Statistics for Copula-based Measures of Multivariate Association

Theory and Applications to Financial Data

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Preface

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List of Symbols and Abbreviations

Chapter 1

Introduction

Concepts of association or dependence play a central role when considering multiple random sources in statistical models as they describe the relationship between two or more random variables. Several questions are of relevance in this context:

1. How can association be measured and detected in empirical data?

2. How can association between random variables be modeled in general?

3. Which estimation and statistical test procedures are available?

Especially in financial applications, the analysis and modeling of association has gained a lot of attention recently and is subject to an increasing research activity. We mention multivariate portfolio theory, risk analysis and management, valuation, hedging, and pricing of complex financial instruments such as basket default options. In particular, the concept of copulas has proven to be useful in those fields of application and research. Before going into detail as far as the modeling and measurement of association by means of copulas is concerned, we briefly describe the role of association within some of the aforementioned fields.

Within portfolio theory, prominent concepts such as the Mean-Variance Markowitz model (Markowitz (1987)), the Capital Asset Pricing Model (CAPM), and the Arbitrage Pricing Theory (APT) (see e.g. Elton et al. (2010)) make use of Pearson's correlation coefficient as a measure of association between the asset returns in order to determine an optimal portfolio choice for a given utility function. In this context, Pearson's correlation coefficient has proven to be a tractable measure while offering an appealing way to describe association between multivariate normally distributed asset returns. The latter distributional assumption, however, is essential for the applicability of Pearson's correlation coefficient as a measure of association, as we will explain below. Pearson's correlation coefficient is also frequently used in the context of risk measurement and management of multivariate asset portfolios. This can partly be put down to the fact that the common risk measure Value-at-Risk (VaR) may be expressed as a simple function of the correlation matrix of the multivariate normally distributed asset returns in the underlying portfolio, representing a coherent risk measure in this case. In particular, the one-factor portfolio model proposed in Pillar I of the regulatory Basel II framework (cf. Basel Committee on Banking Supervision (2006)) makes use of an elegant relationship between the correlation structure of the underlying portfolio and the VaR. We further recall the widely used methodology by RiskMetrics (1996) for measuring market risk, which is based on the assumption of normally distributed returns and the VaR concept. In particular, weighted averages of past observations are used here to forecast and estimate volatility and correlation.

Many researchers and practitioners have extended the above and many other applications to more general concepts of association than Pearson's correlation coefficient. The reasons for those extensions are numerous. For example, Pearson's correlation coefficient is known to be sensitive towards extreme events and, thus, more robust measures of association such as trimmed correlation coefficients have been suggested (see e.g. Maronna et al. (2006)). Further, Pearson's correlation coefficient measures the degree of linear association between two random variables. Usually, however, this does not sufficiently describe association between non-normally or, more generally, non-elliptically distributed random variables (related pitfalls are discussed in Embrechts et al. (2002)). In particular, the concept of correlation does not exist for very heavy-tailed distributions such as alpha-stable distributions where the second moments do not exist (see e.g. Rachev and Mittnik (2000)). In addition, we mention complex non-linear financial products such as basket default options where the correlation coefficient as a second-order approximation to association represents a less adequate measure (cf. Laurent and Gregory (2005) and references therein). Another extension of the correlation coefficient, which measures association between two random variables only, refers to the simultaneous measurement of association between more than two random variables as described by their multivariate distribution function. This type of multivariate measures of association will be developed and investigated in the present thesis.

Amongst those various extensions, the concept of copulas has proven to be the most general and sophisticated concept of describing and modeling association or dependence between the components of a random vector (see e.g. Joe (1997) and Nelsen (2006) for a detailed overview of copulas). Copula techniques are also frequently applied in the quantitative finance literature, we mention Patton (2002), Embrechts et al. (2003), McNeil et al. (2005), Savu and Trede (2008), and Giacomini et al. (2009). Copulas split the multivariate distribution function of a random vector into the univariate marginal distribution functions and the dependence structure represented by the copula. In particular, the copula is invariant with respect to strictly increasing transformations of the components of the random vector. Commonly, it is precisely this property which justifies to call the copula the dependence structure of a random vector. Naturally, measures of association or dependence should be a functional of the copula only. The most prominent measures of this type are Spearman's rho and Kendall's tau. Especially the former measure will be the subject of several analyses and results in this thesis.

Outline and summary

The main aim of this dissertation is the modeling, the estimation and the statistical inference of multivariate versions of copula-based measures of association such as Spearman's rho. Special focus is put on the analysis of the statistical properties of related estimators as well as the derivation of statistical hypothesis tests. The latter may be used to verify specific modeling assumptions on the one hand. On the other hand, statistical tests are developed to test whether association changes over time or whether it differs between random sources such as multivariate asset returns. Only a few statistical tests of those types exist for multivariate measures of association in the copula framework. This thesis addresses this gap and illustrates the theoretical results with applications to financial data. Further, several simulation studies are carried out to investigate the performance of the proposed estimators or statistical hypothesis tests.

All theoretical results in this thesis on the modeling and measuring of association between several random variables use the concept of copulas. In fact, copulas allow to study the dependence structure of a multivariate random vector irrespective of its univariate marginal distribution functions. We consider measures of association which depend on the copula of the underlying random vector only and are invariant with respect to the marginal distribution functions. As a direct functional of the copula, nonparametric estimators for those measures are obtained based on the so called empirical copula, which is derived from the multivariate empirical distribution function. Statistical inference for these measures is established using recent results on the asymptotic weak convergence of the empirical copula process, also for serially dependent observations. The derived results and hypothesis tests are mainly of nonparametric nature and may thus be applied in very general settings. Only weak assumptions on the distribution function, such as continuity of the marginal distributions and continuous partial differentiability of the copula, are made.

One aspect in this thesis is the measurement of multivariate association. As outlined above, measures of multivariate association are naturally based on the copula of the underlying random vector. Various copula-based measures have been proposed in the literature. For example, Wolff (1980) introduces a class of multivariate measures of association which is based on the L_1- and L_∞-norms of the difference between the copula and the independence copula (see also Fernández-Fernández and González-Barrios (2004)). Other authors generalize existing bivariate measures of association to the multivariate case. For example, multivariate extensions of Spearman's rho are considered by Nelsen (1996) and Schmid and Schmidt (2006, 2007a, 2007b). Blomqvist's beta is generalized by Úbeda-Flores (2005) and Schmid and Schmidt (2007c), whereas a multivariate version of Gini's Gamma is proposed by Behboodian et al. (2007). Further, Joe (1990) and Nelsen (1996) discuss multivariate generalizations of Kendall's tau. A multivariate version of Spearman's footrule is considered by Genest et al. (2010). Joe (1989a, 1989b) investigates multivariate measures which are based on the Kullback-Leibler mutual information. Most of these measures have the often undesirable property that they may be zero even though the components of the underlying random vector are not stochastically independent. In chapter 3 of this thesis, we propose a multivariate version of the bivariate measure Hoeffding's Phi-Square which takes the value zero if and only if the components of the random vector are stochastically independent. It is based on a Cramér-von Mises functional and is of importance in the context of tests for (multivariate) stochastic independence. In particular, it is a direct functional of the copula only. The asymptotic distribution of a nonparametric estimator for Hoeffding's Phi-Square is established for the case of independent observations as well as of dependent observations from a strictly stationary strong mixing sequence.

Another important topic of this thesis is the derivation of a weighted version for Spearman's rho, which is addressed in chapter 4. A shared feature of such weighted statistics is to allocate different, non-identical weights to the observations. In a time-dynamic context, for example, different weights are put on past observations to model the evolving correlation over time. A popular representative of this type of weighted statistics is the Exponentially Weighted Moving Average (EWMA) model, introduced by RiskMetrics (1996). Based on observations $X_{t-n+1}, ...X_t$ and $Y_{t-n+1}, \ldots, Y_t$, respectively, the estimator for the (linear) correlation at time t is here given by

$$r_t = \frac{\sum_{j=1}^{n} \lambda^{j-1}(X_{t-j+1} - \bar{X})(Y_{t-j+1} - \bar{Y})}{\sqrt{\sum_{j=1}^{n} \lambda^{j-1}(X_{t-j+1} - \bar{X})^2}\sqrt{\sum_{j=1}^{n} \lambda^{j-1}(Y_{t-j+1} - \bar{Y})^2}} \tag{1.1}$$

with $\bar{X} = 1/n \sum_{i=j}^{n} X_{t-j+1}$ and $\bar{Y} = 1/n \sum_{j=1}^{n} Y_{t-j+1}$ and decay factor $0 < \lambda < 1$. The decay factor λ determines the relative weight which is assigned to each observation. In contrast to a simple moving average model based on equally weighted observations (i.e., with $\lambda = 1$), the above estimator reacts faster to (sudden) changes of the correlation as higher weight is allocated to more recent observations. The RiskMetrics methodology is based on the assumption that the underlying multivariate distribution is (conditionally) normally distributed. As outlined above, other concepts of association than linear correlation are more appropriate when the underlying distributions are non-elliptical. This motivates the introduction of a weighted estimator for the copula-based measure Spearman's rho. As the estimation of Spearman's rho is based on the ranks of the observations, the proposed weighted estimator places different weights to the ranks of the observations and not to the observations themselves, as e.g. in the EWMA model. The asymptotic distribution of this estimator is derived from the weak convergence of weighted empirical processes. Those results allow, for example, to test for significant changes of Spearman's rho over time. A generalization to the multivariate case is also considered.

An assumption frequently made in many financial and statistical models is that pairwise correlations between the underlying random variables are equal. For example, the one-factor portfolio model used in the Basel II framework (BCBS (2006)), which determines the minimum capital requirements for credit risk, is based on the assumption that the asset returns between any two obligors have the same correlation. Engle and Kelly (2009) consider equal pairwise correlations in the context of dynamic conditional correlation modeling. They describe further applications in collateralized debt obligation (CDO) pricing, derivative trading, and portfolio choice. Another field of research where the assumption of equal pairwise correlations plays a central role is the interclass correlation modeling, which is applied within the analysis of familial data. The latter investigates the degree of resemblance between family members and is subject to increasing research activity, see e.g. Helu and Naik (2006), Seo et al. (2006), Naik and Helu (2007), and Wu et al. (2009) and references therein. The assumption of equal pairwise correlations can be verified utilizing adequate statistical tests. Various tests for the null hypothesis of equal linear Pearson's correlation coefficients (or equi linear-correlation) in a multivariate normally distributed random vector have been investigated e.g. by Bartlett (1950, 1951), Anderson (1963), Lawley (1963), and Aitkin et al. (1968). Due to the aforementioned shortcomings of Pearson's correlation coefficient, it is natural to study alternative measures of association such as Spearman's rho. In chapter 5, we develop four (asymptotic) tests for the null hypothesis of equi Spearman's rank-correlation, i.e., that all pairwise Spearman's rho coefficients in a multivariate random vector are equal. The proposed tests for equi rank-correlation are nonparametric and can be applied without further assumptions on the marginal distributions except their continuity. As demonstrated, the proposed tests may also be applied in the context of multivariate distribution modeling. All tests are easy to implement and can be performed with low computational complexity. A simulation study to investigate the power of the tests identifies especially two tests showing a good performance for all considered dimensions and copula models. The test setting also allows the derivation of a test for stochastic independence based on all distinct pairwise Spearman's rho coefficients.

As mentioned before, the analysis of the association in a portfolio of risky assets has attracted increasing interest over the last decade. First, the globalizing and interdependence of financial markets require a thorough portfolio risk modeling and management, which can quickly react to changing market situations. This is particularly important when market conditions deteriorate and the association between asset returns increases – which is also known as the 'correlation breakdown', see e.g. Karolyi and Stulz (1996), Campbell et al. (2002), Bae et al. (2003), Patel (2005), Rodriguez (2007) or Bartram et al. (2007). Simultaneously, the rising awareness of modeling the association in a portfolio may certainly be put down to the recent market turbulence in the entire financial sector. Further, the internal model approach in the context of the regulatory Basel II framework allows banks to use their own portfolio risk models for determining the amount of regulatory capital to be maintained. Hence, a proper understanding of the portfolio's cross correlation structure and diversification may be essential to preserve the financial stability of a bank on the one hand. On the other hand, increasing association between financial asset returns may also lead to increasing association between banks' trading results and, thus, to the risk of simultaneous large losses at several banks. A comprehensive analysis of the association between the banks' trading results may give information about the systemic fragility of the financial system. From a supervisory perspective, we develop two test procedures in chapter 6 to analyze the association between the trading results of a hypothetical portfolio of banks both over time and across banks. In particular, we use Spearman's rho to quantify the association in this supervisory portfolio. Several theoretical results on the asymptotic behavior of the difference of two Spearman's rho coefficients - either between two different samples or for different points in time - form the basis for the formulation of the two test procedures. On the one hand, a time-dynamic two-step test procedure, which is partly based on a nonparametric control chart for Spearman's rho, is designed to detect significant long-term level changes of Spearman's rho. On the other hand, we propose a statistical hypothesis test for significant differences between two Spearman's rho coefficients of different samples by taking into account all respective lower-dimensional Spearman's rho coefficients. This test can be used to simultaneously identify those groups of banks that show significant changes of association around some specific point in time. The theoretical results are applied to real profits and loss data and corresponding Value-at-Risk estimates of eleven German banks which had a regulatory approved internal market-risk model during the years 2001 to 2006. Our empirical study of the supervisory portfolio identifies significant changes in the level of Spearman's rho at three time points during the observation period. At two of those time points the second test procedure reveals a significant change in association for all sub-portfolios comprising more than eight banks. The proposed methods are general and can be applied to any series of multivariate asset returns in finance where the assumption of independent standardized returns holds.

Detailed outline

Chapter 2 deals with the statistical modeling and measurement of multivariate association. We start by describing several properties of association in financial data by introducing different statistical tools and concepts to measure and detect association. The concept of copulas is introduced in section 2.2.1 and several properties of copulas are presented. In section 2.2.2, we discuss the nonparametric estimation of copulas based on the empirical copula. The weak convergence of the empirical copula process is investigated for both independent and serially dependent observations from strictly stationary strong mixing sequences. We further give a brief introduction to the nonparametric bootstrap which can be used to approximate the distribution of the empirical copula process. After introducing several concepts of multivariate association such as concordance or positive dependence in section 2.3.1, section 2.3.2 discusses important properties of multivariate measures of association. Finally, we introduce the (bivariate) copula-based measures of association Spearman's rho, Kendall's tau, and Blomqvist's beta, describe how they can be generalized to the multivariate case, and address their estimation based on the empirical copula in section 2.3.3.

A multivariate version of the measure of association Hoeffding's Phi-Square is discussed in chapter 3. Some of its analytical properties are investigated in section 3.2. We give the explicit value of multivariate Hoeffding's Phi-Square for some copulas of simple form and describe a simulation algorithm to approximate its value when the copula is of a more complicated form. In section 3.3, a nonparametric estimator for multivariate Hoeffding's Phi-Square based on the empirical copula is derived. We establish its asymptotic behavior both in the case of independent observations and dependent observations from strictly stationary strong mixing sequences. The asymptotic variance can consistently be estimated by means of a nonparametric (moving block) bootstrap method. We show how the estimator can be adapted to account for small sample sizes. Section 3.4 finally illustrates the applicability of multivariate Hoeffding's Phi-Square to financial data by using it to analyze financial contagion related to the bankruptcy of Lehman Brothers Inc. in September 2008.

In chapter 4, a weighted nonparametric estimator for multivariate Spearman's rho is proposed and its statistical properties are investigated. After providing relevant definitions and background material in section 4.1, the weighted estimator for multivariate Spearman's rho is introduced in section 4.2. It is derived from the ordinary nonparametric estimator of Spearman's rho based on the empirical copula by allocating nonidentical weights to the ranks (section 4.2.1). In section 4.2.2, we first establish the weak convergence of the weighted empirical copula process under minimal conditions on the weights, which is deduced from the weak convergence properties of general weighted empirical processes. In a second step, the asymptotic behavior of the weighted estimator for Spearman's rho is derived. A bootstrap procedure is described to estimate the asymptotic variance of this estimator. Section 4.3 deals with the application of the weighted estimator for evaluating Spearman's rho over time while placing more weight to recent observations. Several weighting schemes for this purpose are discussed. Finally, the theoretical results are applied to the analysis of association between equity return series of several international banks in section 4.4.

Chapter 5 addresses the statistical testing of the null hypothesis of equi-rank correlation, i.e., that all pairwise Spearman's rho coefficients in a multivariate random vector are equal. After providing relevant definitions and some preliminary results on Spearman's rho in section 5.1, four (asymptotic) nonparametric hypothesis tests for equi rank-correlation are derived in section 5.2. We establish their asymptotic distribution based on empirical process theory. We show that a nonparametric bootstrap method to determine unknown parameters or critical values works. Further, a brief overview of the existing literature on tests for equi linear-correlation is given in section 5.3. The results of a simulation study carried out to investigate the power of the tests are discussed in section 5.4. They are compared to the classical test for equal linear Pearson's correlation coefficients developed by Lawley (1963). Section 5.5 briefly discusses the derivation of a test for stochastic independence based on Spearman's rho before the applicability of the four tests for equi-rank correlation to financial data is demonstrated in section 5.6.

Finally, the last chapter of this thesis, chapter 6, is devoted to the statistical analysis of association in a supervisory portfolio of banks both over time and across banks. After a short motivation in section 6.1, an introduction to the basic theory of control charts is given in section 6.2. The relevant theoretical results on the difference of two Spearman's rho coefficients both over time and across different samples are established in section

6.3. In particular, the test procedure for detecting significant long-term level changes of Spearman's rho is developed in section 6.3.1. The statistical test procedure designed to analyze the statistical properties of multiple Spearman's rho coefficients is derived in 6.3.2. Section 6.4 states relevant definitions and assumptions for the analysis of the supervisory portfolio. The theoretical findings are applied to real profits and loss data and corresponding Value-at-Risk estimates of eleven German banks in section 6.5.

Chapter 2

Statistical modeling and measurement of association

2.1 Association in financial data

As outlined in the previous chapter, the notion of association plays a central role in many applications in financial theory. In the following, we describe some aspects of detecting and measuring association in financial data by using different statistical concepts. Throughout this section, our analysis is based on daily equity (log-) return series from the six international banks BNP Paribas (BNP), Commerzbank (COBA), Barclays (BARC), HSBC, Bank of America (BOA), and Citigroup (CITI) from May 1997 to April 2010.

Figure 2.1 shows a scatter plot matrix for the return series of all pairs of banks. The distinct scatter plots give a first impression of the degree and type of association between the return series. In particular, they show that association in financial data is quite different. This observation is confirmed by figure 2.2, which provides contour plots of the empirical densities of two selected pairs of banks. Whereas the level curves of the empirical density in the right panel of figure 2.2 are rather elliptic, the level curves in the left panel have a quite different shape.

The most common measure to quantify the degree of association between two random variables is Pearson's linear correlation coefficient. For the d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$ whose components are assumed to have nonzero finite variances, the linear correlation coefficient between X_i and X_j is defined as

$$r_{ij} = r_{X_i,X_j} = \frac{Cov(X_i, X_j)}{\sqrt{Var(X_i)}\sqrt{Var(X_j)}}. \tag{2.1}$$

Further, the $d \times d$ matrix $R = (r_{ij})_{1\leq i,j\leq d}$ is called the linear correlation matrix of the random vector $\mathbf{X}$. The linear correlation coefficient measures the degree of linear association between the random variables X_i and X_j. In particular, it can be shown

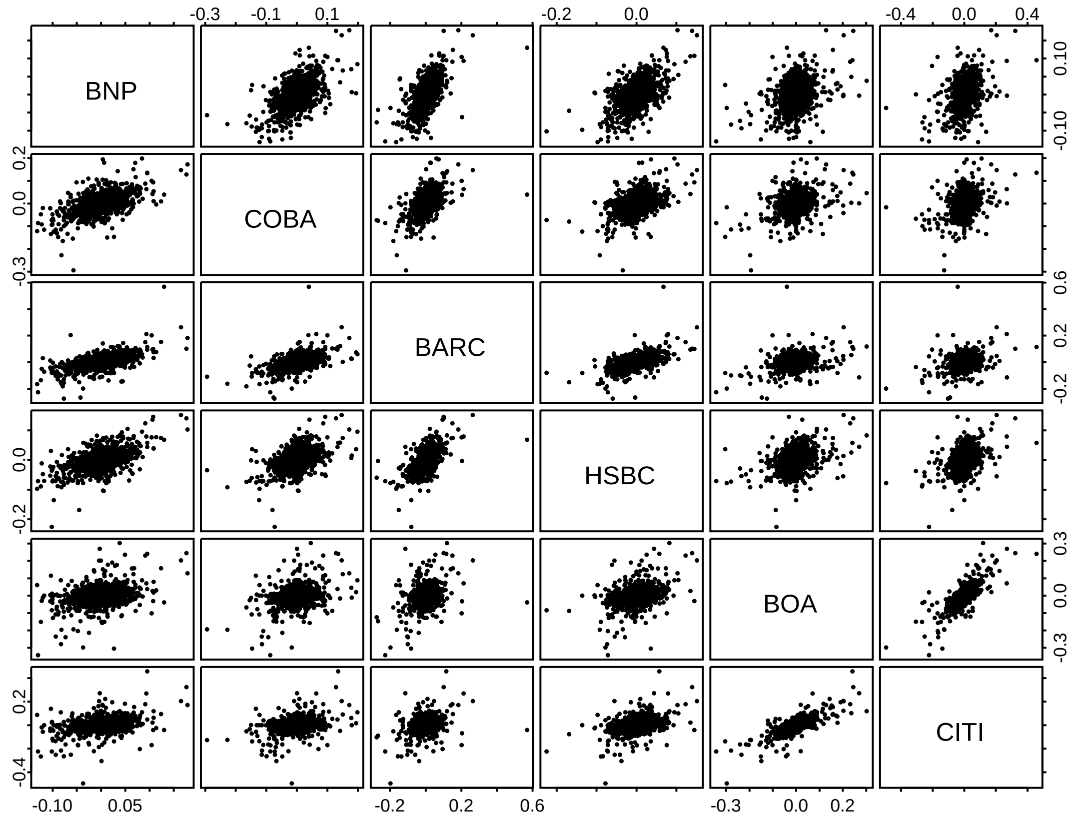


Figure 2.1: Bivariate scatter plots for the daily return series of the banks BNP, COBA, BARC, HSBC, BOA, and CITI for the observation period May 1997 to April 2010.

that $|r_{X_i,X_j}| = 1$ if and only if $X_i = aX_j + b$ almost surely with $a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$, i.e., if there exists a perfect positive or negative linear functional relationship between the random variables. Otherwise, $-1 < r_{X_i,X_j} < 1$. The linear correlation coefficient is invariant with respect to strictly increasing linear transformations of the margins of $\mathbf{X}$, i.e.,

$$r_{aX_i+b,cX_j+d} = \text{sign}(ac) r_{X_i,X_j},$$

with $a, c \in \mathbb{R} \setminus \{0\}$, $b, d \in \mathbb{R}$, and $\text{sign}(x) = 1$ if $x > 0$ and $\text{sign}(x) = -1$ if $x < 0$; cf. Embrechts et al. (2003). Figure 2.3 shows the estimated evolution of the linear correlation coefficient between the returns of selected pairs of banks. In general, the degree of association between the return series (as e.g. measured by linear correlation) can largely differ. Further, association between financial asset returns is not constant but may change over time.

The linear correlation coefficient is frequently used to measure the amount of association between two random variables since it is quite tractable from a user perspective and has numerous applications; cf. chapter 1. It represents the natural measure of association between random variables with a joint normal or elliptical distribution (provided the second moments exist). As already mentioned in the previous chapter, the linear

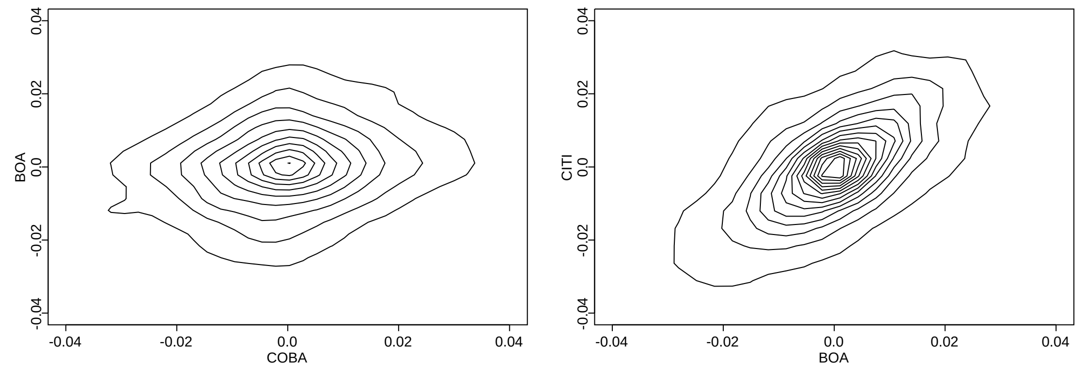


Figure 2.2: Contour plots of the empirical density of the daily returns of the banks BOA and COBA (left panel) and CITI and BOA (right panel) for the observation period May 1997 to April 2010.

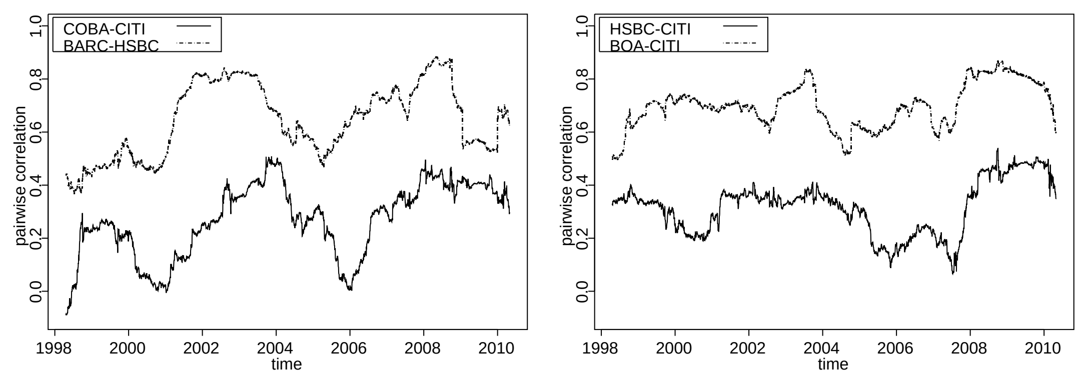


Figure 2.3: Estimated evolution of the linear correlation coefficient between the daily returns of selected pairs of banks for the observation period May 1997 to April 2010, based on a moving window with window size 250.

correlation coefficient is a less appropriate measure of association between two random variables if the underlying distribution is non-elliptical. The theory of copulas allows for a more sophisticated modeling of the dependence structure instead. For example, consider the QQ-Plots in figure 2.4 which graphically compare the empirical quantiles of the returns of banks BNP and BOA with the (theoretical) quantiles of a normal distribution. Together with figure 2.2 (left panel), they give strong evidence that the distributions of the two return series are of different tail behavior. In the context of multivariate distribution modeling, note that all marginal distribution functions of a multivariate elliptical distribution have the same shape (except location and scaling). Copulas allow for the construction of multivariate distribution functions having different marginal distribution functions.

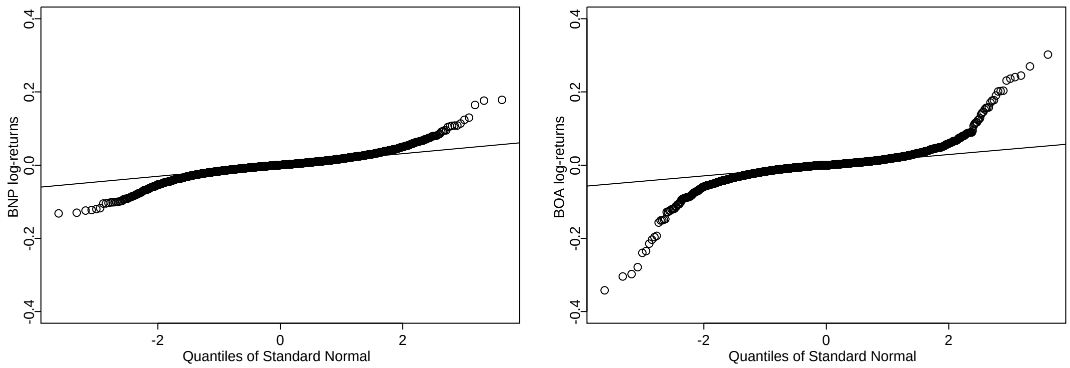


Figure 2.4: QQ-Plots of daily returns of banks BNP and BOA for the observation period May 1997 to April 2010.

As discussed before, measures of association such as Spearman's rho and Kendall's tau represent alternatives to the linear correlation coefficient (see section 2.3.3 for their definition). In contrast to the latter, those measures solely depend on the copula of the underlying random variables and are invariant with respect to the marginal distributions. Figure 2.5 provides contour plots of the empirical densities of realizations from two differently distributed bivariate random vectors. Their distributions have differ-

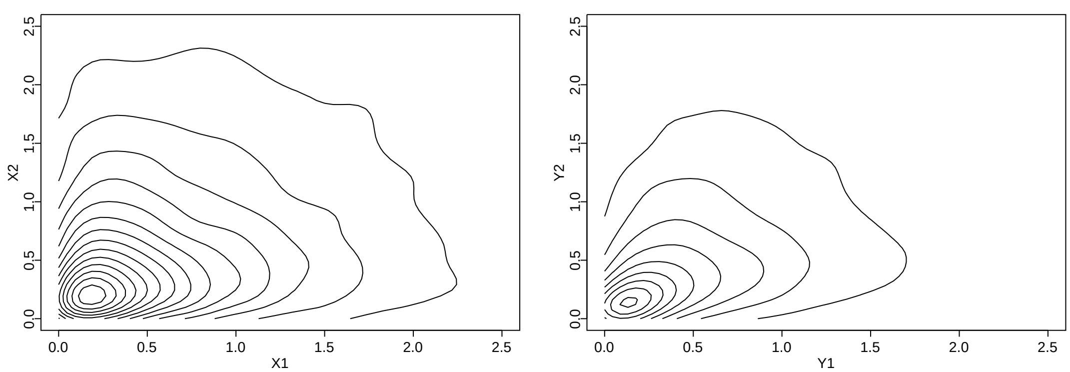


Figure 2.5: Left panel: Contour plot of the empirical density of $10,000$ realizations of a random vector (X_1, X_2) having an equi-correlated Gaussian copula and equally exponentially distributed marginal distributions. Right panel: Contour plot of the empirical density of $10,000$ realizations of a random vector (Y_1, Y_2) having a Clayton copula and equally exponentially distributed marginal distributions.

ent copulas though identical exponentially distributed marginal distribution functions. Though the plots imply that the association between the components of the random vectors differs, this is not reflected by the value of the linear correlation coefficient, which is almost the same in both samples (see table 2.1). For comparison, we additionally give the corresponding values of Spearman's rho and Kendall's tau. In contrast

Table 2.1: Estimated values of the linear correlation coefficient, Spearman's rho, and Kendall's tau of the realizations of the two random vectors (X_1, X_2) and (Y_1, Y_2) as described in figure 2.5.

	Linear correlation	Spearman's rho	Kendall's tau
(X_1, X_2)	0.246	0.293	0.198
(Y_1, Y_2)	0.243	0.403	0.279

to the linear correlation coefficient, both measures indicate a quantitative difference in the degree of association between the components of the random vectors.

Beside the adequate measurement of association between financial asset returns, it is also of interest to analyze and study diversification effects in a portfolio, especially in portfolio theory. For example, it is important to a portfolio manager how the diversification in a portfolio changes if one or several assets are replaced. Such diversification effects can be measured using multivariate versions of Spearman's rho and Kendall's tau. For illustration, figure 2.6 shows the evolution of Spearman's rho and Kendall's tau between the daily returns of two portfolios of five banks each, where the second portfolio results from the first by substituting one bank by another. It becomes apparent that the degree of association in the two distinct portfolios differs in several time periods.

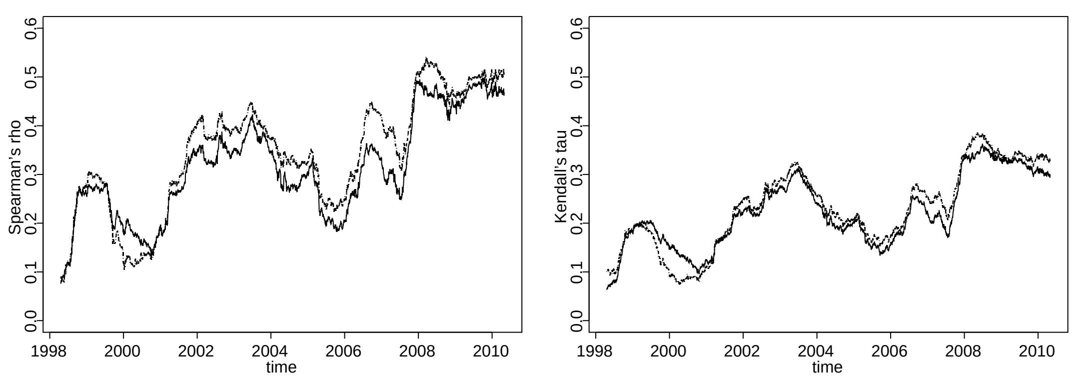


Figure 2.6: Estimated evolution of five-dimensional Spearman's rho (left panel) and Kendall's tau (right panel) of the daily returns of banks COBA, BARC, HSBC, BOA, and CITI (solid) and of the daily returns of banks BNP, COBA, BARC, HSBC, and BOA (dotted) for the observation period May 1997 to April 2010.

Measures of association such as Spearman's rho or Kendall's tau quantify the degree of association between the components of a random vector as determined by its entire distribution function. Another important dependence concept, which we briefly mention for completeness, is tail dependence. Tail dependence is used in the modeling and measurement of association between extreme values such as e.g. extremely negative asset returns and plays a role in financial theory; cf. Joe (1997). In contrast to the mentioned measures of association, measures for tail dependence focus on the tail of the distribution function. For example, the coefficient of upper tail dependence of two random variables X and Y with continuous distribution functions F and G, respectively, is defined as

$$\lambda_U = \lim_{u \to 1-} \mathbb{P}(Y > G^{-1}(u) | X > F^{-1}(u)),$$

provided that the limit $\lambda_U \in (0, 1]$ exists. If $\lambda_U = 0$, then X and Y are said to be asymptotically independent in the upper tail. It can be shown that the tail dependence coefficient is a functional of the copula of the underlying random vector.

2.2 Modeling multivariate association - the concept of copulas

Copulas provide a way to analyze the relationship between a multivariate distribution function and its univariate marginal distribution functions. The notion of copulas has been introduced by Sklar (1959) who showed that copulas are functions which bind or join univariate distribution functions to obtain multivariate distribution functions. In fact, copulas themselves are multivariate distribution functions with univariate marginal distributions being uniform on the interval $[0, 1]$. The study of copulas is of interest in many fields of research and application. In probability and statistics, copulas play an important role basically for the following two reasons: They can be used to construct multivariate distribution functions by modeling each univariate marginal distribution function and the copula separately. Further, copulas allow for a sophisticated analysis and modeling of the association between random variables. Especially in the fields of financial and actuarial sciences, copulas have gained in immense importance over the last two decades as they have opened up many new possibilities to consider association between risky assets.

2.2.1 Definition, properties, and examples

We start with the definition of copulas (see Nelsen (2006), p. 10). Note that although the term copula was established by Sklar (1959), many basic results on copulas can be find earlier in the literature, e.g. in the papers by Hoeffding (see Fisher and Sen (1994) for his collected works).

Definition 2.2.1 *Let $C : [0, 1]^d \to [0, 1]$ be a d-dimensional distribution function on $[0, 1]^d$. Then C is called a copula if it has uniformly distributed univariate marginal distribution functions on the interval* $[0, 1]$.

It immediately follows that all k-dimensional margins of a d-dimensional copula are again copula functions, $2 \leq k \leq d$.

The next theorem gives the representation of a multivariate distribution function in terms of its univariate marginal distribution functions and the copula.

Theorem 2.2.2 (Sklar's theorem) *Let F be a d-dimensional distribution function with univariate marginal distribution functions $F_1, \ldots, F_d$. Then there exists a d-dimensional copula C such that for all $\mathbf{x} = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$,*

$$F(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_n(x_d)\}. \tag{2.2}$$

If $F_1, \ldots, F_d$ are continuous, then C is unique.
Contrary, if C is a d-dimensional copula and $F_1, \ldots, F_d$ are univariate distribution functions, then the right-hand side of (2.2) is a d-dimensional distribution function with univariate marginal distribution functions $F_1, \ldots, F_d$.

The *proof* is given in Sklar (1959). It follows from Sklar's theorem that a multivariate distribution function can be separated into the univariate (continuous) marginal distribution functions and the multivariate dependence structure, which is represented by the copula. Deheuvels (1978) refers to copulas as 'dependence functions'.
For a univariate distribution function G, we define the generalized inverse of G as $G^{-1}(u) = \inf\{x \in I\!R \cup \{\infty\} | G(x) \geq u\}$ for all $u \in (0, 1]$ and $G^{-1}(0) = \sup\{x \in I\!R \cup \{-\infty\} | G(x) = 0\}$.

Corollary 2.2.3 *Let F be a d-dimensional distribution function with univariate marginal distribution functions $F_1, \ldots, F_d$ and corresponding copula C satisfying (2.2). Assuming that $F_1, \ldots, F_d$ are continuous, an explicit representation of C is given by*

$$C(u_1, \ldots, u_d) = F\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\}, \quad \mathbf{u} = (u_1, \ldots, u_d) \in [0, 1]^d. \tag{2.3}$$

This result is a direct consequence from theorem 2.2.2 and is important for the construction of copulas from multivariate distributions. If not stated otherwise, we always assume that the univariate marginal distribution functions $F_1, \ldots, F_d$ are continuous.

Remarks.

1. According to theorem 2.2.7 in Nelsen (2006), the partial derivatives $D_i C(\mathbf{u}) = \partial C(\mathbf{u}) / \partial u_i$ of C exist for almost all $u_i, i = 1, \ldots, d$. As shown in section 2.2.2, weak convergence of the empirical copula process can be established under minimal conditions on those partial derivatives.

2. If a d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$, defined on some probability space $(\Omega, \mathcal{F}, I\!P)$, has distribution function F with univariate marginal distribution functions $F_1, \ldots, F_d$, we also call C as determined by (2.2) the copula of $\mathbf{X}$. It is denoted by $C_{\mathbf{X}}$ or $C_{X_1,\ldots,X_d}$ if necessary.

If the marginal distribution functions $F_1, \ldots, F_d$ of F are continuous, the transformation $X_i \to F_i(X_i)$ is referred to as probability-integral transformation to uniformity since $U_i = F_i(X_i) \sim U(0,1)$ in this case , $i = 1, \dots d$. As implied by (2.2), the copula C (of $\mathbf{X}$ having distribution function F) represents the distribution function of the random vector $\mathbf{U} = (U_1, \dots, U_d)$ and coincides with the copula of the latter random vector. Therefore, many probabilistic investigations concerning copulas can be reduced to the uniform case (see e.g. the proofs of theorems 2.2.8 and 5.2.2 in sections 2.2.2 and 5.2.1, respectively).

We further denote by $\overline{C}$ the survival function of the random vector $\mathbf{U} = (U_1, \dots, U_d)$ whose distribution function is the copula C, i.e.,

$$\overline{C}(\mathbf{u}) = \mathbb{P}(\mathbf{U} > \mathbf{u}) = \mathbb{P}(U_1 > u_1, \dots, U_d > u_d) \quad \text{for all } \mathbf{u} \in [0,1]^d. \tag{2.4}$$

The survival copula is defined as

$$\breve{C}(\mathbf{u}) = \mathbb{P}(\mathbf{U} > \mathbf{1} - \mathbf{u}), \tag{2.5}$$

where $\mathbf{1} - \mathbf{u} = (1 - u_1, \dots, 1 - u_d)$. The copula C is said to be radially symmetric if, and only if, it equals its survival copula, i.e.

$$C(\mathbf{u}) = \mathbb{P}(\mathbf{U} \leq \mathbf{u}) = \mathbb{P}(\mathbf{U} > 1 - \mathbf{u}) = \breve{C}(\mathbf{u}) \text{ for all } \mathbf{u} \in [0,1]^d. \tag{2.6}$$

Every copula is further bounded in the sense that the so called Fréchet-Hoeffding-bounds inequality holds (see e.g. theorem 2.10.12 in Nelsen (2006)): For a d-dimensional copula C, we have

$$W(\mathbf{u}) \leq C(\mathbf{u}) \leq M(\mathbf{u}) \quad \text{for every } \mathbf{u} \in [0,1]^d, \tag{2.7}$$

with functions M and W, defined on $[0,1]^d$ as

$$\begin{aligned} M(u_1, \dots, u_d) &= \min\{u_1, \dots, u_d\} \\ W(u_1, \dots, u_d) &= \max\{u_1 + \dots + u_d - d + 1, 0\}. \end{aligned}$$

The upper bound function M is a d-dimensional copula for all dimensions $d \geq 2$ and is known as the comonotonic copula. If the random vector $\mathbf{X}$ has copula M, each of the random variables $X_1, \dots, X_d$ can (almost surely) be represented as strictly increasing function of any of the others. The copula M is also said to describe perfect positive dependence.
In contrast, the lower bound function W is only a copula for dimension $d = 2$ and is also referred to as the countermonotonic copula in this case. It represents the copula of the bivariate random vector (X_1, X_2) if there exists a strictly decreasing relationship between X_1 and X_2. Here, the copula W describes the case of perfect negative dependence. According to theorem 2.10.13 in Nelsen (2006), there exists for any $d > 2$ and for any $\mathbf{u} \in [0,1]^d$ a d-dimensional copula $C^\star$ such that $C^\star(\mathbf{u}) = W(\mathbf{u})$. The function W in (2.7) represents thus the 'best-possible' lower bound and every dependence structure represented by the copula always lies between those two extreme cases. The fact that W fails to be a copula if $d > 2$ is also closely related to the absence of the concept of perfect negative dependence in this case: For example, if the bivariate random vectors

(X_1, X_2) and (X_2, X_3) are perfectly negatively dependent, respectively, then (X_1, X_3) is perfectly positively dependent and, thus, a three-dimensional perfectly negatively dependent random vector does not exist.

Another important copula is the independence copula Π, defined on $[0,1]^d$ as

$$\Pi(u_1, \dots, u_d) = u_1 \cdot \ldots \cdot u_d.$$

Being a copula for all $d \geq 2$, it describes the dependence structure of stochastically independent random variables $X_1, \dots, X_d$.

The behavior of copulas with respect to strictly monotone transformations is established in the next theorem. Let therefore β_k be a strictly monotone transformation of the kth component X_k of the random vector $\mathbf{X}$ whose domain contains the range of X_k, denoted by $RanX_k$.

Theorem 2.2.4 *Let $\mathbf{X} = (X_1, \dots, X_d)$ be a d-dimensional random vector with distribution function F, continuous marginal distribution functions $F_i, i = 1, \dots, d$, and copula $C_{X_1,\dots,X_d}$.*

(i) If $\beta_1, \dots, \beta_d$ are strictly increasing on $RanX_1, \dots, RanX_d$, respectively, then

$$C_{X_1,\dots,X_d}(\mathbf{u}) = C_{\beta_1(X_1),\dots,\beta_d(X_d)}(\mathbf{u}), \quad \mathbf{u} \in [0,1]^d.$$

(ii) Assume that $\beta_1, \dots, \beta_d$ are strictly monotone on $RanX_1, \dots, RanX_d$, respectively, and let β_k be strictly decreasing for some k, without loss of generality let $k = 1$. Then, for all $\mathbf{u} = (u_1, \dots, u_d) \in [0,1]^d$,

$$\begin{aligned} C_{\beta_1(X_1),\dots,\beta_d(X_d)}(u_1, \dots, u_d) &= C_{\beta_2(X_2),\dots,\beta_d(X_d)}(u_2, \dots, u_d) \\ &\quad - C_{X_1,\beta_2(X_2),\dots,\beta_d(X_d)}(1 - u_1, u_2, \dots, u_d). \end{aligned}$$

For the *proof*, we refer the reader to Embrechts et al. (2003). Part (i) of the theorem implies that the copula is invariant with respect to strictly increasing transformations of the components of $\mathbf{X}$. This behavior together with theorem 2.2.2 forms the basis for the role of copulas in the study of multivariate association. According to Schweizer and Wolff (1981), the copula describes precisely those properties of a joint distribution function which do not change under strictly increasing transformations of the margins. This property is also referred to as 'scale-invariance' since the study of multivariate association based on copulas is, hence, independent of the scale of the margins. Naturally, this is a desirable property of (multivariate) measures of association.
Suppose for the time being that all transformations $\beta_i, i = 1, \dots, d$, in theorem 2.2.4 are strictly decreasing. Applying part (ii) recursively, we obtain the following relationship between the survival copula/survival function and the copula, which we use e.g. in chapter 4:

$$C_{\beta_1(X_1),\dots,\beta_d(X_d)}(\mathbf{1} - \mathbf{u}) = \sum_{A \subseteq S_d} (-1)^{|A|} C(\mathbf{u}^{(A)}) = \breve{C}(\mathbf{1} - \mathbf{u}) = \overline{C}(\mathbf{u}), \qquad (2.8)$$

where $S_d = \{1, \ldots, d\}$ and, in general, $\mathbf{u}^{(A)} = (u_1^{(A)}, \ldots, u_d^{(A)})$ corresponds to the d-dimensional vector $\mathbf{u}$ through $u_j^{(A)} = u_j$ if $j \in A$ and $u_j^{(A)} = 1$ otherwise for all sets $A \subseteq S_d$ with cardinality $0 \leq |A| \leq d$. Note that, if $A = \{i_1, \ldots, i_{|A|}\}$, we also write $\mathbf{u}^{(i_1,\ldots,i_{|A|})}$ instead of $\mathbf{u}^{(A)}$. The first identity on the left-hand side of formula (2.8) is also studied in Wolff (1980), theorem 2. In particular, it holds that $C_{\beta_1(X_1),\ldots,\beta_d(X_d)}$ is independent of the particular choice of the transformations $\beta_i, i = 1, \ldots, d$. For the representation of $\breve{C}$ and $\overline{C}$ in terms of the copula, see also Cherubini et al. (2004), chapter 4.

For more results and background reading on copulas, consult the monographs by Joe (1997) and Nelsen (2006). Regarding their application in finance and risk management, good references are Embrechts et al. (2003), Cherubini et al. (2004), and McNeil et al. (2005). Let us complete this section by briefly discussing two well-known families of copulas, the elliptical and the Archimedean copulas.

Elliptical Copulas. Elliptical copulas are the copulas of the class of elliptical distributions (see Fang et al. (1990) for a discussion of elliptical distributions) and are constructed according to equation (2.3). In the following, we list two important examples.

Examples. (i) The family of the d-dimensional Gaussian copulas is defined as

$$C^G(u_1, \ldots, u_d; K) = \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} (2\pi)^{-\frac{d}{2}} \det(K)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}\mathbf{x}' K^{-1}\mathbf{x}\Big) dx_d \ldots dx_1, \quad (2.9)$$

with $d \times d$ correlation matrix $K = (\kappa_{ij})_{i,j=1,\ldots,d}$ and function Φ denoting the distribution function of the univariate standard normal distribution with generalized inverse function Φ^{-1}. The Gaussian copula C^G is called equi-correlated if $K = K(\kappa) = \kappa \mathbf{1}_d \mathbf{1}_d' + (1-\kappa) I_d$ with parameter κ satisfying $-1/(d-1) < \kappa < 1$. For general $k \in I\!N$, I_k denotes the k-dimensional identity matrix and $\mathbf{1}_k$ and $\mathbf{0}_k$ correspond to the k-dimensional vectors which solely consist of ones or zeroes, respectively.
(ii) The family of d-dimensional t-copulas is given by

$$C^t(u_1, \ldots, u_d; K, \nu) = t_{\nu,K}\{t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\}, \quad (2.10)$$

where $t_{\nu,K}$ denotes the multivariate t-distribution with ν degrees of freedom, location vector zero and correlation matrix $K = (\kappa_{ij})$ (assuming $\nu > 2$) and corresponding univariate marginal distribution function t_ν with generalized inverse function t_ν^{-1}.

Random number generation for elliptical copulas is e.g. described in Embrechts et al. (2003), p. 26-27.

Archimedean Copulas. Consider a continuous and strictly decreasing function $\phi : [0,1] \to [0,\infty]$ such that $\phi(1) = 0$. The function C given by

$$C(u,v) = \phi^{[-1]}\{\phi(u) + \phi(v)\} \tag{2.11}$$

is a bivariate Archimedean copula if and only if ϕ is convex; cf. Nelsen (2006), chapter 4. Here, the function $\phi^{[-1]}$ denotes the pseudo-inverse of ϕ, defined by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) & for \quad 0 \le t \le \phi(0) \\ 0 & for \quad \phi(0) \le t \le \infty \end{cases}$$

The function ϕ is called the generator of the copula C. If $\phi(0) = \infty$, the generator ϕ is is said to be strict and $\phi^{[-1]} = \phi^{-1}$.
The approach in formula (2.11) can naturally be extended to d ($d \ge 2$) dimensions by imposing additional assumptions on ϕ. With continuous, strictly decreasing function ϕ such that $\phi(1) = 0$ and $\phi(0) = \infty$, a d-dimensional Archimedean copula is given by

$$C(u_1, \ldots, u_d) = \phi^{-1}\{\phi(u_1) + \cdots + \phi(u_d)\}$$

if and only if the inverse ϕ^{-1} is completely monotone on $[0,\infty)$, i.e., if it has derivatives of all orders which alternate in sign; formally, $(-1)^k \frac{d^k}{dt^k}\phi^{-1}(t) \ge 0$ for all $t \ge 0$ and all $k \in I\!N$.

Examples. (i) Let $\phi(t) = (-\ln t)^\theta, \theta \ge 1$, which generates the d-dimensional Gumbel family

$$C^{Gu}(u_1, \ldots, u_d; \theta) = \exp\,[-\{(-\ln u_1)^\theta + \cdots + (-\ln u_d)^\theta\}^{1/\theta}]. \tag{2.12}$$

(ii) With $\phi(t) = (t^{-\theta})/\theta, \theta > 0$, we obtain the d-dimensional Clayton family

$$C^{Cl}(u_1, \ldots, u_d; \theta) = (u_1^{-\theta} + \cdots + u_d^{-\theta} - d + 1)^{-1/\theta}. \tag{2.13}$$

A statistical hypothesis test based on the copula-based measure of association Spearman's rho (cf. section 2.3.3) which can be used to verify whether the choice of an Archimedean copula is appropriate in multivariate distribution modeling is discussed in chapter 5. In general, there exist several methods to generate random numbers from a given Archimedean copula; if needed, we use the method proposed by Marshall and Olkin (1988). For the discussion of the general class of hierarchical Archimedean copulas and related random number generation, we refer to Savu and Trede (2008) and Hofert (2008).

2.2.2 Statistical inference: The empirical copula (process)

Depending on the assumptions made on the joint and the univariate marginal distribution functions, we distinguish three general methods for the estimation of copula functions: parametric, semiparametric and nonparametric methods. The parametric and semi-parametric estimation approaches are usually based on maximum-likelihood methods, we mention Genest and Rivest (1993), Genest et al. (1995), Joe and Xu (1996), Joe (2005), Chen and Fan (2006), and Kim et al. (2007). For an overview see also Malevergne and Sornette (2005), chapter 5. In this thesis, we solely consider nonparametric estimation methods for which the joint and the marginal distribution functions are assumed to be unknown. In particular, this method is not exposed to possible misspecifications of the underlying distributions, see Charpentier et al. (2007) for related pitfalls. Nonparametric estimation of copulas was first considered by Rüschendorf (1976) and Deheuvels (1979) who proposed the so called empirical copula as a nonparametric estimator.

Nonparametric estimation

Consider the d-dimensional random vector $\mathbf{X}$ with distribution function F, continuous univariate marginal distribution functions $F_i, i = 1, \ldots, d$, and copula C. Assume that F, C, and F_i are completely unknown and let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a random sample from $\mathbf{X}$. The empirical copula is built in two steps. First, every univariate marginal distribution function F_i is estimated by its univariate empirical distribution function, i.e.,

$$\widehat{F}_{i,n}(x) = \frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{\{X_{ij} \leq x\}} \text{ for } i = 1, \ldots, d \text{ and } x \in \mathbb{R}.$$

The estimated marginal distribution functions are then used to obtain the so called pseudo-observations $\widehat{U}_{ij,n} = \widehat{F}_{i,n}(X_{ij})$ with $\widehat{\mathbf{U}}_{j,n} = (\widehat{U}_{1j,n}, ..., \widehat{U}_{dj,n})$ for $i = 1, ..., d, j = 1, ..., n$. Finally, an estimate of the copula C is given by the empirical distribution function of the sample $\widehat{\mathbf{U}}_{1,n}, \ldots, \widehat{\mathbf{U}}_{n,n}$. The latter is typically called the empirical copula and was introduced by Deheuvels (1979) under the name 'empirical dependence function'.

Definition 2.2.5 *Let $\mathbf{X}$ be a d-dimensional random vector with distribution function F, continuous univariate marginal distribution functions $F_i, i = 1, \ldots, d$, and copula C. Based on a random sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ from $\mathbf{X}$, the empirical copula is defined as*

$$\widehat{C}_n(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\mathbf{1}_{\{\widehat{U}_{ij,n} \leq u_i\}}, \quad \text{for} \quad \mathbf{u} \in [0,1]^d, \tag{2.14}$$

with $\widehat{U}_{ij,n}$ as introduced above.

Since $\widehat{U}_{ij,n} = 1/n(\text{rank of } X_{ij} \text{ in } X_{i1}, ..., X_{in})$, the empirical copula represents a rank-based estimator for the copula C, i.e., only the (normalized) ranks of the observations are included in the estimation. According to definition 2.2.1, the empirical copula itself is a copula. In particular, it is invariant under strictly increasing transformations of the margins (cf. theorem 2.2.4, part (i)) due to the invariance property of the ranks with respect to such transformations. According to Genest and Favre (2007), the ranks associated with the random sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are the statistics that retain the greatest amount of information among all statistics fulfilling this invariance property. For fixed

n, we sometimes suppress the subindex and refer to the pseudo-observations $\widehat{U}_{ij,n}$ as $\widehat{U}_{ij}$ if it is clear from the context.

Weak convergence of the empirical copula process $\sqrt{n}(\widehat{C}_n - C)$ can be established using the functional delta-method, which is introduced next. Note that this technique is also frequently applied in several theoretical considerations later in this thesis, see e.g. the proofs of theorems 4.2.2 and 5.2.2 in sections 4.2.2 and 5.2.1, respectively.

Functional delta-method and Hadamard differentiability

The classical delta-method represents an important technique for deducing the asymptotic distribution of a sequence of transformed random vectors from the asymptotic behavior of the underlying sequence. Let $\ell^\infty([0,1]^d)$ be the space of the collection of all uniformly bounded real-valued functions defined on $[0,1]^d$, equipped with the uniform metric m defined as

$$m(f_1, f_2) = \sup_{\mathbf{t}\in[0,1]^d} |f_1(\mathbf{t}) - f_2(\mathbf{t})|, \quad f_1, f_2 \in \ell^\infty([0,1]^d). \tag{2.15}$$

Assuming that every sample path $\mathbf{u} \to (\widehat{C}_n(\mathbf{u}))(\omega)$ of the empirical copula $\widehat{C}_n$ is a bounded function on $[0,1]^d$, the empirical copula can be viewed as the following map:

$$\widehat{C}_n : \Omega \times [0,1]^d \to \ell^\infty([0,1]^d),$$

i.e. as a random map taking values in the function space $\ell^\infty([0,1]^d)$. In this context, a general version of the delta-method addressing the weak convergence of stochastic processes is needed. This functional delta-method is based on the notion of Hadamard differentiability. Let $\mathbb{D}$ and $\mathbb{E}$ be two metrizable, topological spaces (i.e. vector spaces for which addition and scalar multiplication are continuous operations).

Definition 2.2.6 (Hadamard differentiability) *A map $\phi : \mathbb{D}_\phi \subset \mathbb{D} \to \mathbb{E}$ is called Hadamard-differentiable at $\theta \in \mathbb{D}_\phi$ if there exists a continuous linear map $\phi'_\theta : \mathbb{D} \to \mathbb{E}$ such that*

$$\frac{\phi(\theta + t_n h_n) - \phi(\theta)}{t_n} \longrightarrow \phi'_\theta(h), \quad n \to \infty, \tag{2.16}$$

for all converging sequences $t_n \to 0$ and $h_n \to h$ such that $\theta + t_n h_n \in \mathbb{D}_\phi$ for every n. If ϕ'_θ exists on a subset $\mathbb{D}_0 \subset \mathbb{D}$ only and $h \in \mathbb{D}_0$, the map ϕ is said to be Hadamard-differentiable tangentially to $\mathbb{D}_0$.

The function ϕ'_θ is called the Hadamard derivative of the map ϕ at $\theta \in \mathbb{D}_\phi$. Thereby, the set $\mathbb{D}_\phi$ can be any arbitrary subset of $\mathbb{D}$. For $h \in \mathbb{D}_0$, the derivative $\phi'_\theta(h)$ represents a first order Taylor approximation evaluated at the point θ in direction h. Note that the derivative ϕ'_θ is a continuous function for *fixed* θ, what does not imply that ϕ'_θ is also continuous in θ. For maps $\phi : \mathbb{D}_\phi \subset \mathbb{R}^m \to \mathbb{R}^k$, Hadamard differentiability is equivalent to the usual type of differentiability.

Theorem 2.2.7 (Functional delta-method) *Let $\phi : \mathbb{D}_\phi \subset \mathbb{D} \to \mathbb{E}$ be Hadamard-differentiable at θ tangentially to $\mathbb{D}_0$. Let $X_n : \Omega_n \to \mathbb{D}_\phi$ be maps with $r_n(X_n - \theta) \stackrel{w}{\to} X$ for some sequence of constants $r_n \to \infty$, where X is separable and takes its values in $\mathbb{D}_0$. Then*

$$r_n\{\phi(X_n) - \phi(\theta)\} \stackrel{w}{\longrightarrow} \phi'_\theta(X).$$

with Hadamard derivative ϕ'_θ of ϕ at θ.

A Borel-measurable map $X : \Omega \to \mathbb{D}_0$ is separable if there is a separable, measurable set having probability one under the distribution of X, see van der Vaart and Wellner (1996), p. 17. For a detailed discussion of Hadamard differentiability and related results, we refer to section 3.9 in van der Vaart and Wellner (1996).

Weak convergence of the empirical copula process

Weak convergence of the empirical copula process $\sqrt{n}(\widehat{C}_n - C)$ has been investigated e.g. by Rüschendorf (1976), Gänßler and Stute (1987), van der Vaart and Wellner (1996), and Tsukahara (2005). The following version is established in Fermanian et al. (2004).

Theorem 2.2.8 *Let $\mathbf{X_1}, \ldots, \mathbf{X_n}$ be a random sample from the d-dimensional random vector $\mathbf{X}$ with distribution function F, continuous univariate marginal distribution functions $F_1, \ldots, F_d$, and copula C. Under the assumption that the i-th partial derivatives $D_i C(\mathbf{u})$ of C exist and are continuous for $i = 1, \ldots, d$, we have*

$$\sqrt{n}\{\widehat{C}_n(\mathbf{u}) - C(\mathbf{u})\} \stackrel{w}{\longrightarrow} \mathbb{G}_C(\mathbf{u}).$$

Weak convergence takes place in $\ell^\infty([0,1]^d)$ and

$$\mathbb{G}_C(\mathbf{u}) = \mathbb{B}_C(\mathbf{u}) - \sum_{i=1}^{d} D_i C(\mathbf{u}) \mathbb{B}_C(\mathbf{u}^{(i)}). \tag{2.17}$$

The vector $\mathbf{u}^{(i)}$ denotes the vector where all coordinates, except the ith coordinate of u, are replaced by 1. The process $\mathbb{B}_C$ is a tight centered Gaussian process on $[0,1]^d$ with covariance function

$$E\{\mathbb{B}_C(\mathbf{u})\mathbb{B}_C(\mathbf{v})\} = C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{v}),$$

i.e., $\mathbb{B}_C$ is a d-dimensional Brownian bridge.

For a definition of tightness, see remark 3 on page 32.
Proof of theorem 2.2.8. We outline the single steps of the theorem's proof, which uses the functional delta-method (see theorem 2.2.7).
(i) Let

$$\widehat{F}_n(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^{n} \prod_{i=1}^{d} \mathbf{1}_{\{X_{ij} \le x_i\}}, \quad \mathbf{x} = (x_1, \ldots, x_d) \in \bar{I\!R}^d, \tag{2.18}$$

be the empirical distribution function of the random sample $\mathbf{X_1}, \ldots, \mathbf{X_n}$. Standard results from empirical process theory (see example 2.1.3 in van der Vaart and Wellner (1996)) imply weak convergence of the corresponding empirical process $\sqrt{n}(\widehat{F}_n - F)$ to a d-dimensional Brownian bridge $\mathbb{B}_F$ with covariance function $E\{\mathbb{B}_F(\mathbf{x})\mathbb{B}_F(\mathbf{y})\} = F(\mathbf{x} \wedge \mathbf{y}) - F(\mathbf{x})F(\mathbf{y})$ in $\ell^\infty(\bar{I\!R}^d)$.
(ii) According to van der Vaart and Wellner (1996), p. 389, the copula C of the distribution function F can be represented as a map $\phi : D(\bar{I\!R}^d) \to \ell^\infty([0,1]^d)$ of F via

$$C(\mathbf{u}) = \phi(F)(\mathbf{u}) = F\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\}, \quad \mathbf{u} \in [0,1]^d. \tag{2.19}$$

In general, the space $D(\bar{I\!R}^d)$ comprises all real-valued cadlag functions and the space $C(\bar{I\!R}^d)$ all continuous real-valued functions on $\bar{I\!R}^d$. Both function spaces are equipped with the uniform metric m as defined in (2.15). When inserting the empirical distribution function $\widehat{F}_n$ into (2.19), we obtain the following nonparametric estimator of the copula C :

$$\widetilde{C}_n(\mathbf{u}) = \phi(\widehat{F}_n)(\mathbf{u}) = \widehat{F}_n\{\widehat{F}_{1,n}^{-1}(u_1), \ldots, \widehat{F}_{d,n}^{-1}(u_d)\}. \tag{2.20}$$

(iii) For fixed $0 < p < q < 1$ suppose for the time being that the F_i are continuously differentiable on the intervals $[F_i^{-1}(p) - \varepsilon, F_i^{-1}(q) + \varepsilon]$ for some $\varepsilon > 0$ with strictly positive derivatives $f_i, i = 1, \ldots, d$, and that all partial derivatives $\partial F / \partial x_i, i = 1, \ldots, d$ of F exist and are continuous on the product of these intervals. According to lemma 3.9.28 in van der Vaart and Wellner (1996)), it then follows that the map ϕ in (2.19) is Hadamard-differentiable at F tangentially to $C(\bar{I\!R}^d)$ as a map from $D(\bar{I\!R}^d)$ to $\ell^\infty([p,q]^d)$. To show this, it is used that ϕ can be decomposed as

$$\phi : F \xrightarrow{\phi_1} (F, F_1, \ldots, F_d) \xrightarrow{\phi_2} (F, F_1^{-1}, \ldots, F_d^{-1}) \xrightarrow{\phi_3} F \circ (F_1^{-1}, \ldots, F_d^{-1}).$$

The first map ϕ_1 is Hadamard-differentiable at F tangentially to $C(\bar{I\!R}^d)$ as it is linear and continuous. Its derivative has the form

$$\phi'_{1,F}(h)(x_1, \ldots, x_d) = (h(x_1, \ldots, x_d), h(x_1, \infty, \ldots, \infty), \ldots, h(\infty, \ldots, \infty, x_d)).$$

In particular, note that $\phi'_{1,F}(h) = \phi_1(h)$ for all $h \in C(\bar{I\!R}^d)$.
Further, the second map ϕ_2 is Hadamard-differentiable at $(F, F_1, \ldots, F_d)$ tangentially to $C(\bar{I\!R}^d) \times C([a_1, b_1]) \times \cdots \times C([a_d, b_d])$ according to lemma 3.9.23, part (i), in van der Vaart and Wellner (1996) where $[a_i, b_i] = [F_i^{-1}(p) - \varepsilon, F_i^{-1}(q) + \varepsilon], i = 1, \ldots, d$. Its derivative is given by

$$\begin{aligned} &\phi'_{2,(F,F_1,\ldots,F_d)}(h, h_1, \ldots, h_d)(\mathbf{x}, u_1, \ldots, u_d) \\ &\quad = \left(h(\mathbf{x}), -\left(\frac{h_1}{f_1}\right) \circ F_1^{-1}(u_1), \ldots, -\left(\frac{h_d}{f_d}\right) \circ F_d^{-1}(u_d) \right). \end{aligned}$$

The composition map ϕ_3 is Hadamard-differentiable at $(F, F_1^{-1}, \ldots, F_d^{-1})$ tangentially to $C(\bar{I\!R}^d) \times C([p,q]) \times \cdots \times C([p,q])$ (lemma 3.9.27 in van der Vaart and Wellner (1996)) with derivative

$$\phi'_{3,(F,F_1^{-1},\ldots,F_d^{-1})}(h,g)(\mathbf{u}) = g \circ (F_1^{-1},\ldots,F_d^{-1})(\mathbf{u}) + \left(\frac{\partial F}{\partial x_1}\{F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)\},\ldots,\frac{\partial F}{\partial x_d}\{F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)\}\right)\cdot h(\mathbf{u})$$

Finally, Hadamard differentiability of the map ϕ follows by an application of the chain rule for Hadamard-differentiable functions (lemma 3.9.3 in van der Vaart and Wellner (1996)).

(iv) As a consequence of the functional delta-method (theorem 2.2.7), the process $\sqrt{n}(\widetilde{C}_n - C)$ thus converges weakly in $\ell^\infty([p,q])^d$ to the Gaussian process $\phi'_F(\mathbb{B}_F)$ with

$$\begin{aligned}\phi'_F(h)(\mathbf{u}) &= \phi'_{3,(F,F_1^{-1},\ldots,F_d^{-1})}[\phi'_{2,(F,F_1,\ldots,F_d)}\{\phi'_{1,F}(h)\}](\mathbf{u}) \\ &= h\{F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)\} \\ &\quad -\sum_{i=1}^{d}\frac{\partial F}{\partial x_i}\{F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)\}\frac{h\{\infty,\ldots,\infty,F_i^{-1}(u_i),\infty,\ldots,\infty\}}{f_i\{F_i^{-1}(u_i)\}},\end{aligned} \tag{2.21}$$

cf. van der Vaart and Wellner (1996).

(v) According to Fermanian et al. (2004), it is possible to confine the above analysis to the case when all marginal distributions are uniform on $[0,1]$ by eventually considering the transformed random variables $U_{ij} = F_i(X_{ij})$ with $\mathbf{U}_j = (U_{1j},\ldots,U_{dj})$ for $i = 1,\ldots,d$ and $j = 1,\ldots,n$ (cf. discussions in section 2.2). Namely, with $F^\star = C$ being the distribution function of the random vectors $\mathbf{U}_j, j = 1,\ldots n$ and $\widehat{F}^\star$ the associated empirical distribution function, it can be shown that (lemma 1 in Fermanian et al. (2004))

$$\sqrt{n}(\widetilde{C}_n - C) = \sqrt{n}\{\phi(\widehat{F}) - \phi(F)\} = \sqrt{n}\{\phi(\widehat{F}^\star) - \phi(F^\star)\},$$

with map ϕ as in equation (2.19). By doing so, the above result in (iv) can be extended to obtain weak convergence of $\sqrt{n}(\widetilde{C}_n - C)$ in the space $\ell^\infty([0,1])^d$. Note that the assumption of existing continuous partial derivatives of the copula, as required in the theorem, is sufficient to yield this weak convergence. Moreover, the limiting process in equation (2.21) then coincides with the limiting process $\mathbb{G}_C$ as given in the theorem, cf. formula (2.17).

(vi) Since finally

$$\sup_{\{0\le u_1,\ldots,u_d\le 1\}} |\widetilde{C}_n(u_1,\ldots,u_d) - \widehat{C}_n(u_1,\ldots,u_d)| = O\left(\frac{1}{n}\right), \tag{2.22}$$

weak convergence of the process $\sqrt{n}(\widetilde{C}_n - C)$ implies weak convergence of the empirical copula process $\sqrt{n}(\widehat{C}_n - C)$, as given in the theorem, to the same Gaussian by an application of Slutsky's theorem. This completes the proof. □

Remark. The application of the probability-integral transformation to uniformity yields compact support of the joint and marginal distribution functions of the transformed random variables. In this case, continuous partial differentiability of their joint distribution function $F^\star = C$ is sufficient to obtain Hadamard differentiability of the inverse functions of the marginal distributions in (iii)(cf. lemma 3.9.23 part (ii), in van der Vaart and Wellner (1996) in connection with lemma 2 in Fermanian et al. (2004)). We proceed similarly in the proof of theorem 5.2.2 in section 5.2.1.

The nonparametric estimation of the survival function (see (2.4)), which we consider in chapter 4, can be established analogously. The following result is discussed and proven in Schmid and Schmidt (2007a).

Theorem 2.2.9 *Let $\mathbf{X_1}, \ldots, \mathbf{X_n}$ be a random sample from the d-dimensional random vector $\mathbf{X}$ with distribution function F, continuous marginal distribution functions $F_1, \ldots, F_d$, and copula C. Using the same notation as in theorem 2.2.8, a nonparametric estimator for $\overline{C}$ is given by*

$$\widehat{\overline{C}}_n(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\mathbf{1}_{\{\widehat{U}_{ij,n} > u_i\}}, \quad \textit{for} \quad \mathbf{u} \in [0,1]^d. \tag{2.23}$$

Under the additional assumption that the i-th partial derivatives $D_i\overline{C}(\mathbf{u})$ of $\overline{C}$ exist and are continuous for $i = 1, \ldots, d$, we have

$$\sqrt{n}\{\widehat{\overline{C}}_n(\mathbf{u}) - \overline{C}(\mathbf{u})\} \xrightarrow{w} \mathbb{G}_{\overline{C}}(\mathbf{u}),$$

in $\ell^\infty([0,1]^d)$. Further,

$$\mathbb{G}_{\overline{C}}(\mathbf{u}) = \mathbb{B}_{\overline{C}}(\mathbf{u}) - \sum_{i=1}^{d} D_i\overline{C}(\mathbf{u})\mathbb{B}_C(\mathbf{u}^{(i)}), \tag{2.24}$$

with Brownian Bridge $\mathbb{B}_C$ as defined in theorem 2.2.8. The process $\mathbb{B}_{\overline{C}}$ is a tight centered Gaussian process on $[0,1]^d$ with covariance function

$$E\{\mathbb{B}_{\overline{C}}(\mathbf{u})\mathbb{B}_{\overline{C}}(\mathbf{v})\} = \overline{C}(\mathbf{u} \vee \mathbf{v}) - \overline{C}(\mathbf{u})\overline{C}(\mathbf{v}).$$

Let us complement the above results with some additional remarks.

Remarks.

1. Analogously to (2.20), we may define another version of the empirical survival function by

$$\widetilde{\overline{C}}_n(\mathbf{u}) = \widehat{\overline{F}}_n\{\widehat{F}_{1,n}^{-1}(u_1), \ldots, \widehat{F}_{d,n}^{-1}(u_d)\} = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\mathbf{1}_{\{X_{ij} > \widehat{F}_{i,n}^{-1}(u_i)\}}, \quad \mathbf{u} \in [0,1]^d,$$

with $\overline{\widehat{F}}_n$ being the survival version of the empirical distribution function $\widehat{F}_n$ as given in (2.18), i.e.

$$\overline{\widehat{F}}_n(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\mathbf{1}_{\{X_{ij}>x_i\}}, \quad \mathbf{x}=(x_1,\ldots,x_d)\in I\!R^d.$$

We have $\sup_{\{\mathbf{u}\in[0,1]^d\}}|\widetilde{\overline{C}}_n(\mathbf{u})-\overline{\widehat{C}}_n(\mathbf{u})| = O\left(1/n\right).$

2. Both $\widehat{C}_n$ and $\overline{\widehat{C}}_n$ are strongly consistent estimators for C and $\overline{C}$, respectively, see Schmid and Schmidt (2007c), p. 8.
3. The process $\mathbb{B}_C$ is tight if for every $\varepsilon > 0$ there exists a compact set K such that

$$I\!P(\mathbb{B}_C \notin K) < \varepsilon.$$

Thus, tightness of $\mathbb{B}_C$ and $\mathbb{B}_{\overline{C}}$ implies tightness of $\mathbb{G}_C$ and $\mathbb{G}_{\overline{C}}$, respectively.

The covariance structure of the limiting process $\mathbb{G}_C$ depends on the unknown copula C and can only be calculated explicitly for some special copulas. For example, if $C = \Pi$, direct calculations yield (see e.g. Genest et al. (2007), proposition 2.1)

$$Cov\{\mathbb{G}_C(\mathbf{u}),\mathbb{G}_C(\mathbf{v})\} = \Pi(\mathbf{u}\wedge\mathbf{v}) + \Pi(\mathbf{u})\Pi(\mathbf{v})\left(d-1-\sum_{i=1}^{d}\frac{u_i\wedge v_i}{u_i v_i}\right). \tag{2.25}$$

The nonparametric bootstrap

In general, however, the covariance structure is of complicated form and has to be estimated adequately. In this context, Fermanian et al. (2004) show that the bootstrap methodology can be used to estimate the asymptotic distribution of the empirical copula process $\sqrt{n}(\widehat{C}_n - C)$. The bootstrap was introduced by Efron (1979) and represents a computer-intensive method to estimate or approximate the (unknown) distribution and related functionals such as the mean or the standard error of a given statistic. It has gained significant importance over the last decades, mainly due to the rapidly increasing computer capacities. Before giving the result of Fermanian et al. (2004) in theorem 2.2.10 below, let us briefly describe the general concept of the bootstrap exemplarily for the estimation of the variance of a given statistic (cf. Shao and Tu (1995), chapter 2).
For the time being, let $(\mathbf{X}_j)_{j=1,\ldots,n}$ be a random sample from an unknown d-dimensional distribution function F and let $S_n = S_n(\mathbf{X}_1,\ldots,\mathbf{X}_n)$ be a statistic whose variance, given by

$$\sigma_S^2 = Var(S_n) = \int\left\{S_n(\mathbf{x}_1,\ldots,\mathbf{x}_n) - \int S_n(\mathbf{y}_1,\ldots,\mathbf{y}_n)d\prod_{i=1}^{n}F(\mathbf{y}_i)\right\}^2 d\prod_{i=1}^{n}F(\mathbf{x}_i),$$

shall be estimated. If F were known, the variance of S_n could be approximated using Monte-Carlo simulation by repeatedly drawing observations from F. The idea of

Efron's bootstrap now is to substitute the unknown distribution function F by an adequate estimator. If no further assumptions on F are made, a natural (and sufficient) estimator of F is given by the empirical distribution function $\widehat{F}_n$ of the random sample $(\mathbf{X}_j)_{j=1,\ldots,n}$, see (2.18). Hence, replacing F by $\widehat{F}_n$ in the above equation yields the bootstrap variance estimator of σ_S^2, given by

$$\begin{aligned} v_{Boot} &= \int \left\{ S_n(\mathbf{x}_1, \ldots, \mathbf{x}_n) - \int S_n(\mathbf{y}_1, \ldots, \mathbf{y}_n) d \prod_{i=1}^n \widehat{F}_n(\mathbf{y}_i) \right\}^2 d \prod_{i=1}^n \widehat{F}_n(\mathbf{x}_i) \\ &= Var(S_n(\mathbf{X}_1^B, \ldots, \mathbf{X}_n^B) | \mathbf{X}_1, \ldots, \mathbf{X}_n), \end{aligned} \quad (2.26)$$

where the bootstrap sample $\mathbf{X}_1^B, \ldots, \mathbf{X}_n^B$ denotes an (i.i.d) random sample of size n from $\widehat{F}_n$. Shao and Tu (1995) call the estimator in (2.26) the theoretical form of the bootstrap variance estimator. In general, such theoretical bootstrap estimators are of complicated form and difficult to evaluate. Therefore, a second step is carried out in practice which relies on the approximation of the theoretical bootstrap estimators, here of v_{Boot}, by Monte-Carlo simulation. Since - in contrast to F - the empirical distribution function $\widehat{F}_n$ is known, we can independently draw bootstrap samples $(\mathbf{X}_1^b, \ldots, \mathbf{X}_n^b)$, $b = 1, \ldots, K$, of size n from the empirical distribution function $\widehat{F}_n$ (given the original sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$), compute the corresponding value $S_n^b = S_n(\mathbf{X}_1^b, \ldots, \mathbf{X}_n^b)$ of the statistic and approximate v_{Boot} by the sample variance of those values as follows:

$$v_{Boot}^{(K)} = \frac{1}{K} \sum_{b=1}^{K} \left(S_n^b - \frac{1}{K} \sum_{s=1}^{K} S_n^s \right)^2 .$$

By the strong law of large numbers, it then follows that

$$v_{Boot} = lim_{K \to \infty} v_{Boot}^{(K)} \text{ a.s.}$$

Note that instead of defining the bootstrap sample $(\mathbf{X}_1^b, \ldots, \mathbf{X}_n^b)$ as a random sample from $\widehat{F}_n$, we can equivalently say that the bootstrap sample $(\mathbf{X}_1^b, \ldots, \mathbf{X}_n^b)$ is a random sample of size n drawn with replacement from the original sample $(\mathbf{X}_j)_{j=1,\ldots,n}$; the bootstrap sample thus represents a resampled version from $(\mathbf{X}_j)_{j=1,\ldots,n}$.

Remarks.

1. The bootstrap methodology can also be used to estimate the entire (unknown) distribution of a given statistic. In our example above, an approximation of the distribution of S_n is given by the empirical distribution function of the values $S_n^b = S_n(\mathbf{X}_1^b, \ldots, \mathbf{X}_n^b), b = 1, \ldots, K$. As a by-product, confidence intervals for unknown parameters of the statistic's distribution or associated critical values in the context of hypothesis testing can be obtained by reading off the empirical quantiles of the bootstrap distribution (see also chapter 5). While a number of $K = 250$ bootstrap replications is usually considered large enough to yield adequate estimates of the variance/standard error of a statistic, the number of

bootstrap replications when used for estimating a distribution must be considerably larger; according to Efron and Tibshirani (1993), a number of at least $K = 2500$ is more appropriate in this case.

2. The bootstrap method described above is referred to as the nonparametric bootstrap since no parametric assumptions on the distribution function F are made. If F belongs to a parametric model, for instance $F = F_\theta$ with θ being a vector of unknown parameters, the parametric bootstrap method is commonly applied. In this case, θ is first estimated by an appropriate estimator $\widehat{\theta}_n$ calculated from the original observations and the bootstrap samples are then drawn from $F_{\widehat{\theta}_n}$.

3. Shao and Tu (1995) regard the bootstrap as a mixture of two techniques, the substitution principle (or 'plug-in principle' according to Efron and Tibshirani (1993)) and a numerical approximation. The substitution principle represents a general estimation approach which, for estimating a distribution function F or related functionals, uses the empirical distribution function $\widehat{F}_n$ of a random sample $(\mathbf{X}_j)_{j=1,\dots,n}$ from F as surrogate. For example, the sample mean $1/n \sum_{j=1}^n \mathbf{X}_j$ as estimator for the mean μ_F of F corresponds exactly to the expectation of the random variable $\mathbf{X}$ (having distribution function F) taken with respect to $\widehat{F}_n$.

For an introduction to the bootstrap, see Efron and Tibshirani (1993); for a detailed treatment of the bootstrap theory we refer to the monograph by Shao and Tu (1995).

The study of the bootstrapped empirical copula process is of interest as many statistics, which are considered in this thesis, can be written as a functional of the empirical copula. The related asymptotic behavior can be deduced from the weak convergence properties of the bootstrapped empirical copula process. Given the bootstrap sample $(\mathbf{X}_j^B)_{j=1,\dots,n}$ which is obtained by sampling with replacement from $(\mathbf{X}_j)_{j=1,\dots,n}$, denote by $\widehat{C}_n^B$ the bootstrap version of the empirical copula, i.e., the empirical copula calculated from $(\mathbf{X}_j^B)_{j=1,\dots,n}$ according to equation (2.14). Fermanian et al. (2004) show that the nonparametric bootstrap works, i.e., that the (conditional) distribution of $\sqrt{n}(\widehat{C}_n^B - \widehat{C}_n)$ is an asymptotically consistent estimator of the distribution of $\sqrt{n}(\widehat{C}_n - C)$.

Theorem 2.2.10 *Let $(\mathbf{X}_j^B)_{j=1,\dots,n}$ denote the bootstrap sample which is obtained by sampling from $(\mathbf{X}_j)_{j=1,\dots,n}$ with replacement and let $\widehat{C}_n^B$ be its associated empirical copula calculated according to formula (2.14). Under the assumptions of theorem 2.2.8, the sequence $\sqrt{n}(\widehat{C}_n^B - \widehat{C}_n)$ converges weakly in $\ell^\infty([0,1]^d)$ to the same limiting Gaussian process as the sequence $\sqrt{n}(\widehat{C}_n - C)$ in probability.*

The proof (see Fermanian et al. (2004)) is based on related results by van der Vaart and Wellner (1996) concerning the weak convergence of the bootstrap for general empirical processes. Note that a similar result for the empirical survival function follows from the above theorem by applying the continuous mapping theorem and using relationship (2.8) between the copula and the survival function.

For the applications discussed in this thesis, the nonparametric bootstrap yields satisfactory estimation results. Note that there exist more advanced bootstrap methods such as the weighted or the wild bootstrap, see e.g. Barbe and Bertail (1995) and Davidson and Flachaire (2001).
We also mention another popular resampling method, the (nonparametric) jackknife. Each jackknife sample deletes exactly one observation at a time from the original sample $(\mathbf{X}_j)_{j=1,\ldots,n}$ and is thus based on n determined samples in contrast to the bootstrap. The performance of the jackknife to estimate the variance of an asymptotically normally distributed statistic is explored in chapter 3 by means of a simulation study.

Weak convergence of the empirical copula process for dependent observations

While the nonparametric estimation of the copula for independent observations is well developed in the literature, statistical inference for the copula on the basis of (time-) dependent observations has only been investigated recently. For the sake of completeness, we briefly address the weak convergence properties of the empirical copula process in such a context. Note that, in general, there exist various concepts to describe and quantify temporal dependence, see e.g. Dedecker et al. (2007) for an overview. Our focus here will be on strong mixing sequences (see Bradley (2005) and references therein).

For this purpose, consider the strictly stationary sequence $\{\mathbf{X}_j = (X_{1j}, \ldots, X_{dj})\}_{j\in\mathbb{Z}}$ of d-dimensional random vectors, being defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution function F, continuous univariate marginal distribution functions $F_i, i = 1\ldots, d$, and copula C. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two σ-fields included in $\mathcal{F}$ and define

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A\in\mathcal{A}, B\in\mathcal{B}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

The mixing coefficient $\alpha_{\mathbf{X}}$ associated with the sequence $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ is then given by

$$\alpha_{\mathbf{X}}(r) = \sup_{s\geq 0} \alpha(\mathcal{F}_s, \mathcal{F}^{s+r}), \tag{2.27}$$

where $\mathcal{F}_t = \sigma\{\mathbf{X}_j, j \leq t\}$ and $\mathcal{F}^t = \sigma\{\mathbf{X}_j, j \geq t\}$ denote the σ-fields generated by $\mathbf{X}_j, j \leq t$, and $\mathbf{X}_j, j \geq t$, respectively.

Definition 2.2.11 *Let $\alpha_{\mathbf{X}}$ be the mixing coefficient associated with the strictly stationary sequence $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ as given in (2.27). If*

$$\alpha_{\mathbf{X}}(r) \to 0 \quad \text{for } r \to \infty,$$

the process $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ is said to be strong mixing.

Assume that our observations are realizations of the sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$. Analogously to above, the copula C is estimated by the empirical copula calculated according to formula (2.14). The following theorem establishes weak convergence of the empirical copula process under additional assumptions on the strong mixing coefficient. Note that Fermanian and Scaillet (2003) investigate the asymptotic properties of the smoothed empirical copula in this setting.

Theorem 2.2.12 *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be observations from the strictly stationary strong mixing sequence $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ with coefficient $\alpha_{\mathbf{X}}$ satisfying $\alpha_{\mathbf{X}}(r) = O(r^{-a})$ for some $a > 1$. If the i-th partial derivatives $D_iC(\mathbf{u})$ of C exist and are continuous for $i = 1, ..., d$, we have*

$$\sqrt{n}\{\widehat{C}_n(\mathbf{u}) - C(\mathbf{u})\} \xrightarrow{w} \mathbb{G}^\star(\mathbf{u})$$

in $\ell^\infty([0,1]^d)$. The process $\mathbb{G}^\star$ has the form

$$\mathbb{G}^\star(\mathbf{u}) = \mathbb{B}^\star(\mathbf{u}) - \sum_{i=1}^{d} D_iC(\mathbf{u})\mathbb{B}^\star(\mathbf{u}^{(i)}), \tag{2.28}$$

with tight centered Gaussian process $\mathbb{B}^\star$ in $[0,1]^d$ having covariance function

$$E\{\mathbb{B}^\star(\mathbf{u})\mathbb{B}^\star(\mathbf{v})\} = \sum_{j\in\mathbb{Z}} E\left[\{\mathbf{1}_{\{\mathbf{U}_0\leq\mathbf{u}\}} - C(\mathbf{u})\}\{\mathbf{1}_{\{\mathbf{U}_j\leq\mathbf{v}\}} - C(\mathbf{v})\}\right],$$

where $\mathbf{U}_j = (F_1(X_{1j}), \ldots, F_d(X_{dj})), j \in \mathbb{Z}$.

Proof. Weak convergence of $\sqrt{n}(\widehat{C}_n - C)$ is established analogously as in the proof of theorem 2.2.8 (cf. Fermanian et al. (2004) and Dedecker et al. (2007)) using the functional delta-method (theorem 2.2.7). We thus outline the single steps only briefly. As in the aforementioned proof, we can confine ourselves to the case where the marginal distributions F_i of $F, i = 1, \ldots, d$, are uniform distributions on $[0,1]$ and thus, F has compact support $[0,1]^d$, by considering the random variables $U_{ij} = F_i(X_{ij}), i = 1, \ldots, d, j = 1, \ldots, n$. Further, the functional delta-method is applied based on representation (2.19) of the copula C as Hadamard-differentiable map ϕ of its distribution function F. Rio (2000) shows that under the above assumptions on the mixing coefficient $\alpha_{\mathbf{X}}$, it holds that

$$\sqrt{n}\{\widehat{F}_n(\mathbf{u}) - F(\mathbf{u})\} \xrightarrow{w} \mathbb{B}^\star(\mathbf{u})$$

in $\ell^\infty([0,1]^d)$ with tight centered Gaussian process $\mathbb{B}^\star$. Hence, an application of the functional delta-method yields the weak convergence of $\sqrt{n}\{\phi(\widehat{F}_n) - \phi(F)\}$ to the process $\phi'_F(\mathbb{B}^*) = \mathbb{G}^\star$ where ϕ'_F denotes the derivative of ϕ at F. Since

$$\sup_{\mathbf{u}\in[0,1]^d} |\phi(\widehat{F}_n)(\mathbf{u}) - \widehat{C}_n(\mathbf{u})| = O\left(\frac{1}{n}\right), \tag{2.29}$$

cf. equation (2.22), apply Slutsky's theorem to conclude the proof. □

In contrast to the case of independent observations (cf. theorem 2.2.8), the covariance structure of the limiting process $\mathbb{G}^\star$ in a strong mixing context depends not only on the copula C, but also on the joint distribution of the random vectors $\mathbf{U}_0$ and

$\mathbf{U}_j, j \in \mathbb{Z}$. Further, $\mathbb{G}^\star$ is tight since $\mathbb{B}^\star$ is tight.
Similar results can also be established for β-mixing and weakly dependent sequences of random vectors, see Dedecker et al. (2007) and Doukhan et al. (2009). Note that the nonparametric bootstrap, as introduced in the previous section, is less adequate to estimate the asymptotic covariance structure of the empirical copula process in a (time-) dependent setting as it does not take into account the temporal dependence structure of the observations. In this context, several modified bootstrap approaches have been proposed in the literature such as the (moving) block bootstrap, which we properly introduce in chapter 3. In this section, we also apply the latter theorem to multivariate financial time series most of which are well known to exhibit temporal dependencies such as autocorrelation.

2.3 Measuring multivariate association

The aim of this section is to introduce and discuss some important measures and concepts of multivariate association. All of them share the desirable property of scale-invariance, which, as discussed in section 2.2.1, allows the analysis of multivariate association between the components of a multivariate random vector irrespective of their scale. In the light of theorem 2.2.4, part (i), these concepts and measures can all be represented and described in terms of the copula of the random vector.
For a comprehensive discussion and overview of the following and many further properties, concepts, and measures of association, we refer to Joe (1997) and Nelsen (2006).

2.3.1 Concordance, positive dependence, and comonotonicity

Consider two d-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$ ($d \geq 2$) having distribution function $F_\mathbf{X}$ and $F_\mathbf{Y}$ with copula $C_\mathbf{X}$ and $C_\mathbf{Y}$, respectively, and suppose our aim is to compare the degree of association between the components of $\mathbf{X}$ and those of $\mathbf{Y}$. This could be done e.g. by some measure of multivariate association which assigns a single number to $\mathbf{X}$ and $\mathbf{Y}$ or to their copulas, respectively. In some situations, however, it is of interest to assess whether the components of $\mathbf{X}$ are more dependent than those of $\mathbf{Y}$ according to some dependence ordering. Due to (2.2) and provided that the marginal distribution functions of $\mathbf{X}$ and $\mathbf{Y}$ are continuous, such orderings can be formulated using the copula of the random vectors. The following ordering is referred to as concordance ordering (cf. Joe (1997)):

Definition 2.3.1 *Given two d-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$ with distribution functions $F_\mathbf{X}$ and $F_\mathbf{Y}$ and continuous marginal distribution functions, respectively. Assuming $\mathbf{X}$ has copula $C_\mathbf{X}$ and $\mathbf{Y}$ has copula $C_\mathbf{Y}$, we say that $C_\mathbf{X}$ is smaller than $C_\mathbf{Y}$ (or $C_\mathbf{Y}$ is larger than $C_\mathbf{X}$) if*

$$C_\mathbf{X}(\mathbf{u}) \leq C_\mathbf{Y}(\mathbf{u}) \quad \text{and} \quad \overline{C}_\mathbf{X}(\mathbf{u}) \leq \overline{C}_\mathbf{Y}(\mathbf{u}) \quad \text{for all } \mathbf{u} \in [0,1]^d. \tag{2.30}$$

In short, we write $C_\mathbf{X} \prec C_\mathbf{Y} (C_\mathbf{Y} \succ C_\mathbf{X})$.

If $C_{\mathbf{X}} \prec C_{\mathbf{Y}}$, we also say that $C_{\mathbf{Y}}$ is more concordant than $C_{\mathbf{X}}$ (or $C_{\mathbf{X}}$ is less concordant than $C_{\mathbf{Y}}$). According to Joe (1997), (2.30) intuitively means that the components of $\mathbf{Y}$ are more likely to be simultaneously large (or small) than those of $\mathbf{X}$. The concordance ordering is a partial ordering as not every pair of copulas can be compared in this way. For dimension $d = 2$, (2.30) reduces to

$$C_{\mathbf{X}}(u_1, u_2) \leq C_{\mathbf{Y}}(u_1, u_2) \text{ for all } u_1, u_2 \in [0, 1],$$

since $C_{\mathbf{X}}(u_1, u_2) \leq C_{\mathbf{Y}}(u_1, u_2)$ if and only if $\overline{C}_{\mathbf{X}}(u_1, u_2) \leq \overline{C}_{\mathbf{Y}}(u_1, u_2)$ (cf. Embrechts et al. (2003)). Sets of desirable axioms and properties for a bivariate or multivariate dependence ordering are discussed in Joe (1997).

Remark. Note that (2.30) must hold pointwise, i.e., for each $\mathbf{u} \in [0, 1]^d$, which is usually stronger than comparing the association between the components of $\mathbf{X}$ and $\mathbf{Y}$ based on a single measure of multivariate association.

Closely related to multivariate concordance is the concept of positive and negative (orthant) dependence; see definition 5.6.1 in Nelsen (2006).

Definition 2.3.2 *Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a d-dimensional random vector with distribution function F, continuous univariate marginal distribution functions and copula C.*

1. *$\mathbf{X}$ is positively lower orthant dependent if*

$$C(\mathbf{u}) \geq \Pi(\mathbf{u}) \quad \text{for all } \mathbf{u} \in [0, 1]^d. \tag{2.31}$$

2. *$\mathbf{X}$ is positively upper orthant dependent if*

$$\overline{C}(\mathbf{u}) \geq \overline{\Pi}(\mathbf{u}) \quad \text{for all } \mathbf{u} \in [0, 1]^d. \tag{2.32}$$

3. *$\mathbf{X}$ is positively orthant dependent if both (2.31) and (2.32) hold.*

Negative lower orthant dependence, negative upper orthant dependence, and negative orthant dependence are defined analogously by replacing '$\geq$' with '$\leq$' in equations (2.31) and (2.32). For dimension $d = 2$, positive (negative) upper and lower dependence are the same. Having quadrants rather than orthants in this case, we say that $\mathbf{X} = (X_1, X_2)$ is positively (negatively) quadrant dependent if $C(u_1, u_2) \geq (\leq) \Pi(u_1, u_2)$ for all $(u_1, u_2) \in [0, 1]^2$.
Positive orthant dependence of $\mathbf{X}$ intuitively means that the probability that its components take on large (or small) values simultaneously is at least as large as it would be if the components of $\mathbf{X}$ were independent; see Nelsen (2006). The relationship to multivariate concordance in definition 2.3.1 is straightforward: $\mathbf{X}$ is positively orthant dependent if and only if $\Pi \prec C$.

Recall the Fréchet-Hoeffding bounds inequality in equation (2.7). It states that $C(\mathbf{u}) \leq M(\mathbf{u}), \mathbf{u} \in [0,1]^d$, for any copula C with M being the comonotonic copula. In view of definition 2.3.2, the random vector $\mathbf{X}$ having copula M is thus the most positively lower orthant dependent random vector and the copula of any random vector being positively lower orthant dependent lies between the copulas M and Π. Note further that, for dimension $d = 2$, comonotonicity is the strongest form of concordance and positive (quadrant) dependence, see also related discussions in Embrechts et al. (2002).

2.3.2 Properties of measures of multivariate association

A measure of multivariate association quantifies the degree of association between the components of a d-dimensional ($d \geq 2$) random vector $\mathbf{X}$ with distribution function F and copula C. We think of it as a map

$$\delta : \mathbb{F} \rightarrow \mathbb{R},$$

where $\mathbb{F}$ is a set of d-dimensional distribution functions. Desirable properties of bivariate measures of association are well-established and have been discussed e.g. by Rényi (1959), Lancaster (1963), Schweizer and Wolff (1981), and Scarsini (1984). However, the extension of those properties to the multivariate case is not always straightforward as the study of multivariate association is generally more complex. An example is the fact that - in contrast to perfect positive dependence - the notion of perfect negative dependence does not generalize to the multivariate case (cf. related discussions in section 2.2.1). Several aspects of multivariate measures of association, in particular of measures of concordance, are discussed in Wolff (1980), Joe (1990), Dolati and Úbeda-Flores (2006), and Taylor (2007).
In the following, we confine ourselves to giving a selection of various properties of measures of multivariate association, which are considered desirable in the literature. This list, however, is not intended to be exhaustive.

In the bivariate case, Scarsini (1984) proposes a set of properties to characterize measures of association which are referred to as bivariate measures of concordance. Definitions for multivariate measures of concordance are given in Dolati and Úbeda-Flores (2006) and Taylor (2007). Amongst others, those measures fulfill the following properties (cf. Schmidt (2007)):

P1. δ is well-defined for every d-dimensional $\mathbf{X}$ with continuous margins X_i and is solely determined by the copula C of $\mathbf{X}$,

P2. $-1 \leq \delta_{\mathbf{X}} \leq 1$ and $\delta_{\mathbf{X}} = 1$ if $C = M$,

P3. $\delta_{\mathbf{X}} = \delta_{\pi(\mathbf{X})}$ for any permutation π of the components of $\mathbf{X}$,

P4. if $\mathbf{X}$ has stochastically independent components (i.e., $C = \Pi$), then $\delta_{\mathbf{X}} = 0$,

P5. if $\mathbf{X}$ has copula $C_{\mathbf{X}}$ and $\mathbf{Y}$ copula $C_{\mathbf{Y}}$ such that $C_{\mathbf{X}} \prec C_{\mathbf{Y}}$, then $\delta_{\mathbf{X}} \leq \delta_{\mathbf{Y}}$, P6. if $(\mathbf{X}_n)_{n\in I\!N}$ is a sequence of random vectors with copulas C_n and $\lim_{n\to\infty} C_n(\mathbf{u}) = C(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^d$ and some copula C of the random vector $\mathbf{X}$, then $\lim_{n\to\infty} \delta_{\mathbf{X}_n} = \delta_{\mathbf{X}}$.

Hence, measures of concordance are increasing with respect to the concordance ordering introduced in definition 2.3.1 according to property P5.
Note that a measure of concordance may be zero even though the components of the random vector $\mathbf{X}$ are not stochastically independent. In many applications, however, it would be desirable that the converse of P4 holds, too. In this context, the following properties for multivariate measures of association are considered (see Rényi (1959) and Wolff (1980)):

P2′. $0 \leq \delta_{\mathbf{X}} \leq 1$,

P4′. $\delta_{\mathbf{X}} = 0 \Longleftrightarrow \mathbf{X}$ has stochastically independent components (i.e., $C = \Pi$),

P7. $\delta_{\mathbf{X}} = 1 \Longleftrightarrow C = M$ for $d \geq 3$, and $\delta_{\mathbf{X}} = 1 \Longleftrightarrow C = M$ or $C = W$ if $d = 2$.

If the marginal distributions of $\mathbf{X}$ are continuous, there exist multivariate measures satisfying the properties P1, P2′, P3, P4′, P6, and P7 (cf. chapter 3). While measures of concordance attain their extreme values if the copula is either M or W, the former measures of association have their extremes if the copula is Π (stochastic independence) or M (comonotonicity) for dimension $d \geq 3$. For dimension $d = 2$, they attain the maximal value of one for perfect (positive and negative) dependence, i.e., if C is either M or W. In this case, Lancaster (1963) refers to those measures as measures of dependence.

Properties P2 and P7 imply that, if each component of $\mathbf{X}$ is almost surely a strictly increasing function of any of the others, then $\delta_{\mathbf{X}} = 1$ as $\mathbf{X}$ possess copula M in this case. Further, if $\beta_1, \ldots, \beta_d$ are strictly increasing functions on the range of $X_i, i = 1, \ldots, d$, then $\delta_{\beta_1(X_1),\ldots,\beta_d(X_d)} = \delta_{\mathbf{X}}$ according to property P1 together with theorem 2.2.4, part (i), i.e., the measures inherit the invariance property of the copula under strictly increasing transformations of the margins.

Further properties of measures of multivariate association may be of interest such as e.g. the behavior under strictly monotone transformations of the margins or if an additional independent component is added to $\mathbf{X}$. A comprehensive overview and discussion of properties and characteristics of multivariate measures of association is given in Schmid et al. (2010). In addition to the aforementioned reference, we refer e.g. to Nelsen (1996, 1998, 2002) for a detailed discussion of (multivariate) measures of association and measures of concordance; for further properties of bivariate measures of association, see also Embrechts et al. (2002). Several well-known measures of concordance are discussed in section 2.3.3; a multivariate measure of association satisfying properties P1, P2′, P3, P4′, P6, and P7 is proposed in chapter 3.

2.3.3 Spearman's rho, Kendall's tau, and Blomqvist's beta

In this section, we discuss three well-known measures of association, Spearman's rho, Kendall's tau, and Blomqvist's beta, and describe how they can be generalized to the multivariate case; cf. Schmid et al. (2010). We thereby focus on those multivariate versions which are based on the multivariate dependence structure as represented by the d-dimensional copula of $\mathbf{X}$. Nonparametric estimation and statistical inference for those measures based on the empirical copula is discussed. Since their population versions can be written in terms of the copula of the random vector, the related sample versions represent functionals of the empirical copula and the asymptotic theory can be deduced from the asymptotic behavior of the empirical copula process (cf. theorem 2.2.8).

Note that another type of multivariate measures is given by the average of all (distinct) pairwise bivariate measures.

Spearman's rho

Spearman's rank correlation coefficient (or Spearman's rho) was first studied by Spearman (1904) and represents one of the best-known measures to quantify the degree of association between two random variables. For the two random variables X_1 and X_2 with bivariate distribution function F and continuous univariate margins F_1, F_2, bivariate Spearman's rho is defined as

$$\rho = \frac{Cov\{F_1(X_1), F_2(X_2)\}}{\sqrt{Var\{F_1(X_1)\}}\sqrt{Var\{F_2(X_2)\}}}. \tag{2.33}$$

Assuming X_1 and X_2 have copula C, this is equivalent to

$$\begin{aligned}\rho &= \frac{\int_0^1\int_0^1 u_1u_2dC(u_1,u_2)-\left(\frac{1}{2}\right)^2}{\left(\frac{1}{12}\right)} = 12\int_0^1\int_0^1 C\left(u_1,u_2\right)du_1du_2 - 3\\ &= \frac{\int_{[0,1]^2} C(u_1,u_2)\,du_1du_2 - \int_{[0,1]^2} \mathit{\Pi}(u_1,u_2)\,du_1du_2}{\int_{[0,1]^2} M(u_1,u_2)\,du_1du_2 - \int_{[0,1]^2} \mathit{\Pi}(u_1,u_2)\,du_1du_2},\end{aligned} \tag{2.34}$$

since $\int_{[0,1]^2} M(u_1,u_2)\ du_1du_2 = 1/3$ and $\int_{[0,1]^2} \Pi(u_1,u_2)\ du_1du_2 = 1/4$. Hence, ρ can be interpreted as the normalized average difference between the copula C and the independence copula $\mathit{\Pi}$.

Multivariate extensions of Spearman's rho and their estimation have been discussed e.g. by Ruymgaart and van Zuijlen (1978), Wolff (1980), Joe (1990), Nelsen (1996), Stepanova (2003), and Schmid and Schmidt (2007a). Schmid and Schmidt (2007b) further suggest a related class of multivariate measures of tail dependence based on conditional versions of (multivariate) Spearman's rho. Motivated by equation (2.34), the following multivariate version of ρ can be derived

$$\rho_1 = \frac{\int_{[0,1]^d} C(\mathbf{u})d\mathbf{u} - \int_{[0,1]^d} \mathit{\Pi}(\mathbf{u})d\mathbf{u}}{\int_{[0,1]^d} M(\mathbf{u})d\mathbf{u} - \int_{[0,1]^d} \mathit{\Pi}(\mathbf{u})d\mathbf{u}} = h_\rho(d)\left\{2^d\int_{[0,1]^d} C(\mathbf{u})d\mathbf{u} - 1\right\}, \tag{2.35}$$

with $h_\rho(d) = (d+1)/\{2^d - (d+1)\}$. Using that

$$\frac{\int_{[0,1]^2} C(u_1,u_2)\, du_1 du_2 - \int_{[0,1]^2} \Pi(u_1,u_2)\, du_1 du_2}{\int_{[0,1]^2} M(u_1,u_2)\, du_1 du_2 - \int_{[0,1]^2} \Pi(u_1,u_2)\, du_1 du_2}$$
$$= \frac{\int_{[0,1]^2} u_1 u_2\, dC(u_1,u_2) - \int_{[0,1]^2} u_1 u_2\, d\Pi(u_1,u_2)}{\int_{[0,1]^2} u_1 u_2\, dM(u_1,u_2) - \int_{[0,1]^2} u_1 u_2\, d\Pi(u_1,u_2)},$$

another multivariate version of Spearman's rho can be similarly defined, which is given by

$$\rho_2 = \frac{\int_{[0,1]^d} \Pi(\mathbf{u}) dC(\mathbf{u}) - \int_{[0,1]^d} \Pi(\mathbf{u}) d\Pi(\mathbf{u})}{\int_{[0,1]^d} \Pi(\mathbf{u}) dM(\mathbf{u}) - \int_{[0,1]^d} \Pi(\mathbf{u}) d\Pi(\mathbf{u})} = h_\rho(d) \left\{ 2^d \int_{[0,1]^d} \Pi(\mathbf{u}) dC(\mathbf{u}) - 1 \right\}. \tag{2.36}$$

Nelsen (1996) derives the two versions ρ_1 and ρ_2 from the concept of upper and lower (orthant) dependence; see definition 2.3.2. Recall that the copula C is positively (negatively) lower orthant dependent if $C(\mathbf{u}) \geq (\leq)\ \Pi(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^d$. Hence, ρ_1 can be regarded as a multivariate measure based on the concept of average lower orthant dependence. In a similar fashion, ρ_2 can be viewed as a multivariate measure derived from the concept of average upper orthant dependence since

$$\frac{\int_{[0,1]^d} \Pi(\mathbf{u}) dC(\mathbf{u}) - \int_{[0,1]^d} \Pi(\mathbf{u}) d\Pi(\mathbf{u})}{\int_{[0,1]^d} \Pi(\mathbf{u}) dM(\mathbf{u}) - \int_{[0,1]^d} \Pi(\mathbf{u}) d\Pi(\mathbf{u})} = \frac{\int_{[0,1]^d} \overline{C}(\mathbf{u}) d\Pi(\mathbf{u}) - \int_{[0,1]^d} \overline{\Pi}(\mathbf{u}) d\Pi(\mathbf{u})}{\int_{[0,1]^d} \overline{M}(\mathbf{u}) d\Pi(\mathbf{u}) - \int_{[0,1]^d} \overline{\Pi}(\mathbf{u}) d\Pi(\mathbf{u})}, \tag{2.37}$$

and C is positively (negatively) upper orthant dependent if $\overline{C}(\mathbf{u}) \geq (\leq)\ \overline{\Pi}(\mathbf{u})$, $\mathbf{u} \in [0,1]^d$, with $\overline{C}$ and $\overline{\Pi}$ being the survival functions of C and Π, respectively; see (2.4). Note that equation (2.37) follows from the fact that, for any two d-dimensional copulas C_1 and C_2 with $\mathbf{U}_1 \sim C_1$ and $\mathbf{U}_2 \sim C_2$, it holds that

$$\int_{[0,1]^d} C_1(\mathbf{u}) dC_2(\mathbf{u}) = \int_{[0,1]^d} \mathbb{P}(\mathbf{U}_1 < \mathbf{u}) dC_2(\mathbf{u}) = \mathbb{P}(\mathbf{U}_1 < \mathbf{U}_2) = \mathbb{P}(\mathbf{U}_2 > \mathbf{U}_1)$$
$$= \int_{[0,1]^d} \mathbb{P}(\mathbf{U}_2 > \mathbf{u}) dC_1(\mathbf{u}) = \int_{[0,1]^d} \overline{C}_2(\mathbf{u}) dC_1(\mathbf{u}). \tag{2.38}$$

In particular, ρ_1 and ρ_2 are the same if the copula C is radially symmetric. Nelsen (1996) further considers the average of the two versions, i.e.,

$$\rho_3 = \frac{\rho_1 + \rho_2}{2}, \tag{2.39}$$

which also serves as a basis for our analysis of weighted multivariate measures of association in chapter 4. For $d = 2$, the three versions coincide and reduce to Spearman's rho as given in (2.34).

Statistical inference for $\rho_i,\ i=1,2$, based on the empirical copula is investigated in Schmid and Schmidt (2007a). When replacing the copula C with the empirical copula $\widehat{C}_n$ (see definition 2.2.5), the following nonparametric estimators for $\rho_i,\ i=1,2$, are obtained:

$$\widehat{\rho}_{1,n} = h_\rho(d)\Big\{2^d\int_{[0,1]^d}\widehat{C}_n(\mathbf{u})d\mathbf{u}-1\Big\} = h_\rho(d)\Big\{\frac{2^d}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}(1-\widehat{U}_{ij,n})-1\Big\}, \quad (2.40)$$

$$\widehat{\rho}_{2,n} = h_\rho(d)\Big\{2^d\int_{[0,1]^d}\Pi(\mathbf{u})\,d\widehat{C}_n(\mathbf{u})-1\Big\} = h_\rho(d)\Big\{\frac{2^d}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\widehat{U}_{ij,n}-1\Big\}.$$

Under the assumptions of the theorems 2.2.8 and 2.2.9, it can be shown that

$$\sqrt{n}\,(\widehat{\rho}_{i,n}-\rho_i)\xrightarrow{d} Z_i\sim N(0,\sigma_i^2),\quad n\to\infty,\quad i=1,2.$$

with

$$\sigma_1^2 = 2^{2d}h_\rho(d)^2\int_{[0,1]^d}\int_{[0,1]^d}E\Big\{\mathbb{G}_C(\mathbf{u})\mathbb{G}_C(\mathbf{v})\Big\}d\mathbf{u}d\mathbf{v},$$

$$\sigma_2^2 = 2^{2d}h_\rho(d)^2\int_{[0,1]^d}\int_{[0,1]^d}E\Big\{\mathbb{G}_{\overline{C}}(\mathbf{u})\mathbb{G}_{\overline{C}}(\mathbf{v})\Big\}\,d\mathbf{u}d\mathbf{v},$$

and Gaussian processes $\mathbb{G}_C$ and $\mathbb{G}_{\overline{C}}$ as defined in the aforementioned theorems. Asymptotic normality of $\widehat{\rho}_{3,n}=(\widehat{\rho}_{1,n}+\widehat{\rho}_{2,n})/2$ can analogously be established based on the joint weak convergence of the process $\sqrt{n}(\widehat{C}_n-C,\widehat{\overline{C}}_n-\overline{C})$, see also chapter 4. If the copula C is radially symmetric, it follows that $\sigma_1^2=\sigma_2^2$. For a few copulas of simple form, the asymptotic variances can be explicitly computed, e.g. in the case of stochastic independence (i.e. $C=\Pi$) Schmid and Schmidt (2007a) obtain

$$\sigma_1^2=\sigma_2^2=\frac{(d+1)^2(3(4/3)^d-d-3)}{3(1+d-2^d)^2}.$$

As shown in Schmid and Schmidt (2006), the asymptotic variances can consistently be estimated by the nonparametric bootstrap otherwise (cf. section 2.2.2). Statistical hypothesis tests for the equality of all pairwise Spearman's rho coefficients in a multivariate random vector are developed in chapter 5. Stepanova (2003) and Quessy (2009) investigate statistical hypothesis tests for stochastic independence based on various multivariate versions of Spearman's rho with regard to their asymptotic relative efficiency.

Kendall's tau

Consider the independent and identically distributed bivariate random vectors (X_1,X_2) and (Y_1,Y_2) with distribution function F. The population version of Kendall's tau is defined as the probability of concordance minus the probability of discordance (see Kendall (1938)):

$$\tau = \mathbb{P}\left\{(X_1 - Y_1)(X_2 - Y_2) > 0\right\} - \mathbb{P}\left\{(X_1 - Y_1)(X_2 - Y_2) < 0\right\}. \tag{2.41}$$

If F has the bivariate copula C, this is equal to

$$\tau = 4\int_{[0,1]^2} C(u,v)dC(u,v) - 1, \tag{2.42}$$

see e.g. Nelsen (2006). Multivariate versions of Kendall's tau are discussed in Nelsen (1996, 2002), Joe (1990), and Taylor (2007). Formula (2.42) implies the following multivariate version:

$$\tau = \frac{1}{2^{d-1}-1}\left\{2^d \int_{[0,1]^d} C(\mathbf{u})dC(\mathbf{u}) - 1\right\}. \tag{2.43}$$

A natural nonparametric estimator of τ is given by

$$\begin{aligned}\widehat{\tau}_n =& \frac{1}{2^{d-1}-1}\left\{2^d \int_{[0,1]^d} \widehat{C}_n(\mathbf{u})d\widehat{C}_n(\mathbf{u}) - 1\right\}\\ =& \frac{1}{2^{d-1}-1}\left\{\frac{2^d}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\prod_{i=1}^{d}\mathbf{1}_{\{\widehat{U}_{ij,n}\leq \widehat{U}_{ik,n}\}} - 1\right\},\end{aligned}$$

with empirical copula $\widehat{C}_n$. Nonparametric estimation and statistical inference for τ based on the empirical copula process is the focus of an ongoing work. For further nonparametric statistical analysis of Kendall's tau, which is also frequently considered in the context of tests for stochastic independence, we refer to Barbe et al. (1996), Genest et al. (2002) and references therein.

Blomqvist's beta

A simple measure of association which is commonly referred to as Blomqvist's beta or the medial correlation coefficient was suggested by Blomqvist (1950). With X_1 and X_2 being two continuous random variables having medians $\tilde{x}_1$ and $\tilde{x}_2$, its population version is given by

$$\beta = \mathbb{P}\left\{(X_1 - \tilde{x}_1)(X_2 - \tilde{x}_2) > 0\right\} - \mathbb{P}\left\{(X_1 - \tilde{x}_1)(X_2 - \tilde{x}_2) < 0\right\}.$$

It can be expressed in terms of the copula C of (X_1, X_2) via

$$\begin{aligned}\beta &= 2\mathbb{P}\left\{(X_1 - \tilde{x}_1)(X_2 - \tilde{x}_2) > 0\right\} - 1 = 4C(1/2,1/2) - 1\\ &= \frac{C(1/2,1/2) - \Pi(1/2,1/2) + \overline{C}(1/2,1/2) - \overline{\Pi}(1/2,1/2)}{M(1/2,1/2) - \Pi(1/2,1/2) + \overline{M}(1/2,1/2) - \overline{\Pi}(1/2,1/2)}.\end{aligned} \tag{2.44}$$

Various extensions of Blomqvist's beta to the multivariate case have been considered in Joe (1990), Nelsen (2002), Taskinen et al. (2005), Úbeda-Flores (2005), and Schmid and Schmidt (2007c). The following multivariate version is motivated by equation (2.44):

$$\begin{aligned} \beta &= \frac{C(\mathbf{1/2}) - \Pi(\mathbf{1/2}) + \overline{C}(\mathbf{1/2}) - \overline{\Pi}(\mathbf{1/2})}{M(\mathbf{1/2}) - \Pi(\mathbf{1/2}) + \overline{M}(\mathbf{1/2}) - \overline{\Pi}(\mathbf{1/2})} \\ &= h_\beta(d)\left\{C(\mathbf{1/2}) + \overline{C}(\mathbf{1/2}) - 2^{1-d}\right\}, \end{aligned} \tag{2.45}$$

with $h_\beta(d) := 2^{d-1}/(2^{d-1}-1)$ and $\mathbf{1/2} := (1/2, \ldots, 1/2)$.

A nonparametric estimator of β is given by

$$\widehat{\beta}_n = h_\beta(d)\left\{\widehat{C}_n(\mathbf{1/2}) + \widehat{\overline{C}}_n(\mathbf{1/2}) - 2^{1-d}\right\},$$

when replacing the copula and its survival function by their empirical counterparts, see equations (2.14) and (2.23). Under the assumptions of theorems 2.2.8 and 2.2.9, Schmid and Schmidt (2007c) show that

$$\sqrt{n}(\widehat{\beta}_n - \beta) \xrightarrow{d} Z \sim N(0, \sigma^2), \quad n \to \infty..$$

The variance σ^2 is given by $\sigma^2 = h_\beta(d)^2 E[\{\mathbb{G}_C(\mathbf{1/2}) + \mathbb{G}_{\overline{C}}(\mathbf{1/2})\}^2]$ with Gaussian processes $\mathbb{G}_C$ and $\mathbb{G}_{\overline{C}}$ as defined in the aforementioned theorems. Note that the asymptotic variance can be estimated by the nonparametric bootstrap as described in section 2.2.2, see also Schmid and Schmidt (2007c).

All multivariate measures of association discussed above satisfy properties P1-P6 as defined in the previous section. In the bivariate case, those measures represent measures of concordance according to the definition of Scarsini (1984). In contrast to the linear correlation coefficient, they quantify the degree of monotone association between the components of the random vector in the bivariate case. Their sample versions are based on the ranks of the observations in accordance with the invariance with respect to strictly increasing transformations of the margins.
Note that the asymptotic behavior of the above statistics can alternatively be derived using U-statistics; see e.g. Joe (1990) and Stepanova (2003). For background reading on the theory of U-statistics, consult the monograph by Lee (1990).
We refer to Schmid et al. (2010) and references therein for a more detailed overview of properties of the above measures as well as for other types of measures of multivariate association such as tail dependence or information-based measures. Multivariate extensions of the (copula-based) measures of association Spearman's footrule and Gini's gamma are investigated in Genest et al. (2010) and Behboodian et al. (2007).

Chapter 3

A multivariate version of Hoeffding's Phi-Square

A multivariate measure of association is proposed, which extends the bivariate copula-based measure Phi-Square introduced by Hoeffding (1940). We discuss its analytical properties and calculate its explicit value for some copulas of simple form; a simulation procedure to approximate its value is provided otherwise. A nonparametric estimator for multivariate Phi-Square is derived and its asymptotic behavior is established based on the weak convergence of the empirical copula process both in the case of independent observations and dependent observations from strictly stationary strong mixing sequences. The asymptotic variance of the estimator can be estimated by means of nonparametric bootstrap methods. For illustration, the theoretical results are applied to financial asset return data.

3.1 Preliminaries

Hoeffding (1940) suggests the following measure Φ^2 to quantify the degree of association between the components of a bivariate random vector $\mathbf{X} = (X_1, X_2)$ with copula C :

$$\Phi^2 = 90 \int_{[0,1]^2} \{C(u_1, u_2) - \Pi(u_1, u_2)\}^2 \, du_1 du_2, \tag{3.1}$$

with Π being the independence copula. This measure solely depends on the copula C of the random vector. It is also referred to as 'dependence index' since it attains its lower bound if the random variables X_1 and X_2 are stochastically independent and its upper bound in the case of perfect (positive and negative) dependence, i.e., if there exists a strictly monotone functional relationship between X_1 and X_2 (cf. section 2.3.2).

In this chapter, we suggest a multivariate version of bivariate Hoeffding's Phi-Square as given in (3.1); cf. Gaißer et al. (2010). Like in the bivariate case, this multivariate version is based on a L_2-type distance between the graphs of the copula C of a d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$ and the independence copula Π.

Multivariate Hoeffding's Phi-Square offers a set of properties which are advantageous for many applications. For example, it is zero if and only if the components of $\mathbf{X}$ are stochastically independent. This leads to the construction of statistical tests for multivariate stochastic independence based on Hoeffding's Phi-Square (cf. Genest et al. (2007)). It attains its maximal value in the case of comonotonicity, i.e., when the copula equals the upper Fréchet-Hoeffding bound M. The concept of comonotonicity is of interest, e.g., in actuarial science or finance (see Dhaene et al. (2002a, 2002b)).

A nonparametric estimator $\widehat{\Phi}_n^2$ of multivariate Hoeffding's Phi-Square can be obtained based on the empirical copula. The calculation of the estimator is of low computational complexity, even for large dimension d. Its asymptotic behavior can be derived from the weak convergence of the empirical copula process (theorem 2.2.8). In particular if $C \neq \Pi$, asymptotic normality of $\sqrt{n}(\widehat{\Phi}_n^2 - \Phi^2)$ can be established using the functional delta-method (theorem 2.2.7). When $C = \Pi$, the asymptotic distribution of $\sqrt{n}\widehat{\Phi}_n^2$ is degenerate. In this case, it can be shown that $n\widehat{\Phi}_n^2$ has a non-degenerate limiting distribution. Since it is well-known that financial return series exhibit serial dependencies, we generalize our approach to the case of dependent observations from strictly stationary strong mixing sequences using the weak convergence properties of the empirical copula process in this setting; see theorem 2.2.12. For such (time-) dependent observations, the application of the ordinary nonparametric bootstrap is no longer appropriate to estimate the unknown asymptotic variance of multivariate Hoeffding's Phi-Square as discussed in section 2.2.2. Therefore, we show that the block bootstrap, which was introduced by Künsch (1989) to account for temporal dependence between the observations, can be used instead. This result is deduced from the asymptotic behavior of bootstrapped empirical processes of strictly stationary strong mixing sequences; see Radulović (2002) for an overview. In particular, it enables the construction of (asymptotic) confidence intervals or of simple hypothesis tests for multivariate Hoeffding's Phi-Square.

3.2 Multivariate Hoeffding's Phi-Square

Let $\mathbf{X}$ be a d-dimension random vector with distribution function F, continuous univariate marginal distribution functions $F_i, i = 1, \ldots, d$, and copula C. Motivated by (3.1), we define a multivariate version of Φ^2 by

$$\Phi^2 := \Phi^2(C) = h(d) \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u}, \tag{3.2}$$

with normalization factor

$$h(d) = \left[\int_{[0,1]^d} \{M(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u} \right]^{-1}, \tag{3.3}$$

whose explicit form is calculated later in this section. For dimension $d = 2$, it follows that $h(d) = 90$ and Φ^2 reduces to bivariate Hoeffding's Phi-Square as introduced

in (3.1). In particular, Φ^2 represents a Hadamard-differentiable map on the subset of $l^\infty([0,1]^d)$ for which the integral on the right-hand side of equation (3.2) is well-defined. This relationship is used in section 3.3 where the statistical properties of Φ^2 are derived using the functional delta-method (see proof of theorem 3.3.1). If C is the copula of the random vector $\mathbf{X}$, we also refer to Φ^2 as $\Phi^2_{\mathbf{X}}$.
The measure Φ^2 is one example for a multivariate measure of association which satisfy the properties P1, P2′, P3, P4′, P6, and P7 introduced in section 2.3.2. In particular, it measures association between the components of $\mathbf{X}$ via the squared L_2-distance between the copula and the independence copula and, thus, only takes non-negative values. Some of the aforementioned properties of Φ^2 are discussed in more detail next.

Stochastic independence: The measure Φ^2 fulfills the important property P4′ :

$$\Phi^2 = 0 \quad \text{if and only if} \quad C = \Pi,$$

i.e., a measure's value of 0 implies stochastic independence between the components of the random vector $\mathbf{X}$ in contrast to, e.g., multivariate measures of concordance; cf. section 2.3.2.

Normalization: In order to motivate the choice of $h(d)$ as normalization factor in equation (3.2), we calculate the value of the defining integral for the lower and upper Fréchet-Hoeffding bounds, respectively. For $C = M$, we have

$$h(d)^{-1} = \int_{[0,1]^d} \{M(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u} = \frac{2}{(d+1)(d+2)} - \frac{1}{2^d} \frac{d!}{\prod_{i=0}^{d}\left(i+\frac{1}{2}\right)} + \left(\frac{1}{3}\right)^d, \tag{3.4}$$

and for $C = W$, we obtain

$$g(d)^{-1} = \int_{[0,1]^d} \{W(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u} = \frac{2}{(d+2)!} - 2\sum_{i=0}^{d}\binom{d}{i}(-1)^i \frac{1}{(d+1+i)!} + \left(\frac{1}{3}\right)^d. \tag{3.5}$$

The calculations are outlined in section 3.5.1. Both expressions converge to zero as dimension d tends to infinity. In particular, for $d \geq 3$,

$$\begin{aligned} h(d)^{-1} - g(d)^{-1} &= \frac{2}{(d+1)(d+2)} - \frac{1}{2^d}\frac{d!}{\prod_{i=0}^{d}\left(i+\frac{1}{2}\right)} - \frac{2}{(d+2)!} \\ &\quad + 2\sum_{i=0}^{d}\binom{d}{i}(-1)^i\frac{1}{(d+1+i)!} \\ &= \frac{2d!+2}{(d+2)!} - \frac{1}{2^d}\frac{d!}{\prod_{i=0}^{d}\left(i+\frac{1}{2}\right)} + 2\sum_{i=2}^{d}\binom{d}{i}(-1)^i\frac{1}{(d+1+i)!} \end{aligned} \tag{3.6}$$

$$\geq \frac{2d!+2}{(d+2)!} - \frac{1}{2^d}\frac{d!}{\prod_{i=0}^{d}\left(i+\frac{1}{2}\right)}, \tag{3.7}$$

since the last term in equation (3.6) is non-negative for all $d \geq 3$. Likewise, it can be shown by induction that the expression in equation (3.7) is non-negative in this case. Since $h(2)^{-1} = g(2)^{-1}$, we thus obtain that

$$h(d)^{-1} \geq g(d)^{-1} \quad \text{for all } d \geq 2,$$

such that the range of Φ^2 as defined in (3.2) is restricted to the interval $[0,1]$. That is,

$$0 \leq \Phi^2 \leq 1 \quad \text{for all } \ d \geq 2,$$

and property P2′ is fulfilled.

Comonotonicity: Further, property P7 is satisfied, i.e.,

$$\Phi^2 = 1 \Longleftrightarrow C = M \ \text{ for } \ d \geq 3, \ \text{ and } \ \Phi^2 = 1 \Longleftrightarrow C = M \ \text{ or } \ C = W \ \text{ for } \ d = 2,$$

since $g(2) = h(2) = 90$.

Remark. In view of property P7, bivariate Hoeffding's Phi-Square represents a measure for strictly monotone functional dependence (cf. section 3.1). In consequence, the measure's value of one also implies that the random variables X_1 and X_2 are completely dependent, i.e., that there exists a one-to-one function ψ (which is not necessarily monotone) such that $\mathbb{P}(X_2 = \psi(X_1)) = 1$ (cf. Hoeffding (1942) and Lancaster (1963)). However, the converse does not hold. That is, the value of bivariate Hoeffding's Phi-Square can be made arbitrarily small for completely dependent random variables. For example, two random variables X_1 and X_2 are completely dependent if their copula is a shuffle of M. According to Nelsen (2006), theorem 3.2.2, however, we find shuffles of M which (uniformly) approximate the independence copula arbitrarily closely (see also Mikusinski (1992)).

Continuity: If $\{C_m\}_{m\in\mathbb{N}}$ is a sequence of copulas such that $C_m(\mathbf{u}) \to C(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^d$ and some copula C, then $\Phi^2(C_m) \to \Phi^2(C)$ as a direct consequence of the dominated convergence theorem; cf. property P6.

Invariance with respect to permutations: For every permutation π of the components of $\mathbf{X}$ we have $\Phi^2_{\mathbf{X}} = \Phi^2_{\pi(\mathbf{X})}$ according to Fubini's theorem; cf. property P3.

As mentioned in section 2.3.2, further properties of a measure of multivariate association may be of interest:

Monotonicity: For copulas C_1 and C_2 with $\Pi(\mathbf{u}) \leq C_1(\mathbf{u}) \leq C_2(\mathbf{u}) \leq M(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^d$, we have $\Phi^2(C_1) \leq \Phi^2(C_2)$. For copulas C_3 and C_4 such that

$W(\mathbf{u}) \leq C_3(\mathbf{u}) \leq C_4(\mathbf{u}) \leq \Pi(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^d$, it follows that $\Phi^2(C_3) \geq \Phi^2(C_4)$. This property has also been investigated by Yanagimoto (1970) in the bivariate case.

Behavior with respect to strictly monotone transformations: The behavior of Φ^2 with respect to strictly monotone transformations of one or several components of $\mathbf{X}$ is given in the next proposition.

Proposition 3.2.1 *Let $\mathbf{X}$ be a d-dimensional random vector with distribution function F, continuous univariate marginal distribution functions $F_1, \ldots, F_d$, and copula C.*

(i) For dimension $d \geq 2$, $\Phi^2_{\mathbf{X}}$ is invariant with regard to strictly increasing transformations of one or several components of $\mathbf{X}$.

(ii) For dimension $d = 2$, $\Phi^2_{\mathbf{X}}$ is invariant under strictly decreasing transformations of one or both components of $\mathbf{X}$.
For $d \geq 3$, let β_k be a strictly decreasing transformation of the kth component X_k of $\mathbf{X}$, $k \in \{1, \ldots, d\}$, and let $\boldsymbol{\beta}_k(\mathbf{X}) = (X_1, \ldots, X_{k-1}, \beta_k(X_k), X_{k+1}, \ldots, X_d)$ denote the transformed random vector. Then, $\Phi^2_{\mathbf{X}} = \Phi^2_{\boldsymbol{\beta}_k(\mathbf{X})}$ if one of the following two conditions holds:

- *$(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_d)$ is stochastically independent of X_k, or*
- *$X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_d$ are mutually stochastically independent.*

The *proof* is outlined in section 3.5.3. Note that if part (ii) of the above proposition is not satisfied, equality of $\Phi^2_{\mathbf{X}}$ and $\Phi^2_{\boldsymbol{\alpha}_k(\mathbf{X})}$ does not hold in general for $d \geq 3$.

Irreducibility: Let $\mathcal{C}$ be the set of all $|\mathcal{I}|$-dimensional marginal copulas $C^{\mathcal{I}}$ of C where $\mathcal{I} \subset \{1, \ldots, d\}$ with cardinality $2 \leq |\mathcal{I}| \leq d-1$. For every dimension d and copula C, $\Phi^2(C)$ cannot be expressed as a function of the lower-dimensional Hoeffdings' Phi-Squares $\{\Phi^2(C^{\mathcal{I}})\}_{C^{\mathcal{I}} \in \mathcal{C}}$. A simple example is given by the Farlie-Gumbel-Morgenstern copula, which is defined as

$$C(u_1, ..., u_d) = \prod_{i=1}^{d} u_i + \theta \left\{ \prod_{i=1}^{d} u_i(1 - u_i) \right\}, \quad |\theta| \leq 1. \tag{3.8}$$

Here, every marginal copula $C^{\mathcal{I}}$ of C equals the independence copula for which $\Phi^2(C^{\mathcal{I}}) = 0$ for all $C^{\mathcal{I}} \in \mathcal{C}$ and for all $|\theta| \leq 1$. However, $\Phi^2(C) \neq 0$ for $\theta \neq 0$.

Adding an independent components: The behavior of Hoeffding's Phi-Square in the case that an independent component is added to the d-dimensional random vector $\mathbf{X}$ is investigated next. This may be of interest in portfolio theory when an additional asset is included in the portfolio. Let X_{d+1} be a random variable which is independent of $\mathbf{X}$. Multivariate Hoeffding's Phi-Square of the $(d+1)$-dimensional random vector

$(\mathbf{X}, X_{d+1})$ with copula $\widetilde{C}(u_1, \ldots, u_{d+1}) = C(u_1, \ldots, u_d)u_{d+1}$ has then the form:

$$\begin{aligned}\Phi^2_{(\mathbf{X},X_{d+1})} &= h(d+1)\int_0^1 \int_{[0,1]^d} \left\{ C(\mathbf{u})u_{d+1} - \prod_{i=1}^d u_i u_{d+1} \right\}^2 d\mathbf{u} du_{d+1} \\ &= \frac{1}{3}\,\frac{h(d+1)}{h(d)} \left[h(d) \int_{[0,1]^d} \left\{ C(\mathbf{u}) - \prod_{i=1}^d u_i \right\}^2 d\mathbf{u} \right] \\ &= \frac{1}{3}\,\frac{h(d+1)}{h(d)} \Phi^2_{\mathbf{X}}.\end{aligned}$$

Hence, Hoeffding's Phi-Square changes by a multiplicative factor only which solely depends on the dimension d of $\mathbf{X}$ and which can be determined explicitly (cf. equation (3.4)). Direct calculations further yield that

$$\left| \frac{h(d+1)}{h(d)} \right| < 3 \quad \text{for all } d \geq 2 \text{ and} \quad \lim_{d\to\infty} \left\{ \frac{1}{3}\,\frac{h(d+1)}{h(d)} \right\} = \frac{1}{3},$$

implying that $\Phi^2_{(\mathbf{X},X_{d+1})} < \Phi^2_{\mathbf{X}}$ for all $d \geq 2$ and that $\Phi^2_{(\mathbf{X},X_{d+1})}$ approximates $\Phi^2_{\mathbf{X}}/3$ for large dimension d.

Examples: (i) For the d-dimensional Farlie-Gumbel-Morgenstern copula (equation (3.8)), we have

$$\Phi^2 = h(d)\theta^2 \left(\frac{1}{30} \right)^d, \quad d \geq 2.$$

In particular, $\Phi^2 \leq 1/10$ for $d \geq 2$.
(ii) Let $C(\mathbf{u}) = \theta M(\mathbf{u}) + (1-\theta)\Pi(\mathbf{u})$ with $0 \leqslant \theta \leqslant 1$ for all $\mathbf{u} \in [0,1]^d$. Then

$$\Phi^2 = \theta^2, \quad d \geq 2.$$

Hence, the value of Φ^2 does not depend on the dimension d for this family of copulas.

Remark. The approach of Hoeffding (1940) to define bivariate measures of association based on an adequate notion of distance between the copula and the independence copula has been enhanced by Schweizer and Wolff (1981). In particular, they consider bivariate measures of association based on the L_p-distance for $p = 1$ and $p = \infty$. For the related multivariate case, we refer to Wolff (1980) and Fernández-Fernández and González-Barrios (2004). Based on multivariate Hoeffding's Phi-Square as defined in (3.2), the following multivariate version for $p = 2$ can be defined:

$$\Phi := \Phi(C) = +\sqrt{\Phi^2(C)}.$$

This measure can be interpreted as the normalized average distance between the copula C and the independence copula Π with respect to the L_2-norm. Note that the properties discussed above for Φ^2 can similarly be established for Φ.

If the copula C is of a more complicated structure than in examples (i) and (ii), the value of Φ^2 needs to be determined by simulation. The following equivalent representation of Φ^2 is useful for this purpose:

$$\Phi^2 = h(d) \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u} = h(d) E_{\Pi} \left[\{C(\mathbf{U}) - \Pi(\mathbf{U})\}^2 \right], \tag{3.9}$$

where the random vector $\mathbf{U} = (U_1, \ldots, U_d)$ is uniformly distributed on $[0,1]^d$ with stochastically independent components $U_i, i = 1, \ldots, d$ (which is indicated by the subscript Π). Thus, an approximation of Φ^2 is obtained by estimating the expectation on the right-hand side of equation (3.9) consistently as follows:

$$\widehat{E}_{\Pi} \left[\{C(\mathbf{U}) - \Pi(\mathbf{U})\}^2 \right] = \frac{1}{n} \sum_{i=1}^{n} \{C(\mathbf{U}_i) - \Pi(\mathbf{U}_i)\}^2 \,, \tag{3.10}$$

with $\mathbf{U}_1, \ldots, \mathbf{U}_n$ being independent and identically distributed Monte Carlo replications from $\mathbf{U}$. For illustration we compute the approximated values of Φ^2 and Φ for the equi-correlated Gaussian copula with correlation matrix $K = K(\rho) = \rho \mathbf{1}_d \mathbf{1}_d' + (1-\rho) I_d$ as defined in equation (2.9) for different choices of the parameter ρ and for dimensions $d = 2$, $d = 5$, and $d = 10$; see figure 3.1.

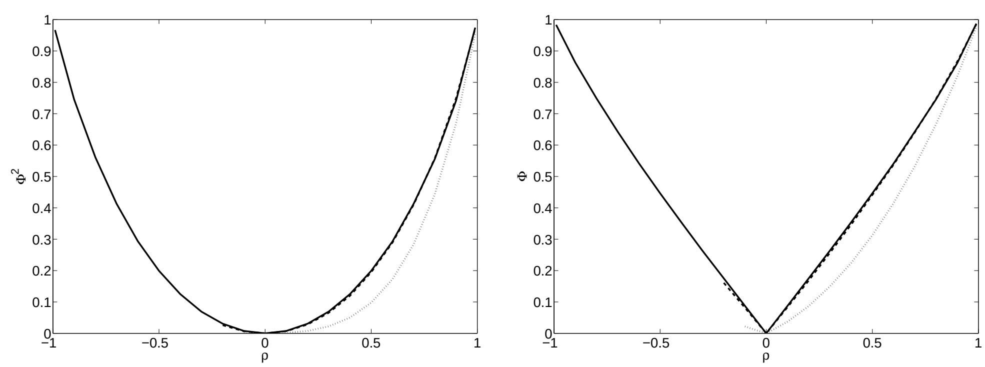


Figure 3.1: Approximated values of Φ^2 (left panel) and Φ (right panel) in the case of a d-dimensional equi-correlated Gaussian copula with parameter ρ for dimension $d = 2$ (solid line), $d = 5$ (dashed line), and $d = 10$ (dotted line); calculations are based on $n = 100,000$ Monte Carlo replications.

3.3 Statistical inference for multivariate Hoeffding's Phi-Square

Statistical inference for multivariate Hoeffding's Phi-Square as introduced in formula (3.2) is based on the empirical copula; see definition 2.2.5. We derive a nonparametric estimator for multivariate Hoeffding's Phi-Square and establish its asymptotic behavior based on the weak convergence of the empirical copula process. After illustrating our approach on the basis of independent observations, we generalize it to the case of dependent observations from strictly stationary strong mixing sequences.

3.3.1 Nonparametric estimation

Consider an (i.i.d.) random sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ from the d-dimensional random vector $\mathbf{X}$ with distribution function F and copula C. We assume that both F and C as well as the (continuous) univariate marginal distribution functions $F_i, i = 1, \ldots, d$, are completely unknown. A nonparametric estimator for Φ^2 is then obtained by replacing the copula C in formula (3.2) by the empirical copula $\widehat{C}_n$ (see (2.14)) i.e.,

$$\widehat{\Phi}_n^2 := \Phi^2(\widehat{C}_n) = h(d) \int_{[0,1]^d} \left\{ \widehat{C}_n(\mathbf{u}) - \Pi(\mathbf{u}) \right\}^2 d\mathbf{u}. \tag{3.11}$$

The estimator is based on a Cramér-von Mises statistic and can explicitly be determined by

$$\widehat{\Phi}_n^2 = h(d) \Big\{ \Big(\frac{1}{n}\Big)^2 \sum_{j=1}^{n} \sum_{k=1}^{n} \prod_{i=1}^{d} (1 - \max\{\widehat{U}_{ij}, \widehat{U}_{ik}\}) - \frac{2}{n} \Big(\frac{1}{2}\Big)^d \sum_{j=1}^{n} \prod_{i=1}^{d} (1 - \widehat{U}_{ij}^2) + \Big(\frac{1}{3}\Big)^d \Big\}. \tag{3.12}$$

The derivation is outlined in section 3.5.2. Obviously, an estimator for the alternative measure Φ is given by $\widehat{\Phi}_n = +\sqrt{\widehat{\Phi}_n^2}$. Asymptotic normality of $\widehat{\Phi}_n^2$ and $\widehat{\Phi}_n$ can be deduced from the asymptotic behavior of the empirical copula process (see theorem 2.2.8).

Theorem 3.3.1 *Under the assumptions of theorem 2.2.8 and if $C \neq \Pi$, it follows that*

$$\sqrt{n}(\widehat{\Phi}_n^2 - \Phi^2) \xrightarrow{d} Z_{\Phi^2} \tag{3.13}$$

where $Z_{\Phi^2} \sim N(0, \sigma_{\Phi^2}^2)$ with

$$\sigma_{\Phi^2}^2 = \{2h(d)\}^2 \int_{[0,1]^d} \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\} E\{\mathbb{G}_C(\mathbf{u}) \mathbb{G}_C(\mathbf{v})\} \{C(\mathbf{v}) - \Pi(\mathbf{v})\} d\mathbf{u} d\mathbf{v}, \tag{3.14}$$

and Gaussian process $\mathbb{G}_C$ as defined in theorem 2.2.8, equation (2.17). Regarding the alternative measure Φ, we have

$$\sqrt{n}(\widehat{\Phi}_n - \Phi) \xrightarrow{d} Z_{\Phi}$$

with $Z_{\Phi} \sim N(0, \sigma_{\Phi}^2)$ and

$$\sigma_{\Phi}^2 = \frac{\sigma_{\Phi^2}^2}{4\Phi^2} = h(d) \frac{\int_{[0,1]^d} \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\} E\{\mathbb{G}_C(\mathbf{u}) \mathbb{G}_C(\mathbf{v})\} \{C(\mathbf{v}) - \Pi(\mathbf{v})\} d\mathbf{u} d\mathbf{v}}{\int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\}^2 d\mathbf{u}}.$$

The *proof* is given in section 3.5.3. Note that the assumption $C \neq \Pi$ guarantees that the limiting random variable is non-degenerated as implied by the form of the variance $\sigma^2_{\Phi^2}$. However, if $C = \Pi$, theorem 2.2.8 and an application of the continuous mapping theorem yield

$$n\widehat{\Phi}^2_n \xrightarrow{d} h(d) \int_{[0,1]^d} \{\mathbb{G}_\Pi(\mathbf{u})\}^2 d\mathbf{u}, \quad \text{as } n \to \infty, \tag{3.15}$$

with

$$E\Big[h(d) \int_{[0,1]^d} \{\mathbb{G}_\Pi(\mathbf{u})\}^2 d\mathbf{u}\Big] = h(d)\Big\{\Big(\frac{1}{2}\Big)^d - \Big(\frac{1}{3}\Big)^d - \frac{d}{6}\Big(\frac{1}{3}\Big)^{d-1}\Big\}.$$

The asymptotic distribution of $\widehat{\Phi}^2_n$ when $C = \Pi$ is important for the construction of tests based on Hoeffding's Phi-Square for stochastic independence between the components of a multivariate random vector. In the bivariate setting, such tests have been studied by Hoeffding (1948) and Blum et al. (1961). Regarding the multivariate case, we mention Genest and Rémillard (2004) and Genest et al. (2007) who consider various combinations of Cramér-von Mises statistics with special regard to their asymptotic local efficiency. In our setting, a hypothesis test for $H_0 : C = \Pi$ against $H_1 : C \neq \Pi$ is performed by rejecting H_0 if the value of $n\widehat{\Phi}^2_n$ exceeds the $(1-\alpha)$-quantile of the limiting distribution in equation (3.15). The latter can be determined by simulation; approximate critical values for the test statistic $\{h(d)\}^{-1}n\widehat{\Phi}^2_n$ are also provided in Genest et al. (2007).

Remark. If the univariate marginal distribution functions F_i are known, Hoeffding's Phi-Square can also be estimated using the theory of U-statistics (cf. the discussions at the end of section 2.3.3). Consider the random variables $U_{ij} = F_i(X_{ij}), i = 1, \ldots, d, j = 1, \ldots, n$ with $\mathbf{U}_j = (U_{1j}, \ldots, U_{dj})$ having distribution function C. Since

$$\begin{aligned}\Phi^2 &= h(d) \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\}^2\, d\mathbf{u} \\ &= \int_{[0,1]^d}\int_{[0,1]^d}\int_{[0,1]^d} h(d)\Big(\prod_{i=1}^d \mathbf{1}_{\{x_i \leq u_i\}} - \prod_{i=1}^d u_i\Big)\Big(\prod_{i=1}^d \mathbf{1}_{\{y_i \leq u_i\}} - \prod_{i=1}^d u_i\Big) d\mathbf{u}dC(\mathbf{x})dC(\mathbf{y}),\end{aligned}$$

an unbiased estimator of the latter based on the random sample $\mathbf{U}_1, \ldots, \mathbf{U}_n$ is given by the U-statistic

$$U_n(\psi) = \binom{n}{2}^{-1} \sum_{1 \leq j < k \leq n} \psi(\mathbf{U_j}, \mathbf{U_k})$$

with kernel ψ of degree 2, defined by

$$\psi(\mathbf{x}, \mathbf{y}) = h(d) \int_{[0,1,]^d} \Big(\prod_{i=1}^d \mathbf{1}_{\{x_i \leq u_i\}} - \prod_{i=1}^d u_i\Big)\Big(\prod_{i=1}^d \mathbf{1}_{\{y_i \leq u_i\}} - \prod_{i=1}^d u_i\Big) d\mathbf{u}, \qquad \mathbf{x}, \mathbf{y} \in [0,1]^d.$$

Results from the theory of U-statistics (see, e.g., Chapter 3 in Lee (1990)) and standard calculations yield that $\sqrt{n}\{U_n(\psi) - \Phi^2\}$ is asymptotically normally distributed with mean zero and variance

$$\begin{aligned}\sigma_U^2 =& Var\{\psi_1(\mathbf{X})\}\\ =& 4\{h(d)\}^2 \int_{[0,1]^d}\int_{[0,1]^d} \{C(\mathbf{u})-\Pi(\mathbf{u})\}\{C(\mathbf{u}\wedge\mathbf{v})-C(\mathbf{u})C(\mathbf{v})\}\{C(\mathbf{v})-\Pi(\mathbf{v})\}d\mathbf{u}d\mathbf{v},\end{aligned}$$

where $\psi_1(\mathbf{x}) = E\{\psi(\mathbf{X},\mathbf{Y})|\mathbf{X}=\mathbf{x}\}$ with independent random vectors $\mathbf{X},\mathbf{Y}$ having distribution function C. The asymptotic variance coincides with the asymptotic variance of $\sqrt{n}(\widehat{\Phi}_n^2-\Phi^2)$ for known marginal distribution functions (cf. equation (3.14)). In particular, $U_n(\psi)$ is degenerate when $C=\Pi$. The fact that both estimators have the same asymptotic distribution in the case of known margins follows also from the relationship

$$\sqrt{n}(\widehat{\Phi}_n^2-\Phi^2) = \frac{1}{n^{3/2}}\sum_{j=1}^{n}\psi(\mathbf{U_j},\mathbf{U_j}) + \sqrt{n}\left\{\frac{n-1}{n}\cdot U_n(\psi)-\Phi^2\right\}$$

where the first term in the right equation converges to zero in probability for $n\to\infty$. In the case of unknown marginal distribution functions, the estimation of Φ^2 by means of U-statistics is more involved in comparison to the above approach based on the empirical copula.

Now let us generalize the previous results to the case of a strong-mixing type of temporal dependence between the observations; cf. section 2.2.2. Assume that $\{\mathbf{X}_j = (X_{1j},\ldots,X_{dj})\}_{j\in\mathbb{Z}}$ is a strictly stationary sequence of d-dimensional random vectors, being defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, with distribution function F, continuous univariate marginal distribution functions F_i, $i=1,\ldots,d$, and copula C. Let $\alpha_{\mathbf{X}}$ denote the mixing coefficient of $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$, which is given by (2.27). In particular, $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ is strong mixing if $\alpha_{\mathbf{X}}(r)\to 0$ for $r\to\infty$ according to definition 2.2.11. We further assume that our observations are realizations of the sample $\mathbf{X}_1,\ldots,\mathbf{X}_n$ and, as before, we denote by $\widehat{\Phi}_n^2$ the corresponding estimator for Hoeffding's Phi-Square calculated according to (3.12). Then, asymptotic normality of $\widehat{\Phi}_n^2$ under additional assumptions on the strong mixing coefficient can be obtained.

Theorem 3.3.2 *Let $\mathbf{X}_1,\ldots,\mathbf{X}_n$ be observations from the strictly stationary strong mixing sequence $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ with mixing coefficient $\alpha_{\mathbf{X}}$ satisfying $\alpha_{\mathbf{X}}(r)=O(r^{-a})$ for some $a>1$. If the i-th partial derivatives $D_iC(\mathbf{u})$ of C exist and are continuous for $i=1,\ldots,d$, and $C\neq\Pi$, we have*

$$\sqrt{n}(\widehat{\Phi}_n^2-\Phi^2) \xrightarrow{d} Z_{\Phi^2}\sim N(0,\sigma_{\Phi^2}^2).$$

The variance is given by

$$\sigma_{\Phi^2}^2 = \{2h(d)\}^2\int_{[0,1]^d}\int_{[0,1]^d} E\Big[\{C(\mathbf{u})-\Pi(\mathbf{u})\}\mathbb{G}^\star(\mathbf{u})\mathbb{G}^\star(\mathbf{v})\{C(\mathbf{v})-\Pi(\mathbf{v})\}\Big]d\mathbf{u}d\mathbf{v} \quad (3.16)$$

with process $\mathbb{G}^\star$ as defined in theorem 2.2.12, equation (2.28).

The result follows from the weak convergence of the empirical copula process for strictly stationary strong mixing sequences to the tight Gaussian process $\mathbb{G}^\star$, as given in theorem 2.2.12, by mimicking the proof of theorem 3.3.1. Note that, in contrast to the case of independent observations (cf. equation (3.14)), the asymptotic variance of multivariate Hoeffding's Phi-Square in equation (3.16) depends also on the joint distribution of $\mathbf{U}_0$ and $\mathbf{U}_j, j \in \mathbb{Z}$. Theorem 3.3.2 can be translated to sequences with temporal dependence structures other than strong mixing using related results on the weak convergence of empirical (copula) processes, see e.g. Arcones and Yu (1994), Dedecker et al. (2007), and Doukhan et al. (2009).

As mentioned in section 2.2.2, the ordinary bootstrap, which draws with replacement n single observations from the sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$, may be inappropriate to approximate the asymptotic variance of $\sqrt{n}(\widehat{\Phi}_n^2 - \Phi^2)$ in the case of dependent observations. Therefore, a modified bootstrap method, the (moving) block bootstrap, has been proposed by Künsch (1989), which is briefly described in the following. Given the sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$, we define blocks of size l, $l < n$, of consecutive observations by

$$B_{s,l} = \{\mathbf{X}_{s+1}, \ldots, \mathbf{X}_{s+l}\}, \quad s = 0, \ldots, n-l.$$

The block bootstrap draws with replacement k blocks from the blocks $B_{s,l}$, $s = 0, \ldots, n-l$ where we assume that $n = kl$ (otherwise the last block is shortened). With $S_1, \ldots, S_k$ being independent and uniformly distributed random variables on $\{0, \ldots, n-l\}$, the bootstrap sample thus comprises those observations from $\mathbf{X}_1, \ldots, \mathbf{X}_n$ which are among the k blocks $B_{S_1,l}, \ldots, B_{S_k,l}$, i.e.,

$$\mathbf{X}_1^B = \mathbf{X}_{S_1+1}, \ldots, \mathbf{X}_l^B = \mathbf{X}_{S_1+l}, \mathbf{X}_{l+1}^B = \mathbf{X}_{S_2+1}, \ldots, \mathbf{X}_n^B = \mathbf{X}_{S_k+l}.$$

The block length l is a function of n, i.e., $l = l(n)$ with $l(n) = o(n)$ and $l(n) \to \infty$ as $n \to \infty$. For a discussion regarding the choice of $l(n)$, see Künsch (1989) and Bühlmann and Künsch (1999). Denote by $\widehat{C}_n^B$ and $\widehat{F}_n^B$ the empirical copula and the empirical distribution function of the block bootstrap sample $\mathbf{X}_1^B, \ldots, \mathbf{X}_n^B$, respectively, and let $\widehat{\Phi}_n^{2,B}$ be the corresponding estimator for Hoeffding's Phi-Square. It follows that the block bootstrap can be applied to estimate the asymptotic variance of $\sqrt{n}(\widehat{\Phi}^2 - \Phi^2)$.

Proposition 3.3.3 *Let $(\mathbf{X}_j^B)_{j=1,\ldots,n}$ be the block bootstrap sample from $(\mathbf{X}_j)_{j=1,\ldots,n}$, which are observations of a strictly stationary, strong mixing sequence $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ of d-dimensional random vectors with distribution function F and copula C whose partial derivatives exist and are continuous. Suppose further that, in probability, $\sqrt{n}(\widehat{C}_n^B - \widehat{C}_n)$ converges weakly in $\ell^\infty([0,1]^d)$ to the same Gaussian limit as $\sqrt{n}(\widehat{C}_n - C)$. If $C \neq \Pi$, the sequences $\sqrt{n}(\widehat{\Phi}_n^2 - \Phi^2)$ and $\sqrt{n}(\widehat{\Phi}_n^{2,B} - \widehat{\Phi}_n^2)$ then converge weakly to the same Gaussian limit in probability.*

The sequence $\sqrt{n}(\widehat{C}_n^B - \widehat{C}_n)$ converges weakly in probability to the same Gaussian limit as $\sqrt{n}(\widehat{C}_n - C)$ if the uniform empirical process $\sqrt{n}(\widehat{F}_n^B - \widehat{F}_n)$ converges weakly in probability to the same Gaussian limit as $\sqrt{n}(\widehat{F}_n - F)$, provided that all partial derivatives of the copula exist and are continuous. This can be seen as follows: Given the map ϕ as defined in (2.19) and due to equation (2.29), it is sufficient to show that $\sqrt{n}\{\phi(\widehat{F}_n^B) - \phi(\widehat{F}_n)\}$ converges weakly in probability in $\ell^\infty([0,1]^d)$ to the process $\phi'_F(\mathbb{B}^\star)$ with process $\mathbb{B}^\star$ as defined in theorem 2.2.12. An equivalent characterization of this weak convergence property (cf. van der Vaart and Wellner (1996), section 3.6, and Giné and Zinn (1990)) is that

$$\sup_{h\in BL_1}\left|E\left(h[\sqrt{n}\{\phi(\widehat{F}_n^B) - \phi(\widehat{F}_n)\}]|\mathbf{X}_1,\ldots,\mathbf{X}_n\right) - E[h\{\phi'_F(\mathbb{B}^\star)\}]\right| \xrightarrow{\mathbb{P}} 0, \quad n\to\infty. \tag{3.17}$$

Here, BL_1 is the set of all functions $h : \ell^\infty([0,1]^d) \to [0,1]$ such that $|h(z_1) - h(z_2)| \le m(z_1,z_2)$ for every $z_1,z_2 \in \ell^\infty([0,1]^d)$ with uniform metric $m(f_1,f_2) = \sup_{\mathbf{t}\in[0,1]^d}|f_1(\mathbf{t}) - f_2(\mathbf{t})|$ as given in (2.15). That formula (3.17) holds can be shown similarly as in the proof of theorem 3.9.11 in van der Vaart and Wellner (1996) using the Hadamard differentiability of the map ϕ at F.
The block bootstrap for empirical processes has been discussed in various settings and for different dependence structures; for an overview see Radulović (2002) and references therein. The following sufficient conditions for $\sqrt{n}(\widehat{F}_n^B - \widehat{F}_n)$ to converge weakly (in the space $D([0,1]^d)$) in probability to the appropriate Gaussian process for strong mixing sequences are derived in Bühlmann (1993):

$$\sum_{r=0}^{\infty}(r+1)^{16(d+1)}\alpha_{\mathbf{X}}^{1/2}(r) < \infty \quad \text{and block length} \quad l(n) = O(n^{1/2-\varepsilon}),\ \varepsilon > 0.$$

The results of a simulation study, which assesses the performance of the bootstrap variance estimator, are presented in section 3.3.2.

Theorem 3.3.2 together with proposition 3.3.3 enables the calculation of an asymptotic $(1-\alpha)$-confidence interval for Hoeffding's Phi-Square $\Phi^2 \in (0,1)$, given by

$$\widehat{\Phi}_n^2 \pm \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\hat{\sigma}^B_{\Phi_n^2}/\sqrt{n}.$$

Here, $(\hat{\sigma}^B_{\Phi_n^2})^2$ denotes the consistent bootstrap variance estimator for $\sigma^2_{\Phi^2}$ in (3.16), obtained by the block bootstrap. Further, an asymptotic hypothesis test for

$$\mathrm{H}_0 : \Phi^2 = \Phi_0^2 \quad \text{against} \quad \mathrm{H}_1 : \Phi^2 \neq \Phi_0^2, \quad \Phi_0^2 \in (0,1),$$

can be constructed by rejecting the null hypothesis at the confidence level α if

$$\left|\sqrt{n}\,\frac{(\hat{\Phi}_n^2 - \Phi_0^2)}{\hat{\sigma}^B_{\Phi_n^2}}\right| > \Phi^{-1}\left(1-\frac{\alpha}{2}\right).$$

Note that in the case $\Phi_0^2 = 1$, the copula corresponds to the upper Fréchet-Hoeffding bound M which does not possess continuous first partial derivatives.

The above results can be extended to statistically analyze the difference of two Hoeffding's Phi-Squares. In a financial context, this may be of interest for assessing whether Hoeffding's Phi-Square of one portfolio of financial assets significantly differs from that of another portfolio (cf. section 3.4). Suppose $\Phi^2_{\mathbf{X}}$ and $\Phi^2_{\mathbf{Y}}$ are multivariate Hoeffding's Phi-Squares associated with the strictly stationary sequences $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ and $\{\mathbf{Y}_j\}_{j\in\mathbb{Z}}$ of d-dimensional random vectors with distribution functions $F_{\mathbf{X}}$ and $F_{\mathbf{Y}}$, continuous marginal distribution functions, and copulas $C_{\mathbf{X}}$ and $C_{\mathbf{Y}}$, respectively. Since the two sequences do not have to be necessarily independent, consider the sequence $\{\mathbf{Z}_j = (\mathbf{X}_j, \mathbf{Y}_j)\}_{j\in\mathbb{Z}}$ of $2d$-dimensional random vectors with joint distribution function $F_{\mathbf{Z}}$, continuous marginal distribution functions $F_{\mathbf{Z},i}, i = 1, \ldots, 2d$, and copula $C_{\mathbf{Z}}$ such that $C_{\mathbf{Z}}(\mathbf{u}, 1 \ldots, 1) = C_{\mathbf{X}}(\mathbf{u})$ and $C_{\mathbf{Z}}(1, \ldots, 1, \mathbf{v}) = C_{\mathbf{Y}}(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in [0,1]^d$.

Theorem 3.3.4 *Let $\mathbf{Z}_1 = (\mathbf{X}_1, \mathbf{Y}_1), \ldots, \mathbf{Z}_n = (\mathbf{X}_n, \mathbf{Y}_n)$ be observations of the strictly stationary, strong mixing sequence $\{\mathbf{Z}_j = (\mathbf{X}_j, \mathbf{Y}_j)\}_{j\in\mathbb{Z}}$ with strong mixing coefficient $\alpha_{\mathbf{Z}}$ satisfying $\alpha_{\mathbf{Z}}(r) = O(r^{-a})$ for some $a > 1$. If the i-th partial derivatives of $C_{\mathbf{Z}}$ exist and are continuous for $i = 1, \ldots, 2d$, and $C_{\mathbf{X}}, C_{\mathbf{Y}} \neq \Pi$, we have*

$$\sqrt{n}\Big\{\hat{\Phi}^2_{\mathbf{X}} - \Phi^2_{\mathbf{X}} - (\hat{\Phi}^2_{\mathbf{Y}} - \Phi^2_{\mathbf{Y}})\Big\} \xrightarrow{d} W \sim N(0, \sigma^2) \quad \text{as } n \to \infty, \tag{3.18}$$

where $\sigma^2 = \sigma^2_{\Phi^2_{\mathbf{X}}} + \sigma^2_{\Phi^2_{\mathbf{Y}}} - 2\sigma_{\Phi^2_{\mathbf{X}},\Phi^2_{\mathbf{Y}}}$ *with*

$$\sigma_{\Phi^2_{\mathbf{X}},\Phi^2_{\mathbf{Y}}} = \{2h(d)\}^2 \int_{[0,1]^d}\int_{[0,1]^d} E\Big[\{C_{\mathbf{X}}(\mathbf{u}) - \Pi(\mathbf{u})\}\mathbb{G}^*_{\mathbf{X}}(\mathbf{u})\mathbb{G}^*_{\mathbf{Y}}(\mathbf{v})\{C_{\mathbf{Y}}(\mathbf{v}) - \Pi(\mathbf{v})\}\Big] d\mathbf{u} d\mathbf{v},$$

and $\sigma^2_{\Phi^2_{\mathbf{X}}} = \sigma_{\Phi^2_{\mathbf{X}},\Phi^2_{\mathbf{X}}}$ *and* $\sigma^2_{\Phi^2_{\mathbf{Y}}} = \sigma_{\Phi^2_{\mathbf{Y}},\Phi^2_{\mathbf{Y}}}$ *(cf. equation (3.16)). The processes* $\mathbb{G}^*_{\mathbf{X}}$ *and* $\mathbb{G}^*_{\mathbf{Y}}$ *are Gaussian processes on* $[0,1]^d$ *as defined in theorem 2.2.12, equation (2.28).*

The *proof* is outlined in section 3.5.3. Analogously to the discussion prior to theorem 3.3.4, an asymptotic confidence interval or a statistical hypothesis test for the difference of two Hoeffding's Phi-Squares can be formulated (cf. section 3.4). For example, under the assumptions of theorem 3.3.4, an (asymptotic) hypothesis test for

$$\mathrm{H}_0: \quad \Phi^2_{\mathbf{X}} = \Phi^2_{\mathbf{Y}} \quad \text{against} \quad \mathrm{H}_1: \quad \Phi^2_{\mathbf{X}} \neq \Phi^2_{\mathbf{Y}} \tag{3.19}$$

is constructed by rejecting H_0 at level α if

$$\left|\sqrt{n}\,\frac{(\hat{\Phi}^2_{\mathbf{X}} - \hat{\Phi}^2_{\mathbf{Y}})}{\hat{\sigma}^B}\right| > \Phi^{-1}\left(1 - \frac{\alpha}{2}\right), \tag{3.20}$$

with $(\hat{\sigma}^B)^2$ being the consistent block bootstrap estimator of the variance σ^2 of the limiting variable W in equation (3.18), obtained by sampling from the sample $\mathbf{Z_1}, \ldots, \mathbf{Z_n}$. In the special case where the sequences $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ and $\{\mathbf{Y}_j\}_{j\in\mathbb{Z}}$ are stochastically independent, we have

$$\sqrt{n}\Big\{\hat{\Phi}^2_{\mathbf{X}} - \Phi^2_{\mathbf{X}} - (\hat{\Phi}^2_{\mathbf{Y}} - \Phi^2_{\mathbf{Y}})\Big\} \xrightarrow{d} W \sim N(0, \sigma^2_{\Phi^2_{\mathbf{X}}} + \sigma^2_{\Phi^2_{\mathbf{Y}}}) \quad \text{as } n \to \infty,$$

with $\sigma^2_{\Phi^2_{\mathbf{X}}}$ and $\sigma^2_{\Phi^2_{\mathbf{Y}}}$ as defined in theorem 3.3.4. An hypothesis test for the hypothesis in (3.19) can be performed analogously.

Remark. The asymptotic distribution of $\widehat{\Phi}_n$ in the case of a strong-mixing type of temporal dependence can be established analogously to the proof of theorem 3.3.1.

3.3.2 Small sample adjustments

The independence copula Π in the definition of $h(d)^{-1}\widehat{\Phi}_n^2$ in equation (3.11) can be replaced by its discrete counterpart $\prod_{i=1}^d U_n(u_i)$ in order to reduce the bias in finite samples, where U_n denotes the (univariate) distribution function of a random variable uniformly distributed on the set $\{\frac{1}{n},\ldots,\frac{n}{n}\}$. This has also been proposed by Genest et al. (2007) in the context of tests for stochastic independence (cf. section 3.3.1). For small samples, we additionally suggest to adjust the normalization factor $h(d)$ in (3.11) to ensure the normalization property of the estimator. In particular, we substitute $h(d)$ by the factor $h(d,n)$, which is obtained by replacing in equation (3.3) the independence copula with $\prod_{i=1}^d U_n(u_i)$ and the upper Fréchet-Hoeffding bound M with its discrete counterpart $M_n(\mathbf{u}) := \min\{U_n(u_1),\ldots,U_n(u_d)\}$, the latter being an adequate upper bound of the empirical copula for given sample size n. A small sample estimator for Φ^2 is then given by

$$\tilde{\Phi}_n^2 = h(d,n)\int_{[0,1]^d}\Big\{\widehat{C}_n(\mathbf{u}) - \prod_{i=1}^d U_n(u_i)\Big\}^2 d\mathbf{u} \tag{3.21}$$

with

$$h(d,n)^{-1} = \int_{[0,1]^d}\Big\{M_n(\mathbf{u}) - \prod_{i=1}^d U_n(u_i)\Big\}^2 d\mathbf{u}.$$

We obtain

$$\begin{aligned}\tilde{\Phi}_n^2 \;=\;& h(d,n)\Big\{\Big(\frac{1}{n}\Big)^2\sum_{j=1}^n\sum_{k=1}^n\prod_{i=1}^d(1-\max\{\widehat{U}_{ij},\widehat{U}_{ik}\})\\ &-\frac{2}{n}\Big(\frac{1}{2}\Big)^d\sum_{j=1}^n\prod_{i=1}^d\Big\{1-\widehat{U}_{ij}^2-\frac{1-\widehat{U}_{ij}}{n}\Big\}+\Big(\frac{1}{3}\Big)^d\Big\{\frac{(n-1)(2n-1)}{2n^2}\Big\}^d\Big\},\end{aligned}$$

with

$$\begin{aligned}h(d,n)^{-1} \;=\;& \Big(\frac{1}{n}\Big)^2\sum_{j=1}^n\sum_{k=1}^n\Big(1-\max\Big\{\frac{j}{n},\frac{k}{n}\Big\}\Big)^d\\ &-\frac{2}{n}\sum_{j=1}^n\Big\{\frac{n(n-1)-j(j-1)}{2n^2}\Big\}^d+\Big(\frac{1}{3}\Big)^d\Big\{\frac{(n-1)(2n-1)}{2n^2}\Big\}^d.\end{aligned}$$

The estimators $\tilde{\Phi}_n^2$ and $\widehat{\Phi}_n^2$ have the same asymptotic distribution, i.e., under the assumptions of theorem 3.3.2 we have

$$\sqrt{n}(\tilde{\Phi}_n^2-\Phi^2)\xrightarrow{d} Z_{\Phi^2}\sim N(0,\sigma_{\Phi^2}^2).$$

This can be shown analogously to the proof of theorem 3.3.2 using the fact that $\lim_{n\to\infty}\sqrt{n}\{h(d,n)-h(d)\} = 0$. Accordingly, it follows that the bootstrap to estimate the asymptotic variance of $\sqrt{n}(\tilde{\Phi}_n^2-\Phi^2)$ works (cf. proposition 3.3.3).

In order to investigate the finite sample performance of the block bootstrap for estimating the asymptotic standard deviation σ_{Φ^2} of $\sqrt{n}(\tilde{\Phi}_n^2 - \Phi^2)$, a simulation study is carried out. For comparative purposes, we additionally provide the corresponding simulation results when using a nonparametric jackknife method to estimate the unknown standard deviation; cf. section 2.2.2. For (time-) dependent observations, Künsch (1989) introduces the delete-l jackknife which is based on systematically deleting one block $B_{s,l}$ of l consecutive observations each time from the original sample, $s = 0, \dots, n-l$. Let $\tilde{\Phi}_n^{2,(s)}$ denote the estimator of Hoeffding's Phi-Square calculated from the original sample where we have deleted block $B_{s,l}, s = 0, \dots, n-l$, and define $\tilde{\Phi}_n^{2,(\cdot)} = (n-l+1)^{-1} \sum_{s=0}^{n-l} \tilde{\Phi}_n^{2,(s)}$. The jackknife estimator of the standard deviation is then given by

$$\hat{\sigma}^J = \sqrt{\frac{(n-l)^2}{nl(n-l+1)} \sum_{s=0}^{n-l} \left(\tilde{\Phi}_n^{2,(s)} - \tilde{\Phi}_n^{2,(\cdot)} \right)^2}.$$

We consider observations from an $AR(1)$-process with autoregressive coefficient β (cf. table 3.2) based on the equi-correlated Gaussian copula as defined in (2.9) with correlation matrix $K = K(\rho) = \rho \mathbf{1}_d \mathbf{1}_d' + (1-\rho) I_d$. To generate these observations, we proceed as follows: Simulate n independent d-dimensional random variates $\mathbf{U}_j = (U_{j1}, \dots, U_{jd}), j = 1, \dots, n$, from the equi-correlated Gaussian copula with parameter ρ. Set $\boldsymbol{\varepsilon}_j = (\Phi^{-1}(U_{j1}), \dots, \Phi^{-1}(U_{jd})), j = 1, \dots, n$. A sample $(\mathbf{X}_j)_{j=1,\dots,n}$ of the $AR(1)$-process is then obtained by setting $\mathbf{X}_1 = \boldsymbol{\varepsilon}_1$ and completing the recursion $\mathbf{X}_j = \beta \mathbf{X}_{j-1} + \boldsymbol{\varepsilon}_j, j = 2, \dots, n$. Additionally, we consider the case of independent observations from the equi-correlated Gaussian copula (cf. table 3.1). To ease comparison, the block bootstrap is used in this case, too.

Tables 3.1 and 3.2 outline the simulation results for dimensions $d = 2, 5$, and 10, sample sizes $n = 50, 100$, and 500, and different choices of the copula parameter ρ. The calculations are based on $1,000$ Monte Carlo simulations of size n and 250 bootstrap replications, respectively. For simplicity, we set the block length $l = 5$ in all simulations. The autoregressive coefficient β of the $AR(1)$-process equals 0.5. The third column of tables 3.1 and 3.2 shows an approximation to the true value of Φ^2, which is calculated from a sample of size $100,000$. Comparing the latter to $m(\tilde{\Phi}_n^2)$ (column 4), we observe a finite-sample bias which depends on the dimension d and the parameter choices, and which decreases with increasing sample size. The standard deviation estimations $s(\tilde{\Phi}_n^2)$ and the empirical means of the block bootstrap estimations, $m(\hat{\sigma}^B)$, as well as the delete-l jackknife estimations, $m(\hat{\sigma}^J)$, for the standard deviation are given in columns 5, 6, and 7. There is a good agreement between their values, especially for the sample sizes $n = 100$ and $n = 500$, implying that the bootstrap and the jackknife procedure to estimate the asymptotic standard deviation of $\sqrt{n}(\tilde{\Phi}_n^2 - \Phi^2)$ perform well for the considered Gaussian copula models. Further, the standard error s of the bootstrap standard deviation estimations is quite small (column 8) and slightly smaller than the obtained jackknife estimates (column 9) in lower dimensions. For large sample size n, however, the jackknife is of a higher computational complexity.

Table 3.1: **Gaussian copula (Independent observations)**. Simulation results for estimating the asymptotic standard deviation of $\sqrt{n}(\tilde{\Phi}_n^2 - \Phi^2)$ by means of the nonparametric block bootstrap with block length l and the delete-l jackknife (for $l = 5$): The table shows the empirical means $m(\cdot)$ and the empirical standard deviations $s(\cdot)$ of the respective estimates, which are calculated based on $1,000$ Monte Carlo simulations of sample size n of a d-dimensional equi-correlated Gaussian copula with parameter ρ and 250 bootstrap samples. The bootstrap estimates are labeled by the superscript B, jackknife estimates by J.

ρ	n	Φ^2	$m(\tilde{\Phi}_n^2)$	$s(\tilde{\Phi}_n^2)$	$m(\hat{\sigma}^B)$	$m(\hat{\sigma}^J)$	$s(\hat{\sigma}^B)$	$s(\hat{\sigma}^J)$
Dimension d=2								
0.2	50	0.032	0.077	0.047	0.055	0.049	0.019	0.026
	100	0.032	0.054	0.034	0.035	0.033	0.012	0.015
	500	0.032	0.035	0.015	0.015	0.015	0.003	0.003
0.5	50	0.197	0.231	0.095	0.089	0.094	0.017	0.022
	100	0.197	0.218	0.070	0.067	0.069	0.010	0.011
	500	0.197	0.202	0.032	0.031	0.032	0.003	0.002
-0.1	50	0.008	0.056	0.035	0.047	0.037	0.015	0.021
	100	0.008	0.032	0.021	0.026	0.022	0.010	0.012
	500	0.008	0.013	0.008	0.008	0.008	0.003	0.003
Dimension d=5								
0.2	50	0.028	0.044	0.023	0.026	0.022	0.010	0.010
	100	0.028	0.036	0.016	0.017	0.015	0.005	0.005
	500	0.028	0.030	0.007	0.007	0.007	0.001	0.001
0.5	50	0.191	0.208	0.065	0.063	0.062	0.013	0.014
	100	0.191	0.202	0.048	0.045	0.046	0.007	0.008
	500	0.191	0.196	0.022	0.021	0.021	0.002	0.002
-0.1	50	0.007	0.015	0.004	0.005	0.004	0.001	0.002
	100	0.007	0.011	0.004	0.003	0.003	0.001	0.001
	500	0.007	0.007	0.002	0.002	0.002	0.000	0.000
Dimension d=10								
0.2	50	0.007	0.014	0.009	0.012	0.008	0.007	0.005
	100	0.007	0.011	0.005	0.007	0.005	0.003	0.003
	500	0.007	0.008	0.002	0.002	0.002	0.001	0.001
0.5	50	0.098	0.111	0.046	0.049	0.043	0.017	0.016
	100	0.098	0.107	0.033	0.035	0.033	0.009	0.009
	500	0.098	0.100	0.015	0.015	0.015	0.002	0.002
-0.1	50	0.001	0.001	0.000	0.000	0.000	0.000	0.000
	100	0.001	0.001	0.000	0.000	0.000	0.000	0.000
	500	0.001	0.001	0.000	0.000	0.000	0.000	0.000

Table 3.2: **Gaussian copula (dependent AR(1) observations)**. Simulation results for estimating the asymptotic standard deviation of $\sqrt{n}(\tilde{\Phi}_n^2 - \Phi^2)$ by means of the nonparametric block bootstrap with block length l and the delete-l jackknife (for $l = 5$): The table shows the empirical means $m(\cdot)$ and the empirical standard deviations $s(\cdot)$ of the respective estimates, which are calculated based on $1,000$ Monte Carlo simulations of sample size n of a d-dimensional equi-correlated Gaussian copula with parameter ρ, AR(1)-processes with standard normal residuals in each margin (with coefficient $\beta = 0.5$ for the first lag) and 250 bootstrap samples. The bootstrap estimates are labeled by the superscript B, jackknife estimates by J.

ρ	n	Φ^2	$m(\tilde{\Phi}_n^2)$	$s(\tilde{\Phi}_n^2)$	$m(\hat{\sigma}^B)$	$m(\hat{\sigma}^J)$	$s(\hat{\sigma}^B)$	$s(\hat{\sigma}^J)$
Dimension d=2								
0.2	50	0.032	0.086	0.059	0.065	0.060	0.022	0.033
	100	0.032	0.059	0.043	0.042	0.040	0.016	0.021
	500	0.032	0.036	0.018	0.017	0.017	0.005	0.005
0.5	50	0.200	0.242	0.114	0.101	0.110	0.022	0.031
	100	0.200	0.222	0.086	0.076	0.081	0.013	0.015
	500	0.200	0.203	0.039	0.037	0.037	0.003	0.003
-0.1	50	0.008	0.067	0.045	0.058	0.050	0.019	0.029
	100	0.008	0.037	0.025	0.032	0.028	0.012	0.016
	500	0.008	0.014	0.010	0.010	0.009	0.004	0.005
Dimension d=5								
0.2	50	0.028	0.047	0.031	0.030	0.026	0.014	0.016
	100	0.028	0.039	0.020	0.020	0.019	0.008	0.008
	500	0.028	0.031	0.009	0.008	0.008	0.002	0.002
0.5	50	0.192	0.212	0.082	0.072	0.074	0.019	0.023
	100	0.192	0.205	0.057	0.053	0.054	0.010	0.011
	500	0.192	0.196	0.026	0.025	0.025	0.003	0.002
-0.1	50	0.007	0.015	0.005	0.006	0.005	0.002	0.002
	100	0.007	0.011	0.004	0.004	0.004	0.001	0.001
	500	0.007	0.008	0.002	0.002	0.002	0.000	0.000
Dimension d=10								
0.2	50	0.007	0.015	0.011	0.013	0.009	0.009	0.008
	100	0.007	0.012	0.007	0.008	0.006	0.005	0.004
	500	0.007	0.008	0.002	0.003	0.002	0.001	0.001
0.5	50	0.099	0.111	0.055	0.053	0.049	0.020	0.023
	100	0.099	0.109	0.042	0.039	0.037	0.014	0.014
	500	0.099	0.100	0.018	0.017	0.017	0.003	0.003
-0.1	50	0.001	0.001	0.000	0.000	0.000	0.000	0.000
	100	0.001	0.001	0.000	0.000	0.000	0.000	0.000
	500	0.001	0.001	0.000	0.000	0.000	0.000	0.000

3.4 Empirical study

The theoretical results are applied to financial data. Specifically, we consider time series of the four major S&P global sector indices Financials, Energy, Industrials, and IT during the period from 1st January 2008 to 8th April 2009 with the aim to analyze the association between their daily (log-)returns before and after the bankruptcy of Lehman Brothers Inc.

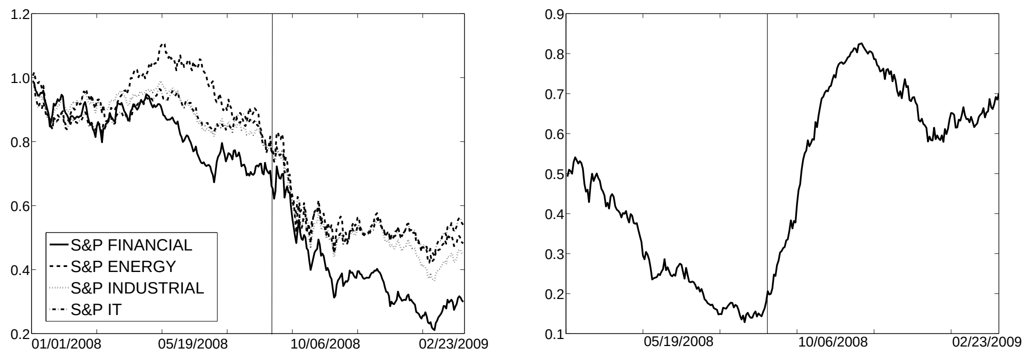


Figure 3.2: Evolution of the S&P global sector indices Financials, Energy, Industrials, and IT with respect to their value on January 1, 2008 (left panel). Estimated evolution of multivariate Hoeffding's Phi-Square $\tilde{\Phi}^2$ of the four indices' returns series, where the estimation is based on a moving window approach with window size 50 (right panel). The vertical line indicates the 15th of September 2008, the day of the bankruptcy of Lehman Brothers Inc.

The evolution of the four indices over the considered time horizon is shown in figure 3.2 (left panel). All series are plotted with respect to their value on January 1, 2008, to ease comparison. The vertical line indicates the 15th of September 2008, the day of the bankruptcy of Lehman Brothers Inc. All series decrease in mid 2008 in the course of deteriorating financial markets; they decline especially sharply after the bankruptcy of Lehman Brothers Inc. Table 3.3 reports the first four moments of the daily returns of the four indices as well as the related results of the Jarque-Bera (JB) test, calculated over the entire time horizon. In addition, the last two rows of the table display the results of the Ljung-Box (LB) Q-statistics, computed from the squared returns of the indices up to lag twenty. All return series show skewness and excess kurtosis. The Jarque-Bera (JB) test rejects the null hypothesis of normality at all standard levels of significance. Further, all squared returns show significant serial correlation as indicated by the Ljung-Box (LB) test, which rejects the null hypothesis of no serial correlation. We fit an ARMA(1,1)-t-GARCH(1,1) model to each return series according to the methodology by Patton (2002) (see also section 4 for a more detailed description of this approach). The model is not rejected by common goodness-of-fit tests. The estimated parameters are consistent with the assumption of strong mixing series (cf. Carrasco and Chen (2002)).

Table 3.3: First four moments (in %) and results of the Jarque-Bera (JB) test, calculated for the returns of the S&P indices, as well as results of the Ljung-Box (LB) Q-statistics, calculated up to lag twenty from the squared returns.

	Financials	Energy	Industrials	IT
Mean	-0.3140	-0.1732	-0.2213	-0.1637
Standard Deviation	3.1696	3.0553	2.1199	2.1037
Skewness	0.1490	-0.2462	-0.1139	0.1148
Kurtosis	5.2523	6.6540	4.8467	5.2024
JB statistics	71.1875	187.4846	47.7495	67.6229
JB p-values	0.0000	0.0000	0.0001	0.0000
LB Q-statistics	141.8456	447.6130	386.8856	266.6008
LB p-values	0.0000	0.0000	0.0000	0.0000

Figure 3.2 (right panel) shows the evolution of multivariate Hoeffding's Phi-Square of the indices' returns, estimated on the basis of a moving window approach with window size 50. Again, the vertical line indicates the day of Lehman's bankruptcy. We observe a sharp increase of Hoeffding's Phi-Square after this date and, hence, an increase of the association between the indices' returns. In order to verify whether this increase is statistically significant, we compare Hoeffding's Phi-Square over two distinct time periods before and after this date using the two-sample test discussed after theorem 3.3.4. Note that the test by Genest and Rémillard (2004) (cf. section 3.3.1) rejects the null hypothesis of stochastic independence (i.e., $C = \Pi$) with a p-value of 0.0005 such that the latter test can be applied. We calculate the estimated values (based on 250 bootstrap samples with block length $l = 5$) of Hoeffding's Phi-Square and the asymptotic variances and covariance as stated in theorem 3.3.4 for both time periods which comprise $n = 100$ observations each:

$$\begin{aligned} \tilde{\Phi}^2_{before} &= 0.1982, & (\tilde{\sigma}^B_{before})^2 &= 0.1663, & \\ \tilde{\Phi}^2_{after} &= 0.7437, & (\tilde{\sigma}^B_{after})^2 &= 0.2064, & \tilde{\sigma}^B_{before,after} = -0.0287. \end{aligned}$$

The choice of the block length $l = 5$ is motivated by the results of the simulation study in section 3.3.2. The value of the test statistic in equation (3.20) is 8.3190 with corresponding p-value 0.0000. Hence, we conclude that there has been a significant increase in association between the returns of the four indices after the bankruptcy of Lehman Brothers Inc.

3.5 Calculations and proofs

3.5.1 Derivation of the functions $h(d)^{-1}$ and $g(d)^{-1}$

We calculate the explicit form of the functions $h(d)^{-1}$ and $g(d)^{-1}$, as stated in equations (3.4) and (3.5). Regarding the function $h(d)^{-1}$, we have

$$\begin{aligned} h(d)^{-1} &= \int_{[0,1]^d} \{M(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u} \\ &= \int_{[0,1]^d} \{M(\mathbf{u})\}^2 \, d\mathbf{u} - 2\int_{[0,1]^d} M(\mathbf{u})\Pi(\mathbf{u})d\mathbf{u} + \int_{[0,1]^d} \{\Pi(\mathbf{u})\}^2 \, d\mathbf{u}. \end{aligned}$$

The first summand on the left-hand side of the above equation can be written as

$$\int_{[0,1]^d} \{M(\mathbf{u})\}^2 \, d\mathbf{u} = E\left([\min\{U_1, \ldots, U_d\}]^2\right) = E\left(X^2\right)$$

where $U_1, \ldots, U_d$ are i.i.d. from $U \sim U(0,1)$ and $X = \min\{U_1, ..., U_d\}$. Therefore,

$$E\left(X^2\right) = d\int_0^1 x^2(1-x)^{d-1}dx = \frac{2}{(d+1)(d+2)}. \tag{3.22}$$

For the second summand, we obtain

$$\begin{aligned} \int_{[0,1]^d} M(\mathbf{u})\Pi(\mathbf{u})d\mathbf{u} &= \frac{1}{2^d}\int_{[0,1]^d} \min\{u_1, \ldots, u_d\}\prod_{i=1}^d 2u_i d\mathbf{u} = \frac{1}{2^d}E\left(\min\{V_1, \ldots, V_d\}\right) \\ &= \frac{1}{2^d}E(Y) \end{aligned}$$

where $V_1, \ldots, V_d$ are i.i.d. from V which has density $f_V(v) = 2v$ for $0 \leqslant v \leqslant 1$ and $Y = \min\{V_1, ..., V_d\}$. Thus,

$$\begin{aligned} \frac{1}{2^d}E(Y) &= \frac{1}{2^d}\int_0^1 xd(1-x^2)^{d-1}2xdx = \frac{1}{2^d}\int_0^1 (1-x^2)^d dx = \frac{1}{2^d}\frac{1}{2}\frac{\Gamma(d+1)\sqrt{\pi}}{\Gamma\left(d+1+\frac{1}{2}\right)} \\ &= \frac{1}{2^d}\frac{d!}{\prod_{i=0}^d\left(i+\frac{1}{2}\right)}. \end{aligned} \tag{3.23}$$

Combining equations (3.22) and (3.23) and using that $\int_{[0,1]^d}\{\Pi(\mathbf{u})\}^2 d\mathbf{u} = (1/3)^d$ yields the asserted form of $h(d)^{-1}$.

Regarding the function $g(d)^{-1}$ as defined in equation (3.5), we have

$$\begin{aligned} g(d)^{-1} &= \int_{[0,1]^d} \{W(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u} \\ &= \int_{[0,1]^d} \{W(\mathbf{u})\}^2 \, d\mathbf{u} - 2\int_{[0,1]^d} W(\mathbf{u})\Pi(\mathbf{u})d\mathbf{u} + \int_{[0,1]^d} \{\Pi(\mathbf{u})\}^2 \, d\mathbf{u}. \end{aligned}$$

For the first summand, it follows that

$$\int_{[0,1]^d} \{W(\mathbf{u})\}^2 \, d\mathbf{u} = \int_0^1 \dots \int_{d-2-\sum_{i=1}^{d-2} u_i}^1 \int_{d-1-\sum_{i=1}^{d-1} u_i}^1 \left(\sum_{i=1}^d u_i - d + 1 \right)^2 du_d \dots du_2 du_1$$
$$= \frac{2}{(d+2)!}. \tag{3.24}$$

Partial integration of the second term further yields

$$\int_{[0,1]^d} W(\mathbf{u}) \Pi(\mathbf{u}) d\mathbf{u} = \int_0^1 \dots \int_{d-2-\sum_{i=1}^{d-2} u_i}^1 u_{d-1} \int_{d-1-\sum_{i=1}^{d-1} u_i}^1 u_d \left(\sum_{i=1}^d u_i - d - 1 \right) du_d \dots du_2 du_1$$
$$= \sum_{i=1}^d \binom{d}{i} (-1)^i \frac{1}{(d+1+i)!}. \tag{3.25}$$

Again, by combining equations (3.24) and (3.25) and using that $\int_{[0,1]^d} \{\Pi(\mathbf{u})\}^2 d\mathbf{u} = (1/3)^d$, we obtain the asserted form of $g(d)^{-1}$.

3.5.2 Derivation of the estimator $\widehat{\Phi}_n^2$

We outline the derivation of the estimator $\widehat{\Phi}_n^2$ as given in (3.12).

$$\begin{aligned}
\{h(d)\}^{-1} \widehat{\Phi}_n^2 &= \int_{[0,1]^d} \left\{ \widehat{C}_n(\mathbf{u}) - \prod_{i=1}^d u_i \right\}^2 d\mathbf{u} \\
&= \int_{[0,1]^d} \left\{ \frac{1}{n} \sum_{j=1}^n \Big(\prod_{i=1}^d \mathbf{1}_{\{\widehat{U}_{ij} \le u_i\}} - \prod_{i=1}^d u_i \Big) \right\}^2 d\mathbf{u} \\
&= \left(\frac{1}{n}\right)^2 \sum_{j=1}^n \sum_{k=1}^n \int_{[0,1]^d} \Big(\prod_{i=1}^d \mathbf{1}_{\{\widehat{U}_{ij} \le u_i\}} - \prod_{i=1}^d u_i \Big) \Big(\prod_{i=1}^d \mathbf{1}_{\{\widehat{U}_{ik} \le u_i\}} - \prod_{i=1}^d u_i \Big) d\mathbf{u} \\
&= \left(\frac{1}{n}\right)^2 \sum_{j=1}^n \sum_{k=1}^n \int_{[0,1]^d} \Big(\prod_{i=1}^d \mathbf{1}_{\{\max\{\widehat{U}_{ij}, \widehat{U}_{ik}\} \le u_i\}} + \prod_{i=1}^d u_i^2 \\
&\quad - \prod_{i=1}^d u_i \mathbf{1}_{\{\widehat{U}_{ij} \le u_i\}} - \prod_{i=1}^d u_i \mathbf{1}_{\{\widehat{U}_{ik} \le u_i\}} \Big) d\mathbf{u} \\
&= \left(\frac{1}{n}\right)^2 \sum_{j=1}^n \sum_{k=1}^n \prod_{i=1}^d (1 - \max\{\widehat{U}_{ij}, \widehat{U}_{ik}\}) - \frac{2}{n} \left(\frac{1}{2}\right)^d \sum_{j=1}^n \prod_{i=1}^d (1 - \widehat{U}_{ij}^2) + \left(\frac{1}{3}\right)^d.
\end{aligned}$$

3.5.3 Proofs

Proof of Proposition 3.2.1. Let $\mathbf{X}$ be a d-dimensional random vector with copula C.
(i) According to theorem 2.2.4, part (i), the copula C is invariant under strictly increasing transformations of one or several components of $\mathbf{X}$ for all $d \geq 2$. As direct functional of the copula, $\Phi^2_{\mathbf{X}}$ inherits this property.
(ii) Let β_k be a strictly decreasing transformation of the kth component X_k of $\mathbf{X}$, $k \in \{1, \ldots, d\}$, defined on the range of X_k. For dimension $d = 2$, we have

$$\begin{aligned}
h(2)^{-1}\Phi^2_{(\beta_1(X_1),\beta_2(X_2))} &= \int_{[0,1]^2} \left\{C_{(\beta_1(X_1),\beta_2(X_2))}(u_1,u_2) - \Pi(u_1,u_2)\right\}^2 du_1du_2 \\
&= \int_{[0,1]^2} \left\{C_{(X_1,X_2)}(1,u_2) - C_{(X_1,\beta_2(X_2))}(1-u_1,u_2) - u_1u_2\right\}^2 du_1du_2 \\
&= \int_{[0,1]^2} \left\{u_1 + u_2 - 1 - C_{(X_1,X_2)}(1-u_1,1-u_2) - u_1u_2\right\}^2 du_1du_2 \\
&= \int_{[0,1]^2} \left\{1 - x + 1 - y - 1 + C_{(X_1,X_2)}(x,y) - (1-x)(1-y)\right\}^2 dxdy \\
&= \int_{[0,1]^2} \left\{C_{(X_1,X_2)}(x,y) - xy\right\}^2 dxdy \\
&= h(2)^{-1}\Phi^2_{\mathbf{X}},
\end{aligned}$$

where the second equation follows from theorem 2.2.4, part (ii). For dimension $d \geq 3$, let $\boldsymbol{\beta}_k(\mathbf{X}) = (X_1, \ldots, X_{k-1}, \beta_k(X_k), X_{k+1}, \ldots, X_d)$ denote the random vector where the kth component of $\mathbf{X}$ is transformed by the function β_k. Without loss of generality set $k = 1$. Consider

$$\begin{aligned}
h(d)^{-1}\Phi^2_{\boldsymbol{\beta}_1(\mathbf{X})} &= \int_{[0,1]^d} \left\{C_{\boldsymbol{\beta}_1(\mathbf{X})}(u_1,u_2,\ldots,u_d) - \Pi(u_1,u_2,\ldots,u_d)\right\}^2 d\mathbf{u} \\
&= \int_{[0,1]^d} \{C_{\mathbf{X}}(1,u_2,\ldots,u_d) - C_{\mathbf{X}}(1-u_1,u_2,\ldots,u_d) - u_1u_2\cdot\ldots\cdot u_d\}^2 d\mathbf{u} \\
&= \int_{[0,1]^d} [\{C_{\mathbf{X}}(1,u_2,\ldots,u_d) - u_2\cdot\ldots\cdot u_d\} \\
&\qquad -\{C_{\mathbf{X}}(x,u_2,\ldots,u_d) - xu_2\cdot\ldots\cdot u_d\}]^2\, dxd\mathbf{u}' \qquad \textit{with } \mathbf{u}' = (u_2,\ldots,u_d) \\
&= \int_{[0,1]^{d-1}} \{C_{\mathbf{X}}(1,u_2,\ldots,u_d) - u_2\cdot\ldots\cdot u_d\}^2 d\mathbf{u}' \\
&\qquad + \int_{[0,1]^d} \{C_{\mathbf{X}}(x,u_2,\ldots,u_d) - xu_2\cdot\ldots\cdot u_d\}^2 dxd\mathbf{u}' \\
&\qquad - \int_{[0,1]^{d-1}} \big[\{C_{\mathbf{X}}(1,u_2,\ldots,u_d) - u_2\cdot\ldots\cdot u_d\}\cdot \\
&\qquad \cdot\{2\int_{[0,1]} C_{\mathbf{X}}(x,u_2,\ldots,u_d)dx - u_2\cdot\ldots\cdot u_d\}\big]d\mathbf{u}'.
\end{aligned} \tag{3.26}$$

According to equation (3.26), it holds that $\Phi^2_{\boldsymbol{\beta}_1(\mathbf{X})} = \Phi^2_{\mathbf{X}}$ if either

- $C_{\mathbf{X}}(1, u_2, \ldots, u_d) = u_2 \cdot \ldots \cdot u_d$, meaning that $X_2, \ldots, X_d$ are mutually stochastically independent, or
- $C_{\mathbf{X}}(1, u_2, \ldots, u_d) = 2 \int_{[0,1]} C_{\mathbf{X}}(x, u_2, \ldots, u_d) dx$, which is fulfilled if X_1 is stochastically independent of $(X_2, \ldots, X_d)$ since $C_{\mathbf{X}}(x, u_2, \ldots, u_d) = x C_{\mathbf{X}}(1, u_2, \ldots, u_d)$ in this case.

□

Proof of theorem 3.3.1. Note that $\Phi^2 = \varphi(C)$ represents a Hadamard-differentiable map φ on $\ell^\infty([0,1]^d)$ of the copula C. Its derivative φ'_C at C, a continuous linear map on $\ell^\infty([0,1]^d)$, is given by

$$\varphi'_C(D) = 2h(d) \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\} D(\mathbf{u}) d\mathbf{u},$$

which can be shown as follows: For all converging sequences $t_n \to 0$ and $D_n \to D$ such that $C + t_n D_n \in \ell^\infty([0,1]^d)$ for every n, we have

$$\begin{aligned} \frac{\varphi(C + t_n D_n) - \varphi(C)}{t_n} &= \frac{h(d) \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u}) + t_n D_n(\mathbf{u})\}^2 d\mathbf{u}}{t_n} \\ &\quad - \frac{h(d) \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\}^2 d\mathbf{u}}{t_n} \\ &= \frac{2h(d) t_n \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\} D_n(\mathbf{u}) d\mathbf{u} + t_n^2 \int_{[0,1]^d} D_n^2(\mathbf{u}) d\mathbf{u}}{t_n} \qquad (3.27) \\ &\to 2h(d) \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\} D(\mathbf{u}) d\mathbf{u}, \end{aligned}$$

for $n \to \infty$, since the second integral in equation (3.27) is bounded for all D_n. An application of the functional delta-method given in theorem 2.2.7 together with theorem 2.2.8 then implies

$$\sqrt{n}(\widehat{\Phi}^2 - \Phi^2) = \sqrt{n}\{\varphi(\widehat{C}_n) - \varphi(C)\} \xrightarrow{d} \varphi'_C(\mathbb{G}_C), \qquad (3.28)$$

where $\varphi'_C(\mathbb{G}_C) = 2h(d) \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\} \mathbb{G}_C(\mathbf{u}) d\mathbf{u}$. Using the fact that $\mathbb{G}_C$ is a tight Gaussian process, lemma 3.9.8 in van der Vaart and Wellner (1996), p. 377, implies that $Z_{\Phi^2} = \varphi'_C(\mathbb{G}_C)$ is normally distributed with mean zero and variance $\sigma^2_{\Phi^2}$ as stated in the theorem. Another application of the delta-method to (3.28) yields the weak convergence of $\sqrt{n}\{\Phi(\widehat{C}_n) - \Phi(C)\}$ to the random variable $Z_\Phi \sim N(0, \sigma^2_\Phi)$.

□

Proof of theorem 3.3.4. Let $\widehat{C}_{\mathbf{Z},n}$ denote the empirical copula based on the sample $\mathbf{Z_1}, \ldots, \mathbf{Z_n}$. Under the assumption of the theorem, weak convergence of the empirical copula process $\sqrt{n}\{\widehat{C}_{\mathbf{Z},n} - C_{\mathbf{Z}}\}$ to the tight Gaussian process $\mathbb{G}^{\star}_{\mathbf{Z}}$ in $\ell^{\infty}([0,1]^{2d})$ follows according to theorem 2.2.12. Since

$$\begin{pmatrix} \Phi^2_{\mathbf{X}} \\ \Phi^2_{\mathbf{Y}} \end{pmatrix} = \begin{pmatrix} \Phi^2\{C_{\mathbf{Z}}(\mathbf{u}, 1 \ldots, 1)\} \\ \Phi^2\{C_{\mathbf{Z}}(1, \ldots, 1, \mathbf{v})\} \end{pmatrix} = g(C_{\mathbf{Z}}),$$

the asymptotic behavior of $(\widehat{\Phi}^2_{\mathbf{X}}, \widehat{\Phi}^2_{\mathbf{Y}})'$ can be established analogously as in the proof of theorem 3.3.1 using the Hadamard differentiability of the map g at $C_{\mathbf{Z}}$ whose derivative is denoted by $g'_{C_{\mathbf{Z}}}$. Hence, $\sqrt{n}\{(\widehat{\Phi}^2_{\mathbf{X}}, \widehat{\Phi}^2_{\mathbf{Y}})' - (\Phi^2_{\mathbf{X}}, \Phi^2_{\mathbf{Y}})'\}$ converges in distribution to the multivariate normally distributed random vector $g'_{C_{\mathbf{Z}}}(\mathbb{G}^{\star}_{\mathbf{Z}})$ given by

$$g'_{C_{\mathbf{Z}}}(\mathbb{G}^{\star}_{\mathbf{Z}}) = \begin{pmatrix} \int_{[0,1]^d}\{C_{\mathbf{X}}(\mathbf{u}) - \mathit{\Pi}(\mathbf{u})\}\mathbb{G}^{\star}_{\mathbf{Z}}(\mathbf{u}, 1 \ldots, 1) d\mathbf{u} \\ \int_{[0,1]^d}\{C_{\mathbf{Y}}(\mathbf{v}) - \mathit{\Pi}(\mathbf{v})\}\mathbb{G}^{\star}_{\mathbf{Z}}(1, \ldots, 1, \mathbf{v}) d\mathbf{v} \end{pmatrix}.$$

With $\mathbb{G}^{\star}_{\mathbf{X}}(\mathbf{u}) = \mathbb{G}^{\star}_{\mathbf{Z}}(\mathbf{u}, 1 \ldots, 1)$ and $\mathbb{G}^{\star}_{\mathbf{Y}}(\mathbf{v}) = \mathbb{G}^{\star}_{\mathbf{Z}}(1, \ldots, 1, \mathbf{v})$, apply the continuous mapping theorem to conclude the proof.

□

Chapter 4

Estimating multivariate association based on weighted observations

So far, a nonparametric estimator for Spearman's rho was presented which is based on the empirical copula where, in particular, all (normalized) ranks are weighted equally. In this chapter, we discuss a more general nonparametric estimator which is obtained by allocating different weights to the ranks. The asymptotic distribution of this estimator is derived from the weak convergence properties of weighted empirical processes under minimal conditions on the weights and on the copula. An important area of application of those weighted estimators lies in the evaluation of Spearman's rho over time while assigning higher weight to more recent observations. In this context, we give examples for possible weighting schemes. For illustration, the theoretical results are applied to financial data.

4.1 Preliminaries

Let $\mathbf{X} = (X_1, X_2)$ be a bivariate random vector with distribution function F and continuous univariate marginal distribution functions F_1, F_2. Recall that bivariate Spearman's rho is defined as the correlation coefficient of the transformed random variables $F_1(X_1)$ and $F_2(X_2)$ (cf. equation (2.33)). The following version of a weighted estimator is then motivated by the EWMA model from RiskMetrics (1996); see (1.1):

$$\widehat{\rho} = \frac{\sum_{j=1}^{n} c_j (R_{1j} - \frac{n+1}{2})(R_{2j} - \frac{n+1}{2})}{\sqrt{\sum_{j=1}^{n} c_j (R_{1j} - \frac{n+1}{2})^2} \sqrt{\sum_{j=1}^{n} c_j (R_{2j} - \frac{n+1}{2})^2}},$$

where $R_{ij}, i = 1, 2, j = 1, \ldots, n$, refer to the ranks of the observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$ from $\mathbf{X}$, i.e., $R_{ij} =$ (rank of X_{ij} in $X_{i1}, \ldots, X_{in}$) and $c_j, j = 1, \ldots, n$, are general, non-negative weights. The derivation of the statistical properties of the above estimator, however, is complicated. Moreover, its generalization to the multivariate case is not straightforward. The weighted estimator considered in this paper is therefore based on the representation of Spearman's rho in terms of the copula C of $\mathbf{X}$, i.e.,

$$\rho = 12 \int_0^1 \int_0^1 C(u_1, u_2)\, du_1 du_2 - 3,$$

cf. equation (2.34). As discussed in section 2.3.3, a nonparametric estimator of Spearman's rho as given above is obtained by replacing the copula with the empirical copula; cf. (2.40). In particular, all (pseudo-) observations (or, equivalently, all normalized ranks) thereby contribute with the equal weight $1/n$ to the estimation. Taking this estimator as a starting point, we generalize it by allowing different weights for each observation; cf. Gaißer (2010). Based on multivariate Spearman's rho as given in section 2.3.3, the weighted estimator can naturally be extended to the multivariate case. The choice of the weights can be quite general such that different weighting schemes are possible. Analogously to the well-known EWMA model by RiskMetrics (1996) (cf. (1.1)), for example, an exponentially weighted estimator for multivariate Spearman's rho can be derived in a time-dynamic context where more weight is assigned to more recent observations.

The asymptotic distribution of the proposed weighted estimator for Spearman's rho can be established under minimal conditions on the copula and the weights. In fact, weighted Spearman's rho can be written as a functional of two empirical processes: the weighted counterpart of the empirical copula process and its survival function. The asymptotic behavior of those processes can be deduced from the weak convergence properties of weighted empirical processes, see e.g. Vanderzanden (1980), Shorack and Wellner (1986), and Koul (2002). We further describe a weighted bootstrap method to estimate the asymptotic variance of the proposed estimator. This enables, for example, the formulation of statistical hypothesis tests based on weighted Spearman's rho as discussed in chapter 6. Similar weighted nonparametric estimators can also be defined for other measures of association such as Kendall's tau or Blomqvist's beta as introduced in section 2.3.3.

Various types of bivariate weighted rank correlation coefficients have already been investigated in the literature. In contrast to our approach, most of those coefficients are constructed in such a way that more weight is placed to higher ranks while, at the same time, lower ranks bear less weight (or vice versa). They are, for example, applied in situations where a certain number of objects is ranked by two independent sources and interest lies in the consistency of top rankings while the disagreement in lower rankings is less relevant. Examples are the comparison of two methods to determine talented students or two methods to rank important documents. In this context, Iman and Conover (1987) introduced a measure of top-down correlation based on Savage scores while a weighted Kendall's tau statistic has been studied by Shieh (1998). Further, Blest (2000) proposed an alternative rank correlation measure which places more emphasis to differences in the top ranks and which was reconsidered by Genest and Plante (2003). We also refer to Pinto da Costa and Roque (2005) and Maturi and Abdelfattah (2008), and Nikitin and Stepanova (2003) for tests for stochastic independence.

Let $\mathbf{X}$ be a d-dimensional random vector with distribution function F, continuous univariate marginal distribution functions $F_i, i = 1, \ldots, d$, and copula C. Our analysis is based on the version ρ_3 of multivariate Spearman's rho as introduced in equation (2.39) though we will refer to it as ρ in the following to simplify notation. Recall its definition

$$\begin{aligned}\rho =& \frac{\rho_1 + \rho_2}{2} = h_\rho(d) \left[2^{d-1} \left\{ \int_{[0,1]^d} C(\mathbf{u}) d\mathbf{u} + \int_{[0,1]^d} \Pi(\mathbf{u}) dC(\mathbf{u}) \right\} - 1 \right] \\ =& h_\rho(d) \left[2^{d-1} \left\{ \int_{[0,1]^d} C(\mathbf{u}) d\mathbf{u} + \int_{[0,1]^d} \overline{C}(\mathbf{u}) d\mathbf{u} \right\} - 1 \right], \end{aligned} \quad (4.1)$$

with $h_\rho(d) = (d+1)/\{2^d - (d+1)\}$, survival function $\overline{C}$ of C (see (2.4)), and ρ_1 and ρ_2 as defined in (2.35) and (2.36), respectively. Note that the last equation follows from equation (2.38). Based on a random sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ from $\mathbf{X}$, an equally weighted nonparametric estimator for ρ is given by

$$\begin{aligned}\widehat{\rho}_n &= h_\rho(d) \left[2^{d-1} \left\{ \int_{[0,1]^d} \widehat{C}_n(\mathbf{u}) d\mathbf{u} + \int_{[0,1]^d} \widehat{\overline{C}}_n(\mathbf{u}) d\mathbf{u} \right\} - 1 \right] \\ &= h_\rho(d) \left[\frac{2^{d-1}}{n} \sum_{j=1}^{n} \left\{ \prod_{i=1}^{d} (1 - \widehat{U}_{ij,n}) + \prod_{i=1}^{d} \widehat{U}_{ij,n} \right\} - 1 \right], \end{aligned} \quad (4.2)$$

where $\widehat{C}_n$ and $\widehat{\overline{C}}_n$ denote the empirical copula and the empirical survival function, respectively, as defined in equations (2.14) and (2.23). As mentioned in section 2.3.3, the asymptotic behavior of $\widehat{\rho}_n$ can be deduced from the (joint) weak convergence of the empirical copula process $\sqrt{n}(\widehat{C}_n - C)$ and its survival version $\sqrt{n}(\widehat{\overline{C}}_n - \overline{C})$. In particular, asymptotic normality and consistency of $\sqrt{n}(\widehat{\rho}_n - \rho)$ can be established though the proof is omitted here since this result follows as a special case from our analysis in the following section (see theorem 4.2.6 in section 4.2.2).

4.2 Multivariate weighted Spearman's rho

This sections derives a weighted nonparametric estimator for multivariate Spearman's rho. The asymptotic distribution of the estimator is established and a nonparametric bootstrap method is described to estimate its asymptotic variance.

4.2.1 Weighted nonparametric estimation

In the defining equation (4.2) of the estimator $\widehat{\rho}_n$, every (pseudo-)observation is weighted equally, i.e., contributes with the equal weight $1/n$ to the estimation. We extend this nonparametric estimator for multivariate Spearman's rho by assigning nonidentical weights to the observations. Specifically, consider a triangular array of non-negative constants $c_{j,n}, j = 1, \dots, n$, with $\mathbf{c}_n = (c_{1,n}, \dots, c_{n,n})'$ such that

$$\sum_{l=1}^{n} c_{l,n} = \mathbf{c}_n' \mathbf{1}_n \quad \text{and} \quad \sum_{l=1}^{n} c_{l,n}^2 = \mathbf{c}_n' \mathbf{c}_n.$$

A weighted nonparametric estimator for ρ is then defined as

$$\widehat{\rho}_n^c = h_\rho(d) \left[2^{d-1} \sum_{j=1}^{n} \frac{c_{j,n}}{\mathbf{c}_n' \mathbf{1}_n} \left\{ \prod_{i=1}^{d} (1 - \widehat{U}_{ij,n}) + \prod_{i=1}^{d} \widehat{U}_{ij,n} \right\} - 1 \right]. \tag{4.3}$$

Without loss of generality, set $c_{1,1} > 0$ such that the estimator is well-defined. Hence, in contrast to the estimator in (4.2), every observation is weighted in such a way that the jth observation contributes with weight $c_{j,n}/\mathbf{c}_n' \mathbf{1}_n$ to the estimation, $j = 1, \dots, n$. Naturally, the weights are normalized such that they sum up to 1, i.e. $\sum_{j=1}^{n} c_{j,n}/\mathbf{c}_n' \mathbf{1}_n = 1$. Note that by setting $c_{j,n} = 1$, we obtain the estimator $\widehat{\rho}_n$ as given in (4.2). By introducing general weights, it is possible to assign more weight to those observations which are considered to be more relevant or which are known to be more precise. Several specific weighting schemes are discussed in section 4.3.

4.2.2 Asymptotic behavior

In order to derive the asymptotic behavior of the proposed estimator $\widehat{\rho}_n^c$, we rewrite it as

$$\begin{aligned} \widehat{\rho}_n^c &= h_\rho(d) \left[2^{d-1} \sum_{j=1}^{n} \frac{c_{j,n}}{\mathbf{c}_n' \mathbf{1}_n} \left\{ \prod_{i=1}^{d} (1 - \widehat{U}_{ij,n}) + \prod_{i=1}^{d} \widehat{U}_{ij,n} \right\} - 1 \right] \\ &= h_\rho(d) \left[2^{d-1} \left\{ \int_{[0,1]^d} \widehat{W}_n(\mathbf{u}) d\mathbf{u} + \int_{[0,1]^d} \widehat{\overline{W}}_n(\mathbf{u}) d\mathbf{u} \right\} - 1 \right], \end{aligned} \tag{4.4}$$

with

$$\widehat{W}_n(\mathbf{u}) = \sum_{j=1}^{n} \frac{c_{j,n}}{\mathbf{c}_n' \mathbf{1}_n} \prod_{i=1}^{d} \mathbf{1}_{\{\widehat{U}_{ij,n} \le u_i\}} \quad \text{and} \quad \widehat{\overline{W}}_n(\mathbf{u}) = \sum_{j=1}^{n} \frac{c_{j,n}}{\mathbf{c}_n' \mathbf{1}_n} \prod_{i=1}^{d} \mathbf{1}_{\{\widehat{U}_{ij,n} > u_i\}}, \ \mathbf{u} \in [0,1]^d. \tag{4.5}$$

The functions $\widehat{W}_n$ and $\widehat{\overline{W}}_n$ represent the weighted counterparts of the empirical copula $\widehat{C}_n$ and the empirical survival function $\widehat{\overline{C}}_n$ as defined in equations (2.14) and (2.23), respectively. We therefore refer to these functions as weighted empirical copula and weighted empirical survival function. Obviously, they coincide with the empirical copula and its survival function if $c_{j,n} = 1, j = 1, \dots, n$. Note however that, in general, $\widehat{W}_n$ is not a copula.

In view of equation (4.4), the asymptotic distribution of $\widehat{\rho}_n^c$ can be derived from the joint asymptotic behavior of the weighted empirical copula and its survival function analogously to the equally weighted case (cf. section 2.3.3). Weak convergence of the weighted empirical copula process is established based on the weak convergence properties of weighted empirical processes, which has been considered in various settings e.g. by Shorack and Wellner (1986) and Koul (2002) in the univariate case and Vanderzanden (1980) and van der Vaart and Wellner (1996) in a multivariate context. The following version forms the basis for the forthcoming results in this section. For this purpose, an additional condition must be imposed on the sequence $c_{j,n},\ j = 1, \ldots, n,$:

$$\textbf{(C)} \quad \max_{\{1\leq j\leq n\}} \frac{c_{j,n}^2}{\mathbf{c}_n'\mathbf{c}_n} \longrightarrow 0 \quad \text{for } n \to \infty.$$

Theorem 4.2.1 *Let $\mathbf{X_1}, \ldots, \mathbf{X_n}$ be a random sample from the d-dimensional random vector $\mathbf{X}$ with distribution function F and continuous univariate marginal distribution functions $F_i, i = 1, \ldots, d$. Consider the triangular array of non-negative constants $c_{j,n}, j = 1, \ldots, n,$ with $\mathbf{c}_n = (c_{1,n}, \ldots, c_{n,n})'$ and define the weighted empirical process as*

$$V_n(\mathbf{x}) = \sum_{j=1}^{n} \frac{c_{j,n}}{\sqrt{\mathbf{c}_n'\mathbf{c}_n}} \left\{ \prod_{i=1}^{d} \mathbf{1}_{\{X_{ij}\leq x_i\}} - F(\mathbf{x}) \right\}, \quad \mathbf{x} = (x_1, \ldots, x_d) \in \bar{I\!R}^d. \tag{4.6}$$

Under the assumption that condition **(C)** *holds, we have*

$$V_n(\mathbf{x}) \xrightarrow{w} \mathbb{B}_F(\mathbf{x}). \tag{4.7}$$

Weak convergence takes place in $\ell^\infty(\bar{I\!R}^d)$ and the process $\mathbb{B}_F$ is a d-dimensional tight centered Gaussian process with covariance function

$$E\{\mathbb{B}_F(\mathbf{x})\mathbb{B}_F(\mathbf{y})\} = F(\mathbf{x} \wedge \mathbf{y}) - F(\mathbf{x})F(\mathbf{y}),$$

i.e., $\mathbb{B}_F$ is a Brownian bridge.

Proof. The assertion follows from example 2.11.8 in van der Vaart and Wellner (1996) provided that the sequence V_n converges marginally. Marginal convergence is given if $(V_n(\mathbf{x}_1), \ldots, V_n(\mathbf{x}_k))$ converges weakly for every finite subset of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k \in [-\infty, \infty]^d$. This is proven via a multivariate version of the Lindeberg-Feller theorem (see Araujo and Giné (1980), p. 41). For $s = 1, \ldots, k$, set therefore

$$Z_{s,n}^{(j)} = \frac{c_{j,n}}{\sqrt{\mathbf{c}_n'\mathbf{c}_n}} \left\{ \prod_{i=1}^{d} \mathbf{1}_{\{X_{ij}\leq x_{is}\}} - F(\mathbf{x}_s) \right\}.$$

Then, $E(Z_{s,n}^{(j)}) = 0$ for all $s = 1, \ldots, k$, and

$$\sum_{j=1}^{n} E(Z_{s,n}^{(j)} \cdot Z_{r,n}^{(j)}) = F(\mathbf{x}_s \wedge \mathbf{x}_r) - F(\mathbf{x}_s)F(\mathbf{x}_r) =: a_{s,r}, \tag{4.8}$$

for all n and $1 \leq s, r \leq k$. For the vector $(Z_{1,n}^{(j)}, \dots, Z_{k,n}^{(j)})$ consider $\|Z_n^{(j)}\|_2^2 = \sum_{l=1}^k (Z_{l,n}^{(j)})^2$ where $\|\cdot\|_2$ denotes the Euclidian norm. We have $\|Z_n^{(j)}\|_2^2 \leq 4kc_{j,n}^2/\mathbf{c}_n'\mathbf{c}_n$ such that

$$\int_{\{\|Z_n^{(j)}\|_2 > \varepsilon\}} \|Z_n^{(j)}\|_2^2 \, d\mathbb{P} \to 0 \quad \text{as } n \to \infty,$$

for every $\varepsilon > 0$ due to condition **(C)**. Thus

$$(V_n(\mathbf{x}_1), \dots, V_n(\mathbf{x}_k)) \xrightarrow{d} N(\mathbf{0}_k, A) \quad \text{as } n \to \infty,$$

with $k \times k$-matrix $A = (a_{s,r})_{s,r=1,\dots,k}$ whose elements $a_{s,r}$ are given in formula (4.8). If A is zero then $(V_n(\mathbf{x}_1), \dots, V_n(\mathbf{x}_k))$ converges to zero in probability as $n \to \infty$. Since convergence in $\ell^\infty(\bar{\mathbb{R}}^d)$ implies marginal convergence, it further follows that the limit process $\mathbb{B}_F$ of V_n must be a zero-mean Gaussian process with covariance function

$$E\{\mathbb{B}_F(\mathbf{x})\mathbb{B}_F(\mathbf{y})\} = F(\mathbf{x} \wedge \mathbf{y}) - F(\mathbf{x})F(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in [-\infty, \infty]^d.$$

This and tightness yield that $\mathbb{B}_F$ is a d-dimensional Brownian bridge according to lemma 1.5.3 in van der Vaart and Wellner (1996). □

In the light of theorem 4.2.1, the weighted empirical copula process is defined as

$$\sum_{j=1}^n \frac{c_{j,n}}{\sqrt{\mathbf{c}_n'\mathbf{c}_n}} \left\{ \prod_{i=1}^d \mathbf{1}_{\{\hat{U}_{ij} \leq u_i\}} - C(\mathbf{u}) \right\}, \quad \mathbf{u} \in [0,1]^d,$$

which can equivalently be written as

$$\frac{\mathbf{c}_n'\mathbf{1}_n}{\sqrt{\mathbf{c}_n'\mathbf{c}_n}} \sum_{j=1}^n \frac{c_{j,n}}{\mathbf{c}_n'\mathbf{1}_n} \left\{ \prod_{i=1}^d \mathbf{1}_{\{\hat{U}_{ij} \leq u_i\}} - C(\mathbf{u}) \right\} = r_n\{\widehat{W}_n(\mathbf{u}) - C(\mathbf{u})\}, \quad \mathbf{u} \in [0,1]^d,$$

in terms of the weighted empirical copula $\widehat{W}_n$ (see equation (4.5)) and with sequence $r_n = \mathbf{c}_n'\mathbf{1}_n/\sqrt{\mathbf{c}_n'\mathbf{c}_n}$. Note that $r_n \to \infty$ for $n \to \infty$ as a consequence of theorem 4.2.1. The following theorem establishes weak convergence of the weighted empirical copula process.

Theorem 4.2.2 *Let $\mathbf{X_1}, \dots, \mathbf{X_n}$ be a random sample from the d-dimensional random vector $\mathbf{X}$ with distribution function F, continuous univariate marginal distribution functions $F_i, i = 1, \dots, d$, and copula C. Consider the triangular array of non-negative constants $c_{j,n}, j = 1, \dots, n$ with $\mathbf{c}_n = (c_{1,n}, \dots, c_{n,n})'$ which satisfy condition* **(C)**, *and define $r_n = \mathbf{c}_n'\mathbf{1}_n/\sqrt{\mathbf{c}_n'\mathbf{c}_n}$. Under the assumptions that the i-th partial derivatives $D_iC(\mathbf{u})$ of C exist and are continuous for $i = 1, \dots, d$, and that $r_n/\sqrt{n} \to q \in [0,1]$ for $n \to \infty$, we have*

$$r_n \left\{\widehat{W}_n(\mathbf{u}) - C(\mathbf{u})\right\} \xrightarrow{w} \mathbb{G}_C^c(\mathbf{u})$$

in $\ell^\infty([0,1]^d)$. *The process* $\mathbb{G}_C^c$ *is a tight centered Gaussian process in* $[0,1]^d$ *of the form*

$$\mathbb{G}_C^c(\mathbf{u}) = \mathbb{B}_C(\mathbf{u}) - q\sum_{i=1}^{d} D_iC(\mathbf{u})\mathbb{B}_C(\mathbf{u}^{(i)}), \tag{4.9}$$

where the process $\mathbb{B}_C$ *is d-dimensional tight centered Gaussian process on* $[0,1]^d$ *with covariance function*

$$E\{\mathbb{B}_C(\mathbf{u})\mathbb{B}_C(\mathbf{v})\} = C(\mathbf{u}\wedge\mathbf{v}) - C(\mathbf{u})C(\mathbf{v}),$$

i.e., $\mathbb{B}_C$ *is a Brownian bridge on* $[0,1]^d$.

We give the *proof* of theorem 4.2.2 after the following lemma.

Lemma 4.2.3 *Consider the following version of the weighted empirical copula*

$$\widetilde{W}_n(\mathbf{u}) = \widehat{F}_n^c\,\{\widehat{F}_{1,n}^{-1}(u_1),\ldots,\widehat{F}_{d,n}^{-1}(u_d)\}, \quad \mathbf{u}\in[0,1]^d, \tag{4.10}$$

where

$$\widehat{F}_n^c(\mathbf{x}) = \sum_{j=1}^{n}\frac{c_j}{\mathbf{c}_n'\mathbf{1}_n}\prod_{i=1}^{d}\mathbf{1}_{\{X_{ij}\le x_i\}}, \quad \mathbf{x}\in I\!R^d, \tag{4.11}$$

denotes the weighted empirical counterpart of the distribution function F *and* $c_{j,n}$ *is defined in theorem 4.2.2. Then, for all* $\mathbf{u}\in[0,1]^d$,

$$\widetilde{W}_n(\mathbf{u}) = \widehat{W}_n(\mathbf{u}) + O\left(\max_{\{1\le j\le n\}}\frac{c_{j,n}}{\mathbf{c}_n'\mathbf{1}_n}\right).$$

In particular,

$$\max_{\{1\le j\le n\}}\frac{c_{j,n}}{\mathbf{c}_n'\mathbf{1}_n} \longrightarrow 0, \quad \text{as } n\to\infty.$$

Proof. Let $c_{j^\star,n} = \max_{\{1\le j\le n\}} c_{j,n}$. Then,

$$\begin{aligned}\sup_{\{0\le u_1,\ldots,u_d\le 1\}} & |\widetilde{W}_n(u_1,\ldots,u_d) - \widehat{W}_n(u_1,\ldots,u_d)| \\ &\le \max_{\{1\le s_1,\ldots,s_d\le n\}}\left|\widetilde{W}_n\left(\frac{s_1}{n},\ldots,\frac{s_d}{n}\right) - \widetilde{W}_n\left(\frac{s_1-1}{n},\ldots,\frac{s_d-1}{n}\right)\right| \\ &\le d\cdot\frac{c_{j^\star,n}}{\mathbf{c}_n'\mathbf{1}_n}.\end{aligned}$$

Further,

$$\frac{c_{j^\star,n}}{\mathbf{c}_n'\mathbf{1}_n} = \frac{c_{j^\star,n}}{\sum_{k=1}^n c_{k,n}} = \frac{c_{j^\star,n}^2}{\sum_{k=1}^n c_{k,n}^2}\cdot\frac{\sum_{k=1}^n c_{k,n}^2}{c_{j^\star,n}\sum_{k=1}^n c_{k,n}} \le \frac{c_{j^\star,n}^2}{\sum_{k=1}^n c_{k,n}^2}\to 0, \quad \text{for } n\to\infty,$$

due to condition **(C)**, which completes the proof. ☐

Proof of theorem 4.2.2. According to lemma 4.2.3 and using the same notation, we have, for all $\mathbf{u} \in [0,1]^d$,

$$r_n\{\widehat{W}_n(\mathbf{u}) - C(\mathbf{u})\} + O\left(c_{j^\star,n}/\sqrt{\mathbf{c}_n'\mathbf{c}_n}\right) = r_n\{\widetilde{W}_n(\mathbf{u}) - C(\mathbf{u})\}, \tag{4.12}$$

and, thus, we can confine ourselves to establishing weak convergence of the process $r_n(\widetilde{W}_n - C)$ as a consequence of Slutsky's theorem (see also proof of theorem 2.2.8). Further, like in the aforementioned proof, it is possible to concentrate on the case when the marginal distributions $F_i, i = 1, \ldots, d$, are uniform distributions on $[0,1]$ and thus $F = C$ has compact support $[0,1]^d$ by considering the random variables $U_{ij} = F_i(X_{ij}), i = 1, \ldots, d, j = 1, \ldots, n$, with $\mathbf{U}_j = (U_{1j}, \ldots, U_{dj})$. Namely, with D being the copula of the random vectors $\mathbf{U}_j, j = 1, \ldots, n$, and $\widetilde{D}_n$ the associated weighted empirical copula calculated according to (4.10), it can be shown that

$$r_n\{\widetilde{W}_n(\mathbf{u}) - C(\mathbf{u})\} = r_n\{\widetilde{D}_n(\mathbf{u}) - D(\mathbf{u})\}, \quad \text{for all } \mathbf{u} \in [0,1]^d,$$

analogously to lemma 1 in Fermanian et al. (2004)). Note, however, that we stick to the previous notation for simplicity's sake. We proceed similarly to the proof of theorem 2 in Schmid and Schmidt (2007a). Based on the following estimator

$$W_n(\mathbf{u}) = \sum_{j=1}^{n} \frac{c_{j,n}}{\mathbf{c}_n'\mathbf{1}_n} \prod_{i=1}^{d} \mathbf{1}_{\{U_{ij} \le u_i\}}, \quad \text{for} \quad \mathbf{u} \in [0,1]^d, \tag{4.13}$$

we can rewrite the weighted empirical copula process on the right-hand side of equation (4.12) as follows:

$$\begin{aligned} r_n\{\widetilde{W}_n(\mathbf{u}) - C(\mathbf{u})\} & \\ = r_n\{W_n(\mathbf{u}) - C(\mathbf{u})\} &+ r_n\{F\,(\widehat{F}_{1,n}^{-1}(u_1), \ldots, \widehat{F}_{d,n}^{-1}(u_d)) - C(\mathbf{u})\} \qquad (4.14) \\ + \sum_{i=1}^{d} \Big[& H_n^c\{F_1^{-1}(u_1), \ldots, F_{i-1}^{-1}(u_{i-1}), \widehat{F}_{i,n}^{-1}(u_i), \ldots, \widehat{F}_{d,n}^{-1}(u_d)\} \qquad (4.15) \\ & - H_n^c\{F_1^{-1}(u_1), \ldots, F_i^{-1}(u_i), \widehat{F}_{i+1,n}^{-1}(u_{i+1}), \ldots, \widehat{F}_{d,n}^{-1}(u_d)\}\Big], \end{aligned}$$

where $H_n^c = r_n(\widehat{F}_n^c - F)$ denotes the weighted empirical process on $[0,1]^d$ with weighted empirical distribution function $\widehat{F}_n^c$ as defined in formula (4.11). According to theorem 4.2.1, the process $r_n(W_n^c - C)$ converges weakly in $\ell^\infty([0,1]^d)$ to a d-dimensional Brownian bridge $\mathbb{B}_C$ with covariance function $E\{\mathbb{B}_C(\mathbf{u})\mathbb{B}_C(\mathbf{v})\} = C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{v})$. Recall that $F = C$. The asymptotic behavior of the second term in line (4.14) can be derived by repeatedly applying the functional delta-method; see theorem 2.2.7. It uses the following weak convergence result:

$$\sqrt{n}(\widehat{F}_{1,n}(u_1) - F_1(u_1), \ldots, \widehat{F}_{d,n}(u_d) - F_d(u_d)) \xrightarrow{w} (\mathbb{B}_{F_1}(u_1), \ldots, \mathbb{B}_{F_d}(u_d)), \tag{4.16}$$

for $n \to \infty$. This follows from the fact that $\sqrt{n}(\widehat{F}_n - F)$ converges weakly in $\ell^\infty([0,1]^d)$ to a d-dimensional Brownian bridge $\mathbb{B}_F$ with covariance function $E\{\mathbb{B}_F(\mathbf{u})\mathbb{B}_F(\mathbf{v})\} =$

$F(\mathbf{u} \wedge \mathbf{v}) - F(\mathbf{u})F(\mathbf{v})$ (see example 2.1.3 in van der Vaart and Wellner (1996)), which implies marginal convergence. As outlined in the proof of theorem 2.2.8, the inverse map, i.e., the map φ_1 with

$$\varphi_1\{F_1(u_1), \ldots, F_d(u_d)\} = (F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)),$$

is Hadamard-differentiable at $(F_1, \ldots, F_d)$ tangentially to $C([0,1]) \times \cdots \times C([0,1])$ under the assumption of continuous partial derivatives of $C = F$ (lemma 3.9.23, part (ii), in van der Vaart and Wellner (1996)). The derivative of φ_1 has the form

$$\varphi'_{1,F_1,\ldots,F_d}(h_1, \ldots, h_d)(u_1, \ldots, u_d) = (-h_1(u_1), \ldots, -h_d(u_d)), \quad h_1, \ldots, h_d \in C([0,1]),$$

such that the functional delta-method together with formula (4.16) yields

$$\sqrt{n}(\widehat{F}_{1,n}^{-1}(u_1) - F_1^{-1}(u_1), \ldots, \widehat{F}_{d,n}^{-1}(u_d) - F_d^{-1}(u_d)) \xrightarrow{w} (-\mathbb{B}_{F_1}(u_1), \ldots, -\mathbb{B}_{F_d}(u_d)), \tag{4.17}$$

as $n \to \infty$. Further note that the map φ_2 with

$$\varphi_2(F_1^{-1}, \ldots, F_d^{-1})(u_1, \ldots, u_d) = F\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\}$$

is Hadamard-differentiable at $(F_1^{-1}, \ldots, F_d^{-1})$ tangentially to $C([0,1]) \times \cdots \times C([0,1])$ with derivative

$$\varphi'_{2,F_1^{-1},\ldots,F_d^{-1}}(h_1, \ldots, h_d)(u_1, \ldots, u_d) = \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\} \cdot h_i(u_i),$$

$h_1, \ldots, h_d \in C([0,1])$. Hence, the functional delta-method applied to formula (4.17) together with Slutsky's theorem finally yields that

$$\begin{aligned} &r_n\{F\ (\widehat{F}_{1,n}^{-1}(u_1), \ldots, \widehat{F}_{d,n}^{-1}(u_d)) - C(\mathbf{u})\} \\ &\quad = \frac{r_n}{\sqrt{n}} \cdot \sqrt{n}\{F\ (\widehat{F}_{1,n}^{-1}(u_1), \ldots, \widehat{F}_{d,n}^{-1}(u_d)) - C(\mathbf{u})\} \xrightarrow{w} -q\sum_{i=1}^{d} D_i C(\mathbf{u})\mathbb{B}_C(\mathbf{u}^{(i)}) \end{aligned}$$

in $\ell^\infty([0,1]^d)$, using that $F = C$ and, thus, $\mathbb{B}_{F_i}(u_i) = \mathbb{B}_C(\mathbf{u}^{(i)})$. Regarding the third term in line (4.15), we know from theorem 4.2.1 that H_n^c converges weakly to the Gaussian process $\mathbb{B}_C$ in $\ell^\infty([0,1]^d)$. Since $\mathbb{B}_C$ is tight, it follows that H_n^c is asymptotically tight according to theorem 1.5.4 in van der Vaart and Wellner (1996). Asymptotical tightness further implies (theorem 1.5.7 of the latter reference) that H_n^c is asymptotically uniformly equicontinuous in probability, i.e. for every $\varepsilon, \eta > 0$ there exists a $\delta > 0$ such that

$$\limsup_n \mathbb{P}\left\{\sup_{||\mathbf{u}-\mathbf{v}||<\delta} |H_n^c(\mathbf{u}) - H_n^c(\mathbf{v})| > \varepsilon\right\} < \eta, \tag{4.18}$$

with $||\cdot||$ referring to the Euclidean norm in $[0,1]^d$. As $F_i, i = 1, \ldots, d$, is uniformly distributed on $[0,1]$, it further holds that (see Csörgö (1983), corollary 1.4.1)

$$\sup_{\{0 \le u \le 1\}} |\widehat{F}_{i,n}^{-1}(u) - F_i^{-1}(u)| \longrightarrow 0, \quad a.s., \ \text{ for } n \to \infty. \tag{4.19}$$

Combining formulas (4.18) and (4.19) yields that the sum in line (4.15) converges to zero in probability. Using almost surely convergent versions of H_n^c, apply an appropriate continuous mapping theorem to conclude weak convergence of the weighted empirical copula process. Finally, tightness of $\mathbb{B}_C$ implies tightness of the process $\mathbb{G}_C^c$. □

Remarks.

1. The fact that $q = \lim_{n\to\infty} r_n/\sqrt{n} \in [0,1]$ can be seen as follows. On the one hand, note that $r_n = \mathbf{c}_n'\mathbf{1}_n/\sqrt{\mathbf{c}_n'\mathbf{c}_n}$ can equivalently be written as $r_n = ||\mathbf{c}_n||_1/||\mathbf{c}_n||_2$ where $||\cdot||_p$ denotes the p-norm in $\mathbb{R}^n, p = 1, 2$. Then,

$$\frac{r_n}{\sqrt{n}} = \frac{||\mathbf{c}_n||_1}{\sqrt{n}||\mathbf{c}_n||_2} \le \frac{\sqrt{n}||\mathbf{c}_n||_2}{\sqrt{n}||\mathbf{c}_n||_2} = 1 \quad \text{for all } n,$$

using that $||\mathbf{y}||_1 \le \sqrt{n}||\mathbf{y}||_2$ for all $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, which can be shown by induction.
On the other hand, consider the specific sequence of constants $c_{j,n} = 1/\sqrt{j}, j = 1, \ldots, n$. Since the harmonic series $\sum_{j=1}^{\infty} 1/j$ tends to infinity, those constants fulfill condition **(C)**. Moreover,

$$\sum_{j=1}^{n} \frac{1}{j} \simeq \ln(n) + \gamma,$$

where γ refers to the Euler-Mascheroni constant; see e.g. Heuser (1998), p. 185. Hence,

$$\frac{r_n}{\sqrt{n}} = \frac{\sum_{j=1}^{n} 1/\sqrt{j}}{\sqrt{n}\sqrt{\sum_{j=1}^{n} 1/j}} \simeq \frac{\sqrt{n}}{\sqrt{n}(\ln(n)+\gamma)} \longrightarrow 0, \quad n \to \infty,$$

since the series $\sum_{j=1}^{\infty} 1/\sqrt{j} = O(\sqrt{n})$.
Hence, for $q = 0$, the limiting process of the weighted empirical copula process coincides with the limiting process $\mathbb{B}_C$. For $q = 1$, the former coincides with the limiting process $\mathbb{G}_C$ of the ordinary empirical copula process; cf. theorem 2.2.8.

2. Note that for proving weak convergence of the weighted empirical copula process we could not proceed in a similar way as for the empirical copula process (cf. theorem 2.2.8), that is, using representation (2.19) as the starting point. The reason is that, in contrast to the empirical copula, the weighted empirical copula can no longer be represented as a map ϕ of the weighted empirical distribution function $\widehat{F}_n^c$; cf. formulas (2.20) and (4.10).

A similar result as in theorem 4.2.2 can be obtained for the process $r_n(\widehat{\overline{W}}_n - \overline{C})$.

Theorem 4.2.4 *Let $\mathbf{X_1}, \ldots, \mathbf{X_n}$ be a random sample from the d-dimensional random vector $\mathbf{X}$ with distribution function F, continuous marginal distribution functions $F_1, \ldots, F_d$, and copula C. Consider the triangular array of non-negative constants*

$c_{j,n}, j = 1, \dots, n$ with $\mathbf{c}_n = (c_{1,n}, \dots, c_{n,n})'$ which satisfy condition **(C)**, *and define $r_n = \mathbf{c}_n' \mathbf{1}_n / \sqrt{\mathbf{c}_n' \mathbf{c}_n}$. Under the additional assumption that the i-th partial derivatives $D_i \overline{C}(\mathbf{u})$ of $\overline{C}$ exist and are continuous for $i = 1, \dots, d$, and that $r_n/\sqrt{n} \to q \in [0,1]$ for $n \to \infty$, it follows that*

$$r_n\{\widehat{\overline{W}}_n(\mathbf{u}) - \overline{C}(\mathbf{u})\} \xrightarrow{w} \mathbb{G}^c_{\overline{C}}(\mathbf{u}), \quad n \to \infty,$$

in $\ell^\infty([0,1]^d)$ and $\mathbb{G}^c_{\overline{C}}$ has the form

$$\mathbb{G}^c_{\overline{C}}(\mathbf{u}) = \mathbb{B}_{\overline{C}}(\mathbf{u}) - q \sum_{i=1}^{d} D_i \overline{C}(\mathbf{u}) \mathbb{B}_C(\mathbf{u}^{(i)}). \tag{4.20}$$

with process $\mathbb{B}_C$ as defined in theorem 4.2.2. The process $\mathbb{B}_{\overline{C}}$ is a d-dimensional tight centered Gaussian process whose covariance function is given by

$$E\{\mathbb{B}_{\overline{C}}(\mathbf{u})\mathbb{B}_{\overline{C}}(\mathbf{v})\} = \overline{C}(\mathbf{u} \vee \mathbf{v}) - \overline{C}(\mathbf{u})\overline{C}(\mathbf{v}).$$

Proof. Consider first the estimator

$$\overline{W}_n(\mathbf{u}) = \sum_{j=1}^{n} \frac{c_j}{\mathbf{c}_n' \mathbf{1}_n} \prod_{i=1}^{d} \mathbf{1}_{\{U_{ij} > u_i\}}, \quad \text{for} \quad \mathbf{u} \in [0,1]^d.$$

with $U_{ij} = F_i(X_{ij}), i = 1, \dots, d, j = 1, \dots, n$. By using relationship (2.8) between the copula C and the survival function $\overline{C}$, i.e.,

$$\overline{C}(\mathbf{u}) = \sum_{A \subseteq S_d} (-1)^{|A|} C(\mathbf{u}^{(A)}),$$

with set $S_d = \{1, \dots, d\}$, the related empirical process $r_n(\overline{W}_n - \overline{C})$ can be written as a linear, continuous function of the process $r_n(W_n - C)$ (cf. formula (4.13)) through

$$r_n\{\overline{W}_n(\mathbf{u}) - \overline{C}(\mathbf{u})\} = \sum_{A \subseteq S_d} (-1)^{|A|} r_n\{W_n(\mathbf{u}^{(A)}) - C(\mathbf{u}^{(A)})\}.$$

The continuous mapping theorem together with theorem 4.2.1 yields the weak convergence of $r_n\{\overline{W}_n(\mathbf{u}) - \overline{C}(\mathbf{u})\}$ in $\ell^\infty([0,1]^d)$ to the process $\sum_{A \subseteq S_d} (-1)^{|A|} \mathbb{B}_C(\mathbf{u}^{(A)})\}$. As finite sum of tight Gaussian processes, this limiting process itself is a tight Gaussian process. Since further

$$E[r_n\{\overline{W}_n(\mathbf{u}) - \overline{C}(\mathbf{u})\}] = 0,$$

and

$$E[r_n\{\overline{W}_n(\mathbf{u}) - \overline{C}(\mathbf{u})\} r_n\{\overline{W}_n(\mathbf{v}) - \overline{C}(\mathbf{v})\}] = \overline{C}(\mathbf{u} \vee \mathbf{v}) - \overline{C}(\mathbf{u})\overline{C}(\mathbf{v}), \tag{4.21}$$

for all n and $\mathbf{u}, \mathbf{v} \in [0,1]^d$, the limiting process is further a centered process with covariance function as in (4.21), and we denote it by $\mathbb{B}_{\overline{C}}$. By writing

$$\begin{aligned} r_n\{\widehat{\overline{W}}_n(\mathbf{u}) - \overline{C}(\mathbf{u})\} + O(c_{j^\star,n}/\sqrt{\mathbf{c}_n'\mathbf{c}_n}) &= r_n\{\widehat{\overline{F}}^c\ (\widehat{F}_{1,n}^{-1}(u_1),\ldots,\widehat{F}_{d,n}^{-1}(u_d)) - \overline{C}(\mathbf{u})\} \\ &= r_n\{\overline{W}_n(\mathbf{u}) - \overline{C}(\mathbf{u})\} + r_n\{\overline{F}\ (\widehat{F}_{1,n}^{-1}(u_1),\ldots,\widehat{F}_{d,n}^{-1}(u_d)) - \overline{C}(\mathbf{u})\} \\ &\quad + \sum_{i=1}^{d} \Big[\overline{H}_n^c\{F_1^{-1}(u_1),\ldots,F_{i-1}^{-1}(u_{i-1}),\widehat{F}_{i,n}^{-1}(u_i),\ldots,\widehat{F}_{d,n}^{-1}(u_d)\} \\ &\quad - \overline{H}_n^c\{F_1^{-1}(u_1),\ldots,F_i^{-1}(u_i),\widehat{F}_{i+1,n}^{-1}(u_{i+1}),\ldots,\widehat{F}_{d,n}^{-1}(u_d)\}\Big], \end{aligned}$$

where $\overline{H}_n^c = r_n(\widehat{\overline{F}}_n^c - \overline{F})$ with

$$\widehat{\overline{F}}_n^c(\mathbf{x}) = \sum_{j=1}^{n} \frac{c_j}{\mathbf{c}_n'\mathbf{1}_n} \prod_{i=1}^{d} \mathbf{1}_{\{X_{ij} > x_i\}}, \quad \mathbf{x} \in I\!R^d,$$

and carrying out the same steps as in the proof of theorem 4.2.2, the asserted weak convergence follows (see also proof of theorem 2 in Schmid and Schmidt (2007a)). □

Note that, when setting $c_{j,n} = 1, j = 1, \ldots, n$, the convergence rate r_n reduces to $\sqrt{n}$ and theorems 4.2.2 and 4.2.4 imply weak convergence of the ordinary empirical copula process and its survival version, respectively, to the Gaussian processes $\mathbb{G}_C$ and $\mathbb{G}_{\overline{C}}$ as $q = 1$; see theorems 2.2.8 and 2.2.9.
The following corollary to theorems 4.2.2 and 4.2.4 characterizes the asymptotic behavior of the process $r_n(\widehat{W}_n - C, \widehat{\overline{W}}_n - \overline{C})$.

Corollary 4.2.5 *Under the assumption of theorems 4.2.2 and 4.2.4, the vector* $r_n\{\widehat{W}_n - C, \widehat{\overline{W}}_n - \overline{C}\}$ *converges weakly in* $\ell^\infty([0,1]^d) \times \ell^\infty([0,1]^d)$ *to a vector* $(\mathbb{G}_C^c, \mathbb{G}_{\overline{C}}^c)$ *of centered Gaussian processes with* $\mathbb{G}_C^c$ *and* $\mathbb{G}_{\overline{C}}^c$ *as stated in formulas (4.9) and (4.20). Its covariance structure is determined by the covariance structure of the vector* $(\mathbb{B}_C, \mathbb{B}_{\overline{C}})$ *of tight centered Gaussian processes (as defined in theorems 4.2.2 and 4.2.4, respectively), which is given by*

$$\begin{aligned} E\{\mathbb{B}_C(\mathbf{u})\mathbb{B}_C(\mathbf{v})\} &= C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{v}), \\ E\{\mathbb{B}_{\overline{C}}(\mathbf{u})\mathbb{B}_{\overline{C}}(\mathbf{v})\} &= \overline{C}(\mathbf{u} \vee \mathbf{v}) - \overline{C}(\mathbf{u})\overline{C}(\mathbf{v}), \\ E\{\mathbb{B}_C(\mathbf{u})\mathbb{B}_{\overline{C}}(\mathbf{v})\} &= \sum_{A \subseteq S_d} (-1)^{|A|} C(\mathbf{u} \wedge \mathbf{v}^{(A)}) - C(\mathbf{u})\overline{C}(\mathbf{v}). \end{aligned}$$

Proof. In a similar way as in the proof of theorem 4.2.4, joint weak convergence can be established using the relationship

$$r_n\{\widehat{\overline{W}}_n(\mathbf{u}) - \overline{C}(\mathbf{u})\} = \sum_{A \subseteq S_d} (-1)^{|A|} r_n\{\widehat{W}_n(\mathbf{u}^{(A)}) - C(\mathbf{u}^{(A)})\}$$

and applying the continuous mapping theorem together with theorem 4.2.2. Straightforward calculations yield the form of the covariance function. □

Now we are ready to establish asymptotic normality of $r_n(\widehat{\rho}_n^c - \rho)$.

Theorem 4.2.6 *Let $\mathbf{X_1}, \ldots, \mathbf{X_n}$ be a random sample from the d-dimensional random vector $\mathbf{X}$ with distribution function F, copula C and continuous univariate marginal distribution functions. Let further $c_{l,n}, l = 1, \ldots, n$ with $\mathbf{c}_n = (c_{1,n}, \ldots, c_{n,n})'$ be a triangular array of non-negative constants, and define $r_n = \mathbf{c}_n' \mathbf{1}_n / \sqrt{\mathbf{c}_n' \mathbf{c}_n}$. Under the assumption that condition* **(C)** *holds, all partial derivatives of C and $\overline{C}$ exist and are continuous, and $r_n/\sqrt{n} \to q \in [0,1]$ for $n \to \infty$, it follows that*

$$r_n(\widehat{\rho}_n^c - \rho) \xrightarrow{d} Z \sim N(0, \sigma^2), \quad \text{for } n \to \infty, \tag{4.22}$$

where the variance σ^2 is given by

$$\sigma^2 = \{h_\rho(d)\}^2 2^{2d-2} \int_{[0,1]^d} \int_{[0,1]^d} E\left[\{\mathbb{G}_C^c(\mathbf{u}) + \mathbb{G}_{\overline{C}}^c(\mathbf{u})\}\{\mathbb{G}_C^c(\mathbf{v}) + \mathbb{G}_{\overline{C}}^c(\mathbf{v})\}\right] d\mathbf{u} d\mathbf{v}. \tag{4.23}$$

Proof. As a consequence of corollary 4.2.5, we obtain that

$$r_n[\widehat{W}_n(\mathbf{u}) - C(\mathbf{u}) + \{\widehat{\overline{W}}_n(\mathbf{u}) - \overline{C}(\mathbf{u})\}] \xrightarrow{w} \mathbb{G}_C^c(\mathbf{u}) + \mathbb{G}_{\overline{C}}^c(\mathbf{u})$$

in $\ell^\infty([0,1]^d)$. The sequence $r_n(\widehat{\rho}_n^c - \rho)$ can be written as a continuous linear map of the above process through

$$r_n(\widehat{\rho}_n^c - \rho) = h_\rho(d) 2^{d-1} \int_{[0,1]^d} r_n \left[\widehat{W}_n(\mathbf{u}) - C(\mathbf{u}) + \{\widehat{\overline{W}}_n(\mathbf{u}) - \overline{C}(\mathbf{u})\}\right] d\mathbf{u},$$

and, thus, converges to $Z = h_\rho(d) 2^{d-1} \int_{[0,1]^d} \{\mathbb{G}_C^c(\mathbf{u}) + \mathbb{G}_{\overline{C}}^c(\mathbf{u})\} d\mathbf{u}$ by an application of the continuous mapping theorem. The process $\mathbb{G}_C^c + \mathbb{G}_{\overline{C}}^c$ is a tight Gaussian process and, thus, Z is normally distributed with mean zero and variance as stated in the theorem according to lemma 3.9.8 in van der Vaart and Wellner (1996), p. 377. □

Again, the asymptotic distribution of the ordinary estimator $\sqrt{n}(\widehat{\rho}_n - \rho)$, see (4.2), is obtained by setting $c_{j,n} = 1, j = 1, \ldots, n$ (and thus $q = 1$).
The asymptotic variance in (4.23) is generally of complicated form. Its structure however simplifies as follows if the copula C is radially symmetric, i.e., $C(\mathbf{u}) = \overline{C}(\mathbf{1} - \mathbf{u})$ for all $\mathbf{u} \in [0,1]^d$; cf. equation (2.6).

Proposition 4.2.7 *Let C be a radially symmetric copula. Under the assumptions of theorem 4.2.6, the asymptotic variance σ^2 has the form*

$$\sigma^2 = \{h_\rho(d)\}^2 2^{2d-1} \left(\int_{[0,1]^d} \int_{[0,1]^d} E\left[\mathbb{G}_C^c(\mathbf{u})\{\mathbb{G}_C^c(\mathbf{v}) + \mathbb{G}_{\overline{C}}^c(\mathbf{v})\}\right] d\mathbf{u} d\mathbf{v} \right).$$

Proof. If C is radially symmetric, the processes $\mathbb{G}_C^c(\mathbf{u})$ and $\mathbb{G}_{\overline{C}}^c(\mathbf{1}-\mathbf{u})$ are equally distributed for all $\mathbf{u} \in [0,1]^d$. This follows according to the same arguments as in the proof of proposition 4 in Schmid and Schmidt (2007a). The asserted form of σ^2 then follows by direct calculations and adequate substitutions. □

For example if $C = \Pi$, we obtain

$$\sigma^2 = \frac{h(d)^2}{2} \left\{ \left(\frac{4}{3}\right)^d + \left(\frac{2}{3}\right)^d - 2 \right\}.$$

In particular, the asymptotic variance does not depend on the parameter q in this case. In particular, it coincides with the asymptotic variance of the ordinary estimator $\sqrt{n}(\widehat{\rho}_n - \rho)$, i.e., if all constants $c_{j,n}$ are set to $1, j = 1, \ldots, n$. Note that this does not hold for general copulas C.

If the copula C is of a more complicated structure, the asymptotic variance must be estimated adequately. In the equally weighted case, i.e., if $c_{j,n} = 1$ for all $j = 1, \ldots, n$, the nonparametric bootstrap as described in section 2.2.2 can be applied by independently drawing bootstrap samples from the ordinary empirical distribution function $\widehat{F}_n$; see (2.18). For nonidentical (non-negative) constants $c_{j,n}$, however, it is no longer possible to sample from the ordinary empirical distribution function. Instead, the bootstrap samples must be independently drawn from the weighted empirical distribution function $\widehat{F}_n^c$ of the original sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ as defined in (4.11); cf. Lahiri (1998). Hence, based on K independent bootstrap samples $\mathbf{X}_1^b, \ldots, \mathbf{X}_n^b, b = 1, \ldots, K$, from $\widehat{F}_n^c$, an estimator for the asymptotic variance in (4.23) is in practice given by

$$r_n^2 \cdot (\widehat{\sigma}^{c,B})^2 = \frac{1}{K-1} \sum_{b=1}^{K} \left\{ r_n \widehat{\rho}_{n,(b)}^c - \overline{r_n \widehat{\rho}_{n,(b)}^c} \right\}^2, \tag{4.24}$$

where $\widehat{\rho}_{n,(b)}^c$ denotes the bootstrap replication of $\widehat{\rho}_n^c$, calculated from the b-th bootstrap sample, $b = 1, \ldots, K$, and $\overline{r_n \widehat{\rho}_{n,(b)}^c} = 1/K \sum_{b=1}^K r_n \widehat{\rho}_{n,(b)}^c$. The results of a simulation study to assess the performance of the bootstrap variance estimator for a particular weighting scheme are presented in section 4.3.

Remark. The relative efficiency provides a way to compare two estimator sequences, see e.g. van der Vaart (1998), chapter 8. If the two sequences have convergence rate $\sqrt{n}$ and are asymptotically normally distributed, the quotient of their asymptotic variances can be used as a measure for their relative efficiency. The asymptotic distribution of the weighted estimator $\widehat{\rho}_n^c$ when having convergence rate $\sqrt{n}$ follows from an application of Slutsky's theorem; namely, according to theorem 4.2.6, we have

$$\sqrt{n}(\widehat{\rho}_n^c - \rho) = \frac{\sqrt{n}}{r_n} r_n (\widehat{\rho}_n^c - \rho) \xrightarrow{d} Z \sim N(0, \sigma^2/q^2), n \to \infty, \tag{4.25}$$

where the weights must be chosen such that $q = \lim_{n\to\infty} r_n/\sqrt{n} \in (0,1]$. Hence, the relative efficiency of the two estimator sequences $r_n(\widehat{\rho}_n^c - \rho)$ and $\sqrt{n}(\widehat{\rho}_n - \rho)$ is

$$\frac{\int_{[0,1]^d}\int_{[0,1]^d} E\left[\{\mathbb{G}_C^c(\mathbf{u}) + \mathbb{G}_{\overline{C}}^c(\mathbf{u})\}\{\mathbb{G}_C^c(\mathbf{v}) + \mathbb{G}_{\overline{C}}^c(\mathbf{v})\}\right] d\mathbf{u}d\mathbf{v}}{q^2\int_{[0,1]^d}\int_{[0,1]^d} E\left[\{\mathbb{G}_C(\mathbf{u}) + \mathbb{G}_{\overline{C}}(\mathbf{u})\}\{\mathbb{G}_C(\mathbf{v}) + \mathbb{G}_{\overline{C}}(\mathbf{v})\}\right] d\mathbf{u}d\mathbf{v}},$$

with Gaussian process $\mathbb{G}_C$ and $\mathbb{G}_{\overline{C}}$ as defined in equations (2.17) and (2.24), respectively; see theorems 2.2.8 and 2.2.9. In particular, if $C = \Pi$, the relative efficiency of $\sqrt{n}(\widehat{\rho}_n^c - \rho)$ with respect to the estimator $\sqrt{n}(\widehat{\rho}_n - \rho)$ is $1/q^2$ as the asymptotic variances of $r_n(\widehat{\rho}_n^c - \rho)$ and $\sqrt{n}(\widehat{\rho}_n - \rho)$ coincide in this case (see discussions after proposition 4.2.7), i.e., it only depends on the parameter q.

4.3 Time-dynamic weighted Spearman's rho

As mentioned in section 4.1, an important area of application of the weighted nonparametric estimator (4.3) lies in the evaluation of Spearman's rho over time using weighted averages of past observations. Two specific weighting schemes for this purpose are discussed next.

Exponentially weighted Spearman's rho. Motivated by the EWMA model from RiskMetrics (1996) (see (1.1)), an exponentially weighted estimator for multivariate Spearman's rho can be obtained by setting

$$c_{j,n} = \lambda^{j-1}, j = 1, \ldots, n, \tag{4.26}$$

with decay factor $0 < \lambda < 1$. With observations $\mathbf{X}_{t-n+1}, \ldots, \mathbf{X}_t$, we can think of them as e.g. representing the returns of d financial assets in a portfolio, an exponentially weighted estimator for multivariate Spearman's rho at time t is given by

$$\widehat{\rho}_{n,t}^c = h_\rho(d)\left[2^{d-1}\sum_{j=1}^{n}\frac{\lambda^{j-1}}{\sum_{k=1}^{n}\lambda^{k-1}}\left\{\prod_{i=1}^{d}(1 - \widehat{U}_{i(t-j+1),n}) + \prod_{i=1}^{d}\widehat{U}_{i(t-j+1),n}\right\} - 1\right], \tag{4.27}$$

with $\widehat{U}_{i(t-j+1),n} = 1/n(\text{rank of } X_{ij} \text{ in } X_{i(t-n+1)}, ..., X_{i(t)}), i = 1, \ldots, d$, and $j = 1, \ldots, n$. Analogously to the EWMA model by RiskMetrics (1996), the weights decrease exponentially such that more weight is assigned to the most recent observations. Hence, it may react faster to changes in Spearman's rho than the equally weighted estimator with $\lambda = 1$. An appropriate value for the decay factor λ must be determined by the statistician; naturally, its choice also depends on the dimension d. Note that instead of the weights $\lambda^{j-1}/\sum_{k=1}^{n}\lambda^{k-1}$, it is also common to apply the weighting scheme $\lambda^{j-1}(1-\lambda), j = 1, \ldots, n$, using that

$$\sum_{k=1}^{n}\lambda^{k-1} \longrightarrow \frac{1}{1-\lambda}, \quad n \to \infty.$$

In this case, the following recursion can be obtained under the assumption of known and constant marginal distribution functions $F_1, \dots, F_d$, which may be of interest in forecasting; cf. RiskMetrics (1996), p. 81. Note, however, that the correct specification of the margins is often unknown in practice.

Lemma 4.3.1 *Assume that the univariate marginal distributions $F_i, i = 1, \dots, d$, are known and constant over time. Set $c_{j,n}/\mathbf{c}_n'\mathbf{1}_n = \lambda^{j-1}(1-\lambda), j = 1, \dots, n$, with $0 < \lambda < 1$. Based on those weights, let $\widehat{\rho}^c_{n,t+1|t}$ denote the forecasted value of Spearman's rho for time $t+1$, given information up to and including time t. Then,*

$$\widehat{\rho}^c_{n,t+1|t} = (1-\lambda)h_\rho(d)2^{d-1}\left\{\prod_{i=1}^d (1-U_{i(t)}) + \prod_{i=1}^d U_{i(t)}\right\} + \lambda\widehat{\rho}^c_{n,t|t-1},$$

with $U_{i(t-j+1)} = F_i(X_{i(t-j+1)}), i = 1, \dots, d$ and $j = 1, \dots, n$.

Proof. The assertion can be shown analogously to RiskMetrics (1996), p. 82, by assuming that an infinite amount of data is available. We then have

$$\begin{aligned}
\widehat{\rho}^c_{n,t+1|t} &= h_\rho(d)\left[2^{d-1}(1-\lambda)\sum_{j=1}^{\infty}\lambda^{j-1}\left\{\prod_{i=1}^d (1-U_{i(t-j+1)}) + \prod_{i=1}^d U_{i(t-j+1)}\right\} - 1\right] \\
&= (1-\lambda)h_\rho(d)2^{d-1}\left\{\prod_{i=1}^d (1-U_{i(t)}) + \prod_{i=1}^d U_{i(t)}\right\} \\
&\quad + \lambda(1-\lambda)h_\rho(d)\left[2^{d-1}\sum_{j=1}^{\infty}\lambda^{j-1}\left\{\prod_{i=1}^d (1-U_{i(t-j)}) + \prod_{i=1}^d U_{i(t-j)}\right\} - 1\right] \\
&= (1-\lambda)h_\rho(d)2^{d-1}\left\{\prod_{i=1}^d (1-U_{i(t)}) + \prod_{i=1}^d U_{i(t)}\right\} + \lambda\widehat{\rho}^c_{n,t|t-1}.
\end{aligned}$$

□

Polynomially weighted Spearman's rho. Another possible choice of weights is

$$c_{j,n} = j^k, j = 1, \dots, n \quad \text{with } k \in \mathbb{N}. \tag{4.28}$$

As in the previous example, this weighting scheme yields the allocation of a higher weight to the most recent observations. The decrease in weights is now of polynomial order and gets more pronounced with increasing parameter k. In particular if k is set to 1, the weights decline linearly. Although the impact of past observations on the estimation is in this case greater than for a larger value of k, we still place more importance on recent observations than we would if the ordinary estimator for multivariate Spearman's rho $\widehat{\rho}_n$ in (4.2) was used.

The differences in the above weighting schemes are illustrated in figure 4.1 together with the equally weighted case where $c_{j,n} = 1$ for all $j = 1, \dots, n$. In addition, figure

4.2 provides a plot of the empirical copula $\widehat{C}_n$ (see equation (2.14)) and the weighted empirical copula $\widehat{W}_n$ (see equation (4.5)) for sample size $n = 10$ and constants $c_{j,n}$ as in formula (4.28) with $k = 2$.

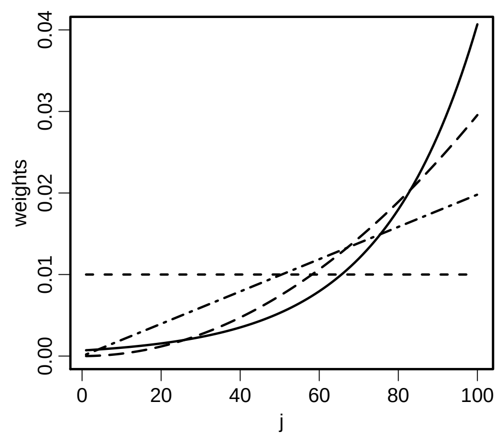


Figure 4.1: **Different weighting schemes**. Weights $c_{j,n}/\mathbf{c}_n'\mathbf{1}_n, j = 1, \ldots, n$, according to formula (4.26) with $\lambda = 0.96$ (solid) and according to formula (4.28) with $k = 2$ (long-dashed) and $k = 1$ (dotted-dashed) together with the equally weighted case where $c_{j,n}/\mathbf{c}_n'\mathbf{1}_n = 1/n$ (dashed) for sample size $n = 100$.

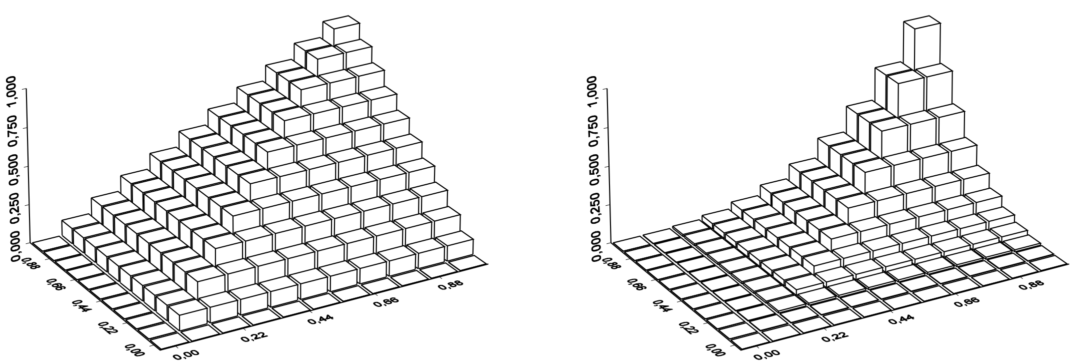


Figure 4.2: Empirical copula $\widehat{C}_n$ (left panel) and weighted empirical copula $\widehat{W}_n$ (right panel) for $n = 10$ and constants $c_{j,n}$ according to formula (4.28) with $k = 2$.

In general, theorem 4.2.6 allows for the construction of statistical hypothesis tests based on weighted multivariate Spearman's rho. In a time-dynamic context, the latter can be used to test for significant changes in Spearman's rho over time; cf. chapter 6. Here, theorem 4.2.6 characterizes the (asymptotic) distribution of the weighted

estimator under the null hypothesis, i.e., that Spearman's rho is constant.
In this context, note that the constants $c_{j,n} = \lambda^{j-1}, j = 1, \dots, n$ in (4.26) for the exponentially weighted Spearman's rho do not fulfill condition **(C)** since

$$\max_{\{1\le j\le n\}} \frac{c_{j,n}^2}{\mathbf{c}_n'\mathbf{c}_n} = \frac{1}{\sum_{k=0}^{n-1}\lambda^{2k}} \longrightarrow 1-\lambda^2, \quad n\to\infty.$$

In contrast, it can be shown that the weights defined by $c_{j,n} = j$ or $c_{j,n} = j^2, j = 1, \dots, n$, (cf. formula (4.28)) fulfill condition **(C)**. In particular, we have (cf. theorem 4.2.2)

$$q = \lim_{n\to\infty} \frac{r_n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sum_{k=1}^n k/\sqrt{\sum_{k=1}^n k^2}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{6}}{2}\cdot\frac{\sqrt{n+1}}{\sqrt{2n+1}} = \frac{\sqrt{3}}{2},$$

and

$$q = \lim_{n\to\infty} \frac{r_n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sum_{k=1}^n k^2/\sqrt{\sum_{k=1}^n k^4}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{30}}{6}\cdot\frac{\sqrt{2n^2+3n-1}}{\sqrt{3n^2+3n-1}} = \frac{\sqrt{5}}{3},$$

for these two weighting schemes.

Tables 4.1 and 4.2 show the results of a simulation study carried out to investigate the finite-sample performance of the bootstrap estimator for the asymptotic standard deviation of $\sqrt{n}(\widehat{\rho}_n^c - \rho)$; see the discussions at the end of the previous section. We consider the d-dimensional equi-correlated Gaussian copula with correlation matrix $K(\kappa) = \kappa\mathbf{1}_d\mathbf{1}_d' + (1-\kappa)I_d$ with $-1/(d-1) < \kappa < 1$ as defined in equation (2.9), and the d-dimensional Clayton copula with parameter $\theta > 0$ as given in equation (2.13). Simulation results are provided for dimensions $d = 2, 5$, and 10, different parameter choices and sample sizes n (see the first and second columns of the tables, respectively). Further, we set $c_{j,n} = j^2, j = 1, \dots, n$, in all simulations. Specifically, column three provides an approximation to the true value of d-dimensional Spearman's rho ρ for the specific copula model, which is calculated from one sample of size 500,000. Columns four and six of the tables show the empirical means $m(\widehat{\rho}_n^c)$ and the standard deviation $\widehat{\sigma}(\widehat{\rho}_n^c)$ of the estimator $\widehat{\rho}_n^c$ with respect to 300 Monte-Carlo simulations of size n. Note that, according to theorem 4.2.6, the estimator $\widehat{\rho}_n^c$ represents a consistent estimator for ρ. The corresponding bootstrap estimates are contained in columns five and seven and are based on 250 bootstrap replications, which are drawn with replacement from each original sample: $m(\widehat{\rho}_n^{c,B})$ denotes the empirical mean of the estimator $\widehat{\rho}_n^{c,B}$ and $m(\widehat{\sigma}^{c,B})$ the empirical mean of the bootstrap estimator $\widehat{\sigma}^{c,B}$; cf. equation (4.24). In addition, column seven provides the standard deviation of the bootstrap estimator $\widehat{\sigma}^{c,B}$ over 300 Monte-Carlo simulations. Finally, the bootstrap estimates for the standard deviation σ as given in equation (4.25) are shown in column eight.

Table 4.1: **Gaussian copula**. Simulation results for estimating the asymptotic standard deviation of weighted Spearman's rho $\widehat{\rho}_n^c$ based on $c_{j,n} = j^2, j = 1, \ldots, n$, (cf. formula (4.28)) by means of the nonparametric bootstrap: The table shows the empirical mean $m(\cdot)$ and the empirical standard deviation $\widehat{\sigma}(\cdot)$ of the respective estimates, which are calculated based on 300 Monte Carlo simulations of sample size n of a d-dimensional equi-correlated Gaussian copula with parameter κ and 250 bootstrap samples. The bootstrap estimates are labeled by the superscript B.

κ	n	ρ	$m(\widehat{\rho}_n^c)$	$m(\widehat{\rho}_n^{c,B})$	$\widehat{\sigma}(\widehat{\rho}_n^c)$	$m(\widehat{\sigma}^{c,B})$	$\widehat{\sigma}(\widehat{\sigma}^{c,B})$	$m(\sqrt{n}\,\widehat{\sigma}^{c,B})$
Dimension d=2								
0.2	100	.190	.196	.176	.135	.131	.011	1.313
	500	.190	.190	.197	.058	.059	.003	1.313
	1000	.190	.192	.192	.041	.042	.002	1.313
0.5	100	.482	.490	.470	.113	.117	.014	1.172
	500	.482	.482	.484	.056	.052	.003	1.169
	1000	.482	.482	.480	.037	.037	.002	1.173
-0.1	100	-.094	-.089	-.089	.142	.133	.011	1.326
	500	-.094	-.096	-.094	.061	.060	.003	1.336
	1000	-.094	-.097	-.092	.040	.042	.002	1.334
Dimension d=5								
0.2	100	.160	.155	.148	.057	.053	.010	.527
	500	.160	.161	.156	.026	.025	.002	.554
	1000	.160	.160	.158	.016	.018	.001	.560
0.5	100	.439	.434	.420	.081	.075	.008	.754
	500	.439	.441	.437	.035	.035	.002	.778
	1000	.439	.439	.437	.026	.025	.001	.782
-0.1	100	-.070	-.069	-.066	.021	.021	.006	.206
	500	-.070	-.069	-.070	.010	.009	.001	.206
	1000	-.070	-.070	-.069	.007	.007	.001	.208
Dimension d=10								
0.2	100	.063	.064	.061	.027	.022	.012	.221
	500	.063	.062	.062	.012	.011	.003	.247
	1000	.063	.062	.063	.008	.008	.001	.257
0.5	100	.285	.276	.259	.069	.062	.016	.619
	500	.285	.284	.279	.030	.031	.004	.694
	1000	.285	.284	.283	.022	.022	.002	.698
-0.1	100	-.009	-.009	-.009	.000	.000	.000	.002
	500	-.009	-.009	-.009	.000	.000	.000	.002
	1000	-.009	-.009	-.009	.000	.000	.000	.002

Table 4.2: **Clayton copula**. Simulation results for estimating the asymptotic standard deviation of weighted Spearman's rho $\widehat{\rho}_n^c$ based on $c_{j,n} = j^2, j = 1, \ldots, n$, (cf. formula (4.28)) by means of the nonparametric bootstrap: The table shows the empirical mean $m(\cdot)$ and the empirical standard deviation $\widehat{\sigma}(\cdot)$ of the respective estimates, which are calculated based on 300 Monte Carlo simulations of sample size n of a d-dimensional Clayton copula with parameter θ and 250 bootstrap samples. The bootstrap estimates are labeled by the superscript B.

θ	n	ρ	$m(\widehat{\rho}_n^c)$	$m(\widehat{\rho}_n^{c,B})$	$\widehat{\sigma}(\widehat{\rho}_n^c)$	$m(\widehat{\sigma}^{c,B})$	$\widehat{\sigma}(\widehat{\sigma}^{c,B})$	$m(\sqrt{n}\,\widehat{\sigma}^{c,B})$
Dimension d=2								
0.1	100	.072	.071	.070	.135	.134	.010	1.338
	500	.072	.072	.073	.063	.060	.003	1.346
	1000	.072	.070	.074	.041	.043	.002	1.349
0.5	100	.293	.281	.279	.137	.130	.013	1.298
	500	.293	.294	.295	.061	.058	.004	1.298
	1000	.293	.297	.297	.041	.041	.002	1.302
1	100	.478	.474	.477	.131	.121	.014	1.208
	500	.478	.477	.473	.052	.054	.004	1.212
	1000	.478	.476	.478	.037	.038	.002	1.209
Dimension d=5								
0.1	100	.057	.053	.053	.043	.041	.010	.407
	500	.057	.058	.055	.019	.019	.003	.427
	1000	.057	.057	.058	.014	.014	.001	.432
0.5	100	.256	.254	.246	.071	.066	.011	.661
	500	.256	.256	.250	.031	.031	.003	.684
	1000	.256	.258	.256	.022	.022	.001	.692
1	100	.437	.430	.416	.084	.078	.009	.784
	500	.437	.434	.432	.035	.036	.002	.816
	1000	.437	.438	.437	.026	.026	.001	.821
Dimension d=10								
0.1	100	.016	.015	.015	.012	.008	.005	.084
	500	.016	.015	.016	.005	.005	.002	.107
	1000	.016	.016	.016	.004	.003	.001	.110
0.5	100	.132	.127	.123	.049	.041	.018	.414
	500	.132	.132	.130	.023	.022	.005	.494
	1000	.132	.131	.131	.016	.016	.003	.508
1	100	.299	.293	.275	.081	.069	.016	.691
	500	.299	.292	.290	.036	.035	.004	.791
	1000	.299	.295	.298	.025	.026	.002	.813

Both $m(\widehat{\rho}_n^c)$ and $m(\widehat{\rho}_n^{c,B})$ exhibit a bias which clearly decreases with increasing sample size. Further, their values are quite close to each other for large sample sizes, especially for $n = 1,000$. This also applies to $\widehat{\sigma}(\widehat{\rho}_n^c)$ and $m(\widehat{\sigma}^{c,B})$, whose values show a great similarity, which suggests that the proposed bootstrap method performs well to estimate the asymptotic variance of $\sqrt{n}(\widehat{\rho}_n^c - \rho)$ for the considered copula models and weights. Further, the standard deviation of the bootstrap estimator $\widehat{\sigma}^{c,B}$ decreases with increasing sample size.

4.4 Empirical study

To illustrate the properties of the proposed weighted estimator for multivariate Spearman's rho, we apply the theoretical results to financial data. We consider time series of daily equity (log-) returns of the four international banks BNP Paribas (BNP), Credit Suisse Group (CS), Deutsche Bank (DBK), and Barclays (BARC) during the period from May 1997 to April 2010 (cf. section 2.1).

Figure 4.3 shows the development of the equity prices of all four banks over the considered time horizon, plotted with respect to their value on May 6, 1997. In general, all series evolve similarly. While equity prices proved to be quite stable in the first half of the observation period, they are much more volatile in the second half. In particular, equity prices from all four banks clearly show a positive trend from the year 2003 onwards, having their peak in mid-2007. Thereafter, they decrease in the course of the beginning financial crisis; especially after the bankruptcy of the investment bank Lehman Brothers in September 2008 we observe a sharp decline. Those observations are emphasized by the development of the series of corresponding daily (log-) returns, which are exemplary provided for banks BARC and CS in figure 4.4. Indeed, the returns of those banks are less volatile in the first years of the observation period while exhibiting a phase of rather high volatility between the years 2008 and 2010. This especially applies to bank BARC. In addition, table 4.3 provides the estimated first four moments (in %) of the return series for all banks. While the returns of all four banks show a non-negative skewness and excess kurtosis, the large estimated kurtosis of bank BARC is particularly noticeable. This does not only go in line with the observed rather high volatile behavior of its equity prices/returns from 2007 onwards, but can also be partly put down to one event in January 2009 where "shares in BARC have jumped by more than 70% after the bank wrote an open letter to reassure investors of its continuing good health"; see BBC News (2009).

In order to apply the theoretical results discussed in the previous section, which are derived under the assumption of independent observations, the returns have to be standardized adequately. Assume therefore that the return of bank $i, i = 1, \ldots 4$, at discrete time t is modeled by a random variable $X_{i,t}$. We proceed according to the methodology proposed by Patton (2002) where the analysis identifies the following

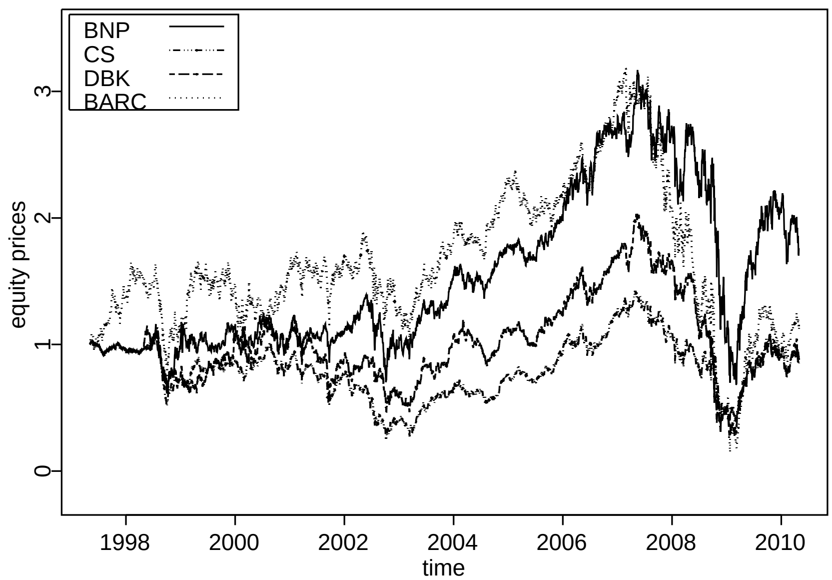


Figure 4.3: Evolution of equity prices of the four banks over the considered time horizon, plotted with respect to their value on May 6, 1997.

ARMA(1,1)-t-GARCH(1,1) specification to model the return series (cf. chapter 3):

$$X_{i,t} = \mu_i + \phi_i X_{i,t-1} + \psi_i \epsilon_{i,t-1} + \epsilon_{i,t} \tag{4.29}$$

$$\sigma^2_{i,t} = \omega_i + \beta_i \sigma^2_{i,t-1} + \alpha_i \epsilon^2_{i,t-1} \tag{4.30}$$

$$\sqrt{\frac{\nu_i}{\sigma^2_{i,t}(\nu_i - 2)}} \cdot \epsilon_{i,t} \sim i.i.d.\ t_{\nu_i}. \tag{4.31}$$

The model assumes that the conditional variances $\sigma^2_{i,t}$ evolve according to a GARCH(1,1) process, and that the (standardized) innovations are independent and identically distributed according to a Student's t-distribution with ν_i degrees of freedom, $i = 1, \ldots, 4$. The unknown parameters of the above models are estimated by means of a Quasi-Maximum Likelihood (QML) method (see table 4.4 in appendix 4.5). Figure 4.5 gives the autocorrelation function of the squared standardized returns of all banks, which exhibit only minor serial correlation; the same applies to the standardized returns themselves. In addition, table 4.5 in appendix 4.5 displays the results of the Ljung-Box (LB) Q-statistics, computed from the squared returns up to lag twenty. The null hypothesis of no serial correlation is not rejected for all series.

Figure 4.6 shows the evolution of weighted multivariate Spearman's rho and ordinary multivariate Spearman's rho of the banks' standardized returns during the years 2004 to 2009, calculated according to equations (4.3) and (4.2). The estimation is based on a moving window approach with window size $n = 250$ while the constants $c_{j,n}$ for weighted Spearman's rho are either $c_{j,n} = j$ (left panel) or $c_{j,n} = j^2, j = 1, \ldots, n$ (right panel); cf. formula (4.28). The differences between the weighted and the ordi-

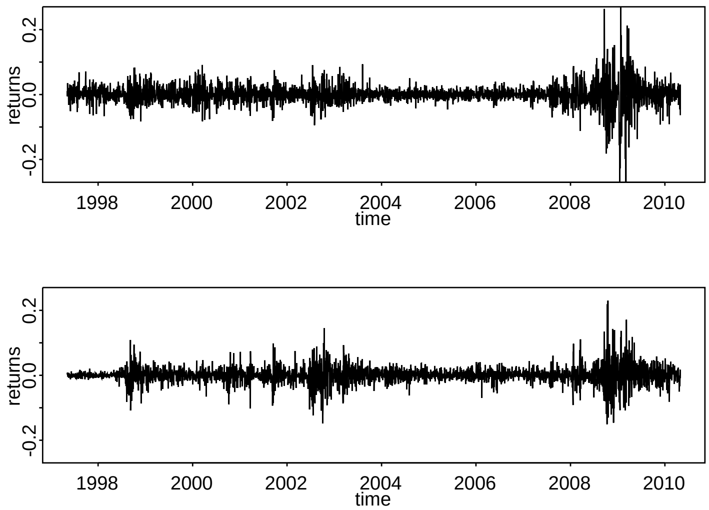


Figure 4.4: Evolution of (log-) returns of equity prices of the banks BARC (upper panel) and CS (lower panel) over the considered time horizon.

Table 4.3: Estimated first four moments (in %) of all four banks.

	Mean	StDev	Kurtosis	Skewness
BNP	0.0242	2.4589	6.4136	0.0054
CS	-0.0021	2.6860	8.0689	0.2995
DBK	-0.0017	2.5861	9.9674	0.2402
BARC	-0.0120	3.2290	40.6446	1.4632

nary Spearman's rho are well identifiable. For example in June 2007, financial markets went into a first turmoil caused by the problems of the investment bank Bear Stearns, which led to an increase in correlation between the banks' returns. Both weighted estimators for Spearman's rho capture this development while the equally weighted estimator takes longer to incorporate this course of events. In this example, the weighted estimators give a more satisfactory estimate of the current value of Spearman's rho. Naturally, the weighted estimator in the right panel of figure 4.6 is more volatile than the estimator in the left panel due to the nature of the weights.
Since both weighting schemes satisfy condition **(C)** defined in the previous section, the asymptotic variance of the weighted estimators as derived in theorem 4.2.6 can be estimated using the nonparametric bootstrap procedure described at the end of section 4.2. Specifically, we consider a time period of 100 observations before June 1, 2007, and assume that Spearman's rho does not change throughout this period. Further, $c_j = j, j = 1, \ldots, n$. The value of $\widehat{\rho}_n^c$ and of its estimated asymptotic variance (as

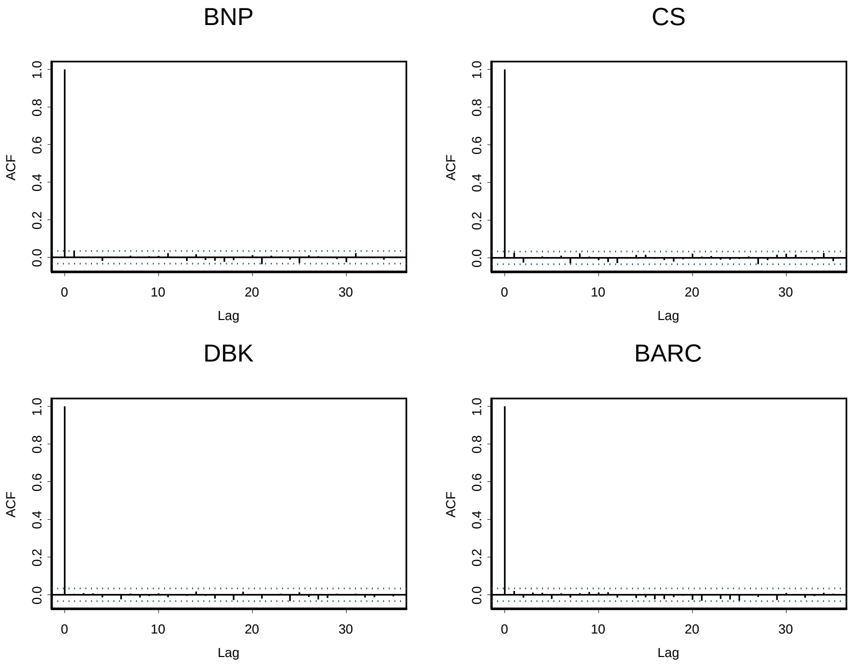


Figure 4.5: Autocorrelation function of the squared standardized returns of banks BNP, CS, DBK, and BARC.

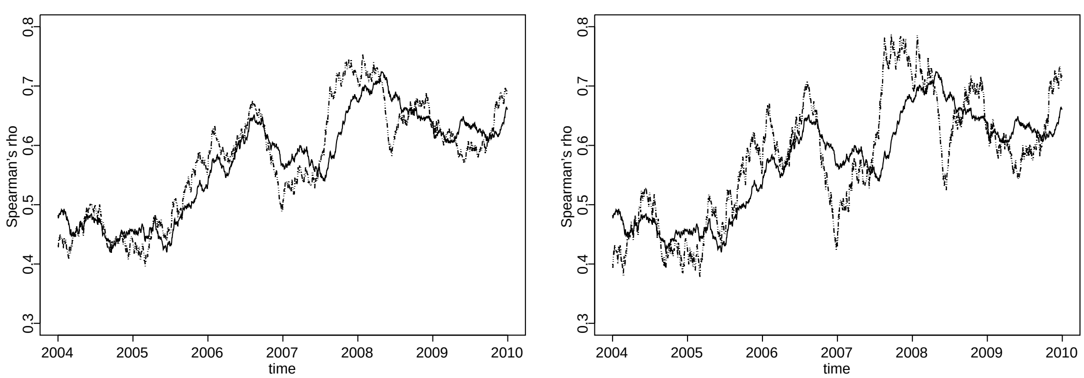


Figure 4.6: Weighted multivariate Spearman's rho $\widehat{\rho}_n^c$ (dotted) and multivariate Spearman's rho $\widehat{\rho}_n$ (solid) of the standardized returns of the four banks for the years 2004 to 2009. The estimation is based on a moving window approach with window size $n = 250$ and weights $c_{j,n} = j$ (left panel) and $c_{j,n} = j^2$ (right panel), cf. formula (4.28).

stated in equation (4.25)) in this period are

$$\widehat{\rho}_n^c = 0.54092 \quad \text{and} \quad \sqrt{n}\,\widehat{\sigma}^{c,B} = 0.731011. \tag{4.32}$$

For comparison, we additionally provide the corresponding values of the equally weighted estimator $\widehat{\rho}_n$ in (4.2):

$$\widehat{\rho}_n = 0.565052 \quad \text{and} \quad \sqrt{n}\,\widehat{\sigma}_n^B = 0.516593. \tag{4.33}$$

In particular, the estimated asymptotic variance of $\widehat{\rho}_n^c$ is larger than that of $\widehat{\rho}_n$ as a result of the different weighting schemes. Based on these estimates, an asymptotic $(1-\alpha)$-confidence interval for Spearman's rho can be calculated. According to theorem 4.2.6, such a confidence interval is given by

$$\widehat{\rho}_n^c \pm \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\widehat{\sigma}^{c,B}/\sqrt{n} \quad \text{and} \quad \widehat{\rho}_n \pm \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\widehat{\sigma}^{B}/\sqrt{n},$$

respectively. Figure 4.7 shows the evolution of weighted multivariate Spearman's rho and ordinary multivariate Spearman's rho of the banks' standardized returns during the years 2007 to 2009 (based on a moving window with samples size 250) together with the asymptotic 95%-confidence intervals for Spearman's rho calculated from the estimated values in (4.32) and (4.33). Naturally, the confidence interval based on the weighted estimator is wider than that derived from the equally weighted estimator due to the larger asymptotic variance of the former. Further, we can conclude that

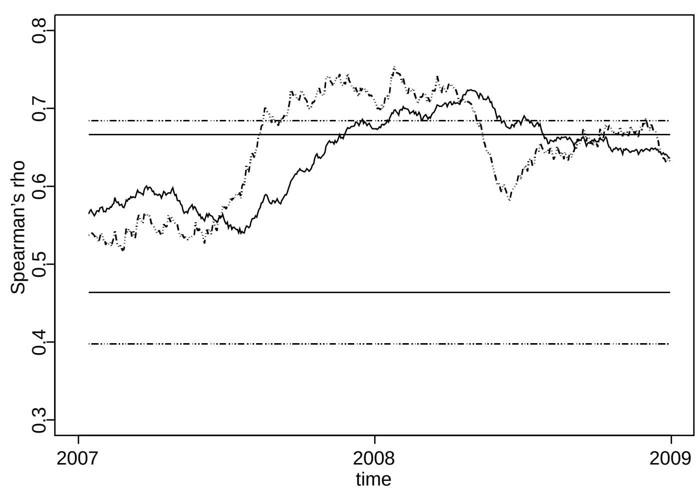


Figure 4.7: Weighted multivariate Spearman's rho $\widehat{\rho}_n^c$ (dotted) and multivariate Spearman's rho $\widehat{\rho}_n$ (solid) of the standardized returns of the banks for the years 2007 to 2009. The estimation is based on a moving window approach with window size $n = 250$ and weights $c_{j,n} = j$; cf. formula (4.28). The horizontal lines indicate asymptotic 95%-confidence intervals for Spearman's rho calculated from the estimated values in (4.32) (dotted lines) and (4.33) (solid lines), respectively, which are estimated from 100 observations before June 1, 2007.

Spearman's rho significantly differs from the value of Spearman's rho in the time period of 100 observations before June 2007, if its estimated values lies outside the confidence intervals. Hence, for both estimators we observe a significant change of Spearman's rho in the course of the financial crisis. This change, however, is indicated much earlier by the weighted estimator, suggesting anew that this estimator captures (sudden) changes in Spearman's rho more satisfactorily.

4.5 Appendix

The following table displays the estimation output when fitting ARMA(1,1)-t-GARCH(1,1) models to the banks' equity return series; cf. equations (4.29)–(4.31).

Table 4.4: Estimates of the coefficients and asymptotic standard errors for the banks' equity returns according to equations (4.29)–(4.31).

	BNP		CS		DBK		BARC	
	Coeff.	SE	Coeff.	SE	Coeff.	SE	Coeff.	SE
Const. (μ_i)	0.0177	0.0703	-0.0064	0.0334	-0.0072	0.0737	0.0001	9.7534
AR(1) (ϕ_i)	-0.1416	0.2610	0.0961	0.4726	-0.1799	0.1547	0.2365	2.5108
MA(1) (ψ_i)	0.1643	0.2578	-0.0913	0.4621	0.2583	0.1445	-0.1735	2.6166
GARCH Const. (ω_i)	0.0129	0.0061	0.0262	0.0111	0.0191	0.0105	0.0590	0.0233
Lag. Var. (β_i)	0.9176	0.0100	0.8885	0.0137	0.8995	0.0133	0.8880	0.0214
Lag. Squ. Err. (α_i)	0.0824	0.0092	0.1115	0.0125	0.1005	0.0118	0.1092	0.0211
D.o.F (ν_i)	6.6600	0.6603	6.4639	0.6057	7.4878	0.9363	7.6297	0.9406

The table below shows the output of the Ljung-Box (LB) test for the banks' standardized return series. A description of the results is given in section 4.4.

Table 4.5: Value of the test statistic and corresponding p-value of the Ljung-Box (LB) test for the squared standardized returns, calculated up to lag twenty.

	BNP	CS	DBK	BARC
LB Q-statistics	15.6468	23.0311	11.3763	15.3863
LB p-values	0.7383	0.2873	0.9359	0.7539

Chapter 5

Testing equality of pairwise rank correlations in a multivariate random vector

In this chapter, we consider statistical tests for the hypothesis that all pairwise Spearman's rank correlation coefficients in a multivariate random vector are equal. The tests are nonparametric and their asymptotic distributions are derived based on the asymptotic behavior of the empirical copula process. Only weak assumptions on the distribution function, such as continuity of the marginal distributions and continuous partial differentiability of the copula, are required for obtaining the results. A nonparametric bootstrap method is suggested for either estimating unknown parameters of the test statistics or for determining the associated critical values. We present a simulation study in order to investigate the power of the proposed tests. The results are compared to a classical parametric test for equal pairwise Pearson's correlation coefficients in a multivariate random vector. The general setting also allows the derivation of a test for stochastic independence based on Spearman's rho. The proposed tests are applied to financial data.

5.1 Preliminaries

Let $\mathbf{X}$ be a d-dimensional random vector with distribution function F, continuous univariate marginal distribution functions $F_1, \ldots, F_d$ and copula C. Bivariate Spearman's rho of the components X_k and X_l of $\mathbf{X}$ is defined as

$$\begin{aligned} \rho_{kl} &= \frac{Cov\{F_k(X_k), F_l(X_l)\}}{\sqrt{Var\{F_k(X_k)\}}\sqrt{Var\{F_l(X_l)\}}} \\ &= 12\int_0^1\int_0^1 C_{kl}(u_k, u_l)du_k du_l - 3, \qquad k, l \in \{1, \ldots, d\}, \end{aligned} \tag{5.1}$$

cf. section 2.3.3. Here, the copula C_{kl} denotes the bivariate copula which corresponds to the kth and lth margin of C, that is, $C_{kl}(u_k, u_l) = C(\mathbf{u}^{(k,l)})$. We develop four (asymptotic) tests for the null hypothesis of equi Spearman's rank-correlation, i.e., that the pairwise Spearman's rho coefficients between all (distinct) components of $\mathbf{X}$ are equal; cf. Gaißer and Schmid (2010). Due to the symmetry of Spearman's rho (i.e. $\rho_{kl} = \rho_{lk}$ for all $k, l \in \{1, \ldots, d\}$) we can confine ourselves to testing the equality of all $m = \binom{d}{2}$ bivariate Spearman's rho coefficients ρ_{kl} with $k < l$ and $k, l \in \{1, \ldots, d\}$. Hence, the null hypothesis of tests for equi rank-correlation is given by

$$\mathrm{H}_0: \quad \rho_{12} = \rho_{13} = \cdots = \rho_{d-1,d}, \tag{5.2}$$

against the alternative that at least two of the ρ_{kl} in (5.2) differ. By defining the m-dimensional vector

$$\boldsymbol{\rho} = (\rho_{12}, \rho_{13}, \ldots, \rho_{1d}, \rho_{23}, \ldots, \rho_{d-1,d})', \tag{5.3}$$

the hypothesis in (5.2) can alternatively be expressed as

$$\mathrm{H}_0: \quad \boldsymbol{\rho} = \rho \mathbf{1}_m \quad \text{versus} \quad \mathrm{H}_1: \quad \boldsymbol{\rho} \neq \rho \mathbf{1}_m \tag{5.4}$$

with unspecified parameter ρ satisfying $-1/(d-1) < \rho < 1$ according to part (i) of proposition 5.7.1; see appendix 5.7.1.

Assume that neither F or C nor the marginal distribution functions $F_i, i = 1, \ldots, d$, of the random vector $\mathbf{X}$ are known and let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a random sample from $\mathbf{X}$. A nonparametric estimator of $\boldsymbol{\rho}$ is given by the random vector

$$\widehat{\boldsymbol{\rho}}_n = (\widehat{\rho}_{12,n}, \widehat{\rho}_{13,n}, \ldots, \widehat{\rho}_{d-1,d,n})', \tag{5.5}$$

with

$$\widehat{\rho}_{kl,n} = 12 \int_0^1 \int_0^1 \widehat{C}_{kl,n}(u_k, u_l) du_k du_l - 3 = \frac{12}{n} \sum_{j=1}^{n} (1 - \widehat{U}_{kj,n})(1 - \widehat{U}_{lj,n}) - 3, \quad k < l, \tag{5.6}$$

which is obtained by replacing the copula C_{kl} in equation (5.1) by the empirical copula $\widehat{C}_{kl,n} = \widehat{C}_n(\mathbf{u}^{(k,l)})$ as defined in (2.14); cf. section 2.3.3. To keep notation simple, we will refer to the i-th element, $i = 1, \ldots, m$, of the vectors $\boldsymbol{\rho}$ and $\widehat{\boldsymbol{\rho}}_n$, respectively, as ρ_i and $\widehat{\rho}_{i,n}$ in the following. The next theorem establishes asymptotic normality of the random vector $\widehat{\boldsymbol{\rho}}_n$ and forms the basis for the forthcoming results on the tests of equi rank-correlation.

Theorem 5.1.1 *Consider the random sample $(\mathbf{X}_j)_{j=1,\ldots,n}$ from the d-dimensional random vector $\mathbf{X}$ with joint distribution function F, continuous univariate marginal distribution functions $F_1, \ldots, F_d$, and copula C. Under the assumption that the i-th partial derivatives $D_i C(\mathbf{u})$ of C exist and are continuous for $i = 1, \ldots, d$, we have*

$$\sqrt{n}\,(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho}) \xrightarrow{d} \mathbf{Z} \sim N(\mathbf{0}_m, \Sigma^{asym}) \quad as \quad n \to \infty.$$

The elements $\Sigma^{asym}_{(k,l)(s,t)}$ *of* Σ^{asym} *are given by*

$$\Sigma^{asym}_{(k,l)(s,t)} = 144 \int_{[0,1]^d} \int_{[0,1]^d} E\{\mathbb{G}_C(\mathbf{u}^{(k,l)})\mathbb{G}_C(\mathbf{v}^{(s,t)})\} d\mathbf{u} d\mathbf{v} \tag{5.7}$$

where $k < l$ *and* $s < t$, *and Gaussian process* $\mathbb{G}_C$ *as defined in equation (2.17); cf. theorem 2.2.8.*

Proof. The assertion is based on the weak convergence of the empirical copula process $\sqrt{n}(\widehat{C}_n - C)$ to the Gaussian process $\mathbb{G}_C$ (see theorem 2.2.8). Note that

$$\sqrt{n}\ (\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho}) = g[\sqrt{n}\ \{\widehat{C}_n(\mathbf{u}) - C(\mathbf{u})\}]$$

can be expressed as a linear and continuous map $g : \ell^\infty([0,1]^d) \to \mathbb{R}^m$ of the empirical copula process. The weak convergence of $\sqrt{n}\ (\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho})$ to the random vector $\mathbf{Z} = g(\mathbb{G}_C)$ follows according to the continuous mapping theorem. Using the fact that $\mathbb{G}_C$ is a tight Gaussian process, lemma 3.9.8 in van der Vaart and Wellner (1996), p. 377, implies that $\mathbf{Z}$ is multivariate normally distributed with mean vector zero and covariance matrix Σ^{asym}. In particular,

$$\Sigma^{asym}_{(k,l)(s,t)} = E\Big\{12 \int_{[0,1]^d} \mathbb{G}_C(\mathbf{u}^{(k,l)}) d\mathbf{u} \cdot 12 \int_{[0,1]^d} \mathbb{G}_C(\mathbf{v}^{(s,t)}) d\mathbf{v}\Big\}$$

and an application of Fubini's theorem yields the asserted form of $\Sigma^{asym}_{(k,l)(s,t)}$. □

Observe that the process $\mathbb{G}_C(\mathbf{u}^{(k,l)})$ in equation (5.7) has the form

$$\mathbb{G}_C(\mathbf{u}^{(k,l)}) = \mathbb{B}_C(\mathbf{u}^{(k,l)}) - D_k C(\mathbf{u}^{(k,l)})\mathbb{B}_C(\mathbf{u}^{(k)}) - D_l C(\mathbf{u}^{(k,l)})\mathbb{B}_C(\mathbf{u}^{(l)}).$$

In particular, the covariance matrix Σ^{asym} of the limiting random vector $\mathbf{Z}$ depends on the unknown copula C. In section 5.2, we make use of the nonparametric bootstrap as described in section 2.2.2 to estimate it. This method has also been utilized by Schmid and Schmidt (2006) in order to estimate the variance of a multivariate version of Spearman's rho, cf. section 2.3.3. Even if the copula is known and although theorem 5.1.1 yields a closed-form expression for Σ^{asym}, the latter can only be calculated explicitly for some special copulas (cf. Schmid and Schmidt (2007a)). For example if C is the independence copula, i.e. $C = \mathit{\Pi}$, it can be shown that $\Sigma^{asym}_{(k,l)(k,l)} = 1$ and $\Sigma^{asym}_{(k,l)(s,t)} = 0$, for all $(k,l) \neq (s,t)$ and, hence, $\Sigma^{asym} = I_m$.

Remark. Note that, in terms of the rank correlation matrix $P = (\rho_{kl})_{1 \leq k,l \leq d}$ of $\mathbf{X}$, hypothesis (5.2) is equivalent to the assertion that $P = \rho \mathbf{1}_d \mathbf{1}_d' + (1-\rho) I_d$ with unknown rank correlation coefficient ρ. It is well-known that a general matrix B of the form $B = r\mathbf{1}_d\mathbf{1}_d' + (1-r)I_d$ with parameter $r \in [-1,1]$ is a Pearson's correlation matrix (i.e. there exists a d-dimensional random vector $\mathbf{X}$ with equal pairwise Pearson's correlation coefficients $r_{ij}, i \neq j$) if and only if $-1/(d-1) < r < 1$. The range of ρ such that a general matrix $B = \rho\mathbf{1}_d\mathbf{1}_d' + (1-\rho)I_d$ is a rank correlation matrix has been discussed in Embrechts et al. (2002) and is restated in proposition 5.7.1 (appendix 5.7.1) together with a shorter proof.

The following test statistics for equi rank-correlation are all formulated in terms of the random vector $\widehat{\boldsymbol{\rho}}_n$. They are thus nonparametric and their asymptotic distribution under the null hypothesis of equi rank-correlation can be derived from the asymptotic behavior of $\sqrt{n}\,(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho})$. In particular, they can be applied without further assumptions on the marginal distribution functions than continuity. We further give theoretical proofs of the validity of the bootstrap procedures, which are used in the tests for either estimating unknown parameters of the test statistics or for determining the associated critical values.
Note that tests for equi rank-correlation also play a role for the choice of an appropriate copula in multivariate distribution modeling, the latter being the central theme of many works; see e.g. Fermanian (2005), Dobrić and Schmid (2007), Savu and Trede (2008), and Genest et al. (2009). Since the bivariate Spearman's rho coefficients of popular copulas such as the multivariate Archimedean copulas (as introduced in section 2.2.1) are equal, tests for equi rank-correlation can be used to verify this copula assumption. Similar (asymptotic) statistical tests based on Kendall's tau (see (2.43)) can be derived once the asymptotic behavior of Kendall's tau using the empirical copula process has been established.

5.2 Statistical tests for equi rank-correlation

We introduce four nonparametric hypothesis tests for equi rank-correlation of a d-dimensional random vector $\mathbf{X}$ with distribution function F and copula C.

5.2.1 Test statistics $\mathcal{T}_{n,1}$ and $\mathcal{T}_{n,2}$

Using the same notation as in the previous section, the random vector $\mathbf{X}$ is equi rank-correlated if and only if the pairwise differences between ρ_i and $\rho_{i+1},\ i = 1, \ldots, m-1$, are zero. Thus, a first test statistic for equi rank-correlation takes the form

$$\mathcal{T}_{n,1} = \frac{n}{m-1} \sum_{i=1}^{m-1} (\widehat{\rho}_{i,n} - \widehat{\rho}_{i+1,n})^2. \tag{5.8}$$

Note that $\mathcal{T}_{n,1}$ is not invariant in the sense that a permutation of the univariate margins of $\mathbf{X}$ may lead to a different test statistic. Instead, it relies on the order of the Spearman's rho coefficients as they are mapped to the vector $\boldsymbol{\rho}$; cf. (5.3). This property is less desirable as it may affect the power of the test (see also discussions in section 5.4.2). The following test statistics are therefore invariant with respect to such permutations. We define a second test statistic for equi rank-correlation by

$$\mathcal{T}_{n,2} = \frac{n}{m} \sum_{i=1}^{m} (\widehat{\rho}_{i,n} - \overline{\widehat{\rho}}_n)^2, \tag{5.9}$$

with $\overline{\widehat{\rho}}_n = 1/m \sum_{i=1}^{m} \widehat{\rho}_{i,n}$. It is based on another sufficient and necessary condition for equi-rank correlation, namely that the variance of all coefficients ρ_i, $i = 1, \ldots, m$, is zero. For both test statistics, we reject H_0 in formula (5.4) for large values of $\mathcal{T}_{n,i}, i = 1, 2$, that is, $\mathcal{T}_{n,i} > c_i$ with c_i being an appropriate critical value. The next theorem establishes their asymptotic null distribution.

Theorem 5.2.1 *Let $(\mathbf{X}_j)_{j=1,\ldots,n}$ denote a random sample from the d-dimensional random vector $\mathbf{X}$ with distribution function F, continuous univariate marginal distribution functions $F_1, \ldots, F_d$, and copula C. Under the null hypothesis (5.4) and under the prerequisites of theorem 5.1.1 with the related notation, the test statistics $\mathcal{T}_{n,i}, i = 1, 2$, have limiting distributions, that is,*

$$\mathcal{T}_{n,i} \xrightarrow{d} W_i, \quad \text{for} \quad n \to \infty,$$

with non-degenerated random variables $W_i, i = 1, 2$. In particular, the W_i are in distribution equivalent to a linear combination of independent χ^2- distributed random variables :

$$W_1 = \sum_{k=1}^{r} \lambda_k \chi^2_{\nu_k}$$

where the weights λ_k are the r distinct, non-zero eigenvalues of the matrix $\frac{1}{m-1} A' A \Sigma^{asym}$ with algebraic multiplicity ν_k. The $(m-1) \times m$ matrix A is defined as

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -1 & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & .. & 0 & 1 & -1 & 0 \\ 0 & .. & 0 & 0 & 1 & -1 \end{pmatrix}, \tag{5.10}$$

and Σ^{asym} denotes the asymptotic covariance matrix of $\sqrt{n}\,(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho})$, cf. theorem 5.1.1. Further,

$$W_2 = \sum_{k=1}^{r} \delta_k \chi^2_{\mu_k}$$

where the weights δ_k are the r distinct, non-zero eigenvalues of the matrix $B\Sigma^{asym}$ with algebraic multiplicity μ_k and matrix $B = \frac{1}{m}(I_m - \frac{1}{m}\mathbf{1}_m \mathbf{1}_m')$.

Proof. Observe that both test statistics $\mathcal{T}_{n,i}, i = 1, 2$, allow for a quadratic form representation under H_0. Namely, with matrix A as defined in formula (5.10),

$$\begin{aligned} \mathcal{T}_{n,1} = \frac{n}{m-1} \sum_{i=1}^{m-1} (\widehat{\rho}_{i,n} - \widehat{\rho}_{i+1,n})^2 &= \frac{1}{m-1} \{\sqrt{n}(A\widehat{\boldsymbol{\rho}}_n - A\boldsymbol{\rho})\}' \{\sqrt{n}(A\widehat{\boldsymbol{\rho}}_n - A\boldsymbol{\rho})\} \\ &= \sqrt{n}(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho})' \frac{1}{m-1} A' A \sqrt{n}(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho}), \end{aligned}$$

since $A\boldsymbol{\rho} = \mathbf{0}_{m-1}$ under H_0, and

$$\begin{aligned}
\mathcal{T}_{n,2} &= \frac{n}{m}\sum_{i=1}^{m}(\widehat{\rho}_{i,n} - \bar{\widehat{\rho}}_n)^2 = n\widehat{\boldsymbol{\rho}}_n'\frac{1}{m}(I_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m')\widehat{\boldsymbol{\rho}}_n \\
&= n\widehat{\boldsymbol{\rho}}_n'\frac{1}{m}(I_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m')\widehat{\boldsymbol{\rho}}_n - 2n\widehat{\boldsymbol{\rho}}_n'\frac{1}{m}(I_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m')\boldsymbol{\rho} \\
&\quad + n\boldsymbol{\rho}'\frac{1}{m}(I_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m')\boldsymbol{\rho} \qquad (5.11) \\
&= \sqrt{n}(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho})'\frac{1}{m}(I_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m')\sqrt{n}(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho}), \qquad (5.12)
\end{aligned}$$

as the last two terms in equation (5.11) vanish under H_0. An application of the continuous mapping theorem together with theorem 5.1.1 yields

$$\mathcal{T}_{n,1} \xrightarrow{d} \mathbf{Z}'\frac{1}{m-1}A'A\mathbf{Z} = W_1 \quad \text{and} \quad \mathcal{T}_{n,2} \xrightarrow{d} \mathbf{Z}'\frac{1}{m}(I_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m')\mathbf{Z} = W_2$$

for $n \to \infty$ and $\mathbf{Z} \sim N(\mathbf{0}_m, \Sigma^{asym})$. The fact that the $W_i, i = 1, 2$, can be expressed as weighted sums of independent χ^2-distributed random variables is a direct consequence of their representation as quadratic forms of normally distributed random vectors; see e.g. Scheffé (1959), p. 418. □

Both limiting distributions depend on the asymptotic covariance matrix Σ^{asym} of $\sqrt{n}\ (\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho})$ (see theorem 5.1.1) and their explicit form can only be determined for some special copulas of simple structure. For example if $C = \Pi$, the weights λ_k and δ_k are the eigenvalues of the matrices $A'A$ and B, respectively, since $\Sigma^{asym} = I_m$. In particular, matrix B possesses the eigenvalues $1/m$ with algebraic multiplicity $m-1$ and 0 with algebraic multiplicity 1. Hence in the case of stochastic independence, $\mathcal{T}_{n,2}$ has asymptotically the same distribution as the random variable $\frac{1}{m}Y$ where Y follows a χ^2-distribution with $m-1$ degrees of freedom (cf. section 5.5). Instead of a direct estimation of the quantiles/critical values of the limiting distributions of $\mathcal{T}_{n,1}$ and $\mathcal{T}_{n,2}$, we make use of a bootstrap technique described in Bickel and Ren (2001).

We denote by $\mathcal{F}_d$ the set of all d-dimensional distribution functions. By writing the vector $\boldsymbol{\rho}$ of the m bivariate Spearman's rho coefficients as a function $\boldsymbol{\rho}(F)$ of $F \in \mathcal{F}_d$ (see proof of theorem 5.2.2 for a description of $\boldsymbol{\rho}(F)$), consider the functionals $T_1 : \mathcal{F}_d \to I\!R^{m-1}$ and $T_2 : \mathcal{F}_d \to I\!R^m$ defined by

$$T_1(F) = A\boldsymbol{\rho}(F) \quad \text{and} \quad T_2(F) = \boldsymbol{\rho}(F) - \frac{\mathbf{1}_m'\boldsymbol{\rho}(F)}{m}\mathbf{1}_m,$$

with matrix A given in formula (5.10). The null hypothesis of equi rank-correlation in (5.4) can then equivalently be written as

$$\mathrm{H}_{0,i}: \quad F \in \mathcal{F}_{0,i} = \{G \in \mathcal{F}_d : T_i(G) = \mathbf{0}\} \quad \text{for} \quad i \in \{1,2\}. \qquad (5.13)$$

We reject $H_{0,i}$ whenever the values of the test statistics $\mathcal{T}_{n,i}, i = 1, 2$, exceed a certain critical value. Let $\widehat{F}_n$ be the empirical distribution function of the random sample $(\mathbf{X}_j)_{j=1,\ldots,n}$ of the random vector $\mathbf{X}$; see (2.18). Note that

$$\mathcal{T}_{n,1} = \tau_1\{\sqrt{n}T_1(\widehat{F}_n)\} \quad \text{and} \quad \mathcal{T}_{n,2} = \tau_2\{\sqrt{n}T_2(\widehat{F}_n)\}$$

with continuous functions $\tau_1 : I\!R^{m-1} \to I\!R^+$ and $\tau_2 : I\!R^m \to I\!R^+$ defined by

$$\tau_1(t_1, \ldots, t_{m-1}) = \frac{1}{m-1}\sum_{i=1}^{m-1} t_i^2 \quad \text{and} \quad \tau_2(t_1, \ldots, t_m) = \frac{1}{m}\sum_{i=1}^{m} t_i^2.$$

Let $\widehat{F}_n^B$ denote the empirical distribution function of the bootstrap sample $(\mathbf{X}_j^B)_{j=1,\ldots,n}$ obtained by sampling from $(\mathbf{X}_j)_{j=1,\ldots,n}$ with replacement. We then use the quantiles of the bootstrap distribution of

$$\tau_i[\sqrt{n}\{T_i(\widehat{F}_n^B) - T_i(\widehat{F}_n)\}]$$

as critical values for the test statistics $\mathcal{T}_{n,i}, i = 1, 2$. The next theorem shows that this approach yields the asymptotically correct critical values (cf. theorem 2.1 in connection with corollary 2.1 in Bickel and Ren (2001)).

Theorem 5.2.2 *Consider the random sample $(\mathbf{X}_j)_{j=1,\ldots,n}$ from the d-dimensional random vector $\mathbf{X}$ with distribution function F, continuous univariate marginal distribution functions $F_1, \ldots, F_d$, and copula C. Let further $(\mathbf{X}_j^B)_{j=1,\ldots,n}$ be the bootstrap sample which is obtained by sampling from $(\mathbf{X}_j)_{j=1,\ldots,n}$ with replacement and suppose that $c_{i,\alpha}^{B,n}$ is the $(1-\alpha)$-quantile of the distribution of $\tau_i[\sqrt{n}\{T_i(\widehat{F}_n^B) - T_i(\widehat{F}_n)\}]$, $i = 1, 2$. Under the assumption that the k-th partial derivatives $D_k C(\mathbf{u})$ of C exist and are continuous for $k = 1, \ldots, d$, we have that*

(i) the test procedure based on $\mathcal{T}_{n,i}$ and critical value $c_{i,\alpha}^{B,n}$ is asymptotically of size α, i.e.,

$$I\!P(\mathcal{T}_{n,i} > c_{i,\alpha}^{B,n}) \longrightarrow \alpha, \quad n \to \infty,$$

if $F \in \mathcal{F}_{0,i}, i = 1, 2$, as defined in formula (5.13), and

(ii) the test procedure based on $\mathcal{T}_{n,i}$ and critical value $c_{i,\alpha}^{B,n}$ is consistent, i.e.,

$$I\!P(\mathcal{T}_{n,i} \leq c_{i,\alpha}^{B,n}) \longrightarrow 0, \quad n \to \infty,$$

if $F \notin \mathcal{F}_{0,i}, i = 1, 2$.

Proof. For the proof of assertions (i) and (ii), we begin by showing that

$$\sqrt{n}\{T_i(\widehat{F}_n^B) - T_i(\widehat{F}_n)\} \tag{5.14}$$

converges weakly to the same limit as

$$\sqrt{n}\{T_i(\widehat{F}_n) - T_i(F)\} \tag{5.15}$$

in probability, $i = 1, 2$. To do so, we utilize the functional delta-method (theorem 2.2.7) that requires Hadamard differentiability of the map T_i. As usual, let $\ell^\infty([0,1]^d)$ be the space of all uniformly bounded real-valued functions defined on $[0,1]^d$, equipped with the uniform metric $m(f_1, f_2) = \sup_{\mathbf{t}\in[0,1]^d} |f_1(\mathbf{t}) - f_2(\mathbf{t})|$; cf. (2.15). Observe that $T_i, i = 1, 2$, can be represented as composition of three maps:

$$T_i(F) = h_i \circ g \circ \phi(F).$$

The map $\phi : D(\overline{I\!R}^d) \to \ell^\infty([0,1]^d)$ transforms the d-dimensional distribution function F into its copula function C and is defined by equation (2.19). The vector $\boldsymbol{\rho}$ of bivariate Spearman's rho coefficients can be represented as a map $g : \ell^\infty([0,1]^d) \to I\!R^m$ of the copula C according to representation (5.1), i.e.,

$$g(C) = \boldsymbol{\rho} = \boldsymbol{\rho}(C).$$

Finally,

$$h_1(\boldsymbol{\rho}) = A\boldsymbol{\rho} \quad \text{and} \quad h_2(\boldsymbol{\rho}) = \boldsymbol{\rho} - \frac{\mathbf{1}_m'\boldsymbol{\rho}}{m}\mathbf{1}_m.$$

The functions g and h_i, $i = 1, 2$, are continuous and linear and, hence, Hadamard-differentiable. Hadamard differentiability of ϕ as a map into $\ell^\infty([p,q]^d)$ with $0 < p < q < 1$ is studied by van der Vaart and Wellner (1996) (lemma 3.9.28 in connection with lemma 3.9.23) under certain assumptions on the joint and marginal distribution functions of $\mathbf{X}$; see also proof of theorem 2.2.8. As in the latter proof, it is possible to concentrate on the case where the marginal distributions $F_1, \ldots, F_d$ are uniform distributions on $[0,1]$ and thus Hadamard differentiability of ϕ as a map into $\ell^\infty([0,1]^d)$ is obtained given the imposed conditions on the copula (cf. Fermanian et al. (2004)). To show this, some more notation is needed:
Consider the random vector $\mathbf{U} = (U_1, \ldots, U_d)$ of the random variables $U_k = F_k(X_k)$, $k = 1, \ldots, d$, with distribution function $F^\star$ and marginal distribution functions $F_k^\star$. Note that $F^\star(\mathbf{u}) = C(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^d$. Let $\widehat{F}_n^\star$ be the empirical distribution function of the random sample $\mathbf{U}_1, \ldots, \mathbf{U}_n$ of $\mathbf{U}$, and $\widehat{F}_n^{\star,B}$ the empirical distribution function of the bootstrap sample $\mathbf{U}_1^B, \ldots, \mathbf{U}_n^B$ with $\mathbf{U}_j^B = (F_1(X_{1j}^B), \ldots, F_d(X_{dj}^B))$, $j = 1, \ldots, n$.
Since $\phi(F)(\mathbf{u}) = \phi(F^\star)(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^d$, we obtain that

$$T_i(F) = h_i[g\{\phi(F)\}] = h_i[g\{\phi(F^\star)\}] = T_i(F^\star).$$

Analogously, it follows that $T_i(\widehat{F}_n) = T_i(\widehat{F}_n^\star)$ and $T_i(\widehat{F}_n^B) = T_i(\widehat{F}_n^{\star,B})$ as $\phi(\widehat{F}_n)(\mathbf{u}) = \phi(\widehat{F}_n^\star)(\mathbf{u})$ and $\phi(\widehat{F}_n^B)(\mathbf{u}) = \phi(\widehat{F}_n^{\star,B})(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^d$ (The latter can be proven along the same lines as in the proof of lemma 1 in Fermanian et al. (2004)).

Hence, weak convergence of $\sqrt{n}\{T_i(\widehat{F}_n^B) - T_i(\widehat{F}_n)\}$ and $\sqrt{n}\{T_i(\widehat{F}_n) - T_i(F)\}$ in equations (5.14) and (5.15) is equivalent to weak convergence of

$$\sqrt{n}\{T_i(\widehat{F}_n^{\star,B}) - T_i(\widehat{F}_n^\star)\} \quad \text{and} \quad \sqrt{n}\{T_i(\widehat{F}_n^\star) - T_i(F^\star)\},$$

where $F^\star$ has compact support $[0,1]^d$. We know that the empirical process $\sqrt{n}(\widehat{F}_n^\star - F^\star)$ converges weakly in $\ell^\infty([0,1]^d)$ to a d-dimensional Brownian bridge $\mathbb{B}_{F^\star}$ with covariance function $E\{\mathbb{B}_{F^\star}(\mathbf{u})\mathbb{B}_{F^\star}(\mathbf{v})\} = F^\star(\mathbf{u} \wedge \mathbf{v}) - F^\star(\mathbf{u})F^\star(\mathbf{v})$; cf. part (i) of the proof of theorem 2.2.8. Note that $\mathbb{B}_{F^\star} = \mathbb{B}_C$. This yields that $\sqrt{n}(\widehat{F}_n^{\star,B} - \widehat{F}_n^\star)$ converges weakly to the same limit in probability due to theorem 3.6.1 in van der Vaart and Wellner (1996), p. 347. According to our discussions above and under the assumption of the theorem, the map ϕ is Hadamard-differentiable at $C = F^\star$ as a map from $D([0,1]^d)$ (tangentially to $C([0,1]^d)$, cf. proof of theorem 2.2.8). As before, the space $C([0,1]^d)$ comprises all continuous real-valued functions and the space $D([0,1]^d)$ all real-valued cadlag functions defined on $[0,1]^d$, both equipped with the uniform metric m as defined in (2.15). As a consequence of the chain rule (lemma 3.9.27 in van der Vaart and Wellner (1996), p. 388), T_i is Hadamard-differentiable with derivative $\dot{T}_{C,i}, i = 1,2$. An application of the functional delta-method together with theorem 3.9.11 in van der Vaart and Wellner (1996) finally yields that, in probability, $\sqrt{n}\{T_i(\widehat{F}_n^{\star,B}) - T_i(\widehat{F}_n^\star)\}$ converges weakly to the same limit $\dot{T}_{C,i}(\mathbb{B}_C)$ as $\sqrt{n}\{T_i(\widehat{F}_n^\star) - T_i(F^\star)\}$, $i = 1,2$.

Let $G_{i,n}^B$ be the distribution function of $\tau_i[\sqrt{n}\{T_i(\widehat{F}_n^B) - T_i(\widehat{F}_n)\}]$ and denote by G_i the distribution function of the limit $\tau_i\{\dot{T}_{C,i}(\mathbb{B}_C)\}$ with corresponding $(1-\alpha)$-quantile $c_{i,\alpha}, i = 1,2$. Since $\mathbb{B}_C$ is a tight Gaussian process and $\dot{T}_{C,i}$ is a continuous linear map (as composition of continuous linear maps), the random vector $\dot{T}_{C,i}(\mathbb{B}_C)$ is normally distributed according to lemma 3.9.8 in van der Vaart and Wellner (1996). This implies that $\tau_i\{\dot{T}_{C,i}(\mathbb{B}_C)\}$ is equivalent in distribution to a finite, weighted sum of independent χ^2-distributed random variables (cf. proof of theorem 5.2.1). Hence, G_i is a continuous and strictly increasing distribution function on $\mathbb{R}^+, i = 1,2$. According to the above weak convergence results and Polyá's Theorem (see e.g. theorem 11.2.9 in Lehmann and Romano (2005), p. 429) we then have that

$$\sup_{x \in \mathbb{R}^+} |G_{i,n}^B(x) - G_i(x)| \to 0 \quad n \to \infty,$$

as τ_i is a continuous function. This finally yields that $c_{i,\alpha}^{B,n}$ converges to $c_{i,\alpha}$ in probability (lemma 11.2.1 in Lehmann and Romano (2005)). Hence, if $F \in \mathcal{F}_{0,i}, i = 1,2$,

$$\begin{aligned}
\mathbb{P}(\mathcal{T}_{n,i} > c_{i,\alpha}^{B,n}) &= \mathbb{P}(\tau_i[\sqrt{n}\{T_i(\widehat{F}_n) - T_i(F)\}] > c_{i,\alpha}^{B,n}) \\
&= \mathbb{P}(\tau_i[\sqrt{n}\{T_i(\widehat{F}_n) - T_i(F)\}] > c_{i,\alpha}) + o(1) \\
&\to 1 - G_i(c_{i,\alpha}) = \alpha, \quad n \to \infty,
\end{aligned}$$

and assertion (i) follows.

For the proof of assertion (ii), let $\| \cdot \|$ denote the Euclidean norm in $\mathbb{R}^m$ or $\mathbb{R}^{m-1}$, respectively. Since $\sqrt{n}\{T_i(\widehat{F}_n) - T_i(F)\}$ converges weakly in $\mathbb{R}^m$ or $\mathbb{R}^{m-1}$, respectively, $\sqrt{n}\{T_i(\widehat{F}_n) - T_i(F)\} =: \mathbf{Y}_n$ is uniformly tight (theorem 2.4 in van der Vaart (1998)), i.e., for every $\varepsilon > 0$ there exists a constant $M > 0$ such that

$$\sup_n \mathbb{P}(\| \mathbf{Y}_n \| > M) < \varepsilon.$$

For $K > 0$, we thus have

$$\begin{aligned}\mathbb{P}(\| \sqrt{n}T_i(\widehat{F}_n) \| > K) &= \mathbb{P}(\| \mathbf{Y}_n + \sqrt{n}T_i(F) \| > K) \\ &= \mathbb{P}(\| \mathbf{Y}_n + \sqrt{n}T_i(F) \| > K \mid \| \mathbf{Y}_n \| > M)\, \mathbb{P}(\| \mathbf{Y}_n \| > M) \\ &\quad + \mathbb{P}(\| \mathbf{Y}_n + \sqrt{n}T_i(F) \| > K \mid \| \mathbf{Y}_n \| \leq M)\, \mathbb{P}(\| \mathbf{Y}_n \| \leq M) \\ &\geq \mathbb{P}(\| \sqrt{n}T_i(F) \| - M > K)(1-\varepsilon).\end{aligned}$$

Since $T_i(F) \neq \mathbf{0}$ if $F \notin \mathcal{F}_{0,i}$, $i = 1, 2$, the last expression converges to $1 - \varepsilon$ for $n \to \infty$. Hence, $\| \sqrt{n}T_i(\widehat{F}_n) \|$ converges in probability to infinity as ε can be chosen arbitrarily small. Due to the continuity of τ_i and the fact that $\tau_i(\mathbf{t}) \to \infty$ as $\| \mathbf{t} \| \to \infty$, assertion (ii) follows from $c_{i,\alpha}^{B,n} \xrightarrow{\mathbb{P}} c_{i,\alpha}$ and an application of the continuous mapping theorem. □

5.2.2 Test statistics $\mathcal{T}_{n,3}$ and $\mathcal{T}_{n,4}$

While in the previous section we made use of the nonparametric bootstrap to estimate the critical values of the test statistics, both tests for equi rank-correlation discussed next involve the estimation of the asymptotic covariance Σ^{asym} of the random vector $\sqrt{n}(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho})$ (cf. theorem 5.1.1) by means of the bootstrap. As above, let $(\mathbf{X}_j^B)_{j=1,\ldots,n}$ be the bootstrap sample which is obtained by sampling from $(\mathbf{X}_j)_{j=1,\ldots,n}$ with replacement and denote by $\widehat{\boldsymbol{\rho}}_n^B$ the corresponding estimator of the vector of pairwise Spearman's rho coefficients calculated according to (5.5). The bootstrap estimator for the asymptotic covariance matrix Σ^{asym} is then given by

$$\widehat{\Sigma}_n^B = n\, Cov(\widehat{\boldsymbol{\rho}}_n^B \mid \mathbf{X}_1, \ldots, \mathbf{X}_n). \tag{5.16}$$

Based on this estimator, the third test statistic $\mathcal{T}_{n,3}$ for equi rank-correlation is similarly constructed as the test statistic $\mathcal{T}_{n,1}$ by using the fact that, under H_0, $A\boldsymbol{\rho} = \mathbf{0}_{m-1}$ with matrix A as defined in formula (5.10). It takes the form

$$\mathcal{T}_{n,3} = n(A\widehat{\boldsymbol{\rho}}_n)'(A\widehat{\Sigma}_n^B A')^{-1}(A\widehat{\boldsymbol{\rho}}_n), \tag{5.17}$$

A further test statistic is given by

$$\mathcal{T}_{n,4} = \sqrt{n}\Big\{ \max_{1\leq j\leq m} \widehat{\rho}_{j,n} - \min_{1\leq j\leq m} \widehat{\rho}_{j,n} \Big\}, \tag{5.18}$$

which is based on the fact that, in the case of equi-rank correlation, the values of the maximum and the minimum Spearman's rho coefficients coincide.
Both test statistics posses a non-degenerated limiting distribution under the null hypothesis as shown next.

Theorem 5.2.3 *Consider the random sample $(\mathbf{X}_j)_{j=1,\ldots,n}$ from the d-dimensional random vector $\mathbf{X}$ with distribution function F, continuous univariate marginal distribution functions $F_1, \ldots, F_d$, and copula C. Let further $(\mathbf{X}_j^B)_{j=1,\ldots,n}$ be the bootstrap sample which is obtained by sampling from $(\mathbf{X}_j)_{j=1,\ldots,n}$ with replacement and denote by $\widehat{\Sigma}_n^B$*

the bootstrap estimator for the asymptotic covariance matrix Σ^{asym} *of* $\sqrt{n}(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho})$ *(cf. theorem 5.1.1), as given in equation (5.16). Under the assumption of theorem 5.1.1 and under the null hypothesis (5.4), we have that*

$$\mathcal{T}_{n,i} \xrightarrow{d} W_i, \quad \text{for} \quad n \to \infty,$$

with non-degenerated random variables $W_i, i = 3, 4$. *In particular,* W_3 *is distributed according to a* χ^2*-distribution with* $m-1$ *degrees of freedom and* $W_4 = \max_{1\leq j\leq m} Z_j - \min_{1\leq j\leq m} Z_j$ *with* $\mathbf{Z} \sim N(\mathbf{0}_m, \Sigma^{asym})$.

Proof. As a by-product of the proof of theorem 5.2.2, it follows that, under the theorem's prerequisites, the sequence $\sqrt{n}(\widehat{\boldsymbol{\rho}}_n^B - \widehat{\boldsymbol{\rho}}_n)$ converges weakly to the same Gaussian limit as $\sqrt{n}(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho})$ in probability. Since in addition the sequences $\{n(\widehat{\rho}^B_{k,n} - \widehat{\rho}_{k,n})(\widehat{\rho}^B_{l,n} - \widehat{\rho}_{l,n})\}$ for $1 \leq k, l \leq m$ are asymptotically uniformly integrable, consistency of $\widehat{\Sigma}_n^B$ is obtained (cf. Shao and Tu (1995), p. 79).
Regarding the limiting behavior of $\mathcal{T}_{n,3}$ under the null hypothesis, we then have that

$$\mathcal{T}_{n,3} = n(A\widehat{\boldsymbol{\rho}}_n)'(A\widehat{\Sigma}_n^B A')^{-1}(A\widehat{\boldsymbol{\rho}}_n) \xrightarrow{d} W_3 \sim \chi^2_{m-1},$$

due to theorem 5.1.1 in connection with Slutsky's theorem. In order to establish the asymptotic distribution of $\mathcal{T}_{n,4}$ under H_0, observe that

$$\mathcal{T}_{n,4} = \sqrt{n}\Big\{\max_{1\leq i\leq m} \widehat{\rho}_{i,n} - \min_{1\leq i\leq m} \widehat{\rho}_{i,n}\Big\} = \max_{1\leq i\leq m}\{\sqrt{n}(\widehat{\rho}_{i,n} - \rho)\} - \min_{1\leq i\leq m}\{\sqrt{n}(\widehat{\rho}_{i,n} - \rho)\},$$

since $\boldsymbol{\rho} = \rho\mathbf{1}_m$. Under H_0, the test statistic $\mathcal{T}_{n,4}$ can thus be represented as a continuous map of the process $\sqrt{n}(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho})$ and an application of the continuous mapping theorem together with theorem 5.1.1 yields the convergence in distribution of $\mathcal{T}_{n,4}$ to W_4 as stated in the theorem. □

We thus reject the null hypothesis at level α whenever $\mathcal{T}_{n,3} > \chi^2_{\alpha,m-1}$, the $(1-\alpha)$-quantile of the χ^2-distribution with $m-1$ degrees of freedom, or the value of $\mathcal{T}_{n,4}$ exceeds the $(1-\alpha)$-quantile of the distribution of W_4, respectively. Since the bootstrap covariance estimator $\widehat{\Sigma}_n^B$ for Σ^{asym} does not depend on $\boldsymbol{\rho}$, the respective quantiles/critical values of $\mathcal{T}_{n,4}$ can directly be determined by Monte Carlo simulation. Note that the bootstrap technique by Bickel and Ren (2001), which we used in section 5.2.1 to determine the critical values of the test statistics $\mathcal{T}_{n,1}$ and $\mathcal{T}_{n,2}$, cannot be applied here since the function $h_4(\boldsymbol{\rho}) = \max_{1\leq j\leq m} \rho_j - \min_{1\leq j\leq m} \rho_j$ fails to be (Hadamard-) differentiable (cf. proof of theorem 5.2.2).

Corollary 5.2.4 *The test procedures based on* $\mathcal{T}_{n,3}$ *and* $\mathcal{T}_{n,4}$ *are asymptotically of size* α *and consistent as* $n \to \infty$.

Note that consistency is obtained since $\mathcal{T}_{n,i} \to \infty$ in probability if $\boldsymbol{\rho} \neq \rho\mathbf{1}_m$, $i = 3, 4$. For completeness, we briefly mention another straightforward test statistic for equi rank-correlation which is equivalent to $\mathcal{T}_{n,4}$ and defined as

$$\mathcal{T}_{n,5} = \sqrt{n}\max_{i<j} |\widehat{\rho}_{i,n} - \widehat{\rho}_{j,n}|.$$

Its asymptotic distribution can be derived similarly to $\mathcal{T}_{n,4}$; however, it is computationally more complex as it involves the comparison of all distinct pairs of Spearman's rho coefficients.

As described in section 2.2.2, we approximate $\widehat{\Sigma}_n^B$ in practice by the sample covariance matrix of K independent bootstrap samples from $(\mathbf{X}_j)_{j=1,\ldots,n}$, i.e. (cf. Efron and Tibshirani (1993))

$$n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n^B) = \frac{1}{K-1}\sum_{b=1}^{K}\{\sqrt{n}\widehat{\boldsymbol{\rho}}_{n,(b)}^B - \overline{\sqrt{n}\widehat{\boldsymbol{\rho}}_{n,(b)}^B}\}\{\sqrt{n}\widehat{\boldsymbol{\rho}}_{n,(b)}^B - \overline{\sqrt{n}\widehat{\boldsymbol{\rho}}_{n,(b)}^B}\}' \tag{5.19}$$

where $\widehat{\boldsymbol{\rho}}_{n,(b)}^B$ denotes the bootstrap replication of $\widehat{\boldsymbol{\rho}}_n$, which corresponds to the bth bootstrap sample, $b = 1, \ldots, K$. The sum in formula (5.19) is applied element-wise. Similarly as in section 4.3 of the previous chapter, tables 5.2 and 5.3 in appendix 5.7.2 display the results of a simulation study which investigates the finite-sample performance of the bootstrap procedure. Specifically, we consider the d-dimensional equi-correlated Gaussian copula with correlation matrix $K(\kappa) = \kappa\mathbf{1}_d\mathbf{1}_d' + (1-\kappa)I_d$ and $-1/(d-1) < \kappa < 1$, as defined in (2.9), and the d-dimensional Clayton copula with parameter $\theta > 0$, as given in (2.13). For dimensions $d = 3, 5$, and 10, different parameter choices and sample sizes n (see the first and third column of tables 5.2 and 5.3), we provide the empirical mean (denoted by $m(\cdot)$), the sample covariance matrix (denoted by $\widehat{\Sigma}(\cdot)$), and the standard deviation (denoted by $\widehat{\sigma}(\cdot)$) of the respective estimates. The estimation is based on 300 Monte Carlo simulations of sample size n and 300 bootstrap samples, respectively, which have been drawn with replacement from each original sample. We display the minimal and the maximal elements of the respective estimated vectors (columns 5 and 6). Regarding the estimated matrices (columns 7 to 12), we show the minimum and the maximum of all diagonal and off-diagonal elements separately (where the respective columns are headed by 'diag' and 'odiag'). The second column of tables 5.2 and 5.3 shows the true value of bivariate Spearman's rho which is constant for all bivariate margins. For the Gaussian copula this value is determined using the following relationship between the parameters κ_{ij} of the Gaussian copula and the bivariate Spearman's rho coefficients ρ_{ij} (cf. McNeil et al. (2005), theorem 5.36, p. 215):

$$\kappa_{ij} = 2\sin\left(\frac{\pi\rho_{ij}}{6}\right). \tag{5.20}$$

For the Clayton copula, ρ_{ij} is calculated numerically. Compared to $m(\widehat{\boldsymbol{\rho}}_n)$ (column5) and $m(\widehat{\boldsymbol{\rho}}_n^B)$ (column 6), there is a finite-sample bias observable which considerably decreases with increasing sample size. The covariance estimates $n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n)$ and the empirical means of the bootstrap covariance estimates $m(n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n^B))$ for Σ^{asym} are given in columns 7 to 10. Their values are close to each other what shows that the bootstrap procedure performs well for the considered copulas. Further, the standard error of the bootstrap covariance estimations (columns 11 and 12) decreases fast with increasing sample size.

Remark. Note that the test setting described in section 5.1 can be generalized: Instead of testing the null hypothesis $\mathrm{H}_0 : \rho_1 = \rho_2 = \cdots = \rho_m$, only a subset of pairwise Spearman's rho coefficients $\{\rho_i, i \in \mathcal{I}\}$ with index set $\mathcal{I} \subset \{1, \ldots, m\}$ of the random vector $\mathbf{X}$ could be tested for equality. The related test statistics are analogously derived and their asymptotic distributions are established using the techniques discussed previously. For example, this generalization may be of interest in the context of interclass correlation models for familial data as mentioned in chapter 1. Within the model of parent-sibling correlation (see Helu and Naik (2006)), one central assumption is that the pairwise correlations of measurements between one parent and the children are equal. Those correlation coefficients form a subset of the vector of pairwise correlations between all family members. The assumption of their equality can be tested (without further assumptions on the marginal distributions than continuity) using the above tests for equal pairwise Spearman's rho coefficients.

5.3 Classical tests for equi linear-correlation

In this section, we give an overview of the existing literature on parametric tests for equi linear-correlation. Statistical tests for equi linear-correlation have been derived by several authors under the assumption of multivariate normality. For the time being, let $\mathbf{X}$ therefore be normally distributed with mean vector $\boldsymbol{\mu}$, covariance matrix $S = (s_{ij})$, and linear correlation matrix $R = (r_{ij})$.

A likelihood ratio test for the null hypothesis of equi linear-correlation and equal variances

$$\mathrm{H}_0 \quad : \quad S = s^2\{(1-r)I_d + r\mathbf{1}_d\mathbf{1}_d'\}$$

with unspecified variance s^2 and linear correlation r is discussed in Wilks (1946). Based on a random sample $(\mathbf{X}_j)_{j=1,\ldots,n}$ from $\mathbf{X}$, he proposes the following test statistic

$$\mathcal{S}_{n,1} = -n \ln \left[\frac{|\widehat{S}_n|}{(\widehat{s}_n^2)^d (1-\widehat{r}_n)^{d-1}\{1+(d-1)\widehat{r}_n\}} \right] \tag{5.21}$$

with sample covariance matrix $\widehat{S}_n = (\widehat{s}_{ij,n})$. Estimates of s^2 and r are

$$\widehat{s}_n^2 = \frac{1}{d}\sum_{i=1}^{d} \widehat{s}_{ii,n} \quad \text{and} \quad \widehat{r}_n = \frac{1}{d(d-1)} \sum_{i \neq j} \widehat{s}_{ij,n}/\widehat{s}_n^2.$$

Under H_0, $\mathcal{S}_{n,1}$ is asymptotically χ^2-distributed with $d(d+1)/2-2$ degrees of freedom and we reject H_0 if $\mathcal{S}_{n,1} > \chi^2_{\alpha,d(d+1)/2-2}$, the latter being the corresponding $(1-\alpha)$-quantile. Regarding further tests for certain structures of the covariance matrix, we refer to Wilks (1946), Arnold (1981), and Rencher (2002).

A likelihood ratio test for the less restrictive hypothesis of equal linear-correlation coefficients (regardless of the value of the variances), i.e., $r_{ij} = r$ $(i \neq j)$ with unspecified r satisfying $-1/(d-1) < r < 1$, is difficult to derive and no closed form solution is available; see Lawley (1963). Approximate likelihood ratio tests for equal linear-correlation coefficients of a multivariate normal distribution have been developed by several authors; we mention Bartlett (1950, 1951), Anderson (1963), and Aitkin et al. (1968). Based on the sample correlation matrix $\widehat{R} = (\widehat{r}_{ij})$, Lawley (1963) considered the statistic

$$\mathcal{S}_{n,2} = \frac{n}{\widehat{\lambda}_n^2}\Big\{\sum_{i<j}(\widehat{r}_{ij,n} - \overline{\widehat{r}}_n)^2 - \widehat{\mu}_n \sum_{k}(\overline{\widehat{r}}_{k,n} - \overline{\widehat{r}}_n)^2\Big\} \tag{5.22}$$

where r is estimated by

$$\overline{\widehat{r}}_n = 2\sum_{j<k}\widehat{r}_{jk,n}/\{d(d-1)\},$$

and

$$\widehat{\lambda}_n = 1 - \overline{\widehat{r}}_n, \quad \widehat{\mu}_n = (d-1)^2(1-\widehat{\lambda}_n^2)/\{d-(d-2)\widehat{\lambda}_n^2\}, \quad \overline{\widehat{r}}_{k,n} = \sum_{i\neq k}\widehat{r}_{ik,n}/(d-1).$$

According to the last mentioned author, the test statistic $\mathcal{S}_{n,2}$ is - under H_0 - asymptotically χ^2-distributed with $(d-2)(d+1)/2$ degrees of freedom. Thus, we reject the null hypothesis whenever the value of $\mathcal{S}_{n,2}$ exceeds the $(1-\alpha)$-quantile of the χ^2-distribution with $(d-2)(d+1)/2$ degrees of freedom. Gleser (1968) shows that the asymptotic null distribution of the above test statistic does not depend on the unknown parameter r. The next section presents a simulation study where the latter test for equi linear-correlation is taken as the benchmark for the four proposed tests for equi rank-correlation derived in sections 5.2.1 and 5.2.2.

5.4 Simulation study

The following simulation study investigates and compares the power of the four proposed tests $\mathcal{T}_{n,i}, i = 1, \dots, 4$, of equi rank-correlation. We start with describing the set of considered alternative hypothesis; thereafter, the simulation results are given in section 5.4.2.

5.4.1 Modeling the set of alternative hypothesis

Consider the m bivariate Spearman's rho coefficients $\rho_i, i = 1, \dots, m$, of a d-dimensional random vector as defined in (5.3). The alternative to equi rank-correlation is that at least two of the m bivariate Spearman's rho coefficients ρ_i differ. Given the fact that there are infinitely many sets of alternative hypothesis, we proceed as follows.

Three different types of common dependence structures are investigated within the simulation study. First, the d-dimensional *Gaussian copula* is considered as defined in formula (2.9) with the general correlation matrix $K = (\kappa_{ij})$. For this copula, the set of alternative hypothesis is defined by the following formula: For fixed $j \in I\!N$ and fixed $\Delta \in I\!R$,

$$\rho_i(k) = \rho\Big\{1 + k\Delta\Big(\frac{i-1}{m}\Big)^j\Big\}, \quad i = 1, \ldots, m, \tag{5.23}$$

where $\rho \in [-1, 1]$ and $k = 0, 1, 2, \ldots, k_{\max}$ with $k_{\max} = \max\{s \in I\!N \mid |\rho_m(s)| \leq 1\}$. The case $k = 0$ corresponds to the null hypothesis of equi rank-correlation, i.e., all coefficients $\rho_i(k)$ equal the parameter ρ. The difference between the bivariate Spearman's rho coefficients gets more pronounced the larger the parameter k is. In the simulation study, each value of k corresponds to a point of the power curve. The corresponding parameters of the Gaussian copula are determined using formula (5.20). For every k it is verified beforehand whether K is a valid correlation matrix.
The difference between the Spearman's rho coefficients is determined by the fixed parameter Δ and the factor $(\frac{i-1}{m})^j$. In particular, the difference increases with increasing indices i and j. Note that k_{max} depends on the dimension d and, thus, on the total number of pairwise Spearman's rho coefficients. The set of alternative hypothesis defined by formula (5.23) is illustrated in figure 5.1 for two different choices of j.

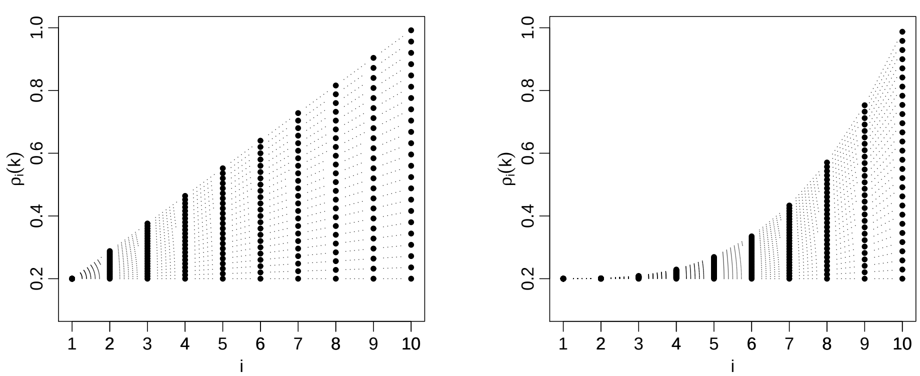


Figure 5.1: **The set of alternative hypothesis**. Bivariate Spearman's rho coefficients $\rho_i(k)$, $k = 1, \ldots, k_{\max}, i = 1, \ldots, m$, according to formula (5.23) for $d = 5$, $\rho = 0.2$, $\Delta = 0.2$, and $j = 1$ (left panel) and $j = 3$ (right panel).

The second copula is the *t-copula* with ν degrees of freedom and correlation matrix K, as defined (2.10). Here, the set of alternative hypothesis is modeled as for the Gaussian copula according to formula (5.23). Since no analytical formula similar to (5.20) exists, the relationship between the parameters of the t-copula and Spearman's rho is determined numerically.

The third considered copula is a *mixture of a Gaussian and a Clayton copula*. It is defined by the convex combination

$$C(u_1, \ldots, u_d; K, \lambda, \theta) = \lambda C^G(u_1, \ldots, u_d; K) + (1 - \lambda)C^{Cl}(u_1, \ldots, u_d; \theta), \tag{5.24}$$

with $0 \leq \lambda \leq 1$, C^G a Gaussian copula with correlation matrix K (see (2.9)), and C^{Cl}

a Clayton copula with parameter θ as defined in (2.13). Here, the set of alternative hypothesis is modeled by stepwise increasing the parameter λ. The case $\lambda = 0$ corresponds to the null hypothesis of equi rank-correlation and each value of λ represents a point on the simulated power curve. The parameters of the Gaussian copula are chosen such that its associated bivariate Spearman's rho coefficients ρ_i are all equal to $\rho_i(k_{max})$ as implied by formula (5.23), $i = 1, \ldots, m$; the parameter of the Clayton copula is determined such that all associated bivariate Spearman's rho coefficients ρ_i are equal to ρ (cf. formula (5.23)).

5.4.2 Simulation results

Lawley's test on equi linear-correlation, based on the test statistic given by formula (5.22), serves as a benchmark for the four tests on equi rank-correlation introduced in sections 5.2.1 and 5.2.2. Figures 5.2, 5.3, and 5.4 show the power curves of those tests together with the benchmark test for the three dependence structures and the alternative hypothesis described in section 5.4.1. We illustrate the power of the tests for the three dimensions $d = 3, 5$, and 10, and the following three different types of marginal distributions: The standard normal, exponential, and Cauchy distribution, which are light, semi-heavy, and heavy-tailed, respectively. Note that equal Pearson's correlation coefficients generally do not imply equal pairwise Spearman's rho coefficients, and vice versa. We obtain equality of all Pearson's correlation coefficients for the considered dependence structures under the null hypothesis of equi rank-correlation since all marginal distributions are of the same type.
Calculations are based on $10,000$ Monte-Carlo simulations of sample size $n = 500$. The number of bootstrap replications is either $3,000$ for determining the critical values for the tests based on $\mathcal{T}_{n,1}$ and $\mathcal{T}_{n,2}$ or 300 for estimating the (asymptotic) covariance matrix Σ^{asym} regarding the test statistics $\mathcal{T}_{n,3}$ and $\mathcal{T}_{n,4}$. The determination of the critical value of the test based on $\mathcal{T}_{n,4}$ is further based on $3,000$ Monte Carlo samples. The significance level α is set to 0.1. For modeling the alternative hypothesis as described by formula (5.23), ρ is set to 0.2 and $j = 1$ in all simulations. The parameter Δ is either 0.05 or 0.1, depending on the location on the curve. For clarity reasons, $k_{\max}$ is chosen such that $k_{\max} = \max\{s \in I\!N \mid |\rho_m(s)| \leq 0.6\}$. The power curve for the mixture of a Gaussian and a Clayton copula, as defined in formula (5.24), is determined for each value of $\lambda = j/10,\ j = 0, \ldots, 10$. In order to allow for comparisons, all power curves are further plotted as a function of the average $\bar{\rho}$ of all pairwise Spearman's rho coefficients under the respective hypothesis.

The following observations can be made: Irrespective of the copula, the test based on $\mathcal{T}_{n,3}$ maintains its significance level only in dimension $d = 3$; the discrepancy gets more pronounced with increasing dimension. Here, the approximation of the exact distribution by the asymptotic χ^2-distribution might be affected by the large number of parameters to be estimated in the test statistic. For example, if the sample size is set to $n = 5,000$ (and all other quantities are left unchanged) the simulated size for $\mathcal{T}_{n,3}$ reduces to 0.34 for dimension $d = 10$ in the case of a Gaussian copula. Likewise, the performance of the test can be improved by increasing the number of bootstrap

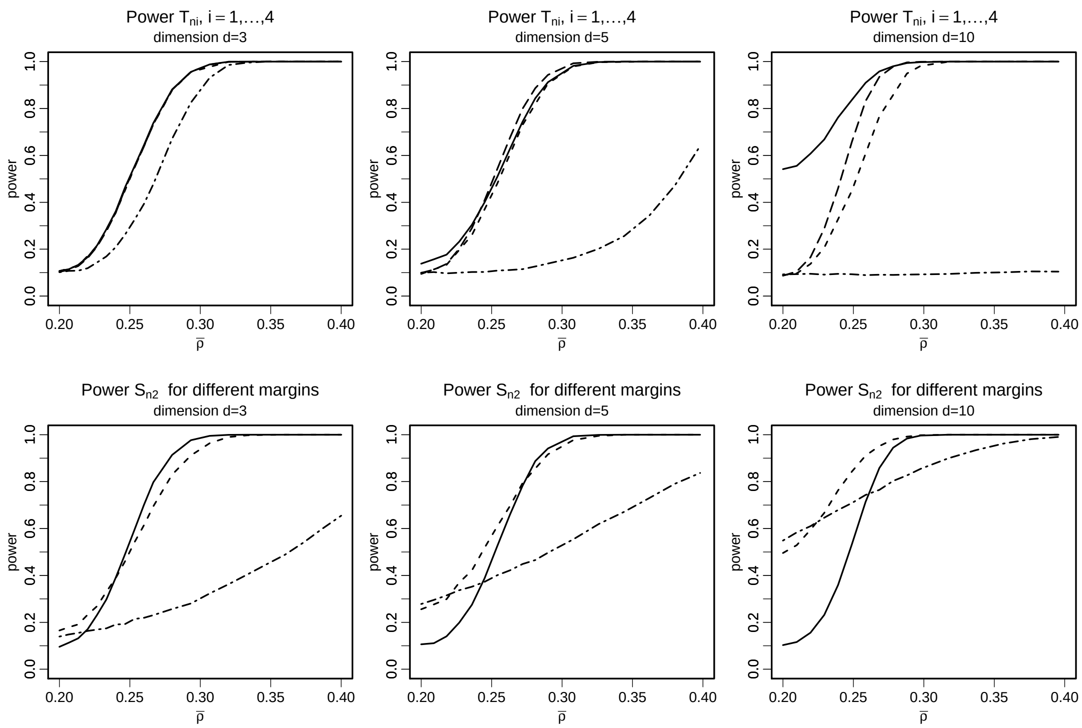


Figure 5.2: **Gaussian copula**. Upper panel: Simulated power curves of $\mathcal{T}_{n,1}$ (dotted-dashed), $\mathcal{T}_{n,2}$ (long-dashed), $\mathcal{T}_{n,3}$ (solid), and $\mathcal{T}_{n,4}$ (dashed). Lower panel: Simulated power curves of $\mathcal{S}_{n,2}$ for standard normal (solid), exponential (dashed), and Cauchy (dotted-dashed) marginal distributions. The power curves are provided for dimension $d = 3, 5, 10$ and plotted as a function of the average Spearman's rho coefficients $\bar{\rho}$; calculations are based on 10,000 independent Monte Carlo samples of size $n = 500$ of a Gaussian copula with parameters according to formula (5.23). The significance level α is set to 0.1.

replications for estimating the asymptotic covariance matrix Σ^{asym} : If, for example, the number of bootstrap replications is 3,000 the simulated size is 0.30 for the Gaussian copula in dimension $d = 10$ and for sample size $n = 500$ (with all other quantities being equal). By contrast, the tests based on $\mathcal{T}_{n,1}$, $\mathcal{T}_{n,2}$, and $\mathcal{T}_{n,4}$ maintain their significance level for every dimension. However, in terms of its power, $\mathcal{T}_{n,1}$ performs poorly in higher dimensions, and is even outperformed by the other three tests for dimension $d = 3$. This behavior in high dimensions can partly be put down to the specific choice of the set of alternative hypothesis (as described in section 5.4.1). Since this test statistic takes into account the order of the Spearman's rho coefficients (cf. discussions in section 5.2.2), the pairwise differences between ρ_i and ρ_{i+1} and, thus, the value of the test statistic decrease with increasing dimension under a given alternative. The power of $\mathcal{T}_{n,2}$, $\mathcal{T}_{n,3}$, and $\mathcal{T}_{n,4}$ is similar for dimension $d = 3$. For dimension $d = 5$, the power curves of the tests based on $\mathcal{T}_{n,2}$ and $\mathcal{T}_{n,4}$ are quite close to each other while, for dimension $d = 10$, the test based on $\mathcal{T}_{n,2}$ exhibits a slightly better power than $\mathcal{T}_{n,4}$. In the case of a Gaussian copula with standard normal margins, the former is even superior

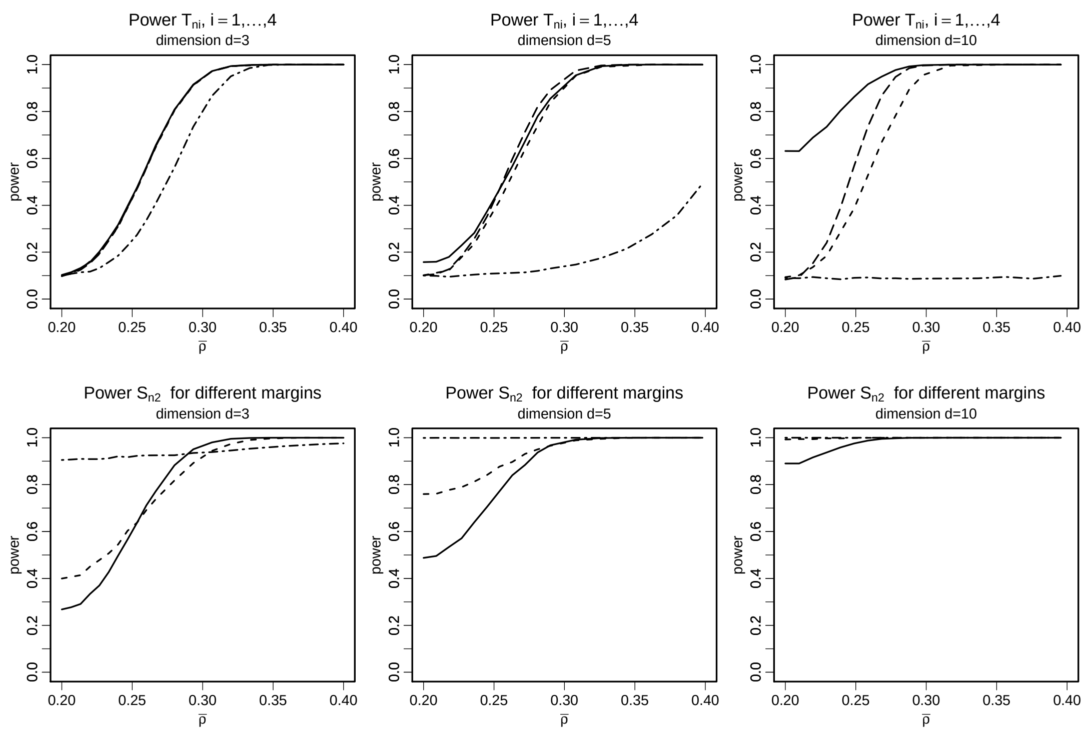


Figure 5.3: **t-copula**. Upper panel: Simulated power curves of $\mathcal{T}_{n,1}$ (dotted-dashed), $\mathcal{T}_{n,2}$ (long-dashed), $\mathcal{T}_{n,3}$ (solid), and $\mathcal{T}_{n,4}$ (dashed). Lower panel: Simulated power curves of $\mathcal{S}_{n,2}$ for standard normal (solid), exponential (dashed), and Cauchy (dotted-dashed) marginal distributions. The power curves are provided for dimension $d = 3, 5, 10$ and plotted as a function of the average Spearman's rho coefficients $\bar{\rho}$; calculations are based on $10,000$ independent Monte Carlo samples of size $n = 500$ of a t-copula with parameters according to formula (5.23) and $\nu = 3$. The significance level α is set to 0.1.

to the classical test based on $\mathcal{S}_{n,2}$ for dimension $d = 10$. Since the derivation of the asymptotic distribution of $\mathcal{S}_{n,2}$ is based on the assumption of multivariate normality, the corresponding test performs well in this case. Otherwise, we often observe a bias in its significance level, especially for dimension $d = 10$.

In order to investigate the behavior of $\mathcal{T}_{n,1}$ under other alternative hypothesis, figure 5.5 displays the simulated power curves of the tests based on the test statistics $\mathcal{T}_{n,i}, i = 1, \ldots, 4$, for the Gaussian copula and the alternative that only one Spearman's rho coefficient changes. In particular, the Spearman's rho coefficients are modeled according to

$$\rho_m(k) = \rho(1 + k\Delta) \quad \text{and} \quad \rho_i(k) = \rho \quad i = 1, \ldots, m-1, \;\; k = 1, \ldots, k_{max}, \qquad (5.25)$$

with all other quantities as described above. For the considered copula model, it turns out that - although the test based on $\mathcal{T}_{n,1}$ is still outperformed by the other tests - the difference in power between the tests has decreased, even in dimension $d = 10$.

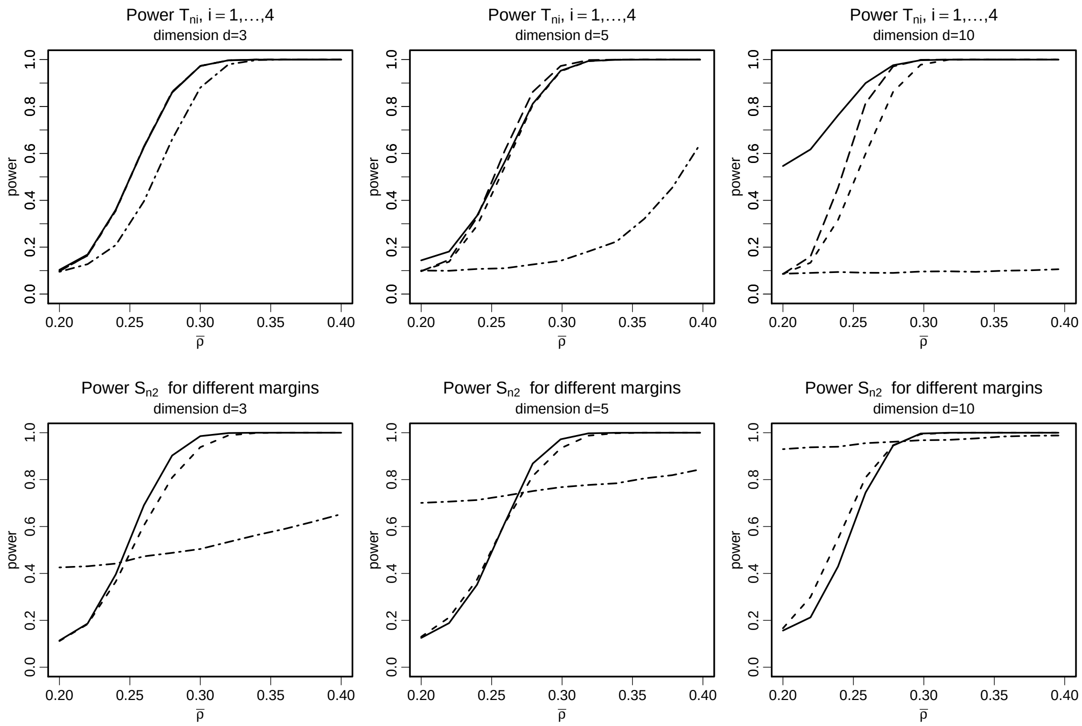


Figure 5.4: **Mixture of Gaussian and Clayton copula**. Upper panel: Simulated power curves of $\mathcal{T}_{n,1}$ (dotted-dashed), $\mathcal{T}_{n,2}$ (long-dashed), $\mathcal{T}_{n,3}$ (solid), and $\mathcal{T}_{n,4}$ (dashed). Lower panel: Simulated power curves of $\mathcal{S}_{n,2}$ for standard normal (solid), exponential (dashed), and Cauchy (dotted-dashed) marginal distributions. The power curves are provided for dimension $d = 3, 5, 10$ and plotted as a function of the average Spearman's rho coefficients $\bar{\rho}$; calculations are based on $10,000$ independent Monte Carlo samples of size $n = 500$ of the mixture copula described in Section 5.4.1. The significance level α is set to 0.1.

Altogether, the simulation study implies that test statistic $\mathcal{T}_{n,2}$ and $\mathcal{T}_{n,4}$ should be favored in terms of statistical power over all other considered tests. In addition, the latter is of lower computational complexity.

5.5 A test for stochastic independence

A simple test for stochastic independence of all components of a multivariate random vector can directly be derived from the asymptotic behavior of $\widehat{\boldsymbol{\rho}}_n$ established in theorem 5.1.1. Although it is not the main focus of this chapter, we briefly outline this approach.

Consider a d-dimensional random vector $\mathbf{X}$ with joint distribution function F and copula C and let $(\mathbf{X}_j)_{j=1,\ldots,n}$ denote a random sample from $\mathbf{X}$. According to theorem

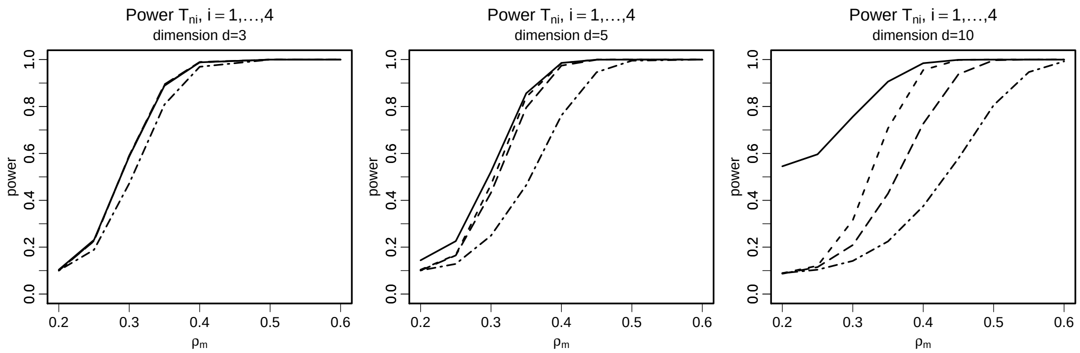


Figure 5.5: **Gaussian copula**. Simulated power curves of $\mathcal{T}_{n,1}$ (dotted-dashed), $\mathcal{T}_{n,2}$ (long-dashed), $\mathcal{T}_{n,3}$ (solid), and $\mathcal{T}_{n,4}$ (dashed) for dimension $d = 3, 5, 10$ and plotted as a function of the Spearman's rho coefficient ρ_m. Calculations are based on $10,000$ independent Monte Carlo samples of size $n = 500$ of a Gaussian copula with parameters according to formula (5.25). The significance level α is set to 0.1.

5.1.1, we have under the hypothesis of stochastic independence (i.e. $C(\mathbf{u}) = \Pi(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^d$)

$$\sqrt{n}\widehat{\boldsymbol{\rho}}_n \xrightarrow{d} \mathbf{Z} \sim N(\mathbf{0}_m, I_m)$$

for $n \to \infty$. It follows that

$$n \sum_{i=1}^{m} \widehat{\rho}_{i,n}^2 \xrightarrow{d} W$$

where W has a χ^2-distribution with m degrees of freedom. A test for stochastic independence is thus performed by rejecting the null hypothesis of stochastic independence if $n \sum_{i=1}^{m} \widehat{\rho}_{i,n}^2 > \chi^2_{\alpha,m}$, the corresponding $(1-\alpha)$-quantile of the χ^2-distribution with m degrees of freedom. This test is statistically tractable since no further unknown parameters have to be estimated. It also complements the set of tests for stochastic independence based on Spearman's rho considered by Quessy (2009) who studied asymptotic local efficiency. Regarding further rank tests for multivariate independence, we refer to Genest and Rémillard (2004) and Genest et al. (2007) and references therein.

5.6 Empirical study

In this section, we apply the four proposed tests for equi rank-correlation to financial data. The analysis is based on the same data as in section 4.4. We consider equity return series of the four banks BNP Paribas (BNP), Credit Suisse Group (CS), Deutsche Bank (DBK), and Barclays (BARC) during the period from May 1997 to April 2010. Since the above tests are derived under the assumption of independent observations, we apply those to the banks' standardized return series. A detailed description of the standardization approach is given in the aforementioned section. In particular, the empirical analysis indicates that all (squared) standardized returns contain minor serial correlation. Further empirical properties of the banks' equity prices and (standardized) returns are listed in section 4.4.

Figure 5.6 shows the evolution of all six pairwise Spearman's rho coefficients of the standardized return series from January 2003 to December 2006 calculated according to (5.6), based on a moving window approach with window size 250. It becomes apparent

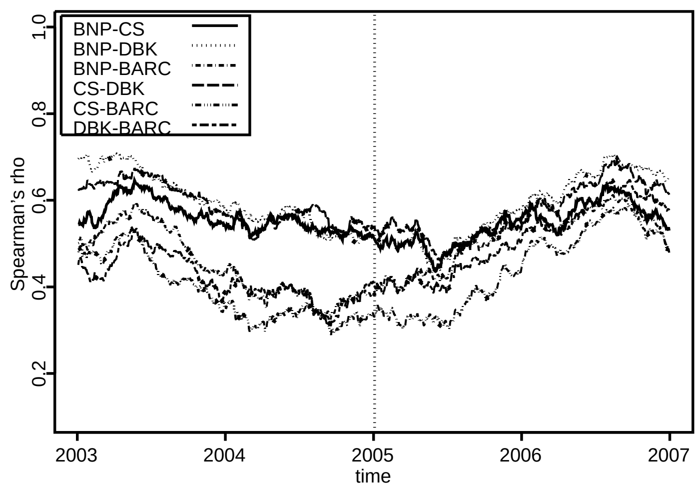


Figure 5.6: **Pairwise Spearman's rho.** Evolution of all six bivariate Spearman's rho coefficients of the standardized return series of BNP Paribas (BNP), Credit Suisse Group (CS), Deutsche Bank (DBK), and Barclays (BARC) from January 2003 to December 2006, calculated according to (5.6). The vertical line indicates January 3rd, 2005. Calculations are based on a moving window approach with window size 250.

that, from year 2005 onwards, all six pairwise Spearman's rho coefficients evolve closely to each other. Before, especially in the year 2004, there is a considerable difference observable. All in all, pairwise Spearman's rho ranges from 0.2890 in September 2004 (between the banks CS and BARC) to 0.7083 in April 2003 (between the banks BNP and DBK) during the considered time horizon. We apply the four proposed tests for equi rank-correlation to two time periods before and after the 3rd January 2005, which is indicated in figure 5.6 by the dotted vertical line. Both time periods comprise 250 observations. The output of all four tests is provided in table 5.1 where we give the values of the test statistics and the corresponding p-values for both periods. It turns out that, in the period before January 3rd, 2005 (period P1), the null hypothesis of equi rank-correlation is rejected by all tests at a significance level of 10%. The tests based on $\mathcal{T}_{n,1}$ and $\mathcal{T}_{n,2}$ thereby exhibit the smallest p-value rejecting the null hypothesis even at a significance level of 1%. For the period after January 3rd, 2005 (period P2), the test results are indifferent. For the tests based on $\mathcal{T}_{n,i}, i = 1, \ldots, 3$, the null hypothesis of equi rank-correlation cannot be rejected at all standard significance levels. In contrast, the test based on $\mathcal{T}_{n,4}$ has a p-value of 0.0827, implying that the null hypothesis for $\mathcal{T}_{n,4}$

Table 5.1: Values of the test statistics $\mathcal{T}_{n,i}, i = 1, \ldots, 4$, and corresponding p-values for the two periods before (period P1) and after (period P2) January 3rd, 2005, which comprises 250 observations each. The number of bootstrap replications is chosen as described in section 5.4.2.

	Period P1 test statistic	p-value	Period P2 test statistic	p-value
$\mathcal{T}_{n,1}$	4.4368	0.0097	0.8155	0.3307
$\mathcal{T}_{n,2}$	1.4982	0.0057	0.5513	0.1090
$\mathcal{T}_{n,3}$	10.7700	0.0562	8.0250	0.1549
$\mathcal{T}_{n,4}$	3.2778	0.0157	2.2963	0.0827

can be rejected at a significance level of 10%. In the light of our discussions at the end of section 5.1, the use of an Archimedean copula to model the dependence structure between the banks' standardized returns in the first period seems thus less appropriate according to the test results.

5.7 Appendix

5.7.1 Rank correlation coefficient ρ

Embrechts et al. (2002) discuss sufficient and necessary conditions on ρ such that the matrix $B = \rho \mathbf{1}_d \mathbf{1}_d' + (1-\rho) I_d$ is a rank correlation matrix. Those conditions are restated in the following proposition along with a shorter proof.

Proposition 5.7.1 *Let $\mathbf{X} = (X_1, ..., X_d)$ be a d-dimensional random vector with $d \geq 3$.*

1. *If the random vector $\mathbf{X}$ is equi rank-correlated with rank correlation coefficient ρ, then $-\frac{1}{d-1} < \rho < 1$.*

2. *If $\frac{6}{\pi} \arcsin(\frac{-\frac{1}{d-1}}{2}) < \rho < 1$, then there exists a random vector $\mathbf{X}$ which is equi rank-correlated with rank correlation coefficient ρ.*

Proof. 1. If the vector $\mathbf{X}$ is equi rank-correlated with rank correlation matrix $P = \rho \mathbf{1}_d \mathbf{1}_d' + (1-\rho) I_d$, then P is automatically a linear correlation matrix since bivariate Spearman's rho is defined as the linear correlation coefficient of the random variables $F_i(X_i), i = 1, \ldots, d$ (cf. formula (5.1)); hence $-\frac{1}{d-1} < \rho < 1$.

2. Let $\frac{6}{\pi} \arcsin(\frac{-\frac{1}{d-1}}{2}) < \rho < 1$ and define $r = 2\sin(\frac{\rho\pi}{6})$. Then, there exists a random vector $\mathbf{X}$ which is multivariate normally distributed with linear correlation matrix $R = r \mathbf{1}_d \mathbf{1}_d' + (1 - r) I_d$. Since, in this case, the relationship (5.20) holds, $\mathbf{X}$ is equi rank-correlated with rank correlation coefficient ρ. □

5.7.2 Simulation results referring to the estimation of the asymptotic covariance matrix

The following two tables display the simulation results of the bootstrap procedure for estimating the asymptotic variance/covariance of pairwise Spearman's rho. A description of the results is given in Section 5.2.2.

Table 5.2: **Gaussian copula**. Simulation results related to the estimation of the asymptotic covariance matrix of the vector of bivariate Spearman's rho coefficients $\widehat{\boldsymbol{\rho}}_n$ (defined in section 5.1) by means of the nonparametric bootstrap: Empirical means $m(\cdot)$, sample covariance matrices $\widehat{\Sigma}(\cdot)$, and (element-wise) standard deviations $\widehat{\sigma}(\cdot)$ of the Spearman's rho estimates. The latter are based on 300 Monte Carlo simulations of sample size n of a d-dimensional equi-correlated Gaussian copula with parameter κ and 300 bootstrap samples. The bootstrap estimates are labeled by the superscript B. We display the minimal and the maximal element of the respective estimated vectors (columns 5 and 6). Regarding the estimated matrices (columns 7 to 12), the minimum and the maximum of all diagonal (columns headed by 'diag') and all off-diagonal entries (columns headed by 'odiag') are shown separately.

κ	ρ	n		$m(\widehat{\boldsymbol{\rho}}_n)$	$m(\widehat{\boldsymbol{\rho}}_n^B)$	$n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n)$		$m(n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n^B))$		$\widehat{\sigma}(n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n^B))$	
						diag	odiag	diag	odiag	diag	odiag
Dimension $d = 3$											
0.5	0.483	100	max	.421	.416	.680	.281	.676	.246	.150	.100
			min	.413	.409	.634	.253	.656	.237	.146	.096
		500	max	.469	.468	.681	.259	.638	.238	.084	.061
			min	.468	.467	.595	.188	.635	.235	.077	.054
		1000	max	.477	.476	.674	.268	.641	.240	.068	.051
			min	.475	.474	.557	.216	.631	.233	.063	.047
0.2	0.191	100	max	.133	.131	1.135	.332	.968	.165	.137	.144
			min	.123	.121	1.032	.268	.956	.153	.132	.136
		500	max	.182	.182	.988	.213	.946	.170	.096	.081
			min	.178	.177	.828	.111	.933	.156	.087	.077
		1000	max	.187	.187	.987	.169	.947	.164	.092	.069
			min	.183	.183	.903	.107	.933	.153	.089	.066
0	0	100	max	-.054	-.055	1.005	.041	1.022	.002	.125	.149
			min	-.066	-.066	.862	-.042	1.020	-.008	.111	.139
		500	max	-.011	-.011	1.135	.059	1.005	.004	.094	.084
			min	-.015	-.015	.976	-.052	.999	-.005	.087	.081
		1000	max	-.004	-.004	1.074	-.020	1.005	.007	.087	.072
			min	-.008	-.008	.847	-.031	.990	-.001	.081	.066
Dimension $d = 5$											
0.5	0.483	100	max	.425	.420	.742	.281	.677	.252	.161	.107
			min	.414	.409	.581	.049	.654	.124	.139	.072
		500	max	.472	.471	.700	.280	.645	.240	.087	.061
			min	.467	.466	.560	.064	.630	.125	.071	.043
		1000	max	.479	.479	.699	.277	.637	.242	.072	.053
			min	.476	.475	.553	.073	.629	.127	.061	.040
0.2	0.191	100	max	.137	.134	1.057	.269	.977	.174	.135	.150
			min	.119	.118	.834	-.018	.954	.034	.116	.104
		500	max	.182	.181	1.003	.255	.953	.171	.094	.084
			min	.174	.174	.869	-.061	.930	.044	.086	.068

Table 5.2: (continued)

κ	ρ	n		$m(\widehat{\boldsymbol{\rho}}_n)$	$m(\widehat{\boldsymbol{\rho}}_n^B)$	$n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n)$ diag	odiag	$m(n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n^B))$ diag	odiag	$\widehat{\sigma}(n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n^B))$ diag	odiag
		1000	max	.187	.187	1.087	.242	.953	.168	.089	.073
			min	.183	.183	.856	-.081	.935	.040	.080	.057
0	0	100	max	-.054	-.054	1.190	.182	1.022	.019	.123	.162
			min	-.069	-.069	.857	-.102	1.012	-.020	.108	.106
		500	max	-.009	-.009	1.189	.161	1.012	.008	.094	.089
			min	-.016	-.016	.891	-.140	.999	-.015	.085	.069
		1000	max	-.004	-.004	1.163	.149	1.011	.009	.091	.076
			min	-.008	-.008	.857	-.093	.994	-.006	.082	.063
Dimension $d = 10$											
0.5	0.483	100	max	.429	.424	.730	.330	.673	.257	.158	.109
			min	.412	.407	.515	.028	.649	.115	.134	.067
		500	max	.477	.476	.750	.355	.641	.246	.089	.062
			min	.468	.467	.504	.041	.622	.124	.074	.041
		1000	max	.480	.480	.813	.342	.641	.247	.076	.056
			min	.475	.474	.554	.061	.622	.126	.062	.038
0.2	0.191	100	max	.139	.137	1.118	.305	.972	.177	.139	.150
			min	.120	.118	.766	-.132	.941	.028	.117	.097
		500	max	.182	.182	1.118	.294	.954	.173	.096	.089
			min	.174	.174	.770	-.136	.934	.035	.082	.061
		1000	max	.187	.187	1.126	.294	.952	.174	.094	.076
			min	.182	.181	.765	-.097	.932	.037	.080	.055
0	0	100	max	-.049	-.049	1.214	.192	1.037	.025	.134	.168
			min	-.072	-.072	.846	-.179	1.005	-.024	.107	.104
		500	max	-.006	-.006	1.195	.181	1.016	.015	.097	.093
			min	-.017	-.017	.872	-.221	.991	-.012	.081	.065
		1000	max	-.003	-.003	1.217	.209	1.014	.011	.092	.080
			min	-.009	-.010	.854	-.195	.994	-.013	.078	.057

Table 5.3: **Clayton copula**. Simulation results related to the estimation of the asymptotic covariance matrix of the vector of bivariate Spearman's rho coefficients $\widehat{\boldsymbol{\rho}}_n$ (defined in section 5.1) by means of the nonparametric bootstrap: Empirical means $m(\cdot)$, sample covariance matrices $\widehat{\Sigma}(\cdot)$, and (element-wise) standard deviations $\widehat{\sigma}(\cdot)$ of the Spearman's rho estimates. The latter are based on 300 Monte Carlo simulations of sample size n of a d-dimensional equi-correlated Gaussian copula with parameter r and 300 bootstrap samples. The bootstrap estimates are labeled by the superscript B. We display the minimal and the maximal element of the respective estimated vectors (columns 5 and 6). Regarding the estimated matrices (columns 7 to 12), the minimum and the maximum of all diagonal (column headed by 'diag') and all off-diagonal entries (column headed by 'odiag') are shown separately.

θ	ρ	n		$m(\widehat{\boldsymbol{\rho}}_n)$	$m(\widehat{\boldsymbol{\rho}}_n^B)$	$n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n)$		$m(n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n^B))$		$\widehat{\sigma}(n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n^B))$	
						diag	odiag	diag	odiag	diag	odiag
Dimension $d = 3$											
0.1	0.072	100	max	.018	.017	.936	.106	1.023	.081	.120	.148
			min	.011	.010	.892	.048	1.017	.075	.115	.135
		500	max	.062	.062	1.005	.096	1.010	.076	.097	.087
			min	.054	.054	.915	.004	1.006	.069	.087	.080
		1000	max	.066	.066	1.015	.107	1.013	.074	.094	.075
			min	.065	.065	.973	.047	1.006	.067	.085	.070
0.5	0.294	100	max	.229	.226	.944	.262	.930	.246	.147	.136
			min	.228	.225	.875	.170	.929	.229	.142	.130
		500	max	.286	.286	.974	.316	.915	.248	.097	.083
			min	.283	.283	.862	.246	.901	.240	.092	.077
		1000	max	.290	.290	1.070	.283	.903	.241	.085	.069
			min	.286	.286	.862	.172	.895	.239	.081	.064
2	0.682	100	max	.623	.617	.398	.192	.421	.211	.135	.088
			min	.616	.611	.360	.162	.410	.199	.127	.082
		500	max	.670	.669	.423	.218	.397	.204	.068	.044
			min	.669	.668	.375	.209	.394	.199	.063	.042
		1000	max	.674	.673	.427	.230	.398	.204	.056	.038
			min	.673	.673	.364	.185	.393	.201	.049	.033
Dimension $d = 5$											
0.1	0.072	100	max	.016	.016	1.139	.174	1.025	.081	.128	.159
			min	.003	.002	.854	-.085	1.008	-.012	.113	.107
		500	max	.063	.063	1.136	.169	1.012	.082	.096	.091
			min	.055	.055	.909	-.128	.997	.003	.086	.069
		1000	max	.069	.069	1.149	.207	1.006	.077	.093	.077
			min	.064	.064	.803	-.049	.996	.000	.080	.062
0.5	0.294	100	max	.248	.245	.969	.313	.937	.265	.154	.145
			min	.219	.216	.813	-.015	.907	.083	.135	.107
		500	max	.284	.284	.961	.273	.911	.246	.098	.082
			min	.279	.278	.806	-.021	.900	.088	.087	.065
		1000	max	.290	.290	.926	.325	.908	.242	.086	.071
			min	.287	.287	.751	-.004	.896	.088	.078	.056
2	0.682	100	max	.628	.622	.467	.272	.418	.210	.142	.090
			min	.618	.612	.364	.125	.397	.120	.130	.057
		500	max	.670	.669	.451	.234	.402	.206	.072	.047
			min	.665	.664	.306	.078	.390	.123	.061	.032
		1000	max	.676	.676	.414	.237	.396	.205	.055	.038
			min	.674	.673	.370	.108	.388	.126	.049	.027

Table 5.3: (continued)

θ	ρ	n		$m(\widehat{\boldsymbol{\rho}}_n)$	$m(\widehat{\boldsymbol{\rho}}_n^B)$	$n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n)$		$m(n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n^B))$		$\widehat{\sigma}(n\widehat{\Sigma}(\widehat{\boldsymbol{\rho}}_n^B))$	
						diag	odiag	diag	odiag	diag	odiag
Dimension $d = 10$											
0.1	0.072	100	max	.029	.028	1.186	.266	1.045	.091	.132	.166
			min	-.004	-.004	.864	-.163	1.001	-.015	.109	.104
		500	max	.063	.062	1.201	.243	1.019	.084	.101	.095
			min	.052	.052	.854	-.171	.994	-.004	.081	.066
		1000	max	.068	.068	1.256	.279	1.017	.082	.094	.080
			min	.061	.061	.819	-.151	.994	-.003	.078	.058
0.5	0.294	100	max	.236	.233	1.109	.389	.937	.250	.160	.152
			min	.220	.217	.777	-.047	.909	.073	.135	.101
		500	max	.287	.287	1.055	.403	.918	.251	.100	.088
			min	.279	.279	.736	-.084	.892	.084	.087	.061
		1000	max	.292	.292	1.017	.394	.911	.249	.089	.077
			min	.286	.285	.778	-.071	.893	.083	.075	.054
2	0.682	100	max	.627	.621	.461	.278	.428	.215	.145	.094
			min	.612	.607	.323	.073	.396	.120	.121	.052
		500	max	.674	.673	.474	.275	.403	.208	.072	.048
			min	.666	.665	.339	.065	.383	.121	.058	.029
		1000	max	.678	.677	.464	.258	.397	.204	.057	.040
			min	.674	.673	.328	.077	.385	.122	.047	.026

Chapter 6

Time dynamic and hierarchical dependence modeling of a supervisory portfolio of banks

Taking the perspective of a supervisor, we develop two (asymptotic) statistical test procedures based on Spearman's rho to analyze the association between the trading books of eleven German banks, having a regulatory approved internal market risk model. Based on real, clean profit and loss data and Value-at-Risk estimates of the eleven banks, we thereby analyze the portfolio's association both over time and across banks. On the one hand, we consider a statistical hypothesis test which is designed to detect significant long-term level changes of the portfolio's association over time. On the other hand, a statistical hypothesis test is proposed to identify the distinct contributions of sub-portfolios towards the overall level of association in a hierarchical manner. Since Spearman's rho is a copula-based measure of association, the tests are nonparametric and invariant with respect to the marginal distribution functions.

6.1 Motivation

Association between financial asset returns changes over time; cf. section 2.1. Especially in the course of deteriorating financial market conditions, this association often increases, which is referred to as the 'correlation breakdown' (cf. chapter 1). Well-established risk measures such as the Value-at-Risk however react quite sensitive towards any changes of a portfolio's correlation structure according to empirical studies; see e.g. Duellmann et al. (2007). This may, for example, affect a bank's internal capital planning which is usually based on Value-at-Risk measures. The internal capital planning process refers to the determination of an adequate level of economic capital for taking market risk or other risk positions. Further, the Basel II capital accord describes the framework under which banks can develop their own Value-at-Risk models in order to determine the amount of regulatory capital to be maintained. Under the assumption that the banks' risk models work correctly, the supervisory authorities can assess the risk stemming from each single bank based on those Value-at-Risk estimates. Increasing association between financial asset returns, however, may also impact the extent of co-movement between the banks' proprietary trading profits and losses (in short: P&L), i.e., the correlation between the banks P&Ls may increase during deteriorating market conditions. This can lead to a rising systemic risk in the banking sector, that is, to the risk of simultaneous large losses at several banks. The identification of such a type of co-movement of the banks' P&L is thus of interest to the supervisory authorities as it gives information about the systemic fragility of the financial system.

In this chapter, we take the perspective of a supervisor. The supervisor aggregates the respective bank trading portfolios into a hypothetical portfolio, the supervisory portfolio, for analyzing its inherent systemic risk. In particular, we develop two (asymptotic) test procedures to analyze the association in the supervisory portfolio both over time and across banks; cf. Gaißer et al. (2009). Regarding the time-dynamic analysis, the interest lies in detecting long-term level shifts of association over time. Possible level shifts should be detected as soon as new information arrives. We develop a two-step test procedure which takes those aspects into account. Based on the concept of control charts, the procedure is of sequential form. For an introduction to the theory of control charts and for different control chart designs, we refer to Wieringa (1999) and Schmid and Knoth (2000). There is a large literature on detecting structural changes in time series, exemplarily we mention Pawlak et al. (2004) or Steland (2002) and Golosnoy and Schmid (2007) who utilize control chart techniques in financial theory. In addition to the time-dynamic aspect, a hypothesis test is developed to analyze the hierarchical dependence structure of the supervisory portfolio at a specific point in time. This aims at identifying those banks that significantly contribute to a change of the overall association in the supervisory portfolio. In other words, this procedure simultaneously determines those groups of banks that show significant changes of association. We use Spearman's rho in order to quantify the degree of association between the P&Ls of the supervisory portfolio (cf. section 2.3.3) and thus, the two proposed test procedures are based on Spearman's rho. As Spearman's rho is a direct functional of the copula, the test procedures are derived from the weak convergence of the empirical copula process; cf. theorem 2.2.8.

Our analysis is based on real, clean P&L data and corresponding Value-at-Risk (VaR) estimates from eleven German banks having a regulatory approved internal market-risk model. Those clean P&L data do not reflect the actual profits and losses of a trading book since they are calculated under the assumption that the bank's trading book positions do not change within one day. However, they are often of more advantage for such empirical studies than e.g. economic P&L data. In general, there are only few studies working with real P&L data and VaR forecasts from such banks. Berkowitz and O'Brien (2002) analyze the daily VaR forecasts and corresponding P&L series of six large US banks to evaluate the performance of the banks' VaR models while Jaschke et al. (2003) provide a comparable analysis of VaR forecasts and P&L series of 13 German banks having a regulatory approved internal market model in the year 2001. The present study is a sequel of the work by Memmel and Wehn (2006), who focus on the VaR estimation of a supervisory portfolio by using different cross-correlation estimates under the assumption of multivariate normally distributed asset returns.

6.2 Control charts

We give a brief introduction to the concept of control charts in the following since the test procedure for Spearman's rho derived in section 6.3.1 is partly based on control-chart techniques; see e.g. Wieringa (1999) and Schmid and Knoth (2000). Control charts are an important tool of statistical process control (SPC). Being developed in the 1920s by Shewhart (1931), they were genuinely applied in the context of quality control management to improve processes in manufacturing and production. Meanwhile, they are used in many other fields, also in financial applications; we refer e.g. to Steland (2002) and Golosnoy and Schmid (2007) for an application of control chart techniques in the context of financial risk and portfolio analysis.

SPC applies statistical methods to control and monitor a given process with the aim to detect significant structural changes e.g. in the form of significant changes of the process' location or its variability. In particular, the aim is to detect such changes as soon as possible. In order to determine whether the process is in 'statistical control' or not, control charts are applied for monitoring the process. Based on observations from the underlying process, the value of a summary statistic, the so called control statistic, is calculated. These values of the control statistic are compared to so called control limits which are determined in such a way that, as long as those values are within the limits, it can be assumed that the process is in statistical control. In contrast, a value of the control statistic being outside the control limits indicates that the process is out of control. Wieringa (1999) refers to such points as out-of-control signals. A control chart is a time plot of the values of the control statistic and the control limits; an illustration is given in figure 6.1.

Remark. Shewhart (1931) originally distinguished between two components of the process' variation. On the one hand, the random, natural variation which has common causes and cannot be tied down to specific influencing factors. This 'common-cause variation' is regarded as to some extent predictable. This means that, if the process is subject to common-cause variation (or, if the process is in-control), it is possible to determine limits to this variation based on past observations of the process. In contrast, the process is subject to non-random, systematic variation (also 'special-cause-variation') if its variation exceeds those limits, i.e., the process is out-of-control. This sort of variation can usually be put down to some specific source. Hence, control charts serve to detect the variation in the process which may be due to 'special-cause-variation'.

Two phases must be distinguished when working with control charts. In a first step, the control chart has to be calibrated, i.e., the control limits have to be determined adequately. This is usually done based on a so called 'pre-sample', a sample of past

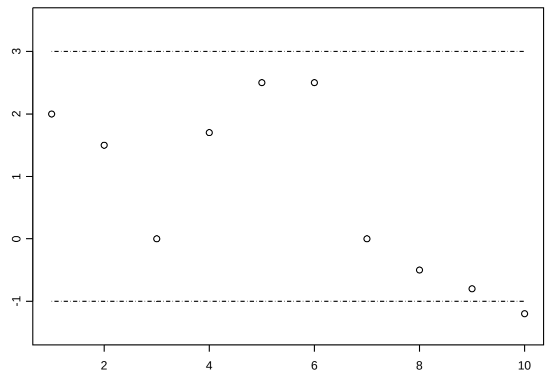


Figure 6.1: Control chart. Control limits (dashed lines) and values of the control statistic.

observations. Note that in order to obtain appropriate control limits, the pre-sample must stem from the in-control process. The second step then comprises the actual usage of the control chart to analyze and monitor the process by drawing sequentially samples from it.

In their original forms, control charts are based on the assumption of independent observations and are often referred to as classical control charts. Since in many areas of application, however, this assumption is not fulfilled, the so called modified control charts have been introduced which generalize the classical control charts to the case of dependent observations; cf. Wieringa (1999) and Schmid and Knoth (2000). As mentioned before, control chart designs further vary with respect to the process' characteristic to be monitored such as the location or the variation. In the following, we give examples for classical control charts for the location of a process since they serve as a basis for the forthcoming test procedure in section 6.3.1.

Classical control charts for the mean of normally distributed observations

Assume that, at discrete time t, the value of the process of interest is modeled by the random variable X_t. The X_t are stochastically independent for all t. Further, it is often assumed in the literature that X_t is normally distributed. In particular, $X_t \sim N(\mu, \sigma^2)$ in the in-control state. Both μ and σ are assumed to be known. As described above, they are usually estimated based on a pre-sample from the in-control process. Let further Z_t be the control statistic whose values are calculated based on observations from $\{X_t\}_{t\in\boldsymbol{Z}}$.

Shewhart-type control charts. Here, the control statistic is defined as

$$Z_t = X_t,$$

i.e., it only depends on the currently observed value of the process X_t. It is concluded that X_t is out-of-control at time t if

$$|Z_t - \mu| > c\sigma, \tag{6.1}$$

where c denotes a pre-specified constant which determines the width of the control limits. In particular, the values of the control statistic are compared to the control limits of the form $\mu \pm c\sigma$. Note that monitoring the process' mean over time as in (6.1) can be formulated as the following test setting; cf. Wieringa (1999). At each time t, we test the hypothesis

$$\mathrm{H}_0: \quad E(X_t) = \mu \quad \text{versus} \quad \mathrm{H}_1: \quad E(X_t) \neq \mu,$$

where H_0 is rejected if $|Z_t - \mu| > c\sigma$. The constant c can be determined using statistical arguments based on the distribution of the control statistic. Setting the probability of observing an out-of-control signal while the process is in-control to α, we obtain the control limits $\mu \pm z_{1-\alpha\setminus 2}\sigma$ with $c = z_{1-\alpha\setminus 2}$ being the $(1-\alpha\setminus 2)$-quantile of the standard normal distribution. Note that Shewhart (1931) originally determined the control limits due to economic considerations, setting $c = 3$.

EWMA control charts. In contrast to Shewhart-type control chart, the control statistic of this chart is not only based on the current, but also on previous observations. It is given by

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t, \quad t \geq 1,$$

where the start value is usually chosen as the target value for the mean, i.e., $z_0 = \mu$, and parameter $\lambda \in (0,1)$; cf. Schmid and Knoth (2000). If $\lambda = 1$, the Shewhart-type control chart is obtained as a special case. Note that Z_t can equivalently be written as

$$Z_t = z_0(1-\lambda)^t + \lambda \sum_{i=1}^{t} (1 - \lambda^{t-i}) X_i.$$

Thus, the parameter λ determines the weight given to the single observations which declines exponentially. Therefore, these control charts are called exponentially weighted moving average (in short: EWMA) control charts.

6.3 Time-dynamic and hierarchical testing for long-term level changes of Spearman's rho

This section develops two test procedures which are applied to analyze the supervisory portfolio's association both over time and across banks. Since the tests can be applied more generally, the results are formulated in a general setting.

As mentioned in section 6.1, both procedures use multivariate Spearman's rho to measure the degree of association in the supervisory portfolio. Specifically, our analysis is based on the d-dimensional version ρ_1 of Spearman's rho which we refer to as ρ_d in the following; cf. section 2.3.3. Recall that, for the d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$ with distribution function F, continuous univariate marginal distribution functions $F_i, i = 1, \ldots, d$, and copula C, it is given by

$$\rho_{d,\mathbf{X}} = h_\rho(d)\Big\{2^d \int_{[0,1]^d} C(\mathbf{u})\, d\mathbf{u} - 1\Big\}, \tag{6.2}$$

with $h_\rho(d) = (d+1)/\{2^d - (d+1)\}$. Note that if we think of $\mathbf{X}$ as representing the returns of d assets in a portfolio, ρ_d quantifies the association between the asset returns as determined by their copula. In the test procedures later, we also consider sub-portfolios, i.e., we are interested in measuring the degree of association between only those components X_i of $\mathbf{X}$ where $i \in \mathcal{I}$ with index set $\mathcal{I} \subseteq \{1, \ldots, d\}$ and cardinality $2 \leq \mathcal{I} \leq d$. Analogously to formula (6.2), we define the $|\mathcal{I}|$-dimensional Spearman's rho as

$$\rho_{|\mathcal{I}|} = h_\rho(|\mathcal{I}|)\Big\{2^{|\mathcal{I}|} \int_{[0,1]^{|\mathcal{I}|}} C_{i_1,\ldots,i_{|\mathcal{I}|}}(\mathbf{u})\, d\mathbf{u} - 1\Big\}, \tag{6.3}$$

for $\mathcal{I} = \{i_1, \ldots, i_{|\mathcal{I}|}\}$. Here, $C_{i_1,\ldots,i_{|\mathcal{I}|}}$ refers to the $|\mathcal{I}|$-dimensional copula which corresponds to the $i_1, \ldots, i_{|\mathcal{I}|}$-margin of C. Obviously, for $\mathcal{I} = S_d = \{1, \ldots, d\}$, $\rho_{|\mathcal{I}|} = \rho_d$. Based on a random sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ from $\mathbf{X}$, a nonparametric estimator for $\rho_{|\mathcal{I}|}$ can be obtained by replacing the copula $C_{i_1,\ldots,i_{|\mathcal{I}|}}$ in equation (6.3) by the empirical copula $\widehat{C}_{i_1,\ldots,i_{|\mathcal{I}|}}(u_{i_1}, \ldots, u_{|\mathcal{I}|}) = \widehat{C}(\mathbf{u}^{(\mathcal{I})})$; cf. definition 2.2.5. This yields the following nonparametric estimator for $\rho_{|\mathcal{I}|}$:

$$\widehat{\rho}_{|\mathcal{I}|,n} = h_\rho(|\mathcal{I}|)\Big\{\frac{2^{|\mathcal{I}|}}{n} \sum_{j=1}^{n} \Big[\,(1 - \widehat{U}_{i_1 j,n}) \ldots (1 - \widehat{U}_{i_{|\mathcal{I}|} j,n})\Big] - 1\Big\}. \tag{6.4}$$

Obviously, the corresponding estimator for ρ_d follows by setting $\mathcal{I} = \{1, \ldots, d\}$.

We establish some theoretical results in the following which form the basis for the forthcoming test procedures. The next theorem extends the results on the asymptotic normality of multivariate Spearman's rho (cf. section 2.3.3) to the difference of two Spearman's rhos. In a second step, a related result is shown for two vectors consisting of multiple Spearman's rho coefficients for various subsets $\mathcal{I}$ of $\{1, \ldots, d\}$. Both results are derived from the weak convergence of the empirical copula process $\sqrt{(}\widehat{C}_n - C)$; cf. theorem 2.2.8.

Theorem 6.3.1 *Consider two stochastically independent random samples* $(\mathbf{X}_s)_{s=1,\ldots,n}$ *and* $(\mathbf{Y}_s)_{s=1,\ldots,m}$ *from the d-dimensional random vectors* $\mathbf{X}$ *and* $\mathbf{Y}$ *with distribution functions* $F_\mathbf{X}$ *and* $F_\mathbf{Y}$*, continuous univariate marginal distribution functions and copulas* $C_\mathbf{X}$ *and* $C_\mathbf{Y}$*. Assume that the i-th partial derivatives of* $C_\mathbf{X}$ *and* $C_\mathbf{Y}$ *exist and are continuous for* $i = 1, \ldots, d$*. Let J be the set of all subsets $\mathcal{I}$ of* $\{1, \ldots, d\}$*. For* $A \subseteq J$ *with cardinality* $|A| = k$*, suppose that* $S_{A,n,\mathbf{X}}$ *and* $S_{A,m,\mathbf{Y}}$*, respectively, denote the k-dimensional random vectors of all sample versions* $\widehat{\rho}_{|\mathcal{I}|,n,\mathbf{X}}$ *and* $\widehat{\rho}_{|\mathcal{I}|,m,\mathbf{Y}}$ *of Spearman's* *rho with $\mathcal{I} \in A$, as calculated from the above random samples according to formula (6.4). Let further $\boldsymbol{\rho}_{A,\mathbf{X}}$ and $\boldsymbol{\rho}_{A,\mathbf{Y}}$ be the corresponding vectors of the true values $\rho_{|\mathcal{I}|,\mathbf{X}}$ and $\rho_{|\mathcal{I}|,\mathbf{Y}}$ of Spearman's rho (cf. equation (6.3)). We denote by $||\cdot||$ an arbitrary matrix norm on the space $[-1,1]^k$. Under the assumption that $\boldsymbol{\rho}_{A,\mathbf{X}} = \boldsymbol{\rho}_{A,\mathbf{Y}}$ and with $m := m(n)$ such that $\sqrt{n/m(n)} \to c \in [0,\infty)$ for $n \to \infty$, we have*

(i) for each $A \subseteq J$ with A being a single set $\mathcal{I}$

$$\sqrt{n}\Big(\widehat{\rho}_{|\mathcal{I}|,n,\mathbf{X}} - \widehat{\rho}_{|\mathcal{I}|,m(n),\mathbf{Y}}\Big) \xrightarrow{d} Z \sim N(0,\sigma^2) \quad \text{as } n \to \infty. \tag{6.5}$$

The variance has the form

$$\begin{aligned}\sigma^2 &= 2^{2|\mathcal{I}|} h_\rho(|\mathcal{I}|)^2 \int_{[0,1]^d}\int_{[0,1]^d} \Big[E\{\mathbb{G}_{C_\mathbf{X}}(\boldsymbol{u}^{(\mathcal{I})})\mathbb{G}_{C_\mathbf{X}}(\boldsymbol{v}^{(\mathcal{I})})\} \\ &+ c^2 E\{\mathbb{G}_{C_\mathbf{Y}}(\boldsymbol{u}^{(\mathcal{I})})\mathbb{G}_{C_\mathbf{Y}}(\boldsymbol{v}^{(\mathcal{I})})\}\Big] d\mathbf{u} d\mathbf{v}, \end{aligned} \tag{6.6}$$

with Gaussian process $\mathbb{G}_{C_\mathbf{X}}$ and $\mathbb{G}_{C_\mathbf{Y}}$ as defined in equation (2.2.8), theorem 2.2.8.

(ii) Further, for each $A \subseteq J$, it follows that

$$\sqrt{n}||S_{A,n,\mathbf{X}} - S_{A,m(n),\mathbf{Y}}|| \xrightarrow{d} W \quad \text{as } n \to \infty,$$

with non-degenerated random variable W.

Proof. (i) Given the theorem's prerequisites, theorem 2.2.8 and the continuous mapping theorem yield that, for a single index set $\mathcal{I}$,

$$\sqrt{n}(\widehat{\rho}_{|\mathcal{I}|,n,\mathbf{X}} - \rho_{|\mathcal{I}|,\mathbf{X}}) \xrightarrow{d} W_\mathbf{X} \sim N(0,\sigma^2_\mathbf{X}),$$

and

$$\sqrt{m(n)}(\widehat{\rho}_{|\mathcal{I}|,m(n),\mathbf{Y}} - \rho_{|\mathcal{I}|,\mathbf{Y}}) \xrightarrow{d} W_\mathbf{Y} \sim N(0,\sigma^2_\mathbf{Y}),$$

for $n \to \infty$, with

$$\sigma^2_\mathbf{X} = 2^{2|\mathcal{I}|} h(|\mathcal{I}|)^2 \int_{[0,1]^d}\int_{[0,1]^d} E\{\mathbb{G}_{C_\mathbf{X}}(\mathbf{u}^{(\mathcal{I})})\mathbb{G}_{C_\mathbf{X}}(\mathbf{v}^{(\mathcal{I})})\} d\mathbf{u} d\mathbf{v},$$

and

$$\sigma^2_\mathbf{Y} = 2^{2|\mathcal{I}|} h(|\mathcal{I}|)^2 \int_{[0,1]^d}\int_{[0,1]^d} E\{\mathbb{G}_{C_\mathbf{Y}}(\mathbf{u}^{(\mathcal{I})})\mathbb{G}_{C_\mathbf{Y}}(\mathbf{v}^{(\mathcal{I})})\} d\mathbf{u} d\mathbf{v},$$

respectively; cf. section 2.3.3. Under the assumption that $\rho_{|\mathcal{I}|,\mathbf{X}} = \rho_{|\mathcal{I}|,\mathbf{Y}}$, we have

$$\sqrt{n}\Big(\widehat{\rho}_{|\mathcal{I}|,n,\mathbf{X}} - \widehat{\rho}_{|\mathcal{I}|,m(n),\mathbf{Y}}\Big) = \sqrt{n}\Big(\widehat{\rho}_{|\mathcal{I}|,n,\mathbf{X}} - \rho_{|\mathcal{I}|,\mathbf{X}}\Big) - \frac{\sqrt{n}}{\sqrt{m(n)}}\sqrt{m(n)}\Big(\widehat{\rho}_{|\mathcal{I}|,m(n),\mathbf{Y}} - \rho_{|\mathcal{I}|,\mathbf{Y}}\Big),$$

and the assertion follows by an application of Slutsky's theorem and the fact that $\widehat{\rho}_{|\mathcal{I}|,n,\mathbf{X}}$ and $\widehat{\rho}_{|\mathcal{I}|,n,\mathbf{Y}}$ are based on stochastically independent random samples.
(ii) Let $\widehat{C}_{n,\mathbf{X}}(\mathbf{u})$ and $\widehat{C}_{m,\mathbf{Y}}, \mathbf{u} \in [0,1]^d$, denote the empirical copula of the random samples $(\mathbf{X}_l)_{l=1,\ldots,n}$ and $(\mathbf{Y}_l)_{l=1,\ldots,m}$, respectively, calculated according to (2.14); cf. definition 2.2.5. For $A \subseteq J$ with $|A| = k$ and $m = m(n)$, consider the k-dimensional random vectors $S_{A,n,\mathbf{X}}$ and $S_{A,m(n),\mathbf{Y}}$ which can be represented as a linear map of the empirical copulas $\widehat{C}_{n,\mathbf{X}}$ and $\widehat{C}_{m(n),\mathbf{Y}}$ into the k-dimensional Euclidean space $I\!R^k$, respectively. Thus, an application of the continuous mapping theorem together with theorem 2.2.8 yields the weak convergence of $\sqrt{n}\{S_{A,n,\mathbf{X}} - \boldsymbol{\rho}_{A,\mathbf{X}}\}$ and $\sqrt{m(n)}\{S_{A,m(n),\mathbf{Y}} - \boldsymbol{\rho}_{A,\mathbf{Y}}\}$ on the space $I\!R^k$. Since, in addition, $S_{A,n,\mathbf{X}}$ and $S_{A,m(n),\mathbf{Y}}$ are based on independent samples, joint weak convergence of

$$\left(\sqrt{n}\{S_{A,n,\mathbf{X}} - \boldsymbol{\rho}_{A,\mathbf{X}}\}, \sqrt{m(n)}\{S_{A,m(n),\mathbf{Y}} - \boldsymbol{\rho}_{A,\mathbf{Y}}\}\right)$$

on the product space $I\!R^{2k}$ is obtained for $n \to \infty$. If $\boldsymbol{\rho}_{A,\mathbf{X}} = \boldsymbol{\rho}_{A,\mathbf{Y}}$, this implies that $\sqrt{n}(S_{A,n,\mathbf{X}} - S_{A,n,\mathbf{Y}})$ converges in $I\!R^k$ by an application of Slutsky's theorem; cf. the proof of part (i). Finally, the fact that the matrix-norm $||\cdot||$ is also a continuous mapping from the space $I\!R^k$ into $I\!R$ and another application of the continuous mapping theorem yields the asserted result. □

Note that, if $S_{A,n,\mathbf{X}} = (\widehat{\rho}_{|\mathcal{I}_1|,n,\mathbf{X}}, \ldots, \widehat{\rho}_{|\mathcal{I}_k|,n,\mathbf{X}})'$ and $\boldsymbol{\rho}_{A,\mathbf{X}} = (\rho_{|\mathcal{I}_1|,\mathbf{X}}, \ldots, \rho_{|\mathcal{I}_k|,\mathbf{X}})'$ with A consisting of the k sets $\mathcal{I}_1, \ldots, \mathcal{I}_k$, $|A| = k$, we have

$$\sqrt{n}(S_{A,n,\mathbf{X}} - \boldsymbol{\rho}_{A,\mathbf{X}}) \xrightarrow{d} \mathbf{W} \sim N(0, \Sigma), \quad n \to \infty,$$

where the elements of the $(k \times k)$-dimensional matrix Σ are given by

$$\Sigma_{i,j} = 2^{|\mathcal{I}_i|+|\mathcal{I}_j|} h_\rho(|\mathcal{I}_i|) h_\rho(|\mathcal{I}_j|) \int_{[0,1]^d} \int_{[0,1]^d} E\{\mathbb{G}_{C_\mathbf{X}}(\mathbf{u}^{(\mathcal{I}_i)}) \mathbb{G}_{C_\mathbf{X}}(\mathbf{v}^{(\mathcal{I}_j)})\} d\mathbf{u} d\mathbf{v}, 1 \leq i, j \leq k.$$

In particular, the sequence $\sqrt{n}(S_{A,n,\mathbf{X}} - S_{A,m(n),\mathbf{Y}})$ is under the assumptions of theorem 6.3.1 asymptotically multivariate normally distributed.

We use the nonparametric bootstrap method as described in section 2.2.2 to estimate the distribution of the limiting variable W in theorem 6.3.1, part (ii), since this distribution usually cannot be derived explicitly and also depends on the choice of the matrix norm. That the nonparametric bootstrap works is shown in the next theorem.

Theorem 6.3.2 *Let $(\mathbf{X}_l^B)_{l=1,\ldots,n}$ and $(\mathbf{Y}_l^B)_{l=1,\ldots,m}$ be the bootstrap samples which are obtained by sampling from the independent random samples $(\mathbf{X}_l)_{l=1,\ldots,n}$ and $(\mathbf{Y}_l)_{l=1,\ldots,m}$ with replacement, respectively. For $A \subseteq J$, let further $S_{A,n,\mathbf{X}}$, $S_{A,m,\mathbf{Y}}$ be the vector of sample versions of Spearman's rho as given in theorem 6.3.1 and let $S^B_{A,n,\mathbf{X}}$, $S^B_{A,m,\mathbf{Y}}$ denote the corresponding estimators calculated from the bootstrap samples $(\mathbf{X}_l^B)_{l=1,\ldots,n}$*

and $(\mathbf{Y}_l^B)_{l=1,\ldots,m}$. *Then, under the assumptions of theorem 6.3.1 and with* $m := m(n)$ *such that* $\sqrt{n/m(n)} \to c \in [0,\infty)$ *for* $n \to \infty$, *the sequences*

$$\sqrt{n}||S^B_{A,n,\mathbf{X}} - S^B_{A,m(n),\mathbf{Y}} - (S_{A,n,\mathbf{X}} - S_{A,m(n),\mathbf{Y}})||$$

converges weakly to the same limit as $\sqrt{n}||S_{A,n,\mathbf{X}} - S_{A,m(n),\mathbf{Y}}||$ *in probability.*

Proof. Let $\widehat{C}^B_{n,\mathbf{X}}(\mathbf{u})$ and $\widehat{C}^B_{m,\mathbf{Y}}(\mathbf{u}), \mathbf{u} \in [0,1]^d$, denote the empirical copula of the bootstrap random sample $(\mathbf{X}_l^B)_{l=1,\ldots,n}$ and $(\mathbf{Y}_l^B)_{l=1,\ldots,m}$, obtained by sampling from the independent random samples $(\mathbf{X}_l)_{l=1,\ldots,n}$ and $(\mathbf{Y}_l)_{l=1,\ldots,m}$ with replacement, respectively. According to theorem 2.2.10, the processes $\sqrt{n}(\widehat{C}^B_{n,\mathbf{X}} - \widehat{C}_{n,\mathbf{X}})$ and $\sqrt{n}(\widehat{C}_{n,\mathbf{X}} - C_{\mathbf{X}})$ as well as $\sqrt{m(n)}(\widehat{C}^B_{m(n),\mathbf{Y}} - \widehat{C}_{m(n),\mathbf{Y}})$ and $\sqrt{m(n)}(\widehat{C}_{m(n),\mathbf{Y}} - C_{\mathbf{Y}})$ converge weakly to the same Gaussian limit in probability, respectively, for $n \to \infty$. An application of the continuous mapping theorem then yields that $\sqrt{n}\{S^B_{A,n,\mathbf{X}} - S_{A,n,\mathbf{X}}\}$ and $\sqrt{n}\{S_{A,n,\mathbf{X}} - \boldsymbol{\rho}_{A,\mathbf{X}}\}$, and $\sqrt{m(n)}\{S^B_{A,m(n),\mathbf{Y}} - S_{A,m(n),\mathbf{Y}}\}$ and $\sqrt{m(n)}\{S_{A,m(n),\mathbf{Y}} - \boldsymbol{\rho}_{A,\mathbf{Y}}\}$ converge in distribution to the same limit, respectively, in probability. Under the assumption that $\boldsymbol{\rho}_{A,\mathbf{X}} = \boldsymbol{\rho}_{A,\mathbf{Y}}$, the assertion follows according to the same reasoning as in the proof of theorem 6.3.1, part (ii). □

In particular, if $|A| = 1$, it follows that the asymptotic variance of $\sqrt{n}(\widehat{\rho}_{|\mathcal{I}|,n,\mathbf{X}} - \widehat{\rho}_{|\mathcal{I}|,m(n),\mathbf{Y}})$ in theorem 6.3.1, part (i), can be estimated using the nonparametric bootstrap.

We will use a moving window to estimate Spearman's rho over time. That is, an estimator for Spearman's rho $\rho^t_{|\mathcal{I}|,\mathbf{X}}$ at time t is calculated based on (past) observations $\mathbf{X}_{t-n+1}, \ldots, \mathbf{X}_t$ from $\mathbf{X}$ according to equation (6.4). The following theorem investigates the statistical properties of the difference of two Spearman's rho coefficients evaluated at two different points in time. To do so, we assume that an infinite amount of (independent) observations from $\mathbf{X}$ is available.

Theorem 6.3.3 *Consider the (i.i.d.) random sample* $(\mathbf{X}_t)_{t\in\mathbb{Z}}$ *from the d-dimensional random vector* $\mathbf{X}$ *with distribution function F, copula C, and continuous univariate marginal distribution functions. For an index set* $\mathcal{I} \subseteq \{1,\ldots,d\}$, *let* $\widehat{\rho}^t_{|\mathcal{I}|,n}$ *denote the estimator for Spearman's rho at time t as defined in formula (6.4) based on a (equally weighted) moving window of size n, i.e., calculated from the sample* $\mathbf{X}_{t-n+1}, \ldots, \mathbf{X}_t$. *We then have*

(i) for any fixed $s \in \mathbb{N}$ *with* $s < n$,

$$n\left(\widehat{\rho}^t_{|\mathcal{I}|,n} - \widehat{\rho}^{t-s}_{|\mathcal{I}|,n}\right) \xrightarrow{d} Z^{t,s}_{|\mathcal{I}|} \quad \text{as } n \to \infty, \tag{6.7}$$

with non-degenerated, bounded and centered random variable $Z^{t,s}_{|\mathcal{I}|}$.

(ii) Further, the limiting variables $Z^{t,s}_{|\mathcal{I}|}$ *and* $Z^{t-r,s}_{|\mathcal{I}|}$ *are stochastically independent for* $n > r > s > 0$. *In particular,*

$$n^2 \, Cov\Big(\widehat{\rho}^{\,t}_{|\mathcal{I}|,n} - \widehat{\rho}^{\,t-s}_{|\mathcal{I}|,n} \; , \; \widehat{\rho}^{\,t-r}_{|\mathcal{I}|,n} - \widehat{\rho}^{\,t-r-s}_{|\mathcal{I}|,n}\Big) \longrightarrow 0 \quad \text{as } n \to \infty, \tag{6.8}$$

for fixed $r, s \in I\!N$ *and* $n > r > s > 0$.

Proof. (i) Due to the same reasoning as in the proof of theorem 2.2.8, we can confine the analysis to the case where the marginal distribution functions F_i are uniform on $[0,1]$. Let $\widehat{F}^t_n$ denote the d-dimensional empirical distribution function of the random sample $\mathbf{X}_{t-n+1}, \ldots, \mathbf{X}_t$, $t \in \boldsymbol{Z}$. For $s < n$, it holds that

$$\begin{aligned} n&\Big\{\widehat{F}^t_n(\mathbf{u}^{(\mathcal{I})}) - \widehat{F}^{t-s}_n(\mathbf{u}^{(\mathcal{I})})\Big\} \\ &= \sum_{j=t-s+1}^{t} \prod_{\substack{i=1\\ i\in\mathcal{I}}}^{d} \mathbf{1}_{\{X_{ij} \le u_i\}} - \sum_{j=t-n-s+1}^{t-n} \prod_{\substack{i=1\\ i\in\mathcal{I}}}^{d} \mathbf{1}_{\{X_{ij} \le u_i\}} \stackrel{d}{=} Y^{t,s}(\mathbf{u}^{(\mathcal{I})}), \mathbf{u} \in [0,1]^d, \end{aligned} \tag{6.9}$$

the distribution of the latter random variable being independent of n. Given the Hadamard-differentiable map ϕ as defined in (2.19), an application of the functional delta-method (theorem 2.2.7) yields

$$n\Big\{\phi(\widehat{F}^t_n)(\mathbf{u}^{(\mathcal{I})}) - \phi(\widehat{F}^{t-s}_n)(\mathbf{u}^{(\mathcal{I})})\Big\} \stackrel{d}{\longrightarrow} \phi'_F(Y^{t,s})(\mathbf{u}^{(\mathcal{I})}), \tag{6.10}$$

where ϕ'_F denotes the Hadamard derivative of ϕ at F. With $\widehat{C}^t_n$ denoting the empirical copula of the random sample $\mathbf{X}_{t-n+1}, \ldots, \mathbf{X}_t$, $t \in \boldsymbol{Z}$, calculated according to (2.14), equation (2.22) then implies

$$n\Big\{\widehat{C}^t_n(\mathbf{u}^{(\mathcal{I})}) - \widehat{C}^{t-s}_n(\mathbf{u}^{(\mathcal{I})})\Big\} \longrightarrow \phi'_F(Y^{t,s})(\mathbf{u}^{(\mathcal{I})}).$$

Using that the difference $\widehat{\rho}^{\,t}_{|\mathcal{I}|,n} - \widehat{\rho}^{\,t-s}_{|\mathcal{I}|,n}$ $(s < n)$ can be written as

$$\widehat{\rho}^{\,t}_{|\mathcal{I}|,n} - \widehat{\rho}^{\,t-s}_{|\mathcal{I}|,n} = h_\rho(|\mathcal{I}|)2^{|\mathcal{I}|} \int_{[0,1]^d} \Big\{\widehat{C}^t_n(\mathbf{u}^{(\mathcal{I})}) - \widehat{C}^{t-s}_n(\mathbf{u}^{(\mathcal{I})})\Big\} d\mathbf{u},$$

an application of the continuous mapping theorem finally yields

$$n\Big(\widehat{\rho}^{\,t}_{|\mathcal{I}|,n} - \widehat{\rho}^{\,t-s}_{|\mathcal{I}|,n}\Big) \stackrel{d}{\longrightarrow} h_\rho(|\mathcal{I}|)2^{|\mathcal{I}|} \int_{[0,1]^d} \phi'(Y^{t,s})(\mathbf{u}^{(\mathcal{I})}) d\mathbf{u} = Z^{t,s}_{|\mathcal{I}|}, \quad n \to \infty. \tag{6.11}$$

(ii) We start with proving that $n\{\widehat{C}^t_n(\mathbf{u}^{(\mathcal{I})}) - \widehat{C}^{t-s}_n(\mathbf{u}^{(\mathcal{I})})\}$ is uniformly bounded in $\mathbf{u} \in [0,1]^d$ for fixed $s < n$. Observe that

$$\begin{aligned} n\Big\{\widehat{C}^t_n(\mathbf{u}^{(\mathcal{I})}) - \widehat{C}^{t-s}_n(\mathbf{u}^{(\mathcal{I})})\Big\} &= \sum_{j=t-s+1}^{t} \prod_{\substack{i=1\\ i\in\mathcal{I}}}^{d} \mathbf{1}_{\{\widehat{U}^t_{ij,n} \le u_i\}} - \\ - \sum_{j=t-n+1}^{t-s} \Big(\prod_{\substack{i=1\\ i\in\mathcal{I}}}^{d} \mathbf{1}_{\{\widehat{U}^t_{ij,n} \le u_i\}} - \prod_{\substack{i=1\\ i\in\mathcal{I}}}^{d} \mathbf{1}_{\{\widehat{U}^{t-s}_{ij,n} \le u_i\}}\Big) &- \sum_{j=t-s-n+1}^{t-n} \prod_{\substack{i=1\\ i\in\mathcal{I}}}^{d} \mathbf{1}_{\{\widehat{U}^{t-s}_{ij,n} \le u_i\}}, \end{aligned} \tag{6.12}$$

where $\widehat{U}_{ij,n}^{t} = 1/n$ (rank of X_{ij} in $X_{i(t-n+1)}, \ldots, X_{it}$) with sample $\mathbf{X}_{t-n+1}, \ldots, \mathbf{X}_t$, and $\widehat{U}_{ij,n}^{t-s} = 1/n$ (rank of X_{ij} in $X_{i(t-s-n+1)}, \ldots, X_{i(t-s)}$), which are based on the sample $\mathbf{X}_{t-s-n+1}, \ldots, \mathbf{X}_{t-s}$, respectively.
Note that the random variables $\widehat{U}_{ij,n}^{t}$ and $\widehat{U}_{ij,n}^{t-s}$ in the middle term of (6.12) deviate by a maximum of s/n only since the underlying rank order statistics are based on the $(n-s-1)$ common random variables $\mathbf{X}_{t-n+1}, \ldots, \mathbf{X}_{t-s}$ for all $i \in \mathcal{I}$. For each fixed $\mathbf{u} \in [0,1]^d$, there exists at most s index values $j_1, \ldots, j_s \in \{t-n+1, \ldots, t-s\}$ for which the middle term does not equal zero due to the bijective mapping of $\widehat{U}_{ij,n}^{t}$ and $\widehat{U}_{ij,n}^{t-s}$ onto $\{\frac{1}{n}, \ldots, \frac{n}{n}\}$. Thus,

$$\Big| \sum_{j=t-n+1}^{t-s} \Big(\prod_{\substack{i=1 \\ i \in \mathcal{I}}}^{d} \mathbf{1}_{\{\widehat{U}_{ij,n}^{t} \leq u_i\}} - \prod_{\substack{i=1 \\ i \in \mathcal{I}}}^{d} \mathbf{1}_{\{\widehat{U}_{ij,n}^{t-s} \leq u_i\}} \Big) \Big| \leq s \tag{6.13}$$

for each $\mathbf{u} \in [0,1]^d$. Including the other terms of formula (6.12) yields

$$|n\{\widehat{C}_n^t(\mathbf{u}^{(\mathcal{I})}) - \widehat{C}_n^{t-s}(\mathbf{u}^{(\mathcal{I})})\}| \leq 3s,$$

and, consequently,

$$|n(\widehat{\rho}_{|\mathcal{I}|,n}^{t} - \widehat{\rho}_{|\mathcal{I}|,n}^{t-s})| \leq 3s2^{|\mathcal{I}|} h_\rho(|\mathcal{I}|).$$

Thus, the bounded convergence theorem (see e.g. theorem 10.32 in Wheeden and Zygmund (1977)) together with part (i) of theorem 6.3.3 yields

$$n^2 \, Cov\Big(\widehat{\rho}_{|\mathcal{I}|,n}^{t} - \widehat{\rho}_{|\mathcal{I}|,n}^{t-s} \, , \, \widehat{\rho}_{|\mathcal{I}|,n}^{t-r} - \widehat{\rho}_{|\mathcal{I}|,n}^{t-r-s}\Big) \longrightarrow Cov\Big(Z_{|\mathcal{I}|}^{t,s}, Z_{|\mathcal{I}|}^{t-r,s}\Big).$$

Finally, formula (6.9) together with formula (6.10) shows that the limiting variables $Z_{|\mathcal{I}|}^{t,s}$ and $Z_{|\mathcal{I}|}^{t-r,s}$ are stochastically independent and, thus, are uncorrelated for $n > r > s > 0$. □

For the remainder of this section, we consider a sequence $(\mathbf{X}_t)_{t \in \mathbb{Z}}$ of d-dimensional random vectors $\mathbf{X}_t = (X_{t,1}, \ldots, X_{t,d})$ with distribution function F_t and continuous univariate marginal distribution functions $F_{t,i}, i = 1, \ldots, d$. According to theorem 2.2.2, there exists a (unique) copula C_t such that

$$F_t(\mathbf{x}) = C_t\{F_{t,1}(x_1), \ldots, F_{t,d}(x_d)\}, \mathbf{x} \in \mathbb{R}^d.$$

In particular, the dependence structure of $\mathbf{X}_t$ is described by the time-varying copula C_t. For an application of time-varying copulas in the context of Value-at-Risk calculations, see Giacomini et al. (2009). We further make the following assumption:

(A1) The random vectors $\mathbf{X}_t$ are stochastically independent for all t. Further, the continuous marginal distribution functions $F_{t,i}$ are constant over time, i.e., $F_{t,i}(x) = F_i(x), x \in \mathbb{R}$, for all $i = 1, \ldots, d$, and all t.

In general, it is difficult to perform hypothesis tests on the copula C_t when no further structural assumptions on the copula are imposed. Since the presumption of a specific parametric copula model would be too restrictive for our purposes, we assume that Spearman's rho determines the dependence structure of the random vectors by making the additional assumption:

(A2) At any time t, the dependence structure is completely described by (an adequate set of $|\mathcal{I}|$-dimensional) Spearman's rho (at time t).

Note that for the majority of parametric families of copulas, there exists a bijective relationship between the copula C and a set of $|\mathcal{I}|$-dimensional Spearman's rho coefficients $\rho_{|\mathcal{I}|}$, which is illustrated by three examples.

1. Let C be a member of the two-dimensional Farlie-Gumbel-Morgenstern family of copulas with parameter $\theta \in [-1, 1]$ as defined in (3.8). Then, bivariate Spearman's rho ρ of a random vector (X_1, X_2) having copula C equals $\theta/3$ (see e.g. Nelsen (2006), p.168).

2. If C is a d-dimensional Gaussian copula with correlation matrix $K = (\kappa_{ij})_{i,j=1,\ldots,d}$ (see (2.9)), it can be fully described by considering the $\binom{d}{2}$-dimensional vector of all bivariate Spearman's rho coefficients $\rho_{ij}, i < j$; cf. (5.1). In particular, relationship (5.20) holds.

3. Let C be a four-dimensional hierarchical Archimedean copula which is constructed by coupling the two-dimensional Archimedean copulas $C_{(1)}$ and $C_{(2)}$, generated by the (strict) generators $\phi_{(1)}$ and $\phi_{(2)}$, respectively, using a third (strict) generator $\phi_{(3)}$. Hence,

$$\begin{aligned} C(u_1, u_2, u_3, u_4) &= C\{C_{(1)}(u_1, u_2), C_{(2)}(u_3, u_4)\} \\ &= {\phi_{(3)}}^{-1}[\phi_{(3)} \circ {\phi_{(1)}}^{-1}\{\phi_{(1)}(u_1) + \phi_{(1)}(u_2)\} \\ &\quad +\phi_{(3)} \circ {\phi_{(2)}}^{-1}\{\phi_{(2)}(u_3) + \phi_{(2)}(u_4)\}]. \end{aligned} \tag{6.14}$$

   The conditions which need to be fulfilled such that the function in (6.14) is a copula function can be found in Joe (1997), section 4.2. For example, if $\phi_{(i)}, i = 1, 2, 3$, are generators of the Gumbel copula (see (2.12)) with parameters $\theta_i \geq 1$, then C is a copula if $\theta_3 < \theta_1$ and $\theta_3 < \theta_2$. The dependence structure of C is completely described by the three-dimensional vector consisting of the pairwise Spearman's rho coefficients $\rho_{(1)}$ and $\rho_{(2)}$ corresponding to the marginal copulas $C_{(1)}$ and $C_{(2)}$, respectively, and multivariate Spearman's rho ρ_4 as defined in (6.2). We refer to Savu and Trede (2008) and Hofert (2008) for further examples of hierarchical copulas and related estimation and simulation techniques.

With a view towards the test procedures derived in sections 6.3.1 and 6.3.2, theorems 6.3.1 and 6.3.3 state the asymptotic distribution of the difference of two Spearman's rho (for different samples or over time) under the assumption that Spearman's rho is constant. Note that all results established above can also be derived for other multivariate versions of Spearman's rho such as the average of distinct pairwise Spearman's rho coefficients.

6.3.1 Detecting long-term level changes of Spearman's rho over time

This section elaborates a procedure to detect level changes in the portfolio dependence over time using multivariate Spearman's rho. Specifically, our interest lies in detecting long-term level changes of Spearman's rho which - in addition - should be indicated as early as possible, i.e., as soon as new information has arrived. The procedure is of sequential form and consists of two (consecutive) steps, which are illustrated in table 6.1. Being based on a control chart for Spearman's rho (cf. section 6.2), Phase 1 sequentially monitors the series in order to detect level shifts of Spearman's rho. As far as long-term changes of Spearman's rho are concerned, it acts like an early indication or warning system. After having been signalled a shift in Spearman's rho in Phase 1, Phase 2 verifies whether a long-term rather than a short-term change is experienced; naturally, further observations need to be awaited for this purpose. The procedure in Phase 2 is therefore of static, retrospective form and can be regarded as a kind of 'dependence backtesting'.

Table 6.1: **Setup of test procedure**.

Test type	Test procedure
Early indication system of change of Spearman's rho	Phase 1: Control chart for Spearman's rho
Detection of sustainable change of Spearman's rho	Phase 2: 'Dependence-backtesting'

Let $(\mathbf{X}_t)_{t\in\boldsymbol{Z}}$ denote a sequence of d-dimensional random vectors with joint distribution function F_t and copula C_t fulfilling assumption (A1) and (A2) as elaborated before. For notational reasons, the description of the two phases is based on the $|\mathcal{I}|$-dimensional Spearman's rho coefficient $\rho^t_{|\mathcal{I}|}$ for arbitrary index set $\mathcal{I} \subseteq \{1,\ldots,d\}$. Note that the following elaborations could be generalized to the case of an adequate vector of Spearman's rho coefficients which should be considered otherwise; cf. assumption (A2)(see also section 6.3.2). The corresponding series of estimators of Spearman's rho based on an equally weighted moving window of size n is denoted by $(\widehat{\rho}^{\,t}_{|\mathcal{I}|,n})_{t\in\boldsymbol{Z}}$.

Phase 1. In this phase, a nonparametric control chart for detecting level changes in multivariate Spearman's rho is developed; cf. section 6.2. Let us therefore assume that, up to time $t^{'}$, there are no changes in Spearman's rho. Formally, $\rho^t_{|\mathcal{I}|} = \rho$ for fixed but unknown parameter ρ and for all $t \leq t^{'}$. We fix a lag parameter $s \in I\!N, s < n$, which allows to choose the time frequency for monitoring Spearman's rho (e.g. daily or weekly). At each time $t = t^{'} + ks, k = 1,2,3,\ldots$, we consider the hypothesis

$$\mathrm{H}_{0,t}: \quad \rho^t_{|\mathcal{I}|} = \rho \quad \text{versus} \quad \mathrm{H}_{1,t}: \quad \rho^t_{|\mathcal{I}|} \neq \rho. \tag{6.15}$$

We thereby reject the null hypothesis at time t if

$$n\,(\widehat{\rho}^{\,t}_{|\mathcal{I}|,n} - \widehat{\rho}^{\,t-s}_{|\mathcal{I}|,n}) > c_2 \quad \text{or} \quad n\,(\widehat{\rho}^{\,t}_{|\mathcal{I}|,n} - \widehat{\rho}^{\,t-s}_{|\mathcal{I}|,n}) < c_1,$$

with predefined constant c_1 and c_2.
Adopting the terminology of control charts, $Y_t := n(\widehat{\rho}^{\,t}_{|\mathcal{I}|,n} - \widehat{\rho}^{\,t-s}_{|\mathcal{I}|,n})$ represents the control statistic while the control limits are given by c_1 and c_2. The process $\widehat{\rho}^{\,t}_{|\mathcal{I}|,n}$ is 'in control' as long as the null hypothesis is not rejected; if it is rejected (i.e., the control chart gives a signal), it is concluded that the process is out of control. Note that Y_t represents an (asymptotically) unbiased estimator of $n(\rho^t_{|\mathcal{I}|} - \rho^{t-s}_{|\mathcal{I}|})$ if the process is in control, i.e., $E(Y_t)$ is asymptotically zero in this case. We thus concentrate on sequentially monitoring the mean or the location of the process $\widehat{\rho}^{\,t}_{|\mathcal{I}|,n}$. In particular, we reject the null hypothesis if Y_t exceeds or is less than the level c_2 or c_1, respectively. According to theorem 6.3.3, part (i), Y_t has a limiting distribution under the null hypothesis H_0 in (6.15), that is, if the process is in control. For a given significance level, the control limits $c_i, i = 1, 2$, can be chosen as the respective quantiles of this distribution due to the second part of theorem 6.3.3 (see also the remark at the end of this section).

Phase 2. The analysis in Phase 1 aims at identifying shifts in the level of Spearman's rho. If, as from the supervisory perspective, the focus lies on detecting long-term, sustaining (level) changes in portfolio dependence, a second phase is added subsequently to the first phase. Thereby, we understand by a long-term change that, after the shift indicated by Phase 1, Spearman's rho stays at this level throughout a specified period.

Assume therefore that Phase 1 gives a signal at time $t^\star > t'$. Based on further $n^\star = n - s$ observations of the process, Phase 2 compares Spearman's rho over distinct time periods before and after $t^\star$. Specifically, we verify whether there is a significant difference between Spearman's rho calculated based on the periods $[t^\star - n + 1, t^\star - s]$ and $[t^\star + 1, t^\star + n^\star]$. We further assume that Spearman's rho does not change throughout the latter period; cf. assumption (A2).
At $t^\star$, we then consider the hypothesis

$$\mathrm{H}_0: \quad \rho^{t^\star + n^\star}_{|\mathcal{I}|} = \rho^{t^\star - s}_{|\mathcal{I}|} \quad \text{versus} \quad \mathrm{H}_1: \rho^{t^\star + n^\star}_{|\mathcal{I}|} \neq \rho^{t^\star - s}_{|\mathcal{I}|}.$$

In this context, the statistic

$$T = \frac{\sqrt{n}(\widehat{\rho}^{\,t^\star + n^\star}_{|\mathcal{I}|,n^\star} - \widehat{\rho}^{\,t^\star - s}_{|\mathcal{I}|,n^\star})}{\hat{\sigma}^B_{|\mathcal{I}|}} \tag{6.16}$$

is under H_0 asymptotically standard normally distributed according to theorem 6.3.1, part (i). Here, $(\hat{\sigma}^B_{|\mathcal{I}|})^2$ represents the consistent bootstrap estimator for the asymptotic variance of $\sqrt{n}(\widehat{\rho}^{\,t^\star + n^\star}_{|\mathcal{I}|,n^\star} - \widehat{\rho}^{\,t^\star - s}_{|\mathcal{I}|,n^\star})$ as given in the latter theorem. We thus reject H_0 at $t^\star$ at level α if $|T| > z_{1-\alpha/2}$, where $z_{1-\alpha/2}$ denotes the $(1 - \alpha/2)$-quantile of the standard normal distribution. Note that by excluding the time point $t^\star$ from the analysis and choosing the window size $n^\star$, it is guaranteed that the above test is independent from the test in Phase 1.

Finally, it is concluded that $\widehat{\rho}^{t}_{|\mathcal{I}|,n}$ is out-of-control at time t if

(B1) both tests of Phase 1 and Phase 2 reject the null hypothesis at time t and

(B2) the control statistic of Phase 1 and the test statistic T of Phase 2 have the same sign.

We refer to such events fulfilling (B1) and (B2) simply as signals; an event in Phase 1 is called alarm in the following.

Remarks.

1. As outlined in section 6.2, the control limits c_1 and c_2 are estimated in a first step based on a sample of past observations from the in-control process to calibrate the control chart in Phase 1.
2. Using the differences $\widehat{\rho}^{t}_{|\mathcal{I}|,n} - \widehat{\rho}^{t-s}_{|\mathcal{I}|,n}$ rather than the actual observations $\widehat{\rho}^{t}_{|\mathcal{I}|,n}$ themselves as control statistic in Phase 1 (cf. the examples in section 6.2) offers several advantages. First, the latter approach would yield control limits involving the parameter ρ, leaving us with an additional parameter to estimate from the pre-sample in order to set up the control chart. Further, note that the series $\widehat{\rho}^{t}_{|\mathcal{I}|,n}$ exhibit a very high serial correlation. In the context of a sequential test, the control limits therefore would have to be adapted and are generally more difficult to determine (see e.g. Golosnoy and Schmid (2007) for the determination of the control limits in this situation in a parametric context). In contrast, the second part of theorem 6.3.3 states that the differences are (asymptotically) independent, allowing to obtain the limits c_1 and c_2 by probabilistic considerations. For completeness, we refer to Pawlak et al. (2004) who mention another basic approach where the control limits are determined subject to the size of a jump which shall be identified with high probability.
3. In chapter 4, section 4.3, we develop an exponentially weighted estimator for multivariate Spearman's rho. In the light of the EWMA control charts described in section 6.2, it would also be possible to design a control chart for detecting level changes in Spearman's rho using this weighted estimator for Spearman's rho as a control statistic.

6.3.2 Hierarchical testing

In contrast to the time-dynamic approach in the previous section, we consider a static approach now which we refer to as hierarchical testing. By fixing a particular time point t, the central question is whether there is a significant difference in the level of Spearman's rho before and after t. The approach differs from Phase 2 above insofar as we now include all lower-dimensional Spearman's rho coefficients into the analysis.

As usual, let $(\mathbf{X}_t)_{t\in\boldsymbol{Z}}$ denote a sequence of d-dimensional random vectors with joint distribution function F_t and copula C_t fulfilling assumption (A1) and (A2). For $l \in I\!N$ with $1 \leq l \leq d-1$, let $A = A(l)$ be the set of all subsets $\mathcal{I}$ of the index set $\{1,\ldots,d\}$ with cardinality $|\mathcal{I}| > l$. We define by $\boldsymbol{\rho}_A^t$ the vector of all $|\mathcal{I}|$-dimensional Spearman's rho coefficients $\rho_{|\mathcal{I}|}^t$ at time $t \in \boldsymbol{Z}$ with $\mathcal{I} \in A$. An estimator of the latter is given by $\widehat{\boldsymbol{\rho}}_{A,n}^t$ based on samples $\mathbf{X}_{t-n+1},\ldots,\mathbf{X}_t$ in a moving window with window size n.
In order to detect whether there is a level change of Spearman's rho at time t, we compare the values of all $|\mathcal{I}|$-dimensional Spearman's rho coefficients with $\mathcal{I} \in A$ between the adjacent and non-overlapping windows of fixed size n before and after t. Let us therefore assume that, throughout both periods, Spearman's rho does not change for any $\mathcal{I} \in A$, respectively (cf. assumption (A2)). We then consider the hypothesis

$$\mathrm{H}_0 : \boldsymbol{\rho}_A^{t-1} = \boldsymbol{\rho}_A^{t+n-1} \quad \text{versus} \quad \mathrm{H}_1 : \boldsymbol{\rho}_A^{t-1} \neq \boldsymbol{\rho}_A^{t+n-1}. \tag{6.17}$$

Note that, for each $\mathcal{I} \in A$, we would reject the null hypothesis of equal $|\mathcal{I}|$-dimensional Spearman's rho in the respective time periods at level $\alpha_{\mathcal{I}}$ if $|Q_{\mathcal{I},n}^t| > z_{1-\alpha_{\mathcal{I}}/2}$ with

$$Q_{\mathcal{I},n}^t = \frac{\sqrt{n}(\widehat{\rho}_{|\mathcal{I}|,n}^{t+n-1} - \widehat{\rho}_{|\mathcal{I}|,n}^{t-1})}{\hat{\sigma}_{|\mathcal{I}|}^B}.$$

Here, $(\hat{\sigma}_{\mathcal{I}}^B)^2$ represents the consistent bootstrap estimator of the variance of $\sqrt{n}(\widehat{\rho}_{|\mathcal{I}|,n}^{t+n-1} - \widehat{\rho}_{|\mathcal{I}|,n}^{t-1})$ as given in theorem 6.3.2, part (i). The latter theorem implies that $Q_{\mathcal{I},n}^t$ is under the null hypothesis (asymptotically) standard normally distributed (cf. also formula (6.16)) since $\widehat{\rho}_{|\mathcal{I}|,n}^{t+n-1}$ and $\widehat{\rho}_{|\mathcal{I}|,n}^{t-1}$ are based on independent samples. The null hypothesis in (6.17) is thus rejected at significance level β_l if $|Q_{\mathcal{I},n}^t| > z_{1-\alpha_{\mathcal{I}}/2}$ for some $\mathcal{I} \in A$, that is

$$I\!P\Big(\bigcup_{\substack{\mathcal{I}\in A\\ |\mathcal{I}|>l}} \Big\{ |Q_{\mathcal{I},n}^t| > z_{1-\alpha_{\mathcal{I}}/2} \Big\} \Big) = \beta_l. \tag{6.18}$$

The interrelationship between β_l and α is complicated – however, it may be approximated by Bonferroni's method as carried out in section 6.5.4. For convenience, we choose $\alpha_{\mathcal{I}} = \alpha$ for all $\mathcal{I}$ and obtain

$$I\!P\Big(\bigcup_{\substack{\mathcal{I}\in A\\ |\mathcal{I}|>l}} \Big\{ |Q_{\mathcal{I},n}^t| > z_{1-\alpha/2} \Big\} \Big) = I\!P(\sup_{\substack{\mathcal{I}\in A\\ |\mathcal{I}|>l}} |Q_{\mathcal{I},n}^t| > z_{1-\alpha/2}) = \beta_l$$

with $\sup_{\{\mathcal{I}\in A\,,\,|\mathcal{I}|>l\}} |Q_{\mathcal{I},n}^t| = \max_{\{\mathcal{I}\in A\,,\,|\mathcal{I}|>l\}} |Q_{\mathcal{I},n}^t|$ as A is finite. Hence, a test statistic for the null hypothesis in (6.17) is given by

$$\max_{\{\mathcal{I}\in A\,,\,|\mathcal{I}|>l\}} |Q_{\mathcal{I},n}^t|,$$

which has a limiting distribution under the null hypothesis according to theorem 6.3.1 with

$$k = |A(l)| = \sum_{\substack{j=l\\ |\mathcal{I}|=j}}^{d} \binom{d}{j}$$

and $||\cdot||$ being the maximum norm, i.e., $||\mathbf{t}|| = \max_{1\leq j\leq l} |t_j|, \mathbf{t} \in I\!R^k$. By changing the parameter l one can move from one portfolio's hierarchy level to another one. Note that for small hierarchical level l the power of the test will decrease due to a larger set of Spearman's rho coefficients included.

6.4 The standardized profits and losses of the supervisory portfolio

This section states the relevant definitions and assumptions for the analysis of the supervisory portfolio.

The daily clean P&L of the trading book of bank $i, i \in \{1,\dots,d\}$, at discrete time t are modeled by a random variable $G_{t,i}$. Since we do not consider economic P&L, we shortly refer to the $G_{t,i}$ as the P&L. Suppose $w_{t,i} = (w^1_{t,i},\dots,w^m_{t,i})'$ represents the positions of bank i on m financial instruments whose corresponding prices at time t are modeled by the random vector $P_t = (P^1_t,\dots,P^m_t)'$. Then $G_{t,i}$ takes the form

$$G_{t,i} = \sum_{j=1}^{m} w^j_{t-1,i}(P^j_t - P^j_{t-1}), \quad i = 1,\dots,d. \tag{6.19}$$

A central objective of a bank's internal risk model is to analyze and predict the future P&L distribution of the trading book by taking all past information into account. If the information flow available up to time t is modeled by the σ-algebra $(\mathcal{F}_{t,i})_{t\geq 0}$, $i = 1,\dots,d$, the interest thus lies in determining the conditional distribution function of $G_{t,i}$, denoted by $F_{t,i}(x) = I\!P(G_{t,i} \leq x|\ \mathcal{F}_{t-1,i})$. If not stated otherwise, without loss of generality we assume that $F_{t,i}(x)$ has infinite support. The Value-at-Risk (VaR) at confidence level α, $V_{t,i}$, is then obtained as the $(1-\alpha)$-quantile of $F_{t,i}$, i.e.

$$V_{t,i} = F^{-1}_{t,i}(1-\alpha), \quad i = 1,\dots,d. \tag{6.20}$$

We consider data $(G_{t,i}, V_{t,i}), i = 1,\dots,d$, which arise within a regulatory approved internal market-risk models. Hence, α is set to 0.99 in line with the supervisory requirement for approval of internal market-risk models. Note that in this setting, VaR is a negative number. For background reading on the VaR, we refer e.g. to Artzner et al. (1999) and Jorion (2006).

The P&Ls are standardized by dividing each bank's P&L by the respective VaR forecast for that day:

$$S_{t,i} = -\frac{G_{t,i}}{V_{t,i}}, \quad i = 1,\dots,d, \tag{6.21}$$

where $S_{t,i}$ is commonly referred to as the standardized P&L or standardized returns of bank i at time t. Assume that the random vector $\mathbf{S}_t = (S_{t,1},\dots,S_{t,d})$ represents the set of the banks' standardized returns in the supervisory portfolio at time t. As shown below, it is possible to concentrate on the modeling of $\mathbf{S}_t$ for our purposes.

Remark. The standardization in (6.21) is motivated by the following: If, conditional on the information $\mathcal{F}_{t-1,i}$, the $G_{t,i}$ are normally distributed, i.e. $G_{t,i} \mid \mathcal{F}_{t-1,i} \sim N(0, \sigma_{t,i}^2)$, the standardized returns $S_{t,i}$ take the form

$$S_{t,i} = -G_{t,i}/V_{t,i} = -\{\Phi^{-1}(1-\alpha)\}^{-1} G_{t,i}/\sigma_{t,i}.$$

That is, the standardization is with respect to the P&L's time-varying volatility in this case and any temporal dependence of the $G_{t,i}$ which is induced by a time-varying volatility (e.g. if asset prices follow a GARCH model) is removed. The standardized returns usually serve as a basis for the validation of a bank's VaR model, see e.g. Jaschke et al. (2003).

The standardized returns have (conditional) joint distribution function $F_{t,\mathbf{S}_t}(\mathbf{x}) = \mathbb{P}(\mathbf{S}_t \leq \mathbf{x} | \mathcal{G}_{t-1})$ with continuous univariate marginal distribution functions $F_{t,\mathbf{S}_{t},i}(x) = \mathbb{P}(S_{t,i} \leq x | \mathcal{G}_{t-1})$, $i = 1, \ldots, d$. Here, the σ-algebra $(\mathcal{G}_t)_{t \geq 0}$ represents the information flow available up to time t. Observe that, in general, $\mathcal{G}_t$ does not coincide with $\mathcal{F}_{t,i}$, $i = 1, \ldots, d$. Further, $V_{t,i}$ is $\mathcal{G}_{t-1}$ measurable. In the following, we always consider conditional distribution functions, taking with respect to the σ-algebra $(\mathcal{G}_t)_{t \geq 0}$ which is in line with the perspective of a supervisor. However, we will omit the conditioning for notational reasons. We assume that the standardized returns fulfill assumptions (A1) and (A2) as elaborated in section 6.3. In particular, the distribution function of $\mathbf{S}_t$ is described by

$$F_{t,\mathbf{S}_t}(\mathbf{x}) = C_t^S\{F_{\mathbf{S}_t,1}(x_1), \ldots, F_{\mathbf{S}_t,d}(x_d)\}, \quad \mathbf{x} \in \mathbb{R}^d, \tag{6.22}$$

with unique copula C_t^S according to theorem 2.2.2.
Note that the standardization of the P&L in (6.21) does not change the dependence structure between the banks' P&Ls as shown in the next corollary.

Corollary 6.4.1 *Suppose $\mathbf{S}_t$ has joint distribution function $F_{t,\mathbf{S}_t}$ with copula C_t^S as in (6.22) and assume that the $\mathbf{G}_{t,i}$, defined in (6.19), have joint distribution $F_{t,\mathbf{G}_t}$ with copula C_t^G and continuous marginal distribution functions and infinite support. Then, conditioned on the information up to time $t-1$, $C_t^S = C_t^G$, i.e., the standardization of $\mathbf{G}_t$ does not change the dependence structure represented by the copula C_t^G.*

Proof. Using the notation in (6.21) and the fact that $V_{t,i}$ is strictly negative, the transformation function $\beta_{t,i}(x) = -x/V_{t,i}$ is, conditioned on the information up to time $t-1$, strictly increasing. According to theorem 2.2.4, part (i), each copula is invariant with respect to strictly increasing transformations of the marginal distributions. □

6.5 Empirical results

In this section, we apply the theoretical results established in section 6.3 to the standardized returns of the supervisory portfolio introduced in section 6.4. Our analysis is based on daily clean P&L data and VaR forecasts of the trading book of eleven German banks which had a regulatory approved internal market risk model during the period from January 2001 to December 2006. The data, which are available on a daily basis, are maintained by the banks and reported in the Basel II framework to the supervisor; altogether we have 1435 observations. According to regulations, the VaR forecasts are calculated at a confidence level of 99% and for a one-day horizon.

6.5.1 Standardized returns

We start with some empirical properties of the standardized returns of the supervisory portfolio; further empirical analysis of the banks' individual standardized returns can be found in Memmel and Wehn (2006) and Jaschke et al. (2003).

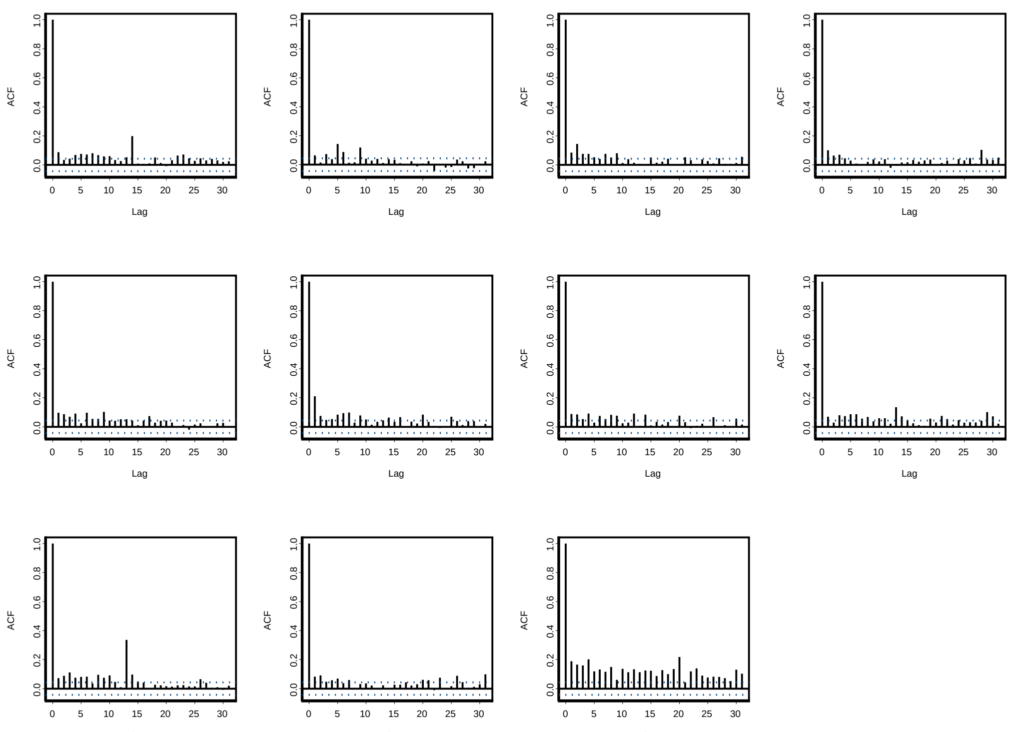


Figure 6.2: Autocorrelation function of the squared standardized returns of each bank of the supervisory portfolio.

The squared standardized returns contain only minor serial correlation; for illustration figure 6.2 gives the autocorrelation function across all banks. The same holds for the standardized returns themselves, too. Thus, the assumption of serial independence of the standardized returns (cf. assumption (A1)) can be justified. Note that this is also consistent with the literature on volatility modeling; see Andersen et al. (2006) for an overview. According to the reasoning in section 6.4, this underpins that the VaR models used by the banks work quite accurate over the considered data time horizon (cf. Jaschke et al. (2003)).

Table 6.2 reports the first four moments of the distribution of the standardized returns across all eleven banks. The results show that the kurtosis varies from bank to bank, ranging from 3.79 to 13.39. In order to verify whether those differences are statistically significant, we perform several pairwise tests on equal kurtosis, which are based on a bootstrap procedure. More specifically, we draw (with replacement) a bootstrap sample from $\mathbf{S}_t, t = 1, \ldots, T$, and determine for all pairs of banks the empirical confidence interval for the difference in kurtosis; an extract is given in table 6.3. The table shows that the difference in kurtosis of 32 out of 55 possible pairs of banks is significantly different from zero. This gives evidence that the tail behavior of (at least two) univariate standardized return distributions significantly differs. Thus, using the theory of copulas to model the joint distribution function of the standardized returns seems reasonable (see also the discussions in section 2.1).

Table 6.2: Descriptive statistics of the standardized returns for each bank of the supervisory portfolio.

Bank	Mean	St. deviation	Skewness	Kurtosis
1	0.0419	0.3225	-0.2422	13.3883
2	0.084	0.3721	0.1502	4.3238
3	0.0115	0.3805	-0.1155	4.5387
4	0.0905	0.3796	0.0284	4.2795
5	0.0046	0.2728	-0.0657	4.5791
6	0.0141	0.3523	-0.0659	4.2136
7	-0.0012	0.2741	-0.1607	5.6778
8	0.0125	0.4344	-0.2966	6.638
9	-0.011	0.4363	-0.3023	8.3636
10	0.0256	0.2786	0.0639	3.7949
11	-0.0532	0.3199	-0.323	5.7466

6.5.2 Multivariate Spearman's rho of the supervisory portfolio

Figure 6.3 (left panel) shows the evolution of multivariate Spearman's of the standardized returns of the supervisory portfolio. The estimation is based on a window size of $n = 150$. In addition, the horizontal line illustrates (multivariate) Spearman's rho calculated for the entire observation period. The figure shows that Spearman's rho fluctuates over time: The situation of deteriorating financial markets after the events of September 11 is accompanied by a steady increase in Spearman's rho. This is in line with the observations of high asset volatilities and correlations during this time period and an increase in medium-term interest rates from October 2001 on (Jaschke et al.

Table 6.3: 90% bootstrap confidence $(\hat{c}_l^B, \hat{c}_u^B)$ intervals for the difference in kurtosis for those pairs of banks where the difference is statistically different from 0, based on 10,000 bootstrap replications.

Bank	Bank	$\hat{c}_l^B$	$\hat{c}_u^B$	Bank	Bank	$\hat{c}_l^B$	$\hat{c}_u^B$
1	2	3.2529	13.8552	4	8	-5.1587	-0.1354
1	3	2.9961	13.6637	4	9	-6.2071	-1.6245
1	4	3.2552	13.9064	4	11	-2.4966	-0.3446
1	5	2.9149	13.6022	5	9	-5.771	-1.4653
1	6	3.3427	13.9024	5	10	0.0943	1.4968
1	7	1.8224	12.6234	5	11	-2.149	-0.1508
1	8	0.4997	12.1646	6	7	-2.6159	-0.3265
1	10	3.7619	14.3313	6	8	-5.1781	-0.2711
1	11	1.7493	12.4315	6	9	-6.1313	-1.7594
2	7	-2.5906	-0.1189	6	11	-2.3826	-0.6456
2	8	-5.1116	-0.1101	7	9	-4.9096	-0.0857
2	9	-6.1154	-1.5992	7	10	0.7575	3.0069
2	11	-2.4091	-0.335	8	10	0.7036	5.5623
3	9	-5.9475	-1.3387	9	10	2.2056	6.5528
3	11	-2.3295	-0.0193	9	11	0.1447	4.7268
4	7	-2.668	-0.1946	10	11	-2.8096	-1.0732

(2003)). After its first peak at the beginning of 2002, Spearman's rho falls sharply. Thereafter, a period of relatively low association in the portfolio is observable, which was characterized by medium-term interest rates at an all-time low and stabilizing markets. The year 2004 reveals a first steady, later sudden rise in Spearman's rho to its second peak in December 2004. During 2005, which proved to be a financial year of rising markets, Spearman's rho peaks off and remains relatively low for the rest of the observation period.

Some developments are particularly noticeable, such as the sudden upwards movement of Spearman's rho on the 6 February 2002 or on the 13 October 2004. At those two days, the P&L or standardized returns, respectively, of all banks in the portfolio proved to be negative, i.e., all banks simultaneously realized losses. For comparison, we provide in figure 6.3 (right panel) the average of all pairwise Spearman's rho coefficients (see also Memmel and Wehn (2006) for an analysis of the supervisory portfolio's VaR based on the average linear correlation coefficient). While multivariate Spearman's rho reacts sensitive to these days of simultaneous negative movements, the latter shows a more gradual and steady increase and does not emphasize those extreme events. However, those events may be of particular interest to the supervisor in general - especially,

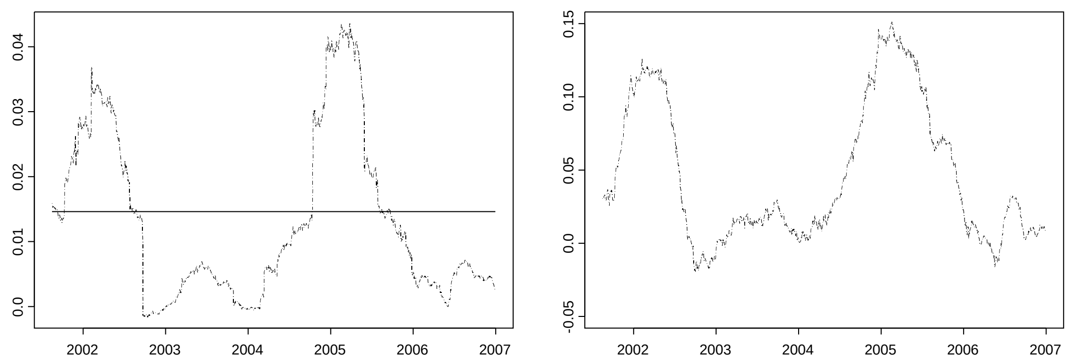


Figure 6.3: Time-varying (dashed line) and cosntant (solid line) multivariate Spearman's rho $\widehat{\rho}_{11,\mathbf{S}}$ (left panel) and the average of all pairwise Spearman's rho coefficients (right panel) of the standardized returns of the supervisory portfolio, based on a moving window of size $n = 150$.

as the simultaneous realization of losses across all banks only happened at altogether four days during the observation period and those days revealed the highest losses.

6.5.3 Level changes of Spearman's rho of the supervisory portfolio over time

The time-dynamic test procedure proposed in section 6.3.1 is applied to the supervisory portfolio, i.e., $\mathcal{I} = \{1, \ldots, d\}$. We further set $s = 1$ and thus focus on monitoring Spearman's rho of the supervisory portfolio on a daily basis.

The main motivation of the control chart design in Phase 1 is the fact that the first differences of Spearman's rho estimates $\widehat{\rho}^{\,t}_{d,n}$ are (asymptotically) serially uncorrelated. The sample autocorrelation functions of the original time series and the first differences are given in figure 6.4. For calibrating the control chart in Phase 1, we use the first 150 observations of the series $\{n(\widehat{\rho}^{\,t}_{d,n} - \widehat{\rho}^{\,t-1}_{d,n})\}$ (denoted as pre-sample which is assumed to be from the in-control process) in order to determine the control limits c_1 and c_2, according to the procedure described in section 6.3.1 and with window size $n = 150$.

Estimates $\hat{c}_1$ and $\hat{c}_2$ of the control limits c_1 and c_2 are given as the empirical $\alpha/2$- and $(1-\alpha/2)$-quantiles of the pre-sample; the confidence level α is set to 0.05 in both Phase 1 and Phase 2. The control chart of Phase 1 as well as the results of the test procedure are given in figure 6.5. Here, $\hat{c}_1$ and $\hat{c}_2$ are -0.03962 and 0.28709, respectively; altogether, we observe 241 alarms in Phase 1. Proceeding with Phase 2, where the estimation of the bootstrap variance is based on 500 bootstrap replications, we obtain two signals at time $t = 308$ and $t = 1038$. The corresponding values of the test

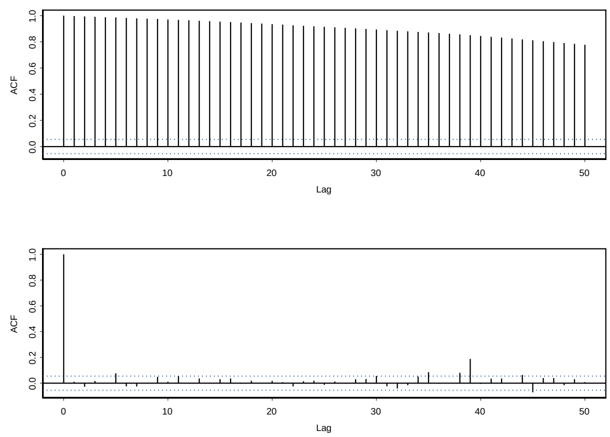
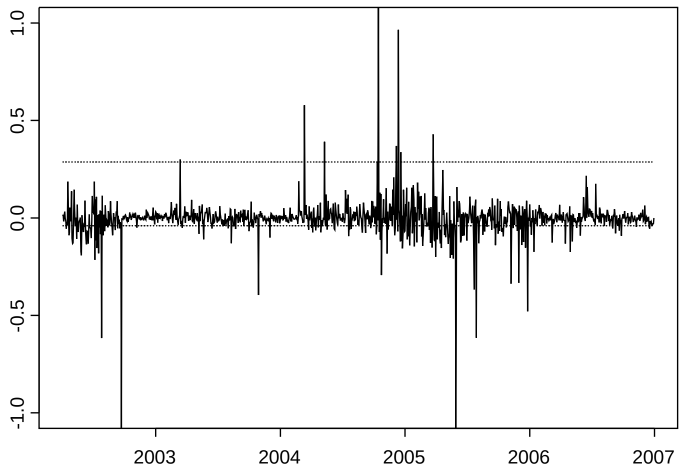


Figure 6.4: Sample autocorrelation function of the time series of Spearman's rho estimates $\widehat{\rho}_{d,n}^{t}$ (upper panel) and of the first differences $\widehat{\rho}_{d,n}^{t} - \widehat{\rho}_{d,n}^{t-1}$ (lower panel) of the supervisory portfolio with window size $n = 150$.

statistic are provided in the table of figure 6.5. It becomes clear from figure 6.3 that both signals occur at the respective global downward movements of Spearman's rho at the beginning of the years 2002 and 2005.

After an alarm has been triggered in Phase 1 and, thus, an early warning of level change in Spearman's rho has occurred, market relevant factors and events should be analyzed around the time the alarm occurred in order to find a possible economic interpretation for the shift in the portfolio's association. As elaborated above, the next observations (in our case 149 observations) shall then be awaited in Phase 2 in order to test for a significant long-term level change of the portfolio's association.

Note that by leaving the control limits unchanged throughout the whole period, we would not use the information provided by the test procedure, i.e., that a signal has occurred. Therefore we apply the test procedure anew - only this time, we recalibrate the control chart of Phase 1 each time after a signal has been observed: The control limits are re-estimated based on the 150 observations following (and including) the signal and the chart is restarted. The corresponding output is given in figure 6.6. We provide the corresponding control chart of Phase 1 together with the re-estimated control limits whose values are explicitly given in the table. This time, 230 alarms are obtained in Phase 1, leaving us with 3 signals in Phase 2. Hence, in addition to the signals obtained from the control chart without recalibration, we observe a signal at $t = 755$. As figure 6.6 shows, this new signal occurs at the global increase of Spearman's rho at the beginning of the year 2004.



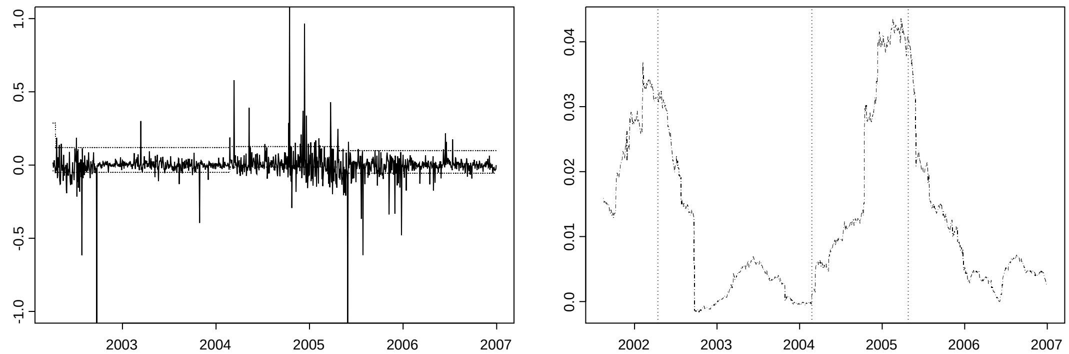
t	date	Phase 1 $[\hat{c}_1, \hat{c}_2]$	$n(\widehat{\rho}^{t}_{d,n} - \widehat{\rho}^{t-1}_{d,n})$	Phase 2 T	p-value
308	12.04.2002	$[-0.03962, 0.28709]$	-0.05490001	-2.73087	0.00632
1038	26.04.2005	$[-0.03962, 0.28709]$	-0.08792838	-2.06398	0.03902

Figure 6.5: Upper panel: Control chart in Phase 1 of differences $n(\widehat{\rho}^{t}_{d,n} - \widehat{\rho}^{t-1}_{d,n})$ with the estimated control limits $\hat{c}_1$ and $\hat{c}_2$ (horizontal lines); lower panel: Summary of the test statistics including those dates with significant signals. Here, T refers to the test statistics given in (6.16). The results are based on $\alpha = 0.05$, 500 bootstrap replications, and window size $n = 150$.

Summarizing the above findings, a decrease in the portfolio's association can be observed in April 2002 which goes in line with the situation of improving financial markets after the events of September 11. We further detect a significant rise in the portfolio's association between February 2004 and April 2005. This increase cannot fully be explained by market events but requires a detailed analysis of the banks' trading portfolios.

6.5.4 Hierarchical considerations for the supervisory portfolio

The hierarchical testing described in section 6.3.2 answers the question which groups of banks of the supervisory portfolio show a significant change of Spearman's rho around a predefined time point when applied to its standardized returns.



		Phase 1		Phase 2	
t	date	$[\hat{c}_1, \hat{c}_2]$	$n(\widehat{\rho}_{d,n}^t - \widehat{\rho}_{d,n}^{t-1})$	T	p-value
308	12.04.2002	$[-0.03962, 0.28709]$	-0.05490001	-2.73087	0.00632
755	23.02.2004	$[-0.04952, 0.11975]$	0.18811689	2.74026	0.00614
1038	26.04.2005	$[-0.01370, 0.12654]$	-0.08792838	-2.06398	0.03902
	>26.04.2005	$[-0.05506, 0.09861]$			

Figure 6.6: Upper left panel: Control chart in Phase 1 of differences $n(\widehat{\rho}_{d,n}^t - \widehat{\rho}_{d,n}^{t-1})$ with the estimated control limits $\hat{c}_1$ and $\hat{c}_2$ (horizontal lines); upper right panel: multivariate Spearman's rho with signals (vertical lines); lower panel: Summary of the test statistics including those dates with significant signals. Here, T refers to the test statistics given in (6.16). The results are based on $\alpha = 0.05$, 500 bootstrap replications, and window size $n = 150$.

According to Bonferronis inequality, we have

$$\mathbb{P}\Big(\bigcup_{\substack{\mathcal{I} \in A \\ |\mathcal{I}|>l}} \Big\{ |Q_{\mathcal{I},n}^t| > z_{1-\alpha/2} \Big\} \Big) \leq \sum_{\substack{\mathcal{I} \in A \\ |\mathcal{I}|>l}} \mathbb{P}(|Q_{\mathcal{I},n}^t| > z_{1-\alpha/2}) = \beta_l,$$

and we may choose α in such a way that

$$\alpha \sum_{\substack{k=l \\ |\mathcal{I}|=k}}^{d} \binom{d}{k} = \beta_l,$$

and thus, $\alpha = \beta_l / \{2^d - \sum_{k=0}^{l-1} \binom{d}{k}\}$. For the supervisory portfolio, we have $d = 11$ and we set the overall test level β_l to 0.1 and $l = 8$. Hence, $\alpha = 0.00149$ and $z_{1-\alpha/2} = 3.176131$, in this case. Further, we concentrate on the three time points identified as level changes of Spearman's rho in the previous section, though any other time point might be possible, too.

The value of the test statistic $\max_{\{I \in J\ ,\ |\mathcal{I}|>8\}} |Q^t_{\mathcal{I},n}|$ at those three time points is given in table 6.4. It follows that the null hypothesis (6.17) is rejected at level β_8 at time

Table 6.4: Results of the hierarchical test procedure of the supervisory portfolio at time points $t = 313$ (19.04.2002), $t = 755$ (23.02.2004), and $t = 1038$ (26.04.2005) for $l = 8$. Value of the test statistic $\max_{\{I \in A,\ |\mathcal{I}|>8\}} |Q^t_{\mathcal{I},n}|$ for testing the overall null hypothesis (6.17). Calculations are based on 500 bootstrap replications and $\beta_8 = 0.1$.

t	date	$\max_{\{I \in A,\ \|\mathcal{I}\|>8\}} \|Q^t_{\mathcal{I},n}\|$	$z_{1-\alpha/2}$
313	19.04.2002	4.53089	3.176131
755	23.02.2004	3.46422	3.176131
1038	26.04.2005	2.71896	3.176131

points $t = 313$ and $t = 755$. This implies that the association has significantly changed in the period before and after those time points among the portfolios with dimension greater than 8. By contrast, we cannot reject the null hypothesis at the time point $t = 1038$.

For the time points $t = 313$ and $t = 755$, table 6.5 further provides all sub-portfolios with dimension greater than 8 showing significant changes of Spearman's rho at level α. Altogether, there are 23 sub-portfolios of dimensions 9 and 10 at $t = 313$. The 9-dimensional sub-portfolio consisting of the banks $2, 3, 4, 5, 6, 7, 8, 10$, and 11 possess the smallest p-value. At $t = 755$, only sub-portfolios of dimension 9 show a significant change in Spearman's rho at level α. Here, the sub-portfolio consisting of the banks $1, 2, 3, 4, 6, 8, 9, 10$, and 11 has the smallest p-value. For illustration, figure 6.7 shows the evolution of Spearman's rho of the sub-portfolio with the highest significant change in Spearman's rho at those two time points together with the sub-portfolio of the same dimension having the largest p-value.

Note that, at $t = 313$, the two smallest banks, banks 1 and 9 (as measured in terms of average VaR), appear considerably less often than any other bank; see table 6.5. This may imply that the changes in association around $t = 313$ are mainly driven by the larger banks.

The present findings of the hierarchical testing do not imply that there is no significant change of Spearman's rho at time point $t = 1038$; it only shows that there is none among the sub-portfolios with dimension greater than 8. Further, the fact that more than three times as many sub-portfolios with dimension greater than 8 show a significant change in Spearman's rho at $t = 313$ than at $t = 755$ might provide an indication of the stability of the financial markets around that time. Financial markets indeed showed a more volatile behavior in 2002 than at the beginning of 2004.

Table 6.5: Output of the hierarchical testing for the supervisory portfolio at the time points $t = 313$ (19.04.2002) and $t = 755$ (23.02.2004) for $l = 8$. Significant sub-portfolio combinations $\mathcal{I}$, corresponding value of the statistic $Q^t_{\mathcal{I},n}$, and p-value. Calculations are based on 500 bootstrap replications and $\beta_l = 0.1$.

t=313			t=755		
$\mathcal{I}$	$Q^t_{\mathcal{I},n}$	p-value	$\mathcal{I}$	$Q^t_{\mathcal{I},n}$	p-value
1 2 3 4 5 6 7 8 10 11	-3.39774	0.00068	1 2 3 4 5 6 8 9 10	3.35484	0.00079
2 3 4 5 6 7 8 9 10 11	-3.82334	0.00013	1 2 3 4 5 6 9 10 11	3.36225	0.00077
1 2 3 4 5 6 7 8 9	-3.23098	0.00123	1 2 3 4 6 7 8 9 11	3.23002	0.00124
1 2 3 4 5 6 7 8 10	-4.25733	0.00002	1 2 3 4 6 8 9 10 11	3.46422	0.00053
1 2 3 4 5 6 7 10 11	-3.54487	0.00039	1 2 3 4 7 8 9 10 11	3.2428	0.00118
1 2 3 4 5 6 8 10 11	-3.60147	0.00032	2 3 4 5 6 8 9 10 11	3.28402	0.00102
1 2 3 4 5 7 8 10 11	-3.33302	0.00086	2 3 4 6 7 8 9 10 11	3.37724	0.00073
1 2 3 4 6 7 8 10 11	-4.00944	0.00006			
1 2 3 5 6 7 8 10 11	-3.99211	0.00007			
1 2 4 5 6 7 8 9 10	-3.29191	0.001			
1 2 4 5 6 7 8 10 11	-4.25264	0.00002			
1 2 5 6 7 8 9 10 11	-3.19147	0.00142			
1 3 4 5 6 7 8 10 11	-3.67182	0.00024			
2 3 4 5 6 7 8 9 10	-3.35125	0.0008			
2 3 4 5 6 7 8 9 11	-3.77839	0.00016			
2 3 4 5 6 7 8 10 11	-4.53089	0.00001			
2 3 4 5 6 7 9 10 11	-3.66836	0.00024			
2 3 4 5 6 8 9 10 11	-3.58624	0.00034			
2 3 4 5 7 8 9 10 11	-3.29662	0.00098			
2 3 4 6 7 8 9 10 11	-3.92365	0.00009			
2 3 5 6 7 8 9 10 11	-3.9123	0.00009			
2 4 5 6 7 8 9 10 11	-4.13888	0.00003			
3 4 5 6 7 8 9 10 11	-3.6275	0.00029			

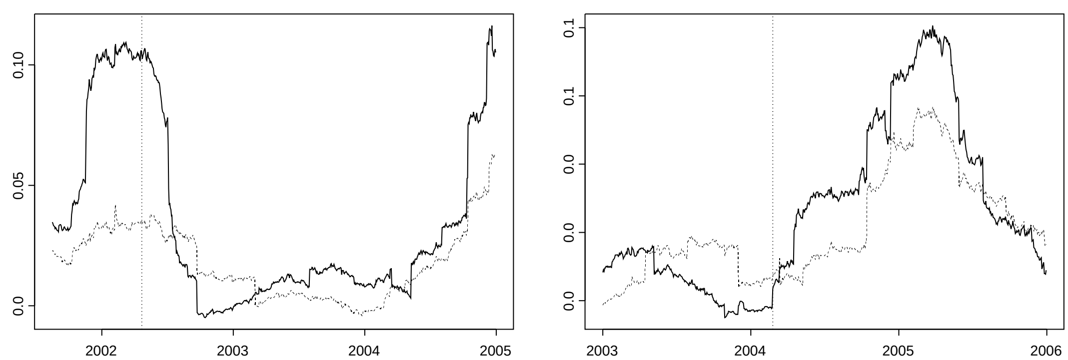


Figure 6.7: Development of Spearman's rho of sub-portfolios. Left panel: Sub-portfolio of banks $2, 3, 4, 5, 6, 7, 8, 10$, and 11 (solid line) against sub-portfolio of banks $1, 2, 3, 4, 6, 7, 8, 9$, and 11 (dotted line) with the vertical line representing the time point $t = 313$ (19.04.2002); Right panel: Sub-portfolio of banks $1, 2, 3, 4, 6, 8, 9, 10$, and 11 (solid line) against sub-portfolio of banks $1, 2, 3, 4, 5, 6, 7, 8$, and 10 (dotted line) with the vertical line representing the time point $t = 755$ (23.02.2004).

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