))^*$ since for $q\geq 2$ every $f\in L^2(\Omega)$ determines a continuous linear mapping on $W^{2m,q}(\Omega)\cap W^{m,q}_0(\Omega)$ through

$$W^{2m,q}(\Omega)\cap W^{m,q}_0(\Omega)\ni h\mapsto\langle f,h\rangle_{L^2_w(\Omega)}, \tag{3.29}$$

see also [1, Paragraph 3.13]. In the following we write f if $f\in L^2(\Omega)$ is meant and $\mathbf{f}$ for the corresponding map $\mathbf{f}(h)=\langle f,h\rangle_{L^2_w(\Omega)}$ in (3.29).

For $f\in L^2(\Omega)$ the symmetry of the kernel implies that

$$\mathcal{G}_{m,0,1}(w\cdot)^*\mathbf{f}=\mathcal{G}_{m,0,1}(wf) \tag{3.30}$$

in the sense that $\mathcal{G}_{m,0,1}(wf)$ determines the continuous linear mapping $\mathcal{G}_{m,0,1}(w\cdot)^*\mathbf{f}$ on $L^2_w(\Omega)$. Indeed, we find for $u\in L^2_w(\Omega)$

$$\begin{aligned}(\mathcal{G}_{m,0,1}(w\cdot)^*\mathbf{f})u&=\int_\Omega f(x)w(x)\left(\int_\Omega G_{m,0,1}(x,y)w(y)u(y)dy\right)dx\\&=\int_\Omega w(y)u(y)\left(\int_\Omega f(x)w(x)G_{m,0,1}(x,y)dx\right)dy\\&=\langle u,\mathcal{G}_{m,0,1}(wf)\rangle_{L^2_w(\Omega)}.\end{aligned}$$

Furthermore, using the inequality

$$|\varphi(x)|\leq\|\varphi\|_{C^m(\overline{\Omega})}d(x)^m\leq\tilde{c}\|\varphi\|_{W^{2m,q}(\Omega)\cap W^{m,q}_0(\Omega)}d(x)^m \tag{3.31}$$

for all $\varphi\in W^{2m,q}(\Omega)\cap W^{m,q}_0(\Omega)$, we obtain for $q>\frac{n}{m}$

$$\begin{aligned}&\|\mathbf{f}\|_{\left(W^{2m,q}(\Omega)\cap W^{m,q}_0(\Omega)\right)^*}:=\\&\sup\left\{\left|\int_\Omega f(x)w(x)\varphi(x)dx\right|;\varphi\in W^{2m,q}(\Omega)\cap W^{m,q}_0(\Omega)\text{ with }\|\varphi\|_{W^{2m,q}(\Omega)}\leq 1\right\}\leq\\&\tilde{c}\sup\left\{\left|\int_\Omega f(x)w(x)\varphi(x)dx\right|;\varphi\in C^m(\overline{\Omega})\cap C^{m-1}_0(\overline{\Omega})\text{ with }\|\varphi\|_{C^m(\overline{\Omega})}\leq 1\right\}.\end{aligned} \tag{3.32}$$

With (3.31) and $c_{w,2}$ as in Remark 3.0.1 we find that for all $f\in L^2(\Omega)$ and

$\varphi \in C^m(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$ with $\|\varphi\|_{C^m(\overline{\Omega})} \leq 1$

$$\left| \int_\Omega f(x)w(x)\varphi(x)dx \right| \leq \int_\Omega |f(x)|w(x)|\varphi(x)|dx \leq c_{w,2} \int_\Omega |f(x)|d(x)^m dx. \tag{3.33}$$

Inequalities (3.32) and (3.33) imply

$$\|\mathbf{f}\|_{\left(W^{2m,q}(\Omega)\cap W_0^{m,q}(\Omega)\right)^*} \leq \tilde{c}\, c_{w,2} \int_\Omega |f(x)|\, d(x)^m\, dx. \tag{3.34}$$

By combining (3.28), (3.30), (3.34) and $(L_w^2(\Omega))^* = L_w^2(\Omega)$, we find a constant $C > 0$ such that

$$\left\| (\mathcal{G}_{m,0,1}(w\,\cdot))^{k_{n,m}} f \right\|_{L_w^2(\Omega)} \leq C \int_\Omega |f(x)|\, d(x)^m\, dx \tag{3.35}$$

holds for all $f \in L^2(\Omega)$. ■

We may apply the results of the last two lemmata and find an estimate from below and above for the infinite sum in (3.15) by the operator $\mathcal{D}_m$. Using the assumption that the p-th eigenfunction is strongly positive, we can also replace $\mathcal{D}_m$ in (3.36) with the projection onto the p-th eigenspace.

Corollary 3.3.3 *Suppose that Condition A is satisfied. Let $k_{n,m}$, $\mathcal{D}_m$ and $\mathcal{P}_{j*,m,w}$ be defined as in* (3.16), (2.17) *and* (2.16) *with j as in* (3.11)*. Then, there exists a constant $C_j > 0$, depending on the domain, m, w, M, δ_1 and δ_2, such that*

$$-C_j\mathcal{D}_m \leq \sum_{k=2k_{n,m}}^{\infty} \lambda^k \mathcal{G}_{m,0,w}^{k+1} \mathcal{P}_{j*,m,w} \leq C_j \mathcal{D}_m \quad \textit{for all } \lambda \in I_{M,\delta_1} \cup I_{\delta_2}, \tag{3.36}$$

where $I_{M,\delta_1}, I_{\delta_2}$ are defined as in (3.6) *and* (3.8)*.*

Proof. We know that

$$f \mapsto \sum_{k=0}^{\infty} \lambda^k \mathcal{G}_{m,0,w}^{k+1} \mathcal{P}_{j*,m,w} f = \mathcal{G}_{m,\lambda,w} \mathcal{P}_{j*,m,w} f \quad \text{for } f \in L^2(\Omega)$$

is a bounded operator from $L_w^2(\Omega)$ to $L_w^2(\Omega)$ for all $\lambda \in I_{M,\delta_1} \cup I_{\delta_2}$. Moreover, we can write

$$\sum_{k=2k_{n,m}}^{\infty} \lambda^k \mathcal{G}_{m,0,w}^{k+1} \mathcal{P}_{j*,m,w} f = \lambda^{2k_{n,m}} \mathcal{G}_{m,0,w}^{k_{n,m}} \left(\sum_{k=0}^{\infty} \lambda^k \mathcal{G}_{m,0,w}^{k+1} \mathcal{P}_{j*,m,w} \right) \mathcal{G}_{m,0,w}^{k_{n,m}} f.$$

Combining Lemma 3.3.1 and 3.3.2, we find constants $C_j, C_j', C_j'' > 0$ such that

$$\begin{aligned}\left|\sum_{k=2k_{n,m}}^{\infty} \lambda^k \left(\mathcal{G}_{m,0,w}^{k+1}\mathcal{P}_{j*,m,w}f\right)(x)\right| &= |\lambda|^{2k_{n,m}} \left|\left(\mathcal{G}_{m,0,w}^{k_{n,m}}\mathcal{G}_{m,\lambda,w}P_{j*,m,w}\mathcal{G}_{m,0,w}^{k_{n,m}}f\right)(x)\right| \\ &\leq C_j \|\mathcal{G}_{m,\lambda,w}P_{j*,m,w}\mathcal{G}_{m,0,w}^{k_{n,m}}f\|_{L^2_w(\Omega)} d(x)^m \leq C_j' \|\mathcal{G}_{m,0,w}^{k_{n,m}}f\|_{L^2_w(\Omega)} d(x)^m \\ &\leq C_j'' d(x)^m \int_\Omega |f(y)| d(y)^m dy = C_j''(\mathcal{D}_m|f|)(x) \text{ for all } f \in L^2(\Omega).\end{aligned}$$

■

So, estimates for *I* and *II* in (3.14) and (3.15) are proven. At this point only an estimate for *III* in (3.15) is missing to complete the proofs of Theorem 3.1.1 and 3.1.3.

3.4 Estimates for the iterated Green operator

In this section we will consider *III* in (3.15). To this end, we will use known estimates for $\mathcal{G}_{m,0,1}$ to derive estimates for powers of the polyharmonic Green operator. Analogous to Lemma 8, Corollary 9 and Lemma 10 in [57], one can prove a similar estimate as in (3.2) for the corresponding iterated Green operator $\mathcal{G}_{m,0,w}^k$ with $k \in \mathbb{N}^+$. Therefore, we look at the operator $\mathcal{H}_{n,m}$ defined in (3.4) and its iterates. For domains with smooth boundary $\partial\Omega \in C^{2m,\gamma}$ one may estimate the operator $\mathcal{H}_{n,m}^k$ through the kernel function $H_{n,m,k} : \overline{\Omega} \times \overline{\Omega} \setminus \left\{(x,x); x \in \overline{\Omega}\right\} \to [0,\infty)$, defined by

$$\mathcal{H}_{n,m}^k f(x) = \int_\Omega H_{n,m,k}(x,y) f(y) dy, \tag{3.37}$$

and one finds:

Lemma 3.4.1 ([34]) *Let $k \in \mathbb{N}^+$. Then there are constants $c_{\Omega,k,m}, C_{\Omega,k,m} > 0$ such that*

$$c_{\Omega,k,m}\tilde{H}_{n,m,k}(x,y) \leq H_{n,m,k}(x,y) \leq C_{\Omega,k,m}\tilde{H}_{n,m,k}(x,y),$$

where

$$\tilde{H}_{n,m,k}(x,y) = \begin{cases} d(x)^m d(y)^m & \text{for } k > 1 + \frac{n}{2m}, \\ d(x)^m d(y)^m \log\left(2 + \frac{1}{|x-y|^2 + d(x)d(y)}\right) & \text{for } k = 1 + \frac{n}{2m}, \\ (d(x)d(y))^{mk-n/2} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}^{m-mk+n/2} & \text{for } \frac{n}{2m} < k < 1 + \frac{n}{2m}, \\ \log\left(1 + \left(\frac{d(x)d(y)}{|x-y|^2}\right)^m\right) & \text{for } k = \frac{n}{2m}, \\ |x-y|^{2mk-n} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}^m & \text{for } k < \frac{n}{2m}. \end{cases} \tag{3.38}$$

In Corollary 9 of [57] we find the following result with analogous proofs for $m = 2$:

Corollary 3.4.2 *Let* $k^*_{n,m} = \left[\frac{n}{2m}\right] + 2$. *Then it holds for* $k \in \mathbb{N}^+$*:*

1. *There exist constants* $C_{3.4.2.1,k} > 0$, *depending on the domain and* m, *such that*

$$\mathcal{D}^k_m = C_{3.4.2.1,k}\mathcal{D}_m. \tag{3.39}$$

2. *There exist constants* $C_{3.4.2.2,k} > 0$, *depending on the domain and* m, *such that*

$$\mathcal{D}_m \leq C_{3.4.2.2,k}\mathcal{H}^k_{n,m} \tag{3.40}$$

and no reverse estimate for $k < k^*_{n,m}$.

3. *There exist contants* $C_{3.4.2.3,k} > 0$, *depending on the domain and* m, *such that*

$$\mathcal{H}^{k+1}_{n,m} \leq C_{3.4.2.3,k}\mathcal{H}^k_{n,m}. \tag{3.41}$$

4. *For* $k \geq k^*_{n,m}$ *there exist* $c_{3.4.2.4,k}, C_{3.4.2.4,k} > 0$, *depending on the domain and* m, *such that*

$$c_{3.4.2.4,k}\mathcal{D}_m \leq \mathcal{H}^k_{n,m} \leq C_{3.4.2.4,k}\mathcal{D}_m. \tag{3.42}$$

5. *There exist* $c_{3.4.2.5,k}, C_{3.4.2.5,k} > 0$, *depending on the domain and* m, *such that*

$$c_{3.4.2.5,k}\mathcal{D}_m \leq \mathcal{D}_m\mathcal{H}_{n,m} \leq C_{3.4.2.5,k}\mathcal{D}_m \text{ and} \tag{3.43}$$

$$c_{3.4.2.5,k}\mathcal{D}_m \leq \mathcal{H}_{n,m}\mathcal{D}_m \leq C_{3.4.2.5,k}\mathcal{D}_m. \tag{3.44}$$

Proof.

1. For $f \in L^2(\Omega)$ it holds that

$$(\mathcal{D}^k_m f)(x) = (\mathcal{D}_m f)(x)\left(\int_\Omega d(y)^{2m}dy\right)^{k-1} \quad \text{for } k \in \mathbb{N}^+ \text{ and } x \in \Omega.$$

So we find $C_{3.4.2.1,k} = \left(\int_\Omega d(y)^{2m} dy\right)^{k-1}$.

2. Inequality (3.40) follows from (3.38). Indeed, for $k > 1 + \frac{n}{2m}$ it follows immediately. For $k = 1 + \frac{n}{2m}$ it holds for all $x, y \in \Omega$ with $x \neq y$

$$\tilde{H}_{n,m,k}(x,y) \geq \log(2) d(x)^m d(y)^m.$$

All other estimates follow from $|x-y|, d(x), d(y) \leq \operatorname{diam}(\Omega)$: For $\frac{n}{2m} < k < 1 + \frac{n}{2m}$ one gets for all $x, y \in \Omega$ with $x \neq y$

$$\begin{aligned}\tilde{H}_{n,m,k}(x,y) &\geq (d(x)d(y))^{mk-n/2} \left(\frac{d(x)d(y)}{\operatorname{diam}(\Omega)^2}\right)^{m-mk+n/2} \\ &= d(x)^m d(y)^m \operatorname{diam}(\Omega)^{-2m+2mk-n}.\end{aligned}$$

Using [21, Lemma 4.5] and the last inequality, we find for $k = \frac{n}{2m}$ a constant $c > 0$ such that for all $x, y \in \Omega$ with $x \neq y$

$$\begin{aligned}\tilde{H}_{n,m,k}(x,y) &\geq c \log\left(2 + \frac{d(y)}{|x-y|}\right) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}^m \\ &\geq c \log(2) \operatorname{diam}(\Omega)^{-2m} d(x)^m d(y)^m.\end{aligned}$$

For $k < \frac{n}{2m}$ we obtain with similar arguments

$$\tilde{H}_{n,m,k}(x,y) \geq \operatorname{diam}(\Omega)^{2mk-n} \left(\frac{d(x)d(y)}{\operatorname{diam}(\Omega)^2}\right)^m = \operatorname{diam}(\Omega)^{2m(k-1)-n} d(x)^m d(y)^m$$

for all $x, y \in \Omega$ with $x \neq y$.

3. For (3.41) we have to prove that there exists a constant $c > 0$ such that $\tilde{H}_{n,m,k_1}(x,y) \leq c\, \tilde{H}_{n,m,k_2}(x,y)$ for $(x,y) \in \Omega \times \Omega$ and $k_1 > k_2$. Analogous to [57, Corollary 9] we show this result for $k_1, k_2 \in \frac{1}{2m}\mathbb{N}^+$ and $k_1 = k_2 + \frac{1}{2m}$. To this end, let $s = |x-y|^2$ and $t = d(x)d(y)$ for a short notation. Then we find:

   - For $k_1 > k_2 > 1 + \frac{n}{2m}$ it holds $\tilde{H}_{n,m,k_1} = \tilde{H}_{n,m,k_2}$.
   - For $k_1 > k_2 = 1 + \frac{n}{2m}$ the inequality follows with $c = \frac{1}{\log(2)}$.
   - For $k_1 = 1 + \frac{n}{2m} > k_2 = \frac{2m-1}{2m} + \frac{n}{2m}$ we find

$$\begin{aligned}t^m \log\left(2 + \frac{1}{s+t}\right) &\leq t^m \left(1 + \min\left\{\frac{1}{s}, \frac{1}{t}\right\}^{\frac{1}{2}}\right) \\ &\leq c_\Omega t^m \min\left\{\frac{1}{s}, \frac{1}{t}\right\}^{\frac{1}{2}} \\ &= c_\Omega t^{m-\frac{1}{2}} \min\left\{1, \frac{t}{s}\right\}^{\frac{1}{2}},\end{aligned}$$

with $c_\Omega = 1+\mathrm{diam}(\Omega)$. We used that $y \mapsto 1+\sqrt{y}-\log(2+y)$ is positive for all $y \geq 0$ and $\frac{1}{s+t} \leq \min\left\{\frac{1}{s}, \frac{1}{t}\right\}$.

– For $\frac{n}{2m} < k_2 < k_1 < 1+\frac{n}{2m}$ the estimate is equivalent to

$$t^{\frac{1}{2}} = \mathrm{diam}(\Omega)\min\left\{1, \frac{t}{\mathrm{diam}(\Omega)^2}\right\}^{\frac{1}{2}} \leq \mathrm{diam}(\Omega)\min\left\{1, \frac{t}{s}\right\}^{\frac{1}{2}}. \tag{3.45}$$

– For $k_1 = \frac{n}{2m}+\frac{1}{2m} > k_2 = \frac{n}{2m}$ we make use of (3.45) and find for $t > s$

$$\begin{aligned} t^{\frac{1}{2}}\min\left\{1, \frac{t}{s}\right\}^{m-\frac{1}{2}} &\leq \mathrm{diam}(\Omega)\min\left\{1, \frac{t}{s}\right\}^{m} = \mathrm{diam}(\Omega) \\ &\leq \frac{\mathrm{diam}(\Omega)}{\log(2)}\log\left(1+\frac{t^m}{s^m}\right), \end{aligned}$$

and for $t \leq s$ we use $\frac{1}{\log(2)}\log(1+y) - y \geq 0$ for $y \in [0,1]$ and get

$$\begin{aligned} t^{\frac{1}{2}}\min\left\{1, \frac{t}{s}\right\}^{m-\frac{1}{2}} &\leq \mathrm{diam}(\Omega)\min\left\{1, \frac{t}{s}\right\}^{m} = \mathrm{diam}(\Omega)\frac{t^m}{s^m} \\ &\leq \frac{\mathrm{diam}(\Omega)}{\log(2)}\log\left(1+\frac{t^m}{s^m}\right). \end{aligned}$$

– For $k_1 = \frac{n}{2m} > k_2 = \frac{n}{2m} - \frac{1}{2m}$ we find for $t \leq s$

$$\log\left(1+\frac{t^m}{s^m}\right) \leq \frac{t^m}{s^m} = \min\left\{1, \frac{t^m}{s^m}\right\} \leq \frac{\mathrm{diam}(\Omega)}{s^{\frac{1}{2}}}\min\left\{1, \frac{t^m}{s^m}\right\}$$

and for $t > s$

$$\begin{aligned} \log\left(1+\frac{t^m}{s^m}\right) &\leq \log\left(1+\frac{\mathrm{diam}(\Omega)^{2m}}{s^m}\right) \leq 2m\frac{\mathrm{diam}(\Omega)}{s^{\frac{1}{2}}} \\ &= 2m\frac{\mathrm{diam}(\Omega)}{s^{\frac{1}{2}}}\min\left\{1, \frac{t^m}{s^m}\right\} \end{aligned}$$

since $y \mapsto \log(1+y^{2m}) - 2my$ is negative for $y = 1$ and decreasing for $y > 1$.

– For $\frac{n}{2m} > k_1 > k_2$ we obtain the result since $s^{\frac{1}{2}} \leq \mathrm{diam}(\Omega)$.

4. Inequality (3.42) follows from (3.38) and (3.40).

5. Inequality (3.42) implies (3.43) since

$$\begin{aligned} C^{-1}_{3.4.2.4,k^*_{n,m}} c_{3.4.2.4,k^*_{n,m}+1}\mathcal{D}_m &\leq C^{-1}_{3.4.2.4,k^*_{n,m}}\mathcal{H}^{k^*_{n,m}+1}_{n,m} = C^{-1}_{3.4.2.4,k^*_{n,m}}\mathcal{H}^{k^*_{n,m}}_{n,m}\mathcal{H}_{n,m} \\ &\leq \mathcal{D}_m\mathcal{H}_{n,m} \leq c^{-1}_{3.4.2.4,k^*_{n,m}}\mathcal{H}^{k^*_{n,m}+1}_{n,m} \\ &\leq c^{-1}_{3.4.2.4,k^*_{n,m}} C_{3.4.2.4,k^*_{n,m}+1}\mathcal{D}_m. \end{aligned}$$

Analogously, we find the inequality in (3.44).

■

This corollary is helpful to find an inequality similar to (3.2) for the iterated Green operator. For the special case $m = 2$ see Lemma 10 in [57]. We recall the steps of the proof in [57] except that we replace 2 with m and include a weight function.

Lemma 3.4.3 *Suppose that Condition A is satisfied. Let* $\mathcal{D}_m$ *and* $\mathcal{H}^k_{n,m}$ *be defined as in* (2.17) *and* (3.37)*. For* $k \in \mathbb{N}^+$ *we find three constants* $C_{3.4.3.1,k}, C_{3.4.3.2,k}, C_{3.4.3.3,k} > 0$*, depending also on the domain and* w*, such that*

$$C_{3.4.3.1,k}\mathcal{H}^k_{n,m} \le \mathcal{G}^k_{m,0,w} + C_{3.4.3.2,k}\mathcal{D}_m \le C_{3.4.3.3,k}\mathcal{H}^k_{n,m}. \tag{3.46}$$

Proof. Using Corollary 3.4.2, we get $\mathcal{D}_m \le C_{3.4.2.2,k}\mathcal{H}^k_{n,m}$. Moreover, with (3.2) we find a constant $c_{n,m} > 0$ such that

$$|G_{m,0,1}(x,y)| \le c_{n,m}H_{n,m}(x,y),$$

so using the estimate for the weight function in (3.1), we obtain

$$\mathcal{G}^k_{m,0,w}f = \left(\mathcal{G}_{m,0,1}(w\cdot)\right)^k f \le \left(|\mathcal{G}_{m,0,1}|(w\cdot)\right)^k f \le c^k_{w,2}c^k_{n,m}\mathcal{H}^k_{n,m}f \text{ for all } 0 \le f \in L^2(\Omega),$$

where $|\mathcal{G}_{m,0,1}|(wf)(x) := \int_\Omega |G_{m,0,1}(x,y)|w(y)f(y)dy$. These estimates imply the estimate on the right-hand side of (3.46). For the left-hand side one uses induction. One finds for $k = 1$ that the assertion holds by (3.2). Supposing that (3.46) holds true for some $k \in \mathbb{N}^+$, we find

$$C_{3.4.3.1,k}\mathcal{H}^{k+1}_{n,m} \le \mathcal{H}_{n,m}\left(\mathcal{G}^k_{m,0,w} + C_{3.4.3.2,k}\mathcal{D}_m\right) \le C_{3.4.3.3,k}\mathcal{H}^{k+1}_{n,m}$$

and with (3.46) for $k = 1$

$$\begin{aligned}
C_{3.4.3.1,1}C_{3.4.3.1,k}\mathcal{H}^{k+1}_{n,m} &\le C_{3.4.3.1,1}\mathcal{H}_{n,m}\left(\mathcal{G}^k_{m,0,w} + C_{3.4.3.2,k}\mathcal{D}_m\right)\\
&\le \left(\mathcal{G}_{m,0,w} + C_{3.4.3.2,1}\mathcal{D}_m\right)\left(\mathcal{G}^k_{m,0,w} + C_{3.4.3.2,k}\mathcal{D}_m\right)\\
&= \mathcal{G}^{k+1}_{m,0,w} + C_{3.4.3.2,k}\mathcal{G}_{m,0,w}\mathcal{D}_m + C_{3.4.3.2,1}\mathcal{D}_m\left(\mathcal{G}^k_{m,0,w} + C_{3.4.3.2,k}\mathcal{D}_m\right)\\
&\le \mathcal{G}^{k+1}_{m,0,w} + C_{3.4.3.2,k}C_{3.4.3.3,1}\mathcal{H}_{n,m}\mathcal{D}_m + C_{3.4.3.2,1}C_{3.4.3.3,k}\mathcal{D}_m\mathcal{H}^k_{n,m} + C_{3.4.3.2,1}C_{3.4.3.2,k}\mathcal{D}^2_m\\
&\le \mathcal{G}^{k+1}_{m,0,w} + C^*\mathcal{D}_m.
\end{aligned}$$

The last inequality with $C^* > 0$ follows from Corollary 3.4.2. ■

The following lemma and a similar proof for $m = 2$ is published in [57, Lemma 11].

Lemma 3.4.4 *For all* $\varepsilon > 0$ *there is a constant* $C_{\varepsilon,3.4.4} > 0$*, depending on the domain and* m*, such that the following inequality holds:*

$$0 \le \mathcal{H}^2_{n,m} \le \varepsilon\mathcal{H}_{n,m} + C_{\varepsilon,3.4.4}\mathcal{D}_m,$$

where $\mathcal{D}_m$ and $\mathcal{H}_{n,m}$ are defined as in (2.17) *and* (3.4).

Proof. Let $\varepsilon > 0$. We prove that there is a constant $C_{\varepsilon,3.4.4} > 0$ such that

$$\tilde{H}_{n,m,2}(x,y) \leq \varepsilon \tilde{H}_{n,m,1}(x,y) + C_{\varepsilon,3.4.4} d(x)^m d(y)^m \text{ for all } x,y \in \Omega, \tag{3.47}$$

where $\tilde{H}_{n,m,k}$ is defined as in (3.38). As in [57, Lemma 11] we will distinguish five cases:

- For $n \in \{2, \ldots, 2m-1\}$, the estimate follows directly from (3.38).
- For $n = 2m$ we use that $a \leq \frac{1}{2}b + \frac{1}{2}\frac{a^2}{b}$ holds for all $a, b > 0$. Setting $a = \tilde{H}_{n,m,2}(x,y)$ and $b = \varepsilon \tilde{H}_{n,m,1}(x,y)$, we get

$$\tilde{H}_{n,m,2}(x,y) \leq \frac{1}{2}\varepsilon \tilde{H}_{n,m,1}(x,y) + \frac{1}{2\varepsilon}\frac{\tilde{H}_{n,m,2}(x,y)^2}{\tilde{H}_{n,m,1}(x,y) d(x)^m d(y)^m} d(x)^m d(y)^m.$$

  If we set

$$s = |x-y|^2 \quad \text{and} \quad t = d(x)d(y) \tag{3.48}$$

  for short notation, we find

$$\begin{aligned} \frac{\tilde{H}_{n,m,2}(x,y)^2}{\tilde{H}_{n,m,1}(x,y) d(x)^m d(y)^m} &= t^m \frac{\log\left(2 + \frac{1}{s+t}\right)^2}{\log\left(1 + \frac{t^m}{s^m}\right)} \leq t^m \frac{\log\left(2 + \frac{1}{s+t}\right)^2}{\min\left\{1, \frac{t^m}{s^m}\right\}\log(2)} \\ &= \frac{\max\left\{s^m, t^m\right\}}{\log(2)} \log\left(2 + \frac{1}{t+s}\right)^2. \end{aligned}$$

  Since this is bounded by a constant $C > 0$, depending on the domain and m, we obtain

$$\tilde{H}_{n,m,2}(x,y) \leq \frac{\varepsilon}{2}\tilde{H}_{n,m,1}(x,y) + \frac{1}{2\varepsilon} C d(x)^m d(y)^m.$$

  Scaling ε, we find (3.47).

- For $n \in \{2m+1, 2m+2, \ldots, 4m-1\}$ we get with s and t as in (3.48) and $-m + \frac{n}{2} \geq m - \frac{2m^2}{n}$ that

$$\begin{aligned} \tilde{H}_{n,m,2}(x,y) &= t^{2m-\frac{n}{2}} \min\left\{1, \frac{t}{s}\right\}^{-m+\frac{n}{2}} \\ &\leq t^{2m-\frac{n}{2}} \min\left\{1, \frac{t}{s}\right\}^{m-\frac{2m^2}{n}} \left(\frac{t}{s}\right)^{-2m+\frac{2m^2}{n}+\frac{n}{2}} \\ &= \left(s^{m-\frac{n}{2}} \min\left\{1, \frac{t}{s}\right\}^m\right)^{1-\frac{2m}{n}} t^{\frac{2m^2}{n}} \\ &= \tilde{H}_{n,m,1}(x,y)^{1-\frac{2m}{n}} \left(d(x)^m d(y)^m\right)^{\frac{2m}{n}}. \end{aligned}$$

- For $n = 4m$ it follows with (3.48)

$$\tilde{H}_{n,m,2}(x,y) = \log\left(1 + \left(\frac{t}{s}\right)^m\right) \le \left(\frac{t}{s}\right)^{\frac{m}{2}} \min\left\{1, \frac{t}{s}\right\}^{\frac{m}{2}}$$
$$= \tilde{H}_{n,m,1}(x,y)^{\frac{1}{2}} \left(d(x)^m d(y)^m\right)^{\frac{1}{2}}.$$

- For $n \ge 4m+1$ we obtain with (3.48)

$$\begin{aligned}\tilde{H}_{n,m,2}(x,y) &= s^{2m-\frac{n}{2}} \min\left\{1, \frac{t}{s}\right\}^m \\ &\le \left(s^{m-\frac{n}{2}} \min\left\{1, \frac{t}{s}\right\}^m\right)^{1-\frac{2m}{n}} t^{\frac{2m^2}{n}} \\ &= \tilde{H}_{n,m,1}(x,y)^{1-\frac{2m}{n}} \left(d(x)^m d(y)^m\right)^{\frac{2m}{n}}.\end{aligned}$$

Hence, using Young's inequality, one gets for $n \ge 2m+1$ and all $x, y \in \Omega$ with $x \ne y$

$$\begin{aligned}\tilde{H}_{n,m,2}(x,y) &\le \tilde{H}_{n,m,1}(x,y)^{1-\frac{2m}{n}} \left(d(x)^m d(y)^m\right)^{\frac{2m}{n}} \\ &\le \left(1 - \tfrac{2m}{n}\right) \varepsilon \tilde{H}_{n,m,1}(x,y) + \tfrac{2m}{n} \varepsilon^{-\frac{1}{2m}(n-2m)} d(x)^m d(y)^m.\end{aligned}$$

■

To complete the proof of Theorem 3.1.1 and 3.1.3 we still have to find estimates for *III* in (3.15) from above and below. Using the previous lemmata and corollary, we get the following proposition:

Proposition 3.4.5 *Suppose that Condition A is satisfied. Let $\mathcal{D}_m$, $\mathcal{H}_{n,m}$ and $\mathcal{P}_{j*,m,w}$ be as defined in* (2.17), (3.4) *and* (2.16) *with $j \in \mathbb{N}^+$ as in* (3.11) *and* (3.12). *There are constants $C_{3.4.5.1}, C_{3.4.5.2}, C_{3.4.5.3}, C_{3.4.5.4} > 0$, depending on the domain, j, m, w, M, δ_1 and δ_2, such that*

$$C_{3.4.5.1}\mathcal{H}_{n,m} - C_{3.4.5.2}\mathcal{D}_m \le \sum_{k=0}^{2k_{n,m}-1} \lambda^k \mathcal{G}_{m,0,w}^{k+1} \mathcal{P}_{j*,m,w} \le C_{3.4.5.3}\mathcal{H}_{n,m} - C_{3.4.5.4}\mathcal{D}_m$$

for all $\lambda \in I_{M,\delta_1} \cup I_{\delta_2}$ with I_{M,δ_1} and I_{δ_2} as in (3.6) *and* (3.8).

Proof.

- Let $\lambda \ge 0$. Analogous to the proof of Lemma 19 in [57], we can use Lemma 3.4.3, (3.41) and (3.42) to find for $k \in \mathbb{N}^+$

$$\begin{aligned}C_{3.4.2.2,k}^{-1} C_{3.4.3.1,k}\mathcal{D}_m \le C_{3.4.3.1,k}\mathcal{H}_{n,m}^k \le \mathcal{G}_{m,0,w}^k + C_{3.4.3.2,k}\mathcal{D}_m \le C_{3.4.3.3,k}\mathcal{H}_{n,m}^k \\ \le C_{3.4.3.3,k}\tilde{C}_{3.4.2.3,k}\mathcal{H}_{n,m}.\end{aligned}$$

  Setting $C := \max\{C_{3.4.3.2,k}; k \in \{1, \ldots, 2k_{n,m}\}\}$, we get

$$\mathcal{G}_{m,0,w}^k + C\mathcal{D}_m \ge 0 \text{ for all } k \in \{1, \ldots, 2k_{n,m}\}.$$

Then we obtain

$$\sum_{k=0}^{2k_{n,m}-1} \lambda^k \mathcal{G}_{m,0,w}^{k+1} + C \left(\sum_{k=1}^{2k_{n,m}-1} \lambda^k \right) \mathcal{D}_m + C_{3.4.3.2,1} \mathcal{D}_m \geq C_{3.4.3.1,1} \mathcal{H}_{n,m} \tag{3.49}$$

for all $\lambda \in I_{M,\delta_1} \cup I_{\delta_2}$ with $\lambda \geq 0$.

Analogously, we find for $\tilde{C} := \min\{C_{3.4.3.2,k}; k \in \{1, \dots, 2k_{n,m}\}\}$ a constant $\hat{C} > 0$ such that

$$\mathcal{G}_{m,0,w}^k + \tilde{C}\mathcal{D}_m \leq \hat{C}\mathcal{H}_{n,m} \text{ for all } k \in \{1, \dots, 2k_{n,m}\}.$$

Hence

$$\sum_{k=0}^{2k_{n,m}-1} \lambda^k \mathcal{G}_{m,0,w}^{k+1} + \tilde{C} \left(\sum_{k=0}^{2k_{n,m}-1} \lambda^k \right) \mathcal{D}_m \leq \hat{C} \left(\sum_{k=0}^{2k_{n,m}-1} \lambda^k \right) \mathcal{H}_{n,m} \tag{3.50}$$

for all $\lambda \in I_{M,\delta_1} \cup I_{\delta_2}$ with $\lambda \geq 0$.

- Let $\lambda < 0$. Then, we get

$$\sum_{k=0}^{2k_{n,m}-1} \lambda^k \mathcal{G}_{m,0,w}^{k+1} = \sum_{k=0}^{k_{n,m}-1} |\lambda|^{2k} \mathcal{G}_{m,0,w}^{2k+1} - \sum_{k=0}^{k_{n,m}-1} |\lambda|^{2k+1} \mathcal{G}_{m,0,w}^{2k+2}.$$

As is (3.49) and (3.50), we may show that there are constants $C_{i,m} > 0$, depending also on w, with $i \in \{1, 2, 3, 4\}$ such that

$$C_{1,m}\mathcal{H}_{n,m} - C_{2,m}\mathcal{D}_m \leq \sum_{k=0}^{k_{n,m}-1} |\lambda|^{2k} \mathcal{G}_{m,0,w}^{2k+1} \leq C_{3,m}\mathcal{H}_{n,m} - C_{4,m}\mathcal{D}_m \tag{3.51}$$

for all $\lambda \in I_{M,\delta_1} \cup I_{\delta_2}$ with $\lambda < 0$. With similar arguments and (3.41), we get constants $\tilde{C}_{i,m} > 0$, depending on w, with $i \in \{1, 2, 3\}$ such that

$$-\tilde{C}_{1,m}\mathcal{D}_m \leq \sum_{k=0}^{k_{n,m}-1} |\lambda|^{2k+1} \mathcal{G}_{m,0,w}^{2k+2} \leq \tilde{C}_{2,m}\mathcal{H}_{n,m}^2 - \tilde{C}_{3,m}\mathcal{D}_m.$$

Let $\varepsilon > 0$. Using Lemma 3.4.4, we then find

$$-\tilde{C}_{1,m}\mathcal{D}_m \leq \sum_{k=0}^{k_{n,m}-1} |\lambda|^{2k+1} \mathcal{G}_{m,0,w}^{2k+2} \leq \tilde{C}_{2,m}\varepsilon\mathcal{H}_{n,m} + (\tilde{C}_{2,m}C_{\varepsilon,3.4.4} - \tilde{C}_{3,m})\mathcal{D}_m \tag{3.52}$$

for all $\lambda \in I_{M,\delta_1} \cup I_{\delta_2}$ with $\lambda < 0$. The inequalities in (3.51) and (3.52) imply

$$\begin{aligned} &(C_{1,m} - \tilde{C}_{2,m}\varepsilon)\mathcal{H}_{n,m} - (C_{2,m} + \tilde{C}_{2,m}C_{\varepsilon,3.4.4} - \tilde{C}_{3,m})\mathcal{D}_m \\ &\quad \le \sum_{k=0}^{2k_{n,m}-1} \lambda^k \mathcal{G}_{m,0,w}^{k+1} \le C_{3,m}\mathcal{H}_{n,m} - (C_{4,m} - \tilde{C}_{1,m})\mathcal{D}_m \end{aligned} \tag{3.53}$$

for all $\lambda \in I_{M,\delta_1} \cup I_{\delta_2}$ with $\lambda < 0$, where ε may be chosen sufficiently small such that $C_{1,m} > \tilde{C}_{2,m}\varepsilon$.

- Moreover we get

$$\sum_{k=0}^{2k_{n,m}-1} \lambda^k \mathcal{G}_{m,0,w}^{k+1} \sum_{i=1}^{j} \mathcal{P}_{i,m,w} = \sum_{i=1}^{j} \sum_{k=0}^{2k_{n,m}-1} \frac{\lambda^k}{\lambda_{i,m,w}^{k+1}} \mathcal{P}_{i,m,w}$$

and therefore

$$\sum_{k=0}^{2k_{n,m}-1} \lambda^k \mathcal{G}_{m,0,w}^{k+1} \sum_{i=1}^{j} \mathcal{P}_{i,m,w} = \sum_{i=1}^{j} \frac{1 - \left(\frac{\lambda}{\lambda_{i,m,w}}\right)^{2k_{n,m}}}{\lambda_{i,m,w} - \lambda} \mathcal{P}_{i,m,w}.$$

Since $\frac{1 - \left(\frac{\lambda}{\lambda_{i,m,w}}\right)^{2k_{n,m}}}{\lambda_{i,m,w} - \lambda}$ is bounded for $\lambda \in I_{M,\delta_1} \cup I_{\delta_2}$ and using (3.24), we obtain two constants $c_{j,m}, \tilde{c}_{j,m} > 0$ such that

$$-c_{j,m}\mathcal{D}_m \le \sum_{k=0}^{2k_{n,m}-1} \lambda^k \mathcal{G}_{m,0,w}^{k+1} \sum_{i=1}^{j} \mathcal{P}_{i,m,w} \le \tilde{c}_{j,m}\mathcal{D}_m. \tag{3.54}$$

Combining (3.49), (3.50), (3.53) and (3.54), we find the result.

■

3.5 Asymptotic formulas for the weighted eigenfunctions and eigenvalues

In order to prove Corollary 3.3.3, we can also use asymptotic estimates for the eigenfunctions and eigenvalues. The eigenfunctions can be estimated in the C^m-norm using the corresponding eigenvalues. So we find an analogous result as in [57, Lemma 13].

Lemma 3.5.1 *Suppose that Condition A is satisfied and let $k_{n,m}$ be defined as in* (3.16)*. Then there exist constants $\tilde{C}_{3.5.1}, C_{3.5.1} > 0$, depending on the domain, w and*

m, such that for all $i \in \mathbb{N}^+$:

$$\|\varphi_{i,m,w}\|_{C^m(\overline{\Omega})} \le C_{3.5.1}\lambda_{i,m,w}^{k_{n,m}} \le \tilde{C}_{3.5.1}\lambda_{i,m,w}^{\frac{n}{4m}+\frac{3}{2}}, \tag{3.55}$$

$$|\varphi_{i,m,w}(x)| \le C_{3.5.1}\lambda_{i,m,w}^{k_{n,m}} d(x)^m \le \tilde{C}_{3.5.1}\lambda_{i,m,w}^{\frac{n}{4m}+\frac{3}{2}} d(x)^m \text{ for all } x \in \Omega. \tag{3.56}$$

Proof. We know that the first eigenvalue of (1.10) is bounded from below by a constant $c > 0$. So, we get for $\alpha < \beta$

$$\lambda_{i,m,w}^{\alpha} = \lambda_{i,m,w}^{\alpha-\beta}\lambda_{i,m,w}^{\beta} \le c^{\alpha-\beta}\lambda_{i,m,w}^{\beta} \text{ for all } i \in \mathbb{N}^+. \tag{3.57}$$

Using the steps in the proof of Lemma 3.3.1, we find constants $\tilde{c}, c > 0$ such that for some $q > \frac{n}{m}$ and all $i \in \mathbb{N}^+$

$$\begin{aligned}\frac{1}{\lambda_{i,m,w}^{k_{n,m}}}\|\varphi_{i,m,w}\|_{C^m(\overline{\Omega})} &= \|\mathcal{G}_{m,0,w}^{k_{n,m}}\varphi_{i,m,w}\|_{C^m(\overline{\Omega})} \\ &\le \tilde{c}\|\mathcal{G}_{m,0,w}^{k_{n,m}}\varphi_{i,m,w}\|_{W^{2m,q}(\Omega)\cap W_0^{m,q}(\Omega)} \\ &\le c\,\tilde{c}\|\varphi_{i,m,w}\|_{L_w^2(\Omega)}.\end{aligned}$$

Since $\|\varphi_{i,m,w}\|_{L_w^2(\Omega)} = 1$ and (3.57) holds, we obtain inequality (3.55). Applying the mean value theorem m-times to $\varphi_{i,m,w}$, we get inequality (3.56). ∎

***Remark* 3.5.2** *Using Taylor, starting from $x^* \in \partial\Omega$ such that $d(x) = |x - x^*|$ and $\varphi_{i,m,w} = \frac{\partial}{\partial\nu}\varphi_{i,m,w} = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1}\varphi_{i,m,w} = 0$ on $\partial\Omega$, we may also find instead of* (3.56) *the inequality*

$$|\varphi_{i,m,w}(x)| \le \tfrac{1}{m!}C_{3.5.1}\lambda_{i,m,w}^{k_{n,m}} d(x)^m \le \tfrac{1}{m!}\tilde{C}_{3.5.1}\lambda_{i,m,w}^{\frac{n}{4m}+\frac{3}{2}} d(x)^m \text{ for all } x \in \Omega.$$

Since $C_{3.5.1}$ is also dependent on m and we do not specify this dependence, we can even use the weaker estimate in (3.56)*.*

***Remark* 3.5.3** *In [57] we have proven for $m = 2$ that*

$$\|\varphi_{i,2,1}\|_{C^2(\overline{\Omega})} \le C\lambda_{i,2,1}^{\frac{3}{4}+\frac{n}{8}}.$$

We note, that

$$k_{n,2}\begin{cases} < \frac{3}{4}+\frac{n}{8} & \text{for } n = 3+8k, k \in \mathbb{N}, \\ = \frac{3}{4}+\frac{n}{8} & \text{for } n = 2+8k, k \in \mathbb{N}, \\ > \frac{3}{4}+\frac{n}{8} & \text{for any other } n \ge 2. \end{cases}$$

Hence, for some dimensions $n \in \mathbb{N}^+$ we may find a sharper result if in addition to regularity results by Agmon-Douglis-Nirenberg and Sobolev imbeddings we also use interpolation theory [1, Theorem 5.8] as described in [57, Lemma 13]. Indeed, one *finds a constant $C > 0$, depending on the domain, such that for all $i \in \mathbb{N}^+$:*

$$\|\varphi_{i,m,w}\|_{C^m(\overline{\Omega})} \leq C\lambda_{i,m,w}^{\frac{n}{4m}+\frac{2m-1}{2m}}. \tag{3.58}$$

In the following, we will use the estimate in Lemma 3.5.1. However, $k_{n,m}$ can be replaced by $\frac{n}{4m}+\frac{2m-1}{2m}$ in the remainder of this section. Since many steps in the proof of this estimate are similar to the proof of Lemma 3.3.1, we skip it here. However, it can be found in Appendix A.1.

Since the orthogonal projections onto the eigenspaces are defined using the eigenfunctions $\varphi_{i,m,w}$, the following estimates follow from (3.56):

Corollary 3.5.4 *Suppose Condition A is satisfied and let $\mathcal{P}_{i,m,w}$ and $\mathcal{D}_m$ be defined as in* (2.15) *and* (2.17). *Then there is a constant $C_{3.5.4} > 0$, depending on the domain, m and w, such that for every $i \in \mathbb{N}^+$ and for all $0 \leq f \in L^2(\Omega)$:*

$$|(\mathcal{P}_{i,m,w}f)(x)| \leq C_{3.5.4}\lambda_{i,m,w}^{2k_{n,m}}(\mathcal{D}_m f)(x) \text{ for all } x \in \Omega.$$

Proof. We find with $\tilde{C}_{3.5.4} := C_{3.5.1}^2$ of Lemma 3.5.1 that for all $0 \leq f \in L^2(\Omega)$

$$\begin{aligned}|(\mathcal{P}_{i,m,w}f)(x)| &= \left|\varphi_{i,m,w}(x)\int_\Omega \varphi_{i,m,w}(y)f(y)w(y)dy\right| \\ &\leq c_{w,2}\tilde{C}_{3.5.4}\lambda_{i,m,w}^{2k_{n,m}}d(x)^m\int_\Omega d(y)^m f(y)dy \\ &= c_{w,2}\tilde{C}_{3.5.4}\lambda_{i,m,w}^{2k_{n,m}}(\mathcal{D}_m f)(x)\end{aligned}$$

for all $x \in \Omega$. ■

Since we assumed in Theorem 3.1.3 that there is a simple eigenvalue $\lambda_{\mathrm{p},m,w}$ with strongly positive eigenfunction $\varphi_{\mathrm{p},m,w}$, we get:

Corollary 3.5.5 *Suppose that Condition A is satisfied. Let $\lambda_{\mathrm{p},m,w}$ be a simple eigenvalue with corresponding strongly positive eigenfunction $\varphi_{\mathrm{p},m,w}$ and $\mathcal{P}_{i,m,w}$ and $\mathcal{P}_{\mathrm{p},m,w}$ be defined as in* (2.15). *Then there are constants $C_{3.5.5.1}, C_{3.5.5.2} > 0$, depending on the domain, m and w and independent of $i \in \mathbb{N}^+$, such that for all $0 \leq f \in L^2(\Omega)$ and $x \in \Omega$*

$$\begin{aligned}|\varphi_{i,m,w}(x)| &\leq C_{3.5.5.1}\lambda_{i,m,w}^{k_{n,m}}\varphi_{\mathrm{p},m,w}(x), \\ |(\mathcal{P}_{i,m,w}f)(x)| &\leq C_{3.5.5.2}\lambda_{i,m,w}^{2k_{n,m}}(\mathcal{P}_{\mathrm{p},m,w}f)(x).\end{aligned}$$

Proof. The estimates are implications of the assumption $\varphi_{\mathrm{p},m,w}(x) \geq C_{SP}\, d(x)^m$, Lemma 3.5.1 and Corollary 3.5.4. ■

The asymptotic behavior of the eigenvalues for problem (1.5) has been studied and known since Weyl's seminal paper [78] and Agmon's article [2] on higher-order problems. However, there is a strong regularity assumption on the boundary of Ω. Since we will only need an estimate for the eigenvalues from below, we can use an inequality proven by Levine and Protter [43] which holds for any domain that fulfills Condition A. With an adapted constant, we may use this estimate for the weighted problem.

Lemma 3.5.6 (Levine-Protter [43, Equation (2.5)]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then it holds that*

$$\lambda_{i,m,1} \geq C_{LP} i^{\frac{2m}{n}} \quad \textit{with} \quad C_{LP} = \frac{n}{n+2m} \left(\frac{(2\pi)^n}{b_n |\Omega|} \right)^{\frac{2m}{n}},$$

where $|\Omega|$ is the volume of Ω and b_n is the volume of the unit ball as described in (2.2)*.*

Corollary 3.5.7 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $c_{w,2} > 0$ be as described in Remark 3.0.1. Then it holds that*

$$\lambda_{i,m,w} \geq C'_{LP} i^{\frac{2m}{n}} \quad \textit{with} \quad C'_{LP} = \frac{n c_{w,2}^{-1}}{n+2m} \left(\frac{(2\pi)^n}{b_n |\Omega|} \right)^{\frac{2m}{n}}, \tag{3.59}$$

where $|\Omega|$ is the volume of Ω and b_n is the volume of the unit ball as described in (2.2)*.*

Proof. Using the Rayleigh Min-Max Principle, see [77, Chapter 2] or [16, Theorem 4.5.1], the sharp estimate in Lemma 3.5.6 and the upper bound for w in Remark 3.0.1, we find for $m \in \mathbb{N}^+$ even and all $i \in \mathbb{N}^+$

$$\begin{aligned}
\lambda_{i,m,w} &= \min_{\substack{E \subset W_0^{m,2}(\Omega) \\ \dim(E)=i}} \max_{\substack{u \in E \\ u \neq 0}} \frac{\int_\Omega \left(\Delta^{\frac{m}{2}} u\right)^2 dx}{\int_\Omega w u^2 dx} \\
&\geq \frac{1}{\sup_{y \in \overline{\Omega}} |w(y)|} \min_{\substack{E \subset W_0^{m,2}(\Omega) \\ \dim(E)=i}} \max_{\substack{u \in E \\ u \neq 0}} \frac{\int_\Omega \left(\Delta^{\frac{m}{2}} u\right)^2 dx}{\int_\Omega u^2 dx} \\
&\geq c_{w,2}^{-1} \lambda_{i,m,1} \geq c_{w,2}^{-1} C_{LP}\, i^{\frac{2m}{n}}.
\end{aligned}$$

Analogously, we find for $m \in \mathbb{N}^+$ odd

$$\lambda_{i,m,w} = \min_{\substack{E \subset W_0^{m,2}(\Omega) \\ \dim(E)=i}} \max_{\substack{u \in E \\ u \neq 0}} \frac{\int_\Omega |\nabla \Delta^{\frac{m-1}{2}} u|^2 dx}{\int_\Omega w u^2 dx} \geq c_{w,2}^{-1} C_{LP}\, i^{\frac{2m}{n}}.$$

■

If we use this estimate, we can also find a growth rate of the $C^m(\overline{\Omega})$-norm of the eigenfunctions and hence prove the convergence of the series considered in *II* of (3.15) and in Corollary 3.3.3.

***Remark* 3.5.8** *With the asymptotic formula in* (3.59)*, we can show the convergence* *in operatornorm of the series*

$$\sum_{k=N}^{\infty} (\lambda \mathcal{G}_{m,0,w})^k \mathcal{G}_{m,0,w} \mathcal{P}_{j*,m,w} \tag{3.60}$$

for $N \in \mathbb{N}^+$ *with* $N > \frac{n}{2m} - 1$ *and* $\lambda \in (-\lambda_{j+1,m,w}, \lambda_{j+1,m,w})$*, where* $\lambda_{j,m,w}$ *is defined as in* (3.11)*. Indeed, using Lemma 3.2.1, we find with* $\|\mathcal{P}_{i,m,w}\|_{BL(L^2_w(\Omega))} = 1$

$$\begin{aligned}\left\|\mathcal{G}^{k+1}_{m,0,w} \mathcal{P}_{j*,m,w}\right\|_{BL(L^2_w(\Omega))} &= \left\| \sum_{i=j+1}^{\infty} \frac{1}{\lambda^{k+1}_{i,m,w}} \mathcal{P}_{i,m,w} \right\|_{BL(L^2_w(\Omega))} \\ &\leq \sum_{i=j+1}^{\infty} \frac{1}{\lambda^{k+1}_{i,m,w}} \|\mathcal{P}_{i,m,w}\|_{BL(L^2_w(\Omega))} = \sum_{i=j+1}^{\infty} \frac{1}{\lambda^{k+1}_{i,m,w}}.\end{aligned}$$

Hence, it holds that

$$\begin{aligned}\sum_{k=N}^{\infty} \left\|(\lambda \mathcal{G}_{m,0,w})^k \mathcal{G}_{m,0,w} \mathcal{P}_{j*,m,w}\right\|_{BL(L^2_w(\Omega))} &= \sum_{k=N}^{\infty} |\lambda|^k \left\|\mathcal{G}^{k+1}_{m,0,w} \mathcal{P}_{j*,m,w}\right\|_{BL(L^2_w(\Omega))} \\ &\leq \sum_{k=N}^{\infty} |\lambda|^k \sum_{i=j+1}^{\infty} \frac{1}{\lambda^{k+1}_{i,m,w}}.\end{aligned}$$

All entries are nonnegative, so we may change the order of summation and obtain

$$\begin{aligned}\sum_{k=N}^{\infty} |\lambda|^k \sum_{i=j+1}^{\infty} \frac{1}{\lambda^{k+1}_{i,m,w}} &= \sum_{i=j+1}^{\infty} \frac{1}{\lambda_{i,m,w}} \left(\frac{|\lambda|}{\lambda_{i,m,w}}\right)^N \sum_{k=0}^{\infty} \left(\frac{|\lambda|}{\lambda_{i,m,w}}\right)^k \\ &= \sum_{i=j+1}^{\infty} \left(\frac{|\lambda|}{\lambda_{i,m,w}}\right)^N \frac{1}{\lambda_{i,m,w} - |\lambda|}.\end{aligned}$$

One finds that $\frac{\lambda_{i,m,w}}{\lambda_{i,m,w} - |\lambda|}$ *is bounded by some constant* $c_j > 0$*, independent of* $i \in \mathbb{N}$ *with* $j + 1 \leq i$*. We note that the constant depends on* λ *and* $c_j \to \infty$ *for* $\lambda \to \lambda_{j+1,m,w}$*. However, since we consider the convergence of the series in* (3.60) *for fixed* λ *with* $|\lambda| < \lambda_{j+1,m,w}$*, this is not a problem. Using Corollary 3.5.7 we get*

$$\begin{aligned}\sum_{k=N}^{\infty} \left\|(\lambda \mathcal{G}_{m,0,w})^k \mathcal{G}_{m,0,w} \mathcal{P}_{j*,m,w}\right\|_{BL(L^2_w(\Omega))} &\leq c_j |\lambda|^N \sum_{i=j+1}^{\infty} \lambda^{-N-1}_{i,m,w} \\ &\leq (C'_{LP})^{-N-1} c_j |\lambda|^N \sum_{i=j+1}^{\infty} i^{-\frac{2m}{n}N - \frac{2m}{n}}.\end{aligned}$$

This series converges for $N > \frac{n}{2m} - 1$*. Since* $2k_{n,m} > \frac{n}{2m} - 1$*, the series in* II *converges in operator norm.*

Using the asymptotic formula for the eigenvalues and eigenfunctions, we find a similar result as Corollary 3.3.3. The only difference is that we have to adjust the starting value of the series.

Lemma 3.5.9 *Let* $N_{n,m} = 2\left[\frac{n+4m}{2m}\right]$ *and Condition A be satisfied. Moreover, let* $\mathcal{D}_m$ *and* $\mathcal{P}_{j*,m,w}$ *be defined as in* (2.17) *and* (2.16) *with* j *as in* (3.11). *Then, there exists a constant* $C_j > 0$, *also depending on the domain,* m, w, M, δ_1 *and* δ_2, *such that*

$$-C_j\mathcal{D}_m \le \sum_{k=N_{n,m}}^{\infty} \lambda^k \mathcal{G}_{m,0,w}^{k+1}\mathcal{P}_{j*,m,w} \le C_j\mathcal{D}_m \tag{3.61}$$

for all $\lambda \in I_{M,\delta_1} \cup I_{\delta_2}$, *where* $I_{M,\delta_1}, I_{\delta_2}$ *are defined as in* (3.6) *and* (3.8).

Proof. Using Lemma 3.2.1 and Corollary 3.5.4, we find for $0 \le f \in L^2(\Omega)$

$$\begin{aligned}\left|\sum_{k=N_{n,m}}^{\infty} \lambda^k \mathcal{G}_{m,0,w}^{k+1}\mathcal{P}_{j*,m,w}f\right| &= \left|\sum_{k=N_{n,m}}^{\infty} \lambda^k \sum_{i=j+1}^{\infty} \frac{1}{\lambda_{i,m,w}^{k+1}}\mathcal{P}_{i,m,w}f\right| \\ &\le \sum_{k=N_{n,m}}^{\infty} |\lambda|^k \sum_{i=j+1}^{\infty} \frac{1}{\lambda_{i,m,w}^{k+1}}|\mathcal{P}_{i,m,w}f| \\ &\le C_{3.5.4}\left(\sum_{k=N_{n,m}}^{\infty} |\lambda|^k \sum_{i=j+1}^{\infty} \frac{1}{\lambda_{i,m,w}^{k+1}}\lambda_{i,m,w}^{2k_{n,m}}\right)\mathcal{D}_m f.\end{aligned}$$

The series in brackets converges, so we can change the order of the summation and get

$$\left|\sum_{k=N_{n,m}}^{\infty} \lambda^k \mathcal{G}_{m,0,w}^{k+1}\mathcal{P}_{j*,m,w}f\right| \le C_{3.5.4}\sum_{i=j+1}^{\infty}\left(\frac{|\lambda|}{\lambda_{i,m,w}}\right)^{N_{n,m}} \frac{\lambda_{i,m,w}^{2k_{n,m}}}{\lambda_{i,m,w}-|\lambda|}\mathcal{D}_m f.$$

Since $\frac{\lambda_{i,m,w}}{\lambda_{i,m,w}-|\lambda|}$ is bounded from above by some constant $C_{\delta_1,\delta_2,M} > 0$, independent of i and $\lambda \in I_{M,\delta_1} \cup I_{\delta_2}$, we obtain with Corollary 3.5.7

$$\begin{aligned}\left|\sum_{k=N_{n,m}}^{\infty} \lambda^k \mathcal{G}_{m,0,w}^{k+1}\mathcal{P}_{j*,m,w}f\right| &\le C_{\delta_1,\delta_2,M}C_{3.5.4}\lambda_{j,m,w}^{N_{n,m}}\sum_{i=j+1}^{\infty}\lambda_{i,m,w}^{2k_{n,m}-N_{n,m}-1}\mathcal{D}_m f \\ &\le C_{\delta_1,\delta_2,M}C_{3.5.4}C''_{LP}\lambda_{j,m,w}^{N_{n,m}}\sum_{i=j+1}^{\infty} i^{\frac{2m}{n}(2k_{n,m}-N_{n,m}-1)}\mathcal{D}_m f,\end{aligned}$$

where $C''_{LP} = (C'_{LP})^{2k_{n,m}-N_{n,m}-1}$. The series converges for

$$N_{n,m} > \frac{n}{2m} + 2k_{n,m} - 1 = \frac{n}{2m} + 1 + 2\left[\frac{n+2m}{4m}\right].$$

Since $N_{n,m} = 2\left[\frac{n+4m}{2m}\right]$ this is fulfilled. ■

Using Lemma 3.5.9 and replacing $2k_{n,m}$ by $N_{n,m}$ in Proposition 3.4.5, we find an alternative proof for Theorem 3.1.1 and 3.1.3.

***Remark* 3.5.10** *If we use the estimate in* (3.58)*, we find the condition* $N_{n,m} > \frac{n-1}{m} + 1$*, so we may also use* $N_{n,m} = \left[\frac{n-1}{m}\right] + 2$ *in Lemma 3.5.9.*

3.6 An anti-maximum principle

Using analogous estimates and arguments as in the last four sections, we find for λ in a small right neighborhood of $\lambda_{\mathrm{p},m,w}$ a similar result as (3.9) for an upper bound of the Green function $G_{m,\lambda,w}$. This implies an anti-maximumm principle, i.e. a sign-reversing property exists: $f \geq 0$ implies $u \leq 0$ for λ in some interval.

First, we prove an estimate from below for the Green function to problem (1.6) if λ is contained in a right neighborhood of a simple eigenvalue with corresponding strongly positive eigenfunction.

Theorem 3.6.1 *Suppose that Condition A is satisfied and let* $\delta_3 > 0$*. Suppose* $0 < w \in C^{0,\gamma}(\overline{\Omega})$ *and that* $\lambda_{\mathrm{p},m,w}$ *is a simple eigenvalue of* (1.10) *with the corresponding eigenfunction* $\varphi_{\mathrm{p},m,w}$ *strongly positive in the sense of (1.11). Moreover, suppose*

$$I_{\delta_3} = (\lambda_{\mathrm{p},m,w}, \lambda_{\mathrm{p},m,w} + \delta_3] \tag{3.62}$$

contains no eigenvalue. Let $G_{m,\lambda,w}$ *be the Green function for* (2.6)*. Then there exist constants* $C_1, C_2, C_3 > 0$*, depending on the domain,* m, δ_3 *and* w*, such that for all* $\lambda \in I_{\delta_3}$ *and* $x, y \in \Omega$*:*

$$G_{m,\lambda,w}(x,y) \leq C_1\ H_{n,m}(x,y) + \left(\frac{C_2}{\lambda_{\mathrm{p},m,w} - \lambda} + C_3\right) \varphi_{\mathrm{p},m,w}(x)\ \varphi_{\mathrm{p},m,w}(y). \tag{3.63}$$

Proof. We have to estimate *I*, *II* and *III*, described in (3.14) and (3.15), from above. Let $\mathcal{H}_{n,m}$ and $\mathcal{P}_{j*,m,w}$ be as defined in (3.4) and (2.16) with $j \in \mathbb{N}^+$ as in (3.11). To find appropriate estimates, we can use results from the previous sections. Using the proof of Lemma 3.2.2, we get a constant $c_{j,1} > 0$ such that

$$\sum_{i=1}^{j} \frac{1}{\lambda_{i,m,w} - \lambda} \mathcal{P}_{i,m,w} \leq \left(\frac{1}{\lambda_{\mathrm{p},m,w} - \lambda} + c_{j,1}\right) \mathcal{P}_{\mathrm{p},m,w} \quad \text{for all } \lambda \in I_{\delta_3}.$$

We find analogous to Corollary 3.3.3 a constant $c_{j,2} > 0$ such that

$$\sum_{k=2k_{n,m}}^{\infty} \lambda^k \left(\mathcal{G}_{m,0,1}(w\cdot)\right)^{k+1} \mathcal{P}_{j*,m,w} \leq c_{j,2} \mathcal{P}_{\mathrm{p},m,w} \quad \text{for all } \lambda \in I_{\delta_3}.$$

Similar to Proposition 3.4.5 we also obtain constants $c_{j,3}, c_{j,4} > 0$ such that

$$\sum_{k=0}^{2k_{n,m}-1} \lambda^k \left(\mathcal{G}_{m,0,1}(w\cdot)\right)^{k+1} \mathcal{P}_{j*,m,w} \leq c_{j,3} \mathcal{H}_{n,m} - c_{j,4} \mathcal{P}_{\mathrm{p},m,w} \quad \text{for all } \lambda \in I_{\delta_3}.$$

Using (3.14) and (3.15), the result in (3.63) follows. ■

Using some known estimates for the integral operator $\mathcal{H}_{n,m}$ and Theorem 3.6.1, we obtain an anti-maximum principle:

Theorem 3.6.2 *Suppose that Condition A is satisfied. Suppose* $0 < w \in C^{0,\gamma}(\overline{\Omega})$ *and* $\lambda_{\mathrm{p},m,w}$ *is a simple eigenvalue of* (1.10) *with the corresponding eigenfunction* $\varphi_{\mathrm{p},m,w}$ *strongly positive as in (1.11). Let* $0 \le f \in L^q(\Omega)$ *with* f *nontrivial and* $q > \max\{1, \frac{n}{m}\}$*. Then, there exists* $\delta_f > 0$*, such that for all* $\lambda \in (\lambda_{\mathrm{p},m,w}, \lambda_{\mathrm{p},m,w} + \delta_f)$ *the following holds: There is a constant* $c_{f,\lambda,q} > 0$ *such that the solution* $u_{m,\lambda,w} \in W^{2m,q}(\Omega) \cap W_0^{m,q}(\Omega)$ *of* (1.6) *satisfies*

$$u_{m,\lambda,w}(x) \le -c_{f,\lambda,q}\, d(x)^m \quad \text{for all } x \in \Omega.$$

Proof. We may consider $\mathcal{H}_{n,m}$, $\mathcal{D}_m$ or $\mathcal{P}_{i,m,w}$ as operators on $L^q(\Omega)$ with $q > \max\{1, \frac{n}{m}\}$ instead of $L^2(\Omega)$. Since the Green operator $\mathcal{G}_{m,\lambda,w} : L^q(\Omega) \to W^{2m,q}(\Omega) \cap W_0^{m,q}(\Omega)$ is defined through the kernel function $G_{m,\lambda,w}$ which does not depend on q, we may use inequality (3.63).

In the proof of Lemma 2 in [33] it is shown that for all $f \in L^q(\Omega)$ with $q > \max\{1, \frac{n}{m}\}$, one finds a constant $c_q > 0$, depending also on the domain and m, such that

$$|(\mathcal{H}_{n,m} f)(x)| \le c_q \|f\|_{L^q(\Omega)} d(x)^m \quad \text{for all } x \in \Omega. \tag{3.64}$$

Indeed, this result can be proven using the definition of the kernel function $H_{n,m}$ in (3.3), the estimates in [21, Lemma 4.5] and the Hölder inequality. Since $\varphi_{\mathrm{p},m,w}$ is strongly positive in the sense of (1.11), the inequality in (3.64) implies that there exists $c_{f,q} > 0$ such that

$$|(\mathcal{H}_{n,m} f)(x)| \le c_{f,q} \varphi_{\mathrm{p},m,w}(x) \quad \text{for all } x \in \Omega.$$

Then, one finds the following estimate for the solution $u_{m,\lambda,w}$ of (1.6):

$$u_{m,\lambda,w}(x) \le \varphi_{\mathrm{p},m,w}(x) \left[C_1 c_{f,q} + \left(\frac{C_2}{\lambda_{\mathrm{p},m,w} - \lambda} + C_3 \right) \int_\Omega f(y) \varphi_{\mathrm{p},m,w}(y) dy \right],$$

where C_1, C_2, C_3 are chosen as in Theorem 3.6.1. Since $\varphi_{\mathrm{p},m,w}$ is strongly positive and $f \ge 0$ with $f \not\equiv 0$, it holds that $\int_\Omega f(y)\varphi_{\mathrm{p},m,w}(y)dy > 0$, so the constant in square brackets becomes negative if $\lambda_{\mathrm{p},m,w} - \lambda < 0$ and λ is close enough to $\lambda_{\mathrm{p},m,w}$. Accordingly, there exists a $\delta_f > 0$ such that the value in square brackets is less than zero for $\lambda \in (\lambda_{\mathrm{p},m,w}, \lambda_{\mathrm{p},m,w} + \delta_f)$. ■

***Remark* 3.6.3** *We notice that in this result the positivity of the kernel function* $H_{n,m}$*, respectively the singularity of the Green function, is not used. We only need an estimate as in* (3.64) *and the strong positivity of an eigenfunction with corresponding simple eigenvalue.*

Chapter 4

Construction of a weighted problem with simple eigenvalue and positive eigenfunction

In this chapter we make use of Condition B to find a positive and Hölder continuous weight function such that a strongly positive eigenfunction of the weighted eigenvalue problem with corresponding simple eigenvalue exists. In [58] we found an explicit weight function for the weighted biharmonic problem. The idea and the steps in the following proofs are similar to [58, Sections 2, 4] with small changes concerning the additional assumptions in Condition B.

4.1 Idea of the construction

Let Condition A be satisfied. Moreover, let $u_0 \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$ be as described in Condition B and

$$(-\Delta)^m u_0 = f_0 \, d(\cdot)^{m_0}, \tag{4.1}$$

with $m_0 \in \mathbb{N}$, $m_0 \le m$ and $f_0 \in C^{0,\gamma}(\overline{\Omega})$ strictly positive. If u_0 is an eigenfunction to problem (1.10) with $w \equiv 1$ and eigenvalue $\lambda_{\mathrm{p},m,1} > 0$, we are done and can choose $w \equiv \lambda_{\mathrm{p},m,1}$ as a weight function. Then, u_0 is an eigenfunction to the weighted eigenvalue problem (1.10) with weight w and eigenvalue $\lambda_{\mathrm{p},m,w} = 1$. If u_0 is not an eigenfunction, then the idea is to take

$$w = \frac{f_0 d(\cdot)^{m_0}}{u_0} \tag{4.2}$$

as the weight function. Then u_0 satisfies

$$\begin{cases} (-\Delta)^m u_0 = \lambda w u_0 & \text{in } \Omega, \\ u_0 = \frac{\partial}{\partial \nu} u_0 = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{4.3}$$

with $\lambda = 1$, so it would be a strongly positive eigenfunction to a weighted eigenvalue problem. However, the problem is that the weight function has to be Hölder continuous and bounded so that we can apply the converse to Krein-Rutman. For $m_0 < m$, the quotient of $f_0 d(\cdot)^{m_0}$ and u_0 would be unbounded near the boundary $\partial\Omega$. Therefore, the idea is to change $d(\cdot)^{m_0} f_0$ in a way that it behaves like the distance function $d(\cdot)^m$ in a small neighborhood of the boundary $\Omega(\varepsilon)$, which is defined by

$$\Omega(\varepsilon) := \{x \in \Omega;\, d(x) < \varepsilon\}. \tag{4.4}$$

Then, we use a similar idea as in (4.2) and (4.3) to achieve an appropriate weight function. After we have found a weighted problem with positive eigenfunction, we use small perturbations of this function so that the corresponding eigenvalue becomes simple.

4.2 Construction of the weight function

In this section we follow the steps in [58, Section 2], except that, instead of $m = 2$ and f_0, we consider $m \geq 2$ and $f_0 d(\cdot)^{m_0}$ from (1.13) in Condition B. In the following, we investigate the function $f_{0,\varepsilon} : \overline{\Omega} \to \mathbb{R}$ defined by

$$f_{0,\varepsilon}(x) = \chi_\varepsilon(d(x))^{m-m_0} d(x)^{m_0} f_0(x) \tag{4.5}$$

for $\varepsilon > 0$ but small. We choose $\chi_\varepsilon \in C^\infty(\mathbb{R})$ such that it is an ε-sized mollification of the sign-function, see Figure 4.1.

***Remark* 4.2.1 ([58, Remark 8])** *The function χ_ε is constructed with the mollifiers from Friedrichs $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ with support in $[-\varepsilon, \varepsilon]$ and defined by $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right)$, where*

$$\varphi(t) = \begin{cases} c^{-1}\exp\left(\frac{-1}{1-t^2}\right) & \text{for } |t| < 1, \\ 0 & \text{for } |t| \geq 1, \end{cases} \qquad \text{and} \qquad c = \int_{-1}^{1} \exp\left(\tfrac{-1}{1-s^2}\right) ds.$$

With $\operatorname{sign}(t) = t/|t|$ *for $t \neq 0$ we define the function*

$$\chi_\varepsilon(t) = (\varphi_\varepsilon * \operatorname{sign})(t) = -\int_{-\infty}^{0} \varphi_\varepsilon(t-y)dy + \int_{0}^{\infty} \varphi_\varepsilon(t-y)dy \text{ for } t \in \mathbb{R}.$$

Note that $\chi_\varepsilon \in C^\infty(\mathbb{R})$ satisfies $\chi_\varepsilon(0) = 0$ and $\chi_\varepsilon(t) = 1$ for $t > \varepsilon$, see Figure 4.1. Moreover, we find $\chi_\varepsilon'(t) = 2\varphi_\varepsilon(t)$, so $\chi_\varepsilon'(0) = \frac{2}{ce}\varepsilon^{-1}$ and

$$\min\left\{\tfrac{t}{\varepsilon}, 1\right\} \leq \chi_\varepsilon(t) \leq \min\left\{\tfrac{2}{ce}\tfrac{t}{\varepsilon}, 1\right\} \text{ for } t \geq 0. \tag{4.6}$$

***Remark* 4.2.2** *In [22, Lemma 14.16] one finds that for the distance function it holds $d \in C^{2m,\gamma}$ near $\partial\Omega$ follows from $\partial\Omega \in C^{2m,\gamma}$. Also one finds $\frac{\partial}{\partial\nu}d = -1$ on $\partial\Omega$.*

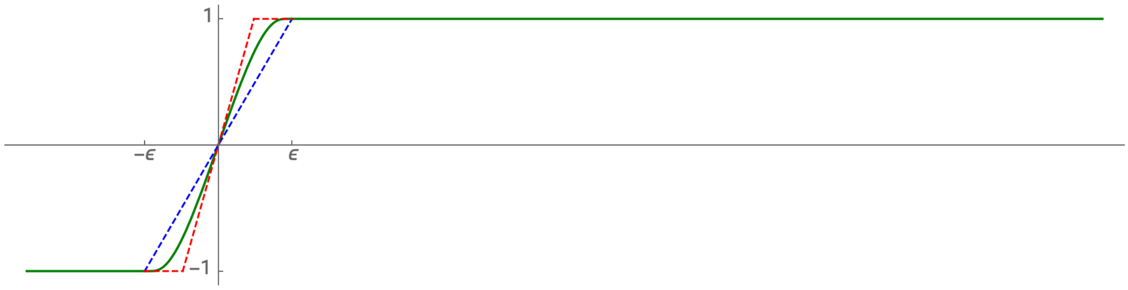


Figure 4.1: Sketch of χ_ε as mollified sign-function with the estimates from (4.6).

So, $\chi_\varepsilon(d(\cdot))^m \in C^{2m,\gamma}(\overline{\Omega})$ for sufficiently small $\varepsilon > 0$. Moreover, it holds that

$$\chi_\varepsilon(d(\cdot))^m = \frac{\partial}{\partial\nu}\chi_\varepsilon(d(\cdot))^m = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1}\chi_\varepsilon(d(\cdot))^m = 0 \text{ and} \tag{4.7}$$

$$\left(\frac{\partial}{\partial\nu}\right)^{m}\chi_\varepsilon(d(\cdot))^m \neq 0 \tag{4.8}$$

on $\partial\Omega$. In addition one gets that $d(\cdot)^{m_0}\chi_\varepsilon(d(\cdot))^{m-m_0} \in C^{2m,\gamma}(\overline{\Omega(\varepsilon)})$ and that (4.7) and (4.8) hold when we replace $\chi_\varepsilon(d(\cdot))^m$ by $d(\cdot)^{m_0}\chi_\varepsilon(d(\cdot))^{m-m_0}$.

Furthermore, instead of u_0, we consider the solution $u_{0,\varepsilon}$ to the polyharmonic Dirichlet problem

$$\begin{cases} (-\Delta)^m u_{0,\varepsilon} = f_{0,\varepsilon} & \text{in } \Omega, \\ u_{0,\varepsilon} = \frac{\partial}{\partial\nu}u_{0,\varepsilon} = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} u_{0,\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, we can construct a weight function such that $u_{0,\varepsilon}$ is a strongly positive eigenfunction to the corresponding weighted eigenvalue problem. To this end we will use the following lemma:

Lemma 4.2.3 *Suppose that Condition A is satisfied. Let $u \in C^{2m,\gamma}(\overline{\Omega})$ satisfy Condition B. Then we find that*

$$\left(\frac{\partial}{\partial\nu}\right)^{m} u(x) \neq 0 \text{ for all } x \in \partial\Omega.$$

Proof. If $\left(\frac{\partial}{\partial\nu}\right)^m u(x_0) = 0$ for some $x_0 \in \partial\Omega$, then we find a contradiction: Since we assume that (1.11) holds, there is a constant $C_{SP} > 0$ such that

$$\frac{u(x)}{d(x)^m} \geq C_{SP} \text{ for all } x \in \Omega.$$

Let $y_x \in \partial\Omega$ be such that $d(x) = |x - y_x|$. Using the mean value theorem and the fact that u is m-times continuously differentiable, one obtains

$$|u(x)| \leq \sup_{\xi_x \in [x,y_x]} \|D^m u(\xi_x)\| d(x)^m,$$

where $[x, y_x] = \{\theta x + (1-\theta)y_x; \theta \in (0,1)\}$. Then one gets

$$0 < C_{SP} \leq \liminf_{x \to x_0} \frac{|u(x)|}{d(x)^m} \leq \liminf_{x \to x_0} \sup_{\xi_x \in [x,y_x]} \|D^m u(\xi_x)\| = 0,$$

a contradiction. ■

In addition, we will use the following auxiliary lemma concerning the Hölder continuity of some function in one dimension. It provides an idea why the subsequent construction of the weight function yields a Hölder continuous function.

Lemma 4.2.4 *Let* $f \in C^{m,\gamma}([0,1])$ *with* $f(0) = \frac{\partial}{\partial \nu} f(0) = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} f(0) = 0$. *Then* $g : [0,1] \to \mathbb{R}$ *defined by*

$$g(t) := \begin{cases} \frac{f(t)}{t^m} & \text{for } t \in (0,1], \\ \frac{1}{m!} f^{(m)}(0) & \text{for } t = 0, \end{cases}$$

is Hölder continuous, i.e. $g \in C^{0,\gamma}([0,1])$.

Proof. Using Taylor's formula with Lagrange form of the remainder, one finds that $\lim_{t \downarrow 0} \frac{f(t)}{t^m} = \frac{1}{m!} f^{(m)}(0)$, so $g \in C([0,1])$. Hence, it remains to show that

$$[g]_\gamma := \sup_{0 \leq t < s \leq 1} \frac{|g(t) - g(s)|}{|t-s|^\gamma} < \infty.$$

Let $0 < t < s \leq 1$. Using Taylor's formula again, we find

$$f(s) = \sum_{k=0}^{m-1} \frac{(s-t)^k f^{(k)}(t)}{k!} + \frac{(s-t)^m}{m!} f^{(m)}(\xi_{t,s})$$

for some $\xi_{t,s} \in (t,s)$. So we obtain

$$\frac{\left|\frac{f(t)}{t^m} - \frac{f(s)}{s^m}\right|}{|t-s|^\gamma} = \frac{1}{|s-t|^\gamma s^m} \left| \frac{f(t) s^m}{t^m} - \sum_{k=0}^{m-1} \frac{(s-t)^k f^{(k)}(t)}{k!} - \frac{(s-t)^m}{m!} f^{(m)}(\xi_{t,s}) \right|. \tag{4.9}$$

We may also rewrite $f^{(k)}(t)$ for $k \in \{0, \ldots, m-1\}$ using Taylor's formula and get with the assumption $f(0) = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} f(0) = 0$ that

$$f^{(k)}(t) = \frac{t^{m-k}}{(m-k)!} f^{(m)}(\xi_{t,k}) \tag{4.10}$$

for some $\xi_{t,k} \in (0,t)$. Then, (4.9) and (4.10) imply

$$\frac{\left|\frac{f(t)}{t^m} - \frac{f(s)}{s^m}\right|}{|t-s|^\gamma}$$
$$= \frac{1}{|s-t|^\gamma s^m} \left| \frac{f^{(m)}(\xi_{t,0}) s^m}{m!} - \sum_{k=0}^{m-1} \frac{(s-t)^k t^{m-k} f^{(m)}(\xi_{t,k})}{(m-k)!k!} - \frac{(s-t)^m}{m!} f^{(m)}(\xi_{t,s}) \right|.$$

Applying the binomial theorem, we find

$$\frac{(s-t)^m}{m!} = \frac{s^m}{m!} - \sum_{k=0}^{m-1} \frac{(s-t)^k t^{m-k}}{(m-k)!k!}. \tag{4.11}$$

Hence using the triangle inequality it follows

$$\frac{\left|\frac{f(t)}{t^m} - \frac{f(s)}{s^m}\right|}{|t-s|^\gamma}$$
$$\leq \frac{s^m - t^m}{m!|s-t|^\gamma s^m} \left| f^{(m)}(\xi_{t,0}) - f^{(m)}(\xi_{t,s}) \right| \tag{4.12}$$
$$+ \sum_{k=1}^{m-1} \frac{(s-t)^k t^{m-k}}{(m-k)!k!|s-t|^\gamma s^m} \left| f^{(m)}(\xi_{t,k}) - f^{(m)}(\xi_{t,s}) \right|. \tag{4.13}$$

Since $s \geq |\xi_{t,k} - \xi_{t,s}|$ for all $k \in \{0, \ldots, m-1\}$, $s \geq t$ and $s \geq |s-t|$, we obtain with (4.12), (4.13) and (4.11)

$$\frac{\left|\frac{f(t)}{t^m} - \frac{f(s)}{s^m}\right|}{|t-s|^\gamma}$$
$$\leq \sum_{k=1}^{m} \frac{1}{(m-k)!k!} \frac{\left| f^{(m)}(\xi_{t,0}) - f^{(m)}(\xi_{t,s}) \right|}{|\xi_{t,0} - \xi_{t,s}|^\gamma} + \sum_{k=1}^{m-1} \frac{1}{(m-k)!k!} \frac{\left| f^{(m)}(\xi_{t,k}) - f^{(m)}(\xi_{t,s}) \right|}{|\xi_{t,k} - \xi_{t,s}|^\gamma}$$
$$\leq 2 \sum_{k=1}^{m} \frac{1}{(m-k)!k!} \left[f^{(m)} \right]_\gamma. \tag{4.14}$$

For $0 = t < s \leq 1$, we find a value $\xi_{0,s} \in (0,s)$ such that

$$\frac{\left|\frac{f^{(m)}(0)}{m!} - \frac{f(s)}{s^m}\right|}{|0-s|^\gamma} = \frac{\left|\frac{f^{(m)}(0)}{m!} - \frac{f^{(m)}(\xi_{s,0})}{m!}\right|}{s^\gamma}$$
$$\leq \frac{1}{m!} \frac{\left| f^{(m)}(0) - f^{(m)}(\xi_{s,0}) \right|}{\xi_{0,s}^\gamma} \leq \frac{1}{m!} [f^{(m)}]_\gamma. \tag{4.15}$$

With (4.14) and (4.15), we obtain an upper bound for $[g]_\gamma$ and thus g is Hölder continuous. ∎

Similar to [58, Proposition 9] we find a strictly positive weight function w_ε with strongly positive eigenfunction. The following proof is similar to the one in [58], however we consider the regularity of the weight function in more detail.

Proposition 4.2.5 *Suppose that Condition A and B are satisfied. Let f_0, $f_{0,\varepsilon}$ and $u_{0,\varepsilon}$ be as defined in* (1.14), (4.1) *and* (4.5). *Then, there exists a value $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ it holds:*

1. $w_\varepsilon := \frac{f_{0,\varepsilon}}{u_{0,\varepsilon}} \in C^{0,\gamma}(\overline{\Omega})$ *and* $\min\{w_\varepsilon(x); x \in \overline{\Omega}\} > 0$.
2. $\varphi := u_{0,\varepsilon}$ *is a strongly positive eigenfunction with eigenvalue $\lambda = 1$ for the weighted eigenvalue problem*

$$\begin{cases} (-\Delta)^m \varphi = \lambda w_\varepsilon \varphi & \text{in } \Omega, \\ \varphi = \frac{\partial}{\partial \nu}\varphi = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

Proof. For simplicity, we assume that $m_0 = 0$. Because of the arguments in Remark 4.2.2, we may follow the steps analogously for $0 < m_0 \leq m$. First, we show estimates for $u_{0,\varepsilon}$, then the existence of a positive lower bound for w_ε, and after that we prove the Hölder continuity of w_ε by using Taylor expansions.

- Let $\Omega(\varepsilon)$ be defined as in (4.4). Then, we see that for all $q \in [1, \infty)$

$$\|f_{0,\varepsilon} - f_0\|_{L^q(\Omega)} \leq \|f_0\|_{L^\infty(\Omega)} |\Omega(\varepsilon)|^{1/q} \to 0 \text{ for } \varepsilon \downarrow 0,$$

where $|\Omega(\varepsilon)|$ is the volume of $\Omega(\varepsilon)$. Using Agmon-Douglis-Nirenberg results, see [21, Theorem 2.20], we also find for $q \in (1, \infty)$ a constant $C_{\mathrm{ADN},q} > 0$ such that

$$\|u_{0,\varepsilon} - u_0\|_{W^{2m,q}(\Omega)} \leq C_{\mathrm{ADN},q} \|f_{0,\varepsilon} - f_0\|_{L^q(\Omega)} \to 0 \text{ for } \varepsilon \downarrow 0.$$

By Sobolev imbeddings in (2.11), one gets that $W^{2m,q}(\Omega)$ imbeds in $C^m(\overline{\Omega})$ for $q > \frac{n}{m}$. This implies

$$\|u_{0,\varepsilon} - u_0\|_{C^m(\overline{\Omega})} \to 0 \text{ for } \varepsilon \downarrow 0.$$

Hence, using the mean value theorem, we obtain

$$u_0(x) - u_{0,\varepsilon}(x) \leq \|u_{0,\varepsilon} - u_0\|_{C^m(\overline{\Omega})} d(x)^m \text{ for all } x \in \Omega.$$

Then we can use that u_0 is strongly positive, so there is a constant $c_1 > 0$ such that $u_0(x) \geq c_1 d(x)^m$ for all $x \in \Omega$. One finds

$$u_{0,\varepsilon}(x) \geq u_0(x) - \|u_{0,\varepsilon} - u_0\|_{C^m(\overline{\Omega})} d(x)^m \geq \left(c_1 - \|u_{0,\varepsilon} - u_0\|_{C^m(\overline{\Omega})}\right) d(x)^m.$$

So, there exists $\varepsilon_0 > 0$ and a constant $\tilde{c}_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$u_{0,\varepsilon}(x) \geq \tilde{c}_1 d(x)^m \text{ in } \Omega.$$

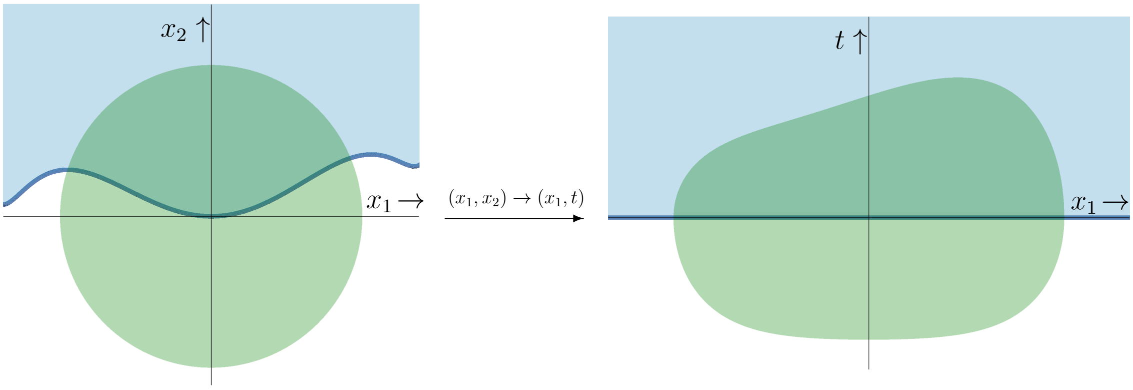


Figure 4.2: The coordinate transformation with ψ as in (4.18) "flattens out" the boundary: Ω is displayed in blue, and in green you can see $B_\delta(0) \subset \mathbb{R}^2$ on the left and U on the right. The boundary of Ω is shown in the left figure in dark blue. The "flattened" boundary can be seen in the right figure.

Applying the mean value theorem, we find a constant $\tilde{c}_2 > 0$ such that

$$\tilde{c}_1 d(x)^m \leq u_{0,\varepsilon}(x) \leq \tilde{c}_2 d(x)^m \text{ in } \Omega. \tag{4.17}$$

Hence $u_{0,\varepsilon}$ is a strongly positive eigenfunction to (4.16) for $\varepsilon \in (0, \varepsilon_0)$.

- Using (4.17) and (4.6), we find an upper and lower bound for the weight function w_ε. It holds that

$$0 < \frac{\min\{f_0(x); x \in \overline{\Omega}\}}{\tilde{c}_2 \max\{\operatorname{diam}(\Omega), \varepsilon\}^m} \leq \frac{f_{0,\varepsilon}}{u_{0,\varepsilon}} = w_\varepsilon \leq \frac{2^m \max\{f_0(x); x \in \overline{\Omega}\}}{\tilde{c}_1 c^m e^m \varepsilon^m} < \infty.$$

  So, it remains to prove that $w_\varepsilon \in C^{0,\gamma}(\overline{\Omega})$.

- Next, we show that $w_\varepsilon \in C(\overline{\Omega})$. Since for sufficiently small $\varepsilon > 0$ it holds that $u_{0,\varepsilon}, \chi_\varepsilon(d(\cdot))^m \in C^{2m,\gamma}(\overline{\Omega})$, $f_0 \in C^{0,\gamma}(\overline{\Omega})$ and $u_{0,\varepsilon} > 0$ in Ω, we find that $\frac{f_{0,\varepsilon}}{u_{0,\varepsilon}} \in C(\Omega)$. Let $x_0 \in \partial\Omega$. For simplicity, we may assume that $x_0 = 0$. Now, we prove that w_ε is continuous in x_0. Since $\partial\Omega \in C^{2m,\gamma}$ we find $\delta > 0$ and $\psi \in C^{2m,\gamma}(\mathbb{R}^{n-1})$ (after relabeling and reorientating the coordinate axes if necessary) such that

$$\Omega \cap B_\delta(0) = \{x \in B_\delta(0); x_n > \psi(x_1, \ldots, x_{n-1})\}.$$

  Then, we can "flatten out" the boundary near $x_0 = 0$ and define the new coordinates $(x_1, \ldots, x_{n-1}, t)$, where

$$t = x_n - \psi(x_1, \ldots, x_{n-1}), \tag{4.18}$$

  see Figure 4.2. In the following we use the short notation $x = (x_1, \ldots, x_n) =$

(x', x_n) and $(x_1, \dots, x_{n-1}, t) = (x', t)$. We set

$$\begin{aligned}\tilde{f}_0(x', t) &:= f_0(x', t + \psi(x')),\\ \chi_\varepsilon(\tilde{d}(x', t))^m &:= \chi_\varepsilon(d(x', t + \psi(x')))^m,\\ \tilde{u}_{0,\varepsilon}(x', t) &:= u_{0,\varepsilon}(x', t + \psi(x')).\end{aligned}$$

Then

$$\tilde{u}_{0,\varepsilon}, \chi_\varepsilon(\tilde{d}(\cdot))^m \in C^{2m,\gamma}(\overline{U \cap ((\mathbb{R})^{n-1} \times \mathbb{R}^+)}) \text{ and } \tilde{f}_0 \in C^{0,\gamma}(\overline{U \cap ((\mathbb{R})^{n-1} \times \mathbb{R}^+)})$$

for some neighborhood U of $0 \in \mathbb{R}^n$, and we can extend the functions to negative values for t by symmetric or antisymmetric extension:

$$\hat{u}_{0,\varepsilon}(x', t) := \begin{cases} \tilde{u}_{0,\varepsilon}(x', t) & \text{for } t \geq 0,\\ (-1)^m \tilde{u}_{0,\varepsilon}(x', -t) & \text{for } t < 0 \end{cases}$$

and analogous we find $\chi_\varepsilon(\hat{d}(\cdot))^m$ and $\hat{f}_0$. Then $\hat{u}_{0,\varepsilon}, \chi_\varepsilon(\hat{d}(\cdot)) \in C^m(\overline{V})$ and $\hat{f}_0 \in C(\overline{V})$, where $V \subset \mathbb{R}^n$ is some neighborhood of 0. We may assume that V is some small ball with center 0. Using Taylor's theorem in $(x', 0) \in V$ with respect to t, we find

$$\begin{aligned}&\chi_\varepsilon(\hat{d}(x', t))^m\\ &= \sum_{k=0}^{m-1} \frac{t^k}{k!} \left(\left(\frac{\partial}{\partial s}\right)^k \chi_\varepsilon(\hat{d}(x', s))^m \right)_{|s=0} + \frac{t^m}{m!} \left(\left(\frac{\partial}{\partial s}\right)^m \chi_\varepsilon(\hat{d}(x', s))^m \right)_{|s=\xi_{(x',t)} t}\\ &= \frac{t^m}{m!} \left(\left(\frac{\partial}{\partial s}\right)^m \chi_\varepsilon(\hat{d}(x', s))^m \right)_{|s=\xi_{(x',t)} t}\end{aligned}$$

for some $\xi_{(x',t)} \in (0, 1)$ and a similar formula for $\hat{u}_{0,\varepsilon}$ and some $\eta_{(x',t)} \in (0, 1)$. Then, we obtain that

$$\begin{aligned}\lim_{x \to 0} \frac{f_{0,\varepsilon}(x)}{u_{0,\varepsilon}(x)} &= \lim_{(x',t)\to(0,0)} \frac{\hat{f}_0(x', t)\chi_\varepsilon(\hat{d}(x', t))^m}{\hat{u}_{0,\varepsilon}(x', t)}\\ &= \lim_{(x',t)\to(0,0)} \frac{\hat{f}_0(x', t) \left(\left(\frac{\partial}{\partial s}\right)^m \chi_\varepsilon(\hat{d}(x', s))^m \right)_{|s=\xi_{(x',t)} t}}{\left(\left(\frac{\partial}{\partial s}\right)^m \hat{u}_{0,\varepsilon}(x', s) \right)_{|s=\eta_{(x',t)} t}}\\ &= \frac{\hat{f}_0(0, 0) \left(\left(\frac{\partial}{\partial s}\right)^m \chi_\varepsilon(\hat{d}(0, s))^m \right)_{|s=0}}{\left(\left(\frac{\partial}{\partial s}\right)^m \hat{u}_{0,\varepsilon}(0, s) \right)_{|s=0}}\end{aligned}$$

and the right-hand side exists because $f_0, \left(\frac{\partial}{\partial \nu}\right)^m \chi_\varepsilon(d(\cdot))^m, \left(\frac{\partial}{\partial \nu}\right)^m u_{0,\varepsilon} \neq 0$ on $\partial\Omega$, see Lemma 4.2.3. So, $w_\varepsilon \in C(\overline{\Omega})$. We could also have used Taylor's formula with integral form of the remainder to find

$$\frac{\chi_\varepsilon(\hat{d}(x',t))^m}{\hat{u}_{0,\varepsilon}(x',t)} = \frac{\int_0^t (t-s)^{m-1} \frac{\partial^m}{\partial s^m} \chi_\varepsilon(\hat{d}(x',s))^m ds}{\int_0^t (t-s)^{m-1} \frac{\partial^m}{\partial s^m} \hat{u}_{0,\varepsilon}(x',s) ds} \quad \text{in } V.$$

This expression also implies the continuity of the weight function.

- To show the Hölder continuity of w_ε one notices that $w_\varepsilon \in C^{0,\gamma}(K)$ for all compact $K \subset \Omega$. So, it remains to examine the behavior of w_ε near the boundary $\partial\Omega$. Therefore, one can proceed similar to the proof of $w_\varepsilon \in C(\overline{\Omega})$ and again use a transformation to "flatten out" the boundary. Then one may use Hölder continuity of $u_{0,\varepsilon}$, $\chi_\varepsilon(d(\cdot))^m$ and f_0 to show Hölder continuity of the weight function. The main idea of the proof leads back to the one-dimensional result in Lemma 4.2.4.

■

***Remark* 4.2.6** *One may suspect that generically all eigenvalues are simple for $\varepsilon \in (0, \varepsilon_0)$, see for example some similar investigations in [3]. We do not want a generic result and we only need the simplicity of the eigenvalue $\lambda = 1$. To this end, we may fix $\varepsilon = \frac{1}{2}\varepsilon_0$ and proceed by an appropriate perturbation of $f_{0,\varepsilon}$ for this fixed ε. This is done in Section 4.4 and yields a simple eigenvalue $\lambda = 1$.*

4.3 Unique continuation

We will construct an appropriate weight function to find a weighted polyharmonic eigenvalue problem with strongly positive eigenfunction and corresponding simple eigenvalue. To prove simplicity of the eigenvalue with corresponding positive eigenfunction, we need the unique continuation principle. There are many results of this kind, see for example [51, 48, 65] and references therin. We recall some result proven by Protter in 1960:

Lemma 4.3.1 (Protter [51, p. 90]) *Let $x_0 \in \Omega \subset \mathbb{R}^n$ and $u \in C^{2m}(\overline{\Omega})$ satisfy the inequality*

$$|\Delta^m u| \leq g(x, u, Du, \ldots, D^k u) \ \text{in } \Omega, \tag{4.19}$$

where $k = \left[\frac{3m}{2}\right]$ and $(x, u, p_1, \ldots, p_k) \mapsto g(x, u, p_1, \ldots, p_k)$ is Lipschitzian in

$$(u, p_1, \ldots, p_k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \cdots \times \mathbb{R}^{n^k},$$

i.e. there is a constant $L > 0$ such that

$$|g(x, u, p_1, \ldots, p_k) - g(x, v, q_1, \ldots, q_k)| \leq L|(u, p_1, \ldots, p_k) - (v, q_1, \ldots, q_k)|$$

for all $x \in \Omega$ *and* $(u, p_1 \ldots, p_k), (v, q_1, \ldots, q_k) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \cdots \times \mathbb{R}^{n^k}$. *Suppose that*

$$e^{2|x-x_0|^{-\beta}} u(x) \to 0 \text{ as } x \to x_0 \tag{4.20}$$

for every positive β. *Then,* u *vanishes identically in* Ω.

A weaker formulation of this principle and a special case is the following result:

Corollary 4.3.2 *Let* u *and* g *be as defined in Lemma 4.3.1 such that* u *satisfies* (4.19). *Suppose that there exists an open subset* $U \subset \Omega$, *such that* u *vanishes identically in* U. *Then* u *vanishes identically in* Ω.

Proof. Choose some $x_0 \in U$. Then (4.20) is fulfilled and we can apply Lemma 4.3.1. ∎

To show the simplicity of a weighted eigenvalue in the next section, we will use the unique continuation principle as well as the following lemma:

Lemma 4.3.3 *Suppose that* $f, g \in C(\Omega)$ *satisfy*

$$f(x)g(x) = 0 \text{ for all } x \in \Omega.$$

Moreover, let

$$\{x \in \Omega; f(x) \neq 0\} \cap B_\delta(x_0) \neq \emptyset \text{ for all } x_0 \in \Omega, \delta > 0. \tag{4.21}$$

Then, it holds that g *vanishes identically in* Ω.

Proof. This follows directly from the continuity of f and g. Assume that $g(x_1) \neq 0$ for some $x_1 \in \Omega$. Because g is continuous, there exists a value $\delta_1 > 0$ such that $g(x) \neq 0$ for all $x \in B_{\delta_1}(x_1) \cap \Omega$. Since $f(x)g(x) = 0$ for $x \in \Omega$, we find that $f(x) = 0$ for all $x \in B_{\delta_1}(x_1) \cap \Omega$. This is a contradiction to (4.21). ∎

4.4 Simplicity of the weighted eigenvalue

We can prove the simplicity of the eigenvalue $\lambda = 1$ of (4.16) analogously to the case $m = 2$ in [58]. The only difference is that in [58] we used the unique continuation theorem of Shirota [65]. Since this is a result only for fourth order equations, we cannot apply it in the following investigation. If we use Corollary 4.3.2 and Lemma 4.3.3 instead, we find the result in [58] for higher order problems ($m \geq 2$) with a similar proof. We recall the following definition of a small perturbation of the weight function w_ε and some results and properties for the perturbated eigenvalues and eigenfunctions, see [58, Section 4]:

Definition 4.4.1 *Let* $f_{0,\varepsilon}$ *and* $u_{0,\varepsilon}$ *be as in Proposition 4.2.5 for a sufficiently small and fixed* $\varepsilon > 0$. *For* $q \in C_c^\infty(\Omega)$ *and* $t \in \mathbb{R}$ *with* $|t|$ *small set*

$$w_{tq,\varepsilon} = \frac{f_{0,\varepsilon} + tq}{u_{0,\varepsilon} + t\mathcal{G}_{m,0,1}(q)}, \tag{4.22}$$

where $\mathcal{G}_{m,0,1}$ is the solution operator for (1.6) with $w \equiv 1$ and $\lambda = 0$, and define $A(tq) : W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega) \to L^2(\Omega)$ by

$$A(tq) = (-\Delta)^m - w_{tq,\varepsilon}. \tag{4.23}$$

***Remark* 4.4.2** *For small t the weight function $w_{tq,\varepsilon}$ can be expressed using the following series:*

$$w_{tq,\varepsilon} = w_{0,\varepsilon} + \sum_{k=1}^{\infty} t^k (-1)^k \left(\frac{\mathcal{G}_{m,0,1} q}{u_{0,\varepsilon}} \right)^{k-1} \frac{1}{u_{0,\varepsilon}} \left(w_{0,\varepsilon} \mathcal{G}_{m,0,1} q - q \right),$$

where $w_{0,\varepsilon} = w_\varepsilon = \frac{f_{0,\varepsilon}}{u_{0,\varepsilon}}$.

***Remark* 4.4.3** *Generic simplicity of the spectrum for the biharmonic eigenvalue problem was proven in [46, 47, 49]. The difference to our method is that the authors used perturbations of the underlying domain instead of perturbations of the differential operator.*

In the following we investigate the eigenvalue problem

$$\mathcal{A}(tq) : \begin{cases} \left((-\Delta)^m - w_{tq,\varepsilon}\right) \varphi = \lambda \varphi & \text{in } \Omega, \\ \varphi = \frac{\partial}{\partial \nu} \varphi = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.24}$$

If one compares (4.16) with (4.24), one notices that the multiplicity of $\lambda = 1$ in problem (4.16) coincides with the multiplicity of $\lambda = 0$ in problem (4.24). So, since we consider the changed eigenvalue problem, we have to reduce the multiplicity of the eigenvalue $\lambda = 0$.

The following description can be found in [58, Section 4] for $m = 2$. Assuming that $\lambda = 0$ is an eigenvalue of multiplicity $M \geq 2$ for (4.24) with $t = 0$, one finds by Kato [39, Theorem 3.9, Chapter 7] or Rellich [55, pp. 76–100] the existence of an interval $(-t_0, t_0) \subset \mathbb{R}$ and M real analytic functions

$$t \mapsto \left(\tilde{\lambda}_{i,t,q}, \tilde{\varphi}_{i,t,q}\right) : (-t_0, t_0) \to \mathbb{R} \times C_0^{m-1}(\overline{\Omega}) \cap C^{2m,\gamma}(\overline{\Omega}) \text{ for } i \in \{1, \ldots, M\},$$

with:

1. $\left(\tilde{\lambda}_{i,t,q}, \tilde{\varphi}_{i,t,q}\right)$ are eigenvalue and eigenfunction for $\mathcal{A}(tq)$ for all $i \in \{1, \ldots, M\}$;
2. $\{\tilde{\varphi}_{i,t,q}\}_{i=1}^M$ is an orthogonal system in $L^2(\Omega)$ and so $\{\tilde{\varphi}_{i,t,q}\}_{i=1}^M$ is independent for $|t|$ small;
3. $\tilde{\lambda}_{i,0,q} = 0$ for all $i \in \{1, \ldots, M\}$;
4. For every open interval $(a, b) \subset \mathbb{R}$ such that $0 \in (a, b)$ is the only eigenvalue of $\mathcal{A}(0)$ in $[a, b]$, there exist exactly M eigenvalues $\tilde{\lambda}_{1,t,q}, \ldots, \tilde{\lambda}_{M,t,q}$ of $\mathcal{A}(tq)$ in (a, b), assuming $|t|$ is small enough.

***Remark* 4.4.4** *We use the notation with an additionl tilde in* $\tilde{\lambda}_{i,t,q}$ *and* $\tilde{\varphi}_{i,t,q}$ *to make a distinction between the eigenvalues and eigenfunctions of* (1.10) *and to avoid more indices.*

***Remark* 4.4.5** *One calls a function* $t \mapsto \tilde{\lambda}_{i,t,q}$ *real analytic in* $t = 0$*, if there are* $\tilde{\lambda}_{i,q}^{(j)} \in \mathbb{R}$ *for* $j \in \mathbb{N}$ *such that*

$$\tilde{\lambda}_{i,t,q} = \sum_{j=0}^{\infty} t^j \, \tilde{\lambda}_{i,q}^{(j)}$$

converges for t *in a neighborhood of* 0*. And analogously* $t \mapsto \tilde{\varphi}_{i,t,q}$ *is real analytic in* $t = 0$*, if there are* $\tilde{\varphi}_{i,q}^{(j)} \in L^2(\Omega)$ *such that*

$$\tilde{\varphi}_{i,t,q} = \sum_{j=0}^{\infty} t^j \, \tilde{\varphi}_{i,q}^{(j)}$$

converges in $L^2(\Omega)$*. Since* $\tilde{\varphi}_{i,t,q}$ *are eigenfunctions to* (4.24)*, we find that* $\tilde{\varphi}_{i,t,q}, \tilde{\varphi}_{i,q}^{(j)} \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$ *for every* $j \in \mathbb{N}$*.*

The following explanation of the idea of the proof can be found in [58, p. 12]: We may set

$$\tilde{\varphi}_{1,t,q} = u_{0,\varepsilon} + t\mathcal{G}_{m,0,1}(q) \tag{4.25}$$

for the first eigenfunction to problem (4.24). This function is analytic in t and the other $M-1$ eigenfunctions can be chosen orthogonally to this function in $L^2(\Omega)$-sense. Then, we find

$$\tilde{\lambda}_{1,t,q} = 0 \text{ for all } t \in (-t_0, t_0)\,.$$

We will prove the existence of a smooth function q_1 such that

$$\tilde{\lambda}'_{k,0,q_1} := \left(\frac{\partial}{\partial t} \tilde{\lambda}_{k,t,q_1} \right)_{|t=0} \neq 0$$

for at least one $k \in \{2, \ldots, M\}$. If this holds true, one finds a small interval $(0, t^*) \subset (0, \infty)$, such that for $t_1 \in (0, t^*)$ we obtain $\tilde{\lambda}_{k,t_1,q_1} \neq 0$ and hence that 0 is an eigenvalue of multiplicity at most $M-1$ for $\mathcal{A}(t_1 q_1)$. If the multiplicity of the eigenvalue 0 for $\mathcal{A}(t_1 q_1)$ is 1, we have found a suitable weight function, a simple eigenvalue and a strongly positive eigenfunction. Otherwise we repeat our arguments for $\mathcal{A}(t_1 q_1 + tq)$. After $k \leq M-1$ steps we have found an eigenvalue problem $\mathcal{A}(t_1 q_1 + \cdots + t_k q_k)$ having 0 as a simple eigenvalue. The idea of the proof was inspired by Albert [3] and Teytel [72] and the following lemma can be found in [58, Lemma 19] for $m = 2$.

Lemma 4.4.6 *Suppose that* 0 *is an eigenvalue of multiplicity* $M \geq 2$ *for problem*

(4.24) *with* $t = 0$. *Then there exist* $k \in \{2, \dots, M\}$ *and* $q_1 \in C_c^\infty(\Omega)$ *such that*

$$\left(\frac{\partial}{\partial t} \tilde{\lambda}_{k,t,q_1} \right)_{|t=0} \neq 0.$$

Proof. For the first step, we can proceed as in the case $m = 2$ and therefore repeat the first part of the proof in [58, Lemma 19]. We assume that $\left(\frac{\partial}{\partial t} \tilde{\lambda}_{k,t,q} \right)_{|t=0} = 0$ for all $k \in \{1, \dots, M\}$ and $q \in C_c^\infty(\Omega)$ and show in two steps that this leads to a contradiction.

1. Differentiation with respect to t of

$$A(tq)\tilde{\varphi}_{k,t,q} = \tilde{\lambda}_{k,t,q}\, \tilde{\varphi}_{k,t,q} \qquad \text{for all } k \in \{1, \dots, M\}$$

yields

$$\left(A(tq) - \tilde{\lambda}_{k,t,q} \right) \tfrac{\partial}{\partial t} \tilde{\varphi}_{k,t,q} = \left(\tfrac{\partial}{\partial t} w_{tq,\varepsilon} + \tfrac{\partial}{\partial t} \tilde{\lambda}_{k,t,q} \right) \tilde{\varphi}_{k,t,q}.$$

Setting $t = 0$ and using (4.22), (4.23) and $\left(\frac{\partial}{\partial t} \tilde{\lambda}_{k,t,q} \right)_{|t=0} = 0$, we find

$$A(0) \left(\tfrac{\partial}{\partial t} \tilde{\varphi}_{k,t,q} \right)_{|t=0} = \frac{1}{u_{0,\varepsilon}} \left(q - w_{0,\varepsilon} \mathcal{G}_{m,0,1}(q) \right) \tilde{\varphi}_{k,0,q}.$$

Hence, we obtain that $\frac{1}{u_{0,\varepsilon}} \left(q - w_{0,\varepsilon} \mathcal{G}_{m,0,1}(q) \right) \tilde{\varphi}_{k,0,q}$ is in the range of $A(0)$ for all $q \in C_c^\infty(\Omega)$. Since every eigenfunction in $\ker(A(0))$ can be written in the form $\sum_{k=1}^m c_k\, \tilde{\varphi}_{k,0,q}$ with $c_k \in \mathbb{R}$ and $A(0)$ is self-adjoint, it follows that

$$\frac{1}{u_{0,\varepsilon}} \left(q - w_{0,\varepsilon} \mathcal{G}_{m,0,1}(q) \right) \phi_1 \perp \ker(A(0)) \text{ for all } \phi_1 \in \ker(A(0)),$$

or in other words

$$\int_\Omega \frac{1}{u_{0,\varepsilon}} \left(q - w_{0,\varepsilon} \mathcal{G}_{m,0,1}(q) \right) \phi_1\, \phi_2\, dx = 0 \text{ for all } \phi_1, \phi_2 \in \ker(A(0)).$$

Using the symmetry of the Green function $G_{m,0,1}(x,y) = G_{m,0,1}(y,x)$, we obtain

$$\begin{aligned} 0 &= \int_\Omega \frac{1}{u_{0,\varepsilon}} \left(q - w_{0,\varepsilon} \mathcal{G}_{m,0,1}(q) \right) \phi_1\, \phi_2\, dx \\ &= \int_\Omega \left(q(x) - w_{0,\varepsilon}(x) \int_\Omega G_{m,0,1}(x,y) q\,(y)\, dy \right) \frac{\phi_1(x)\, \phi_2(x)}{u_{0,\varepsilon}(x)} dx \\ &= \int_\Omega q(x) \left(\frac{\phi_1(x)\, \phi_2(x)}{u_{0,\varepsilon}(x)} - \mathcal{G}_{m,0,1} \left(w_{0,\varepsilon} \frac{\phi_1\, \phi_2}{u_{0,\varepsilon}} \right)(x) \right) dx \end{aligned}$$

and we can apply the fundamental lemma of calculus of variation to find for all $\phi_1, \phi_2 \in \ker(A(0))$ that

$$\frac{\phi_1(x)\ \phi_2(x)}{u_{0,\varepsilon}(x)} - \mathcal{G}_{m,0,1}\left(w_{0,\varepsilon}\frac{\phi_1\ \phi_2}{u_{0,\varepsilon}}\right)(x) = 0 \quad \text{for all } x \in \Omega.$$

So if ϕ_1 and ϕ_2 are eigenfunctions of $\mathcal{A}\,(0)$ with $\lambda = 0$ in (4.24), then also

$$\widetilde{\phi}_{1,2} := \frac{\phi_1\ \phi_2}{u_{0,\varepsilon}} \tag{4.26}$$

is an eigenfunction for $\mathcal{A}\,(0)$ with $\lambda = 0$. If we set $\phi_1 = u_{0,\varepsilon}$, then $\widetilde{\phi}_{1,2} = \phi_2$, so in this case it is obvious that $\widetilde{\phi}_{1,2}$ is an eigenfunction. For arbitrary $\phi_1, \phi_2 \in \ker(A(0))$ this is not to be expected. Let $\psi \in C^{2m}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$ be an eigenfunction for $\mathcal{A}(0)$ and orthogonal to $u_{0,\varepsilon}$ in $L^2(\Omega)$-sense. If we now set ϕ_1 and ϕ_2 equal to ψ, we find that $\frac{\psi^2}{u_{0,\varepsilon}}$ is an eigenfunction to the eigenvalue $\lambda = 0$. After setting $\phi_1 = \frac{\psi^2}{u_{0,\varepsilon}}$ and $\phi_2 = \psi$ and repeating this step, we find the eigenfunctions

$$\psi_n(x) := \left(\frac{\psi(x)}{u_{0,\varepsilon}(x)}\right)^n \psi(x) \quad \text{for all } n \in \mathbb{N}. \tag{4.27}$$

2. Using unique continuation, we show that this cannot be true. So, we find the same result as in [58, Lemma 19] for the polyharmonic eigenvalue problem. The only difference is that we apply the results in Section 4.3 instead of the unique continuation result proven by Shirota. Hence, the second part of the proof deviates from the proof in [58].

   By induction we find infinite multiplicity of the eigenvalue $\lambda = 0$ for $\mathcal{A}(0)$: Let ψ_n be defined as in (4.27). It holds true that ψ_0 and ψ_1 are linearly independent eigenfunctions. Indeed, using (4.27), we obtain that if

$$\psi(x) = \psi_0(x) = c\psi_1(x) = c\frac{\psi(x)^2}{u_{0,\varepsilon}(x)} \text{ for all } x \in \Omega$$

   and some $c \in \mathbb{R}$, then $\psi(x)(u_{0,\varepsilon}(x) - c\psi(x)) = 0$ for all $x \in \Omega$. If $\psi(x) = 0$ for x in some ball $B_\delta(x_0) \subset \Omega$ with $\delta > 0$, then $\psi \equiv 0$ by unique continuation, see Corollary 4.3.2. Since the function ψ is a nontrivial eigenfunction, this cannot be true. If there exists no open set where ψ vanishes, then it follows from Lemma 4.3.3 that $u_{0,\varepsilon} - c\psi \equiv 0$. Since ψ has a nodal line and $u_{0,\varepsilon} > 0$ this cannot be true for any $c \in \mathbb{R}$. Therefore, ψ_0 and ψ_1 are linearly independent.

   By induction, we find that $\{\psi_n\}_{n=0}^N$ is a set of linearly independent eigenfunctions with corresponding eigenvalue $\lambda = 0$ for every $N \in \mathbb{N}^+$. Indeed, when rewriting

$$c_0\psi_0(x) + c_1\psi_1(x) + \cdots + c_{N-1}\psi_{N-1}(x) = \psi_N(x) \text{ for all } x \in \Omega,$$

we find

$$\frac{\psi(x)}{u_{0,\varepsilon}(x)}\left(c_0 u_{0,\varepsilon}(x) + c_1\psi_0(x) + \cdots + c_{N-1}\psi_{N-2}(x) - \psi_{N-1}(x)\right) = 0$$

for all $x \in \Omega$. Again, either $\psi \equiv 0$, or

$$c_0 u_{0,\varepsilon}(x) + c_1\psi_0(x) + \cdots + c_{N-1}\psi_{N-2}(x) - \psi_{N-1}(x) = 0 \text{ for all } x \in \Omega.$$

Since $c_1\psi_0 + \cdots + c_{N-1}\psi_{N-2} - \psi_{N-1}$ has a nodal line and $u_{0,\varepsilon} > 0$, one finds $c_0 = 0$, so

$$c_1\psi_0(x) + \cdots + c_{N-1}\psi_{N-2}(x) = \psi_{N-1}(x) \text{ for all } x \in \Omega.$$

Using the induction hypothesis, we obtain the result. Hence $\lambda = 0$ has infinite multiplicity, a contradiction.

■

As mentioned above, using Lemma 4.4.6, we can find a perturbation of the weight function w_ε such that the eigenvalue $\lambda = 0$ becomes simple. The next corollary can be stated and proven analogously to the case $m = 2$ and can be found in [58, Corollary 20].

Corollary 4.4.7 *Suppose that Condition A and B are satisfied. Let ε be fixed as described in Remark* 4.2.6 *and $f_{0,\varepsilon}, u_{0,\varepsilon}$ be defined as in Proposition 4.2.5. Then there is $q^* \in C_c^\infty(\Omega)$ such that*

1. $w^* = \dfrac{f_{0,\varepsilon} + q^*}{u_{0,\varepsilon} + \mathcal{G}_{m,0,1}(q^*)} \in C^{0,\gamma}(\overline{\Omega})$ *is strictly positive on $\overline{\Omega}$, and*
2. $\varphi = u_{0,\varepsilon} + \mathcal{G}_{m,0,1}(q^*)$ *is a strongly positive eigenfunction in the sense of (1.11) for*

$$\begin{cases} \left((-\Delta)^m - w^*\right)\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = \frac{\partial}{\partial\nu}\varphi = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1}\varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.28}$$

with simple eigenvalue $\lambda = 0$.

Proof. If the multiplicity of the eigenfunction $\varphi = u_{0,\varepsilon}$ for the weight function $w = f_{0,\varepsilon}/u_{0,\varepsilon}$ is $M \geq 2$, we may proceed as in Lemma 4.4.6 and find q_1 such that for $t_1 > 0$ but small enough, problem $\mathcal{A}(t_1 q_1)$ contains a positive weight function and has a positive eigenfunction $\tilde{\varphi}_{1,t_1,q_1}$ with eigenvalue 0 of multiplicity at most $M-1$. Then, repeating the argument now starting with $\mathcal{A}(t_1 q_1)$ as in (4.23) and considering $\mathcal{A}_1(tq) = \mathcal{A}(t_1 q_1 + tq)$ we may again reduce the multiplicity. After at most $K \leq M-1$ steps the multiplicity for $\mathcal{A}(q^*)$ with

$$q^* = t_1 q_1 + t_2 q_2 + \cdots + t_K q_K$$

and $t_1 \gg t_2 \gg \cdots \gg t_K > 0$ has reduced to 1. ■

If we transfer the result to our original eigenvalue problem, we have found that

$$\begin{cases} (-\Delta)^m \varphi = \lambda w^* \varphi & \text{in } \Omega, \\ \varphi = \frac{\partial}{\partial \nu}\varphi = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.29}$$

is an eigenvalue problem with simple eigenvalue $\lambda_{\mathrm{p},m,w^*} = 1$ and corresponding positive eigenfunction $\varphi_{\mathrm{p},m,w^*} = u_{0,\varepsilon} + \mathcal{G}_{m,0,1}(q^*)$. This completes the proof of Theorem 1.2.8.

Using Lemma 3.1.7, Theorem 3.1.3 and Theorem 3.6.2, we also find the positivity preserving property and an anti-maximum principle for the weighted Dirichlet problem with weight function w^* and therefore the results in Theorem 1.2.10 and Theorem 1.2.12:

Corollary 4.4.8 *Let Ω, w^* and $\lambda_{\mathrm{p},m,w^*} = 1$ be as in Corollary 4.4.7. Then there is $\lambda_c < \lambda_{\mathrm{p},m,w^*}$ such that for $0 \leq f \in L^2(\Omega)$ with f nontrivial and u the weak solution to*

$$\begin{cases} (-\Delta)^m u - \lambda w^* u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial \nu}u = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.30}$$

a Hopf type result holds if $\lambda \in (\lambda_c, \lambda_{\mathrm{p},m,w^*})$: There exists $c_{f,\lambda} > 0$ such that

$$u(x) \geq c_{f,\lambda}\, d(x)^m \text{ for almost every } x \in \Omega.$$

Moreover, if $\lambda_{\mathrm{p},m,w^*}$ is not the first eigenvalue of (4.29), then it holds

$$\lambda_c \geq \lambda_{\mathrm{p}-1,m,w^*} + \frac{\lambda_{\mathrm{p},m,w^*} - \lambda_{\mathrm{p}-1,m,w^*}}{2}.$$

Corollary 4.4.9 *Let Ω, w^* and $\lambda_{\mathrm{p},m,w^*} = 1$ be as in Corollary 4.4.7. Let $0 \leq f \in L^q(\Omega)$ with f nontrivial and $q > \max\{1, \frac{n}{m}\}$. Then, there exists $\delta_f > 0$, such that for all $\lambda \in (\lambda_{\mathrm{p},m,w^*}, \lambda_{\mathrm{p},m,w^*} + \delta_f)$ the following holds: There is a constant $\tilde{c}_{f,\lambda} > 0$ such that the weak solution u of (4.30) satisfies*

$$u(x) \leq -\tilde{c}_{f,\lambda}\, d(x)^m \text{ for every } x \in \Omega.$$

Chapter 5

Some special cases and examples

In this chapter, we consider some examples and special cases and apply Theorem 1.2.10. For some domains one can calculate explicit functions that fulfill Condition B, see Example 5.0.1, and therefore one obtains a positivity preserving property for a weighted problem.

Example 5.0.1 *Let Ω_c be defined as follows:*

$$\Omega_c = \{(x_1, x_2) \in \mathbb{R}^2; x_1^8 + c x_2^2 < 1\} \text{ with } c \geq 12 \tag{5.1}$$

and let $v_0 : \Omega_c \to \mathbb{R}$ be $v_0(x_1, x_2) = 1 - x_1^8 - c x_2^2$. Then, v_0 solves

$$\begin{cases} -\Delta v_0 = 2c + 56 x_1^6 \geq 24 & \text{in } \Omega_c, \\ v_0(x_1, x_2) = 0 & \text{on } \partial\Omega_c. \end{cases}$$

Using Hopf's boundary point lemma, we find $v_0(x_1, x_2) \geq c_d\, d(x_1, x_2)$ for some constant $c_d > 0$ and all $(x_1, x_2) \in \overline{\Omega}_c$. So, the function

$$u_0(x_1, x_2) = v_0(x_1, x_2)^2 = (1 - x_1^8 - c x_2^2)^2 \tag{5.2}$$

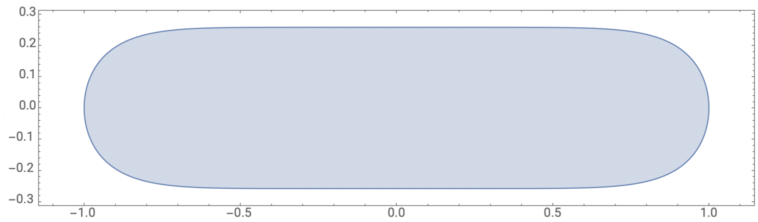
satisfies $u_0(x_1, x_2) \geq c_d^2 d(x_1, x_2)^2$ for all $(x_1, x_2) \in \overline{\Omega}_c$, see Figure 5.1. Moreover, we calculate

$$\begin{aligned} (-\Delta)^2 u_0(x_1, x_2) &= 8\left(3c^2 - 420 x_1^4 + 5460 x_1^{12} + 56 c x_1^6 + 420 c x_1^4 x_2^2\right) \\ &\geq 24(c^2 - 140), \end{aligned}$$

and this expression is greater than zero for $c \geq 12$. Therefore, we have found a function that satisfies Condition B in the biharmonic case.

Analogously, we find that for $c \geq 30$ the function $\tilde{u}_0 = v_0^3$ fulfills Condition B for $m = 3$ in Ω_c.

For general domains it is rather difficult to construct an explicit function u_0 that is a strongly positive m-polyharmonic Dirichlet supersolution in the sense of Condition B. In the following sections we will show the validity of the condition



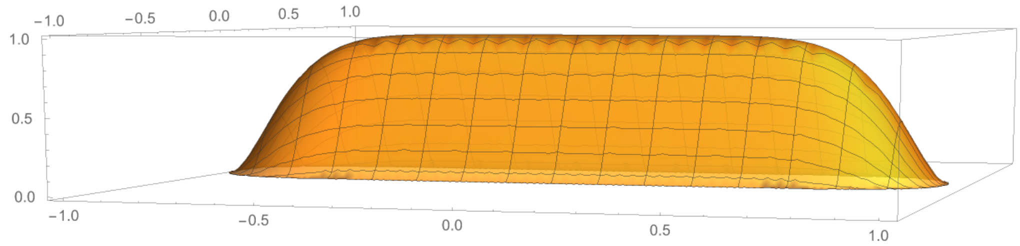


Figure 5.1: top: Ω_c as defined in (5.1) with $c = 15$; bottom: u_0 as defined in (5.2) with $c = 15$.

described in Remark 1.2.7 for some smooth domains and some values for $m \in \mathbb{N}^+$. First, we consider the case of the weighted biharmonic Dirichlet problem.

5.1 Biharmonic Dirichlet problem on smooth domains

Using Theorem 1.2.10, we can prove the validity of the positivity preserving property of a weighted fourth order problem. With additional assumptions, we can also show properties of λ_c in Theorem 1.2.10.

5.1.1 Positivity preserving property

In joint work with Guido Sweers, I have recently proven that in every smooth domain the positivity preserving property is valid for a weighted fourth order Dirichlet problem. In this subsection we present the content of [58], published in *Pure and Applied Analysis* by Mathematical Sciences Publishers.

We assume that Condition A is fulfilled. Then we may prove that for $m = 2$ one can find a function $u_0 \in C^{4,\gamma}(\overline{\Omega})$ which satisfies Condition B using the solution $\mathbf{e}$ to the Dirichlet problem as described in Remark 1.2.7:

$$\begin{cases} -\Delta \mathbf{e} = 1 & \text{in } \Omega, \\ \mathbf{e} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Using the maximum principle, we find that $\mathbf{e}$ is positive and we obtain $\mathbf{e} \in C^{4,\gamma}(\overline{\Omega}) \cap C_0(\overline{\Omega})$, see [22, Theorem 6.19]. When applying the bilaplace operator to $\mathbf{e}^2$, we get

$$\begin{aligned}\Delta^2\mathbf{e}^2 &= 2(-\Delta)\left((-\Delta\mathbf{e})\,\mathbf{e}-\sum_{i=1}^{n}\left(\frac{\partial\mathbf{e}}{\partial x_i}\right)^2\right) = 2(-\Delta)\left(\mathbf{e}-\sum_{i=1}^{n}\left(\frac{\partial\mathbf{e}}{\partial x_i}\right)^2\right)\\ &= 2+4\sum_{i=1}^{n}\left(\frac{\partial\mathbf{e}}{\partial x_i}\frac{\partial\Delta\mathbf{e}}{\partial x_i}\right)+4\sum_{i,j=1}^{n}\left(\frac{\partial^2\mathbf{e}}{\partial x_i\partial x_j}\right)^2 = 2+4\sum_{i,j=1}^{n}\left(\frac{\partial^2\mathbf{e}}{\partial x_i\partial x_j}\right)^2. \qquad (5.4)\end{aligned}$$

So, $(-\Delta)^2\mathbf{e}^2 > 0$ and $(-\Delta)^2\mathbf{e}^2 \in C^{2,\gamma}(\overline{\Omega})$. Using Theorem 1.2.8, we obtain for every smooth domain a weighted biharmonic problem, such that the positivity preserving property holds.

***Remark* 5.1.1** *In general, for $m > 2$ one cannot choose $u_0 = \mathbf{e}^m$ with $\mathbf{e}$ as in (5.3). For example one finds for the annulus $\Omega = B_1(0)\backslash\overline{B_\delta(0)} \subset \mathbb{R}^n$ with $n > 2$ and $\delta \in (0,1)$ the radial symmetric solution $\tilde{\mathbf{e}}(r) := \tilde{\mathbf{e}}(|x|) = \mathbf{e}(x)$ as follows:*

$$\tilde{\mathbf{e}}(r) = \frac{r^2(-\delta^n+\delta^2)+(\delta^{n+2}-\delta^2)+r^{2-n}(-\delta^{n+2}+\delta^n)}{2n\,(\delta^n-\delta^2)} \quad \text{for } r \in (\delta,1). \qquad (5.5)$$

For $n = 2$ one finds

$$\tilde{\mathbf{e}}(r) = \frac{1}{4}-\frac{r^2}{4}-\frac{(1-\delta^2)\log(r)}{4\log(\delta)} \quad \text{for } r \in (\delta,1). \qquad (5.6)$$

Since the maximum principle holds, the functions are positive in Ω, but in general $(-\Delta)^3\mathbf{e}(x)^3$ is not positive for all $x \in \Omega$. Indeed, applying the trilaplacian to (5.6), we obtain

$$\begin{aligned}(-\Delta)^3\mathbf{e}^3(x) &= \left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right)^3\tilde{\mathbf{e}}^3(r)\\ &= 3\frac{6(-1+\delta^2)^3-4(-1+\delta^2)^3\log(r)+(-4+r^2)(-1+\delta^2)^2\log(\delta)+24r^6\log(\delta)^3}{2r^6\log(\delta)^3}\end{aligned}$$

and this is not strictly positive for small δ, see Fig. 5.2 for $\delta = \frac{1}{20}$.

So the construction in [58] is not necessarily possible for $m > 2$. Even if the approach $u_0 = \mathbf{e}^m$ cannot be used for all higher order problems and every domain, one can find exceptions as in Example 5.1.2.

Example 5.1.2 *We have seen that the solution $\mathbf{e}$ to (5.3) does not fulfill $(-\Delta)^3\mathbf{e}^3 > 0$ for all annuli. Let $\Omega = B_1(0)\backslash\overline{B_\delta(0)} \subset \mathbb{R}^2$ be the annulus with inner radius $\delta \in (0,1)$ as in Remark 5.1.1 and $\mathbf{e}$ be the radial symmetric solution to problem (5.3) with $\tilde{\mathbf{e}}(r) := \tilde{\mathbf{e}}(|x|) = \mathbf{e}(x)$ and $\tilde{\mathbf{e}} : (\delta,1) \to \mathbb{R}$. We define $g_\delta(r) := (-\Delta)^3\tilde{\mathbf{e}}^3(r)$ and find for $r \in (\delta,1)$*

$$g_\delta(r) := 3\frac{6(-1+\delta^2)^3-4(-1+\delta^2)^3\log(r)+(-4+r^2)(-1+\delta^2)^2\log(\delta)+24r^6\log(\delta)^3}{2r^6\log(\delta)^3}.$$

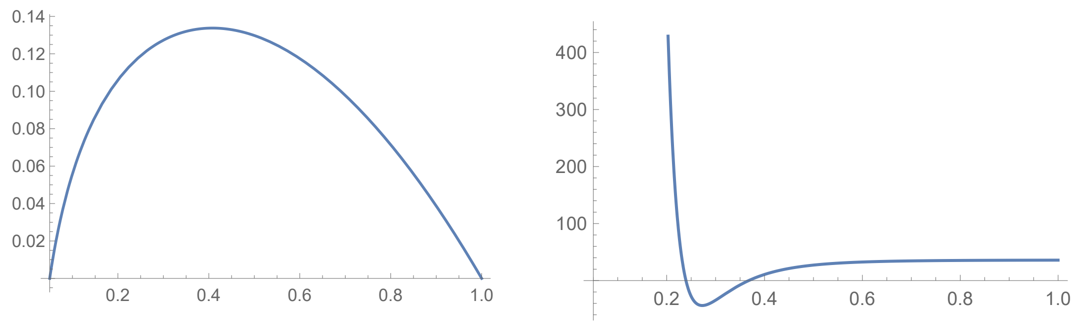


Figure 5.2: Left: $\tilde{\mathbf{e}}$ as defined in (5.6) with $\delta = \frac{1}{20}$; right: $(-\Delta)^3\tilde{\mathbf{e}}^3$.

Then

$$g_\delta'(r) = \frac{6(-1+\delta^2)^2\,(-10(-1+\delta^2)+6(-1+\delta^2)\log(r)-(-6+r^2)\log(\delta))}{r^7\log(\delta)^3} \quad \text{for } r \in (\delta,1)$$

is negative for sufficiently large $\delta < 1$*. So we find a value* $\delta_1 \in (0,1)$ *such that for* $\delta \in (\delta_1, 1)$

$$g_\delta(r) \geq g_\delta(1) = \frac{9\,(2(-1+\delta^2)^3-(-1+\delta^2)^2\log(\delta)+8\log(\delta)^3)}{2\log(\delta)^3} \quad \text{for all } r \in (\delta,1)$$

and

$$\lim_{\delta\uparrow 1} g_\delta(1) = 90.$$

Since $g_\delta(1)$ *is continuous in* $\delta \in (\delta_1,1)$*, there is a value* $\delta_2 \in (\delta_1,1)$ *such that* $g_\delta(r) \geq g_\delta(1) > 0$ *for* $\delta \in (\delta_2,1)$ *and all* $r \in (\delta,1)$*, so* $(-\Delta)^3\mathbf{e}^3 > 0$ *on* $B_1(0)\backslash\overline{B_\delta(0)}$ *for* $\delta \in (\delta_2,1)$*.*

5.1.2 Anti-eigenvalue problem

As mentioned in Remark 1.2.11, it is known that the one-dimensional fourth order problem

$$\begin{cases} u'''' - \lambda u = f & \text{in } (0,1), \\ u(0) = u'(0) = 0, \\ u(1) = u'(1) = 0 \end{cases} \tag{5.7}$$

is positivity preserving iff $\lambda \in [\lambda_c, \lambda_{1,2,1})$, where $\lambda_{1,2,1}$ is the principle eigenvalue to the biharmonic Dirichlet problem in $\Omega = (0,1)$ and $\lambda_c = -4\mu_c^4$ with μ_c the first positive solution of $\tan(\mu) = \tanh(\mu)$.

For $\lambda = 0$, problem (5.7) is positivity preserving. Using methods proven by Schröder in [61, 62, 63] for the clamped bar, one finds that by decreasing λ one reaches a value $\lambda_c < 0$ such that the Green function is sign-changing for $\lambda < \lambda_c$, and that the negative part comes in through the boundary. Then, one obtains that λ_c is the first negative eigenvalue for the 'switched' eigenvalue problem, see also [68, Lemma 2.3]

$$\begin{cases} \varphi'''' = \lambda\,\varphi & \text{in } (0,1)\,, \\ \varphi\,(0) = 0, \\ \varphi\,(1) = \varphi'\,(1) = \varphi''\,(1) = 0. \end{cases} \tag{5.8}$$

One calls this problem a 'switched' eigenvalue problem, since the highest order boundary condition on one side is replaced by the next available lowest order boundary condition on the other side. The number λ_c is also called an 'anti'-eigenvalue for (5.7), see for example [40, p. 1025].

***Remark* 5.1.3** *One notices, that an eigenfunction to* (5.8) *with corresponding eigenvalue* λ_c *is*

$$\varphi(x) = \lim_{y\downarrow 0} \frac{G_{2,\lambda_c,1}(x,y)}{y^2} = \tfrac{1}{2}\left(\tfrac{\partial}{\partial y}\right)^2 G_{2,\lambda_c,1}(x,0), \tag{5.9}$$

where $G_{2,\lambda_c,1}$ *is the Green function for* (5.7) *with* $\lambda = \lambda_c$.

In higher dimensions and higher order problems no such result is known, but if we assume that for decreasing λ the sign-change of the Green function for (1.6) comes in through the boundary, we expect a similar result.

So, in this subsection, we suppose that Condition A is satisfied and the Green function $G_{2,\lambda,w}$ for (1.6) with $m = 2$ is nonnegative iff $\lambda \in [\lambda_c, \lambda_{\mathrm{p},2,w})$. Moreover, we suppose that the sign-change of the Green function $G_{2,\lambda,w}$ comes in through the boundary for decreasing λ. Let $(x_0, y_0) \in \partial\Omega \times \partial\Omega \backslash \{(x,x); x \in \partial\Omega\}$ be such that an additional zero for the Green function $G_{2,\lambda,w}$ with $\lambda \downarrow \lambda_c$ occurs in (x_0, y_0). More precisely, analogous to the description in [25] this means that for some sequence $\{\lambda_k\}_{k\in\mathbb{N}^+} \subset (\lambda_{\mathrm{p}-1,2,w}, \lambda_c)$ if $\mathrm{p} > 1$ or $\{\lambda_k\}_{k\in\mathbb{N}^+} \subset (-\infty, \lambda_c)$ if $\mathrm{p} = 1$ such that $\lambda_k \uparrow \lambda_c$, there exists a sequence $\{(x_k, y_k)\}_{k\in\mathbb{N}^+} \subset \Omega \times \Omega$ with

$$G_{2,\lambda_k,w}(x_k, y_k) = 0 \text{ for all } k \in \mathbb{N}^+,$$

and $x_k \to x_0 \in \partial\Omega$, $y_k \to y_0 \in \partial\Omega$.

Similar to (5.9), we consider the function $g_\lambda : \overline{\Omega} \to \mathbb{R}$ for $\lambda \in [\lambda_c, \lambda_{\mathrm{p},2,w})$ defined by

$$g_\lambda(x) = \lim_{\Omega\ni y\to y_0} \frac{G_{2,\lambda,w}(x,y)}{d(y)^2}. \tag{5.10}$$

Using Taylor's formula, we find that for $x \in \overline{\Omega}\backslash\{y_0\}$

$$g_\lambda(x) = \tfrac{1}{2}\left(\frac{\partial}{\partial\nu_y}\right)^2 G_{2,\lambda,w}(x,y_0)$$

and $g_\lambda(x) \geq 0$ for all $x \in \Omega$, since $G_{2,\lambda,w}(x,y), d(y) \geq 0$ for all $x, y \in \Omega$ with $x \neq y$. Moreover, with $G_{2,\lambda,w}(x,y_0) = \frac{\partial}{\partial\nu_y} G_{2,\lambda,w}(x,y_0) = 0$ for all $x \in \Omega$, we get

$$g_\lambda(x) = \tfrac{1}{2}\left(\frac{\partial}{\partial\nu_y}\right)^2 G_{2,\lambda,w}(x,y_0) = \tfrac{1}{2}\Delta_y G_{2,\lambda,w}(x,y_0).$$

Using [25, Proposition 3], one finds $g_\lambda \in C^4(\overline{\Omega}\backslash\{y_0\})$ and g_λ satisfies

$$\begin{cases} \Delta^2 g_\lambda - \lambda w\, g_\lambda = 0 & \text{in } \Omega, \\ g_\lambda = 0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu} g_\lambda = 0 & \text{on } \partial\Omega\backslash\{y_0\}. \end{cases}$$

Using [25, Theorem 3], we also find

$$\left(\tfrac{\partial}{\partial\nu}\right)^2 g_{\lambda_c}(x_0) = \Delta_x g_{\lambda_c}(x_0) = \tfrac{1}{2}\Delta_x\Delta_y G_{2,\lambda_c,w}(x_0,y_0) = 0.$$

Therefore, $g_{\lambda_c} \in C^4(\overline{\Omega}\backslash\{y_0\})$ is nonzero and fulfills

$$\begin{cases} \Delta^2 g_{\lambda_c} - \lambda_c w\, g_{\lambda_c} = 0 & \text{in } \Omega, \\ g_{\lambda_c} = 0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu} g_{\lambda_c} = 0 & \text{on } \partial\Omega\backslash\{y_0\}, \\ \left(\frac{\partial}{\partial\nu}\right)^2 g_{\lambda_c}(x_0) = 0. \end{cases} \tag{5.11}$$

One should notice that this is not an obviously well-defined eigenvalue problem, i.e. it is not clear in which space one should search for pairs of solutions (λ, φ).

Example 5.1.4 *Let $\Omega = B_1(0) \subset \mathbb{R}^n$. Then, we may derive informations about the regularity of the function g_{λ_c}, since we know an explicit formula for the polyharmonic Green function. We obtain with $w \equiv 1$ that the Green function $G_{2,\lambda_c,1}$ can be written using the biharmonic Green function $G_{2,0,1}$, see [25, Proof of Proposition 2]:*

$$G_{2,\lambda_c,1}(x,y) = G_{2,0,1}(x,y) + \sum_{j=1}^{\ell} \Gamma_{\lambda_c,j}(x,y) + v_{\lambda_c,x}(y), \tag{5.12}$$

where we choose $\ell > 1 + \frac{n}{4}$ and $\Gamma_{\lambda_c,j}$ can be defined inductively

$$\Gamma_{\lambda_c,1}(x,y) = \lambda_c \int_{B_1(0)} G_{2,0,1}(x,z) G_{2,0,1}(z,y)\,dz,$$

$$\Gamma_{\lambda_c,j+1}(x,y) = \lambda_c \int_{B_1(0)} \Gamma_{\lambda_c,j}(x,z) G_{2,0,1}(z,y)\,dz.$$

We find

$$|\Gamma_{\lambda_c,\ell}|, |\nabla\Gamma_{\lambda_c,\ell}| \leq C \tag{5.13}$$

for some $C > 0$, *dependent on* λ_c *and* n *see [25, Equation (31)]. Furthermore,* $v_{\lambda_c,x} \in C^{4,\gamma}(\overline{B_1(0)})$ *is the solution to*

$$\begin{cases} \Delta^2 v_{\lambda_c,x}(y) - \lambda_c v_{\lambda_c,x}(y) = \lambda_c \Gamma_{\lambda_c,\ell}(x,y) & \text{in } B_1(0), \\ v_{\lambda_c,x}(y) = \frac{\partial}{\partial\nu} v_{\lambda_c,x}(y) = 0 & \text{on } \partial B_1(0). \end{cases} \tag{5.14}$$

Using (5.13), *differentiating* (5.14) *with respect to* x, *applying regularity theory and Sobolev imbeddings, one finds two constants* $C_1, C_2 > 0$, *independent of* x, *such that*

$$|\Delta_y v_{\lambda_c,x}(y_0)| < C_1, \quad |\nabla_x \Delta_y v_{\lambda_c,x}(y_0)| < C_2.$$

With (5.12), *we obtain for* $x \in \Omega$

$$g_{\lambda_c}(x) = \tfrac{1}{2}\Delta_y G_{2,0,1}(x,y_0) + \tfrac{1}{2}\sum_{j=1}^{\ell} \Delta_y \Gamma_{\lambda_c,j}(x,y_0) + \tfrac{1}{2}\Delta_y v_{\lambda_c,x}(y_0),$$

where $\Delta_y G_{2,0,1}(x,y)$ *is the Poisson kernel for the biharmonic problem and using Remark 2.2.2, one may calculate*

$$\tfrac{1}{2}\Delta_y G_{2,0,1}(x,y_0) = \frac{1}{nb_n 4} \frac{(1-|x|^2)^2}{|x-y_0|^n}$$

with b_n *as in* (2.2). *It follows that* $g_{\lambda_c} \in L^\infty(B_1(0))$ *if* $n = 2$ *and* $g_{\lambda_c} \in L^{q_1}(\Omega)$ *for all* $q_1 \in \left[1, \frac{n}{n-2}\right)$ *if* $n \geq 3$. *Moreover, we find* $g_{\lambda_c} \in W^{1,q_2}(B_1(0))$ *for all* $q_2 \in \left[1, \frac{n}{n-1}\right)$ *and* $n \geq 2$, *but* $g_{\lambda_c} \notin W^{2,q_3}(B_1(0))$ *and* $g_{\lambda_c} \in W^{2,q_3}(K)$ *for all* $q_3 \in [1,\infty)$ *and compact subsets* $K \subset B_1(0)$.

***Remark* 5.1.5** *Let Condition A be satisfied. By Green function estimates of Krasovskiĭ or Pulst, see [41], [53, Theorem 2.4], one finds for* $\alpha \in \mathbb{N}^n$ *with* $0 \leq |\alpha| \leq 2$

$$|D_x^\alpha \Delta_y G_{2,\lambda_c,w}(x,y_0)| \leq c_{\alpha,\lambda_c,w}|x-y_0|^{2-n-|\alpha|}$$

if $2 - n < |\alpha|$ *and using [21, Theorem 4.29] one obtains*

$$|\Delta_y G_{2,\lambda_c,w}(x,y_0)| \leq c_{\alpha,\lambda_c,w}$$

if $n = 2$. *So, we find similar regularity results as in Example 5.1.4 for arbitrary bounded domains* $\Omega \subset \mathbb{R}^n$ *with* $\partial\Omega \in C^{4,\gamma}$:

- $g_{\lambda_c} \in L^\infty(\Omega)$ *if* $n = 2$ *and* $g_{\lambda_c} \in L^{q_1}(\Omega)$ *for all* $q_1 \in \left[1, \frac{n}{n-2}\right)$ *if* $n \geq 3$,
- $g_{\lambda_c} \in W^{1,q_2}(\Omega)$ *for all* $q_2 \in \left[1, \frac{n}{n-1}\right)$ *and* $n \geq 2$,

- $g_{\lambda_c} \in W^{2,q_3}(K)$ *for all compact subsets* $K \subset \Omega$, $q_3 \in [1,\infty)$ *and* $n \geq 2$.

Moreover, we may prove that $\lambda = \lambda_c$ is the largest value smaller than $\lambda_{\mathrm{p},2,w}$, such that $g_\lambda \in C^{4,\gamma}(\overline{\Omega}\backslash\{y_0\})$ fulfills (5.11). Indeed, we find for $\mu \in (\lambda_c, \lambda_{\mathrm{p},2,w})$ that $(\frac{\partial}{\partial\nu_x})^2(\frac{\partial}{\partial\nu_y})^2 G_{2,\mu,w}(x,y) = \Delta_x\Delta_y G_{2,\mu,w}(x,y) \neq 0$ for all $x,y \in \partial\Omega$ with $x \neq y$. The following proof is inspired by [32, Lemma 2].

Lemma 5.1.6 *Suppose that* $G_{2,\lambda_c,w}(x,y) \geq 0$ *for all* $x,y \in \Omega$ *with* $x \neq y$*. Then, for all* $\mu \in \mathbb{R}$ *with* $\lambda_c < \mu < \lambda_{\mathrm{p},2,w}$ *one finds*

$$\left(\frac{\partial}{\partial\nu_x}\right)^2 \left(\frac{\partial}{\partial\nu_y}\right)^2 G_{2,\mu,w}(x,y) > 0 \text{ for all } x,y \in \partial\Omega \text{ with } x \neq y. \tag{5.15}$$

Proof. We fix some $y^* \in \partial\Omega$. Using [32, Lemma 2] one finds

$$\left(\frac{\partial}{\partial\nu_y}\right)^2 G_{2,\mu,w}(x,y^*) > 0 \text{ for all } x \in \Omega. \tag{5.16}$$

Since $\left(\frac{\partial}{\partial\nu_y}\right)^2 G_{2,\mu,w}(x,y^*) = \frac{\partial}{\partial\nu_x}\left(\frac{\partial}{\partial\nu_y}\right)^2 G_{2,\mu,w}(x,y^*) = 0$ for $x \in \partial\Omega\backslash\{y^*\}$, one gets

$$\left(\frac{\partial}{\partial\nu_x}\right)^2 \left(\frac{\partial}{\partial\nu_y}\right)^2 G_{2,\mu,w}(x,y^*) \geq 0 \text{ for all } x \in \partial\Omega\backslash\{y^*\}.$$

Analogously, one obtains

$$\left(\frac{\partial}{\partial\nu_x}\right)^2 \left(\frac{\partial}{\partial\nu_y}\right)^2 G_{2,\lambda_c,w}(x,y^*) \geq 0 \text{ for all } x \in \partial\Omega\backslash\{y^*\}. \tag{5.17}$$

With the resolvent formula $\mathcal{G}_{2,\mu,w} = \mathcal{G}_{2,\lambda_c,w}(\mathcal{I} + (\mu-\lambda_c)\mathcal{G}_{2,\mu,w})$ one gets

$$G_{2,\mu,w}(x,y) = G_{2,\lambda_c,w}(x,y) + (\mu-\lambda_c)\int_\Omega G_{2,\lambda_c,w}(x,z)G_{2,\mu,w}(z,y)w(z)dz.$$

Then, we find for arbitrary $x^* \in \partial\Omega$ with $x^* \neq y^*$

$$\begin{aligned} \left(\frac{\partial}{\partial\nu_x}\right)^2 \left(\frac{\partial}{\partial\nu_y}\right)^2 G_{2,\mu,w}(x^*,y^*) &= \left(\frac{\partial}{\partial\nu_x}\right)^2 \left(\frac{\partial}{\partial\nu_y}\right)^2 G_{2,\lambda_c,w}(x^*,y^*) \\ +(\mu-\lambda_c)\int_\Omega \left(\frac{\partial}{\partial\nu_x}\right)^2 G_{2,\lambda_c,w}(x^*,z)&\left(\frac{\partial}{\partial\nu_y}\right)^2 G_{2,\mu,w}(z,y^*)w(z)dz. \end{aligned} \tag{5.18}$$

Using (5.18), (5.16), (5.17) and $\left(\frac{\partial}{\partial\nu_x}\right)^2 G_{2,\lambda_c,w}(x^*,z) \geq 0$ for all $z \in \Omega$, we obtain the result in (5.15) if we prove that

$$\left(\frac{\partial}{\partial\nu_x}\right)^2 G_{2,\lambda_c,w}(x^*,z) > 0 \text{ for some } z \in \Omega.$$

Suppose that $\left(\frac{\partial}{\partial\nu_x}\right)^2 G_{2,\lambda_c,w}(x^*,z) = 0$ for all $z \in \Omega$. Then, it holds for every

$f \in C(\overline{\Omega})$ that the solution u_f of

$$\begin{cases} \Delta^2 u_f - \lambda_c w\, u_f = f & \text{in } \Omega, \\ u_f = \frac{\partial}{\partial\nu} u_f = 0 & \text{on } \Omega \end{cases} \tag{5.19}$$

fulfills $\left(\frac{\partial}{\partial\nu}\right)^2 u_f(x^*) = 0$, see also [32, Equation (10)]. Using the arguments of the proof in [32, Lemma 2], one finds that this leads to a contradiction. Indeed, since $\partial\Omega \in C^{4,\gamma}$ we obtain that there exists some $\varepsilon > 0$ such that $d(\cdot) \in C^{4,\gamma}(\Omega(\varepsilon))$, see [22, Lemma 14.16], where $\Omega(\varepsilon)$ is defined as in (4.4). Let $h \in C^\infty(\overline{\Omega})$ be such that $h = 1$ in $\Omega(\frac{1}{2}\varepsilon)$ and $h = 0$ in $\Omega \backslash \Omega(\varepsilon)$. Moreover, let

$$\begin{aligned} u^*(x) &:= d(x)^2 h(x), \\ f^*(x) &:= \Delta^2 u^*(x) - \lambda_c w(x) u^*(x). \end{aligned}$$

Then u^* solves (5.19) with right-hand side f^* and using Remark 4.2.2 we find

$$\left(\tfrac{\partial}{\partial\nu}\right)^2 u^*(x^*) = 2\left(\tfrac{\partial}{\partial\nu} d(x^*)\right)^2 \neq 0,$$

a contradiction. ■

Corollary 5.1.7 *Suppose that Condition A is satisfied. Let w, λ_c and $\lambda_{\mathrm{p},2,w}$ be as in Theorem 1.2.10 such that $G_{2,\lambda,w} \geq 0$ iff $\lambda \in [\lambda_c, \lambda_{\mathrm{p},2,w})$. Assume that the sign-change of the Green function for decreasing λ comes in through $(x_0, y_0) \in \partial\Omega \times \partial\Omega \backslash \{(x,x); x \in \partial\Omega\}$. Then $\lambda = \lambda_c$ is the largest real value with $\lambda < \lambda_{\mathrm{p},2,w}$ such that $g_\lambda \in C^4(\overline{\Omega} \backslash \{y_0\})$ in (5.10) is a nontrivial solution of*

$$\begin{cases} \Delta^2 \varphi = \lambda\, w\, \varphi & \textit{in } \Omega, \\ \varphi = 0 & \textit{on } \partial\Omega, \\ \frac{\partial}{\partial\nu}\varphi = 0 & \textit{on } \partial\Omega \backslash \{y_0\}, \\ \left(\frac{\partial}{\partial\nu}\right)^2 \varphi(x_0) = 0. \end{cases} \tag{5.20}$$

***Remark* 5.1.8** *Analogous to the one-dimensional case,* (5.20) *can be understood as a 'switched' eigenvalue problem, since $\frac{\partial}{\partial\nu}\varphi(y_0) = 0$ is replaced by $\left(\frac{\partial}{\partial\nu}\right)^2 \varphi(x_0) = 0$.*

***Remark* 5.1.9** *There are still some unanswered questions about the 'anti'-eigenvalue problem for dimensions $n \geq 2$. It is not known whether the additional zero of the Green function, respectively the sign-change, comes through the boundary of the domain. Grunau and Robert proved in [25] that if the transition from positivity to sign-change occurs for $(x_0, y_0) \in \partial\Omega \times \partial\Omega$ and $\lambda_{\mathrm{p},2,w} = \lambda_{1,2,w}$, then it holds $x_0 \neq y_0$ for $n \geq 3$. In addition, since the eigenfunction g_{λ_c} is not necessarily an element of $W^{2,2}(\Omega)$, one cannot use the theory of weak solutions. So, it is not obvious in which function space the eigenfunctions of problem* (5.20) *would be well defined.*

5.2 Positivity preserving on an ellipsoid

In two dimensions it is known that the polyharmonic problem in (1.4) with $\lambda = 0$ is not positivity preserving on some eccentric ellipses, see [69, 70] for $m = 2, 3, 4$. Since there is no positive eigenfunction for problem (1.5) with $m = 2$ on domains with corners, see for example [13], it is possible that ellipses with a large ratio do not have a positive eigenfunction either. This is an open problem. But even if it is true and we cannot apply Theorem 3.1.3 directly, we find positivity for a weighted bilaplace Dirichlet problem. This result is a special case of the result in Section 5.1. Moreover, we can show a positivity preserving property for the weighted polyharmonic problem on an ellipsoid in every dimension since there is a function $u_0 \in C^{2m,\gamma}(\overline{\Omega})$ that satisfies Condition B.

Definition 5.2.1 *Let $a \in (0, \infty)^n$. An ellipsoid $E_a \subset \mathbb{R}^n$ is defined as*

$$E_a = \left\{ x \in \mathbb{R}^n;\ \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} < 1 \right\}. \tag{5.21}$$

To find a positivity preserving property for a weighted polyharmonic problem on E_a, we have to show that Condition B is fulfilled. Therefore, fix $a \in (0, \infty)^n$ and consider the function

$$\mathbf{e}(x) = \left(1 - \sum_{i=1}^{n} \frac{x_i^2}{a_i^2}\right) \left(\sum_{i=1}^{n} \frac{2}{a_i^2}\right)^{-1}. \tag{5.22}$$

We find that $u_0 := \mathbf{e}^m$ satisfies Condition B:

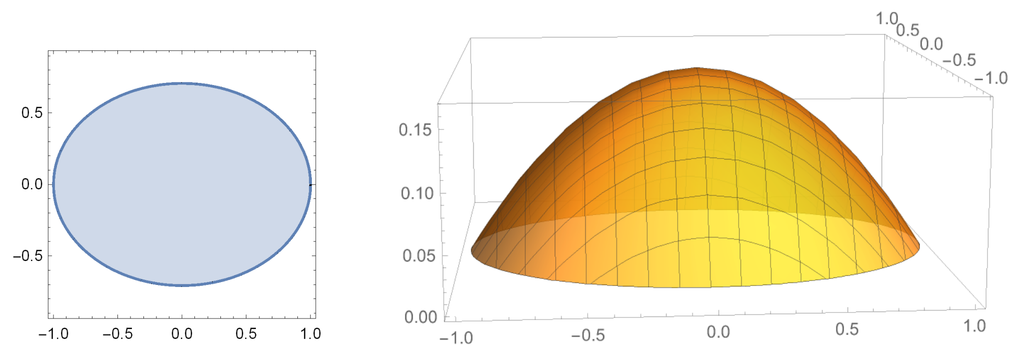


Figure 5.3: An ellipse with $a_1 = 1$ and $a_2 = \frac{\sqrt{2}}{2}$ and the corresponding function $\mathbf{e}$ as defined in (5.22).

Lemma 5.2.2 *Let $\mathbf{e}$ be defined as in (5.22). Then, $\mathbf{e}$ is the strongly positive solution to*

$$\begin{cases} -\Delta \mathbf{e} = 1 & \text{in } E_a, \\ \mathbf{e} = 0 & \text{on } \partial E_a, \end{cases} \tag{5.23}$$

so there exists a constant $c_a > 0$ *such that* $\mathbf{e}(x) \geq c_a\, d(x)$ *for all* $x \in E_a$*. Moreover, there is a constant* $C_a > 0$ *such that* $(-\Delta)^m \mathbf{e}^m = C_a$ *in* E_a*.*

Proof. A direct computation shows that $\mathbf{e}$ satisfies problem (5.23). Furthermore, we find that $\mathbf{e}^m$ is a polynomial of order $2m$. So, there exists a constant $C_a \in \mathbb{R}$ such that $(-\Delta)^m \mathbf{e}^m = C_a$. If $C_a = 0$, then $\mathbf{e}^m$ satisfies

$$\begin{cases} (-\Delta)^m \mathbf{e}^m = 0 & \text{in } E_a, \\ \mathbf{e}^m = \frac{\partial}{\partial \nu}\mathbf{e}^m = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} \mathbf{e}^m = 0 & \text{on } \partial E_a. \end{cases}$$

Since the problem has a unique solution, it holds that $\mathbf{e}^m \equiv 0$, which is a contradiction. Hence $C_a \in \mathbb{R}\backslash\{0\}$. Since $a \mapsto C_a : (0,\infty)^n \to \mathbb{R}$ is a continuous function, we find that either $C_a > 0$ for all $a \in (0,\infty)^n$ or $C_a < 0$ for all $a \in (0,\infty)^n$. For arbitrary $a_1 \in (0,\infty)$ we obtain

$$\lim_{a_2,\ldots,a_n \to \infty} C_a = \lim_{a_2,\ldots,a_n \to \infty} \left((-\Delta)^m \mathbf{e}^m(0)\right) = \frac{(2m)!}{a_1^{2m}} \left(\frac{2}{a_1^2}\right)^{-m} = \frac{(2m)!}{2^m} > 0.$$

So $C_a > 0$ for all $a \in (0,\infty)^n$. ∎

We proved that Condition B is fulfilled, so we find a strictly positive weight function $w \in C^{0,\gamma}(\overline{E_a})$ and an interval $I \subset \mathbb{R}$ for λ such that problem (1.6) is positivity preserving for all $\lambda \in I$ on $\Omega = E_a$.

Example 5.2.3 *For an ellipse* $E_a \subset \mathbb{R}^2$ *with* $a \in (0,\infty)^2$ *we find*

$$(-\Delta)^2 \mathbf{e}^2 = 2 + \left(\tfrac{1}{a_1^2} + \tfrac{1}{a_2^2}\right)^{-2} \left(\tfrac{4}{a_1^4} + \tfrac{4}{a_2^4}\right) > 2;$$
$$(-\Delta)^3 \mathbf{e}^3 = 90 - \tfrac{216}{a_1^2 a_2^2} \left(\tfrac{1}{a_1^2} + \tfrac{1}{a_2^2}\right)^{-2} = 90 - \frac{216}{\left(\frac{a_2}{a_1}\right)^2 + \left(\frac{a_1}{a_2}\right)^2 + 2} > 18.$$

5.3 Small perturbations of ellipses in two dimensions

Let $\mathbf{e}$ be the solution to (5.3). For the polylaplacian, we cannot use the function $u_0 = \mathbf{e}^m$ in Condition B for all domains as we mentioned in Remark 5.1.1. But for some domains in two dimensions this approach works. We cannot compute $\mathbf{e}$ for a general domain Ω, so it is difficult to predicate informations about the sign of $(-\Delta)^m \mathbf{e}^m$. However, we can use the results from the previous section and consider small perturbations of ellipses $E_a \subset \mathbb{R}^2$, where E_a for $a \in (0,\infty)^2$ is defined as in (5.21). Indeed, using the positivity of $\mathbf{e}$ on E_a and biholomorphic mappings, we obtain positivity results for sufficiently small perturbations of these ellipses. The following result and proof is inspired by [28].

Lemma 5.3.1 *Let* $E_a \subset \mathbb{R}^2$ *be defined as in* (5.21)*. Then there exists* $\varepsilon_0 > 0$ *such that the following result holds for all* $\varepsilon \in (0, \varepsilon_0)$*: Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected,* *bounded domain that satisfies Condition A. Let $h : \Omega \to E_a$ be a biholomorphic mapping such that $h \in C^{2m,\gamma}(\overline{\Omega};\mathbb{R}^2)$ and $h^{-1} \in C^{2m,\gamma}(\overline{E_a};\mathbb{R}^2)$. Let $\mathbf{e}$ be the solution to* (5.3) *on E_a and $\mathrm{Id} : \overline{\Omega} \to \overline{\Omega}$ the identical map. If*

$$\|h - \mathrm{Id}\,\|_{C^m(\overline{\Omega};\mathbb{R}^2)} \leq \varepsilon,$$

then $\mathbf{e}^m \circ h \in C^{2m,\gamma}(\overline{\Omega})$ fulfills the following properties:

1. *It holds that $(-\Delta)^m(\mathbf{e}^m \circ h)(x) > 0$ for all $x \in \overline{\Omega}$.*

2. *There exists a constant $C > 0$ such that $(\mathbf{e}^m \circ h)(x) \geq C\, d(x, \partial\Omega)^m$ for all $x \in \Omega$.*

***Remark* 5.3.2** *A biholomorphic function $h : \Omega \to E_a$ is a conformal mapping and the Cauchy-Riemann equations hold:*

$$\frac{\partial}{\partial x_1} h_1(x_1, x_2) = \frac{\partial}{\partial x_2} h_2(x_1, x_2) \text{ and } \frac{\partial}{\partial x_1} h_2(x_1, x_2) = -\frac{\partial}{\partial x_2} h_1(x_1, x_2)$$

and therefore

$$\begin{aligned} |\nabla h_1|^2 &= \left(\frac{\partial h_1}{\partial x_1}\right)^2 + \left(\frac{\partial h_1}{\partial x_2}\right)^2 = \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}, \\ |\nabla h_2|^2 &= \left(\frac{\partial h_2}{\partial x_1}\right)^2 + \left(\frac{\partial h_2}{\partial x_2}\right)^2 = \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}, \\ &\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_1} + \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_2} = 0, \\ &\Delta h_1 = \Delta h_2 = 0. \end{aligned}$$

The existence of biholomorphic functions with regularity properties as described in Lemma 5.3.1 was proven by Kellogg and Warschawski, see [50, p. 4, Theorem 3.6].

Proof. Since $\mathbf{e} \in C^{2m,\gamma}(\overline{E_a})$ and $h \in C^{2m,\gamma}(\overline{\Omega};\mathbb{R}^2)$, we find $\mathbf{e}^m \circ h \in C^{2m,\gamma}(\overline{\Omega})$.

1. Using Lemma (5.2.2) we obtain

$$\mathbf{e}(y) = \left(1 - \frac{y_1^2}{a_1^2} - \frac{y_2^2}{a_2^2}\right)\left(\frac{2}{a_1^2} + \frac{2}{a_2^2}\right)^{-1}$$

and a constant $C_a > 0$ such that

$$(-\Delta_y)^m \mathbf{e}^m(y) = C_a > 0 \text{ for all } y \in E_a. \tag{5.24}$$

Hence $\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_k}\mathbf{e}(y) = 0$ for all $y \in E_a$ and $i, j, k \in \{1, 2\}$. Since

$$\|h - \mathrm{Id}\,\|_{C^m(\overline{\Omega},\mathbb{R}^2)} \leq \varepsilon,$$

we find

$$\|h_j - \mathrm{Id}_j\|_{C(\overline{\Omega})} \leq \varepsilon, \quad \|\tfrac{\partial}{\partial x_i} h_j - \delta_{ij}\|_{C(\overline{\Omega})} \leq \varepsilon, \quad \|\tfrac{\partial}{\partial x_{k_1}} \tfrac{\partial}{\partial x_{k_2}} \cdots \tfrac{\partial}{\partial x_{k_\ell}} h_j\|_{C(\overline{\Omega})} \leq \varepsilon \tag{5.25}$$

for all $\ell \in \{2, \ldots, m\}$ and $i, j, k_1, \ldots, k_\ell \in \{1, 2\}$, where δ_{ij} is the Kronecker delta. Using the properties of the biholomorphic mapping h in Remark 5.3.2, we can calculate

$$\begin{aligned}(-\Delta_x)(\mathbf{e}^m \circ h)(x) &= \left(\left(\tfrac{\partial}{\partial x_1} h_1(x)\right)^2 + \left(\tfrac{\partial}{\partial x_2} h_1(x)\right)^2\right)(-\Delta_y \mathbf{e}^m)(h(x)) \\ &= |\nabla h_1|^2 (-\Delta_y \mathbf{e}^m)(h(x)).\end{aligned}$$

After m steps, we find

$$(-\Delta_x)^m (\mathbf{e}^m \circ h)(x) = |\nabla h_1(x)|^{2m} ((-\Delta_y)^m \mathbf{e}^m)(h(x)) + R(x), \tag{5.26}$$

where $R : \overline{\Omega} \to \mathbb{R}$ is a sum of products of partial derivatives of h_1, h_2 and $\mathbf{e}$. One notices that $(-\Delta_x)^m(\mathbf{e}^m \circ h)(x)$ contains no derivative of h_1 or h_2 of order larger than m. This can be shown analogously to [56, Lemma 1]. Indeed, for $m = 1$ or $m = 2$ it follows from direct calculation. Using $\Delta h_1 = \Delta h_2 = 0$ and induction, it can be proven for all $m \in \mathbb{N}^+$.

Moreover, using $|\frac{\partial}{\partial x_1} h_1(x)| \geq 1 - \varepsilon$ and $|\frac{\partial}{\partial x_2} h_1(x)| \geq 0$ for all $x \in \overline{\Omega}$, we obtain

$$|\nabla h_1(x)|^{2m} = \left(\left(\tfrac{\partial}{\partial x_1} h_1(x)\right)^2 + \left(\tfrac{\partial}{\partial x_2} h_1(x)\right)^2\right)^m \geq (1 - \varepsilon)^{2m}. \tag{5.27}$$

Therefore (5.24), (5.26) and (5.27) imply

$$(-\Delta_x)^m (\mathbf{e}^m \circ h)(x) \geq C_a (1 - \varepsilon)^{2m} + R(x) \text{ for all } x \in \overline{\Omega}.$$

Since each summand in R contains a factor $\frac{\partial}{\partial x_{k_1}} \frac{\partial}{\partial x_{k_2}} \cdots \frac{\partial}{\partial x_{k_\ell}} h_1$ or $\frac{\partial}{\partial x_{k_1}} \frac{\partial}{\partial x_{k_2}} \cdots \frac{\partial}{\partial x_{k_\ell}} h_2$ with $k_1, k_2, \ldots k_\ell \in \{1, 2\}$ and $\ell \in \{2, \ldots, m\}$ and (5.25) holds, we find a value $\varepsilon_0 \in (0, 1)$ such that R becomes so small that

$$(-\Delta_x)^m (\mathbf{e}^m \circ h)(x) > 0 \text{ for all } x \in \overline{\Omega}$$

if $\|h - \mathrm{Id}\,\|_{C^m(\overline{\Omega})} \leq \varepsilon$ and $\varepsilon \in (0, \varepsilon_0)$.

2. Using Lemma 5.2.2, we obtain that there exists a constant $\tilde{C} > 0$ such that

$$\mathbf{e}^m(h(x)) \geq \tilde{C}\, d(h(x), \partial E_a)^m \text{ for all } x \in \overline{\Omega}.$$

Since $h^{-1} \in C^{2m,\gamma}(\overline{E_a}; \mathbb{R}^2)$, h^{-1} is Lipschitz continuous. So, there is a constant

$L > 0$ such that for all $x, \tilde{x} \in \Omega$

$$|x - \tilde{x}| = |h^{-1}(h(x)) - h^{-1}(h(\tilde{x}))| \leq L|h(x) - h(\tilde{x})|.$$

Therefore, we get for all $x \in \Omega$

$$\begin{aligned} d(h(x), \partial E_a)^m &= \left(\inf_{y \in \partial E_a} |h(x) - y| \right)^m = \left(\inf_{\tilde{x} \in \partial \Omega} |h(x) - h(\tilde{x})| \right)^m \\ &\geq L^{-m} \left(\inf_{\tilde{x} \in \partial \Omega} |x - \tilde{x}| \right)^m = L^{-m} d(x, \partial \Omega)^m. \end{aligned}$$

So, we find

$$(\mathbf{e}^m \circ h)(x) \geq \tilde{C} L^{-m} d(x, \partial \Omega)^m \text{ for all } x \in \Omega.$$

■

***Remark* 5.3.3** *In [28, Theorem 1.5, Lemma 2.1] and [30, Section 5] an analogous result is shown. The authors prove that if one considers small perturbations of the unit disk, then the Green function to the polyharmonic Dirichlet problem* (1.4) *with* $\lambda = 0$ *is positive, so the problem is positivity preserving. The difference to the result in Lemma 5.3.1 is that in [28, Lemma 2.1] the authors used closeness in* C^{2m-1}*-sense with respect to biholomorphic mappings. Sassone improved this result in [56] and he showed that closeness in* $C^{m,\gamma}$*-sense is sufficient.*

We proved that Condition B is fulfilled, so we can apply Theorem 1.2.8 and again find a positivity preserving property for a weighted polyharmonic Dirichlet problem on small perturbations of ellipses with respect to biholomorphic mappings.

Chapter 6

Classical solutions to some higher order semilinear Dirichlet problems

In this chapter, we consider classical solvability of some semilinear Dirichlet problem, where the principle part of the differential operator is of the form $(-\Delta)^m$. In the following, we closely follow [59]. As mentioned in the introduction, we investigate the problem

$$\begin{cases} (-\Delta)^m u(x) + g(x,u(x)) = f(x) & \text{for } x \in \Omega, \\ u(x) = \frac{\partial}{\partial\nu}u(x) = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{6.1}$$

where $f \in C^{0,\gamma}(\overline{\Omega})$ and $g \in C^{0,\gamma}(\overline{\Omega} \times \mathbb{R})$ satisfies the sign condition

$$g(x,t) \cdot t \geq 0 \text{ for all } x \in \Omega, t \in \mathbb{R}. \tag{6.2}$$

One may include lower order derivatives in the partial differential equation. However, the differential operator L has to be coercive in the sense that there is a constant $c > 0$ such that

$$\int_\Omega Lu(x)u(x)dx \geq c\,\|u\|^2_{W^{m,2}(\Omega)} \text{ for all } u \in C^{2m}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega}),$$

and the principle part has to be the polylaplace operator $(-\Delta)^m$, see Remark 1.2.16. Otherwise, we cannot use Pulst's results concerning the Green function estimates, and one doubts if such estimates hold true if the principle part is not a product of second order operators, see [27]. We proved in the previous chapters that if we consider the operator $(-\Delta)^m - \lambda w$ for appropriately chosen $\lambda \in \mathbb{R}$ and $w \in C^{0,\gamma}(\overline{\Omega})$ instead of $(-\Delta)^m$, we find a positivity preserving property for the corresponding Dirichlet problem. In the following sections, we will only use the estimates for the Green operator $\mathcal{G}_{m,0,1}$ in (3.2) and regularity results instead of a maximum or comparison principle, so the Green operator does not have to be positivity preserving. Accordingly, no weight function is necessary and we will only examine problem (6.1).

The objective of this chapter is to prove the result in Theorem 1.2.15. First, we recall regularity results for the linear polyharmonic Dirichlet problem in Section 6.1. Then we describe an approximation with bounded functions for the nonlinear term g and prove the existence of a weak solution to the changed problem with bounded nonlinear part. Then we use regularity results to find uniform bounds for these weak solutions. The existence of a classical solution to (6.1) follows using a bootstrapping argument. In Sections 6.3 and 6.4, we prove Theorem 1.2.15. We devide it into two parts. First, we show the special cases of Theorem 1.2.15, where g satisfies a one-sided growth condition. Then, we prove the remaining case of Theorem 1.2.15. The proof can be done iteratively using similar arguments as in the first and second case, where g fulfills a one-sided growth condition.

6.1 Linear regularity

First, we recall a regularity result for the linear polyharmonic Dirichlet problem. We assume that Condition A is satisfied. It is known that if $f \in L^p(\Omega)$ with $p \in (1, \infty)$, then there exists a unique solution $u \in W^{2m,p}(\Omega)$ for problem (1.4) with $\lambda = 0$. Moreover it holds, see [21, Theorem 2.20], that there is a constant $C_{\Omega,m,p} > 0$, independent of f, such that

$$\|u\|_{W^{2m,p}(\Omega)} \leq C_{\Omega,m,p} \|f\|_{L^p(\Omega)}.$$

As mentioned and described in [59] and [60], when investigating the polyharmonic Dirichlet problem, one may consider separately the solutions u_+ and u_- of

$$\begin{cases} (-\Delta)^m u_\pm(x) = f^\pm(x) & \text{for } x \in \Omega, \\ u_\pm(x) = \frac{\partial}{\partial \nu} u_\pm(x) = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} u_\pm(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{6.3}$$

where $f^+ := \max\{0, f\}$ and $f^- = \max\{0, -f\}$. One notices that u_+ and u_- do not have to be nonnegative. Then, using estimates for the polyharmonic Green function as in (3.2), one finds functions $u^\oplus, u^\ominus \geq 0$ such that $u = u^\oplus - u^\ominus$, and one may prove the following sign-dependent regularity estimates, see [60, Theorem 1]:

Theorem 6.1.1 *Let Condition A be fulfilled, $p_\pm \in (1, \infty)$ and $p = \min\{p_+, p_-\}$. Suppose that $f = f^+ - f^-$ with $f^+ \in L^{p_+}(\Omega)$ and $f^- \in L^{p_-}(\Omega)$. Then there are constants $c_{\Omega,p_+,m}, c_{\Omega,p_-,m} > 0$, independent of f^+ and f^-, such that the following holds: There is a unique solution $u \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$ of*

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial \nu} u = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.4}$$

with $u = u^\oplus - u^\ominus$, $u^\oplus, u^\ominus \geq 0$, and such that $u^\oplus \in W^{2m,p_+}(\Omega) \cap W_0^{m,p_+}(\Omega)$ and

$u^{\ominus} \in W^{2m,p_-}(\Omega) \cap W_0^{m,p_-}(\Omega)$ *with*

$$\left\|u^{\oplus}\right\|_{W^{2m,p_+}(\Omega)} \leq c_{\Omega,p_+,m}\left(\left\|f^+\right\|_{L^{p_+}(\Omega)} + \left\|f^-\right\|_{L^1(\Omega)}\right), \tag{6.5}$$

$$\left\|u^{\ominus}\right\|_{W^{2m,p_-}(\Omega)} \leq c_{\Omega,p_-,m}\left(\left\|f^-\right\|_{L^{p_-}(\Omega)} + \left\|f^+\right\|_{L^1(\Omega)}\right). \tag{6.6}$$

Indeed, using the Green function estimates in (3.2) and the solution $0 \leq \mathbf{e} \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0(\overline{\Omega})$ of the Dirichlet Laplace problem (5.3), one finds the integral operator

$$(\tilde{\mathcal{D}}_m f)(x) = \mathbf{e}^m(x)\int_\Omega \mathbf{e}^m(y)f(y)dy$$

and a constant $c_{\Omega,m} > 0$ such that

$$\mathcal{G}_{m,0,1} + c_{\Omega,m}\tilde{\mathcal{D}}_m \geq 0,$$

where $\mathcal{G}_{m,0,1}$ is the polyharmonic Green operator. Moreover, the operator $\tilde{\mathcal{H}}_{n,m}$ defined by

$$\tilde{\mathcal{H}}_{n,m} := \mathcal{G}_{m,0,1} + c_{\Omega,m}\tilde{D}_m$$

fulfills for some $\tilde{c}_{1,\Omega,m}, \tilde{c}_{2,\Omega,m} > 0$ the inequality

$$\tilde{c}_{1,\Omega,m}\mathcal{H}_{n,m} \leq \tilde{\mathcal{H}}_{n,m} \leq \tilde{c}_{2,\Omega,m}\mathcal{H}_{n,m},$$

where $\mathcal{H}_{n,m}$ is defined as in (3.4). Since $\mathbf{e}^m$ is bounded and $\tilde{\mathcal{H}}_{n,m}$ satisfies a Riesz potential estimate, see [60, Lemma 4], the operators $\tilde{\mathcal{D}}_m$ and $\tilde{\mathcal{H}}_{n,m}$ are defined for all $f \in L^p(\Omega)$ with $p \in (1,\infty)$.

Hence, the solutions u_+ and u_- of (6.3) can be written as

$$u_+(x) = (\tilde{\mathcal{H}}_{n,m}f^+)(x) - c_{\Omega,m}(\tilde{\mathcal{D}}_m f^+)(x), \quad u_-(x) = (\tilde{\mathcal{H}}_{n,m}f^-)(x) - c_{\Omega,m}(\tilde{\mathcal{D}}_m f^-)(x),$$

and one may choose

$$u^{\oplus}(x) = (\tilde{\mathcal{H}}_{n,m}f^+)(x) + c_{\Omega,m}(\tilde{\mathcal{D}}_m f^-)(x), \quad u^{\ominus}(x) = (\tilde{\mathcal{H}}_{n,m}f^-)(x) + c_{\Omega,m}(\tilde{\mathcal{D}}_m f^+)(x).$$

Then, the estimates in (6.5) and (6.6) follow, see [60, Section 3].

Note that in general $u^{\oplus} \neq u^+$, but

$$u^+ \leq (u^{\oplus} - u^{\ominus})^+ \leq u^{\oplus} \text{ and } u^- \leq (u^{\oplus} - u^{\ominus})^- \leq u^{\ominus},$$

where $u^+ = \max\{0, u\}$ and $u^- = \max\{0, -u\}$.

Using Sobolev imbeddings, (6.5) and (6.6) imply norm estimates for u^+ and u^-.

6.2 Approximation and weak solutions

The content and some paragraphs of this section can be found in [59, Section 3]. The paragraphs that are adopted from [59] are formulated by me.

Let f and g be as described in Theorem 1.2.8. We define for $M \in \mathbb{R}^+$ the function $g_M : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by cutting off g as follows:

$$g_M(x,t) = \begin{cases} \min\{g(x,t), M\} & \text{for } t \geq 0, \\ \max\{g(x,t), -M\} & \text{for } t < 0. \end{cases} \tag{6.7}$$

Then, the function g_M is bounded. We consider the nonlinear Dirichlet problem

$$\begin{cases} (-\Delta)^m u(x) + g_M(x, u(x)) = f(x) & \text{for } x \in \Omega, \\ u(x) = \frac{\partial}{\partial \nu} u(x) = \cdots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{6.8}$$

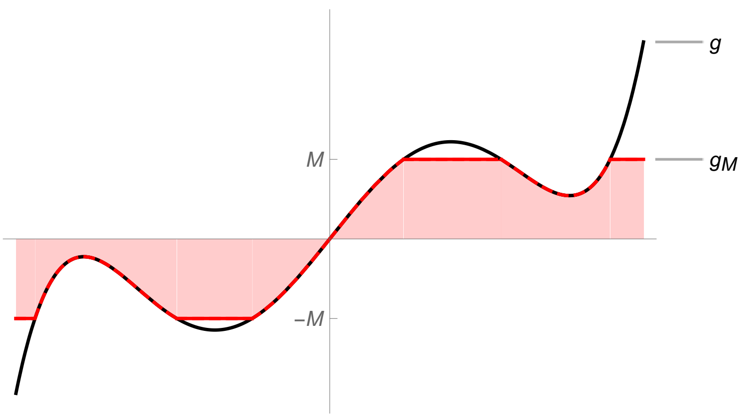


Figure 6.1: Cut-off of some Hölder continuous function $u \mapsto g(u)$ as described in (6.7); this figure appears in [59] and was created by Guido Sweers.

In the following we prove the existence of a weak solution to (6.8) as well as some norm estimates independent of M. Here a weak solution to (6.8) is defined by $u \in W_0^{m,2}(\Omega)$ satisfying

$$\langle u, \varphi \rangle_{W_0^{m,2}(\Omega)} + \int_\Omega g_M(x, u(x))\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx \quad \text{for all } \varphi \in W_0^{m,2}(\Omega),$$

with

$$\langle u, \varphi \rangle_{W_0^{m,2}(\Omega)} := \begin{cases} \displaystyle\int_\Omega \left(\Delta^{m/2} u(x)\right) \left(\Delta^{m/2} \varphi(x)\right) dx & \text{for } m \in \mathbb{N}^+ \text{ even}, \\ \displaystyle\int_\Omega \nabla \Delta^{(m-1)/2} u(x) \cdot \nabla \Delta^{(m-1)/2} \varphi(x)\, dx & \text{for } m \in \mathbb{N}^+ \text{ odd}. \end{cases}$$

Since the function g_M is bounded, the existence of a weak solution u_M to (6.8) for each M directly follows from minimizing a variational problem. Each such u_M is also a classical solution. Even if we consider problem (1.19) in Remark 1.2.16 with

g_M instead of g, one finds with [44] that a classical solution exists. We will show a priori estimates for u_M, that do not depend on M. Then, we are able to obtain uniform bounds for $\|u_M\|_\infty$. Hence for M large enough the function u_M will not depend on M and therefore it will be a classical solution to (6.1).

The following lemma can be found in [59, Lemma 4].

Lemma 6.2.1 *There exist constants $C_{\Omega,m}, C'_{\Omega,m} > 0$, such that for each $M > 0$ and $f \in L^2(\Omega)$ there exists a weak solution u_M of (6.8) with*

$$\|u_M\|_{W_0^{m,2}(\Omega)} \leq C_{\Omega,m} \|f\|_{L^2(\Omega)} \quad \text{and} \quad \|g_M(\cdot, u_M)\, u_M\|_{L^1(\Omega)} \leq C'_{\Omega,m} \|f\|^2_{L^2(\Omega)},$$

where $\|u_M\|_{W_0^{m,2}(\Omega)} := \langle u_M, u_M \rangle^{1/2}_{W_0^{m,2}(\Omega)}$.

Proof. Let $J_M : W_0^{m,2}(\Omega) \to \mathbb{R}$ be defined by

$$J_M(u) := \frac{1}{2} \|u\|^2_{W_0^{m,2}(\Omega)} + \int_\Omega \left(\int_0^{u(x)} g_M(x,t)\, dt - f(x)u(x) \right) dx.$$

Since g_M is bounded, one finds that the operator J_M is well defined. Moreover, J_M is coercive on $W_0^{m,2}(\Omega)$. Indeed, using the Poincaré-Friedrichs inequality, one finds a constant $C_{PF} > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C_{PF} \|u\|_{W_0^{m,2}(\Omega)} \text{ for all } u \in W_0^{m,2}(\Omega). \tag{6.9}$$

Applying Cauchy-Schwarz and using the sign condition $g_M(x,t)t \geq 0$ for all $x \in \Omega$ and $t \in \mathbb{R}$, one gets

$$J_M(u) \geq \frac{1}{2}\|u\|^2_{W_0^{m,2}(\Omega)} - \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \geq \frac{1}{2}\|u\|^2_{W_0^{m,2}(\Omega)} - C_{PF}\|f\|_{L^2(\Omega)}\|u\|_{W_0^{m,2}(\Omega)}.$$

Hence $J_M(u) \to \infty$ for $\|u\|_{W_0^{m,2}(\Omega)} \to \infty$ and J_M is bounded from below. Therefore, there is a minimizing sequence $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{m,2}(\Omega)$ that is bounded since J_M is coercive. Since $W_0^{m,2}(\Omega)$ is a Hilbert space and therefore a reflexive Banach space, we get with Kakutani's theorem a weakly convergent subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ with weak limit $u_M \in W_0^{m,2}(\Omega)$:

$$u_{k_j} \rightharpoonup u_M \quad \text{in } W_0^{m,2}(\Omega) \text{ for } j \to \infty. \tag{6.10}$$

Using that $\{u_{k_j}\}_{j\in\mathbb{N}}$ is bounded and the compactness of the Sobolev imbedding $W^{m,2}(\Omega) \hookrightarrow L^2(\Omega)$, we find that this subsequence again has a subsequence $\{u_{k_{j_\ell}}\}_{\ell\in\mathbb{N}}$ such that

$$u_{k_{j_\ell}} \to u_M \quad \text{in } L^2(\Omega) \text{ for } \ell \to \infty. \tag{6.11}$$

Using (6.10) we obtain

$$\liminf_{\ell\to\infty} \|u_{k_{j_\ell}}\|_{W_0^{m,2}(\Omega)} \geq \|u_M\|_{W_0^{m,2}(\Omega)}$$

and the convergence in (6.11) as well as $|g_M(x,u(x))| \leq M$ for all $x \in \Omega$ and $u \in W_0^{m,2}(\Omega)$ imply

$$\lim_{\ell\to\infty}\int_\Omega f(x)u_{k_{j_\ell}}(x)dx = \int_\Omega f(x)u_M(x)dx,$$
$$\lim_{\ell\to\infty}\int_\Omega\int_0^{u_{k_{j_\ell}}(x)} g_M(x,t)dtdx = \int_\Omega\int_0^{u_M(x)} g_M(x,t)dtdx.$$

Hence

$$\inf_{u\in W_0^{m,2}(\Omega)} J_M(u) = \lim_{\ell\to\infty} J_M(u_{k_{j_\ell}}) \geq J_M(u_M) \geq \inf_{u\in W_0^{m,2}(\Omega)} J_M(u),$$

which implies that $u_M \in W_0^{m,2}(\Omega)$ is a minimizer of J_M.

This minimizer satisfies the weak Euler-Lagrange equation

$$\langle u_M,\varphi\rangle_{W_0^{m,2}(\Omega)} + \int_\Omega \left(g_M\left(x,u_M(x)\right) - f\left(x\right)\right)\varphi(x)dx = 0 \text{ for all } \varphi \in W_0^{m,2}(\Omega). \tag{6.12}$$

By the sign condition for g in (6.2) we find that for all $u \in W_0^{m,2}(\Omega)$

$$\int_\Omega g_M\left(x,u(x)\right)u(x)dx \geq 0.$$

Taking $\varphi = u_M$ in (6.12), we get

$$\|u_M\|^2_{W_0^{m,2}(\Omega)} + \int_\Omega g_M\left(x,u_M(x)\right)u_M(x)dx = \int_\Omega f(x)u_M(x)dx.$$

One notices that on the left-hand side two positive terms appear. Hence each of them can be estimated from above by

$$\max\left\{\|u_M\|^2_{W_0^{m,2}(\Omega)}, \int_\Omega g_M\left(x,u_M(x)\right)u_M(x)dx\right\}$$
$$\leq \int_\Omega f(x)u_M(x)dx \leq \|f\|_{L^2(\Omega)}\|u_M\|_{L^2(\Omega)}.$$

Since (6.9) holds true, we find a constant $C_{\Omega,m} > 0$, independent of M, such that

$$\|u_M\|_{W_0^{m,2}(\Omega)} \leq C_{\Omega,m}\|f\|_{L^2(\Omega)}.$$

We also get with (6.9) that

$$\|g_M(\cdot,u_M)u_M\|_{L^1(\Omega)} \leq \|f\|_{L^2(\Omega)}\|u_M\|_{L^2(\Omega)} \leq C_{PF}C_{\Omega,m}\|f\|^2_{L^2(\Omega)}.$$

■

Using the results in Lemma 6.2.1, we obtain additional norm estimates which we use in the following two sections.

The following corollary can be found in [59, Corollary 6].

Corollary 6.2.2 *Let $M > 0$, $f \in L^2(\Omega)$, g_M as defined in* (6.7) *and $u_M \in W_0^{m,2}(\Omega)$ be a weak solution to* (6.8) *as described in Lemma 6.2.1. Then the following estimates hold:*

1. *There is a constant $C_{\Omega,m,g,f} > 0$, independent of M, such that*

$$\|g_M(\cdot, u_M)\|_{L^1(\Omega)} \leq C_{\Omega,m,g,f}. \tag{6.13}$$

2. *There exists a constant $C_{\Omega,m,p} > 0$, independent of M, such that*

$$\|u_M\|_{L^p(\Omega)} \leq C_{\Omega,m,p} \|f\|_{L^2(\Omega)} \quad \text{for all} \quad \begin{cases} p \in \left[1, \frac{2n}{n-2m}\right] & \text{if } n > 2m, \\ p \in [1, \infty) & \text{if } n = 2m. \end{cases} \tag{6.14}$$

3. *Let g fulfill*

$$g(x,t) \leq c_1(1 + |t|^\sigma) \text{ for } t \geq 0. \tag{6.15}$$

If $n > 2m$ and $\sigma \in \left[1, \frac{2n}{n-2m}\right]$, then one finds $C_{\Omega,m,\sigma} > 0$, independent of M, such that

$$\left\|g_M(\cdot, u_M^+)\right\|_{L^{\frac{2n}{n-2m}\frac{1}{\sigma}}(\Omega)} \leq C_{\Omega,m,\sigma}\left(1 + \|f\|^\sigma_{L^2(\Omega)}\right), \tag{6.16}$$

and if $n = 2m$, $q \in [1, \infty)$ and $\sigma \in [1, \infty)$, there exists $C_{\Omega,m,\sigma,q} > 0$ such that

$$\left\|g_M(\cdot, u_M^+)\right\|_{L^q(\Omega)} \leq C_{\Omega,m,\sigma,q}\left(1 + \|f\|^\sigma_{L^2(\Omega)}\right). \tag{6.17}$$

4. *Let g fulfill*

$$g(x,t) \geq -c_2(1 + |t|^\tau) \text{ for } t \leq 0. \tag{6.18}$$

If $n > 2m$ and $\tau \in \left[1, \frac{2n}{n-2m}\right]$, then one finds $C_{\Omega,m,\tau} > 0$, independent of M, such that

$$\left\|g_M(\cdot, -u_M^-)\right\|_{L^{\frac{2n}{n-2m}\frac{1}{\tau}}(\Omega)} \leq C_{\Omega,m,\tau}\left(1 + \|f\|^\tau_{L^2(\Omega)}\right) \tag{6.19}$$

and if $n = 2m$, $q \in [1, \infty)$ and $\tau \in [1, \infty)$, there exists $C_{\Omega,m,\tau,q} > 0$ such that

$$\left\|g_M(\cdot, -u_M^-)\right\|_{L^q(\Omega)} \leq C_{\Omega,m,\tau,q}\left(1 + \|f\|^\tau_{L^2(\Omega)}\right). \tag{6.20}$$

Proof. All estimates are consequences of Lemma 6.2.1:

1. Using the results in Lemma 6.2.1, one obtains that

$$\begin{aligned}\|g_M(\cdot,u_M)\|_{L^1(\Omega)} &\le |\Omega| \max_{x\in\overline{\Omega},t\in[-1,1]} |g(x,t)| + \|g_M(\cdot,u_M)u_M\|_{L^1(\Omega)}\\ &\le |\Omega| \max_{x\in\overline{\Omega},t\in[-1,1]} |g(x,t)| + C'_{\Omega,m}\|f\|^2_{L^2(\Omega)},\end{aligned}$$

   where $|\Omega| = \int_\Omega 1dx$ is the Lebesgue-measure of Ω.

2. Since we have proven that $\|u_M\|_{W_0^{m,2}(\Omega)} \le C_{\Omega,m}\,\|f\|_{L^2(\Omega)}$ with $C_{\Omega,m}$ independent of M in Lemma 6.2.1, inequality (6.14) follows by Sobolev imbeddings.

3. Using $|g_M(x,t)| \le |g(x,t)|$ for all $(x,t)\in\overline{\Omega}\times\mathbb{R}$, one gets, if g fulfills (6.15), $n > 2m$ and $\sigma \le \frac{2n}{n-2m}$, that

$$\begin{aligned}\left\|g_M(\cdot,u_M^+)\right\|_{L^{\frac{2n}{n-2m}\frac{1}{\sigma}}(\Omega)} &\le c_1\|1+|u_M^+|^\sigma\|_{L^{\frac{2n}{n-2m}\frac{1}{\sigma}}(\Omega)}\\ &\le c_1|\Omega|^{\sigma\frac{n-2m}{2n}} + c_1\||u_M^+|^\sigma\|_{L^{\frac{2n}{n-2m}\frac{1}{\sigma}}(\Omega)}\\ &= c_1|\Omega|^{\sigma\frac{n-2m}{2n}} + c_1\|u_M^+\|^\sigma_{L^{\frac{2n}{n-2m}}(\Omega)}.\end{aligned}$$

   Using (6.14), one finds a constant $C'_{\Omega,m,\sigma}>0$ independent of M such that

$$\left\|g_M(\cdot,u_M^+)\right\|_{L^{\frac{2n}{n-2m}\frac{1}{\sigma}}(\Omega)} \le c_1|\Omega|^{\sigma\frac{n-2m}{2n}} + c_1C'_{\Omega,m,\sigma}\|f\|^\sigma_{L^2(\Omega)}.$$

   Similar arguments for $n=2m$, $\sigma\in[1,\infty)$, (6.15) and $q\in[1,\infty)$ provide the inequality

$$\left\|g_M(\cdot,u_M^+)\right\|_{L^q(\Omega)} \le c_1\|1+|u_M^+|^\sigma\|_{L^q(\Omega)} \le c_1|\Omega|^{\frac{1}{q}} + c_1C'_{\Omega,m,\sigma,q}\|f\|^\sigma_{L^2(\Omega)}$$

   for some $C'_{\Omega,m,\sigma,q}>0$ indendent of M.

4. Analogously, one obtains, if g fulfills (6.18), $n>2m$ and $\tau\le\frac{2n}{n-2m}$, that

$$\begin{aligned}\left\|g_M(\cdot,-u_M^-)\right\|_{L^{\frac{2n}{n-2m}\frac{1}{\tau}}(\Omega)} &\le c_2\|1+|u_M^-|^\tau\|_{L^{\frac{2n}{n-2m}\frac{1}{\tau}}(\Omega)}\\ &\le c_2|\Omega|^{\tau\frac{n-2m}{2n}} + c_2C'_{\Omega,m,\tau}\|f\|^\tau_{L^2(\Omega)},\end{aligned}$$

   for some $C'_{\Omega,m,\tau}>0$ independent of M. For $n=2m$, $\tau\in[1,\infty)$, (6.18) and $q\in[1,\infty)$ we find with (6.14) a constant $C'_{\Omega,m,\tau,q}>0$ such that

$$\left\|g_M(\cdot,-u_M^-)\right\|_{L^q(\Omega)} \le c_2\|1+|u_M^-|^\tau\|_{L^q(\Omega)} \le c_2|\Omega|^{\frac{1}{q}} + c_2C'_{\Omega,m,\tau,q}\|f\|^\tau_{L^2(\Omega)}.$$

■

6.3 Classical solution with one-sided growth condition

The content and some paragraphs of this section can be found in [59, Section 4]. The result and proof are the same as in [59], except that the notation has been adjusted. The paragraphs that are adopted from [59] are formulated by me.

In the following, we assume that g fulfills the growth condition from above, that is, there exists a constant $c_1 > 0$ such that

$$g(x,t) \le c_1(1+t^\sigma) \text{ for all } t>0, x\in\Omega, \text{ with } \begin{cases} \sigma = 1 & \text{if } n \ge 6m, \\ \sigma \in \left[1, \frac{4m}{n-2m}\right) & \text{if } n \in (2m, 6m), \\ \sigma \in [1,\infty) & \text{if } n = 2m, \end{cases} \tag{6.21}$$

which combines the first and second case in Theorem 1.2.15. Then we obtain that the semilinear Dirichlet problem has a classical solution. The following result can be found in [59, Theorem 7].

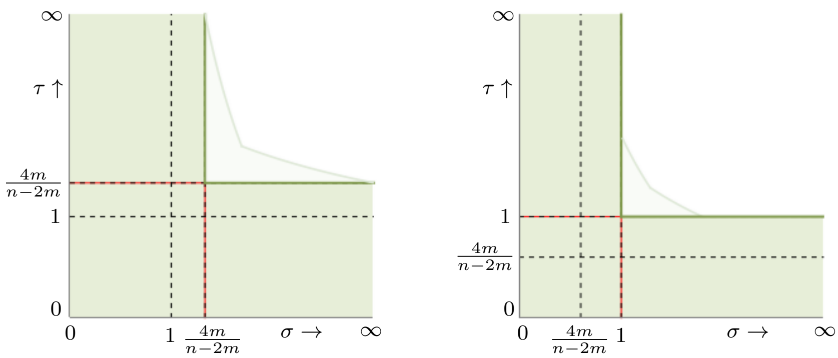


Figure 6.2: Range of admissable growth rates proven in Theorem 6.3.1 for some $n \in (2m, 6m)$ (left) and some $n > 6m$ (right), when $g(x,t) \le c_1(1+t^\sigma)$ or $-c_2(1+t^\tau) \le g(x,t)$ and σ, respectively τ, as in (6.21). The missing sections compared to Figure 1.1 are displayed in light green.

Theorem 6.3.1 *Let $n \ge 2m$ and Condition A be fulfilled. Suppose that $g \in C^{0,\gamma}(\overline{\Omega} \times \mathbb{R})$ satisfies the sign condition* (6.2) *and the growth condition* (6.21)*. Then for any $f \in C^{0,\gamma}(\overline{\Omega})$ the Dirichlet problem in* (6.1) *has a classical solution $u \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$.*

Proof. Let u_M be a weak solution to (6.8) as described in Lemma 6.2.1.

Case 1, $\mathbf{n \in [2m, 6m)}$: By Theorem 6.1.1 with right-hand side $-g_M(\cdot, u_M) + f$ instead of f in (6.4), we find that there exist $u_M^\oplus, u_M^\ominus \ge 0$ such that $u_M = u_M^\oplus - u_M^\ominus$

and using Sobolev imbeddings and (6.6), we get for $n > 2m$

$$\begin{aligned}\|u_M^-\|_{L^\infty(\Omega)} &\le \|u_M^\ominus\|_{L^\infty(\Omega)} \le C_{m,n,\sigma}\|u_M^\ominus\|_{W^{2m,\frac{2n}{n-2m}\frac{1}{\sigma}}(\Omega)} \\ &\le C'_{m,n,\sigma}\left(\|f\|_{L^\infty(\Omega)} + \|g_M(\cdot,u_M^+)\|_{L^{\frac{2n}{n-2m}\frac{1}{\sigma}}(\Omega)} + \|g_M(\cdot,-u_M^-)\|_{L^1(\Omega)}\right).\end{aligned}$$

For $n = 2m$ we find

$$\begin{aligned}\|u_M^-\|_{L^\infty(\Omega)} &\le \|u_M^\ominus\|_{L^\infty(\Omega)} \le C_{m,n,\sigma}\|u_M^\ominus\|_{W^{2m,2}(\Omega)} \\ &\le C'_{m,n,\sigma}\left(\|f\|_{L^\infty(\Omega)} + \|g_M(\cdot,u_M^+)\|_{L^2(\Omega)} + \|g_M(\cdot,-u_M^-)\|_{L^1(\Omega)}\right).\end{aligned}$$

With the inequalities (6.16), (6.17) and (6.13) it follows for $n \ge 2m$

$$\|u_M^-\|_{L^\infty(\Omega)} \le C'_{m,n,\sigma}\left(\|f\|_{L^\infty(\Omega)} + C_{\Omega,m,\sigma}\left(1 + \|f\|^\sigma_{L^2(\Omega)}\right) + C_{\Omega,m,g,f}\right),$$

where the right-hand side does not depend on M. Then, using Sobolev imbeddings, (6.5) and

$$\|g_M(\cdot,-u_M^-)\|_{L^\infty(\Omega)} \le \max_{-\|u_M^-\|_{L^\infty(\Omega)}\le t\le 0;\, x\in\overline{\Omega}} |g(x,t)|,$$

we also find an upper bound for $\|u_M^+\|_{L^\infty(\Omega)}$ which is independent of M. Hence, for a sufficiently large $M_1 \in \mathbb{R}^+$ it holds that $\|g(\cdot,u_{M_1})\|_{L^\infty(\Omega)} \le M_1$, so $g_{M_1}(\cdot,u_{M_1}) = g(\cdot,u_{M_1})$. Therefore, u_{M_1} is a weak solution of (6.1). Since $-g(\cdot,u_{M_1})+f$ is bounded, we obtain by Agmon-Douglis-Nirenberg results, see [21, Theorems 2.19, 2.20], and Sobolev imbeddings that $u_{M_1} \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$ is a classical solution of (6.1).

Case 2 n $\ge$ 6m: We rewrite problem (6.8) with $c_1 > 0$ such that

$$g(x,t) \le c_1(1+t) \text{ for } t > 0$$

and investigate

$$\begin{cases}(-\Delta)^m u + c_1 u = -g_M(\cdot,u) + f + c_1 u & \text{in } \Omega,\\ u = \frac{\partial}{\partial\nu}u = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1}u = 0 & \text{on } \partial\Omega.\end{cases} \tag{6.22}$$

Then, we find for the right-hand side in (6.22) for suitable functions u that

$$\begin{aligned}-g_M(x,u(x)) + f(x) + c_1u(x) &\ge -g_M(x,u^+(x)) - f^-(x) + c_1u(x)\\ &\ge -c_1(1+u^+(x)) - f^-(x) + c_1u^+(x) - c_1u^-(x)\\ &= -c_1 - f^-(x) - c_1u^-(x) \quad \text{for } x\in\Omega\end{aligned}$$

and

$$-g_M(x,u(x)) + f(x) + c_1u(x) \le -g_M(x,-u^-(x)) + f^+(x) + c_1u^+(x) \quad \text{for } x \in \Omega.$$

So we get

$$(-g_M(\cdot,u)+f+c_1u)^- \le f^- + c_1u^- + c_1, \tag{6.23}$$
$$(-g_M(\cdot,u)+f+c_1u)^+ \le f^+ - g_M(\cdot,-u^-) + c_1u^+. \tag{6.24}$$

One finds a similar result as in Theorem 6.1.1 for the linear Dirichlet problem

$$\begin{cases} (-\Delta)^m u(x) + c_1 u(x) = h(x) & \text{for } x \in \Omega, \\ u(x) = \frac{\partial}{\partial\nu}u(x) = \cdots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

with $h = h^+ - h^-$ and $h^+ \in L^{p_+}(\Omega)$, $h^- \in L^{p_-}(\Omega)$ for some $p_\pm \in (1,\infty)$. Therefore, one obtains with $u_M^- \in L^{\frac{2n}{n-2m}}(\Omega)$, (6.23), (6.24) and Sobolev imbeddings that there exists a constant $C_{m,n,q_1} > 0$ such that

$$\begin{aligned} &\|u_M^-\|_{L^{q_1}(\Omega)} \\ &\le C_{m,n,q_1}\left(\|f\|_{L^\infty(\Omega)} + \|u_M^-\|_{L^{\frac{2n}{n-2m}}(\Omega)} + \|u_M^+\|_{L^1(\Omega)} + 1 + \|g_M(\cdot,-u_M^-)\|_{L^1(\Omega)}\right) \end{aligned} \tag{6.25}$$

holds for all

$$\begin{cases} q_1 \in \left[1, \frac{2n}{n-6m}\right] & \text{if } n > 6m, \\ q_1 \in [1,\infty) & \text{if } n = 6m. \end{cases}$$

With (6.14) and (6.13) one gets an upper bound for $\|u_M^-\|_{L^{q_1}(\Omega)}$ independent of M:

$$\begin{aligned} &\|u_M^-\|_{L^{q_1}(\Omega)} \\ &\le C_{m,n,q_1}\left(\|f\|_{L^\infty(\Omega)} + C_{\Omega,m,\frac{2n}{n-2m}}\|f\|_{L^2(\Omega)} + C_{\Omega,m,1}\|f\|_{L^2(\Omega)} + 1 + C_{\Omega,m,g,f}\right). \end{aligned}$$

Using a bootstrapping argument for u_M^- through (6.23) and analogous arguments as in (6.25), regularity results and Sobolev imbeddings, we find after k steps that there exists an upper bound independent of M for $\|u_M^-\|_{L^{q_k}(\Omega)}$ with

$$\begin{cases} q_k \in \left[1, \frac{2n}{n-2m(1+2k)}\right] & \text{if } n > 2m(1+2k), \\ q_k \in [1,\infty) & \text{if } n \in [6m, 2m(1+2k)]. \end{cases}$$

Hence, after finitely many steps we find $k \ge \frac{n-2m}{4m}$ and therefore u_M^- lies in $L^q(\Omega)$ for any $q \in [1,\infty)$ with $\|u_M^-\|_{L^q(\Omega)}$ bounded independently of M.

One more iteration leads to $u_M^- \in L^\infty(\Omega)$ with $\|u_M^-\|_{L^\infty(\Omega)}$ bounded by a constant independent of M.

With the norm estimates for $u_M^- \in L^\infty(\Omega)$, similar arguments as in (6.25) for the positive part u_M^+, (6.24) and bootstrapping again, we also find $u_M^+ \in L^\infty(\Omega)$ and a uniform upper bound for $\|u_M^+\|_{L^\infty(\Omega)}$. Analogous to the case $n \in [2m, 6m)$ it follows for a sufficiently large $M_2 \in \mathbb{R}^+$ that the function u_{M_2} fulfills $\|g(\cdot,u_{M_2})\|_{L^\infty(\Omega)} \le M_2$, so $g_{M_2}(\cdot,u_{M_2}) = g(\cdot,u_{M_2})$. Therefore, u_{M_2} is a classical solution of (6.1). ∎

One notices that the growth condition from above for $g(\cdot, t)$ with $t \geq 0$ may also be changed to a condition from below for $g(\cdot, t)$ with $t \leq 0$ and we obtain the same result, see Figure 6.2.

6.4 Classical solution with two-sided growth condition

Similar to the last section, the content of this section can be found in [59, Section 5]. The notation has been adjusted and some paragraphs have been adopted. However the steps of the following proof have been formulated by me.

As remarked in [24, Section 3], one may improve the growth condition from above by adding a constraint from below. We assume that $n > 2m$, since the case $n = 2m$ is contained in the previous section. It is known, see [24], that if $g \in C^{0,\gamma}(\overline{\Omega} \times \mathbb{R})$ satisfies the sign condition (6.2) and there are two constants $c_1, c_2 > 0$ such that

$$-c_2(1 + |t|^\tau) \leq g(x, t) \leq c_1(1 + |t|^\sigma) \text{ for } t \in \mathbb{R}, x \in \Omega, \tag{6.26}$$

with

$$\frac{n + 2m}{n - 2m} \leq \tau < \infty \quad \text{and} \quad 1 \leq \sigma < \frac{4m}{n - 2m} + \frac{1}{\tau}\frac{n + 2m}{n - 2m},$$

then there exists a solution $u \in C^{2m,\gamma}(\Omega) \cap W_0^{m,2}(\Omega)$ to (6.1). Using Theorem 6.1.1 and Lemma 6.2.1, we may show that there exists a classical solution. So the boundary conditions are satisfied in classical sense. The following result can be found in [59, Theorem 8].

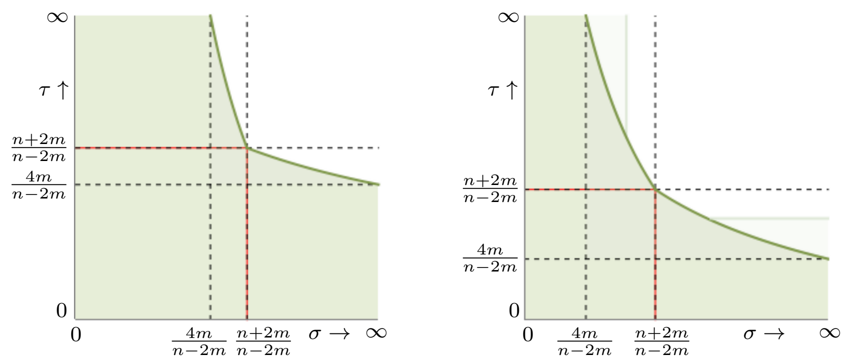


Figure 6.3: Range of admissable growth rates proven in Theorem 6.4.1 for some $n \in (2m, 6m]$ (left) and some $n > 6m$ (right). The missing section compared to Figure 1.1 is shown in light green.

Theorem 6.4.1 *Let $n > 2m$, Condition A be fulfilled, $f \in C^{0,\gamma}(\overline{\Omega})$ and $g \in C^{0,\gamma}(\overline{\Omega} \times \mathbb{R})$ satisfies (6.2) and (6.26). Then, the semilinear Dirichlet problem in (6.1) has a classical solution $u \in C^{2m,\gamma}(\overline{\Omega}) \cap C_0^{m-1}(\overline{\Omega})$.*

Proof. We assume that

$$\max\left\{1, \frac{4m}{n-2m}\right\} \leq \sigma < \frac{4m}{n-2m} + \frac{1}{\tau}\frac{n+2m}{n-2m}.$$

Indeed, the case that $0 \leq \sigma < \max\left\{1, \frac{4m}{n-2m}\right\}$ is contained in Theorem 6.3.1.

Let u_M be a weak solution to (6.8) as described in Lemma 6.2.1. Using similar arguments as in the proof of Theorem 6.3.1, we note that it is sufficient to show that there are upper bounds for $\|u_M^+\|_{L^\infty(\Omega)}$ and $\|u_M^+\|_{L^\infty(\Omega)}$ independent of M. Then, for a sufficiently large $M_3 \in \mathbb{R}^+$ the function u_{M_3} would be a classical solution to (6.1).

As mentioned in (6.14), it holds that $u_M \in L^{p_0}(\Omega)$ with

$$p_0 = \frac{2n}{n-2m}$$

and $\|u_M\|_{L^{p_0}(\Omega)} \leq C'_{\Omega,m,p_0}\|f\|_{L^2(\Omega)}$. By Theorem 6.1.1 we find that there are functions $u_M^\oplus, u_M^\ominus \geq 0$ with $u_M = u_M^\oplus - u_M^\ominus$ and such that the regularity estimates in (6.5) and (6.6) hold for suitable $p_\pm \in (1,\infty)$. Next, we will show iteratively that $\|u_M\|_{L^\infty(\Omega)}$ is bounded by a constant independent of M:

Step 1. Using Sobolev imbeddings and (6.6), we obtain

$$\begin{aligned}\|u_M^-\|_{L^{p_1^\ominus}(\Omega)} &\leq \|u_M^\ominus\|_{L^{p_1^\ominus}(\Omega)} \leq C_{m,n,\sigma,p_1^\ominus}\|u_M^\ominus\|_{W^{2m,\frac{2n}{n-2m}\frac{1}{\sigma}}(\Omega)} \\ &\leq C'_{m,n,\sigma,p_1^\ominus}\left(\|f\|_{L^\infty(\Omega)} + \|g_M(\cdot,u_M^+)\|_{L^{\frac{2n}{n-2m}\frac{1}{\sigma}}(\Omega)} + \|g_M(\cdot,-u_M^-)\|_{L^1(\Omega)}\right)\end{aligned} \tag{6.27}$$

for all

$$\begin{cases} p_1^\ominus \in \left[1, \frac{2n}{\sigma(n-2m)-4m}\right] & \text{if } \frac{4m}{n-2m} < \sigma < \frac{4m}{n-2m} + \frac{1}{\tau}\frac{n+2m}{n-2m}, \\ p_1^\ominus \in [1,\infty) & \text{if } \sigma = \frac{4m}{n-2m}. \end{cases} \tag{6.28}$$

Since $\sigma < \frac{2n}{n-2m}$, inequalities (6.27), (6.13) and (6.16) imply

$$\|u_M^-\|_{L^{p_1^\ominus}(\Omega)} \leq C'_{m,n,\sigma,p_1^\ominus}\left(\|f\|_{L^\infty(\Omega)} + C_{\Omega,m,\sigma}\left(1 + \|f\|^\sigma_{L^2(\Omega)}\right) + C_{\Omega,m,g,f}\right)$$

and the right-hand side does not depend on M. Analogous to (6.19), we then find for $p_1^\ominus \geq \tau$ a constant $C_{\Omega,m,\tau,p_1^\ominus} > 0$ independent of M such that

$$\|g_M(\cdot,-u_M^-)\|_{L^{p_1^\ominus\frac{1}{\tau}}(\Omega)} \leq C_{\Omega,m,\tau,p_1^\ominus}\left(1 + \|f\|^\tau_{L^2(\Omega)}\right). \tag{6.29}$$

Step 2. We may choose $p_1^\ominus > \tau$. Similarly as in Step 1 we obtain with Sobolev imbeddings and (6.5)

$$\begin{aligned}\|u_M^+\|_{L^{p_1^\oplus}(\Omega)} &\leq \|u_M^\oplus\|_{L^{p_1^\oplus}(\Omega)} \leq C_{m,n,\tau,p_1^\oplus,p_1^\ominus}\|u_M^\oplus\|_{W^{2m,p_1^\ominus\frac{1}{\tau}}(\Omega)} \\ &\leq C'_{m,n,\tau,p_1^\oplus,p_1^\ominus}\left(\|f\|_{L^\infty(\Omega)} + \|g_M(\cdot,-u_M^-)\|_{L^{p_1^\ominus\frac{1}{\tau}}(\Omega)} + \|g_M(\cdot,u_M^+)\|_{L^1(\Omega)}\right)\end{aligned}$$

for

$$\begin{cases} p_1^{\oplus} \in \left[1, \frac{np_1^{\ominus}/\tau}{n-2mp_1^{\ominus}/\tau}\right] & \text{if } n > 2mp_1^{\ominus}/\tau, \\ p_1^{\oplus} \in [1,\infty) & \text{if } n \leq 2mp_1^{\ominus}/\tau. \end{cases}$$

Hence, using (6.13), (6.29) and (6.28), we find for all

$$\begin{cases} p_1^{\oplus} \in \left[1, \frac{2n}{\tau(\sigma(n-2m)-4m)-4m}\right] & \text{if } \tau\left(\sigma(n-2m)-4m\right) > 4m, \\ p_1^{\oplus} \in [1,\infty) & \text{if } \tau\left(\sigma(n-2m)-4m\right) \leq 4m, \end{cases} \tag{6.30}$$

an upper bound for $\|u_M^+\|_{L^{p_1^{\oplus}}(\Omega)}$ independent of M. Since

$$\sigma < \frac{4m}{n-2m} + \frac{1}{\tau}\frac{n+2m}{n-2m}$$

is equivalent to

$$\tau(\sigma(n-2m)-4m)-4m < n-2m,$$

we found an upper bound independent of M for $\|u_M^+\|_{L^q(\Omega)}$ for all $q \in [1,\infty)$ or have gained some regularity: From a uniform bound for $\|u_M^+\|_{L^{\frac{2n}{n-2m}}(\Omega)}$ we derived a uniform bound for $\|u_M^+\|_{L^{\frac{2n}{\tau(\sigma(n-2m)-4m)-4m}}(\Omega)}$.

Let $\varepsilon > 0$ be such that

$$\sigma = \frac{4m}{n-2m} + \frac{1}{\tau}\frac{n+2m}{n-2m} - \varepsilon. \tag{6.31}$$

We note that (6.30) can be rewritten in

$$\begin{cases} p_1^{\oplus} \in \left[1, \frac{2n}{(n-2m)(1-\varepsilon\tau)}\right] & \text{if } 1 > \varepsilon\tau, \\ p_1^{\oplus} \in [1,\infty) & \text{if } 1 \leq \varepsilon\tau. \end{cases}$$

In the following, we repeat the arguments in Step 1 and Step 2 and attain after k-times that $\|u_M^{\oplus}\|_{L^{p_k^{\oplus}}(\Omega)}$ is bounded by a constant independent of M, where

$$L^{p_k^{\oplus}}(\Omega) \subset L^{p_k}(\Omega) \tag{6.32}$$

with $k \in \mathbb{N}^+$ and

$$\begin{cases} p_k = \frac{2n}{(n-2m)(1-\varepsilon\tau)^k} & \text{if } 1 > \varepsilon\tau, \\ p_k \in [1,\infty) & \text{if } 1 \leq \varepsilon\tau. \end{cases} \tag{6.33}$$

Indeed, one finds (6.32) with induction. For $k=1$ it holds true. So let (6.32) be satisfied for a $k \in \mathbb{N}^+$ and $\|u_M^+\|_{L^{p_k^{\oplus}}(\Omega)}$ be bounded by a constant independent of M. Then, either

- $\varepsilon\tau \geq 1$ and for all $q \in [1, \infty)$ the norm $\|u_M^+\|_{L^q(\Omega)}$ is bounded independently of M,
- or $\varepsilon\tau < 1$ and for $q = \frac{2n}{(n-2m)(1-\varepsilon\sigma)^k}$ the norm $\|u_M^+\|_{L^q(\Omega)}$ is bounded independently of M.

Using Step 1 and Step 2 again, we then find for all $q \in [1, \infty)$ an upper bound for $\|u_M^+\|_{L^q(\Omega)}$ independent of M or an upper bound for $\|u_M^+\|_{L^{p_{k+1}^{\oplus}}(\Omega)}$ with

$$p_{k+1}^{\oplus} = \frac{2n}{\tau(\sigma(n-2m)(1-\varepsilon\tau)^k - 4m) - 4m}.$$

Since for $\varepsilon\tau < 1$ one finds with (6.31)

$$\begin{aligned}
&\tau(\sigma(n-2m)(1-\varepsilon\tau)^k - 4m) - 4m \\
&= 4m\tau\left((1-\varepsilon\tau)^k - 1\right) - 4m + (n+2m)(1-\varepsilon\tau)^k - \varepsilon\tau(n-2m)(1-\varepsilon\tau)^k \\
&\leq (n-2m)(1-\varepsilon\tau)^k - \varepsilon\tau(n-2m)(1-\varepsilon\tau)^k \\
&= (n-2m)(1-\varepsilon\tau)^{k+1},
\end{aligned}$$

one obtains $L^{p_{k+1}^{\oplus}}(\Omega) \subset L^{p_{k+1}}(\Omega)$ with p_{k+1} as in (6.33).

Hence, for $k \in \mathbb{N}^+$ sufficiently large, we find $p_k > \frac{n\sigma}{2m}$. Then, using Step 1 and Step two again, one obtains for all $q \in [1, \infty)$ an upper bound for $\|u_M\|_{L^q(\Omega)}$ independent of M. One more iteration leads to the existence of a constant $C_{\Omega,m,f,g,\sigma,\tau} > 0$ independent of M such that

$$\|u_M\|_{L^\infty(\Omega)} \leq C_{\Omega,m,f,g,\sigma,\tau}.$$

It follows for M sufficiently large that u_M is a classical solution to (6.1). ■

So if, instead of applying local maximum principles as in [29] and [24], we use the Green function and regularity estimate for the polyharmonic Dirichlet problem in (3.2) and Theorem 6.1.1, we can improve the results in [29]. If we combine Theorem 6.3.1 and 6.4.1, we find the result stated in Theorem 1.2.15. This is also shown in Figures 1.1, 6.2 and 6.3. The parts in Figure 6.2 that are missing compared to Figure 1.1 (displayed in light green) are contained in Figure 6.3 and vice versa. So, the range of admissable growth rates can be represented as in Figure 1.1.

***Remark* 6.4.2** *The condition* $\sigma < \frac{4m}{n-2m} + \frac{1}{\tau}\frac{n+2m}{n-2m}$ *is necessary so that* $p_1^{\oplus}$ *in* (6.30) *fulfills* $p_1^{\oplus} > \frac{2n}{n-2m}$. *Otherwise we would not obtain an increasing sequence* $p_1^{\oplus}, p_2^{\oplus}, \ldots, p_k^{\oplus}$. *For* $\tau = \frac{n+2m}{n-2m}$ *we find a classical solution if* $0 \leq \sigma < \frac{n+2m}{n-2m}$. *This is the known result which has already been proven by von Wahl [76] and Luckhaus [44]. As already noted in [24], the result in Theorem 6.4.1 can be understood as an interpolation between this standard case and the result in Theorem 6.3.1.*

***Remark* 6.4.3** *We have shown that under some growth and sign conditions there is a classical solution to problem* (6.1). *An open problem is the question whether one can assume arbitrary growth of g and still achieve the same result. We note* *that one has to assume at least monotonicity or a sign condition as in* (6.2). *Indeed, Luckhaus constructed in [44] an example in which g fulfills the condition*

$$|g(x,t)| \leq C\left(1+|t|^{\frac{n+2m}{n-2m}+\varepsilon}\right) \text{ for some } \varepsilon > 0,$$

but not necessarily (6.2). *He has found a function that solves problem* (6.1) *in weak but not in classical sense. Moreover, Reichel and Weth constructed a semilinear term that does not fulfill the sign condition, such that there is no solution* $u \in C^{2m,\gamma}(\overline{\Omega})$, *see [54, Theorem 3].*

Appendix

A.1 Upper bound for the C^m-norm of the weighted eigenfunctions

In this section we prove the estimate in (3.58). The result and proof are an adapted version of Lemma 13 in [57], so it coincides with the proof in [57] in some paragraphs except that we replace 2 with m.

Lemma A.1.1 *Suppose that Condition A is satisfied and let* $\{\lambda_{i,m,w}\}_{i\in\mathbb{N}^+}$ *be the eigenvalues to problem* (1.10) *and* $\{\varphi_{i,m,w}\}_{i\in\mathbb{N}^+}$ *the corresponding eigenfunctions with* $\|\varphi_{i,m,w}\|_{L^2_w(\Omega)} = 1$. *Then there exists a constant* $C_{A.1.1} > 0$, *depending on the domain,* w *and* m, *such that for all* $i \in \mathbb{N}^+$*:*

$$\begin{aligned} \|\varphi_{i,m,w}\|_{C^m(\overline{\Omega})} &\leq C_{A.1.1}\lambda_{i,m,w}^{\frac{n}{4m}+\frac{2m-1}{2m}}, \\ |\varphi_{i,m,w}(x)| &\leq C_{A.1.1}\lambda_{i,m,w}^{\frac{n}{4m}+\frac{2m-1}{2m}} d(x)^m \text{ for all } x \in \Omega. \end{aligned} \tag{A.1}$$

Proof. All constants that we use in this proof depend on the domain and on m. We have assumed that the eigenfunctions are normalized in $L^2_w(\Omega)$, so $\|\varphi_{i,m,w}\|_{L^2_w(\Omega)} = 1$. As in [57, Lemma 13] we recall the three main arguments:

1. Regularity: Using Agmon-Douglis-Nirenberg for (1.10) we find some constant $C_{ADN,w,p} > 0$ such that

$$\|\varphi_{i,m,w}\|_{W^{2m,p}(\Omega)} \leq \begin{cases} C_{ADN,w,p}\,\lambda_{i,m,w}\|\varphi_{i,m,w}\|_{L^p(\Omega)} \text{ for } p \in (2,\infty), \\ C_{ADN,w,2}\,\lambda_{i,m,w}\|\varphi_{i,m,w}\|_{L^2_w(\Omega)} \text{ for } p = 2. \end{cases} \tag{A.2}$$

So, for $p = 2$ we find $\|\varphi_{i,m,w}\|_{W^{2m,2}(\Omega)} \leq C_{ADN,w,2}\lambda_{i,m,w}$.

2. Imbeddings: With the Sobolev imbeddings in (2.11)i. and (2.11)ii., we get constants $C_{I,p} > 0$ such that

$$\|u\|_{C^m(\overline{\Omega})} \leq C_{I,p}\|u\|_{W^{2m,p}(\Omega)} \text{ for all } u \in W^{2m,p}(\Omega) \tag{A.3}$$

and with (2.11)iii.-v. we find $C_{I,p,q} > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq C_{I,p,q}\|u\|_{W^{2m,p}(\Omega)} \text{ for all } u \in W^{2m,p}(\Omega) \tag{A.4}$$

with p, q as in Theorem 2.5.1.

3. Interpolation: By Theorem 5.8 of [1] we find for $q \in [p, p_n^*] \cap \mathbb{R}$ and $\theta = \frac{n}{2m}\left(\frac{1}{p} - \frac{1}{q}\right)$, with p_n^* as in (2.11), that there exist constants $C_{p,q} > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq C_{p,q}\|u\|^{\theta}_{W^{2m,p}(\Omega)}\|u\|^{1-\theta}_{L^p(\Omega)} \text{ for all } u \in W^{2m,p}(\Omega). \tag{A.5}$$

We distinguish several cases depending on the dimension n.

- $n \in \{2, \ldots, 2m-1\}$: Using (A.3) and (A.2), we find constants such that

$$\|\varphi_{i,m,w}\|_{C^m(\overline{\Omega})} \leq C_{I,2}\|\varphi_{i,m,w}\|_{W^{2m,2}(\Omega)} \leq C'_{I,2}\lambda_{i,m,w}. \tag{A.6}$$

- $n \in \{2m, \ldots, 6m-1\}$: As mentioned in the proof of Lemma 3.3.1, we get

$$\begin{cases} 2_n^* = \infty & \text{for } 2m \leq n \leq 4m, \\ 2_n^* = \frac{2n}{n-4m} \geq \frac{2n}{2m-1} & \text{for } 4m+1 \leq n \leq 6m-1. \end{cases}$$

  We want to proceed with $q \in \left(\frac{n}{m}, 2_n^*\right)$ and since $2_n^* \leq \frac{n}{m}$ for $n \geq 6m$, we need the restriction $n \leq 6m-1$. Setting $q = \frac{4n}{4m-1} \in \left(\frac{n}{m}, 2_n^*\right)$ and $p = 2$, we obtain

$$\theta = \frac{n}{2m}\left(\frac{1}{2} - \frac{4m-1}{4n}\right) = \frac{n}{4m} - \frac{4m-1}{8m}$$

  and using (A.5) and (A.2), we find

$$\begin{aligned} \|\varphi_{i,m,w}\|_{L^q(\Omega)} &\leq C_{2,q}\|\varphi_{i,m,w}\|^{\theta}_{W^{2m,2}(\Omega)}\|\varphi_{i,m,w}\|^{1-\theta}_{L^2(\Omega)} \leq C'_{2,q}\lambda^{\theta}_{i,m,w} \\ &= C'_{2,q}\lambda_{i,m,w}^{\frac{n}{4m}-\frac{4m-1}{8m}}. \end{aligned}$$

  So, applying (A.3) and (A.2), we get

$$\|\varphi_{i,m,w}\|_{C^m(\overline{\Omega})} \leq C_{I,q}\|\varphi_{i,m,w}\|_{W^{2m,q}(\Omega)} \leq C'_{I,q}C'_{2,q}\lambda_{i,m,w}^{\frac{n}{4m}+\frac{4m+1}{8m}}. \tag{A.7}$$

- $n \geq 6m$: Here we follow analogous steps as in the proof of Lemma 3.3.1. Again, we set $\ell := \left[\frac{n-2m}{4m}\right]$ and for $k \leq \ell$ we define iteratively $p_0 = 2$ and $p_{k+1} = (p_k)_n^*$. Then we find for $k \leq \ell$

$$p_k = \frac{2n}{n-4mk}.$$

  Using (3.25) and (A.4), we get that $W^{2m,p_k}(\Omega)$ imbeds in $L^{p_{k+1}}(\Omega)$ for $k < \ell$, so

$$\|u\|_{L^{p_{k+1}}(\Omega)} \leq C_{I,p_k,p_{k+1}}\|u\|_{W^{2m,p_k}(\Omega)} \text{ for all } u \in W^{2m,p_k}(\Omega).$$

  With $u = \varphi_{i,m,w}$ and (A.2) it holds that

$$\|\varphi_{i,m,w}\|_{L^{p_{k+1}}(\Omega)} \leq C_{I,p_k,p_{k+1}}\|\varphi_{i,m,w}\|_{W^{2m,p_k}(\Omega)} \leq C'_{I,p_k,p_{k+1}}\lambda_{i,m,w}\|\varphi_{i,m,w}\|_{L^{p_k}(\Omega)}.$$

We can set $C_\ell := \prod_{k=0}^{\ell-1} C'_{I,p_k,p_{k+1}}$ and find

$$\|\varphi_{i,m,w}\|_{L^{p_\ell}(\Omega)} \leq C_\ell \lambda_{i,m,w}^\ell \|\varphi_{i,m,w}\|_{L^2_w(\Omega)} \leq C_\ell \lambda_{i,m,w}^\ell. \tag{A.8}$$

If we take $p = p_\ell$ and $q = \frac{n}{m-\delta}$ with $\delta > 0$ so small that $q < p_n^*$, we get

$$\theta = \frac{n}{2m}\left(\frac{1}{p_\ell} - \frac{1}{q}\right) = \frac{n-2m}{4m} - \left[\frac{n-2m}{4m}\right] + \frac{\delta}{2m}$$

and therefore (A.5) and (A.2) imply

$$\|\varphi_{i,m,w}\|_{L^q(\Omega)} \leq C_{p_\ell,q} \|\varphi_{i,m,w}\|^\theta_{W^{2m,p_\ell}(\Omega)} \|\varphi_{i,m,w}\|^{1-\theta}_{L^{p_\ell}(\Omega)} \leq C'_{p_\ell,q} \lambda_i^\theta \|\varphi_{i,m,w}\|_{L^{p_\ell}(\Omega)}.$$

Combining this inequality with (A.8), we find

$$\|\varphi_{i,m,w}\|_{L^q(\Omega)} \leq C'_{p_\ell,q} C_\ell \lambda_{i,m,w}^{\theta+\ell} = C'_{p_\ell,q} C_\ell \lambda_{i,m,w}^{\frac{n-2m}{4m}+\frac{\delta}{2m}}.$$

Using (A.3) and (A.2), one gets

$$\begin{aligned} \|\varphi_{i,m,w}\|_{C^m(\overline{\Omega})} &\leq C_{I,q} \|\varphi_{i,m,w}\|_{W^{2m,q}(\Omega)} \leq C'_{I,q} \lambda_{i,m,w} \|\varphi_{i,m,w}\|_{L^q(\Omega)} \\ &\leq C'_{I,q} C'_{p_\ell,q} C_\ell \lambda_{i,m,w}^{\frac{n}{4m}+\frac{m+\delta}{2m}}. \end{aligned} \tag{A.9}$$

In (A.6), (A.7) and (A.9) we have shown that there is a constant $C_n > 0$ and $\alpha_n \in \left(0, \frac{n}{4m} + \frac{2m-1}{2m}\right]$ such that

$$\|\varphi_{i,m,w}\|_{C^m(\overline{\Omega})} \leq C_n \lambda_{i,m,w}^{\alpha_n} \text{ for all } i \in \mathbb{N}^+.$$

With (3.57) the result in (A.1) follows. As in the proof of Lemma 3.5.1, the mean value theorem implies

$$|\varphi_{i,m,w}(x)| \leq \|\varphi_{i,m,w}\|_{C^m(\overline{\Omega})} d(x)^m \leq C_{A.1.1} \lambda_{i,m,w}^{\frac{n}{4m}+\frac{2m-1}{2m}} d(x)^m \text{ for all } x \in \Omega.$$

∎

Bibliography

[1] R.A. Adams, J. Fournier, Sobolev Spaces, 2nd edn. Academic Press, San Diego, 2003.

[2] S. Agmon, On kernels, eigenvalues and eigenfunctions of operators related to elliptic problems, Commun. Pure Appl. Math. 18 (1965), 627–663.

[3] J.H. Albert, Genericity of simple eigenvalues for elliptic PDE's, Proc. Amer. Math. Soc. 48 (1975), 413–418.

[4] H.W. Alt, Lineare Funktionalanalysis, Eine anwendungsorientierte Einführung, Springer-Verlag Berlin Heidelberg, 2012.

[5] T. Boggio, Sulle funzioni di Green d'ordine m, Rendiconti del Circolo Matematico di Palermo 20 (1905), 97–135.

[6] A. Bressan, Lecture Notes on Functional Analysis. With applications to linear partial differential equations. Graduate Studies in Mathematics, 143, American Mathematical Society, Providence, RI, 2013.

[7] H. Brezis, F.E. Browder, Strongly nonlinear elliptic boundary value problems. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 5 (1978), no. 3, 587–603.

[8] H. Brezis, X. Cabre, Some simple nonlinear PDE's without solutions, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1998), no. 2, 223–262.

[9] F.E. Browder, Existence theory for boundary value problems for quasilinear elliptic systems with strongly nonlinear lower order terms, Partial differential equations (Proc. Sympos. Pure Math., Vol. XXII, Univ. California, Berkeley, Calif., 1971), Amer. Math. Soc., Providence, R.I., 1973, 269–286.

[10] Ph. Clément, L.A. Peletier, An anti-maximum principle for second-order elliptic operators, J. Differential Equations 34 (1979), no. 2, 218–229.

[11] Ph. Clément, G. Sweers, Uniform anti-maximum principle, J. Differential Equations 164 (2000), no. 1, 118–154.

[12] Ph. Clément, G. Sweers, Uniform anti-maximum principle for polyharmonic boundary value problems, Proc. Amer. Math. Soc. 129 (2001), no. 2, 467–474.

[13] C.V. Coffman, On the structure of solutions to $\Delta^2 u = \lambda u$ which satisfy the clamped plate conditions on a right angle, SIAM J. Math. Anal. 13 (1982), no. 5, 746–757.

[14] C.V. Coffman, R.J. Duffin, D.H. Shaffer, The fundamental mode of vibration of a clamped annular plate is not of one sign. In: Constructive Approaches to Mathematical Models (Proc. Conf. in honor of R.J. Duffin, Pittsburgh, Pa., 1978), pp. 267–277. Academic Press, New York, 1979.

[15] A. Dall'Acqua, G. Sweers, The clamped plate equation for the limaçon, Annali di Matematica Pura ed Applicata. (4) 184 (2005), no. 3, 361–374.

[16] E.B. Davies, Spectral theory and differential operators, Cambridge Studies in Advanced Mathematics, 42, Cambridge University Press, Cambridge, 1995.

[17] R.J. Duffin, On a question of Hadamard concerning super-biharmonic functions, J. Math. Phys. 27 (1949), 253–258.

[18] R.J. Duffin, D.H. Shaffer, On the modes of vibration of a ring-shaped plate. Bull. Am. Math. Soc. 58 (1952), 652.

[19] M. Engliš, J. Peetre, A Green's function for the annulus, Ann. Mat. Pura Appl. (4) 171 (1996), 313–377.

[20] P.R. Garabedian, A partial differential equation arising in conformal mapping, Pacific J. Math. 1 (1951), 485–524.

[21] F. Gazzola, H.-Ch. Grunau, G. Sweers, Polyharmonic boundary value problems, Springer Lecture Notes Series 1991, 2010.

[22] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order, Second edition, Grundlehren der Mathematischen Wissenschaften 224, Springer-Verlag, Berlin, 1983.

[23] H.-Ch. Grunau, Das Dirichletproblem für semilineare elliptische Differentialgleichungen höherer Ordnung, Doctoral thesis, Göttingen: Georg-August-Universität, 1990.

[24] H.-Ch. Grunau, The Dirichlet problem for some semilinear elliptic differential equations of arbitrary order, Analysis 11 (1991), no. 1, 83–90.

[25] H.-Ch. Grunau, F. Robert, Positivity and almost positivity of biharmonic Green's functions under Dirichlet boundary conditions, Arch. Ration. Mech. Anal. 195 (2010), no. 3, 865–898.

[26] H.-Ch. Grunau, F. Robert, G. Sweers, Optimal estimates from below for biharmonic Green functions, Proc. Amer. Math. Soc. 139 (2011), no. 6, 2151–2161.

[27] H.-Ch. Grunau, G. Romani, G. Sweers, Differences between fundamental solutions of general higher order elliptic operators and of products of second-order operators, Math. Ann. (2020). https://doi.org/10.1007/s00208-020-02015-3

[28] H.-Ch. Grunau, G. Sweers, Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions, Math. Nachr. 179 (1996), 89–102.

[29] H.-Ch. Grunau, G. Sweers, Classical solutions for some higher order semilinear elliptic equations under weak growth conditions, Nonlinear Anal. 28 (1997), no. 5, 799–807.

[30] H.-Ch. Grunau, G. Sweers, Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions, Math. Ann. 307 (1997), no. 4, 589–626.

[31] H.-Ch. Grunau, G. Sweers, The maximum principle and positive principal eigenfunctions for polyharmonic equations, in Reaction Diffusion systems, Marcel Dekker Inc., New York (1997), 163-182.

[32] H.-Ch. Grunau, G. Sweers, Sign change for the Green function and for the first eigenfunction of equations of clamped-plate type, Arch. Ration. Mech. Anal. 150 (1999), no. 2, 179–190.

[33] H.-Ch. Grunau, G. Sweers, Optimal conditions for anti-maximum principles, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 30 (2001), no. 3-4, 499–513.

[34] H.-Ch. Grunau, G. Sweers, Sharp estimates for iterated Green functions, Proc. Roy. Soc. Edinburgh Sect. A 132 (2002), no. 1, 91–120.

[35] J. Hadamard, Mémoire sur le problème d'analyse relatif à l'équilibrie des plaques élastiques encastrées. In: Œuvres de Jacques Hadamard, Tomes II, pp. 515–641, Centre National de la Recherche Scientifique, Paris 1968, reprint of: Mémoire couronne par l'Académie des Sciences (Prix Vaillant), Mém. Sav. Étrang. 33 (1907).

[36] J. Hadamard, Sur certain cas intéressants du problème biharmonic, in : Œuvres de Jaques Hadamard, Tome III, pp. 1297–1299, Centre National de la Recherche Scientifique: Paris, 1968 reprint of: Atti IVe Congr. Intern. Mat. Rome (1908), 12–14.

[37] P. Hess, On strongly nonlinear elliptic problems, Functional analysis (Proc. Brazilian Math. Soc. Sympos., Inst. Mat. Univ. Estad. Campinas, Sao Paulo, 1974), pp. 91–109, Lecture Notes in Pure and Appl. Math., 18, Dekker, New York, 1976.

[38] R. Jentzsch, Über Integralgleichungen mit positivem Kern, Journal für die reine und angewandt Mathematik 141 (1912), 235–244.

[39] T. Kato, Perturbation theory for linear operators, Reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[40] B. Kawohl, G. Sweers, On 'anti'-eigenvalues for elliptic systems and a question of McKenna and Walter, Indiana Univ. Math. J. 51 (2002), no. 5, 1023–1040.

[41] Yu.P. Krasovskiĭ, Isolation of singularities of the Green's function, Math. USSR Izvestiya 1 No. 5 (1967), 935–966.

[42] M.G. Kreĭn, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, Amer. Math. Soc. Translation (1950), no. 26, 128 pp., Russian original: Uspehi Matem. Nauk (N.S.) 23 (1948), 3–95.

[43] H.A. Levine, M.H. Protter, Unrestricted lower bounds for eigenvalues for classes of elliptic equations and systems of equations with applications to problems in elasticity, Math. Methods Appl. Sci. 7 (1985), no. 2, 210–222.

[44] S. Luckhaus, Existence and regularity of weak solutions to the Dirichlet problem for semilinear elliptic systems of higher order, J. Reine Angew. Math. 306 (1979), 192–207.

[45] M. Nakai, L. Sario, Green's function of the clamped plate punktured disk, J. Aust. Math. Soc. 20 (Series B) (1977), no. 2, 175–181.

[46] J.H. Ortega, E. Zuazua, Generic simplicity of the spectrum and stabilization for a plate equation, SIAM J. Control Optim. 39 (2000), no. 5, 1585–1614.

[47] J.H. Ortega, E. Zuazua, Addendum to "Generic simplicity of the spectrum and stabilization for a plate equation", SIAM J. Control Optim. 42 (2003), no. 5, 1905–1910.

[48] R.N. Pederson, On the unique continuation theorem for certain second and fourth order elliptic equations, Comm. Pure Appl. Math. 11 (1958), 67–80.

[49] M.C. Pereira, Generic simplicity of eigenvalues for a Dirichlet problem of the bilaplacian operator, Electron. J. Differential Equations 114 (2004), 1–21.

[50] Ch. Pommerenke, Boundary behaviour of conformal maps, Grundlehren der Mathematischen Wissenschaften 299, Springer-Verlag, Berlin, 1992.

[51] M.H. Protter, Unique continuation for elliptic equations, Trans. Amer. Math. Soc. 95 (1960), 81–91.

[52] M.H. Protter, H.F. Weinberger, Maximum principles in differential equations, Prentice Hall, Englewood Cliffs N.J., 1967.

[53] L. Pulst, Dominance of Positivity of the Green's Function associated to a Perturbed Polyharmonic Dirichlet Boundary Value Problem by Pointwise Estimates, Dissertation (2014). http://dx.doi.org/10.25673/4208

[54] W. Reichel, T. Weth, Existence of solutions to nonlinear, subcritical higher order elliptic Dirichlet problems, J. Differential Equations 248 (2010), no. 7, 1866–1878.

[55] F. Rellich, Perturbation theory of eigenvalue problems, Assisted by J. Berkowitz. With a preface by Jacob T. Schwartz, Gordon and Breach Science Publishers, New York-London-Paris, 1969.

[56] E. Sassone, Positivity for polyharmonic problems on domains close to a disk, Ann. Mat. Pura Appl. (4) 186 (2007), no. 3, 419–432.

[57] I. Schnieders, G. Sweers, A biharmonic converse to Krein-Rutman: a maximum principle near a positive eigenfunction, Positivity 24 (2020), no. 3, 677–710. https://doi.org/10.1007/s11117-019-00702-3

[58] I. Schnieders, G. Sweers, A maximum principle for a fourth order Dirichlet problem on smooth domains, Pure Appl. Anal. 2 (2020), no. 3, 685–702. https://doi.org/10.2140/paa.2020.2.685

[59] I. Schnieders, G. Sweers, Classical solutions up to the boundary to some higher order semilinear Dirichlet problems, to appear in Nonlinear Analysis.

[60] I. Schnieders, G. Sweers, Note on a sign-dependent regularity for the polyharmonic Dirichlet problem, to appear in J. Differential Equations. https://arxiv.org/abs/2009.09747

[61] J. Schröder, Randwertaufgaben vierter Ordnung mit positiver Greenscher Funktion, Math. Z. 90 (1965), 429–440.

[62] J. Schröder, Hinreichende Bedingungen bei Differential-Ungleichungen vierter Ordnung, Math. Z. 92 (1966), 75–94.

[63] J. Schröder, Zusammenhängende Mengen inverspositiver Differentialoperatoren vierter Ordnung, Math. Z. 96 (1967), 89–110.

[64] H.S. Shapiro, M. Tegmark, An elementary proof that the biharmonic Green function of an eccentric ellipse changes sign, SIAM Rev. 36 (1994), 99–101.

[65] T. Shirota, A remark on the unique continuation theorem for certain fourth order elliptic equations, Proc. Japan Acad. 36 (1960), no. 1, 571–573.

[66] G. Sweers, L^N is sharp for the anti-maximum principle, J. Differential Equations 134 (1997), no. 1, 148–153.

[67] G. Sweers, When is the first eigenfunction for the clamped plate equation of fixed sign?, Electron. J. Diff. Eqns., Conf. 06 (2001), 285–296.

[68] G. Sweers, On sign preservation for clotheslines, curtain rods, elastic membranes and thin plates, Jahresber. Dtsch. Math.-Ver. 118 (2016), no. 4, 275–320.

[69] G. Sweers, An elementary proof that the triharmonic Green function of an eccentric ellipse changes sign, Arch. Math. (Basel) 107 (2016), no. 1, 59–62.

[70] G. Sweers, Correction to: An elementary proof that the triharmonic Green function of an eccentric ellipse changes sign, Arch. Math. (Basel) 112 (2019), no. 2, 223–224.

[71] G. Sweers, Bilaplace eigenfunctions compared with Laplace eigenfunctions in some special cases. In: Buskes, G., et al. (eds.) Positivity and Noncommutative Analysis, Birkhäuser, London (2019), 537–561.

[72] M. Teytel, How rare are multiple eigenvalues?, Comm. Pure Appl. Math. 52 (1999), no. 8, 917–934.

[73] S. Timoshenko, S. Woinowsky-Krieger, Theory of plates and shells, 2nd Edition, McGraw-Hill, 1959.

[74] F. Tomi, Über elliptische Differentialgleichungen 4. Ordnung mit einer starken Nichtlinearität, Nachr. Akad. Wiss. Göttigen Math.-Phys. Kl. II (1976), no. 3, 33–42.

[75] M. Ulm, The interval of resolvent-positivity for the biharmonic operator, Proc. Amer. Math. Soc. 127 (1999), no. 2, 481–489.

[76] W. von Wahl, Semilinear elliptic and parabolic equations of arbitrary order, Proc. Roy. Soc. Edinburgh Sect. A 78 (1977/78), no. 3-4, 193–207.

[77] A. Weinstein, W. Stenger, Methods of intermediate problems for eigenvalues, Theory and ramification, Mathematics in Science and Engineering, Vol. 89, Academic Press, New York-London, 1972.

[78] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen, Math. Ann. 71 (1912), no. 4, 441–479.

[79] Z. Zhao, Green function for Schrödinger operator and conditioned Feynman-Kac gauge, J. Math. Anal. Appl. 116 (1986), no. 2, 309–334.

[80] Z. Zhao, Green functions and conditioned gauge theorem for a 2-dimensional domain, In: Seminar on Stochastic Processes, 1987 (Princeton, NJ, 1987), vol. 15 of Progr. Probab. Statist., Birkhäuser Boston, Boston, 1988, 283–294.

Acknowledgments

I am grateful to my advisor Prof. Guido Sweers for suggesting the interesting subject of this thesis. I am thankful for his advise and encouragement during the last years. His hints and comments as well as our discussions were very helpful in the assembly of this dissertation.

I would like to express my thanks to my mentor apl. Prof. Dirk Horstmann for helpful remarks and his encouragement during my undergraduate studies.

Moreover thanks go to Michael and my parents for their great support.

Erklärung zur Dissertation

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne die Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Werken dem Wortlaut oder dem Sinn nach entnommen wurden, sind als solche kenntlich gemacht. Ich versichere an Eides statt, dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen und eingebundenen Artikeln und Manuskripten - noch nicht veröffentlicht worden ist sowie, dass ich eine Veröffentlichung der Dissertation vor Abschluss der Promotion nicht ohne Genehmigung des Promotionsausschusses vornehmen werde. Die Bestimmungen dieser Ordnung sind mir bekannt. Darüber hinaus erkläre ich hiermit, dass ich die Ordnung zur Sicherung guter wissenschaftlicher Praxis und zum Umgang mit wissenschaftlichem Fehlverhalten der Universität zu Köln gelesen und sie bei der Durchführung der Dissertation zugrundeliegenden Arbeiten und der schriftlich verfassten Dissertation beachtet habe und verpflichte mich hiermit, die dort genannten Vorgaben bei allen wissenschaftlichen Tätigkeiten zu beachten und umzusetzen. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.

Teilpublikationen

1. I. Schnieders, G. Sweers, A biharmonic converse to Krein-Rutman: a maximum principle near a positive eigenfunction, Positivity 24 (2020), no. 3, 677–710. https://doi.org/10.1007/s11117-019-00702-3

2. I. Schnieders, G. Sweers, A maximum principle for a fourth order Dirichlet problem on smooth domains, Pure Appl. Anal. 2 (2020), no. 3, 685–702. https://doi.org/10.2140/paa.2020.2.685

3. I. Schnieders, G. Sweers, Classical solutions up to the boundary to some higher order semilinear Dirichlet problems, to appear in Nonlinear Analysis (accepted January 2021).

Verwandte Artikel:

4. I. Schnieders, G. Sweers, Note on a sign-dependent regularity for the polyharmonic Dirichlet problem, to appear in J. Differential Equations (accepted January 2021). https://arxiv.org/abs/2009.09747

Köln, 18.01.2021

Ort, Datum

Inka Schnieders