The Color-Flavor Transformation and its Applications to Quantum Chromodynamics

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Köln 2002

Abstract

The color-flavor transformation mediates between two equivalent formulations of a quantum field theory. The corresponding partition function is expressed in two different ways, namely as an integral over one of the two Lie groups of a dual pair in the group of canonical transformations.

In context of quantum chromodynamics one of the groups is connected with the color degrees of freedom and the other one with the flavor and spin degrees of freedom. The original color-flavor transformation (Zirnbauer, 1996) applies to an integral over the unitary group. We extend it to a version for the special unitary group, the gauge group of quantum chromodynamics.

In this dissertation we apply the color-flavor transformation to two limits of quantum chromodynamics: to gluodynamics, a Yang-Mills theory of the gauge field without coupling to the quarks, and to the strong coupling limit of chromodynamics, corresponding to the neglection of the kinetic energy of the gluons. In both cases, we proceed from a formulation of quantum chromodynamics on a lattice.

In case of gluodynamics, the color-flavor transformation does not apply directly to the common lattice action which was introduced by Wilson. We replace Wilson's action by a physically equivalent expression that can be generated by integration over heavy bosons.

In the strong coupling limit, the color-flavor transformation can be applied immediately: The partition function decomposes into a sum of contributions belonging to the possible distributions of baryons on the lattice. We consider the vacuum configuration and a further configuration with the intention to model a static baryon.

In the vacuum action the color degrees of freedom are completely decoupled. After integration over the quark fields they enter the partition function only through a factor N_c (number of colors) in front of the action. This structure organizes the theory in a perturbation series with $1/N_c$ as parameter. The lowest order of this large-N_c expansion is the saddle point approximation, where the partition function is approximated by its value at the saddle point. The solutions of the corresponding saddle point equations are the low energy modes of the theory. By considering space-time fluctuations of this modes we derive an effective low energy theory of quantum chromodynamics.

In case of the static baryon, the situation is more complicated. We study an approximate solution of the saddle point equation that is valid far away from the baryon. We obtain a Klein-Gordon equation and reproduce the Yukawa potential for the nuclear force.

Zusammenfassung

Die Color-Flavor-Transformation vermittelt zwischen zwei äquivalenten Formulierungen einer Quantenfeldtheorie. Die dieser Transformation zugrundeliegende Struktur ist ein duales Paar von Liegruppen in der Gruppe der kanonischen Transformationen. Basierend darauf läßt sich die Zustandssumme der Feldtheorie einerseits als Integral über den einen Partner und andererseits als Integral über den anderen Partner des dualen Paares ausdrücken.

Im Zusammenhang mit der Quantenchromodynamik identifizieren wir die eine der beiden Gruppen mit der Eichgruppe, die auf die Farbfreiheitsgrade wirkt, die andere mit der Symmetriegruppe der Flavor- und Spinfreiheitsgrade. Die Color-Flavor-Transformation in ihrer ersten Version (Zirnbauer, 1996), ist auf den Fall der unitären Gruppe als Farbgruppe anwendbar. Wir verallgemeinern diese auf eine Version für die spezielle unitäre Gruppe, die Eichgruppe der Quantenchromodynamik.

In dieser Arbeit wenden wir die Color-Flavor-Transformation auf zwei Grenzfälle der Quantenchromodynamik an: auf die Gluodynamik, die das Eichfeld alleine, d. h. ohne Kopplung an die Quarkfreiheitsgrade beschreibt, und den Limes starker Kopplung, bei dem die kinetische Energie der Gluonen vernachlässigt wird. In beiden Fällen bedienen wir uns einer Gitterformulierung der Chromodynamik.

Für den Fall der Gluodynamik ist eine direkte Anwendung der Color-Flavor-Transformation auf die übliche, von Wilson eingeführte Gitterwirkung nicht möglich. Wir ersetzen Wilsons Wirkung durch einen physikalisch äquivalenten Term, der sich aus einem Integral über schwere Bosonenfelder erzeugen lässt.

Im Starkkopplungslimes kann die Color-Flavor-Transformation unmittelbar angewendet werden: Die Zustandssumme zerfällt in eine Summe von Beiträgen, die den möglichen Verteilungen von Baryonen auf dem Gitter entsprechen. Wir betrachten die Vakuumkonfiguration – in diesem Fall sind überhaupt keine Baryonen vorhanden – und eine weitere Konfiguration, die ein einzelnes, statisches Baryon modelliert.

Für die Vakuumkonfiguration tritt nach Integration über die Quarkfelder die Anzahl der Farben N_c als Faktor vor der effektiven Wirkung auf. Dieser organisiert die Zustandssumme in eine Störungsreihe in $1/N_c$, deren niedrigste Ordnung durch die Sattelpunktsnäherung gegeben ist. Die Lösungen der zugehörigen Sattelpunktsgleichungen bilden die niederenergetischsten Moden der Theorie. Wir betrachten Raumzeit-Fluktuationen dieser Moden und leiten so eine effektive Niederenergietheorie der Quantenchromodynamik ab.

Im Fall des statischen Baryons ist die Situation komplizierter: Hier gelingt uns die Lösung der Sattelpunktsgleichungen nur in einer weit entfernt vom Baryon gültigen Nährung. Wir werden auf eine Klein-Gordon-Gleichung geführt und können das Yukawa-Potential für das die Kernkräfte vermittelnde Feld reproduzieren.

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Chapter 1

Introduction

1.1 From Atoms to Quarks

What is matter? This question has challenged the human mind since the beginning of physics and philosophy. About 400 b.c. the Greek philosopher Demokrit advanced the theory that matter is built of small, indivisible particles (later called atoms). First experimental evidence for the existence of atoms came from chemistry around 1800. Dalton developed his atom theory according to which a substance is composed of different kinds of atoms which are in a constant proportion. It was found that the different atom sorts can be systematically arranged in the periodic table of elements (Mendelejew 1870). After the discovery of the electron (Thomson 1897) Rutherford's scattering experiment (1910) revealed the inner structure of the atoms: They are built of a nucleus (consisting of protons and neutrons) and electrons moving around it.

It has become clear only in the last 50 years that protons and neutrons have a substructure: they are composed of three quarks. The quarks were introduced to bring order in a large set of new particles (today called hadrons) which had been discovered meanwhile. These new particles, unstable under normal conditions on earth, were first observed in cosmic rays and later produced in colliders. The hadrons consist of either three (baryons) or two quarks (mesons). The nucleons (proton and neutron) belong to the baryons. There are $N_f = 6$ different kinds (called flavors) of quarks, named up (u), down (d), strange (s), charm (c), bottom (b) and top (t). Their mass, electric charge and spin are listed in Table I.

Different from the nucleons, the electrons have (as far as we know) no substructure and are therefore considered as elementary. They belong to another family of particles, the leptons, that besides the electron e^- contains the myon μ and the tauon τ. Associated with each of these three particles are three further leptons, the neutrinos. The neutrinos are electrically neutral and very light (very recent experiments [20] suggest that they are not massless).

Let us return to the hadrons. A short period after the quark model was proposed, an enthusiastic search for free quarks – based on their fractional electric charge – was beginning. The result was disappointing, no such particle could be found. So far, if they exist, they seemed never to leave the hadrons, being forever confined inside them.

Later, evidence for the existence of quarks could be obtained from the following experiments which rest on dynamic properties of the quarks: deep inelastic scattering of leptons (for example of electrons) on hadrons and positron-electron annihilation into hadrons. The results of the first experiment could be interpreted in terms of the elastic scattering of the lepton by

quark	electric charge	spin	mass [MeV]
u	$+2/3$	$1/2$	1 to 5
d	$-1/3$	$1/2$	3 to 9
s	$-1/3$	$1/2$	75 to 170
c	$+2/3$	$1/2$	1,150 to 1,350
b	$-1/3$	$1/2$	4,000 to 4,400
t	$+2/3$	$1/2$	$174{,}300 \pm 5{,}100$

Table I: Quarks – electric charge (in units of the electron charge), spin and mass [20].

a quasi-free, point-like constituent of the hadron. This led to a picture of the hadron as a loosely bound assemblage of constituents which can be identified with the quarks. The second experiment, the annihilation of positron-electron into hadrons "jets" could be understood by the production of a quark-antiquark pair as intermediate state.

The cross section of the annihilation $e^+e^- \to hadrons$ was compared with that of the lepton-lepton scattering $e^+e^- \to \mu^+\mu^-$. It turned out that – under consideration of the contributing flavors – the former cross section is three times larger than expected. This was one of the reasons to introduce – in addition to the flavor degree of freedom – a color degree of freedom of the quarks. The quarks were postulated to come in three "colors": red, green, and blue. The introduction of the color also allows to put right the antisymmetric quark statistics which is connected with their non-integer spin. Colored particles have never been directly observed. The hadrons are colorless: the mesons are quark-antiquarks pairs of two quarks with the same color, and baryons consist of three quarks, one of each color. The question for the reason of this color confinement has to be addressed to a theory of the strong interaction.

We have briefly recapitulated the history of elementary particles physics with a focus on the hadrons and their structure. Having discussed the static quark model, now we turn to a dynamical theory of quark interaction.

Today, four kinds of forces acting between the elementary particles are known: gravitation, electromagnetism, the weak interaction and the strong interaction. We disregard gravitation which is connected with the geometry of space-time. Since the mid-1970 there is a wide agreement that the three remaining interactions are mediated through the exchange of vector bosons between the elementary particles. The widely accepted theoretical model is a gauge theory, where the exchange bosons are identified with the gauge field. We now focus on the strong interaction which is responsible for nuclear binding and the interaction of the quarks.

1.2   Quantum Chromodynamics

The gauge group of quantum chromodynamics (QCD) is the color group $SU(N_c)$, where the number of colors in nature is $N_c = 3$. The corresponding gauge bosons are called gluons and couple to the color degrees of freedom of the quarks.

The most important property of quantum chromodynamics is the so-called *asymptotic freedom* which is connected with its non-Abelian gauge group. As a quantum field theory, QCD is ultraviolet divergent and the divergences have to be removed by regularization and a renormalization of the coupling constants. From the one-loop order of perturbation theory [38] one

obtains the running coupling constant

$$g^2(k) = \frac{g^2}{1 + \frac{g^2}{(4\pi)^2}\left(\frac{11}{3}\,N_c - \frac{2}{3}\,N_f\right)\ln\left(k^2/M^2\right)}, \tag{1.1}$$

where M is an arbitrary renormalization scale. In the real world, the number of colors and the number of flavors are $N_c = 3$ and $N_f = 6$. As a consequence, the factor in front of the logarithm is positive and the coupling $g(k)$ flows to zero, when the momentum is increased. Quantum field theories with such a behavior of the coupling constant are called asymptotically free. This behavior of the coupling between the quarks fits to the experimental results from deep inelastic scattering. Indeed, the constituents of the hadron do not behave exactly like free particles, but in a way which is in agreement with the logarithmic scaling (1.1).

QCD is the widely accepted theory of strong interaction and at weak coupling it can be evaluated by perturbation theory. Due to asymptotic freedom weak coupling belongs to high energies. In contrast to the perturbative high energy sector, the low energy sector is non-perturbative and worse understood. Experimentally, the color degrees of freedom are confined inside the hadrons. However – in spite of intensive research in that area – there is no convincing theoretical proof of color-confinement based on QCD.

In order to reach the non-perturbative low energy sector, in 1974 Wilson proposed a lattice formulation of QCD. His guiding idea was to implement the gauge symmetry on the lattice. In doing so, the gauge field having values in the Lie algebra of the gauge group is replaced by a field U which has values in the gauge group and is placed on the lattice links (see Chapter 4 for details).

In this formulation QCD is open to an application of the *color-flavor transformation*. This transformation was recently developed by Zirnbauer in the of context disordered systems [53]; later – in view of applications to QCD – it got its present name.

1.3   The Color-Flavor Transformation

The color-flavor transformation connects an integral over the unitary group $U(N_c)$ with an integral over complex $N_f \times N_f$ matrices which parametrize the symmetric space $U(2N_f)/U(N_f) \times U(N_f)$,

$$\int_{U(N_c)} dU \, \exp\left(\bar{q}^i_{+a} U^{ij} q^j_{+a} - \bar{q}^i_{-b} U^{\dagger ij} q^j_{-b}\right)$$

$$= \text{const.} \times \int_{\mathbb{C}^{N_f \times N_f}} \frac{dZ dZ^\dagger}{\text{Det}(1 + ZZ^\dagger)^{2N_f + N_c}} \, \exp\left(\bar{q}^i_{+a} Z_{ab} q^i_{-b} + \bar{q}^j_{-b} Z^\dagger_{ba} q^j_{+a}\right). \tag{1.2}$$

In this dissertation we will develop another version of this transformation where the unitary group is replaced by the special unitary group. One of the central ideas of this work is to build a bridge between two different formulations of QCD: a formulation in terms of the quarks and gluons which are the high energy degrees of freedom and a formulation in terms of hadrons, the low energy degrees of freedom. Indeed, the color-flavor transformation replaces the gauge field U by a colorless field Z that couples quarks which are in a color singlet. After integration over the quark fields, one obtains an effective action of Z-field with a factor N_c in front of it which comes from the color content of the quarks. Thus, in the large-N_c limit the theory is organized in a $1/N_c$-perturbation series.

The color-flavor transformation is an exact identity, but in the form (1.2) it applies only to the *strong coupling limit* of QCD. In this dissertation we will show that it applies also to gluodynamics and, in principle, to full QCD.

1.4   Outline

Our work is organized as follows: In the first two chapters we discuss different versions of the color-flavor transformation. We start off with a review of the canonical transformations of a multi-particle quantum system. To each "dual pair" of groups in the group of canonical transformations there is an associated color-flavor transformation. We work out in detail the color-flavor transformation for the dual pair $(\mathrm{U}(N_c), \mathrm{U}(2N_f))$ for a system of bosons and a system of fermions (Chapter 2).

The color group of QCD is the special unitary group $\mathrm{SU}(N_c)$ and not the full unitary group $\mathrm{U}(N_c)$. For the smaller color group the set of colorless states is larger: It does not only contain mesons, but also baryons. This gives rise to a more complicated version of the color-flavor transformation which is discussed in Chapter 3.

In Chapter 4 we review the lattice formulation of QCD. Special attention is paid to how the symmetries of the original continuum theory are realized on the lattice. We discuss in detail the chiral symmetry and the chiral symmetry breaking mechanisms.

The next three chapters deal with applications of the color-flavor transformation to QCD. In Chapter 5 we apply it to gluodynamics which is modeled by a $\mathrm{U}(N_c)$ Yang-Mills theory on a lattice. The color-flavor transformation does not apply to Wilson's lattice action which is a sum over traces of plaquette terms. We introduce a new lattice action which can be generated by Gaussian integration over a number of auxiliary bosons. We study the physics behind the new action in $d = 2$ spacetime dimensions. Further, we present first results concerning the color-flavor transformation of our gluodynamic action.

The last two chapters are devoted to the color-flavor transformation of strong coupling QCD. The strong coupling limit is reached by neglecting the kinetic energy of the gluons in the QCD action. It can be employed as an approximation to QCD, when one wants to calculate a low energy effective theory. In Chapter 6 we consider the mesonic sector of strong coupling QCD which can be reached by the $\mathrm{U}(N_c)$ color-flavor transformation. The color-flavor transformed action comes up with a factor N_c in front of it which organizes the partition function in a $1/N_c$ perturbation series. From a saddle point approximation, which is valid in the large-N_c limit, we obtain a low energy effective action. In the spirit of chiral perturbation theory, this action is an expansion in the momenta p of the mesons and the quark masses m, where the order $O(m)$ is treated as $O(p^2)$.

Chapter 7 deals with full strong coupling QCD, taking into account mesons and baryons. We work with the more complicated $\mathrm{SU}(N_c)$ color-flavor transformation. The color-flavor transformed partition function decomposes into a sum over the different possible distributions of baryons on the lattice. Modeling a static baryon, we consider a special baryonic flux and calculate the corresponding effective action. We try to get information about the shape of the mesonic background field around the baryon in the large-N_c limit. In order to obtain the behavior of this field far away from the baryon, we linearize the saddle point equations around its vacuum value.

Chapter 2

Canonical Transformations and the Color-Flavor Transformation

Zirnbauer's color-flavor transformation [53] is an exact identity which connects two formulations of a certain field theory. The purpose of this chapter is to discuss the algebraic structures which underlie this transformation.

The field theory we are going to consider is connected with a corresponding multi particle quantum system. While the original version of the color-flavor transformation [53] deals with a supersymmetric system, we consider a pure bosonic and a pure fermionic system. The basic structure of such a quantum system are the canonical (anti-) commutation relations which are satisfied by the creation and annihilation operators. The transformations, which keep the canonical commutation relations invariant, form the group of canonical transformations.

To each dual pair (K, G) of subgroups inside the group of canonical transformation there is a associated color-flavor transformation. The notion of dual pairs inside a group goes back to Howe [23] and will be explained later.

In view of applications to QCD we work out the color-flavor transformation for the dual pair consisting of $K = \mathrm{U}(N_c)$ and $G = \mathrm{U}(N_f, N_f)$ for the bosonic system, as well as $K = \mathrm{U}(N_c)$ and $G = \mathrm{U}(2N_f)$ for the fermionic system.

2.1 Multi Particle Quantum Systems

The basic operators for the description of a quantum system with N degrees of freedom are the creation operators $\bar{a}_i$ and the annihilation operators a_i ($i = 1, ..., N$). For a system of bosons they satisfy the canonical commutation relations

$$[a_i, \bar{a}_j] = \delta_{ij}, \quad [a_i, a_j] = [\bar{a}_i, \bar{a}_j] = 0, \tag{2.1}$$

while for a system of fermions they obey the canonical anticommutation relations

$$\{a_i, \bar{a}_j\} = \delta_{ij}, \quad \{a_i, a_j\} = \{\bar{a}_i, \bar{a}_j\} = 0. \tag{2.2}$$

The algebra generated by the creators and annihilators is called Weyl algebra (bosons) or Clifford algebra (fermions). We consider a linear transformations of the basic operators,

$$\begin{aligned}
\bar{a}_j' &= \bar{a}_i Q_{ij} + a_i S_{ij} \\
a_j' &= \bar{a}_i R_{ij} + a_i T_{ij},
\end{aligned} \tag{2.3}$$

where Q, R, S, T denote arbitrary complex $N \times N$ matrices. Here and thereafter we assume the usual summation convention, i. e. one has to sum over repeated indices as long as nothing else is demanded. (The transformation (2.3) does not necessarily preserve the commutation relations (2.1), (2.2). Canonical transformations are considered in the next subsection.) The composition of two transformations of that kind is given by matrix multiplication, if we arrange the four block to a $2N \times 2N$ matrix $X = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$. Thus we deal with a representation of $\mathfrak{gl}(2N, \mathbb{C})$ on the complex vector space which is spanned by the creators $\bar{a}_i$ and the annihilators a_i.

Our aim is to show that some of the transformations (2.3) can be realized as interior operations inside the Weyl or Clifford algebra [6, 37, 54]. For that purpose we define quadratic operators

$$
\begin{array}{ll}
\hat{Q}_{ij} = \bar{a}_i a_j, & \hat{R}_{ij} = \varepsilon\, \bar{a}_i \bar{a}_j, \\
\hat{S}_{ij} = a_i a_j, & \hat{T}_{ij} = \varepsilon\, a_i \bar{a}_j,
\end{array}
\qquad
\text{where } \varepsilon := \left\{
\begin{array}{ll}
-1 & \text{for the bosonic system,} \\
+1 & \text{for the fermionic system.}
\end{array}
\right.
\tag{2.4}
$$

and map the complex $2N \times 2N$ matrix X to an operator $\hat{X}$ by

$$
X = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}
\quad \mapsto \quad
\hat{X} := \frac{1}{2} \sum_{i,j=1}^{N} \left(Q_{ij} \hat{Q}_{ij} + R_{ij} \hat{R}_{ij} + S_{ij} \hat{S}_{ij} + T_{ij} \hat{T}_{ij} \right).
\tag{2.5}
$$

We make use of the general formulas for commutators (anticommutators) inside an associative algebra

$$
[ab, c] = a[b, c] + [a, c]b,
\tag{2.6}
$$
$$
\{ab, c\} = a\{b, c\} - \{a, c\}
\tag{2.7}
$$

and obtain straightforwardly

$$
\begin{aligned}
[\hat{X}, \bar{a}_j] &= \frac{1}{2}\bar{a}_i(Q - T^t)_{ij} + \frac{1}{2}a_i(S - \varepsilon S^t)_{ij}, \\
[\hat{X}, a_j] &= \frac{1}{2}\bar{a}_i(R - \varepsilon R^t)_{ij} + \frac{1}{2}a_i(T - Q^t)_{ij}.
\end{aligned}
\tag{2.8}
$$

From now on we restrict the matrices X by the constraint

$$
Q^t = -T, \quad R^t = -\varepsilon R, \quad S^t = -\varepsilon S,
\tag{2.9}
$$

so that the last two equations take the simple form

$$
\begin{aligned}
[\hat{X}, \bar{a}_j] &= \bar{a}_i Q_{ij} + a_i S_{ij}, \\
[\hat{X}, a_j] &= \bar{a}_i R_{ij} + a_i T_{ij}.
\end{aligned}
\tag{2.10}
$$

To sum up, we have implemented the representation (2.3) by the adjoint action of the quadratic operator $\hat{X}$ for matrices X which fulfill the conditions (2.9).

Let us have a closer look at the constraint. It can be described by passing from $\mathfrak{gl}(2N, \mathbb{C})$ to the symplectic (orthogonal) complex Lie algebra,

$$
\mathfrak{sp}(2N, \mathbb{C}) := \{ X \in \mathfrak{gl}(2N, \mathbb{C}) \mid X^t J + J X = 0 \} \quad \text{or}
\tag{2.11}
$$

$$
\mathfrak{o}(2N, \mathbb{C}) := \{ X \in \mathfrak{gl}(2N, \mathbb{C}) \mid X^t \Sigma_x + \Sigma_x X = 0 \}
\tag{2.12}
$$

for the bosonic system (fermionic system). As defined above J and Σ_x denote the $2N \times 2N$ block matrices

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \Sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.13}$$

For X, Y inside the two subalgebras the map (2.5) respects the Lie bracket, [1]

$$[\hat{X}, \hat{Y}] = \widehat{[X, Y]}, \tag{2.14}$$

and the maps

$$w : \mathfrak{sp}(2N, \mathbb{C}) \to \operatorname{span}\{\bar{a}_i a_j + a_j \bar{a}_i, a_i a_j, \bar{a}_i \bar{a}_j\} \text{ and} \tag{2.15}$$
$$X \mapsto w_X := \hat{X}$$

$$w : \mathfrak{o}(2N, \mathbb{C}) \to \operatorname{span}\{\bar{a}_i a_j - a_j \bar{a}_i, a_i a_j, \bar{a}_i \bar{a}_j\} \tag{2.16}$$
$$X \mapsto w_X := \hat{X}$$

are Lie algebra isomorphisms [37].

2.2 Canonical Transformations and Real Transformations

We turn to the study of *canonical* transformations, that are invertible transformations, which respect the canonical commutation or anticommutation relations. We switch from a Lie algebra to a group formulation: Corresponding to (2.3) the matrix of an invertible transformation $G = \exp X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ acts on the basic operators by

$$\bar{a}'_j = \bar{a}_i A_{ij} + a_i C_{ij}$$
$$a'_j = \bar{a}_i B_{ij} + a_i D_{ij}. \tag{2.17}$$

The transformation G preserves the (anti-) commutation relations, iff [2]

$$\varepsilon B^t C + D^t A = I,$$
$$\varepsilon B^t D + D^t B = 0, \tag{2.18}$$
$$\varepsilon A^t C + C^t A = 0.$$

Thus G gives rise to a canonical transformation, iff it belongs to the following complex symplectic (orthogonal) group,

$$\operatorname{Sp}(2N, \mathbb{C}) := \{G \in \operatorname{GL}(2N, \mathbb{C}) \mid G^t J G = J\} \text{ and} \tag{2.19}$$

$$\operatorname{O}(2N, \mathbb{C}) := \{G \in \operatorname{GL}(2N, \mathbb{C}) \mid G^t \Sigma_x G = \Sigma_x\}. \tag{2.20}$$

We make use of the general formula $\exp[\hat{X}, \bullet] b = \exp(\hat{X}) b \exp(-\hat{X})$ for two arbitrary operators $\hat{X}$ and b, and we obtain by exponentiating the representation (2.10)

$$\exp(\hat{X}) \bar{a}_j \exp(-\hat{X}) = \bar{a}_i A_{ij} + a_i C_{ij}$$
$$\exp(\hat{X}) a_j \exp(-\hat{X}) = \bar{a}_i B_{ij} + a_i D_{ij}, \tag{2.21}$$

---

[1]$[[\hat{X}, \hat{Y}], b] = [\hat{X}, [\hat{Y}, b]] - [\hat{Y}, [\hat{X}, b]] = [\widehat{[X, Y]}, b]$ for all basic operators b follows directly from (2.10), but to prove (2.14) additional considerations are necessary.

[2]We write "*iff*" as an abbreviation for *if and only if*.

for canonical transformations $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \exp \begin{pmatrix} Q & R \\ S & T \end{pmatrix} = \exp X$.
There is an involution $\dagger$ on the Weyl algebra defined by its action

$$a_i^\dagger = \bar{a}_i, \quad \bar{a}_i^\dagger = a_i \tag{2.22}$$

on the basic operators and the rules

$$(zb)^\dagger = \bar{z} b^\dagger, \quad (bc)^\dagger = c^\dagger b^\dagger \tag{2.23}$$

for all complex numbers z and all operators b, c. (In the next section we construct a representation of the operator algebra on a Hilbert space, the Fock space. In that context $\dagger$ will become the hermitian conjugation with respect to the scalar product of the Hilbert space.) Therefore it is natural to call a transformation (2.3) *real*, if it is compatible with the "hermitian conjugation", that is

$$(a_i')^\dagger = \bar{a}_i', \quad \text{for } i = 1, ..., N. \tag{2.24}$$

It is easy to see that a transformation $X = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$ is real, iff it satisfies $S = \bar{R}$ and $T = \bar{Q}$, that is, it belongs to the subalgebra

$$\mathfrak{r} := \left\{ X \in \mathfrak{gl}(2N, \mathbb{C}) \mid X\Sigma_x = \Sigma_x \bar{X} \right\} \tag{2.25}$$

of $\mathfrak{gl}(2N, \mathbb{C})$. Note that the algebra $\mathfrak{r}$ is closed, not only under the Lie bracket, but also under the matrix multiplication of two of its elements.
The intersection between the complex Lie algebras $\mathfrak{sp}(2N, \mathbb{C})$ and $\mathfrak{o}(2N, \mathbb{C})$ and the algebra $\mathfrak{r}$ of real transformations are the following real forms of the Lie algebras,

$$\mathfrak{sp}(2N, \mathbb{R}) := \{ X \in \mathfrak{r} \mid X^\dagger \Sigma_z + \Sigma_z X = 0 \}, \quad \text{where } \Sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \tag{2.26}$$

$$\mathfrak{so}(2N, \mathbb{R}) := \{ X \in \mathfrak{r} \mid X^\dagger + X = 0 \}, \tag{2.27}$$

Therefore the set of *real* and *canonical* transformations is given by the following real forms of the groups $\mathrm{Sp}(2N, \mathbb{R})$, $\mathrm{O}(2N, \mathbb{R})$,

$$\mathrm{Sp}(2N, \mathbb{R}) := \{ G \in \mathfrak{r} \mid G^\dagger \Sigma_z G = \Sigma_z \} \quad \text{and} \tag{2.28}$$

$$\mathrm{O}(2N, \mathbb{R}) := \{ G \in \mathfrak{r} \mid G^\dagger G = I \}. \tag{2.29}$$

Up to now the creation and annihilation operators were considered as elements of an abstract algebra. In the next section we realize them as operators of the Fock space.

2.3   Representations on the Fock Space

The Fock space $\mathcal{H}$ is generated by the action of the creators and annihilators on a vacuum state $|0\rangle$. The vacuum is exterminated by the annihilators, $a_j |0\rangle = 0$, the transpose of the vacuum by the creators, $\langle 0| \bar{a}_j = 0$. The bosonic Fock space is infinite dimensional, while the dimension of the fermionic Fock space is 2^N. The maps w (2.15) and (2.16) can be considered as representations of the symplectic or the orthogonal Lie algebra on the Hilbert space,

$$\begin{aligned} w : \mathfrak{sp}(2N, \mathbb{R}) &\to \mathfrak{gl}(\mathcal{H}) & w : \mathfrak{o}(2N, \mathbb{R}) &\to \mathfrak{gl}(\mathcal{H}) \\ X &\mapsto w(X) := \hat{X}, & X &\mapsto w(X) := \hat{X}. \end{aligned} \tag{2.30}$$

(By $\mathfrak{gl}(\mathcal{H})$ we denote the algebra of all linear maps $\mathcal{H} \to \mathcal{H}$.)   The Lie groups $\mathrm{Sp}(2N, \mathbb{R})$ and $\mathrm{O}(2N, \mathbb{R})$ corresponding to the above Lie algebras are not simply connected.   Simple connectivity is a condition to apply a general theorem of Lie group theory (cf. any book on Lie group theory, e.g. [43]) to press a homomorphism of Lie algebras down to the corresponding groups.   It turns out that there is an obstruction in our case: One has to pass the twofold following coverings, the metaplectic group $\mathrm{Mp}(2N)$ for the symplectic group and the Pin group $\mathrm{Pin}(2N)$ for the orthogonal group to press the representations down [37].   In doing so we get representations

$$\omega : \mathrm{Mp}(2N, \mathbb{C}) \to \mathrm{Gl}(\mathcal{H}) \qquad\qquad \omega : \mathrm{Pin}(2N, \mathbb{C}) \to \mathrm{Gl}(\mathcal{H}) \qquad (2.31)$$
$$G \mapsto \omega(G), \qquad\qquad\qquad\qquad G \mapsto \omega(G)$$

with the property

$$\omega(\widetilde{\exp(X)}) = \exp\left(w(X)\right) \qquad (2.32)$$

for all elements X of the Lie algebra. (By $\mathfrak{gl}(\mathcal{H})$ we denote the group of all invertible linear maps $\mathcal{H} \to \mathcal{H}$.) Here $\widetilde{\exp(X)}$ denotes one of the two elements of the covering group which is mapped to $\exp(X)$ by the covering homomorphism. The representation ω is called *oscillator representation* (*spin representation*) in the case of the bosonic system (fermionic system). From the definition (2.5) we get $\hat{X}^\dagger = -\hat{X}$ for both $X \in \mathfrak{sp}(2N, \mathbb{R})$ and $X \in \mathfrak{o}(2N, \mathbb{R})$ and conclude

$$w^\dagger = -w, \qquad \omega^\dagger = \omega^{-1}, \qquad (2.33)$$

that is ω is a *unitary representation* in both cases.

To sum up, an element $\tilde{G}$ of the covering group, which is mapped to $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ by the covering homomorphism, acts on the basic operator by

$$\omega(\tilde{G})\bar{a}_j\omega(\tilde{G})^{-1} = \bar{a}_i A_{ij} + a_i C_{ij}$$
$$\omega(\tilde{G})a_j\omega(\tilde{G})^{-1} = \bar{a}_i B_{ij} + a_i D_{ij}. \qquad (2.34)$$

The oscillator and the spin representation are built by exponentiating quadratic operators. Therefore the Hilbert space is not irreducible under these representations, but it decomposes according to the parity of the number of particles, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. The subspaces of even and odd particle numbers are irreducible under the oscillator and the spin representation [37].

We define standard coherent states by

$$\langle \bar{\Psi} | := \langle 0 | \exp\left(\bar{\psi}_j a_j\right), \quad |\Psi\rangle := \exp\left(\bar{a}_j \psi_j\right) |0\rangle. \qquad (2.35)$$

In the expression above $\bar{\psi}$ and ψ denote two independent vectors of complex numbers (bosonic case) or Grassmann numbers (fermionic case). The coherent states are eigenstates of annihilators or the creators,

$$\langle \bar{\Psi} | \bar{a}_j = \bar{\psi}_j \langle \bar{\Psi} |, \quad a_j | \Psi \rangle = |\Psi\rangle \psi_j. \qquad (2.36)$$

From $[\bar{\psi}_i a_i, \bar{a}_j \psi_j] = \bar{\psi}_i \psi_i$ we obtain with the help of the Baker-Campbell-Hausdorff formula [3]

$$\exp(\bar{a}_i \psi_i + \bar{\psi}_i a_i) = \exp\left(\frac{1}{2}\bar{\psi}\psi\right) \exp(\bar{a}_i \psi_i) \exp(\bar{\psi}_i a_i), \qquad (2.37)$$

$$\exp(\bar{\psi}_i a_i) \exp(\bar{a}_i \psi_i) = \exp(\bar{\psi}\psi) \exp(\bar{a}_i \psi_i) \exp(\bar{\psi}_i a_i). \qquad (2.38)$$

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[3] $\exp(\hat{A}+\hat{B}) = \exp\left(-\frac{1}{2}[\hat{A}, \hat{B}]\right) \exp\hat{A} \exp\hat{B}$ for operators $\hat{A}$, $\hat{B}$, which both commute with their commutator $[\hat{A}, \hat{B}]$.

To calculate the action of the oscillator (spin) representation, we use the generalized Gaussian decomposition of G (cf. Appendix A) into an upper diagonal matrix Z_+, a diagonal matrix H and a lower diagonal matrix Z_-,

$$G = Z_+ H Z_- = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}. \tag{2.39}$$

The vacuum state and its transpose are eigenstates of H,

$$\langle 0 | \omega(H) = (\mathrm{Det}\, D)^{\frac{\varepsilon}{2}} \langle 0 |, \qquad \omega(H)|0\rangle = |0\rangle (\mathrm{Det}\, D)^{\frac{\varepsilon}{2}}, \tag{2.40}$$

while the coherent state $\langle \bar{\Psi} |$ is an eigenstate of Z_+ and the coherent state $|\Psi\rangle$ is an eigenstate of Z_-,

$$\langle \bar{\Psi} | \omega(Z_+) = \exp\left(\frac{\varepsilon}{2} \bar{\psi} B D^{-1} \bar{\psi}\right) \langle \bar{\Psi} |, \qquad \omega(Z_-)|\Psi\rangle = |\Psi\rangle \exp\left(\frac{1}{2} \psi D^{-1} C \psi\right). \tag{2.41}$$

The formulas above can be used to calculate the matrix elements of the oscillator representation (spin representation) with respect to the bose (fermi) coherent states,

$$\langle \bar{\Psi} | \omega(\widetilde{G}) | \Psi \rangle = \pm (\mathrm{Det}\, D)^{\varepsilon/2} \exp\left(\frac{\varepsilon}{2} \bar{\psi} B D^{-1} \bar{\psi}\right) \exp\left(-\varepsilon \psi D^{-1} \bar{\psi}\right) \exp\left(\frac{1}{2} \psi D^{-1} C \psi\right). \tag{2.42}$$

In deriving the last formula we made use of $A - BD^{-1}C = (D^{-1})^t$, which follows from (2.18).

2.4   The Color-Flavor Transformation for the Unitary Group

2.4.1   Charge Conservation

We consider a system of $N = p + q$ *charged* species, where all particles carry the same amount of charge and p particles are positively, q particles negatively charged. We label the positive charged particles by $i = 1, ..., p$ and the negative particles by $i = -1, ..., -q$. The charge operator counts the positive charged particles minus the number of negatively charged particles and is given by

$$\hat{Q} = \sum_{i=1}^{p} \bar{a}_{+i} a_{+i} - \sum_{i=1}^{q} \bar{a}_{-i} a_{-i}. \tag{2.43}$$

Corresponding to the particles with positive and negative charge, we split each block of the matrix $X = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$ into four blocks, $Q = \begin{pmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{pmatrix}$ and in the same way for R, S, T. We say a transformation X conserves the charge, iff $\hat{X}$ commutes with the charge operator $\hat{Q}$. The charge conserving transformations form the algebra

$$\mathfrak{c} := \left\{ X \in \mathfrak{gl}(2N, \mathbb{C}) \mid X \tilde{\Sigma}_z = \tilde{\Sigma}_z X \right\}, \qquad \text{where } \tilde{\Sigma}_z = \mathrm{diag}(1, -1, -1, 1). \tag{2.44}$$

Note that $\hat{Q} = \hat{\tilde{\Sigma}}_z + \frac{\varepsilon}{2}(p - q)$ and a charge conserving transformation $X \in \mathfrak{c}$ has the block structure

$$X = \begin{pmatrix} Q_{++} & 0 & 0 & R_{+-} \\ 0 & Q_{--} & R_{-+} & 0 \\ 0 & S_{+-} & T_{++} & 0 \\ S_{-+} & 0 & 0 & T_{--}. \end{pmatrix}. \tag{2.45}$$

Therefore the representation of $\mathfrak{c}$ as $2N \times 2N$-matrices decomposes into a direct sum of two sub-representations,

$$X \leftrightarrow \left(X' = \begin{pmatrix} Q_{++} & R_{+-} \\ S_{-+} & T_{--} \end{pmatrix}, X'' = \begin{pmatrix} Q_{--} & R_{-+} \\ S_{+-} & T_{++} \end{pmatrix} \right), \tag{2.46}$$

where X' and X'' are $N \times N$-matrices. For transformations $X \in \mathfrak{sp}(2N, \mathbb{C}) \cap \mathfrak{c}$ and $X \in \mathfrak{o}(2N, \mathbb{C}) \cap \mathfrak{c}$ the matrices X' and X'' are connected via

$$X'' = - \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} X'^t \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}^{-1}. \tag{2.47}$$

Because (2.9) put no further constraints on X' and X'', we obtain

$$\mathfrak{sp}(2N, \mathbb{C}) \cap \mathfrak{c} \cong \mathfrak{gl}(N, \mathbb{C}), \qquad \mathfrak{o}(2N, \mathbb{C}) \cap \mathfrak{c} \cong \mathfrak{gl}(N, \mathbb{C}) \tag{2.48}$$

and switching to the group level we conclude

$$\mathrm{Sp}(2N, \mathbb{C}) \cap \mathfrak{c} \cong \mathrm{Gl}(N, \mathbb{C}), \quad \mathrm{O}(2N, \mathbb{C}) \cap \mathfrak{c} \cong \mathrm{Gl}(N, \mathbb{C}). \tag{2.49}$$

The representations of the former groups decompose into a sum of the fundamental representation of $\mathrm{Gl}(N, \mathbb{C})$ and a representation equivalent to the contragredient representation,

$$G \leftrightarrow \left(G', G'' \right), \text{ where } G'' = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} (G'^{-1})^t \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}^{-1}. \tag{2.50}$$

A restriction of the charge conserving transformations to real transformations leads the following real forms of the Lie algebras and Lie groups,

$$\mathfrak{sp}(2N, \mathbb{R}) \cap \mathfrak{c} \cong \mathfrak{u}(p, q), \qquad \mathfrak{o}(2N, \mathbb{R}) \cap \mathfrak{c} \cong \mathfrak{u}(p + q), \tag{2.51}$$

$$\mathrm{Sp}(2N, \mathbb{R}) \cap \mathfrak{c} \cong \mathrm{U}(p, q), \qquad \mathrm{O}(2N, \mathbb{R}) \cap \mathfrak{c} \cong \mathrm{U}(p + q). \tag{2.52}$$

Finally we note that the operator corresponding to a charge conserving transformation $X \in \mathfrak{sp}(2N, \mathbb{C}) \cap \mathfrak{c}$ and $X \in \mathfrak{o}(2N, \mathbb{C}) \cap \mathfrak{c}$ can be described in terms of X' alone,

$$\hat{X} = \sum_{i,j} (Q_{+i,+j} \bar{a}_{+i} a_{+j} + \varepsilon R_{+i,-j} \bar{a}_{+i} \bar{a}_{-j} + S_{-i,+j} a_{-i} a_{+j} + \varepsilon T_{-i,-j} a_{-i} \bar{a}_{-j})$$
$$- \frac{\varepsilon}{2} \operatorname{Tr} Q_{++} - \frac{\varepsilon}{2} \operatorname{Tr} T_{--}. \tag{2.53}$$

2.4.2   Representation on the Fock Space

For charge conserving transformations it is convenient to modify the representations (2.30) a little bit: We add the term $\frac{\varepsilon}{2} \operatorname{Tr} X' = \frac{\varepsilon}{2} \operatorname{Tr} Q_{++} + \frac{\varepsilon}{2} \operatorname{Tr} T_{--}$ to $\hat{X}$ to cancel the last line of (2.53) and we define

$$t : \mathfrak{gl}(N, \mathbb{C}) \to \mathfrak{gl}(\mathcal{H}) \tag{2.54}$$
$$X' \mapsto t_{X'} := \hat{X} + \frac{1}{2} \operatorname{Tr} X'.$$

The modified representation has the property $\exp\left(2\pi i\, t_{E_{ii}}\right) = \exp\left(\pm 2\pi i\,\bar{a}_i a_i\right) = 1$ (no sum over i), because the spectrum of the counting operator $\bar{a}_i a_i$ is the set $\{0, 1\}$. Therefore (cf. Appendix B) the Lie algebra representation can be pressed down to a group representation [4],

$$T : \mathrm{Gl}(N, \mathbb{C}) \rightarrow \mathrm{Gl}(\mathcal{H}) \tag{2.55}$$
$$G' \mapsto T_{G'}.$$

The representation of the Lie algebra and the group representation are connected via

$$T_{\exp X'} = \exp t_{X'}. \tag{2.56}$$

From now on we write $G' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$, $G'' = \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix}$ and suppress the signs $+$ and $-$ in front of the indices of the blocks, more explicitly $A' = A_{++}$, $B' = B_{+-}$, $C' = C_{-+}$, $D' = D_{--}$ and $A'' = A_{--}$, $B'' = B_{-+}$, $C'' = C_{+-}$ and $D'' = D_{++}$. The operator $T_{G'}$ acts on the basic operators of the Weyl algebra by

$$T_{G'}\bar{a}_{+j}T_{G'}^{-1} = \bar{a}_{+i}A'_{ij} + a_{-i}C'_{ij}, \qquad T_{G'}\bar{a}_{-j}T_{G'}^{-1} = \bar{a}_{-i}A''_{ij} + a_{+i}C''_{ij}, \tag{2.57}$$
$$T_{G'}a_{+j}T_{G'}^{-1} = \bar{a}_{+i}B''_{ij} + a_{+i}D''_{ij}, \qquad T_{G'}a_{-j}T_{G'}^{-1} = \bar{a}_{+i}B'_{ij} + a_{-i}D'_{ij}, \tag{2.58}$$

where, according to (2.50), the primed variables are given by

$$A'' = \left((D' - C'A'^{-1}B')^{-1}\right)^t = \bar{D}' \tag{2.59}$$

$$B'' = \varepsilon\left((C' - D'B'^{-1}A')^{-1}\right)^t = \bar{C}' \tag{2.60}$$

$$C'' = \varepsilon\left((B' - A'C'^{-1}D')^{-1}\right)^t = \bar{B}' \tag{2.61}$$

$$D'' = \left((A' - B'D'^{-1}C')^{-1}\right)^t = \bar{A}'. \tag{2.62}$$

The left equalities are valid in general, provided that A', B', C' and D' are invertible, while the right equalities are only valid, if G' is pseudounitary (unitary). Because the transformations $T_{G'}$ conserve the charge, it is clear that the Hilbert space decomposes into subsectors characterized by a fixed charge Q under the representation T. These subspaces are irreducible under the representation T.

Following the line of section 2.3 we decompose $G' \in \mathrm{Gl}(p + q, \mathbb{C})$ into

$$G' = Z'_+ H' Z'_- = \begin{pmatrix} 1 & B'D'^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A' - B'D'^{-1}C' & 0 \\ 0 & D' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ D'^{-1}C' & 1 \end{pmatrix}. \tag{2.63}$$

and get

$$\langle 0|T_{H'} = (\mathrm{Det}\, D')^\varepsilon \langle 0|, \qquad T_{H'}|0\rangle = |0\rangle(\mathrm{Det}\, D')^\varepsilon, \tag{2.64}$$

$$\langle\bar{\Psi}|T_{Z'_+} = \exp\left(\varepsilon\bar{\psi}_+ B'D'^{-1}\bar{\psi}_-\right)\langle\bar{\Psi}|, \qquad T_{Z'_-}|\Psi\rangle = |\Psi\rangle\exp\left(\psi_- D'^{-1}C'\psi_+\right). \tag{2.65}$$

Finally we get for the matrix element of the representation T with respect to the standard coherent states

$$\langle\bar{\Psi}|T_{G'}|\Psi\rangle = (\mathrm{Det}\, D)^\varepsilon \exp\left(\varepsilon\bar{\psi}_+ B'D'^{-1}\bar{\psi}_-\right)\exp\left(\psi_- D'^{-1}C'\psi_+\right)$$
$$\exp\left(\bar{\psi}_+(A' - B'D'^{-1}C')\psi_+ - \varepsilon\psi_- D'^{-1}\bar{\psi}_-\right). \tag{2.66}$$

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[4] Note that $\mathrm{Gl}(N, \mathbb{C})$ is not simply connected and there could be a topological obstruction.

For a real transformation (i.e. G' is pseudounitary (unitary)) the last line of the previous expression simplifies to

$$\exp\left(\bar{\psi}_+(A' - B'D'^{-1}C')\psi_+ - \varepsilon\psi_- D'^{-1}\bar{\psi}_-\right) = \exp\left(-\varepsilon\left(\psi_+\bar{A}'^{-1}\bar{\psi}_+ + \psi_- D'^{-1}\bar{\psi}_-\right)\right).$$
(2.67)

2.4.3 The Color-Flavor Transformation

We consider a set of creation and annihilation operators $\bar{a}^i_A$ and a^i_A which carry two different indices. The lower index takes values $+a$ or $-a$, with range $a = 1, ..., N_f$, while the upper index takes values $i = 1, ..., N_c$. Having QCD in mind [5], we interpret the operators $\bar{a}^i_{+a}$ and $\bar{a}^i_{-a}$ as creation operators for quarks and antiquarks, respectively; the index i corresponds to the color (i.e. gauge) degrees of freedom, while the index a labels the different flavors of the quarks. (The quarks are regarded as being spinless.) The connection to the preceding subsections is built by the identifications $p = q = N_f N_c$, that is $N = p + q = 2N_f N_c$.

The Lie algebra $\mathfrak{gl}(2N_f N_c)$ has two subalgebras $\mathfrak{gl}(N_c)$ and $\mathfrak{gl}(2N_f)$, which are embedded in a natural way: A matrix $X \in \mathfrak{gl}(N_c)$ is identified with $I_{2N_f} \otimes X$, and a matrix $Y \in \mathfrak{gl}(2N_f)$ with $Y \otimes I_{N_c}$. Through these embeddings, $\mathfrak{gl}(N_c)$ and $\mathfrak{gl}(2N_f)$ form a pair of maximal commuting subalgebras of $\mathfrak{gl}(2N_f N_c)$. The subgroups $GL(N_c)$ and $GL(2N_f)$ are embedded into $GL(2N_f N_c)$ in the same way. We define the *color group* to be the subgroup $U(N_c)$ of $GL(N_c)$, and the *flavor group* to be the subgroup $U(N_f, N_f)$ (bosonic system) or $U(2N_f)$ (fermionic system) of $GL(2N_f)$.

Let now ψ^i_A and $\bar{\psi}^i_A$ be two independent vectors of complex variables (bosonic system) or Grassmann variables (fermionic system). We consider the following integral over the color group,

$$\mathcal{Z}(\psi, \bar{\psi}) := \int_{U(N_c)} dU \, \exp(\bar{\psi}^i_{+a} U^{ij} \psi^j_{+a} + \bar{\psi}^i_{-b} \bar{U}^{ij} \psi^j_{-b}).$$
(2.68)

The integration measure dU is the *Haar measure*, which is invariant under left- and right-multiplication and normalized by $\int_{U(N_c)} dU = 1$.

A key step towards the color-flavor transformation is the interpretation of $\mathcal{Z}(\psi, \bar{\psi})$ as matrix element of the *projector on the colorless sector* of the Fock space. We denote this projector by $\mathcal{P}$. An example for a colorless state is the vacuum, which transforms under the color group as $T_U|0\rangle = |0\rangle(\text{Det } U)^{\varepsilon N_f}$. We define the *colorless sector* (or flavor sector) to contain all states which transform under the color group like the vacuum, that is

$$T_U|\text{flavor}\rangle = (\text{Det } U)^{\varepsilon N_f}|\text{flavor}\rangle, \qquad \text{for all } U \in U(N_c).$$
(2.69)

Making use of the representation T, the projector $\mathcal{P}$ can be expressed as an integral over the color group,

$$\mathcal{P} = \int_{U(N_c)} \frac{dU}{(\text{Det } U)^{\varepsilon N_f}} T_U.$$
(2.70)

To prove (2.70) we first observe, that due to the normalization of the Haar measure, $\mathcal{P}$ acts as the identity on the colorless sector. Secondly, we get from the invariance of the Haar measure $T_U\mathcal{P} = (\text{Det } U)^{\varepsilon N_f}\mathcal{P}$, that is, $\mathcal{P}$ projects on the colorless sector.

---

[5]Nevertheless we still consider fermionic and bosonic systems.

From (2.66) and (2.67) we can read off the matrix elements of the representation of the color group with respect to the standard coherent states,

$$\langle \bar{\Psi}|T_U|\Psi\rangle = (\text{Det}\, U)^{\varepsilon N_f} \exp\left(\bar{\psi}^i_{+a} U^{ij}\psi^j_{+a} + \bar{\psi}^i_{-b}\bar{U}^{ij}\psi^j_{-b}\right) \tag{2.71}$$

and conclude that

$$\mathcal{Z}(\psi,\bar{\psi}) = \langle\bar{\Psi}|\mathcal{P}|\Psi\rangle. \tag{2.72}$$

By formula (2.70) we have expressed the projector $\mathcal{P}$ as an integral over the color group. The color-flavor transformation is established by expressing the projector in an alternative way. It turns out that an investigation of the colorless sector of the Fock space allows us to express $\mathcal{P}$ as an integral over the flavor group. The color group acts an the Fock space by

$$T_U \bar{a}_{+j} T_U^{-1} = \bar{a}_{+i} U_{ij}, \qquad T_U \bar{a}_{-j} T_U^{-1} = \bar{a}_{-i}\bar{U}_{ij}, \tag{2.73}$$

$$T_U a_{+j} T_U^{-1} = a_{+i}\bar{U}_{ij}, \qquad T_U a_{-j} T_U^{-1} = a_{-i} U_{ij}, \tag{2.74}$$

which are special cases of (2.57) and (2.58). Based on this formulas, one can describe the colorless sector of the Fock space, as it was done in [53]: It consists of the vacuum and *mesonic* excitations above it. A prototype of such an excitation is the one-meson state

$$|m_{+a,-b}\rangle = \sum_{i=1}^{N_c} \bar{a}^i_{+a}\bar{a}^i_{-b}|0\rangle. \tag{2.75}$$

By multiple action of the $\mathfrak{gl}(2N_f)$ generators $\bar{a}^i_{+a}\bar{a}^i_{-b}$ (we implicitly sum over the color indices), one can build states containing two and more mesons with different flavors. In the case of fermions the number of mesons is limited by $N_f N_c$. These states are automatically $U(N_c)$-invariant and conversely all $U(N_c)$-invariant states are linear combinations of such multi-meson states. The flavor group $U(N_f, N_f)$ (bosonic system) or $U(2N_f)$ (fermionic system) keeps the flavor sector invariant and acts irreducible on it.

Having analysed the flavor sector, we will express the projector $\mathcal{P}$ as an integral over the flavor group. For that purpose we will use coherent states in the spirit of Perelomov [37]. On the flavor sector, we build *generalized coherent states* by acting with the flavor group $G = U(N_f, N_f)$ (bosonic system) or $G = U(2N_f)$ (fermionic system) on the vacuum state,

$$|g\rangle := T_g|0\rangle, \qquad \langle g| := \langle 0|T_g^\dagger. \tag{2.76}$$

The crucial property of coherent states we will use now, is that they supply a resolution of unity. Because the flavor sector is irreducible under the action of the flavor group,

$$\mathcal{P} = \alpha_0 \int_G dg\, |g\rangle\langle g| \tag{2.77}$$

is the orthogonal projector on that sector, when the normalization constant α_0 is chosen appropriately. Indeed the operator on the r.h.s. commutes with all element of flavor group. (That can be shown by a simple calculation, which makes use of the invariance of the Haar measure.) Thus Schur's lemma ensures that it is proportional to the identity on the flavor sector, which is an irreducible space of this group. Because $\langle g|$ is orthogonal to the states

outside the flavor sector, it vanishes there. Finally $\mathcal{P}$ becomes the identity operator on the flavor sector, when we fix the normalization constant at

$$\alpha_0 = \left(\int_G dg \, \langle 0|T_g|0\rangle^2 \right)^{-1}. \tag{2.78}$$

For fermionic version of the color-flavor transformation, the value of the normalization constant is calculated in Chapter 3.

We have expressed the projector on the flavor space as an integral over the flavor group and can rewrite the partition function (2.68) as

$$\mathcal{Z}(\psi, \bar{\psi}) = \alpha_0 \int_G dg \, \langle \bar{\Psi}|g\rangle \langle g|\Psi\rangle. \tag{2.79}$$

On an open dense subset of G (defined by $\mathrm{Det}\, D \neq 0$) we have the generalized Gaussian decomposition

$$g = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix},$$
$$g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} g^\dagger \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon Y^\dagger \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F^\dagger & 0 \\ 0 & D^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon Z^\dagger & 1 \end{pmatrix}. \tag{2.80}$$

Using the decompositions of g and g^{-1} we can directly read off the overlaps between the standard coherent states and the coherent states of the flavor sector from (2.66),

$$\langle \bar{\Psi}|g\rangle = \langle \bar{\Psi}|T_g|0\rangle = (\mathrm{Det}\, D)^{\varepsilon N_c} \exp\left(\varepsilon \bar{\psi}^i_{+a} Z_{ab} \bar{\psi}^i_{-b} \right) \tag{2.81}$$

$$\langle g|\Psi\rangle = \langle 0|T_g^{-1}|\Psi\rangle = (\mathrm{Det}\, D^\dagger)^{\varepsilon N_c} \exp\left(\varepsilon \psi^j_{-b} Z^\dagger_{ab} \psi^j_{+a} \right). \tag{2.82}$$

We note that $DD^\dagger = (1 + \varepsilon Z^\dagger Z)^{-1}$ (cf. Appendix A) and therefore the integrand of (2.79) depends on the group elements g only via Z and $Z^\dagger$, i.e. on the right coset gH of g, where H is the subgroup $H = \mathrm{U}(N) \times \mathrm{U}(N)$ of G. Thus the integral in (2.79) can be restricted to the coset space G/H, which is parameterized by the complex matrices $Z \in \mathbb{C}^{N_f \times N_f}$. The matrices Z are arbitrary for the fermionic system and constrained for the bosonic system, namely by the requirement that the eigenvalues of the positive semidefinite matrix $ZZ^\dagger$ are less than unity. The invariant measure on the coset space G/H is (cf. Appendix A),

$$D\mu(Z, Z^\dagger) = C_{N_f} \, \mathrm{Det}(1 + \varepsilon ZZ^\dagger)^{-2N_f} \prod_{i,j} dZ_{ij} \, d\bar{Z}_{ij}. \tag{2.83}$$

Finally we get the following expressions for the bosonic and the fermionic partition function,

$$\mathcal{Z}_{\mathrm{B}}(\psi, \bar{\psi}) = \alpha_0 \int_{\mathbb{C}^{N_f \times N_f}} D\mu(Z, Z^\dagger) \, \mathrm{Det}(1 - ZZ^\dagger)^{N_c} \exp\left(\bar{\psi}^i_{+a} Z_{ab} \bar{\psi}^i_{-b} + \psi^j_{-b} Z^\dagger_{ba} \psi^j_{+a} \right), \tag{2.84}$$

$$\mathcal{Z}_{\mathrm{F}}(\psi, \bar{\psi}) = \alpha_0 \int_{\mathbb{C}^{N_f \times N_f}} D\mu(Z, Z^\dagger) \, \mathrm{Det}(1 + ZZ^\dagger)^{-N_c} \exp\left(\bar{\psi}^i_{+a} Z_{ab} \bar{\psi}^i_{-b} + \psi^j_{-b} Z^\dagger_{ba} \psi^j_{+a} \right). \tag{2.85}$$

To bring the bosonic integral into the form (2.84), we have applied the substitution $(Z, Z^\dagger) \mapsto (-Z, -Z^\dagger)$. While fermionic partition is a convergent integral for all values of N_f and N_c,

the number of flavors has to be restricted by $2N_f \leq N_c$ for the bosonic field theory. (In polar coordinates with radius r the integrand of (2.84) behaves like $(1 - r)^{N_c - 2N_f}$ for $r \to 1$.) Note that we have expressed the original partition function in terms of the matrix U and the complex conjugated matrix $\bar{U}$, while the transformed partition function is written down as a function of the matrix Z and the Hermitian conjugated matrix $Z^\dagger$. This convention allows us to avoid minus signs in the expressions of both, the bosonic and the fermionic version of the transformation.

The scheme which converts the partition function (2.68) to the expression $\mathcal{Z}_\mathrm{B}$ or $\mathcal{Z}_\mathrm{F}$ is called *color-flavor transformation*. Indeed it transforms an action $S(U) = \bar{\psi}^i_{+a} U^{ij} \psi^j_{+a} + \bar{\psi}^i_{-b} \bar{U}^{ij} \psi^j_{-b}$ which couples the color degrees of freedom of the quark fields through the gauge field U into an action $S(Z, Z^\dagger) = \psi^i_{+a} Z_{ab} \psi^i_{-b} + \bar{\psi}^j_{-b} Z^\dagger_{ba} \bar{\psi}^j_{+a}$ which couples the flavor degrees of freedom through the mesonic [6] field Z. The scheme was discovered in 1996 by Zirnbauer and initially formulated in a supersymmetric setting [53]. The pure bosonic and pure fermionic versions can be derived from the supersymmetric color-flavor transformation by putting the fermionic or the bosonic components of the field to zero. After that it remains to do the Berezin integral. We have done so (cf. Appendix C) and obtained a result which is consistent with (2.84) and (2.85).

The most general formulation of the color-flavor transformation involves a number $N_{f,\mathrm{B}}$ of bosons and a number $N_{f,\mathrm{F}}$ of fermions. For reasons of convergence these numbers have to be restricted to $2(N_{f,\mathrm{B}} - N_{f,\mathrm{F}}) \leq N_c$.

2.5 Outlook: Dual Pairs

The color-flavor transformation relates an integral over the color group $K = \mathrm{U}(N_c)$ to an integral over the flavor group $G = \mathrm{U}(N_f, N_f)$ or $G = \mathrm{U}(2N_f)$. Both groups are subgroups of the group of real, canonical transformations $S = \mathrm{Sp}(2N_f N_c, \mathbb{R})$ or $S = \mathrm{O}(2N_f N_c, \mathbb{R})$. The pair of subgroups (K, G) of S has the property that K is the centralizer [7] of G and vice versa. According to Howe such a pair of groups is called a *dual pair* in S and there is a decomposition of the oscillator representation in terms of representations of the members of each dual pair in the symplectic group [23]. The color-flavor transformation, which rests on two different descriptions of the projector on the flavor sector, is intimately related to that structure. The classifications of irreducible dual pairs in the symplectic group is an interesting but manageable problem [23]. We are interested in dual pairs where one member (the color group) is compact. A complete list of these dual pairs is shown in Table II. The

(K, G) dual pair in	K	G	H	class of G/H
$\mathrm{Sp}(2n(p + q), \mathbb{R})$	$\mathrm{U}(n)$	$\mathrm{U}(p, q)$	$\mathrm{U}(p) \times \mathrm{U}(q)$	AIII
$\mathrm{Sp}(2nm, \mathbb{R})$	$\mathrm{O}(n)$	$\mathrm{Sp}(2m, \mathbb{R})$	$\mathrm{U}(m)$	CI
$\mathrm{Sp}(4nm, \mathbb{R})$	$\mathrm{Sp}(2n)$	$\mathrm{O}^*(2m)$	$\mathrm{U}(m)$	DIII

Table II: Dual pairs in the symplectic group

corresponding dual pairs in the orthogonal group are listed in Table III. There is a color-flavor transformation corresponding to each dual pair of the two above lists, that is the color group

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[6] This term is justified, because Z couples to a color singlet built by quark-antiquark pair.

[7] The centralizer K of G in S contains all elements of S which commute with each element of G.

(K, G) dual pair in	K	G	H	class of G/H
$O(2n(p+q))$	$U(n)$	$U(p+q)$	$U(p) \times U(q)$	AIII
$O(2nm)$	$O(n)$	$O(2m)$	$U(m)$	DIII
$O(4nm)$	$Sp(2n)$	$Sp(2m)$	$U(m)$	CI

Table III: Dual pairs in the orthogonal group

$U(N_c)$ can be replaced by each of the compact groups $O(N_c)$ and $Sp(2N_c)$. The integral over the group is replaced by an integral over the symmetric space G/H formed by quotienting the flavor group by a subgroup. The class printed in the tables refers to Cartan's complete list of the symmetric spaces [22]. Recently this Cartan's classification was extended to a superanalytic setting, leading to a complete list of Riemannian symmetric superspaces [52]. Some of the latter spaces are involved in the color-flavor transformations of systems with bosons *and* fermions.

Finally we point out that the gauge group of quantum chromodynamics, the *special unitary group* $SU(N_c)$, does not occur as a member of a dual pair and the derivation of the corresponding color-flavor transformation involves a special technique.

Chapter 3

The Color-Flavor Transformation for the Special Unitary Group

In the last chapter we have reviewed the color-flavor transformation for the unitary group. The purpose of this chapter is to generalize the transformation to the *special unitary group*. We define the color group now to be the smaller group $\mathrm{SU}(N_c)$.

Again, the central idea is to express the projector on the colorless sector of the Fock space in two different ways. In doing so we have to take into account that the colorless sector for the special unitary group is larger than the colorless sector for the unitary group. Restricting ourselves to the case of fermions, two kinds of colorless particles can be composed of the quarks: *mesons* which are quark-antiquark pairs and the *baryons* which are totally antisymmetric combinations of N_c quarks. While the mesonic excitations over the vacuum of the Fock space were already considered in Chapter 2, we additionally have to take into consideration the baryons when we replace the color group $\mathrm{U}(N_c)$ by $\mathrm{SU}(N_c)$.

3.1 The Partition Function as an Integral over the Color Group

We restrict ourselves to a fermionic system and let ψ_A^i, $\bar{\psi}_A^i$ be two vectors of Grassmann variables. We consider the following partition function which is defined as an integral over the color group,

$$\mathcal{Z}(\psi, \bar{\psi}) = \int_{\mathrm{SU}(N_c)} dU \, \exp(\bar{\psi}_{+a}^i U^{ij} \psi_{+a}^j + \bar{\psi}_{-b}^i \bar{U}^{ij} \psi_{-b}^j). \tag{3.1}$$

Again, the first step towards the color-flavor transformation is to express the partition function as a matrix element of the projector $\mathcal{P}$ on the colorless sector,

$$\mathcal{Z}(\psi, \bar{\psi}) = \langle \bar{\Psi} | \mathcal{P} | \Psi \rangle. \tag{3.2}$$

The colorless sector (or flavor sector) is given by the set of states which transform trivially under the color group, that is

$$T_U |\text{flavor}\rangle = |\text{flavor}\rangle, \qquad \text{for all } U \in \mathrm{SU}(N_c). \tag{3.3}$$

Equation (3.2) can be established by expressing the projector $\mathcal{P}$ as an integral over the color group,

$$\mathcal{P} = \int_{\mathrm{SU}(N_c)} dU \, T_U. \tag{3.4}$$

Thanks to the restriction from the unitary group to the special unitary group there is no factor $(\operatorname{Det} U)^{\varepsilon N_f}$ in the last two formulas.

3.2   The Flavor Sector

To express the projector $\mathcal{P}$ as an integral over the flavor group, we investigate the colorless sector of the Fock space. The set of $\mathrm{SU}(N_c)$-colorless states contains the mesonic excitations, which were already considered in Chapter 2, but is larger and, moreover, contains states with *baryons*. A prototype of the latter states is a baryon with flavors $a_1, \ldots, a_{N_c}$ which is defined as

$$|b_{A_1 \ldots A_{N_c}}\rangle := \frac{1}{N_c!} \varepsilon_{i_1 \ldots i_{N_c}} \bar{a}^{i_1}_{A_1} \cdot \ldots \cdot \bar{a}^{i_{N_c}}_{A_{N_c}} |0\rangle. \tag{3.5}$$

The indices $A_k = \pm a_k$ have to be taken either all positive (baryon), or all negative (antibaryon). A matrix $U \in \mathrm{GL}(N_c)$ acts on this state by multiplication with $\operatorname{Det}(U)$ (baryon) or $\operatorname{Det}^{-1}(U)$ (antibaryon). Therefore these states are invariant under the color group $\mathrm{SU}(N_c)$. We define the *baryon number* operator by

$$\hat{B} := \frac{1}{N_c} \hat{Q} = \frac{1}{N_c} \left(\sum_{i=1}^{N_c} \bar{a}_{+i} a_{+i} - \sum_{i=1}^{N_c} \bar{a}_{-i} a_{-i} \right). \tag{3.6}$$

The above baryon (antibaryon) is an eigenstate of the baryon number operator $\hat{B}$ with eigenvalue $B = 1$ (-1). More generally, $\hat{B}$ counts the number of baryons minus the number of antibaryons contained in a colorless state. Acting on the state (3.5) with the generators of the flavor algebra $\mathfrak{gl}(2N_f)$ one builds other colorless states with the same baryon number. These states form an irreducible subspace for the flavor group $\mathrm{U}(2N_f)$: the one-baryon and the one-antibaryon sector, respectively.

The one-baryon sector can be generated from the state (3.5) with all $a_j = 1$. One can similarly build B-baryon and B-antibaryon states by

$$|\mathcal{B}_B\rangle := \prod_{a=1}^{B} \bar{f}^1_{+a} \cdot \ldots \cdot \bar{f}^{N_c}_{+a} |0\rangle, \quad |\mathcal{B}_0\rangle := |0\rangle, \quad |\mathcal{B}_{-B}\rangle := \prod_{a=1}^{B} \bar{f}^1_{-a} \cdot \ldots \cdot \bar{f}^{N_c}_{-a} |0\rangle. \tag{3.7}$$

In doing so the value of the baryon number B is restricted by Pauli's exclusion principle and ranges from $-N_f$ to N_f. As with the one-baryon state, acting on $|\mathcal{B}_B\rangle$ with the algebra $\mathfrak{gl}(2N_f)$ builds the full B-baryon part of the flavor sector, so that the group $\mathrm{U}(2N_f)$ acts irreducibly on this part. This can be proved by making use of the dual-pair property of the subalgebras $\mathfrak{gl}(2N_f)$ and $\mathfrak{gl}(N_c)$, as exposed in [24].

To summarize, the flavor sector of Fock space decomposes into $2N_f + 1$ subsectors, characterized by the baryon number B. Each sector carries an irreducible unitary representation of the flavor group $\mathrm{U}(2N_f)$.

3.3   The Partition Function as an Integral over the Flavor Group

Having explored the structure of the flavor sector, we are now in position to express the projector $\mathcal{P}$ as an integral over the flavor group. For this purpose we will use the coherent

states, in the spirit of Perelomov [37]. On each subsector with a fixed baryon number B, we consider the *generalized coherent states* built by the action of $G := \mathrm{U}(2N_f)$ on the reference state $|\mathcal{B}_B\rangle$, i. e. the states

$$|g_B\rangle := T_g |\mathcal{B}_B\rangle, \qquad \langle g_B| := \langle\mathcal{B}_B|T_g^\dagger, \qquad \text{for all } B = -N_f, ..., N_f. \tag{3.8}$$

The crucial property of coherent states is that they supply a resolution of unity. Because of the irreducibility of the $\mathrm{U}(2N_f)$ action on each subsector of baryon number B, the operator

$$\mathcal{P}_B := \alpha_B \int_G dg \, |g_B\rangle\langle g_B| \tag{3.9}$$

coincides with the orthogonal projector on that subsector, provided that the normalization constant α_B is chosen in an appropriate way. Indeed, by the invariance of the Haar measure the operator $\mathcal{P}_B$ commutes with every element of the flavor group; Schur's lemma then ensures that it is proportional to the identity on each irreducible space of this group, thus on each subsector of a fixed baryon number. Owing to orthogonality, $\mathcal{P}_B$ vanishes on all subsectors with $B' \neq B$, whereas it is the identity on the B-subsector if the normalization constant is fixed at

$$\alpha_B = \left(\int_G dg \, |\langle\mathcal{B}_B|T_g|\mathcal{B}_B\rangle|^2 \right)^{-1}. \tag{3.10}$$

Some particular values of the constant (namely α_0 and $\alpha_{\pm 1}$) are computed in Subsection 3.6. The projector $\mathcal{P}$ on the full flavor sector is an orthogonal sum of the projectors on the subspaces with fixed baryon number B,

$$\mathcal{P} = \bigoplus_{B=-N_f}^{N_f} \mathcal{P}_B, \tag{3.11}$$

and its matrix elements can be written as

$$\mathcal{Z}(\psi, \bar\psi) = \sum_{B=-N_f}^{N_f} \alpha_B \int_G dg \, \langle\bar\Psi|g_B\rangle\langle g_B|\Psi\rangle. \tag{3.12}$$

With equation (3.12) we have obtained a representation of the partition function as an integral over the flavor group.

3.4   Gaussian Decomposition

To compute the overlaps $\langle\bar\Psi|g_B\rangle$ and $\langle g_B|\Psi\rangle$, we again make use of the Gaussian decomposition of $G = \mathrm{U}(2N_f)$: almost any matrix $g \in G$ can be factored into

$$g = z_+ h z_- = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}, \tag{3.13}$$

where the relations $Z = BD^{-1}$, $Y = D^{-1}C$, and $F = A - BD^{-1}C$ hold. The decomposition becomes singular if D does, but this happens only on a submanifold of codimension one (and

hence measure zero) of G. The unitarity of g implies $\tilde{Z} = -D^{-1}Z^\dagger A$ and allows to write the central matrix in the form

$$\begin{pmatrix} F & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} (1 + ZZ^\dagger)^{1/2} & 0 \\ 0 & (1 + Z^\dagger Z)^{-1/2} \end{pmatrix} \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{V} \end{pmatrix}, \tag{3.14}$$

with two unitary $N_f \times N_f$-matrices $\mathcal{U}$ and $\mathcal{V}$. $\begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{V} \end{pmatrix}$ is an element of the diagonal $U(N_f) \times U(N_f)$ subgroup of G, which we call H. It can thus be shown that the elements g of an open dense subset of G are in one-to-one correspondence with the triplets $(Z, \mathcal{U}, \mathcal{V})$, where the pair $\mathrm{diag}(\mathcal{U}, \mathcal{V})$ is an element of H, while Z represents a point in the coset space G/H and can be any complex $N_f \times N_f$ matrix. Moreover, the Haar measure dg of G factorizes into

$$\int_G dg = \int_{G/H} d(gH) \int_H dh = \int_{\mathbb{C}^{N_f \times N_f}} d\mu(Z, Z^\dagger) \int_H d\mathcal{U}\, d\mathcal{V}. \tag{3.15}$$

Both $d\mathcal{U}$ and $d\mathcal{V}$ are normalized Haar measures on $U(N_f)$, and

$$d\mu(Z, Z^\dagger) = C_{N_f} \mathrm{Det}(1 + ZZ^\dagger)^{-2N_f} \prod_{i,j} dZ_{ij} d\bar{Z}_{ij} \tag{3.16}$$

is the normalized invariant measure on G/H. The normalization factor C_{N_f} is computed in Section 3.6; see Eq. (3.46).

We now explain how to use this decomposition to compute the overlaps. The Gaussian decomposition (3.13) carries over to any representation of G, so for every $g \in G$ we can write the operator T_g as

$$T_g = T_{z_+} T_h T_{z_-}. \tag{3.17}$$

According to the relations (2.57) and (2.58), the factors T_ζ and $T_{\tilde{\zeta}}$ act trivially on the reference states:

$$T_{\tilde{\zeta}}|\mathcal{B}_B\rangle = |\mathcal{B}_B\rangle, \text{ and } \langle\mathcal{B}_B|T_\zeta = \langle\mathcal{B}_B|, \qquad \text{for all } B = -N_f, ..., +N_f. \tag{3.18}$$

The action of the block-diagonal operator is slightly more subtle. [1]

$$T_h|0\rangle = (\mathrm{Det}\, D)^{N_c}|0\rangle,$$

$$T_h|\mathcal{B}_1\rangle = (\mathrm{Det}\, D)^{N_c} \prod_{i=1}^{N_c} F_{a1} \bar{f}^i_{+a}|0\rangle, \tag{3.19}$$

$$T_h|\mathcal{B}_{-1}\rangle = (\mathrm{Det}\, D)^{N_c} \prod_{i=1}^{N_c} (D^{-1})_{1a} \bar{f}^i_{-a}|0\rangle.$$

(To make sense of these formulas one must remember that we are using the summation convention: the flavor index a under the product is understood to be summed over.) These

---

[1] $T_h|\mathcal{B}_1\rangle = T_h \prod_{i=1}^{N_c} \bar{a}^i_{+1}|0\rangle = \prod_{i=1}^{N_c} T_h \bar{a}^i_{+1} T_h^{-1}|0\rangle =_{(2.57)} \prod_{i=1}^{N_c} \bar{a}^i_{+a} F_{a1}|0\rangle$

formulas directly yield the desired overlaps with $\langle\bar{\Psi}|$:

$$\langle\bar{\Psi}|g_0\rangle = (\text{Det }D)^{N_c} \prod_{i=1}^{N_c} \exp(\bar{\psi}^i_{+a} Z_{ab} \bar{\psi}^i_{-b}) \,,$$

$$\langle\bar{\Psi}|g_1\rangle = (\text{Det }D)^{N_c} \prod_{i=1}^{N_c} \bar{\psi}^i_{+c} F_{c1} \exp(\bar{\psi}^i_{+a} Z_{ab} \bar{\psi}^i_{-b}) \,,$$

$$\langle\bar{\Psi}|g_{-1}\rangle = (\text{Det }D)^{N_c} \prod_{i=1}^{N_c} (D^{-1})_{1c} \bar{\psi}^i_{-c} \exp(\bar{\psi}^i_{+a} Z_{ab} \bar{\psi}^i_{-b}) \,,$$

as well as the overlaps with $|\Psi\rangle$:

$$\langle g_0|\Psi\rangle = (\text{Det }D^\dagger)^{N_c} \prod_{i=1}^{N_c} \exp(\psi^i_{-a} Z^\dagger_{ab} \psi^i_{+b}) \,,$$

$$\langle g_1|\Psi\rangle = (\text{Det }D^\dagger)^{N_c} \prod_{i=1}^{N_c} F^\dagger_{1c} \psi^i_{+c} \exp(\psi^i_{-a} Z^\dagger_{ab} \psi^i_{+b}) \,,$$

$$\langle g_{-1}|\Psi\rangle = (\text{Det }D^\dagger)^{N_c} \prod_{i=1}^{N_c} \psi^i_{-c} (D^{-1})^\dagger_{c1} \exp(\psi^i_{-a} Z^\dagger_{ab} \psi^i_{+b}) \,.$$

The overlaps with the coherent states $|g_B\rangle$ containing more than one baryon ($|B| > 1$) can be computed in the same way; in front of the exponential factors, there will be $|B|$ similar products, with flavor indices $1, \ldots, |B|$.

We now insert the above expressions for the overlaps into (3.12), and use the factorization (3.15) to arrive at an integral over triples $(Z, \mathcal{U}, \mathcal{V})$. Leaving the Z-integral for later, we next carry out the integrations over the unitary matrices $\mathcal{U}$ and $\mathcal{V}$. They enter in the overlaps via the matrix elements of F and D; see Eq. (3.14). To simplify the notation, we first perform a flavor rotation on the Grassmann fields:

$$\phi^i_{+b} = (1 + ZZ^\dagger)^{1/2}_{ba} \psi^i_{+a}, \qquad \phi^i_{-b} = \psi^i_{-a}(1 + Z^\dagger Z)^{1/2}_{ab}, \tag{3.20}$$

$$\bar{\phi}^i_{+b} = \bar{\psi}^i_{+a}(1 + ZZ^\dagger)^{1/2}_{ab}, \qquad \bar{\phi}^i_{-b} = (1 + Z^\dagger Z)^{1/2}_{ba} \bar{\psi}^i_{-a}. \tag{3.21}$$

The integrals we need to compute then read as follows (assuming $B > 0$):

$$\chi_B(\bar{\phi}_+, \phi_+) := \alpha_B \int_{\mathrm{U}(N_f)} d\mathcal{U} \prod_{c=1}^{B} \prod_{i=1}^{N_c} (\bar{\phi}^i_{+a} \mathcal{U}_{ac})(\phi^i_{+b} \mathcal{U}^{-1}_{cb}), \tag{3.22}$$

$$\chi_{-B}(\bar{\phi}_-, \phi_-) := \alpha_{-B} \int_{\mathrm{U}(N_f)} d\mathcal{V} \prod_{c=1}^{B} \prod_{i=1}^{N_c} (\bar{\phi}^i_{-a} \mathcal{V}^{-1}_{ca})(\phi^i_{-b} \mathcal{V}_{bc}). \tag{3.23}$$

We also set $\chi_0 := \alpha_0$, $\chi_B(\bar{\psi}, \psi; Z) := \chi_B(\bar{\phi}_+, \phi_+)$, and $\chi_{-B}(\bar{\psi}, \psi; Z) := \chi_{-B}(\bar{\phi}_-, \phi_-)$. The function $\chi_1(\bar{\phi}_+, \phi_+)$ will play a distinguished role in the lattice gauge theory application in Chapter 7, and we therefore evaluate it explicitly in the next section.

Having done the integration over H, we are left with an integral over G/H, i.e. over a Z-dependent integrand, in each B-subsector. Putting everything together, we finally arrive at the following identity:

$$
\int_{\mathrm{SU}(N_c)} dU \ \exp(\bar{\psi}^i_{+a} U^{ij} \psi^j_{+a} + \bar{\psi}^i_{-b} \bar{U}^{ij} \psi^j_{-b})
$$

$$
= \sum_{B=-N_f}^{N_f} \int_{\mathbb{C}^{N_f \times N_f}} \frac{d\mu(Z, Z^\dagger)}{\mathrm{Det}(1 + ZZ^\dagger)^{N_c}} \ \chi_B(\bar{\psi}, \psi; Z) \exp(\bar{\psi}^i_{+a} Z_{ab} \bar{\psi}^i_{-b} + \psi^j_{-b} Z^\dagger_{ba} \psi^j_{+a}),
$$

(3.24)

which is called the color-flavor transformation for $\mathrm{SU}(N_c)$, and it is the central result of the present chapter. Note that the right-hand side of the transformation has the attractive feature of organizing the contributions according to the different baryonic sectors.

3.5   The Functions $\chi_1(\bar{\phi}_+, \phi_+)$ and $\chi_{-1}(\bar{\phi}_-, \phi_-)$

We consider the coefficient in front of the contribution from the one-baryon subsector to the partition function,

$$
\chi_{+1}(\bar{\phi}_+, \phi_+) = \alpha_1 \int_{\mathrm{U}(N_f)} d\mathcal{U} \ \prod_{i=1}^{N_c} (\bar{\phi}^i_{+a} \mathcal{U}_{a1})(\phi^i_{+b} \bar{\mathcal{U}}_{b1}).
$$

(3.25)

Note that only the first column of the unitary matrix $\mathcal{U}$ occurs in the integrand, so the integral is effectively over an unit sphere in N_f-dimensional complex space, $S^{2N_f - 1} = \mathbb{C}^{N_f}/\mathbb{R}_+$. Parametrizing the latter by a complex vector $z = (z_1, \ldots, z_{N_f})$ with unit norm $|z| = 1$, we have

$$
\chi_{+1}(\bar{\phi}_+, \phi_+) = \alpha_{+1} \frac{\int_{|z|=1} d\Omega(z, \bar{z}) \prod_{i=1}^{N_c} (\bar{\phi}^i_{+a} z_a)(\phi^i_{+b} \bar{z}_b)}{\int_{|z|=1} d\Omega(z, \bar{z})},
$$

(3.26)

where $d\Omega(z, \bar{z})$ is an $\mathrm{U}(N_f)$-invariant measure on the unit sphere $|z| = 1$. Because of the homogeneity in z and $\bar{z}$, we may apply the trick of replacing the numerator by an integral over $\mathbb{C}^{N_f}$, with a Gaussian weight function $\exp(-|z|^2)$ included in the integrand. This integral can be generated by differentiating the function

$$
I(J, \bar{J}) := \pi^{-N_f} \int_{\mathbb{C}^{N_f}} dz d\bar{z} \ = \exp(\bar{J}J)
$$

(3.27)

with respect to the bosonic currents J, $\bar{J}$ which are two independent vectors in flavor space,

$$
\prod_{i=1}^{N_c} \left(\bar{\phi}^i_{+a} \frac{\partial}{\partial \bar{J}_a} \phi^i_{+b} \frac{\partial}{\partial J_b} \right) \bigg|_{\bar{J}=J=0} I(J, \bar{J}) = \pi^{-N_f} \int_{\mathbb{C}^{N_f}} dz d\bar{z} \ \exp(-\bar{z}z) \prod_{i=1}^{N_c} \left(\bar{\phi}^i_{+a} z_a \phi^i_{+b} z_b \right)
$$

$$
= \pi^{-N_f} \left(\int_0^\infty dr \ r^{2N_c + 2N_f - 1} \right) \left(\int_{|z|=1} d\Omega(z, \bar{z}) \prod_{i=1}^{N_c} \left(\bar{\phi}^i_{+a} z_a \phi^i_{+b} z_b \right) \right)
$$

(3.28)

$$
= \pi^{-N_f} \left(\int_0^\infty dr \ r^{2N_c + 2N_f - 1} \right) \left(\int_{|z|=1} d\Omega(z, \bar{z}) \right) \frac{1}{\alpha_1} \chi_{+1}(\bar{\phi}_+, \phi_+)
$$

$$
= \pi^{-N_f} \frac{1}{2} \Gamma(N_c + N_f) \frac{2 N_f \pi^{N_f}}{\Gamma(N_f + 1)} \frac{1}{\alpha_1} \chi_{+1}(\bar{\phi}_+, \phi_+) = \frac{(N_c + N_f - 1)!}{(N_f - 1)!} \frac{1}{\alpha_1} \chi_{+1}(\bar{\phi}_+, \phi_+).
$$

The crucial point in the calculations above is the homogeneity of degree $2N_c$ of the integrand, which allows to split the integral over $\mathbb{C}^{N_f}$ into an integral over $\mathbb{R}_+$ and the sphere $S^{2N_f - 1}$.

On the other hand the derivative of the generating function can be calculated by differentiating $\exp(\bar{J}J) = \prod_{c=1}^{N_f} \exp(\bar{J}_c J_c)$,

$$\prod_{i=1}^{N_c} \left(\bar{\phi}_{+a}^i \frac{\partial}{\partial \bar{J}_a} \phi_{+b}^i \frac{\partial}{\partial J_b} \right) \Bigg|_{\bar{J}=J=0} I(J, \bar{J}) = \prod_{i=1}^{N_c} \left(\bar{\phi}_{+a}^i \frac{\partial}{\partial \bar{J}_a} \phi_{+b}^i \frac{\partial}{\partial J_b} \right) \Bigg|_{\bar{J}=J=0} \prod_{c=1}^{N_f} \bar{J}_c J_c$$
$$= \sum_{\sigma \in \mathfrak{S}_{N_c}} \mathrm{sgn}(\sigma) \prod_{i=1}^{N_c} \left(\bar{\phi}_{+a}^i \phi_{+a}^{\sigma(i)} \right), \tag{3.29}$$

where $\mathfrak{S}_{N_c}$ denotes the group of permutations of the numbers $1, \ldots, N_c$.

Recognizing that the last expression is Leibniz' formula for a determinant, we obtain the final result

$$\chi_{+1}(\bar{\phi}_+, \phi_+) = \alpha_{+1} \frac{(N_f - 1)!}{(N_c + N_f - 1)!} \mathrm{Det}(\bar{\phi}_+ \phi_+^t). \tag{3.30}$$

The $N_c \times N_c$-matrix under the determinant is given by

$$(\bar{\phi}_+ \phi_+^t)^{ij} := \bar{\phi}_{+a}^i \phi_{+a}^j = \bar{\psi}_{+a}^i (1 + ZZ^\dagger)_{ab} \bar{\psi}_{+b}^j \tag{3.31}$$

In an analogous way we obtain for the coefficient in front of the contribution from the one-antibaryon sector

$$\chi_{-1}(\bar{\phi}_-, \phi_-) = \alpha_{-1} \frac{(N_f - 1)!}{(N_c + N_f - 1)!} \mathrm{Det}(\bar{\phi}_- \phi_-^t), \tag{3.32}$$

where

$$(\bar{\phi}_- \phi_-^t)^{ij} := \bar{\phi}_{-a}^i \phi_{-a}^j = \bar{\psi}_{-a}^i (1 + Z^\dagger Z)_{ab} \bar{\psi}_{-b}^j. \tag{3.33}$$

After our work on the $\mathrm{SU}(N_c)$ generalization had been completed and presented on a conference [11], we learned that it also has been worked out by Schlittgen and Wettig [39]. Their result includes the evaluation of the coefficient $\chi_B(\bar{\psi}, \psi)$ for all baryon numbers $B = -N_f, \ldots, +N_f$ and coincides with (3.30) and (3.32) for $B = +1$ and $B = -1$.

3.6   The Normalization Constants α_{-1}, α_0, α_1

We are going to calculate the normalization constants $\alpha_B^{-1} = \int_G dg \, |\langle B_B | T_g | B_B \rangle|^2$ introduced in Eq. (3.10) – for the values $B = 1, 0, -1$. To that end, we employ the decomposition of the group $G = \mathrm{U}(2N_f)$ given by Eqs. (3.13) and (3.14). This yields

$$|\langle B_1 | T_g | B_1 \rangle|^2 = \frac{|(\sqrt{1 + ZZ^\dagger})_{1a} \mathcal{U}_{a1}|^{2N_c}}{\mathrm{Det}(1 + ZZ^\dagger)^{N_c}} \tag{3.34}$$

for $B = 1$, and similar expressions for the other two cases. The first step now is to do the integral over $\mathcal{U} \in \mathrm{U}(N_f)$, which for $B = \pm 1$ is effectively an integral over a $(2N_f - 1)$-dimensional sphere. Carrying it out by the method of Section 3.5, we get the preliminary expressions

$$\alpha_0^{-1} = C_{N_f} \int_{\mathbb{C}^{N_f \times N_f}} \frac{dZ dZ^\dagger}{\mathrm{Det}(1 + ZZ^\dagger)^{2N_f + N_c}}, \tag{3.35}$$

$$\alpha_1^{-1} = \alpha_{-1}^{-1} = C_{N_f} \frac{(N_f - 1)! N_c!}{(N_c + N_f - 1)!} \int_{\mathbb{C}^{N_f \times N_f}} \frac{[(1 + Z^\dagger Z)_{11}]^{N_c} dZ dZ^\dagger}{\mathrm{Det}(1 + ZZ^\dagger)^{2N_f + N_c}}, \tag{3.36}$$

where C_{N_f} is defined by

$$C_{N_f}^{-1} = \int_{\mathbb{C}^{N_f \times N_f}} \frac{dZ dZ^\dagger}{\mathrm{Det}(1 + ZZ^\dagger)^{2N_f}}. \tag{3.37}$$

For later convenience, we have made a change of integration variables $Z \leftrightarrow Z^\dagger$ in the numerator of the integral in (3.36).

In the second step we perform the integration over the $N_f \times N_f$ matrix Z using a recursion procedure similar to that in [25]. From here on we use the simplified notation $n = N_f$. The recursion consists of slicing the matrix Z into vertical vectors, step by step. We now detail the first step of the recursion. We decompose Z as $Z = (Z_{n,n-1}, z_1)$, where z_1 is a (column) n-vector, and $Z_{n,n-1}$ is a $n \times (n-1)$ matrix. We then have the expressions

$$ZZ^\dagger = Z_{n,n-1}Z_{n,n-1}^\dagger + z_1 z_1^\dagger,$$
$$(Z^\dagger Z)_{11} = (Z_{n,n-1}^\dagger Z_{n,n-1})_{11}. \tag{3.38}$$

Using the (positive definite) $n \times n$ matrix Γ_1 which is defined as the square root of

$$\Gamma_1^2 = 1 + Z_{n,n-1}Z_{n,n-1}^\dagger, \tag{3.39}$$

we make a change of variables, from z_1 to $w_1 = \Gamma_1^{-1} z_1$. From $1 + ZZ^\dagger = \Gamma_1(1 + w_1 w_1^\dagger)\Gamma_1$, we get the relation

$$\mathrm{Det}(1 + ZZ^\dagger) = (1 + w_1^\dagger w_1)\,\mathrm{Det}(1 + Z_{n,n-1}Z_{n,n-1}^\dagger). \tag{3.40}$$

The change of variables from Z to $\{Z_{n,n-1}, w_1\}$ has the Jacobian $\mathrm{Det}(1 + Z_{n,n-1}Z_{n,n-1}^\dagger)$. Each of the integrals (3.35), (3.36), and (3.37) can now be written as the product of a $Z_{n,n-1}$-integral times a w_1-integral.

The former can in turn be expressed as the product of a $Z_{n,n-2}$-integral times a w_2-integral (with w_2 a n-vector), which can be decomposed in turn, and so on, until we reach, at the n-th step, a $Z_{n,1}$-integral, i.e. an integral over the first column of the original matrix Z. We call this column vector w_n for reasons of homogeneity.

The successive Jacobians multiply to give the following integration measure:

$$dZdZ^\dagger = dw_1^\dagger dw_1\,(1 + w_2^\dagger w_2)dw_2^\dagger dw_2 \cdots (1 + w_n^\dagger w_n)^{n-1} dw_n dw_n^\dagger. \tag{3.41}$$

The integrands in (3.35), (3.36), (3.37) also have simple expressions in the new variables, due to the identities $(Z^\dagger Z)_{11} = w_n^\dagger w_n$ and

$$\mathrm{Det}(1 + ZZ^\dagger) = (1 + w_1^\dagger w_1)(1 + w_2^\dagger w_2) \cdots (1 + w_n^\dagger w_n). \tag{3.42}$$

The w_i-integrals to be performed are all of the type $(N \geq n)$

$$\int_{\mathbb{C}^n} \frac{dw^\dagger dw}{(1 + w^\dagger w)^{N+1}} = \pi^n \frac{(N-n)!}{N!}. \tag{3.43}$$

The resulting expressions for the normalization constants are

$$\alpha_0 = \frac{1}{C_{N_f}\pi^{N_f^2}}\frac{(2N_f + N_c - 1)! \cdots (N_f + N_c)!}{(N_c + N_f - 1)! \cdots N_c!}, \tag{3.44}$$

$$\alpha_1 = \alpha_{-1} = \frac{1}{C_{N_f}\pi^{N_f^2}}\frac{N_f(2N_f + N_c - 1)! \cdots (N_f + N_c + 1)!}{(N_c + N_f - 2)! \cdots N_c!}, \tag{3.45}$$

$$C_{N_f} = \frac{1}{\pi^{N_f^2}}\frac{(2N_f - 1)! \cdots N_f!}{(N_f - 1)! \cdots 0!}, \tag{3.46}$$

where we have reinstated $n = N_f$. The quantity of physical interest is the ratio

$$\frac{\alpha_1}{\alpha_0} = \frac{N_f}{N_f + N_c}. \tag{3.47}$$

Chapter 4

Lattice Quantum Chromodynamics

In this chapter we review the lattice formulation of QCD which was introduced by Wilson. Special attention is paid to the symmetries of QCD: we compare their realization in the continuum and the lattice theory. We discuss in detail the chiral symmetry and chiral symmetry breaking mechanisms.

4.1   Continuum QCD

Quarks, the elementary building blocks of hadronic matter, appear in N_c colors ($N_c = 3$ in the real word) and N_f flavors ($N_f = 6$ in the real world). The strong interaction that confines them to hadrons is mediated by the exchange of gluons which couple to their color degrees of freedom. Quantum chromodynamics (QCD) is a SU(N_c) gauge theory, which identifies the gluons with the gauge field and couples it with the quarks in its fundamental representation. QCD is believed to be the theory of strong interaction. The QCD continuum action in d-dimensional Euclidean space-time is given by

$$S_{\text{cont.}}[A, \bar{q}, q] = \int d^d x \left(\frac{1}{2} \operatorname{Tr} \left(F^{\mu\nu} F_{\mu\nu} \right) + \sum_{\text{flavours } f} \bar{q}_f (\slashed{D} + m_f) q_f \right). \tag{4.1}$$

The gauge field A takes values in the hermitian traceless matrices [1] and the corresponding field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]. \tag{4.2}$$

The first term in the action describes the kinetic energy of the gluons, while the second term corresponds to the kinetic energy of the quarks and the coupling between the gluons and quarks. The quarks fields q are considered as N_c-component vectors in color space, N_f-component vectors in flavor space and Dirac spinors with $N_s = 2^{[d/2]}$ components [2] ($N_s = 4$ in four space-time dimensions). Their dynamics is described by the antihermitian Dirac operator

$$\slashed{D} = \gamma^\mu (\partial_\mu + ig A_\mu), \tag{4.3}$$

---

[1] In our convention A takes values in the hermitian matrices and the coupling constant appears explicitly in (4.2) "physicists convention"). Another possibility is to absorb the factor ig into the definition of A and F ("mathematician's convention"). Then the 1-form $A = A_\mu dx^\mu$ and the 2-form $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$ are dimensionless geometrical quantities, which take values in the Lie algebra $\mathfrak{su}(N_c)$.

[2] $[d/2]$ = greatest integer smaller or equal to $d/2$.

where the Euclidean γ-matrices satisfy the Clifford algebra relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}. \tag{4.4}$$

We refer to Appendix D for a derivation of the Dirac action in Euclidean space-time.

For the action we can read off the following *physical dimensions* of the different quantities,

$$[F_{\mu\nu}] = [\text{length}]^{-d/2},\ [A_\mu] = [\text{length}]^{-d/2+1},\ [q_f \bar{q}_f] = [\text{length}]^{-d+1},\ [m_f] = [\text{length}]^{-1}. \tag{4.5}$$

We work in a system of units, where $[\text{time}] = [\text{length}]$ and $[\text{mass}] = [\text{length}]^{-1}$. From the physical dimensions of the gauge field and its potential, we get the physical dimensions of the coupling constant,

$$[g] = [F_{\mu\nu}][A_\mu]^{-2} = [A_\mu]^{-1}[\text{length}]^{-1} = [\text{length}]^{d/2-2}, \tag{4.6}$$

in particular it is dimensionless in four space-time dimensions.

How does one derive a quantum theory from the classical action (4.1)? In principle, there are two methods, canonical quantization and the path integral formulation. The first method leads to a quantum field theory in $d - 1$ space dimensions and one time dimension, while the result of the second method is a system of statistical mechanics in d dimensions. Both methods can be shown to be equivalent by a transfer matrix argument [30]. Further, both methods are plagued by the same difficulty: The invariance of the theory under gauge transformations of the gauge potential A. In the first case it leads to a constrained system (i. e. it is impossible to express the velocities in terms of the canonical momenta), in the second case it leads to a diverging partition function (infinite contribution of the gauge volume). On the level of the continuum theory the only way to overcome this difficulty on the level is to break the gauge invariance and fix the gauge.

Lattice gauge theory, which is considered in the next chapter, is an elegant way to keep the gauge invariance and to get a well-defined partition function at the same time. Passing from the continuum to the lattice theory corresponds to the transition from the Lie algebra to the corresponding compact Lie group, where the gauge volume is no longer infinite. As an extra benefit the discretisation of space-time introduces an ultraviolett cutoff and serves as a renormalization scheme. Lattice gauge theory is the main non-perturbative approach to QCD.

4.2   Quarks and Gluons on a Lattice

4.2.1   Construction of a Gauge Invariant Lattice Action

Consider a d-dimensional hypercubic lattice with lattice constant a. A gauge invariant lattice action with continuum limit (4.1) was introduced by Wilson in 1974, [48]. In his formulation the quarks are put on the lattice sites, while the gluons (or the gauge field) are put on the lattice links, see Figure I. We label the lattice sites with a multi-index n, the d directions with a positive integer μ, the unit vector in positive μ-direction with $\hat{\mu}$ and the lattice links with their middle points. To put the quark field on the lattice one has to do nothing more than to restrict the space-time coordinates to $x = an$, while the gauge field on the lattice is obtained by exponenting the gauge potential to the Lie group,

$$U(n + \hat{\mu}/2) := \exp\left(iag\, A_\mu((n + \hat{\mu}/2)a)\right). \tag{4.7}$$
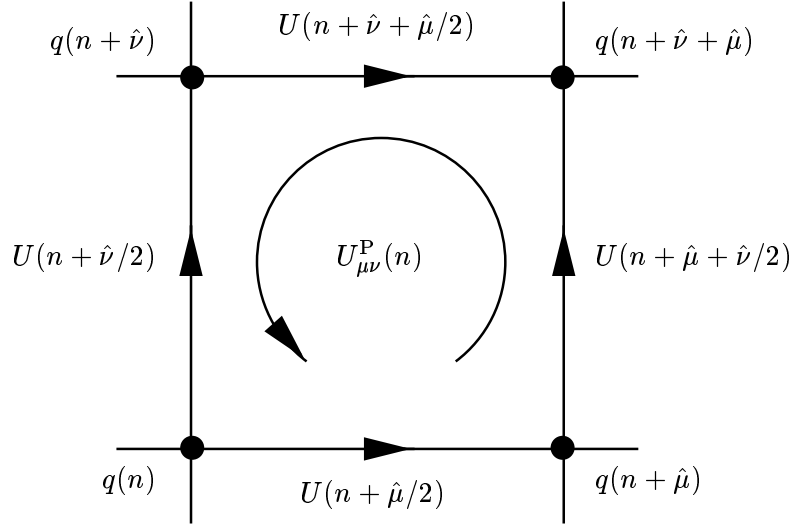
Figure I: Quarks and Gluons on a Lattice

A lattice gauge transformation is defined as a set of unitary transformations $\mathcal{U}(n) \in \mathrm{U}(N_c)$, which act on the lattice fields as

$$q(n) \; \rightarrow \; \mathcal{U}(n)\, q(n) \tag{4.8}$$

$$\bar{q}(n) \; \rightarrow \; \bar{q}(n)\, \mathcal{U}^{\dagger}(n) \tag{4.9}$$

$$U(n + \hat{\mu}/2) \; \rightarrow \; \mathcal{U}(n + \hat{\mu})\, U(n + \hat{\mu}/2)\, \mathcal{U}^{\dagger}(n). \tag{4.10}$$

Out of these ingredients Wilson constructed a lattice action

$$S_{\text{lattice}} = S_{\text{gluons}} + S_{\text{quarks}}, \tag{4.11}$$

where the two terms have the first and second summand of (4.1) as continuum limit.
The action for the gauge field is to be constructed of gauge invariant expressions. The simplest such expression is the product of the gauge field along a elementary plaquette

$$U^{\mathrm{P}}_{\mu\nu}(n) := U(n + \hat{\mu}/2)U(n + \hat{\mu} + \hat{\nu}/2)U^{-1}(n + \hat{\nu} + \hat{\mu}/2)U^{-1}(n + \hat{\nu}/2). \tag{4.12}$$

Using the Baker-Campbell-Hausdorff formula [3] we obtain

$$U^{\mathrm{P}}_{\mu\nu}(n) = \exp(iga^2\, F_{\mu\nu}(n) + O(a^3)). \tag{4.13}$$

Looking at excitations of long wavelengths compared to a, $ga^2 F_{\mu\nu} \ll 1$, one can expand the exponential up to second order,

$$\mathrm{Tr}\, U^{\mathrm{P}}_{\mu\nu}(n) \approx N_c + iga^2\, \mathrm{Tr}\, F_{\mu\nu}(n) - \frac{1}{2}g^2 a^4\, \mathrm{Tr}\, F^2_{\mu\nu}(n). \tag{4.14}$$

---

[3]A version for more than two operators reads $\exp(A_1)\exp(A_2)...\exp(A_n) = \exp(A_1 + A_2 + ... + A_n + \sum_{i<j}[A_i, A_j] + ...)$, where the dots at the end of the second expression stand for higher commutators.

Note that the second term is purely imaginary, because the field strength $F_{\mu\nu}(n)$ is hermitian. It cancels when one adds its complex conjugate, and we conclude that the gauge lattice action introduced by Wilson,

$$S_{\text{gluons}} = -\frac{a^{d-4}}{2g^2} \sum_{\text{plaquettes } (n,\mu\nu)} \left(\text{Tr}\left(U^{\text{P}}_{\mu\nu}(n)\right) + \text{Tr}\left(U^{\text{P}}_{\mu\nu}{}^{\dagger}(n)\right) \right) \qquad (4.15)$$

has the desired continuum limit.

To construct gauge invariant expressions, a term with a product of quark fields at two different lattice sites must contain a product of gluons fields along a path of lattice links connecting the two sites. In doing so the orientation of the path has to be considered: If it runs through a link in negative orientation, the corresponding gluon field has to be conjugated. A quark lattice action can be constructed of terms containing the quark field on two neighboring sites and the gluon field on the link between them,

$$S_{\text{quarks}} = \sum_{\text{sites } n} \left(\frac{a^{d-1}}{2} \sum_{\mu=1}^{d} \left(\bar{q}(n)\gamma_\mu U(n + \hat{\mu}/2)q(n + \hat{\mu}) \right. \right.$$
$$\left. \left. - \bar{q}(n + \hat{\mu})\gamma_\mu U^{\dagger}(n + \hat{\mu}/2)q(n) \right) + a^d\, \bar{q}(n)\mathcal{M}q(n) \right). \qquad (4.16)$$

In the action above, $\mathcal{M} := \text{diag}(m_1, ..., m_{N_f})$ is the quark mass matrix, which contains the masses of the quarks of the different flavors. Inserting the expression (4.7) for the gluons into the last equation and expanding it up to first order in the lattice constant, it is straightforward to prove that (4.16) has the Dirac action as continuum limit.

4.2.2   The Doubling Problem

After Wilson's pioneering work, it was soon realized that putting Dirac fermions on a lattice is plagued by a problem: In the *naive formulation* of lattice fermions, one can identify not only one Dirac particle in the continuum limit, but for each dimension a doubled number of alltogether 2^d Dirac particles ("species doubling"). To understand this phenomenon, we consider a system of massless Dirac fermions on a d-dimensional lattice with a length L,

$$\mathcal{S}_{\text{quarks}} = \frac{a^{d-1}}{2} \sum_{\text{sites } n} \sum_{\mu=1}^{d} \left(\bar{q}(n)\gamma_\mu q(n + \hat{\mu}) - \bar{q}(n + \hat{\mu})\gamma_\mu q(n) \right). \qquad (4.17)$$

We apply a lattice Fourier transformation (cf. Appendix F) and express the action in terms of the fields in momentum space,

$$\mathcal{S}_{\text{quarks}} = a^{d-1} \sum_{k} \sum_{\mu=1}^{d} \bar{q}(k)\, \gamma_\mu \sin(k_\mu)\, q(k). \qquad (4.18)$$

Because the action is diagonal in momentum space, we can easily evaluate the propagator,

$$\langle \bar{q}(n)q(m) \rangle = a^{1-d} \sum_{k} e^{ik \cdot (m-n)}\, D(k), \quad \text{with } D(k) = \frac{1}{\sum_{\mu=1}^{d} \gamma_\mu \sin k_\mu}. \qquad (4.19)$$

For small wave vectors $k \ll 1$ this reduces to the usual Dirac propagator $1/k\!\!\!/$. But because of the properties of the sine function, the propagator has not only poles at $k = 0$, but also at some point at the boundary of the Brillouin zone $]-\pi, \pi]^d$. To be more precise, the propagator has poles, if all components of the wave vector take values $k_\mu \in \{0, \pi\}$, and we deal with 2^d fermion species, even though we initially seemed to have only one.

In the early days of lattice QCD the species doubling was regarded as lattice artifact and much effort has gone in studying more complex fermion actions. There are two popular approaches to overcome the doubling problem: The first one [40] staggers the spin components of the fermions over the lattice (Kogut-Susskind fermions).

The other traditional approach [49] (Wilson fermions) completely destroys the chiral symmetry. In this approach one introduces a mass to the doublers at nonzero wave vectors. This "mass" depends on the wave-vector in such a way that it does not affect the fermion at $k = 0$. The choice $r(1 - \cos k_\mu)$ for this extra "mass" term, where r is an arbitrary parameter, leads to a local form of the modified action in position space.

Later, in the early 1980s, Nielsen and Ninomiya presented their no-go theorems [36] for chiral fermions on a lattice and for the regularization of chiral fermions independently of a lattice. Nevertheless, in recent days there is an exciting new approach (overlap Dirac operators), which might completely solve the doubling problem. It is connected with the introduction of an *infinite dimensional flavor space*, for details of the still ongoing research see [35].

4.3 The Symmetries of QCD

This section is devoted to the study of the symmetries of lattice QCD. While the gauge symmetry was already discussed in the last section, we consider here the space-time symmetry and the chiral symmetry. The main aspect of our investigations will be the comparison between lattice symmetries and the corresponding continuum symmetries.

4.3.1 Space-time Symmetry

The Euclidean formulation of continuum QCD has the Euclidean group in d dimensions E(d) as space-time symmetry group. It consists of rotations which form the orthogonal group O(d) and d-dimensional translations. Switching from the continuum formulation to a formulation on a lattice the space-time symmetry group is broken down to a discrete subgroup of the continuum symmetry group.

Hypercubic Lattice

The rotational symmetry group $O_{hc}(d)$ of an d-dimensional hypercubic (hc) lattice can be constructed as follows: It consists of all maps which are permutations of the basis $B_1 = \{e_1, ..., e_d\}$ of $\mathbb{R}^d$ and in addition may map some of the basis elements to their negative. Thus $O_{hc}(d)$ is a $(2^d \, d!)$-element subgroup of O(d), in particular has 384 elements in four dimensions.

In some of the next chapters of this dissertation we will follow the strategy to start with lattice QCD, to apply the color-flavor transformation, and then to transform to a continuum theory of the considered degrees of freedom. In doing so it is desirable to recover the full symmetry of the original continuum theory. This is not always possible; the resulting action can lack the full continuum symmetry. For example on the hc lattice the recovering of the

continuum symmetry can only be guaranteed for tensors up to second order. Indeed there are tensors of fourth order, for example $\sum_\mu \partial_\mu^4$, which are invariant on the hc lattice, but do not have the full continuum symmetry. Therefore we would like to work on a lattice with the highest possible symmetry. In two and three dimensions the hc lattice has the largest symmetry group, but in four dimensions there is a lattice with a larger one, the bodycentered hypercubic (bhc) lattice.

Bodycentered Hypercubic Lattice

The existence of a lattice with a higher symmetry than the hc lattice in four dimensions is based on the fact that the 16 vectors $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ from the origin to the centers of the neighboring hypercubes have the same length as the hc lattice vectors $e_1, ..., e_d$. This is a special feature of four space-time dimensions.

To construct the symmetry group of the bhc lattice, we consider two more orthonormal bases $B_2 = \{f_1, f_2, f_3, f_4\}$ and $B_3 = \{f, f_{12}, f_{13}, f_{14}\}$ of $\mathbb{R}^4$. In this notation a vector $f_{ijk...}$ has the entries $-1/2$ at the positions $i, j, k, ...$ and $1/2$ elsewhere, in particular $f = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The symmetry transformations of the four-dimensional bhc lattice can be generated as follows: The elements of B_1 can be mapped bijectively to the elements of B_1, B_2 or B_3 and after that each basis vector can acquire an arbitrary sign. This leads to a symmetry group $O_{bhc}(4)$, which is three times (1152 elements) larger than the one of the hc lattice. One can prove that the more symmetric bhc lattice in four dimensions guarantees the restoration of the continuum symmetry for tensors up to fourth order [34]. The proof is based on the identification of the four-dimensional bhc lattice as the lattice formed by the roots of the exceptional Lie algebra $\mathfrak{f}_4$.

There is another parametrisation of the bhc lattice [31, 32] which represents the vectors from a lattice site to a nearest neighbor in a more symmetric way. It is connected with the formulation above from switching to the new orthonormal basis $e_1' = \frac{1}{\sqrt{2}}(1, 1, 0, 0)$, $e_2' = \frac{1}{\sqrt{2}}(-1, 1, 0, 0)$, $e_3' = \frac{1}{\sqrt{2}}(0, 0, 1, 1)$, $e_4' = \frac{1}{\sqrt{2}}(0, 0, -1, 1)$. In the new basis, the vectors to the 24 nearest neighbors from a lattice site are given by

$$v_{\mu\nu}^{\alpha\beta} = \frac{\alpha e_\mu' + \beta e_\nu'}{\sqrt{2}}, \quad 1 \le \mu < \nu \le 4, \ \alpha = \pm 1, \ \beta = \pm 1. \tag{4.20}$$

Among these vectors we call the ones with $\alpha > 0$ *positive*. In doing so we have defined an *orientation* on the bhc lattice.

In the context of lattice gauge theory the four-dimensional bhc lattice was first considered by Celmaster [12, 45]. The generalizations of the gluon action and the quark action from sect. 4.2 to an arbitrary oriented lattice [31, 32] read

$$S_{\text{gluons}} = -K_1 \frac{a^{d-4}}{g^2} \sum_{\text{plaquettes } (n,vw)} \left(\text{Tr} \left(U_{vw}^{\text{P}}(n) \right) \right), \tag{4.21}$$

$$S_{\text{quarks}} = \sum_{\text{sites } n} \left(K_2 a^{d-1} \sum_v \left(\bar{q}(n) \gamma_v U(n + v/2) q(n + v) \right. \right.$$

$$\left. \left. - \bar{q}(n + v) \gamma_v U^\dagger(n + v/2) q(n) \right) + K_3 a^d \, \bar{q}(n) m q(n) \right). \tag{4.22}$$

The sum extends over all lattice sites and the links to the next neighbors in positive direction. (For the four-dimensional bhc lattice the sum runs over the 12 vectors v from (4.20) with positive α). The quarks are coupled through the γ-matrix in v-direction $\gamma_v := \sum_\mu v_\mu \gamma_\mu$. The values of the constants K_1, K_2, K_3 depend on the type of lattice, see Table IV. They are obtained from the continuum limit of the corresponding lattice formulation; for the continuum limit of quantities on the hc lattice we refer to Appendix G.

lattice type	K_1	K_2	K_3
hc lattice	1/2	1/2	1
bhc lattice	1/12	1/12	1/2

Table IV: Values of the Constants K_1, K_2, K_3 for the hypercubic and the body-centered hypercubic lattice

4.3.2 Chiral Symmetry

The continuum QCD Lagrangian with N_f massless flavors (chiral limit of QCD) possesses a global chiral $U(N_f)_L \times U(N_f)_R$ symmetry. The symmetry group is a product of left and right chiral transformations, which is based on the fact that the flavor transformations can act on the left- and right-handed components of the spinors separately. The basics of the chiral structure of the spinor space are laid down in Appendix D.1. Left- and right-handed components of the quark fields can be defined with the help of the projectors $P_\pm = \frac{1}{2}(1 \pm \Gamma_5)$ introduced there,

$$
\begin{aligned}
q_L &:= P_- q, & q_R &:= P_+ q, \\
\bar{q}_L &:= \bar{q} P_+, & \bar{q}_R &:= \bar{q} P_-.
\end{aligned}
\tag{4.23}
$$

Under a chiral transformation $(V_L, V_R) \in U(N_f)_L \times U(N_f)_R$ they transform according to

$$
\begin{aligned}
q_R &\to V_R q_R, & q_L &\to V_L q_L, \\
\bar{q}_R &\to \bar{q}_R V_R^{-1}, & \bar{q}_L &\to \bar{q}_L V_L^{-1}.
\end{aligned}
\tag{4.24}
$$

Writing down the Dirac operator in terms of the left- and right-handed components of the spinors,

$$
\bar{q}\, \slashed{D} q = \bar{q}_L \slashed{D} q_L + \bar{q}_R \slashed{D} q_R
\tag{4.25}
$$

the chiral invariance becomes transparent. In the chiral limit the continuum action (4.1) is invariant under the chiral transformations, while a mass term

$$
\bar{q}\mathcal{M}q = \bar{q}_L \mathcal{M} q_R + \bar{q}_R \mathcal{M} q_L
\tag{4.26}
$$

explicitly breaks the chiral symmetry.

Chiral Transformations on the Lattice

Chiral transformations act on the spin and the flavor degrees of freedom of the quarks. Allowing an independent transformation for each lattice site, we consider two independent sets

of matrices $\mathcal{V}(n)$, $\bar{\mathcal{V}}(n) \in \mathrm{U}(N_s N_f)$ and the transformations

$$
\begin{aligned}
q(n) &\rightarrow \mathcal{V}(n)q(n), \\
\bar{q}(n) &\rightarrow \bar{q}(n)\bar{\mathcal{V}}(n).
\end{aligned}
\tag{4.27}
$$

In the chiral limit the lattice action is invariant under the transformation (4.27), iff for all lattice sites n and all vectors v to next neighbors in positive direction,

$$
\begin{aligned}
\bar{\mathcal{V}}(n) &= \gamma_v \mathcal{V}(n+v)^{-1}\gamma_v, \\
\bar{\mathcal{V}}(n+v) &= \gamma_v \mathcal{V}(n)^{-1}\gamma_v.
\end{aligned}
\tag{4.28}
$$

The last two conditions are equivalent to

$$
\begin{aligned}
\mathcal{V}(n+v+w) &= \gamma_w\gamma_v\mathcal{V}(n)\gamma_v\gamma_w \\
\bar{\mathcal{V}}(n) &= \gamma_v\mathcal{V}(n+v)^{-1}\gamma_v
\end{aligned}
\tag{4.29}
$$

for all n and positive v, w. We remark that the right hande side of the second equation (4.29) is automatically independent of v, if the first equation is satisfied. Further these equations imply $\mathcal{V}(n+2v) = \mathcal{V}(n)$.

The set of solutions of (4.29) depends very much on the structure of the underlying lattice. We distinguish the following cases:

(i) *General Lattice*

Independently of the structure of the lattice, a set of solutions of equations (4.29) is given by the homogeneous ansatz

$$
\begin{aligned}
\mathcal{V}(n) &= P_- \otimes V_L + P_+ \otimes V_R, \\
\bar{\mathcal{V}}(n) &= P_- \otimes V_R^{-1} + P_+ \otimes V_L^{-1}.
\end{aligned}
\tag{4.30}
$$

These solutions correspond to the usual continuum chiral transformations (4.24). However, on some lattices, equation (4.29) admits additional solutions.

(ii) *Bipartite Lattice*

A lattice is called *bipartite*, if there is a decomposition of the lattice into two nested sublattices A and B such that, whenever two sites are nearest neighbors, one of the sites belongs to A and the other site belongs to B. On bipartite lattices there is another solution of (4.29), where the quark fields on the sites of lattice A transform independently of the quarks on the lattice B. The existence of that solution is based on the fact, that in all terms of lattice action the quark field $\bar{q}$ on one of the sublattices is coupled with the quark field q on the other lattice. Labeling the sites of the sublattice A by a and the sites of B by b this solution is given by

$$
\begin{aligned}
\mathcal{V}(a) &= \bar{\mathcal{V}}^{-1}(b) = 1 \otimes V_A \\
\mathcal{V}(b) &= \bar{\mathcal{V}}^{-1}(a) = 1 \otimes V_B.
\end{aligned}
\tag{4.31}
$$

This kind of chiral structure which is still present if one neglects the spinor degrees of freedom, was considered in [3], [2]. Indeed, analogous to the Clifford element Γ_5, for

bipartite lattices there is a chiral operator $\tilde{\Gamma}_5$ which acts on the lattice and not on the spin degrees of freedom,

$$
\begin{aligned}
(\tilde{\Gamma}_5 q)(a) &= q(a), & (\tilde{\Gamma}_5 q)(b) &= -q(b) \\
(\tilde{\Gamma}_5 \bar{q})(a) &= -\bar{q}(a), & (\tilde{\Gamma}_5 \bar{q})(b) &= \bar{q}(b).
\end{aligned}
\tag{4.32}
$$

The left- and right-handed components of the quarks with respect to this chiral operator are their components on the sublattice A or B, respectively.

(iii) *Hypercubic Lattice*

For the bipartite hc lattice the group of chiral transformations is enlarged to $\mathrm{U}(N_s N_f)_A \times \mathrm{U}(N_s N_f)_B$. Indeed, one may choose a site a_0 and a site b_0 from each of the sublattices and rotate the quarks in an arbitrary way in spin and flavor space,

$$
\mathcal{V}(a_0) = \mathcal{V}_A, \qquad \mathcal{V}(b_0) = \mathcal{V}_B \qquad \text{with } \mathcal{V}_A, \mathcal{V}_B \in \mathrm{U}(N_s N_f).
\tag{4.33}
$$

Having fixed the transformation for two lattice sites, the transformations on all other lattice sites are determined by (4.29). Note, that because of the square structure of the elementary plaquettes on the hc lattice, the first equation of (4.29) in combination with the existence of closed paths on the lattice does not lead to any conditions on $\mathcal{V}_A$ and $\mathcal{V}_B$.

(iv) *Bodycentered Hypercubic Lattice*

On the four-dimensional bhc lattice the group of chiral transformations is *not* enlarged compared to the continuum. To prove that, we derive in a first step the structure of a chiral transformation on the spin sector and show in a second step that the transformation is homogeneous. To learn something about the structure of a chiral transformation on the spin sector, we calculate

$$
\begin{aligned}
\mathcal{V}(n + v + w) &= \gamma_w \gamma_v \mathcal{V}(n) \gamma_v \gamma_w \\
&= (\langle v, w \rangle - \gamma_v \gamma_w) \, \mathcal{V}(n) \, (\langle v, w \rangle - \gamma_w \gamma_v) \\
&= \langle v, w \rangle \left(\langle v, w \rangle \mathcal{V}(n) - \gamma_v \gamma_w \mathcal{V}(n) - \mathcal{V}(n) \gamma_w \gamma_v \right) + \gamma_v \gamma_w \mathcal{V}(n) \gamma_w \gamma_v \\
&= \langle v, w \rangle (\gamma_w \gamma_v \mathcal{V}(n) - \mathcal{V}(n) \gamma_w \gamma_v) + \mathcal{V}(n + w + v)
\end{aligned}
\tag{4.34}
$$

and conclude

$$
\gamma_w \gamma_v \mathcal{V}(n) = \mathcal{V}(n) \gamma_w \gamma_v, \qquad \text{whenever } \langle v, w \rangle \neq 0.
\tag{4.35}
$$

For v and w we consider the following three positive elementary lattice vectors of the bhc lattice,

$$
v = \frac{1}{\sqrt{2}}(e'_\mu + e'_\nu), \ \ w_1 = \frac{1}{\sqrt{2}}(e'_\mu + e'_\omega), \ \ w_2 = \frac{1}{\sqrt{2}}(e'_\mu - e'_\omega)
\tag{4.36}
$$

with $\mu, \nu, \omega \in 1, ..., 4$ and $\nu \neq \omega$, which have the properties

$$
\langle v, w_1 \rangle \neq 0, \ \langle v, w_2 \rangle \neq 0, \qquad \text{and } \gamma_{w_1} \gamma_v + \gamma_{w_2} \gamma_v = 1 + \gamma'_\mu \gamma'_\nu.
\tag{4.37}
$$

Then we apply condition (4.35) to both (v, w_1) as well to (v, w_2) and add the corresponding equations, getting

$$
\gamma'_\mu \gamma'_\nu \mathcal{V}(n) = \mathcal{V}(n) \gamma'_\mu \gamma'_\nu \qquad \text{for all } \mu, \nu.
\tag{4.38}
$$

Decomposing $\mathcal{V}(n)$ with respect to a basis of the Clifford algebra, cf. (D.13), this implies that $\mathcal{V}(n)$ can only contain contributions proportional to basis elements 1 and γ_5, that is

$$\mathcal{V}(n) = P_+ \otimes V_L(n) + P_- \otimes V_R(n). \tag{4.39}$$

with suitable matrices $V_R(n), V_L(n) \in \mathrm{U}(N_f)$. In terms of these matrices condition (4.29) takes the form

$$\mathcal{V}_{L/R}(n + v + w) = \mathcal{V}_{L/R}(n). \tag{4.40}$$

We turn to the second step and apply the last equation twice, to get a similar equation for differences $(v - w)$ of positive lattice vectors,

$$\mathcal{V}_{L/R}(n + v - w) = \mathcal{V}_{L/R}(n + (v + w) - (w + w)) = \mathcal{V}_{L/R}(n). \tag{4.41}$$

Now we make use of the triangular structure of the elementary plaquettes of the bhc lattice and find that each of the positive vectors (4.20) can be written as a sum $v + w$ or a difference $v - w$ of positive vectors v, w.

Applying (4.40) and (4.41) in forward and backward direction, we deduce that $\mathcal{V}_{L/R}$ does not depend on the lattice site n. We conclude that the only transformations (4.27) that leave invariant the quark action on the bhc lattice are the continuum chiral transformations (4.30).

4.4   Chiral Symmetry Breaking

4.4.1   The Lightest Mesons as Goldstone Bosons

Realistic QCD has an approximate chiral symmetry when one considers the two or three lightest flavors. Indeed, the (current) quark masses $m_u \approx 4$ MeV, $m_d \approx 7$ MeV (and $m_s \approx 150$ MeV) are light compared to (most of) the hadron masses.

However, the full chiral symmetry group $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ is not a symmetry of the experimental observed particle spectrum. In fact, we do not observe a parity degeneracy of the hadron states, which would be caused by the axial chiral transformations $\mathrm{U}(N_f)_{L-R}$ (these transformations are defined by the identity $V_L = V_R^{-1}$). The conclusion is that the approximate chiral symmetry of the QCD Lagrangian is broken *spontaneously*; to be more precise the vacuum is only invariant under the vector chiral transformations $\mathrm{U}(N_f)_{L+R}$ (these transformations satisfy $V_L = V_R$).

A spontaneously broken symmetry leads to the appearance of massless modes (Goldstone bosons) in the spectrum of the theory. Due to the Goldstone theorem there is a massless particle corresponding to each broken generator of the symmetry group. Because chiral symmetry is only an approximate symmetry of QCD, the corresponding Goldstone bosons acquire a small mass. These pseudo Goldstone bosons should be identified with the lightest mesons.

Let us have a look at the zoo of the lightest mesons, see Table V. By far the lightest mesons are the pions; they form an isospin ($N_f = 2$) triplet. The pions, kaons and the η-particle built a flavor ($N_f = 3$) octet, while the η' -particle is a flavor singlet. There is a non-trivial mixing between the η and the η' particle.

The table contains nine particles and it is tempting to identify them with the Goldstone bosons of a spontaneous symmetry breaking pattern $\mathrm{U}(3)_L \times \mathrm{U}(3)_R \to \mathrm{U}(3)_{L+R}$. The higher masses of the kaons and the η'-particle compared to the pions can be attributed to the high

particle	quark content	mass [MeV]
π_0	$u\bar{u} - d\bar{d}$	135
π_+	$u\bar{d}$	140
π_-	$d\bar{u}$	140
K^+	$s\bar{u}$	494
K^-	$u\bar{s}$	494
K^0	$d\bar{s}$	498
$\bar{K}^0$	$s\bar{d}$	498
η	$\approx u\bar{u} + d\bar{d} - 2s\bar{s}$	547
η'	$\approx u\bar{u} + d\bar{d} + s\bar{s}$	958

Table V: Masses of the light pseudoscalar Mesons [20]. The quark contents of the η and the η' particle is given approximately; the real quark contents are a mixing between the listed ones.

mass of the strange quark. However, the mass of the η'-particle is much higher than the masses of all other listed mesons. The η'-meson badly fits into the spontaneous symmetry breaking pattern mentioned above; this is called the *strong* U(1)-*problem*.

A first step toward the resolution of the strong U(1)-problem is to realize the axial anomaly. Due to Adler, Bell and Jackiv [1], [5] the axial U(1)$_A$-symmetry of the Lagrangian does not lead to a conserved current; the divergence of the axial vector current is different from zero due to quantum effects. Based on that, there is a widely accepted reasoning, which makes the η and η' particles behave as they should [42]. 't Hooft's reasoning is connected with instantons, which are topologically nontrivial configurations of the gauge field. Their presence leads to a violation of the axial current conservation and provides the η- and the η'-particle with an anomalous contributions to their masses.

Taking the axial anomaly into account, we get the following spontaneous symmetry breaking patterns: The pions are the pseudo Goldstone bosons due to the spontaneous symmetry breaking of the isospin group, SU(2)$_L$ × SU(2)$_R$ → SU(2)$_{L+R}$. The particles of the flavor octet (π, K, η) are the pseudo Goldstone bosons of the spontaneous symmetry breaking SU(3)$_L$ × SU(3)$_R$ → SU(3)$_{L+R}$. Note that the vector U(1)$_V$-symmetry which corresponds to the baryon number is not spontaneously broken.

4.4.2   Chiral Condensate

As an indicator for chiral symmetry breaking, one usually makes use of the chiral condensate,

$$\langle \bar{q}q \rangle = \langle \bar{q}_L q_R \rangle + \langle \bar{q}_R q_L \rangle. \tag{4.42}$$

It takes a value different from zero if the axial chiral symmetry is broken. In the real world chiral symmetry is explicitly broken by the finite quark masses. We will show that this explicit symmetry breaking gives rise to a nonzero chiral condensate by an argument of perturbation theory with the quark masses as small parameters. To do this, we decompose the QCD Lagrangian $\mathcal{L}$ [4] into a chiral invariant part plus a part containing the symmetry breaking

---

[4]Because we are working in Euclidean space-time the Lagrangian is the energy density of the system.

quark masses,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1,$$
$$\mathcal{L}_0 = \frac{1}{2}\operatorname{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right) + \sum_{\text{flavors } f} \bar{q}_f\,\slashed{D}q_f, \qquad \mathcal{L}_1 = \sum_{\text{flavors } f} m_f\bar{q}_f q_f. \qquad (4.43)$$

We consider the following chiral transformation, which transforms a quark of one flavor f_0 with a transformation T_α generated by Γ_5 and leaves all other quarks invariant,

$$T_\alpha q_{f_0} = e^{i\alpha\Gamma_5}\,q_{f_0}, \qquad \bar{q}_{f_0}T_\alpha = \bar{q}_{f_0}\,e^{i\alpha\Gamma_5} \qquad\qquad (4.44)$$
$$T_\alpha q_f = q_f, \qquad\quad \bar{q}_f T_\alpha = \bar{q}_f, \qquad \text{for all flavors } f \neq f_0. \qquad (4.45)$$

The transformation T_α, which acts on the classical variables, is implemented as a transformation on Hilbert space by an unitary operator $\mathcal{T}_\alpha$,

$$\mathcal{T}_\alpha \hat{q}_\mu \mathcal{T}_\alpha^{-1} = (T_\alpha)_{\mu\nu}q_\nu, \qquad \mathcal{T}_\alpha \hat{\bar{q}}_\mu \mathcal{T}_\alpha^{-1} = \hat{\bar{q}}_\nu (T_\alpha)_{\nu\mu}. \qquad (4.46)$$

Let $|0\rangle$ be the ground state of the complete system $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$. The chiral rotated state $|\alpha\rangle = \mathcal{T}_\alpha|0\rangle$ is still a ground state of the chiral limit of QCD $\mathcal{L}_0$, but not of the quark mass term $\mathcal{L}_1$. To the lowest order of perturbation theory the correction to the vacuum energy is given by

$$\Delta E(\alpha) = \langle\alpha|\mathcal{L}_1|\alpha\rangle = \sum_f m_f\langle\alpha|\hat{\bar{q}}_f\hat{q}_f|\alpha\rangle$$
$$= \sum_f m_f\langle 0|\mathcal{T}_\alpha^{-1}\hat{\bar{q}}_f\mathcal{T}_\alpha\,\mathcal{T}_\alpha^{-1}\hat{q}_f\mathcal{T}_\alpha|0\rangle \qquad\qquad (4.47)$$
$$= m_{f_0}\langle 0|\hat{\bar{q}}_{f_0}e^{-2i\alpha\Gamma_5}\hat{q}_{f_0}|0\rangle + \sum_{f\neq f_0} m_f\langle 0|\hat{\bar{q}}_f\hat{q}_f|0\rangle.$$

The energy has to take a minimum for $\alpha = 0$, which gives the conditions

$$\left.\frac{\partial\Delta E}{\partial\alpha}\right|_{\alpha=0} = 0, \qquad \left.\frac{\partial^2\Delta E}{\partial\alpha^2}\right|_{\alpha=0} > 0. \qquad (4.48)$$

We conclude, that for all flavors f the corresponding chiral condensates satisfy

$$\langle 0|\bar{q}_f\Gamma_5 q_f|0\rangle = 0$$
$$\langle 0|\bar{q}_f q_f|0\rangle < 0. \qquad\qquad (4.49)$$

This is in agreement with the negative sign of the experimental value $<\bar{q}_f q_f>_{\exp} = -((250 \pm 35)\ \text{MeV})^3$ for the chiral condensate.

Chapter 5

Gluodynamics

This chapter deals with a new approach to gluodynamics that is connected with the color-flavor transformation. Wilson introduced a gauge invariant lattice action $S_{\text{gauge}}[U]$ which has gluodynamics as formal continuum limit (cf. Chapter 4.2). However, Wilson's action does not allow a direct application of the color-flavor transformation. Our approach to gluodynamics is based on the replacement of Wilson's action by a new lattice action which can be generated as a Gaussian integral over an auxiliary field. In doing so we profit from a freedom in the choice of the lattice theory; indeed there are several lattice theories with the same continuum limit.

For sake of simplicity we work with the unitary group $U(N_c)$ instead of $SU(N_c)$ as gauge group. Physically this simplification means that we restrict the hadronic matter to meson and allow no baryons. However, in the $N_c \to \infty$ limit, the choice of $U(N_c)$ instead of $SU(N_c)$ is irrelevant [41].

5.1   A New Lattice Action for the Gauge Field

In Chapter 4.2 we have already introduced Wilsons gauge action on a hc lattice,

$$S_{\text{gluons}}[U] = -\frac{a^{d-4}}{2g^2} \sum_{\text{plaquettes } p} \left(\operatorname{Tr} U^{\text{P}}(p) + \operatorname{Tr} U^{\text{P}\dagger}(p) \right). \tag{5.1}$$

It is a gauge-invariant functional of the plaquette variables $U^{\text{P}}(p)$, which is given by the product of the gauge field along a plaquette, see (4.12). We replace Wilson's action by a new action, which is still a gauge invariant functional of the plaquette terms,

$$S'_{\text{gluons}}[U] = N_a \sum_{\text{plaquettes } p} \left(\ln \operatorname{Det}(1 - \beta U^{\text{P}}(p)) + \ln \operatorname{Det}(1 - \beta U^{\text{P}\dagger}(p)) \right). \tag{5.2}$$

The parameter $\beta \in [0, 1[$ is a dimensionless coupling constant, which will be set into correspondence to the coupling constant g of QCD. There is a second dimensionless parameter N_a which will be fixed at a positive integer. Only for integer values of N_a the action $S'_{\text{gluons}}[U]$ can generated by Gaussian integration over an auxiliary field.) The partition function of the
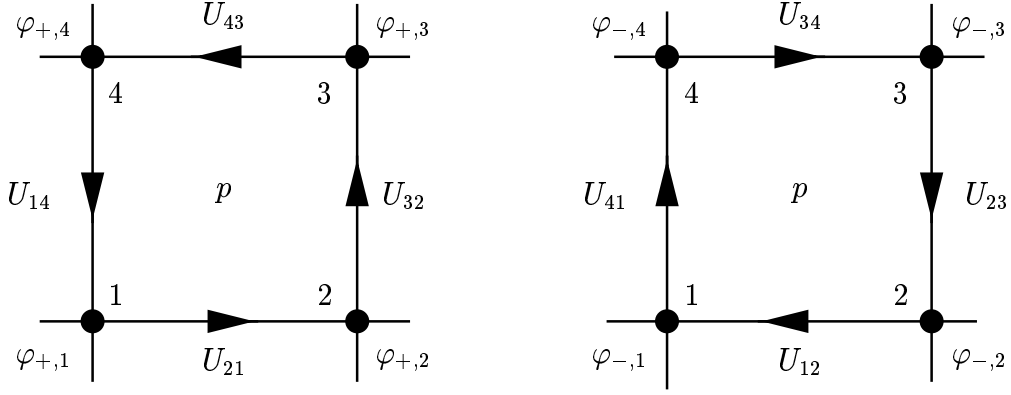
Figure II: Movement of the auxiliary bosons and their coupling to the gauge field.

new action reads

$$
\begin{aligned}
\mathcal{Z}' &:= \int [dU]\, e^{-S'_{\text{gluons}}[U]} \\
&= \int [dU] \prod_{\text{plaquettes } p} \mathrm{Det}^{-N_a}(1 - \beta U^{\mathrm{P}}(p))\, \mathrm{Det}^{-N_a}(1 - \beta U^{\mathrm{P}\dagger}(p)).
\end{aligned}
\tag{5.3}
$$

In the next subsection we will present a way to generate the action (5.2) by a Gaussian integral over auxiliary fields. The remaining subsections deal with the connection between the lattice theory defined by the new action $S'_{\text{gluons}}[U]$ and gluodynamics.

5.1.1   Generation of the Action by a Gaussian Integral over an auxiliary field

The motivation to replace Wilsons action by a new action is to represent it in a way that the color-flavor transformation can be applied. Indeed, the action $S'_{\text{gluons}}[U]$ can be generated as a Gaussian integral over a *bosonic auxiliary field*. For each plaquette p we introduce two bosons, one hopping counterclockwise and the other one hopping clockwise around the plaquette, see Figure II. We denote the corresponding fields by $\varphi_{+,n}(p)$ and $\varphi_{-,n}(p)$, where the index n labels the four sites of the plaquette. Further we assume the auxiliary fields to have the following inner degrees of freedom: both fields are vectors in color space as well in an auxiliary space of dimension N_a. From the counterclockwise hopping auxiliary field the gauge field we construct the action

$$
S_+[U, \bar{\varphi}_+, \varphi_+] := \sum_p \sum_{n=1}^{4} \left(-\bar{\varphi}_{+,n+1}(p) U_{n+1,n}(p) \varphi_{+,n}(p) + \hat{m}\bar{\varphi}_{+,n}(p)\varphi_{+,n}(p) \right),
\tag{5.4}
$$

where the indices are evaluated modulo 4. Being suppressed in the notation above, the indices labeling the spin and the auxiliary space are implicitly summed over. The action contains a hopping term and mass term for the auxiliary field. The mass $\hat{m}$ is measured in units of the inverse lattice constant. We are using a new labeling of the gauge field, which we set into correspondence to the notations in Chapter 4: For a plaquette p labeled by the lattice site n

and the directions μ, ν with $\mu < \nu$ we define

$$U_{21}(p) := U(n + \hat{\mu}/2), \qquad U_{32}(p) := U(n + \hat{\mu} + \hat{\nu}/2), \tag{5.5}$$

$$U_{14}(p) := U^{\dagger}(n + \hat{\nu}/2), \qquad U_{43}(p) := U^{\dagger}(n + \hat{\nu} + \hat{\mu}/2). \tag{5.6}$$

The gauge field on the links of opposite direction is defined by $U_{n,n+1} = U^{\dagger}_{n+1,n}$. By making use of the new labeling the product of the gauge field around a plaquette p in respective counterclockwise or clockwise direction can be expressed as

$$
\begin{aligned}
U^{\mathrm{P}}(p) = U^{\mathrm{P}}_{\mu\nu}(n) = U_{21}(p)U_{32}(p)U_{43}(p)U_{14}(p), \\
U^{\mathrm{P}\dagger}(p) = U^{\mathrm{P}\dagger}_{\mu\nu}(n) = U_{41}(p)U_{34}(p)U_{23}(p)U_{12}(p).
\end{aligned}
\tag{5.7}
$$

Neglecting an irrelevant factor, the integration over the gauge field yields

$$
\begin{aligned}
\int [d\bar{\varphi}_{+}d\varphi_{+}] \exp\left(-S_{+}[U, \bar{\varphi}_{+}, \varphi_{+}]\right) &= \prod_{p} \mathrm{Det}^{-N_a} \begin{pmatrix} \hat{m} & 0 & 0 & -U_{14}(p) \\ -U_{21}(p) & \hat{m} & 0 & 0 \\ 0 & -U_{32}(p) & \hat{m} & 0 \\ 0 & 0 & -U_{43}(p) & \hat{m} \end{pmatrix} \\
&= \prod_{p} \mathrm{Det}^{-N_a} \left(\begin{pmatrix} \hat{m} & 0 \\ -U_{21}(p) & \end{pmatrix} - \begin{pmatrix} 0 & -U_{14}(p) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{m} & 0 \\ U_{43}(p) & \hat{m} \end{pmatrix} \begin{pmatrix} 0 & -U_{32}(p) \\ 0 & 0 \end{pmatrix} \right) \\
&= \prod_{p} \mathrm{Det}^{-N_a} \begin{pmatrix} \hat{m}^2 & -^3 U_{14}(p)U_{43}(p)U_{32}(p) \\ -bU_{21}(p) & \hat{m}^2 \end{pmatrix} \\
&= \prod_{p} \mathrm{Det}^{-N_a} \left(\hat{m}^4 - U_{21}(p)U_{14}(p)U_{43}(p)U_{32}(p) \right).
\end{aligned}
\tag{5.8}
$$

When we identify $\beta := \hat{m}^{-4}$, this can be rewritten as

$$\int [d\bar{\varphi}_{+}d\varphi_{+}] \exp\left(-S_{+}[U, \bar{\varphi}_{+}, \varphi_{+}]\right) = \prod_{p} \mathrm{Det}^{-N_a}(1 - \beta U^{\mathrm{P}}(p)), \tag{5.9}$$

where we have droped a $\hat{m}$-depended constant. The clockwise hopping auxiliary field is governed by the action

$$S_{-}[U, \bar{\varphi}_{-}, \varphi_{-}] := \sum_{p} \sum_{n=1}^{4} \left(-\bar{\varphi}_{-,n}(p)U_{n,n+1}(p)\varphi_{-,n+1}(p) + \hat{m}\bar{\varphi}_{-,n}(p)\varphi_{-,n}(p) \right) \tag{5.10}$$

and the result of the Gaussian integration is

$$\int [d\bar{\varphi}_{-}d\varphi_{-}] \exp(-S_{-}[U, \bar{\varphi}_{-}, \varphi_{-}]) = \prod_{p} \mathrm{Det}^{-N_a}(1 - \beta U^{\mathrm{P}\dagger}(p)). \tag{5.11}$$

Putting the two auxiliary fields together, we obtain

$$
\begin{aligned}
\int [d\bar{\varphi}_{+}d\varphi_{+}d\bar{\varphi}_{-}d\varphi_{-}] &\exp(-S_{+}[U, \bar{\varphi}_{+}, \varphi_{+}] - S_{-}[U, \bar{\varphi}_{-}, \varphi_{-}]) \\
&= \prod_{p} \mathrm{Det}^{-N_a}(1 - \beta U^{\mathrm{P}}(p)) \, \mathrm{Det}^{-N_a}(1 - \beta U^{\mathrm{P}\dagger}(p)) \\
&= \exp(-S'_{\mathrm{gluons}}[U]).
\end{aligned}
\tag{5.12}
$$

To sum up, the action $S'_{\text{gluons}}[U]$ can be written as a Gaussian integral over $2N_a$ bosons, which hop clockwise and counterclockwise around the elementary plaquettes of the lattice. In this formulation the lattice gauge theory defined by $S'_{\text{gluons}}[U]$ is open to an application of the color-flavor transformation.

5.1.2 Strong Coupling Limit

By an expansion of the logarithm for small β we obtain

$$
\begin{aligned}
S'_{\text{gluons}}[U] &= N_a \sum_{\text{plaquettes } p} \Big(\ln \text{Det}(1 - \beta U^{\text{P}}(p)) + \ln \text{Det}(1 - \beta U^{\text{P}\dagger}(p)) \Big) \\
&= N_a \sum_{\text{plaquettes } p} \Big(\text{Tr} \ln(1 - \beta U^{\text{P}}(p)) + \text{Tr} \ln(1 - \beta U^{\text{P}\dagger}(p)) \Big) \\
&\approx -N_a \beta \sum_{\text{plaquettes } p} \Big(\text{Tr}\, U^{\text{P}}(p) + \text{Tr}\, U^{\text{P}\dagger}(p) \Big) + O(\beta^2).
\end{aligned}
\tag{5.13}
$$

Comparison with (5.1) shows that the new action converges to Wilson's action in the limit $\beta \to 0$. In that limit the coupling parameters are related by

$$
g = \frac{1}{\sqrt{2N_a\beta}} a^{d/2-2} \qquad (\text{for } \beta \ll 1).
\tag{5.14}
$$

We conclude that the limit $\beta \to 0$ (infinite heavy auxiliary bosons) corresponds to the strong coupling limit $g \to \infty$ of QCD.

5.1.3 Continuum limit

We generalize the correspondence between the plaquette variables and the field strength (4.14) by raising them to the power of a positive integer k,

$$
\begin{aligned}
(U^{\text{P}}_{\mu\nu})^k &= \exp(ikga^2 F_{\mu\nu} + O(a^3)) \\
&\approx 1 + ikga^2 F_{\mu\nu} - \frac{1}{2} k^2 g^2 a^4 F^2_{\mu\nu}.
\end{aligned}
\tag{5.15}
$$

This expansion up to second order is a good approximation as long as the argument of the exponential function is small. We assume that this condition is fulfilled for all k lower or equal to a fixed integer l,

$$
|kga^2 F_{\mu\nu}| \ll 1, \text{ for all } k \le l.
\tag{5.16}
$$

We split the expansions of the logarithms in the action $S'_{\text{gluons}}[U]$ into a sum up to l and a remainder. The terms in the sum up to l, by making use of (5.15), can be replaced by continuum expressions,

$$
\begin{aligned}
&\text{Tr} \ln(1 - \beta U^{\text{P}}_{\mu\nu}) + \text{Tr} \ln(1 - \beta U^{\text{P}\dagger}_{\mu\nu}) \\
&= -\sum_{k=1}^{\infty} \frac{1}{k} \beta^k \Big(\text{Tr}(U^{\text{P}}_{\mu\nu})^k + \text{Tr}(U^{\text{P}\dagger}_{\mu\nu})^k \Big) \\
&\approx g^2 a^4 \, \text{Tr}\, F^2_{\mu\nu} \sum_{k=1}^{l} k\beta^k - \sum_{k=l+1}^{\infty} \frac{1}{k} \beta^k \Big(\text{Tr}(U^{\text{P}}_{\mu\nu})^k + \text{Tr}(U^{\text{P}\dagger}_{\mu\nu})^k \Big) \\
&= g^2 a^4 \big(a_0(\beta) - a_l(\beta) \big) \, \text{Tr}\, F^2_{\mu\nu} + R_l(\beta, U^{\text{P}}).
\end{aligned}
\tag{5.17}
$$

Here and thereafter we make use of the following abbreviations for two infinite series

$$a_l(\beta) := \sum_{k=l+1}^{\infty} k\beta^k, \qquad b_l(\beta) := \sum_{k=l+1}^{\infty} \frac{1}{k}\beta^k \tag{5.18}$$

and for the remainder term

$$R_l(\beta, U^{\mathrm{P}}) := -\sum_{k=l+1}^{\infty} \frac{1}{k}\beta^k \left(\mathrm{Tr}\, U^{\mathrm{P}}_{\mu\nu} + \mathrm{Tr}(U^{\mathrm{P}\dagger}_{\mu\nu})^k\right). \tag{5.19}$$

The remainder term has an upper bound which involves the second infinite series,

$$|R_l(\beta, U^{\mathrm{P}})| \le 2N_c b_l(\beta), \qquad \text{for all } U^{\mathrm{P}} \in \mathrm{U}(N_c). \tag{5.20}$$

In order to drive the remainder term to zero, l should be sent to infinity. However, this would lead to a violation of (5.16) which would spoil the expansion in power of the field strength $F_{\mu\nu}$. To solve this problem, we introduce a dependence of l on the coupling parameter,

$$l(\beta) := \left[\frac{1}{(1-\beta)^{1+\varepsilon}}\right] \tag{5.21}$$

where ε is a small positive number. [1] By the relation (5.21) the integer l is driven to infinity in the limit $\beta \to 1$ which we expect to correspond to the weak coupling limit $g \to 0$. In Appendix E we have shown that

$$\lim_{\beta \to 1} a_{l(\beta)}(\beta) = 0, \qquad \lim_{\beta \to 1} |R_{l(\beta)}(\beta, U^{\mathrm{P}})| = \lim_{\beta \to 1} b_{l(\beta)}(\beta) = 0 \tag{5.22}$$

and

$$a_0(\beta) = \frac{\beta}{(1-\beta)^2}. \tag{5.23}$$

We conclude that the lattice action converges to the continuum action

$$S'_{\mathrm{gluons}}[A] = N_a g^2 a^{4-d} \frac{\beta}{(1-\beta)^2} \int d^d x\, \mathrm{Tr}\, F^2_{\mu\nu} \tag{5.24}$$

in the limit $\beta \to 1$ and we recover the gluodynamics action provided that

$$g = \frac{1-\beta}{\sqrt{2N_a\beta}} a^{d/2-2} \qquad (\text{for } a \to 0,\ \beta \to 1). \tag{5.25}$$

To summarize, the lattice theory defined by the action $S'_{\mathrm{gluons}}[U]$ has gluodynamics as a formal continuum limit. Passing from one theory to the other, the coupling constant of the lattice theory $\beta \in]0,1[$ is connected with the continuum coupling constant $g \in]0,\infty[$ via (5.25). It remains to check condition (5.16), which is crucial for our derivation.

Recall that the lattice constant can be thought as an ultraviolet cutoff of the divergent continuum theory. Recovering the continuum theory one must address the question of renormalization. The divergences of the continuum theory must be removed in the calculation of the physical observables. Through the process of renormalization the bare couplings acquire an

---

[1]$[x]$ denotes the greatest integer smaller or equal to x.

cutoff dependence chosen in such a manner that the physical quantities have a finite limit when the cutoff is removed. Thus the couplings become functions $g(a)$ and $\beta(a)$ of the lattice constant and (5.16) should be read as

$$l\big(\beta(a)\big)g(a)a^2 \to 0, \qquad \text{for } a \to 0. \tag{5.26}$$

While the evaluation of the four-dimensional lattice theory is difficult, as drastic simplification occurs it in two space-time dimensions. We will show that condition the (5.26) is fulfilled in that case.

5.2   Gluodynamics in Two Space-Time Dimensions

In two space-time dimensions lattice gauge theories turn out to have a much simpler structure than in higher dimensions. Indeed, having solved the gauge theory in a space-time consisting of only one plaquette one has essentially solved the full two-dimensional lattice gauge theory.

5.2.1   Factorization of Partition Function and Wilson Loop

In this section we review that the calculation of the partition function and the Wilson loop of the two-dimensional gauge theory can be reduced to the calculation of a single integral over the unitary group [21]. This is true for both, the theory defined by Wilsons action $S_{\text{gauge}}[U]$ and the one defined by the action $S'_{\text{gauge}}[U]$.

Starting point is the gauge invariance of the theory, that is the invariance under

$$U(n + \hat{\mu}/2) \; \to \; \mathcal{U}(n + \hat{\mu})\, U(n + \hat{\mu}/2)\, \mathcal{U}^\dagger(n). \tag{5.27}$$

We exploit the gauge freedom as follow:

- We chose the temporal gauge $U((x,t) + \hat{0}/2) = I$ for all x and t; this gauge choice corresponds to $A_0 = 0$ for the continuum theory.

- Further we fix a time t_0 and assume $U((x,t_0) + \hat{1}/2) = I$ for all x, corresponding to $A_1(t_0, x) = 0$ for all x for the continuum theory.

The physical degrees of freedom, which remain after this gauge fixing, are drawn as circles in Figure III.

In the temporal gauge the plaquette variables reduce to

$$U_{01}^{\text{P}}(n) = U(n + \hat{0} + \hat{1}/2)U^{-1}(n + \hat{1}/2). \tag{5.28}$$

As a consequence the partition function can be evaluated by a change of variables from the link variables to the plaquette variables. The crucial point is that both actions have the structure

$$S[U] = \sum_p S^{\text{P}}(U^{\text{P}}(p)), \qquad S'[U] = \sum_p S'^{\text{P}}(U^{\text{P}}(p)), \tag{5.29}$$

i. e. they are sums over the plaquettes p of a local action

$$\begin{aligned}
S_{\text{gluons}}^{\text{P}}(U) &= -\frac{1}{2a^2g^2}\left(\operatorname{Tr} U^{\text{P}} + \operatorname{Tr} U^{\text{P}\dagger}\right), \text{ resp.} \\
S_{\text{gluons}}'^{\text{P}}(U) &= -N_a\left(\operatorname{Tr}\ln(1 - \beta U^{\text{P}}) + \operatorname{Tr}\ln(1 - \beta U^{\text{P}\dagger})\right)
\end{aligned} \tag{5.30}$$
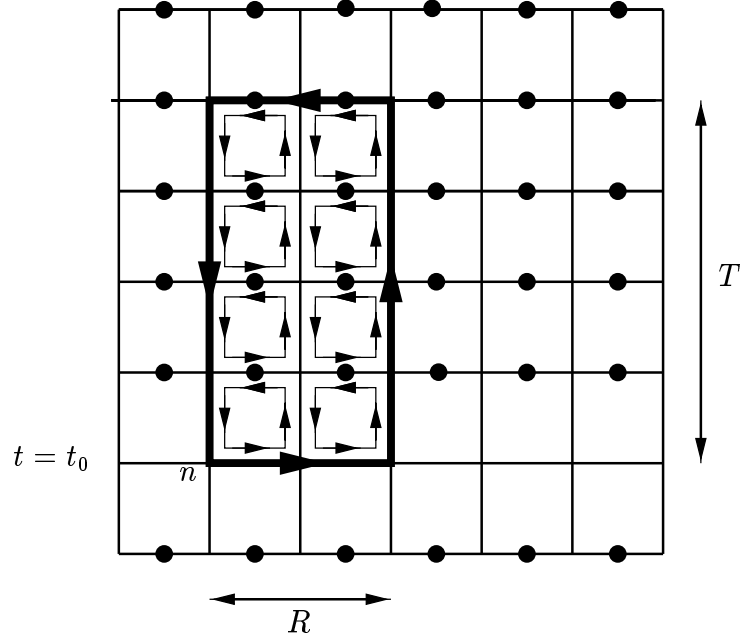
Figure III: Fixing of the gauge and Wilson loop: the remaining degrees of freedom of the gauge field are depicted by filled circles on the links.

evaluated at the corresponding plaquette variable $U^{\mathrm{P}}(p)$. As a result the partition functions decompose into a product of independed factors coming from the different plaquettes,

$$Z = (Z^{\mathrm{P}})^{\mathcal{N}}, \qquad Z' = (Z'^{\mathrm{P}})^{\mathcal{N}}. \tag{5.31}$$

In the expression above we have introduced the one-plaquette partition functions

$$Z^{\mathrm{P}} = \int dU \, \exp(-S^{\mathrm{P}}(U)), \qquad Z'^{\mathrm{P}} = \int dU \, \exp(-S'^{\mathrm{P}}(U)) \tag{5.32}$$

and $\mathcal{N}$ denotes the number of plaquettes.

Ground-state expectation values of physical observables O, which are gauge invariant functionals of the gauge field, are given by

$$\langle O[U] \rangle := \frac{1}{Z} \int [dU] \, \exp(-S_{\mathrm{gluons}}[U]) O[U], \qquad \langle O[U] \rangle' := \frac{1}{Z'} \int [dU] \, \exp(-S'_{\mathrm{gluons}}[U]) O[U]. \tag{5.33}$$

We consider the Wilson loop, which is a product of the gauge field along a closed loop L,

$$W_L := \frac{1}{N_c} \Big\langle \mathrm{Tr} \Big(\prod_L U \Big) \Big\rangle, \qquad W'_L := \frac{1}{N_c} \Big\langle \mathrm{Tr} \Big(\prod_L U \Big) \Big\rangle'. \tag{5.34}$$

We chose the loop to be rectangular with a spatial extend Ra and a time extend Ta. Passing to the plaquette variables, the Wilson loop can be expressed as

$$\begin{aligned} W_L &= \frac{1}{N_c} \Big\langle \mathrm{Tr} \prod_{j=R-1}^{0} U\big(n + T\hat{0} + (j + 1/2)\big) \\ &= \frac{1}{N_c} \Big\langle \mathrm{Tr} \prod_{j=R-1}^{0} \prod_{i=T-1}^{0} U^{\mathrm{P}}(n + i\hat{0} + j\hat{1}) \Big\rangle. \end{aligned} \tag{5.35}$$

The last equality is illustrated in Figure III: The contributions from two adjacent plaquettes inside the Wilson loop to the common link cancel each other. Further we took advantage from the temporal gauge and the gauge at the fixed time $t = t_0$.

We now consider the integral over a particular plaquette variable U^{P} in (5.35). Making use of the invariance of the Haar measure and the invariance of the action $S^{\mathrm{P}}(VUV^\dagger) = S^{\mathrm{P}}(U)$ for $V \in \mathrm{U}(N_c)$, we obtain

$$\int dU \, \exp(-S^{\mathrm{P}}(U)) \, \mathrm{Tr}(AUB) = \int dU \, \exp(-S^{\mathrm{P}}(U)) \, \mathrm{Tr}(AVUV^\dagger B). \qquad (5.36)$$

Here A and B represent the product of the remaining plaquette variables. We integrate (5.36) over V with respect to the Haar measure dV of $\mathrm{U}(N_c)$. From the orthogonality

$$\int_{\mathrm{U}(N_c)} dV \, V_{ij} V_{kl}^\dagger = \frac{1}{N_c} \delta_{il} \delta_{jk} \qquad (5.37)$$

we conclude that $\mathrm{Tr}(AU^{\mathrm{P}}B)$ can be replaced by $1/N_c \, \mathrm{Tr}(U^{\mathrm{P}}) \, \mathrm{Tr}(AB)$. Therefore we can iterate the argument and obtain the result

$$W_L = (W^{\mathrm{P}})^{RT}, \qquad W_L' = (W'^{\mathrm{P}})^{RT} \qquad (5.38)$$

where we have introduced the one-plaquette Wilson loops

$$W^{\mathrm{P}} = \frac{1}{Z^{\mathrm{P}}} \int_{\mathrm{U}(N_c)} dU \, \exp(-S^{\mathrm{P}}(U)) \frac{1}{N_c} \mathrm{Tr}\, U, \qquad W'^{\mathrm{P}} = \frac{1}{Z'^{\mathrm{P}}} \int_{\mathrm{U}(N_c)} dU \, \exp(-S'^{\mathrm{P}}(U)) \frac{1}{N_c} \mathrm{Tr}\, U \qquad (5.39)$$

and RT is the number of plaquettes inside the Wilson loop L.

The expectation value of the Wilson loop is directly related to an observable quantity. In fact, it measures the *force law* between two static sources which are placed in the system [48], [30]. A transfer matrix argument shows that for large T

$$W_L \sim \exp(-\epsilon(R)Ta), \qquad W_L' \sim \exp(-\epsilon'(R)Ta) \qquad (5.40)$$

where $\epsilon(R)$ and $\epsilon'(R)$ respectively is the gauge field energy associated with static quark-antiquark sources separated by a distance R.

$$\epsilon(R) = -\frac{\ln(W^{\mathrm{P}})}{a}R, \qquad \epsilon(R) = -\frac{\ln(W'^{\mathrm{P}})}{a}R. \qquad (5.41)$$

For both lattice gauge theories, the energy rises linearly with the separation of the quark-antiquarks-pair. Since $W^{\mathrm{P}} < 1$ and $W'^{\mathrm{P}} < 1$ for $\beta > 0$ the constant of proportionality is always positive. To summarize, both two-dimensional lattice gauge theories confine quark-antiquarks-pairs in a linear potential.

5.2.2   Partition Function and Wilson Loop of an Elementary Plaquette

This subsection is devoted to the calculation of the one-plaquette partition function Z'^{P} and the one-plaquette Wilson loop W'^{P}.

Monte Carlo Simulations

Monte Carlo simulations can be used to estimate the expectation value of an observable

$$\langle O(U) \rangle' = \frac{1}{Z'^{\mathrm{P}}} \int dU \, \exp(-S'^{\mathrm{P}}(U)) O(U) \tag{5.42}$$

with respect to the action $S'^{\mathrm{P}}(U)$. The method works effectively, when the weight $\exp(-S'^{\mathrm{P}}(U))$ is sharply peaked around a matrix U_0. In case of our action

$$S'^{\mathrm{P}}(U) = -N_a \left(\mathrm{Tr}\ln(1 - \beta U) + \mathrm{Tr}\ln(1 - \beta U^\dagger) \right) \tag{5.43}$$

the weight $\exp(-S'^{\mathrm{P}}(U))$ is distributed more and more tightly around $U_0 = I$ in the limit $\beta \to 1$. (In a thermodynamical analog $\beta \to 1$ corresponds to the low temperature and $\beta \to 0$ to the high temperature limit.) The idea of the Monte Carlo method [14] is to replace the integral (5.42) by a sum over a random chosen configuration [2] $\{U\}$ which is typical for the distribution $\exp(-S'^{\mathrm{P}}(U))$:

$$\langle O(U) \rangle' \approx \frac{1}{\#\{U\}} \sum_{\{U\}} \exp(-S'^{\mathrm{P}}(U)) O(U). \tag{5.44}$$

To generate such a configuration we make use of the *Metropolis algorithm* which is implemented in the following way:

(i) Generate a list $\{U\}$ of unitary matrices which are weighted towards the unity I.

(ii) Choose a start matrix U.

(iii) Build a trial matrix V by multiplying U with a random matrix from the list.

(iv) If the action is lowered ($S'^{\mathrm{P}}(V) < S'^{\mathrm{P}}(U)$) take V as new matrix. If the action is raised, ($S'^{\mathrm{P}}(V) \geq S'^{\mathrm{P}}(U)$) accept a change from U to V with the probability $\mathrm{e}^{-\left(S'^{\mathrm{P}}(V) - S'^{\mathrm{P}}(U)\right)}$

(v) Add the new matrix to the list $\{U\}$, designate it as U and go on with (iii).

By its definition the algorithm satisfies the *detailed balance* relation

$$P(V, U) \exp(-S'^{\mathrm{P}}(V)) = P(U, V) \exp(-S'^{\mathrm{P}}(U)) \tag{5.45}$$

where $P(V, U)$ is the probability to change the matrix U into V in one step of the stochastic process. Detailed balance is a sufficient condition to obtain a list $\{U\}$ with the distribution $\exp(-S'^{\mathrm{P}}(U))$, see [14].
The results of the Monte Carlo simulation for $N_c = 3$ colors and different values of the parameter N_a are presented in Figure IV. The diagram shows the expectation value of the one-plaquette Wilson loop in dependence of the coupling parameter β. The number of Monte-Carlo steps (that is the number of contributions to the sum (5.44)) was chosen as $80,000$ for all values of the constants.

---

[2] The list $\{U\}$ is not a set in the mathematical sense, because it may contain the same element twice or several times
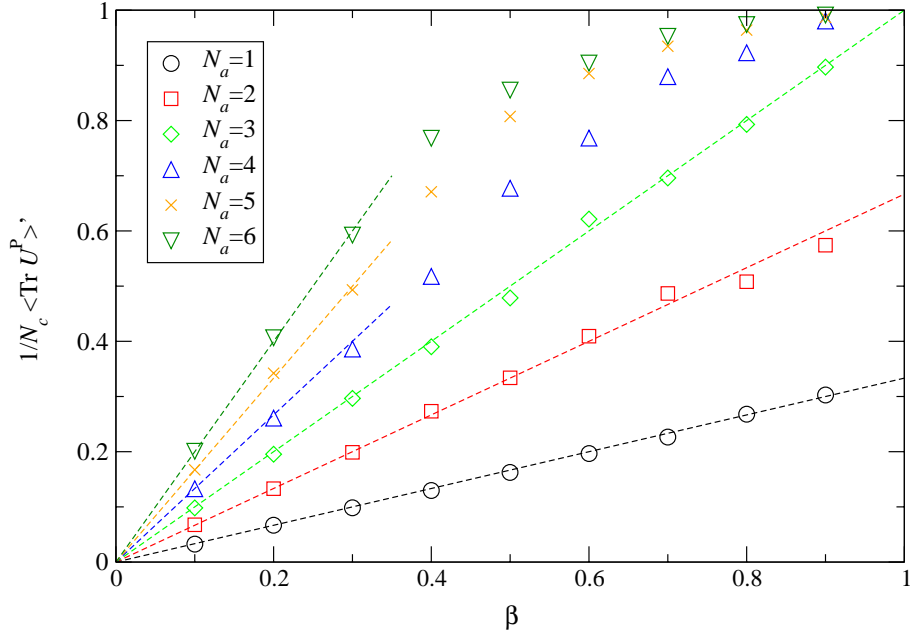
Figure IV: Expectation value of the Wilson loop operator in dependence of the coupling constant β. Results of Monte Carlo simulations (symbols) and analytical results (lines) for $N_c = 3$ colors and $N_a = 1, ..., 6$. The lines are given by the functions $f(\beta) = \frac{N_a}{N_c}\beta$.

The Wilson loop is a monotonically increasing function of the coupling parameter β. The slope of this function is equal to N_a/N_c, in general for small values of β and for arbitrary β if $N_a \leq N_c$. The slope for small values of β can be estimated from the expansion

$$\int dU \left| \mathrm{Det}\,(1 - \beta U) \right|^{-2N_a} \mathrm{Tr}\, U \approx \int dU \left| 1 - \beta \, \mathrm{Tr}\, U \right|^{-2N_a} \mathrm{Tr}\, U$$

$$\approx \int dU \left(1 + N_a\beta(\mathrm{Tr}\, U + \mathrm{Tr}\, U^\dagger) \right) \mathrm{Tr}\, U = N_a\beta \tag{5.46}$$

which gives the result

$$W'^{\mathrm{P}} = \frac{1}{N_c}\langle \mathrm{Tr}\, U \rangle' = \frac{N_a}{N_c}\beta + O(\beta^2) \tag{5.47}$$

for the Wilson loop. The next subsection is devoted to an analytical calculation of the Wilson loop. We are able to obtain an exact result for all N_c and all $N_a \leq N_c$ which confirms the linear dependence of the one-plaquette Wilson loop on the coupling parameter β with slope N_a/N_c over the whole range of the coupling parameter.

Analytical Approach

The integrand of the partition function Z'^{P} depends on U only through its eigenvalues. In a first step we simplify the integration by a change of the integration variables to the eigenvalues

of U and certain angular variables, which can be integrated out,

$$Z'^{\mathrm{P}} = \int_{\mathrm{U}(N_c)} dU \, | \operatorname{Det}(1 - \beta U)|^{-2N_a}$$

$$\sim \int_0^{2\pi} d\phi_1 \cdots \int_0^{2\pi} d\phi_{N_c} \frac{\prod_{k<l} |\exp(i\phi_k) - \exp(i\phi_l)|^2}{\prod_{k=1}^{N_c} |1 - \beta \exp(i\phi_k)|^{2N_a}}. \tag{5.48}$$

With the sign $\sim$ we denote equality up to a β-independent factor. The factor can be restored at the end of the calculations by the condition $Z'^{\mathrm{P}}(\beta = 0) = 1$. To rewrite (5.48) as a multiple complex contour integral over the unit circle, we put $z_k = \exp(i\phi_k)$ and $dz_k = iz_k d\phi_k$ for $k = 1, ..., N_c$. Noting that $\bar{z}_k = \frac{1}{z_k}$ on the contour, we obtain

$$Z'^{\mathrm{P}} \sim \frac{1}{\beta^{N_a N_c}} \int_{S^1} dz_1 \cdots \int_{S^1} dz_{N_c} \frac{\prod_{k<l} (z_k - z_l)^2}{\prod_{k=1}^{N_c} z_k^{N_c - N_a} (z_k - \beta)^{N_a} (z_k - \frac{1}{\beta})^{N_a}}. \tag{5.49}$$

Our approach to calculate the partition function is based on the theorem of residues, which is applied iterated to the variables $z_1, ..., z_{N_c}$. Evaluating the residues, we have to take into account the poles of the integrand in the unit disc. They occur at 0 and β. Recall the following formula for the residuum: If $f : D_f \to \mathbb{C}$ is analytic at z_0, the corresponding residuum is given by

$$Res_{z_0} \frac{f(z)}{(z - z_0)^n} = \frac{f^{(n-1)}(z_0)}{(n-1)!}. \tag{5.50}$$

We are now ready to evaluate the integral (5.49): We separately consider the cases $N_a = N_c$ and $N_a < N_c$. The case $N_a > N_c$ is more involved and not considered further.

Case $N_a = N_c$: In this case the integrand has a N_c-fold poles at $z_k = \beta$ for all k and no poles at $z_k = 0$. Due to (5.50) the partition function is proportional to

$$\partial_{z_1}^{N_c - 1} ... \partial_{z_{N_c}}^{N_c - 1} \frac{\prod_{k<l} (z_k - z_l)^2}{\beta^{N_c^2} \prod_{k=1}^{N_c} (z_k - \frac{1}{\beta})^{N_c}} \bigg|_{z_1 = ... = z_k = \beta} \sim \frac{1}{\beta^{N_c^2} (\beta - \frac{1}{\beta})^{N_c^2}} \tag{5.51}$$

Under consideration of $Z'^{\mathrm{P}}(\beta = 0) = 1$ we conclude

$$Z'^{\mathrm{P}} = \frac{1}{(1 - \beta^2)^{N_c^2}}. \tag{5.52}$$

Case $N_a < N_c$: To label the different contributions to the partition function, we divide the set $\{1, ..., N_c\}$ in two subsets A and B of cardinality a and b respectively. The contribution belonging to the partition (A, B) is defined by taking iterated residues of the integrand at $z_k = 0$ for $k \in A$ and at $z_k = \beta$ for $k \in B$. Without loss of generality we assume $A = \{z_1, ..., z_a\}$ and $B = \{z_{a+1}, ..., z_{N_c}\}$. Now we again make use of the formula (5.50) for the

residuum. As a result we find contribution to the partition function corresponding to the partition (A, B) to be proportional to

$$
\partial_{z_1}^{N_c-N_a-1}...\partial_{z_a}^{N_c-N_a-1}\partial_{z_{a+1}}^{N_a-1}...\partial_{z_{N_c}}^{N_a-1} \frac{\prod_{k<l}(z_k - z_l)^2}{\beta^{N_a N_c} \prod_{k=a+1}^{N_c} z_k^{N_c-N_a} \prod_{k=1}^{a}(z_k - \beta)^{N_a} \prod_{k=1}^{N_c}(z_k - \frac{1}{\beta})^{N_a}} \tag{5.53}
$$

evaluated at $z_1 = ... = z_a = 0$ and $z_{a+1} = ... = z_{N_c} = \beta$. We split the numerator of (5.53) into contributions from the set A, cross-terms, and contributions from the set B,

$$
\prod_{k<l}(z_k - z_l)^2 = \prod_{k,l\in A, k<l}(z_k - z_l)^2 \prod_{k\in A, l\in B}(z_k - z_l)^2 \prod_{k,l\in B, k<l}(z_k - z_l)^2. \tag{5.54}
$$

To pick the partitions (A, B) with a non-vanishing contribution to the partition function we remark that

$$
\partial_{z_1}^{j_1}...\partial_{z_a}^{j_a}\Big|_{z_1=...=z_n} \prod_{k,l\in A, k<l}^{N_c}(z_k - z_l)^2 = 0, \text{ if } j_1 + ... + j_n \neq a(a + 1),
$$
$$
\partial_{z_{a+1}}^{j_{a+1}}...\partial_{z_{N_c}}^{j_{N_c}}\Big|_{z_{a+1}=...=z_{N_c}} \prod_{k,l\in B, k<l}^{N_c}(z_k - z_l)^2 = 0, \text{ if } j_{a+1} + ... + j_{N_c} \neq b(b + 1). \tag{5.55}
$$

Thus, to get a contribution, the first factor in (5.54) needs to be differentiated $a(a-1)$ times, and the last factor $b(b-1)$ times. Comparing with (5.53) we get $a(a-1) \leq a(N_c - N_a - 1)$ and $b(b-1) \leq b(N_a - 1)$. Because of $a + b = N_c$ the only non-vanishing contributions come from $a = N_c - N_a$ and $b = N_a$. We fix the normalization constant by $Z'^{\mathrm{P}}(\beta = 0) = 1$ and get the result

$$
Z'^{\mathrm{P}} = \frac{1}{(1 - \beta^2)^{N_a^2}}. \tag{5.56}
$$

The expression for the one-plaquette Wilson loop can be rewritten in the same way,

$$
\langle \mathrm{Tr}\, U \rangle' = N_c \int_{S^1} dz_1 \cdots \int_{S^1} dz_{N_c} \frac{z_1 \prod_{k<l}(z_k - z_l)^2}{\prod_{k=1}^{N} z_k^{N_c-N_a}(z_k - \beta)^{N_a}(z_k - \frac{1}{\beta})^{N_a}} \Bigg/
$$
$$
\int_{S^1} dz_1 \cdots \int_{S^1} dz_{N_c} \frac{\prod_{k<l}(z_k - z_l)^2}{\prod_{k=1}^{N} z_k^{N_c-N_a}(z_k - \beta)^{N_a}(z_k - \frac{1}{\beta})^{N_a}}. \tag{5.57}
$$

We employ the method developed above to evaluate this expression for $N_a \leq N_c$. The extra factor z_1 in the numerator needs to be evaluated at β. Thus the ratio of the two integrals gives the combinatorial factor $\binom{N_c-1}{N_a-1}/\binom{N_c}{N_a} = \frac{N_a}{N_c}$ times β:

$$
W'^{\mathrm{P}} = \frac{1}{N_c}\langle \mathrm{Tr}\, U \rangle' = \frac{N_a}{N_c}\beta \tag{5.58}
$$

The analytical result $(N_a \leq N_c)$ is in agreement with the Monte Carlo data, see Figure IV.

5.2.3   String Tension and Running Coupling Constant

We have already shown that the two-dimensional lattice gauge theory defined by the action $S'^{\mathrm{P}}[U]$ confines the quarks in a linear potential. We write

$$\epsilon'(R) = \sigma' R a, \qquad (5.59)$$

where we introduced the string tension

$$\sigma' = -\frac{1}{a^2} \ln W_L^{'\mathrm{P}}. \qquad (5.60)$$

We now restrict ourselves to the theory where the parameter N_a is lower or equal to the number of colors, $N_a \leq N_c$. In that case we can work with the analytical result for the Wilson loop from the last subsection and obtain for the string tension

$$\sigma' = -\frac{1}{a^2} \ln \left(\frac{N_a}{N_c} \beta \right). \qquad (5.61)$$

Passing to the continuum with the aim to reach gluodynamics, one has to renormalize the coupling parameter in such a way that the physical observables like the string tension σ' is kept constant. This is impossible in the case $N_a < N_c$ and we conclude the lattice gauge theory is ill defined in that case in the sense that it cannot be considered as a lattice regularization of gluodynamics. In the case $N_a = N_c$ we obtain the renormalization group equation

$$\beta(a) = \exp(-\sigma' a^2). \qquad (5.62)$$

Having calculated the renormalization of the coupling parameter β we are in position to rigorously justify the way we obtained gluodynamics from the lattice theory for the two-dimensional theories. The validity of the calculations in Subsection 5.1.3 follows from

$$lga^2 \approx \frac{a^{d/2}}{\sqrt{2N_a\beta}(1-\beta)^\varepsilon} = \frac{a^{d/2-2\varepsilon}}{\sqrt{2N_a}\sigma'^\varepsilon} \left(1 + O(a^2)\right), \qquad (5.63)$$

that is $lga^2 \to 0$ in the continuum limit.

5.3   Color-Flavor Transformation of Gluodynamics

In this section we consider lattice gluodynamics in an arbitrary number of space-time dimensions. Our idea is to apply the color-flavor transformation to the action $S_+[U, \bar\phi_+, \phi_+] + S_-[U, \bar\phi_-, \phi_-]$, defined by (5.4) and (5.10). However, there is a problem with the convergence of the color-flavor transformation: for bosonic fields it only converges if the number of flavors, which we identify with the dimension of the auxiliary space N_a, and the number of colors N_c satisfy $N_a \leq 2N_c$, cf. the discussion after Eq. (2.84). This condition is in conflict to the inequality $N_a \geq N_c$ that has to be satisfied for physical reasons. We overcome this problem by the introduction of a fermion as a partner to each of the bosons. For the resulting supersymmetric theory the color-flavor transformation is always convergent. Note that one can keep the influence of the fermions small by making their mass large. In order to apply the color-flavor transformation, we write the partition function of the theory as a product over the lattice links

$$\mathcal{Z}[\Phi, \bar\Phi] = \prod_{n,\mu} \int_{\mathrm{U}(N_c)} dU(n + \hat\mu/2) \; \mathrm{e}^{-S_{U,\Phi,\bar\Phi}(n+\hat\mu/2) - S_{\hat m_{\mathrm{B}}, \hat m_{\mathrm{F}}, \Phi, \bar\Phi}(n+\hat\mu/2)} . \qquad (5.64)$$

We introduce a labeling of the $2(d-1)$ plaquettes neighboring a lattice link $n + \hat{\mu}/2$. The plane of such a plaquette can be characterized by $\hat{\mu}$ and a second vector $\hat{\nu}$. We denote the plaquette spanned by attaching the vectors $\hat{\mu}$ and $\hat{\nu}$ to the site n by $(n; +\mu\nu)$ and the one spanned by attaching the vectors $\hat{\mu}$ and $-\hat{\nu}$ to this site by $(n; -\mu\nu)$. (Note that the order of μ and ν is important here.) With this preparation we are ready to write down the action belonging to a link $n + \hat{\mu}/2$,

$$
S_{U,\Phi,\bar{\Phi}}(n + \hat{\mu}/2) =
$$
$$
\sum_{\nu > \mu} \Big(\bar{\Phi}_{+,n+\hat{\mu}}(n;\mu\nu) U(n + \hat{\mu}/2) \Phi_{+,n}(n;\mu\nu) + \bar{\Phi}_{-,n}(n;\mu\nu) U^{\dagger}(n + \hat{\mu}/2) \Phi_{-,n+\hat{\mu}}(n;\mu\nu)
$$
$$
+ \bar{\Phi}_{+,n}(n;-\mu\nu) U^{\dagger}(n + \hat{\mu}/2) \Phi_{+,n+\hat{\mu}}(n;-\mu\nu) + \bar{\Phi}_{-,n+\hat{\mu}}(n;-\mu\nu) U(n + \hat{\mu}/2) \Phi_{-,n}(n;-\mu\nu) \Big)
$$
$$
+ \sum_{\nu < \mu} \Big(\bar{\Phi}_{+,n}(n;\mu\nu) U^{\dagger}(n + \hat{\mu}/2) \Phi_{+,n+\hat{\mu}}(n;\mu\nu) + \bar{\Phi}_{-,n+\hat{\mu}}(n;\mu\nu) U(n + \hat{\mu}/2) \Phi_{-,n}(n;\mu\nu)
$$
$$
+ \bar{\Phi}_{+,n+\hat{\mu}}(n;-\mu\nu) U(n + \hat{\mu}/2) \Phi_{+,n}(n;-\mu\nu) + \bar{\Phi}_{-,n}(n;-\mu\nu) U^{\dagger}(n + \hat{\mu}/2) \Phi_{-,n+\hat{\mu}}(n;-\mu\nu) \Big).
$$
$$(5.65)$$

The corresponding mass term is given by

$$
S_{\hat{m}_{\mathrm{B}},\hat{m}_{\mathrm{F}},\Phi,\bar{\Phi}}(n + \hat{\mu}/2) = \frac{1}{d}\hat{m}_{\mathrm{B}} \bar{\Phi}_{\mathrm{B}}(n)\Phi_{\mathrm{B}}(n) + \frac{1}{d}\hat{m}_{\mathrm{F}} \bar{\Phi}_{\mathrm{F}}(n)\Phi_{\mathrm{F}}(n), \tag{5.66}
$$

where we have split the supersymmetric field into the bosonic components and the fermionic components Φ_{B} and Φ_{F} with different masses $\hat{m}_{\mathrm{F}}$ and $\hat{m}_{\mathrm{B}}$. Now it is straightforward to apply the color-flavor transformation, separately to each of the links. The color-flavor transformed action reads

$$
S_{Z,\Phi,\bar{\Phi}}(n + \hat{\mu}/2) = \sum_{\nu,\nu' > \mu} \big(\bar{\Phi}_{+,n+\hat{\mu}}(n;+\mu\nu), \bar{\Phi}_{-,n+\hat{\mu}}(n;-\mu\nu) \big) \, Z_{\nu\nu'}(n + \hat{\mu}/2) \begin{pmatrix} \Phi_{+,n+\hat{\mu}}(n;-\mu\nu) \\ \Phi_{-,n+\hat{\mu}}(n,+\mu\nu) \end{pmatrix}
$$
$$
+ \big(\bar{\Phi}_{+,n}(n;-\mu\nu), \bar{\Phi}_{-,n}(n,+\mu\nu) \big) \, \tilde{Z}^t_{\nu\nu'}(n + \hat{\mu}/2) \begin{pmatrix} \Phi_{+,n}(n;+\mu\nu) \\ \Phi_{-,n}(n,-\mu\nu) \end{pmatrix}
$$
$$
\sum_{\nu > \mu > \nu'} \big(\bar{\Phi}_{+,n+\hat{\mu}}(n;+\mu\nu), \bar{\Phi}_{-,n+\hat{\mu}}(n;-\mu\nu) \big) \, Z_{\nu\nu'}(n + \hat{\mu}/2) \begin{pmatrix} \Phi_{+,n+\hat{\mu}}(n;+\mu\nu) \\ \Phi_{-,n+\hat{\mu}}(n,-\mu\nu) \end{pmatrix}
$$
$$
+ \big(\bar{\Phi}_{+,n}(n;+\mu\nu), \bar{\Phi}_{-,n}(n,-\mu\nu) \big) \, \tilde{Z}^t_{\nu\nu'}(n + \hat{\mu}/2) \begin{pmatrix} \Phi_{+,n}(n;+\mu\nu) \\ \Phi_{-,n}(n,-\mu\nu) \end{pmatrix}
$$
$$
\sum_{\nu < \mu < \nu'} \big(\bar{\Phi}_{+,n+\hat{\mu}}(n;-\mu\nu), \bar{\Phi}_{-,n+\hat{\mu}}(n;+\mu\nu) \big) \, Z_{\nu\nu'}(n + \hat{\mu}/2) \begin{pmatrix} \Phi_{+,n+\hat{\mu}}(n;-\mu\nu) \\ \Phi_{-,n+\hat{\mu}}(n,+\mu\nu) \end{pmatrix}
$$
$$
+ \big(\bar{\Phi}_{+,n}(n;-\mu\nu), \bar{\Phi}_{-,n}(n,+\mu\nu) \big) \, \tilde{Z}^t_{\nu\nu'}(n + \hat{\mu}/2) \begin{pmatrix} \Phi_{+,n}(n;-\mu\nu) \\ \Phi_{-,n}(n,+\mu\nu) \end{pmatrix}
$$
$$
\sum_{\nu,\nu' < \mu} \big(\bar{\Phi}_{+,n+\hat{\mu}}(n;-\mu\nu), \bar{\Phi}_{-,n+\hat{\mu}}(n;+\mu\nu) \big) \, Z_{\nu\nu'}(n + \hat{\mu}/2) \begin{pmatrix} \Phi_{+,n+\hat{\mu}}(n;+\mu\nu) \\ \Phi_{-,n+\hat{\mu}}(n,-\mu\nu) \end{pmatrix}
$$
$$
+ \big(\bar{\Phi}_{+,n}(n;+\mu\nu), \bar{\Phi}_{-,n}(n,-\mu\nu) \big) \, \tilde{Z}^t_{\nu\nu'}(n + \hat{\mu}/2) \begin{pmatrix} \Phi_{+,n}(n;-\mu\nu) \\ \Phi_{-,n}(n,+\mu\nu) \end{pmatrix}.
$$
$$(5.67)$$

Figure V: Coupling of the Z-field to the auxiliary field at the site n. The filled circles inside the plaquettes symbolize the auxiliary field Φ_n belonging to that plaquette.

As the gauge field, the new field Z is located on the lattice links. For a link $n + \hat{\mu}/2$ it is a matrix in the space of the links $\nu \neq \mu$ and two-dimensional space of $+$ and $-$,

$$Z_{\nu\nu'}(n + \hat{\mu}/2) = \begin{pmatrix} Z_{\nu\nu',++}(n + \hat{\mu}/2) & Z_{\nu\nu',+-}(n + \hat{\mu}/2) \\ Z_{\nu\nu',--}(n + \hat{\mu}/2) & Z_{\nu\nu',--}(n + \hat{\mu}/2). \end{pmatrix} \tag{5.68}$$

In terms of the color-flavor transformed action, the partition function (5.64) reads

$$\mathcal{Z}[\Phi, \bar{\Phi}] = \prod_{n,\mu} \int_{U(N_c)} dU(n + \hat{\mu}/2) \; e^{-S_{Z,\Phi,\bar{\Phi}}(n+\hat{\mu}/2) - S_{\hat{m}_B, \hat{m}_F, \Phi, \bar{\Phi}}(n+\hat{\mu}/2)} \,. \tag{5.69}$$

The coupling of the auxiliary fields $\bar{\Phi}_n$, Φ_n to the Z-field in the $\mu\nu$-plane is shown in Figure V. Following the arrows of this picture, one could perform the integration over the quarks in two space-time dimensions. However, the two-dimensional theory is trivial and we have already evaluated it in the last chapter. The situation in higher dimensions is much more involved: in this case the Z-field couples to Φ-fields in different planes.

5.4   Conclusion

We have found a way to put gluodynamics on a lattice that allows an application of the color-flavor transformation. The idea is to replace Wilson's lattice action by another action which can be generated by an integral over bosonic auxiliary fields. The new action depends on a coupling parameter β, where $\beta \to 1$ corresponds to $g \to 0$ and $\beta \to 0$ to $g \to \infty$ (g is the coupling constant of gluodynamics). If one wants to work with a finite number of auxiliary particles ($N_a < \infty$), it is decisive to employ bosons (and not fermions), because only in this case the integrand of the partition function shows the right behavior. It needs to develop a pole at $U^{\mathrm{P}} = I$ in the limit $\beta \to 1$. This pole ensures that the main contribution to the partition function comes from matrices near the unity matrix. If this effect is strong enough, the fluctuations of the large contributions to the partition function can be controlled by sending $\beta \to 1$ and a continuum limit exists.

It turns out that this is true only when the order of the pole in the integrand of the partition function is higher than the number of colors. This is the case, when the dimension of the auxiliary space is greater than the number of colors, $N_a \geq N_c$.

On the other hand the bosonic color-flavor transformation needs $2N_a \leq N_c$ in order to be convergent. We have lifted this conflict by passing to a supersymmetric theory and applied the color-flavor transformation. Further evaluation of the color-flavor transformed partition function is left for the future. The next step would be to perform the integration over the auxiliary fields.

Chapter 6

Strong Coupling Quantum Chromodynamics (Mesons Sector)

This and the next chapter deal with the *strong coupling limit* of QCD. It is reached by neglecting the gluon term (4.15) in the lattice action $S_{\text{lattice}} = S_{\text{gluons}} + S_{\text{quarks}}$, which is suppressed as $1/g^2$. The lattice approach should be thought as an ultraviolet regularization of QCD with lattice constant as cutoff. Through the process of renormalization the coupling constant acquires a dependence on the lattice constant a. In case of QCD the running coupling constant $g(a)$ goes to zero in the continuum limit $a \to 0$, which reflect the asymptotic freedom. In the opposite limit of strong coupling $g(a) \gg 1$ the lattice constant has to be kept at a finite value and the theory can be thought as an approximation to low energy physics.

In the present chapter we restrict ourselves to a simplified theory of mesons and leave the full theory with baryons to the next chapter. This corresponds to a replacement of the special unitary group $\text{SU}(N_c)$ by the full unitary group $\text{U}(N_c)$ as gauge group, a fact that can be explicitly seen in the formalism of the color-flavor transformation: Taking $\text{SU}(N_c)$ as gauge group, the color-flavor transformed partition decomposes into a sum over contributions characterized by different baryon numbers, as exposed in Chapter 2. The contribution belonging to baryon number zero is exactly the color-flavor transform of the partitions function with $\text{U}(N_c)$ as gauge group.

6.1   Color-Flavor Transformation of the Lattice Action

We consider a $\text{U}(N_c)$ gauge theory in d-dimensional Euclidean space-time placed on an orientated lattice. The coordination number of the lattice (the number of nearest neighbours of a lattice site) is denoted by 2δ. On a hypercubic lattice we have $\delta = d$, but we will also consider the four-dimensional bhc lattice, where coordination number is bigger than twice the number of dimensions. The theory is defined by the partition function, which can be written as a product over all lattice links,

$$\mathcal{Z} = \int [dq d\bar{q}] \, \mathcal{Z}[q, \bar{q}], \qquad \text{where}$$

$$\mathcal{Z}[q, \bar{q}] = \prod_{n,v} \int_{\text{U}(N_c)} dU(n + v/2) \, \mathrm{e}^{-S_{U,q,\bar{q}}(n+v/2) - S_{\mathcal{M},q,\bar{q}}(n)} \, . \tag{6.1}$$

The corresponding lattice action $S_{\text{quarks}} = \sum_{n,v} \left(S_{U,q,\bar{q}}(n + v/2) + S_{\mathcal{M},q,\bar{q}}(n) \right)$ was already introduced in Chapter 4. Recall that the quarks $q(n)$ are put on the lattice sites n, while the gluons represented by a matrix field $U(n + v/2)$ are placed on the lattice links. The sum runs over all lattice links, which are parametrized by the lattice sites n and the vectors to the nearest neighbors in positive direction v. The quarks on two neighboring sites are coupled through the gauge field on the connecting link,

$$
\begin{aligned}
S_{U,q,\bar{q}}(n + v/2) = K_2 a^{d-1} \Big(& \bar{q}_a^i(n) \gamma_v U^{ij}(n + v/2) q_a^j(n + v) \\
& - \bar{q}_b^i(n + v) \gamma_v U^{\dagger ij}(n + v/2) q_b^j(n) \Big)
\end{aligned}
\tag{6.2}
$$

Additionally there is a mass term for the quarks,

$$
S_{\mathcal{M},q,\bar{q}}(n) = \frac{K_3}{\delta} a^d \bar{q}_a(n) \mathcal{M}_{ab} q_b(n),
\tag{6.3}
$$

which is diagonal in flavor space, $\mathcal{M} = \text{diag}(m_1, ..., m_{N_f})$. There is a factor $1/\delta$ in front of this term to cancel the summation over the δ links which correspond to each lattice site. The upper indices i, j label the color degrees of freedom, while the lower indices a, b refer to the flavor degrees of freedom. We employ the naive description of lattice fermions and ignore the doubling problem.

On each link we perform a $\text{U}(N_c)$ color-flavor transform which replaces the gauge field $U(n + v/2)$ by a complex field $Z(n + v/2)$. We recall that we are considering Dirac matter with flavor *and* spin degrees of freedom complementing the color degrees of freedom; therefore the field Z becomes a matrix in spin *and* flavor space, $Z(n + v/2) \in \mathbb{C}^{N_s N_f \times N_s N_f}$. To link the notations above to the corresponding ones of Chapter 2, we put

$$
\bar{\psi}_+ = -\bar{q}(n)\gamma_v, \ \psi_+ = q(n + v), \ \bar{\psi}_- = q(n), \ \psi_- = -\bar{q}(n + v)\gamma_v
\tag{6.4}
$$

and obtain the color-flavor transformed partition function

$$
\mathcal{Z}[q, \bar{q}] = \alpha_0 \prod_{n,v} \int_{\mathbb{C}^{N_s N_f \times N_s N_f}} \frac{D\mu(Z, Z^\dagger(n + v/2))}{\text{Det}(1 + (ZZ^\dagger)(n + v/2))^{N_c}} \ e^{-S_{Z,q,\bar{q}}(n + v/2) - S_{\mathcal{M},q,\bar{q}}(n)} .
\tag{6.5}
$$

The action corresponding to the link $n + v/2$ is given by

$$
\begin{aligned}
S_{Z,q,\bar{q}}(n + v/2) = K_2 a^{d-1} \Big(& \bar{q}_a^i(n) \gamma_v Z_{ab}(n + v/2) q_b^i(n) \\
& + \bar{q}_a^j(n + v) \gamma_v Z_{ab}^\dagger(n + v/2) q_b^j(n + v) \Big).
\end{aligned}
\tag{6.6}
$$

The field Z parametrizes the symmetric space $\text{U}(2N_s N_f)/\text{U}(N_s N_f) \times \text{U}(N_s N_f)$ equipped with its $\text{U}(2N_s N_f)$-invariant measure

$$
D\mu(Z, Z^\dagger) = C_{N_f} \ \text{Det}(1 + ZZ^\dagger)^{-2N_f} \prod_{i,j} dZ_{ij} \, d\bar{Z}_{ij}.
\tag{6.7}
$$

The quarks are now coupled through their *flavor* indices, whereas in the original action the coupling had been mediated by the color degrees of freedom. Moreover, the coupling has become ultralocal: A quark at the site n is coupled to another one at the same site via $Z(n + \hat{\mu}/2)$, a quark at the site $n + \hat{\mu}$ via $Z^\dagger(n + \hat{\mu}/2)$. Correlations between neighboring quarks are solely due to the relation between Z and $Z^\dagger$, Hermitian conjugation. A graphical description of the change of the coupling scheme is given in Figure VI. Note that the quarks enter the color-flavor transformed action only through color singlets built by quark-antiquark pairs, where the field Z mediates the coupling between these "mesons".
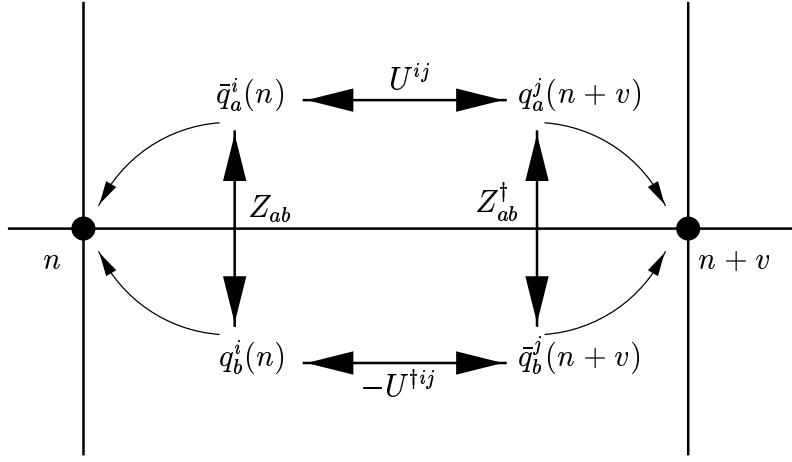
Figure VI: coupling of the quarks fields before (horizontal arrows) and after (vertical arrows) color-flavor transformation.

6.2 Chiral Symmetry

In Chapter 4 we have extensively studied chiral transformations on the lattice. They are defined by two sets of matrices $\mathcal{V}(n)$, $\bar{\mathcal{V}}(n) \in \mathrm{U}(N_s N_f)$, which satisfy

$$
\begin{aligned}
\mathcal{V}(n + v + w) &= \gamma_w \gamma_v \mathcal{V}(n) \gamma_v \gamma_w \\
\bar{\mathcal{V}}(n) &= \gamma_v \mathcal{V}(n + v)^{-1} \gamma_v.
\end{aligned}
\tag{6.8}
$$

These transformations act on the quark field by

$$
\begin{aligned}
q(n) &\rightarrow \mathcal{V}(n) q(n), \\
\bar{q}(n) &\rightarrow \bar{q}(n) \bar{\mathcal{V}}(n),
\end{aligned}
\tag{6.9}
$$

while the gauge field is left unchanged. In order to keep the color-flavor transformed action invariant, the meson field Z has to be transformed in the following way,

$$
\begin{aligned}
\gamma_v Z(n + v/2) &\rightarrow \bar{\mathcal{V}}^{-1}(n) \gamma_v Z(n + v/2) \mathcal{V}^{-1}(n), \\
\gamma_v Z^\dagger(n + v/2) &\rightarrow \bar{\mathcal{V}}^{-1}(n + v) \gamma_v Z^\dagger(n + v/2) \mathcal{V}^{-1}(n + v).
\end{aligned}
\tag{6.10}
$$

Making use of condition (6.8) this transformation can be rewritten as

$$
\begin{aligned}
Z(n + v/2) &\rightarrow \mathcal{V}(n + v) Z(n + v/2) \mathcal{V}^{-1}(n) \\
Z^\dagger(n + v/2) &\rightarrow \mathcal{V}(n) Z^\dagger(n + v/2) \mathcal{V}^{-1}(n + v).
\end{aligned}
\tag{6.11}
$$

One easily checks that the determinant coming from the color-flavor transformations is invariant under the chiral transformation,

$$
\begin{aligned}
\mathrm{Det}\left(1 + (ZZ^\dagger)(n + v/2)\right) &\rightarrow \mathrm{Det}\left(1 + \mathcal{V}(n + v)(ZZ^\dagger)(n + v/2)\mathcal{V}^{-1}(n + v)\right) \\
&= \mathrm{Det}\left(\mathcal{V}(n + v)\left(1 + (ZZ^\dagger)(n + v/2)\right)\mathcal{V}^{-1}(n + v)\right) \\
&= \mathrm{Det}\left(1 + (ZZ^\dagger)(n + v/2)\right).
\end{aligned}
\tag{6.12}
$$

To sum up, chiral symmetry at the level of the color-flavor transformed action is realized by simultaneous transformations of the quarks (6.9) and the field Z (6.11).

6.3   Integration over the Fermions

The integration over the quark fields yields

$$\mathcal{Z} = \int [dq d\bar{q}] \, \mathcal{Z}[q, \bar{q}] = \int D\mu[Z, Z^\dagger] \, \exp(-N_c S_{\text{vacuum}}[Z]), \tag{6.13}$$

where we have sent the result of the integration back to the exponent. The factor N_c in the exponent comes from the color content of the quarks: since the action $S_{Z,q,\bar{q}}(n + v/2)$ is diagonal in the color degrees of freedom, the Grassmann integral factorizes into N_c identical integrals. The resulting effective action reads

$$\begin{aligned}
S_{\text{vacuum}}[Z] = &- \sum_n \text{Tr} \ln \left(\sum_v \gamma_v \left(Z^\dagger(n - v/2) + Z(n + v/2) \right) + \hat{\mathcal{M}} \right) \\
&+ \sum_{n,v} \text{Tr} \ln \left(1 + Z(n + v/2) Z^\dagger(n + v/2) \right),
\end{aligned} \tag{6.14}$$

where we have introduced a dimensionless mass matrix

$$\hat{\mathcal{M}} := \frac{K_3}{K_2} a\mathcal{M}, \qquad \left(\hat{\mathcal{M}} = 2a\mathcal{M} \text{ on a hc lattice }, \quad \hat{\mathcal{M}} = 6a\mathcal{M} \text{ on a bhc lattice} \right). \tag{6.15}$$

and neglected an additive constant in the action.

The action $S_{\text{vacuum}}[Z]$ is a complicated functional of the meson field Z which is invariant under the chiral transformations (6.11) by construction. The color degrees of freedom are completely decoupled and appear only through the factor N_c in front of the action. This structure organizes the partition function in a perturbation series with $1/N_c$ as small parameter.

6.4   Saddle Point Approximation

The lowest order of the large-N_c expansion is the saddle point approximation, where the partition function is approximated by the value of the action on its saddle point, $\mathcal{Z} \approx \exp(-N_c S_{\text{vacuum}}[Z_{\text{saddle}}])$. The saddle point can be indentified by solving the saddle point equations

$$\begin{aligned}
\gamma_v \left(\frac{1}{Z^\dagger(n + v/2)} + Z(n + v/2) \right) &= \sum_w \gamma_w \left(Z^\dagger(n - w/2) + Z(n + w/2) \right) + \hat{\mathcal{M}}, \\
\gamma_v \left(\frac{1}{Z(n - v/2)} + Z^\dagger(n - v/2) \right) &= \sum_w \gamma_w \left(Z^\dagger(n - w/2) + Z(n + w/2) \right) + \hat{\mathcal{M}}.
\end{aligned} \tag{6.16}$$

This system of equations, labeled by the lattice links (n runs over all lattice sites and w over all positive directions), is obtained by setting the first variation of the effective action with respect to Z and $Z^\dagger$ to zero.

Our results of the search for solutions of the saddle point equations are presented in the next subsections. There is a saddle point for all values of the quark masses, which is proportional to the identity on the flavor sector (Section 6.4.1). The study of deformations of this saddle point (Section 6.4.2) shows that it blows up to a saddle point manifold in the chiral limit (Section 6.4.3), while no deformations are possible away from the chiral limit. In Section

6.4.4 we show that the occurence of a saddle point manifold in the chiral limit is directly connected to chiral symmetry and calculate it for different lattices. For the theory placed on the four-dimensional bhc lattice, the saddle point manifold is diffeomorphic to $U(N_f)$.

The pattern of symmetry groups fits perfectly with the one predicted by the *Goldstone theorem*: In the chiral limit the theory on the four-dimensional bhc lattice has a $U(N_f)_L \times U(N_f)_R$ chiral symmetry. The saddle point (6.22) breaks the symmetry to the subgroup $U(N_f)_{L+R}$, defined by the pairs $(V_L, V_R) \in U(N_f)_L \times U(N_f)_R$ with $V_L = V_R$. In fact there are zero modes $g \in U(N_f) \cong U(N_f)_L \times U(N_f)_R / U(N_f)_{L+R}$, which occurs as the parametrization of the saddle point manifold. In case of the bhc lattice the pattern of chiral symmetry breaking exactly corresponds to the continuum theory.

By the way of contrast, the theory on a hc lattice has an enlarged chiral symmetry group: Qualitatively the same symmetry breaking pattern occurs in that case, but the number of involved degrees of freedom changes ($N_f \to N_s N_f$). This enlargement of the symmetry group should be seen as an artifact connected with a too small lattice symmetry group.

6.4.1 Saddle Point

We start with a homogeneous ansatz for the collective field, where $Z(n + v/2)$ is given by γ_v in the spin sector tensored with an arbitrary position-independed matrix X in the flavor sector,

$$Z_{\text{saddle}}(n + v/2) = \gamma_v X, \qquad Z^\dagger_{\text{saddle}}(n + v/2) = \gamma_v X^\dagger, \tag{6.17}$$

Inserting the ansatz into the saddle point equations (6.16), they reduce to

$$1 + X^\dagger X = X^\dagger \left(\delta X^\dagger + \delta X + \hat{\mathcal{M}} \right),$$
$$1 + X^\dagger X = \left(\delta X^\dagger + \delta X + \hat{\mathcal{M}} \right) X. \tag{6.18}$$

Subtraction the second equation of from the first one we obtain

$$\delta(X^{\dagger 2} - X^2) + X^\dagger \hat{\mathcal{M}} - \hat{\mathcal{M}} X = 0. \tag{6.19}$$

We now assume that X is a *diagonal matrix* and obtain by commutating the matrices of the last equation

$$\left(X^\dagger - X \right) \left(X^\dagger + X + \hat{\mathcal{M}}/\delta \right) = 0. \tag{6.20}$$

We discuss the solutions of this equation: The solution where $X^\dagger + X + \hat{\mathcal{M}}/\delta = 0$ is unphysical, because in that case the argument of the first logarithm of the action (6.14) vanishes and the action grows to infinity. The solution with $X = X^\dagger$ yields the quadratic equation

$$(2\delta - 1)X^2 + \hat{\mathcal{M}} X - 1 = 0 \tag{6.21}$$

with the real solutions

$$X_\pm = X^\dagger_\pm = x \left(\pm\sqrt{1 + \left(\frac{1}{2} x \hat{\mathcal{M}} \right)^2} - \frac{1}{2} x \hat{\mathcal{M}} \right), \quad \text{where } x := \frac{1}{\sqrt{2\delta - 1}}. \tag{6.22}$$

The expression for the saddle point $X_\pm$ contains the square root of a diagonal matrix. For each of the diagonal elements one may choose the positive or the negative square root. In

our notation X_+ refers to the term, where the positive root is taken for all diagonal elements, while X_- is a generic notation for a term, where at least one negative square root is taken. At the saddle points (6.22) the action (6.14) takes the values

$$S_\pm^{(0)} = \mathcal{N} \operatorname{Tr} \ln \frac{(1 + X_\pm^2)^\delta}{2\delta X_\pm + \hat{\mathcal{M}}},$$

(6.23)

where $\mathcal{N}$ is the total number of lattice sites. The solutions (6.22) of the saddle point equations are trivial in the spin sector and diagonal in the flavor sector. To decide which of the two saddle points X_+ and X_- is dominant, we compare the corresponding values of the action. For one of the saddle points (X_-) the argument under the logarithm might become negative; in that case we look at the real part of the action. In the chiral limit the real part of the action takes the same value for both X_+ and X_-. But away from this limit we have $\operatorname{Re} S_{\text{vacuum}}[X_+] < \operatorname{Re} S_{\text{vacuum}}[X_-]$ and conclude that $X := X_+$ is the dominant saddle point. In what follows we will often make use of its expansion around the chiral limit,

$$X = xI - \frac{1}{2}x^2\hat{\mathcal{M}} + \frac{1}{8}x^3\hat{\mathcal{M}}^2 + \cdots,$$

(6.24)

where I is the identity matrix in flavor space. In the next subsection we look for further solutions of the saddle point equations which are continuous deformations of the ansatz (6.17).

6.4.2 Linearized Saddle Point Equations

Consider a non-homgenious deformation of the saddle point,

$$\begin{aligned}
Z(n + v/2) &= \gamma_v X \left(1 + \zeta(n + v/2)\right), \\
Z^\dagger(n + v/2) &= \left(1 + \zeta^\dagger(n + v/2)\right) X\gamma_v.
\end{aligned}$$

(6.25)

The parameters of the deformation, a set of matrices in flavor space $\zeta(n+v/2)$ and $\zeta^\dagger(n+v/2)$, are assumed to be small. We expand the saddle point equations (6.16) up to linear order in these parameters,

$$\begin{aligned}
-X^{-1}\zeta^\dagger(n + v/2) + X\zeta(n + v/2) &= \Sigma(n), \\
-\zeta(n - v/2)X^{-1} + \zeta^\dagger(n - v/2)X &= \gamma_v\Sigma(n)\gamma_v.
\end{aligned}$$

(6.26)

The new field $\Sigma(n)$ on the r.h.s. is a sum over the fields ζ and $\zeta^\dagger$ on the links adjacent to the site n, defined by

$$\Sigma(n) := \sum_v \left(\gamma_v\zeta^\dagger(n - v/2)\gamma_v X + X\zeta(n + v/2)\right).$$

(6.27)

The linearized saddle point equations (6.26) and their spatially shifted version

$$\begin{aligned}
-X^{-1}\zeta^\dagger(n - v/2) + X\zeta(n - v/2) &= \Sigma(n - v), \\
-\zeta(n + v/2)X^{-1} + \zeta^\dagger(n + v/2)X &= \gamma_v\Sigma(n + v)\gamma_v
\end{aligned}$$

(6.28)

build a system of linear equations which can be solved for variables $\zeta(n+v/2)$ and $\zeta^\dagger(n-v/2)$, namely

$$\begin{aligned}
X^2\zeta(n + v/2) - \zeta(n + v/2)X^{-2} &= X\Sigma(n) + \gamma_v\Sigma(n + v)\gamma_v X^{-1}, \\
\zeta^\dagger(n - v/2)X^2 - X^{-2}\zeta^\dagger(n - v/2) &= X^{-1}\Sigma(n - v) + \gamma_v\Sigma(n)\gamma_v X.
\end{aligned}$$

(6.29)

Writing the saddle point as $X = \mathrm{diag}(x_1, ..., x_n)$, we obtain explicit expressions for the matrix elements of the fields ζ and $\zeta^\dagger$ (the indices $a, b = 1, ..., N_f$ refer to the flavor degrees of freedom),

$$\zeta_{ab}(n + v/2) = \frac{1}{x_a^2 - x_b^{-2}} \left(x_a \Sigma_{ab}(n) + x_b^{-1} \gamma_v \Sigma_{ab}(n + v) \gamma_v \right)$$

$$\gamma_v \zeta_{ab}^\dagger(n - v/2) \gamma_v, = \frac{1}{x_b^2 - x_a^{-2}} \left(x_a^{-1} \gamma_v \Sigma_{ab}(n - v) \gamma_v + x_b \Sigma_{ab}(n) \right). \tag{6.30}$$

In the case of one space-time dimension and only two next neighbors the solution above is not useful, because in that case the r.h.s. of (6.30) is not well-defined in the chiral limit, where $x_a = x_b = 1$. From here on we assume the number of space-time dimensions to be bigger than one. [1]

Inserting the last two equations into the definition of the sum variable $\Sigma(n)$, the saddle point equations can be rewritten as set of linear equations for the new variable,

$$\left(\frac{1}{x_a x_b} + (2\delta - 1) x_a x_b \right) \Sigma_{ab}(n) + \sum_v \left(\gamma_v \Sigma_{ab}(n - v) \gamma_v + \gamma_v \Sigma_{ab}(n + v) \gamma_v \right) = 0. \tag{6.31}$$

Calculation of the coefficient in front $\Sigma(n)$ enables us to bring this equation into the remarkable simple form

$$\sum_v \left(\gamma_v \Sigma_{ab}(n - v) \gamma_v + 2\Sigma_{ab}(n) + \gamma_v \Sigma_{ab}(n + v) \gamma_v \right) + M_{ab} \Sigma_{ab}(n) = 0, \tag{6.32}$$

where we introduced the abbreviation

$$M_{ab} := \left(1 - \frac{x_a x_b}{x^2} \right) \left(\frac{1}{x_a x_b} - 1 \right) = \frac{\delta - 1}{\sqrt{2\delta - 1}} (\hat{m}_a + \hat{m}_b) + O(\hat{m}^2). \tag{6.33}$$

The matrix M vanishes in the chiral limit and its matrix entries are proportional to the sum of two quark masses in first order of an expansion in the quark masses. The equations above in terms of the sum variable Σ are completely equivalent to the linearized saddle point equations (6.26) in terms of ζ and $\zeta^\dagger$. Note that up to now ζ and $\zeta^\dagger$ were treated as independent variables.

---

[1] In the case of one space-time dimensions $\delta = d = 1$ the linearized saddle point equations read

$$\zeta(n + v/2) = -X\zeta(n - v/2)X, \qquad \zeta^\dagger(n - v/2) = -X^{-1}\zeta^\dagger(n + v/2)X^{-1}.$$

Taking into account that ζ and $\zeta^\dagger$ are the hermitian conjugated matrix of each other, this equations have only nontrivial solutions, if $x_a = \pm 1$ for all flavors a that is in the chiral limit. The solution in the chiral limit is given by

$$\zeta(n + v/2) = -\zeta(n - v/2), \qquad \zeta^\dagger(n + v/2) = -\zeta^\dagger(n - v/2)$$

and exponenting it back to the group yields

$$Z(n + v/2) = Z^{-1}(n - v/2), \qquad Z^\dagger(n + v/2) = Z^{\dagger^{-1}}(n - v/2).$$

The general solution of saddle point equations in $d = 1$ space-time dimensions is an staggered configuration of the Z-field along a one-dimensional chain. The zero modes belong to the general linear group $\mathrm{Gl}(N_f)$ in difference to the theory in higher space-time dimensions, where the zero modes belong to an unitary group.

6.4.3   Hermitian Structure

In the color-flavor transformed partition function the variables Z and $Z^\dagger$ are related by Hermitian conjugation. Thus, looking for a saddle point, one would assume the deformation parameters ζ and $\zeta^\dagger$ to be the conjugated matrix of each other. (There are situations, when – assuming that Z and $Z^\dagger$ are related by Hermitian conjugation – no solutions of the saddle point equations exists. This happens in context of a model of a static baryon, see Chapter 7. In that case one can lift the relation between Z and $Z^\dagger$ and try to find a solution of the saddle point equations treating Z and $Z^\dagger$ as independend variables.) By making use of (6.26) and (6.28) we translate the relation between Z and $Z^\dagger$ into an equation for the sum variable,

$$\Sigma(n + v) = \gamma_v \Sigma^\dagger(n)\gamma_v. \tag{6.34}$$

As a consequence of the last equation there is only one free variable (for example $\Sigma(n = 0)$) that determines all other $\Sigma(n)$.
Inserting the last equation into the saddle point equation (6.32) we get the condition

$$\Sigma^\dagger_{ab}(n) + \left(1 + \frac{M_{ab}}{2\delta}\right)\Sigma_{ab}(n) = 0 \tag{6.35}$$

for all flavors a and b. Because hermitian conjugation is an involution, a non-trivial solution of this equation requires

$$1 + \frac{M_{ab}}{2\delta} = \pm 1. \tag{6.36}$$

This equation can only be fulfilled in the chiral limit, where $M_{ab} = 0$. [2] We conclude that away from the chiral limit the saddle point can not be deformed without enlarging the action. This result is in agreement with a uniqueness theorem for the solutions of the saddle point equations [33].
From here up to the end of the section we consider the *chiral limit* $\hat{\mathcal{M}} = 0$. In this limit equation (6.35) reduces to

$$\Sigma^\dagger(n) = -\Sigma(n). \tag{6.37}$$

As a consequence equation (6.34) reads $\gamma_v \Sigma(n \pm v)\gamma_v = -\Sigma(n)$ and from (6.30) we obtain that $\zeta(n + v/2)$ and $\gamma_v \zeta^\dagger(n - v/2)\gamma_v$ are independent of the direction v, namely

$$\zeta(n + v/2) = \gamma_v \zeta^\dagger(n - v/2)\gamma_v = \frac{x}{1 + x^2}\Sigma(n) = \frac{\sqrt{2\delta - 1}}{2\delta}\Sigma(n). \tag{6.38}$$

Let us exponentiate the small deformation $\Sigma(n)$ and define

$$\mathcal{G}(n) := e^{\frac{\sqrt{2\delta - 1}}{2\delta}\Sigma(n)}. \tag{6.39}$$

In doing so we pass from the Lie algebra of antihermitian matrices $\Sigma(n)$ to the unitary group $U(N_s N_f)$ of matrices $\mathcal{G}(n)$. The condition (6.34), which is equivalent to the saddle point equations, can rewritten in the form

$$\mathcal{G}(n + v) = \gamma_v \mathcal{G}^{-1}(n)\gamma_v. \tag{6.40}$$

---

[2] The condition can be only fulfilled if $M_{ab} \leq 0$ for all flavors a and b, especially for $b = a$. Let us assume that one of the quark masses is different from zero, for example $m_a \neq 0$.
We conclude $m_{aa} = m_a x_a(x_a^{-2} - 1) \leq 0 \Leftrightarrow x_a^2 \geq 1 \Leftrightarrow 2\delta - 1 + m_a x_a - 1 \leq 0 \Leftrightarrow m_a x_a \leq -2(\delta - 1) < 0$, which is a contradiction to $m_a > 0$ and $x_a > 0$. Applying this argument to all flavors a, we conclude that the condition can be only fulfilled in the chiral limit.

The equations (6.38) can be considered as the beginning of an expansion in $\zeta(n + v/2)$, $\zeta^\dagger(n + v/2)$ and $\Sigma(n)$ of

$$e^{\zeta(n+v/2)} = \mathcal{G}(n), \qquad e^{\zeta^\dagger(n+v/2)} = \mathcal{G}^{-1}(n). \tag{6.41}$$

Further the ansatz (6.25) can be read as the linearized version of

$$Z(n + v/2) = \gamma_v X \mathcal{G}(n), \qquad Z^\dagger(n + v/2) = \mathcal{G}^{-1}(n) X \gamma_v. \tag{6.42}$$

In the next subsection we proove that (6.42), when $\mathcal{G}(n)$ satisfies the condition (6.40), is an exact solution of the saddle point equations.

6.4.4 Saddle Point Manifold

Inserting the ansatz (6.42) into the effective action (6.14) yields

$$S[\mathcal{G}] = -\sum_n \operatorname{Tr} \ln \left(\sum_v \left(\gamma_v \mathcal{G}^{-1}(n - v)\gamma_v X + X\mathcal{G}(n) \right) + \hat{\mathcal{M}} \right) + \sum_{n,v} \operatorname{Tr} \ln \left(1 + X^2 \right). \tag{6.43}$$

Making use of relation (6.40) and assuming the chiral limit, this expression simplifies to

$$S[\mathcal{G}] = S^{(0)} - \sum_n \operatorname{Tr} \ln \mathcal{G}(n). \tag{6.44}$$

A further application of (6.40) allows to rewrite the term under the sum as follows,

$$
\begin{aligned}
\operatorname{Tr} \ln \mathcal{G}(n) &= \frac{1}{2} \operatorname{Tr} \ln \mathcal{G}(n) + \frac{1}{2\delta} \sum_w \operatorname{Tr} \ln(\gamma_w \mathcal{G}(n + w)^{-1}\gamma_w) \\
&= \frac{1}{2} \operatorname{Tr} \ln \mathcal{G}(n) - \frac{1}{2\delta} \sum_w \operatorname{Tr} \ln \mathcal{G}(n + w).
\end{aligned}
\tag{6.45}
$$

This epression vanishes, when we perform the sum over the lattice sites. In the chiral limit we deal with a saddle point manifold: It is parameterized by a set of matrices $\mathcal{G}(n) \in \mathrm{U}(N_s N_f)$ which satisfies (6.40).

The replacement of the saddle point by a saddle point manifold is related to the invariance of the action under the chiral transformations (6.11). Indeed, the saddle point Z_{saddle} transforms under this transformations like

$$
\begin{aligned}
Z_{\mathrm{saddle}}(n + v/2) &\rightarrow Z_{\mathrm{saddle}}(n + v/2)(\gamma_v \mathcal{V}(n + v)\gamma_v \mathcal{V}^{-1}(n)) \\
Z_0^\dagger(n + v/2) &\rightarrow (\mathcal{V}(n)\gamma_v \mathcal{V}^{-1}(n + v)\gamma_v)Z_0^\dagger(n + v/2)
\end{aligned}
\tag{6.46}
$$

and with the identification

$$\mathcal{G}(n) = \gamma_v \mathcal{V}(n + v)\gamma_v \mathcal{V}^{-1}(n) \tag{6.47}$$

we obtain the saddle point manifold. In this framework (6.40) is consequence of the relation $\mathcal{V}(n + 2v) = \mathcal{V}(n)$ for chiral transformations.

Some of the chiral transformations leave the saddle point invariant: This happens if and only if

$$\mathcal{V}(n + v) = \gamma_v \mathcal{V}^{-1}(n)\gamma_v. \tag{6.48}$$

We are faced with the following symmetry breaking pattern: The saddle point breaks the symmetry of the chiral group, defined by the condition

$$\mathcal{V}(n + v + w) = \gamma_w \gamma_v \mathcal{V}(n) \gamma_v \gamma_w, \tag{6.49}$$

to a subgroup, defined by the stronger condition (6.48).

We calculate the saddle point manifold for different types of lattices. It turns out that the possible solutions of (6.40) depend on the structure of the underlying lattice. This is related to the dependence of the group of chiral transformations on the lattice, see Chapter 4. Again we consider the following lattices:

(i) *General Lattice*
   Independent of the lattice there is a homogeneous solution of the saddle point equations in the chiral limit,
   $$\mathcal{G}(n) = P_- \otimes g^{-1} + P_+ \otimes g \tag{6.50}$$
   with an arbitrary matrix $g \in \mathrm{U}(N_f)$.

(ii) *Bipartite Lattice*
   On a bipartite lattice there is further solution of (6.40), where the matrix $\mathcal{G}$ staggers from a lattice site a to its neighbor b in the way
   $$\mathcal{G}(a) = 1 \otimes g^{-1}, \qquad \mathcal{G}(b) = 1 \otimes g. \tag{6.51}$$
   In the equation above $g \in \mathrm{U}(N_f)$ and $a \in A$ and $b \in B$ label the sites of the two nested sublattices.

(iii) *Hypercubic Lattice*
   On the d-dimensional hypercubic (HC) lattice, the saddle point manifold is parameterized by $\mathrm{U}(N_s N_f)$. Indeed, we may choose a site n_0 of the lattice and let the parameter at this site take an arbitrary value $\mathcal{G}(n_0) = \mathcal{G} \in \mathrm{U}(N_s N_f)$. Then (6.40) is satisfied, iff

   $$\mathcal{G}\left(n_0 + \sum_{\mu=1}^{d} k_\mu e_\mu\right) = \gamma_1^{k_1} ... \gamma_d^{k_d} \mathcal{G}^{(-1)^{\sum_\mu k_\mu}} \gamma_d^{k_d} ... \gamma_1^{k_1}. \tag{6.52}$$

   The choice of boundary conditions can possibly spoil the existence of this solution. We assume an infinite lattice or a lattice with an even number of sites in each direction and periodic boundary conditions.

(iv) *Bodycentered Hypercubic Lattice*
   A twofold application of (6.40) shows that the parameters of the saddle point manifold satisfy
   $$\mathcal{G}(n + v + w) = \gamma_w \gamma_v \mathcal{G}(n) \gamma_v \gamma_w. \tag{6.53}$$
   On the four-dimensional bhc lattice this relation implies
   $$\mathcal{G}(n) = P_- \otimes g_L + P_+ \otimes g_R \tag{6.54}$$
   with g_L, $g_R \in \mathrm{U}(N_f)$, as we have proved in Chapter 4.3.2. The l. h. s. of the last equation has to be equal to
   $$\gamma_v \mathcal{G}^{-1}(n) \gamma_v = P_+ \otimes g_L^{-1} + P_- \otimes g_R^{-1}, \tag{6.55}$$

and we conclude $g_R = g_L^{-1} =: g$. The saddle point manifold is given by

$$\mathcal{G}(n) = P_- \otimes g^{-1} + P_+ \otimes g, \tag{6.56}$$

where the parameter $g \in \mathrm{U}(N_f)$ runs over the unitary matrices in flavor space.

6.5 Gradient Expansion (Hypercubic Lattice)

Our aim is to derive an effective theory describing the long range behavior of the zero modes. We obtain an effective continuum action by a long distance approximation in combination with a gradient expansion around the saddle point manifold, as it was developed in [3, 2]. The effective action replaces the color-flavor-transformed lattice action,

$$S_{\mathrm{vacuum}}[Z] \longrightarrow S[\mathcal{G}] = S^{(0)} + S^{(2)}[\mathcal{G}] + S^{(4)}[\mathcal{G}] + \cdots . \tag{6.57}$$

The low energy expansion is carried out in the momenta of the zero modes; the indices $k = 0, 2, 4$ refer to the corresponding order $O(p^k)$ of the low energy expansion. The contributions to the effective action come from the fluctuations of the zero modes but also from the breaking of chiral symmetry by the quark masses m_f. The masses of the light quarks are considered as small. To be more precise, the momenta of the zero modes p and the quark masses m_f are treated as small parameters, while the quotient p^2/m_f may take any value. In the low energy expansion, the quarks masses count like $O(p^2)$. We will justify this *chiral power counting* later. The expansion (6.57) involves only terms of even order, because contributions which are odd in the momenta would violate the space-time parity.

There is another way to split the action into two terms, which have a different origin from the calculational point of view,

$$S[\mathcal{G}] = S_{\mathrm{zero}}[\mathcal{G}] + S_{\mathrm{massive}}[\mathcal{G}]. \tag{6.58}$$

In passing to the low energy theory, the physical degrees of freedom are divided into massless modes, whose dynamics is described by the effective action, and massive modes, which are integrated out. The first term comes from the fluctuations of the massless modes and their coupling to the quark masses. The second term comes from the coupling of the massless modes to massive modes; in the lowest order of the saddle point approximation this contribution is given by a Gaussian integral over the massive modes. This section is devoted to the calculation of $S_{\mathrm{zero}}[\mathcal{G}]$, while some of the contributions to $S_{\mathrm{massive}}[\mathcal{G}]$ are calculated in the next section. Recall that the zero modes can be written as

$$Z(n + v/2) = \gamma_v X \mathcal{G}(n), \qquad Z^\dagger(n + v/2) = \mathcal{G}^{-1}(n) X \gamma_v, \tag{6.59}$$

where the field $\mathcal{G}(n)$ is placed on the lattice sites and has to satisfy the relation

$$\mathcal{G}(n + v) = \gamma_v \mathcal{G}^{-1}(n)\gamma_v. \tag{6.60}$$

In the chiral limit they parameterize the saddle point manifold. As pointed out before, the solutions of equation (6.60) and the shape of the saddle point manifold depend on the underlying lattice.

On the hc lattice the solutions of (6.60) are staggered configurations of the field $\mathcal{G}$. In view of the continuum limit it is convenient to employ (instead of $\mathcal{G}(n)$) a new field $G(n)$, which
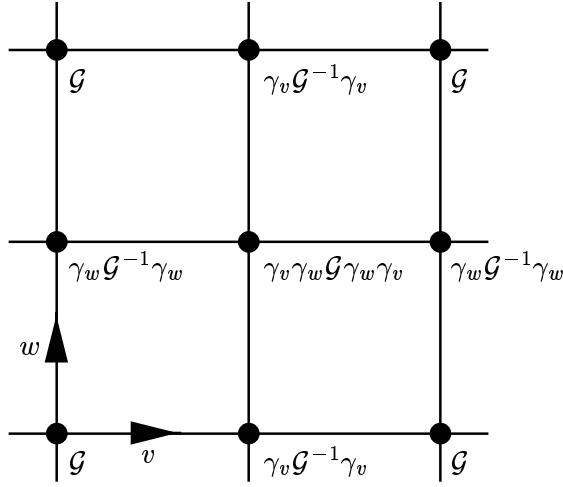
Figure VII: Construction of a smooth parametrization of the massless modes

gives rise to a smooth parametrization of the saddle point manifold. To define the new field, we view the original lattice as a coarser lattice (with lattice constant $2a$) with a non-trivial elementary cell containing 2^d sites of the original lattice. On the sites of the coarser lattice n' the new field coincides with the old one,

$$G(n') := \mathcal{G}(n'), \tag{6.61}$$

but the two fields differ on the other $2^d - 1$ sites of the elementary cell,

$$G(n' + v_1 + ... + v_k) := \gamma_1...\gamma_k \, \mathcal{G}^{\pm 1}(n' + v_1 + ... + v_k) \, \gamma_k...\gamma_1. \tag{6.62}$$

On the r. h. s. of the last formula on has to take $\mathcal{G}$ if k is even and $\mathcal{G}^{-1}$ if k is odd. The construction of the field G is visualized in Figure VII.

We insert the ansatz (6.59) into the vacuum action and obtain

$$S_{\text{zero}}[\mathcal{G}] = - \sum_n \text{Tr} \ln \left(\sum_v \left(\gamma_v \mathcal{G}^{-1}(n - v)\gamma_v X + X\mathcal{G}(n) \right) + \hat{\mathcal{M}} \right). \tag{6.63}$$

The ansatz does not reach all possible field configurations; it belongs to the massless modes, which are now allowed to fluctuate in space-time and are not constrained by (6.60). In the next step we pass to the variable $G(n)$, which is the appropriate field in the continuum limit. If $n = n' + v_1 + ... + v_k$, where n' is a site of the coarser lattice the connection is built by

$$\gamma_v \mathcal{G}^{-1}(n - v)\gamma_v + \mathcal{G}(n) = \gamma_1...\gamma_k \left(G^{\pm 1}(n - v) + G^{\pm 1}(n) \right) \gamma_k...\gamma_1. \tag{6.64}$$

On the r.h.s. of the expression above, one has to take G for even k and G^{-1} for odd k. Because of the dependence of this expression on the parity of k, it is useful to divide the hc lattice in two nested sublattices A and B to express the action in terms of the new field,

$$\begin{aligned}
S_{\text{zero}}[G] = &- \sum_{a \in A} \text{Tr} \ln \left(\sum_v \left(G(a - v)X + XG(a) \right) + \hat{\mathcal{M}} \right) \\
&- \sum_{b \in B} \text{Tr} \ln \left(\sum_v \left(G^{-1}(b - v)X + XG^{-1}(b) \right) + \hat{\mathcal{M}} \right).
\end{aligned} \tag{6.65}$$

Our aim is to expand the action $S_{\text{zero}}[G]$ up to order $O(p^4)$ in the momenta of the massless modes. In doing so we make use of the Taylor expansions

$$
\begin{aligned}
G(n) = {}& G(n - v/2) + \frac{a}{2}\partial_v G(n - v/2) + \frac{1}{2!}\left(\frac{a}{2}\right)^2 \partial_v^2 G(n - v/2) \\
& + \frac{1}{3!}\left(\frac{a}{2}\right)^3 \partial_v^3 G(n - v/2) + \frac{1}{4!}\left(\frac{a}{2}\right)^4 \partial_v^4 G(n - v/2) + ..., \\
G(n - v) = {}& G(n - v/2) - \frac{a}{2}\partial_v G(n - v/2) + \frac{1}{2!}\left(\frac{a}{2}\right)^2 \partial_v^2 G(n - v/2) \\
& - \frac{1}{3!}\left(\frac{a}{2}\right)^3 \partial_v^3 G(n - v/2) + \frac{1}{4!}\left(\frac{a}{2}\right)^4 \partial_v^4 G(n - v/2) - +...
\end{aligned}
\tag{6.66}
$$

and the expansion of the saddle point in powers of the quark masses (6.24).

Inserting these expression into the action $S_{\text{zero}}[G]$, we obtain the following contribution from sublattice A,

$$
\begin{aligned}
S_{\text{zero},A}[G] = {}& -\frac{1}{2a^d}\int d^d x \, \text{Tr}\ln\left(d\{X, G\} + \hat{\mathcal{M}}\right. \\
& + \sum_v \left(\frac{a}{2}[X, \partial_v G] + \frac{1}{2!}\left(\frac{a}{2}\right)^2 \{X, \partial_v^2 G\} + \frac{1}{3!}\left(\frac{a}{2}\right)^3 [X, \partial_v^3 G] + \frac{1}{4!}\left(\frac{a}{2}\right)^4 \{X, \partial_v^4 G\}\right)\bigg) \\
= {}& -\frac{1}{2a^d}\int d^d x \, \text{Tr}\ln(2dxG) \\
& -\frac{1}{2a^d}\int d^d x \, \text{Tr}\ln\left(1 + \frac{G^{-1}}{2dx}\left(\hat{\mathcal{M}} - \frac{d}{2}x^2\{\hat{\mathcal{M}}, G\} + \frac{d}{8}x^3\{\hat{\mathcal{M}}^2, G\}\right.\right. \\
& \left.\left. -\frac{a}{4}x^2[\hat{\mathcal{M}}, \partial G] + \frac{a^2}{4}x\Delta G - \frac{a^2}{16}x^2\{\hat{\mathcal{M}}, \Delta G\} + \frac{a^4}{192}x\partial^4 G\right)\right).
\end{aligned}
\tag{6.67}
$$

For the differential operators which are acting on the massless modes we have introduced the abbreviations $\partial := \sum_v \partial_v$ and $\partial^4 := \sum_v \partial_v^4$. The contribution from sublattice B is obtained from the result for the sublattice A by the replacement $G \to G^{-1}$. When we sum both contributions the integral in the third line of (6.67) cancels against the corresponding term from sublattice B.

The next step is to expand the logarithm; the straightforward but cumbersome calculations are presented in Appendix H.3. We arrange the result in the form

$$
S_{\text{zero}}[G] = S_{\text{fl}}[G] + S_{\mathcal{M}}[G] + S_{\text{fl},\mathcal{M}}[G],
\tag{6.68}
$$

where the first term belongs to the fluctuations of the massless modes, the second term to the breaking of chiral symmetry due to finite quark masses $\mathcal{M}$ and the third term to a coupling between the fluctuations and the quarks masses. Expressed in terms of the *left currents* $L_\mu := G^{-1}\partial_\mu G$ the three contributions are given by

$$
\begin{aligned}
S_{\text{fl}}[G] = {}& -\frac{a^{2-d}}{8d}\int d^d x \sum_\mu \text{Tr}\, L_\mu^2 \\
& + \frac{a^{4-d}}{128d}\int d^d x \left(\frac{1}{d}\sum_{\mu,\nu}\left(\text{Tr}(L_\mu^2 L_\nu^2) + \text{Tr}(\partial_\mu L_\mu \partial_\nu L_\nu)\right) + \frac{1}{3}\sum_\mu\left(\text{Tr}(\partial_\mu L_\mu)^2 - \text{Tr}\, L_\mu^4\right)\right),
\end{aligned}
\tag{6.69}
$$

$$S_{\mathcal{M}}[G] = -\frac{a^{-d}}{4dx} \int d^d x \, \mathrm{Tr} \left(\hat{\mathcal{M}}(G + G^{-1}) \right)$$

$$+ \frac{a^{-d}}{16d^2 x^2} \int d^d x \, \mathrm{Tr} \left(\hat{\mathcal{M}} G^{-1} \hat{\mathcal{M}} G^{-1} + \hat{\mathcal{M}} G \hat{\mathcal{M}} G \right) \tag{6.70}$$

$$- \frac{a^{-d}}{8d} \int d^d x \, \mathrm{Tr} \left(\hat{\mathcal{M}}^2 (G + G^{-1}) \right) + \frac{x^2 a^{-d}}{16} \int d^d x \, \mathrm{Tr} \left(\hat{\mathcal{M}} G^{-1} \hat{\mathcal{M}} G \right),$$

$$S_{\mathrm{fl},\mathcal{M}}[G] = + \frac{x a^{1-d}}{8d} \int d^d x \, \mathrm{Tr} \left(\hat{\mathcal{M}} (G^{-1} \partial G - \partial G G^{-1}) \right)$$

$$- \frac{a^{2-d}}{32 d^2 x} \int d^d x \sum_\mu \left(\mathrm{Tr} \left(L_\mu^2 (G^{-1} \hat{\mathcal{M}} + \hat{\mathcal{M}} G) \right) + \mathrm{Tr} \left(\partial_\mu L_\mu (G^{-1} \hat{\mathcal{M}} - \hat{\mathcal{M}} G) \right) \right).$$

$$\tag{6.71}$$

Our result for the action $S_{\mathrm{zero}}[G]$ contains two terms which do not have the full $O(d)$ continuum symmetry: The last term of (6.69) which is proportional to $\sum_\mu \partial_\mu^4$ and the first term of (6.71) which involves a first derivative in direction $e_1 + ... + e_d$. The occurrence of the former term is an artifact of the hc lattice. As we will see later it does not occur on the bhc lattice.

6.6 Massive Modes (Hypercubic Lattice)

Starting point is the color-flavor transformed action of strong-coupling QCD on a d-dimensional hc lattice,

$$S_{\mathrm{vacuum}}[Z, Z^\dagger] = -\sum_n \mathrm{Tr} \ln \left(\sum_{\mu=1}^d \gamma_\mu \left(Z^\dagger(n - \hat{\mu}/2) + Z(n + \hat{\mu}/2) \right) + \hat{\mathcal{M}} \right)$$

$$+ \sum_n \sum_{\mu=1}^d \mathrm{Tr} \ln \left(1 + Z(n + \hat{\mu}/2) Z^\dagger(n + \hat{\mu}/2) \right). \tag{6.72}$$

In this section we assume all quarks masses to have the equal value, $m := m_1 = ... = m_{N_f}$. In this case the mass matrix $\hat{\mathcal{M}}$ and the saddle point X proportional to the identity matrix in flavor space and they can be treated as scalars. Further we restrict ourselves to an *even* number of space-time dimensions. In this case the structure of the Clifford algebra is analogous to the one of the 4-dimensional Clifford algebra (cf. Appendix D).

6.6.1 Second Order Expansion

We introduce a set of coordinates $\zeta(n + \hat{\mu}/2)$, $\zeta^\dagger(n + \hat{\mu}/2)$ which parameterize the configuration manifold relative to the saddle point,

$$Z(n + \hat{\mu}/2) = \gamma_\mu X \left(1 + \zeta(n + \hat{\mu}/2) \right)$$

$$Z^\dagger(n + \hat{\mu}/2) = \left(1 + \zeta^\dagger(n + \hat{\mu}/2) \right) X \gamma_\mu. \tag{6.73}$$

In these coordinates the action reads

$$S[\zeta,\zeta^\dagger] = S^{(0)} - \sum_n \mathrm{Tr}\ln\left(1 + D\sum_{\mu=1}^d \left(\gamma_\mu \zeta_\mu^\dagger(n-\hat\mu/2)\gamma_\mu + \zeta(n+\hat\mu/2)\right)\right)$$
$$+ \sum_n \sum_{\mu=1}^d \mathrm{Tr}\ln\left(1 + D\left(\zeta(n+\hat\mu/2) + \zeta^\dagger(n+\hat\mu/2) + \zeta(n+\hat\mu/2)\zeta^\dagger(n+\hat\mu/2)\right)\right),$$

$$(6.74)$$

where we have introduced the mass dependent parameter

$$D := \frac{X^2}{1+X^2} = \frac{1}{2d}\frac{1}{1+\hat{\mathcal{M}}/2dX}.$$

$$(6.75)$$

We assume that the deviations ζ, $\zeta^\dagger$ from the saddle point are small and expand the action in these parameters,

$$S = S^{(0)} + S^{(2)}[\zeta,\zeta^\dagger] + \cdots.$$

$$(6.76)$$

In this expansion the zeroth order $S^{(0)}$ is the value of the action at the saddle point, the first order of the expansion vanishes, because we are expanding around a saddle point, and the second order is given by

$$S^{(2)}[\zeta,\zeta^\dagger] = \sum_n \frac{1}{2}D^2 \sum_{\mu,\nu}\left(\mathrm{Tr}\left(\zeta^\dagger(n+\hat\mu/2)\zeta(n+\hat\nu/2)\right)\right.$$
$$+ 2\,\mathrm{Tr}\left(\zeta(n+\hat\mu/2)\gamma_\nu\zeta^\dagger(n-\hat\nu/2)\gamma_\nu\right) + \mathrm{Tr}\left(\gamma_\mu\zeta^\dagger(n-\hat\mu/2)\gamma_\mu\gamma_\nu\zeta^\dagger(n-\hat\nu/2)\gamma_\nu\right)\right)$$
$$+ D\sum_\mu \mathrm{Tr}\left(\zeta_\mu^\dagger(n)\zeta_\mu(n)\right)$$
$$- \frac{1}{2}D^2\sum_\mu\left(\mathrm{Tr}\left(\zeta_\mu^2(n)\right) + 2\,\mathrm{Tr}\left(\zeta_\mu(n)\zeta^\dagger(n-\hat\mu/2)\right) + \mathrm{Tr}\left(\zeta_\mu^{\dagger 2}(n)\right)\right)$$

$$(6.77)$$

The traces in the action above include flavor and spin degrees of freedom. In the next section we will evaluate the trace over the spin space.

6.6.2 Decomposition of the Spin Sector

In this subsection we make use of some properties of the Clifford algebra in even dimensions (cf. Appendix D). Consider a matrix ζ in spin and flavor space. Concerning the spin components it can be decomposed into a sum of the Clifford basis elements $\{\gamma_m\}$, namely

$$\zeta = \sum_m \zeta_m \gamma_m.$$

$$(6.78)$$

The sum extends over all subsets m of $\{1,...,d\}$ and the coefficients coefficients $\zeta^{(m)}$ are matrices in flavor space. In particular, ζ_m with $m = \emptyset$ is the scalar part of ζ, and ζ_m with $m = \{1,...,d\}$ the pseudoscalar part.

We insert the decomposition (6.78) into the second order action (6.77). In an even dimensional space, the traces of all basic Clifford elements except the unity matrix vanish. We conclude that there is no coupling between the sectors of different indices m and the action decomposes into a sum over these sectors,

$$S^{(2)}[\zeta, \zeta^\dagger] = N_s \sum_m S_m^{(2)}[\zeta_m, \zeta_m^\dagger]. \tag{6.79}$$

For an index $m = \{\mu_1, ..., \mu_k\}$ we define

$$|m| := \#m = k \tag{6.80}$$

$$\delta_{\nu m} := \begin{cases} 1 & \text{if } \nu \in m \\ 0 & \text{otherwise.} \end{cases} \tag{6.81}$$

With the help of this definitions the combinations of γ-matrices appearing in the action can be expressed as

$$\gamma_\nu \gamma_m \gamma_\nu = (-1)^{\delta_{\nu m} + |m|} \gamma_m, \tag{6.82}$$

where the action belonging sector m is given by

$$
\begin{aligned}
S_m^{(2)}(\zeta, \zeta^\dagger) = \sum_n \frac{1}{2} D^2 \sum_{\mu, \nu} \Bigg(& \operatorname{Tr}(\zeta(n + \hat{\mu}/2)\zeta(n + \hat{\nu}/2)) \\
& + 2(-1)^{|m| + \delta_{\nu m}} \operatorname{Tr}(\zeta(n + \hat{\mu}/2)\zeta^\dagger(n - \hat{\nu}/2)) \\
& + (-1)^{\delta_{\mu m} + \delta_{\nu m}} \operatorname{Tr}(\zeta^\dagger(n - \hat{\mu}/2)\zeta^\dagger(n - \hat{\nu}/2)) \Bigg) \\
& + D \sum_\mu \operatorname{Tr}(\zeta_\mu^\dagger(n)\zeta_\mu(n)) \\
& - \frac{1}{2} D^2 \sum_\mu \Big(\operatorname{Tr}(\zeta_\mu^2(n)) + 2\operatorname{Tr}(\zeta_\mu(n)\zeta^\dagger(n - \hat{\mu}/2)) + \operatorname{Tr}(\zeta_\mu^{\dagger 2}(n)) \Big).
\end{aligned}
\tag{6.83}
$$

In order to calculate the eigenmodes of the action (6.83), we will exploit its periodicity over the lattice.

6.6.3   Fourier Transformation

Fourier Transformation of the Fields

Note, that the field is located on a lattice built by the centers of the links of the d-dimensional hypercubic lattice (the line graph of the later). It can be easily seen that in dimensions $d > 2$ this lattice is no Bravais lattice, because it is not homogeneous with respect to its sites. Thus we will apply the Fourier transformation to an hypercubic lattice with lattice constant $\frac{a}{2}$, which includes all the points of the line graph. Its first Brillouin zone is given the wave vectors k with

$$-2\pi\hat{\mu} < k_\mu \le 2\pi\hat{\mu}. \tag{6.84}$$

Further we assume periodic boundary conditions in each of the d directions,

$$\zeta(n + L\hat{\mu}) = \zeta(n), \tag{6.85}$$

which lead to a quantization of the wave vector,

$$k \in \text{span} \left\{ \frac{2\pi}{L}\hat{1}, ..., \frac{2\pi}{L}\hat{d} \right\}. \tag{6.86}$$

Having done all the preparations we label the sites of the finer lattice by the variable l and define Fourier coefficients through

$$\zeta_m(k) := L^{-d/2} \frac{1}{N^d} \sum_l e^{-ikl} \zeta_m(l). \tag{6.87}$$

Using the Fourier theorem, the spacetime fields can be expressed as

$$\zeta_m(l) = L^{-d/2} \sum_k e^{ikl} \zeta_m(k), \qquad \zeta_m^\dagger(l) = L^{-d/2} \sum_k e^{ikl} \zeta_m^\dagger(-k). \tag{6.88}$$

Fourier Transformation of the Action

Inserting the Fourier transformed fields into the second order action, we get

$$S_m^{(2)} = N_s D^2 \sum_m \sum_k \left(A_m(k) \, \text{Tr} \, \zeta_m(k)\zeta_m(-k) \right.$$

$$\left. + B_m(k) \, \text{Tr} \, \zeta_m(k)\zeta_m^\dagger(k) + C_m(k) \, \text{Tr} \, \zeta_m^\dagger(k)\zeta_m^\dagger(-k) \right), \tag{6.89}$$

where coefficients A_m, B_m, C_m are given by

$$A_m(k) = \sum_{\mu<\nu} \cos \frac{k_\mu - k_\nu}{2}$$

$$\text{Re}\, B_m(k) = (-1)^{|m|} \sum_{\mu,\nu} (-1)^{\delta_{m\nu}} \cos \frac{k_\mu + k_\nu}{2} + \frac{d}{X^2}$$

$$\text{Im}\, B_m(k) = -(-1)^{|m|} \sum_{\mu,\nu} (-1)^{\delta_{m\nu}} \sin \frac{k_\mu + k_\nu}{2} \tag{6.90}$$

$$C_m(k) = \sum_{\mu<\nu} (-1)^{\delta_{m\mu}+\delta_{m\nu}} \cos \frac{k_\mu - k_\nu}{2}$$

In the next step we decompose the field in the configuration space $\zeta(l)$ into an hermitian part $\xi(l)$ and an antihermitian part $\eta(l)$, namely

$$\zeta(l) = \xi(l) + \eta(l), \qquad \text{where } \xi^\dagger(n) = \xi(n) \text{ and } \eta^\dagger(n) = -\eta(n). \tag{6.91}$$

The Fourier coefficients of ξ and η satisfy

$$\xi^\dagger(k) = \xi(-k), \quad \eta^\dagger(k) = -\eta(-k). \tag{6.92}$$

In Fourier space ξ and η are no longer restricted to be hermitian or antihermitian, but may take arbitrary complex values. In term of the new variables the action takes the form

$$S^{(2)}[\xi,\eta] = N_s D^2 \sum_m \sum_k \left(A_m'(k) \, \text{Tr} \, \xi_m^\dagger(k)\xi_m(k) + B_m'(k) \, \text{Tr} \, \xi_m^\dagger(k)\eta_m(k) + \right.$$

$$\left. C_m'(k) \, \text{Tr} \, \eta_m^\dagger(k)\xi_m(k) + D_m'(k) \, \text{Tr} \, \eta_m^\dagger(k)\eta_m(k) \right), \tag{6.93}$$

where the primed coefficients are given by

$$
\begin{aligned}
A'(k) &= A(k) + B(k) + C(k), \\
B'(k) &= A(k) + B(k) - C(k), \\
C'(k) &= -A(k) + B(k) + C(k), \\
D'(k) &= -A(k) + B(k) - C(k).
\end{aligned}
\tag{6.94}
$$

Hermitian conjugation has to be combined with PT (parity and time reversion) transformation to be a symmetry of the field $\xi(k)$ resp. $\eta(k)$. Thus we can restrict the summation over the Fourier vectors k to the half of the first Brillouin zone. We define a subset of the first Brillouin zone by $M := \{k \in 1.\ \mathrm{BZ} \mid (k_1 > 0) \text{ or } (k_1 = 0 \text{ and } k_2 > 0) \text{ or } \ldots \text{ or } (k_1 = 0, \ldots, k_{d-1} = 0 \text{ and } k_d > 0)\}$. In doing so we get a decomposition of the first Brillouin zone, $1.\ \mathrm{BZ} = \{0\} \cup M \cup (-M)$. In what follows for a function $f(k)$ on Fourier space we make use of the restricted sum

$$
{\sum_k}' f(k) := \frac{1}{2} f(0) + \sum_{k \in M} f(k).
\tag{6.95}
$$

Switching to the fields ξ, η and restricting the values of k to the half of the first Brillouin zone we get for the second order action

$$
\begin{aligned}
S^{(2)}[\xi, \eta] = N_s D^2 \sum_m {\sum_k}' &\left(A''_m(k) \operatorname{Tr} \xi_m^\dagger(k) \xi_m(k) + B''_m(k) \operatorname{Tr} \xi_m^\dagger(k) \eta_m(k) + \right. \\
&\left. C''_m(k) \operatorname{Tr} \eta_m^\dagger(k) \xi_m(k) + D''\_m(k) \operatorname{Tr} \eta_m^\dagger(k) \eta_m(k) \right) \\
= N_s D^2 \sum_m {\sum_k}' &\ Q_{m,k}(\xi_m(k), \eta_m(k)),
\end{aligned}
\tag{6.96}
$$

where $Q_k^{(m)}$ denotes the quadratic form

$$
Q_{m,k}(\xi, \eta) = (\xi^\dagger, \eta^\dagger) \begin{pmatrix} A''_m(k) & B''_m(k) \\ C''_m(k) & D''_m(k) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.
\tag{6.97}
$$

The coefficients of the quadratic form Q_k follow from (6.90) and are given by

$$
\begin{aligned}
A''_m(k) &= 2A_m(k) + 2\operatorname{Re} B_m(k) + 2C_m(k), \\
B''_m(k) &= 2A_m(k) + 2i \operatorname{Im} B_m(k) - 2C_m(k), \\
C''_m(k) &= -2A_m(k) + 2i \operatorname{Im} B_m(k) + 2C_m(k), \\
D''_m(k) &= -2A_m(k) + 2\operatorname{Re} B_m(k) - 2C_m(k).
\end{aligned}
\tag{6.98}
$$

6.6.4   Masseless and Massive Modes

Positive Semidefiniteness and Zero Modes

$$
\begin{aligned}
\operatorname{Det}(Q_k - \lambda_k I) &= \operatorname{Det} \begin{pmatrix} A''(k) - \lambda_k & B''(k) \\ C''(k) & D''(k) - \lambda_k \end{pmatrix} \\
&= \lambda^2 - (A''(k) + D''(k))\lambda + A''(k)D''(k) - B''(k)C''(k) \\
&= \lambda_k^2 - 4\operatorname{Re}(B)\lambda_k + 4(|B|^2 - 4AC).
\end{aligned}
\tag{6.99}
$$

$$
\lambda_k = 2\left(\operatorname{Re} B(k) \pm \sqrt{4A(k)C(k) - (\operatorname{Im} B(k))^2} \right)
\tag{6.100}
$$

We specialize to the scalar ($m = \emptyset$) and the pseudoscalar ($m = \{1, ..., d\}$) sector. In this sectors two off the coefficients are equal, $A_m(k) = C_m(k)$, and we obtain the following estimate for the eigenvalue with the smaller real part,

$$
\begin{aligned}
\operatorname{Re} \lambda &\geq 2 \operatorname{Re} B - 4A \\
&= \pm 2 \sum_{\nu,\mu} \cos \frac{k_\nu + k_\mu}{2} + \frac{2d}{X^2} - 4 \sum_{\mu < \nu} \cos \frac{k_\mu - k_\nu}{2} \\
&\geq -2d^2 + \frac{2d}{X^2} - 2d(d-1) \\
&= 2d(2d-1) \left(\frac{1}{1 - \hat{\mathcal{M}} X} - 1 \right) \\
&\geq 0.
\end{aligned}
\tag{6.101}
$$

The plus sign (minus sign) in the second line corresponds to the scalar (pseudoscalar) sector. The last inequality is valid, because we expanded the action around $z = z_- < 0$.

Having proved that $Q_m(k)$ is positive semidefinite in the considered sectors we now look for zero eigenvalues of the quadratic form. In doing so we have to distinguish between the scalar and the pseudoscalar sector. In the scalar sector equality at the first and the third line of (6.101) occurs if and only if $\sin \frac{k_\mu + k_\nu}{2} = 0$, $\cos \frac{k_\mu + k_\nu}{2} = -1$ and $\cos \frac{k_\mu - k_\nu}{2} = 1$ that is $k_\mu + k_\nu \in 2\pi + 4\pi\mathbb{Z}$ and $k_\mu - k_\nu \in 4\pi\mathbb{Z}$. In the pseudoscalar sector a zero mode appears for $\sin \frac{k_\mu + k_\nu}{2} = 0$, $\cos \frac{k_\mu + k_\nu}{2} = 1$ and $\cos \frac{k_\mu - k_\nu}{2} = 1$ that is $k_\mu + k_\nu \in 4\pi\mathbb{Z}$ and $k_\mu - k_\nu \in 4\pi\mathbb{Z}$. Restricting the values of k to the half of the first Brillouin zone there are zero modes for

$$
k_1 = ... = k_d = \pi \quad \text{(scalar sector)}, \tag{6.102}
$$
$$
k_1 = ... = k_d = 0 \quad \text{(pseudoscalar sector)}. \tag{6.103}
$$

(In $d = 2$ there is another zero mode in the pseudoscalar sector: $k_1 = k_2 = 2\pi$.) Equality at the last line of (6.101) occurs only in the chiral limit $m = 0$, i.e. massless excitations in the two considered sectors exist only in the chiral limit.

Integration over the Massive Modes (Pseudoscalar Sector)

In the pseudo-scalar sector the coefficients of the second-order quadratic form take the values (we drop the subscripts $m = \{1, 2, ..., d\}$)

$$
A''(k) = 4 \left(\sum_\mu \sin \frac{k_\mu}{2} \right)^2 - 2d + \frac{2d}{X^2} \tag{6.104}
$$

$$
B''(k) = C''(k) = 2i \sum_{\mu,\nu} \sin \frac{k_\mu + k_\nu}{2} \tag{6.105}
$$

$$
D''(k) = -4 \left(\sum_\mu \cos \frac{k_\mu}{2} \right)^2 - 2d + \frac{2d}{X^2}. \tag{6.106}
$$

Thus the quadratic form at zero wave vector is given by

$$
Q(k = 0) = \begin{pmatrix} 4d(d-1) + \mu(m) & 0 \\ 0 & \mu(m) \end{pmatrix}, \tag{6.107}
$$

where we introduced a function of the quark masses, which for $aM \ll 1$ is linear in that masses, namely

$$\mu(m) := 2d(2d-1) \left(\frac{1}{1 - \hat{\mathcal{M}}/X} - 1 \right)$$

$$\approx 4d\sqrt{2d-1}\,am.$$

(6.108)

The next step is to do the Gaussian integral over the massive field ξ, which gives the result

$$\int d\xi(k)d\xi^\dagger(k) \exp\left(-N_c N_s D^2 Q_k(\xi(k), \eta(k)) \right)$$

$$\approx \text{const.} \times \exp\left(-N_c N_s D^2 \left(D''(k) - \frac{B''(k)C''(k)}{A''(k)} \right) \text{Tr}\,\eta^\dagger(k)\eta(k) \right)$$

$$= \text{const.} \times \exp\left(-N_c (D_0(k) + D_2(k)) \text{Tr}\,\eta^\dagger(k)\eta(k) \right),$$

(6.109)

where the coefficient in front of the quadratic term in η

$$D_0(k) := N_s D^2 D''(k) = N_s D^2 \left(4d^2 - 4 \left(\sum_\mu \cos\frac{k_\mu}{2} \right)^2 + \mu(M) \right)$$

$$\approx N_s D^2 \left(a^2 d \sum_\mu k_\mu^2 + \mu(M) \right)$$

(6.110)

picks up the correction

$$D_2(k) := -N_s D^2 \frac{B''(k)C''(k)}{A''(k)} = N_s D^2 \frac{\left(\sum_{\mu=1}^d \sin\frac{k_\mu}{2} \right)^2 \left(\sum_{\mu=1}^d \cos\frac{k_\mu}{2} \right)^2}{\left(\sum_{\mu=1}^d \sin\frac{k_\mu}{2} \right)^2 + d(d-1) + \frac{\mu(m)}{4}}$$

$$\approx a^2 N_s D^2 \frac{d^2}{4d(d-1) + \mu(m)} \left(\sum_\mu k_\mu \right)^2.$$

(6.111)

The term $\left(\sum_\mu k_\mu \right)^2$ is not invariant under the lattice symmetry group: its breaks the symmetry under reflections on hyperplanes parallel to the coordinate axis. In what follows we replace it by the invariant term $\sum_\mu k_\mu^2$.
In the chiral limit the correction is

$$D_2(k) = \frac{1}{4(d-1)} D_0(k),$$

(6.112)

that are about 8.3% of $D_0(k)$ in $d = 4$ dimensions.

6.6.5   Long Distance Approximation

There are two equivalent ways to get a continuum action in space time from the discrete action in Fourier space

$$S_2[\eta, \eta^\dagger] = \frac{1}{2} \sum_k (D_0(k) + D_2(k)) \text{Tr}\,\eta^\dagger(k)\eta(k)$$

(6.113)

One can now proceed in two different way: start with the inverse Fourier transformation and then do the continuum limit or begin with the continuum limit and then do the inverse Fourier transformation. Applying the latter method, we replace the sum over the Fourier space by an integral,

$$\sum_k \quad \longrightarrow \quad \left(\frac{La}{2\pi}\right)^d \int_{\mathbb{R}^d} d^d p. \tag{6.114}$$

Note that lattice Fourier transformed and the continuum Fourier transformed functions are not identical (see Appendix F), but differ by a factor,

$$\eta(k) \quad \longrightarrow \quad a^{-d}\left(\frac{2\pi}{L}\right)^{d/2} \eta(p) \tag{6.115}$$

Because we are dealing with quadratic expression, we will pick up an overall factor a^{-d}, when we pass from the lattice to an continuum formulation.

We exponentiate the hermitian field η to an unitary field $g := \exp \eta \in \mathrm{U}(N_f)$. Making use of Parseval's relation and the expansion $g \approx 1 + \eta + \frac{1}{2}\eta^2$, we obtain

$$\int d^d p \ \mathrm{Tr} \ \eta^\dagger(p)\eta(p) = \int d^d x \ \mathrm{Tr} \ \eta^\dagger(x)\eta(x)$$
$$= -\int d^d x \ \mathrm{Tr}(g(x) + g^{-1}(x)). \tag{6.116}$$

Using Parseval's relation again and an expansion $g \approx 1 + \eta$ we get

$$\int d^d p \ p^2 \ \mathrm{Tr} \ \eta^\dagger(p)\eta(p) = \int d^d p \ \mathrm{Tr}(ip\eta(k))^\dagger(ip\eta(k))$$
$$= \int d^d x \ \mathrm{Tr} \ \partial\eta^\dagger(x)\partial\eta(x) \tag{6.117}$$
$$= \int d^d x \ \mathrm{Tr} \ \partial g^{-1}(x)\partial g(x).$$

Finally we obtain a continuum action in spacetime which consists of two contributions,

$$S^{(2)}(g) = S_{\mathrm{fl}}(g) + S_M(M). \tag{6.118}$$

The first summand contains the kinetic energy of fluctuations up to second order in the chiral limit and is given by

$$S_{\mathrm{fl}}(g) = N_s \frac{a^{2-d}}{8d}\left(1 + \frac{1}{4(d-1)}\right)\int d^d x \ \mathrm{Tr}\left(\partial g^{-1}(x)\partial g(x)\right). \tag{6.119}$$

The second part describes the effect of nonzero quarks masses M in first order,

$$S_m(g) = -N_s \frac{a^{1-d}}{2d}\sqrt{2d-1}\,m \int d^d x \ \mathrm{Tr}\left(g(x) + g^{-1}(x)\right). \tag{6.120}$$

The results are in agreement with calculations in spacetime done before. New is the replacement $1 \to 1 + \frac{1}{4(d-1)}$ due to the contribution of the Gaussian integration.

6.7   Gradient Expansion (Bodycentered Hypercubic Lattice)

We follow the line of the gradient expansion on the hc lattice (6.5) to obtain an effective action on the bhc lattice. Again, the massless modes are given by

$$Z(n + v/2) = \gamma_v X \mathcal{G}(n), \qquad Z^\dagger(n + v/2) = \mathcal{G}^{-1}(n) X \gamma_v, \tag{6.121}$$

where the field $\mathcal{G}(n)$ has to satisfy $\mathcal{G}(n + v) = \gamma_v \mathcal{G}^{-1}(v)\gamma_v$. We have already studied the solutions of this equation on the bhc lattice: On this lattice the field $\mathcal{G}(n)$ has the shape

$$\mathcal{G}(n) = P_- \otimes g^{-1}(n) + P_+ \otimes g(n), \tag{6.122}$$

with $g(n) \in \mathrm{U}(N_f)$ and the saddle point manifold is given by the homogeneous configurations $g(n) = \mathrm{const}$. Thus the above parametrisation of the saddle point manifold corresponds to a smooth configuration of the field g and the dynamics of the massless modes can be described by an continuum action $S[g]$, which replaces the color flavor-transformed lattice action,

$$S_{\mathrm{vacuum}}[Z] \longrightarrow S[g] = S^{(0)} + S^{(2)}[g] + S^{(4)}[g] + \cdots . \tag{6.123}$$

The three contributions refer to the order of the expansion in the momenta of the massless modes, as in the case of the hc lattice.

Neglecting the coupling between the massless and the massive modes, we only consider the contribution to the effective action $S[g]$, which comes from inserting the massless modes (6.121) into the color-flavor transformed action. By making use of the property $\gamma_v \mathcal{G}^{-1}(n)\gamma_v = \mathcal{G}(n)$ which follows from (6.122), we obtain the action

$$S_{\mathrm{zero}}[g] = - \sum_n \mathrm{Tr} \ln \left(\sum_v (\mathcal{G}(n - v)X + X\mathcal{G}(n)) + \hat{\mathcal{M}} \right). \tag{6.124}$$

The next step is to perform an expansion in the momenta of the zero modes and the quark masses in complete analogy to what we have done for the theory on the hc lattice; we again arrange the result in the form $S_{\mathrm{zero}}[g] = S_{\mathrm{fl}}[g] + S_{\mathcal{M}}[g] + S_{\mathrm{fl},\mathcal{M}}[g]$. In doing so we pass from the theory on the four-dimensional bhc lattice to a continuum formulation. The connection between these formulations is built by the formulas

$$\sum_n \rightarrow \frac{2}{a^4} \int d^4x, \qquad \sum_v \partial_v^2 \rightarrow 3\Delta, \qquad \sum_v \partial_v^4 \rightarrow \frac{3}{2}\Delta^2, \tag{6.125}$$

which are derived in G.

Different from the theory on the hc lattice, where the Goldstone field is a $N_s N_f \times N_s N_f$ matrix, the Goldstone field obtained from the theory on the bhc lattice is a $N_f \times N_f$ matrix and the trace over the spin sector can be performed,

$$\mathrm{Tr}\,\mathcal{G} = \frac{N_s}{2}(\mathrm{Tr}\,g + \mathrm{Tr}\,g^{-1}) = 2(\mathrm{Tr}\,g + \mathrm{Tr}\,g^{-1}). \tag{6.126}$$

Having paid attention to the special qualities of the theory on the bhc lattice, we are now in position to derive the corresponding low energy action directly from the expression which we have obtained in appendix H.3. The three contributions to the effective action are given by

$$\begin{aligned}
S_{\mathrm{fl}}[g] = &- \frac{1}{4a^2} \int d^4x \sum_\mu \mathrm{Tr}\, L_\mu^2 \\
&+ \frac{1}{768} \int d^4x \sum_{\mu,\nu} \left(\mathrm{Tr}(L_\mu^2 L_\nu^2) + 5\,\mathrm{Tr}(\partial_\mu L_\mu \partial_\nu L_\nu) \right),
\end{aligned} \tag{6.127}$$

$$S_{\hat{\mathcal{M}}}[g] = -\frac{\sqrt{23}}{6a^4} \int d^4x \; \text{Tr}\left(\hat{\mathcal{M}}(g+g^{-1})\right)$$
$$+\frac{23}{288a^4} \int d^dx \; \text{Tr}\left(\hat{\mathcal{M}}g^{-1}\hat{\mathcal{M}}g^{-1} + \hat{\mathcal{M}}g\hat{\mathcal{M}}g\right) \tag{6.128}$$
$$-\frac{1}{12a^4} \int d^dx \; \text{Tr}\left(\hat{\mathcal{M}}^2(g+g^{-1})\right) + \frac{1}{46a^4} \int d^dx \; \text{Tr}\left(\hat{\mathcal{M}}g^{-1}\hat{\mathcal{M}}g\right),$$

$$S_{\text{fl},\mathcal{M}}[g] = +\frac{1}{12\sqrt{23}a^3} \int d^dx \; \text{Tr}\left(\hat{\mathcal{M}}(g^{-1}\partial g - \partial g g^{-1})\right)$$
$$-\frac{\sqrt{23}}{192a^2} \int d^dx \sum_\mu \Big(\text{Tr}\left(L_\mu^2(g^{-1}\hat{\mathcal{M}} + \hat{\mathcal{M}}g)\right) + \text{Tr}\left(\partial_\mu L_\mu(g^{-1}\hat{\mathcal{M}} - \hat{\mathcal{M}}g)\right). \tag{6.129}$$

For the bhc lattice all terms except the first term of $S_{\text{fl},\mathcal{M}}[g]$ have the full O(4) continuum symmetry. There are no fourth order contributions to $S_{\text{fl}}[g]$ that break the continuum symmetry, as for the hc lattice. As we discussed before (cf. Chapter 4.3.1), this is a consequence of the fact that the bhc lattice has a larger symmetry group than the hc lattice.

6.8   Comparison with Experimental Data

We have applied the color-flavor transformation to the strong coupling limit of QCD. The transformed partition function was evaluated in the large-N_c limit and an effective low energy theory was derived. This section deals with the calculation of observable quantities from the effective theory and comparison of their values with experimental data. We start with an estimate for the value for the chiral condensate. Then we compare the coefficients in front of the different terms of the effective Lagrangian with experimental values.

Having worked with a lattice formulation of QCD, one would expect that the lattice constant a has to be driven to zero to recover a continuum theory. But the second order fluctuation action is proportional to a^{2-d} and would diverge in that limit. We conclude that the lattice constant has to be fixed at a finite value and a continuum action can be obtained in a long distance approximation, where fluctuations of the effective field on scales smaller than the lattice constant are neglected. The impossibility to reach the continuum limit $a \to 0$ is connected with the strong coupling approximation of QCD which we employed right from the beginning: QCD is an asymptotically free theory, i. e. it is weakly coupled at small distances. Thus a continuum limit would contradict the strong coupling approximation.

The effective field of the low energy theory is a collective field of quarks and gluons which should describe the lightest mesons. Fitting the effective theory to their properties, the lattice constant a should be seen as a free parameter, which sets the energy scale of the theory. Different estimates for this parameter will be obtained in the next sections from comparison with experimental data.

6.8.1   Chiral Condensate

The chiral condensate is the expectation value of the quantity $\bar{q}_f q_f$ of a quark field of flavor $f = u, d, s, \dots$ in the ground state. With help of the partition function (2.68) and the volume of space-time V it can be expressed as

$$\langle \bar{q}_f q_f \rangle = \frac{1}{V} \int [d\bar{q}dq] \, \bar{q}_f q_f \mathcal{Z}(\bar{q}, q) \Big/ \int [d\bar{q}dq] \, \mathcal{Z}(\bar{q}, q). \tag{6.130}$$

The chiral condensate of the lightest flavors $f = u, d$ should be well approximated by its value in the chiral limit, which can be expressed as a logarithmic derivate,

$$\langle \bar{q}_f q_f \rangle \approx -\frac{1}{V}\frac{d}{dm_f}\bigg|_{m_f=0} \ln \mathcal{Z} = -\frac{K_3 a}{K_2 V}\frac{d}{d\hat{m}_f}\bigg|_{\hat{m}_f=0} \ln \mathcal{Z}. \tag{6.131}$$

Because we do not have an exact expression for the dependence of the partition function on the quark masses $\hat{m}_f$, we make use of the saddle point approximation $\mathcal{Z} \approx e^{-N_c S_{\text{saddle}}}$ which is exact in the large N_c limit,

$$\begin{aligned}
\langle \bar{q}_f q_f \rangle &\approx \frac{N_c K_3 a}{K_2 V}\frac{d}{d\hat{m}_f}\bigg|_{\hat{m}_f=0} S_{\text{saddle}} \\
&= -\frac{N_c N_s}{K_2 a^{d-1}}\frac{d}{d\hat{m}_f}\bigg|_{\hat{m}_f=0} \ln(2\delta x_f + \hat{m}_f) \\
&= -\frac{N_c N_s}{K_2 a^{d-1}}\frac{\delta - 1}{2\delta\sqrt{2\delta - 1}}.
\end{aligned} \tag{6.132}$$

Recall that 2δ is the coordination number of the lattice. In the calculations above the volume of space-time was expressed in terms of the number of lattice sites, $V = K_3 \mathcal{N} a^d$. Note the presence of the factor K_3 in this formula, which takes into account that the bhc lattice consists of two nested sublattices.

The negative value for the chiral condensate is in agreement with general theoretical considerations (cf. Section 4.3). We compare it with the experimental value $\langle \bar{q}_f q_f \rangle_{\text{exp}} = -((250 \pm 35)\text{ MeV})^3$ to get an estimate for the lattice constant a. The result is

$$a_{\bar{q}q,\text{ hc}} = (166\text{ MeV})^{-1} \tag{6.133}$$

for the theory on the hc lattice and

$$a_{\bar{q}q,\text{ bhc}} = (104\text{ MeV})^{-1} \tag{6.134}$$

for the theory on the bhc lattice. We conclude that for a realistic theory, the lattice constant has to be fixed at about 1 fm (200 MeV $\approx$ 1 fm).

6.8.2   Low Energy Effective Lagrangians

Chiral Lagrangians have been shown to work well in the description of low energy properties of mesons and baryons since their introduction by Weinberg. They were first introduced to recover the relations of the current algebra [46] and later justified by symmetry arguments [47]. Based on the chiral symmetry, Gasser and Leutwyler developed a technique which allows to expand the QCD Green's functions in terms of momenta p of the low energy modes and the masses of the lightest quarks m [17], [18]. This formalism is known as *chiral perturbation theory* and treats p^2 and m as small, while p^2/m is allowed to have any value (chiral power counting). The expansion was carried out up to fourth order in p and the coefficients in front of the different terms were determined by experimental data [18], [7]. More recently the expansion was extended up to sixth order [16].

Different attempts were made to obtain chiral Lagrangians as low energy approximation of more fundamental theories. Low energy effective Lagrangians were obtained from the Nambu-Jona-Lasino model [15] or the strong coupling limit of QCD [28], [29], [26], [31]. We employed

the color-flavor transformation to connect the usual formulation of QCD in terms of quarks and gluons with an equivalent, but in view of low energy physics more suitable formulation. The color-flavor transformation is an exact identity, but we are working in the strong coupling limit and make use of a large N_c approximation.

We identified the Goldstone modes in the color-flavor transformed formulation and obtained an effective Lagrangian which describes their dynamics. We have neglected the axial anomaly; therefore we expect the symmetry breaking pattern $U(N_f)_L \times U(N_f)_R \to U(N_f)_{L+R}$ and N_f^2 Goldstone modes. In fact the expected number of Goldstone particles appears in the theory on the bhc lattice, but the theory on the hc lattice has $N_s^2 N_f^2$ Goldstone modes. This enlarged number of Goldstone modes should be regarded as a lattice artifact; QCD on the hc lattice has an enlarged chiral symmetry group compared with the continuum theory (cf. Chapter 4). We therefore restrict ourselves to the theory on the bhc lattice, when we try to identify the Goldstone particles with the experimentally observed light mesons.

Second Order Chiral Perturbation Theory

A general second order Lagrangian is given by the kinetic energy of the Goldstone bosons plus symmetry breaking term which is of first order in the quark masses,

$$\mathcal{L}^{(2)}[g] = \mathcal{L}^{(2)}_{\mathrm{fl}}[g] + \mathcal{L}^{(2)}_{\mathcal{M}}[g] = \frac{F^2}{4} \left(\mathrm{Tr}(\partial g^{-1} \partial g) - 2B \, \mathrm{Tr}\,\mathcal{M}(g^{-1} + g) \right). \tag{6.135}$$

Comparing this expression with our result (bhc lattice, $N_c = 3$ colors), the coupling constants come out as

$$F = \sqrt{N_c} \frac{1}{a}, \qquad B = 2\sqrt{23} \frac{1}{a}. \tag{6.136}$$

The nine Goldstone bosons arising from the spontaneous symmetry breaking are collected in the U(3) matrix

$$g(x) = \exp(i\phi(x)/F), \tag{6.137}$$

which is set into correspondence to the physical mesons by the identification

$$\phi = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta + \frac{1}{\sqrt{3}}\eta' & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta + \frac{1}{\sqrt{3}}\eta' & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta + \frac{1}{\sqrt{3}}\eta' \end{pmatrix}. \tag{6.138}$$

We expand the Lagrangian in terms of the physical fields, to obtain the masses of the mesons. In doing so we split the charged pions in real and imaginary part $\pi^\pm = \pi_r \pm i\pi_i$ and put all pions together to an isospin vector $\pi = (\pi_r, \pi_i, \pi_0)$. Analogously we put the charged kaons together to a spinor $K_0 = (K_{0,r}, K_{0,i})$ as well as the neutral kaon and its antiparticle to a spinor $K = (K_r^0, K_i^0)$. An expansion of the Lagrangian around $\phi = 0$ gives

$$\begin{aligned} \mathcal{L}^{(2)}[g] &\approx \frac{1}{4} \mathrm{Tr}(\partial \phi)^2 - B \, \mathrm{Tr}\,\mathcal{M} \left(F^2 - \frac{1}{2}\phi^2 \right) \\ &= \frac{1}{2} \left((\partial \pi)^2 + (\partial \eta)^2 + (\partial K)^2 + (\partial K_0)^2 + (\partial \eta')^2 \right) \\ &\quad - BF^2 \left(m_u + m_d + m_s \right) \\ &\quad + \frac{1}{2} \left(M_\pi^2 \pi^2 + M_\eta^2 \eta^2 + M_K^2 K^2 + M_{K_0}^2 K_0^2 + M_{\eta'}^2 \eta'^2 \right) \end{aligned} \tag{6.139}$$
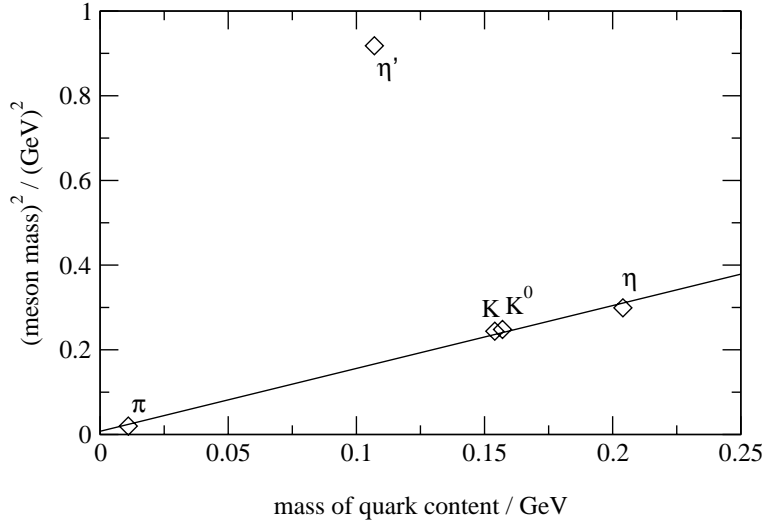
Figure VIII: Masses of the light pseudoscalar mesons versus the masses of their quark content. Data (diamonds) from [20]. The mass of the η'-particle is anomalous and left out of the fit. Slope of the fitted line: $B = 1.48 \pm 0.08$ GeV.

with the masses

$$
\begin{aligned}
M_\pi^2 &= B(m_u + m_d) \\
M_\eta^2 &= \frac{1}{3}B\left(m_u + m_d + 4m_s\right) \\
M_K^2 &= B(m_u + m_s) \\
M_{K_0}^2 &= B(m_d + m_s) \\
M_{\eta'}^2 &= \frac{2}{3}B\left(m_u + m_d + m_s\right).
\end{aligned}
\tag{6.140}
$$

As result we obtain the standard kinetic energy and a mass terms for all the mesons. The square of the meson masses turn out to be linear in the quark masses. That is the reason why the quark masses have to be considered as $O(p^2)$ in the framework of chiral perturbation theory. The constant B can be obtained from a linear fit, see Figure VIII.

Further, in the chiral limit, we get for the chiral condensate of a flavor $f = u, d, s$

$$
\langle \bar{q}_f q_f \rangle \approx \left.\frac{d}{d\hat{m}_f}\right|_{\hat{m}_f = 0} \left(-BF^2\left(m_u + m_d + m_s\right)\right) = -BF^2.
\tag{6.141}
$$

Solving this equation for the constant B and inserting it into the equation for the pion mass (6.140) we recover the Gellmann-Oakes-Renner relation [19],

$$
M_\pi^2 F^2 = -\frac{m_u + m_d}{2}\left(\langle \bar{q}_u q_u \rangle + \langle \bar{q}_d q_d \rangle\right) + O(m^2).
\tag{6.142}
$$

We conclude that the constant F has to be identified with the pion decay constant F_π. Having fixed the two low energy constants at their physical values

$$
F = F_\pi = 93 \text{ MeV}, \qquad B = 1480 \text{ MeV},
\tag{6.143}
$$

we make use of (6.136) to get an estimate for the lattice constant from each of the two values,

$$a_{F,\,\text{bhc}} = (54 \text{ MeV})^{-1}, \qquad a_{B,\,\text{bhc}} = (154 \text{ MeV})^{-1}. \tag{6.144}$$

Fourth Order Chiral Perturbation Theory

Neglecting the axial anomaly, the chiral Lagrangian at order p^4 contains two different classes of contributions:

(i) The most general fourth order Lagrangian $\mathcal{L}^{(4)}[g]$ which is compatible with gauge and Poincaré invariance.

(ii) One-loop graphs associated with the Lagrangian of order $O(p^2)$; they are also of order $O(p^4)$.

Gasser and Leutwyler obtained the most general form of the fourth order Lagrangian, which contains ten low energy constants $\ell_1, ..., \ell_{10}$ as parameters [18]. Their result is related to a $SU(3)$ meson field g which corresponds to the octet of the lightest mesons without the heavier η' particle. The first eight terms of their Lagrangian read

$$\begin{aligned}
\mathcal{L}^{(4)}_{\text{GL}}[g] = &\, \ell_1 \big(\text{Tr}(\partial_\mu g^\dagger \partial_\mu g) \big)^2 + \ell_2 \, \text{Tr}(\partial_\mu g^\dagger \partial_\nu g) \, \text{Tr}(\partial_\mu g^\dagger \partial_\nu g) + \ell_3 \, \text{Tr}(\partial_\mu g^\dagger \partial_\mu g \partial_\nu g^\dagger \partial_\nu g) \\
&+ \ell_4 \, \text{Tr}(\partial_\mu g^\dagger \partial_\mu g) \, \text{Tr}(\chi^\dagger g + \chi g^\dagger) + \ell_5 \, \text{Tr}(\partial_\mu g^\dagger \partial_\mu g(\chi^\dagger g + g^\dagger \chi)) \\
&+ \ell_6 \big(\text{Tr}(\chi^\dagger g + \chi g^\dagger) \big)^2 + \ell_7 \big(\text{Tr}(\chi^\dagger g - \chi g^\dagger) \big)^2 + \ell_8 \, \text{Tr}(\chi^\dagger g \chi^\dagger g + \chi g^\dagger \chi g^\dagger).
\end{aligned} \tag{6.145}$$

We have omitted two more terms which contain couplings of the meson field to external vector currents. The field $\chi(x)$ has physical dimension $[\text{length}]^{-1}$ and in absence of external perturbations it is given by

$$\chi(x) = 2B\mathcal{M}. \tag{6.146}$$

The calculation of the loops associated with the second order Lagrangian gives a divergent result. The method of *dimensional regularization* was applied [18] to absorb the divergences into the coupling constants $\ell_1, ..., \ell_{10}$ and to renormalize them. The renormalized low energy constants are given by

$$\ell_i^{\text{r}} = \ell_i - \Gamma_i \lambda, \tag{6.147}$$

where the scale dependent divergence enters through the parameter

$$\lambda = \frac{1}{32\pi^2} \mu^{d-4} \left(\frac{2}{d-4} - \ln 4\pi - 1 + \gamma \right). \tag{6.148}$$

When the renormalization scale μ changes from μ_1 to μ_2 the renormalized coupling constants experience a change

$$\ell_i^{\text{r}}(\mu_2) = \ell_i^{\text{r}}(\mu_1) + \frac{\Gamma_i}{16\pi^2} \ln \left(\frac{\mu_1}{\mu_2} \right). \tag{6.149}$$

The renormalization coefficients Γ_i for the low energy constants $\ell_1, ...\ell_8$ are listed in Table VI. The table also contains estimates of the low energy constants from experimental data.

i	Γ_i	low energy constants	
		experiment ℓ_i^{r}	theory ℓ_i
1	$3/32$	0.9 ± 0.3	0
2	$3/16$	1.7 ± 0.7	0
3	0	-4.4 ± 2.5	3.9
4	$1/8$	0 ± 0.5	0
5	$3/8$	2.2 ± 0.5	15.0
6	$11/144$	0 ± 0.3	0
7	0	-0.4 ± 0.2	0
8	$5/48$	1.1 ± 0.3	26.8

Table VI: Low energy coupling constants in units of 10^{-3}. Experimental values at the scale $\mu = M_\eta$. Predictions of our theory for $a^{-1} = 100$ MeV.

Let us collect the terms of order $O(p^4)$ which we have obtained from our approach to strong coupling QCD. The result is a Lagrangian

$$
\begin{aligned}
\mathcal{L}^{(4)}[g] = N_c \Big(& l_1 \operatorname{Tr}(L_\mu^2 L_\nu^2) + l_2 \operatorname{Tr}(\partial_\mu L_\mu \partial_\nu L_\nu) \\
& + l_3 a^{-1} \operatorname{Tr}\left(L_\mu^2 (g^{-1}\mathcal{M} + \mathcal{M}g)\right) + l_4 a^{-1} \operatorname{Tr}\left(\partial_\mu L_\mu (g^{-1}\mathcal{M} - \mathcal{M}g)\right) \\
& + l_5 a^{-2} \operatorname{Tr}\left(\mathcal{M}g^{-1}\mathcal{M}g^{-1} + \mathcal{M}g\mathcal{M}g\right) \\
& + l_6 a^{-2} \operatorname{Tr}\left(\mathcal{M}^2(g + g^{-1})\right) + l_7 a^{-2} \operatorname{Tr}\left(\mathcal{M}g^{-1}\mathcal{M}g\right) \Big),
\end{aligned}
\tag{6.150}
$$

with the dimensionless constants

$$
l_1 = \frac{1}{768}, \ l_2 = \frac{5}{768}, \ l_3 = l_4 = -\frac{\sqrt{23}}{32}, \ l_5 = \frac{23}{8}, \ l_6 = -3, \ l_7 = \frac{18}{23}.
\tag{6.151}
$$

Our Lagrangian does not contain contributions, which are a product of two traces,

$$
\ell_1 = \ell_2 = \ell_4 = \ell_6 = \ell_7 = 0.
\tag{6.152}
$$

(Such terms can arise, when one calculates the mass action, which describes the influence of the massive modes on the Goldstone modes, to higher order than we did. As on example of such a term, in Ref. [2] the Garde term (H.22) was obtained.) Not all of the above coefficient are independent. At order $O(p^4)$ of chiral perturbation theory we have to consider the classical equation of motion

$$
\partial_\mu L_\mu + B g^{-1}\mathcal{M} - B\mathcal{M}g = 0
\tag{6.153}
$$

which follows from the second order Lagrangian (6.139). Applying this relations, we observe that all contributions of our Lagrangian except the two last ones appear in the general Lagrangian of Gasser and Leutwyler. The predictions for the corresponding coefficients are

$$
\begin{aligned}
\ell_3 &= N_c l_1, \\
\ell_5 &= -\frac{1}{2} N_c l_3 (aB)^{-1}, \\
\ell_8 &= \frac{1}{2} N_c l_2 - \frac{1}{4} N_c l_4 (aB)^{-1} + \frac{1}{4} N_c l_5 (aB)^{-2}.
\end{aligned}
\tag{6.154}
$$

Our values for the low energy constants ℓ_3, ℓ_5, ℓ_8 with $a = (100 \text{ MeV})^{-1}$ are plotted in Table VI.

6.8.3   Discussion

Our predictions for the low energy constants are in bad agreement with the experimental values. This must be connected with the approximations we have done: the strong coupling limit and the saddle point approximation. In the strong coupling limit one loses all information about the dynamics of the quarks. For that reason one should not expect quantitative agreement between the predictions of the theory and the experimental values. While the $1/N_c$ expansion in principle could be extended to higher orders, QCD is definitely much more complicated than its strong coupling limit.

Chapter 7

Strong Coupling Quantum Chromodynamics (Static Baryon)

In this chapter we calculate the color-flavor transformation of strong coupled QCD and discuss a model of a static baryon. In contrast to the last chapter, where we restricted ourselves to the mesonic sector of the theory, we now treat the full theory with baryons. As a consequence of the structure of the color-flavor transformation for the special unitary group $\mathrm{SU}(N_c)$, the color-flavor-transformed partition function decomposes into a sum of contributions which correspond to mesonic excitations over the baryonic vacuum as well as over configurations with baryons and antibaryons.

7.1 Color-Flavor Transformation of the Lattice Action

We consider the strong coupling limit of QCD in the formulation of a $\mathrm{SU}(N_c)$ lattice gauge theory. With the notations of Chapter 6, the partition function is given by

$$
\mathcal{Z} = \int [dq d\bar{q}]\, \mathcal{Z}[q, \bar{q}], \qquad \text{where}
$$
$$
\mathcal{Z}[q, \bar{q}] = \prod_{n,v} \int_{\mathrm{SU}(N_c)} dU(n + v/2)\, \mathrm{e}^{-S_U(n+v/2) - S_{\mathcal{M}, q, \bar{q}}(n)} .
\tag{7.1}
$$

We perform the color-flavor transformation in the same way as in the last chapter. Now make use of its more complicated version for the special unitary group $\mathrm{SU}(N_c)$ which was stated in Chapter 3. The result of the transformation reads

$$
\mathcal{Z}[q, \bar{q}] = \sum_{\{B\}} \prod_{n,v} \int \frac{D\mu(Z, Z^\dagger(n + v/2))}{\mathrm{Det}(1 + (ZZ^\dagger)(n + v/2))^{N_c}}\, \chi^B_{Z, q, \bar{q}}(n + v/2)\, \mathrm{e}^{-S_{Z, q, \bar{q}}(n+v/2) - S_{\mathcal{M}, q, \bar{q}}(n)},
\tag{7.2}
$$

where the sum extends over all possible distributions $\{B\}$ of the baryon number (actually, the baryonic flux) over the links of the lattice. As before, the color-flavor transformed action on a link $n + v/2$ given by

$$
\begin{aligned}
S_{Z, q, \bar{q}}(n + v/2) = K_2 a^{d-1} \Big(&\bar{q}^i_a(n) \gamma_v Z_{ab}(n + v/2) q^i_b(n) \\
&+ \bar{q}^j_b(n + v) \gamma_v Z^\dagger_{ab}(n + v/2) q^j_b(n + v) \Big).
\end{aligned}
\tag{7.3}
$$

The only new ingredients are the coefficients $\chi^0(n+\hat{\mu}/2) := \alpha_0$ and (with $B = B(n+\hat{\mu}/2) > 0$)

$$\chi^B_{Z,q,\bar{q}}(n + v/2) :=$$
$$\chi_{B(n+v/2)}\left(-\bar{q}(n)\gamma_v\sqrt{1 + ZZ^\dagger(n + v/2)}, \sqrt{1 + ZZ^\dagger(n + v/2)}q(n + v)\right), \qquad (7.4)$$

$$\chi^{-B}_{Z,q,\bar{q}}(n + v/2) :=$$
$$\chi_{-B(n+v/2)}\left(\sqrt{1 + Z^\dagger Z(n + v/2))}q(n), -\bar{q}(n + v)\gamma_v\sqrt{1 + Z^\dagger Z(n + v/2)}\right). \qquad (7.5)$$

with functions χ_B, $B = -N_f, ..., +N_f$ from Chapter 3. The partition function (7.2) is a sum over all configurations of the baryonic flux $\{B(n + v/2)\}$ on the lattice. For most of these configurations, the Grassmann integral over the quarks vanishes identically. To see that, we expand the integrand for a given configuration into a polynomial in the Grassmann fields, and count (for each site n) the number of fermions $q(n)$, $\bar{q}(n)$ in the various monomials:

- For every direction v, the coefficient $\chi^B_{Z,q,\bar{q}}(n+v/2)$ contains $N_c|B|$ Grassmann variables $\bar{q}^i_a(n)$ if $B > 0$, and the same number of Grassmann variables $q^i_a(n)$ if $B < 0$.

- For the coefficients $\chi^B_{Z,q,\bar{q}}(n - v/2)$ the situation is the same, except that $q(n)$ and $\bar{q}(n)$ change their roles.

- Each term of the expansion of $e^{-S_{Z,q,\bar{q}}(n+v/2)-S_{\mathcal{M},q,\bar{q}}(n)}$ involves as many $\bar{q}(n)$ as $q(n)$.

The Grassmann integral $\int dq(n)d\bar{q}(n)$ extracts the coefficient of the top–monomial,

$$\int d\bar{q}(n)dq(n) \prod_{i=1}^{N_c}\prod_{a=1}^{N_f} q^i_a(n)\bar{q}^i_a(n) = 1 , \qquad (7.6)$$

setting all others to zero. This monomial contains just as many $\bar{q}(n)$ as $q(n)$. Hence, in view of the counting above, the contribution from a configuration $\{B(n + v/2)\}$ vanishes unless the following condition is met:

$$\sum_v B(n + v/2) = \sum_v B(n - v/2). \qquad (7.7)$$

The physical meaning of this equation is conservation of the baryon current: the (algebraic) number of baryons "arriving" at the site n (from the links $n - v/2$) must equal the number of baryons "leaving" the site (via the links $n + v/2$).

The general structure of the partition function (7.2) corresponds to the hadronic correlation function written in terms of colorless N_c–quark currents [27], [13].

7.2   Static Baryon: Integration over the Fermions

We consider the color-flavor-transformed $SU(N_c)$ lattice gauge theory on a d-dimensional hypercubic lattice. We introduce a split into space and time: The time direction is labeled by $\mu = 0$ and the $d - 1$ space directions by $\mu = 1, .., d - 1$, the corresponding unit vectors by $\hat\mu$. The static baryon is modeled [1] by the following distribution of baryonic flux over the lattice: $B(n + \hat\mu/2) = 1$ along the links of the "world line" (or "string") $n = (t, 0, \ldots, 0) \in \mathbb{Z}^d$ in direction $\hat\mu = \hat0$, with $t = 0, \ldots, T - 1$; on all other links $B = 0$. This distribution satisfies the current conservation law (7.7) at all sites but the ends $t = 0$ and $t = T$ of the world line. There it also does, if we impose periodic (or antiperiodic) boundary conditions on the Grassmann fields: for a lattice of length T in the time direction, we set $q(T, \vec r) = q(0, \vec r)$ and $\bar q(T, \vec r) = \bar q(0, \vec r)$. As for the meson sector (Chapter 6), we do the integration over the quarks and write the partition function in the form

$$\mathcal{Z}_{\text{baryon}} = \int [d\bar q dq]\, \mathcal{Z}_{\text{baryon}}[q, \bar q] = \int D[Z, Z^\dagger]\, \exp(-N_c S_{\text{baryon}}[Z]). \tag{7.8}$$

The effective action $S_{\text{baryon}}[Z]$ contains the "sea" term $S_{\text{vacuum}}[Z]$, plus an extra part coming from the factors χ_1 along the world line of the baryon. The sea term can be written in the form

$$S_{\text{vacuum}}[Z] = -\sum_n \operatorname{Tr} \ln M(n) + \sum_n \sum_{\mu=0}^{d-1} N(n + \mu/2), \tag{7.9}$$

where we have introduced the short-hand notations

$$M(n) := \sum_{\mu=0}^{d-1} \gamma_\mu \big(Z^\dagger(n - \hat\mu/2) + Z(n + \hat\mu/2) \big) + \hat{\mathcal{M}}, \tag{7.10}$$

$$N(n + \mu/2) := 1 + Z(n + \hat\mu/2) Z^\dagger(n + \mu/2). \tag{7.11}$$

To calculate the extra part, we need to integrate polynomials in the quark fields along the world line of the baryon, weighted by the same Gaussian as in the vacuum sector. From the explicit form of the χ-coefficient for $B = 1$, (3.30), we find

$$\chi^1_{Z,q,\bar q}(n + \hat0/2) = \chi_1(\bar q(n), \tilde N(n + \hat0/2) q(n)), \tag{7.12}$$

where $\tilde N(n + \hat0/2) := -\gamma_0 N(n + \hat0/2)$ The part of the integrand containing the quark fields located on the string, namely $q(t\hat0)$, $\bar q(t\hat0)$ for $t = 0, \ldots, T - 1$, then reads

$$\chi_1\big(\bar q(0), \tilde N(\tfrac{1}{2}\hat0) q(1\hat0)\big)\ \chi_1\big(\bar q(1\hat0), \tilde N(\tfrac{3}{2}\hat0) q(2\hat0)\big) \cdot \ldots$$

$$\ldots \cdot \chi_1\big(\bar q((T-1)\hat0), \tilde N((T - \tfrac{1}{2})\hat0) q(T\hat0)\big) \times \exp\left(-\sum_{t=0}^{T-1} \bar q(t\hat0) M(t\hat0) q(t\hat0) \right). \tag{7.13}$$

(We use the abbreviation $t\hat0 := (0, ..., 0) + t\hat0$ to denote the sites and link on the world line of the baryon.)

---

[1]This is a very simple model of a baryon. More correct the real baryon should be thought as an excitation to which different distributions of the baryonic flux can contribute.

Isolating the terms with fermions at the site $n = t\hat{0}$, we are faced with the integral

$$\int d\bar{q}(n)dq(n)\,\chi_1\big(\bar{q}(n-\hat{0}), \tilde{N}(n-\hat{0}/2)q(n)\big)\,\mathrm{e}^{-\bar{q}(n)M(n)q(n)}\,\chi_1\big(\bar{q}(n), \tilde{N}(n+\hat{0}/2)q(n+\hat{0})\big)$$

$$= \left(\alpha_1 \frac{(N_f-1)!}{(N_c+N_f-1)!}\right)^2 \int \prod_{k,c} d\bar{q}_c^k(n)dq_c^k(n)\,\mathrm{e}^{-\bar{q}_a^k(n)M_{ab}(n)q_b^k(n)}$$

$$\sum_{\sigma,\tau\in\mathfrak{S}_{N_c}} \mathrm{sgn}\,\sigma\,\mathrm{sgn}\,\tau \prod_i \bar{q}_a^i(n-\hat{0})\tilde{N}_{ab}(n-\hat{0}/2)q_b^{\sigma(i)}(n) \prod_j \bar{q}_{a'}^j(n)\tilde{N}_{a'b'}(n+\hat{0}/2)q_{b'}^{\tau(j)}(n+\hat{0})$$

$$= \left(\alpha_1 \frac{(N_f-1)!}{(N_c+N_f-1)!}\right)^2 \sum_{\sigma,\tau\in\mathfrak{S}_{N_c}} \mathrm{sgn}(\sigma\tau) \prod_i \Big(\bar{q}_{a_i}^{\sigma^{-1}(i)}(n-\hat{0})\gamma_0\tilde{N}_{a_ib_i}(n-\hat{0}/2)$$

$$\left\{\int d\bar{q}^i(n)dq^i(n)\,q_{b_i}^i(n)\bar{q}_{a_i'}^i(n)\,\mathrm{e}^{-\bar{q}^i(n)M(n)q^i(n)}\right\}\tilde{N}_{a_i'b_i'}(n+\hat{0}/2)q_{b_i'}^{\tau(i)}(n+\hat{0})\Big),$$

$$\tag{7.14}$$

where the first equality sign uses expression (3.30) for the function χ_1. Note that the integral between curly brackets involves only fermions of color i. The fermionic version of Wick's theorem yields for it the value $M_{b_ia_i'}^{-1}(n)\,\mathrm{Det}\,M(n)$, so after combining the permutations σ and τ, the above expression becomes

$$\alpha_1^2 \frac{(N_f-1)!^2}{(N_c+N_f-1)!^2}\,N_c!\,\mathrm{Det}\,M(n)^{N_c} \sum_{\rho\in\mathfrak{S}_{N_c}} \mathrm{sgn}\,\rho \prod_i \bar{q}_{a_i}^i(n-\hat{0})G_{a_ib_i'}(n-\hat{0}\to n+\hat{0})q_{b_i'}^{\rho(i)}(n+\hat{0})$$

$$= \alpha_1\,\mathrm{Det}\,M(n)^{N_c}\binom{N_c+N_f-1}{N_f-1}^{-1}\chi_1\big(\bar{q}(n-\hat{0}), G(n-\hat{0}\to n+\hat{0})q(n+\hat{0})\big),$$

$$\tag{7.15}$$

with the "propagator" $G(n-\hat{0}\to n+\hat{0}) := \tilde{N}(n-\hat{0}/2)M(n)^{-1}\tilde{N}(n+\hat{0}/2)$.

Repeating the procedure, we successively integrate over the quark fields along the string. By this process the matrices N and M get organized into a single propagator. In the final integration step, we need to take into account the periodic boundary conditions for the quark fields: $q(T\hat{0}) = q(0)$. The final integral over $q(0)$ then reads

$$\int d\bar{q}(0)dq(0)\,\chi_1\big(\bar{q}(0), G(0\to T\hat{0})q(0)\big)\,\mathrm{e}^{-\bar{q}(0)M(0)q(0)}.\tag{7.16}$$

We now make use of an alternative expression for the χ-coefficient,

$$\chi_1(\bar{\phi}, \phi) = \alpha_1 \frac{(N_f-1)!}{(N_c+N_f-1)!}\sum_{\{a_i\}}\sum_{\sigma\in\mathfrak{S}_{N_c}}\prod_{i=1}^{N_c}\bar{\phi}_{a_i}^i\phi_{a_{\sigma(i)}}^i,\tag{7.17}$$

which is easily obtained from Eq. (3.30) by interchanging the product over colors with the sum over flavors. Wick's theorem then yields for the $q(0)$-integral the result

$$\alpha_1 \frac{(N_f-1)!}{(N_c+N_f-1)!}\,\mathrm{Det}\,M(0)^{N_c}(-1)^{N_c}\sum_{\{a_i,b_i\}}\sum_{\sigma\in\mathfrak{S}_{N_c}}\prod_i G_{a_{\sigma(i)}b_i}(0\to T\hat{0})M_{b_ia_i}^{-1}(0).\tag{7.18}$$

The last matrix product may also be expressed in terms of the propagator G along the closed string which we define as $G := G(0\to T\hat{0})M(0)^{-1}$.

What is the interpretation of the sign factor $(-1)^{N_c}$? To answer this question, recall that we evaluated the Grassmann field integral using time–*periodic* boundary conditions (instead of the conventional time-antiperiodic ones). In a d–dimensional quantum mechanical frame work with Hamiltonian H and inverse temperature β, this would mean that we are computing not the usual partition function but rather the *super*trace $\mathrm{Tr}\,(-1)^{N_\mathrm{F}}\mathrm{e}^{-\beta H}$ with N_F the total fermion number. The overall sign factor $(-1)^{N_c}$ originates from that very fermion number, and is simply telling us that the baryon is a fermion (boson) if N_c is odd (respective even). Dropping the factor $(-1)^{N_c}$ and collecting the contributions from all the sites of the string, we obtain

$$\mathrm{e}^{-N_c\,S_{\mathrm{baryon}}[Z]} = \frac{1}{N_c!}\left\{\frac{\alpha_0}{\alpha_1}\binom{N_c+N_f-1}{N_c}\right\}^{-T}\sum_{\{a\}}\sum_{\sigma\in\mathfrak{S}_{N_c}}\prod_i G_{a_{\sigma(i)}a_i}\,\mathrm{e}^{-N_c\,S_{\mathrm{vacuum}}[Z]}, \quad (7.19)$$

The extra term coming from the static baryon depends on the values of the field Z on its world line and enters the partition function through the matrix

$$G := \tilde{N}(^{1}/_{2}\,\hat{0})M(1\hat{0})^{-1}\tilde{N}(^{3}/_{2}\,\hat{0})\cdots M((T-1)\hat{0})^{-1}\tilde{N}((T-^{1}/_{2})\hat{0})M(T\hat{0})^{-1}. \quad (7.20)$$

This product of matrices runs over all sites n on the closed baryon world line (it is expressed as a "quark propagator" along that line).

Let us take a closer look at the contributions from the sum over permutations $\sigma\in\mathfrak{S}_{N_c}$. Each permutation σ can be uniquely decomposed into a product of independent cycles. Denoting by $c_l(\sigma)$ the number of cycles of length l in this decomposition, the contribution from σ to the partition function can be written as

$$\sum_{\{a_i\}}\prod_{i=1}^{N_c}G_{a_i a_{\sigma(i)}} = \prod_{l=1}^{N_c}(\mathrm{Tr}\,G^l)^{c_l(\sigma)}. \quad (7.21)$$

The permutation group $\mathfrak{S}_{N_c}$ may be partitioned into disjoint classes with respect to conjugation (σ,σ' are said to be conjugate to each other iff there exists a permutation τ such that $\sigma'=\tau^{-1}\sigma\tau$). Two permutations σ and σ' are in the same conjugacy class iff they have the same cycle structure, i. e. $c_l(\sigma)=c_l(\sigma')$ for all l. This allows to rewrite the sum over σ as a sum over the conjugacy classes $\hat{\sigma}\in\hat{\mathfrak{S}}_{N_c}$, taking into account the cardinality $\mathcal{N}(\hat{\sigma})$ of each class. The result is

$$\sum_{\sigma\in\mathfrak{S}_{N_c}}\sum_{\{a_i\}}\prod_{i=1}^{N_c}G_{a_i a_{\sigma(i)}} = \sum_{\hat{\sigma}\in\hat{\mathfrak{S}}_{N_c}}\mathcal{N}(\hat{\sigma})\prod_{l=1}^{N_c}\left(\mathrm{Tr}\,G^l\right)^{c_l(\hat{\sigma})}. \quad (7.22)$$

Each class $\hat{\sigma}$ is uniquely specified by the sequence $\{c_l\}$, or equivalently by a Young diagram. The weight factor $\mathcal{N}(\hat{\sigma})$ is the number of different permutations which can be obtained by distributing N_c numbers in the Young corresponding to $\hat{\sigma}$. Dividing the number of all permutations by the number of (in the sense of the last sentence) equivalent permutations, we obtain

$$\mathcal{N}(\hat{\sigma}) = \frac{N_c!}{\displaystyle\prod_{l=1}^{N_c}l^{c_l(\hat{\sigma})}c_l(\hat{\sigma})!}. \quad (7.23)$$

The main result of this section is the effective action of the static baryon,

$$S_{\text{baryon}}[Z] = S_{\text{vacuum}}[Z] + S_{N_c}(G[Z]) + C_{N_c,N_f} \qquad (7.24)$$

which consists of the following contributions: of the sea term (7.9), the string term [2]

$$S_{N_c}(G) := - \sum_{\hat{\sigma} \in \hat{\mathfrak{S}}_{N_c}} \mathcal{N}(\hat{\sigma}) \prod_{l=1}^{N_c} \left(\text{Tr } G^l \right)^{c_l(\hat{\sigma})} \qquad (7.25)$$

and the constant

$$C_{N_c,N_f} := \frac{1}{N_c} \left(T \ln \left(\frac{\alpha_0}{\alpha_1} \binom{N_c + N_f - 1}{N_c} \right) - \ln N_c! \right). \qquad (7.26)$$

The N_c- and N_f-dependent constant needs to be taken into account when one wants to compare the energy of the static baryon with the energy of the vacuum. Thus it makes contribution to the baryon mass which was worked out in [10].

For the lowest numbers of colors the explicit expressions for the string term are

$$\begin{aligned} S_1(G) &= - \ln \text{Tr } G \\ S_2(G) &= -\frac{1}{2} \ln \left((\text{Tr } G)^2 + \text{Tr } G^2 \right) \\ S_3(G) &= -\frac{1}{3} \ln \left((\text{Tr } G)^3 + 3 \, \text{Tr } G^2 \, \text{Tr } G + 2 \, \text{Tr } G^3 \right). \end{aligned} \qquad (7.27)$$

In the following section, we look for the saddle-point configurations of the effective action $S_{\text{baryon}}[Z]$.

7.3   Saddle Point Equations

In Chapter 6 we have used the saddle point approximation to evaluate the partition function $Z_{\text{vacuum}} = \exp(-N_c S_{\text{vacuum}}[Z])$. In the mesonic sector, where N_c appears explicitly as a factor in front of the action, the saddle-point approximation is fully justified in the large-N_c limit. The situation is less transparent for the effective action of the static baryon (7.24). However, if the matrix G is proportional to the unity matrix $G = g \, I_{N_s N_f}$ with scalar field g, the static-baryon action simplifies to

$$S_{\text{baryon}}[Z] = S_{\text{vacuum}}[Z] - \ln g + \text{const.} \qquad (7.28)$$

Therefore the large-N_c saddle-point expansion of $Z_{\text{baryon}} = \exp(-N_c S_{\text{baryon}}[Z])$ is rigorously justified if the partition function is restricted to these configurations. However, we will use it to approximate the full integral.

The effective action $S_{\text{baryon}}[Z]$ for the static-baryon sector (7.24) contains a string term $S_{N_c}(G[Z])$ in addition to the sea term $S_{\text{vacuum}}[Z]$. While the sea term depends on each of the matrices $Z(n + \hat{\mu}/2)$, the string term involves only those matrices Z and $Z^\dagger$ that are situated in the near vicinity of the string. More precisely, what enters into the baryon world

---

[2]The string term can be written in an elegant way, $S_{N_c}(G) = -\frac{1}{N_c} \left(\frac{d}{d\lambda} \right)^{N_c} \Big|_0 \text{Det}^{-1}(1 - \lambda G)$ which follows from the expansion $\text{Det}(1 - \lambda G)^{-1} = \sum_{N_c=0}^{\infty} \frac{\lambda^{N_c}}{N_c!} e^{-N_c S_{N_c}(G)}$.

line propagator, G, are the matrices $N((t + \frac{1}{2})\hat{0})$ and $M(t\hat{0})$. Of these, the former depend only on Z and $Z^\dagger$ along the string, whereas the latter also involve the matrices $Z(t\hat{0} + \hat{\mu}/2)$ and $Z^\dagger(t\hat{0} - \hat{\mu}/2)$ on the spacelike links next to the string. Taking into account this structure of the baryon action, we have to distinguish three different cases deriving the saddle point equations: variation with respect to the Z-field on the string, variation with respect to the Z-field on the spacelike links next to the string and variation with respect to the Z-field on all other links.

Starting with the last case, the only contribution to the variation of the static baryon action comes from the sea term $S_{\text{vacuum}}[Z]$. Making use of expression (7.9) for this term, we obtain

$$
\begin{aligned}
\delta S_{\text{baryon}}/\delta Z(n + \hat{\mu}/2) &= Z^\dagger(n + \hat{\mu}/2)N^{-1}(n + \hat{\mu}/2) - M^{-1}(n)\gamma_\mu = 0, \\
\delta S_{\text{baryon}}/\delta Z^\dagger(n - \hat{\mu}/2) &= N^{-1}(n - \hat{\mu}/2)Z(n - \hat{\mu}/2) - M^{-1}(n)\gamma_\mu = 0
\end{aligned}
\tag{7.29}
$$

and recover the vacuum saddle point equations (6.16).

Turning to the two other cases we additionally have to take into account the string term $S_{N_c}[Z]$. For sake of simplicity, we restrict ourselves to a low number of colors, $N_c = 1, 2, 3$. The most general variation of the expressions (7.27) yields

$$
\begin{aligned}
\delta S_1 &= -\frac{\text{Tr}(\delta G)}{\text{Tr } G} \\
\delta S_2 &= -\frac{\text{Tr}(G\delta G) + \text{Tr}(G)\,\text{Tr}(\delta G)}{\text{Tr}(G^2) + (\text{Tr } G)^2}, \\
\delta S_3 &= -\frac{(\text{Tr } G)^2\,\text{Tr}\,\delta G + 2\,\text{Tr } G\,\text{Tr}(G\delta G) + \text{Tr}(G^2)\,\text{Tr}(\delta G) + 2\,\text{Tr}(G^2\delta G)}{(\text{Tr } G)^3 + 3\,\text{Tr } G\,\text{Tr}(G^2) + 2\,\text{Tr } G^3}.
\end{aligned}
\tag{7.30}
$$

In the next step, we work out how the various traces of powers of G respond to variations of the matrices Z and $Z^\dagger$ entering the definition of G. For instance, variations of $Z(t\hat{0} + \hat{\mu}/2)$ with $\mu \neq 0$ affect only the matrix $M(t\hat{0})$, whereas varying $Z((t + \frac{1}{2})\hat{0})$ affects both $M(t\hat{0})$ and $N((t + \frac{1}{2})\hat{0})$. These computations are simplified by the use of cyclicity properties: given any decomposition $G = G_1 G_2$, we may replace G in the static–baryon action by the matrix $G' = G_2 G_1$, as G always appears under a trace.

We provide detailed calculations for the variation with respect to $\zeta := Z((t + \frac{1}{2})\hat{0})$. The modified factors of G in this case are $M(t\hat{0})^{-1}$ and $N((t + \frac{1}{2})\hat{0})$, and the modified matrix G reads

$$
G + \delta G = \cdots M(t\hat{0})^{-1}\gamma_0 \left\{ 1 + \delta\zeta \left(-M(t\hat{0})^{-1}\gamma_0 + \zeta^\dagger N((t + \tfrac{1}{2})\hat{0})^{-1} \right) \right\} \gamma_0 \tilde{N}((t + \tfrac{1}{2})\hat{0}) \cdots . \tag{7.31}
$$

It is now natural to conjugate $G + \delta G$ into

$$
S\,(G + \delta G)\,S^{-1} = G' + \delta\zeta \left(-M(t\hat{0})^{-1}\gamma_0 + \zeta^\dagger N((t + \tfrac{1}{2})\hat{0})^{-1} \right) G', \tag{7.32}
$$

where $G' := \gamma_0 \tilde{N}((t + \frac{1}{2})\hat{0})M((t + 1)\hat{0})^{-1} \cdots \tilde{N}((t - \frac{1}{2})\hat{0})M(t\hat{0})^{-1}\gamma_0$.

The variation with respect to $\zeta'^\dagger := Z^\dagger((t - \frac{1}{2})\hat{0})$ can be worked out in the same way. In this case it is convenient to conjugate $G + \delta G$ into

$$
T\,(G + \delta G)\,T^{-1} = G'' + G'' \left(N^{-1}((t - \tfrac{1}{2})\hat{0})\zeta' - M^{-1}(n)\gamma_0 \right)\delta\zeta'^\dagger, \tag{7.33}
$$

where $G'' = M(t\hat{0})^{-1}\tilde{N}((t + \frac{1}{2})\hat{0}) \cdots M((t - 1)\hat{0})^{-1}\tilde{N}((t - \frac{1}{2})\hat{0})$.

As result of the variations, we obtain the following saddle point equations: for all sites n on the string,

$$
\begin{aligned}
\left(Z^\dagger(n + \hat{0}/2)N^{-1}(n + \hat{0}/2) - M^{-1}(n)\gamma_0 \right) \left(I_{N_s N_f} + F_{N_c}(G') \right) &= 0, \\
\left(N^{-1}(n - \hat{0}/2)Z(n - \hat{0}/2) - M^{-1}(n)\gamma_0 \right) \left(I_{N_s N_f} + F_{N_c}(G'') \right) &= 0,
\end{aligned}
\tag{7.34}
$$

where matrix-valued function $F_{N_c}(G)$ is given by

$$
\begin{aligned}
F_1(G) &:= -\frac{G}{\operatorname{Tr}(G)}, \\
F_2(G) &:= -\frac{G^2 + G\operatorname{Tr} G}{\operatorname{Tr}(G^2) + (\operatorname{Tr} G)^2}, \\
F_3(G) &:= -\frac{G(\operatorname{Tr} G)^2 + 2G^2\operatorname{Tr} G + G\operatorname{Tr}(G^2) + 2G^3}{(\operatorname{Tr} G)^3 + 3\operatorname{Tr} G\operatorname{Tr}(G^2) + 2\operatorname{Tr}(G^3)}.
\end{aligned}
\tag{7.35}
$$

The saddle point equations for the timelike links on the string and away from it differ only by the factor $I_{N_s N_f} + F_{N_c}(G')$ or $I_{N_s N_f} + F_{N_c}(G'')$. We conclude that the saddle point equations for the timelike links equivalent to the vacuum saddle point equations, iff the matrices $I_{N_s N_f} + F_{N_c}(G')$ and $I_{N_s N_f} + F_{N_c}(G'')$ are non-singular.

In the last step consider the variation of the static baryon action with respect to the Z-field on the spacelike links next to the baryon. This components of the Z-field enter the string term of the action only through $M(n)$. In analogy to the variations done before, we obtain the following saddle point equations: for all sites n on the string and for all spacelike directions $\mu = 1, .., d-1$,

$$
\begin{aligned}
Z^\dagger(n + \hat{\mu}/2)N^{-1}(n + \hat{\mu}/2) - M^{-1}(n)\gamma_\mu \left(I_{N_s N_f} + F_{N_c}(G') \right) &= 0, \\
N^{-1}(n - \hat{\mu}/2)Z(n - \hat{\mu}/2) - M^{-1}(n)\gamma_\mu \left(I_{N_s N_f} + F_{N_c}(G'') \right) &= 0.
\end{aligned}
\tag{7.36}
$$

where the matrices G', G'' are the same as before. To get an idea of the matrix $I_{N_s N_f} + F_{N_c}(G)$, we compute in the vacuum configuration $Z(n + \hat{\mu}/2) = Z(n + \hat{\mu}/2) = x\gamma_\mu$. In this case all $M(n)$ and all $N(n + \hat{\mu})$ are scalar matrices. Then, assuming that the extension of the lattice in time direction T is an *even* number of sites, the propagator is also a scalar matrix $G = G' = G'' = gI_{N_f}$. (Otherwise, if the lattice extended over an odd number of lattice sites, the propagator would be proportional to γ_0.) For any scalar matrix gI_{N_f} the factor that occurs in the saddle point equations (7.34) and (7.36) takes the value

$$
I_{N_s N_f} + F_{N_c}(g I_{N_s N_f}) = (1 - 1/N_f) I_{N_s N_f}.
\tag{7.37}
$$

This equation holds for all number of colors N_c. Note that in the limit $N_f \to \infty$ the static baryon saddle point equations for a scalar configuration reduce to the vacuum saddle point equations.

Let us consider a configuration of the Z-field that yields scalar propagators G' and G''. In that case the saddle point equations obtained by variation on the spacelike links next to the string (7.36) represent the only obstruction that prevents the vacuum configuration from being also a saddle point in the static-baryon sector.

7.4 Solutions of the Saddle Point Equations

Let us consider the first set of saddle point equations (7.36) which are identical to the vacuum saddle point equations. Recall from Chapter 6 that treating $Z^\dagger$ as the Hermitian conjugate of Z leads to a homogeneous configuration, where the fields are either constant over the whole lattice for the bhc lattice (cf. Eq. (6.56)) or staggered over 2^d sublattices for the d-dimensional hc lattice (cf. Eq. (6.52)). Such a homogeneous configuration *cannot* satisfy the last set of saddle point equations (7.36) (as long as $G' = G'' \neq 0$). In Ref. [33] it was shown that the homogeneous configuration is the unique solution of the saddle point equations, for a theory of spinless fermions similar to our one. We expect that the uniqueness of solution carries over to the theory we are considering. It seems that there are no solutions of the static baryon saddle point equations, as long as we treat $Z^\dagger$ as the Hermitian conjugate of Z. To get a solution describing a static baryon, we need to relax this Hermiticity relation.

By its construction via the color–flavor transformation, the integrand $\exp(-S_{\text{baryon}}[Z]$ is to be viewed primarily as a function of the *real* variables $\{(Z_{ab}+Z_{ba}^\dagger)(n+\hat{\mu}/2), \mathrm{i}(Z_{ab}-Z_{ba}^\dagger)(n+\hat{\mu}/2)\}$, the total number of which is $D := 2N_s^2 N_f^2 d\mathcal{N}$. (Recall the $\mathcal{N}$ denotes the number of lattice sites and $d\mathcal{N}$ is the number of lattice links.) If this function does not have a saddle point on $\mathbb{R}^D$, one can try to analytically continue it into $\mathbb{C}^D$, where a *complex saddle point* may exist. For such a saddle point configuration there must exist at least one link $(n + \hat{\mu}/2)$ where the matrix $Z^\dagger$ differs from the Hermitian conjugate of Z. If a complex saddle point is not "too far" from the original contour of integration, it contributes to the vacuum-sector partition function, upon deforming the contour of integration so as to reach that point.

7.4.1 Linearized Saddle Point Equations

We want the baryon to be a localized object, in the sense that a baryonic saddle point configuration should differ significantly from a vacuum configuration only in some neighborhood of the baryon world line. The baryonic saddle point configuration should converge to the vacuum configuration for large distances from the baryon. Therefore it makes sense to linearize the saddle point equations around the vacuum configuration: In doing so one obtains the behavior of the Z-field far away from the baryon.

In Chapter 6, considering the meson sector, we have linearized the saddle point equation around the saddle point and studied its solutions. Under consideration of the Hermiticity relation of Z and $Z^\dagger$ we have found a saddle manifold in the chiral limit. This solution does not satisfy the set of saddle point equations for the static baryon which are corresponding to the spacelike links next to the string (7.36). In order to obtain a solution of the whole system of saddle point equations in the static-baryon sector, we drop the Hermiticity condition. Recall that the linearized saddle point equations can be written as a set of equations (6.31) for the sum variable $\Sigma(n)$ which is placed on the sites of the lattice:

$$\sum_\mu \left(\gamma_\mu \Sigma_{ab}(n - \hat{\mu})\gamma_\mu + 2\Sigma_{ab}(n) + \gamma_\mu \Sigma_{ab}(n + \hat{\mu})\gamma_\mu \right) + M_{ab}\Sigma_{ab}(n) = 0, \qquad (7.38)$$

for all $a, b = 1, ..., N_f$. This system of equations is decoupled with respect the flavor degrees of freedom. The constant M is a matrix in flavor space that depends on the quarks masses and is given by expression (6.33). The field Σ is in one-to-one correspondence to the linearized versions ζ, ζ of the original fields $Z, Z^\dagger$.

Lets us consider the pseudoscalar ansatz

$$\Sigma_{ab}(n) = \Gamma_5 \otimes \sigma_{ab}(n), \tag{7.39}$$

where Γ_5 is proportional to the product of all γ-matrices, see Eq. (D.5), and $\sigma(n)$ is an arbitrary $N_f \times N_f$ matrix. For this ansatz the linearized saddle point equations (7.38) read

$$\sum_\mu \left(\sigma_{ab}(n + \hat{\mu}) - 2\sigma_{ab}(n) + \sigma_{ab}(n - \hat{\mu}) \right) = \mathcal{M}_{ab}\, \sigma_{ab}(n). \tag{7.40}$$

Assuming that the fluctuations of the field $\Sigma(n)$ on scales of order of lattice constant are small, we may convert this difference equation into a differential equation

$$\Delta \sigma_{ab} = M_{ab}^2\, \sigma_{ab}. \tag{7.41}$$

Here Δ denotes the d-dimensional Laplace operator and the mass M_{ab} on r. h. s. is defined by

$$M_{ab} := \sqrt{\mathcal{M}_{ab}}\, \frac{1}{a}. \tag{7.42}$$

Eq. (7.41) is the Klein-Gordon equation for a particle σ_{ab} of mass M_{ab} in d-dimensional Euclidean space-time. Making use of (6.33) the mass can be expanded in powers of the quark masses (which are assumed to be small compared with the scale a of the theory),

$$M_{ab}^2 = 2 \frac{d-1}{\sqrt{2d-1}} \frac{1}{a}(m_a + m_b) + O(m^2). \tag{7.43}$$

Recall that in linear approximation, the field $\Sigma(n)$ is in one-to-one correspondence to the field $Z(n)$. The latter is a collective, colorless field which was introduced by the color-flavor transformation and has replaced the quark and gauge degrees of freedom. We have worked out the strong coupling approximation of lattice QCD with an cutoff a in order to reach the low energy regime. It is tempting to identify σ with fluctuations of a mesonic background around the static baryon.

To the lowest order in m square of the mass of σ_{ab} is proportional to the sum of two quark masses $(m_a + m_b)$. Similar formulas are know from the second order of chiral perturbation theory (cf. Chapter 6), for example $M_\pi^2 = B(m_u + m_d)$. The pions are the lightest mesons; one can expect that they give the main contribution to the fluctuations of the mesonic background and set their mass into correspondence to the mass of σ_{ud}, $M_\pi = M_{ud}$. As a consequence the prefactor in front of the quark masses in (7.43) must be equal to the constant $B = 1480$ MeV. To check the consistency of this considerations, we calculate the lattice constant from this correspondence,

$$a_{\Delta,\mathrm{hc}} = (653\ \mathrm{MeV})^{-1} \tag{7.44}$$

The result is of the same order as the estimates in Chapter 6, but the agreement is bad. However, the Klein-Gordon equation (7.41) fits perfectly an old idea in connection with the nuclear forces. Yukawa suggested in 1935 that the nuclear forces are mediated by exchange of a massive particle between the nucleons [51]. The exchange particle should satisfy the Klein-Gordon equation, which is the quantized version of the formula for the energy of a relativistic particle,

$$(\Box + M^2)\sigma(\vec{x}, t) = 0 \tag{7.45}$$

On the one hand side the Klein-Gordon equation has plain wave solutions describing propagating particles of mass M. On the other hand side it has a static solution

$$\sigma(\vec{x}, t) = C \frac{\mathrm{e}^{-M|\vec{x}|}}{4\pi|\vec{x}|}. \tag{7.46}$$

This solution is singular at the origin and decreases exponentially. According to Yukawa this solution is identified with the field of the exchange particle around the nucleon. As a consequence the range l of the exchange particle is set into correspondence to its mass, $l = 1/M$. In that way Yukawa concluded from the short range of the nuclear force that the corresponding exchange particle should be massive. The typical range $l \approx 1$ fm of the nuclear force also allows to estimate the mass of the corresponding exchange particle, $M \approx 200$ MeV. By way of contrast the exchange particle of the long range electromagnetic interaction, the photon, is massless.

7.4.2 Static Baryons in $1 + 1$ dimensions

In this subsection we consider the simplest nontrivial case for solutions of the static baryon saddle point equations, the two-dimensional Euclidean square lattice ($d = 2$). Assuming a static configuration, the field depends only on an integer n which labels the distance from the baryon in units of the lattice constant. We consider a pseudoscalar and a scalar ansatz for the field $\Sigma(n)$,

$$\Sigma_{ab}(n) = \Gamma_5 \otimes \sigma_{ab}(n), \tag{7.47}$$
$$\Sigma'_{ab}(n) = I_{N_s} \otimes \sigma'_{ab}(n), \tag{7.48}$$

where $\sigma(n)$ and $\sigma'(n)$ are matrices in flavor space. The corresponding saddle point equations (7.38) read

$$\sigma_{ab}(n+1) - (2 + M_{ab})\sigma_{ab}(n) + \sigma_{ab}(n-1) = 0 \tag{7.49}$$
$$\sigma'_{ab}(n+1) + (6 + M_{ab})\sigma'_{ab}(n) + \sigma'_{ab}(n-1) = 0 \tag{7.50}$$

As a benefit of the low number of dimensions, we can exactly solve these difference equations. In the chiral limit, where $M_{ab} = 0$, the solutions are given by

$$\sigma(n) = C^{(1)}n + C^{(0)}, \tag{7.51}$$
$$\sigma'(n) = C'(-1)^n \, \mathrm{e}^{-\lambda' n}. \tag{7.52}$$

The parameters $C^{(1)}$, $C^{(0)}$, C' are arbitrary complex $N_f \times N_f$ matrices, while λ' is determinated by the equation $\mathrm{e}^{-\lambda'} - 6 + \mathrm{e}^{\lambda'} = 0$. This condition has the solutions

$$\lambda' = \pm \operatorname{arcosh} 3 = \pm \ln(3 + 2\sqrt{2}). \tag{7.53}$$

Thus, we have found two linear independent solutions for each of the saddle point equations (7.49). They span the full solution space of these difference equations. Away from the chiral limit, we obtain the following solutions of the saddle point equations,

$$\sigma_{ab}(n) = C_{ab} \, \mathrm{e}^{-\lambda_{ab} n}, \qquad \text{with } \lambda_{ab} = \pm \operatorname{arcosh}(1 + M_{ab}/2) \tag{7.54}$$
$$\sigma'_{ab}(n) = C'_{ab}(-1)^n \, \mathrm{e}^{-\lambda'_{ab} n}, \qquad \text{with } \lambda'_{ab} = \pm \operatorname{arcosh}(3 + M_{ab}/2). \tag{7.55}$$

Thus, giving the quarks a mass m leads to corrections of order ma in the exponents.

We expect the static baryon configuration to converge to the vacuum configuration, when the distance from the baryon is large: $\sigma_{ab}(x), \sigma'_{ab} \to 0$ for $x \to \pm\infty$. We conclude that only the exponential decreasing solutions are physical and the parameters mentioned above have to satisfy $C^{(1)} = C^{(0)} = 0$, $\lambda_{ab} > 0$, $\lambda'_{ab} > 0$. Note that for the pseudoscalar ansatz (7.47) a static baryon configuration exists only away from the chiral limit.

To get the expression for the meson field Z, corresponding to the two approximate solutions of the saddle point equations, we make use of equations (6.30) and (6.25). For the pseudoscalar solution we obtain

$$
\begin{aligned}
Z_{ab}(n + \hat{0}/2) &= \gamma_0 X_{ab} + \gamma_1 C_{ab} A_{ab} e^{-\lambda_{ab} n}, \\
Z_{ab}(n + \hat{1}/2) &= \gamma_1 X_{ab} + \gamma_0 C_{ab} B_{ab} e^{-\lambda_{ab} n}, \\
Z^\dagger_{ab}(n + \hat{0}/2) &= \gamma_0 X_{ab} + \gamma_1 C_{ab} A_{ab} e^{-\lambda_{ab} n}, \\
Z^\dagger_{ab}(n + \hat{1}/2) &= \gamma_1 X_{ab} - \gamma_0 C_{ab} \tilde{B}_{ab} e^{-\lambda_{ab} n},
\end{aligned}
\tag{7.56}
$$

where we have introduced the constants

$$
A_{ab} := \frac{x_a x_b}{1 + x_a x_b}, \quad B_{ab} := \frac{x_a x_b \left(x_a x_b - e^{-\lambda_{ab}}\right)}{(1 + x_a x_b)(1 - x_a x_b)}, \quad \tilde{B}_{ab} := \frac{x_a x_b \left(1 - x_a x_b e^{-\lambda_{ab}}\right)}{(1 + x_a x_b)(1 - x_a x_b)}.
\tag{7.57}
$$

We have absorbed a factor i into the constants C_{ab} that comes from $\Gamma_5 = i\gamma_0\gamma_1$. For the scalar solution we obtain

$$
\begin{aligned}
Z'_{ab}(n + \hat{0}/2) &= \gamma_0 \left(X_{ab} - C'_{ab} A'_{ab} (-1)^n e^{-\lambda'_{ab} n} \right), \\
Z'_{ab}(n + \hat{1}/2) &= \gamma_1 \left(X_{ab} - C'_{ab} B'_{ab} (-1)^n e^{-\lambda'_{ab} n} \right), \\
Z'^\dagger_{ab}(n + \hat{0}/2) &= \gamma_0 \left(X_{ab} - C'_{ab} A'_{ab} (-1)^n e^{-\lambda'_{ab} n} \right), \\
Z'^\dagger_{ab}(n + \hat{1}/2) &= \gamma_1 \left(X_{ab} - C'_{ab} \tilde{B}'_{ab} (-1)^n e^{-\lambda'_{ab} n} \right)
\end{aligned}
\tag{7.58}
$$

with the constants

$$
A'_{ab} := \frac{x_a x_b}{1 - x_a x_b}, \quad B'_{ab} := \frac{x_a x_b \left(x_a x_b - e^{-\lambda'_{ab}}\right)}{(1 + x_a x_b)(1 - x_a x_b)}, \quad \tilde{B}'_{ab} := \frac{x_a x_b \left(1 - x_a x_b e^{-\lambda'_{ab}}\right)}{(1 + x_a x_b)(1 - x_a x_b)}.
\tag{7.59}
$$

The scalar solution can be specialized to the chiral limit: In this case, the rate of decrease is given by (7.53) and the above constants have the values

$$
A'_{ab}(\mathcal{M} = 0) = \frac{1}{2}, \quad B'_{ab}(\mathcal{M} = 0) = \frac{3}{4}\sqrt{2} - 1, \quad \tilde{B}'_{ab}(\mathcal{M} = 0) = \frac{1}{4}\sqrt{2}.
\tag{7.60}
$$

To summarize, we have obtained a pseudoscalar and scalar (7.58) solution of the saddle point equations, printed in formula (7.56) and (7.58). Calculated from small perturbations of the vacuum configuration, they describe the behavior of the meson field Z far away from the baryon. Both solutions contain an undetermined constant, called C'_{ab} or C'_{ab}, respectively. This constant has to be fixed by matching to solutions of the saddle point equations at and near the baryon.

7.4.3   Discussion

There is an important difference between the pseudoscalar and the scalar lattice solution, when one approaches the chiral limit. For small quarks masses, the rate of decrease of the two solutions behaves like

$$\lambda_{ab} = \operatorname{arcosh}(1 + M_{ab}/2) \approx \sqrt{M_{ab}} + O(M_{ab}), \tag{7.61}$$

$$\lambda'_{ab} = \operatorname{arcosh}(3 + M_{ab}/2) \approx \operatorname{arcosh} 3 + \frac{1}{8}\sqrt{2}M_{ab} + O(M_{ab}^2). \tag{7.62}$$

For the scalar solution there is a fixed rate, $\lambda'_{ab}(\mathcal{M} = 0) = \operatorname{arcosh} 3$, which acquires a small correction from the quark masses. By the way of contrast, for pseudoscalar solution – in the lowest order of the quark masses – the rate is proportional the square root of the quark masses.

The decrease rate (7.61) of the pseudoscalar lattice solution near the chiral limit is in agreement with the predictions from the continuum method developed in Section 7.4.1: Static solutions of the $(1 + 1)$-dimensional Euclidean Klein-Gordon equation (7.41) decrease with a rate $\sqrt{M_{ab}}$. Non-vanishing quarks masses are essential for the short-range character of the pseudoscalar solution: In the chiral limit there are no static, spherical symmetric short range solutions of the Klein-Gordon equation. The solutions increase linearly or logarithmically $(d = 1 + 1,\ d = 1 + 2)$ with the distance of the baryon or decrease $(d = 1 + 3)$ inverse proportionally to the distance from the baryon.

So far we have only considered the saddle point equation far away from the baryon which are identical with the vacuum saddle point equations. In Ref. [10] we have searched for solutions of the full system of saddle point equations (7.29), (7.34), and (7.36). In this paper we have considered a simplified model without spin degrees of freedom. All field were chosen scalar, that is proportional to the unity matrix in flavor space I_{N_f}. We have employed Newton's method and obtained a numerical solution of the saddle point equations on a finite lattice. We have confirmed this solution by its agreement with an approximate analytical solution.

The structure of the solution of the full system of saddle point equations is exactly the one predicted by the *scalar* asymptotic solution Eq. (7.58). We compare the above predictions for the constants appearing in (7.58) with the corresponding values from [10]. We consider the chiral limit. There is agreement about the rate of decrease which takes the value $\lambda = \ln(3 + 2\sqrt{2})$. Further all the amplitudes $(C'_{ab}A'_{ab},\ C'_{ab}B'_{ab},\ C'_{ab}\tilde{B}'_{ab})$ can be brought in agreement, if the value of the undeterminated constant C'_{ab} is fixed at

$$C'_{ab} \approx 0.92\,\delta_{ab}. \tag{7.63}$$

Further investigations are necessary to decide, if and how the asymptotic *pseudoscalar* solution can be continued to a solution of the full saddle point equations.

Chapter 8

Conclusion

In this dissertation we have stated different versions of the color-flavor transformation. They were applied to two limits of quantum chromodynamics, to gluodynamics and to the strong coupling limit. In both cases we have worked with the formulation as a lattice gauge theory that was introduced by Wilson.

Color-Flavor Transformation

In Chapter 2 we reviewed some mathematical structures which are fundamental for the color-flavor transformation. We consider a quantum multi-particle system characterized by the canonical (anti-)commutation relations of a set of creation and annihilation operators. The color-flavor transformation is based on a structure in the corresponding group of canonical transformations: For each "dual pair" of subgroups in the group of canonical transformations there is an associated version of the color-flavor transformation. In view of quantum chromodynamics we have considered the case of a dual pair consisting of two unitary groups. In our applications we connect one of the two groups with the color degrees of freedom, and the other one with the flavor and spin degrees of freedom. We have worked out this version of the color-flavor transformation for a pure bosonic and a pure fermionic system.
However, the gauge group of chromodynamics is not a unitary group, but the special unitary group SU(3). For that reason we have generalized the color-flavor transformation to the special unitary groups (Chapter 3). The result is a sum over contributions coming from disconnected sectors characterized by the baryon number $B = -N_f, ..., N_f$. The contribution belonging to $B = 0$ is exactly the result of the color-flavor transformation for the unitary group.

Gluodynamics

Wilson lattice action for gluodynamics, which is a sum over traces of plaquette terms, does not allow an application of the color-flavor transformation. We have replaced it by a physically equivalent action that can be generated by an integral over bosonic auxiliary fields (Chapter 5). In this formalism each plaquette has its own auxiliary field that additionally carries inner degrees of freedom. It turned out that the dimension of the inner space N_a and the number of colors N_c should satisfy $N_a \geq N_c$. We have argued that only if this condition is fulfilled one is in the position to recover gluodynamics as continuum limit of the lattice theory.

Two-dimensional gauge theories have a much simpler structure than higher dimensional ones. In this case the calculation of the general Wilson loop can be reduced to the calculation of the one-plaquette Wilson loop. As a well-known conclusion, two-dimensional gauge theories are confining in the sense that the quark-antiquark potential is linear in the distance. Concerning our model, we have obtained an exact result for the string tension, in the case $N_a \leq N_c$. From Monte Carlo simulations also some results for $N_a > N_c$ are available.

In four space-time dimensions gauge theories are more complicated. In that case we hope to make progress with the help of the color-flavor transformation. However, there are sometimes problems with the convergence of the bosonic color-flavor transformation: In our case it works only for $N_c \geq 2N_a$. This is in conflict with the physical condition discussed above. One can overcome the convergence problems by passing to a supersymmetric theory. In this framework the influence of the fermions can be kept small by making them heavy. We have applied the color-flavor transformation to the supersymmetric theory and obtained a result, which can be the starting point for further investigations.

QCD in strong coupling approximation

In the strong coupling limit, the color-flavor transformation can be applied immediately: The partition function decomposes into a sum of contributions belonging to the different possible distributions of baryons on the lattice. We have studied the contribution of the vacuum and static baryon configuration in detail.

In the vacuum action the color degrees of freedom are completely decoupled. After integration over the quark fields they enter the partition function only through a factor N_c (number of colors) in front of the action. This structure organizes the theory in a perturbation series with $1/N_c$ as parameter. The lowest order of this large-N_c expansion is the saddle point approximation. We have solved the corresponding saddle point equations and obtained a single saddle point for non-vanishing quarks masses and a saddle point manifold in the chiral limit. We have shown that this result – ignoring the strong U(1)-problem – perfectly fits the common picture of the spontaneous chiral symmetry breaking. The saddle point manifold parameterizes the Goldstone mode which are set into correspondence with the lightest pseudoscalar mesons.

Chiral perturbation theory is a modern approach to describe the low energy properties of the lightest mesons. It is an expansion of a low energy effective Lagrangian in the momenta p of the low energy modes and the quark masses m, where p^2 and m are treated as small, but p^2/m may take any value. Starting from the color-flavor transformed action of strong coupling QCD, we have obtained such an expansion up to order $O(p^4)$. The resulting theory has the lattice constant as a free parameter. From comparison with experimentally know quantities, we have estimated the lattice constant in three different ways: By the chiral Lagrangian of order $O(p^2)$ the lattice constant is connected with the pion decay constant on the one hand side and with the pion mass on the other hand side. A third, independent estimate can be obtained from the experimental value of the chiral condensate. All methods yield a lattice constant of order $a \approx 1$ fm, but are in bad agreement. In order $O(p^4)$ we were able to recover some of the terms of the general chiral Lagrangian introduced by Gasser and Leutwyler. However, the values for the coefficients in front of different terms are in bad agreement with their phenomenological values. Recall that from the beginning we were working in the strong coupling limit. This is a crude approximation to QCD. It is useful to understand the structure of the low energy sector, but one would not expect to obtain quantitative good results.

Let us summarize our results for the static baryon configuration: In this case effective action comes out as a sum of a "sea term"and a "string term". The sea term is identical to the vacuum effective action, while the string term depends only on fields in the nearest vicinity of the static baryon. Therefore, the saddle point equations for the static baryon are identical with the vacuum saddle point equations unless we are in the vicinity of the static baryon. To get information about the background field far away from the baryon we have linearized the vacuum saddle point equations around the vacuum solution. We have shown that – in the pseudoscalar sector of the theory – the deviations from the vacuum obey a Klein-Gordon equation with a mass proportional to the square root of the quark masses. In this way one obtains a Yukawa potential, with a range inverse proportional to the square root of the quark masses.

Bodycentered Hypercubic Lattice

In this dissertation we have discussed field theories on different kind of lattices. They are thought as a model for the isotropic and continuous physical world. Remarkably, in four space-time dimension there is a lattice with a larger symmetry group than the one of the hypercubic (hc) lattice. The bodycentered hypercubic (bhc) lattice is obtained by introducing an extra site at the middle of each elementary cell of the hypercubic lattice. In four dimensions this construction enlarges the symmetry group by a factor three. The following aspects of our work are connected with the bhc lattice: The chiral symmetry group of lattice QCD is exactly the continuum chiral group for the bhc lattice, while it is enlarged for the hc lattice. This was proven in Chapter 4. Another fact was helpful in the context of the calculation of the fourth order of chiral perturbation theory: For tensors up to fourth order, the bhc lattice symmetry implies continuum symmetry. This is only true for tensors up to order two in case of the hc lattice.

Appendix A

The Hermitian Symmetric Spaces
$\mathrm{U}(2N)/\mathrm{U}(N) \times \mathrm{U}(N)$ **and** $\mathrm{U}(N,N)/\mathrm{U}(N) \times \mathrm{U}(N)$

In this appendix we examine the analysis and geometry of the coset spaces $\mathrm{U}(2N)/\mathrm{U}(N) \times \mathrm{U}(N)$ and $\mathrm{U}(N,N)/\mathrm{U}(N) \times \mathrm{U}(N)$. In the first section we introduce suitable coordinates Z, $Z^\dagger$ on it and embed it into the vector space of complex $2N \times 2N$ matrices, see for example [53]. In the second section we describe the generalized Gaussian decomposition for the complex extension of respective $\mathrm{U}(2N)$ and $\mathrm{U}(N,N)$. The coset space considered is an hermitian symmetric space [22], in particular it is carrying an invariant Kähler metric. Section A.3 is devoted to the Kähler metric and the corresponding symplectic structure, which can be used to construct the invariant measure of the coset space. In the last section we introduce canonical coordinates for the symplectic structure, in which the invariant measure appears as a flat measure. These coordinates were already considered in [44].

A.1   Coordinates and Embedding into a Matrix Space

We consider the unitary group $G = \mathrm{U}(2N)$ and the pseudounitary group $G = \mathrm{U}(N,N)$ in $2N$ dimensions. Making use of a decomposition of a complex $2N \times 2N$ matrix into $N \times N$ blocks, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the group G is defined by the condition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \text{ where } \varepsilon = \left\{ \begin{array}{ll} -1 & \text{pseudounitary case,} \\ +1 & \text{unitary case.} \end{array} \right. \tag{A.1}$$

The subgroup $H = \mathrm{U}(N) \times \mathrm{U}(N)$ of G consists of the matrices h, which commute with a matrix Σ_z,

$$h = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \text{ and } \Sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.2}$$

The next step is to pass from the group G to the right coset space G/H and to introduce coordinates on it. We begin with the observation that the invariants for the right action of $h \in H$ on G are

$$Z := BD^{-1} \text{ and } \tilde{Z} := CA^{-1}. \tag{A.3}$$

They provide coordinates for the right coset space. The condition A.1 in terms of the four

blocks takes the form

$$A^\dagger = (A - BD^{-1}C)^{-1}$$
$$B^\dagger = \varepsilon(B - AC^{-1}D)^{-1}$$
$$C^\dagger = \varepsilon(C - DB^{-1}A)^{-1}$$
$$D^\dagger = (D - CA^{-1}B)^{-1}. \tag{A.4}$$

Therefore Z and $\tilde{Z}$ are connected via

$$\tilde{Z} = CA^{-1} = -\varepsilon Z^\dagger. \tag{A.5}$$

In the case $\varepsilon = -1$ we get further

$$ZZ^\dagger = BD^{-1}CA^{-1} = 1 - (AA^\dagger)^{-1}. \tag{A.6}$$

Thus $ZZ^\dagger$ is restricted to have eigenvalues less than unity in the case $G = \mathrm{U}(N, N)$, while there is no such restriction in the case $G = \mathrm{U}(2N)$.

The coordinates $(Z, \tilde{Z})$ of the coset space transform under the left action of G on G/H ($g \cdot g'H := gg'H$ for $g \in G$) like

$$g \cdot Z = (AZ + B)(CZ + D)^{-1} \tag{A.7}$$
$$g \cdot \tilde{Z} = (C + D\tilde{Z})(A + B\tilde{Z})^{-1}. \tag{A.8}$$

To embed G/H into the space of complex $2N \times 2N$ matrices we look at the action of G on itself by conjugation: $g' \mapsto c_g(g') := gg'g^{-1}$. We make use of the general fact, that the orbit of a point is isomorphic to the group factored by the isotropy group of that point, in our case $c_G(\Sigma_z) \cong G/G_{\Sigma_z}$. The isotropy group of Σ_z is $G_{\Sigma_z} = H$, and we get

$$\{Q = g\Sigma_z g^{-1} \mid g \in \mathrm{U}(2N)\} \cong G/H. \tag{A.9}$$

For later use we note that

$$
\begin{aligned}
Q &= g\Sigma_z g^{-1} \\
&= \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}^{-1} \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & Z \\ -\varepsilon Z^\dagger & 1 \end{pmatrix} \Sigma_z \begin{pmatrix} 1 & Z \\ -\varepsilon Z^\dagger & 1 \end{pmatrix}^{-1}.
\end{aligned} \tag{A.10}
$$

A.2 Generalized Gaussian Decomposition

The generalized Gaussian decomposition of a unitary or pseudounitary matrix leads to factors, which lie in the complex extensions $G^{\mathbb{C}}$ of the unitary or pseudounitary group G. The complex extension $G^{\mathbb{C}}$ is obtained from the real form G of the group by exponenting the complex extension of the Lie algebra of G. Passing to the complex extension $G^{\mathbb{C}} = \mathrm{GL}(2N, \mathbb{C})$ of G, every complex matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with an invertible D can be decomposed into

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}, \tag{A.11}$$

where

$$Z = BD^{-1}, \quad Y = D^{-1}C, \quad F = A - BD^{-1}C. \tag{A.12}$$

Thus the determinant can be calculated as

$$\mathrm{Det} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Det}(A - BD^{-1}C)\,\mathrm{Det}\,D. \tag{A.13}$$

Restricting again to the unitary (pseudounitary) group G we get from (A.4) the relations

$$Y = D^{-1}\tilde{Z}A = -\varepsilon D^{-1}Z^\dagger A \tag{A.14}$$

and we define

$$\begin{aligned} V &:= AA^\dagger = (1 + \varepsilon ZZ^\dagger)^{-1}, \\ W &:= DD^\dagger = (1 + \varepsilon Z^\dagger Z)^{-1}. \end{aligned} \tag{A.15}$$

A.3 $\mathrm{U}(2N)$-invariant Geometry

In section A.1 we have embedded the coset space G/H into the vector space of complex $2N \times 2N$ matrices. The canonical metric on the latter space is

$$\tilde{g} = \frac{1}{2}\,\mathrm{Tr}\,dZ^\dagger \odot dZ, \tag{A.16}$$

where $dZ^\dagger \odot dZ = \frac{1}{2}(dZ^\dagger \otimes dZ + dZ \otimes dZ^\dagger)$ denotes the symmetric tensor product. More explicitly the metric is given by

$$\tilde{g}(X,Y) = \mathrm{Re}\,\mathrm{Tr}(X^\dagger Y) = \sum_{i,j=1}^{2N} (\mathrm{Re}\,X_{ij}\,\mathrm{Re}\,Y_{ij} + \mathrm{Im}\,X_{ij}\,\mathrm{Im}\,Y_{ij}). \tag{A.17}$$

Pulling back $\tilde{g}$ to G/H with the help of the embedding Q we get a metric for the coset space (note: $Q^\dagger = Q$):

$$g = Q^*\tilde{g} = \mathrm{Tr}\,dQ \odot dQ. \tag{A.18}$$

The metric $\tilde{g}$ of the matrix space is bi-invariant under the action of G. It is easy to see that g inherits the invariance property from $\tilde{g}$: From $Q \circ l_g = r_{g^{-1}} \circ l_g \circ Q$ we conclude

$$l_g^*g = l_g^*Q^*\tilde{g} = (Q \circ l_g)^*\tilde{g} = (r_{g^{-1}} \circ l_g \circ Q)^*\tilde{g} = Q^*l_g^*r_{g^{-1}}^*\tilde{g} = Q^*\tilde{g} = g, \tag{A.19}$$

i.e. g is left-invariant under the action of G. To express g in the coordinates Z and $Z^\dagger$, we note that

$$dQ = dg\,\Sigma_z g^{-1} - g\Sigma_z g^{-1}dg\,g^{-1}, \tag{A.20}$$

where

$$g = \begin{pmatrix} 1 & Z \\ -\varepsilon Z^\dagger & 1 \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} V & -ZW \\ \varepsilon Z^\dagger V & W \end{pmatrix} \tag{A.21}$$

with V, W as in (A.15). Further calculations lead to

$$g^{-1}dg = \begin{pmatrix} \varepsilon ZW\,dZ^\dagger & V\,dZ \\ \varepsilon W\,dZ^\dagger & \varepsilon Z^\dagger V\,dZ \end{pmatrix}, \qquad \Sigma_z g^{-1}dg = \begin{pmatrix} \varepsilon ZW\,dZ^\dagger & V\,dZ \\ \varepsilon W\,dZ^\dagger & -\varepsilon Z^\dagger V\,dZ \end{pmatrix} \tag{A.22}$$

and the result

$$
\begin{aligned}
g &= \mathrm{Tr}(dQ \odot dQ) = \mathrm{Tr}(\Sigma_z g^{-1} dg \odot \Sigma_z g^{-1} dg) - \mathrm{Tr}(g^{-1} dg \odot g^{-1} dg) \\
&= 4\varepsilon\, \mathrm{Tr}\, V\, dZ \odot W\, dZ^\dagger.
\end{aligned}
\tag{A.23}
$$

In terms of the parametrization Z, $Z^\dagger$ of the the coset space G/H, the complex structure is the multiplication with i. We may choose $\omega(X,Y) = \frac{\varepsilon}{4i} g(iX,Y)$ as an invariant symplectic structure on the coset space, that is

$$
\omega = \mathrm{Tr}\, V\, dZ \wedge W\, dZ^\dagger.
\tag{A.24}
$$

From the symplectic structure we can construct the invariant volume element for G/H,

$$
\begin{aligned}
\Omega = \omega^N &= \mathrm{Det}\, V^N\, \mathrm{Det}\, W^N \bigwedge_{i,j=1}^{2N} dZ_{ij} \wedge d\bar{Z}_{ij} \\
&= \mathrm{Det}(1 \pm ZZ^\dagger)^{-2N} \bigwedge_{i,j=1}^{2N} dZ_{ij} \wedge d\bar{Z}_{ij},
\end{aligned}
\tag{A.25}
$$

where we have to put the plus (minus) sign for the coset space $\mathrm{U}(2N)/\mathrm{U}(N)\times\mathrm{U}(N)$ $(\mathrm{U}(N,N)/\mathrm{U}(N)\times\mathrm{U}(N))$ of the unitary (pseudounitary) group.

A.4   Canonical coordinates for the Symplectic Structure

The purpose of this section is to introduce coordinates in which the volume form $\Omega = \omega^N$ takes the standard form

$$
d\Omega = \bigwedge_{i,j=1}^{2N} d\zeta_{ij} \wedge d\bar{\zeta}_{ij}.
\tag{A.26}
$$

The Darboux theorem guarantees the existence of coordinates, in which the symplectic structure takes the canonical form

$$
\omega = \mathrm{Tr}\, d\zeta \wedge d\zeta^\dagger
\tag{A.27}
$$

A possible choice for the canonical coordinates was given in [44],

$$
\begin{aligned}
\zeta &= V^{1/2} Z = (1 \pm ZZ^\dagger)^{1/2} Z, \\
\zeta^\dagger &= Z^\dagger V^{1/2} = Z^\dagger (1 \pm ZZ^\dagger)^{1/2},
\end{aligned}
\tag{A.28}
$$

with plus (minus) sign for the coset space of the unitary (pseudounitary) group. Due to the non-commutativity (for $N > 1$) the calculations are cumbersome and we will give them explicitly. For example there is no simply expression for the differential of the square-root $dV^{1/2}$, but only for

$$
dV^{1/2} V^{1/2} + V^{1/2} dV^{1/2} = dV = -V(ZdZ^\dagger + dZZ^\dagger)V.
\tag{A.29}
$$

Making use of (A.29) and the relation $\mathrm{Tr}\,\alpha \wedge \alpha = 0$ for an arbitrary matrix-valued 1-form, we get

$$
\begin{aligned}
\mathrm{Tr}\,d\zeta \wedge d\zeta^\dagger &= \mathrm{Tr}\,V^{1/2}dV^{1/2} \wedge ZdZ^\dagger - \mathrm{Tr}\,dV^{1/2}ZZ^\dagger dV^{1/2} \wedge dV^{1/2} \\
&\quad + \mathrm{Tr}\,VdZ \wedge dZ^\dagger - \mathrm{Tr}\,dV^{1/2}V^{1/2} \wedge dZZ^\dagger \\
&= \mathrm{Tr}\,V^{1/2}dV^{1/2} \wedge (ZdZ^\dagger + dZZ^\dagger) - \mathrm{Tr}\,dV^{1/2}ZZ^\dagger dV^{1/2} \wedge dV^{1/2} \\
&\quad + \mathrm{Tr}\,VdZ \wedge dZ^\dagger - \mathrm{Tr}\,VZdZ^\dagger V \wedge dZ^\dagger \\
&= \mathrm{Tr}(V^{-1} - ZZ^\dagger)dV^{1/2} \wedge dV^{1/2} + \mathrm{Tr}\,VdZ \wedge (1 - Z^\dagger V Z)dZ^\dagger \\
&= \mathrm{Tr}\,VdZ \wedge WdZ^\dagger = \omega.
\end{aligned}
\tag{A.30}
$$

Thus the volume element Ω takes the canonical from (A.26), when it is expressed in the coordinates ζ, $\zeta^\dagger$.

Appendix B

Can a representation of the Lie Algebra of $\mathrm{GL}(N, \mathbb{C})$ pressed down to the Group?

The purpose of this appendix is to give a criterion for the existence of a group representation $T : \mathrm{GL}(N, \mathbb{C}) \to \mathrm{GL}(\mathcal{H})$ with the property $T \circ \exp = \exp \circ t$, when a representation of the corresponding Lie algebra $t : \mathfrak{gl}(N, \mathbb{C}) \to \mathfrak{gl}(\mathcal{H})$ is given. For simply connected groups the existence of such a representation is guaranteed by a general theorem of Lie group theory [43]. But the general linear group is not simply connected. In fact the path $c : [0,1] \to \mathrm{GL}(N, \mathbb{C}), t \mapsto I - E^{11} + e^{2\pi i t} E^{11}$ cannot be deformed into a single point. Here I denotes the identity matrix and E^{11} the matrix with one at the place $(1,1)$ and zero elsewhere. Thus $\mathrm{GL}(N, \mathbb{C})$ has a nontrivial topological structure, which we examine in sect. B.1. In doing so we make use of some basic facts from homotopy theory [8].

In sect. B.2 a necessary and sufficient condition for the existence of the group representation is given: $\exp(2\pi i\, t_{E^{11}}) = \mathrm{id}_{\mathcal{H}}$. Note that there is a connection between the occurrence of E^{11} in the condition and the path mentioned above, which generates the nontrivial fundamental group of $\mathrm{GL}(N, \mathbb{C})$. The criterion is quite easy to check for concrete realizations of $t : \mathfrak{gl}(N, \mathbb{C}) \to \mathfrak{gl}(\mathcal{H})$. In cases where $t_{E^{11}}$ is diagonalizable, it is fulfilled iff the eigenvalues of $t_{E^{11}}$ are integers.

B.1 Fundamental Groups

The fundamental groups of the special linear group and the general linear group over the complex numbers in N dimensions are

$$\pi_1(\mathrm{SL}(N, \mathbb{C})) = 0, \tag{B.1}$$

$$\pi_1(\mathrm{GL}(N, \mathbb{C})) = \mathbb{Z}. \tag{B.2}$$

To prove statement (B.1) we let $\mathrm{SL}(N+1, \mathbb{C})$ act on $\mathbb{C}^{N+1}$. For a group action, the coset space with respect to the isotropy group of a fixed vector is diffeomorphic to its orbit. In our case we obtain $\mathrm{SL}(N+1, \mathbb{C})/\mathrm{SL}(N, \mathbb{C}) \cong \mathbb{C}^{N+1} \setminus \{0\}$. From the fibration $\mathrm{SL}(N, \mathbb{C}) \hookrightarrow \mathrm{SL}(N+1, \mathbb{C}) \to \mathrm{SL}(N+1, \mathbb{C})/\mathrm{SL}(N, \mathbb{C})$ we pass to the corresponding exact sequence of homotopy groups

$$\pi_1(\mathrm{SL}(N, \mathbb{C})) \to \pi_1(\mathrm{SL}(N+1, \mathbb{C})) \to \pi_1(\mathbb{C}^{N+1} \setminus \{0\}). \tag{B.3}$$

But $\pi_1(\mathbb{C}^{N+1} \setminus \{0\}) = 0$ for $N \geq 1$ and from $\pi_1(\mathrm{SL}(1, \mathbb{C})) = 0$ we conclude inductively $\pi_1(\mathrm{SL}(N, \mathbb{C})) = 0$ for all N.

To prove (B.2) we consider the determinant homomorphism $\mathrm{Det} : \mathrm{GL}(N, \mathbb{C}) \to \mathbb{C}^*$. Its kernel is $\mathrm{SL}(N, \mathbb{C})$ and so there is a fibration $\mathrm{SL}(N, \mathbb{C}) \hookrightarrow \mathrm{GL}(N, \mathbb{C}) \to \mathbb{C}^*$. The corresponding exact sequence of homotopy groups reads

$$\pi_1(\mathrm{SL}(N, \mathbb{C})) \to \pi_1(\mathrm{GL}(N, \mathbb{C})) \to \pi_1(S^1) \to \pi_0(\mathrm{SL}(N, \mathbb{C})) \tag{B.4}$$

and from $\pi_1(\mathrm{SL}(N, \mathbb{C})) = 0$ and $\pi_0(\mathrm{SL}(N, \mathbb{C})) = 0$ we conclude $\pi_1(\mathrm{GL}(N, \mathbb{C})) = \pi_1(S^1) = \mathbb{Z}$. The universal covering group of $\mathrm{GL}(N, \mathbb{C})$ can be modeled as

$$\widetilde{\mathrm{GL}}(N, \mathbb{C}) = \mathrm{SL}(N, \mathbb{C}) \times \mathbb{C}, \tag{B.5}$$

where the group multiplication is given by $(A, b) \circ (A', b') = (AA', b + b')$ and the covering homomorphism reads

$$\pi : \widetilde{\mathrm{GL}}(N, \mathbb{C}) \to \mathrm{GL}(N, \mathbb{C}) \tag{B.6}$$
$$(A, b) \mapsto e^b A.$$

In fact, the differential at the unity element

$$\pi_*(X, y) = \frac{d}{dt}\Big|_0 \pi(\exp(tX), ty) = \frac{d}{dt}\Big|_0 \left(e^{ty} \exp(tX)\right) = X + yI \tag{B.7}$$

is a Lie algebra isomorphism and therefore $\pi : \widetilde{\mathrm{GL}}(N, \mathbb{C}) \to \mathrm{GL}(N, \mathbb{C})$ is a covering. Its kernel

$$\ker \pi = \left\{ \left(e^{-2\pi i k/N} I, 2\pi i(k/N + l)\right) \,\Big|\, k, l \in \mathbb{Z} \right\} \tag{B.8}$$

is isomorphic to $\mathbb{Z}$ in agreement with the fundamental group of $\mathrm{GL}(N, \mathbb{C})$, which we calculated before.

B.2   Representations

Proposition: A representation of the Lie algebra of the general linear group over the complex numbers $t : \mathfrak{gl}(N, \mathbb{C}) \to \mathfrak{gl}(\mathcal{H})$ can be pressed down to a representation of the group, iff $\exp(2\pi i\, t_{E^{11}}) = \mathrm{id}_{\mathcal{H}}$, where

$$E^{11} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}. \tag{B.9}$$

To prove the proposition we consider the commutative diagram

$$\begin{array}{ccc}
\mathfrak{gl}(N, \mathbb{C}) & \xrightarrow{\ t\ } & \mathfrak{gl}(\mathcal{H}) \\
{\scriptstyle \widetilde{\exp}}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\
\widetilde{\mathrm{GL}}(N, \mathbb{C}) & \xrightarrow{\ \widetilde{T}\ } & \mathrm{GL}(\mathcal{H}) \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{GL}(N, \mathbb{C}) & \xdashrightarrow{\ T\ } & \mathrm{GL}(\mathcal{H}).
\end{array} \tag{B.10}$$

The first step is to press down the representation of the Lie algebra to a representation $\widetilde{T} : \widetilde{\mathrm{Gl}}(N, \mathbb{C}) \to \mathcal{H}$ of the simply connected covering group $\widetilde{\mathrm{GL}}(N, \mathbb{C})$. This is guaranteed by a general theorem of Lie group theory [43]. The representations of the Lie algebra and the Lie group are linked by $\widetilde{T} \circ \widetilde{\exp} = \exp \circ t$. The exponential map of $\widetilde{\mathrm{Gl}}(N, \mathbb{C})$ can be expressed in terms of the decomposition of an element $M \in \mathfrak{gl}(n, \mathbb{C})$ into $M = X + yI$, where X is the traceless part of M and $y = \frac{1}{N} \operatorname{Tr} X$, namely

$$\widetilde{\exp} : \mathfrak{gl}(N, \mathbb{C}) \to \widetilde{\mathrm{GL}}(N, \mathbb{C}) \tag{B.11}$$
$$X + yI \mapsto (\exp X, y).$$

The second step is to press down the representation of the covering group to a representation of the general linear group $T : \mathrm{GL}(N, \mathbb{C}) \to \mathrm{GL}(\mathcal{H})$. This is possible, iff the former representation maps the kernel of the covering map to the identity of the Hilbert space. To get a more explicit criterion, note that, by decomposing E^{11} into its traceless and its trace part,

$$\widetilde{\exp} \left(2\pi i E^{11} \right) = \left(e^{-2\pi i/N} I, 2\pi i/N \right). \tag{B.12}$$

Because the matrix on the r.h.s. belongs to the kernel of the covering map, the "only if" part follows immediately from $\exp(2\pi i\, t_{E^{11}}) = \widetilde{T} \left(\widetilde{\exp}(2\pi i\, E^{11}) \right)$.

Turning to the "if" part we start with the observation, that the matrix E^{11} in equation (B.12) can be replaced by each of the matrices E^{aa} ($a = 2, ..., N$), which have one at the place (a, a) and zero elsewhere. In doing so the r.h.s. of (B.12) does not change and we conclude

$$\widetilde{\exp} \left(2\pi i E^{aa} \right) = \widetilde{\exp} \left(2\pi i E^{11} \right). \tag{B.13}$$

Thus, by making use of the decomposition of the identity into pairwise commuting matrices $I = \sum_{a=1}^{N} E^{aa}$, we conclude

$$\widetilde{\exp} \left(2\pi i I \right) = \widetilde{\exp} \left(2\pi i N E^{11} \right). \tag{B.14}$$

Finally we consider an arbitrary element of the kernel of the covering map and let the representation of the covering group act on it:

$$\begin{aligned}
\widetilde{T} \left(e^{-2\pi i\, k/N} I, 2\pi i(k/N + l) \right) &= \widetilde{T} \left(\widetilde{\exp} \left(2\pi i \left(k E^{11} + lI \right) \right) \right) \\
&= \widetilde{T} \left(\widetilde{\exp} \left(2\pi i\, k E^{11} \right) \circ \widetilde{\exp} \left(2\pi i\, lI \right) \right) \\
&= \widetilde{T} \left(\widetilde{\exp} \left(2\pi i\, k E^{11} \right) \circ \widetilde{\exp} \left(2\pi i\, lN E^{11} \right) \right) \\
&= \left(\exp \left(2\pi i\, t_{E^{11}} \right) \right)^k \circ \left(\exp \left(2\pi i\, t_{E^{11}} \right) \right)^{lN} \\
&= \mathrm{id}_{\mathcal{H}}.
\end{aligned} \tag{B.15}$$

In the last step the condition on the representation of the Lie algebra entered the argumentation. Indeed, the kernel elements are mapped to the identity on the Hilbert space and we conclude that representation of the covering group can be pressed down to $\mathrm{GL}(N, \mathbb{C})$, q.e.d.

Appendix C

From the Supersymmetric Color-Flavor Transformation to Bosonic and Fermionic Versions

The first version of the color flavor transformation [53] was proven in a supersymmetric setting. It transforms an integral over $\mathrm{U}(N)$ into an integral over Efetov's σ-model space with unitary symmetry,

$$
\begin{aligned}
&\int_{\mathrm{U}(N)} dU \, \exp\left(\bar{\psi}^i_{+a} U^{ij} \psi^j_{+a} + \bar{\psi}^j_{+a} \bar{U}^{ij} \psi^i_{-b}\right) \\
&= \int D(Z, \tilde{Z}) \, \mathrm{SDet}(1 - \tilde{Z}Z)^N \, \exp\left(\bar{\psi}^i_{+a} Z_{ab} \psi^i_{-b} + \bar{\psi}^j_{-b} \tilde{Z}_{ba} \psi^j_{+a}\right).
\end{aligned}
\tag{C.1}
$$

The new integration variables are $2N_f \times 2N_f$ supermatrices with boson-fermion decomposition

$$
Z = \begin{pmatrix} Z_{BB} & Z_{BF} \\ Z_{FB} & Z_{FF} \end{pmatrix}, \qquad \tilde{Z} = \begin{pmatrix} \tilde{Z}_{BB} & \tilde{Z}_{BF} \\ \tilde{Z}_{FB} & \tilde{Z}_{FF} \end{pmatrix}.
\tag{C.2}
$$

The integration measure is the flat Berezin measure $D(Z, \tilde{Z})$ and the integration domain is given by

$$
\tilde{Z}_{BB} = Z^\dagger_{BB} \ \text{ and } \ 1 - \tilde{Z}_{BB} Z_{BB} > 0, \qquad \tilde{Z}_{FF} = -Z^\dagger_{FF}.
\tag{C.3}
$$

The purpose of this appendix is to rederive the bosonic and the fermionic color-flavor transformation (which were derived in chapter 2) as special cases of the supersymmetric formula (C.1). For simplicity we restrict ourselves to the case $N_f = 1$, while $N := N_c$ is arbitrary.

Thus the fields Z_{BB} and Z_{FF} are just numbers and evaluation of the superdeterminant yields

$$
\begin{aligned}
\mathrm{SDet}\,(1 - \tilde{Z}Z) = \mathrm{SDet}\,&\begin{pmatrix} 1 - \tilde{Z}_{BB}Z_{BB} - \tilde{Z}_{BF}Z_{FB} & \tilde{Z}_{BB}Z_{BF} + \tilde{Z}_{BF}Z_{FF} \\ \tilde{Z}_{FB}Z_{BB} + \tilde{Z}_{FF}Z_{FB} & 1 - \tilde{Z}_{FB}Z_{BF} - \tilde{Z}_{FF}Z_{FF} \end{pmatrix} \\
= &\bigg(\left(1 - \tilde{Z}_{BB}Z_{BB} - \tilde{Z}_{BF}Z_{FB} \right) \left(1 - \tilde{Z}_{FB}Z_{BF} - \tilde{Z}_{FF}Z_{FF} \right) \\
& - \left(\tilde{Z}_{BB}Z_{BF} + \tilde{Z}_{BF}Z_{FF} \right) \left(\tilde{Z}_{FB}Z_{BB} + \tilde{Z}_{FF}Z_{FB} \right) \bigg) \bigg/ \left(1 - \tilde{Z}_{FB}Z_{BF} - \tilde{Z}_{FF}Z_{FF} \right)^2 \\
= &\bigg(\left(1 - \tilde{Z}_{BB}Z_{BB} \right) \left(1 - \tilde{Z}_{FF}Z_{FF} \right) - \tilde{Z}_{BF}Z_{FB} - \left(1 - 2\tilde{Z}_{BB}Z_{BB} \right) \tilde{Z}_{FB}Z_{BF} \\
& - \tilde{Z}_{BB}\tilde{Z}_{FF}Z_{BF}Z_{FB} - Z_{FF}Z_{BB}\tilde{Z}_{BF}\tilde{Z}_{FB} + \tilde{Z}_{BF}Z_{FB}\tilde{Z}_{FB}Z_{BF} \bigg) \bigg/ \left(1 - \tilde{Z}_{FB}Z_{BF} - \tilde{Z}_{FF}Z_{FF} \right)^2 .
\end{aligned}
\tag{C.4}
$$

To remove the Grassmann variables from the denominator of the last expression, we manipulate it as follows,

$$
\begin{aligned}
\left(1 - \tilde{Z}_{FB}Z_{BF} - \tilde{Z}_{FF}Z_{FF} \right)^{-2} &= \left(1 - \tilde{Z}_{FF}Z_{FF} \right)^{-2} \left(1 - \frac{\tilde{Z}_{FB}Z_{BF}}{1 - \tilde{Z}_{FF}Z_{FF}} \right)^{-2} \\
&= \left(1 - \tilde{Z}_{FF}Z_{FF} \right)^{-2} \left(1 + 2\frac{\tilde{Z}_{FB}Z_{BF}}{1 - \tilde{Z}_{FF}Z_{FF}} \right) .
\end{aligned}
\tag{C.5}
$$

Now we are in position to rewrite the superdeterminant as a power series in the Grassmann variables,

$$
\begin{aligned}
\mathrm{SDet}\,(1 - \tilde{Z}Z) =& \frac{1 - \tilde{Z}_{BB}Z_{BB}}{1 - \tilde{Z}_{FF}Z_{FF}} \\
& + \frac{-\tilde{Z}_{BF}Z_{FB} + \tilde{Z}_{FB}Z_{BF} - \tilde{Z}_{BB}\tilde{Z}_{FF}Z_{BF}Z_{FB} - Z_{FF}Z_{BB}\tilde{Z}_{BF}\tilde{Z}_{FB}}{\left(1 - \tilde{Z}_{FF}Z_{FF} \right)^2} \\
& - \frac{\left(1 + \tilde{Z}_{FF}Z_{FF} \right) \tilde{Z}_{BF}Z_{FB}\tilde{Z}_{FB}Z_{BF}}{\left(1 - \tilde{Z}_{FF}Z_{FF} \right)^3} .
\end{aligned}
\tag{C.6}
$$

We start to calculate the integral over the Berezin measure,

$$
D(Z, \tilde{Z}) = \mathrm{const.} \times dZ_{BB}\,d\tilde{Z}_{BB}\,dZ_{FF}\,d\tilde{Z}_{FF} \, \frac{\partial^4}{\partial Z_{BF}\tilde{Z}_{BF}Z_{FB}\tilde{Z}_{FB}} ,
\tag{C.7}
$$

by performing the "integration" over the Grassmann variables,

$$
\Psi(Z_{BB}, \tilde{Z}_{BB}, Z_{FF}, \tilde{Z}_{FF}) := \frac{\partial^4}{\partial Z_{BF} \tilde{Z}_{BF} Z_{FB} \tilde{Z}_{BF}} \, \mathrm{SDet}(1 - \tilde{Z}Z)^N
$$

$$
= N \left(\frac{1 - \tilde{Z}_{BB} Z_{BB}}{1 - \tilde{Z}_{FF} Z_{FF}} \right)^{N-1} \frac{1 + \tilde{Z}_{FF} Z_{FF}}{\left(1 - \tilde{Z}_{FF} Z_{FF}\right)^3}
$$

$$
- N(N-1) \left(\frac{1 - \tilde{Z}_{BB} Z_{BB}}{1 - \tilde{Z}_{FF} Z_{FF}} \right)^{N-2} \left(\frac{-1 + \tilde{Z}_{BB} Z_{BB} \tilde{Z}_{FF} Z_{FF}}{\left(1 - \tilde{Z}_{FF} Z_{FF}\right)^4} \right) \tag{C.8}
$$

$$
= \frac{N \left(N - \tilde{Z}_{BB} Z_{BB} + \tilde{Z}_{FF} Z_{FF} - N \tilde{Z}_{BB} Z_{BB} \tilde{Z}_{FF} Z_{FF} \right)}{\left(1 - \tilde{Z}_{BB} Z_{BB}\right)^{2-N} \left(1 - \tilde{Z}_{FF} Z_{FF}\right)^{2+N}}.
$$

To get the color-flavor transformation for the pure bosonic and the pure fermionic case, we have to do the integral over Z_{FF} and Z_{BB}, respectively. These integrals can be done by making use of the identities

$$
\int_{\mathbb{C}} \frac{dZ d\bar{Z}}{(1 + Z\bar{Z})^{2+N}} = \frac{\pi}{N+1} = \int_{|Z|<1} \frac{dZ d\bar{Z}}{(1 - Z\bar{Z})^{2-N}}, \tag{C.9}
$$

which can be derived by passing to polar coordinates in the complex plane. The corresponding integrals with additional $Z\bar{Z}$ in the numerator can be obtained from (C.9) via

$$
\begin{aligned}
\int \frac{Z\bar{Z} \, dZ d\bar{Z}}{(1 + Z\bar{Z})^{2+N}} &= \int \frac{dZ d\bar{Z}}{(1 + Z\bar{Z})^{1+N}} - \int \frac{dZ d\bar{Z}}{(1 + Z\bar{Z})^{2+N}}, \\
\int \frac{Z\bar{Z} \, dZ d\bar{Z}}{(1 - Z\bar{Z})^{2-N}} &= - \int \frac{dZ d\bar{Z}}{(1 - Z\bar{Z})^{1-N}} + \int \frac{dZ d\bar{Z}}{(1 - Z\bar{Z})^{2-N}}.
\end{aligned} \tag{C.10}
$$

Finally we get the results

$$
\int_{\mathbb{C}} dZ_{FF} d\bar{Z}_{FF} \, \Psi(Z_{BB}, \tilde{Z}_{BB}, Z_{FF}, \tilde{Z}_{FF}) = \frac{(1 - 1/N)\,\pi}{\left(1 + Z_{BB} \bar{Z}_{BB}\right)^{2-N}}, \tag{C.11}
$$

$$
\int_{|Z|<1} dZ_{BB} d\bar{Z}_{BB} \, \Psi(Z_{BB}, \tilde{Z}_{BB}, Z_{FF}, \tilde{Z}_{FF}) = \frac{(1 + 1/N)\,\pi}{\left(1 + Z_{FF} \bar{Z}_{FF}\right)^{2+N}}, \tag{C.12}
$$

which are in agreement with the factors in front of the exponentials in equations (2.84) and (2.85).

Appendix D

Clifford Algebras and Dirac operators

The first section of this appendix is devoted to Clifford algebras and γ-matrices. We point out the *chiral structure* of the spinor space over a space-time of even dimensions. Again in an even number of dimensions, we calculate the traces of products of γ-matrices.

The remaining sections deal with the derivation of the correct expression for the Dirac operator, the Dirac action and the partition function in Euclidean space-time. This will be done starting with the formulation in Minkowski space-time, where the physics takes place, and then performing a Wick rotation to switch to Euclidean space-time. The Dirac operator in Minkowski space-time turns out to be *hermitian* with respect to an indefinite product on the space of wave functions, while the Dirac operator in Euclidean space-time comes out *antihermitian* with respect to the usual scalar product on the space of wave functions. To understand the structure of the Dirac operators, it is useful to start with a study of invariant quadratic forms on spinor spaces.

D.1   Clifford Algebra

Let $g_{\mu\nu} = \text{diag}(+1, ..., +1, -1, ..., -1)$ be a metric with r positive and s negative eigenvalues. We consider the $d = r+s$-dimensional Clifford algebra associated to the pseudoeuclidean space $(\mathbb{R}^d, g)$ and its representation on the complex spinor space $\mathbb{C}^{N_s}$ of the dimension $N_s = 2^{[d/2]}$ [1]. The corresponding γ-matrices satisfy the relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}. \tag{D.1}$$

Left and Right Chirality

The *volume element* of the Clifford algebra is defined to be

$$\omega := \gamma_1 \cdot ... \cdot \gamma_d. \tag{D.2}$$

It is elementary to check that the volume element has the properties

$$\omega^2 = (-1)^{d(d-1)/2+s} \tag{D.3}$$

$$\gamma_\mu \omega + (-1)^d \omega \gamma_\mu = 0 \quad \text{for } \mu = 1, ..., d. \tag{D.4}$$

---

[1] $[d/2] =$ greatest integer smaller or equal to $d/2$.

From the last relation, we read of that the volume element in even dimensions anticommutes with the γ-matrices, while it belongs to the center of the Clifford algebra in odd dimensions. Therefore the volume element is proportional to the identity in odd dimensions, if the Clifford representation is irreducible. [2] Some of the following results are based on the anticommutation of the volume element and the γ-matrices; in such cases we have to restrict ourselves to an even number of space-time dimensions.

To discuss the *chiral structure* of the spinor space, we define a Clifford element, which is idempotent and proportional to the volume element,

$$\Gamma_5 := i^{d(d-1)/2}\, \gamma_1 \cdot \ldots \cdot \gamma_d. \tag{D.5}$$

This definition reads for two special cases of a Euclidean space-time in low dimensions

$$\Gamma_5 = i\gamma_1\gamma_2 \qquad\qquad (d=2), \tag{D.6}$$

$$\Gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4 = -\gamma_5 \quad (d=4). \tag{D.7}$$

A Dirac spinor ψ can always be decomposed into a left- and a right-handed part according to

$$\psi = \psi_L + \psi_R \qquad \text{with } \psi_L := P_-\psi, \quad \psi_R := P_+\psi, \tag{D.8}$$

where

$$P_- := \frac{1}{2}(1-\gamma_5), \qquad P_+ := \frac{1}{2}(1+\gamma_5). \tag{D.9}$$

Indeed, P_+ and P_- are orthogonal projectors, which sum up to the identity,

$$P_\pm^2 = P_\pm, \qquad P_+P_- = P_-P_+ = 0, \qquad P_+ + P_- = 1. \tag{D.10}$$

The customary decomposition of the spinor in left- and a right-handed parts works only in *even dimensions*, where the projectors satisfy

$$\gamma_\mu P_\pm = P_\mp \gamma_\mu \qquad \text{for all } \mu. \tag{D.11}$$

In odd dimensions the Spinor space has no chiral structure. In that case, one of the projectors vanishes, as long as the spinor space is irreducible.

Basis of the Clifford Algebra

To label a basis of the Clifford algebra we introduce the sets $m = \{\mu_1, ..., \mu_k\}$ with $k = 0, ...d$, and $\mu_j \in \{1, ..., d\}$ for $j = 1, ..., k$. THe corresponding Clifford elements are given by

$$\gamma_m := \gamma_{\mu_1} \cdot \ldots \cdot \gamma_{\mu_k}, \qquad \text{where } 1 \le \mu_1 < \mu_2 < ... < \mu_d \le d. \tag{D.12}$$

For the empty set this definition (D.12) should be read as $\gamma_\phi := 1$. From the defining relation of the Clifford (D.1) it follows that the set

$$\{\gamma_m\}_m = \{1, \gamma_{\mu_1}, \gamma_{\mu_1}\gamma_{\mu_2}, ..., \gamma_1 \cdot \ldots \cdot \gamma_d\} \tag{D.13}$$

contains a basis of the Clifford algebra. In *even dimensions d* the spinor space has dimension $N_s = 2^{d/2}$ and we conclude, that for even d the 2^d matrices (D.13) form a basis of the complex $N_s \times N_s$ matrices, considered as a complex vector space. In an odd number of dimensions, the set (D.13) still contains a basis of the matrix space, but it is linear dependent.

---

[2]The Clifford algebra itself decomposes into a sum of two subalgebras, iff $s - r \equiv 3$ modulo 4, as it can be seen from the classification of Clifford algebras, see e.g. [9].

Traces of Clifford Elements in Even Dimensions

Finally we will show that in *even dimensions*, the *traces* of the basis elements (D.13) except the unitary matrix 1, vanish. That can be seen by looking at the even and odd basis elements separately: For even k we make use of the anticommutation relations (D.1) and the cyclicity of the trace to get

$$
\begin{aligned}
\mathrm{Tr}(\gamma_{\mu_1} \cdot \ldots \cdot \gamma_{\mu_k}) &= \frac{1}{2}\,\mathrm{Tr}(\gamma_{\mu_1} \cdot \ldots \cdot \gamma_{\mu_k}) - \frac{1}{2}\,\mathrm{Tr}(\gamma_{\mu_2} \cdot \ldots \cdot \gamma_{\mu_k}\gamma_{\mu_1}) \\
&= \frac{1}{2}\,\mathrm{Tr}(\gamma_{\mu_1} \cdot \ldots \cdot \gamma_{\mu_k}) - \frac{1}{2}\,\mathrm{Tr}(\gamma_{\mu_1} \cdot \ldots \cdot \gamma_{\mu_k}) \\
&= 0.
\end{aligned}
\tag{D.14}
$$

For odd k, we may write the basis element as a product of the volume element (D.2) and a matrix α that consists of a product of an odd number of γ-matrices,

$$
\gamma_{\mu_1} \cdot \ldots \cdot \gamma_{\mu_k} = \omega\alpha.
\tag{D.15}
$$

By anticommutation relations (D.4) (n even!) and the cyclicity of the trace, we conclude

$$
\begin{aligned}
\mathrm{Tr}(\gamma_{\mu_1} \cdot \ldots \cdot \gamma_{\mu_k}) &= \mathrm{Tr}(\omega\alpha) \\
&= \frac{1}{2}\,\mathrm{Tr}(\omega\alpha) - \frac{1}{2}\,\mathrm{Tr}(\alpha\omega) \\
&= \frac{1}{2}\,\mathrm{Tr}(\omega\alpha) - \frac{1}{2}\,\mathrm{Tr}(\omega\alpha) \\
&= 0.
\end{aligned}
\tag{D.16}
$$

Combining both cases we have shown that the traces of γ-matrices in even dimensions vanish, as long as they do not cancel each other to the identity matrix.

D.2   Invariant Forms on Spinor Spaces (Euclidean Case)

Let us first consider the case of Euclidean signature $g_{\mu\nu} = \delta_{\mu\nu}$. Choose an arbitrary positive definite hermitian form $(,)$ on $\mathbb{C}^{N_s}$. We will construct an invariant form by averaging over the basis $B = \{\gamma_m\}_m = \{1, \gamma_{\mu_1}, \gamma_{\mu_1}\gamma_{\mu_2}, ..., \gamma_1 \cdot \ldots \cdot \gamma_d\}$ of the Clifford algebra,

$$
\langle v, w \rangle = \frac{1}{\#B} \sum_{\gamma \in B} (\gamma v, \gamma w)
\tag{D.17}
$$

Here $\#B = 2^d$ denotes the number of elements in B. Multiplying the elements of B from the right with a matrix γ_μ, we get a new set $B_\mu = B\gamma_\mu$. Up to signs, B_μ contains the same elements as B. Thus the sum in (D.17) does not change its value, when extended over B_μ instead of B, so that

$$
\langle \gamma_\mu v, \gamma_\mu w \rangle = \langle v, w \rangle.
\tag{D.18}
$$

More generaly, the form is invariant under $\gamma_a = \sum_{\mu=1}^{d} a_\mu \gamma_k$ with $g(a,a) = 1$:

$$
\begin{aligned}
\langle \gamma_a v, \gamma_a w \rangle &= \sum_{\mu,\nu=1}^{d} a_\mu a_\nu \langle \gamma_\mu v, \gamma_\nu w \rangle \\
&= \sum_{\mu=1}^{d} a_\mu^2 \langle \gamma_\mu v, \gamma_\mu w \rangle + \sum_{\mu<\nu} a_\mu a_\nu \left(\langle \gamma_\nu \gamma_\mu v, \gamma_\nu^2 w \rangle + \langle \gamma_\mu \gamma_\nu v, \gamma_\mu^2 w \rangle \right) \\
&= \sum_{\mu=1}^{d} a_\mu^2 \langle v, w \rangle + \sum_{\mu<\nu} a_\mu a_\nu \langle (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) v, w \rangle \\
&= \langle v, w \rangle.
\end{aligned}
\tag{D.19}
$$

To summarize, $\langle , \rangle$ is a positive definite $Pin(n)$-invariant hermitian product on the spinor space. Recall that $Pin(n) = \{ \gamma_{a_1} \cdot ... \cdot \gamma_{a_k} \mid a_1, ..., a_k \in S_n \}$, where $S_n = \{ a \mid g(a,a) = 1 \}$. The *gamma*-matrices are hermitian with respect to that product:

$$
\langle \gamma_a v, w \rangle = \langle v, \gamma_a w \rangle.
\tag{D.20}
$$

Denoting hermitian conjugation with respect to $\langle , \rangle$ with $\dagger$, we may write $\gamma_a^\dagger = \gamma_a$.

D.3　Invariant Form (General Case)

Now we assume a pseudoeuclidean signature with an odd number r of positive and an arbitrary number s of negative eigenvalues. Now there are indices μ with $\gamma_\mu^2 = 1$ and other indices ν with $\gamma_\nu^2 = -1$, and therefore one can no longer drop the sum $\sum_{\mu<\nu}$ in (D.19). The quadratic from needs to be modified to be invariant. Defining $\gamma_t = i^{r(r-1)/2} \gamma_1 ... \gamma_r$, one gets

$$
\gamma_t \gamma_\mu = \gamma_\mu \gamma_t \quad \text{for } \mu \leq r
\tag{D.21}
$$
$$
\gamma_t \gamma_\mu = -\gamma_\mu \gamma_t \text{ for } \mu > r.
\tag{D.22}
$$

A new sesquilinear form, which is *not* positive definite, is introduced by

$$
\langle\langle v, w \rangle\rangle = \langle \gamma_t v, w \rangle.
\tag{D.23}
$$

For $\gamma_a = \sum_{\mu=1}^{d} a_\mu \gamma_\mu$ with an arbitrary $a \in \mathbb{C}^d$ one gets

$$
\begin{aligned}
\langle\langle \gamma_a v, \gamma_a w \rangle\rangle &= \sum_{\mu,\nu=1}^{d} a_\mu a_\nu \langle \gamma_t \gamma_\mu v, \gamma_\nu w \rangle \\
&= \sum_{\mu=1}^{d} a_\mu^2 \langle \gamma_t \gamma_\mu v, \gamma_\mu w \rangle + \sum_{\mu<\nu} a_\mu a_\nu \left(\langle \gamma_\nu \gamma_t \gamma_\mu v, \gamma_\mu^2 w \rangle + \langle \gamma_\mu \gamma_t \gamma_\nu v, \gamma_\nu^2 w \rangle \right) \\
&= \sum_{\mu=1}^{d} a_\mu^2 g_{\mu\mu} \langle \gamma_\mu \gamma_t v, \gamma_\mu w \rangle + \sum_{\mu<\nu} a_\mu a_\nu \left(\langle \gamma_t \gamma_\mu \gamma_\nu v, w \rangle + \langle \gamma_t \gamma_\nu \gamma_\mu v, w \rangle \right) \\
&= g(a,a) \langle\langle v, w \rangle\rangle.
\end{aligned}
\tag{D.24}
$$

To summarize, $\langle\langle,\rangle\rangle$ is a $Pin(r,s)^+$-invariant sesquilinear form on the spinor space. Here $Pin(r,s)^+ = \{\gamma_{a_1} \cdot ... \cdot \gamma_{a_k} \mid a_1, ..., a_k \in S_{r,s}$ and for an even number of $a_\mu : g(a_\mu, a_\mu) = -1\}$, where $S_{r,s} = \{a \mid |g(a,a)| = 1\}$. For $a \in S_{r,s}$

$$\langle\langle \gamma_a v, w \rangle\rangle = \langle\langle v, \gamma_a w \rangle\rangle. \tag{D.25}$$

Further

$$\langle \gamma_\mu v, w \rangle = \pm\langle v, \gamma_\mu w \rangle, \tag{D.26}$$

where the plus sign occurs for $\mu \leq r$ and the minus sign for $\mu > r$.

D.4   Dirac Operator

The following considerations take place in the 4-dimensional Minkowski space-time with signature $(+ - - -)$. Greek indices run from $0, ..., 3$ and Latin indices from $1, ..., 3$. The dimension of the spinor space on which the γ-matrices are acting is $N_s = 4$. Specializing the results of the preceding section to $r = 1$ and $s = 3$, we get $\gamma_t = \gamma_0$ and

$$\gamma_0^\dagger = \gamma_0, \quad \gamma_k^\dagger = -\gamma_k = \gamma_0 \gamma_k \gamma_0. \tag{D.27}$$

Now the representation of the Clifford algebra is chosen in such a way, that $\langle,\rangle$ is the canonical inner product on $\mathbb{C}^4$ ($\langle v, w \rangle = \sum_{\mu=0}^3 v_\mu^* w_\mu$). (This can be done by performing an equivalence transformation with a matrix which maps an orthonormal basis related to $\langle,\rangle$ to an orthonormal basis related to the usual scalar product.) As usual the Dirac conjugation is defined by $\bar{\psi} = \psi^\dagger \gamma_0$. Thus the scalar products of the preceding sections translate to $\langle \phi, \psi \rangle = \phi^* \psi$ and $\langle\langle \phi, \psi \rangle\rangle = \bar{\phi}\psi$. The Dirac operator, defined by

$$\slashed{D} = i\gamma_\mu(\partial^\mu + iA^\mu), \tag{D.28}$$

is *hermitian* with respect to the product

$$\langle\langle \phi, \psi \rangle\rangle_{\mathbb{R}^4} = \int d^4x \, \langle\langle \phi(x), \psi(x) \rangle\rangle = \int d^4x \, \phi^\dagger(x)\gamma_0\psi(x) \tag{D.29}$$

of spinor wave functions. Proof (arrows over partial derivatives mark, whether they act to the left or to the right):

$$\begin{aligned}
\langle\langle \slashed{D}\phi, \psi \rangle\rangle_{\mathbb{R}^4} &= \int d^4x \, (\slashed{D}\phi)^\dagger \gamma_0 \psi \\
&= \int d^4x \, \phi^\dagger (-i\gamma_\mu^\dagger)(\overleftarrow{\partial^\mu} - iA^\mu)\gamma_0\psi \\
&= \int d^4x \, \phi^\dagger \gamma_0(-i\gamma_\mu)(\overleftarrow{\partial^\mu} - iA^\mu)\psi \\
&= \int d^4x \, \phi^\dagger \gamma_0(-i\gamma_\mu)(-\overrightarrow{\partial^\mu} - iA^\mu)\psi \\
&= \int d^4x \, \phi^\dagger \gamma_0 \, \slashed{D}\psi \\
&= \langle\langle \phi, \slashed{D}\psi \rangle\rangle_{\mathbb{R}^4}.
\end{aligned} \tag{D.30}$$

Integration by parts was employed to let the partial derivatives act to the right instead to the left. Action and partition function of the Dirac operator on Minkowski space read

$$Z = \int d\psi d\bar{\psi} \, \exp(iS), \quad S = \int d^4x \, \bar{\psi}(\slashed{D} - m)\psi. \tag{D.31}$$

D.5   Euclidean Dirac Operator

To switch to the Euclidean formulation, we perform a Wick rotation $t \to -i\tau$, $\partial_t \to i\partial_\tau$. Since in Euclidean space, the covering group of the Lorentz group, $Pin(1,3)$, has to be replaced by the covering group of the four-dimensional Euclidean rotations $Pin(4)$, it is convenient to express the Dirac operator in terms of Euclidean γ-matrices. They are given by $\gamma_k^E = -i\gamma_k$, $\gamma_0^E = \gamma_0$ and satisfy

$$\gamma_\mu^E \gamma_\nu^E + \gamma_\nu^E \gamma_\mu^E = 2\delta_{\mu\nu} \tag{D.32}$$

as well as

$$\gamma_\mu^{E\dagger} = \gamma_\mu^E. \tag{D.33}$$

Carrying out the Wick rotation, we have to do the replacement $\slashed{D} \to -\slashed{D}^E$, where

$$\slashed{D}^E = \gamma_\mu^E(\partial^\mu + iA^\mu) \tag{D.34}$$

is the Euclidean Dirac operator. It is *antihermitian* with respect to the Hermitian product

$$\langle \phi, \psi \rangle_{\mathbb{R}^4} = \int d^4x \, \langle \phi(x), \psi(x) \rangle = \int d^4x \, \phi^\dagger(x)\psi(x) \tag{D.35}$$

Proof:

$$\begin{aligned}
\langle \slashed{D}^E \phi, \psi \rangle_{\mathbb{R}^4} &= \int d^4x \, (\slashed{D}^E \phi)^\dagger \psi \\
&= \int d^4x \, \phi^\dagger \gamma_\mu^E (\overleftarrow{\partial^\mu} - iA^\mu)\psi \\
&= \int d^4x \, \phi^\dagger \gamma_\mu^E (-\overrightarrow{\partial^\mu} - iA^\mu)\psi \\
&= -\langle \phi, \slashed{D}^E \psi \rangle_{\mathbb{R}^4}
\end{aligned} \tag{D.36}$$

The Euclidean partition function and the Euclidean action are given by $(d^4x \to -id^4x)$

$$Z^E = \int d\psi d\bar{\psi} \, \exp(-S^E), \quad S^E = \int d^4x \, \bar{\psi}(\slashed{D}^E + m)\psi. \tag{D.37}$$

Note that ψ and $\bar{\psi}$ are Grassmann variables, and therefore independent variables. Thus the replacement $\bar{\psi} \to \psi^\dagger = \bar{\psi}\gamma_0$ can be done by the theorem of substitution for integrals.

Appendix E

Two Infinite Series

In this appendix we calculate the limit of two infinite series. They arise in context with the continuum limit of a lattice gauge theory, which is considered in Chapter 5. The first series can be expressed as derivative of the geometric series,

$$
\begin{aligned}
a_l(\beta) := \sum_{k=l+1}^{\infty} k\beta^k &= \beta \frac{d}{dx}\Big|_\beta \sum_{k=l+1}^{\infty} x^k \\
&= \beta \frac{d}{dx}\Big|_\beta \frac{x^{l+1}}{1-x} \\
&= \frac{\beta^{l+1}}{(1-\beta)^2}(1 + l(1-\beta)).
\end{aligned}
\tag{E.1}
$$

The second series can be expressed as an integral over the geometric series,

$$
\begin{aligned}
b_l(\beta) := \sum_{k=l+1}^{\infty} \frac{1}{k}\beta^k &= \int_0^\beta dx \sum_{k=l}^{\infty} x^k \\
&= \int_0^\beta dx\, \frac{x^l}{1-x}.
\end{aligned}
\tag{E.2}
$$

In Chapter 5 we have introduced a dependence of the integer index l on parameter β. It is defined by the unequalities

$$
l(\beta) \le \frac{1}{(1-\beta)^{1+\varepsilon}}, \qquad l(\beta) + 1 \ge \frac{1}{(1-\beta)^{1+\varepsilon}},
\tag{E.3}
$$

where ε is a fixed small positive number. To connect the lattice gauge theory with quenched QCD, it turns out to be crucial that

$$
\lim_{\beta \to 1} a_{l(\beta)}(\beta) = 0, \qquad \lim_{\beta \to 1} b_{l(\beta)}(\beta) = 0.
\tag{E.4}
$$

To prove that both sequences converge to zero, we note that they are positive and have the following upper bounds,

$$
a_{l(\beta)}(\beta) \le \frac{\beta^{1/(1-\beta)^{1+\varepsilon}}}{(1-\beta)^2}\left(1 + \frac{1}{(1-\beta)^\varepsilon}\right), \qquad b_{l(\beta)}(\beta) \le \beta \frac{\beta^{l(\beta)}}{1-\beta} \le \frac{\beta^{1/(1-\beta)^{1+\varepsilon}}}{1-\beta}.
\tag{E.5}
$$

To obtain the upper bound for the second sequence we have estimated the integral (E.2) by the length of the integration interval times the maximum of the integrand.

The simplest way to show that the right hand sides of the inequalities (E.5) converge to zero for $\beta \to 1$, is to take the logarithm these expressions:

$$
\begin{aligned}
\ln a_{l(\beta)}(\beta) &\leq \frac{1}{(1-\beta)^{1+\varepsilon}} \ln(1 - (1-\beta)) - (2+\varepsilon)\ln(1-\beta) + \ln(1 + (1-\beta)^{\varepsilon}) \\
&= -(1-\beta)^{-\varepsilon} + O((1-\beta)^{1-\varepsilon}) + O(\ln(1-\beta)) + O((1-\beta)^{\varepsilon}) \\
&\to -\infty.
\end{aligned}
\tag{E.6}
$$

The convergence to minus infinity follows from the fact that the power $(1-\beta)^{-\varepsilon}$ dominates the logarithm $\ln(1-\beta)$ in the limit $\beta \to 1$. For the second sequence we obtain

$$
\begin{aligned}
\ln b_{l(\beta)}(\beta) &\leq \frac{1}{(1-\beta)^{1+\varepsilon}} \ln(1 - (1-\beta)) - \ln(1-\beta) \\
&= -(1-\beta)^{-\varepsilon} + O((1-\beta)^{1-\varepsilon}) + O(\ln(1-\beta)) \\
&\to -\infty.
\end{aligned}
\tag{E.7}
$$

Again we took into account that a logarithm is increasing slower than any power.

Appendix F

Fourier Transform

In this appendix we shortly discuss the Fourier transform and some related formulas. For the omitted proofs we refer to [50] for the continuum version and to [4] for the lattice version of the Fourier transform.

The Continuum Fourier Transform

The Fourier transform of a sufficiently fast decreasing function is defined by

$$f(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} d^d x \, e^{-i(p \cdot x)} f(x). \tag{F.1}$$

One can get back the original function by Fourier's inversion theorem,

$$f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} d^d p \, e^{i(p \cdot x)} f(p). \tag{F.2}$$

Parseval's relation states that the hermitian product between two functions can be calculated in real space and in Fourier space,

$$\int_{\mathbb{R}^d} d^d x \, \bar{f}(x) g(x) = \int_{\mathbb{R}^d} d^d p \, \bar{f}(p) g(p). \tag{F.3}$$

The normalization constant in the definition of the Fourier transform is chosen such, that no additional factors appear in Parseval's relation. The Fourier transform of the first partial derivatives of a function is given by

$$\left(\frac{\partial f}{\partial x_k} \right)(p) = ip_k f(p). \tag{F.4}$$

This formula can be derived from (F.1) by differentiation under the integral followed by a partial integration.

The Lattice Fourier Transform

On a d-dimensional hypercubic lattice with lattice constant a and a length L in lattice sites in each direction we define the Fourier transform as

$$f(k) := L^{-d/2} \sum_n e^{-i(k \cdot n)} f(n). \tag{F.5}$$

We use the convention that the position on the lattice and the wave vectors in Fourier space are labeled by dimensionless variables n and k. The wave vector takes discrete values and is restricted to the first Brillouin zone,

$$k_\mu = \frac{2\pi}{L}l, \quad \text{with } l \in \mathbb{Z} \text{ and } k_\mu \in]-\pi, \pi].\tag{F.6}$$

Again the original function can be recovered by Fourier's inversion theorem,

$$f(n) = L^{-d/2} \sum_k e^{i(k\cdot n)} f(k).\tag{F.7}$$

Parseval's relation on the lattice reads

$$\sum_n \bar{f}(n)g(n) = \sum_k \bar{f}(k)g(k).\tag{F.8}$$

Again we have chosen the normalization in the definition of the Fourier transform such that Parseval's relation takes this simple form.

Connection between the Continuum and the Lattice version of Fourier Transform

The multi-indices n and k which label the lattices, are related to the d-dimensional continuum variables x and p via

$$x = na, \quad p = \frac{k}{a}.\tag{F.9}$$

Integrals over the position space and the momentum space can be translated in sum on the lattices via

$$\int d^d x \quad \longleftrightarrow \quad a^d \sum_n \tag{F.10}$$

$$\int d^d p \quad \longleftrightarrow \quad \left(\frac{2\pi}{La}\right)^d \sum_k.\tag{F.11}$$

For the position space the correspondence between the integral and the sum becomes an equation in the continuum limit $a \to 0$. For the momentum space this is the case in the limit $La \to \infty$ of an infinitely large system. To connect the continuum Fourier transformation with the discrete version, one has to combine both limits. The functions on position and momentum space are connected with the corresponding quantities on the lattice by

$$f_{\text{cont.}}(x = na) = f_{\text{lattice}}(n)\tag{F.12}$$

$$f_{\text{cont.}}\left(p = \frac{k}{a}\right) = a^d \left(\frac{L}{2\pi}\right)^{d/2} f_{\text{lattice}}(k).\tag{F.13}$$

Appendix G

Continuum Limit on the Four-Dimensional Bodycentered Hypercubic Lattice

The purpose of this appendix is to put difference operators on the bhc lattice into correspondence to continuum differential operators. Sums over the sites of a four-dimensional bhc lattice translate in the continuum limit into an integral by

$$\frac{a^d}{2} \sum_n \qquad \longleftrightarrow \qquad \int d^4x. \tag{G.1}$$

Note the extra factor $1/2$ on the l. h. s. compared to the corresponding formula for the hc lattice. It takes into account that the bhc lattices consist of two nested hc lattices.

In performing the continuum limit, the differences on the lattice are replaced by derivatives in direction of the elementary lattice vectors

$$v_{\mu\nu}^{\alpha\beta} = \frac{\alpha e_\mu' + \beta e_\nu'}{\sqrt{2}}, \quad 1 \le \mu < \nu \le 4,\ \alpha = \pm1,\ \beta = \pm1. \tag{G.2}$$

We will calculate the expressions for a first, a second and a third order differential operator. All sums $\sum_v$ run over the *positive* lattice vectors, that is the vectors (G.2) with $\alpha = 1$. We start with the calculation of a Dirac operator,

$$
\begin{aligned}
\sum_v \gamma_v \partial_v &= \frac{1}{2} \sum_{\mu<\nu, \beta=\pm1} (\gamma_\mu + \beta\gamma_\nu)(\partial_\mu + \beta\partial_\nu) \\
&= \frac{1}{2} \sum_{\mu<\nu, \beta=\pm1} (\gamma_\mu\partial_\mu + \gamma_\nu\partial_\nu + \beta(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)) \\
&= \sum_{\mu<\nu} (\gamma_\mu\partial_\mu + \gamma_\nu\partial_\nu) \\
&= 3\,\slashed{D}
\end{aligned}
\tag{G.3}
$$

In a completely analogous way, one obtains the following expression for the Laplace operator,

$$\sum_v \partial_v^2 = \frac{1}{2} \sum_{\mu < \nu, \beta = \pm 1} \left(\partial_\mu^2 + \partial_\nu^2 + 2\beta \partial_\mu \partial_\nu \right)$$
$$= \sum_{\mu < \nu} \left(\partial_\mu^2 + \partial_\nu^2 \right) \tag{G.4}$$
$$= 3\Delta.$$

Finally we calculate the expression for the following differential operator of fourth order:

$$\sum_v \partial_v^4 = \frac{1}{4} \sum_{\mu < \nu, \beta = \pm 1} \left(\partial_\mu^2 + \partial_\nu^2 + 2\beta \partial_\mu \partial_\nu \right)^2$$
$$= \frac{1}{4} \sum_{\mu < \nu, \beta = \pm 1} \left((\partial_\mu^2 + \partial_\nu^2)^2 + 4\beta \partial_\mu \partial_\nu (\partial_\mu^2 + \partial_\nu^2) + 4\partial_\mu^2 \partial_\nu^2 \right)$$
$$= \frac{1}{2} \sum_{\mu < \nu} \left(\partial_\mu^4 + 6\partial_\mu^2 \partial_\nu^2 + \partial_\nu^4 \right) \tag{G.5}$$
$$= 3 \sum_{\mu < \nu} \partial_\mu^2 \partial_\nu^2 + \frac{3}{2} \sum_\mu \partial_\mu^4$$
$$= \frac{3}{2} \Delta^2.$$

Appendix H

Invariant Lagrangian Densities

In non-linear field theories the field cannot take arbitrary values in a vector space, but is restricted by certain constraints. We consider the case, where it is a map from space-time into a matrix group, $g : \mathbb{R}^d \to G$. In our applications this group is either the unitary group $G = \mathrm{U}(N)$ or the special unitary group $G = \mathrm{SU}(N)$.

The field theory is defined by an action functional, which is given by an space-time integral over a Lagrangian density,

$$S[g] = \int_{\mathbb{R}^d} d^d x \, \mathcal{L}(g, \partial_\mu g, \partial_\mu \partial_\nu g). \tag{H.1}$$

The Lagrangian density has to respect the symmetries of the physical problem; we assume it to be invariant under

  (i) the Euclidean space time symmetry, that is under the translations $\mathbb{R}^4$ as well as the rotations $\mathrm{O}(d)$,

  (ii) left as well as right multiplication with elements of the group G.

It is convenient to built the Lagrangian density of the *left currents* [1]

$$L_\mu := g^{-1} \partial_\mu g. \tag{H.2}$$

The left currents are left invariant under group transformations; under a general transformation $g \mapsto h_1 g h_2$ with h_1, $h_2 \in G$ they transforms like $L_\mu \to h_2^{-1} L_\mu h_2$. By taking a trace over products of left currents one can obtain expressions which are left- and rightinvariant. The left currents take values in the Lie algebra of the group G. In particular if $G = \mathrm{U}(N)$ the left currents are antihermitian ($L_\mu^\dagger = -L_\mu$) and if $G = \mathrm{SU}(N)$ they are antihermitian and traceless.

H.1 Higher Derivatives

Left invariant expressions containing higher derivatives can be written in terms of the left currents. To show that, we make use of the relation $L_\mu = g^{-1} \partial_\mu g = -(\partial_\mu g^{-1}) g$, which

---

[1] Alternatively one could make use of the right currents $R_\mu := (\partial_\mu g) g^{-1}$.

follows from differentiation of the equation $g^{-1}g = I$. *Under a trace* the following set of equations hold:

$$L_\mu^2 = -\partial_\mu g^{-1}\partial_\mu g, \tag{H.3}$$

$$g^{-1}(\partial_\mu^2 g) = L_\mu^2 + \partial_\mu L_\mu, \tag{H.4}$$

$$(\partial_\mu^2 g^{-1})g = L_\mu^2 - \partial_\mu L_\mu, \tag{H.5}$$

$$g^{-1}(\partial_\mu^3 g) = L_\mu^3 + 3L_\mu\partial_\mu L_\mu + \partial_\mu^2 L_\mu, \tag{H.6}$$

$$g^{-1}(\partial_\mu^4 g) = L_\mu^4 + 6L_\mu^2\partial_\mu L_\mu + 4L_\mu\partial_\mu^2 L_\mu + 3(\partial_\mu L_\mu)^2 + \partial_\mu^3 L_\mu, \tag{H.7}$$

$$= L_\mu^4 + 2\partial_\mu(L_\mu^3) + 4\partial_\mu(L_\mu\partial_\mu L_\mu) - (\partial_\mu L_\mu)^2 + \partial_\mu^3 L_\mu. \tag{H.8}$$

H.2   Invariant Lagrangians

We built Lagrangian densities of the left currents, which are compatible with space-time and group symmetries. In doing so, each space-time index should be contracted with a partner, to get a scalar with respect to space-time transformations. Invariance under the group transformations can be achieved by building a trace over the group space.

A general invariant Lagrangian density $\mathcal{L}(g)$ can be expanded in powers of the momentum,

$$\mathcal{L}(g) = \mathcal{L}^{(0)}(g) + \mathcal{L}^{(2)}(g) + \mathcal{L}^{(4)}(g) + ... \tag{H.9}$$

In the expansion above, the first term is of order $O(p^0)$ and contains no derivatives, the second term is of order $O(p^2)$ and contains two derivatives, the third term is of order $O(p^4)$ and contains four derivatives, and so on. We have left out terms of an odd order because they are in conflict with space-time symmetry, to be more precise with reflections on hyperplanes and the parity transformation.

Firstly we construct Lagrangian densities for a $SU(N)$-valued field. Later we show some additional terms, which are admissible for a $U(N)$-valued but not for a $SU(N)$-valued field.

Special Unitary Group

At order $O(p^0)$, the group invariant terms are constants, which do not contribute to the equations of motion. There can be symmetry breaking terms, for example

$$\mathcal{L}^{(0)}(g) = \operatorname{Tr}\mathcal{M}(g + g^{-1}), \tag{H.10}$$

where the symmetry breaking parameter $\mathcal{M}$ is a $N \times N$-matrix.

At order $O(p^2)$, there is only one term, which is compatible with the required invariance properties,

$$\mathcal{L}^{(2)}(g) = -\alpha_1 \sum_\mu \operatorname{Tr} L_\mu^2, \tag{H.11}$$

where α_1 is a constant. Making use of (H.4) and a partial integration, it can be rewritten as

$$\mathcal{L}^{(2)}(g) = \alpha_1 \operatorname{Tr}(\nabla g^{-1}\nabla g) \cong -\alpha_1 \operatorname{Tr}(g^{-1}\Delta g). \tag{H.12}$$

With the sign "$\cong$" we denote equality up to a divergence term.

At order $O(p^4)$ there are ten possible terms at first sight,

$$\mathcal{L}^{(4)}(g) = \sum_{\mu,\nu} \Bigg(\beta_1 \operatorname{Tr}(L_\mu^2) \operatorname{Tr}(L_\nu^2) + \beta_2 \operatorname{Tr}(L_\mu L_\nu) \operatorname{Tr}(L_\mu L_\nu)$$
$$+\beta_3 \operatorname{Tr}(L_\mu^2 L_\nu^2) + \beta_4 \operatorname{Tr}(L_\mu L_\nu L_\mu L_\nu)$$
$$+\beta_5 \operatorname{Tr}(L_\mu^2 \partial_\nu L_\nu) + \beta_6 \operatorname{Tr}(L_\mu L_\nu \partial_\mu L_\nu) + \beta_7 \operatorname{Tr}(L_\mu L_\nu \partial_\nu L_\mu) \tag{H.13}$$
$$+\beta_8 \operatorname{Tr}(\partial_\mu L_\mu \partial_\nu L_\nu) + \beta_9 \operatorname{Tr}(\partial_\mu L_\nu \partial_\mu L_\nu) + \beta_{10} \operatorname{Tr}(\partial_\mu L_\nu \partial_\nu L_\mu) \Bigg).$$

Four of the contributions to $\mathcal{L}^{(4)}(g)$ can be transformed into other terms due to partial integration and the Maurer-Cartan equation

$$\partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu, L_\nu] = 0; \tag{H.14}$$

the result is that we can set $\beta_6 = \beta_7 = \beta_9 = \beta_{10} = 0$. For a low number of group degrees of freedom not all of the left contribution are independent. They are connected through trace relations; there are two such relations for the special unitary group in two dimensions and one such relation for the special unitary in three dimensions: For two traceless 2×2 matrices L_μ and L_ν it is elementary to derive the identity

$$\operatorname{Tr}(L_\mu^2) \operatorname{Tr}(L_\nu^2) = \operatorname{Tr}(L_\mu^2 L_\nu^2) + \operatorname{Tr}(L_\mu L_\nu L_\mu L_\nu)$$
$$\operatorname{Tr}(L_\mu L_\nu) \operatorname{Tr}(L_\mu L_\nu) = 2 \operatorname{Tr}(L_\mu^2 L_\nu^2). \tag{H.15}$$

For two traceless 3×3 matrices L_μ and L_ν the following identity holds,

$$\operatorname{Tr}(L_\mu L_\nu L_\mu L_\nu) = -2 \operatorname{Tr}(L_\mu^2 L_\nu^2) + \frac{1}{2} \operatorname{Tr}(L_\mu^2) \operatorname{Tr}(L_\nu^2) + \operatorname{Tr}(L_\mu L_\nu) \operatorname{Tr}(L_\mu L_\nu). \tag{H.16}$$

For a proof of this relation see [16]. To sum up, the general form of a fourth order Lagrangian density for a SU(3)-valued field has the form

$$\mathcal{L}^{(4)}(g) = \sum_{\mu,\nu} \Bigg(\beta_1 \operatorname{Tr}(L_\mu^2) \operatorname{Tr}(L_\nu^2) + \beta_2 \operatorname{Tr}(L_\mu L_\nu) \operatorname{Tr}(L_\mu L_\nu) + \beta_3 \operatorname{Tr}(L_\mu^2 L_\nu^2)$$
$$+\beta_5 \operatorname{Tr}(L_\mu^2 \partial_\nu L_\nu) + \beta_8 \operatorname{Tr}(\partial_\mu L_\mu \partial_\nu L_\nu) \Bigg). \tag{H.17}$$

The first three terms are identical to the first terms of the Gasser-Leutwyler expansion [18]. Both other terms contain second derivatives of the field, namely $\sum_\mu \partial_\mu L_\mu$.
Finally we consider the transformation behavior of the different contributions under the group inversion $g \rightarrow g^{-1}$. It elementary to prove that

$$\operatorname{Tr}(L_\mu^2 \partial_\nu L_\nu) \rightarrow -\operatorname{Tr}(L_\mu^2 \partial_\nu L_\nu) \qquad \text{for } g \rightarrow g^{-1}, \tag{H.18}$$

while all other contributions are invariant under the group inversion.

Unitary Group

In case of an $U(N)$-valued field the currents the corresponding Lie algebra may be decomposed into

$$\mathfrak{u}(N) = \mathfrak{u}(1) \oplus \mathfrak{su}(N). \tag{H.19}$$

One easily can write down a term, which involves only the subgroup $U(1)$ of $U(N)$, namely

$$\mathcal{L}_{\mathfrak{u}(1)}^{(0)}[g] = m' \operatorname{Det}(g + g^{-1}) \tag{H.20}$$

with a scalar parameter m'. It is of order $O(1)$ and breaks the $U(1)$ symmetry.

The decomposition (H.19) is orthogonal with respect to the bi-invariant inner product $\langle L_\mu, L_\nu \rangle = -\operatorname{Tr}(L_\mu^\dagger L_\nu)$ of the Lie algebra. As a consequence, one can consider contributions, which correspond to the $\mathfrak{u}(1)$ subalgebra inside $\mathfrak{u}(N)$ and do not depend on the value of the currents in the $\mathfrak{su}(N)$ subalgebra. A terms of such a kind is the trace of a single left current,

$$\operatorname{Tr} L_\mu = \operatorname{Tr}(g^{-1} \partial_\mu g) = \operatorname{Tr}(\partial_\mu \ln g) = \partial_\mu \operatorname{Tr} \ln g = \partial_\mu \ln \operatorname{Det} g. \tag{H.21}$$

At order $O(p^2)$ the list of the last section has to be extended by one extra term, the so-called Garde term

$$\mathcal{L}_{\mathfrak{u}(1)}^{(2)}[g] = \alpha_2 \sum_\mu (\operatorname{Tr} L_\mu)^2. \tag{H.22}$$

Making use of (H.21) and a partial integration, it can be written as

$$\mathcal{L}_{\mathfrak{u}(1)}^{(2)}[g] = \alpha_2 (\nabla \ln \operatorname{Det} g)^2 \cong -\alpha_2 (\ln \operatorname{Det} g) \Delta (\ln \operatorname{Det} g). \tag{H.23}$$

At order $O(p^4)$ there are different terms due to possible mixing of $\mathfrak{u}(1)$ and $\mathfrak{su}(N)$ depended contributions.

H.3 Calculation of an Effective Action for Strong Coupling QCD (hc Lattice)

In this section we display details of the calculation of a QCD low energy action. Starting point for a expansion in the momenta of the light mesons p and the quark masses m_f is the action (6.67). The expansion is carried out up to order $O(p^4)$; in doing so the quark masses count as $O(p^2)$ in the spirit of chiral power counting. The first contribution is the fluctuation action, which is obtained be setting the quark masses to zero,

$$
\begin{aligned}
S_{\mathrm{fl}}[G] =& -\frac{a^{-d}}{2} \int d^d x \, \operatorname{Tr} \ln \left(1 + \frac{a^2}{8d} G^{-1} \Delta G + \frac{a^4}{384d} G^{-1} \partial^4 G \right) + G \leftrightarrow G^{-1} \\
=& -\frac{a^{2-d}}{16d} \int d^d x \, \operatorname{Tr}(G^{-1} \Delta G) \\
&+ \frac{a^{4-d}}{256d} \int d^d x \left(\frac{1}{d} \operatorname{Tr}(G^{-1} \Delta G)^2 - \frac{1}{3} \operatorname{Tr}(G^{-1} \partial^4 G) \right) + G \leftrightarrow G^{-1} \\
=& -\frac{a^{2-d}}{8d} \int d^d x \sum_\mu \operatorname{Tr} L_\mu^2 \\
&+ \frac{a^{4-d}}{128d} \int d^d x \left(\frac{1}{d} \sum_{\mu,\nu} \left(\operatorname{Tr}(L_\mu^2 L_\nu^2) + \operatorname{Tr}(\partial_\mu L_\mu \partial_\nu L_\nu) \right) + \frac{1}{3} \sum_\mu \left(\operatorname{Tr}(\partial_\mu L_\mu)^2 - \operatorname{Tr} L_\mu^4 \right) \right).
\end{aligned}
\tag{H.24}
$$

In the last step of the calculation, we made use of the identities $G^{-1}\Delta G = \sum_\mu \left(L_\mu^2 + \partial_\mu L_\mu \right)$ and $G^{-1}\partial^4 G \cong \sum_\mu \left(L_\mu^4 - (\partial_\mu L_\mu)^2 \right)$. This relations where derived in H.1; the second identity is valid up to divergence terms. Performing the square of the first expression we get a mixed term $2\sum_{\mu,\nu} \mathrm{Tr}(L_\mu^2 \partial_\nu L_\nu)$. This terms is odd with respect to the group inversion $G \leftrightarrow G^{-1}$ and cancels when the symmetrization is done.

The second contribution is the mass action, which is obtained by setting the derivatives of the Goldstone field to zero. During the calculation we neglect constant terms, which do not contain the Goldstone field,

$$
\begin{aligned}
S_\mathcal{M}[\mathcal{G}] &= -\frac{a^{-d}}{2} \int d^d x \, \mathrm{Tr}\ln\left(1 + \frac{1}{2dx}G^{-1}\hat{\mathcal{M}} - \frac{x}{4}G^{-1}\{\hat{\mathcal{M}}, G\} + \frac{x^2}{16}G^{-1}\{\hat{\mathcal{M}}^2, G\} \right) + G \leftrightarrow G^{-1} \\
&= -\frac{a^{-d}}{4dx} \int d^d x \, \mathrm{Tr}\,(G^{-1}\hat{\mathcal{M}}) \\
&\quad + \frac{a^{-d}}{4} \int d^d x \, \mathrm{Tr}\left(\frac{1}{2dx}G^{-1}\hat{\mathcal{M}} - \frac{x}{4}G^{-1}\{\hat{\mathcal{M}}, G\} \right)^2 + G \leftrightarrow G^{-1} \\
&= -\frac{a^{-d}}{4dx} \int d^d x \, \mathrm{Tr}\,(\hat{\mathcal{M}}(G + G^{-1})) \\
&\quad + \frac{a^{-d}}{16d^2 x^2} \int d^d x \, \mathrm{Tr}\,(\hat{\mathcal{M}}G^{-1}\hat{\mathcal{M}}G^{-1} + \hat{\mathcal{M}}G\hat{\mathcal{M}}G) \\
&\quad - \frac{a^{-d}}{8d} \int d^d x \, \mathrm{Tr}\,(\hat{\mathcal{M}}^2(G + G^{-1})) + \frac{x^2 a^{-d}}{16} \int d^d x \, \mathrm{Tr}\,(\hat{\mathcal{M}}G^{-1}\hat{\mathcal{M}}G).
\end{aligned}
$$

$$(H.25)$$

Last but not least we turn to the third contribution which couples the fluctuations of the Goldstone field and the quark masses. During the calculation we neglect all terms which do not contain both derivatives and the quark masses,

$$
\begin{aligned}
S_{\mathrm{fl},\mathcal{M}}[G] &= -\frac{a^{-d}}{2} \int d^d x \, \mathrm{Tr}\ln\left(1 + \frac{1}{2dx}G^{-1}\hat{\mathcal{M}} - \frac{x}{4}G^{-1}\{\hat{\mathcal{M}}, G\} \right. \\
&\quad \left. - \frac{xa}{8d}G^{-1}[\hat{\mathcal{M}}, \partial G] + \frac{a^2}{8d}G^{-1}\Delta G - \frac{xa^2}{32d}G^{-1}\{\hat{\mathcal{M}}, \Delta G\} \right) + G \leftrightarrow G^{-1} \\
&= +\frac{xa^{1-d}}{16d} \int d^d x \, \mathrm{Tr}\,(\hat{\mathcal{M}}(G^{-1}\partial G - \partial G G^{-1})) \\
&\quad - \frac{xa^{2-d}}{64d} \int d^d x \, \mathrm{Tr}\,(\hat{\mathcal{M}}(G^{-1}\Delta G + \Delta G G^{-1})) \\
&\quad - \frac{a^{-d}}{4} \int d^d x \, \mathrm{Tr}\left(\frac{1}{2dx}G^{-1}\hat{\mathcal{M}} - \frac{x}{4}G^{-1}\{\hat{\mathcal{M}}, G\} + \frac{a^2}{8d}G^{-1}\Delta G \right)^2 + G \leftrightarrow G^{-1} \\
&= +\frac{xa^{1-d}}{8d} \int d^d x \, \mathrm{Tr}\,(\hat{\mathcal{M}}(G^{-1}\partial G - \partial G G^{-1})) \\
&\quad - \frac{a^{2-d}}{32d^2 x} \int d^d x \, \mathrm{Tr}\,(\hat{\mathcal{M}}(G^{-1}\Delta G G^{-1} + G\Delta G^{-1}G)) \\
&= +\frac{xa^{1-d}}{8d} \int d^d x \, \mathrm{Tr}\,(\hat{\mathcal{M}}(G^{-1}\partial G - \partial G G^{-1})) \\
&\quad - \frac{a^{2-d}}{32d^2 x} \int d^d x \sum_\mu \left(\mathrm{Tr}\,(L_\mu^2(G^{-1}\hat{\mathcal{M}} + \hat{\mathcal{M}}G)) + \mathrm{Tr}\,(\partial_\mu L_\mu(G^{-1}\hat{\mathcal{M}} - \hat{\mathcal{M}}G)) \right).
\end{aligned}
$$

$$(H.26)$$

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Zusammenfassung

Das heutige Bild der mikroskopischen physikalischen Welt, ihrer Grundbestandteile und der Kräfte zwischen ihnen, ist das Standardmodell der Elementarteilchenphysik. Im Standardmodell werden die Kräfte zwischen den Grundbausteinen der Materie, den Quarks und den Leptonen, durch den Austausch von Eichbosonen vermittelt. Die mathematische Modellierung dieser Vorstellung geschieht im Rahmen von quantisierten Eichtheorien, die den verschiedenen Wechselwirkungen zugeordnet sind.

Diese Arbeit ist einem neuen Zugang zur Quantenchromodynamik (QCD), der Theorie der starken Wechselwirkung, gewidmet. Die QCD ist die Eichtheorie der Gluonen und Quarks, die über die Farbfreiheitsgrade aneinander koppeln. Komplementär zu der Farbe besitzen die Quarks Spin und Flavor als weitere Freiheitsgrade, die nicht an die Gluonen koppeln.

Die Color-Flavor-Transformation

Diese Struktur bildet den Ausgangspunkt für die Anwendung der Color-Flavor-Transformation, einer kürzlich entdeckten Äquivalenz zwischen zwei Formulierungen bestimmter Quantenfeldtheorien [53]. Einer der im Hinblick auf die Quantenchromodynamik interessanten Aspekte an der Color-Flavor-Transformation ist, daß diese die durch das Eichfeld vermittelte Kopplung der Farbfreiheitsgrade der Quarks durch eine Kopplung der dazu komplementären Spin- und Flavorfreiheitsgrade ersetzt. Das diese Kopplung vermittelnde Feld bezeichnen wir als „Mesonenfeld", da es an Farbsinguletts von Quarks koppelt und mit den physikalischen Mesonen in Verbindung gebracht werden kann. Die zentrale Idee dieser Arbeit ist, mit Hilfe der Color-Flavor-Transformation eine Verbindung zwischen den Hochenergie- und den Niederenergiesektor der Quantenchromodynamik herzustellen. Im Hochenergiesektor ist die Kopplungskonstante der QCD klein und die Theorie störungstheoretisch auswertbar, der Niederenergiesektor ist hingegen nicht störungstheoretisch zugänglich. Für niedrige Energien sollte die Chromodynamik in eine Theorie von schwach gekoppelten Mesonen und Baryonen übergehen. Ziel dieser Arbeit ist es, zum Verständnis dieses Übergangs beizutragen.

In Kapitel 2 dieser Arbeit haben wir uns mit den mathematischen Grundlagen der Color-Flavor-Transformation auseinandergesetzt. Wir betrachten ein Quantenvielteilchensystem, das durch die kanonischen Vertauschungsrelationen von Erzeugungs- und Vernichtungsoperatoren charakterisiert ist. Die Color-Flavor-Transformation baut auf einer Struktur in der Gruppe der kanonischen Transformationen der Erzeugungs- und Vernichtungsoperatoren auf: Jedem dualen Paar von Untergruppen dieser Gruppe ist eine Version der Color-Flavor-Transformation zugeordnet. Die für uns in Zusammenhang mit der Quantenchromodynamik interessante Color-Flavor-Transformation gehört zu einem aus zwei unitären Gruppen bestehenden dualen Paar. Eine diesem Fall entsprechende Version haben wir für ein rein bosonisches und rein fermionisches System ausgearbeitet.

Nun ist die Eichgruppe der Quantenchromodynamik aber keine unitäre Gruppe, sondern die spezielle unitäre Gruppe in drei Dimensionen. Dies führt dazu, daß farblose Zustände nicht nur aus Quark-Antiquark-Paaren sondern auch aus drei Quarks, von denen jedes eine andere Farbe besitzt, gebildet werden können. Anders ausgedrückt: Neben Mesonen gibt es auch Baryonen. Es war daher nötig, die Color-Flavor-Transformation auf die spezielle unitäre Gruppe zu verallgemeinern (Kapitel 3). Als Ergebnis erhalten wir Summe von Beiträgen, von denen jeder durch eine Baryonenzahl B gekennzeichnet ist. Der zu $B = 0$ gehörende Beitrag entspricht genau dem Ergebnis der Color-Flavor-Transformation für die unitäre Gruppe. Die anderen Terme enthalten zusätzlich einen Vorfaktor, der aus einem Produkt von Quarkfeldern besteht. Die Anzahl der Baryonen, zu denen sich diese zusammensetzen, kennzeichnet den jeweiligen Beitrag.

Gitterformulierung der Quantenchromodynamik

Die QCD ist als relativistische Feldtheorie invariant unter Poincaré-Transformationen, den Symmetrietransformationen der relativistischen Raumzeit. Wir arbeiten stets in der euklidischen Formulierung der Feldtheorie, die den Vorteil hat, daß Raum und Zeit völlig symmetrisch behandelt werden. Die Color-Flavor-Transformation ist aber nicht auf die Kontinuumsformulierung der QCD, sondern auf ihre Gitterformulierung anwendbar. Der Übergang zum Gitter bricht die kontinuierliche Raumzeitsymmetrie zu einer endlichen Gittersymmetriegruppe. Wir arbeiten mit zwei unterschiedlichen Gittern: mit dem hyperkubischen (hc) Gitter und dem raumzentrierten hyperkubischen (bhc) Gitter. Bemerkenswerter Weise gibt es nämlich in vier Raumzeitdimensionen (anders als in zwei oder drei Dimensionen) ein Gitter mit einer größeren Symmetriegruppe als das hc Gitter: das bhc Gitter. Dabei ist der Fall von vier Raumzeitdimensionen besonders ausgezeichnet, da das bhc Gitter nur in diesem Fall ein Bravais-Gitter ist.
Wichtig für das Verständnis der starken Wechselwirkung bei niedrigen Energien ist die approximativ gültige chirale Symmetrie der QCD. Man geht davon aus, daß diese vom Grundzustand der Quantenchromodynamik gebrochen wird. Nach dem Goldstonetheorem besitzt die Theorie dann massenlose Anregungen, die den gebrochenen Symmetriegeneratoren zugeordnet sind. Im Fall der chiralen Symmetriebrechung identifiziert man diese Goldstonebosonen mit den leichtesten experimentell beobachteten Mesonen. Die Goldstonebosonen wären masselos, wenn die chiralen Transformationen eine exakte Symmetrie der QCD wären. In der QCD wird die chirale Symmetrie aber explizit durch die endlichen Quarkmassen gebrochen. So erklärt man die nicht-verschwindenden Massen der leichtesten Mesonen, die proportional zur Wurzel aus den Quarkmassen sind. Um die Color-Flavor-Transformation anwenden zu können, sind wir auf die Gitterformulierung der QCD angewiesen. Daher haben wir untersucht, welchen Einfluß der Übergang zum Gitter auf die chirale Symmetriegruppe hat (Kapitel 4). Es stellt sich heraus, daß im Fall des hc Gitters eine vergrößerte chirale Symmetriegruppe auftritt, während die Theorie auf dem bhc Gitter dieselbe chirale Symmetriegruppe wie die Kontinuumstheorie besitzt.

Gluodynamik

In dieser Arbeit haben wir die Color-Flavor-Transformation auf zwei Grenzfälle der Quantenchromodynamik angewendet: auf die Gluodynamik und auf den Starkkopplungslimes der QCD. Für den Fall der Gluodynamik, formuliert mit Hilfe der von Wilson eingeführten eichin-

varianten Gitterwirkung [48], ist eine direkte Anwendung der Color-Flavor-Transformation nicht möglich. Wir haben Wilsons Wirkung durch einen physikalisch äquivalenten Term ersetzt, der sich aus einem Integral über massive Hilfsbosonenfelder erzeugen läßt (Kapitel 5). Es hat sich dabei gezeigt, daß in unserem Ansatz die Zahl der Hilfsfelder mindestens gleich der Zahl der Farben gewählt werden muß, damit unsere Gittertheorie tatsächlich die Gluodynamik als Kontinuumslimes besitzt.

Eine vollständige Auswertung unserer Theorie gelingt in zwei Raumzeitdimensionen, in denen Gittereichtheorien eine wesentlich einfachere Struktur als in höheren Dimensionen besitzen. Bekanntlich tritt in den zweidimensionalen Theorien stets Confinement mit einem im Abstand linearen Potential zwischen Quark und Antiquark auf. Die zu diesem Potential gehörende Seitenspannung haben wir für unsere Theorie einerseits mit Hilfe von Monte-Carlo-Simulationen und andererseits analytisch berechnet. Durch die analytische Rechnung konnten wir ein exaktes Ergebnis erzielen, das für eine beliebige Anzahl von Farben und Hilfsbosonen gültig ist, solange die Zahl der Hilfsbosonen kleiner oder gleich der Zahl der Farben ist. Beim Übergang zum Kontinuum, das heißt dem Übergang von der Gittertheorie zur Gluodynamik, streben Gitterkonstante und Kopplungskonstante gleichzeitig gegen Null. Die beiden Konstanten müssen so miteinander verknüpft werden, daß sich die physikalischen Observablen bei der Durchführung des Kontinuumslimes nicht ändern. Dadurch wird die Abhängigkeit der Kopplungskonstanten von der Gitterkonstante, d. h. die Renormierungsgruppengleichung definiert. Für unsere Theorie hat sich gezeigt, daß die Seitenspannung im Kontinuumslimes nur konstant gehalten werden kann, falls die Zahl der Hilfsbosonen größer oder gleich der Zahl der Farben ist. In diesem Fall haben wir die Renormierungsgruppengleichung für die Kopplungskonstante hergeleitet und gezeigt, daß unsere Gittertheorie tatsächlich die Gluodynamik als Kontinuumslimes besitzt.

Die grundlegende Idee ist, den komplizierteren Fall der vierdimensionalen Gluodynamik mit der Color-Flavor-Transformation zu bearbeiten. Hier ergibt sich ein Problem mit der Konvergenz der Color-Flavor-Transformation: Die Color-Flavor-Transformation konvergiert für bosonische Felder nur, falls die Zahl der Farben größer oder gleich der Zahl der doppelten Zahl der Bosonen ist. Andererseits muß aber die Zahl der Hilfsbosonen größer als die Zahl der Farben sein, um die Gluodynamik als Kontinuumslimes der Gittertheorie zu erhalten. Als Ausweg aus dieser Situation führen wir für jedes Hilfsboson einen fermionischen Partner ein und gelangen so zu einer supersymmetrischen Theorie, für die die Color-Flavor-Transformation immer konvergiert. Der Einfluß der Hilfsfermionen kann beliebig gering gehalten werden, indem man sie beliebig schwer werden läßt. Auf diese supersymmetrischen Theorie haben wir die Color-Flavor-Transformation angewandt und so einen Ausdruck für die Zustandssumme der Gluodynamik hergeleitet, der als Ausgangspunkt für weitere Rechnungen dienen kann.

Starkkopplungslimes der Chromodynamik

Auf den Starkkopplungslimes der QCD kann die Color-Flavor-Transformation unmittelbar angewandt werden. Sie organisiert die Zustandssumme der Theorie in eine Summe von Beiträgen, von denen jeder zu einer Verteilung von Baryonen auf dem Gitter (d. h. zu einem baryonischen Fluß in der Raumzeit) gehört. Zwei physikalisch besonders interessante Beiträge haben wir weiter ausgewertet: Das „Vakuum" – hier existiert zu keiner Zeit ein Baryon im System – und ein Modell eines statischen Baryons, bei dem sich jeweils ein Baryon auf den Kanten einer fest gewählten Gerade in Zeitrichtung befindet.

Im Fall der Vakuumskonfiguration tritt nach Integration über die Quarkfelder die Anzahl der

Farben N_c als Faktor vor der effektiven Wirkung auf. Dieser organisiert die Zustandssumme in eine Störungsreihe in $1/N_c$, deren niedrigste Ordnung durch die Sattelpunktsnäherung gegeben ist. Die Lösungen der zugehörigen Sattelpunktsgleichungen bilden die niederenergetischsten Moden der Theorie. Die Analyse der Sattelpunktsgleichungen ergab im Fall nichtverschwindender Quarkmassen einen isolierten Sattelpunkt, während im chiralen Limes eine ganze Sattelpunktsmannigfaltigkeit existiert. Diese Sattelpunktsmannigfaltigkeit ensteht durch die Wirkung der chiralen Symmetriegruppe auf den Sattelpunkt. Diese Strukturen können mit dem Schema der spontanen chiralen Symmetriebrechung in Einklang gebracht werden. Dabei lassen wir das starke U(1)-Problem außer acht. (Dieses besteht in der anomal hohen Masse des η'-Teilchens, die nach einer Theorie von 't Hooft aus der axialen Anomalie der QCD folgt.) In unserer Theorie tritt folgendes Schema der chiralen Symmetriebrechung wie folgt auf: Der Sattelpunkt bricht die chirale Symmetrie zu einer Untergruppe, wobei die gebrochenen Symmetriegeneratoren die im chiralen Limes auftretende Sattelpunktsmannigfaltigkeit parametrisieren. Die Sattelpunktsmannigfaltigkeit sollte daher mit den Goldstonemoden der spontanen chiralen Symmetriebrechung identifiziert werden.

Seit langem ist bekannt, daß die Niederenergieeigenschaften der leichtesten Mesonen erfolgreich durch phänomenologische Lagrangedichten beschrieben werden können. Diese Technik, bei der die chirale Symmetrie eine entscheidende Rolle spielt, wurde von Weinberg [46] eingeführt und von Gasser und Leutwyler [18] weiterentwickelt. Die „chirale Störungstheorie" besteht in einer Entwicklung der Niederenergie-Lagrangedichte in den Impulsen p der Niederenergiemoden und den Quarkmassen m. Dabei werden p^2 und m als klein behandelt, während p^2/m jeden Wert annehmen darf. Es sind verschiedene Versuche unternommen worden (siehe die Referenzen in Kapitel 6), Niederenergie-Lagrangedichten aus der QCD herzuleiten. Ausgehend von der color-flavor-transformierten Zustandssumme der QCD im Starkkopplungslimes haben wir eine chirale Entwicklung bis zur Ordnung $O(p^4)$ durchgeführt. Bei unserem Zugang erhält man die Niederenergiemoden als Lösung der Sattelpunktsgleichungen. Durch eine Gradientenentwicklung haben wir eine auf großen Längenskalen gültige Kontinuumstheorie hergeleitet, die die Dynamik dieser Moden beschreibt. Es handelt sich um eine Entwicklung in der Art der chiralen Störungstheorie, die wir bis zur Ordnung $O(p^4)$ durchgeführt haben. Durch den Vergleich dieses Ergebnisses mit experimentellen Daten wird der Wert des einzigen freien Parameters der Theorie, der Gitterkonstanten, fixiert. Insgesamt schätzen wir die Gitterkonstante auf drei verschiedenen Arten ab: Über die zwei Terme in der chiralen Lagrangedichte der Ordnung $O(p^2)$ steht die Gitterkonstante einerseits im Zusammenhang mit der Pionenzerfallskonstante und andererseits mit der Pionenmasse. Eine dritte, unabhängige Abschätzung erhalten wir aus dem experimentellen Wert für das chirale Kondensat. Dieses können wir theoretisch direkt aus der Zustandssumme in Sattelpunktsnäherung berechnen. Alle drei Abschätzungen ergeben für die Gitterkonstante eine Größenordnung von $a \approx 1$ fm, stimmen untereinander aber schlecht überein. In der Ordnung $O(p^4)$ der chiralen Störungstheorie können wir einige Terme der allgemeinen Lagrangedichte von Gasser und Leutwyler reproduzieren. Unsere Werte für die Kopplungkonstanten in der Ordnung $O(p^4)$ weichen erheblich von den aus experimentellen Daten gewonnenen phänomenologischen Werten ab. Daraus läßt sich schließen, daß die angewandten Näherungen – Starkkopplungslimes und Sattelpunktsnäherung – bereits in der Ordnung $O(p^2)$ der chiralen Störungstheorie erhebliche Fehler hervorrufen und zu roh sind, um vernünftige Werte für die Kopplungskonstanten der Terme von der Ordnung $O(p^4)$ zu liefern.

Abschließend kommen wir zu den Ergebnissen, die wir für das Modell des statischen Baryons erzielt haben (Kapitel 7). Auch in diesem Fall konnten wir die Integration über die Quarks

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ausführen und eine effektive Wirkung herleiten. Diese besteht aus einem „Seeterm", der identisch mit der effektiven Wirkung für die Vakuumskonfiguration ist und einem Term, der nur von den Feldern längs der Weltlinie des Baryons abhängt. Von den Sattelpunktsgleichungen weichen daher nur diejenigen, die aus der Variation nach den Feldern in unmittelbarer Nähe des Baryons resultieren, von den Vakuumsattelpunktsgleichungen ab. Das nutzen wir aus, um näherungsweise das Verhalten des Mesonenfelds weit entfernt vom Baryon zu beschreiben. Dazu linearisieren wir die Sattelpunktsgleichungen um die Vakuumskonfiguration. In dieser Näherung werden wir im pseudoskalaren Sektor auf eine Klein-Gordon-Gleichung mit einer zur Wurzel aus Quarksmassen proportionalen Masse geführt. (Die physikalischen Dimensionen kommen hier wie folgt aus: die in der Klein-Gordon-Gleichung auftretende Masse ist proportional zur Wurzel aus Quarkmassen und umgekehrt proportional zur Wurzel aus der Gitterkonstanten.) Die Klein-Gordon-Gleichung besitzt eine statische, kugelsymmetrische, mit dem Abstand vom Ursprung exponentiell abfallende Lösung, die am Ursprung singulär wird. In dieser Weise erhalten wir ein Yukawa-Potential, dessen typische Längenskala umgekehrt proportional zur Wurzel aus der Quarkmasse ist.

Erklärung

Ich versichere, daß ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; daß diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; daß sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist sowie, daß ich eine solche Veröffentlichung vor Abschluß des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Martin R. Zirnbauer betreut worden.

Köln, den 28. February 2002

Teilpublikationen:

J. Budczies, Y. Shnir, *Color-Flavor Transformation and QCD Low Energy Effective Action*, Proceedings of the Workshop "Hadron Physics", Coimbra 1999, Eds. A.H. Blin et al., AIP Press, N.-Y. (2000), p.172-181.

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Danksagung

Zuallererst danke ich Prof. Dr. M. Zirnbauer für die Vergabe des interessanten und aktuellen Themas. Seine stete Diskussionsbereitschaft und seine Ideen haben wesentlich zum Gelingen dieser Arbeit beigetragen.

Bei Prof. Dr. A. Altland bedanke ich mich für hilfreiche Diskussionen in der Frühphase dieses Projekts und die Übernahme des Koreferats.

Bei meinen beiden Kollegen Dr. Ya. Shnir und Dr. S. Nonnenmacher möchte ich mich für die lehrreiche und fruchtbare Zusammenarbeit bedanken. Ihnen beiden und Dr. R. Klesse gilt darüber hinaus Dank für die Durchsicht von Teilen des Manuskripts.

Ich danke Herrn A. Sindermann für seine stete Bereitschaft mir bei Computerfragen- und problemen zu helfen.

Desweiteren geht mein Dank an Herrn Dr. P. Rottländer, der meine Arbeit als Physiker und dennoch Fachfremder durchgesehen hat.

Schließlich danke ich den Herren S. Draganis und T. Nussbaumer für das Korrekturlesen in sprachlicher Hinsicht und den damit verbundenen Zeitaufwand.

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