The path model and Bott–Samelson manifolds in the context of loop groups

Inaugural-Dissertation

zur

Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Fakultät

der Universität zu Köln

vorgelegt von

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Köln, 02. November 2020

Diese Arbeit wurde von der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln zur Dissertation angenommen.

Berichterstatter (Gutachter)

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Tag der mündlichen Prüfung: 26.01.2021

Zusammenfassung

Wir realisieren das Littelmann Pfad Modell und die zugehörigen Wurzeloperatoren auf der Schleifengruppe des Torus einer kompakten, einfachen Lie Gruppe. Für integrale Schleifen - solche mit guter kombinatorischer Beschreibung der Wurzeloperatoren - geben wir eine geometrische Interpolation der Wurzeloperatoren mittels Bott–Samelson Mannigfaltigkeiten, deren Definition wir für diesen Zweck erweitern. Wir betten diese Mannigfaltigkeiten in die Schleifengruppe der einfachen Gruppe ein und geben ein Kriterium an, unter dem die symplektische Struktur der Schleifengruppe zu einer symplektischen Struktur der Bott–Samelson Mannigfaltigkeit einschränkt. Für Schleifen in dominante Richtung berechnen wir das Bild unter der Impulsabbildung. Um eine komplexe Struktur zu etablieren, geben wir einen Diffeomorphismus zwischen den Bott–Samelson Mannigfaltigkeiten und den Bott–Samelson–Demazure–Hansen Varietäten assoziiert zu einer Gallerie im affinen Gebäude an. Diese Abbildung ist verträglich mit Wurzeloperatoren, und wir interpretieren die Ergebnisse von Gaussent und Littelmann im Rahmen des Gallerienmodells neu. Durch diese Interpretation definieren wir isotope Einbettungen der Mirković-Vilonen Zykel in die differentialgeometrische Schleifengruppe. Wir untersuchen dazu das Verhalten der Bott–Samelson Mannigfaltigkeit unter Homotopien der zugrundeliegenden Schleifen. Eine Folgerung davon ist ein weiteres Kriterium, um das Bild der Impulsabbildung zu bestimmen.

Abstract

We realize the Littelmann path model and the associated root operators on the loop groups of the torus in a compact, simple Lie group. For integral loop - those loops with a good combinatorial description of the root operators - we define a geometric interpolation of the root operators through Bott–Samelson manifolds, whose definition we generalize for this purpose. We embed these manifolds in the loop group of the simple group and give a criterion under which the symplectic structure of the loop group restricts to a symplectic structure of the Bott–Samelson manifold. For loops in dominant direction we compute the image under the moment map. To establish a complex structure we give a diffeomorphism between the Bott–Samelson manifolds and the Bott–Samelson–Demazure–Hansen variety associated to a gallery in the affine building. This map is compatible with the root operators, and we interpret the results of Gaussent and Littelmann in the context of the gallery model anew. By means of this interpretation we define isotopic embeddings of Mirković-Vilonen cycles into the differential-geometric loop group. For this purpose we investigate the behavior of the Bott–Samelson manifold under homotopies of the underlying loop. A consequence of this is another criterion to determine the image of the moment map.

Danksagung

Mein Dank geht an viele Menschen ohne die diese Arbeit nicht möglich gewesen wäre.
Ein großer Dank geht an Prof. Dr. Peter Littelmann für seine Betreuung, Diskussionen und Ratschläge.
Ich möchte den Mitgliedern meiner Arbeitsgruppe Dr. Lara Bossinger, Dr. Xin Fang, Dr. Bea Schumann und Dr. Christian Steinert für ein inspirierendes und herzliches Umfeld während meines Promotionsstudiums danken.
Insbesondere möchte ich Dr. Xin Fang, Dr. Bea Schumann, Daniel Teutsch und meiner Mutter Claudia Rappel danken, die mir durch Korrektur lesen und Kommentare bei der Fertigstellung dieser Dissertation geholfen haben.
Ein weiterer Dank geht an Dr. Reda Chhaibi für seine Gastfreundschaft während meines Aufenthalts an der Université Paul Sabatier in Toulouse im Mai 2019.
Und insgesamt geht mein Dank an meine Freunde und meine Familie für ihre Unterstützung im akademischen und nicht-akademischen Leben.
Die vorliegende Arbeit wurde unterstützt durch den SFB/TRR 191 "Symplectic Structures in Geometry, Algebra and Dynamics".

Contents

Introduction

The aim of the present thesis is to link the path model and Mirković–Vilonen cycles directly by approaching the problem from the perspective of compact Lie groups avoiding the gallery model. Our approach consists of two steps: The first step is to show that the root operators of the path model descend to the loop group of a compact torus. This enables us to construct crystal graphs from any loop γ with image in the compact torus. In the second step we construct a manifold from the path model. It has a cell decomposition by which the Mirković–Vilonen cycles are obtained. The manifold constructed is the Bott–Samelson manifold Γ_γ, which we generalize to suit our purpose. A torus of the Lie group acts on Γ_γ and the fixed points can be identified with elements of the path model. We prove symplecticness of Γ_γ for a well behaved class of loops and compute its moment map image. The connection to MV cycles is realized through an explicit diffeomorphism of Γ_γ and the Bott–Samelson–Demazure–Hansen variety respecting the path and gallery model. We realize the affine Schubert varieties and MV cycles as subsets of the loop group without computation of the Iwasawa decomposition. Using this result we make use of the flexibility of the topological category and define isotopic embeddings of MV cycles and the affine Schubert variety into the loop group. Let us describe the above in more detail and give some context. Since the classification results of unitary representations of compact Lie groups in the beginning of the 20th century researchers have been focused on the construction of such representations and their bases. Through means of Lie correspondence and complexification the question can be treated also from the angle of Lie algebras and algebraic groups, perspectives which have proven useful. In more recent years various bases of representations have been constructed. Important to our story will be the global crystal basis [Kas90] as well as Mirković–Vilonen cycles [MV07]. Kashiwara encoded the combinatorial structure of the crystal basis in the crystal graph; a graph with edges colored by simple roots. The path model developed by Littelmann can construct for all symmetrizable Kac–Moody algebras, the crystal graph by means of root combinatorics [Lit95]. The key object are paths in the dual of the Lie algebra of a compact torus. Littelmann constructed so called root operators, maps crucial to the construction of a crystal. Each of the root operators acts by cutting the path into segments, identifying a certain subset of segments

which are translated by the Weyl group and reassembled into a new path. The Littelmann path model is a powerful tool in the construction of crystal graphs, while still adhering to combinatorics explicit enough to implement it in Sagemath.
Of course the story does not end here. Constructing the crystal graph for a representation via the path model is useful but provides only a shadow of the representation. A combination of Mirković–Vilonen cycles and geometrization of the path model lifts this restriction.
The Mirković–Vilonen cycles provide a basis for the representation. More precisely: For a complex, reductive algebraic group G denote the Langlangs dual group by $G^\vee$. Then the Mirković–Vilonen cycles are certain finite-dimensional subvarieties of the affine Grassmannian $G(\mathbb{C}((t)))/G(\mathbb{C}[|t|])$. The closure of a $G(\mathbb{C}[|t|])$-orbit in the affine Grassmanian is called an affine Schubert variety. The Mirković–Vilonen cycles are the irreducible components of the intersection of the affine Schubert variety with certain Białynicki-Birula cells. Their classes form a basis for the intersection homology of these affine Schubert varieties. Mirković and Vilonen showed that the intersection homology is a finite-dimensional irreducible representation of the group $G^\vee$. Using Tannakian reconstrution $G^\vee$ is reconstructable from its representation making this the first approach to define $G^\vee$ without reliance on the classification of complex algebraic groups.
Gaussent and Littelmann introduced the gallery model, a coarser version of the path model. A gallery is a sequence of faces in the affine Coxeter complex of G (or rather the Weyl group of G), and one can partition the set of galleries into finite pieces separated by their type. From a gallery one can construct an irreducible, finite-dimensional, smooth variety Σ, which comes with a natural map to the affine Schubert variety. This map is a resolution of singularities, and in addition one can compute Mirković–cycles with it. A torus of G acts on Σ and the resulting Białynicki–Birula cell decomposition is centered around the torus fixed points. These fixed points are in one-to-one correspondence with galleries of a fixed type. The root operators of the gallery model act on a certain subset of these galleries, called LS-galleries. Gaussent and Littelmann described an open, dense subset of a cell centered around an LS-gallery which is mapped to an open dense subset of a Mirković–Vilonen cycle under the resolution of singularities. Combining the transitions from path model to gallery model to cycles interlocks the path model with an explicit basis of a representation.
There are two natural questions to be asked here:

1. Can the relationship of MV cycles and the path model be stated without the gallery model?

2. Does the differential geometric or compact Lie group point of view yield any merits?

The natural object to consider is the loop group $\Omega(K)$ of a compact Lie group K. It has natural connections to the path model (via the exponential map) and to the affine Grassmannian, which is a dense subset of the loop group. The main motivation of this thesis is to make this change of viewpoint with the variety Σ as its center piece. For this approach we first remold the path model into the compact Lie group setting. This includes the descending of the root operators from paths in the Cartan subalgebra to the loop group of a compact torus S of K. In this context the weight function can be reinterpreted as winding numbers around certain components of S. Key to this endeavor is proposition 2.1.2, which roughly states:

Proposition. *The root operators of the path model descend to the loop group $\Omega(S)$ of a compact torus $S \subseteq K$. The resulting path models parametrize bases of irreducible $K^\vee$-representations.*

What we are looking for is a suitable geometric object to embed a path model into. We propose a generalization of the classical Bott–Samelson manifold [BS58] for this task. Given a loop $\gamma \in \Omega(S)$ the Bott–Samelson manifold Γ_γ is a fibered product of subgroups of K associated to γ. It can be embedded into the loop group $\Omega(K)$. Our main tool in computing the Bott–Samelson manifold is Borel-de Siebenthal theory, which classifies maximal compact subgroups of maximal rank in K by means of the extended Dynkin diagram. These groups arise as the stabilizers of certain elements in a torus of K, the situation in which we will encounter them. The loop group is a symplectic manifold and we compute a formula for the restriction of the symplectic form to Γ_γ. We can give a necessary condition on γ to conclude non-degeneracy of the restricted form. This shows that the class of loops in *dominant direction* have symplectic Bott–Samelson manifolds attached. In this case we compute the moment map image of Γ_γ: It turns out to be the Weyl polytope. This is summarized in the following metatheorem joining lemma 3.5.4 and corollary 3.5.12.

Theorem. *Let $\gamma \in \Omega(S)$ be in dominant direction. The manifold Γ_γ, seen as a submanifold of $\Omega(K)$, is a symplectic S-submanifold, and the image of the moment map is the Weyl polytope of highest weight* $\mathrm{wt}(\gamma)$.

By results of Demazure [Dem74] and Hansen [Han73] the classical Bott–Samelson manifolds are complex varieties, but not in a unique way. As a remedy we propose a unique complexificitation by means of the *maximally folded loops* and parahoric subgroups of $G(\mathbb{C}((t)))$. To a maximally folded loop we associate a generalized gallery $\delta(\gamma)$ and a generalized Bott–Samelson–Demazure–Hansen variety $\Sigma(\delta(\gamma))$. We summarize theorem 3.6.12 and lemma 3.6.13:

Theorem. *Given a maximally folded loop $\gamma \in \Omega(S)$ the generalized Bott–Samelson–Demazure–Hansen variety $\Sigma(\delta(\gamma))$ and the Bott–Samelson manifold Γ_γ are diffeomorphic. The map realizing this diffeomorphism can be given in a natural way, it is S-equivariant and respects the path respectively the gallery model.*

The diffeomorphism between Γ_γ and $\Sigma(\delta(\gamma))$ gives a different point of view to the desingularization of the affine Schubert variety by $\Sigma(\delta(\gamma))$. The representative of $[g_0 : \cdots : g_k] \in \Sigma(\delta(\gamma))$ obtained from the diffeomorphism is componentwise contained in the free loop group of K. In conjuction with the results by Gaussent and Littelmann we can thus use Γ_γ to compute the MV cycles and affine Schubert varieties as subsets of $\Omega(K)$ without the Iwasawa decomposition. This gives an affirmative answer to the first question we asked. To prove the results we rely on the gallery model. Nevertheless, the construction of Γ_γ, the map from Γ_γ to the affine Schubert variety and the identification of the good paths in Γ_γ does not involve the gallery model.
One advantage of working in the differential geometric category is its extra flexibility. For us this materializes in terms of homotopies, by which we deduce three more results. We introduce the notion of a homotopy *fitted to the path model*, those homotopies which induce maps between the associated Bott–Samelson manifolds. Assume that Γ_γ is not symplectic as a subset of $\Omega(K)$. We define a procedure to obtain $\eta \in \Omega(S)$ close to γ such that Γ_η is symplectic and there exists a homotopy from γ to η fitted to the path model. The induced map $\Gamma_\gamma \to \Gamma_\eta$ is surjective. By introducing a shrinking algorithm we can give a larger class of loops for which the moment map image is the Weyl polytope. The shrinking algorithm produces a homotopy fitted to the path model from γ to a loop η such that η is contained in the 1-skeleton of the Coxeter complex. Using the flexibility of the differential geometric category we give a partial answer to the second question we asked: The loop group $\Omega(K)$ allows different explicit embeddings of the affine Schubert variety and MV cycles using homotopies.

Theorem (Theorem 4.4.4)**.** *Given a maximally folded loop γ in S such that Γ_γ contains a loop η in dominant direction, the map*

$$\begin{aligned} \pi_\gamma : \Gamma_\gamma \to \Sigma(\delta(\gamma)) &\to \Omega(K) \\ [g] &\mapsto g.\nu \end{aligned}$$

is well-defined for a large class of loops ν. The image $\mathrm{Im}(\pi_\nu)$ *is diffeomorphic to the affine Schubert variety at* $\mathrm{wt}(\eta)$*. Moreover the embedding of the affine Schubert variety as* $\mathrm{Im}(\pi_\nu)$ *and the identity map are isotopic. If* $\eta \in \Gamma_\gamma$ *is an element of the crystal generated by* γ *and we denote by* $\Gamma_{\gamma,\eta}$ *the cell centered at* η*, then* $\pi(\overline{\Gamma_{\gamma,\eta}})$ *is homeomorphic to an MV cycle and the associated embedding is isotopic to the identiy.*

In chapter 1 we give necessary notations and facts about the objects used. This includes the notations and conventions we use for compact Lie groups and algebraic groups in section 1.1, Borel–de Siebenthal theory in section 1.2, the loop group and its different structures in section 1.3. We continue with the algebraic setting and introduce the path model in 1.4 and versions of the gallery model in section 1.5. This includes 1-skeleton galleries and the Bott–Samelson–Demazure–Hansen variety $\Sigma(\delta)$. We conclude with the definition of the affine Schubert varieties, semi-infinite orbits, MV cycles and MV polytopes in section 1.6.
Chapter 2 contains the descend of the root operators to the loop group of S and a review of the Birkhoff decomposition in light of the descended path model.
Chapter 3 contains our first two main results. We introduce the Bott–Samelson manifolds in section 3.1 and review the classical theory via partial flag manifold and coadjoint orbits in sections 3.2 and 3.3. In section 3.4 we relate Γ_γ to the path model. Section 3.5 is devoted to our result on the symplecticness of Γ_γ and computation of the moment map image, while we construct the diffeomorphism $\Gamma_\gamma \to \Sigma(\delta)$ in section 3.6.
In chapter 4 we examine the relation between homotopies of γ and the associated Bott–Samelson manifolds. We define the notion of *homotopy fitted to the path model* and introduce the method to obtain from a non-symplectic Γ_γ a close but symplectic Γ_η in section 4.1. We introduce the shrinking algorithm in section 4.2. Section 4.3 is a sidenote on MV polytopes via results of Ehrig in relation to Γ_γ. We describe the embeddings of MV cycles and affine Schubert varieties into $\Omega(K)$ in section 4.4.
We end the thesis with chapter 5, where we discuss possible future research directions.

1 Preliminaries

1.1 Notation

Let K be a compact simple Lie group of rank n. Its maximal compact torus is denoted by $S \cong (S^1)^n$. The complexification $K_{\mathbb{C}}$ of K is a simple complex, algebraic group. It has a maximal, complex torus $S_{\mathbb{C}} \cong (\mathbb{C}^*)^n$, the complexification of S. The Lie algebra of K will be denoted $\mathrm{Lie}(K)$ and for other groups likewise. The character lattice $X^*(S) := \mathrm{Hom}(S, S^1)$ of S can be identified as a subset of $\mathrm{Lie}(S)^*$. Similarly we can identify the cocharacters $X_*(S)$ as a subset of $\mathrm{Lie}(S)$. We denote the evalution of $\lambda \in \mathrm{Lie}(S)^*$ on $X \in \mathrm{Lie}(S)$ by $\langle X, \lambda \rangle$. The Lie algebra $\mathrm{Lie}(K_{\mathbb{C}})$ decomposes into the direct sum of root subspaces $\mathrm{Lie}(K_{\mathbb{C}})_\alpha$ where $\alpha \in X^*(S)$ is extended to $\mathrm{Lie}(S_{\mathbb{C}})$ linearly. The non-zero characters α for which the root subspace is non-zero are called roots. The set of all roots will be denoted by Φ. The set of simple roots, denoted by $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subseteq X^*(S)$, is a $\mathbb{Z}$-basis of the root lattice R such that every root can be expressed as a non-negative, integral linear combination or a non-positive one. The simple coroots $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ achieve the same for the coroot lattice Q. The positive (co-)roots are those (co-)roots which can be written as a non-negative linear combination of the simple (co-)roots and will be denoted by $\Phi^{(\vee),+}$. Write a root $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ then the *height* of β is $\mathrm{ht}(\beta) = \sum_{\alpha \in \Delta} k_\alpha$. The highest root is the unique positive root of maximal height and analogous for the highest coroot. A partial order $\leq$ is defined on $X^*(S)$ by prescribing $\lambda \leq \beta$ if $\lambda - \beta$ is a non-negative linear combination of simple roots. We use the same notation for the analogously defined partial order on $X_*(S)$.

The Weyl group $W = W_K$ is defined as the normalizer of the torus modulo the torus itself,

$$W_K = \mathrm{N}_K(S)/S = \mathrm{N}_{K_{\mathbb{C}}}(S_{\mathbb{C}})/S_{\mathbb{C}}.$$

It acts faithfully on $\mathrm{Lie}(S)$ and is generated by reflections in the hyperplanes defined by the simple roots. These reflections will be denoted by s_α. The length $l(w)$ of $w \in W_K$ is the length of the shortest word in the $s_\alpha, \alpha \in \Delta$ that represents w. For every positive root α there exist $\mathcal{H}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha$ such that the map

$$\varphi_\alpha : \mathfrak{su}_2 \to \mathrm{Lie}(K)$$
$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mapsto \mathcal{H}_\alpha, \; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \mathcal{J}_\alpha, \; \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \mapsto \mathcal{K}_\alpha$$

is a monomorphism of Lie algebras. It integrates to a group homomorphism $\Phi_\alpha : \mathrm{SU}_2 \to K$ which is either injective or has kernel $\{\pm 1\}$.

On every compact Lie group exists a bi-invariant Riemannian metric $(\, , \,)$ induced by the *Killing form* on $\mathrm{Lie}(K)$. The Lie group K is uniquely identified by the quadruple of simple roots, simple coroots, character and cocharacter lattice $(\Delta, \Delta^\vee, X^*(S), X_*(S))$. More precisely the quotient $R/X^*(S)$ is isomorphic to the fundamental group of K while the quotient $Q/X_*(S)$ is isomorphic to the center of K. By switching roots with coroots and characters with cocharacters one obtains the data $(\Delta^\vee, \Delta, X_*(S), X^*(S))$ for the *Langlands dual group* $K^\vee$. Existence of $K^\vee$ is provided by the classification of simple compact Lie groups.

The Lie group exponential of S is a covering map and group homomorphism $\exp : \mathrm{Lie}(S) \to S$ which has kernel $X_*(S)$. The Weyl vector ρ is the halfsum of all positive roots, in the same way $\rho^\vee$ is defined.
We briefly recall the connection of representations of a compact simple Lie group and its Lie algebra. The elements dual to the simple coroots are called *fundamental weights*, we will denote the element dual to $\alpha^\vee$ by $\varpi_{\alpha^\vee}$. As we also fixed an enumeration of Δ, we can enumerate the fundamental weights by the same Indices. Elements of the $\mathbb{Z}$-module spanned by the fundamental weights are called *infinitesimal weights* of K. We denote the set of infinitesimal weights by P. For K simply-connected P coincides with the character lattice. The dominant Weyl chamber $\mathfrak{C}$ is defined by the choice of simple roots as $\mathfrak{C} = \{X \in \mathrm{Lie}(S)^* \mid \langle \alpha_i^\vee, X \rangle \geq 0 \; i = 1, \ldots, n\}$. The antidominant Weyl chamber is defined by swapping the simple roots with their negatives, we denote it by $\mathfrak{C}_{-\infty}$. An infinitesimal weight is called *dominant* if it is an element of the dominant Weyl chamber. A weight is dominant if and only if it is an integral, positive linear combination of fundamental weights. The set of dominant weights is denoted by P_+ and similiary for subsets of P. Given an infinitesimal dominant weight λ, there exists a simple, finite-dimensional $\mathrm{Lie}(K)$ representation, denoted by $V(\lambda)$; every finite-dimensional $\mathrm{Lie}(K)$ representation decomposes as a direct sum of such representations. The representation $V(\lambda)$ is a representation of K if and only if λ is a cocharacter. All notions for roots and weights are valid for coroots and coweights and will be denoted similarly.

Let Φ' be a subset of the positive (co-)roots. We say that a (co-)weight λ is regular with respect to Φ' if $\alpha^\vee(\lambda)$ is positive for all $\alpha \in \Phi'$.

A *parabolic subgroup* of $K_{\mathbb{C}}$ is a subgroup P of $K_{\mathbb{C}}$ such that the quotient $K_{\mathbb{C}}/P$ is a projective manifold. The quotient is called a *partial flag manifold*, it is smooth. A minimal parabolic subgroup is called a *Borel subgroup*. Our choice of a torus $S_{\mathbb{C}}$ of $K_{\mathbb{C}}$ and of positive roots Φ^+ fixes a standard Borel subgroup which we will denote by B. It is the Lie group associated to the Lie subalgebra

$$\mathrm{Lie}(B) = \mathrm{Lie}(S_{\mathbb{C}}) \oplus \bigoplus_{\alpha\in\Phi_+} \mathrm{Lie}(K_{\mathbb{C}})_\alpha.$$

A parabolic subgroup containing B is called *standard*, and every parabolic subgroup is conjugate to a standard parabolic subgroup. The standard parabolic subgroups are in bijection with subsets I of the nodes of the Dynkin diagram of K. Denote by $\Phi(I)$ the set of roots which are linear combinations of the simple roots in I and the superscript $+$ the subset of positive roots. Then the parabolic subgroup P_I associated to I is the group with Lie algebra

$$\mathrm{Lie}(P_I) = \mathrm{Lie}(B) \oplus \bigoplus_{\alpha\in\Phi(I)^+} \mathrm{Lie}(K_{\mathbb{C}})_{-\alpha}.$$

The parabolic subgroup P_I contains a subgroup L_I with Lie algebra

$$\mathrm{Lie}(L_I) = \mathrm{Lie}(S_{\mathbb{C}}) \oplus \bigoplus_{\beta\in\Phi(I)} \mathrm{Lie}(K_{\mathbb{C}})_\beta$$

called *standard Levi subgroup* and is a reductive complex algebraic group itself. Every subgroup conjugate to a standard Levi subgroup is called a *Levi subgroup*. Sometimes these subgroups are called reductive Levi subgroups while the maximal semisimple algebraic subgroup of L_I is called a Levi subgroup in this case. We will not use this variation.

We obtain an inclusion of the Weyl group W_{L_I} in the Weyl group W_K. It is the subgroup generated by s_α for $\alpha \in I$.

The connected algebraic subgroup U of B with Lie algebra

$$\mathrm{Lie}(U) = \bigoplus_{\alpha\in\Phi^+} \mathrm{Lie}(K_{\mathbb{C}})_\alpha$$

is called the unipotent radical of B.

1.2 Borel–de Siebenthal theory

In the definition of the Bott–Samelson variety down the road we will be interested in subgroups of the compact group K which contain the maximal torus S. These subgroups are called subgroups of maximal rank and can be classified by subsets of the extended Dynkin diagram of K by *Borel–de Siebenthal theory*. In this section we will review this theory and deduce some consequences.

Definition 1.2.1. *The fundmantal alcove $\Delta_f \subseteq \mathrm{Lie}(S)$ is defined as*

$$\Delta_f = \left\{ X \in \mathrm{Lie}(S) \mid 0 \leq \beta(X) \leq 1, \beta \in \Phi^+ \right\}.$$

When we study the gallery model we will look at the fundamental alcove more in depth. At this point we are only interested in the vertices of Δ_f. As the highest root is positive we can write it as $\sum m_i \alpha_i$ with non-negative m_i. The vertices of the fundamental alcove can be deduced as $v_0 = 0$ and $v_i = m_i^{-1} \varpi_i^\vee$, where $\varpi_i^\vee$ are the cofundamental weights.

Definition 1.2.2. *The extended Dynkin diagram of K is the Dynkin diagram of K with an added node for the highest root, with edges to this node given by the same rules as for the other nodes. To every node we attach as a label m_i and the highest root gets the label $m_0 = 1$. We will refer to the node with the label m_i as the i-th node.*

Theorem 1.2.3 ([BS49]). *Any maximal connected subgroup of maximal rank in K is conjugate to (the connected component of the neutral element of) a subgroup which stabilizes either the edge connecting v_0 with v_i where $m_i = 1$ or it stabilizes a vertex for which m_i is prime. Furthermore the resulting group is a reductive, compact group and its type is deducible from the extended Dynkin diagram. In the first case the resulting group is given by the diagram in which the i-th and 0-th node are deleted, in the second case the i-th node is deleted.*

In our examples the case m_i not prime will never occur because of the following fact.

Remark 1.2.4. *The only groups with $m_i \neq 1$ not prime are of type E_7, E_8 and F_4.*

We will not only be interested in the maximal connected subgroup of maximal rank, but all connected subgroups of maximal rank.

Remark 1.2.5. *By passing from K to a maximal connected subgroup of maximal rank, one can inductively find all connected subgroups of maximal rank in K.*

To illustrate let us give an example.

Example 1.2.6. *Let $K = \mathrm{SU}(3)$ then all maximal, compact subgroups are conjugate to*

$$K_1 = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix} \right\}, \; K_2 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix} \right\}.$$

The extended Dynkin diagram of K is a 3-loop and the corresponding subgraphs for K_1 and K_2 are the diagrams consisting of just the second respectively first node. One can already observe here that the list one obtains contains duplicate entries as also K_1 and K_2 are conjugate. There is a third group we could associate via deletion of two nodes in the extended Dynkin diagram, which is the stabilizer of the hyperplane defined by the highest root. This is conjugate to both of the above groups by a lift of the simple reflections s_1 resp. s_2, but will play an important role later on.

Let us record the last observation in general.

Lemma 1.2.7. *There exists a compact, connected subgroup to every subset of nodes of the extended Dynkin diagram of K by deleting the nodes outside of the subset and applying Borel–de Siebenthal.*

We will reobserve this behaviour in the context of parahoric subgroups of the algebraic loop group.

1.3 The loop group

In this section we introduce the based loop group $\Omega(K)$ as well as the free loop group $\mathrm{L}(K)$, subgroups, different topologies and additional structures.

Definition 1.3.1. *The based loop group $\Omega(K)$ consists of Sobolev class 1 loops in K, i.e. absolutely continuous loops of finite energy, see for example [Kli78]. Our main reference concerning loop groups will be [PS86].*
Equipped with pointwise multiplication $\Omega(K)$ becomes a group. The based loop group is an infinite dimensional manifolds modeled on the space $\Omega(\mathrm{Lie}(K))$ of Sobolev class 1 loops in $\mathrm{Lie}(K)$ based at 0. These structures are compatible making $\Omega(K)$ into an infinite dimensional Lie group. The Lie algebra of $\Omega(K)$ is $\Omega(\mathrm{Lie}(K))$.

It can also be realized as the quotient of the space of free loops.

Definition 1.3.2. *By $\mathrm{L}(K)$ we denote the space of free loops of Sobolev class 1 in K; it is also a group via pointwise multiplication. The constant loops* *form a subgroup isomorphic to K and $\Omega(K)$ is a subgroup as well. The free loop group $L(K)$ is an infinite dimensional manifold modeled on $L(\mathrm{Lie}(K))$, the space of loops of Sobolev class 1 in $\mathrm{Lie}(K)$. The natural map*

$$\begin{aligned}\Omega(K) &\to L(K)/K\\ \gamma &\mapsto \gamma K\end{aligned}$$

is a diffeomorphism. As a group $L(K)$ is the semidirect product of K and $\Omega(K)$.

One can choose a different regularity for the loops one wants to consider and obtain subgroups of $L(K)$ or $\Omega(K)$. For example the polynomial loop group.

Definition 1.3.3. *The polynomial loop group $\Omega^{pol}(K)$ is the subset of $\Omega(K)$ consisting of loops which have a finite Fourier expansion.*

It has the analogous realization as $L^{\mathrm{pol}}(K)/K$. There is another realization, which we will be able to define in subsection 1.5.2. This is the reason for our interest in $\Omega^{\mathrm{pol}}(K)$.
We will need to introduce on $\Omega^{\mathrm{pol}}(K)$ a topology different from its subspace topology. We assume K is embedded into some special unitary group $\mathrm{SU}(N)$ which is always possible. Let $H = \mathrm{L}^2(S^1, \mathbb{C}^N)$ be the space of square-integrable functions $S^1 \to \mathbb{C}^N$ and denote by $\mathrm{Gr}(H)$ the Grassmannian of H. It consists of subspaces of H with some complementary dimension conditions which are discussed in detail in [PS86][chapter 7]. One such subspace is given by H_+ the closed span of $z^k v$ for $k \geq 0$ and $v \in \mathbb{C}^N$. Define Gr_j to be the set of vector spaces W in $\mathrm{Gr}(H)$ subject to the conditions

$$\begin{aligned}z^j H_+ \subseteq &W \subseteq z^{-j} H_+\\ \dim(W/z^j H_+) &= \dim(z^{-j} H_+/W).\end{aligned}$$

The map

$$\begin{aligned}\mathrm{Gr}_j &\to \mathrm{Gr}(jN, 2jN)\\ W &\mapsto W/z^j H_+\end{aligned}$$

identifies the space Gr_j with the finite-dimensional Grassmanian $\mathrm{Gr}(jN, 2jN)$ of jN-dimensional subspaces of $\mathbb{C}^{2jN} \cong z^{-j}H_+/z^j H_+$. It is straightforward to compute $\mathrm{Gr}_j \subseteq \mathrm{Gr}_{j+1}$.

Proposition 1.3.4 ([PS86]). *The map*

$$\begin{aligned}\Omega(K) &\to \mathrm{Gr}(H)\\ \gamma &\mapsto \gamma H_+\end{aligned}$$

is an embedding and the image of the polynomial loop group $\Omega^{pol(K)}$ *is contained in* $\bigcup_{j\geq 0} \mathrm{Gr}_j$.

Now we can define $\Omega_j = \Omega^{\mathrm{pol}}(K) \cap \mathrm{Gr}_j$ and obtain the polynomial loop group as the direct limit of the Ω_j. This gives rise to the direct limit topology on $\Omega^{\mathrm{pol}}(K)$.

Proposition 1.3.5 ([Mar10, HHJM06])**.** *We choose the analytic topology on* Gr_j. *The direct limit topology on* $\Omega^{pol}(K)$ *is finer than the subspace topology.*

Remark 1.3.6. *In [HHJM06] the authors give an example which provides that in general the direct limit topology is strictly finer than the subspace topology.*

Of course there is also the Zariski topology on Gr_j. The Ω_j are subvarities of Gr_j and thus inhibit a Zariski topology of their own [AP83]. Hence we also obtain the direct limit Zariski topology.

We want to make an immediate observation about the coweight and weight lattice in relation to the loop group.

Remark 1.3.7. *The coweight lattice* X_* *embeds into* $\Omega(K)$.

Remark 1.3.8. *Every* $\lambda \in X^*(S)$ *defines a map* $\Omega(S) \to \Omega(S^1)$; $\gamma \mapsto \lambda \circ \gamma$.

The based loop group carries the structure of a symplectic manifold as follows.

Definition 1.3.9. *A left invariant symplectic form* ω *on* $\Omega(K)$ *is defined by left translation of the skew-form on* $\mathrm{Lie}(\Omega(K)) = \Omega(\mathrm{Lie}(K))$ *given by*

$$S(X,Y) = \frac{1}{2\pi}\int_0^{2\pi} \langle X'(e^{i\varphi}), Y(e^{i\varphi})\rangle \,\mathrm{d}\,\varphi$$

The torus S acts via conjugation and S^1 acts via rotation of the loop.

$$e^{i\theta}.\gamma(e^{i\varphi}) := \gamma(e^{i(\varphi+\theta)})$$

This definition is to be understood in the realization $L(K)/K$; otherwise a factor of $\gamma(e^{i\theta})^{-1}$ needs to be added.
The actions of S and S^1 commute and thus the bigger torus $S \times S^1$ acts on $\Omega(K)$ with resulting moment map $\mu : \Omega(K) \to \mathrm{Lie}(S) \times \mathbb{R}$ [AP83]. The components of the map are given by the formulas

$$\mu_{\mathrm{Lie}(S)}(\gamma) := \mathrm{pr}_{\mathrm{Lie}(S)}\left(\frac{1}{2\pi}\int_0^{2\pi} \gamma(e^{i\varphi})^{-1}\gamma'(e^{i\varphi})\,\mathrm{d}\varphi\right)$$
$$\mu_{\mathbb{R}}(\gamma) := \frac{1}{4\pi}\int_0^{2\pi} \|\gamma(e^{i\varphi})^{-1}\gamma'(e^{i\varphi})\|^2\,\mathrm{d}\varphi$$

where the latter one is the energy function and $\mathrm{Lie}(S)^*$ has been identified with $\mathrm{Lie}(S)$ via the Killing form.

The symplectic structure of $\Omega(K)$ is part of a Kähler structure. We will define the complex structure J following [Pre82]. Consider the case $K = S^1$ first. We can identify $\mathrm{Lie}(\Omega(S^1)) = H^1(S^1, \mathbb{R})/\mathbb{R}$. Thus we can write every $f \in \mathrm{Lie}(\Omega(S^1))$ as its Fourier expansion $f = \sum_{k \neq 0} f_k z^k$. Now we can define

$$J(f) = \sum_{k<0} -i f_k z^k + \sum_{k>0} i f_k z^k.$$

For the general case note $\mathrm{Lie}(\Omega(K)) \cong \mathrm{Lie}(K) \otimes \mathrm{Lie}(\Omega(S^1))$.

Definition 1.3.10. *For a sublattice $L \subseteq P^\vee$ denote by $\Pi(L)$ the set of Sobolev 1 paths in $\mathrm{Lie}(S)$ with endpoint in L, i.e.*

$$\Pi(L) := \left\{\pi : [0,1] \to \mathrm{Lie}(S) \mid \pi(0) = 0, \pi(1) \in L, \pi \in H^1([0,1], \mathrm{Lie}(S))\right\}$$

We will refer to elements of $\Pi(L)$ simply as paths. On the set $\Pi(L)$ we have the operation $*$ of concatenation of paths

$$\pi_1 * \pi_2(t) = \begin{cases} \pi_1(2t) \text{ for } t \in [0, \frac{1}{2}] \\ \pi_2(2t-1) + \pi_1(1) \text{ for } t \in [\frac{1}{2}, 1] \end{cases}$$

Of course one could define the same operation on $\Omega(K)$, but the two operations are homotopy equivalent.

Lemma 1.3.11. *The map*

$$\psi : \Pi(X_*(S)) \to \Omega(S)$$
$$\pi \mapsto \exp \circ \pi$$

is well-defined, continuous and a bijection.

Proof. Note that $\pi(0), \pi(1) \in X_*(S)$. As exp is a covering map $\mathrm{Lie}(S) \to S$ and $X_*(S)$ is the kernel of exp, it is clear that the map is well-defined. Bijectivity follows from the usual lifting property of coverings. $\square$

1.4 The Littelmann path model

In this section we will define the root operators and how to construct a Littelmann path model with their help. We define crystals and crystal graphs and remind the reader about properties of the path model.

Definition 1.4.1. *To define what a Littelmann path model is, we need the maps*

$$\varepsilon_\alpha(\pi) = -\inf_{0\le s\le 1} \alpha(\pi(s)) \tag{1.1}$$

$$\varphi_\alpha(\pi) = \alpha(\pi(1)) - \inf_{0\le s\le 1} \alpha(\pi(s)) \tag{1.2}$$

$$e_\alpha(\pi)(t) = \pi(t) - \varepsilon_\alpha(\pi)\alpha^\vee - \min(-\varepsilon_\alpha(\pi) - 1, \inf_{t\le s\le 1}(\alpha(\pi(s))))\alpha^\vee, \tag{1.3}$$

$$f_\alpha(\pi)(t) = \pi(t) - \varepsilon_\alpha(\pi)\alpha^\vee - \min(-\varepsilon_\alpha(\pi) + 1, \inf_{t\le s\le 1}(\alpha(\pi(s))))\alpha^\vee. \tag{1.4}$$

The element $f_\alpha(\pi)$ is well-defined if and only if $\varphi_\alpha(\pi) \ge 1$, while $e_\alpha(\pi)$ is well-defined if and only if $\varepsilon_\alpha(\pi) \ge 1$ as continuity at 0 fails otherwise. We will take the stance of f_α, e_α as partial functions which are just not defined for π whenever the image is not a well-defined element of $\Pi(X_*)$. The f_α are called lowering operators, while the e_α are called raising operators. Both are referred to as root operators.

The explicit form of the formulas is taken from the thesis of Chhaibi [Chh14]. For a path π such that $\rho^\vee * \pi$ is contained in the dominant Weyl chamber we define $\mathcal{A}\pi$ as the smallest set which contains π and is stable under the root operators. By [Lit97] we know $\mathcal{A}\pi$ is finite. It is the set of all monomials in the root operators applied to π.

Definition 1.4.2. *The crystal graph $\mathcal{G}_\pi$ of $\pi \in \Pi_+$ is a colored, directed graph with vertices $\mathcal{A}\pi$. There is an arrow of color $\alpha \in \Delta$ from $\eta \to \eta'$ if and only if $f_\alpha(\eta) = \eta'$.*

Littelmann proved the following theorems [Lit97] relating the path model to the representation theory of $\mathrm{Lie}(K_\mathbb{C})$.

Theorem 1.4.3 (Isomorphism Theorem)**.** *The graphs $\mathcal{G}_\pi$ and $\mathcal{G}_{\pi'}$ are isomorphic if and only if $\pi(1) = \pi'(1)$.*

Theorem 1.4.4 (Character Formula)**.** *Define the character of a path model $\mathcal{A}\pi$ as* $\mathrm{char}(\mathcal{A}\pi) := \sum_{\eta\in\mathcal{A}\pi} e^{\eta(1)}$. *We have that the character* $\mathrm{char}(\mathcal{A}\pi)$ *is equal to the character of the irreducible* $\mathrm{Lie}(K)$*-module $V_{\pi(1)}$ of highest weight $\pi(1)$.*

Theorem 1.4.5 (Generalized Littlewood-Richardson rule)**.** *Let $\lambda, \mu \in P_+$ be dominant integral weights of* $\mathrm{Lie}(K_\mathbb{C})$ *and $\pi_1, \pi_2 \in \Pi(P)$ be such that $\pi_1(1) = \lambda$ and $\pi_2(1) = \mu$. Then the tensor product $V(\lambda) \otimes V(\mu)$ decomposes into simple modules as*

$$V(\lambda) \otimes V(\mu) \cong \bigoplus V(\lambda + \eta(1))$$

where the sum ranges over all $\eta \in \mathcal{A}\pi_2$ such that the concatenation $\pi_1 * \eta$ is contained in the dominant Weyl chamber.

Theorem 1.4.6 (Restriction formula)**.** *Let $L \subset \mathrm{Lie}(K_{\mathbb{C}})$ be a Levi subalgebra. Denote by $U(\lambda)$ the simple L-module of highest weight λ. Then $V(\mu)$ decomposes as an L-module as*

$$V(\mu) = \bigoplus_{\nu} U(\nu)$$

where ν runs over all paths in $\mathcal{A}\mu$ such that the projection of ν to the weight space of L is contained in the dominant Weyl chamber of L.

The following theorem connecting the path model to the theory of crystal graphs was proven by Kashiwara and Joseph.

Theorem 1.4.7 ([Jos95, Kas96])**.** *The crystal graph $\mathcal{G}_\pi$ is isomorphic to Kashiwara's crystal graph of $V_q^{\pi(1)}$ the irreducible representation of the quantum group $\mathrm{Lie}(K_{\mathbb{C}})_q^\vee$ of highest weight $\pi(1)$.*

We will at this place remind the reader what a crystal is. This is for completeness and will not occur later.

Definition 1.4.8. *Let $\mathbb{B}$ be a finite set with a* weight map $\mathrm{wt} : \mathbb{B} \to P^\vee$ *and for each simple root maps $\varepsilon_\alpha, \varphi : \mathbb{B} \to \mathbb{Z}$ and partial maps*

$$e_\alpha, f_\alpha : \mathbb{B} \to \mathbb{B}.$$

The set $\mathbb{B}$ with the structural maps is called a finite dimensional crystal *of the group $K_{\mathbb{C}}^\vee$ if the structural maps fulfill the following axioms (i)-(iv):*

(i) For each simple root α it holds $\varphi_\alpha(b) = \varepsilon_\alpha(b) + \langle \alpha, \mathrm{wt}(b) \rangle$.

(ii) If $e_\alpha(b) \neq 0$, then

$$\begin{aligned} \varepsilon_\alpha(e_\alpha(b)) &= \varepsilon_\alpha(b) - 1 \\ \varphi_\alpha(e_\alpha(b)) &= \varphi(b) + 1 \\ \mathrm{wt}(e_\alpha(b)) &= \mathrm{wt}(b) + \alpha^\vee. \end{aligned}$$

(iii) If $f_\alpha(b) \neq 0$, then

$$\begin{aligned} \varepsilon_\alpha(f_\alpha(b)) &= \varepsilon_\alpha(b) + 1 \\ \varphi_\alpha(f_\alpha(b)) &= \varphi(b) - 1 \\ \mathrm{wt}(e_\alpha(b)) &= \mathrm{wt}(b) - \alpha^\vee. \end{aligned}$$

(iv) *The maps* e_α, f_α *are partial inverses, e.g.* $f_\alpha(b_1) = b_2$ *if and only if* $b_1 = e_\alpha(b_2)$.

For later treatment of the path model we will also introduce the set of integral paths. This definition is not the same as in [Lit97], but rather from [BGL20].

Definition 1.4.9. *A path* $\pi \in \Pi(X_*)$ *is called* integral *if for every simple root* α *every local minimum of the function* $\alpha \circ \tilde{\pi}$ *is in* $\mathbb{Z}$ *for all* $\tilde{\pi} \in \mathcal{A}\pi$. *The set of integral paths will be denoted by* $\Pi(X_*)_{int}$.

Proposition 1.4.10 ([BGL20])**.**

1. *A path which is contained in the dominant Weyl chamber is integral if the function* $\alpha_i(\pi(t))$ *is weakly increasing for all* $i \in \{1, \dots, n\}$.
2. *If* π *is an integral path, there exists a unique path* $\tilde{\pi} \in \mathcal{A}\pi$ *contained in the dominant Weyl chamber.*

On integral paths the action of the root operators can be described in simpler terms.
Denote by m_α the minimal value of $\alpha \circ \pi$ and $s \in [0,1]$ the maximum such that m_α is attained. If $\alpha \circ \pi(1) - m_\alpha \geq 1$, then there exists $t \in [0,1]$ such that $\alpha \circ \pi(t) = m_\alpha + 1$. We proceed by cutting π into three pieces π_1, π_2, π_3, where $\pi_1 = \pi|_{[0,s]}$, $\pi_2 = \pi|_{[s,t]}$ and $\pi_3 = \pi|_{[t,1]}$. Of course it holds $\pi = \pi_1 * \pi_2 * \pi_3$.

Lemma 1.4.11. *The lowering operator acts on* π *by the reflection* s_α *on* π_2 *and translation by* α *on* π_3

$$f_\alpha(\pi) = \pi_1 * s_\alpha(\pi_2) * \tau_\alpha(\pi_3).$$

There is a similar formula for the raising operator.

The lift of the root operators to the loop group is immediate by the map ψ of lemma 1.3.11 and its inverse.

Definition 1.4.12. *For every* $\alpha \in \Delta$ *define the lowering operator*

$$\begin{aligned} F_\alpha : \Omega(S) &\to \Omega(S) \\ \gamma &\mapsto \psi \circ f_\alpha \circ \psi^{-1}(\gamma) \end{aligned}$$

where f_α *denotes the usual lowering operator for the Littelmann path model. Similarly define the raising operator*

$$\begin{aligned} E_\alpha : \Omega(S) &\to \Omega(S) \\ \gamma &\mapsto \psi \circ e_\alpha \circ \psi^{-1}(\gamma) \end{aligned}$$

where e_α *denotes the usual raising operator for the Littelmann path model.*

Proposition 1.4.13. *The partial maps F_α, E_α are well-defined.*

Proof. As $\psi^{-1}(\gamma)(1) \in X_*$ by the proof of 1.3.11 and f_α, e_α change the end-point by $\pm\alpha$ we have

$$\left(f_\alpha \circ \psi^{-1}(\gamma)\right)(1), \left(e_\alpha \circ \psi^{-1}(\gamma)\right)(1) \in X_*(S)$$

whenever these are defined. □

1.5 The gallery model

The gallery model can be understood as a roughed out version of the path model. The replacement for the paths are galleries in the affine Coxeter complex of K. Our treatment here follows [GL05].

Definition 1.5.1. *The affine Weyl group $W^{aff} := W \ltimes R^\vee$ acts on* $\mathrm{Lie}(S)$ *via affine reflections. The affine reflection hyperplanes are of the form*

$$H_{\beta,m} := \{X \in \mathrm{Lie}(S) \mid \langle X, \beta\rangle = m\}$$

for β a positive root and m an integer. We will denote the reflection with respect to such a hyperplane by $s_{\beta,m}$. The symbol $\boldsymbol{H}$ denotes the union of all these hyperplanes. For every affine hyperplane we also have the corresponding closed affine half-spaces

$$H^+_{\beta,m} := \{X \in \mathrm{Lie}(S) \mid \langle X, \beta\rangle \geq m \},$$
$$H^-_{\beta,m} := \{X \in \mathrm{Lie}(S) \mid \langle X, \beta\rangle \leq m\}.$$

The connected components of $\mathrm{Lie}(S) \setminus \boldsymbol{H}$ *are called* open alcoves*; the closure of an open alcove is called an* alcove*. A face F is a subset of* $\mathrm{Lie}(S)$ *obtained by choosing for every pair (β, m) of a positive root β and an integer m one of the associated affine half-spaces or the affine hyperplane and taking the intersection over all pairs. By replacing the closed half-spaces with the open ones the* open face *F^o is defined. The set of all faces defines a polyhedral complex, called the* (affine) Coxeter complex*.*

The open faces are a partition of $\mathrm{Lie}(S)$.

Definition 1.5.2. *The affine span $\langle F^o\rangle_{aff} = \langle F\rangle_{aff}$ is the* support *of the (open) face. We will refer to the support of a codimension-one face as a* wall of an alcove *following the respective literature. More generally we will refer to hyperplanes as* walls*, as is usual in the theory of buildings.*

Definition 1.5.3. *A fundamental domain for the action of the affine Weyl group is given by the* fundamental alcove

$$\Delta_f := \left\{X \in \mathrm{Lie}(S) \mid 0 \leq \beta(X) \leq 1, \beta \in \Phi^+\right\}$$

defined already in section 1.2. Furthermore the affine Weyl group W^{aff} is generated by the reflections in the walls of the fundamental alcove. They are the elements s_α for $\alpha \in \Delta$ of the finite Weyl group and the element $s_0 = s_{\beta,1}$ for β the highest root. The type *of a face F of the fundamental alcove is the set of reflection hyperplanes it is contained in,*

$$\mathrm{type}(F) = \{s_{\beta,m} \mid F \subseteq H_{\beta,m}\}.$$

The type *of a face F in the Coxeter complex is defined as the type of the unique face of the fundamental alcove in the affine Weyl group orbit of F. The* length *$l(w)$ of $w \in W^{aff}$ is the length of the shortest word in the letters $s_\alpha, \alpha \in \Delta$ and s_0 which represents w.*

Example 1.5.4. *We will consider* $\mathrm{SU}(3)$ *as our example.*

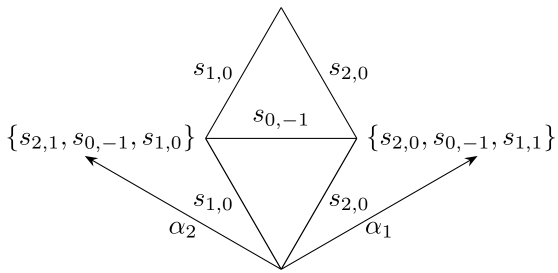

Figure 1.1: Types of faces for the fundamental alcove and one neighboring alcove

Definition 1.5.5. *Let $A, B \subseteq \mathrm{Lie}(S)$.*

1. *A W^{aff}-translate of a Weyl chamber is called a* sector*. The translate of* 0 *by the same element of W^{aff} is called the* vertex *of the sector.*
2. *Two sectors $\mathfrak{s}, \mathfrak{s}_0$ are called* equivalent *if there is a sector $\mathfrak{s}_1$ contained in their intersection: $\mathfrak{s}_1 \subseteq \mathfrak{s} \cap \mathfrak{s}_0$.*
3. *A hyperplane $H_{\alpha,m}$ separates A and B if $A \subset H^+_{\alpha,m}$ and $B \subset H^{-,o}_{\alpha,m}$ or $A \subset H^-_{\alpha,m}$ and $B \subset H^{+,o}_{\alpha,m}$, where we denote by a superscript o the open halfspace.*

4. *For an equivalence class $\mathfrak{C}$ of sectors we say A is separated from $\mathfrak{C}$ by a hyperplane $H_{\alpha,m}$ if there exists a representative $\mathfrak{s} \in \mathfrak{C}$ such that A is separated from $\mathfrak{s}$ by the hyperplane $H_{\alpha,m}$.*

Now we define the objects on which the root operators will act: The galleries.

Definition 1.5.6. *A combinatorial gallery δ joining 0 with $\lambda \in X_*$ is a sequence of faces of* $\mathrm{Lie}(S)$,

$$\delta = (0 \subset \Gamma_0' \supset \Gamma_1 \subset \cdots \supset \Gamma_p \subset \Gamma_p' \supset \lambda)$$

such that all Γ_j' have dimension $\dim H_\lambda$, all Γ_j are common faces of Γ_{j-1}' and Γ_j' of relative codimension one. Here H_λ denotes the intersection of hyperplanes $H_{\alpha,0}$, which contain λ. We say δ has target λ.

Remark 1.5.7. *As Gaussent and Littelmann remarked in [GL05] the dimension conditions are not necessary to define the gallery model. It simplifies the definition of the root operators for the gallery model; thus we will follow their approach. Later on we will associate a gallery to a loop. As this procedure does not always produce a gallery in the sense we defined here, we will drop the dimension conditions and take an ad hoc approach to the gallery model.*

Definition 1.5.8. *We denote by $\mathcal{M}(F, E)$ the set of hyperplanes which separates the faces E and F. A combinatorial gallery δ joining 0 and λ is called* minimal *if every face of δ is contained in H_λ and if the number of large faces $p+1$ is minimal in the sense: For $j \in \{0, \ldots, p\}$ denote by H_j the set of affine hyperplanes H such that the small face $\Gamma_j \subseteq H_j$, but the large face $\Gamma_j' \not\subseteq H$. The sets H_j are pairwise distinct and $\bigcup_{j\in\{0,\ldots,p-1\}} H_j = \mathcal{M}(0, \lambda)$.*

Definition 1.5.9. *The* gallery of types t_δ *is the tuple of types of the faces of the gallery δ.*

$$t_\delta := \mathrm{type}(\delta) := (t_0 \supset t_0' \subset t_1 \supset \cdots \subset t_p \supset t_p' \subset t_\lambda)$$

where t_i is the type of Γ_i, t_i' the type of Γ_i' and t_λ of the face λ. The set of galleries of the same type as δ will be denoted by $\Gamma(\delta)$ and $\Gamma(\delta, \nu)$ for $\nu \in X_*(S)$ is the subset of galleries with target ν.

Such a gallery of types gives rise to sequence of subgroups of W^{aff}

$$W_0 \supset W_0' \subset W_1 \supset \cdots \subset W_p \supset W_p' \subset W_\lambda$$

where W_j is generated by the reflections along hyperplanes in t_i and likewise for W_j'. Then it is easy to parametrize the set of combinatorial galleries of a fixed type.

Proposition 1.5.10. *Let W_δ be the quotient of $W_0 \times \cdots \times W_p$ by the right action of $W_0' \times \cdots \times W_p'$ defined by*

$$(w_0, \ldots, w_p)(w_0', \ldots, w_p') = (w_0 w_0', w_0'^{-1} w_1 w_1', \ldots, w_{p-1}'^{-1} w_p w_p)$$

Then define the map

$$\begin{aligned} W_\delta &\to \Gamma(\delta) \\ [w_0, \ldots, w_p] &\mapsto (\{0\} \subseteq \Sigma_0' \supseteq \cdots \subseteq \Sigma_p' \supseteq \Sigma_\lambda) \end{aligned}$$

where $\Sigma_k' = w_0 \cdots w_k(F_k')$ for F_k' the unique face of Δ_f of type t_k. The small faces are then prescribed by the large faces and the type. The above map is a bijection. We will use $[w_0, \ldots, w_p]$ also for the gallery defined by means of this map.

Let δ_λ be a minimal gallery joining 0 and λ. Then in the parametrization W_{δ_λ} the gallery δ_λ is given by $[1, \tau_1, \ldots, \tau_p]$ where τ_i is the unique representative of the largest class in W_j/W_j' in the induced Bruhat order.

Definition 1.5.11. *We call a combinatorial gallery $\delta = [w_0, \ldots, w_p] \in \Gamma(\delta)$ folded around the j-th small face if $w_j \neq \tau_j$.*

Proposition 1.5.12. *Let Σ_j be a small face of a gallery $[w_0, \ldots, w_p]$. We denote by Ω_j the j-th large face of the gallery $[w_0, \ldots, w_j, \tau_{j+1}, \ldots, \tau_p]$. Then there exist positive roots $\beta_1, \ldots, \beta_q$ and integers $m_1, \ldots, m_q$ such that Σ_j is contained in the affine hyperplanes H_{β_i, m_i} and for the large faces Σ_j' it holds*

$$\Sigma_j = s_{\beta_q, m_q} \cdots s_{\beta_1, m_1}(\Omega_j)$$

We use the notation of this proposition in the following definition.

Definition 1.5.13. *We say a gallery δ is* positively folded *at the small face Σ_j if the large face Σ_j is contained in the positive halfspace $H^+_{\beta_i, m_i}$ for all $i \in \{1, \ldots, q\}$. In case a gallery is not folded at a small face, the condition is empty and therefore positively folded at the small face. We call a gallery* positively folded *if it is positively folded at all small faces.*

As we have seen integral paths are better suited for the path model and similarly some galleries are better suited to build a model for a representation. Gaussent and Littelmann gave two combinatorial conditions which guarantee good constructions. These conditions are the already defined *positive folding* and a dimension condition; a gallery subject to both is called an *LS-gallery*.

Definition 1.5.14. *Let H_j be the set of affine reflection hyperplanes H such that $\Gamma_j \subset H$ and $\Gamma'_j \not\subset H$. We say a hyperplane $H \in H_j$ is a* load-bearing wall *if H seperates Γ'_j from the equivalence class of the sector $\mathfrak{C}_{-\infty}$.*

For a positively folded gallery all folding hyperplanes are load-bearing walls.

Definition 1.5.15. *The* dimension *of a positively folded combinatorial gallery δ is the number of pairs (H, Γ_j) such that H is a load-bearing wall for δ at Γ_j.*

Proposition 1.5.16. *If $\delta \in \Gamma^+(\delta_\lambda, \nu)$, then* $\dim \delta \leq \langle \lambda + \nu, \rho \rangle$.

This inequality leads to the definition of an LS-gallery.

Definition 1.5.17. *A combinatorial gallery $\delta \in \Gamma^+(\delta_\lambda, \nu)$ is called an* LS-gallery (of type λ) *if* $\dim = \langle \lambda + \nu, \rho \rangle$. *To denote LS-galleries, we will use a subscript LS.*

As for the path model a formula for the character of a representation can be deduced.

Proposition 1.5.18 ([GL05])**.** *The character of the irreducible $K^\vee_{\mathbb{C}}$-representation $V(\lambda)$ is given by*

$$\sum_{\nu \in X_*} |\Gamma^+_{LS}(\delta_\lambda, \nu)| \exp(\nu).$$

The root operators in the gallery model

As for the path model Gaussent and Littelmann define partial maps f_α, e_α on the set of combinatorial LS-galleries to construct a gallery model from one LS-gallery.
Let α be a simple root and m the smallest integer such that the hyperplane $H_{\alpha,m}$ contains one of the small faces Γ'_k. As all combinatorial galleries start in 0, it follows $m \leq 0$.

Definition 1.5.19. *If $m \leq -1$, let j be the maximal integer between m and 0 such that the small face Γ'_j is contained in the hyperplane $H_{\alpha,m+1}$. Then we define $e_\alpha \delta$ as the gallery which has faces Δ_j resp. Δ'_j, where*

$$\Delta_j = \begin{cases} \Gamma_i & \text{for } i \leq j-1, \\ s_{\alpha,m+1}(\Gamma_j) & \text{for } j \leq i \leq k-1, \\ \tau_{\alpha^\vee}(\Gamma_i) & \text{for } i \geq k. \end{cases}$$

The operator $\tau_{\alpha^\vee}$ is again the translation by $\alpha^\vee$.

We define a partial inverse f_α.

Definition 1.5.20. *If $\langle \nu, \alpha \rangle - m \geq 1$, let j be the maximal integer such that $\Gamma'_j \subset H_{\alpha,m}$ and fix the integer k to be minimal such that $\Gamma'_k \subset H_{\alpha,m+1}$. Then we define $f_\alpha \delta$ as the gallery which has faces Δ_j, where*

$$\Delta_j = \begin{cases} \Gamma_i & \text{for } i \leq j-1, \\ s_{\alpha,m}(\Gamma_j) & \text{for } j \leq i \leq k-1, \\ t_{-\alpha^\vee}(\Gamma_i) & \text{for } i \geq k. \end{cases}$$

We refer to [GL05][pp.60, 61] for a pictorial explanation of the situation.

Proposition 1.5.21. *The set of LS-galleries in $\Gamma^+(\delta_\lambda)$ is the subset generated from δ_λ by application of the root operators f_α.*

1.5.1 1-Skeleton galleries

A direct connection of the gallery model and the path model can be easily seen for 1-skeleton galleries. They were studied by Gaussent and Littelmann in [GL12].

Definition 1.5.22. *A gallery δ is called a* 1-skeleton gallery *if all its large faces are* 1*-dimensional faces.*

Definition 1.5.23. *A ray of the dominant Weyl chamber generated by a fundamental weight ϖ crosses edges and vertices of the Coxeter complex. The edges and vertices obtained by this up until reaching ϖ define a* 1*-skeleton gallery joining* 0 *and ϖ. These edges are called* fundamental faces, *even though they are not necessarily part of the fundmanental alcove.*

Definition 1.5.24. *We call a* 1*-skeleton gallery a* dominant combinatorial gallery joining 0 and λ *if all its large faces are translations of fundamental faces.*

Given a 1-skeleton gallery, associate to it the path, which is the concatenation of the large faces. This gives a direct link between the path model and the gallery model.

Definition 1.5.25. *A* 1*-skeleton gallery δ is called* minimal *if there exists an equivalence class of sector $\mathfrak{C}$ and representatives $\mathfrak{s}_0, \ldots, \mathfrak{s}_p \in \mathfrak{C}$ such that the small face V_j of δ is the vertex of $\mathfrak{s}_j$ and the large face E_j is contained in $\mathfrak{s}_j$ for all $j \in \{0, \ldots, p\}$.*

For 1-skeleton galleries the notion of positive folding and LS-gallery do exist, but are more complicated. We refer to [GL12] for definitions and properties. A dominant combinatorial gallery δ joining 0 and λ is always positively folded and an LS-gallery. To obtain all other LS-galleries of the same type, we use the identification of δ with a path and the path model.

1.5.2 The Bott-Samelson manifold for a gallery

With our presentation of the gallery model up until now the benefit over the path model is unclear. The potential of the gallery model is not in its combinatorics but in a natural generalization of galleries to the affine building of K. For a definition of the affine building see [GL05]. While the theory of buildings is rich and beautiful, it is also rich in its notations. As the advantages of introducing the affine building are outweighed by the difficulties in our case, we refrain from doing it. Denote by F the ring of Laurent series with complex coefficients $\mathbb{C}((t))$ and by $A = \mathbb{C}[[t]]$ its ring of integers, the power series ring. We obtain the algebraic loop group K_F and its subgroup K_A. The algebraic loop group K_F is not to be confused with the polynomial loop group $\Omega^{\mathrm{pol}}(K)$. The nomenclature for these is a bit muddled as $\Omega^{\mathrm{pol}(K)}$ is also called the algebraic loop group by some authors, which is a fact one need to be aware of.
We can identify both K_F and K_A as the set of matrices with entries in F respectively A subject to the complex polynomial equations defining $K_{\mathbb{C}}$. There is an action of K_F on the affine building, and it allows to extend the definition of the type of gallery to galleries in the affine building. What we are interested in is a parametrization of the galleries of a fixed type. This parametrization by the Bott–Samelson–Demazure–Hansen variety was used by Gaussent and Littelmann [GL05]. To state it we give some more notations and facts for the algebraic loop group.

The quotient $\mathcal{G} = K_F/K_A$ is called the *affine Grassmanian* and in our context is best thought of as the loop group $\Omega^{\mathrm{pol}}(K)$ of loops with a Laurent polynomial expansion.

Lemma 1.5.26. *In this lemma we will regard $\Omega^{pol}(K)$ topologized by either the direct limit of the Zariski topology or the direct limit of the analytical topology. The map*

$$\Omega^{pol}(K) \to \mathcal{G};\ u(e^{i\varphi}) \mapsto u(t)K_A$$

is a homeomorphism.

Proof. The map is bijective by the Iwasawa decomposition, which states

$$K_F = \Omega^{\mathrm{pol}}(K) \cdot K_A$$

and the intersection of $\Omega^{\mathrm{pol}}(K)$ and K_A is just 1.
Regarding continuity: The finite-dimensional model Ω_j for $\Omega^{\mathrm{pol}}(K)$ coincides with the one of $\mathcal{G}$ given in [Kum02]. It follows the map is a homeomorphism for every piece of the filtration, which means it is a homeomorphism for the whole space in the direct limit topology. ☐

Via the evaluation map $\mathrm{ev}_0 : K_A \to K_{\mathbb{C}}, A(t) \mapsto A(0)$ we can define the standard Iwahori subgroup $\mathcal{B}$ as the preimage of the Borel subgroup B of $K_{\mathbb{C}}$, fixed by our choice of simple roots. We define a parahoric subgroup to be any subgroup which is a finite union of double coset of $\mathcal{B}$. The double cosets are in one-to-one correspondence with elements of the affine Weyl group W^{aff} via the Bruhat decomposition which states

$$K_F = \bigsqcup_{w \in W_a} \mathcal{B} w \mathcal{B}.$$

The parahoric subgroups which contain the standard Iwahori are called *standard.*

Lemma 1.5.27 ([Kum02])**.** *The standard parahoric subgroups are in one-to-one correspondence with subsets of nodes of the extended Dynkin diagram of K.*

The choice of nodes in the subgraph describes a subset of roots of K, and we can thus also associate to a face of the fundamental alcove a standard parahoric subgroup. This piece of data can be extracted from the type as the set of reflections $s_{\alpha,0}$ and $s_{\alpha_0,1}$ which are in the type of a face. We will denote the parahoric subgroup to a face F as $\mathcal{P}_F$.

Remark 1.5.28. *As in the finite-dimensional case $\mathcal{P}_I$ contains a Levi subgroup which is a finite-dimensional algebraic subgroup [Kum02]. The associated Weyl group embeds into W^{aff} as the subgroup generated by the reflections s_α for $\alpha \in I$.*

To define the Bott-Samelson variety in this setting we will be using the gallery of types.

Definition 1.5.29. *The* Bott–Samelson–Demazure–Hansen *variety $\Sigma(\delta)$ is the fibered product of parahoric subgroups*

$$\mathcal{P}_0 \times_{\mathcal{Q}_0} \cdots \times_{\mathcal{Q}_{p-1}} \mathcal{P}_p / \mathcal{Q}_p,$$

where the $\mathcal{P}_i$ are the parahoric subgroups associated to the type of the i-th small face and the $\mathcal{Q}_i$ to the i-th large face. It is a finite-dimensional, complex, smooth, projective variety [Kum02] [GL05]. As is custom we will refer to it as the Bott–Samelson variety.

We use the same definition for 1-skeleton galleries.

1.5.3 Comparison of the models

As stated at the beginning of section 1.5 the gallery model is a roughed out version of the path model. In the following we will explain what we mean by this.

Example 1.5.30. *Let $\pi_s(t)$ be the piecewise defined path in* $\mathrm{Lie}(S)$*, where S is the torus of diagonal matrices in* $\mathrm{SU}(3)$*, given by*

$$\pi_s(t) = \begin{cases} 2t(s\varpi_1^\vee + (1-s)\varpi_2^\vee) & \text{for } t \in [0, \frac{1}{2}] \\ (2t-1)((1-s)\varpi_1^\vee + s\varpi_2^\vee) + s\varpi_1^\vee + (1-s)\varpi_2^\vee & \text{for } t \in [\frac{1}{2}, 1], \end{cases}$$

for $s \in (0,1)$. All considered paths π_s define a path model for the crystal of highest weight $\rho^\vee = \varpi_1^\vee + \varpi_2^\vee$, the common endpoint of all π_s. Define $\Delta(\pi_s)$ to be the sequence of faces in the affine Coxeter complex, which π_s traverses. It is independent of the choice of s. The gallery model obtained from $\Delta(\pi_s)$ is also a crystal of highest weight $\rho^\vee$.

For a given gallery model there are many different path models resulting in this gallery model, which leads us to our interpretation of the gallery model as rougher than the path model.

1.6 Mirković–Vilonen cycles and polytopes

The Mirković–Vilonen cycles, or short MV cycles, are the intersection of two naturally defined subsets of the affine Grassmannian $\mathcal{G}$. They occur as basis for the representations of $K_{\mathbb{C}}^\vee$. Their image under the moment map of $\mathcal{G}$ are the MV polytopes. In this section we will give their definition and facts about both the MV cycles and polytopes. First the affine Schubert varieties C_λ.

Definition 1.6.1. *The* affine Schubert cell $\mathcal{G}_\lambda$ *is defined as the K_A-orbit of a dominant coweight λ on the affine Grassmannian. They partition the affine Grassmannian and this partition is called the* Bruhat decomposition

$$\mathcal{G} = \bigsqcup_{\lambda \in X_{*+}} \mathcal{G}_\lambda.$$

Its closure $C_\lambda = \overline{\mathcal{G}_\lambda}$ is called the affine Schubert variety*. The closure is given by*

$$C_\lambda = \bigsqcup_{\nu \leq \lambda} \mathcal{G}_\nu.$$

The affine Schubert variety is a complex variety of dimension $\mathrm{ht}(\lambda - w_0\lambda)$.

The notation C_λ might be unusual for the algebraically fluent reader and stems from the context of loop groups. The following observation is an immediate consequence of the definition of the affine Schubert cell and the description of the stable manifolds of the energy flow by Pressely [Pre82].

Lemma 1.6.2. *The affine Schubert cell are the stable manifolds of the energy flow on $\Omega(K)$.*

The second ingredient for the definition of MV cycles are the semi-infinite orbits S_λ.

Definition 1.6.3. *Let U_F be the set of F points of the unipotent radical of B. It acts on $\mathcal{G}$ via left multiplaction and its orbits, which are called* semi-infinite orbits*, can be indexed by weights,*

$$\mathcal{G} = \bigsqcup_{\lambda \in X_*} U_F \lambda.$$

The orbit through λ will be denoted by S_λ. Such an orbit can further be described as a kind of Białynicki-Birula cell (see definition 1.6.8), i.e.

$$S_\lambda = \left\{ g \in \mathcal{G} \mid \lim_{a \to 0} \rho^\vee(a) g = \lambda \right\}.$$

We will denote the set of irreducible components of a topological space X by $\mathrm{Irr}(X)$.

Definition 1.6.4 ([MV07])**.** *For $\lambda, \mu \in X_*$ we set*

$$\mathcal{Z}(\lambda)_\mu = \mathrm{Irr}(\overline{S_\mu \cap \mathcal{G}_\lambda}).$$

An element of $\mathcal{Z}(\lambda)_\mu$ is called a Mirković-Vilonen cycle. We denote by $\mathcal{Z}(\lambda)$ the union of all $\mathcal{Z}(\lambda)_\mu$.

We will not dwell on why these objects are called cycles more than we already did in the introduction.

Theorem 1.6.5 ([MV07]). *Let $\lambda \in X_*(S)$ be a chocharacter, then the representation $V(\lambda)$ of $K_{\mathbb{C}}^{\vee}$ can be decomposed as*

$$V(\lambda) = \bigoplus_{X \in \mathcal{Z}(\lambda)} \mathbb{C} \cdot X,$$

and if $X \in \mathcal{Z}(\lambda)_\mu$ then it is a weight vector of weight μ. More importantly the action of $K_{\mathbb{C}}^{\vee}$ can be constructed naturally, and thus one can construct $K_{\mathbb{C}}^{\vee}$ without relying on the classification of simple complex Lie groups.

The affine Schubert varieties are in general singular, but as in the classical case the Bott–Samelson varieties can be used to desingularize them.

Lemma 1.6.6. *Let δ be a minimal combinatorial gallery joining 0 and λ, where λ is a dominant coweight. The map*

$$\begin{aligned} \pi : \Sigma(\delta) &\to C_\lambda \\ [g_0 : \cdots : g_k] &\mapsto g_0 \cdots g_k \lambda^f \end{aligned}$$

is a resolution of singularities, i.e. birational and proper. Here λ^f denotes the element of T_F/T_A, which is the unique vertex of Δ_f in the affine Weyl group orbit of λ.

As one would hope the same holds true for 1-skeleton galleries.

Lemma 1.6.7. *Let δ be a minimal combinatorial 1-skeleton gallery joining 0 and λ, where λ is a dominant coweight. The map*

$$\begin{aligned} \pi : \Sigma(\delta) &\to C_\lambda \\ [g_0 : \cdots : g_k] &\mapsto g_0 \cdots g_k \lambda^f \end{aligned}$$

is a resolution of singularities.

To state the connection of the Bott–Samelson variety and the MV cycles we need another tool from algebraic geometry, the Białynicki-Birula cell decomposition.

Definition 1.6.8 ([BB73]). *Let X be a projective variety over $\mathbb{C}$ with an action of the complex torus $\mathbb{C}^*$. We assume the set of $\mathbb{C}^*$-fixed points to be finite. If $p \in X^{\mathbb{C}^*}$ is a fixed point and we define the attractor set X_p as*

$$X_p := \left\{ x \in X \mid \lim_{t \to 0} t.x = p \right\},$$

then the decomposition

$$X = \bigsqcup_{p \in X^{\mathbb{C}*}} X_p$$

is a decomposition into complex cells.

On the variety $\Sigma(\delta)$ there is an action of the complex torus $S_{\mathbb{C}}$ defined by left multiplying the first factor of $\Sigma(\delta)$. By choice of a regular coweight in the Weyl chamber $\mathfrak{C}_{-\infty}$ it can be restricted to an action of $\mathbb{C}^*$ such that the fixed point sets of both actions are the same.

Lemma 1.6.9. *The $S_{\mathbb{C}}$-fixed points of $\Sigma(\delta)$ can be identified with the set of galleries of type* $\mathrm{type}(\delta)$.

We explain the identification. Given $\delta_0 = [w_0 : \cdots : w_k] \in \Gamma(\delta)$ we know by remark 1.5.28 that we can find lifts p_j in $\mathcal{P}_j$ of w_j. The point $[p_0 : \cdots : p_k]$ of $\Sigma(\delta)$ is a $S_{\mathbb{C}}$ fixed point and corresponds to δ.

Theorem 1.6.10 ([GL05])**.** *The image under π of the Białynicki-Birula cell $\Sigma(\delta)_\mu$ for μ an LS-gallery is a dense subset of an MV cycle of weight* $\mathrm{target}(\mu)$*, and every MV cycle contains the image of such a Białynicki-Birula cell.*

The same holds true if we replace δ by a 1-skeleton gallery [GL12]. The map π is proper and thus closed which implies that the MV cycle is the same as the image of $\overline{\Sigma(\delta)_\mu}$.
Anderson and Kamnitzer [And03, Kam10, Kam07] defined and researched the MV polytopes. Let us give the definition first.

Definition 1.6.11. *The affine Grassmannian can be given a symplectic form and together with the $S_{\mathbb{C}}$-action by left multiplication we obtain a moment map*

$$\mu : \mathcal{G} \to \mathrm{Lie}(S_{\mathbb{C}})^*.$$

If we identify $\mathrm{Lie}(S_{\mathbb{C}})^*$ *via the Killing form with* $\mathrm{Lie}(S_{\mathbb{C}})$*, we can describe the image of an MV cycle as the convex hull of its torus fixed points*

$$\mu(X) = \mathrm{conv}(\{\lambda \mid \lambda \in X \cap X_*(S)\}$$

for $X \in \mathcal{Z}(\lambda)$.

By works of Kamnitzer descriptions of the MV polytopes via inequalities are at hand.
To compute MV-polytopes Ehrig introduced the vertex gallery [Ehr09].

Definition 1.6.12. *To an LS-gallery δ and an element $w \in W$ we define the* vertex gallery $\Xi_w(\delta)$ *as follows. For a simple root α let $e_\alpha^{max}(\delta) = e_\alpha^{\varepsilon(\delta)}(\delta)$, where $\varepsilon(\delta)$ is the maximal applicable number of e_α for δ. For $w = s_{i_1} \cdots s_{i_k}$ a reduced decomposition into simple reflections define $e_w = e_{i_k}^{max} \cdots e_{i_1}^{max}$. Then $\Xi_w(\delta) = w(e_w(\delta))$. It is independent of the reduced decomposition.*

There are different formulas for the vertex gallery in terms of folding operators. Using these formulas one can interpret the vertex gallery as an LS-gallery for a different choice of simple roots, fixed by the choice of w. We will not consider this interpretation. By the given formulation it is immediately clear that the endpoint of $\Xi_w(\delta)$ can be computed in terms of the crystal graph without knowledge of the gallery model. As the following theorem by Ehrig gives a combinatorial construction of MV-polytopes with the help of the gallery model and the vertex gallery, it is clear that MV-polytopes can be computed from the crystal graph. The proof still relies on the gallery model and the associated Bott–Samelson variety.

Theorem 1.6.13 ([Ehr09])**.** *Let δ be an LS-gallery such that every large face is an alcove. Then*

$$\operatorname{conv}\{\operatorname{wt}(\Xi_w(\delta)) \mid w \in W\}$$

is the MV-polytope corresponding to the MV-cycle which contains δ.

From this one easily deduces that the MV-polytope of the pair of weights (λ, λ) is the *Weyl polytope* of weight λ defined as $\operatorname{conv}(W.\lambda)$.

2 Root operators and the weight function

2.1 Descend to the loop group

Definition 2.1.1. *For every $\alpha \in \Delta$ define the function*

$$\begin{aligned}\mathrm{wind}_\alpha : \Omega(S) &\to \mathbb{Z}\\ \gamma &\mapsto \mathrm{wind}(\alpha \circ \gamma),\end{aligned}$$

where wind *denotes the winding number for loops on S^1.*

The following proposition is a reinterpretation of the Littelmann path model inside of the loop group via the exponential map. It gives the setup in which we will work later on. We will still use the original approach by Littelmann as necessary to avoid notational debt.

Proposition 2.1.2. *Let $\gamma \in \Omega(S)$ with $\rho^\vee * \pi_\gamma$ contained in the interior of the dominant Weyl chamber and denote by $\mathcal{A}\pi$ the smallest subset of $\Omega(S)$ stable under the root operators, then the following holds:*

1. *The weight of a loop μ is given by*

$$wt(\mu) = \sum_{\alpha \in \Delta} \mathrm{wind}_\alpha(\gamma) \varpi_\alpha^\vee. \tag{2.1}$$

2. *The crystal graph obtained from $\mathcal{A}\gamma$ parametrizes a basis for the irreducible $K^\vee$-module with highest weight* $\mathrm{wt}(\gamma)$.

3. *To every $\gamma_1 \in \Omega(S)$ there exist $\mu_+, \mu_- \in \Omega(S)$ such that*

$$\begin{aligned}F_\alpha(\gamma_1) &= \gamma_1 \mu_+\\ E_\alpha(\gamma_1) &= \gamma_1 \mu_-\end{aligned}$$

whenever the left hand side is defined.

Corollary 2.1.3. *The root operators fix the connected components of the loop group $\Omega(K)$. Equivalently the homotopy classes of $\gamma, F_\alpha(\gamma), E_\alpha(\gamma)$ in K are the same.*

Proof. 1. Part: We compute

$$\begin{aligned}\mathrm{wind}_\alpha(\mu) &= \mathrm{wind}(\alpha \circ \mu) \\ &= \pi_{\alpha\circ\mu}(1) \\ &= (\mathrm{d}\,\alpha)_1(\pi_\mu)(1)\end{aligned}$$

which is equal to the coefficient of $\varpi_\alpha^\vee$ in $\pi_\mu(1)$ written in the fundamental coweights. The third equality holds as $\mathrm{d}\,\alpha(\pi_\mu)$ is a lifting of $\alpha \circ \mu$ to a path in $\mathrm{T}_1\, S^1$ starting in 0. By uniqueness of lifts with fixed starting points equality follows. It follows immediately that $\sum \mathrm{wind}_\alpha(\gamma)\varpi_\alpha^\vee$ is a formula for the endpoint of π_μ. The weight is defined as the endpoint.

2. Part: The crystal graph $\mathcal{A}\gamma$ parametrizes a basis for a highest representation of the simply connected form of $K_\mathbb{C}$ following [Lit97]. As irreducible representations [FH91, p. 438] of the maximal compact group and the complex group are the same, we only need to check whether $\mathrm{wt}(\gamma) \in X_*(S^\vee)$ or equivalently $\mathrm{wt}(\gamma) \in X_*(S)$. By part 1 the weight is given by the endpoint of a lift to $\mathrm{Lie}(S)$ and this must be an element of $X_*(S)$.

3. Part: Choose

$$\mu_\pm = \psi\left(-\varepsilon_\alpha(\pi_\gamma)\alpha^\vee - \min\left(-\varepsilon_\alpha(\pi_\gamma) \pm 1, \inf_{t\le s\le 1}\left(\alpha(\pi_\gamma(s))\right)\right)\alpha^\vee\right).$$

Now part 3 follows from equation 1.1 and because exp is a group homomorphism.

□

Proof of Corollary. The endpoint of $\psi^{-1}(\mu_\pm)$ is the root $\pm\alpha^\vee$. The homotopy class of $\mu_\pm$ depends only on this endpoint and is trivial if it is in the root lattice [Sep07, p. 173]. The action of such an element on the set of connected components is trivial. □

Remark 2.1.4. *As proposition 2.1.2 is in essence a translation of the path model to the language of loop groups, the Character Formula, Generalized Littlewood-Richardson rule and the Restriction formula still hold in this setting and for compact groups.*

Example 2.1.5. *First we will give the example of the adjoint represenation for SU_3. We will write (f_1, f_2, f_3) for the diagonal matrix with entries f_1, f_2, f_3.*

$$\begin{tikzcd}
 & (1,z,z^{-1}) \arrow[r, "\alpha_2"] & \gamma_1 \arrow[r, "\alpha_2"] & (1,z^{-1},z) \arrow[dr, "\alpha_1"] & \\
\gamma \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"'] & & & & \gamma^{-1} \\
 & (z,z^{-1},1) \arrow[r, "\alpha_1"] & \gamma_2 \arrow[r, "\alpha_1"] & (z^{-1},z,1) \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

The loop $\gamma_1(z)$ is equal to $-\alpha_2^\vee$ if the imaginary part of z is ≥ 0 and $\alpha_2^\vee$ otherwise. The loop $\gamma_2(z)$ is equal to $-\alpha_1^\vee$ if the imaginary part of z is ≥ 0 and $\alpha_1^\vee$ otherwise. We will adopt the notation $\gamma_1(z) = -\frac{1}{2}\alpha_2^\vee * \frac{1}{2}\alpha_2^\vee$ even though neither part on the right-hand side is a loop.

Example 2.1.6. *As a second example we will consider $K = PU_3$ and the loop $\gamma = (z, 1, 1)$. The resulting crystal is*

$$\gamma \xrightarrow{\quad\alpha_1\quad} (1, z, 1) \xrightarrow{\quad\alpha_2\quad} (1, 1, z).$$

Application the weight function yields

$$\mathrm{wt}(\gamma) = \varpi_1^\vee, \mathrm{wt}((1, z, 1)) = \varpi_2^\vee - \varpi_1^\vee, \mathrm{wt}((1, 1, z)) = -\varpi_2^\vee.$$

Observe that this is a crystal for the first fundamental representation of SU_3 which is only a projective representation of PU_3.

With proposition 2.1.2 the translation of the path model to the loop group on the compact torus is complete. We will use the notions introduced for the path model now also in the context of the loop group. Depending on the problem we will use the paths and loops interchangeably but will make it clear, which one is used at which point.

2.2 Application of the Birkhoff decomposition

In special cases proven by Birkhoff [Bir09] and in general proven by Pressley and Segal [Pre82] [PS86, p.120 - 142] there exists the Birkhoff decomposition for the based loop group. We will first give the necessary notations and then state the Birkhoff decomposition.

Definition 2.2.1. *The free loop group $L(K_\mathbb{C})$ has subgroups of holomorphic loops in the sense*

$$L_+(K_\mathbb{C}) := \{\gamma \in L(K_\mathbb{C}) \mid \gamma \text{ extends to } \{z \in \mathbb{C} \mid |z| \leq 1\} \to K_\mathbb{C}\},$$
$$L_-(K_\mathbb{C}) := \{\gamma \in L(K_\mathbb{C}) \mid \gamma \text{ extends to } \{z \in \mathbb{C} \mid |z| \geq 1\} \cup \{\infty\} \to K_\mathbb{C}\}.$$

These subgroups perform quite analogous to the parabolic subgroups of a complex Lie group.

Theorem 2.2.2 ([Pre82]). *For every $\gamma \in \Omega(S)$ there exist a unique $\lambda \in X_*(S)$ and $p_- \in L_-(T), p_+ \in L_+(T)$ such that*

$$\gamma = p_- \lambda p_+.$$

The theorem as stated in [Pre82] is stated in the more general context where S is replaced by any compact Lie group. The proof relies on the good properties of the energy function on the loop group. This theorem is not true if we replace loops of finite energy with continuous loops.

Corollary 2.2.3. *The Birkhoff decomposition is unique up to multiplication by a constant loop.*

This statement is not true if one replaces S by K, in which case it is only true if $\lambda = 1$. Such loops form a dense subset of $\Omega(K)$.

Proof. Suppose $\gamma = p_- \lambda p_+ = q_- \lambda q_+$ are decompositions as in 2.2.2. The loop group of S is commutative so we can cancel λ and invert q_- and p_+. Then $p_- q_-^{-1} = p_+^{-1} q_+$ on S^1. The left-hand side is given by a Laurent series with only nonpositive exponents while the right-hand side has only nonnegative exponents. The corollary follows. □

The loop group $\Omega(S)$ is stable under the root operators. Thus the first question to answer should be how the Birkhoff decomposition behaves with respect to the root operators and the weight function. At least for the $X_*(S)$ factor this is easily answered.

Proposition 2.2.4. *Let $\gamma \in \Omega(S)$, then the Birkhoff decomposition is of the form $p_- \operatorname{wt}(\gamma) p_+$. An immediate consequence is that the Birkhoff decomposition, after application of a root operator, has the form:*

$$f_\alpha(\gamma) = q_-(\lambda - \alpha^\vee) q_+ \qquad\qquad e_\alpha(\gamma) = r_-(\lambda + \alpha^\vee) r_+,$$

Proof. Let $\gamma = p_- \lambda p_+$. By [Pre82] the energy flow applied to γ has limit point λ. The connected components of $\Omega(S)$ are the exactly the homotopy classes. These facts imply: The homotopy class of γ and the one of λ must be the same. However the homotopy class (in S) determines the weight, thus $\operatorname{wt}(\gamma) = \lambda$. □

3 Bott-Samelson Manifolds

3.1 Definition and properties

The Bott–Samelson manifold as defined by Bott and Samelson in [BS58] plays a key role to determine the homology ring of $\Omega(K)$. The complex version of the Bott-Samelson manifold has been used by Gaussent and Littelmann to give a link between MV cycles and the path model [GL05]. Our approach redefines this link inside of the loop group. In this section we define a generalization of the Bott–Samelson manifold. Through examples we explain our choices for the direction the generalization takes. We relate the Bott–Samelson manifold to Borel–de Siebenthal theory using the latter to compute the dimension of the former.

Definition 3.1.1. *Parametrize S^1 via $\varphi \mapsto e^{i\varphi}$ and let $\gamma \in \Omega(S)$. For $\varphi \in [0, 2\pi]$ denote by K_φ the maximal connected subgroup of K stabilizing $\gamma(e^{i\varphi})$ under the conjugation action of K on itself. By continuity of γ we know that there exist*

$$t_0 = 0 \leq t_1 \leq \cdots \leq t_k \leq t_{k+1} = 2\pi$$

such that $K_{t_j - \varepsilon}$ is a proper subgroup of K_{t_j}. We will refer to $\gamma\Big|_{[t_j, t_{j+1}]}$ as the j-th segment *of γ. The connected component of the neutral element of the pointwise stabilizer of the j-th segment will be denoted by K'_j. For brevity we write K_j for K_{t_j}. Define the Bott–Samelson manifold as the fibered product*

$$\Gamma_\gamma = K_0 \times_{K'_0} \cdots \times_{K'_{k-1}} K_k / K'_k,$$

where the right action of $(q_0, \ldots, q_k) \in K'_0 \times \cdots \times K'_k$ is given by

$$(p_0, \ldots, p_k) \cdot (q_0, \ldots, q_k) = (p_0 q_0, q_0^{-1} p_1 q_2, \ldots, q_{k-1}^{-1} p_k q_k).$$

It embeds into the loop group

$$\begin{aligned} h_\gamma : \Gamma_\gamma &\to \Omega(K) \\ (g_1, \ldots, g_k) &\mapsto \left(\varphi \mapsto (g_1 \cdots g_j).\gamma(e^{2\pi i \varphi}) \text{ for } \varphi \in [t_j, t_{j+1}]\right). \end{aligned}$$

We introduce the shorthand notation $\pi_j([g_0, \dots, g_k]) = g_0 \cdots g_j$ which is only well-defined up to a factor in K'_j. If it is clear from context, we will also write $\pi_j([g_0, \dots, g_k]) = \pi_j$. In summary, the map h_γ sends a point of Γ_γ to a piecewise-defined loop with pieces the conjugated segments of γ.

Remark 3.1.2. *In general, the stabilizer of $\gamma(e^{i\varphi})$ is not connected, e.g. the stabilizer of the matrix $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ in PSU_2 is $\pm 1S$. In the definition of Γ_γ in [BS58] all connected components are considered. While most of the following is also true for a non-connected Γ_γ, we need to assume connectedness at least for our treatment of Γ_γ as a complex manifold.*

Remark 3.1.3. *The K_j can be deduced from the lift π_γ by the set of affine hyperplanes in which $\pi_\gamma(t_j)$ is contained. A priori the lift contains more information as also the value of the root α defining the affine hyperplane can be read off. In the loop group setting the value can be computed as the winding number of $\alpha \circ \gamma|_{[0,t_j]}$. Thus we can speak of a loop* crossing *the affine hyperplane $\alpha = n$, when $\alpha \circ \gamma(t_j) = 1$ and $\mathrm{wind}_\alpha(\gamma|_{[0,t_j]}) = n$.*

By Borel–de Siebenthal theory we conclude the structure of the K_j.

Lemma 3.1.4. *Let F be the smallest face of Δ_f which contains an affine Weyl group conjugate of $\gamma(e^{t_j})$, then K_j is conjugate to the subgroup of K which is associated to F iteratively using Borel-de Siebenthal theory, i.e. it is the subgroup associated to the set of simple roots (or highest root) which vanish on F.*

Definition 3.1.5. *We can define to every K_j a subset of Φ^+. The positive roots of K which vanish at $\gamma(e^{it_j})$ will be denoted $\Phi(K_j)$. We follow the same notation for K'_j, where the roots have to vanish along the j-th segment.*

The above lemma allows us to compute the dimension of K_j.

Lemma 3.1.6. *The dimension of K_j is*

$$\mathrm{rank}(K) + 2|\Phi(K_j)|.$$

The dimension of K'_j computes similarly as

$$\mathrm{rank}(K) + 2|\Phi(K'_j)|.$$

The lemma follows from [Hel62][p.261, Lemma 5.1]. Let us give examples.

Example 3.1.7. *The group is PSU_3. With this example we want to clarify our choice of definition of the Bott–Samelson manifold. As loop we choose*

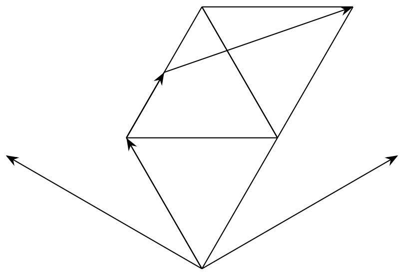

Figure 3.1: Loop with no classical Bott-Samelson manifold defined

the concatenation $\pi = \varpi_1^\vee * (\frac{1}{2}\varpi_2^\vee) * (\alpha_2^\vee - \frac{1}{2}\varpi_2^\vee)$ *depicted in figure 3.1. The Bott–Samelson manifold is*

$$\Gamma_\gamma = \mathrm{PSU}_3 \times_{\mathrm{PSU}(3,\alpha_2)} \mathrm{PSU}_3 \times_S \mathrm{PSU}(3,\alpha_2)/S,$$

where $\mathrm{PSU}(3,\alpha_2)$ *is the subgroup associated to the subset of the Dynkin diagram consisting only of* α_2*.*

Example 3.1.8. *Choose* $K = \mathrm{SU}_3$ *and* γ *the piecewise defined loop*

$$\gamma(e^{i\varphi}) = \begin{cases} (3\alpha_1^\vee + 3\alpha_2^\vee)(e^{i\varphi}) & \text{for } \varphi \in [0, \frac{2\pi}{3}] \\ (3\alpha_1^\vee)(e^{i\varphi}) & \text{for } \varphi \in [\frac{2\pi}{3}, \frac{4\pi}{3}] \\ (3\alpha_2^\vee)(e^{i\varphi}) & \text{for } \varphi \in [\frac{4\pi}{3}, 2\pi] \end{cases}$$

In table 3.1 we collect the necessary data to define the Bott–Samelson manifold. Every submatrix of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is a unitary matrix in the table. There does not exists a Borel subgroup* B_0*, such that* $K_i \subseteq L_i \supseteq B_0$ *for some Levi subgroup* L_i*. The groups* K_3, K_5 *already fix the Borel subgroup to be the upper triangular matrices inside* $\mathrm{SL}_3(\mathbb{C})$*. The group* K_1 *is not maximal compact in a Levi subgroup for the Borel subgroup* B *of upper triangular matrices. Any parabolic containing* K_1 *and* B*, contains the highest root and thus is already* $\mathrm{SL}_3(\mathbb{C})$*.*

Lemma 3.1.9. *The Bott-Samelson manifold is an* S*-manifold under the action via left multiplication of the first factor and the embedding* h_γ *is* S*-equivariant, where the* S*-action on* $\Omega(K)$ *is conjugation.*

3.2 The classical algebraic side

In this section we remind the reader of the notions of Bott–Samelson varieties in the finite-dimensional complex algebraic setting. By restricting to

i	t_i	$\gamma(t_i)$	K_i
0	0	$\mathrm{diag}(1,1,1)$	SU_3
1	$\frac{1}{6}2\pi$	$\mathrm{diag}(-1,1,-1)$	$\begin{pmatrix} a & 0 & b \\ 0 & (ad-bc)^{-1} & 0 \\ c & 0 & d \end{pmatrix}$
2	$\frac{1}{3}2\pi$	$\mathrm{diag}(1,1,1)$	SU_3
3	π	$\mathrm{diag}(-1,-1,1)$	$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & (ad-bc)^{-1} \end{pmatrix}$
4	$\frac{4}{6}2\pi$	$\mathrm{diag}(1,1,1)$	SU_3
5	$\frac{5}{6}2\pi$	$\mathrm{diag}(1,-1,-1)$	$\begin{pmatrix} (ad-bc)^{-1} & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$

Table 3.1: Data for the Bott-Samelson manifold in example 3.1.8.

a certain subclass of loops (or paths) in the definition of the Bott-Samelson manifold Demazure [Dem74] and Hansen [Han73] showed that the occuring Bott-Samelson manifolds carry the structure of projective, complex varieties. We will recall this structure.

Definition 3.2.1. *Let $w \in W$ a Weyl group element. Let π be a straight line in $\mathrm{Lie}(S)$ which connects a fixed generic point of the dominant Weyl chamber $\mathfrak{C}$ with a fixed generic point in the Weyl chamber $w.\mathfrak{C}$. We assume that π crosses no two hyperplanes at the same time. Even though π is not a loop, we can still define the groups K_j and K_j' and thus a Bott–Samelson manifold Γ_π. If we denote the hyperplanes which π crosses by $H_1, \ldots, H_k$, then $w = s_k \cdots s_1$, where s_i is the reflection in the hyperplane H_i. We define $\Gamma_w = \Gamma_\pi$.*

Lemma 3.2.2. *Each H_j defines a parabolic group P_j of $K_\mathbb{C}$, and K_j is a maximal compact subgroup of the Levi subgroup of P_j. The intersection of two consecutive parabolics $P_j \cap P_{j+1}$ is a Borel subgroup B_j, and we choose a subgroup B_k contained in P_k. The map*

$$\begin{aligned} \Gamma_w &\to P_0 \times_{B_0} \cdots \times_{B_{k-1}} P_k/B_k \\ [g_0, \ldots, g_n] &\mapsto [g_0 : g_1 : \cdots : g_k] \end{aligned}$$

is a diffeomorphism, where the action of $B_0 \times \cdots \times B_k$ is defined with the same formulas as Γ_γ. The map

$$\begin{aligned} \Gamma_w &\to K_\mathbb{C}/P_0 \times \cdots \times K_\mathbb{C}/P_k \\ [g_0, \ldots, g_k] &\mapsto (g_0P_0, g_0g_1P_1, \ldots, g_0 \cdots g_kP_k) \end{aligned}$$

is an injective, smooth embedding, and the image is a subvariety of the codomain.

The product of partial flag manifolds is a smooth, projective variety and thus a symplectic manifold. Pulling the restriction of the symplectic form through the embedding map gives a sympletic structure on Γ_w. Note that there are different choices for the symplectic form. We will give an explicit description of some symplectic structures on partial flag manifolds in section 3.3. There is a well known generalization of the classical Bott–Samelson variety.

Lemma 3.2.3. *For $j = 0, \dots, k$ let I_j be a set of nodes in the Dynkin diagram of K and $I'_j \subseteq I_j \cap I_{j+1}$ for $j = 0, \dots, k-1$ and $I'_k \subseteq I_k$. Denote by P_j the standard parabolic subgroup of $K_\mathbb{C}$ associated to I_j and P'_j be the subgroup associated to I'_j. Similary let K_j be the subgroup of K defined by I_j by Borel-de Siebenthal and K'_j for I'_j. We denote by Γ the Bott–Samelson manifold formed by the K_j and K'_j even though we have not given an associate path, which is possible. Then the map*

$$\Gamma \to P_0 \times_{P'_0} \cdots \times_{P'_{k-1}} P_k/P'_k$$

is a diffeomorphism.

There is another generalization, which is less typical but still well known.

Definition 3.2.4. *Let $P_i \subseteq K_\mathbb{C}$ be a sequence of parabolic subgroups for $i \in \{0, \dots, n\}$ such that in every consecutive intersection $P_i \cap P_{i+1}$ another parabolic subgroup P'_i is contained. We can form the Bott–Samelson variety*

$$P_0 \times_{P'_0} P_1 \times_{P'_1} \cdots \times_{P'_{n-1}} P_n/P'_n.$$

All Borel subgroups are conjugate to the Borel subgroup we fixed by our choice of torus and simple roots. From this it follows that the last two definitions are in essence equivalent.

Remark 3.2.5. *Every Bott–Samelson variety in the style of definition 3.2.4 is diffeomorphic to one in the style of lemma 3.2.3.*

These diffeomorphisms will in general fail to be $S_\mathbb{C}$-equivariant and thus give rise to different Białynicki-Birula cell decompositions.

Remark 3.2.6. *Let $\gamma \in \Omega(S)$ be a torus loop. There exists a sequence of parabolic subgroups P_i such that the Bott–Samelson variety defined by the P_i is diffeomorphic to Γ_γ.*

There is no canonical choice for this in this setting, but we will relate Γ_γ to the Bott–Samelson–Demazure–Hansen variety of Gaussent and Littelmann in a canonical way giving rise to a $S_\mathbb{C}$-action which behaves well with respect to the path model.

Example 3.2.7. *The group considered is* $\mathrm{SU}(2)$ *and* $\gamma = \alpha^\vee$. *The Bott–Samelson manifold is* $\mathrm{SU}(2) \times_S \mathrm{SU}(2)/S$. *There are a total of four choices for the complexification*

$$\begin{aligned} \mathrm{SL}(2) \times_B \mathrm{SL}(2)/B \\ \mathrm{SL}(2) \times_B \mathrm{SL}(2)/B_- \\ \mathrm{SL}(2) \times_{B_-} \mathrm{SL}(2)/B \\ \mathrm{SL}(2) \times_{B_-} \mathrm{SL}(2)/B_-, \end{aligned}$$

where B *is the Borel subgroup of upper triangular matrices and* B_- *the opposite Borel subgroup of lower triangular matrices.*

3.3 Partial flag varieties as coadjoint orbits

Continuing from the last section we want to define a symplectic structure on the Bott-Samelson manifold via the embedding of lemma 3.2.2. There are different approaches to this. We will identify the partial flag manifolds $K_{\mathbb{C}}/P_i$ as coadjoint orbits inside of $Lie(\mathbb{K}_{\mathbb{C}})$.

The groups K and $K_{\mathbb{C}}$ act on $\mathrm{Lie}(K_{\mathbb{C}})^*$ via the contragredient representation of the adjoint representation. In general their orbits are different, as the K-orbits are necessarily compact, while the action of $K_{\mathbb{C}}$ gives non-compact orbits. We restrict ourselves to orbits through points in $\mathrm{Lie}(K)^*$. Every orbit of K intersects $\mathrm{Lie}(S_{\mathbb{C}})^*$, and the intersection is a Weyl group orbit. In the compact setting the coadjoint action can be replaced by the adjoint action with help of the negative of the Killing form. As this form is positive definit in the compact setting, it gives an isomorphism of adjoint and coadjoint orbits. It is thus usual to restrict to the adjoint representation in the compact setting.

Definition 3.3.1. *A* coadjoint orbit $\mathcal{O}_p$ *is the orbit of any point* $p \in \mathrm{Lie}(K)^*$ *under the coadjoint action of* K. *It is isomorphic to the quotient* K/K_p, *where* K_p *is the stabilizer of* p.

We are interested in the symplectic form a coadjoint orbit obtains. For this we realize $\mathcal{O}_p$ in $\mathrm{Lie}(K)$ as described in the beginning of this subsection.

Lemma 3.3.2. *We identify* $T_p\mathcal{O}_p \cong \mathrm{Lie}(K)/\mathrm{Lie}(K_p)$ *using the orbit map. Any coadjoint orbit* $\mathcal{O}_p$ *has a* K*-invariant symplectic form via left-translation of the form induced by*

$$\begin{aligned} T_p\mathcal{O}_p \times T_p\mathcal{O}_p &\to \mathbb{R} \\ (X,Y) &\mapsto B(\alpha, [X,Y]), \end{aligned}$$

where α is any element of $\mathrm{Lie}(S)^* \cap \mathcal{O}_p$ *and B denotes the negative of the Killing form to avoid confusion of different parenthesis.*

We can go one step further and see there is a Kähler form induced by the Killing form.

Lemma 3.3.3. *There exists $P \subseteq K_{\mathbb{C}}$ a parabolic subgroup such that $K_p \subseteq P$ and the inclusion map $K/K_p \to K_{\mathbb{C}}/P$ is a diffeomorphism.*

The partial flag variety $K_{\mathbb{C}}/P$ is a smooth variety and as such is a Kähler manifold in its analytic topology. The symplectic form obtained from the coadjoint orbit construction is part of this Kähler structure [Kir][Ch. 5].

Lemma 3.3.4. *The Bott-Samelson variety Γ_w is symplectic.*

Proof. By lemma 3.2.2 the Bott-Samelson manifold is a (smooth) subvariety of the product of partial flag manifolds and as such inherits the symplectic structure by restriction. $\square$

Remark 3.3.5. *The same holds true for more general definitions using the Bott–Samelson variety of definition 3.2.4 and embedding it into an appropriate version of the partial flag manifolds. We obtain symplectic structures on Γ_γ for $\gamma \in \Omega(S)$.*

This approach via coadjoint orbits gives rise to different symplectic structures on Γ_γ depending on which points p are chosen for the isomorphism of a coadjoint orbit and the partial flag manifold.

Even though this fact is well known, we were unable to find a closed formula for the resulting symplectic form. Let us do the bookkeeping and record:

Lemma 3.3.6. *Let $p_i \in \mathrm{Lie}(S)$ such that the stabilizer of p_i is K_i. Denote the form induced by p_i on $\mathcal{O}_{p_i}$ as ω_i. The skew form $\omega_{[g_0,\dots,g_k]}$ on the tangent space $\mathrm{T}_{[g_0,\dots,g_k]}\Gamma_\gamma$ induced by the ω_i is given by*

$$\omega(g_0v_0,\dots,g_kv_k,g_0w_0,\dots,g_kw_k) = \sum_{i=0}^{k}\sum_{j,l\leq i}(p_i,[\mathrm{Ad}(\pi_i^{-1}\pi_j)v_j,\mathrm{Ad}(\pi_i^{-1}\pi_l)w_l]),$$

where v_j and w_j are tangent vectors of K_i at 1.

Proof. The computation of the differential of the embedding ι is straightforward and analogous to the computation we will do for lemma 3.5.1. Therefore we only state the result

$$\mathrm{D}_{[g_0,\dots,g_k]}\,\iota(g_0v_0,\dots,g_kv_k) = (g_0v_0, g_0v_0g_1 + g_0g_1v_1,\dots,\sum_{j=0}^{k}\pi_jv_j\pi_j^{-1}\pi_k),$$

where the vector at position l is given by

$$\sum_{j=0}^{l} \pi_j v_j \pi_j^{-1} \pi_l.$$

We left multiply to $[1, \dots, 1]$ in the product of partial flag manifolds to obtain the l-th position as

$$\sum_{j=0}^{l} \pi_l^{-1} \pi_j v_j \pi_j^{-1} \pi_l = \sum_{j=0}^{l} \mathrm{Ad}(\pi_l^{-1} \pi_j)(v_j).$$

Inserting into the formula for the symplectic form on the product of partial flag manifolds proves the claim. □

3.4 Bott-Samelson manifolds and path models

As shown by Gaussent and Littelmann using the gallery model there is a connection of the path model and Bott-Samelson manifolds. The class of loops for which this connection works are the integral loops. We will give arguments for these loops and against non-integral loops in this section.

Lemma 3.4.1. *For every integral loop γ it holds: The Bott–Samelson manifold Γ_γ contains the crystal generated by γ.*

Proof. We cut γ according to 1.4.11 into three paths $\gamma = \gamma_1 * \gamma_2 * \gamma_3$. By definition of an integral loop we know $\alpha \circ \gamma(e^{is}) = \alpha \circ \gamma(e^{it}) = 1$. Thus $\overline{s_\alpha} \in K_s, K_t$. Then $f_\alpha(\gamma) = h_\gamma([1 : \dots : s_\alpha : 1 : \dots : 1 : s_\alpha : 1 : \dots : 1])$, where the s_α are at the positions corresponding to s and t. A similar argument holds in the case of the raising operator. As the image of Γ_γ and the image of $\Gamma_{f_\alpha(\gamma)}$ inside of the loop group coincide, the lemma follows by induction on the length of a monomial in the root operators. □

This is already an indication of the useful connection between integral loops, their path model and the Bott–Samelson manifolds. Let us give an example for the obstructions one faces when considering loops which are not integral.

Example 3.4.2. *The following example shows that there cannot be a simple generalization of lemma 3.4.1 to non-integral loops.*

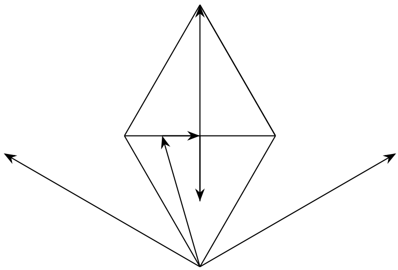

Figure 3.2: Non-integral loop with different Bott-Samelson manifolds

The considered group is $K = \mathrm{SU}(3)$. The loop γ is defined by its lift to $\mathrm{Lie}(S)$.

$$\pi_\gamma(t) = \begin{cases} t(3\varpi_1^\vee + \varpi_2^\vee) & \textit{for } t \in [0, \frac{1}{4}] \\ t(\varpi_2 \vee - \varpi_1^\vee) + \varpi_1^\vee & \textit{for } t \in [\frac{1}{4}, \frac{1}{2}] \\ t(-\varpi_1^\vee - \varpi_2^\vee) + \varphi_1^\vee + \varpi_2^\vee & \textit{for } t \in [\frac{1}{2}, \frac{3}{4}] \\ t(3\varpi_1^\vee + 3\varpi_2^\vee) - 2\varpi_1^\vee - 2\varpi_2^\vee & \textit{for } t \in [\frac{3}{4}, 1] \end{cases}$$

This is a loop of weight $\rho^\vee$ which is non-integral because of the local minimum at $t = \frac{3}{4}$. We compute the Bott-Samelson manifold as

$$\Gamma_\gamma = \mathrm{SU}(3) \times_{\mathrm{S}} \mathrm{SU}(3, \alpha_0) \times_{\mathrm{S}} \mathrm{SU}(3, \alpha_0) / \mathrm{S}\,.$$

If we compute the second root operator acting on π_γ, we obtain

$$f_{\alpha_2}(\pi_\gamma) = \begin{cases} t(4\varpi_1^\vee - \varpi_2^\vee) & \textit{for } t \in [0, \frac{1}{4}] \\ t(\varpi_2^\vee - \varpi_1^\vee) - \frac{5}{4}\varpi_1^\vee - \frac{1}{2}\varpi_2^\vee & \textit{for } t \in [\frac{1}{4}, \frac{1}{2}] \\ t(-\varpi_1^\vee - \varpi_2^\vee) + \frac{5}{4}\varpi_1^\vee + \frac{1}{2}\varpi_2^\vee & \textit{for } t \in [\frac{1}{2}, \frac{3}{4}] \\ t(6\varpi_1^\vee - 3\varpi_2^\vee) - 4\varpi_1^\vee + 2\varpi_2^\vee & \textit{for } t \in [0, \frac{1}{4}]. \end{cases}$$

This loop has the Bott-Samelson manifold

$$\Gamma_{f_2(\gamma)} = \mathrm{SU}(3) \times_{\mathrm{S}} \mathrm{SU}(3, \alpha_0) \times_{\mathrm{S}} \mathrm{SU}(3, \alpha_2) \times_{\mathrm{S}} \mathrm{SU}(3, \alpha_0) / \mathrm{S}\,.$$

In conclusion, the Bott-Samelson manifolds of γ and $f_2(\gamma)$ are not even of the same dimension.

3.5 The moment map image

In this section we compute the moment map image for the Bott–Samelson manifold under the condition that it is a symplectic submanifold. We give a

condition on the loop γ for which Γ_γ is symplectic.
As the energy function is constant on $h_\gamma(\Gamma_\gamma)$, we will discard it and compute the component $\mu_{\mathrm{Lie}(S)}$ of the moment map. To compute the moment map image $\mu(\Gamma_\gamma)$ it is sufficient to know the image of the fixed points of the torus action. By [Ati82] and independently by [GS82] the moment map image $\mu(\Gamma_\gamma)$ is the convex hull of $\mu(\Gamma_\gamma^S)$ if the considered manifold is symplectic. The next two lemmata fit the image of the Bott-Samelson manifold into this framework under the aforementioned non-degeneracy condition on γ.

Lemma 3.5.1. *The differential at a point $p = [g_0, \ldots, g_k] \in \Gamma_\gamma$ of h_γ is given by*

$$\begin{aligned} &h_\gamma(p)^{-1} D_p(h_\gamma)(g_0 v_0, \ldots, g_k v_k) \\ &\qquad = \sum_{l=0}^{k} \sum_{j \leq l} \chi_{[t_l, t_{l+1}]} (\mathrm{Ad}(h_\gamma(p)^{-1} \pi_j(p))(v_j) - \mathrm{Ad}(\pi_j)(v_j)), \end{aligned}$$

where we left translated $h_\gamma(p)$ to the neutral element for convenience of notation, $v_j \in \mathrm{Lie}(K_j)$, and $\chi_{[t_l, t_{l+1}]}$ denotes the indicator function of the interval $[t_l, t_{l+1}]$.

Proof. Denote by $g_j(t)$ a left-translated 1-parameter subgroup in K_j with $g_j(0) = g_j$ and $g_j'(0) = g_j v_j$. We compute

$$\begin{aligned} D_p(h_\gamma)(v_j) &= \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} h_\gamma(g_0, \ldots, g_j(t), \ldots, g_k)(e^{i\varphi}) \\ &= \pi_j v_j g_{j+1} \cdots g_l \gamma(e^{i\varphi}) \pi_l^{-1} - \pi_l \gamma(e^{i\varphi}) g_l^{-1} \cdots g_{j+1}^{-1} v_j \pi_j^{-1} \end{aligned}$$

as long as $\varphi \in [t_l, t_{l+1}]$ and $l \geq j$. If $l \leq j$, the expression is independent of t and thus 0. Left translating results in

$$h_\gamma(p)^{-1} D_p(h_\gamma)(v_j)(e^{i\varphi}) = \mathrm{Ad}(h_\gamma(p)^{-1}(e^{i\varphi})\pi_j(p))(v_j) - \mathrm{Ad}(\pi_j(p))(v_j).$$

Now the formula follows by linearity of the differential. □

Proposition 3.5.2. *For $\gamma \in \Omega(S)$ the restriction of the symplectic form ω of $\Omega(K)$ to Γ_γ is non-degenerate if and only if $\int_{t_j}^{t_j+1} \gamma(e^{i\varphi})^{-1} \gamma'(e^{i\varphi})\, \mathrm{d}\varphi$ is regular for $\Phi^+ \setminus \Phi(K_j')$ for all j.*

Proof. In the following we will use the shorthands $v = (v_0, \ldots, v_k)$, $w = (w_0, \ldots, w_k)$ and $h_\gamma(p) = h_\gamma$. We will also omit dependence on $e^{i\varphi}$ and use i

as an index instead of the complex unit.

$$
\begin{aligned}
&2\pi\omega_{h_\gamma}(\mathrm{D}_p(h_\gamma)(v), \mathrm{D}(h_\gamma)(w)) = S(h_\gamma^{-1}\,\mathrm{D}_p(h_\gamma)(v), h_\gamma^{-1}(\mathrm{D}_p(h_\gamma)(w)))\\
&= \sum_{i=0}^{k}\sum_{j,l\le i}\int_{t_i}^{t_{i+1}} \big((\mathrm{Ad}(h_\gamma^{-1}\pi_j)(v_j) - \mathrm{Ad}(\pi_j)(v_j))', \mathrm{Ad}(h_\gamma^{-1}\pi_l)(w_l) - \mathrm{Ad}(\pi_l)(w_l)\big)\, d\varphi\\
&= \sum_{i=0}^{k}\sum_{j,l\le i}\left(\int_{t_i}^{t_{i+1}} \big(\mathrm{Ad}(\pi_i\gamma^{-1})\circ \mathrm{ad}(-\gamma^{-1}\gamma')\circ \mathrm{Ad}(\pi_i^{-1}\pi_j)(v_j), \mathrm{Ad}(h_\gamma^{-1}\pi_l)(w_l)\big)\, d\varphi\right.\\
&\qquad\left. - \int_{t_i}^{t_{i+1}} \big((\mathrm{Ad}(h_\gamma^{-1}\pi_j)(v_j) - \mathrm{Ad}(\pi_j)(v_j))', \mathrm{Ad}(\pi_l)(w_l)\big)\, d\varphi\right)\\
&= \sum_{i=0}^{k}\sum_{j,l\le i}\left(\int_{t_i}^{t_{i+1}} \big(\mathrm{ad}(-\gamma^{-1}\gamma')\circ \mathrm{Ad}(\pi_i^{-1}\pi_j)(v_j), \mathrm{Ad}(\pi_i^{-1}\pi_l)(w_l)\big)\, d\varphi\right.\\
&\qquad\left. - \int_{t_i}^{t_{i+1}} \big((\mathrm{Ad}(h_\gamma^{-1}\pi_j)(v_j) - \mathrm{Ad}(\pi_j)(v_j))', \mathrm{Ad}(\pi_l)(w_l)\big)\, d\varphi\right)
\end{aligned}
$$

We first consider the second term for fixed j and l with $j \geq l$. Using partial integration we obtain

$$
\sum_{i\ge j}\Big[\big(\mathrm{Ad}(h_\gamma^{-1}\pi_j)(v_j) - \mathrm{Ad}(\pi_j)(v_j), \mathrm{Ad}(\pi_l)(w_l)\big)\Big]\Big|_{t_i}^{t_{i+1}};
$$

which simplifies to

$$
\begin{aligned}
&\big(\mathrm{Ad}(h_\gamma^{-1}(1)\pi_j)(v_j) - \mathrm{Ad}(\pi_j)(v_j), \mathrm{Ad}(\pi_l)(w_l)\big)\\
&- \big(\mathrm{Ad}(h_\gamma^{-1}(e^{it_j}))\pi_j)(v_j) - \mathrm{Ad}(\pi_j)(v_j), \mathrm{Ad}(\pi_l)(w_l)\big)\,,
\end{aligned}
$$

where we reintroduce the dependence on $e^{i\varphi}$. The first term vanishes as $h_\gamma(1) = 1$. For the second term note $h_\gamma(e^{it_j})^{-1}\pi_j = \pi_j\gamma(e^{it_j})^{-1}$, and furthermore $\gamma(e^{it_j})^{-1}$ acts trivially on $\mathrm{Lie}(K_j)$.
Now we consider the case $j \leq l$. By the same arguments as before the resulting term is

$$
\begin{aligned}
&\big(\mathrm{Ad}(h_\gamma(e^{it_l})^{-1}\pi_j)(v_j) - \mathrm{Ad}(\pi_j)(v_j), \mathrm{Ad}(\pi_l)(w_l)\big)\\
&= \big(\mathrm{Ad}(\pi_j)(v_j), \mathrm{Ad}(h_\gamma(e^{it_l})\pi_l)(w_l)\big) - (\mathrm{Ad}(\pi_j)(v_j), \mathrm{Ad}(\pi_l)(w_l))\,.
\end{aligned}
$$

And by the same arguments as before this term vanishes.
So we are left with

$$\sum_{i=0}^{k}\sum_{j,l\leq i}\int_{t_i}^{t_{i+1}}\big(\mathrm{ad}(-\gamma(e^{i\varphi})^{-1}\gamma'(e^{i\varphi}))\circ\mathrm{Ad}(\pi_i^{-1}\pi_j)(v_j),\mathrm{Ad}(\pi_i^{-1}\pi_l)(w_l)\big)\,d\varphi$$

$$=\sum_{i=0}^{k}\sum_{j,l\leq i}\left(\int_{t_i}^{t_{i+1}}-\gamma(e^{i\varphi})^{-1}\gamma(e^{i\varphi})'\,\mathrm{d}\varphi,\big[\mathrm{Ad}(\pi_i^{-1}\pi_j)(v_j),\mathrm{Ad}(\pi_i^{-1}\pi_l)(w_l)\big]\right),$$

which is the symplectic form already found in lemma 3.3.6 up to a scalar. □

Let us state the last remark seperately due to its importance.

Corollary 3.5.3. *The symplectic form defined by restriction of ω to Γ_γ is the form obtained from lemma 3.3.6, induced by the choice*

$$p_j=\frac{1}{2\pi}\int_{t_j}^{t_j+1}\gamma(e^{i\varphi})^{-1}\gamma'(e^{i\varphi})\,\mathrm{d}\varphi\,.$$

We have already showen that integral loops are well-fitted for the Bott–Samelson manifolds. Not quite the same is true for symplecticness of the Bott–Samelson manifold; however another condition we encountered earlier is sufficient. We will record this in the following lemma.

Lemma 3.5.4. *For any dominant loop γ such that the function $\alpha(\gamma^{-1}\gamma')$ is non-negative, the image of the Bott-Samelson manifold Γ_γ is a symplectic, S-invariant submanifold of $\Omega(K)$.*

Remark 3.5.5. *The condition directly translates to condition 1 in proposition 1.4.10 as we are working with absolutely continuous loops. We will denote loops subject to this condition as having* dominant direction.

Proof. We can write

$$\alpha\left(\int_{t_i}^{t_{i+1}}\gamma^{-1}(e^{i\varphi})\gamma'(e^{i\varphi})d\varphi\right)=\int_{t_i}^{t_{i+1}}\alpha(\gamma^{-1}(e^{i\varphi})\gamma'(e^{i\varphi}))d\varphi,$$

and the integrand of the right-hand side is non-negative. If the integral vanishes so must the integrand, so $\alpha\in\Phi(K_j')$. For all other roots the integral is positive and the regularity follows. □

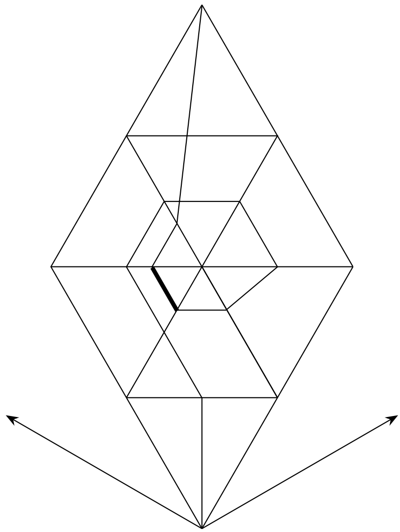

Figure 3.3: A non-integral loop with symplectic Bott-Samelson manifold

Example 3.5.6. *The loop given by its lift seen in figure 3.3 is non-integral. There is a local minimum of the function h_{α_1} along the 9-th segment - marked by extra thickness - of γ which is non-integral. Still the Bott–Samelson manifold is symplectic.*

Integrality is not sufficient to conclude symplecticness as another example shows.

Example 3.5.7. *Let $\pi(t) = t\varpi_1^\vee + -(t^2 - 2\pi t)\varpi_2^\vee$ the lift of γ for the group* $\mathrm{PSU}(3)$*. It is the straight line joining 0 and ϖ_1 with an added perturbation moving its only segment into the fundamental alcove. Then Γ_γ is the flag manifold $\mathrm{PSU}(3)/S$, and the form $\omega|_{\Gamma_\gamma}$ is induced by $\varpi_1^\vee$. This form is degenerate on the flag manifold, and the loop γ is integral.*

To determine the moment map image, we still need to know the fixed points of S on Γ_γ. Fortunately we can derive the general case from the gallery model Bott-Samelson case.

Lemma 3.5.8. *The Bott-Samelson manifold contains as a subset the fibered product of Weyl groups*

$$W_\gamma := W_0 \times_{W_0'} \cdots \times_{W_{k-1}'} W_k/W_k'.$$

where W_i is the Weyl group of K_i and W_i' the Weyl group of K_i'.

Proof. By definition W_γ acts on γ by the same formula as Γ_γ. This gives an injective map $W_\gamma \to h_\gamma(\Gamma_\gamma)$ and thus $W_\gamma \to \Gamma_\gamma$. The explicit form of the map follows as the same formula is used for the action on γ. $\square$

Lemma 3.5.9. *The set of torus fixed points is given by the fibered product of Weyl groups.*

$$h_\gamma(\Gamma_\gamma)^S = h_\gamma(\Gamma_\gamma) \cap \Omega(S) = h_\gamma(W_\gamma)$$

Proof. A loop is invariant under the conjugation action of S if and only if its image is contained in S. Thus the first equality holds.
Every W_i normalizes the torus, thus every segment of a loop in $h_\gamma(W_\gamma)$ is contained in the torus, therefore $h_\gamma(W_\gamma) \subseteq h_\gamma(\Gamma_\gamma)^S$. The other direction is a direct implication of the proof of [Kna02][Chapter 7 Prop. 2.1]. $\square$

Lemma 3.5.10. *Given a loop γ in dominant direction, it holds*

$$\mathrm{wt}(\gamma) \geq \mathrm{wt}(\mu) \textit{ for all } \mu \in \Gamma_\gamma.$$

Proof. We will proof this lemma later when we have established the connection between $\Sigma(\delta)$ and Γ_γ in the next section. $\square$

Proposition 3.5.11. *The image of Γ_γ^S under the moment map are exactly the lattice points in $\mathrm{conv}(W.\,\mathrm{wt}(\gamma))$, which are in the class of γ in the fundamental group of K.*

Proof. The crystal $\mathcal{A}\gamma$ is a subset of $h_\gamma(W_\gamma)$. The weight map and the moment map agree, and as $\mathrm{wt}(\mathcal{A}\gamma))$ contains all weights of the irreducible representation $V(\mathrm{wt}(\gamma))$ of $K^\vee$ it follows $\mu_{\mathrm{Lie}(S)}(\Gamma_\gamma^S) \supseteq (W.\,\mathrm{wt}(\gamma) \cap (R^\vee + \mathrm{wt}(\gamma)))$. Lemma 3.5.10 implies

$$\mu_{\mathrm{Lie}(S)}(\Gamma_\gamma^S) \subseteq \mu_{\mathrm{Lie}(S)}(\mathcal{A}\gamma) \subseteq (W.\,\mathrm{wt}(\gamma) \cap (R^\vee + \mathrm{wt}(\gamma))).$$

$\square$

Corollary 3.5.12. *The moment map image of Γ_γ is the Weyl polytope of* $\mathrm{wt}(\gamma)$.

Proof. Combining proposition 3.5.11 with lemma 3.5.4 we conclude

$$\mu_{\mathrm{Lie}(S)}(\Gamma_\gamma) = \mathrm{conv}(\mu_{\mathrm{Lie}(S)}(\Gamma_\gamma^S)) = \mathrm{conv}(W.\,\mathrm{wt}(\gamma)).$$

$\square$

3.6 The complex structure

For $\eta \in h_\gamma(\Gamma_\gamma)^S$ we have $h_\eta(\Gamma\eta) = h_\gamma(\Gamma_\gamma)$; so we can choose freely any of the torus fixed points to study the embedded Bott-Samelson manifold.

Definition 3.6.1. *We call a loop maximally folded if its lift to* $\mathrm{Lie}(S)$ *is contained in the fundamental alcove.*

Following Gaussent and Littelmann [GL05] we choose as the initial loop the maximally folded one. This gives the advantage that the groups K_i are a priori known to be contained in the set of standard parabolic groups associated to the chosen complexification $K_{\mathbb{C}}$ with the addition of the groups associated to the affine root $(-\alpha, -1)$, where α is the highest root.

Lemma 3.6.2. *Every Bott–Samelson manifold contains a maximally folded loop.*

Proof. Via induction on the segments of γ and the inclusion of W_γ in the Bott-Samelson manifold. □

In the following $\gamma \in \Omega(T)$ denotes the maximally folded loop.

Lemma 3.6.3. *For $\eta \in \Gamma_\gamma^S$ there is a natural isomorphism*

$$\begin{aligned} h_{\gamma,\eta} &: \Gamma_\gamma \to \Gamma_\eta; \\ [g_0, \ldots, g_k] &\mapsto [g_0 p_0, p_0^{-1} g_1 p_1 p_0, \ldots, p_0^{-1} \cdots p_{k-1}^{-1} g_k p_k \cdots p_0] \end{aligned}$$

which is nothing else but $h_{\gamma,\eta} = h_\eta^{-1} \circ h_\gamma$.

Definition 3.6.4. *Let $\gamma \in \Omega(S)$. Denote by F_j the smallest face of the affine Coxeter complex which contains $\gamma(t_j)$ and by F_j' the smallest face which contains the j-th segment of γ. Then the loop γ gives rise to a sequence of faces $\delta(\gamma) = (F_0, F_0', \ldots, F_k)$. By adding the weight of γ to the end of the sequence we obtain an object similar to a gallery, where the usual dimension and codimension conditions on the faces are dropped. In the case $\delta(\gamma)$ is a gallery, we call γ a* gallery walk *of the gallery $\delta(\gamma)$. If it is a gallery of alcoves, we say γ is an alcove walk and if the gallery is a 1-skeleton gallery, we say γ is a 1-skeleton walk.*

Remark 3.6.5. *Even in the case $\delta(\gamma)$ is not a gallery, the Bott–Samelson variety $\Sigma(\delta(\gamma))$ is still well-defined.*

Remark 3.6.6. *If γ is an integral loop, we can use the root operators acting on γ to define the action on*

$$\delta(\mathcal{A}\gamma) = \{\delta(\tilde{\gamma}) \mid \tilde{\gamma} \in \mathcal{A}\gamma\}.$$

This is what we meant by ad hoc approach in section 1.5.

We want to show that the Bott–Samelson variety and manifold obtained from $\delta(\gamma)$ and γ are naturally isomorphic. We will be using the maximally folded loop to construct the isomorphism.

Lemma 3.6.7. *For every proper subset I of the extendend Dynkin diagram of K let $K_I \subseteq K$ be the connected subgroup of maximal rank associated to I using Borel–de Siebenthal and $\mathcal{P}_I$ the standard parahoric associated to I. Then there exists an injective group homomorphism $\varphi_I : K_I \to \mathcal{P}_I$. If we take a subset $J \subseteq I$, then the map φ_J is the restriction of φ_I to K_J.*

Proof. By [Kum02] the parahoric subgroup $\mathcal{P}_I$ can be realized as the semidirect product of a finite-dimensional complex reductive group G_I with a pro-unipotent progroup. The group G_I can be thought of as a Levi component of $\mathcal{P}_I$. The type of G_I is given by the subgraph of the extended Dynkin diagram of K induced by the set of nodes I. Thus it is of the same type as the subgroup K_I of K. By the choice of a torus $K_{\mathbb{C}}$ we have a torus of G_I and by this a uniquely prescribed maximal compact subgroup $\tilde{K}_I$ containing $S_{\mathbb{C}}$ of the same type as G_I. As we have fixed a set of simple roots for $K_{\mathbb{C}}$ and by this also for G_I, we have a unique Lie algebra homomorphism $\mathrm{Lie}(K_I) \to \mathrm{Lie}(\tilde{K}_I)$. To check whether it lifts to a homomorphism of the Lie groups, we can check that the exponential maps of the tori of K_I and $\tilde{K}_I$ have the same kernel. However this is clear as S is the torus for both.
For a subset $J \subseteq I$ we note that by the construction in [Kum02] we have $G_J \subseteq G_I$. □

Remark 3.6.8. *For a special vertex v of the fundamental alcove with associated parahoric subgroup $\mathcal{P}_v$ the isomorphism described can be made more explicit. The exponential of the straight line joining* 0 *and v is an element γ_v of the loop group of the adjoint group $K^{ad} = K/Z(K)$. The parahoric subgroup associated to $\{0\}$ is K_A and has a natural embedding $K \to K_A$. Conjugating K_A by γ_v gives the group $\mathcal{P}_v$ and by this the embedding $K \to \mathcal{P}_v$.*
In the case of an alcove walk this further simplifies. In this case K_I is the stabilizer of a root hyperplane $H_{\alpha,n}$, where $n = 0$ if α is simple and $n = 1$ if α is the highest root. Then K_I can be obtained using a standard $\mathrm{Lie}(\mathrm{SU}(2))$*-triple, and φ_I is Φ_α if α is simple. If α is the highest root, then φ_I is the* $\mathrm{Lie}(\mathrm{SU}(2))$*-triple associated to the affine root $(-\alpha, -1)$.*

It is also worthwile mentioning the nature of the image $\varphi_I(K_I)$.

Proposition 3.6.9. *The image $\varphi_I(K_I)$ is a subset of the free polynomial loop group of K.*

Proof. Every group K_I is generated by the stabilizers of the facets adjacent to the face defined by I. These groups are contained in the free polynomial loop group by remark 3.6.8. □

The groups K_I are a small part of $\mathcal{P}_I$ as they are finite-dimensional while the latter is infinite-dimensional. Nevertheless, K_I contains the essential information regarding $\mathcal{P}_I$'s role in the gallery model.

Lemma 3.6.10. *The Bott–Samelson variety* $\Sigma(\gamma)$ *is connected in its analytic topology.*

Proof. First consider the case of an alcove walk. In this case the map

$$\begin{aligned} \Sigma(\delta(\gamma)) &\to \mathcal{P}_0 \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \mathcal{P}_{k-1}/\mathcal{B} \\ [g_0 : \cdots : g_k] &\mapsto [g_0 : \cdots : g_{k-1}] \end{aligned}$$

is a $\mathbb{P}_1$-fiber bundle [Kum02][p.206]. The fiber is connected; the base space is connected by induction. Thus also the total space $\Sigma(\gamma)$ is connected. For the general case we note that we can adapt the above map to

$$\begin{aligned} \Sigma(\gamma) &\to \mathcal{P}_0 \times_{\mathcal{Q}_0} \cdots \times_{\mathcal{Q}_{k-2}} \mathcal{P}_{k-1}/\mathcal{Q}_{k-1} \\ [g_0 : \cdots : g_k] &\mapsto [g_0 : \cdots : g_{k-1}], \end{aligned}$$

and the proof of [Kum02] can be adapted in order to show that this map is a fiber bundle with fiber the (partial) flag variety $\mathcal{P}_k/\mathcal{Q}_k$. The connection of those cases is the flag variety of the unique SL(2)-triple in $\mathcal{P}_k$; the $\mathbb{P}^1$ from above. □

The final piece we need is the dimension of the Bott–Samelson manifold and the variety.

Lemma 3.6.11. *The dimensions of the Bott–Samelson manifold* Γ_γ *and* $\Sigma(\delta_\gamma)$ *are the same.*

Proof. The complex dimension of $\Sigma(\delta_\gamma)$ can be computed by the fiber bundle introduced in the last proof. Dimension of the fibers can be computed via root combinatorics. The dimension of the i-th fiber in the inductive fiber bundle construction is the number of affine hyperplanes containing $\gamma(t_{k-i+1})$ from which the number of hyperplanes containing the segment after $\gamma(t_{k-i+1})$ needs to be subtracted. This is easily seen from the decomposition of $\mathcal{P}_I$ as a semidirect product.
As Γ_γ is the quotient of a finite-dimensional space by a finite-dimensional group, it is enough to compute the dimension of these, which was done in lemma 3.1.6. The result coincides with the above one. □

Theorem 3.6.12. *The map*

$$\begin{aligned} \Gamma_\gamma &\to \mathcal{P}_0 \times_{\mathcal{Q}_0} \mathcal{P}_1 \times_{\mathcal{Q}_1} \cdots \times_{\mathcal{Q}_{k-1}} \mathcal{P}_k/\mathcal{Q}_k \\ [g_0 : \cdots : g_k] &\mapsto [\varphi_0(g_0) : \cdots : \varphi_k(g_k)] \end{aligned}$$

is a diffeomorphism.

Proof. The defined map is injective as it follows from lemma 3.6.7 that the intersection $\mathcal{Q}_j \cap K_j = K_j'$. Thus the map is an injective, closed map between compact, connected manifolds of the same dimension forcing it to be a diffeomorphism. □

For record keeping we state the following translation from loops to galleries.

Lemma 3.6.13. *Let $\gamma \in \Omega(S)$ be an integral loop. Then it holds for a loop $\eta \in h_\gamma(\Gamma_\gamma)$:*

1. *The weight $\mathrm{wt}(\eta)$ is the same as the target of $\delta(\eta)$.*

2. *If η is an element of a crystal which is contained in Γ_γ, then the root operators commute with the diffeomorphism $\Gamma_\gamma \to \Sigma(\delta(\gamma))$.*

Proof. The first statement is only a translation from loops to galleries. For the second statement we refer to the description of the root operators in section 1.4 and section 1.5. □

Example 3.6.14. *We continue example 2.1.5. The maximally folded path is $h_{\alpha^\vee}([\mathbb{1} : w_0])$, where w_0 is a representative of the longest word in the Weyl group. There are two singular points $\alpha^\vee(1), \alpha^\vee(-1)$ with attached groups SU_3 and $\mathrm{SU}_3(\alpha_1+\alpha_2)$, where the second group is the same as K_1 in example 3.1.8. Thus the map*

$$\mathrm{SU}_3 \times_S \mathrm{SU}_3(\alpha_1 + \alpha_2)/S \to \mathrm{SL}_3(\mathcal{O}) \times_{\mathcal{B}} \mathcal{P}_0/\mathcal{B}$$

$$\left[A : \begin{pmatrix} a & 0 & b \\ 0 & (ad-bc)^{-1} & 0 \\ c & 0 & d \end{pmatrix}\right] \mapsto \left[A : \begin{pmatrix} a & 0 & t^{-1}b \\ 0 & (ad-bc)^{-1} & 0 \\ tc & 0 & d \end{pmatrix}\right]$$

is a diffeomorphism. Via the diffeomorphism we obtain an action of the complex torus $S_{\mathbb{C}}$ which complexifies the action of S. There are a total of 12 fixed points given by the (trivially) fibered product of Weyl groups

$$W(\mathrm{SU}_3) \times W(\mathrm{SU}_2(\alpha_1 + \alpha_2)).$$

Using the diffeomorphism of theorem 3.6.12, we can define an action of $S_{\mathbb{C}}$ complexifying the action by S. We can translate the results of Gaussent and Littelmann on the Białynicki-Birula cells of $\Sigma(\delta(\gamma))$ to Γ_γ. For this we extend the definition of the weight function wt to the whole loop group $\Omega(S)$ via the first part of proposition 2.1.2.

Proposition 3.6.15. *Let γ be in dominant direction. The dimension of a Białynicki-Birula cell centered at a torus fixed point p is bounded from above by $\langle \mathrm{wt}(h_\gamma(p)), \rho\rangle$. The bound is equal to the dimension if $h_\gamma(p)$ is an element of the crystal obtained in 3.5.10 contained in $h_\gamma(\Gamma_\gamma)$.*

Example 3.6.16. *Continuing example 3.6.14 we find that the crystal of highest weight (in the sense of 3.5.10) inside of the Bott-Samelson $\Gamma_{\alpha^\vee}$ consists of the elements $[\mathbb{1},\mathbb{1}],[\mathbb{1},w]$ for $w \in W(\mathrm{SU}(3))$ and the two paths of weight zero $[s_1,w_0],[s_2,w_0]$.*

Let us recall that the purpose for which Gaussent and Littelmann used the Bott–Samelson variety was as a desingularization of the affine Schubert variety C_λ and the description of the MV cycles. Our approach via the compact setting opens another point of view.

Proposition 3.6.17. *The affine Schubert variety C_λ realized in $\Omega(K)$ is the same as the image of*

$$\begin{aligned}\Gamma_\gamma &\to \Omega(K)\\ [g_0,\dots,g_n] &\mapsto \varphi_0(g_0)\cdots\varphi_n(g_n)\lambda^f,\end{aligned}$$

where λ^f is the exponential of the straight edge joining 0 and a special vertex of the fundamental alcove which defines the same class as λ in the fundamental group of K.

While this description of C_λ is by no means minimal (the map has nontrivial fibers in general), it is a direct description which does not rely on the Iwasawa decomposition of K_F.

Proof. By [Kum02] and [GL05] the image of the map is the affine Schubert variety. What remains is that it is in the slice $\Omega(K)$ which we identify as $L(K)/K$; this is clear by proposition 3.6.9. $\square$

Corollary 3.6.18. *Let $\gamma \in \Omega(S)$ in dominant direction and $\eta \in \mathcal{A}\gamma$. Let X be the MV cycle associated to the gallery $\delta(\eta)$ and $\Gamma_{\gamma,\eta}$ the Białynicki-Birula cell of Γ_γ centered around η. Then X realized in $\Omega(K)$ is given by*

$$\varphi_0(g_0)\cdots\varphi_k(g_k)\lambda^f\,\mathrm{K}$$

for $[g_0:\dots:g_k] \in \overline{\Gamma_{\gamma,\eta}}$.

Example 3.6.19. *Consider again the loop $\alpha^\vee$ inside the torus of* $\mathrm{SU}(2)$. *The image of Γ_γ in $\Omega(\mathrm{SU}(2))$ under the resolution of singularities π consists of loops of the form*

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} c & dt^{-1} \\ -\bar{d}t & \bar{c} \end{pmatrix} \mathrm{SU}(2).$$

We equip Γ_γ with the $S_{\mathbb{C}}$-action from the diffeomorphism $\Gamma_\gamma \to \Sigma(\delta(\gamma))$ and compute the Białynicky-Birula cells. There are a total of four cells

$$\begin{aligned} C_{[1,s_0]} &= D(a) \cap D(d) \\ C_{[1,1]} &= D(a) \cap \mathcal{V}(d) \\ C_{[s_1,1]} &= \mathcal{V}(a) \cap D(c) \\ C_{[s_1,s_0]} &= \mathcal{V}(a) \cap \mathcal{V}(c), \end{aligned}$$

where $\mathcal{V}$ denotes vanishing of the coordinate and D non-vanishing of the coordinate. The index of C is the torus fixed point around which the cell is centered. The elements of the crystal are $[1, s_0]$,$[s_1, 1]$ and $[s_1, s_0]$. Thus the MV cycles in $\Omega(\mathrm{SU}(2))$ are given by the closures of

$$\begin{aligned} X_{\alpha^\vee,-\alpha^\vee} &= \{-\alpha^\vee\} \\ X_{\alpha^\vee,0} &= X_{\alpha^\vee,-\alpha^\vee} \cup \left\{ \begin{pmatrix} |c|^2 + z|d|^2 & -b^2\bar{c}\bar{d}(1-z) \\ -\bar{b}^2cd(z^{-1}-1) & |c|^2 + z^{-1}|d|^2 \end{pmatrix} \right\} \\ X_{\alpha^\vee,\alpha^\vee} &= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} c & dz^{-1} \\ -\bar{d}z & \bar{c} \end{pmatrix} \begin{pmatrix} \bar{c} & -d \\ \bar{d} & c \end{pmatrix} \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} \right\}. \end{aligned}$$

As we have defined in section 1.3 the loop group $\Omega(K)$ carries a complex structure. We have realized Γ_γ as a symplectic submanifold of the loop group. Unfortunately, it is in general not a complex submanifold as the following example shows.

Example 3.6.20. *Let $K = \mathrm{SU}(2)$ and take $\gamma = \alpha^\vee$ the single positive root. We obtain $\Gamma_\gamma = \mathrm{SU}(2) \times_S \mathrm{SU}(2)/S$. For an element $(0, B)$ of the tangent space at $[1:1]$ we obtain the Fourier coefficients*

$$a_k = \begin{cases} 0 & \text{for } k \in 2\mathbb{Z} \setminus \{2, -2\} \\ 2i\left(\frac{4\operatorname{Re}(b) - 2ik\operatorname{Im}(b)}{\pi(k-2)(2+k)k}\right) & \text{for } k \in 2\mathbb{Z}+1 \\ \frac{\bar{b}}{2} & \text{for } k = -2 \\ \frac{b}{2} & \text{for } k = 2, \end{cases}$$

where we choose B to be represented by $\left(\begin{smallmatrix} 0 & b \\ -\bar{b} & 0 \end{smallmatrix}\right)$. Choose $b = 1$ and assume $J(0, B) \in \mathrm{T}_{[1,1]}\Gamma_\gamma$. From the first Fourier coefficient of $J(0, B) = (\tilde{A}, \tilde{B})$ one *can deduce* $\tilde{b} = -2i$ *however, then the third coefficient of the left-hand side is* $\frac{-8}{\pi 15}$ *while the right-hand side has coefficient* $\frac{24}{\pi 15}$.

The example combines the cases γ is an alcove walk and a 1-skeleton walk. These are the best studied and well behaved cases. It is reasonable to assume that the instances where Γ_γ is a complex submanifold of $\Omega(K)$ are rare.
We will now turn to the proof of 3.5.10.

Proof of lemma 3.5.10. Let $\gamma \in \Omega(S)$ be in dominant direction and $\eta \in h_\gamma(\Gamma_\gamma)$ the maximally folded loop. We choose $w = [w_0 : \cdots : w_k] \in \Gamma_\eta$ to be represented by elements of the Weyl group such that $h_\eta(w) = \gamma$. The element $\pi_k(w)$ can be decomposed as $\pi_k(w) = w_0 \cdots w_k$. We claim $l(w) = \sum_j l(w_j)$. The length of w_j is the number of hyperplanes entered at $\gamma(e^{it_j})$ and left at some time during the $j-th$ segment. As γ is in dominant direction, it crosses every hyperplane at most once. If γ crosses a hyperplane, it crosses from the negative to the positive halfspace associated to the hyperplane. Thus the set of hyperplanes crossed is $\mathcal{M}(0, \mathrm{wt}(\gamma))$. As w translates the fundamental alcove to an alcove containing $\mathrm{wt}(\gamma)$, it follows $l(w) \geq \mathcal{M}(0, \mathrm{wt}(\gamma))$; and from this it follows that $\pi(\Sigma(\delta(\gamma)))$ is contained in the affine Schubert variety $C_{\mathrm{wt}(\gamma)}$ [Kum02][Theorem 5.1.3]. By the closure relations described in definition 1.6.1 we know that

$$\mathrm{wt}(\tilde{\gamma}) = \mathrm{wt}(\delta(\tilde{\gamma}) \leq \mathrm{wt}(\delta(\gamma)) = \mathrm{wt}(\gamma).$$

$\square$

4 Implications for the path model

When we speak of a homotopy, we will always use the unit intervall $[0, 1]$ as the time parameter.

Definition 4.0.1. *If γ_s is a homotopy of loops in the torus S, we will denote the Bott–Samelson manifold Γ_{γ_s} by Γ_s and likewise for the embeddings into the loop group.*

Definition 4.0.2. *A homotopy γ_s is fitted to the path model if Γ_s is Γ_0 for all $s < 1$.*

Let us emphasize that we want Γ_s to be the same and not just isomorphic.

Lemma 4.0.3. *A homotopy which is fitted to the path model induces a map*

$$\Psi : \Gamma_s \to \Gamma_1$$

which is compatible with embedding maps h_s in the sense that

$$\Psi = h_1^{-1} \circ \lim_{s \to 1} h_s.$$

Proof. We need only check, whether the map Ψ is a well-defined map $\Gamma_0 \to \Gamma_1$, and we will furthermore only need to check whether the image of $\lim h_s$ lies inside of Γ_1. As $\Gamma_s = \Gamma_0$ for $s \leq 1$, we can deduce that there exist continuous functions $t_j(s)$ such that $\alpha(\gamma_s(e^{it_j(s)})) = 0$ for $s \leq 1$ and $\alpha \in \Phi^+(K_j)$. Then $\alpha(\gamma_s(e^{it_j(1)})) = 0$ is also true. The statement holds similarly for segments, but it might happen that segments of γ_0 collaps in the limit $s \to 1$. □

More directly the map can be achieved by a sequence of quotient maps and multiplication of several factors. We refer to the examples in the upcoming sections for this statement.

4.1 Homotopies and the symplectic form

Equipped with the homotopies fitted to the path model, we want to apply them to the Bott–Samelson manifolds. Let us first record how the form $\omega|_{\Gamma_\gamma}$ changes when Ψ is reduced to the identity map.

Corollary 4.1.1. *If $\gamma : S^1 \times [0,1] \to K; (z,t) \mapsto \gamma_t(z)$ is a homotopy of loops such that the Bott-Samelson manifold Γ_{γ_t} is the same for all t, then the pullback $h^*_{\gamma_t}\omega$ is a continuous family of 2-forms on Γ_{γ_0}.*

Proof. This is seen by looking at our formula for $\omega|_{\Gamma_\gamma}$ and the fact that the element $\int_{t_j(s)}^{t_{j+1}(s)} \gamma_s^{-1}\gamma_s'$ is continuously dependent on s. □

Remark 4.1.2. *Given a loop $\gamma \in \Omega(S)$ we can obtain the same induced form $\omega_\gamma := \omega|_{\Gamma_\gamma}$ via a loop which is segmentwise polynomial. Among these loops we can choose the one of minimal energy to get a natural realization for ω_γ. The resulting loop is segmentwise a geodesic, and we can even choose it to be parametrized to have constant speed.*

Degeneracy of the 2-form can be remedied via homotopy.

Proposition 4.1.3. *Let $\gamma \in \Omega(S)$ be a maximally folded loop such that ω is degenerate if restricted to Γ_γ. Then there exists a homotopy γ_s fitted to the path model such that ω is non-degenerate if restricted to Γ_1 and the induced map $\Gamma_0 \to \Gamma_1$ is surjective.*

Proof. We use induction on the number of segments such that $\int_{t_j}^{t_{j+1}} \gamma^{-1}\gamma'$ is not-regular for $\Phi^+ \setminus \Phi(K_j')$. If there is no such segment, then the restriction of ω is non-degenerate. Now suppose j is such that $\int_{t_j}^{t_{j+1}} \gamma^{-1}\gamma'$ is not-regular for $\Phi^+ \setminus \Phi(K_j')$. We can rephrase this to: There exists $\alpha \in \Phi^+ \setminus \Phi(K_j')$ such that

$$\alpha(\int_{t_j}^{t_{j+1}} \gamma^{-1}\gamma') = 0,$$

which is to say that γ touches the hyperplane $H_{\alpha,m}$ both at t_j and t_{j+1}, where

$$m = \int_0^{t_j} \gamma^{-1}\gamma'.$$

As all faces of the Coxeter complex are topologically trivial, we move the j-th segment of γ into the intersection of all such $H_{\alpha,m}$. If $\alpha_0 \notin \Phi^+ \setminus \Phi(K_j)$, we can decompose the segment into a linear combination of cofundamental weights

$$\gamma|_{[t_j,t_{j+1}]} = \sum_{l=1}^{n} q_l \varpi_l^\vee$$

for some functions q_l. The homotopy we want to choose is achieved by replacing q_l by $(1-s)q_l$. If $\alpha_0 \in \Phi^+ \setminus \Phi(K_j)$, we choose a vertex v of Δ_f stabilized

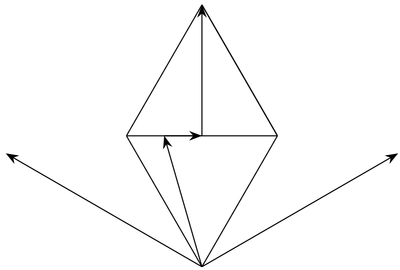

Figure 4.1: The loop $\tilde{\gamma}$ of example 4.1.4

by K_j instead. By Borel-de Siebenthal this givestm a subsystem of the root system of rank n. We choose a simple root system for this subsystem by declaring the Weyl chamber dominant which contains Δ_f. This set of simple roots is obtained by the simple roots which vanish at v and the negative of the highest root. With this choice of simple roots for this subsystem we can again write the segment in terms of the cofundamental weights of $\Phi(\mathrm{Stab}(v))$. Using the same procedure as in the first case we arrive at an explicit homotopy. Note also that $\Phi(K_j) \subseteq \Phi(\mathrm{Stab}(V))$.
The explicit form of the homotopy implies that in the limit $s \to 1$ the group K_j is unchanged and only K_j' is enlarged to contain exactly those roots which caused the degeneracy of ω_{Γ_γ}. □

Example 4.1.4. *We continue the example of figure 3.2. We will remove the local minimum via a homotopy of γ to the loop $\tilde{\gamma}$, see figure 4.1. The map that is achieved on the level of Bott–Samelson manifolds is*

$$\begin{aligned} \mathrm{SU}(3) \times_S \mathrm{SU}(\alpha_0) \times_S \mathrm{SU}(\alpha_0)/S &\to \mathrm{SU}(3) \times \mathrm{SU}(\alpha_0)/S \\ [g_0 : g_1 : g_2] &\mapsto [g_0 : g_1 g_2], \end{aligned}$$

which is surjective. This example also gives us the chance to examine a slight misfit of notation in the proof. Following the definition of the Bott–Samelson manifold, the loop γ has the groups K_0, K_1, K_2 attached while $\tilde{\gamma}$ has only the groups K_0, K_1 attached. In the proof we claimed that K_2 would not change under the described homotopy. If we follow the steps of the proof, the manifold we will obtain is

$$\mathrm{SU}(3) \times_S \mathrm{SU}(\alpha_0) \times_{\mathrm{SU}(\alpha_0)} \mathrm{SU}(\alpha_0)/S,$$

which does not quite fit our definition of the Bott–Samelson manifold for $\tilde{\gamma}$. The reason for this is that $\Phi(K_1')$ is exactly the same as the set of roots for *which $\int_{t_1}^{t_2} \gamma^{-1}\gamma'$ vanishes. This is no problem as the multipliction map*

$$\begin{aligned} \mathrm{SU}(3) \times_S \mathrm{SU}(\alpha_0) \times_{\mathrm{SU}(\alpha_0)} \mathrm{SU}(\alpha_0)/S &\to \mathrm{SU}(3) \times \mathrm{SU}(\alpha_0)/S \\ [g_0 : g_1 : g_2] &\mapsto [g_0 : g_1 g_2] \end{aligned}$$

is an isomorphism to the actual Bott–Samelson of $\tilde{\gamma}$ and is compatible with the action on $\tilde{\gamma}$. All things considered, this is just an oddity of notation.

4.2 The shrinking algorithm

We will now explain the shrinking algorithm. We also describe a closely related method to obtain from a loop γ an alcove walk connected to γ through a homotopy fitted to the path model.

Theorem 4.2.1. *Let $\gamma \in \Omega(S)$ be a loop.*

1. *There exists a homotopy γ_s fitted to the path model such that γ_1 is a 1-skeleton gallery walk.*
2. *There exists $\eta \in \Omega(S)$ an alcove walk and a homotopy η_s fitted to the path model such that $\eta_1 = \gamma$.*

Proof. As reference loop to construct the Bott–Samelson manifold we will take γ to be maximally folded. We will coarsen the partition of $[0, 2\pi]$ by the t_i. Let H_i be the set of affine hyperplanes that contain $\gamma(t_i)$. Let $t'_0 = 0$, and define t'_{i+1} as the t_j such that there is a hyperplane H, in which $\gamma(t_j)$ is contained, that has no common intersection with the union of the H_k for $t'_i \leq t_k < t_j$. If $t'_i = t_k$, we will denote $k-1$ by $i^\flat$. The concatenation of segments $\gamma_{[t'_i, t_{(i+1)^\flat}]}$ can be homotoped into any vertex in the intersection of all H_k such that no K_k changes until $s = 1$. We choose a vertex and denote it by v_i. The $i^\flat$-th segment can be homotoped into the edge $[v_{i-1}, v_i]$ (this is of course only necessary for $i \neq 0$). By choice of v_i it is fixed by all K_k for $i^\flat + 1 \leq k \leq (i+1)^\flat$, and as such the map Ψ is well-defined. Starting for $i = 0$ and homotoping segment by segment a gallery walk of a 1-skeleton gallery is obtained. By definition it is fitted to the path model.
We will prove the second part of the theorem. We still assume γ to be maximally folded. For every $j \in \{1, \dots, k\}$ choose a positive root β_j such that $\beta_j(\gamma(t_j))$ vanishes. The β_j are necessarily either simple roots or the highest root. They define facets F_j of the fundamental alcove Δ_f. We choose points p_j in the open face F_j^o. Define η to be the loop with j-th segment a path joining the point p_j with the point $p_j + 1$ inside of the interior of the fundamental alcove Δ_f^o for $j \in \{1, \dots, k-1\}$. It might very well occur that $p_j = p_{j+1}$ and

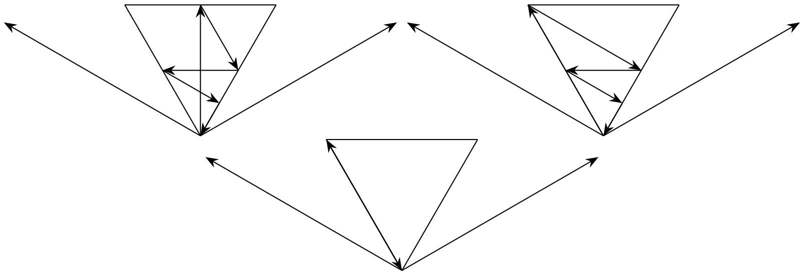

Figure 4.2: Application of the shrinking algorithm

need to be joined by a non-trivial path. The 0-th segment is a path joining 0 with p_1, while the $k-1$-st segment joins p_k and the endpoint of γ. Now a homotopy from η to γ can again be given in two steps. First move p_j to $\gamma(t_j)$ inside the face F_j. Then the segments can be homotoped to the segments of γ. Again this homotopy can be realized in a way such that it is fitted to the path model. □

Let us work through the first algorithm in an example.

Example 4.2.2. *Consider the loop depicted to the left in figure 4.2. We obtain $t'_0 = 0 < t'_1 = t_1 < t'_2 = 2\pi$. In the first step we only need to consider the hyperplane $H_{\alpha_0,1}$ which contains as vertices the cofundamental weights. We choose $\varpi_1^\vee$ and obtain the loop in the top right of figure 4.2. The points $\gamma(t_2), \gamma(t_3), \gamma(t_4), \gamma(t_5)$ have common intersection in 0 and we can homotope the multisegment $\gamma|_{[t_2,t_5]}$ into this vertex of Δ_{fund}. The final result is seen at the bottom of figure 4.2. It is the concatenation of $\varpi_1^\vee$ with its negative.*

The crux of these maps is of course that there is no guarantee for surjectivity.

Lemma 4.2.3. *Let γ be a maximally folded loop and γ_s a homotopy fitted to the path model. Then $\mathrm{wt}(\Psi(\eta)) = \mathrm{wt}(\eta)$ for every torus loop η in Γ_γ.*

Proof. As the homotopy is fitted to the path model, if $h_\gamma(w) = \eta$ then $h_s(w)$ defines a homotopy in S starting in η ending in $\Psi(\eta)$. □

We were already able to give the moment map image of Γ_γ for a loop γ in dominant direction. Using lemma 4.2.3 we can do better.

Proposition 4.2.4. *If γ is an integral loop such that Γ_γ is symplectic and there exists a homotopy γ_s fitted to the path such that:*

1. *γ_1 is a 1-skeleton walk and the type of $\delta(\gamma_1)$ is the type of a dominant combinatorial 1-skeleton gallery,*

2. *the image of the map* Ψ *contains a minimal gallery* D *(and by this all minimal galleries in* $\Sigma(\delta(\gamma_1))$*),*

then $\mu_{\mathrm{Lie}(S)}(\Gamma_\gamma) = \mathrm{conv}(W.\,\mathrm{wt}(D))$.

Proof. By condition 1) the map $\Gamma_1 \to C_{\mathrm{wt}(D)}$ is well-defined and surjective. Let $w \in \Gamma_\gamma$ such that $\delta(\Psi(w)) = D$ then $\mathrm{wt}(w) = \mathrm{wt}(D)$ by 4.2.3. Again by the closure relations of affine Schubert varieties we know that for $w' \in \Gamma_\gamma^S$

$$\mathrm{wt}(w') = \mathrm{wt}(\Psi(w')) \leq \mathrm{wt}\,(D) = \mathrm{wt}(w).$$

□

Example 4.2.5. *We remind the reader of the loop depicted in figure 3.3. The shrinking algorithm takes three steps in this case. We will describe the result in terms of the loop seen in the figure instead of the maximally folded one.*
In the first step the first segment is moved to the line connecting 0 *and* $\varpi_1^\vee$ *and the* 2*-nd segment is shrunk into* $\varpi_1^\vee$*. In the second step segments* 3 *to* 10 *are shrunk into the line connecting* $\varpi_1^\vee$ *with* $\alpha_1^\vee + \alpha_2^\vee$*. In the third step segment* 11 *is homotoped to the segment joining* $\alpha_1^\vee + \alpha_2^\vee$ *with* $2\varpi_1^\vee + \varpi_2^\vee$*, forcing the last segment to connect to* $2\alpha_1^\vee + 2\alpha_2^\vee$*.*
The result is the concatenation $\varpi_1^\vee * \varpi_2^\vee * \varpi_1^\vee * \varpi_2^\vee$ *a* 1*-skeleton walk of weight* $\lambda = 2\alpha_1^\vee + 2\alpha_2^\vee$*. This is also the weight of corresponding minimal dominant combinatorial* 1*-skeleton gallery and we obtain*

$$\mu_{\mathrm{Lie}(S)}(\Gamma_\gamma) = \mathrm{conv}(W.\lambda).$$

Example 4.2.6. *The considered group is* $\mathrm{SO}(5)$ *and we fix the concatenation of edges* $\gamma = \frac{1}{2}\varpi_1^\vee * (\varpi_2^\vee + \frac{1}{2}\varpi_1^\vee)$*. This loop is contained in the dominant Weyl chamber, it is not in dominant direction, but it is integral. It is a* 1*-skeleton walk, but* $\mathrm{type}(\delta(\gamma))$ *is not the type of a dominant combinatorial* 1*-skeleton gallery. Still it holds that*

$$\mu_{\mathrm{Lie}(S)}(\Gamma_\gamma) = \mathrm{conv}(W.\,\mathrm{wt}(\varpi_2^\vee)).$$

4.3 MV-polytopes via minimal loops

Following Ehrig [Ehr09] we define $E_\alpha^{\max}(\gamma) = E_\alpha^{\varepsilon_\alpha(\gamma)}(\delta)$. For $w \in W$ with reduced decomposition $w = s_{i_1} \cdots s_{i_k}$ we define $E_w(\gamma) := E_{i_k}^{\max} \cdots E_{i_1}^{\max}(\gamma)$. By [Lit97] the resulting loop $E_w(\gamma)$ is independent of the chosen reduced decomposition of w.

Definition 4.3.1. *Let γ be an integral loop and $w = s_{i_1} \dots s_{i_k}$ an element of the Weyl group W. We define the vertex loop $\Xi_w(\gamma)$ as*

$$\Xi_w(\gamma) = wE_w^{max}(\gamma).$$

Corollary 4.3.2. *Let γ be an integral loop which is also an alcove walk, then:*

(i) The loop $\Xi_w(\gamma)$ is an element of Γ_γ

(ii) The MV-polytope corresponding to δ_γ is given by

$$\operatorname{conv}\left\{\operatorname{wt}(\Xi_w(\gamma) \mid w \in W\right\}.$$

By application of the shrinking algorithm we see that the same statement holds for a 1-skeleton gallery. For this just note that every 1-skeleton gallery walk is a shrinking of an alcove walk.

4.4 A family of affine Schubert varieties

Let us remind the reader of two facts about affine Schubert varieties C_λ. First, they are the closure of the K_A-orbit of λ, an element of T_F/T_A. Second, they are the image of an appropriate Bott–Samelson variety under the desingularization map $[g_0 : \cdots : g_k] \mapsto g_0 \cdots g_k.\lambda^f$. The loop group setting opens up another possibility as we can think of T_F/T_A as the group of polynomial loops of the torus S. It is a discrete group in the non-discrete group $\Omega^{\mathrm{pol}}(K)$. The corresponding subgroup $\Omega(S)$ of $\Omega(K)$ is much larger than T_F/T_A.

Definition 4.4.1. *Let $\eta \in \Omega(S)$ be in dominant direction and $\gamma \in \Gamma_\eta$ be maximally folded. Let ν be any loop in S which is stabilized by the stabilizer K'_k of the last segment of γ and has weight $\operatorname{wt}(\eta)^f$. We define the map*

$$\begin{aligned} \pi_\nu : \Sigma(\delta(\gamma)) &\to \Omega(K) \\ p = [g_0, \dots, g_k] &\mapsto g_0 \cdots g_k \nu, \end{aligned}$$

where we choose p as being represented by elements which are in the image of Γ_γ. We denote the image of π_ν by C_ν.

Proposition 4.4.2. *The set C_ν can be continuously mapped to the affine Schubert variety $C_{\operatorname{wt}(\eta)}$.*

Proof. Let ν_s be a homotopy of ν with $\nu_1 = \operatorname{wt}(\eta)^f$ inside of S, where again $\operatorname{wt}(\eta)^f$ is the unique loop inside of Δ_f of the same type as $\operatorname{wt}(\eta)^f$. The homotopy can be chosen in a such a way that ν_s is also stabilized by K'_k.

Such a homotopy exists as the fundamental group of S is isomorphic to $X_*(S)$ and the class of ν in the fundamental group of S is $\mathrm{wt}(\nu)$. We obtain maps $\pi_s : \Sigma(\delta(\gamma)) \to \Omega(K)$ and because of our choice of γ and by [GL05] the image of π_1 is the affine Schubert variety $C_{\mathrm{wt}(\eta)}$. □

We restate this result using the proper language.

Theorem 4.4.3. *Denote the embedding of the affine Schubert variety C_λ as C_ν by ψ. Then the embedding ψ and the usual embedding $C_\lambda \to \Omega(K)$ are isotopic.*

Using corollary 3.6.18 this theorem holds true if one replaces the affine Schubert variety by an MV cycle. We condens this into the follwing theorem:

Theorem 4.4.4. *Given a maximally folded loop γ in S such that Γ_γ contains a loop η in dominant direction, the map*

$$\begin{aligned} \pi_\nu : \Gamma_\gamma \to \Sigma(\delta(\gamma)) &\to \Omega(K) \\ [g] &\mapsto g.\nu \end{aligned}$$

is well-defined for any loop $\nu \in \Omega(S)$, which is stabilized by the stabilizer K'_k of the last segment of γ and has weight $\mathrm{wt}(\eta)^f$. The image $\mathrm{Im}(\pi_\nu)$ is homeomorphic to the affine Schubert variety at $\mathrm{wt}(\eta)$. Moreover, the embedding of the affine Schubert variety as $\mathrm{Im}(\pi_\nu)$ and the identity map are isotopic. If $\eta \in \Gamma_\gamma \cap \mathcal{A}\gamma$ and we denote by $\Gamma_{\gamma,\eta}$ the cell centered at η, then $\pi(\overline{\Gamma_{\gamma,\eta}})$ is homeomorphic to an MV cycle and the associated embedding is isotopic to the identity map. Every MV cycle is obtained from such a cell.

5 Further Questions

In section 3.5 we were able to deduce the moment map image of Γ_γ and the result was the Weyl polytope. This is the expected result in view of the diffeomorphism $\Gamma_\gamma \cong \Sigma(\Delta(\gamma))$ as the Weyl polytope is the MV polytope associated to the pair of weights (λ, λ). The natural question to ask is whether the closure of Białinycki-Birula cells are mapped to the other MV polytopes under the moment map.
We have seen that the symplectic structure of Γ_γ can be manipulated via homotopies of γ. Assume now that the restriction of ω to Γ_γ is degenerate. It still gives Γ_γ the structure of a presymplectic manifold. Under certain conditions it is possible to form the quotient of Γ_γ by the group of diffeomorphisms obtained as flows of vector fields in the kernel of ω. This should be computable by *straightening* the parts of γ which give a contribution to ω which is not regular in the sense of 3.5.2. We make this precise.

Conjecture 5.0.1. *Let γ be a loop such that Γ_γ is not-symplectic when equipped with the restriction of ω. A vector field X in the kernel of $\omega|_{\Gamma_\gamma}$ defines a global flow Φ_X of Γ_γ. Denote by γ_s a homotopy from γ to the loop where every segment of γ is replaced by the straight line joining the endpoints of the segment. From the existence of maximal folded loops it follows that the homotopy can be chosen to be fitted to the path model. Then the map $\Gamma_\gamma \to \Gamma_1$ is the quotient map induced by the action of all global flows Φ_X.*

Another interesting object for further study is the Birkhoff decomposition in relation to the root operators. While it was straightforward to compute the cocharacter appearing in the Birkhoff decomposition after application of a root operator, the other two factors p_- and p_+ remain unknown. As the cocharacter only determines the weight of an element in $\Omega(S)$, the action of the root operators is largely determined by p_- and p_+. To extend the root operators to (some subset of) $\Omega(K)$, a precise knowledge of the full decomposition might prove crucial.
Relating to section 4.4: There are two known bases for the homology of $\Omega(K)$. The original approach from Bott and Samelson proves that the homology $H(\Omega(K))$ has a basis consisting of the fundamental classes of Γ_λ for λ ranging over the dominant cocharacters, pushed forward to $\Omega(K)$ using the map

h_λ. Another classical theorem is that the inclusion $\mathcal{G} \to \Omega(K)$ is a homotopy equivalence. The homology of the affine Grassmannian is generated by the affine Schubert varieties as $\mathcal{G}$ is a cell complex with the affine Schubert varieties as closures of cells. To our knowledge there are no relations known between these two bases. Our maps π_γ and the map h_γ are a first link between the two and could prove useful.

Bibliography

[And03] J. E. Anderson. A polytope calculus for semisimple groups. *Duke Math. J. 116(3): 567 - 588*, 2003.

[AP83] M. F. Atiyah and Andrew Pressley. *Arithmetic and Geometry Volume II Geometry*. Birkhäuser, 1983.

[Ati82] M.F. Atiyah. Convexity and commuting hamiltonians. *Bull. London Math. Soc. 14*, 1982.

[BB73] A. Białynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math., 98, no. 2*, 1973.

[BGL20] Pierre Baumann, Stéphane Gaussent, and Peter Littelmann. Bases of tensor products and geometric satake correspondence. 2020.

[Bir09] G. D. Birkhoff. Singular points of ordinary differential equations. *Transactions of the American mathematical Society, 10, 436-470*, 1909.

[BS49] A. Borel and J. De Siebenthal. Les sous-groupes fermés de rang maximum des lie clos. *Commentarii Mathematici Helvetici, 23: 200-221*, 1949.

[BS58] Raoul Bott and Hans Samelson. Applications of the theory of morse to symmetric spaces. *American Journal of Mathematics, Vol 80, No.4*, 1958.

[Chh14] Reda Chhaibi. *Littelmann path model for geometric crystal*. PhD thesis, 2014.

[Dem74] M. Demazure. Désingularisation des variétés de schubert généralisées. *Annales Scientifiques de l'École Normale Supérieure, 7: 53–88*, 1974.

[Ehr09] M. Ehrig. Mv-polytopes via affine buildings. *Duke Mathematical Journal 155(3)*, 2009.

[FH91] William Fulton and Joe Harris. *Representation Theory - A First Course*. Springer, 1991.

[GL05] Stéphane Gaussent and Peter Littelmann. Ls galleries, the path model, and mv cycles. *Duke Mathematical Journal Vol. 127, No. 1*, 2005.

[GL12] Stéphane Gaussent and Peter Littelmann. One-skeleton galleries, the path model, and a generalization of macdonald's formula for hall-littlewood polynomials. *Int. Math. Res. Not. IMRN*, 2012.

[GS82] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping. *Invent. Math. 67*, 1982.

[Han73] H. C. Hansen. On cycles in flag manifolds. *Math. Scand. 33*, 1973.

[Hel62] Sigurdur Helgason. *Differential Geometry and Symmetric Spaces*. Academic Press, 1962.

[HHJM06] Megumi Harada, Tara S. Holm, Lisa C. Jeffrey, and Augustin-Liviu Mare. Connectivity properties of moment maps on based loop groups. *Geometry and Topology 10, 1607–1634*, 2006.

[Jos95] A. Joseph. *Quantum Groups and Their Primitive Ideals*. Springer-Verlag, Berlin, 1995.

[Kam07] J. Kamnitzer. The crystal structure on the set of mirković-vilonen polytopes. *Adv. Math., 215(1): 66-93*, 2007.

[Kam10] J. Kamnitzer. Mirković-vilonen cycles and polytopes. *Annals of Mathematics, Volume 171, pp. 245-294*, 2010.

[Kas90] Masaki Kashiwara. Crystalizing the q-analogue of universal enveloping algebras. *Communications in Mathematical Physics, 133 (2)*, 1990.

[Kas96] M. Kashiwara. Similarity of crystal bases, lie algebras and their representations. *Contemporary Mathematics, vol. 194*, 1996.

[Kir] A. A. Kirillov. *Lectures on the Orbit Method (Graduate Studies in Mathematics Vol. 64).*

[Kli78] Wilhelm Klingenberg. *Lectures on Closed Geodesics*. Springer-Verlag, 1978.

[Kna02] Anthony W. Knapp. *Lie Groups: Beyond an Introduction.* Birkhäuser, 2002.

[Kum02] Shrawan Kumar. *Kac–Moody Groups, their Flag Varieties and Representation Theory.* Birkhäuser, 2002.

[Lit95] Peter Littelmann. Paths and root operators in representation theory. *Annals of Mathematics, 142, 499-525*, 1995.

[Lit97] Peter Littelmann. Characters of representations and paths. *Representation Theory and Automorphic Forms, Proceedings of Symposia in Pure Mathematics*, 80(4):30–49, 1997.

[Mar10] Augustin-Liviu Mare. Connectedness of levels for moment maps on various classes of loop groups. *Osaka J. Math. 47, 609-626*, 2010.

[MV07] I. Mirković and K. Vilonen. Geometric langlands duality and representations of algebraic groups over commutative rings. *Annals of Mathematics 166*, 2007.

[Pre82] Andrew Pressley. The energy flow on the loop space of a compact lie group. *Journal of the London Mathematical Society 26, 557-566*, 1982.

[PS86] Andrew Pressley and Graeme Segal. *Loop Groups.* Oxford University Press, 1986.

[Sep07] Mark R. Sepanski. *Compact Lie Groups.* Springer-Verlag, 2007.

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