On the weight distribution in Demazure modules of $\widehat{\mathfrak{sl}}_2$

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Stavros Kousidis

aus Oelde

Berichterstatter:	Prof. Dr. Peter Littelmann, Prof. Dr. George Marinescu, Prof. Dr. Evgeny Feigin.
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Kurzzusammenfassung. Wir berechnen den erwarteten Grad eines zufällig gewählten Elements in einer Basis von Gewichtsvektoren eines beliebigen Demazure Moduls von $\widehat{\mathfrak{sl}}_2$ durch Induktion über Demazures Charakterformel. Entlang unseres Argumentationsweges erhalten wir einen neuen Beweis für Sandersons Dimensionsformel für diese Demazure Moduln. Zusätzlich berechnen wir die Kovarianz der vollen Gewichtsverteilung in Level 1 Demazure Moduln von $\widehat{\mathfrak{sl}}_2$. Der Schwerpunkt liegt dabei auf der Berechnung der Varianz der Gradverteilung. Die Kenntniss der Kovarianz erlaubt es uns, das schwache Gesetz der großen Zahlen mittels Chebyshevs Ungleichung zu beweisen. Wir führen zwei Beweise für unsere Resultate bezüglich Level 1 Demazure Moduln, der Erste durch Induktion über Demazures Charakterformel, der Zweite mittels Quantum Analysis und der Tatsache, dass die Charaktere von Level 1 Demazure Moduln in Verbindung zu Macdonald und Rogers–Szegő Polynomen stehen.

Abstract. We compute the expected degree of a randomly chosen element in a basis of weight vectors of an arbitrary Demazure module of $\widehat{\mathfrak{sl}}_2$ by induction along Demazure's character formula. Along those lines we obtain a new proof of Sanderson's dimension formula for these Demazure modules. Furthermore, we compute the covariance of the full weight distribution in level 1 Demazure modules of $\widehat{\mathfrak{sl}}_2$. The crucial step is to compute the variance of the degree distribution. The knowledge of the covariance allows us to prove the weak law of large numbers for the degree and full weight distribution using Chebyshev's inequality. We give two proofs of our results concerning level 1 Demazure modules, one by induction along Demazure's character formula, and one by using quantum calculus and the fact that the characters of level 1 Demazure modules are related to Macdonald and Rogers–Szegő polynomials.

Contents

1. Introduction, organization, and notation

1.1. **Introduction.** The traditional way to study the dimensions of weight spaces of a given representation is by means of their generating function, the character. While the character comprehends all weight multiplicities, it may be difficult to extract meaningful information from it. For example, the characters of irreducible representations of semisimple Lie algebras are explicitly given by Weyl's character formula. But to estimate weight multiplicities in representations with a large highest weight, more meaningful information can possibly be obtained from the fact that for $N \to \infty$ the weight distribution of $V(N\lambda)$ converges weakly to an absolutely continuous measure with piecewise polynomial density. Similarly, given the character of a representation, one can immediately write down the character of its tensor powers $T^N(V)$, namely $\mathrm{ch}_{T^N(V)} = (\mathrm{ch}_V)^N$. To extract meaningful information, one could interpret this probabilistically as a convolution product of measures, saying that weights in tensor powers are distributed like sums of independent random variables, identically distributed according to the weight distribution of V. Then, by the central limit theorem and careful analysis, one can derive statistical information and estimates of weight multiplicities in large tensor powers [34].

In this thesis, we take a probabilistic point of view on weight multiplicities in Demazure modules of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. The weight distributions of those Demazure modules are discrete measures on the plane. Some examples are shown in Figure 1[1].

Even though initially studied for semisimple Lie algebras [1, 6], Demazure modules in our context are most interesting in the case of Kac–Moody algebras of affine or indefinite type [21, 27, 28], as they provide an exhaustion of the infinite dimensional integrable highest weight modules by finite dimensional vector spaces. The characters of Demazure modules are in principle explicitly known by Demazure's character formula. Yet, the recursive and non-positive nature of the latter makes it hard to extract explicit meaningful information about individual weight multiplicities, asymptotics of weight multiplicities, or overall features of the weight distribution. Other character formulas for Demazure modules exist, e.g. [11, 13, 15, 22, 23, 24, 31, 33]. As these express Demazure characters in terms of functions which are themselves subject to current research, it is again not obvious how to extract the above mentioned information from them.

For affine Kac–Moody algebras, certain specializations of Demazure characters have been extensively and successfully studied, the so-called *real characters*, i.e., characters of the underlying semisimple Lie algebra. The main statement is known as the *factorization phenomenon*, and says that

[1]The software used to produce most of the figures in this thesis is available at https://sourceforge.net/projects/demazure.

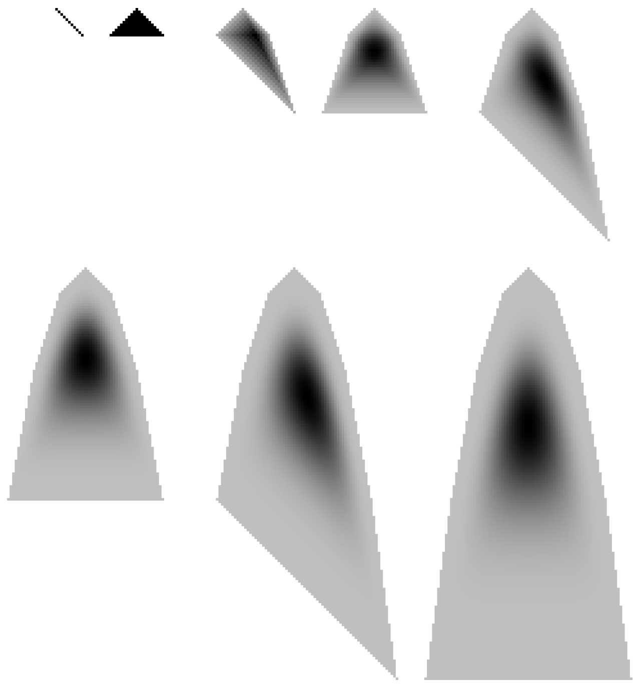

FIGURE 1. Weight distribution of $V_{w_{N,0}}(10\Lambda_0)$ for $N = 1,\dots,8$. The horizontal axis corresponds to the finite weight, the vertical axis to the degree. Light gray corresponds to the weight multiplicity 1, black to the maximal occurring weight multiplicity in a given Demazure module.

these specialized versions allow a product decomposition ([12, 30] and many others). When formulated in probabilistic terms, the product decomposition translates into a convolution product decomposition of the corresponding weight distribution. Hence they can be studied by a great number of basic and advanced techniques, such as the law of large numbers and the central limit theorem for independent identically distributed random variables, large deviations techniques, and the method of stationary phase [34]. Unfortunately, the factorization phenomenon and all those tools only apply to the real characters. Specializing to the real characters means that all degree information contained in the Demazure modules is lost, i.e., any two weights are identified if they differ only by a multiple of the null root δ. In probabilistic terms, one only studies a marginal distribution of the full weight distribution, which we will call the finite weight distribution.

The study of the full weight distribution should start with the determination of its most basic statistical quantities, the expected value and the covariance matrix. We compute the expected value of the full weight distribution for all Demazure modules of the affine Kac–Moody algebra $\widehat{\mathfrak{sl}}_2$,

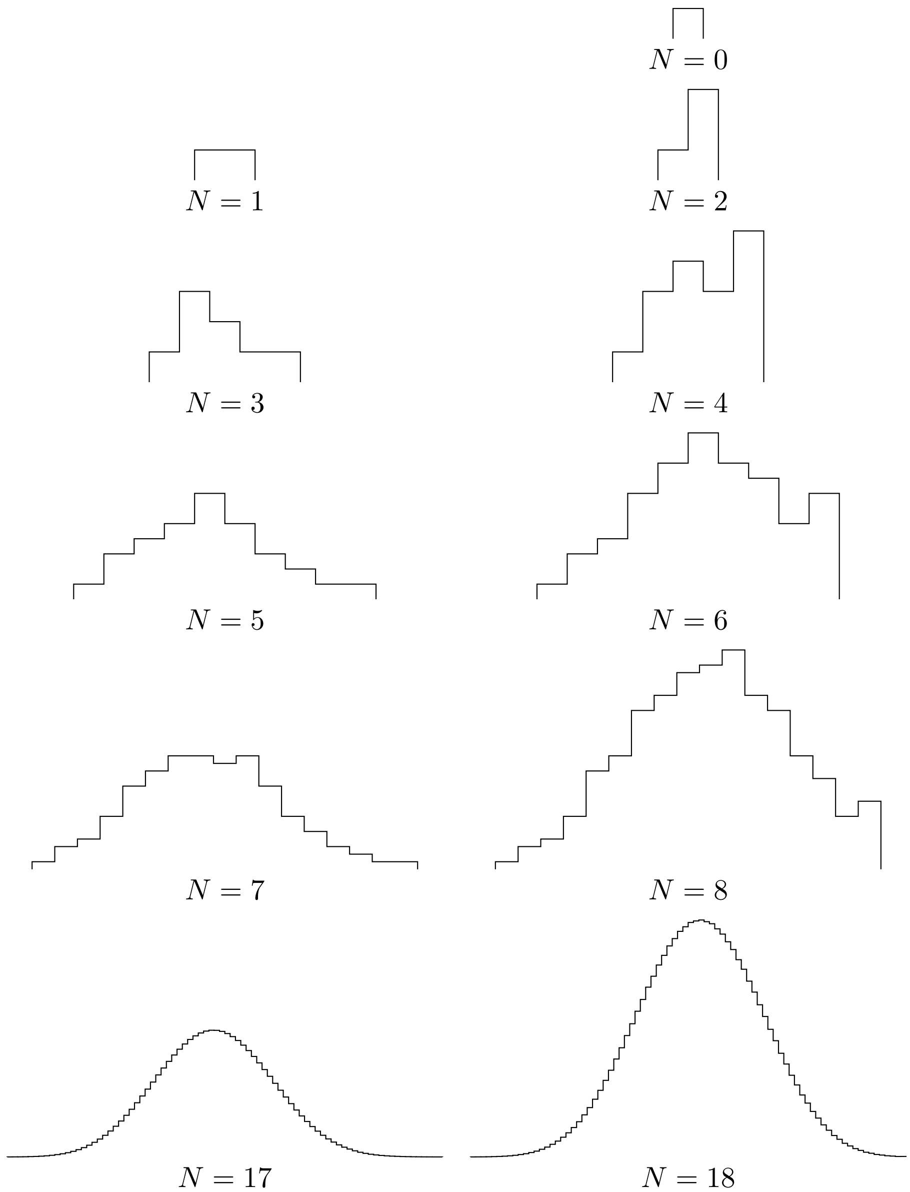

FIGURE 2. Degree distribution of $V_{w_{N,0}}(\Lambda_0)$ for $N = 0, 1, \ldots, 8; 17, 18$. Each picture displays degree 0, the degree of the highest weight, on the left, and the maximal occurring degree in the given Demazure module on the right.

and continue this investigation by computing the covariance matrix of the full weight distribution for its level 1 Demazure modules. This is enough to obtain the weak law of large numbers as a corollary, to our knowledge the first result to give an idea about the overall weight distribution in large Demazure modules.

The key problem is to understand the other important marginal distribution, the degree distribution, which seems to have escaped attention so far. This is especially surprising since it corresponds to the basic specialization of Demazure characters, and in the case of infinite dimensional integrable highest weight modules the basic specialization yields Macdonald's identities for Dedekind's η-function [16, §12.2]. Some examples of degree distributions associated to Demazure modules are shown in Figure 2 and 3. In fact, although the pictures suggest that the central limit theorem holds, what makes its analysis particularly difficult is that the factorization phenomenon does not occur in this situation, i.e, the degree distribution does not decompose as a convolution product.

Let us elaborate on the main results presented in this thesis. First, note that almost all main results are initially formulated and proven for Demazure modules indexed by elements in the affine Weyl group W^{aff}. Nevertheless, it is possible to extend and formulate them in terms of Demazure modules indexed by elements in the extended affine Weyl group $\widetilde{W}^{\mathrm{aff}}$. Let us do this here for reasons of brevity, that is to avoid case considerations. Denote the non-trivial automorphism of the Dynkin diagram of $\widehat{\mathfrak{sl}}_2$ by σ, and the scaling element (giving the degree of weights) by d.

Now, the expected value of the overall weight distribution of a Demazure module is fully described by the individual expected values of the (marginal) finite weight and degree distribution, respectively. Due to the factorization phenomenon and Weyl group symmetry the expected value of the former can easily be determined. Therefore we are only concerned about the expected value of the latter.

Theorem (Expected degree, Cf. Theorem 2.2.2). *Let $\Lambda = m\Lambda_0 + n\Lambda_1$ be a dominant integral weight, $N \geq 1$, and $j \in \{0, 1\}$. Denote by $\mu_{(\sigma s_j)^N}$ the weight distribution of $V_{(\sigma s_j)^N}(\Lambda)$. Consider $-d \in \mathfrak{h}$ as a function on $\mathfrak{h}^*$. Choose a basis of weight vectors in the Demazure modules $V_{(\sigma s_0)^N}(\Lambda)$ and $V_{(\sigma s_1)^N}(\Lambda)$. Then the expected degrees of a randomly chosen basis element are given by the following formulas, respectively.*

$$\begin{aligned}
\mathrm{E}_{\mu_{(\sigma s_0)^N}}[-d] &= \frac{2(N-1)m(m+2) + (N-1)(N-2)(m+n)(m+n+2)}{12(m+n+1)} \\
&\quad + \left\lfloor \frac{N-1}{2} \right\rfloor \frac{n}{2} + \left\lfloor \frac{N}{2} \right\rfloor \frac{m}{2}, \\
\mathrm{E}_{\mu_{(\sigma s_1)^N}}[-d] &= \frac{2(N-1)n(n+2) + (N-1)(N-2)(m+n)(m+n+2)}{12(m+n+1)} \\
&\quad + \left\lceil \frac{N-1}{2} \right\rceil \frac{m}{2} + \left\lceil \frac{N}{2} \right\rceil \frac{n}{2}.
\end{aligned}$$

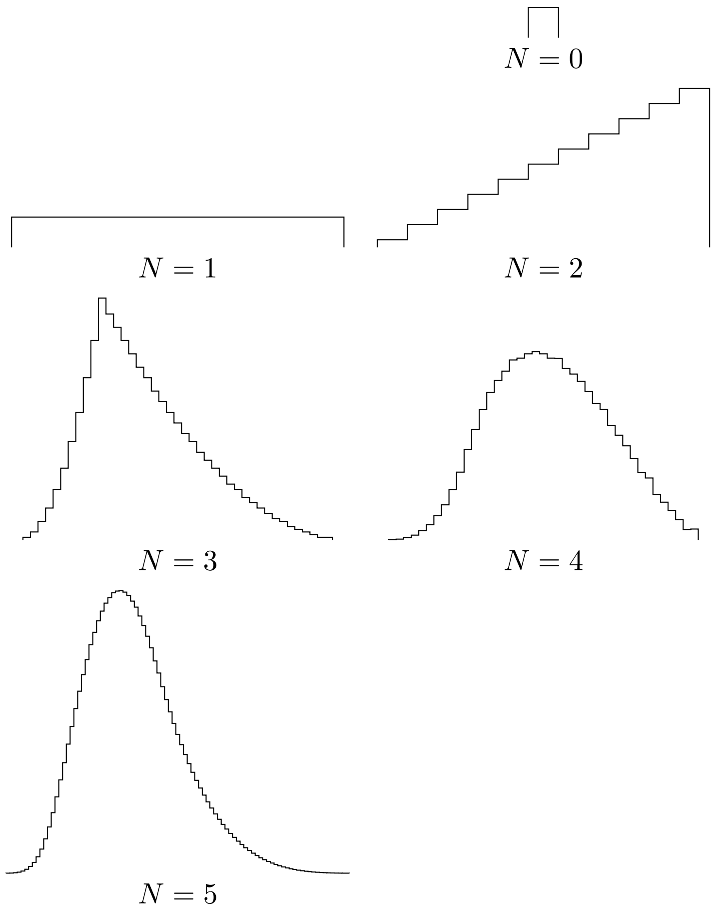

FIGURE 3. Degree distribution of $V_{w_{N,0}}(10\Lambda_0)$ for $N = 0, \ldots, 5$. Again, each individual picture displays degree 0, the degree of the highest weight, on the left, and the maximal occurring degree in the given Demazure module on the right.

This theorem already allows us to derive asymptotic statements when the parameters become large. For example, one can compute the maximal occurring degree $B_{N,j}^{m,n}$ in $V_{(\sigma s_j)^N}(\Lambda)$ (Cf. Lemma 2.3.4) and obtain:

Corollary (Limit ratio, Cf. Corollary 2.3.5)**.** *Let $\Lambda = m\Lambda_0 + n\Lambda_1$ be a dominant integral weight and $j \in \{0, 1\}$. Consider $-d \in \mathfrak{h}$ as a function on $\mathfrak{h}^*$. Then the limit ratio of the expected and maximal degree in $V_{(\sigma s_j)^N}(\Lambda)$, as N tends to infinity, is given by*

$$\lim_{N \to \infty} \frac{\mathrm{E}_{\mu_{(\sigma s_j)^N}}[-d]}{B_{N,j}^{m,n}} = \frac{m+n+2}{3(m+n+1)}.$$

Now, for level 1 Demazure modules of $\widehat{\mathfrak{sl}}_2$, i.e., with highest weight being equal to Λ_0 or Λ_1, we compute the variance of the degree distribution, the covariance matrix of the full weight distribution, and prove the weak law of large numbers.

Theorem (Variance of the degree, Cf. Theorem 3.4.5 and Corollary 3.4.7)**.** *Let $\mu_{(\sigma s_0)^N}$ and $\mu_{(\sigma s_1)^N}$ be the weight distributions of the Demazure modules $V_{(\sigma s_0)^N}(\Lambda_0)$ and $V_{(\sigma s_1)^N}(\Lambda_1)$, respectively. Consider $-d \in \mathfrak{h}$ as a function on $\mathfrak{h}^*$. Then,*

$$\mathrm{Var}_{\mu_{(\sigma s_0)^N}}(-d) = \frac{N(N-1)(2N+5)}{96},$$

and

$$\mathrm{Var}_{\mu_{(\sigma s_1)^N}}(-d) = \frac{N(N-1)(2N+5)}{96} + \frac{N}{4}.$$

The information gathered so far allows us to determine the covariance matrix of the full weight distribution.

Theorem (Covariance of the weight distribution, Cf. Theorem 4.2.1)**.** *Let $\mu_{(\sigma s_0)^N}$ and $\mu_{(\sigma s_1)^N}$ be the weight distributions of the Demazure modules $V_{(\sigma s_0)^N}(\Lambda_0)$ and $V_{(\sigma s_1)^N}(\Lambda_1)$, respectively. Consider $-d, \alpha_1^\vee \in \mathfrak{h}$ as functions on $\mathfrak{h}^*$. The covariance matrix of $-d$ and $\alpha_1^\vee$ with respect to $\mu_{(\sigma s_0)^N}$ is given by*

$$\begin{pmatrix} \frac{N(N-1)(2N+5)}{96} & 0 \\ 0 & N \end{pmatrix},$$

and the covariance matrix of $-d$ and $\alpha_1^\vee$ with respect to $\mu_{(\sigma s_1)^N}$ is

$$\begin{pmatrix} \frac{N(N-1)(2N+5)}{96} + \frac{N}{4} & 0 \\ 0 & N \end{pmatrix}.$$

The covariance matrix can be visualized by the covariance ellipse (see §4.2 for details). In Figure 4 the covariance ellipses have been centered at the expected weight.

Finally, we are able to prove the weak law of large numbers using Chebyshev's inequality (see Lemma 4.3.1 or [4, (5.32)]).

Theorem (Weak law of large numbers, Cf. Theorem 4.3.3)**.** *Let $\mu_{(\sigma s_0)^N}$ and $\mu_{(\sigma s_1)^N}$ be the weight distributions of the Demazure modules $V_{(\sigma s_0)^N}(\Lambda_0)$ and $V_{(\sigma s_1)^N}(\Lambda_1)$, respectively, and let $\bar{\mu}_{(\sigma s_0)^N}, \bar{\mu}_{(\sigma s_1)^N}$ be their normalizations. Consider $-d, \alpha_1^\vee \in \mathfrak{h}$ as functions on $\mathfrak{h}^*$. Then, for $j \in \{0,1\}$ we have*

$$\left(D_{(\lfloor N^2/4 \rfloor + j\lceil N/2 \rceil)^{-1}}\right)_* (-d)_* \bar{\mu}_{(\sigma s_j)^N} \xrightarrow{\mathrm{w}} \delta_{\frac{1}{2}},$$

and consequently

$$\left(D_{(N^{-1}, (\lfloor N^2/4 \rfloor + j\lceil N/2 \rceil)^{-1})}\right)_* (\alpha_1^\vee, -d)_* \bar{\mu}_{(\sigma s_j)^N} \xrightarrow{\mathrm{w}} \delta_{(0,\frac{1}{2})}.$$

In view of the limit ratio of the expected and maximal degree, as stated for an arbitrary Demazure module in the above corollary, we conjecture the following:

Conjecture (Cf. Conjecture 4.3.4). *Let $\Lambda = m\Lambda_0 + n\Lambda_1$ be a dominant integral weight and $j \in \{0,1\}$. Denote by $B^{m,n}_{N,j}$ the maximal occurring degree in $V_{(\sigma s_j)^N}(\Lambda)$, by $\mu_{(\sigma s_j)^N}$ its weight distribution, and by $\bar{\mu}_{(\sigma s_j)^N}$ its normalization. Consider $-d, \alpha_1^\vee \in \mathfrak{h}$ as functions on $\mathfrak{h}^*$. Then,*

$$(D_{(B^{m,n}_{N,j})^{-1}})_*(-d)_*\bar{\mu}_{(\sigma s_j)^N} \xrightarrow{\text{w}} \delta_{\frac{m+n+2}{3(m+n+1)}},$$

and consequently

$$(D_{(((m+n)N)^{-1},(B^{m,n}_{N,j})^{-1})})_*(\alpha_1^\vee, -d)_*\bar{\mu}_{(\sigma s_j)^N} \xrightarrow{\text{w}} \delta_{\left(0,\frac{m+n+2}{3(m+n+1)}\right)}.$$

See Figure 5 for an illustration of the weak law of large numbers, and likewise Figure 6 for the conjecture.

Let me close by mentioning that even though we prefer a probabilistic language, our results concerning the degree distribution can equivalently be phrased as follows: Let $\mathrm{ch}_{V_w(\Lambda)}$ be the character of a Demazure module $V_w(\Lambda)$ of $\widehat{\mathfrak{sl}}_2$. We compute the Taylor expansion at $0 \in \mathfrak{h}$ of the basic specialization of $\mathrm{ch}_{V_w(\Lambda)}$ up to order 1 when Λ is arbitrary, and its Taylor expansion up to order 2 when Λ is of level 1 (see §4.1).

1.2. **Organization.** This thesis is organized as follows:

In §2, we calculate the expected degree of a weight in (arbitrary) Demazure modules by induction on the number of Demazure operators in Demazure's character formula. The actual induction follows a snake-like pattern (Figure 8 and 9). Our strategy dictates that we must express the expected degree of a weight in a given Demazure module in terms of statistical information about the weight distribution of the previous Demazure module. It turns out that this involves not only the expected value, but also a second moment (Lemma 2.1.6). For this reason, we cannot apply an induction argument at this point. By what appears to be a coincidence to us, the necessary second moment can be expressed purely in terms of the variance of the finite weight (Lemma 2.1.7). This variance is known by [30] (see Lemma 2.1.8), thereby yielding a recurrence relation Lemma 2.1.9 and consequently an explicit formula Theorem 2.2.1. We conclude this section by some asymptotic statements in §2.3.

In §3 we give two proofs of our main result Theorem 3.4.3 on the variance of the degree of weights in level 1 Demazure modules. The first proof is by induction along Demazure's character formula §3.1–§3.4. In §3.1, we show that one recursion step in Demazure's character formula expresses the second moment of the degree distribution of a given Demazure module in terms of the third moments of the weight distribution of a smaller Demazure module

					1					
				1	1	1				
				1	2	1				
				2	3	2				
			1	3	5	3	1			
			1	5	7	5	1			
			2	6	9	6	2			
			3	9	11	9	3			
			4	10	14	10	4			
		1	5	13	16	13	5	1		
		1	7	14	18	14	7	1		
		2	8	16	19	16	8	2		
		2	9	16	20	16	9	2		
		3	10	18	20	18	10	3		
		3	10	16	19	16	10	3		
		4	10	16	18	16	10	4		
	1	4	10	14	16	14	10	4	1	
	1	5	9	13	14	13	9	5	1	
	1	4	8	10	11	10	8	4	1	
	1	4	7	9	9	9	7	4	1	
	1	3	5	6	7	6	5	3	1	
	1	3	4	5	5	5	4	3	1	
	1	2	3	3	3	3	3	2	1	
	1	2	2	2	2	2	2	2	1	
	1	1	1	1	1	1	1	1	1	
1	1	1	1	1	1	1	1	1	1	1

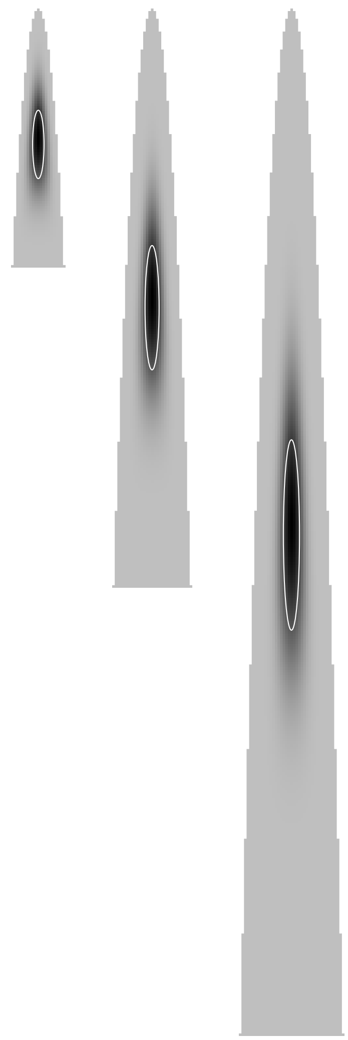

FIGURE 4. Weight distribution of $V_{(\sigma s_0)^N}(\Lambda_0)$ for $N = 10$ (indicated by numbers) and $N = 20, 30, 40$ (indicated by shades of gray). See §4.2 for the definition of the covariance ellipses included in the figure.

(Lemma 3.1.1). We try to express these third moments in terms of the (known) third moments of the distribution of the finite weight, but succeed not quite (Corollary 3.1.6). In §3.2, we show that the weight multiplicities are symmetric in each string of weights differing only by a multiple of δ (Lemma 3.2.1). We use this in §3.3 to show that the covariance between two specific quadratic functions vanishes (Corollary 3.3.4). This allows us to

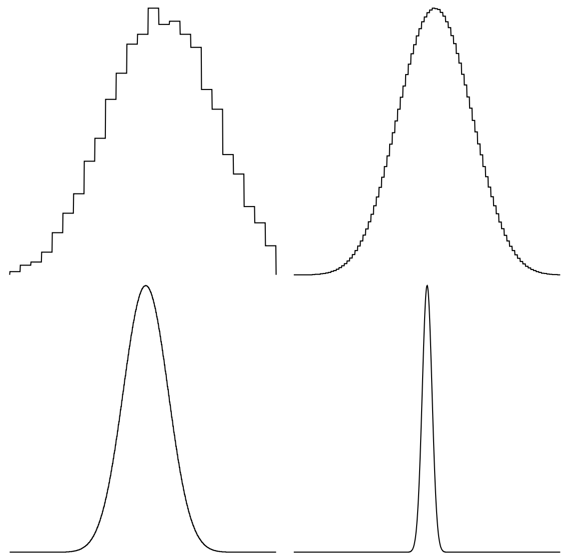

FIGURE 5. Degree distribution of $V_{w_N}(\Lambda_0)$ for $N = 10, 20, 50, 1000$. Theorem 4.3.3 states that this distribution converges weakly to $\delta_{\frac{1}{2}}$ as $N \to \infty$.

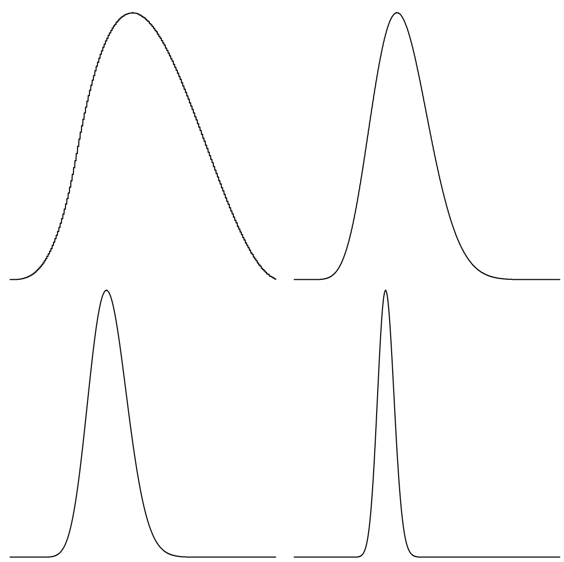

FIGURE 6. Degree distribution of $V_{w_N}(50\Lambda_0)$ for $N = 4, 10, 20, 100$. Conjecture 4.3.4 asserts that this distribution converges weakly to $\delta_{\frac{51}{156}}$ as $N \to \infty$.

explicitly compute the previously problematic third moments (Lemma 3.3.5). This yields an explicit recurrence relation, stated in §3.4, for the second moment of the degree distribution (Lemma 3.4.2), which is easy to solve and culminates in the determination of the variance of the degree distribution Theorem 3.4.3. The second proof in §3.5 exploits a relation between (level 1) Demazure characters, Macdonald polynomials, and Rogers–Szegő polynomials, i.e., generating functions of Gaussian polynomials. This allows us to obtain our main result Theorem 3.4.3 again, this time by quantum calculus (Theorem 3.5.4).

The following section §4 contains applications: In §4.1 our results from §2 and §3 are expressed in terms of the basic specialization of Demazure characters. In §4.2 we complement our result on the variance of the degree distribution in level 1 Demazure modules by determining the full covariance matrix (Theorem 4.2.1). In §4.3 we deduce the weak law of large numbers for weight distributions in level 1 Demazure modules Theorem 4.3.3 and assert the immediate Conjecture 4.3.4 for arbitrary Demazure modules.

We conclude in §5 with an outlook on further questions to be discussed.

1.3. **Notation.** For general notation about Kac–Moody algebras we mostly follow [16]. Let $\mathfrak{h}$ be a 3-dimensional complex vector space and $\alpha_0^\vee, \alpha_1^\vee \in \mathfrak{h}$, $\alpha_0, \alpha_1 \in \mathfrak{h}^*$ a realization of the generalized Cartan matrix $A = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. Let $\widehat{\mathfrak{sl}}_2 = \mathfrak{g}(A)$ be the associated affine Lie algebra. We denote the canonical pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$ by $\langle \alpha, h \rangle = \alpha(h)$. Choose $d \in \mathfrak{h}$ such that $\langle \alpha_0, d \rangle = 1$ and $\langle \alpha_1, d \rangle = 0$. Such an element d is called a *scaling element*. The *degree* of $\lambda \in \mathfrak{h}^*$ is defined as $\langle \lambda, d \rangle$. The set $\{\alpha_0^\vee, \alpha_1^\vee, d\}$ is a basis of $\mathfrak{h}$. Let $\{\Lambda_0, \Lambda_1, \delta\}$ be the corresponding dual basis of $\mathfrak{h}^*$. Then $\delta = \alpha_0 + \alpha_1$, and Λ_0, Λ_1 are called *fundamental weights*.

Integrable highest weight modules of $\widehat{\mathfrak{sl}}_2$ are parametrized up to isomorphism by dominant integral weights

$$\Lambda = m\Lambda_0 + n\Lambda_1 + c\delta$$

for $m, n \in \mathbf{N}$ and $c \in \mathbf{C}$. We denote the integrable highest weight module corresponding to Λ by $V(\Lambda)$. As a change in c simply corresponds to the choice of a different scaling element $d \in \mathfrak{h}$, it is customary to suppose $c = 0$ and only consider dominant integral weights of the form $\Lambda = m\Lambda_0 + n\Lambda_1$, which we do from now on.

Let $K = \alpha_0^\vee + \alpha_1^\vee$ be the canonical central element. The *level* of a weight $\lambda \in \mathfrak{h}^*$ is defined as $\langle \lambda, K \rangle$. If $\lambda = m\Lambda_0 + n\Lambda_1$, then its level is $\langle \lambda, K \rangle = m + n$. Hence the dominant integral weights of level 1 (which we will consider exclusively in §3) are exactly the fundamental weights Λ_0, Λ_1.

A weight $\lambda \in \mathfrak{h}^*$ is said to *occur* in a given integrable highest weight module $V(\Lambda)$ if the weight space $V(\Lambda)_\lambda$ is nontrivial. The set of weights occuring in $V(\Lambda)$ is contained in the (affine) lattice

$$\Gamma = \Lambda + \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1.$$

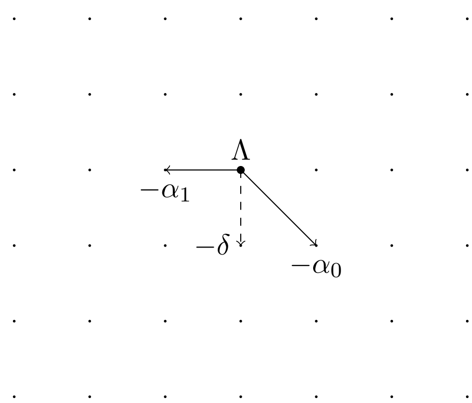

FIGURE 7. The (affine) lattice Γ.

We define coordinates a, b on Γ by

$$\lambda = \Lambda - a(\lambda)\alpha_0 - b(\lambda)\alpha_1$$

for all $\lambda \in \Gamma$. Note that a, b depend on Λ. In Figure 4, each matrix component resp. pixel represents a point in Γ. The highest weight Λ is located at the apex of the parabola. The other elements of Γ are arranged as shown in Figure 7. We write $\Gamma_j = \Lambda_j + \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1$ for $j \in \{0,1\}$ to refer to the two lattices corresponding to the fundamental weights.

For $j \in \{0,1\}$ define the *simple reflections* $s_j : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_j(\lambda) = \lambda - \langle \lambda, \alpha_j^\vee \rangle \alpha_j.$$

The Weyl group W^{aff} of $\widehat{\mathfrak{sl}}_2$ is by definition the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by s_0 and s_1. All elements of W^{aff} have the form

$$w_{N,0} = \underbrace{\cdots s_0 s_1 s_0}_{N \text{ factors}} \quad \text{or} \quad w_{N,1} = \underbrace{\cdots s_1 s_0 s_1}_{N \text{ factors}}$$

for $N \geq 0$.

Let $\mathfrak{n}_+ \subset \widehat{\mathfrak{sl}}_2$ be the sum of the positive root spaces. For $w \in W^{\mathrm{aff}}$ and Λ a dominant integral weight, define the *Demazure module* $V_w(\Lambda)$ to be the $(\mathfrak{h} \oplus \mathfrak{n}_+)$-module generated by $V(\Lambda)_{w\Lambda}$. Demazure's character formula for Kac–Moody algebras [21, 27, 28] allows the computation of the character of $V_w(\Lambda)$ by an iterated application of certain operators on the monomial e^Λ as follows. Associated with a simple reflection s_j we define the operator D_j to act on monomials e^λ, $\lambda \in \Gamma$, by

$$D_j e^\lambda = \sum_{i=0}^{\langle \lambda, \alpha_j^\vee \rangle} e^{\lambda - i\alpha_j}.$$

Here we use the conventions that

$$\sum_{i=0}^{-1} a_i = 0, \quad \text{and} \quad \sum_{i=0}^{k} a_i = -a_{-1} - \ldots - a_{k+1}$$

for $k < -1$. Note that this is natural in the sense that Gauss's summation formula $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ extends to all $k \in \mathbf{Z}$, as does the identity $\sum_{i=0}^{k} 1 = k+1$. Now, Demazure's character formula states that

$$\mathrm{ch}_{V_{w_{N,0}}(\Lambda)} = \underbrace{\cdots D_0 D_1 D_0}_{N \text{ factors}} e^{\Lambda}, \quad \text{and} \quad \mathrm{ch}_{V_{w_{N,1}}(\Lambda)} = \underbrace{\cdots D_1 D_0 D_1}_{N \text{ factors}} e^{\Lambda}.$$

As $V_w(\Lambda)$ is in particular an $\mathfrak{h}$-module, it has a weight space decomposition

$$V_w(\Lambda) = \bigoplus_{\lambda \in \mathfrak{h}^*} V_w(\Lambda)_\lambda.$$

Let $\mathrm{Meas}_{\mathrm{c}}(\Gamma)$ denote the set of measures on Γ with compact (hence finite) support. We define the *weight distribution* of $V_{w_{N,j}}(\Lambda)$ to be

$$\mu_{N,j} = \sum_{\lambda \in \mathfrak{h}^*} \dim(V_w(\Lambda)_\lambda) \cdot \delta_\lambda \in \mathrm{Meas}_{\mathrm{c}}(\Gamma).$$

Note that the dependence on Λ is important, but only implicit in the notation.

Given Λ, the space $\mathrm{Meas}_{\mathrm{c}}^{\pm}(\Gamma)$ of signed measures on Γ with compact support is isomorphic to the $\mathbf{Z}$-module generated by $\{e^\lambda : \lambda \in \Gamma\}$ via the mapping $\delta_\lambda \mapsto e^\lambda$. The Demazure operators D_0, D_1 act on the latter hence via this isomorphism on the former by

$$D_j \delta_\lambda = \sum_{i=0}^{\langle \lambda, \alpha_j^\vee \rangle} \delta_{\lambda - i\alpha_j}$$

for $j \in \{0, 1\}$ and $\lambda \in \Gamma$. Demazure's character formula now becomes

$$\mu_{N,0} = \underbrace{\cdots D_0 D_1 D_0}_{N \text{ factors}} \delta_\Lambda, \quad \text{and} \quad \mu_{N,1} = \underbrace{\cdots D_1 D_0 D_1}_{N \text{ factors}} \delta_\Lambda.$$

Consider elements of $\mathfrak{h}$ as functions on $\mathfrak{h}^*$. We refer to the push-forward measure $(-d)_* \mu_{N,j}$ as the *degree distribution*, and to $(\alpha_1^\vee)_* \mu_{N,j}$ as the *distribution of the finite weight* of $V_{w_{N,j}}(\Lambda)$. Note that in terms of the coordinates a, b we have

$$-d = a \quad \text{and} \quad \alpha_1^\vee = \begin{cases} -2(a-b) & \text{if } N \text{ is even,} \\ -2(a-b)+1 & \text{if } N \text{ is odd.} \end{cases}$$

as functions on $\mathfrak{h}^*$. Hence the degree distribution is $a_* \mu_{N,j}$, and the distribution of the finite weight is $(a-b)_* \mu_{N,j}$ up to translation and scaling.

Let us introduce the extended affine Weyl group $\widetilde{W}^{\mathrm{aff}}$. Let Σ be the automorphism group of the Dynkin diagram of $\widehat{\mathfrak{sl}}_2$, then $\widetilde{W}^{\mathrm{aff}} = \Sigma \ltimes W^{\mathrm{aff}}$. Note that $\Sigma = \langle \sigma \rangle$ and $\sigma^2 = 1$ in the case of $\widehat{\mathfrak{sl}}_2$. The element σ maps $\Lambda = m\Lambda_0 + n\Lambda_1$ to $\sigma(\Lambda) = n\Lambda_0 + m\Lambda_1$ and the lattice Γ to $\sigma(\Gamma) = \sigma(\Lambda) + \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1$

via $\sigma(\Lambda + x\alpha_0 + y\alpha_1) = \sigma(\Lambda) + y\alpha_0 + x\alpha_1$. The Demazure operators D_w with $w \in W^{\mathrm{aff}}$ extend to Demazure operators indexed by elements in $\widetilde{W}^{\mathrm{aff}}$ by $D_\sigma e^\lambda = e^{\sigma(\lambda)}$ and $D_\sigma \delta_\lambda = \delta_{\sigma(\lambda)}$, respectively.

Recall that the expected value of a function $f : \Gamma \to \mathbf{R}$ with respect to a nonzero measure $\mu \in \mathrm{Meas}_{\mathrm{c}}(\Gamma)$ is

$$\mathrm{E}_\mu[f] = \frac{1}{\mu(\Gamma)} \sum_{\lambda \in \Gamma} \mu(\{\lambda\}) f(\lambda).$$

The covariance of two functions f and g is

$$\mathrm{Cov}_\mu(f, g) = \mathrm{E}_\mu[(f - \mathrm{E}_\mu[f])(g - \mathrm{E}_\mu[g])],$$

and the variance of f is $\mathrm{Var}_\mu(f) = \mathrm{Cov}_\mu(f, f)$.

Let $\mathbf{N}$ be the non-negative integers $\{0, 1, 2, \ldots\}$, and denote the parity of an integer N by

$$\pi_N = \begin{cases} 0 & \text{if } N \text{ is even,} \\ 1 & \text{if } N \text{ is odd.} \end{cases}$$

The following notations will be used in §3.2 and §3.5. Let q be a variable. For $a \in \mathbf{C}(q)$, $N \in \mathbf{N}$, and $k \in \{0, \ldots, N\}$ let

$$(a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i) \quad \text{and} \quad \begin{bmatrix} N \\ k \end{bmatrix}_q = \frac{(q; q)_N}{(q; q)_k (q; q)_{N-k}}.$$

Note that $\begin{bmatrix} N \\ k \end{bmatrix}_q \in \mathbf{C}[q]$ is a polynomial in q for every N and k, a *Gaussian polynomial*. For general facts about those we refer to [2] and [17].

We continue by defining the Macdonald polynomials [25], see also [26, Chapter VI]. Let Λ be the ring of symmetric functions. For a partition λ, let p_λ denote the corresponding product of power sums and m_λ the corresponding monomial symmetric function. Let q and t be variables. On $\Lambda \otimes_{\mathbf{Z}} \mathbf{Q}(q, t)$ consider the nondegenerate symmetric bilinear form determined by imposing

$$(p_\lambda, p_\mu) = \delta_{\lambda\mu} \prod_{r \geq 1} (r^{m_r} r!) \prod_{i \geq 1} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}},$$

where m_r is the multiplicity of the part r in λ, and $\delta_{\lambda\mu}$ is the Kronecker symbol. Let $\leq$ denote the dominance ordering on partitions. Then there are unique $P_\lambda \in \Lambda \otimes_{\mathbf{Z}} \mathbf{Q}(q, t)$ parametrized by all partitions, such that $P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu$ for some $u_{\lambda\mu} \in \mathbf{Q}(q, t)$, and that $(P_\lambda, P_\mu) = 0$ whenever $\lambda \neq \mu$ [25, Theorem 2.3]. These symmetric functions are called *Macdonald polynomials*. We denote the bivariate image of P_λ by $P_\lambda(z_1, z_2; q, t) \in \mathbf{Q}(q, t)[z_1, z_2]^{S_2}$. We will only need the specializations $P_\lambda(z, z^{-1}; q, 0) \in \mathbf{Q}[z^{\pm 1}, q^{\pm 1}]$.

2. Expected degree of weights

2.1. **Lemmata.** For this section we fix a dominant integral weight $\Lambda = m\Lambda_0 + n\Lambda_1$, and the lattice $\Gamma = \Lambda + \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1$. Define coordinates a, b on Γ such that $\lambda = \Lambda - a(\lambda)\alpha_0 - b(\lambda)\alpha_1$ for all $\lambda \in \Gamma$. With these coordinates

$$\langle \,\bullet\,, \alpha_0^\vee \rangle = m - 2(a-b), \text{ and} \tag{2.1.1}$$

$$\langle \,\bullet\,, \alpha_1^\vee \rangle = n + 2(a-b). \tag{2.1.2}$$

We start by gathering information about the operation of the Demazure operators associated with simple reflections on measures in $\mathrm{Meas_c}(\Gamma)$.

Lemma 2.1.1. *Let* $\mu \in \mathrm{Meas_c}(\Gamma)$. *Then* $\mathrm{E}_{D_0\mu}[a-b] = \frac{m}{2}$ *and* $\mathrm{E}_{D_1\mu}[a-b] = -\frac{n}{2}$.[2]

Proof. Let us first compute $\mathrm{E}_{D_0\mu}[a-b] = \frac{m}{2}$. With $\mu = \sum_{\lambda\in\Gamma} p_\lambda \delta_\lambda$ we have

$$\begin{aligned}
D_0\mu(\Gamma) \cdot \mathrm{E}_{D_0\mu}[a-b] &= \sum_{\lambda\in\Gamma} p_\lambda \sum_{i=0}^{\langle\lambda,\alpha_0^\vee\rangle} (a-b)(\lambda - i\alpha_0) \\
&= \sum_{\lambda\in\Gamma} p_\lambda \sum_{i=0}^{\langle\lambda,\alpha_0^\vee\rangle} ((a-b)(\lambda) + i) \\
&= \sum_{\lambda\in\Gamma} p_\lambda \cdot (\langle\lambda,\alpha_0^\vee\rangle + 1)((a-b)(\lambda) + \frac{1}{2}\langle\lambda,\alpha_0^\vee\rangle) \\
&\overset{(2.1.1)}{=} \sum_{\lambda\in\Gamma} p_\lambda \cdot (\langle\lambda,\alpha_0^\vee\rangle + 1) \cdot \frac{m}{2} \\
&= D_0\mu(\Gamma) \cdot \frac{m}{2}.
\end{aligned}$$

The last equation holds since each $\lambda \in \Gamma$ produces exactly $\langle\lambda,\alpha_0^\vee\rangle + 1$ successors via the operation of D_0 on δ_λ. To verify $\mathrm{E}_{D_1\mu}[a-b] = -\frac{n}{2}$, one pursues the same computation. □

We want to prove Theorem 2.2.1 by induction on the length N of the Weyl group element w. To that end we investigate how certain expected values change under the operation of the Demazure operators D_0, D_1.

Lemma 2.1.2. *Let* $\mu \in \mathrm{Meas_c}(\Gamma)$ *and* $k \geq 0$. *Then*

$$\mathrm{E}_{D_1\mu}[a^k] = \frac{\mu(\Gamma)}{D_1\mu(\Gamma)} \mathrm{E}_\mu[a^k(n+1+2(a-b))], \text{ and}$$

$$\mathrm{E}_{D_0\mu}[b^k] = \frac{\mu(\Gamma)}{D_0\mu(\Gamma)} \mathrm{E}_\mu[b^k(m+1-2(a-b))].$$

[2] Of course, we have to assume that $D_0\mu(\Gamma) \neq 0$ and $D_1\mu(\Gamma) \neq 0$, respectively, which we silently do here and in future similar situations.

Proof. Let $\mu = \sum_{\lambda\in\Gamma} p_\lambda \delta_\lambda$. Then

$$\begin{aligned}
D_1\mu(\Gamma)\cdot \mathrm{E}_{D_1\mu}[a^k] &= \sum_{\lambda\in\Gamma} p_\lambda \sum_{i=0}^{\langle\lambda,\alpha_1^\vee\rangle} a^k(\lambda - i\alpha_1)\\
&= \sum_{\lambda\in\Gamma} p_\lambda \sum_{i=0}^{\langle\lambda,\alpha_1^\vee\rangle} a^k(\lambda)\\
&= \sum_{\lambda\in\Gamma} p_\lambda \cdot (\langle\lambda,\alpha_1^\vee\rangle + 1)\cdot a^k(\lambda)\\
&= \sum_{\lambda\in\Gamma} p_\lambda \cdot (n + 2(a-b)(\lambda) + 1)\cdot a^k(\lambda)\\
&= \mu(\Gamma)\cdot \mathrm{E}_\mu[a^k(n+1+2(a-b))].
\end{aligned}$$

The computation is analogous for $\mathrm{E}_{D_0\mu}[b^k]$. □

By setting $k=0$ in Lemma 2.1.2 we obtain:

Corollary 2.1.3. *Let $\mu \in \mathrm{Meas_c}(\Gamma)$. Then, the total mass of $D_0\mu$ and $D_1\mu$ is given by*

$$\begin{aligned}
D_0\mu(\Gamma) &= \mu(\Gamma)(m+1-2\,\mathrm{E}_\mu[a-b]), \text{ and}\\
D_1\mu(\Gamma) &= \mu(\Gamma)(n+1+2\,\mathrm{E}_\mu[a-b]).
\end{aligned}$$

Resolving those equations for the weight distribution μ_w we derive the following dimension formulas for the Demazure module $V_w(\Lambda)$.

Corollary 2.1.4. *Let $N \geq 0$. Then the following dimension formulas hold:*

$$\mu_{N,0}(\Gamma) = \begin{cases} 1 & N=0\\ (m+1)(m+n+1)^{N-1} & N\geq 1,\end{cases}$$

$$\mu_{N,1}(\Gamma) = \begin{cases} 1 & N=0\\ (n+1)(m+n+1)^{N-1} & N\geq 1.\end{cases}$$

Proof. We restrict to the Weyl group elements starting in s_0, for the argumentation in the other case is similar. For $N=0$ one has $\mu_{0,0}(\Gamma) = \delta_\Lambda(\Gamma) = 1$, and if $N=1$, then $\mu_{1,0}(\Gamma) = D_0\delta_\Lambda(\Gamma) = \langle\Lambda,\alpha_0^\vee\rangle + 1 = m+1$. We proceed by induction on $N\geq 2$.

$$\begin{aligned}
\mu_{N,0}(\Gamma) &= \begin{cases} D_1\mu_{N-1,0}(\Gamma) & \text{if } N \text{ even,}\\ D_0\mu_{N-1,0}(\Gamma) & \text{if } N \text{ odd}\end{cases}\\
&= \begin{cases} \mu_{N-1,0}(\Gamma)(n+1+2\,\mathrm{E}_{D_0\mu_{N-2,0}}[a-b]) & \text{if } N \text{ even,}\\ \mu_{N-1,0}(\Gamma)(m+1-2\,\mathrm{E}_{D_1\mu_{N-2,0}}[a-b]) & \text{if } N \text{ odd}\end{cases}\\
&= \begin{cases} \mu_{N-1,0}(\Gamma)(n+1+m) & \text{if } N \text{ even,}\\ \mu_{N-1,0}(\Gamma)(m+1+n) & \text{if } N \text{ odd.}\end{cases}
\end{aligned}$$

The second equation follows from Corollary 2.1.3 and the third by replacing the expected values by their actual values computed in Lemma 2.1.1. Hence by induction $\mu_{N,0}(\Gamma) = (m+1)(m+n+1)^{N-1}$. $\square$

Remark 2.1.5. Corollary 2.1.4 is Sanderson's dimension formula [29, Theorem 1]. Her original proof uses the path model for highest weight representations of Kac–Moody algebras.

If we consider two consecutive applications of Demazure operators associated with simple reflections, Lemma 2.1.2 for $k = 1$ can be stated as follows.

Lemma 2.1.6. *Let $\mu \in \mathrm{Meas_c}(\Gamma)$. Then we have the following equations:*

$$\mathrm{E}_{D_1D_0\mu}[a] = \frac{D_0\mu(\Gamma)}{D_1D_0\mu(\Gamma)}\Big((m+n+1)\,\mathrm{E}_{D_0\mu}[a] + 2\,\mathrm{Cov}_{D_0\mu}(a, a-b)\Big), \text{ and}$$

$$\mathrm{E}_{D_0D_1\mu}[b] = \frac{D_1\mu(\Gamma)}{D_0D_1\mu(\Gamma)}\Big((m+n+1)\,\mathrm{E}_{D_1\mu}[b] - 2\,\mathrm{Cov}_{D_1\mu}(b, a-b)\Big).$$

Proof. We prove the first assertion, the second being analogous. From Lemma 2.1.2 we obtain $\mathrm{E}_{D_1D_0\mu}[a] = \frac{D_0\mu(\Gamma)}{D_1D_0\mu(\Gamma)}\,\mathrm{E}_{D_0\mu}[a(n+1+2(a-b))]$. As $\mathrm{E}_{D_0\mu}[a-b] = \frac{m}{2}$ by Lemma 2.1.1, the second factor of the right-hand side is

$$\begin{aligned}\mathrm{E}_{D_0\mu}[a(n+1+2(a-b))] &= \mathrm{E}_{D_0\mu}[a(n+1)] + 2\,\mathrm{E}_{D_0\mu}[a(a-b)]\\ &= (n+1)\,\mathrm{E}_{D_0\mu}[a] + 2\,\mathrm{E}_{D_0\mu}[a-b]\,\mathrm{E}_{D_0\mu}[a]\\ &\quad + 2\,\mathrm{Cov}_{D_0\mu}(a, a-b)\\ &= (m+n+1)\,\mathrm{E}_{D_0\mu}[a] + 2\,\mathrm{Cov}_{D_0\mu}(a, a-b).\end{aligned}$$ $\square$

Note that Lemma 2.1.6 is not a recurrence relation for the expected degree, as covariances of the previous distribution are involved.

Lemma 2.1.7. *Let $\mu \in \mathrm{Meas_c}(\Gamma)$. Then*

$$\begin{aligned}&\mathrm{Cov}_{D_0\mu}(a, a-b) = \mathrm{Var}_{D_0\mu}(a-b) && \text{and } \mathrm{Cov}_{D_0\mu}(b, a-b) = 0,\\ &\mathrm{Cov}_{D_1\mu}(a, a-b) = 0 && \text{and } \mathrm{Cov}_{D_1\mu}(b, a-b) = -\mathrm{Var}_{D_1\mu}(a-b).\end{aligned}$$

Proof. Let us first treat $\mathrm{Cov}_{D_0\mu}(a-b, b) = 0$. Indeed $\mathrm{Cov}_{D_0\mu}(a-b, b) = \mathrm{E}_{D_0\mu}[(a-b)b] - \mathrm{E}_{D_0\mu}[a-b]\,\mathrm{E}_{D_0\mu}[b]$ and with $\mu = \sum_{\lambda\in\Gamma} p_\lambda\delta_\lambda$ we have

$$\begin{aligned}D_0\mu(\Gamma)\cdot\mathrm{E}_{D_0\mu}[(a-b)b] &= \sum_{\lambda\in\Gamma} p_\lambda \sum_{i=0}^{\langle\lambda,\alpha_0^\vee\rangle}(a-b)(\lambda - i\alpha_0)\cdot b(\lambda - i\alpha_0)\\ &= \sum_{\lambda\in\Gamma} p_\lambda\cdot b(\lambda)\sum_{i=0}^{\langle\lambda,\alpha_0^\vee\rangle}((a-b)(\lambda)+i)\\ &\overset{(2.1.1)}{=} \sum_{\lambda\in\Gamma} p_\lambda\cdot b(\lambda)\cdot(\langle\lambda,\alpha_0^\vee\rangle + 1)\cdot\frac{m}{2}\end{aligned}$$

$$= \frac{m}{2}\sum_{\lambda\in\Gamma} p_\lambda \sum_{i=0}^{\langle\lambda,\alpha_0^\vee\rangle} b(\lambda - i\alpha_0)$$
$$= \mathrm{E}_{D_0\mu}[a-b]\cdot D_0\mu(\Gamma)\cdot \mathrm{E}_{D_0\mu}[b].$$

For the last equation see Lemma 2.1.1. The same argument shows that $\mathrm{Cov}_{D_1\mu}(a, a-b) = 0$. The remaining claims follow from the bilinearity of the covariance. To be precise,

$$\begin{aligned}\mathrm{Cov}_{D_0\mu}(a, a-b) &= \mathrm{Cov}_{D_0\mu}((a-b)+b, a-b)\\ &= \mathrm{Cov}_{D_0\mu}(a-b, a-b) + \mathrm{Cov}_{D_0\mu}(b, a-b)\\ &= \mathrm{Cov}_{D_0\mu}(a-b, a-b).\end{aligned}$$

The computation of $\mathrm{Cov}_{D_1\mu}(b, a-b)$ is essentially the same. □

The variances appearing in Lemma 2.1.7 can be computed via Sanderson's formula [30, Theorem 1] for the real character of the Demazure module $V_w(\Lambda)$.

Lemma 2.1.8. *For $N \geq 1$ we have*

$$\mathrm{Var}_{\mu_{N,0}}(a-b) = \frac{m(m+2) + (N-1)(m+n)(m+n+2)}{12}, \text{ and}$$
$$\mathrm{Var}_{\mu_{N,1}}(a-b) = \frac{n(n+2) + (N-1)(m+n)(m+n+2)}{12}.$$

Proof. We only compute $\mu_{N,0}$, the computation for $\mu_{N,1}$ being analogous. Let q be a variable. For $k \geq 0$ we define the q-integer $[k]_q = \sum_{i=0}^{k-1} q^i$. Sanderson's formula [30, Theorem 1] for the real character of $V_{w_{N,0}}(\Lambda)$ states that

$$\sum_{\lambda\in\Gamma} \dim(V_{w_{N,0}}(\Lambda)_\lambda) q^{(a-b)(\lambda)} = q^{-(m+n)\lfloor N/2\rfloor}[m+1]_q[m+n+1]_q^{N-1}.$$

For $k \geq 0$ we define $\delta_{[k]} = \sum_{i=0}^{k-1}\delta_i \in \mathrm{Meas_c}(\mathbf{Z})$. The linear map $\mathbf{C}[q, q^{-1}] \to \mathrm{Meas}_c^{\mathbf{C}}(\mathbf{Z})$ given by $q^k \mapsto \delta_k$ is an isomorphism of algebras, the multiplication of measures being convolution. This isomorphism maps $[k]_q$ to $\delta_{[k]}$ and hence

$$\sum_{\lambda\in\Gamma} \dim(V_{w_{N,0}}(\Lambda)_\lambda)\delta_{(a-b)(\lambda)} = \delta_{-(m+n)\lfloor N/2\rfloor} * \delta_{[m+1]} * \delta_{[m+n+1]}^{*(N-1)}.$$

The measure on the left-hand side is by definition the push-forward measure $(a-b)_*\mu$. By straightforward computation $\mathrm{Var}(\delta_{[k]}) = \frac{1}{k}\sum_{i=0}^{k-1}\left(i - \frac{k-1}{2}\right)^2 = \frac{(k-1)(k+1)}{12}$. Hence

$$\begin{aligned}\mathrm{Var}_{\mu_{N,0}}(a-b) &= \mathrm{Var}((a-b)_*\mu)\\ &= \mathrm{Var}\left(\delta_{-(m+n)\lfloor N/2\rfloor} * \delta_{[m+1]} * \delta_{[m+n+1]}^{*(N-1)}\right)\\ &= \mathrm{Var}(\delta_{[m+1]}) + (N-1)\,\mathrm{Var}(\delta_{[m+n+1]})\\ &= \frac{m(m+2)}{12} + (N-1)\frac{(m+n)(m+n+2)}{12}.\end{aligned}$$
□

Combining Lemma 2.1.6, Lemma 2.1.7, and Lemma 2.1.8, we finally obtain recurrence relations.

Lemma 2.1.9 (Recurrence relation)**.** *Let $N \geq 2$. Then the following recurrence relations hold:*

$$\mathrm{E}_{\mu_{N,0}}[a] = \frac{\mu_{N-1,0}(\Gamma)}{\mu_{N,0}(\Gamma)}\Big((m+n+1)\,\mathrm{E}_{\mu_{N-1,0}}[a] + \frac{m(m+2)+(N-2)(m+n)(m+n+2)}{6}\Big) \text{ if } N \text{ even,} \tag{2.1.3}$$

$$\mathrm{E}_{\mu_{N,0}}[b] = \frac{\mu_{N-1,0}(\Gamma)}{\mu_{N,0}(\Gamma)}\Big((m+n+1)\,\mathrm{E}_{\mu_{N-1,0}}[b] + \frac{m(m+2)+(N-2)(m+n)(m+n+2)}{6}\Big) \text{ if } N \text{ odd,} \tag{2.1.4}$$

$$\mathrm{E}_{\mu_{N,1}}[b] = \frac{\mu_{N-1,1}(\Gamma)}{\mu_{N,1}(\Gamma)}\Big((m+n+1)\,\mathrm{E}_{\mu_{N-1,1}}[b] + \frac{n(n+2)+(N-2)(m+n)(m+n+2)}{6}\Big) \text{ if } N \text{ even,} \tag{2.1.5}$$

$$\mathrm{E}_{\mu_{N,1}}[a] = \frac{\mu_{N-1,1}(\Gamma)}{\mu_{N,1}(\Gamma)}\Big((m+n+1)\,\mathrm{E}_{\mu_{N-1,1}}[a] + \frac{n(n+2)+(N-2)(m+n)(m+n+2)}{6}\Big) \text{ if } N \text{ odd.} \tag{2.1.6}$$

Proof. The equations follow immediately by replacing the covariance in Lemma 2.1.6 with the variance as computed in Lemma 2.1.7. Subsequently, substitute this variance by its value as computed in Lemma 2.1.8. Depending on the parity of N one has to keep track during this procedure of the leftmost simple reflection in the Weyl group element $w_{N,j}$ defining the measure $\mu_{N,j} = \mu_{w_{N,j}}$. □

2.2. **Main theorems.** Our recurrence relations from Lemma 2.1.9 allow us to prove:

Theorem 2.2.1 (Expected degree)**.** *Let $\Lambda = m\Lambda_0 + n\Lambda_1$ be a dominant integral weight and $N \geq 1$. Choose a basis of weight vectors in the Demazure modules $V_{w_{N,0}}(\Lambda)$ and $V_{w_{N,1}}(\Lambda)$. Then the expected degrees of a randomly chosen basis element are given by the following formulas, respectively.*

$$\mathrm{E}_{\mu_{N,0}}[a] = \frac{2(N-1)m(m+2)+(N-1)(N-2)(m+n)(m+n+2)}{12(m+n+1)} + \left\lfloor\frac{N-1}{2}\right\rfloor\frac{m+n}{2} + \frac{m}{2}, \tag{2.2.1}$$

(2.2.2)

$$\mathrm{E}_{\mu_{N,1}}[a] = \frac{2(N-1)n(n+2)+(N-1)(N-2)(m+n)(m+n+2)}{12(m+n+1)} + \left\lfloor \frac{N}{2} \right\rfloor \frac{m+n}{2}.$$

Proof. We start by showing (2.2.1). One directly sees that $\mathrm{E}_{\mu_{1,0}}[a] = \frac{m}{2}$. By Corollary 2.1.4 we have $\frac{\mu_{N-1,0}(\Gamma)}{\mu_{N,0}(\Gamma)} = \frac{1}{m+n+1}$ for $N \geq 2$. Hence by (2.1.3) and (2.1.4) we have

$$\mathrm{E}_{\mu_{N,0}}[a] = \mathrm{E}_{\mu_{N-1,0}}[a] + \frac{m(m+2)+(N-2)(m+n)(m+n+2)}{6(m+n+1)} \tag{2.2.3}$$

for even $N \geq 2$, and

$$\mathrm{E}_{\mu_{N,0}}[b] = \mathrm{E}_{\mu_{N-1,0}}[b] + \frac{m(m+2)+(N-2)(m+n)(m+n+2)}{6(m+n+1)} \tag{2.2.4}$$

for odd $N \geq 2$. By Lemma 2.1.1 we have

$$\mathrm{E}_{\mu_{N,0}}[b] = \mathrm{E}_{\mu_{N,0}}[a] + \frac{n}{2} \tag{2.2.5}$$

for even $N \geq 2$, and

$$\mathrm{E}_{\mu_{N,0}}[a] = \mathrm{E}_{\mu_{N,0}}[b] + \frac{m}{2} \tag{2.2.6}$$

for odd $N \geq 2$. In order to recursively compute $\mathrm{E}_{\mu_{N,0}}[a]$ from $\mathrm{E}_{\mu_{1,0}}[a] = \frac{m}{2}$ we must apply – in this order – (2.2.3), (2.2.5), (2.2.4) and (2.2.6) periodically to compute $\mathrm{E}_{\mu_{2,0}}[a]$, $\mathrm{E}_{\mu_{2,0}}[b]$, $\mathrm{E}_{\mu_{3,0}}[b]$, $\mathrm{E}_{\mu_{3,0}}[a]$ etc. The reader is referred to Figure 8 for an illustration of these recursion steps. The contributions from (2.2.3) and (2.2.4) add up to

$$\sum_{i=2}^{N} \frac{m(m+2)+(i-2)(m+n)(m+n+2)}{6(m+n+1)}$$

which is the first summand of (2.2.1) by Gauss's summation formula. The contributions from (2.2.5) and (2.2.6) add up to $\left\lfloor \frac{N-1}{2} \right\rfloor \frac{n+m}{2}$, which is the second summand of (2.2.1). The third summand is the initial value of the recursion, $\mathrm{E}_{\mu_{1,0}}[a] = \frac{m}{2}$.

The proof of (2.2.2) is similar. Here we deduce

$$\mathrm{E}_{\mu_{N,1}}[b] = \mathrm{E}_{\mu_{N-1,1}}[b] + \frac{n(n+2)+(N-2)(m+n)(m+n+2)}{6(m+n+1)} \tag{2.2.7}$$

for even $N \geq 2$, and

$$\mathrm{E}_{\mu_{N,1}}[a] = \mathrm{E}_{\mu_{N-1,1}}[a] + \frac{n(n+2)+(N-2)(m+n)(m+n+2)}{6(m+n+1)} \tag{2.2.8}$$

N	$w_{N,0}$	$\mathrm{E}_{\mu_{N,0}}[a]$		$\mathrm{E}_{\mu_{N,0}}[b]$
1	s_0	$\frac{m}{2}$		0
		$\downarrow$ (2.2.3)		
2	$s_1 s_0$	$*$	$\xrightarrow{(2.2.5)}$	$*$
				$\downarrow$ (2.2.4)
3	$s_0 s_1 s_0$	$*$	$\xleftarrow{(2.2.6)}$	$*$
		$\downarrow$ (2.2.3)		
4	$s_1 s_0 s_1 s_0$	$*$	$\xrightarrow{(2.2.5)}$	$*$
				$\downarrow$ (2.2.4)

FIGURE 8. Recursion steps for $\mathrm{E}_{\mu_{N,0}}[a]$.

N	$w_{N,1}$	$\mathrm{E}_{\mu_{N,1}}[a]$		$\mathrm{E}_{\mu_{N,1}}[b]$
1	s_1	0		$\frac{n}{2}$
				$\downarrow$ (2.2.7)
2	$s_0 s_1$	$*$	$\xleftarrow{(2.2.9)}$	$*$
		$\downarrow$ (2.2.8)		
3	$s_1 s_0 s_1$	$*$	$\xrightarrow{(2.2.10)}$	$*$
				$\downarrow$ (2.2.7)
4	$s_0 s_1 s_0 s_1$	$*$	$\xleftarrow{(2.2.9)}$	$*$
		$\downarrow$ (2.2.8)		

FIGURE 9. Recursion steps for $\mathrm{E}_{\mu_{N,1}}[a]$.

for odd $N \geq 2$ from (2.1.5) and (2.1.6), respectively. From Lemma 2.1.1 we get

$$\mathrm{E}_{\mu_{N,1}}[a] = \mathrm{E}_{\mu_{N,1}}[b] + \frac{m}{2} \tag{2.2.9}$$

for even $N \geq 2$ and

$$\mathrm{E}_{\mu_{N,1}}[b] = \mathrm{E}_{\mu_{N,1}}[a] + \frac{n}{2} \tag{2.2.10}$$

for odd $N \geq 2$. Now, starting from $\mathrm{E}_{\mu_{1,1}}[b] = \frac{n}{2}$ we recursively compute $\mathrm{E}_{\mu_{2,1}}[b]$, $\mathrm{E}_{\mu_{2,1}}[a]$, $\mathrm{E}_{\mu_{3,1}}[a]$, $\mathrm{E}_{\mu_{3,1}}[b]$ etc. by periodic application – again in this order – of (2.2.7), (2.2.9), (2.2.8), and (2.2.10), as illustrated in Figure 9. The first summand of (2.2.2) collects the contributions of the applications of (2.2.7) and (2.2.8). The second summand collects both the initial value and the contributions of the applications of (2.2.9) and (2.2.10). □

It remains to extend Theorem 2.2.1 to the case of Demazure modules indexed by elements in the extended affine Weyl group $\widetilde{W}^{\mathrm{aff}}$. Recall from §1.3 that σ denotes the non-trivial automorphism of the Dynkin diagram of $\widehat{\mathfrak{sl}}_2$.

Theorem 2.2.2. *Let $\Lambda = m\Lambda_0 + n\Lambda_1$ be a dominant integral weight, $N \geq 1$, and $j \in \{0, 1\}$. Denote by $\mu_{(\sigma s_j)^N}$ the weight distribution of $V_{(\sigma s_j)^N}(\Lambda)$. Consider $-d \in \mathfrak{h}$ as a function on $\mathfrak{h}^*$. Choose a basis of weight vectors in the Demazure modules $V_{(\sigma s_0)^N}(\Lambda)$ and $V_{(\sigma s_1)^N}(\Lambda)$. Then the expected degrees of a randomly chosen basis element are given by the following formulas, respectively.*

$$\begin{aligned}
\mathrm{E}_{\mu_{(\sigma s_0)^N}}[-d] &= \frac{2(N-1)m(m+2) + (N-1)(N-2)(m+n)(m+n+2)}{12(m+n+1)} \\
&\quad + \left\lfloor \frac{N-1}{2} \right\rfloor \frac{n}{2} + \left\lfloor \frac{N}{2} \right\rfloor \frac{m}{2}, \\
\mathrm{E}_{\mu_{(\sigma s_1)^N}}[-d] &= \frac{2(N-1)n(n+2) + (N-1)(N-2)(m+n)(m+n+2)}{12(m+n+1)} \\
&\quad + \left\lceil \frac{N-1}{2} \right\rceil \frac{m}{2} + \left\lceil \frac{N}{2} \right\rceil \frac{n}{2}.
\end{aligned}$$

Proof. Let us first treat $V_{(\sigma s_0)^N}(\Lambda)$. We have $(\sigma s_0)^N = \sigma^{(N \bmod 2)} w_{N,0}$. If N is even, the expected degree is given by $\mathrm{E}_{\mu_{N,0}}[a]$, and a slight modification of (2.2.1) in Theorem 2.2.1 immediately proves the claim. If N is odd, note that $\sigma s_0 = s_1 \sigma$ implies $\sigma w_{N,0} = w_{N,1}\sigma$. Since $V_{w_{N,1}\sigma}(\Lambda) = V_{w_{N,1}}(\sigma(\Lambda))$ we now have to consider the weight distribution $\mu_{N,1}$ of $V_{w_{N,1}}(\sigma(\Lambda))$ which is supported on the lattice $\Gamma' = \sigma(\Lambda) + \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1$. Therefore the claimed formula follows from the second part, that is (2.2.2), of Theorem 2.2.1, now applied with the highest weight $\sigma(\Lambda) = n\Lambda_0 + m\Lambda_1$. For the Demazure module $V_{(\sigma s_1)^N}(\Lambda)$ the situation is completely analogous. Here one notes that $(\sigma s_1)^N = \sigma^{(N \bmod 2)} w_{N,1}$ and $\sigma w_{N,1} = w_{N,0}\sigma$. The interesting case is again for N odd. Now one has to consider the weight distribution $\mu_{N,0}$ of $V_{w_{N,0}}(\sigma(\Lambda))$ to compute the expected degree of a randomly chosen basis weight vector in $V_{(\sigma s_1)^N}(\Lambda)$. □

Note 2.2.3. The linearity of the expected value and Lemma 2.1.1 easily allow the reformulation of Theorem 2.2.1 in terms of the coordinate b. Likewise, if one regards the automorphism σ as a function on $\mathfrak{h}^*$ (by exchanging Λ_0 and Λ_1, and acting trivially on δ) one can rephrase Theorem 2.2.2 in terms of the coordinate $-d \circ \sigma = b$.

2.3. **Asymptotic statements.** From Theorem 2.2.1 we can immediately derive asymptotic statements when the parameters N, m or n become large. First, we compute the maximal occurring degree in a given Demazure module for comparison with the expected degree.

Lemma 2.3.1 (Cf. [16, (6.5.2), (6.5.3)])**.** *Let $\Lambda = m\Lambda_0 + n\Lambda_1$ be a dominant integral weight and $N \geq 0$. The highest degree of a weight is*

$$A_{N,0}^{m,n} = \left\lceil \frac{N}{2} \right\rceil m + (m+n)\left(\left\lceil \frac{N}{2} \right\rceil - 1\right)\left\lceil \frac{N}{2} \right\rceil \qquad \text{in } V_{w_{N,0}}(\Lambda), \text{ and}$$

$$A_{N,1}^{m,n} = \left\lfloor \frac{N}{2} \right\rfloor (m+2n) + (m+n)\left(\left\lfloor \frac{N}{2} \right\rfloor - 1\right)\left\lfloor \frac{N}{2} \right\rfloor \qquad \text{in } V_{w_{N,1}}(\Lambda).$$

Proof. By an easy induction on even N, one proves that the coefficients of $w_{N,0}\Lambda = \Lambda - x\alpha_0 - y\alpha_1$ satisfy $x = \sum_{i=0}^{N/2-1}(m + 2i(m+n))$ and $x - y = -\frac{N}{2}(m+n)$. As the operator D_1 preserves the degree we have $A_{N-1,0}^{m,n} = A_{N,0}^{m,n}$. Hence replacing $\frac{N}{2}$ with $\lceil \frac{N}{2} \rceil$ extends the formula to all N, thereby implying the lemma in the case of $V_{w_{N,0}}(\Lambda)$. Note that the proof in the case of $V_{w_{N,1}}(\Lambda)$ is completely analogous if one starts with $\Lambda' = s_1\Lambda = \Lambda - n\alpha_1$ instead. This first step introduces the $2n$ and switches the $\lceil \frac{N}{2} \rceil$ to $\lfloor \frac{N}{2} \rfloor$ in the formula claimed above. □

Now, let us start with the case when the length of the Weyl group element, that is N, tends to infinity.

Corollary 2.3.2 (Limit ratio)**.** *Let $\Lambda = m\Lambda_0 + n\Lambda_1$ be a dominant integral weight and $j \in \{0,1\}$. Then the limit ratio of the expected and maximal degree in $V_{w_{N,j}}(\Lambda)$, as N tends to infinity, is given by*

$$\lim_{N\to\infty} \frac{\mathrm{E}_{\mu_{N,j}}[a]}{A_{N,j}^{m,n}} = \frac{m+n+2}{3(m+n+1)}.$$

Similar asymptotic statements hold with respect to the coefficients of the fundamental weights Λ_0, Λ_1, i.e. m and n, respectively.

Corollary 2.3.3. *Let $N \geq 1$ and $j \in \{0,1\}$. Then the limit ratios of the expected and maximal degree in $V_{w_{N,j}}(m\Lambda_0 + n\Lambda_1)$, as m or n tend to infinity, are given by*

$$\frac{\mathrm{E}_{\mu_{N,0}}[a]}{A_{N,0}^{m,n}} \to \begin{cases} \frac{N^2 - N + 6\lceil \frac{N}{2} \rceil}{12\lceil \frac{N}{2} \rceil^2} & (m \to \infty,\ n \text{ fixed}) \\ \frac{N^2 - 3N - 4 + 6\lceil \frac{N}{2} \rceil}{12\lceil \frac{N}{2} \rceil(\lceil \frac{N}{2} \rceil - 1)} & (n \to \infty,\ m \text{ fixed}), \end{cases}$$

and

$$\frac{\mathrm{E}_{\mu_{N,1}}[a]}{A_{N,1}^{m,n}} \to \begin{cases} \frac{N^2 - 3N + 2 + 6\lfloor \frac{N}{2} \rfloor}{12\lfloor \frac{N}{2} \rfloor^2} & (m \to \infty,\ n \text{ fixed}) \\ \frac{N^2 - N + 6\lfloor \frac{N}{2} \rfloor}{12\lfloor \frac{N}{2} \rfloor(\lfloor \frac{N}{2} \rfloor + 1)} & (n \to \infty,\ m \text{ fixed}). \end{cases}$$

From Theorem 2.2.2 we can derive similar asymptotic statements in the extended affine Weyl group case. First let us compute the maximal occurring degree.

Lemma 2.3.4. *Let $\Lambda = m\Lambda_0 + n\Lambda_1$ be a dominant integral weight and $N \geq 0$. Consider the Demazure modules $V_{(\sigma s_0)^N}(\Lambda)$ and $V_{(\sigma s_1)^N}(\Lambda)$. The highest degree of a weight is*

$$B_{N,0}^{m,n} = \left\lfloor \frac{N}{2} \right\rfloor m + (m+n) \left\lfloor \frac{N-1}{2} \right\rfloor \left\lfloor \frac{N}{2} \right\rfloor \qquad \text{in } V_{(\sigma s_0)^N}(\Lambda), \text{ and}$$

$$B_{N,1}^{m,n} = \left\lceil \frac{N}{2} \right\rceil n + (m+n) \left\lfloor \frac{N}{2} \right\rfloor \left\lceil \frac{N}{2} \right\rceil \qquad \text{in } V_{(\sigma s_1)^N}(\Lambda).$$

Proof. Both claims can be derived from Lemma 2.3.1 and we will demonstrate this in the case of the Demazure module $V_{(\sigma s_0)^N}(\Lambda)$. Note that $(\sigma s_0)^N = w_{N,0}$ if N is even, and equals $w_{N,1}\sigma$ when N is odd, and $\sigma(\Lambda) = n\Lambda_0 + m\Lambda_1$. By Lemma 2.3.1 the maximal degree of a weight in $V_{w_{N,0}}(\Lambda)$ is $\lceil N/2 \rceil m + (m + n)(\lceil N/2 \rceil - 1)\lceil N/2 \rceil$, and is equal to $\lfloor N/2 \rfloor (n+2m) + (m+n)(\lfloor N/2 \rfloor - 1)\lfloor N/2 \rfloor$ in $V_{w_{N,1}}(\sigma(\Lambda))$. The formula claimed above for $V_{(\sigma s_0)^N}(\Lambda)$ unifies those case considerations. □

As expected, for the Demazure modules $V_{(\sigma s_0)^N}(\Lambda)$ and $V_{(\sigma s_1)^N}(\Lambda)$ one obtains the same limit ratio for large N as in Corollary 2.3.2. Let us state it here for the sake of completeness.

Corollary 2.3.5 (Limit ratio)**.** *Let $\Lambda = m\Lambda_0 + n\Lambda_1$ be a dominant integral weight and $j \in \{0, 1\}$. Consider $-d \in \mathfrak{h}$ as a function on $\mathfrak{h}^*$. Then the limit ratio of the expected and maximal degree in $V_{(\sigma s_j)^N}(\Lambda)$, as N tends to infinity, is given by*

$$\lim_{N \to \infty} \frac{\mathrm{E}_{\mu_{(\sigma s_j)^N}}[-d]}{B_{N,j}^{m,n}} = \frac{m+n+2}{3(m+n+1)}.$$

However, the limit ratios with respect to the coefficients of the fundamental weights Λ_0 and Λ_1 are slightly different.

Corollary 2.3.6. *Let $N \geq 1$ and $j \in \{0, 1\}$. Consider $-d \in \mathfrak{h}$ as a function on $\mathfrak{h}^*$. Then the limit ratios of the expected and maximal degree in $V_{(\sigma s_j)^N}(m\Lambda_0 + n\Lambda_1)$, as m or n tend to infinity, are given by*

$$\frac{\mathrm{E}_{\mu_{(\sigma s_0)^N}}[-d]}{B_{N,0}^{m,n}} \to \begin{cases} \frac{N^2 - N + 6\lfloor \frac{N}{2} \rfloor}{12\lfloor \frac{N}{2} \rfloor \lceil \frac{N}{2} \rceil} & (m \to \infty,\ n \text{ fixed}) \\ \frac{N^2 - 3N - 4 + 6\lceil \frac{N}{2} \rceil}{12\lfloor \frac{N}{2} \rfloor (\lceil \frac{N}{2} \rceil - 1)} & (n \to \infty,\ m \text{ fixed}), \end{cases}$$

and

$$\frac{\mathrm{E}_{\mu_{(\sigma s_1)^N}}[-d]}{B_{N,1}^{m,n}} \to \begin{cases} \frac{N^2 - 3N + 2 + 6\lfloor \frac{N}{2} \rfloor}{12\lfloor \frac{N}{2} \rfloor \lceil \frac{N}{2} \rceil} & (m \to \infty,\ n \text{ fixed}) \\ \frac{N^2 - N + 6\lceil \frac{N}{2} \rceil}{12\lceil \frac{N}{2} \rceil (\lfloor \frac{N}{2} \rfloor + 1)} & (n \to \infty,\ m \text{ fixed}). \end{cases}$$

3. Variance of the degree of weights

Let us abbreviate the Weyl group elements $w_{N,0} = \cdots s_0 s_1 s_0$ by w_N, as from now on these will be the elements we will mostly discuss. Analogously we abbreviate the weight distribution $\mu_{N,0}$ of a Demazure module $V_{w_N}(\Lambda)$ by μ_N.

3.1. Technical results.

We start with some immediate generalizations of results in §2.1.

Lemma 3.1.1 (generalized Lemma 2.1.6)**.** *Let* $\Lambda = m\Lambda_0 + n\Lambda_1$ *and* $\Gamma = \Lambda + \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1$*. Then, for any* $\mu \in \mathrm{Meas_c}(\Gamma)$ *and* $k \geq 0$ *we have*

$$\mathrm{E}_{D_1 D_0 \mu}[a^k] = \frac{D_0\mu(\Gamma)}{D_1 D_0 \mu(\Gamma)} \Big((m+n+1)\,\mathrm{E}_{D_0\mu}[a^k] + 2\,\mathrm{Cov}_{D_0\mu}(a^k, a-b) \Big),$$

$$\mathrm{E}_{D_0 D_1 \mu}[b^k] = \frac{D_1\mu(\Gamma)}{D_0 D_1 \mu(\Gamma)} \Big((m+n+1)\,\mathrm{E}_{D_1\mu}[b^k] - 2\,\mathrm{Cov}_{D_1\mu}(b^k, a-b) \Big).$$

Lemma 3.1.2 (partially generalized Lemma 2.1.7)**.** *Let* $\Lambda = m\Lambda_0 + n\Lambda_1$ *and* $\Gamma = \Lambda + \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1$*. Then, for any* $\mu \in \mathrm{Meas_c}(\Gamma)$ *and* $k \geq 0$ *we have*

$$\mathrm{Cov}_{D_0\mu}(b^k, a-b) = 0 \quad \textit{and} \quad \mathrm{Cov}_{D_1\mu}(a^k, a-b) = 0.$$

We want to use Lemma 3.1.1 to inductively compute the variance of the degree distribution (Theorem 3.4.3). This corresponds to the case where $D_o\mu$ (resp. $D_1\mu$) is the weight distribution of the Demazure module $V_{w_N}(\Lambda_0)$ and $k = 2$. Then the summands $\mathrm{Cov}_{D_0\mu}(a^k, a-b)$ and $\mathrm{Cov}_{D_1\mu}(b^k, a-b)$ in Lemma 3.1.1 are *third* moments of the weight distribution, so we cannot suppose them known by induction. We can express them differently as follows:

Lemma 3.1.3. *Let* $\Lambda = m\Lambda_0 + n\Lambda_1$ *and* $\Gamma = \Lambda + \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1$*. Then, for any* $\mu \in \mathrm{Meas_c}(\Gamma)$ *we have*

$$\begin{aligned}\mathrm{Cov}_{D_0\mu}(a^2, a-b) &= \mathrm{Cov}_{D_0\mu}((a-b)^2, a-b) + 2\,\mathrm{Var}_{D_0\mu}(a-b)\,\mathrm{E}_{D_0\mu}[b] \\ &\quad + 2\,\mathrm{Cov}_{D_0\mu}(b, (a-b)^2), \\ \mathrm{Cov}_{D_1\mu}(b^2, a-b) &= \mathrm{Cov}_{D_1\mu}((a-b)^2, a-b) - 2\,\mathrm{Var}_{D_1\mu}(a-b)\,\mathrm{E}_{D_1\mu}[a] \\ &\quad - 2\,\mathrm{Cov}_{D_1\mu}(a, (a-b)^2).\end{aligned}$$

Proof. We will only discuss the formula for $\mathrm{Cov}_{D_0\mu}(a^2, a-b)$ since the second part is completely analogous. First write $a^2 = (a-b)^2 + 2ab - b^2$ and note that $\mathrm{Cov}_{D_0\mu}(b^2, a-b) = 0$ by Lemma 3.1.2. Consequently,

$$\mathrm{Cov}_{D_0\mu}(a^2, a-b) = \mathrm{Cov}_{D_0\mu}((a-b)^2, a-b) + 2\,\mathrm{Cov}_{D_0\mu}(ab, a-b).$$

By definition

$$\mathrm{Cov}_{D_0\mu}(ab, a-b) = \mathrm{E}_{D_0\mu}[ab(a-b)] - \mathrm{E}_{D_0\mu}[ab] \cdot \mathrm{E}_{D_0\mu}[a-b].$$

Now

$$\mathrm{E}_{D_0\mu}[ab(a-b)] = \mathrm{E}_{D_0\mu}[((a-b)+b)b(a-b)]$$

$$
\begin{aligned}
&= \mathrm{E}_{D_0\mu}[(a-b)^2 b] + \mathrm{E}_{D_0\mu}[b^2(a-b)] \\
\text{(Lemma 3.1.2)} \quad &= \mathrm{E}_{D_0\mu}[(a-b)^2 b] + \mathrm{E}_{D_0\mu}[b^2] \cdot \mathrm{E}_{D_0\mu}[a-b],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{E}_{D_0\mu}[ab] \cdot \mathrm{E}_{D_0\mu}[a-b] &= \mathrm{E}_{D_0\mu}[((a-b)+b)b]\, \mathrm{E}_{D_0\mu}[a-b] \\
&= (\mathrm{E}_{D_0\mu}[(a-b)b] + \mathrm{E}_{D_0\mu}[b^2]) \cdot \mathrm{E}_{D_0\mu}[a-b] \\
\text{(Lemma 3.1.2)} \quad &= (\mathrm{E}_{D_0\mu}[a-b] \cdot \mathrm{E}_{D_0\mu}[b] + \mathrm{E}_{D_0\mu}[b^2]) \cdot \mathrm{E}_{D_0\mu}[a-b] \\
&= \mathrm{E}_{D_0\mu}[a-b]^2 \cdot \mathrm{E}_{D_0\mu}[b] + \mathrm{E}_{D_0\mu}[b^2] \cdot \mathrm{E}_{D_0\mu}[a-b].
\end{aligned}
$$

Subtracting the two expressions yields

$$
\begin{aligned}
\mathrm{Cov}_{D_0\mu}(ab, a-b) &= \mathrm{E}_{D_0\mu}[(a-b)^2 \cdot b] - \mathrm{E}_{D_0\mu}[a-b]^2 \cdot \mathrm{E}_{D_0\mu}[b] \\
&= \mathrm{Var}_{D_0\mu}(a-b)\, \mathrm{E}_{D_0\mu}[b] + \mathrm{Cov}_{D_0\mu}(b, (a-b)^2). \qquad \square
\end{aligned}
$$

Lemma 3.1.3 reveals that, in the case of the weight distribution μ_N of the Demazure module $V_{w_N}(\Lambda_0)$, the quantities in question are partially given by higher moments of the finite weight distribution $(a-b)_*\mu_N$ and the expected value of the degree distribution. These are known by [30] and Theorem 2.2.1, respectively. The main part of the proof of Theorem 3.4.3 is therefore concerned with the computation of the a priori unknown quantities $\mathrm{Cov}_{\mu_N}(b, (a-b)^2)$ and $\mathrm{Cov}_{\mu_N}(a, (a-b)^2)$ when N is odd and even, respectively. Let us first recollect the quantities we claim to know so far.

Let $B(N, \frac{1}{2})$ denote the binomial distribution on $\mathbf{R}$ for N trials with success probability $\frac{1}{2}$, and let $X = \mathrm{id}_{\mathbf{R}}$. Recall that

$$
\mathrm{E}_{B(N,\frac{1}{2})}[X] = \frac{N}{2}, \tag{3.1.1}
$$

and its first central moments are

	k	$\mathrm{E}_{B(N,\frac{1}{2})}[(X - \mathrm{E}_{B(N,\frac{1}{2})}[X])^k]$
(3.1.2)	2	$\frac{N}{4}$
(3.1.3)	3	0
(3.1.4)	4	$\frac{N(3N-2)}{16}$.

The main result in [30] implies that if μ_N is the weight distribution of the Demazure module $V_{w_N}(\Lambda_0)$, then

$$
\left(a - b + \left\lfloor \tfrac{N}{2} \right\rfloor\right)_* \mu_N = 2^N B(N, \tfrac{1}{2}). \tag{3.1.5}
$$

Lemma 3.1.4. *Let μ_N be the weight distribution of the Demazure module $V_{w_N}(\Lambda_0)$. Then,*

$$
\begin{aligned}
\mathrm{E}_{\mu_N}[a-b] &= \frac{\pi_N}{2}, \\
\mathrm{E}_{\mu_N}[b] &= \frac{(N-1)(N+2)}{8} \qquad \textit{if } N \textit{ is odd,}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}_{\mu_N}[a] &= \frac{N(N+1)}{8} && \textit{if } N \textit{ is even,}\\
\mathrm{Var}_{\mu_N}(a-b) &= \frac{N}{4}, && \textit{and}\\
\mathrm{Cov}_{\mu_N}((a-b)^2, a-b) &= \frac{N\pi_N}{4}.
\end{aligned}
$$

Proof. The first three assertions are immediate from Lemma 2.1.1 and Theorem 2.2.1, respectively. The variance $\mathrm{Var}_{\mu_N}(a-b)$ equals the second central moment of the binomial distribution and is hence given by (3.1.2). Finally, by definition

$$
\mathrm{Cov}_{\mu_N}((a-b)^2, a-b) = \mathrm{E}_{\mu_N}[(a-b)^3] - \mathrm{E}_{\mu_N}[(a-b)^2]\,\mathrm{E}_{\mu_N}[a-b],
$$

and by Equation 3.1.3

$$
\begin{aligned}
0 &= \mathrm{E}_{\mu_N}[(a-b-\mathrm{E}_{\mu_N}[a-b])^3]\\
&= \mathrm{E}_{\mu_N}[(a-b)^3] + 2\,\mathrm{E}_{\mu_N}[a-b]^3 - 3\,\mathrm{E}_{\mu_N}[(a-b)^2]\,\mathrm{E}_{\mu_N}[a-b]\\
&= \mathrm{Cov}_{\mu_N}((a-b)^2, a-b) + 2\cdot\frac{\pi_N}{8} - 2\left(\frac{N}{4} + \frac{\pi_N}{4}\right)\frac{\pi_N}{2}\\
&= \mathrm{Cov}_{\mu_N}((a-b)^2, a-b) - \frac{N\pi_N}{4}.
\end{aligned}
$$

□

Lemma 3.1.3 and Lemma 3.1.4 imply:

Corollary 3.1.5. *Let μ_N be the weight distribution of the Demazure module $V_{w_N}(\Lambda_0)$. Then,*

$$
\begin{aligned}
\mathrm{Cov}_{\mu_N}(a^2, a-b) &= \frac{N(N^2+N+2)}{16} + 2\,\mathrm{Cov}_{\mu_N}(b, (a-b)^2) && \textit{for odd } N,\\
\mathrm{Cov}_{\mu_N}(b^2, a-b) &= -\frac{N^2(N+1)}{16} - 2\,\mathrm{Cov}_{\mu_N}(a, (a-b)^2) && \textit{for even } N.
\end{aligned}
$$

Note that $\mu_N(\Gamma_0) = \dim V_{w_N}(\Lambda_0) = 2^N$. Consequently Corollary 3.1.5 implies the following specialization of Lemma 3.1.1.

Corollary 3.1.6. *Let μ_N be the weight distribution of the Demazure module $V_{w_N}(\Lambda_0)$ and $k \geq 2$. Then, for N odd*

$$
\mathrm{E}_{\mu_{N+1}}[a^2] = \mathrm{E}_{\mu_N}[a^2] + \frac{N(N^2+N+2)}{16} + 2\,\mathrm{Cov}_{\mu_N}(b, (a-b)^2),
$$

and for N even

$$
\mathrm{E}_{\mu_{N+1}}[b^2] = \mathrm{E}_{\mu_N}[b^2] + \frac{N^2(N+1)}{16} + 2\,\mathrm{Cov}_{\mu_N}(a, (a-b)^2).
$$

In §3.2 and §3.3 we deal with the computation of $\mathrm{Cov}_{\mu_N}(b, (a-b)^2)$ and $\mathrm{Cov}_{\mu_N}(a, (a-b)^2)$, respectively. We identify Γ_0 with $\mathbf{Z}^2$ by means of the coordinates a, b. This endows Γ_0 with a $\mathbf{Z}$-module structure. The covariance $\mathrm{Cov}_{\mu_N}(\cdot, (a-b)^2)$ is a linear form on the symmetric algebra $\mathrm{Sym}((\Gamma_0 \otimes_{\mathbf{Z}} \mathbf{R})^*)$. We use a symmetry property of the Demazure module $V_{w_N}(\Lambda_0)$ described in §3.2 to compute elements in its kernel. It turns out, that those elements involve higher moments of the finite weight distribution and (multiples of) the linear forms $b, a \in (\Gamma_0 \otimes_{\mathbf{Z}} \mathbf{R})^*$, respectively. See Corollary 3.3.4 for the precise statement. Let us first describe the mentioned symmetry property.

3.2. **Palindromicity of the string functions.**

Lemma 3.2.1. *Let $\lambda \in \Gamma_0 = \Lambda_0 + \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1$ and q be a variable. For even N, the string function*

$$p_{\lambda,N}(q) = \sum_{\mu \in \lambda + \mathbf{Z}\delta} \dim V_{w_N}(\Lambda_0)_\mu \cdot q^{a(\mu)}$$

satisfies

$$p_{\lambda,N}(q) = p_{\lambda,N}(q^{-1}) \cdot q^{\frac{1}{4}N^2 + (a-b)^2(\lambda)}.$$

In other words,

$$\dim V_{w_N}(\Lambda_0)_\lambda = \dim V_{w_N}(\Lambda_0)_{\lambda - \left(\frac{1}{4}N^2 + (a-b)^2(\lambda) - 2a(\lambda)\right)\delta}.$$

For odd N, the string function

$$p_{\lambda,N}(q) = \sum_{\mu \in \lambda + \mathbf{Z}\delta} \dim V_{w_N}(\Lambda_0)_\mu \cdot q^{b(\mu)}$$

satisfies

$$p_{\lambda,N}(q) = p_{\lambda,N}(q^{-1}) \cdot q^{\frac{1}{4}(N^2-1) + (a-b)^2(\lambda) - (a-b)(\lambda)}.$$

In other words,

$$\dim V_{w_N}(\Lambda_0)_\lambda = \dim V_{w_N}(\Lambda_0)_{\lambda - \left(\frac{1}{4}(N^2-1) + (a-b)^2(\lambda) - (a-b)(\lambda) - 2b(\lambda)\right)\delta}.$$

The palindromicity of the string functions can be derived from the fact that Demazure characters are related to Macdonald polynomials [31, Theorem 6 and Theorem 7], which in turn are related to Rogers–Szegő polynomials, i.e. generating functions of Gaussian polynomials [13, (3.4)]. That is, those statements combined show that the string functions are (translated) Gaussian polynomials.

Proof. Recall the definition of the Gaussian polynomials $\begin{bmatrix} N \\ k \end{bmatrix}_q$ and the specialized bivariate Macdonald polynomials $P_\lambda(z, z^{-1}; q, 0)$ from §1.3.

By [31, Theorem 6 and Theorem 7]

$$\mathrm{ch}_{V_{w_{N,\pi_N}}(\Lambda_{\pi_N})} = e^{\Lambda_0 - \lfloor \frac{1}{4}N^2 \rfloor \delta} \cdot P_\lambda(e^{\frac{1}{2}\alpha_1}, e^{-\frac{1}{2}\alpha_1}; e^\delta, 0). \tag{3.2.1}$$

Furthermore, by [13, (3.4)]

$$P_\lambda(z, z^{-1}; q, 0) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q z^{2k-N}. \tag{3.2.2}$$

Combining (3.2.1) and (3.2.2) we obtain

$$\operatorname{ch}_{V_{w_{N,\pi_N}}(\Lambda_{\pi_N})} = e^{\Lambda_0} \cdot e^{-\lfloor \frac{1}{4}N^2 \rfloor \delta} \cdot \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_{e^{\delta}} e^{(2k-N)\frac{1}{2}\alpha_1}. \tag{3.2.3}$$

This translates the verification of the palindromicity of the string functions into quantum calculus. That is, consider a fixed $\lambda \in \Gamma_0$, and let $k = \left\lfloor \frac{N}{2} \right\rfloor + (a-b)(\lambda) \in \{0, \dots, N\}$ and $q = e^{\delta}$. Then (3.2.3) yields the relations

$$p_{\lambda,N}(q) = q^{\lfloor \frac{1}{4}N^2 \rfloor} \cdot \begin{bmatrix} N \\ k \end{bmatrix}_{q^{-1}}, \quad \text{and} \tag{3.2.4}$$

$$p_{\lambda,N}(q^{-1}) = q^{-\lfloor \frac{1}{4}N^2 \rfloor} \cdot \begin{bmatrix} N \\ k \end{bmatrix}_{q}. \tag{3.2.5}$$

Now, for $k \in \{0, \dots, N\}$ by [17, p. 19] we have

$$\begin{bmatrix} N \\ k \end{bmatrix}_{q} = q^{k(N-k)} \cdot \begin{bmatrix} N \\ k \end{bmatrix}_{q^{-1}}. \tag{3.2.6}$$

Hence, by (3.2.4)–(3.2.6) we obtain for N even

$$\begin{aligned} p_{\lambda,N}(q) &= q^{\frac{1}{4}N^2} \cdot \begin{bmatrix} N \\ k \end{bmatrix}_{q^{-1}} \\ &= q^{\frac{1}{4}N^2} \cdot \left(q^{-\frac{1}{4}N^2 + (a-b)^2(\lambda)} \cdot \begin{bmatrix} N \\ k \end{bmatrix}_{q} \right) \\ &= q^{(a-b)^2(\lambda)} \cdot \begin{bmatrix} N \\ k \end{bmatrix}_{q} \\ &= q^{\frac{1}{4}N^2 + (a-b)^2(\lambda)} \cdot p_{\lambda,N}(q^{-1}). \end{aligned}$$

Note that in the second step, for our choice of k depending on the weight λ,

$$\begin{aligned} k(N-k) &= \left(\frac{1}{2}N + (a-b)(\lambda) \right) \left(N - \frac{1}{2}N - (a-b)(\lambda) \right) \\ &= \frac{1}{4}N^2 - (a-b)^2(\lambda). \end{aligned}$$

For N odd the equations (3.2.4)–(3.2.6) yield

$$\begin{aligned} p_{\lambda,N}(q) &= q^{\frac{1}{4}(N^2-1)} \cdot \begin{bmatrix} N \\ k \end{bmatrix}_{q^{-1}} \\ &= q^{\frac{1}{4}(N^2-1)} \cdot \left(q^{-\frac{1}{4}(N^2-1) + (a-b)^2(\lambda) - (a-b)(\lambda)} \cdot \begin{bmatrix} N \\ k \end{bmatrix}_{q} \right) \\ &= q^{(a-b)^2(\lambda) - (a-b)(\lambda)} \cdot \begin{bmatrix} N \\ k \end{bmatrix}_{q} \\ &= q^{\frac{1}{4}(N^2-1) + (a-b)^2(\lambda) - (a-b)(\lambda)} \cdot p_{\lambda,N}(q^{-1}). \end{aligned}$$

Note that in the second step, again for our specific k,

$$k(N-k) = \left(\frac{1}{2}(N-1)+(a-b)(\lambda)\right)\left(N-\frac{1}{2}(N-1)-(a-b)(\lambda)\right)$$
$$= \frac{1}{4}(N^2-1)-(a-b)^2(\lambda)+(a-b)(\lambda).$$

The explicit interpretations of the two symmetries in terms of weight multiplicities are immediate. □

3.3. **The stretching trick.** The following two propositions follow directly from the definitions.

Proposition 3.3.1. *Let $\mu \in \mathrm{Meas}(\mathbf{R}^2)$. Let $X, Y : \mathbf{R}^2 \to \mathbf{R}$ be the projection on the first and second component, respectively. Let $q : \mathbf{R}^2 \to \mathbf{R}^2$ be given by $q(x,y) = (x^2, y)$. Then*

$$\mathrm{Cov}_\mu(X^2, Y) = \mathrm{Cov}_{q_*\mu}(X, Y).$$

By saying that $s : \mathbf{R}^2 \to \mathbf{R}^2$ is a *reflection at H_1 along H_2* we mean that H_1 is the 1-eigen space and H_2 is the (-1)-eigen space of s. A measure $\mu \in \mathrm{Meas}(\mathbf{R}^2)$ is said to be *symmetric* with respect to s if $s_*\mu = \mu$.

Proposition 3.3.2. *Let $X, Y : \mathbf{R}^2 \to \mathbf{R}$ be the projection on the first and second component, respectively. Let $\mu \in \mathrm{Meas}(\mathbf{R}^2)$ be symmetric at $\{Y = 0\}$ along $\{X = 0\}$. Then $\mathrm{Cov}_\mu(X, Y) = 0$.*

Now we are ready to exploit the symmetry of the string functions associated with the Demazure module $V_{w_N}(\Lambda_0)$.

Lemma 3.3.3. *Let μ_N be the weight distribution of the Demazure module $V_{w_N}(\Lambda_0)$ supported on the lattice $\Gamma_0 = \Lambda_0 + \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1$. Define coordinates $X, Y : \Gamma_0 \to \mathbf{Z}$ as follows: If N is odd, let*

$$X = a - b - \frac{1}{2} \quad \text{and} \quad Y = b - \frac{N^2-2}{8}.$$

If N is even, let

$$X = a - b \quad \text{and} \quad Y = a - \frac{N^2}{8}.$$

Let $q : \Gamma_0 \to \Gamma_0$ such that $X(q(\lambda)) = X(\lambda)^2$ and $Y(q(\lambda)) = Y(\lambda)$ for all $\lambda \in \Gamma_0$. Then

$$\mathrm{Cov}_{q_*\mu_N}(X - 2Y, X) = 0.$$

While reading the proof, see Figure 10 and 11 for an illustration.

Proof. If N is odd, then by Lemma 3.2.1 the strings in μ_N are symmetric around

$$b = \frac{1}{2}\left(\frac{1}{4}(N^2-1)+(a-b)^2-(a-b)\right)$$

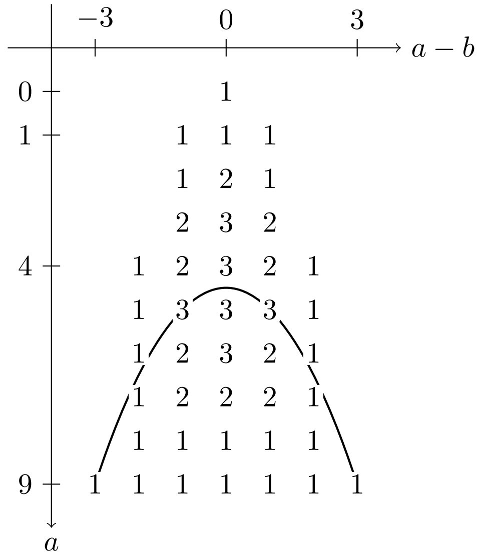

FIGURE 10. Weight distribution μ_6 of $V_{w_6}(\Lambda_0)$ and the parabola of the string symmetry points.

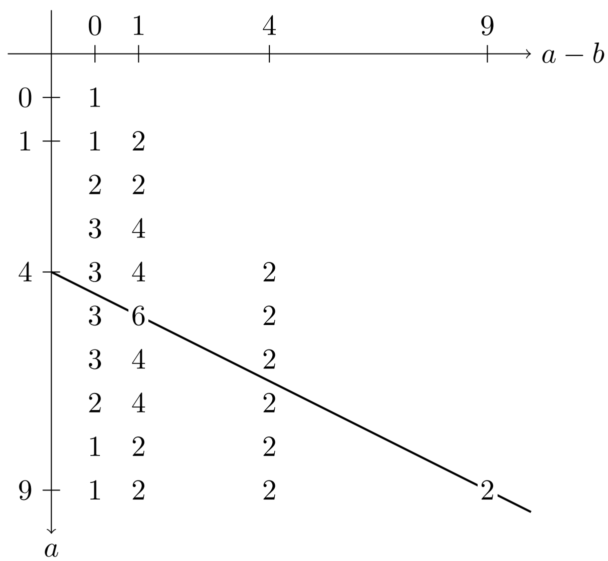

FIGURE 11. Stretched weight distribution $q_*\mu_6$ of $V_{w_6}(\Lambda_0)$ and the line of the string symmetry points.

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{1}{4}(N^2-1) + (a-b-\frac{1}{2})^2 - \frac{1}{4}\right) \\
&= \frac{1}{8}(N^2-2) + \frac{1}{2}(a-b-\frac{1}{2})^2.
\end{aligned}
$$

In other words, they are symmetric around $Y = \frac{1}{2}X^2$. If N is even, then by Lemma 3.2.1 the strings in μ_N are symmetric around

$$\begin{aligned} a &= \frac{1}{2}\left(\frac{1}{4}N^2 + (a-b)^2\right) \\ &= \frac{1}{8}N^2 + \frac{1}{2}(a-b)^2. \end{aligned}$$

In other words, they are symmetric around $Y = \frac{1}{2}X^2$. In both cases, the string midpoints in μ_N are $(x, \frac{1}{2}x^2)$ in coordinates X, Y, so the string midpoints in $q_*\mu_N$ are $(x^2, \frac{1}{2}x^2)$. Hence $q_*\mu_N$ is symmetric at $\{X - 2Y = 0\}$ along $\{X{=}0\}$. The lemma follows by Proposition 3.3.2. □

By formulating Lemma 3.3.3 in terms of the weight distribution of $V_{w_N}(\Lambda_0)$ via Proposition 3.3.1 we obtain:

Corollary 3.3.4. *Let μ_N be the weight distribution of the Demazure module $V_{w_N}(\Lambda_0)$. If N is odd, then*

$$\mathrm{Cov}_{\mu_N}((a-b)^2 - (a-b) - 2b, (a-b)^2 - (a-b)) = 0.$$

If N is even, then

$$\mathrm{Cov}_{\mu_N}((a-b)^2 - 2a, (a-b)^2) = 0.$$

Consequently we can now determine the values of $\mathrm{Cov}_{\mu_N}(\cdot, (a-b)^2)$ at b and a respectively:

Lemma 3.3.5. *Let μ_N be the weight distribution of the Demazure module $V_{w_N}(\Lambda_0)$. Then,*

$$\begin{aligned} \mathrm{Cov}_{\mu_N}(b, (a-b)^2) &= \frac{N(N-1)}{16} && \textit{if } N \textit{ is odd,} \\ \mathrm{Cov}_{\mu_N}(a, (a-b)^2) &= \frac{N(N-1)}{16} && \textit{if } N \textit{ is even.} \end{aligned}$$

Proof. By Corollary 3.3.4 we obtain for odd N

$$\begin{aligned} \mathrm{Cov}_{\mu_N}(b, (a-b)^2) = \frac{1}{2}\big(&\mathrm{Var}_{\mu_N}((a-b)^2) - 2\,\mathrm{Cov}_{\mu_N}((a-b)^2, a-b) \\ &+ \mathrm{Var}_{\mu_N}(a-b) + 2\,\mathrm{Cov}_{\mu_N}(b, a-b)\big), \end{aligned}$$

and for even N

$$\mathrm{Cov}_{\mu_N}(a, (a-b)^2) = \frac{1}{2}\,\mathrm{Cov}_{\mu_N}((a-b)^2, (a-b)^2).$$

We know the values on the right-hand sides of those equations. Let us recollect them. If N is odd, we know by Lemma 3.1.2 and Lemma 3.1.4 that

$$\begin{aligned} \mathrm{Cov}_{\mu_N}(b, a-b) &= 0, \\ \mathrm{Cov}_{\mu_N}((a-b)^2, a-b) &= \frac{N}{4}, \quad \text{and} \\ \mathrm{Var}_{\mu_N}(a-b) &= \frac{N}{4}. \end{aligned}$$

From (3.1.1)–(3.1.4) one can derive that for any N we have

$$\operatorname{Var}_{\mu_N}((a-b)^2) = \frac{N(N-1)+2N\pi_N}{8} = \begin{cases} \frac{N(N-1)}{8} & \text{if } N \text{ is even,} \\ \frac{N(N+1)}{8} & \text{if } N \text{ is odd.} \end{cases} \tag{3.3.1}$$

Now, Equation 3.3.1 immediately proves the claim in the even case. Finally, for N odd

$$\operatorname{Cov}_{\mu_N}(b,(a-b)^2) = \frac{1}{2}\left(\frac{N(N+1)}{8} - 2\frac{N}{4} + \frac{N}{4} - 2\cdot 0\right) = \frac{N(N-1)}{16}. \qquad \square$$

By means of Lemma 3.3.5, we can describe recurrence relations in order to compute the variance of the degree distribution. We do this in the following section.

3.4. **Variance of the degree distribution.** Due to Corollary 3.1.6 and Lemma 3.3.5 the following recurrence relations are immediate.

Lemma 3.4.1 (Recurrence relations)**.** *Let μ_N be the weight distribution of the Demazure module $V_{w_N}(\Lambda_0)$. Then,*

$$\mathrm{E}_{\mu_{N+1}}[a^2] = \mathrm{E}_{\mu_N}[a^2] + \frac{N^2(N+3)}{16} \qquad \textit{if } N \textit{ is odd,}$$
$$\mathrm{E}_{\mu_{N+1}}[b^2] = \mathrm{E}_{\mu_N}[b^2] + \frac{N(N^2+3N-2)}{16} \qquad \textit{if } N \textit{ is even.}$$

In order to resolve the recurrence relations, we need to switch between the coordinates a and b depending on the parity of N. Therefore, the following version of Lemma 3.4.1 is more practical.

Lemma 3.4.2 (Modified recurrence relations)**.** *Let μ_N be the weight distribution of the Demazure module $V_{w_N}(\Lambda_0)$. Then,*

$$\mathrm{E}_{\mu_{N+1}}[a^2] = \mathrm{E}_{\mu_N}[b^2] + \frac{N(N+2)(N+3)}{16} \qquad \textit{if } N \textit{ is odd,}$$
$$\mathrm{E}_{\mu_{N+1}}[b^2] = \mathrm{E}_{\mu_N}[a^2] + \frac{N(N+1)(N+2)}{16} \qquad \textit{if } N \textit{ is even.}$$

Proof. Write $a^2-b^2=(a-b)(a+b)$ and consider

$$\operatorname{Cov}_{\mu_N}(a-b,a+b) = \mathrm{E}_{\mu_N}[a^2-b^2] - \mathrm{E}_{\mu_N}[a-b]\,\mathrm{E}_{\mu_N}[a+b].$$

For odd N we obtain

$$\begin{aligned}\mathrm{E}_{\mu_N}[a^2] &= \mathrm{E}_{\mu_N}[b^2] + \operatorname{Cov}_{\mu_N}(a-b,a) + \operatorname{Cov}_{\mu_N}(a-b,b) \\ &\quad + \mathrm{E}_{\mu_N}[a-b]\,\mathrm{E}_{\mu_N}[a+b] \\ &= \mathrm{E}_{\mu_N}[b^2] + \frac{N}{4} + 0 + \frac{1}{2}\left(\frac{1}{2} + 2\frac{(N-1)(N+2)}{8}\right)\end{aligned}$$

$$= \mathrm{E}_{\mu_N}[b^2] + \frac{N(N+3)}{8}$$

by Lemma 3.1.4 and Lemma 2.1.7, and hence

$$\begin{aligned}\mathrm{E}_{\mu_{N+1}}[a^2] &= \mathrm{E}_{\mu_N}[b^2] + \frac{N(N+3)}{8} + \frac{N^2(N+3)}{16} \\ &= \mathrm{E}_{\mu_N}[b^2] + \frac{N(N+2)(N+3)}{16}.\end{aligned}$$

For even N one similarly derives

$$\mathrm{E}_{\mu_N}[b^2] = \mathrm{E}_{\mu_N}[a^2] + \frac{N}{4},$$

and consequently

$$\begin{aligned}\mathrm{E}_{\mu_{N+1}}[b^2] &= \mathrm{E}_{\mu_N}[a^2] + \frac{N}{4} + \frac{N(N^2+3N-2)}{16} \\ &= \mathrm{E}_{\mu_N}[a^2] + \frac{N(N+1)(N+2)}{16}.\end{aligned}$$

□

The (modified) recurrence relations plus some additional case considerations give:

Theorem 3.4.3 (Variance of the degree distribution)**.** *Let μ_N be the weight distribution of the Demazure module $V_{w_N}(\Lambda_0)$ and denote the parity of N by π_N. Then, for $N \geq 1$ we have*

$$\begin{aligned}\mathrm{Var}_{\mu_N}(a) &= \frac{N(N-1)(2N+5)}{96} + \pi_N \cdot \frac{N}{4}, \\ \mathrm{Var}_{\mu_N}(b) &= \frac{N(N-1)(2N+5)}{96} + \pi_{N+1} \cdot \frac{N}{4}.\end{aligned}$$

Proof. Solving the modified recurrence relations in Lemma 3.4.2 yields

$$\begin{aligned}\mathrm{E}_{\mu_N}[a^2] &= \frac{1}{16}\sum_{i=0}^{\frac{N}{2}-1} 2i(2i+1)(2i+2) + \frac{1}{16}\sum_{j=0}^{\frac{N}{2}-1}(2t+1)(2t+3)(2t+4) \\ &= \frac{N(3N^3+10N^2+9N-10)}{192}\end{aligned}$$

for even N, and

$$\begin{aligned}\mathrm{E}_{\mu_N}[b^2] &= \frac{1}{16}\sum_{i=0}^{\frac{N-1}{2}} 2i(2i+1)(2i+2) + \frac{1}{16}\sum_{j=0}^{\frac{N-3}{2}}(2t+1)(2t+3)(2t+4) \\ &= \frac{(N-1)(3N^3+13N^2+10N-12)}{192}\end{aligned}$$

for odd N. Hence, for even N we get

$$\mathrm{Var}_{\mu_N}(a) = \mathrm{E}_{\mu_N}[a^2] - \mathrm{E}_{\mu_N}[a]^2$$

$$(3.4.1)\qquad = \frac{N(3N^3+10N^2+9N-10)}{192} - \left(\frac{N(N+1)}{8}\right)^2 = \frac{N(N-1)(2N+5)}{96}.$$

Similarly, for odd N,

$$\begin{aligned}\mathrm{Var}_{\mu_N}(b) &= \mathrm{E}_{\mu_N}[b^2] - \mathrm{E}_{\mu_N}[b]^2 \\ &= \frac{(N-1)(3N^3+13N^2+10N-12)}{192} - \left(\frac{(N-1)(N+2)}{8}\right)^2 \qquad (3.4.2)\\ &= \frac{N(N-1)(2N+5)}{96}.\end{aligned}$$

□

It remains to compute $\mathrm{Var}_{\mu_N}(a)$ for odd N, and $\mathrm{Var}_{\mu_N}(b)$ for even N. Now, for all $N \geq 1$, the bilinearity of the covariance yields

$$(3.4.3)\qquad \mathrm{Var}_{\mu_N}(a-b) = \mathrm{Var}_{\mu_N}(a) - 2\,\mathrm{Cov}_{\mu_N}(a,b) + \mathrm{Var}_{\mu_N}(b).$$

Our Lemma 2.1.7 allows us to substitute $\mathrm{Cov}_{\mu_N}(a,b)$ as follows. For N odd we have

$$(3.4.4)\qquad \mathrm{Cov}_{\mu_N}(a,b) = \mathrm{Var}_{\mu_N}(a) - \mathrm{Var}_{\mu_N}(a-b),$$

and for N even

$$(3.4.5)\qquad \mathrm{Cov}_{\mu_N}(a,b) = \mathrm{Var}_{\mu_N}(b) - \mathrm{Var}_{\mu_N}(a-b).$$

The equations (3.4.3)–(3.4.5) and our previous computations yield

$$\begin{aligned}\mathrm{Var}_{\mu_N}(a) &= \mathrm{Var}_{\mu_N}(b) + \mathrm{Var}_{\mu_N}(a-b)\\ &= \frac{N(N-1)(2N+5)}{96} + \mathrm{Var}_{\mu_N}(a-b)\end{aligned}$$

for N odd, and likewise

$$\begin{aligned}\mathrm{Var}_{\mu_N}(b) &= \mathrm{Var}_{\mu_N}(a) + \mathrm{Var}_{\mu_N}(a-b)\\ &= \frac{N(N-1)(2N+5)}{96} + \mathrm{Var}_{\mu_N}(a-b)\end{aligned}$$

for N even. Note that $\mathrm{Var}_{\mu_N}(a-b) = \frac{N}{4}$ for all $N \geq 1$ by Lemma 2.1.8 which finishes the proof.

Note 3.4.4. The single summands in (3.4.1) and (3.4.2) are quartics but their highest order terms cancel, leading to a cubic variance. This fact will allow us to prove the weak law of large numbers in §4.3.

We extend Theorem 3.4.3 to Demazure modules indexed by elements in the extended affine Weyl group $\widetilde{W}^{\mathrm{aff}}$ as we have done it for the expected degree in Theorem 2.2.2. Recall from §1.3 that σ denotes the non-trivial automorphism of the Dynkin diagram of $\widehat{\mathfrak{sl}}_2$, and note that σ induces a bijection $\sigma : \Gamma_0 \to \Gamma_1$ of the weight lattices of $V(\Lambda_0)$ and $V(\Lambda_1)$ via $\sigma(\Lambda_0 + x\alpha_0 + y\alpha_1) = \Lambda_1 + y\alpha_0 + x\alpha_1$. The weight distribution $\mu_{(\sigma s_0)^N}$ of $V_{(\sigma s_0)^N}(\Lambda_0)$ is supported on Γ_0 if N is even and on $\sigma(\Gamma_0) = \Gamma_1$ if N is odd. The bijection

σ gives $\sigma_*\mu_{w_N} = \mu_{\sigma w_N}$ and $\sigma_*^{-1}\mu_{\sigma w_N} = \mu_{w_N}$. Recall the definition of the scaling element d from §1.3, and note that $\langle -d, \lambda\rangle$ is the $(-\alpha_0)$-coefficient of λ for $\lambda \in \Gamma_0$ *and* $\lambda \in \Gamma_1$.

Theorem 3.4.5. *Let $\mu_{(\sigma s_0)^N}$ be the weight distribution of the Demazure module $V_{(\sigma s_0)^N}(\Lambda_0)$. Consider $-d \in \mathfrak{h}$ as a function on $\mathfrak{h}^*$. Then,*

$$\mathrm{Var}_{\mu_{(\sigma s_0)^N}}(-d) = \frac{N(N-1)(2N+5)}{96}.$$

Proof. Note that $V_{(\sigma s_0)^N}(\Lambda_0) \subset V(\Lambda_{\pi_N})$. Hence its weight distribution $\mu_{(\sigma s_0)^N} \in \mathrm{Meas_c}(\Gamma_{\pi_N})$ is supported on Γ_{π_N}. Denote by a_0, b_0 and a_1, b_1 the coefficients of $-\alpha_0$ and $-\alpha_1$ in Γ_0 and Γ_1 respectively. Then,

$$\begin{aligned}
\mathrm{Var}_{\mu_{(\sigma s_0)^N}}(-d) &= \mathrm{Var}_{\mu_{(\sigma s_0)^N}}(a_{\pi_N}) \\
&= \mathrm{Var}_{\mu_{\sigma^{(\pi_N)} w_N}}(a_{\pi_N}) \\
&= \mathrm{Var}_{\sigma_*^{\pi_N}\mu_{w_N}}(a_{\pi_N}) \\
&= \mathrm{Var}_{\mu_N}(a_{\pi_N} \circ \sigma^{\pi_N}) \\
&= \begin{cases} \mathrm{Var}_{\mu_N}(a_0) & \text{if } N \text{ is even,} \\ \mathrm{Var}_{\mu_N}(b_0) & \text{if } N \text{ is odd,} \end{cases} \\
&= \frac{N(N-1)(2N+5)}{96}
\end{aligned}$$

by Theorem 3.4.3. □

Based on Theorem 3.4.3 and 3.4.5 we can derive formulas for the Demazure module $V_{w_{N,1}}(\Lambda_1)$, finishing the computation of the variance of the degree distribution for level one Demazure modules.

Corollary 3.4.6. *Let $\mu_{N,1}$ be the weight distribution of the Demazure module $V_{w_{N,1}}(\Lambda_1)$ and denote the parity of N by π_N. Then, for $N \geq 1$ we have*

$$\begin{aligned}
\mathrm{Var}_{\mu_{N,1}}(a) &= \frac{N(N-1)(2N+5)}{96} + \pi_{N+1} \cdot \frac{N}{4}, \\
\mathrm{Var}_{\mu_{N,1}}(b) &= \frac{N(N-1)(2N+5)}{96} + \pi_N \cdot \frac{N}{4}.
\end{aligned}$$

Proof. Let us only treat $\mathrm{Var}_{\mu_{N,1}}(a)$, as the argumentation for the coordinate b is completely analogous. If N is odd, $V_{w_{N,1}}(\Lambda_1) = V_{(\sigma s_0)^N}(\Lambda_0)$, i.e., $\mathrm{Var}_{\mu_{N,1}}(a) = \mathrm{Var}_{\mu_{(\sigma s_0)^N}}(-d)$ and Theorem 3.4.5 applies. If N is even, we write $V_{w_{N,1}}(\Lambda_1) = V_{w_{N,1}\sigma\sigma}(\Lambda_1) = V_{\sigma w_{N,0}}(\Lambda_0)$ and hence $\mathrm{Var}_{\mu_{N,1}}(a) = \mathrm{Var}_{\mu_{N,0}}(b)$ since the automorphism σ interchanges the coordinates referring to the $(-\alpha_0)$- and $(-\alpha_1)$-coefficient, respectively. Therefore, Theorem 3.4.3 finishes the proof. □

Again, we can extend Corollary 3.4.6 to Demazure modules indexed by elements in the extended affine Weyl group $\widetilde{W}^{\mathrm{aff}}$.

Corollary 3.4.7. *Let $\mu_{(\sigma s_1)^N}$ be the weight distribution of the Demazure module $V_{(\sigma s_1)^N}(\Lambda_1)$. Consider $-d \in \mathfrak{h}$ as a function on $\mathfrak{h}^*$. Then,*

$$\mathrm{Var}_{\mu_{(\sigma s_1)^N}}(-d) = \frac{N(N-1)(2N+5)}{96} + \frac{N}{4}.$$

Note 3.4.8 (Cf. Note 2.2.3)**.** In the extended affine Weyl group case the function $-d$ on $\mathfrak{h}^*$ always gives the coefficient of $-\alpha_0$. If one regards the automorphism σ as a function on $\mathfrak{h}^*$ (by exchanging Λ_0 and Λ_1, and acting trivially on δ) one can rephrase Theorem 3.4.5 and Corollary 3.4.7 easily in terms of the coordinate $-d \circ \sigma$ which refers to the $(-\alpha_1)$-coefficient. Then, as expected, the values in the two statements simply interchange.

3.5. **Computing the variance by quantum calculus.** Sanderson [31] for type A and Ion [15] in the general case showed that characters of level 1 Demazure modules are related to Macdonald polynomials [25]. The latter are related to Rogers–Szegő polynomials, i.e., generating functions of Gaussian polynomials (see e.g. [13] for the connection). The knowledge of total mass, expected value, and covariance of a weight distribution is equivalent to the knowledge of the Taylor expansion of the corresponding character up to order 2. Hence, by studying derivatives of Gaussian polynomials, we have a second way of calculating the covariance matrix of Demazure modules in level 1.

Recall the definition of the Gaussian polynomials from §1.3.

Lemma 3.5.1. *Let $N \in \mathbf{N}$ and $k \in \{0, \dots, N\}$. The Taylor expansion of $\begin{bmatrix} N \\ k \end{bmatrix}_q$ at $q = 1$ is*

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = \binom{N}{k} \cdot \Bigg(1 + \frac{k(N-k)}{2}(q-1) + \frac{k(N-k)(3Nk-3k^2+N-5)}{24}(q-1)^2 + \cdots\Bigg).$$

Proof. Let $\tilde{q} = q - 1$ be the local coordinate at 1. For rational functions $A, B \in \mathbf{C}(q)$ we write $A \sim B$ if A and B have the same pole order at 1, and the first 3 nonzero coefficients in their Laurent expansion at 1 coincide. Then

$$\begin{aligned}(q;q)_k &= (1+\tilde{q}; 1+\tilde{q})_k \\ &= \prod_{i=1}^{k} \left(1 - (1+\tilde{q})^i\right) \\ &\sim \prod_{i=1}^{k} \left(1 - \left(1 + i\tilde{q} + \binom{i}{2}\tilde{q}^2 + \binom{i}{3}\tilde{q}^3\right)\right) \\ &= \prod_{i=1}^{k} \left(-i\tilde{q} - \frac{i(i-1)}{2}\tilde{q}^2 - \frac{i(i-1)(i-2)}{6}\tilde{q}^3\right) \\ &= (-1)^k k! \tilde{q}^k \prod_{i=1}^{k} \left(1 + \frac{i-1}{2}\tilde{q} + \frac{(i-1)(i-2)}{6}\tilde{q}^2\right)\end{aligned}$$

$$\sim (-1)^k k! \tilde{q}^k \left(1 + p_1(k)\tilde{q} + p_2(k)\tilde{q}^2\right),$$

where

$$p_1(k) = \sum_{i=1}^{k} \frac{i-1}{2} = \frac{k(k-1)}{4}$$

and

$$\begin{aligned} p_2(k) &= \sum_{i=1}^{k} \frac{(i-1)(i-2)}{6} + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \frac{(i-1)(j-1)}{4} \\ &= \frac{k(k-1)(k-2)(9k+13)}{288}. \end{aligned}$$

Hence

$$\begin{aligned} \begin{bmatrix} N \\ k \end{bmatrix}_q &= \frac{(q;q)_N}{(q;q)_k (q;q)_{N-k}} \\ &\sim \binom{N}{k} \frac{1 + p_1(N)\tilde{q} + p_2(N)\tilde{q}^2}{(1 + p_1(k)\tilde{q} + p_2(k)\tilde{q}^2)(1 + p_1(N-k)\tilde{q} + p_2(N-k)\tilde{q}^2)} \\ &\sim \binom{N}{k} \cdot \left(1 + p_1(N)\tilde{q} + p_2(N)\tilde{q}^2\right) \\ &\qquad \cdot \left(1 - p_1(k)\tilde{q} + \left(p_1(k)^2 - p_2(k)\right)\tilde{q}^2\right) \\ &\qquad \cdot \left(1 - p_1(N-k)\tilde{q} + \left(p_1(N-k)^2 - p_2(N-k)\right)\tilde{q}^2\right). \end{aligned}$$

We have

$$p_1(k)^2 - p_2(k) = \frac{k(k-1)(9k^2 - 13k + 26)}{288}.$$

Expanding the product we obtain

$$\begin{bmatrix} N \\ k \end{bmatrix}_q \sim \binom{N}{k} \cdot \left(1 + \frac{k(N-k)}{2}\tilde{q} + \frac{k(N-k)(3Nk - 3k^2 + N - 5)}{24}\tilde{q}^2\right). \quad \square$$

For $N \in \mathbf{N}$, the N-th *Rogers–Szegő polynomial* $H_N(z,q) \in \mathbf{C}[z,q]$ is defined by

$$H_N(z,q) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q z^k.$$

Proposition 3.5.2. *Let $N \in \mathbf{N}$. The Taylor expansion of $H_N(z,q)$ at $z = q = 1$ is*

$$\begin{aligned} \frac{H_N(z,q)}{2^N} = 1 &+ \frac{N}{2}(z-1) + \frac{N(N-1)}{8}(q-1) + \frac{N(N-1)}{8}(z-1)^2 \\ &+ \frac{N^2(N-1)}{16}(z-1)(q-1) \\ &+ \frac{N(N-1)(N-2)(3N+7)}{384}(q-1)^2 + \cdots. \end{aligned}$$

Proof. As $H_N(1,q) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q$, the value and the partial derivatives $\frac{\partial}{\partial q}$ and $\frac{\partial^2}{\partial q^2}$ can be computed by summation from Lemma 3.5.1. The partial derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial^2}{\partial z^2}$ can be computed from the fact that $H_N(z,1) = (1+z)^N$. For the mixed partial derivative, let $\tilde{q} = q-1$ and $\tilde{z} = z-1$ be the local coordinates at $(1,1)$, and note that

$$\begin{aligned} H_N(z,q) &= \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q (1+\tilde{z})^k \\ &= \sum_{k=0}^{N} \binom{N}{k} \cdot \left(1 + \frac{k(N-k)}{2}\tilde{q} + \cdots\right) \cdot (1 + k\tilde{z} + \cdots), \end{aligned}$$

the $\tilde{q}\tilde{z}$-coefficient of which is

$$\sum_{k=0}^{N} \binom{N}{k} \frac{k^2(N-k)}{2} = 2^N \cdot \frac{N^2(N-1)}{16}. \qquad \square$$

Recall the definition of the specialized bivariate Macdonald polynomials $P_\lambda(z, z^{-1}; q, 0)$ from §1.3.

Proposition 3.5.3. *Let $\lambda = (\lambda_1, \lambda_2)$ be a partition and $N = \lambda_1 - \lambda_2$. Then the Taylor expansion of $P_\lambda(z, z^{-1}; q, 0)$ at $z = q = 1$ is*

$$\begin{aligned} \frac{P_\lambda(z, z^{-1}; q, 0)}{2^N} &= 1 + \frac{N(N-1)}{8}(q-1) + \frac{N}{2}(z-1)^2 \\ &\quad + \frac{N(N-1)(N-2)(3N+7)}{384}(q-1)^2 + \cdots. \end{aligned}$$

Proof. If $\lambda = (\lambda_1, \lambda_2)$ is a partition and $N = \lambda_1 - \lambda_2$, then

$$P_\lambda(z, z^{-1}; q, 0) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q z^{2k-N} = H_N(z^2, q) z^{-N} \tag{3.5.1}$$

by [13, (3.4)].[3] The result follows from Proposition 3.5.2 by change of variables. $\square$

Theorem 3.5.4. *Let $N \in \mathbf{N}$, and let $\mu = \mu_{(\sigma s_0)^N}$ be the weight distribution of $V_{(\sigma s_0)^N}(\Lambda_0)$. Consider $-d, \alpha_1^\vee \in \mathfrak{h}$ as functions on $\mathfrak{h}^*$. Then,*

$$\begin{aligned} \mu(\mathfrak{h}^*) &= 2^N, \\ \mathrm{E}_\mu[-d] &= \frac{N^2 + N - 2\pi_N}{8}, \\ \mathrm{E}_\mu[\alpha_1^\vee] &= 0, \\ \mathrm{Var}_\mu(-d) &= \frac{N(N-1)(2N+5)}{96}, \end{aligned}$$

[3]Hikami considers homogeneous Rogers–Szegő and Macdonald polynomials. Our statement follows by evaluating at (z, z^{-1}).

$$\mathrm{Var}_\mu(\alpha_1^\vee) = N,$$
$$\mathrm{Cov}_\mu(-d, \alpha_1^\vee) = 0.$$

Proof. By [31, Theorem 6 and Theorem 7] we have that

$$\begin{aligned} P_\lambda(e^{\frac{1}{2}\alpha_1}, e^{-\frac{1}{2}\alpha_1}; e^\delta, 0) &= e^{\lfloor \frac{1}{4}N^2 \rfloor \delta - \Lambda_0} \cdot \mathrm{ch}_{V_{w_{N,\pi_N}}(\Lambda_{\pi_N})} \\ &= e^{\lfloor \frac{1}{4}N^2 \rfloor \delta - \Lambda_0} \cdot \mathrm{ch}_{V_{(\sigma s_0)^N}(\Lambda_0)} \end{aligned} \tag{3.5.2}$$

for all $N \geq 0$.[4]

If μ is a measure on $\mathbf{Z}^2$, we denote its generating function by

$$f_\mu(z,q) = \frac{1}{\mu(\mathbf{Z}^2)} \sum_{k,l \in \mathbf{Z}} \mu(\{(k,l)\}) \cdot z^k q^l = \mathrm{E}_\mu[z^X q^Y],$$

where X and Y are the canonical coordinates on $\mathbf{Z}^2$. Then $f_\mu(1,1) = 1$ and it is immediately verified that

$$\left. \frac{\partial}{\partial z} f_\mu(z,q) \right|_{z=q=1} = \mathrm{E}_\mu[X], \tag{3.5.3}$$

$$\left. \frac{\partial^2}{\partial z^2} f_\mu(z,q) \right|_{z=q=1} = \mathrm{Var}_\mu(X) + \mathrm{E}_\mu[X](\mathrm{E}_\mu[X] - 1) \tag{3.5.4}$$

and analogously for $\frac{\partial}{\partial q}$, and

$$\left. \frac{\partial^2}{\partial z \partial q} f_\mu(z,q) \right|_{z=q=1} = \mathrm{Cov}_\mu(X,Y) + \mathrm{E}_\mu[X]\,\mathrm{E}_\mu[Y]. \tag{3.5.5}$$

Note that generally

$$\left. \frac{\partial}{\partial q} f(q^{-1}) \right|_{q=1} = -f'(1) \quad \text{and} \quad \left. \frac{\partial^2}{\partial q^2} f(q^{-1}) \right|_{q=1} = f''(1) + 2f'(1).$$

Then (3.5.2) in our new notation states that for $\mu = \mu_{(\sigma s_0)^N}$ we have

$$f_{(\alpha_1^\vee, -d)_* \mu}(z,q) = \frac{q^{\lfloor \frac{1}{4}N^2 \rfloor} P_\lambda(z, z^{-1}; q^{-1}, 0)}{2^N}.$$

Hence it follows from Proposition 3.5.3 that

$$\begin{aligned} \mathrm{E}_\mu[\alpha_1^\vee] &= \left. \frac{\partial}{\partial z} \frac{q^{\lfloor \frac{1}{4}N^2 \rfloor} P_\lambda(z, z^{-1}; q^{-1}, 0)}{2^N} \right|_{z=q=1} \\ &= 0, \end{aligned}$$

[4]There seems to be a missprint in [31]. Namely, the specialization should be at $q = e^\delta$, not $q = e^{-\delta}$. The factor $e^{-\Lambda_0}$ does not occur in her paper, as she implicitly restricts characters to $\mathbf{C}\alpha_1^\vee \oplus \mathbf{C}d \subset \mathfrak{h}$, and $e^{-\Lambda_0} = 1$ on $\mathbf{C}\alpha_1^\vee \oplus \mathbf{C}d$.

$$\begin{aligned}
\mathrm{E}_\mu[-d] &= \frac{\partial}{\partial q}\frac{q^{\lfloor \frac{1}{4}N^2\rfloor}P_\lambda(z,z^{-1};q^{-1},0)}{2^N}\bigg|_{z=q=1} \\
&= \left\lfloor \frac{N^2}{4}\right\rfloor - \frac{N(N-1)}{8} \\
&= \frac{N^2+N-2\pi_N}{8}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathrm{Var}_\mu(\alpha_1^\vee) &= \mathrm{Var}_\mu(\alpha_1^\vee) + \mathrm{E}_\mu[\alpha_1^\vee](\mathrm{E}_\mu[\alpha_1^\vee]-1) \\
&= \frac{\partial^2}{\partial z^2}\frac{q^{\lfloor \frac{1}{4}N^2\rfloor}P_\lambda(z,z^{-1};q^{-1},0)}{2^N}\bigg|_{z=q=1} \\
&= N.
\end{aligned}$$

From

$$\begin{aligned}
&\mathrm{Var}_\mu(-d) + \mathrm{E}_\mu[-d](\mathrm{E}_\mu[-d]-1) \\
&\quad = \frac{\partial^2}{\partial q^2}\frac{q^{\lfloor \frac{1}{4}N^2\rfloor}P_\lambda(z,z^{-1};q^{-1},0)}{2^N}\bigg|_{z=q=1} \\
&\quad = \left\lfloor \frac{N^2}{4}\right\rfloor\left(\left\lfloor \frac{N^2}{4}\right\rfloor - 1\right) - 2\left\lfloor \frac{N^2}{4}\right\rfloor\frac{N(N-1)}{8} \\
&\qquad + \frac{N(N-1)(N-2)(3N+7)}{192} + 2\frac{N(N-1)}{8}
\end{aligned}$$

it follows that

$$\mathrm{Var}_\mu(-d) = \frac{N(N-1)(2N+5)}{96}.$$

Finally

$$\begin{aligned}
\mathrm{Cov}_\mu(d,\alpha_1^\vee) &= \mathrm{Cov}_\mu(-d,\alpha_1^\vee) + \mathrm{E}_\mu[-d]\,\mathrm{E}_\mu[\alpha_1^\vee] \\
&= \frac{\partial^2}{\partial z\partial q}\frac{q^{\lfloor \frac{1}{4}N^2\rfloor}P_\lambda(z,z^{-1};q^{-1},0)}{2^N}\bigg|_{z=q=1} \\
&= 0.
\end{aligned}$$

□

Similarly one proves:

Theorem 3.5.5. *Let $N \in \mathbf{N}$, and let $\mu = \mu_{(\sigma s_1)^N}$ be the weight distribution of $V_{(\sigma s_1)^N}(\Lambda_1)$. Consider $-d, \alpha_1^\vee \in \mathfrak{h}$ as functions on $\mathfrak{h}^*$. Then,*

$$\begin{aligned}
\mu(\mathfrak{h}^*) &= 2^N, \\
\mathrm{E}_\mu[-d] &= \frac{N^2+N+2\pi_N}{8}, \\
\mathrm{E}_\mu[\alpha_1^\vee] &= 0, \\
\mathrm{Var}_\mu(-d) &= \frac{N(N-1)(2N+5)}{96} + \frac{N}{4},
\end{aligned}$$

$$\begin{aligned} \mathrm{Var}_\mu(\alpha_1^\vee) &= N, \\ \mathrm{Cov}_\mu(-d, \alpha_1^\vee) &= 0. \end{aligned}$$

Note that while we find it convenient to work with [31], the connection between Demazure characters and Gaussian polynomials was earlier described by Kuniba et al. [22].

4. Implications and conclusions

4.1. **Basic specialization of Demazure characters.** By the **basic specialization** of a function on $\mathfrak{h}$, we mean its restriction to the space $\mathbf{C}d$ generated by the scaling element d. Consider the coordinate q on $\mathbf{C}d$ which is the restriction of $e^{-\alpha_0}$ (or equivalently of $e^{-\delta}$ with $\delta = \alpha_0 + \alpha_1$) to $\mathbf{C}d$. Then, for a Demazure module $V_w(\Lambda)$ the basic specialization of its character

$$\mathrm{ch}_{V_w(\Lambda)} = \sum_{\lambda \in \mathfrak{h}^*} \dim(V_w(\Lambda)_\lambda) \cdot e^{\lambda}$$

is

$$f_w = \sum_{\lambda \in \mathfrak{h}^*} \dim(V_w(\Lambda)_\lambda) \cdot q^{a(\lambda)}.$$

In the following we consider $f_w = f_w(q)$ as a polynomial in q, so $f_w(1)$ means the evaluation at $q = 1$, i.e., at $0 \in \mathfrak{h}$. Similarly, $f'(q) = \frac{d}{dq} f(q)$. (Cf. [16, §§1.5, 10.8, and 12.2].)

Corollary 4.1.1. *Let $\Lambda = m\Lambda_0 + n\Lambda_1$ be a dominant integral weight and $j \in \{0, 1\}$. Let $f_{N,j}(q) \in \mathbf{C}[q]$ be the basic specialization of $\mathrm{ch}_{V_{w_{N,j}}}(\Lambda)$. Then*

$$f_{N,j}(1) = \begin{cases} (m+1)(m+n+1)^{N-1} & \text{if } j = 0, \\ (n+1)(m+n+1)^{N-1} & \text{if } j = 1. \end{cases} \tag{4.1.1}$$

Furthermore,

(4.1.2)

$$f'_{N,0}(1) = \Bigg(\frac{2(N-1)m(m+2) + (N-1)(N-2)(m+n)(m+n+2)}{12(m+n+1)} + \left\lfloor \frac{N-1}{2} \right\rfloor \frac{m+n}{2} + \frac{m}{2} \Bigg) \cdot (m+1)(m+n+1)^{N-1},$$

and

(4.1.3)

$$f'_{N,1}(1) = \Bigg(\frac{2(N-1)n(n+2) + (N-1)(N-2)(m+n)(m+n+2)}{12(m+n+1)} + \left\lfloor \frac{N}{2} \right\rfloor \frac{m+n}{2} \Bigg) \cdot (n+1)(m+n+1)^{N-1}.$$

Proof. Equation (4.1.1) is clear since $f_{N,j}(1) = \dim(V_{w_{N,j}}(\Lambda))$ and these dimensions are determined in Corollary 2.1.4 (Cf. [29, Theorem 1]).

If μ is a measure on $\mathbf{Z}$, we denote its generating function by

$$f_\mu(q) = \frac{1}{\mu(\mathbf{Z})} \sum_{k \in \mathbf{Z}} \mu(\{k\}) \cdot q^k = \mathrm{E}_\mu[q^X],$$

where $X = \mathrm{id}_{\mathbf{Z}}$ is the canonical coordinate on $\mathbf{Z}$. Then $f_\mu(1) = 1$ and as in (3.5.3) it is immediately verified that

$$f'_\mu(1) = \mathrm{E}_\mu[X]. \tag{4.1.4}$$

Hence

$$\begin{aligned} f'_{N,j}(1) &= f_{N,j}(1) \cdot f'_{a_*\mu_{N,j}}(1) \\ &= \dim(V_{w_{N,j}}(\Lambda)) \cdot \mathrm{E}_{\mu_{N,j}}[a] \end{aligned}$$

which verifies (4.1.2) and (4.1.3) through Theorem 2.2.1. □

For level 1 Demazure modules we have:

Corollary 4.1.2. *Let* $f_N(q) \in \mathbf{C}[q]$ *be the basic specialization of* $\mathrm{ch}_{V_{w_{N,0}}}(\Lambda_0)$. *Then*

$$f_N(1) = 2^N. \tag{4.1.5}$$

If N is even, then

$$f'_N(1) = \frac{2^N N(N+1)}{8}, \tag{4.1.6}$$

$$f''_N(1) = \frac{2^N N(N-2)(3N^2+16N+17)}{192}. \tag{4.1.7}$$

If N is odd, then

$$f'_N(1) = \frac{2^N (N^2+N+2)}{8}, \tag{4.1.8}$$

$$f''_N(1) = \frac{2^N (N-1)(3N^3+13N^2+10N+36)}{192}. \tag{4.1.9}$$

Proof. Equation (4.1.5) is clear since $f_N(1) = \dim(V_{w_{N,0}}(\Lambda_0)) = 2^N$.

Again, we denote the generating function of a measure μ on $\mathbf{Z}$ by

$$f_\mu(q) = \frac{1}{\mu(\mathbf{Z})} \sum_{k\in\mathbf{Z}} \mu(\{k\}) \cdot q^k = \mathrm{E}_\mu[q^X],$$

where $X = \mathrm{id}_{\mathbf{Z}}$ is the canonical coordinate on $\mathbf{Z}$. By definition, $f_\mu(1) = 1$ and as in (3.5.3), (3.5.5) and (4.1.4) it is immediately verified that

$$f'_\mu(1) = \mathrm{E}_\mu[X], \tag{4.1.10}$$

$$f''_\mu(1) = \mathrm{Var}_\mu(X) + \mathrm{E}_\mu[X](\mathrm{E}_\mu[X]-1). \tag{4.1.11}$$

Suppose that N is even. Then $\mathrm{E}_{\mu_{N,0}}[a] = \frac{1}{8}N(N+1)$ by Theorem 2.2.1 and $\mathrm{Var}_{\mu_{N,0}}(a) = \frac{1}{96}N(N-1)(2N+5)$ by Theorem 3.4.3. Hence

$$\begin{aligned} f'_N(1) &= 2^N f'_{a_*\mu_{N,0}}(1) \\ &= 2^N\, \mathrm{E}_{\mu_{N,0}}[a] \\ &= \frac{2^N N(N+1)}{8} \end{aligned}$$

and

$$f''_N(1) = 2^N f''_{a_*\mu_{N,0}}(1)$$

$$= 2^N\big(\mathrm{Var}_{\mu_{N,0}}(a) + \mathrm{E}_{\mu_{N,0}}[a](\mathrm{E}_{\mu_{N,0}}[a]-1)\big)$$
$$= \frac{2^N N(N-2)(3N^2+16N+17)}{192},$$

so we have verified (4.1.6) and (4.1.7).

The verification of (4.1.8) and (4.1.9) for odd N is similar. For this case we use $\mathrm{E}_{\mu_{N,0}}[a] = \frac{1}{8}(N^2+N+2)$ by Theorem 2.2.1 and $\mathrm{Var}_{\mu_{N,0}}(a) = \frac{1}{96}N(N-1)(2N+5) + \frac{1}{4}N$ by Theorem 4.2.1. □

In the same fashion one proves:

Corollary 4.1.3. *Let $f_N(q) \in \mathbf{C}[q]$ be the basic specialization of $\mathrm{ch}_{V_{w_{N,1}}}(\Lambda_1)$. Then*

$$f_N(1) = 2^N. \tag{4.1.12}$$

If N is even, then

$$f_N'(1) = \frac{2^N N(N+1)}{8}, \tag{4.1.13}$$

$$f_N''(1) = \frac{2^N N(3N^3+10N^2-15N+14)}{192}. \tag{4.1.14}$$

If N is odd, then

$$f_N'(1) = \frac{2^N (N^2+N-2)}{8}, \tag{4.1.15}$$

$$f_N''(1) = \frac{2^N (N-1)(3N^3+13N^2-14N-60)}{192}. \tag{4.1.16}$$

4.2. **Covariance of the weight distribution.** For a distribution $\mu \in \mathrm{Meas_c}(\mathbf{Z}^2)$ and coordinates $X, Y : \mathbf{Z}^2 \to \mathbf{Z}$ we define the **covariance matrix** of X and Y with respect to μ to be the 2×2 matrix

$$\begin{pmatrix} \mathrm{Cov}_\mu(X,X) & \mathrm{Cov}_\mu(X,Y) \\ \mathrm{Cov}_\mu(Y,X) & \mathrm{Cov}_\mu(Y,Y) \end{pmatrix}.$$

Note that the covariance matrix is symmetric.

Theorem 4.2.1 (Covariance of the weight distribution)**.** *Let $\mu_{(\sigma s_0)^N}$ and $\mu_{(\sigma s_1)^N}$ be the weight distributions of the Demazure modules $V_{(\sigma s_0)^N}(\Lambda_0)$ and $V_{(\sigma s_1)^N}(\Lambda_1)$, respectively. Consider $-d, \alpha_1^\vee \in \mathfrak{h}$ as functions on $\mathfrak{h}^*$. The covariance matrix of $-d$ and $\alpha_1^\vee$ with respect to $\mu_{(\sigma s_0)^N}$ is given by*

$$\begin{pmatrix} \frac{N(N-1)(2N+5)}{96} & 0 \\ 0 & N \end{pmatrix},$$

and the covariance matrix of $-d$ and $\alpha_1^\vee$ with respect to $\mu_{(\sigma s_1)^N}$ is

$$\begin{pmatrix} \frac{N(N-1)(2N+5)}{96} + \frac{N}{4} & 0 \\ 0 & N \end{pmatrix}.$$

Proof. We just have to recollect the facts we have proven so far. Theorem 3.5.4 gives a complete overview of the data one needs in the case of $V_{(\sigma s_0)^N}(\Lambda_0)$. For $V_{(\sigma s_1)^N}(\Lambda_1)$ one has to make use of its analog, Theorem 3.5.5. □

For visualization purposes, it is convenient to represent the covariance matrix by the associated **covariance ellipse**, defined as follows: Let μ be a measure on $\mathbf{R}^2$ with nondegenerate covariance matrix C. Then the covariance ellipse of μ is

$$S_\mu = \{x \in \mathbf{R}^2 : x^t C^{-1} x = 1\}.$$

Some statistical quantities can be easily read off S_μ. For example, if $\varphi : \mathbf{R}^2 \to \mathbf{R}$ is a linear form, then the standard deviation of φ with respect to μ is

$$\mathrm{Var}_\mu(\varphi)^{1/2} = \max_{x \in S_\mu} \varphi(x).$$

In particular, the level line $\varphi^{-1}(\{1\})$ is tangential to S_μ if, and only if, $\mathrm{Var}_\mu(\varphi) = 1$. Similarly, let $\varphi, \psi : \mathbf{R}^2 \to \mathbf{R}$ be nonzero linear forms. Let d_φ be the diameter of S_μ with endpoints

$$\arg\max_{x \in S_\mu} \varphi(x) \quad \text{and} \quad \arg\min_{x \in S_\mu} \varphi(x),$$

and analogously for ψ. Then $\mathrm{Cov}_\mu(\varphi, \psi) = 0$ if, and only if, d_φ and d_ψ are conjugate diameters of S_μ. In Figure 4, the covariance ellipses have been translated to be centered at the expected weight.

For example, for the Demazure module $V_{w_N}(\Lambda_0)$ when N is even, Theorem 4.2.1 gives that the a-diameter of the covariance ellipse (i.e., the height of the ellipses in Figure 4) is

$$2\sqrt{\mathrm{Var}_{\mu_N}(a)} = 2\sqrt{\frac{N(N-1)(2N+5)}{96}}.$$

When we divide this diameter by the maximal value of a as computed in Lemma 2.3.1, we obtain that the relative height of the covariance ellipse is

$$\sqrt{\frac{(N-1)(2N+5)}{6N^3}}.$$

This converges to 0 as $N \to \infty$, illustrating our weak law of large numbers (Theorem 4.3.3).

4.3. **Law of large numbers.** Recall the following useful basic lemma in probability theory (see e.g. [4, (5.32)]).

Lemma 4.3.1 (Chebyshev's inequality)**.** *Let P be a probability distribution on $\mathbf{R}$ with finite expected value and variance. Then, for any $k > 0$*

$$P(\mathbf{R} \setminus (\mathrm{E}[P] - k\sqrt{\mathrm{Var}(P)}, \mathrm{E}[P] + k\sqrt{\mathrm{Var}(P)})) \leq \frac{1}{k^2}.$$

We write $\nu_N \xrightarrow{\text{w}} \nu$ if the sequence of measures ν_N converges weakly to ν as $N \to \infty$. We will use the following abstract version of the weak law of large numbers.

Lemma 4.3.2 (Weak law of large numbers)**.** *Let P_N be a sequence of probability distributions on $\mathbf{R}$ such that*

$$\mathrm{E}[P_N] \to c \in \mathbf{R} \quad \textit{and} \quad \mathrm{Var}(P_N) \to 0.$$

Then,

$$P_N \xrightarrow{\text{w}} \delta_c.$$

This version of the weak law of large numbers can be derived from Chebychev's inequality in the usual way:

Proof. We need to prove for any $\varepsilon > 0$ that $P_N([c-\varepsilon, c+\varepsilon]) \to 1$ as $N \to \infty$. Choose $N' \in \mathbf{N}$ such that $|c - \mathrm{E}[P_N]| < \frac{\varepsilon}{2}$ for all $N \geq N'$. Then, for such N

$$\begin{aligned} P_N([c-\varepsilon, c+\varepsilon]) &= 1 - P_N(\mathbf{R} \setminus (c-\varepsilon, c+\varepsilon)) \\ &\geq 1 - P_N(\mathbf{R} \setminus (\mathrm{E}[P_N] - \frac{\varepsilon}{2}, \mathrm{E}[P_N] + \frac{\varepsilon}{2})) \\ &\geq 1 - \frac{4\,\mathrm{Var}(P_N)}{\varepsilon^2} \qquad \longrightarrow 1 \quad \text{as } N \to \infty. \end{aligned}$$

The last inequality follows from Lemma 4.3.1 with $k = \frac{\varepsilon}{2\sqrt{\mathrm{Var}(P_N)}}$. This proves the claim since by definition of a probability measure $1 \geq P_N([c - \varepsilon, c+\varepsilon])$. $\square$

If μ is a nonzero measure, denote by $\bar{\mu}$ the probability measure on $\mathbf{R}$ obtained by normalizing μ with its total mass. For $c \in \mathbf{R}$, let $D_c : \mathbf{R} \to \mathbf{R}$ denote the corresponding dilation operator, given by $D_c(x) = cx$. Similarly, for $c \in \mathbf{R}^2$, let $D_c : \mathbf{R}^2 \to \mathbf{R}^2$ denote the operator given by $D_c(x) = (c_1x_1, c_2x_2)$.

Theorem 4.3.3 (Weak law of large numbers)**.** *Let $\mu_{(\sigma s_0)^N}$ and $\mu_{(\sigma s_1)^N}$ be the weight distributions of the Demazure modules $V_{(\sigma s_0)^N}(\Lambda_0)$ and $V_{(\sigma s_1)^N}(\Lambda_1)$, respectively, and let $\bar{\mu}_{(\sigma s_0)^N}, \bar{\mu}_{(\sigma s_1)^N}$ be their normalizations. Consider $-d, \alpha_1^\vee \in \mathfrak{h}$ as functions on $\mathfrak{h}^*$. Then, for $j \in \{0,1\}$ we have*

$$\left(D_{(\lfloor N^2/4\rfloor + j\lceil N/2\rceil)^{-1}}\right)_* (-d)_* \bar{\mu}_{(\sigma s_j)^N} \xrightarrow{\text{w}} \delta_{\frac{1}{2}},$$

and consequently

$$\left(D_{(N^{-1}, (\lfloor N^2/4\rfloor + j\lceil N/2\rceil)^{-1})}\right)_* (\alpha_1^\vee, -d)_* \bar{\mu}_{(\sigma s_j)^N} \xrightarrow{\text{w}} \delta_{(0,\frac{1}{2})}.$$

See Figure 5 for an illustration.

Proof. Because of similarity let us only treat the weight distribution of $V_{(\sigma s_0)^N}(\Lambda_0)$. For the first assertion denote $P_N = \left(D_{\lfloor N^2/4\rfloor^{-1}}\right)_* (-d)_* \bar{\mu}_{(\sigma s_0)^N}$. By Theorem 2.2.2 and Theorem 3.4.5

$$\mathrm{E}[P_N] = \frac{1}{2}\frac{N(N+1)}{N^2} \quad \text{and} \quad \mathrm{Var}(P_N) = \frac{1}{6}\frac{N(N-1)(2N+5)}{N^4}.$$

Now apply Lemma 4.3.2. The second part follows likewise. That is, [30, Theorem 1] implies that $(\alpha_1^\vee)_*\bar{\mu}_{(\sigma s_0)^N}$ is the binomial distribution with success probability $\frac{1}{2}$ centered at 0 and dilated by 2. Therefore, for $Q_N = (D_{N^{-1}})_*(\alpha_1^\vee)_*\bar{\mu}_{(\sigma s_0)^N}$ we have

$$\mathrm{E}[Q_N] = 0 \quad \text{and} \quad \mathrm{Var}(Q_N) \overset{(3.1.2)}{=} \frac{1}{N}. \qquad \square$$

In view of the limit ratio of the expected and maximal degree stated in Corollary 2.3.5, it is tempting to conjecture that the weak law of large numbers also holds for the normalized degree distribution associated to a Demazure module $V_{(\sigma s_j)^N}(\Lambda)$, $j \in \{0,1\}$, with arbitrary highest weight $\Lambda = m\Lambda_0 + n\Lambda_1$.

Conjecture 4.3.4. *Let $\Lambda = m\Lambda_0 + n\Lambda_1$ be a dominant integral weight and $j \in \{0,1\}$. Denote by $B_{N,j}^{m,n}$ the maximal occurring degree in $V_{(\sigma s_j)^N}(\Lambda)$, by $\mu_{(\sigma s_j)^N}$ its weight distribution, and by $\bar{\mu}_{(\sigma s_j)^N}$ its normalization. Consider $-d, \alpha_1^\vee \in \mathfrak{h}$ as functions on $\mathfrak{h}^*$. Then,*

$$(D_{(B_{N,j}^{m,n})^{-1}})_*(-d)_*\bar{\mu}_{(\sigma s_j)^N} \xrightarrow{\mathrm{w}} \delta_{\frac{m+n+2}{3(m+n+1)}},$$

and consequently

$$(D_{(((m+n)N)^{-1},(B_{N,j}^{m,n})^{-1})})_*(\alpha_1^\vee, -d)_*\bar{\mu}_{(\sigma s_j)^N} \xrightarrow{\mathrm{w}} \delta_{\left(0,\frac{m+n+2}{3(m+n+1)}\right)}.$$

See Figure 6 for an illustration.

5. Further questions

According to Chebyshev's inequality, any sequence of probability measures individually dilated by a factor greater than their standard deviation converges weakly to the singular Dirac distribution. Consequently, the next interesting step is to determine the limiting distribution of the degree distributions of $V_{(\sigma s_0)^N}(\Lambda_0)$ when each of them is dilated by its standard deviation. Although each individual degree distribution is not distributed like the sum of independent identically distributed (i.i.d.) random variables, Figure 5 is a glimpse of Gaussian normal distributedness centered at $\frac{1}{2}$ in the case of $\Lambda = \Lambda_0$.

Furthermore, it is immediate to ask about possible generalizations of our results. One possible direction is to consider level 1 Demazure modules of higher rank Kac–Moody algebras (including types different from A). Our second approach in §3.5 is based on quantum calculus and the fact that Demazure characters are related to Macdonald and Rogers–Szegő polynomials (see [11, 13, 15, 22, 25, 31]). The connection between those polynomials is valid for level 1 Demazure modules in the higher rank cases, described via q-multinomial coefficients. Even though the computations will become more complicated, it should be possible to generalize our second approach this way.

Another possible generalization is to study the weight distribution in Demazure modules of higher level. A first step towards this would be to prove or disprove Conjecture 4.3.4. Combining terminology introduced in [32] and results obtained in [9, 10] one can deduce a graded character formula for higher level Demazure modules of $\widehat{\mathfrak{sl}}_2$ based on q-supernomials, i.e., finite sums of specific products of Gaussian polynomials. Therefore, in view of our second approach via quantum calculus, a proof of the stated conjecture seems immediate via the computation of Taylor expansions of q-supernomials. Another possible approach could be [33, Theorem 1.2] where Shimozono proves a type A higher level generalization of a graded character formula for Demazure characters stated in [23, Theorem 5] (see also [24]). Note that those graded character formulas are expressed in terms of Kostka–Foulkes polynomials (level 1) and a generalization of Kostka polynomials (higher level), and Schur functions. There is a connection between Kostka–Foulkes polynomials and Gaussian polynomials (see e.g. [8, Theorem 8.7] and [18, Exercise 7.C]). Again, in view of our second approach via quantum calculus, it seems reasonable to investigate further if and how one can derive statistical data about the higher level Demazure modules from those graded versions of their characters.

Once enough statistical data has been gathered (such as limiting distributions) one could try to to establish asymptotic formulæ for the weight multiplicities in Demazure modules. Tate and Zelditch have applied this procedure successfully to the weight distributions associated to tensor powers of irreducible representations [34]. Those weight distributions are distributed like i.i.d. random variables. This enables them to apply the i.i.d. version of the central limit theorem, large deviations techniques, and the method of stationary phase (see e.g. [3, 7], and [14] for the terminology) to derive statistical data and (pointwise) asymptotic formulæ for the weight multiplicities inside those tensor powers. Adapting the line of questioning in [34] to the context of Demazure modules, it is interesting to determine whether one can derive asympotic formulæ for the weight multiplicities of weights in the central limit region (that is, where one deviates O(one standard deviation) from the expected value). This seems reasonable under the hypothesis that one can prove Gaussian distributedness as described above. To our knowledge though, we are in a nonstandard setting. The degree distribution associated to Demazure modules is not distributed like the sum of i.i.d. random variables. Therefore, the usual central limit theorem and resulting asymptotic expansions do not apply. Even more ambitious is the question whether large deviation techniques can be applied to the degree and consequently weight distributions of Demazure modules.

In view of [19] (which has been continued in [20]) there is another interesting aspect coming up. Following [19] we can associate a Duistermaat–Heckman measure to the Demazure module $V_{w_{N,0}}(\Lambda_0)$ (and likewise to $V_{w_{N,1}}(\Lambda_1)$ which we omit here) via

$$\mathrm{DH}_N = \sum_{C \subset [1, w_{N,0}]} (\Phi_C)_*(\lambda_{\Delta_C}).$$

To explain the notation: we denote by C a maximal chain of Weyl group elements (in the Bruhat order) in the interval $[1, w_{N,0}] \subset W^{\mathrm{aff}}$, Δ_C is an N-simplex associated to the maximal chain C, λ_{Δ_C} is Lebesgue measure on that simplex and $(\Phi_C)_*$ denotes the push-forward of that Lebesgue measure via an affine-linear map Φ_C mapping the simplex Δ_C to $\mathfrak{h}^*_{\mathbf{R}}$. The Duistermaat–Heckman measure DH_N is supported on the convex hull $\mathrm{Conv}([1, w_{N,0}].\Lambda_0) \subset \Gamma_0 \otimes_{\mathbf{Z}} \mathbf{R}$. Here Γ_0 is endowed with a $\mathbf{Z}$-module structure by its identification with $\mathbf{Z}^2$ via the coordinates a, b. Let a denote the degree coordinate on Γ_0 and $\Gamma_0 \otimes_{\mathbf{Z}} \mathbf{R}$. It is interesting to contrast the distributions $a_*\mu_{w_{N,0}}$ and $a_*\,\mathrm{DH}_N$ with each other. Having the degree distribution and the weak law of large numbers Theorem 4.3.3 in mind, one expects the normalized measure $a_*\overline{\mathrm{DH}}_N$ to tend weakly to the Dirac distribution $\delta_{1/2}$ when scaled to the fixed support $[0, 1]$. Note that there is a discrete version of the measure DH_N resulting from [5], which has to be taken into account. That is, Knutson originally defines the simplices Δ_C in [19] in terms of the symplectic structure and Morse decompositions of the symplectic manifold he considers. In the setting of a coadjoint orbit one can show that those geometrically defined simplices coincide with the (maximal) simplices defined combinatorially by Dehy [5].

Let us also mention that [11] and [22] seem to be a good place to start in order to find physical applications of our results. They exhibit that one-dimensional configuration sums in solvable lattice models in statistical mechanics are closely related to Demazure characters.

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List of Figures

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