The Dressing method: Application to selected integrable models

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Kurzzusammenfassung

Das *AKNS System*, ein *integrables* System partieller Differentialgleichungen (PDEs), ist 1974 von Mark J. Ablowitz, David J. Kaup, Alan C. Newell und Harvey Segur eingeführt und nach diesen benannt worden. Folgt man dem Schemata, das für diese Systeme entwickelt worden ist, so lässt sich ein integrables Anfangswertproblem (AWP) auf der reellen Linie zu einer Kompatibilitätsbedingung, oder einer *Nullkrümmungsbedingung*, bezüglich zweier linearer gewöhnlicher Differentialgleichungen umschreiben. Wichtige Beispiele, die in diese Kategorie fallen, sind die nichtlineare Schrödinger-Gleichung (NLS) und die sinus-Gordon-Gleichung (sG). Die NLS-Gleichung ist bekannt für ihre Beschreibung von Lichtwellen und dem Bose-Einstein-Kondensat, wohingegen die sG-Gleichung bekannt ist für ihre Beschreibung von der Bewegung von Bloch-Wänden, der Versetzungsbewegung in Kristallen und dem magnetischen Fluss auf einer Josephson-Kreuzung. Durch ihre Verbindung zum AKNS System sind beide Gleichungen für die Anwendung der *inversen Streutransformation* geeignet und daher ist es möglich exakte Lösungen herzuleiten. Eine interessante Ansichtsweise, die dabei natürlicherweise aufkommt, ist das Betrachten von geringen Störungen in der jeweiligen PDE, die dazu führen können, dass das AWP nicht mehr integrabel ist. Eine bestimmte Klasse von internen Randbedingungen, die *Defektbedingungen*, ist untersucht worden und dabei ist festgestellt worden, dass in besonderen Fällen die Integrabilität erhalten werden kann. Des Weiteren hat sich die Kombination einer solchen Defektbedingung mit einer Randbedingung in speziellen Fällen als hilfreich in der Herleitung von integrablen Anfangsrandwertproblemen (ARWP) in den erwähnten PDEs auf der reellen Halbgeraden herausgestellt. Insbesondere sind mit diesem Ansatz die *neuen Randwertbedingungen* für die NLS-Gleichung konstruiert worden.

Folglich ist es von besonderem Interesse eine Methode zu entwicklen um exakte Lösungen in diesen integrablen Modellen zu finden. Eine Methode, die diese Aufgabe bezüglich der ARWP übernimmt, ist das *nichtlineare Analogon der Methode der Spiegelladung* aus der Elektrostatik. Dazu wird die *Bäcklund-Transformation* genutzt um die Lösung bezüglich der reellen Halbgeraden auf die Lösung bezüglich der reellen Linie so zu erweitern, dass die Randwertbedingung automatisch erfüllt ist. Ein anderer, als „*dressing the boundary*“ bekannter Ansatz ist entwickelt worden und basiert auf der Methode der *vereinheitlichten Transformation* und der *„Dressing“ Methode*, die für sich genommen neben inverser Streutransformation und Bäcklund-Transformation eine weitere Methode liefert um Lösungen für integrable AWP, die als AKNS System darstellbar sind, zu konstruieren. Bezüglich der Konstruktion exakter Lösungen für ARWP auf der reellen Halbgeraden ist diese Methode genauso effizient wie das nichtlineare Analogon der Methode der Spiegelladung. Des Weiteren birgt der Ansatz dressing the boundary den strukturellen Vorteil, den räumlichen Bereich nicht auf die reelle Linie erweitern zu müssen. Daher eignet sich dieser Ansatz für die Anwendung bezüglich integrabler Modelle auf zwei Halbgeraden, die einer Defektbedingung folgen.

In der vorliegenden Dissertation entwickeln wir zunächst ebendiesen Ansatz weiter um alle im ersten Absatz erwähnten integrablen Modelle miteinbeziehen zu können. Anschließend nutzen wir diese Resultate um in den Modellen *Solitone*, spezielle exakte Lösungen, zu konstruieren, die bei der Betrachtung von den jeweiligen physikalischen Phänomenen hilfreich sein könnten.

Abstract

The *AKNS system*, an *integrable* system of partial differential equations (PDEs), has been introduced in 1974 by and named after Mark J. Ablowitz, David J. Kaup, Alan C. Newell and Harvey Segur. Following the scheme developed for these systems, the integrable initial value problem on the line can be rewritten as a compatibility condition, or as a *zero curvature condition*, of two linear ordinary differential equations. Important examples falling into this category are the nonlinear Schrödinger (NLS) equation and the sine-Gordon (sG) equation. The NLS equation is known for its application to the propagation of light and Bose–Einstein condensates, whereas the sG equation is known for its application to Bloch-Wall motion, the propagation of a crystal dislocation and magnetic flux on a Josephson junction. Due to their description as AKNS systems, these two equations are suited for the application of the *inverse scattering method* implying that exact solutions can be derived. An interesting viewpoint, which naturally arises in that context, is the occurrence of small perturbations in the respective partial differential equation, which may or may not leave the initial value problem integrable. A particular class of internal boundary conditions, the *defect conditions*, have been investigated for which in some cases it can be verified that integrability is preserved. Further, the combination of such a defect condition with a boundary condition has in specific cases proven instructive in the derivation of integrable initial-boundary value problems regarding the mentioned PDEs on the half-line. Particularly, the *new boundary conditions* for the NLS equation on the half-line have been constructed through this approach.

Thus, the development of a method in order to obtain exact solutions in these integrable models is of particular interest. One approach with regards to initial-boundary value problems on the half-line is the *nonlinear analog of the method of images*, where the idea is to utilize the *Bäcklund transformation* to extend the half-line solution to a solution on the whole line while the boundary condition is automatically satisfied. A different approach, called *dressing the boundary*, has been developed based on the ideas of the *unified transform method* combined with the *Dressing method*, which is yet another method in addition to the inverse scattering method and the Bäcklund transformation commonly used to construct exact solutions for initial value problems associated with the AKNS system. When it comes to the construction of exact solutions for integrable initial-boundary value problems on the half-line this approach seems just as powerful as the nonlinear analog of the method of images. Further, due to the structural advantage that it is not necessary to extend the spatial domain to the whole line for the dressing the boundary method, this approach may therefore also be applied to integrable models on two half-lines connected via the defect condition.

The present thesis provides the application of the method of dressing the boundary to the integrable models mentioned in the first paragraph. By this application, it is then possible to construct *solitons*, special exact solutions, which may prove useful in the corresponding physical models.

Contents

Chapter 1

Introduction

1.1 Nonlinear Schrödinger equation with a delta-potential

The nonlinear Schrödinger equation (NLS) with an external potential $W(x)$, also known as the Gross–Pitaevskii equation, given by

$$iu_t(t,x) + W(x)u(t,x) + \Delta u(t,x) + 2\epsilon|u(t,x)|^2 u(t,x) = 0, \qquad \epsilon = \pm 1 \tag{1.1.1}$$

is a model characterizing a great number of phenomena in physics. It provides, for instance, a description of the evolution of Bose–Einstein condensates in dilute boson gases at very low temperatures, which has first been realized experimentally in 1995 and whereupon its significance and importance ultimately has come to light through the 2001 Nobel prize in Physics, see [17, 37] and the references therein. Moreover, the Gross–Pitaevskii equation appears to be one of the most commonly utilized models in the theories of superfluidity and superconductivity, where it provides a fairly solid basis for the understanding on a microscopic level of some of the fundamental properties of the superfluid and superconducting states [31]. In theory, exact solutions of the corresponding initial value problem are sought, simply because of the practical applications to real life physical systems. However, apart from the spatially homogeneous case, where the external potential is assumed to be constant $W(x) \equiv c$, it turns out to be difficult to find such solutions, even in one dimension.

Nonetheless, if the one-dimensional case is reduced even further to the delta-potential at $x = 0$, $W(x) = \alpha\delta_0(x)$ with $\alpha \in \mathbb{R}$, and combined with an initial condition $u_0(x) = u(0,x)$ being an even function, the resulting model of (1.1.1) becomes *integrable*. This is due to the fact that the delta-potential is to be understood as introducing a jump in the first x-derivative at $x = 0$,

$$u_x(t,0+) - u_x(t,0-) + 2\alpha u(t,0) = 0, \qquad t > 0, \tag{1.1.2}$$

which if $u(t,x)$ is even with respect to space, reduces to the (homogenous) Robin boundary condition $u_x(t,0+) + \alpha u(t,0) = 0$. Therefore, the initial value problem with an even initial condition can be reduced to the initial-boundary value problem for the NLS equation on a half-line and this problem with the Robin boundary condition has already been proven to be integrable in 1987, see [39]. In [17], the authors use the integrability of this very initial-boundary value problem together with a method for Riemann–Hilbert problems, developed by Deift and Zhou, which emerge in the context of the so-called *inverse scattering method*, in order to state some remarkable results on the long time asymptotic behavior of solutions. If, however, the assumption of an even initial condition $u_0(x)$ is dropped, then the approach they have developed can not be applied. In that regard, advances have been made in order to analyze initial-boundary value problems on the half-line more naturally. A method developed by Fokas, known as *unified transform method* has in this context been successfully applied to the linear Schrödinger equation with a point singular potential in the case of a general initial condition $u_0(x)$ resulting not only in an expression of the solution in terms of Fourier transforms of the initial condition, but also in their long time asymptotic behavior, see [37]. To expand on this idea, let us first give an overview of the mentioned methods.

1.2 Inverse scattering method *vs.* unified transform method

Both methods rely on the fact that the equation of interest can be written as a so-called *Lax system*, a system of linear ordinary differential equations involving an additional spectral parameter λ. In particular, establishing the compatibility condition of the derived Lax system is then equivalent to the initial equation. In the inverse scattering method used for solving initial value problems of such equations, one then proceeds to change the relevant variable with the help of the spatial equation of the Lax system, passing from functions of the spatial variable to functions of the spectral parameter. Thus, one particular function of the spectral parameter $\rho(\lambda; 0)$, together with *simple eigenvalues* $\lambda_j(0)$ of another function and correlated *norming constants* $C_j(0)$, $j = 1, \dots, N$, constitute the so-called *scattering data* for the initial condition $u_0(x)$, while the process is known as the direct scattering. Further, the time equation of the Lax system then induces a linear time evolution for the scattering data. Afterwards, the solution $u(t, x)$ needs to be recovered from the evolved scattering data, for instance, through a Riemann–Hilbert problem, which therefore realizes the inverse scattering [2]. The well known visualization of this procedure is given by:

$$\begin{tikzcd}
u_0(x) \arrow[r, "\text{direct scattering}"'] \arrow[d, dashed, "\text{NLS}"] & \mathcal{S}(u_0) = (\rho(\lambda;0), \{\lambda_j(0), C_j(0)\}_{j=1}^N) \arrow[d, "\text{linear time evolution}"] \\
u(t,x) & \mathcal{S}(u) = (\rho(\lambda;t), \{\lambda_j(0), C_j(t)\}_{j=1}^N) \arrow[l, "\text{inverse scattering}"]
\end{tikzcd}$$

We perform a more precise implementation of this method in Chapter 2, where not only the scattering map $\mathcal{S}$, but also the resulting scattering data is described in great detail.

The unified transform method, similar to the method for initial value problems, is based on the representation of the equation as a compatibility condition for a Lax system. However, the structural innovation of the unified transform method lies in the simultaneous use of the spatial and time equation of this system in the direct scattering process, which are directly connected to the given initial and boundary condition, respectively. Afterwards, the resulting scattering data is again put into a Riemann–Hilbert problem in order to recover the solution $u(t, x)$ on the half-line. Even though Fokas' approach seems to be an appropriate generalization of the inverse scattering method to initial-boundary value problems, where both an initial and a boundary condition are given, in practice, it is difficult to obtain the solutions of the corresponding Riemann–Hilbert problem and therefore to give explicit solutions of the model in question. In that context, a particular class of boundary conditions has been identified, the so-called *linearizable* boundary condition, for which it is possible to bypass the additional intricacies and therefore to solve the Riemann–Hilbert problem as effectively as the problem on the line, see [23] for the relevant treatment in the case of the NLS equation. Moreover, an effective description of the long time asymptotic behavior of $u(t,x)$ can be provided as explained in [22].

1.3 Initial-boundary value problem

With these two methods in mind, different approaches have been developed in order to tackle the derivation of explicit solutions for initial-boundary value problems of not only the NLS equation, but also other partial differential equations (PDEs) for which it is possible to state an appropriate Lax system.

1.3.1 Nonlinear method of images

One such approach is the nonlinear analog of the method of images or for short nonlinear method of images: An extension $u_e(t,x)$ of the half-line solution $u(t,x)$ is sought such that it solves the NLS equation on the whole line and the boundary condition is automatically satisfied. Subsequently, the inverse scattering method may be used to solve the initial value problem for the extended solution $u_e(t,x)$, which at the same time serves as a solution of the initial-boundary value problem, see [5, 17]. An essential ingredient in this approach is the notion of a *Bäcklund transformation* which serves as the means to extend the solution under these particular conditions.

Originally, the transformation introduced by Bäcklund in 1882 is meant to be used to iteratively construct pseudospherical surfaces, that is, surfaces with constant negative Gaussian curvature. The application to PDEs has been established much later, even after the impressive breakthrough, by which Bianchi demonstrates that the Bäcklund transformation admits a commutativity property. Thus, in 1974, Lamb constructs an (auto-)Bäcklund transformation for the NLS equation, a mapping B of a solution u of the NLS equation to a solution $\tilde{u} = B(u)$ of the NLS equation. In general, a Bäcklund transformation may map a solution of an equation to a solution of a different equation. Since then, the Bäcklund transformation has been utilized in a more elegant version to Lax systems, associated to an AKNS system, which have been developed at fairly the same time, see [1]. Furthermore, the commutativity property has once more been established in connection with the application to PDEs resulting in the concept of a nonlinear superposition principle for PDEs of that type.

The investigation of the same problem through the unified transform method may seem more natural in the sense that the initial domain prescribed by the problem is retained, rather than extended. In that context, with respect to specific linearizable boundary conditions it has been shown that this approach ultimately results in a Riemann–Hilbert problem coinciding with the one derived from the nonlinear method of images, see [28]. What is more, this framework, while further insisting on the restriction to integrable boundary conditions, has then been successfully combined with a purely algebraic algorithm to construct explicit solutions, the *Dressing method*.

1.3.2 Integrability

On one hand, integrability for initial value problems, which can be expressed in the form of an AKNS system, is well-established. By analyzing relations of the spectral functions, see [1], it is possible to give an infinite set of conserved quantities. Thus, for equations solvable by the inverse scattering method, the method may be interpreted as a canonical transformation from physical variables to an infinite set of action-angle variables. On the other hand, integrability for initial-boundary value problems is not as imminent. Initialized by Sklyanin [39], one formalism to derive integrable boundary conditions on an interval is based on the Hamiltonian formulation of integrable PDEs, see [21], where the half-line can be realized as a special case of the interval setting one end to zero and the other to infinity. Following the ideas therein, given the classical r matrix and boundary matrices associated to either end of the interval, the so-called *classical reflection equation* is used to give rise to a generating function of commuting integrals of motion, which leads to the generation of infinitely many conserved quantities implying integrability. Note that the boundary matrices are usually associated with the formulation of the respective PDE as a Lax system.

1.3.3 Dressing the boundary

The Dressing method is based on the *Darboux transformation*, which has been introduced in the late nineteenth century by Darboux to study Sturm–Liouville problems. Then, after it has been shown by Crum that it could be applied iteratively to Sturm–Liouville problems, it has been successfully applied to integrable equations with Lax systems to generate so-called *multi-soliton* solutions. Based on the fundamental solution of the Lax system for a given solution $u[0](t,x)$ of the PDE and given a spectral parameter $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$, which is distinct from the set of simple eigenvalues of the scattering data, and a solution $\psi_1(t,x)$ of the Lax system at this value, it is possible to algebraically construct a new solution $u[1](t,x)$ of the PDE by the application of a one-fold dressing matrix while simultaneously updating the Lax system and fundamental solution which are then associated to the new solution. As indicated before, this process can be iterated which essentially boils down to incrementing by 1 the number indexing the previous solution and effectively leads to the realization of an N-fold dressing matrix $D[N](t,x,\lambda)$, where N is finite. Subsequently, various researchers have been motivated to make attempts in analyzing the connection of Bäcklund and Darboux transformations with respect to soliton theory, see for example [14, 32, 33, 36].

Hence, it is not surprising that in the pursuit of explicit solutions of integrable initial-boundary value problems, the Dressing method turns out to be a promising alternative approach along with the nonlinear method of images incorporating the Bäcklund transformation. Having said that, the first well documented implementation of this idea goes back to [42], where it is called *dressing the boundary*. It has successfully been applied to the NLS equation on the half-line with the Robin boundary condition and therefore, it constitutes a method at least equally as effective as the nonlinear method of images [5]. Prior to that, dressing the boundary has been applied in an abbreviated manner to the sine-Gordon (sG) equation on the half-line [43] with the sin-boundary condition. The sG equation is a model, which describes numerous physical phenomena including for example Bloch-wall motion, the propagation of a crystal dislocation and magnetic flux on a Josephson junction and fits into the framework of AKNS systems. Therefore, the application of dressing the boundary in both the NLS and sG equation on the half-line are fundamentally related, which is further supported by the fact that the associated boundary matrices representing the boundary condition are structurally similar, that is, diagonal and time independent 2×2-matrices. In this thesis, we pursue the goal of generalizing the dressing the boundary method to incorporate a broader spectrum of integrable problems while retaining the models already covered.

1.3.4 The NLS equation with a new integrable boundary

In [41], particular boundary conditions for the NLS equation on the half-line, see [30], have been revisited pursuing a different approach. Namely, they are derived by the combination of the so-called *defect conditions* with a (Dirichlet) boundary condition. In the course of this approach, a time dependent boundary matrix including off-diagonal entries is obtained, which then ultimately corresponds to the *new boundary condition*

$$u_x(t,0) = \frac{iu_t(t,0)}{2\Omega(t,0)} - \frac{u(t,0)\Omega(t,0)}{2} + \frac{u(t,0)|u(t,0)|^2}{2\Omega(t,0)} - \frac{u(t,0)\alpha^2}{2\Omega(t,0)}$$

for the NLS equation on the half-line, that is, (1.1.1) with $W(x) \equiv 0$ and $\varepsilon = 1$. At the same time, it has been shown with the classical r matrix method that the corresponding initial-boundary value problem is integrable in the sense of the existence of infinitely many conserved quantities.

1.4 PDEs with defect conditions

A different approach to the idea of generalizing the analysis of initial-boundary value problems is to lift the initial-boundary value problem on one half-line to one on a finite number of half-infinite edges, which corresponds with respect to the unified transform method to lifting the relevant spectral functions and consequently the scattering data to diagonal-matrix valued spectral functions and scattering data enabling the analysis of integrable PDEs on a star-graph [12]. In that context, the one-dimensional NLS equation with the delta-potential at $x = 0$ therefore has an equivalent expression in this framework. By dividing the whole line at $x = 0$ into two half-lines and denoting the potential to the left and right of the partition by $\tilde{u}$ and u, respectively, the jump (1.1.2) takes the form $u_x(t,0) + \tilde{u}_x(t,0) + 2\alpha u(t,0) = 0$ for $t > 0$. In combination with the potential being continuous across $x = 0$, i.e. $\tilde{u}(t,0) = u(t,0)$ for $t > 0$, and an even initial condition $\tilde{u}_0(x) = u_0(x)$, it is obvious that the situation is the same as the above described integrable model of the NLS on the half-line with the Robin boundary condition. Alternatively, with these notations the conditions on the boundary imply that a specific symmetry relation can be applied to the diagonal-matrix valued spectral functions, which, in turn, simplifies the Riemann–Hilbert problem originating from the unified transform method to the one found in [17] by the nonlinear method of images.

This idea sheds light on the principle of having an internal boundary which links a potential on $x < 0$ with a potential on $x > 0$, commonly known as a defect condition. Again, it is of interest to find defect conditions corresponding to an integrable model, which has been pursued in [7]. In that context, an approach based on a Lagrangian formalism has been utilized and moreover, a connection between defect conditions and Bäcklund transformations *frozen* at the location of the defect has been indicated. Based on this observation, general results on defect conditions for integrable PDEs with corresponding AKNS systems have been established in [11], which particularly encompasses the modification of the generating functionals for the conserved quantities implying integrability. Hence, it can be expected that in this context the theory of solitons can be applied to a certain extent. The foundation for that idea has been put forward in [15], where the one- and two-soliton solutions for the NLS equation on two half-lines connected by the defect condition have been calculated by a direct ansatz.

1.5 Scope of this thesis

As indicated by the argumentation above, the subject matter of integrability has been widely covered in the literature not only with the mentioned methods but also with other developed methods, see for example [40]. The derivation of exact solutions seems to be of interest, considering that the validation by a direct ansatz of a two-soliton solution satisfying the defect condition is by no means easy. Hence, the goal of this thesis is to generalize the dressing the boundary method introduced in [42, 43] by the following means: Firstly, to enable the analysis of more than just diagonal and time independent boundary matrices. Secondly, to lift the dressing the boundary method to more than just one half-line, particularly, including (integrable) defect conditions, thereby making it possible to introduce multi-soliton solutions in the presented integrable models of initial-boundary value problems and PDEs with defect conditions.

In the case of the NLS equation, the set of all solutions $u(t,\cdot) \in H^{1,1}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \colon xf, f_x \in L^2(\mathbb{R})\}$, for which the scattering map admits the following data $\mathcal{S}(u) = (\rho(\lambda;t), \{\lambda_j(0), C_j(t)\}_{j=1}^N)$ with distinct $\lambda_j \in \mathbb{C} \setminus \mathbb{R}$, is denoted by $\mathcal{G}_N$. On one hand, if the spectral function $\rho(\lambda;t) \equiv 0$, then the corresponding solution is a pure N-soliton solution. On the other hand, if $u(t,\cdot) \in \mathcal{G}_0$, then the solution is soliton free. Hence, with regard to the Dressing method, the connection to the construction of solitons can be made, see Section 3.3. Without giving the complete explanation on soliton solutions in advance, let us give the main results worked out in this thesis or rather in [25, 26].

1.5.1 NLS and sG with defect conditions

Given so-called *seed solutions* $\tilde{u}[0]$ and $u[0]$ subject to the NLS equation on either side of the defect location and the defect condition, it is possible to construct the matrix $\mathcal{B}_0(t,0,\lambda)$ representing the frozen Bäcklund transformation and thus the defect condition. Further, assuming $\tilde{u}[0](\cdot,0), u[0](\cdot,0), \tilde{u}_x[0](\cdot,0), u_x[0](\cdot,0) \in H_t^{1,1}(\mathbb{R})$, where $H_t^{1,1}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \colon tf, f_t \in L^2(\mathbb{R})\}$, and taking a specific parameter λ_0 constructed from constant known parameters of the defect condition, the following statement holds.

Proposition A. *Applying N-fold dressing matrices $\widetilde{D}[N](t,x,\lambda)$ and $D[N](t,x,\lambda)$ to the seed solutions on either side of the defect location constructed by distinct $\lambda = \lambda_j \in \mathbb{C} \setminus \big(\mathbb{R} \cup \{\lambda_0, \lambda_0^*\}\big)$ and associated solutions of the Lax systems given by $\widetilde{\psi}_j(t,x)$ and $\psi_j(t,x)$, which need to satisfy*

$$\widetilde{\psi}_j\big|_{x=0} = \mathcal{B}_0(t,0,\lambda_j)\psi_j\big|_{x=0}, \quad j = 1,\dots,N,$$

leads to solutions $\tilde{u}[N]$ and $u[N]$ of the NLS equation on either side of the defect location preserving the defect condition if for the matrix $\mathcal{B}_N(t,0,\lambda) = \widetilde{D}[N](t,0,\lambda)\mathcal{B}_0(t,0,\lambda)(D[N])^{-1}(t,0,\lambda)$ the following holds:

$$\operatorname{Im}\Big(\lim_{\lambda\to 0}[2\lambda(\mathcal{B}_N(t,0,\lambda) - \mathbb{1})]_{11}\Big)$$

is greater than or equal to or rather less than or equal to 0 for all $t \in \mathbb{R}$ depending on its limit as $|t| \to \infty$.

Similarly, for zero seed solutions, $\tilde{\theta}[0] \equiv 0$ and $\theta[0] \equiv 0$, subject to the sG equation, which we specify in Section 2.2, on either side of the defect location and the defect condition, it is possible to construct the matrix $\mathbb{B}_0(\lambda)$ representing the frozen Bäcklund transformation and thus the defect condition. Further, taking a specific parameter λ_0 constructed from a constant known parameter of the defect condition, the following statement holds.

Proposition B. *Applying N-fold dressing matrices $\widetilde{D}[N](t,x,\lambda)$ and $D[N](t,x,\lambda)$ to the seed solutions on either side of the defect location constructed by distinct $\lambda = \lambda_j \in \mathbb{C} \setminus \big(\mathbb{R} \cup \{\lambda_0, \lambda_0^*\}\big)$ and associated solutions of the Lax systems given by $\widetilde{\psi}_j(t,x)$ and $\psi_j(t,x)$, which need to satisfy*

$$\widetilde{\psi}_j\big|_{x=0} = \mathbb{B}_0(\lambda_j)\psi_j\big|_{x=0}, \quad j = 1,\dots,N,$$

leads to solutions $\tilde{\theta}[N]$ and $\theta[N]$ of the sG equation on either side of the defect location preserving the defect condition.

1.5.2 NLS and sG with boundary conditions

Given the seed solution $u[0]$ subject to the NLS equation on the half-line and the Robin boundary condition, it is possible to construct the boundary matrix $\mathcal{K}_0(\lambda)$ representing the spectral version of the boundary condition. Further, taking a specific parameter λ_0 constructed from a constant known parameter of the boundary condition and dividing the number of solitons, which are envisaged to be constructed, into solitons N_s and *boundary-bound* solitons N_{bbs}, the following statement holds.

Proposition C. *Applying a $(2N_s + N_{bbs})$-fold dressing matrix $D[N_d](t,x,\lambda)$ to the seed solution constructed by distinct $\lambda = \lambda_j \in \mathbb{C} \setminus \big(\mathbb{R} \cup \{\lambda_0, \lambda_0^*\}\big)$, $j = 1, \dots, N_s + N_{bbs}$, as well as additionally distinct $\lambda = -\lambda_j$ (only if $\mathrm{Im}(\lambda_j) \neq 0$ corresponding to N_s), and associated solutions of the Lax system given by $\psi_j(t,x)$ as well as $\widehat{\psi}_j(t,x)$, which need to satisfy*

$$\begin{aligned} \widehat{\psi}_j\big|_{x=0} &= \mathcal{K}_0((-1)^{N_{bbs}}\lambda_j)\psi_j\big|_{x=0}, && \textit{only if } \mathrm{Im}(\lambda_j) \neq 0, \\ \varphi_j\big|_{x=0} &= \mathcal{K}_0((-1)^{N_{bbs}}\lambda_j)\psi_j\big|_{x=0}, && \textit{only if } \mathrm{Im}(\lambda_j) = 0, \end{aligned}$$

where $\varphi_j(t,x) = -i\sigma_2\psi_j^*(t,x)$ is the solution of the same Lax system at $\lambda = \lambda_j^*$, leads to a solution $u[N_d]$ of the NLS equation on the half-line preserving the Robin boundary condition.

Assuming, alternatively, that the seed solution and its x-derivative $u[0](\cdot,0), u_x[0](\cdot,0) \in H_t^{1,1}(\mathbb{R})$ and the seed solution is subject to the NLS equation on the half-line and the new boundary condition, it is possible to construct the boundary matrix $\mathcal{K}_0(t,0,\lambda)$, again, representing the spectral version of the boundary condition. Further, taking a specific parameter λ_0 constructed from constant known parameters of the boundary condition, the following statement holds.

Proposition D. *Applying a $2N$-fold dressing matrix $D[2N](t,x,\lambda)$ to the seed solution constructed by distinct $\lambda = \lambda_j \in \mathbb{C} \setminus \big(\mathbb{R} \cup i\mathbb{R} \cup \{\lambda_0, \lambda_0^*, -\lambda_0, -\lambda_0^*\}\big)$, $j = 1, \dots, N$, as well as additionally distinct $\lambda = -\lambda_j$ and associated solutions of the Lax system given by $\psi_j(t,x)$ as well as $\widehat{\psi}_j(t,x)$, which need to satisfy*

$$\widehat{\psi}_j\big|_{x=0} = \mathcal{K}_0(t,0,\lambda_j)\psi_j\big|_{x=0},$$

leads to a solution $u[2N]$ of the NLS equation on the half-line preserving the new boundary condition if for the matrix $\mathcal{K}(t,0,\lambda) = D[2N](t,0,\lambda)\mathcal{K}_0(t,0,\lambda)(D[2N](t,0,\lambda))^{-1}$ the following holds:

$$\mathrm{Im}\bigg(\lim_{\lambda\to 0}\frac{1}{\lambda}\big[\big((\lambda - i|\mathrm{Im}(\lambda_0)|)^2 - (\mathrm{Re}(\lambda_0))^2\big)\mathcal{K}_N(t,0,\lambda) + \big((\mathrm{Re}(\lambda_0))^2 + (\mathrm{Im}(\lambda_0))^2\big)\mathbb{1}\big]_{11}\bigg)$$

is greater than or equal to or rather less than or equal to 0 for all $t \in \mathbb{R}_+$ depending on its limit as $t \to \infty$.

Finally, given a zero seed solution $\theta[0] \equiv 0$ subject to the sG equation on the half-line and the sin-boundary condition, it is possible to construct the boundary matrix $\mathbb{K}(\lambda)$. Further, taking specific parameters $\lambda_0^{\pm}$ constructed from a constant known parameter of the boundary condition and dividing the number of solitons, which are envisaged to be constructed, into single solitons N_s, *breathers* N_b and boundary-bound breathers N_{bbb}, the following statement holds.

Proposition E. *Applying an $(2N_s + 4N_b + 2N_{bbb})$-fold dressing matrix $D[N_d](t,x,\lambda)$ to the seed solution constructed by distinct $\lambda = \lambda_j \in \mathbb{C} \setminus \big(\mathbb{R} \cup \{-i, i, \lambda_0^{\pm}, (\lambda_0^{\pm})^*\}\big)$, $j = 1, \dots, N_s + N_b + N_{bbb}$ as well as additionally distinct $\lambda = \lambda_j^{-1}$ (only if $|\lambda_j| \neq 1$ corresponding to N_s and N_b) and $\lambda = -\lambda_j^*$*

(only if $\mathrm{Im}(\lambda_j) \neq 0$ *corresponding to* N_b *and* N_{bbb}*) and associated solutions of the Lax system given by* $\psi_j(t,x)$ *as well as* $\widehat{\psi}_j(t,x)$ *and* $\Phi_j(t,x)$*, which need to satisfy*

$$
\begin{aligned}
\widehat{\psi}_j\big|_{x=0} &= \mathbb{K}_0(\lambda_j^{(-1)^{N_{bbb}}})\psi_j\big|_{x=0}, && \text{only if } |\lambda_j| \neq 1,\\
\widehat{\Phi}_j\big|_{x=0} &= \mathbb{K}_0((-\lambda_j^*)^{(-1)^{N_{bbb}}})\Phi_j\big|_{x=0}, && \text{only if } |\lambda_j| \neq 1 \text{ and } \mathrm{Im}(\lambda_j) \neq 0,\\
\Phi_j\big|_{x=0} &= \mathbb{K}_0((-\lambda_j)^{(-1)^{N_{bbb}}})\varphi_j\big|_{x=0}, && \text{only if } |\lambda_j| = 1 \text{ and } \mathrm{Im}(\lambda_j) \neq 0,
\end{aligned}
$$

where $\varphi_j(t,x) = \sigma_1\psi_j(t,x)$ *is the solution of the same Lax system at* $\lambda = -\lambda_j$*, leads to a solution* $\theta[N_d]$ *of the sG equation on the half-line preserving the* sin*-boundary condition.*

The notion of dividing the simple eigenvalues λ_j by their spectral properties such as $\mathrm{Im}(\lambda_j) \neq 0$, $j = 1, \ldots, N$, is closely related to their role in the scattering data and is here only given superficially. For a more detailed depiction of these necessities, the inverse scattering method proves to be instructive.

The thesis is structured as follows. We present an analysis of the inverse scattering method regarding the NLS and sG equation in Chapter 2, which is sufficient for our purposes. This enables us to compare the inverse scattering method to other solution construction methods, that is, the Dressing method and the Bäcklund transformation in Chapter 3. Then, in Chapter 4, we introduce the defect conditions for both PDEs, which are related to the Bäcklund transformation, and the boundary matrices associated with the relevant integrable boundary conditions. Further, we give some insight into preliminary considerations in order for the dressing the boundary method to be smoothly applicable. Subsequently, in Chapter 5, we state the propositions, presented in an abbreviated form in the Introduction, with more insight and prove them explicitly. Chapter 6 contains the application of the aforementioned propositions starting from a detailed consideration of multi-soliton solutions.

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Janin

Chapter 2

Inverse scattering method

2.1 Inverse scattering method for the NLS equation

We begin with a brief summary of the inverse scattering method of the focusing NLS equation. As in [6, 24], it serves as a guideline for the implementation of additional results on top of the construction of solutions. Let us state the NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0 \tag{2.1.1}$$

for $u(t,x)\colon \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$. Formulated as a Cauchy problem, we look for solutions of (2.1.1) with $u(0,x) = u_0(x)$ and given initial condition $u_0(x)$ for $x \in \mathbb{R}$. Mainly, we follow the analysis provided in [2, Sec. 2.2].

2.1.1 Lax pair

As suggested in the Introduction, an important concept in the context of applying the inverse scattering method is the existence of a so-called *Lax pair*, a pair of 2×2-matrices $\mathcal{U}(t,x,\lambda)$ and $\mathcal{V}(t,x,\lambda)$, which enables us to restate the NLS equation as a compatibility condition of the following linear spectral problems

$$\begin{aligned} \psi_x(t,x,\lambda) &= \mathcal{U}(t,x,\lambda)\psi(t,x,\lambda), \\ \psi_t(t,x,\lambda) &= \mathcal{V}(t,x,\lambda)\psi(t,x,\lambda), \end{aligned} \tag{2.1.2}$$

where the function $\psi(t,x,\lambda)$ is used as an auxiliary 2×2-matrix and the newly introduced parameter $\lambda \in \mathbb{C}$ is the so-called *spectral parameter*, which itself is independent of t and x. We call (2.1.2) the *Lax system* corresponding to the potential $u(t,x)$, whereas in the literature it is more generally referred to as 2×2 AKNS system [1]. For a solution $\psi(t,x,\lambda)$ of the Lax system (2.1.2) with an appropriate Lax pair, it can be shown that the compatibility condition $\psi_{tx}(t,x,\lambda) = \psi_{xt}(t,x,\lambda)$ for all $\lambda \in \mathbb{C}$ holds if and only if $u(t,x)$ satisfies the NLS equation (2.1.1). Since the choice of the Lax pair is by no means unique, it is important to carefully select the right matrices in order for the inverse scattering method to be applicable. Here, the Lax pair takes the form

$$\mathcal{U}(t,x,\lambda) = -i\lambda\sigma_3 + \mathcal{Q}, \quad \mathcal{V}(t,x,\lambda) = -2i\lambda^2\sigma_3 + \mathcal{Q}_1, \tag{2.1.3}$$

where the potentials $\mathcal{Q}$ and $\mathcal{Q}_1$ of $\mathcal{U}$ and $\mathcal{V}$ and the third Pauli matrix σ_3 are defined by

$$\mathcal{Q}(t,x) = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \quad \mathcal{Q}_1(t,x,\lambda) = \begin{pmatrix} i|u|^2 & 2\lambda u + iu_x \\ -2\lambda u^* + iu_x^* & -i|u|^2 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Throughout this thesis, we write $u = u(t,x)$, $\mathcal{Q} = \mathcal{Q}(t,x)$ and $\mathcal{Q}_1 = \mathcal{Q}_1(t,x,\lambda)$ as well as $\mathcal{U} = \mathcal{U}(t,x,\lambda)$ and $\mathcal{V} = \mathcal{V}(t,x,\lambda)$ to simplify notation, unless specified otherwise. Moreover, we shall refer to $\mathcal{U}$ and $\mathcal{V}$ as the x and t part of the Lax pair, respectively. The connection between the Lax pair $\mathcal{U}$ and $\mathcal{V}$ and the NLS equation can also be made without the auxiliary function $\psi(t,x,\lambda)$ in terms of the *zero curvature condition*, that is:

$$\mathcal{U}_t - \mathcal{V}_x + [\mathcal{U},\mathcal{V}] = 0 \text{ for all } \lambda \in \mathbb{C},$$

which holds if and only if u satisfies the NLS equation (2.1.1). In the literature, the existence of a Lax pair for an equation means that the equation is *integrable* [21]. The Lax pair is not unique, however, one essential property, among others, of our particular choice for the Lax pair is that the matrices $\mathcal{U}$ and $\mathcal{V}$ admit the following symmetry relation

$$\mathcal{U}(t,x,\lambda) = \sigma_2\big(\mathcal{U}(t,x,\lambda^*)\big)^*\sigma_2, \quad \mathcal{V}(t,x,\lambda) = \sigma_2\big(\mathcal{V}(t,x,\lambda^*)\big)^*\sigma_2, \tag{2.1.4}$$

where the second Pauli matrix σ_2 is given by

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

From here, the goal is to describe the spectrum and the eigenfunctions of the Lax pair.

2.1.2 Jost functions and direct scattering

In order to study the Lax system in more detail, we assume that the potential $u(t,x) \to 0$ with respect to x as well as its derivative $u_x(t,x) \to 0$ decay sufficiently fast as $|x| \to \infty$. Therefore, it is natural to assume for $\lambda \in \mathbb{R}$ that there exist 2×2-matrix-valued solutions ψ_- and ψ_+, also known as *Jost functions*, of the Lax system with the asymptotic behavior

$$\psi_\pm(t,x,\lambda) \sim e^{-i(\lambda x + 2\lambda^2 t)\sigma_3}, \quad \text{as } x \to \pm\infty$$

derived in accordance with the limits of the potential $\mathcal{Q}$ and $\mathcal{Q}_1$, where the *phase* is $\Theta(t,x,\lambda) = \lambda x + 2\lambda^2 t$ in the case of the NLS equation. For a function $f(t,x,\lambda)$, the term $e^{if(t,x,\lambda)\sigma_3}$ is defined by

$$e^{if(t,x,\lambda)\sigma_3} := \begin{pmatrix} e^{if(t,x,\lambda)} & 0 \\ 0 & e^{-if(t,x,\lambda)} \end{pmatrix}.$$

Further, we define the *modified Jost functions* under time evolution as $\widehat{\psi}(t,x,\lambda) = \psi(t,x,\lambda) \cdot e^{i\Theta(t,x,\lambda)\sigma_3}$, which then serve as solutions of the *modified Lax system* given by

$$\widehat{\psi}_x + i\lambda[\sigma_3,\widehat{\psi}] = \mathcal{Q}\widehat{\psi}, \qquad \widehat{\psi}_t + 2i\lambda^2[\sigma_3,\widehat{\psi}] = \mathcal{Q}_1\widehat{\psi}.$$

Then, the modified Jost functions admit constant limits as $x \to \pm\infty$ and for all $\lambda \in \mathbb{R}$, i.e.

$$\widehat{\psi}_\pm(t,x,\lambda) \to \mathbb{1}, \qquad \text{as } x \to \pm\infty,$$

where $\mathbb{1} = \operatorname{diag}(1,1)$ is the identity matrix and thus they are solutions of the following Volterra integral equations:

$$\begin{aligned} \widehat{\psi}_-(t,x,\lambda) &= \mathbb{1} + \int_{-\infty}^{x} e^{-i\Theta(0,x-y,\lambda)\sigma_3}\mathcal{Q}(t,y)\widehat{\psi}_-(t,y,\lambda)e^{i\Theta(0,x-y,\lambda)\sigma_3}\,\mathrm{d}y, \\ \widehat{\psi}_+(t,x,\lambda) &= \mathbb{1} - \int_{x}^{\infty} e^{-i\Theta(0,x-y,\lambda)\sigma_3}\mathcal{Q}(t,y)\widehat{\psi}_+(t,y,\lambda)e^{i\Theta(0,x-y,\lambda)\sigma_3}\,\mathrm{d}y. \end{aligned} \tag{2.1.5}$$

Writing the modified Jost functions $\widehat{\psi}_\pm = (\widehat{\psi}_\pm^{(1)}, \widehat{\psi}_\pm^{(2)})$ in terms of column vectors $\widehat{\psi}_\pm^{(1)}$ and $\widehat{\psi}_\pm^{(2)}$, the following theorem provides information on the possible continuations for the column vectors in terms of the spectral parameter.

Theorem 2.1.1 (Deift & Zhou, [18]). *Let $u(t,\cdot) \in H^{1,1}(\mathbb{R}) = \{f \in L^2(\mathbb{R})\colon xf, f_x \in L^2(\mathbb{R})\}$. Then, for every $\lambda \in \mathbb{R}$, there exist unique solutions $\widehat{\psi}_\pm(t,\cdot,\lambda) \in L^\infty(\mathbb{R})$ satisfying the integral equations* (2.1.5)*. Therefore, the two column vectors $\widehat{\psi}_-^{(2)}(t,x,\lambda)$ and $\widehat{\psi}_+^{(1)}(t,x,\lambda)$ of the modified Jost functions can be continued analytically in $\lambda \in \mathbb{C}_-$ and continuously in $\lambda \in \mathbb{C}_- \cup \mathbb{R}$, while the two column vectors $\widehat{\psi}_-^{(1)}(t,x,\lambda)$ and $\widehat{\psi}_+^{(2)}(t,x,\lambda)$ of the modified Jost functions can be continued analytically in $\lambda \in \mathbb{C}_+$ and continuously in $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.*

Proof. It suffices to prove the statement for the column function $\widehat{\psi}_-^{(1)}(t,x,\lambda)$. The fact that $u(t,\cdot) \in L^1(\mathbb{R})$ ensures that each entry of $\mathcal{Q}(t,\cdot)$ is in $L^1(\mathbb{R})$ and therefore the operator

$$T[f](t,x,\lambda) = \int_{-\infty}^{x} \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\lambda(x-y)} \end{pmatrix} \mathcal{Q}(t,y)\, f(t,y,\lambda)\, \mathrm{d}y$$

is a bounded operator mapping functions with respect to x from $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$ for any fixed λ such that $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$, since $x - y \geq 0$. Now, defining

$$T_j[f](t,x,\lambda) = \int_{x_{j-1}}^{x} \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\lambda(x-y)} \end{pmatrix} \mathcal{Q}(t,y)\, f(t,y,\lambda)\, \mathrm{d}y,$$

where we fix λ such that $\lambda \in \mathbb{R}$, for an arbitrary interval $(x_{j-1}, x_j) \subset \mathbb{R}$ we obtain the estimate

$$||T_j[f](t,\cdot,\lambda)||_{L^\infty(x_{j-1},x_j)} \leq ||\mathcal{Q}(t,\cdot,\lambda)||_{L^1(x_{j-1},x_j)} ||f(t,\cdot,\lambda)||_{L^\infty(x_{j-1},x_j)}.$$

Then, we can choose x_j, $j = 1, \ldots, \ell$, in such a way that the operator T_j is a contraction from $L^\infty(x_{j-1}, x_j)$ to $L^\infty(x_{j-1}, x_j)$. Repeating this argument starting from $x_0 = -\infty$ and appropriately chosen x_1, …, to $x_{\ell-1}$ and $x_\ell = \infty$, we can obtain finitely many intervals so that T_j is contraction from $L^\infty(x_{j-1}, x_j)$ to $L^\infty(x_{j-1}, x_j)$, $j = 1, \ldots, \ell$. Setting $f_0(t,x,\lambda) \equiv e_1$ on (x_0, x_1), $e_1 = (1,0)^\intercal$, where $\cdot^\intercal$ indicates taking the transpose, we can find a unique function $f_j(t,\cdot,\lambda) \in L^\infty(x_{j-1}, x_j)$ by the Banach Fixed Point Theorem such that it solves the equation

$$f_j(t,x,\lambda) = f_{j-1}(t,x_j,\lambda) + T_j[f_j](t,x,\lambda), \qquad x \in (x_{j-1}, x_j)$$

for every $j = 2, \ldots, \ell$. Combining these functions, we find a continuous function in $L^\infty(\mathbb{R})$ satisfying the first column of the first Volterra integral equation (2.1.5). The exponential in t comes from the additional assumption that $u(t,\cdot) \in H^{1,1}(\mathbb{R})$. Given that, the entries of $\mathcal{Q}_1(t,x,\lambda)$ go to zero and moreover $\mathcal{V} \to -2i\lambda^2\sigma_3$ as $|x| \to \infty$. Hence, the time dependent Jost functions have the supplementary exponential term $e^{2i\lambda^2 t\sigma_3}$ in order for the limit to be consistent. Here, the first column of the Jost function $\psi_-(t,x,\lambda)$ in (2.1.5) takes the form

$$\psi_-^{(1)}(t,x,\lambda) = \widehat{\psi}_-^{(1)}(t,x,\lambda) e^{-i\Theta(t,x,\lambda)}.$$

Now, for the continuation of $\widehat{\psi}_-^{(1)}(t,x,\lambda)$ to $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$. The Neumann series $\widehat{\psi}_-^{(1)}(t,x,\lambda) = \sum_{j=0}^\infty T^j[f_0](t,x,\lambda)$, where $f_0(t,x,\lambda) \equiv e_1$, is formally a solution of the first column of the first Volterra integral equation (2.1.5). We can derive the bound $|T^j[f_0](t,x,\lambda)| \leq c||f_0(t,\cdot,\lambda)||_{L^\infty(\mathbb{R})} \cdot \frac{(h(t,x))^j}{j!}$ with a positive constant c for all λ such that $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$ and all $j \in \mathbb{N}$, where

$$h(t,x) = \int_{-\infty}^{x} |\mathcal{Q}(t,y)|\, \mathrm{d}y \leq \int_{-\infty}^{\infty} |\mathcal{Q}(t,y)|\, \mathrm{d}y \leq \|\mathcal{Q}(t,\cdot)\|_{L^1(\mathbb{R})}.$$

By induction, we have

$$\begin{aligned}
|T^{j+1}[f_0](t,x,\lambda)| &\le c\frac{\|f_0(t,\cdot,\lambda)\|_{L^\infty(\mathbb{R})}}{j!}\int_{-\infty}^{x}|\mathcal{Q}(t,y)|\,(h(t,y))^j\,\mathrm{d}y\\
&= c\frac{\|f_0(t,\cdot,\lambda)\|_{L^\infty(\mathbb{R})}}{j!}\int_0^{h(t,x)}s^j\,\mathrm{d}s\\
&= c\|f_0(t,\cdot,\lambda)\|_{L^\infty(\mathbb{R})}\frac{(h(t,x))^{j+1}}{(j+1)!},
\end{aligned}$$

where we put $s = h(t,y)$. Thus, as $\sum_{j=0}^{\infty}T^j[f_0](t,x,\lambda)$ is majorized in norm by a uniformly convergent power series, the series itself is uniformly convergent for $\lambda\in\mathbb{C}_+\cup\mathbb{R}$. The analyticity and continuity domains of the series transfer to its limit and therefore $\widehat{\psi}_-^{(1)}(t,x,\lambda)$ can be continued analytically in $\lambda\in\mathbb{C}_+$ and continuously in $\lambda\in\mathbb{C}_+\cup\mathbb{R}$.

Having two solutions $g_1(t,x,\lambda)$ and $g_2(t,x,\lambda)$ to the first column of the first Volterra integral equation (2.1.5), their difference $f(t,x,\lambda)=g_1(t,x,\lambda)-g_2(t,x,\lambda)$ satisfies

$$f(t,x,\lambda)=\int_{-\infty}^{x}e^{-i\Theta(0,x-y,\lambda)\sigma_3}\mathcal{Q}(t,y)f(t,y,\lambda)e^{i\Theta(0,x-y,\lambda)\sigma_3}\,\mathrm{d}y$$

or, individually,

$$\begin{aligned}
[f(t,x,\lambda)]_1 &= -\int_{-\infty}^{x}u(t,y)\int_{-\infty}^{y}e^{2i\lambda(y-y')}u^*(t,y')[f(t,y',\lambda)]_1\,\mathrm{d}y'\,\mathrm{d}y,\\
[f(t,x,\lambda)]_2 &= -\int_{-\infty}^{x}e^{2i\lambda(x-y)}u^*(t,y)[f(t,y,\lambda)]_1\,\mathrm{d}y.
\end{aligned}$$

Estimating $[f(t,x,\lambda)]_1$ and iterating this estimate j times, similarly to the estimation for $T^{j+1}[f_0]$, we get that in the case $[f(t,x,\lambda)]_1$ is bounded, i.e. $|[f(t,x,\lambda)]_1|\le C$, the following estimate

$$|[f(t,x,\lambda)]_1|\le C\frac{\left(\int_{-\infty}^{\infty}|u(t,y)|\,\mathrm{d}y\right)^{(2j)}}{(2j)!},$$

which goes to 0 as $j\to\infty$. Hence, $f(t,x,\lambda)$ is identically zero and the solution of the Volterra integral equation (2.1.5) is unique in the space of continuous functions. □

Analogously, the columns of $\psi_\pm(t,x,\lambda)=(\psi_\pm^{(1)},\psi_\pm^{(2)})$ can be continued analytically and continuously into the complex λ-plane. That is, $\psi_-^{(2)}$ and $\psi_+^{(1)}$ can be continued analytically in $\lambda\in\mathbb{C}_-$ and continuously in $\lambda\in\mathbb{C}_-\cup\mathbb{R}$, while $\psi_-^{(1)}$ and $\psi_+^{(2)}$ can be continued analytically in $\lambda\in\mathbb{C}_+$ and continuously in $\lambda\in\mathbb{C}_+\cup\mathbb{R}$.

The limits of the Jost functions imply

$$\lim_{x\to\pm\infty}\det\psi_\pm=1$$

and the zero trace of the matrix $\mathcal{U}$, which is another essential property of the selected Lax pair, then gives $\det\psi_\pm=1$ for all $x\in\mathbb{R}$. Further, ψ_+ and ψ_- are both fundamental matrix solutions of the Lax system (2.1.2), so there exists an x and t independent matrix $\mathcal{A}(\lambda)$ such that

$$\psi_-(t,x,\lambda)=\psi_+(t,x,\lambda)\mathcal{A}(\lambda),\qquad \lambda\in\mathbb{R}. \tag{2.1.6}$$

If we consider an arbitrary matrix $\mathcal{A}(t, x, \lambda)$, then taking the derivative with respect to x and using the x part of the Lax system leads to $\mathcal{U}\psi_- = \mathcal{U}\psi_+\mathcal{A} + \psi_+\mathcal{A}_x$. With the initial relation, we then find that $0 = \psi_+\mathcal{A}_x$ and analogously, differentiating with respect to t and using the t part of the Lax system, we find that $0 = \psi_+\mathcal{A}_t$. Taking the determinant and the limit value of ψ_+ into account, we obtain the equality (2.1.6) with a t and x independent matrix $\mathcal{A}(\lambda)$, which, in turn, determines the so-called *scattering matrix* uniquely, since we have

$$\mathcal{A}(\lambda) = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{pmatrix} = \psi_+^{-1}(t, x, \lambda)\psi_-(t, x, \lambda)$$

and additionally it can be deduced from the determinants of $\psi_\pm$ that $\det \mathcal{A}(\lambda) = 1$ for $\lambda \in \mathbb{R}$. Moreover, its entries can be written in terms of Wronskians determinants. In particular, the diagonal entries of the scattering matrix are $a_{11}(\lambda) = \det[\psi_-^{(1)}|\psi_+^{(2)}]$ and $a_{22}(\lambda) = -\det[\psi_-^{(2)}|\psi_+^{(1)}]$ implying that they can be continued in $\lambda \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_-$, respectively. On the other hand, the off-diagonal entries of the scattering matrix can be derived by $a_{12}(\lambda) = \det[\psi_-^{(2)}|\psi_+^{(2)}]$ and $a_{21}(\lambda) = -\det[\psi_-^{(1)}|\psi_+^{(1)}]$, which can in general not be continued off $\lambda \in \mathbb{R}$.

Lemma 2.1.2. *The Jost functions satisfy the symmetry relation*

$$\psi_\pm(t, x, \lambda) = \sigma_2\big(\psi_\pm(t, x, \lambda^*)\big)^*\sigma_2.$$

Proof. By the symmetry of the Lax pair (2.1.4), we have that $\psi_\pm(t, x, \lambda)$ and $\sigma_2\big(\psi_\pm(t, x, \lambda^*)\big)^*\sigma_2$ are solutions of the Lax system, since

$$(\psi_\pm(t, x, \lambda))_x = \mathcal{U}(t, x, \lambda)\psi_\pm(t, x, \lambda),$$

multiplied by the second Pauli matrix σ_2 from the left and the right, complex conjugated and λ^* inserted, can be written as

$$\big(\sigma_2\big(\psi_\pm(t, x, \lambda^*)\big)^*\sigma_2\big)_x = \sigma_2(\mathcal{U}(t, x, \lambda^*))^*\sigma_2 \cdot \sigma_2\big(\psi_\pm(t, x, \lambda^*)\big)^*\sigma_2 = \mathcal{U}(t, x, \lambda)\sigma_2\big(\psi_\pm(t, x, \lambda^*)\big)^*\sigma_2.$$

Further, both respective solutions have the same normalization as $x \to \pm\infty$, which is

$$\begin{aligned} \lim_{x\to\pm\infty} \psi_\pm(t, x, \lambda)e^{i\Theta(t,x,\lambda)\sigma_3} &= \mathbb{1}, \\ \lim_{x\to\pm\infty} \sigma_2\big(\psi_\pm(t, x, \lambda^*)\big)^*\sigma_2 e^{i\Theta(t,x,\lambda)\sigma_3} &= \lim_{x\to\pm\infty} \sigma_2\big(\psi_\pm(t, x, \lambda^*)\big)^*\big(e^{i\Theta(t,x,\lambda^*)\sigma_3}\big)^*\sigma_2 = \mathbb{1}. \end{aligned}$$

Consequently, the assertion $\psi_\pm(t, x, \lambda) = \sigma_2\big(\psi_\pm(t, x, \lambda^*)\big)^*\sigma_2$ holds. □

Therefore, the entries of the scattering matrix satisfy the following relations.

Proposition 2.1.3. *The elements of the scattering matrix $\mathcal{A}(\lambda)$ are related by $a_{11}(\lambda) = a_{22}^*(\lambda^*)$ for $\lambda \in \mathbb{C}_+$ and $a_{12}(\lambda) = -a_{21}^*(\lambda)$ for $\lambda \in \mathbb{R}$.*

Proof. We have by definition

$$\mathcal{A}(\lambda) = \psi_+^{-1}(t, x, \lambda)\psi_-(t, x, \lambda)$$

and with the symmetry relation of the Jost solutions

$$\begin{aligned} &= \sigma_2\big(\psi_+^{-1}(t, x, \lambda^*)\big)^*\big(\psi_-(t, x, \lambda^*)\big)^*\sigma_2 \\ &= \sigma_2\mathcal{A}^*(\lambda^*)\sigma_2 \end{aligned}$$

for $\lambda \in \mathbb{R}$. Solely for the diagonal entries the relation can be continued to the upper/lower half plane corresponding to the continuations of $a_{11}(\lambda)$ and $a_{22}(\lambda)$, thereby proving the assertion. □

For $\lambda \in \mathbb{R}$, we then have $|a_{11}(\lambda)|^2 + |a_{12}(\lambda)|^2 = 1$ due to $\det \mathcal{A}(\lambda) = 1$ and Proposition 2.1.3. The asymptotic behavior of the modified Jost functions and scattering matrix as $|\lambda| \to \infty$ is

$$\begin{aligned} \widehat{\psi}_- &= \mathbb{1} + \frac{1}{2i\lambda}\sigma_3 \mathcal{Q} + \frac{1}{2i\lambda}\sigma_3 \int_{-\infty}^{x} |u(t,y)|^2 \,\mathrm{d}y + \mathcal{O}(1/\lambda^2), \\ \widehat{\psi}_+ &= \mathbb{1} + \frac{1}{2i\lambda}\sigma_3 \mathcal{Q} - \frac{1}{2i\lambda}\sigma_3 \int_{x}^{\infty} |u(t,y)|^2 \,\mathrm{d}y + \mathcal{O}(1/\lambda^2), \end{aligned} \tag{2.1.7}$$

which can be shown using integration by parts and the Riemann–Lebesgue lemma. Exemplary, $\widehat{\psi}_-^{(1)}$ has the following integral expressions, see proof of Theorem 2.1.1, with regard to $[\widehat{\psi}_-]_{11}$:

$$[\widehat{\psi}_-(t,x,\lambda)]_{21} = -\int_{-\infty}^{x} u^*(t,y)[\widehat{\psi}_-(t,y,\lambda)]_{11} e^{2i\lambda(x-y)} \,\mathrm{d}y, \tag{2.1.8}$$

$$[\widehat{\psi}_-(t,x,\lambda)]_{11} = 1 - \int_{-\infty}^{x} u(t,y) \int_{-\infty}^{y} u^*(t,z)[\widehat{\psi}_-(t,z,\lambda)]_{11} e^{2i\lambda(y-z)} \,\mathrm{d}z \,\mathrm{d}y. \tag{2.1.9}$$

Here, $[\widehat{\psi}_-]_{ij}$ means that we take the (ij)-entry of the matrix $\widehat{\psi}_-$. Now, integration by parts used for the inner integral, the property $u(t,x) \to 0$ as $x \to -\infty$ and applying the Riemann–Lebesgue lemma to the remaining integral to replace it with $\mathcal{O}(1/\lambda^2)$, we obtain for (2.1.9) the following

$$[\widehat{\psi}_-(t,x,\lambda)]_{11} = 1 + \frac{1}{2i\lambda} \int_{-\infty}^{x} |u(t,y)|^2 [\widehat{\psi}_-(t,y,\lambda)]_{11} \,\mathrm{d}y + \mathcal{O}(1/\lambda^2).$$

Utilizing essentially the same steps, (2.1.8) amounts to

$$[\widehat{\psi}_-(t,x,\lambda)]_{21} = \frac{1}{2i\lambda} u^*(t,x)[\widehat{\psi}_-(t,x,\lambda)]_{11} + \mathcal{O}(1/\lambda^2)$$

and therefore considering a power series ansatz with respect to λ, the asymptotic behavior (2.1.7) proves well-founded. Note that $\mathcal{A}(\lambda) = \mathbb{1} + \mathcal{O}(1/\lambda)$.

2.1.3 Scattering data

Now, we are prepared to introduce the second crucial concept in the context of applying the inverse scattering method—the scattering data, which consists of particular properties of the (modified) Jost functions in combination with the scattering matrix. First off, it is easy to see that a zero of $a_{11}(\lambda)$ leads to $\det[\psi_-^{(1)} | \psi_+^{(2)}] = 0$ at a particular spectral parameter, say $\lambda = \lambda_1$, which implies that the two vector-valued functions $\psi_-^{(1)}(t,x,\lambda_1), \psi_+^{(2)}(t,x,\lambda_1)$ are linearly dependent. Therefore, we define the following.

Definition 2.1.4. For $N \in \mathbb{N}$, the function u admits *simple eigenvalues* if $a_{11}(\lambda)$ is nonzero in $\mathbb{C}_+ \cup \mathbb{R}$ except at a finite number of points $\lambda_1, \dots, \lambda_N \in \mathbb{C}_+$, where it has simple zeros, i.e. $a_{11}(\lambda_j) = 0$, $\frac{\mathrm{d}a_{11}}{\mathrm{d}\lambda}(\lambda_j) \neq 0$, $j = 1, \dots, N$. Moreover, the relation $a_{11}(\lambda) = a_{22}^*(\lambda^*)$ from Proposition 2.1.3 implies that if $\lambda_1, \dots, \lambda_N$ are simple eigenvalues, then $a_{22}(\lambda)$ is nonzero in $\mathbb{C}_- \cup \mathbb{R}$ except at the points $\lambda_1^*, \dots, \lambda_N^* \in \mathbb{C}_-$. Then, by $\mathcal{G}_N$, $N \in \mathbb{N}_0$, let us denote all functions $u(t,\cdot) \in H^{1,1}(\mathbb{R})$ that admit exactly N simple eigenvalues in the upper half-plane, where the infinite union of these sets

$$\mathcal{G} := \bigcup_{N=0}^{\infty} \mathcal{G}_N$$

gives the set of *generic* functions, which is, particularly, an open dense subset of $H^{1,1}(\mathbb{R})$, see [4, 44].

Beside the linear dependence of the vector-valued functions $\psi_-^{(1)}(t,x,\lambda_j), \psi_+^{(2)}(t,x,\lambda_j)$ at simple eigenvalues λ_j, $j \in \{1,\dots,N\}$, of u, we therefore also have in accordance with the zeros of $a_{22}(\lambda)$ and its equality to $-\det[\psi_-^{(2)}|\psi_+^{(1)}]$ that the vector-valued functions $\psi_-^{(2)}(t,x,\lambda_j^*), \psi_+^{(1)}(t,x,\lambda_j^*)$ are linearly dependent. Hence, there exist constants $b_j(t,x)$ and $\bar{b}_j(t,x)$ such that

$$\psi_-^{(1)}(t,x,\lambda_j) = b_j(t,x)\psi_+^{(2)}(t,x,\lambda_j), \quad \psi_-^{(2)}(t,x,\bar{\lambda}_j) = \bar{b}_j(t,x)\psi_+^{(1)}(t,x,\bar{\lambda}_j), \tag{2.1.10}$$

where we write $\bar{\lambda}_j = \lambda_j^*$. These constants are independent of t and x, which can be demonstrated by differentiating (2.1.10) and using either the x or t part of the Lax system (2.1.2) similar to the argument for the t and x independence of $\mathcal{A}(\lambda)$. Thus, the relations (2.1.10) can be reduced to the following relations for the modified Jost functions:

$$\widehat{\psi}_-^{(1)}(t,x,\lambda_j) = b_j\widehat{\psi}_+^{(2)}(t,x,\lambda_j)e^{2i\Theta(t,x,\lambda_j)}, \quad \widehat{\psi}_-^{(2)}(t,x,\bar{\lambda}_j) = \bar{b}_j\widehat{\psi}_+^{(1)}(t,x,\bar{\lambda}_j)e^{-2i\Theta(t,x,\bar{\lambda}_j)}. \tag{2.1.11}$$

For $j = 1,\dots,N$, the relations (2.1.11) then provide residue conditions

$$\begin{aligned} \operatorname*{Res}_{\lambda=\lambda_j}\Big(\frac{\widehat{\psi}_-^{(1)}}{a_{11}}\Big) &= C_j e^{2i\Theta(t,x,\lambda_j)}\widehat{\psi}_+^{(2)}(t,x,\lambda_j), \\ \operatorname*{Res}_{\lambda=\bar{\lambda}_j}\Big(\frac{\widehat{\psi}_-^{(2)}}{a_{22}}\Big) &= \bar{C}_j e^{-2i\Theta(t,x,\bar{\lambda}_j)}\widehat{\psi}_+^{(1)}(t,x,\bar{\lambda}_j), \end{aligned} \tag{2.1.12}$$

which are used in the inverse scattering method. The *norming constants* are defined by

$$C_j = b_j\Big(\frac{\mathrm{d}a_{11}}{\mathrm{d}\lambda}\big|_{\lambda=\lambda_j}\Big)^{-1}, \qquad \bar{C}_j = \bar{b}_j\Big(\frac{\mathrm{d}a_{22}}{\mathrm{d}\lambda}\big|_{\lambda=\bar{\lambda}_j}\Big)^{-1} \tag{2.1.13}$$

and they satisfy the symmetry relations $\bar{b}_j = -b_j^*$ and $\bar{C}_j = -C_j^*$.

Definition 2.1.5. For $N \in \mathbb{N}$, let the initial condition $u_0 \in \mathcal{G}_N$ and define the *reflection coefficient* $\rho(\lambda) = a_{21}(\lambda)/a_{11}(\lambda)$, where $\rho\colon \mathbb{R} \to \mathbb{C}$. Further, let $\lambda_1,\dots,\lambda_N$ be the pairwise distinct simple eigenvalues of u_0 in the upper half-plane and $C_1,\dots,C_N$ the corresponding norming constants defined in (2.1.13). Then, the scattering data of u_0 is given by

$$\mathcal{S}(u_0) = (\rho(\lambda;0), \{\lambda_j(0), C_j(0)\}_{j=1}^N). \tag{2.1.14}$$

In fact, one can prove that $u \in H^{1,1}(\mathbb{R})$ implies that $\rho \in H_1^{1,1}(\mathbb{R}) = \{f \in H^{1,1}(\mathbb{R}) : \|f\|_{L^\infty(\mathbb{R})} < 1\}$, see [18]. Effectively, this can be used to show that the scattering map satisfies the following.

Theorem 2.1.6 (Zhou, [44])**.** *For each $N \in \mathbb{N}_0$, the scattering maps $\mathcal{S}\colon \mathcal{G}_N \to H_1^{1,1}(\mathbb{R}) \times (\mathbb{C}_+)^N \times (\mathbb{C}\setminus\{0\})^N$ are Lipschitz continuous.*

2.1.4 Inverse scattering

The idea of inverse scattering is to recover the function u from given scattering data. Hence, we want to construct an inverse map $(\rho, \{\lambda_j, C_j\}_{j=1}^N) \mapsto u$. With the relation (2.1.6) containing the scattering matrix in mind, we define the sectionally meromorphic functions

$$M_-(t,x,\lambda) = (\widehat{\psi}_+^{(1)}, \widehat{\psi}_-^{(2)}/a_{22}), \qquad M_+(t,x,\lambda) = (\widehat{\psi}_-^{(1)}/a_{11}, \widehat{\psi}_+^{(2)}).$$

The functions M_- and M_+ enable us to rewrite the relation with the reflection coefficient $\rho(\lambda)$ as

$$M_+(t,x,\lambda) = M_-(t,x,\lambda)\begin{pmatrix} 1+|\rho(\lambda)|^2 & e^{-2i\Theta(t,x,\lambda)}\rho^*(\lambda) \\ e^{2i\Theta(t,x,\lambda)}\rho(\lambda) & 1 \end{pmatrix} \quad \text{for } \lambda \in \mathbb{R}, \tag{2.1.15}$$

where we define the so-called *jump matrix* by

$$\mathcal{J}(t,x,\lambda) := \begin{pmatrix} |\rho(\lambda)|^2 & e^{-2i\Theta(t,x,\lambda)}\rho^*(\lambda) \\ e^{2i\Theta(t,x,\lambda)}\rho(\lambda) & 0 \end{pmatrix}.$$

In particular, let us note that the jump matrix $\mathcal{J}$ is written in terms of the reflection coefficient $\rho(\lambda)$ and the phase $\Theta(t,x,\lambda)$. Considering the meromorphic functions, we obtain the following Riemann–Hilbert problem for $M(t,x,\cdot)$ from the residue conditions (2.1.12) and the discontinuity condition (2.1.15) only relying on the scattering data (2.1.14), which is essential in the reconstruction of the function $u(t,x)$.

Riemann–Hilbert problem 1. For given scattering data $(\rho, \{\lambda_j, C_j\}_{j=1}^N)$ as well as $t, x \in \mathbb{R}$, find a 2×2-matrix-valued function $\mathbb{C} \setminus \mathbb{R} \ni \lambda \mapsto M(t,x,\lambda)$ satisfying

1. $M(t,x,\cdot)$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$.
2. $M(t,x,\lambda) = \mathbb{1} + \mathcal{O}(1/\lambda)$ as $|\lambda| \to \infty$.
3. Non-tangential boundary values $M_\pm(t,x,\lambda)$ exist, satisfying the following jump condition $M_+(t,x,\lambda) = M_-(t,x,\lambda)(\mathbb{1} + \mathcal{J}(t,x,\lambda))$ for $\lambda \in \mathbb{R}$.
4. $M(t,x,\lambda)$ has simple poles at $\lambda_1, \dots, \lambda_N, \bar{\lambda}_1, \dots, \bar{\lambda}_N$ with

$$\operatorname*{Res}_{\lambda=\lambda_j} M(t,x,\lambda) = \lim_{\lambda\to\lambda_j} M(t,x,\lambda) \begin{pmatrix} 0 & 0 \\ C_j e^{2i\Theta(t,x,\lambda_j)} & 0 \end{pmatrix},$$
$$\operatorname*{Res}_{\lambda=\bar{\lambda}_j} M(t,x,\lambda) = \lim_{\lambda\to\bar{\lambda}_j} M(t,x,\lambda) \begin{pmatrix} 0 & \bar{C}_j e^{-2i\Theta(t,x,\bar{\lambda}_j)} \\ 0 & 0 \end{pmatrix}.$$

In that regard, note that the scattering data $(\rho(\lambda;0), \{\lambda_j(0), C_j(0)\}_{j=1}^N)$ given in Definition 2.1.5 are commonly understood as being derived from the known initial condition $u_0(x)$ in the context of the inverse scattering process. Then, at any time $t \in \mathbb{R}$, the evolution of the scattering data $\mathcal{S}(u)$ can be derived by observing the additional time dependent terms multiplied with each component of the scattering data. Hence, due to the coefficient $a_{11}(\lambda)$ being time independent, we have

$$\mathcal{S}(u) = (\rho(\lambda;0)e^{4i\lambda^2 t}, \{\lambda_j(0), C_j(0)e^{4i\lambda_j^2 t}\}_{j=1}^N).$$

In comparison to the usual asymptotic behavior of the Jost functions, which are taken to be $e^{-i\lambda x\sigma_3}$ as x goes to plus or minus infinity, in the asymptotic behavior we choose, the time dependence is not as apparent. Respecting this difference in the normalization, the usual scattering matrix $\mathcal{A}(\lambda)$ is not time independent, since under the assumption $\psi_\pm^{(1)} \sim e^{-i\lambda x}$ as $x \to \pm\infty$, the (12)-entry then has the following time dependency

$$a_{21}(\lambda) = -\det[\psi_-^{(1)} e^{2i\lambda^2 t}, \psi_+^{(1)} e^{2i\lambda^2 t}] = -e^{4i\lambda^2 t}\det[\psi_-^{(1)}, \psi_+^{(1)}].$$

However, due to the fact that we are primarily interested in the inverse scattering method in order to elaborate on a few key features the method has to offer, we conclude that it is sufficiently clarified by this treatment.

With regards to the asymptotic expansions (2.1.7) of the modified Jost functions as $|\lambda| \to \infty$, the reconstruction formula for the function $u(t,x)$ in terms of the solution of the Riemann–Hilbert problem 1 can be derived by

$$u(t,x) = 2i \lim_{|\lambda|\to\infty} \lambda [M(t,x,\lambda)]_{12}.$$

Therefore, the missing link to recover u from the scattering data (2.1.14) is to actually evaluate the solution of the Riemann–Hilbert problem which involves the Cauchy operator $\mathcal{C}$ defined as

$$\mathcal{C}[f](\lambda) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(\zeta)}{\zeta - \lambda} \, \mathrm{d}\zeta, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$$

for $f \in L^p(\mathbb{R})$ with $1 \leq p < \infty$. Further, we need to introduce the projection operators $\mathcal{P}^{\pm}$ which are given by

$$\mathcal{P}^{\pm}[f](\lambda) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - (\lambda \pm i\varepsilon)} \, \mathrm{d}\zeta, \quad \lambda \in \mathbb{R},$$

corresponding to the Cauchy operator in the case λ approaches the real line transversely from $\mathbb{C}^{\pm}$. We need the following results for these projection operators in order to establish the inverse scattering map.

Proposition 2.1.7.

(i) (Plemelj formulae) For $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, the limits of the projection operators $\mathcal{P}^{\pm}$ satisfy

$$\mathcal{P}^{\pm}[f](\lambda) = \frac{\pm f(\lambda) + iH[f](\lambda)}{2} \quad \text{for } \lambda \in \mathbb{R},$$

where $H \colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is a special case of the principal value integral, namely, the Hilbert transform *given by*

$$H[f](\lambda) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \Big(\int_{-\infty}^{\lambda - \varepsilon} + \int_{\lambda + \varepsilon}^{\infty} \Big) \frac{f(\zeta)}{\zeta - \lambda} \, \mathrm{d}\zeta, \quad \lambda \in \mathbb{R}.$$

(ii) If $f_{\pm}(\lambda)$ is analytic for λ in $\mathbb{C}_{\pm}$ and $f_{\pm}(\lambda) \to 0$ as $|\lambda| \to \infty$, then

$$\mathcal{P}^{\pm}[f_{\mp}](\lambda) = 0, \qquad \mathcal{P}^{\pm}[f_{\pm}](\lambda) = \pm f_{\pm}(\lambda) \quad \text{for } \lambda \in \mathbb{R}.$$

(iii) If $f \in L^1(\mathbb{R})$, $\mathcal{C}[f](\lambda)$ decays to zero as $|\lambda| \to \infty$ and it admits the asymptotic

$$\lim_{|\lambda| \to \infty} \lambda \mathcal{C}[f](\lambda) = -\frac{1}{2\pi i} \int_{\mathbb{R}} f(\zeta) \, \mathrm{d}\zeta,$$

taking the limit in either the upper or lower half-plane.

Proof. (i) A proof for the Plemelj formulae is standard in complex analysis and can be found for example in [35].

(ii) The Plemelj formulae together with Cauchy's integral theorem prove the assumption, since $H[f_+](\lambda) = -if_+(\lambda)$ and $H[f_-](\lambda) = if_-(\lambda)$.

(iii) For the asymptotic, one can calculate

$$\begin{aligned} \lim_{|\lambda| \to \infty} \lambda \mathcal{C}[f](\lambda) &= \lim_{|\lambda| \to \infty} \frac{\lambda}{2\pi i} \int_{\mathbb{R}} \frac{f(\zeta)}{\zeta - \lambda} \, \mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} \lim_{|\lambda| \to \infty} \lambda \frac{f(\zeta)}{\zeta - \lambda} \, \mathrm{d}\zeta \\ &= -\frac{1}{2\pi i} \int_{\mathbb{R}} f(\zeta) \, \mathrm{d}\zeta. \end{aligned}$$

Considering similar steps for $\mathcal{C}[f](\lambda)$, we obtain zero as the limit for $|\lambda| \to \infty$.

□

Now, let us assume that the solution of the Riemann–Hilbert problem has no simple poles, i.e. $u \in \mathcal{G}_0$. Applying $\mathcal{P}^-$ and $\mathcal{P}^+$ to the jump condition of the Riemann–Hilbert problem 1 yields

$$M_-(t,x,\lambda) = \mathbb{1} + \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{M_-(t,x,\zeta)\mathcal{J}(t,x,\zeta)}{\zeta-(\lambda-i\varepsilon)}\,\mathrm{d}\zeta,$$

since $M_+(t,x,\lambda)-\mathbb{1}$ is analytic in $\mathbb{C}_+$ and tends to 0 as $|\lambda|$ goes to infinity, and

$$M_+(t,x,\lambda) = \mathbb{1} + \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{M_-(t,x,\zeta)\mathcal{J}(t,x,\zeta)}{\zeta-(\lambda+i\varepsilon)}\,\mathrm{d}\zeta,$$

since $M_-(t,x,\lambda)-\mathbb{1}$ is analytic in $\mathbb{C}_-$ and tends to 0 as $|\lambda|$ goes to infinity, respectively. Hence,

$$M(t,x,\lambda) = \mathbb{1} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{M_-(t,x,\zeta)\mathcal{J}(t,x,\zeta)}{\zeta-\lambda}\,\mathrm{d}\zeta$$

is formally a solution of the Riemann–Hilbert problem and using the third property proven in Proposition 2.1.7, we find the asymptotic expansion of $M(t,x,\lambda)$ to be

$$M(t,x,\lambda) = \mathbb{1} - \frac{1}{2\pi i\lambda}\int_{-\infty}^{\infty} M_-(t,x,\zeta)\mathcal{J}(t,x,\zeta)\,\mathrm{d}\zeta + \mathcal{O}(\lambda^{-2}).$$

Comparing this to the asymptotics of the modified Jost functions (2.1.7), we obtain the potential in terms of the scattering data. Particularly,

$$\begin{aligned} u(t,x) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\Theta(t,x,\lambda)}\rho^*(\lambda)[M(t,x,\lambda)]_{11}\,\mathrm{d}\lambda \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\Theta(t,x,\lambda)}\rho^*(\lambda)[\widehat{\psi}_+^*(t,x,\lambda)]_{22}\,\mathrm{d}\lambda. \end{aligned}$$

In general, assuming that the function u is generic with $N\in\mathbb{N}$ simple eigenvalues, i.e. $u\in\mathcal{G}_N$, the Riemann–Hilbert problem 1 also comprises of N simple poles. Writing (2.1.6) as

$$\frac{\widehat{\psi}_-^{(1)}(t,x,\lambda)}{a_{11}(\lambda)} = \widehat{\psi}_+^{(1)}(t,x,\lambda) + \rho(\lambda)e^{2i\Theta(t,x,\lambda)}\widehat{\psi}_+^{(2)}(t,x,\lambda), \tag{2.1.16}$$

$$\frac{\widehat{\psi}_-^{(2)}(t,x,\lambda)}{a_{22}(\lambda)} = \widehat{\psi}_+^{(2)}(t,x,\lambda) - \rho^*(\lambda)e^{-2i\Theta(t,x,\lambda)}\widehat{\psi}_+^{(1)}(t,x,\lambda), \tag{2.1.17}$$

we can utilize the projection operators $\mathcal{P}^-$ and $\mathcal{P}^+$ on both sides of equations (2.1.16) and (2.1.17), respectively, to obtain the following system

$$\begin{aligned} \widehat{\psi}_+^{(1)}(t,x,\lambda) &= \begin{pmatrix}1\\0\end{pmatrix} + \sum_{j=1}^{N}\frac{\operatorname{Res}_{\zeta=\lambda_j}\left(\frac{\widehat{\psi}_-^{(1)}}{a_{11}}\right)}{\lambda-\lambda_j} + \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{2i\Theta(t,x,\zeta)}\rho(\zeta)\widehat{\psi}_+^{(2)}(t,x,\zeta)}{\zeta-(\lambda-i\varepsilon)}\,\mathrm{d}\zeta, \\ \widehat{\psi}_+^{(2)}(t,x,\lambda) &= \begin{pmatrix}0\\1\end{pmatrix} + \sum_{j=1}^{N}\frac{\operatorname{Res}_{\zeta=\lambda_j}\left(\frac{\widehat{\psi}_-^{(2)}}{a_{22}}\right)}{\lambda-\lambda_j} + \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-2i\Theta(t,x,\zeta)}\rho^*(\zeta)\widehat{\psi}_+^{(1)}(t,x,\zeta)}{\zeta-(\lambda+i\varepsilon)}\,\mathrm{d}\zeta, \end{aligned} \tag{2.1.18}$$

where the residues are as in the Riemann–Hilbert problem 1. Inserting $\lambda=\lambda_\ell$ into (2.1.18), the resulting system of equations together with (2.1.18) constitute a linear algebraic-integral system of equations, which, in principle, solves the inverse problem for the eigenfunctions $\widehat{\psi}_+^{(1)}(t,x,\lambda)$ and $\widehat{\psi}_+^{(2)}(t,x,\lambda)$. Comparing (2.1.18) to the asymptotic expansion (2.1.7) yields the reconstruction formula

$$u(t,x) = -2i\sum_{j=1}^{N} C_j^* e^{-2i\Theta(t,x,\lambda_j^*)}[\widehat{\psi}_+^*(t,x,\lambda_j)]_{22} + \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\Theta(t,x,\lambda)}\rho^*(\lambda)[\widehat{\psi}_+^*(t,x,\lambda)]_{22}\,\mathrm{d}\lambda. \quad (2.1.19)$$

In the reflectionless case we have $\rho(\lambda)=0$ for $\lambda\in\mathbb{R}$ and then the Riemann–Hilbert problem can be reduced to an algebraic system

$$\begin{aligned}\widehat{\psi}_+^{(1)}(t,x,\lambda_\ell) &= e_1 + \sum_{j=1}^{N}\frac{C_j e^{2i\Theta(t,x,\lambda_j)}\widehat{\psi}_+^{(2)}(t,x,\lambda_j)}{(\bar{\lambda}_\ell-\lambda_j)},\\ \widehat{\psi}_+^{(2)}(t,x,\lambda_j) &= e_2 + \sum_{m=1}^{N}\frac{\bar{C}_m e^{-2i\Theta(t,x,\bar{\lambda}_j)}\widehat{\psi}_+^{(1)}(t,x,\lambda_m)}{(\lambda_j-\bar{\lambda}_m)}\end{aligned}$$

for $\ell,j=1,\dots,N$. This particular restriction of the scattering data is the third important concept in the context of applying the inverse scattering method. Especially in connection with the second concept, the scattering data, this is the foundation of further considerations in this thesis. First off, let us continue the analysis of the algebraic system. For $N=1$, the modified Jost function $\widehat{\psi}_+^{(2)}(t,x,\lambda)$ at $\lambda=\lambda_1$ takes the form

$$\begin{aligned}[\widehat{\psi}_+(t,x,\lambda_1)]_{21} &= -\frac{C_1^*}{\lambda_1-\lambda_1^*}e^{-2i\Theta(t,x,\lambda_1^*)}\left[1-\frac{|C_1|^2 e^{2i(\Theta(t,x,\lambda_1)-\Theta(t,x,\lambda_1^*))}}{(\lambda_1-\lambda_1^*)^2}\right]^{-1},\\ [\widehat{\psi}_+(t,x,\lambda_1)]_{22} &= \left[1-\frac{|C_1|^2 e^{2i(\Theta(t,x,\lambda_1)-\Theta(t,x,\lambda_1^*))}}{(\lambda_1-\lambda_1^*)^2}\right]^{-1}. \end{aligned} \quad (2.1.20)$$

Therefore, inserting (2.1.20) into (2.1.19), we obtain the so-called *one-soliton* solution, which can be written with $\lambda_1=\xi_1+i\eta_1$ as

$$u(t,x) = -2i\eta_1\frac{C_1^*}{|C_1|}e^{-i(2\xi_1 x+4(\xi_1^2-\eta_1^2)t)}\operatorname{sech}\Big(2\eta_1(x+4\xi_1 t)-\log\frac{|C_1|}{2\eta_1}\Big).$$

We adapt the notation $u(t,x)=u_{sol}(t,x;\{\lambda_1,C_1\})$ resulting in

$$u_{sol}(t,x;\{\xi_1+i\eta_1, 2\eta_1 e^{2\eta_1 x_1+i\phi_1}\}) = 2\eta_1 e^{-i(2\xi_1 x+4(\xi_1^2-\eta_1^2)t+(\phi_1+\pi/2))}\operatorname{sech}(2\eta_1(x+4\xi_1 t-x_1)), \quad (2.1.21)$$

where $\phi_1=\arg(C_1)$ and $x_1=\frac{1}{2\eta_1}\log\frac{|C_1|}{2\eta_1}$. We invite the reader to make a mental note of the relations between the scattering data and the parameters of the solution and bear in mind that the inverse scattering transform is by no means exclusive to derive these kind of solutions. We shall come back to these points later on in the thesis. Further, it is worth mentioning that in [44], it has also been proven that the inverse scattering map maps $\rho\in H_1^{1,1}(\mathbb{R})$ to $u\in H^{1,1}(\mathbb{R})$, which we capture without the proof in the following statement.

Theorem 2.1.8 (Zhou, [44]). *For each $N\in\mathbb{N}_0$, the inverse scattering maps $\mathcal{S}^{-1}\colon H_1^{1,1}(\mathbb{R})\times(\mathbb{C}_+)^N\times(\mathbb{C}\setminus\{0\})^N\to H^{1,1}(\mathbb{R})$ are Lipschitz continuous.*

2.2 Inverse scattering method for the sG equation

The second PDE of interest in this thesis is the sine-Gordon equation which in laboratory coordinates has the form

$$\theta_{tt} - \theta_{xx} + \sin\theta = 0 \tag{2.2.1}$$

for $\theta(t, x)\colon \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$. Formulated as a Cauchy problem, we look for solutions of (2.2.1) with $\theta(0, x) = \theta_0(x)$ and $\theta_t(0, x) = \theta_1(x)$, where the functions $\theta_0(x)$ and $\theta_1(x)$ are the given initial data for $x \in \mathbb{R}$.

2.2.1 Lax pair

Similarly to the direct scattering for the NLS equation, the sG equation is another candidate for which there exists a Lax pair. Given the 2×2-matrices $\mathbb{U}$ and $\mathbb{V}$ of the form

$$\begin{aligned}
\mathbb{U} &= -\frac{i}{4}(\theta_t - \theta_x)\sigma_1 - \frac{i\lambda}{4}\sigma_3 + \frac{i}{4\lambda}\begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}, \\
\mathbb{V} &= -\frac{i}{4}(\theta_x - \theta_t)\sigma_1 + \frac{i\lambda}{4}\sigma_3 + \frac{i}{4\lambda}\begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix},
\end{aligned} \tag{2.2.2}$$

where we again have the spectral parameter $\lambda \in \mathbb{C}$ as well as the third Pauli matrix $\sigma_3 = \operatorname{diag}(1, -1)$ and the first Pauli matrix σ_1 which is defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the sG equation can be written as a compatibility condition of the following linear spectral problems

$$\begin{aligned}
\psi_x(t, x, \lambda) &= \mathbb{U}(t, x, \lambda)\psi(t, x, \lambda), \\
\psi_t(t, x, \lambda) &= \mathbb{V}(t, x, \lambda)\psi(t, x, \lambda).
\end{aligned} \tag{2.2.3}$$

In that regard, for a 2×2-matrix solution $\psi(t, x, \lambda)$, the compatibility condition $\psi_{tx}(t, x, \lambda) = \psi_{xt}(t, x, \lambda)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ is satisfied if and only if $\theta(t, x)$ is a solution of the sG equation (2.2.1). Without the auxiliary function, the equivalent zero curvature condition is again

$$\mathbb{U}_t - \mathbb{V}_x + [\mathbb{U}, \mathbb{V}] = 0 \text{ for all } \lambda \in \mathbb{C} \setminus \{0\}.$$

Note that a solution $\psi(t, x, \lambda)$ of the Lax system of the NLS equation is not the same as a solution of the Lax system of the sG equation. However, we denote solutions of either of these systems by ψ, since it should be clear from the context which setting the solution belongs to. Further, under the condition that θ is real, the Lax pair $\mathbb{U}$, $\mathbb{V}$ satisfies the symmetry relations

$$\begin{aligned}
&\mathbb{U}(t, x, \lambda) = \sigma_1 \mathbb{U}(t, x, -\lambda)\sigma_1, \quad \mathbb{U}(t, x, \lambda) = \sigma_2\big(\mathbb{U}(t, x, \lambda^*)\big)^*\sigma_2, \quad \mathbb{U}(t, x, \lambda) = \sigma_3\big(\mathbb{U}(t, x, -\lambda^*)\big)^*\sigma_3, \\
&\mathbb{V}(t, x, \lambda) = \sigma_1 \mathbb{V}(t, x, -\lambda)\sigma_1, \quad \mathbb{V}(t, x, \lambda) = \sigma_2\big(\mathbb{V}(t, x, \lambda^*)\big)^*\sigma_2, \quad \mathbb{V}(t, x, \lambda) = \sigma_3\big(\mathbb{V}(t, x, -\lambda^*)\big)^*\sigma_3.
\end{aligned} \tag{2.2.4}$$

2.2.2 Jost functions and direct scattering

Under the assumption that $\lim_{x\to-\infty}\theta = 0$, $\lim_{x\to\infty}\theta = 2\pi C$ sufficiently fast and $C \in \mathbb{Z}$, the *topological charge*, it is reasonable to assume that there exist 2×2-matrix-valued functions ψ_- and ψ_+, solutions of the equations (2.2.3), with asymptotic behavior

$$\psi_\pm(t, x, \lambda) \sim e^{(-\frac{i}{4}(\lambda - \frac{1}{\lambda})x + \frac{i}{4}(\lambda + \frac{1}{\lambda})t)\sigma_3} \quad \text{as } x \to \pm\infty,$$

for $\lambda \in \mathbb{R}$. Similarly to the case of the NLS equation, we can define the modified Jost functions $\widehat{\psi}(t,x,\lambda) = \psi(t,x,\lambda)e^{i\Theta(t,x,\lambda)}$, where for the sG equation the phase $\Theta(t,x,\lambda)$ is equal to $\frac{1}{4}(\lambda - \frac{1}{\lambda})x - \frac{1}{4}(\lambda + \frac{1}{\lambda})t$. Again, as in the case of the solutions of the Lax system, we use the same notation as for the phase of the NLS equation and the context should be clear. In particular, the modified Jost functions satisfy the modified Lax system

$$\widehat{\psi}_x + \frac{i}{4}\Big(\lambda - \frac{1}{\lambda}\Big)[\sigma_3, \widehat{\psi}] = \mathbb{Q}\widehat{\psi}, \qquad \widehat{\psi}_t - \frac{i}{4}\Big(\lambda + \frac{1}{\lambda}\Big)[\sigma_3, \widehat{\psi}] = \mathbb{Q}_1\widehat{\psi}, \tag{2.2.5}$$

where

$$\mathbb{Q} = \frac{i}{4}\begin{pmatrix} \lambda^{-1}(\cos\theta - 1) & -i\lambda^{-1}\sin\theta - \theta_t + \theta_x \\ i\lambda^{-1}\sin\theta - \theta_t + \theta_x & \lambda^{-1}(1-\cos\theta) \end{pmatrix},$$
$$\mathbb{Q}_1 = \frac{i}{4}\begin{pmatrix} \lambda^{-1}(\cos\theta - 1) & -i\lambda^{-1}\sin\theta + \theta_t - \theta_x \\ i\lambda^{-1}\sin\theta + \theta_t - \theta_x & \lambda^{-1}(1-\cos\theta) \end{pmatrix}.$$

Therefore, the modified Jost solutions have constant limits for all $\lambda \in \mathbb{R}$,

$$\widehat{\psi}_\pm(t,x,\lambda) \to \mathbb{1}, \qquad \text{as } x \to \pm\infty$$

and are solutions of the following Volterra integral equations:

$$\begin{aligned} \widehat{\psi}_-(t,x,\lambda) &= \mathbb{1} + \int_{-\infty}^{x} e^{-i\Theta(0,x-y,\lambda)\sigma_3}\mathbb{Q}(t,y,\lambda)\widehat{\psi}_-(t,y,\lambda)e^{i\Theta(0,x-y,\lambda)\sigma_3}\,\mathrm{d}y, \\ \widehat{\psi}_+(t,x,\lambda) &= \mathbb{1} - \int_{x}^{\infty} e^{-i\Theta(0,x-y,\lambda)\sigma_3}\mathbb{Q}(t,y,\lambda)\widehat{\psi}_+(t,y,\lambda)e^{i\Theta(0,x-y,\lambda)\sigma_3}\,\mathrm{d}y, \end{aligned} \tag{2.2.6}$$

which can be derived from the modified Lax system. Here again, we continue to denote the Jost functions in terms of their column vectors as $\psi_\pm = (\psi_\pm^{(1)}, \psi_\pm^{(2)})$ and also the modified Jost functions in terms of their column vectors as $\widehat{\psi}_\pm = (\widehat{\psi}_\pm^{(1)}, \widehat{\psi}_\pm^{(2)})$.

Lemma 2.2.1. *Let $1-\cos(\theta(t,\cdot)), \sin(\theta(t,\cdot)), \theta_t(t,\cdot), \theta_x(t,\cdot) \in L^1(\mathbb{R})$. Then for each $x \in \mathbb{R}$, the columns $\widehat{\psi}_-^{(2)}(t,x,\lambda)$ and $\widehat{\psi}_+^{(1)}(t,x,\lambda)$ of the modified Jost functions are analytic for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in (\mathbb{C}_- \cup \mathbb{R}) \cap \{\lambda \in \mathbb{C} \colon |\lambda| \geq \varepsilon\}$, while the columns $\widehat{\psi}_-^{(1)}(t,x,\lambda)$ and $\widehat{\psi}_+^{(2)}(t,x,\lambda)$ are analytic for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in (\mathbb{C}_+ \cup \mathbb{R}) \cap \{\lambda \in \mathbb{C} \colon |\lambda| \geq \varepsilon\}$ for each $\varepsilon > 0$.*

Proof. The proof is analogous to the second part of the proof of Theorem 2.1.1 with the exception that the operator T is uniformly bounded for $|\lambda| \geq \varepsilon$: An iteration of the operator

$$T[f](t,x,\lambda) = \int_{-\infty}^{x} \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i}{2}\left(\lambda - \frac{1}{\lambda}\right)(x-y)} \end{pmatrix} \mathbb{Q}(t,y,\lambda)\, f(t,y,\lambda)\,\mathrm{d}y,$$

starting with the unit vector $f_0(t,x,\lambda) \equiv e_1$, formally provides a solution in the form of an infinite series $\sum_{j=0}^{\infty} T^j[f_0](t,x,\lambda)$ to the first column of the first Volterra integral equation (2.2.6), which we denote by $\widehat{\psi}_-^{(1)}$. Observing that $\lambda \in \mathbb{C}_+$ implies that $\lambda - \lambda^{-1} \in \mathbb{C}_+$ and therefore if we additionally take $x - y \geq 0$, then the kernel function with respect to y is bounded by a linear combination of the entries $1-\cos(\theta(t,y)), \sin(\theta(t,y)), \theta_t(t,y), \theta_x(t,y)$ of $\mathbb{Q}(t,y,\lambda)$ with constant coefficients independent of y and uniformly bounded for $|\lambda| \geq \varepsilon > 0$. As in the case of the NLS equation, if we take $h(t,x) = \int_{-\infty}^{x} |\mathbb{Q}(t,y,\lambda)|\,\mathrm{d}y$, we have

$$|T^j[f_0](t,x,\lambda)| \leq c\|f_0(t,\cdot,\lambda)\|_{L^\infty(\mathbb{R})}\frac{(h(t,x))^j}{j!}.$$

Hence, it follows that the partial sums of $\widehat{\psi}_-^{(1)}$ are majorized by those of an exponential series implying the convergence of the partial sums. Since the partial sums are uniformly convergent, analyticity for $\lambda \in \mathbb{C}_+$ and continuity for $\lambda \in (\mathbb{C}_+ \cup \mathbb{R}) \cap \{\lambda \in \mathbb{C}\colon |\lambda| \geq \varepsilon\}$ extend from the partial sums to the limit $\widehat{\psi}_-^{(1)}$. The argument can essentially be repeated for the other columns of the modified Jost functions. $\square$

Even though the argument which is sufficient for the NLS equation fails for the sG equation, it is still possible to prove that the (modified) Jost solutions are continuous on the complete half-planes. Using a gauge transformation, a new set of functions with respect to the Jost functions can be defined by

$$\Psi_\pm(t,x,\lambda) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix} \psi_\pm(t,x,\lambda). \tag{2.2.7}$$

As before, the new set of functions can be written in terms of its column vectors as $\Psi_\pm(t,x,\lambda) = (\Psi_\pm^{(1)}(t,x,\lambda), \Psi_\pm^{(2)}(t,x,\lambda))$ and it can be calculated that these matrices $\Psi_\pm(t,x,\lambda)$ satisfy the modified eigenvalue equation

$$\Psi_x = \Big(-\frac{i}{4}\Big(\lambda - \frac{1}{\lambda}\Big)\sigma_3 + \mathbb{Q}_2\Big)\Psi,$$

where

$$\mathbb{Q}_2 = \frac{i}{4}\begin{pmatrix} \lambda(1-\cos\theta) & i\lambda\sin\theta + \theta_t + \theta_x \\ -i\lambda\sin\theta + \theta_t + \theta_x & \lambda(\cos\theta - 1) \end{pmatrix}.$$

Note that the terms containing a power of λ changed from λ^{-1} in $\mathbb{Q}_1$ to λ in $\mathbb{Q}_2$. Under the same assumption that $\lim_{x\to-\infty}\theta = 0$, $\lim_{x\to\infty}\theta = 2\pi C$ as before, it is reasonable to assume that there exist functions $\Psi_\pm$ with asymptotic behavior

$$\begin{aligned} \Psi_-(t,x,\lambda) &\sim \sigma_3 e^{-i\Theta(t,x,\lambda)\sigma_3} && \text{as } x \to -\infty, \\ \Psi_+(t,x,\lambda) &\sim (-1)^C \sigma_3 e^{-i\Theta(t,x,\lambda)\sigma_3} && \text{as } x \to +\infty \end{aligned}$$

for $\lambda \in \mathbb{R}$. Then, we define $\widehat{\Psi}(t,x,\lambda) = \Psi(t,x,\lambda)e^{i\Theta(t,x,\lambda)\sigma_3}$ denoted in terms of its column vectors by $\widehat{\Psi}(t,x,\lambda) = (\widehat{\Psi}^{(1)}(t,x,\lambda), \widehat{\Psi}^{(2)}(t,x,\lambda))$. Their Volterra integral equations are of the form

$$\begin{aligned} \widehat{\Psi}_-(t,x,\lambda) &= \sigma_3 + \int_{-\infty}^x e^{-i\Theta(0,x-y,\lambda)\sigma_3}\mathbb{Q}_2(t,x,y)\widehat{\Psi}_-(t,y,\lambda)e^{i\Theta(0,x-y,\lambda)\sigma_3}\,\mathrm{d}y, \\ \widehat{\Psi}_+(t,x,\lambda) &= (-1)^C\sigma_3 - \int_x^{\infty} e^{-i\Theta(0,x-y,\lambda)\sigma_3}\mathbb{Q}_2(t,x,y)\widehat{\Psi}_+(t,y,\lambda)e^{i\Theta(0,x-y,\lambda)\sigma_3}\,\mathrm{d}y. \end{aligned} \tag{2.2.8}$$

Given these modified integral equations for the transformed solutions $\widehat{\Psi}_\pm(t,x,\lambda)$, we can prove that the columns of $\widehat{\psi}_\pm(t,x,\lambda)$ are continuous in a neighborhood of $\lambda = 0$ in the upper or lower half-plane.

Lemma 2.2.2. *Let $1 - \cos(\theta(t,\cdot)), \sin(\theta(t,\cdot)), \theta_t(t,\cdot), \theta_x(t,\cdot) \in L^1(\mathbb{R})$. Then for each $x \in \mathbb{R}$, the columns $\widehat{\psi}_-^{(2)}(t,x,\lambda)$ and $\widehat{\psi}_+^{(1)}(t,x,\lambda)$ of the modified Jost functions are continuous for $\lambda \in (\mathbb{C}_- \cup \mathbb{R}) \cap \{\lambda \in \mathbb{C}\colon |\lambda| \leq \varepsilon\}$, while the columns $\widehat{\psi}_-^{(1)}(t,x,\lambda)$ and $\widehat{\psi}_+^{(2)}(t,x,\lambda)$ are continuous for $\lambda \in (\mathbb{C}_+ \cup \mathbb{R}) \cap \{\lambda \in \mathbb{C}\colon |\lambda| < \varepsilon\}$ for each $\varepsilon > 0$.*

Proof. Following the proof of Lemma 2.2.1, we have the initial vector $f_0(t,x,\lambda) \equiv e_1$ and the operator $T[f](t,x,\lambda)$ with $\mathbb{Q}$ replaced by $\mathbb{Q}_2$. This means that every factor of λ^{-1} which occurred in T through $\mathbb{Q}$ is now a factor of λ coming from $\mathbb{Q}_2$. Hence, instead of the condition $|\lambda| \geq \varepsilon$, we have for $\widehat{\Psi}_-^{(1)}$ the same analysis under the condition $|\lambda| < \varepsilon$. Thus, the first column $\widehat{\Psi}_-^{(1)}$ of the modified transformed Jost function $\widehat{\Psi}_-$ is analytic for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$, where $|\lambda| < \varepsilon$. Further, the gauge transformation (2.2.7) is independent of λ and therefore it is not affecting the continuity with respect to λ, which enables to transfer the results to the columns of the modified Jost functions. This procedure can be applied in a similar way to each column. $\square$

By Lemmas 2.2.1 and 2.2.2, we have that the columns of the modified Jost functions in the case of the sG equation can be continued analytically in $\lambda \in \mathbb{C}_\pm$ and continuously in $\lambda \in \mathbb{C}_\pm \cup \mathbb{R}$, which we summarize in the following statement.

Theorem 2.2.3 (Kaup, [9, 29]). *Let $1-\cos(\theta(t,\cdot)), \sin(\theta(t,\cdot)), \theta_t(t,\cdot), \theta_x(t,\cdot) \in L^1(\mathbb{R})$. Then, for every $\lambda \in \mathbb{R}$, there exist unique solutions $\widehat{\psi}_\pm(t,\cdot,\lambda) \in L^\infty(\mathbb{R})$ satisfying the integral equations* (2.2.6). *The two column vectors $\widehat{\psi}_-^{(1)}(t,x,\lambda)$ and $\widehat{\psi}_+^{(2)}(t,x,\lambda)$ of the modified Jost functions can be continued analytically in $\lambda \in \mathbb{C}_-$ and continuously in $\lambda \in \mathbb{C}_- \cup \mathbb{R}$, while the two column vectors $\widehat{\psi}_-^{(2)}(t,x,\lambda)$ and $\widehat{\psi}_+^{(1)}(t,x,\lambda)$ of the modified Jost functions can be continued analytically in $\lambda \in \mathbb{C}_+$ and continuously in $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.*

Proof. Repeating the first part of the proof of Theorem 2.1.1, we have, since each entry of the matrix $\mathbb{Q}(t,\cdot,\lambda)$ is in $L^1(\mathbb{R})$, that we can find a unique continuous function in $L^\infty(\mathbb{R})$ satisfying for example the first column of the first Volterra integral equation (2.2.6).

Then, to continue the function in $\lambda \in \mathbb{C}_+$, we utilize Lemmas 2.2.1 and 2.2.2 under consideration of the sign of the real part of the exponential factors including $\mathrm{Im}(\lambda - \lambda^{-1})$ in the Volterra integral equations (2.2.6). An analogous result holds for the other columns of the modified Jost functions. $\square$

Now, these properties imply that the columns of the Jost functions can be continued analytically and continuously into the complex λ-plane as their modified counter part. The limits of the Jost solutions and the zero trace of the matrix $\mathbb{U}$ gives $\det \psi_\pm = 1$ for all $x \in \mathbb{R}$. Moreover, $\psi_\pm$ are both fundamental matrix solutions of the Lax system (2.2.3), so there exists an x and t independent matrix $\mathbb{A}(\lambda)$, see Section 2.1, such that

$$\psi_-(t,x,\lambda) = \psi_+(t,x,\lambda)\mathbb{A}(\lambda), \qquad \lambda \in \mathbb{R}. \tag{2.2.9}$$

The scattering matrix $\mathbb{A}$ is determined by this system and therefore we can write the scattering matrix as $\mathbb{A}(\lambda) = \psi_+^{-1}(t,x,\lambda)\psi_-(t,x,\lambda)$, whose entries can be written in terms of Wronskians. In particular, $a_{11}(\lambda) = \det[\psi_-^{(1)}|\psi_+^{(2)}]$ and $a_{22}(\lambda) = -\det[\psi_-^{(2)}|\psi_+^{(1)}]$ implying that they can be continued in $\lambda \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_-$, respectively. The eigenfunctions inherit the symmetry relation of the Lax pair:

Lemma 2.2.4. *Assuming that θ is real, the Jost functions $\psi_\pm(t,x,\lambda)$ satisfy the three symmetry relations $\psi_\pm(t,x,\lambda) = \sigma_1\psi_\pm(t,x,-\lambda)\sigma_1, \psi_\pm(t,x,\lambda) = \sigma_2\big(\psi_\pm(t,x,\lambda^*)\big)^*\sigma_2$ and $\psi_\pm(t,x,\lambda) = \sigma_3\big(\psi_\pm(t,x,-\lambda^*)\big)^*\sigma_3$.*

Proof. As in Lemma 2.1.2, it is possible to show that $\psi_\pm(t,x,\lambda)$, $\sigma_1\psi_\pm(t,x,-\lambda)\sigma_1$ as well as $\sigma_2\big(\psi_\pm(t,x,\lambda^*)\big)^*\sigma_2$ and $\sigma_3\big(\psi_\pm(t,x,-\lambda^*)\big)^*\sigma_3$ are all solutions of the same Lax system. This means

$$\begin{aligned}
(\psi_\pm(t,x,\lambda))_x &= \mathbb{U}(t,x,\lambda)\psi_\pm(t,x,\lambda),\\
(\sigma_1\psi_\pm(t,x,-\lambda)\sigma_1)_x &= \sigma_1\mathbb{U}(t,x,-\lambda)\psi_\pm(t,x,-\lambda)\sigma_1 = \mathbb{U}(t,x,\lambda)(\sigma_1\psi_\pm(t,x,-\lambda)\sigma_1),\\
(\sigma_2\big(\psi_\pm(t,x,\lambda^*)\big)^*\sigma_2)_x &= \sigma_2\big(\mathbb{U}(t,x,\lambda^*)\big)^*\big(\psi_\pm(t,x,\lambda^*)\big)^*\sigma_2 = \mathbb{U}(t,x,\lambda)\sigma_2\big(\psi_\pm(t,x,\lambda^*)\big)^*\sigma_2,\\
(\sigma_3\big(\psi_\pm(t,x,-\lambda^*)\big)^*\sigma_3)_x &= \sigma_3\big(\mathbb{U}(t,x,-\lambda^*)\big)^*\big(\psi_\pm(t,x,-\lambda^*)\big)^*\sigma_3 = \mathbb{U}(t,x,\lambda)\sigma_3\big(\psi_\pm(t,x,-\lambda^*)\big)^*\sigma_3.
\end{aligned}$$

This also holds for the t part of the Lax system, since both matrices $\mathbb{U}$ and $\mathbb{V}$ of the Lax pair satisfy the same symmetry relations. Further, the normalization of each of these matrices can be derived by

$$\begin{aligned}
\lim_{x\to\pm\infty} \psi_\pm(t,x,\lambda)e^{i\Theta(t,x,\lambda)\sigma_3} &= \mathbb{1},\\
\lim_{x\to\pm\infty} \sigma_1\big(\psi_\pm(t,x,-\lambda)\big)\sigma_1 e^{i\Theta(t,x,\lambda)\sigma_3} &= \lim_{x\to\pm\infty} \sigma_1\big(\psi_\pm(t,x,-\lambda)\big)e^{i\Theta(t,x,-\lambda)\sigma_3}\sigma_1 = \mathbb{1},\\
\lim_{x\to\pm\infty} \sigma_2\big(\psi_\pm(t,x,\lambda^*)\big)^*\sigma_2 e^{i\Theta(t,x,\lambda)\sigma_3} &= \lim_{x\to\pm\infty} \sigma_2\big(\psi_\pm(t,x,\lambda^*)\big)^*\big(e^{i\Theta(t,x,\lambda^*)\sigma_3}\big)^*\sigma_2 = \mathbb{1},\\
\lim_{x\to\pm\infty} \sigma_3\big(\psi_\pm(t,x,-\lambda^*)\big)^*\sigma_3 e^{i\Theta(t,x,\lambda)\sigma_3} &= \lim_{x\to\pm\infty} \sigma_3\big(\psi_\pm(t,x,-\lambda^*)\big)^*\big(e^{i\Theta(t,x,-\lambda^*)\sigma_3}\big)^*\sigma_3 = \mathbb{1}.
\end{aligned}$$

We use the fact that $-\Theta(t,x,\lambda) = \Theta(t,x,-\lambda)$ and $\Theta(t,x,\lambda) = \Theta^*(t,x,\lambda^*)$. Hence, the assertion is proven. $\square$

As in the case of the NLS equation, these symmetries can be used to derive relations between the elements of the scattering matrix $\mathbb{A}(\lambda)$, which then can be utilized to specify particularities in the scattering data.

Proposition 2.2.5. *Assuming that θ is real, the elements of the scattering matrix $\mathbb{A}(\lambda)$ are related by $a_{11}(\lambda) = a_{22}(-\lambda) = a_{22}^*(\lambda^*) = a_{11}^*(-\lambda^*)$ for $\lambda \in \mathbb{C}_+$ and $a_{12}(\lambda) = a_{21}(-\lambda) = -a_{21}^*(\lambda) = -a_{12}^*(-\lambda)$ for $\lambda \in \mathbb{R}$.*

Proof. We have by definition

$$\mathbb{A}(\lambda) = \psi_+^{-1}(t,x,\lambda)\psi_-(t,x,\lambda)$$

and with the symmetry relations of Lemma 2.2.4 regarding the Jost functions, we have

$$\begin{aligned}
&= \sigma_1\mathbb{A}(-\lambda)\sigma_1\\
&= \sigma_2\mathbb{A}^*(\lambda^*)\sigma_2\\
&= \sigma_3\mathbb{A}^*(-\lambda^*)\sigma_3
\end{aligned}$$

for $\lambda \in \mathbb{R}$. Solely for the diagonal entries the relation can be continued to the upper/lower half-plane corresponding to the continuations of $a_{11}(\lambda)$ and $a_{22}(\lambda)$. $\square$

For the NLS equation, the relations of the elements of the scattering matrix resulted in the pairing of zeros in the scattering data, i.e. if $\lambda_1 \in \mathbb{C}_+$ is a zero for $a_{11}(\lambda)$, then $\lambda_1^* \in \mathbb{C}_-$ is a zero for $a_{22}(\lambda)$, see Definition 2.1.4. In the case of the sG equation, the derived relations of the elements of the scattering matrix cause the zeros to come in quadruples: Given a zero $\lambda_1 \in \mathbb{C}_+$ of $a_{11}(\lambda)$, $-\lambda_1^* \in \mathbb{C}_+$ is a zero of $a_{11}(\lambda)$ and $-\lambda_1, \lambda_1^* \in \mathbb{C}_-$ are zeros of $a_{22}(\lambda)$; in the special case $\lambda_1 \in i\mathbb{R}$, the pairs in either half-plane (upper/lower) coincide, leaving only two zeros $\lambda_1 \in i\mathbb{R}_+$ of $a_{11}(\lambda)$ and $\lambda_1^* = -\lambda_1 \in i\mathbb{R}_-$ of $a_{22}(\lambda)$.

If $\lambda \in \mathbb{R}$, we have $|a_{11}(\lambda)|^2 + |a_{12}(\lambda)|^2 = 1$ since $\det\mathbb{A}(\lambda) = 1$ and Proposition 2.2.5. From the integral equations (2.2.6), one can show that the asymptotic behavior of the modified Jost functions and scattering coefficient $a_{11}(\lambda)$ satisfy

$$\begin{aligned}
\widehat{\psi}_\pm(t,x,\lambda) &= \mathbb{1} + \mathcal{O}(1/\lambda),\\
a_{11}(\lambda) &= 1 + \mathcal{O}(1/\lambda)
\end{aligned}$$

as $|\lambda| \to \infty$ in the appropriate half-planes and considering the gauge transformed modified Jost functions, we have

$$\begin{aligned} \widehat{\psi}_+(t,x,\lambda) &= (-1)^C \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} + \mathcal{O}(\lambda), \\ \widehat{\psi}_-(t,x,\lambda) &= \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix} + \mathcal{O}(\lambda), \\ a(\lambda) &= (-1)^C + \mathcal{O}(\lambda) \end{aligned} \tag{2.2.10}$$

as $|\lambda| \to 0$ in the appropriate half-planes, see [9, 29].

2.2.3 Scattering data

As for the NLS equation, the scattering data can be derived from particular properties of the functions dependent on the spectral parameter λ introduced in the last section. As mentioned in the last subsection, the simple eigenvalues for the function θ introduce up to four zeros of the functions $a_{11}(\lambda)$ and $a_{22}(\lambda)$ of the scattering matrix. Hence, we have the following definition.

Definition 2.2.6. For θ real and $N \in \mathbb{N}$, the function θ admits *simple eigenvalues* if $a_{11}(\lambda)$ is nonzero in $\mathbb{C}_+ \cup \mathbb{R}$ except at a finite number of points $\lambda_1, \dots, \lambda_N \in \mathbb{C}_+$, where it has simple zeros, i.e. $a_{11}(\lambda_j) = 0$, $\frac{\mathrm{d}a_{11}}{\mathrm{d}\lambda}(\lambda_j) \neq 0$, $j = 1, \dots, N$. Moreover, the relation $a_{11}(\lambda) = a_{22}(-\lambda) = a_{22}^*(\lambda^*) = a_{11}(-\lambda^*)$ from Proposition 2.2.5 implies that if $\lambda_1, \dots, \lambda_N$ are simple eigenvalues, then $-\lambda_1^*, \dots, -\lambda_N^* \in \mathbb{C}_+$ are simple zeros of $a_{11}(\lambda)$ and $a_{22}(\lambda)$ is nonzero in $\mathbb{C}_- \cup \mathbb{R}$ except at the points $-\lambda_1, \dots, -\lambda_N, \lambda_1^*, \dots, \lambda_N^* \in \mathbb{C}_-$. Then, we define by $\mathcal{G}_N$, $N \in \mathbb{N}_0$, the set of all functions $\theta(t,x)$ with $1 - \cos(\theta(t,\cdot)), \sin(\theta(t,\cdot)), \theta_t(t,\cdot), \theta_x(t,\cdot) \in L^1(\mathbb{R})$ that admit exactly N simple eigenvalues in the upper half-plane. The infinite union of these sets

$$\mathcal{G} := \bigcup_{N=0}^{\infty} \mathcal{G}_N$$

gives the set of *generic* functions.

When connecting simple zeros, two distinct cases can occur.

Definition 2.2.7. Assuming that θ is real, we split the number of simple eigenvalues into $N = N_s + N_b$. Here,

(i) N_s is the number of simple eigenvalues lying on the imaginary axis $\lambda_j = i\eta_j$, $\eta_j > 0$ and therefore corresponding to single solitons.

(ii) N_b is the number of simple eigenvalues not lying on the imaginary axis $\lambda_j = \xi_j + i\eta_j$, $\xi_j \neq 0$ and therefore corresponding to the so-called *breather solutions*.

The topological charge C can be related to the number of single solitons N_s. Basically, an even number of single solitons corresponds to 0 as charge, whereas an odd number of single solitons corresponds to a value of 1 or -1 for the charge.

From Lemma 2.2.4, we obtain the following relations for the column vectors of the modified Jost functions

$$\begin{aligned} \psi_\pm^{(1)}(t,x,\lambda) &= \sigma_1 \psi_\pm^{(2)}(t,x,-\lambda) = i\sigma_2 \big(\psi_\pm^{(2)}(t,x,\lambda^*)\big)^* = \sigma_3 \big(\psi_\pm^{(1)}(t,x,-\lambda^*)\big)^*, \\ \psi_\pm^{(2)}(t,x,\lambda) &= \sigma_1 \psi_\pm^{(1)}(t,x,-\lambda) = -i\sigma_2 \big(\psi_\pm^{(1)}(t,x,\lambda^*)\big)^* = -\sigma_3 \big(\psi_\pm^{(2)}(t,x,-\lambda^*)\big)^*. \end{aligned} \tag{2.2.11}$$

Now, given a simple eigenvalue λ_j, $j \in \{1,\dots,N\}$, of a real valued θ, we have that the column vectors $\psi_-^{(1)}(t,x,\lambda_j)$ and $\psi_+^{(2)}(t,x,\lambda_j)$ are linearly dependent, i.e. there exists a constant b_j such that

$$\psi_-^{(1)}(t,x,\lambda_j) = b_j\psi_+^{(2)}(t,x,\lambda_j),$$

where the constant is independent of t and x, see Subsection 2.1.3. Then, the relations (2.2.11) imply that

$$\psi_-^{(2)}(t,x,-\lambda_j) = b_j\psi_+^{(1)}(t,x,-\lambda_j), \quad \psi_-^{(2)}(t,x,\lambda_j^*) = -b_j^*\psi_+^{(1)}(t,x,\lambda_j^*)$$

and also

$$\psi_-^{(1)}(t,x,-\lambda_j^*) = -b_j^*\psi_+^{(2)}(t,x,-\lambda_j^*).$$

In the case λ_j lies on the imaginary axis $\lambda_j = i\eta_j$ with $\eta_j > 0$, we particularly have $\lambda_j = -\lambda_j^*$ and $-\lambda_j = \lambda_j^*$. From which $-b_j^* = b_j$ follows and therefore b_j is as λ_j necessarily purely imaginary.

As for the NLS equation, these facts enable us to provide residue conditions which are essential for the Riemann–Hilbert problem. We have

$$\begin{aligned}
&\operatorname*{Res}_{\lambda=\lambda_j}\Big(\frac{\widehat{\psi}_-^{(1)}}{a_{11}}\Big) = C_j e^{2i\Theta(t,x,\lambda_j)}\widehat{\psi}_+^{(2)}(t,x,\lambda_j), \qquad \operatorname*{Res}_{\lambda=-\lambda_j}\Big(\frac{\widehat{\psi}_-^{(2)}}{a_{22}}\Big) = C_j e^{-2i\Theta(t,x,-\lambda_j)}\widehat{\psi}_+^{(1)}(t,x,-\lambda_j),\\
&\operatorname*{Res}_{\lambda=\lambda_j^*}\Big(\frac{\widehat{\psi}_-^{(2)}}{a_{22}}\Big) = \bar{C}_j e^{-2i\Theta(t,x,\lambda_j^*)}\widehat{\psi}_+^{(1)}(t,x,\lambda_j^*), \quad \operatorname*{Res}_{\lambda=-\lambda_j^*}\Big(\frac{\widehat{\psi}_-^{(1)}}{a_{11}}\Big) = \bar{C}_j e^{2i\Theta(t,x,-\lambda_j^*)}\widehat{\psi}_+^{(2)}(t,x,-\lambda_j^*),
\end{aligned} \tag{2.2.12}$$

where the norming constant

$$C_j = b_j\Big(\frac{\mathrm{d}a_{11}}{\mathrm{d}\lambda}\big|_{\lambda=\lambda_j}\Big)^{-1} \tag{2.2.13}$$

is defined similarly to the one for the NLS equation (2.1.13). By this definition, we also have

$$\bar{C}_j = -b_j^*\Big(\frac{\mathrm{d}a_{22}}{\mathrm{d}\lambda}\big|_{\lambda=\lambda_j^*}\Big)^{-1} = -b_j^*\Big(\frac{\mathrm{d}a_{11}^*}{\mathrm{d}\lambda}\big|_{\lambda=\lambda_j}\Big)^{-1} = -C_j^*.$$

Definition 2.2.8. Let $N \in \mathbb{N}$ and suppose $a_{11}(\lambda)$ has only simple zeros in $\mathbb{C}_+$. Then, the scattering data associated to the initial data $\theta_0(x)$, $\theta_1(x)$ is given by the reflection coefficient $\rho(\lambda) = a_{21}(\lambda)/a_{11}(\lambda)$, where $\rho\colon \mathbb{R} \to \mathbb{C}$, the simple eigenvalues $\lambda_1,\dots,\lambda_N$ which are pairwise distinct in $\mathbb{C}_+$ and the norming constants $C_1,\dots,C_N$ as in (2.2.13). We write

$$\mathcal{S}(\theta_0,\theta_1) = (\rho(\lambda;0), \{\lambda_j, C_j(0)\}_{j=1}^N).$$

Following the same ideas we motivated for the NLS equation, the time dependence of the scattering data can be observed to be influenced by the phase for both the norming constant and the reflection coefficient. Thus, the scattering data corresponding to the potential $\theta \in \mathcal{G}_N$ is in general given by

$$\mathcal{S}(\theta) = (\rho(\lambda;0)e^{-\frac{i}{2}(\lambda+\frac{1}{\lambda})t}, \{\lambda_j, C_j(0)e^{-\frac{i}{2}(\lambda_j+\frac{1}{\lambda_j})t}\}_{j=1}^N). \tag{2.2.14}$$

2.2.4 Inverse scattering

As for the NLS equation, we want to establish the inverse scattering to recover the function θ from the scattering data, which can be expressed as a mapping $(\rho, \{\lambda_j, C_j\}_{j=1}^N) \mapsto \theta$. Therefore, we define the sectionally meromorphic functions

$$M_-(t,x,\lambda) = (\widehat{\psi}_+^{(1)}, \widehat{\psi}_-^{(2)}/a_{22}), \qquad M_+(t,x,\lambda) = (\widehat{\psi}_-^{(1)}/a_{11}, \widehat{\psi}_+^{(2)}),$$

which are used to rewrite the relation (2.2.11) into

$$M_+(t,x,\lambda) = M_-(t,x,\lambda)(\mathbb{1} + \mathbb{J}(t,x,\lambda)), \tag{2.2.15}$$

where the jump matrix for the sG equation is given by

$$\mathbb{J}(t,x,\lambda) = \begin{pmatrix} |\rho(\lambda)|^2 & e^{-2i\Theta(t,x,\lambda)}\rho^*(\lambda) \\ e^{2i\Theta(t,x,\lambda)}\rho(\lambda) & 0 \end{pmatrix}.$$

Then, the Riemann–Hilbert problem for $M(t,x,\cdot)$ is obtained with regard to the residue conditions (2.2.12) and the jump condition (2.2.15) from the scattering data (2.2.14) and therefore we have a method of recovering the solution $\theta(t,x)$ from the scattering data.

Riemann–Hilbert problem 2. For given scattering data $(\rho, \{\lambda_j, C_j\}_{j=1}^N)$ as well as $t, x \in \mathbb{R}$, find a 2×2-matrix-valued function $\mathbb{C} \setminus \mathbb{R} \ni \lambda \mapsto M(t,x,\lambda)$ satisfying

1. $M(t,x,\cdot)$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$.
2. $M(t,x,\lambda) = \mathbb{1} + \mathcal{O}(1/\lambda)$ as $|\lambda| \to \infty$.
3. Non-tangential boundary values $M_\pm(t,x,\lambda)$ exist, satisfying the following jump condition $M_+(t,x,\lambda) = M_-(t,x,\lambda)(\mathbb{1} + \mathbb{J}(t,x,\lambda))$ for $\lambda \in \mathbb{R}$.
4. $M(t,x,\lambda)$ has simple poles at $\lambda_j, -\lambda_j^*, -\lambda_j, \lambda_j^*$, $j = 1,\dots,N$, with

$$\operatorname*{Res}_{\lambda=\lambda_j} M(t,x,\lambda) = \lim_{\lambda\to\lambda_j} M(t,x,\lambda) \begin{pmatrix} 0 & 0 \\ C_j e^{2i\Theta(t,x,\lambda_j)} & 0 \end{pmatrix},$$
$$\operatorname*{Res}_{\lambda=-\lambda_j^*} M(t,x,\lambda) = \lim_{\lambda\to-\lambda_j^*} M(t,x,\lambda) \begin{pmatrix} 0 & 0 \\ \bar{C}_j e^{2i\Theta(t,x,-\lambda_j^*)} & 0 \end{pmatrix},$$
$$\operatorname*{Res}_{\lambda=-\lambda_j} M(t,x,\lambda) = \lim_{\lambda\to-\lambda_j} M(t,x,\lambda) \begin{pmatrix} 0 & C_j e^{-2i\Theta(t,x,-\lambda_j)} \\ 0 & 0 \end{pmatrix},$$
$$\operatorname*{Res}_{\lambda=\lambda_j^*} M(t,x,\lambda) = \lim_{\lambda\to\lambda_j^*} M(t,x,\lambda) \begin{pmatrix} 0 & \bar{C}_j e^{-2i\Theta(t,x,\lambda_j^*)} \\ 0 & 0 \end{pmatrix}.$$

Then, for the reconstruction formula we expand the solution $M(t,x,\cdot)$ of the Riemann–Hilbert problem 2 as

$$M(t,x,\lambda) = M_0(t,x) + \lambda M_1(t,x) + \mathcal{O}(\lambda^2), \quad \text{as } \lambda \to 0$$

in contrast to the asymptotic expansion for the NLS equation. Hence, if we set

$$M_0(t,x) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

the solution of the sG equation can be reconstructed as in the following proposition.

Proposition 2.2.9 (Cheng, Venakides & Zhou, [13]). *Assume that $M(t,x,\lambda)$ is the solution of the Riemann–Hilbert problem 2. Then, the solution $\theta(t,x)$ of the sG equation* (2.2.1) *can be expressed as*

$$\cos\theta = 1 + 2M_{12}M_{21},$$
$$\sin\theta = -2iM_{21}M_{22}.$$

Proof. The equality $\widehat{\psi}_x = -\frac{i}{4}(\lambda - \lambda^{-1})[\sigma_3, \widehat{\psi}] + \mathbb{Q}\widehat{\psi}$ for the solution M of the Riemann–Hilbert problem 2 gives

$$\begin{aligned}(M_0)_x + \lambda(M_1)_x + \mathcal{O}(\lambda^2) = &-\frac{i}{4}\lambda[\sigma_3, M_0 + \lambda M_1 + \mathcal{O}(\lambda^2)]\\ &- \frac{i}{4}(\theta_t - \theta_x)\sigma_1(M_0 + \lambda M_1 + \mathcal{O}(\lambda^2))\\ &+ \frac{i}{4}\lambda^{-1}[\sigma_3, M_0 + \lambda M_1 + \mathcal{O}(\lambda^2)]\\ &+ \frac{i}{4}\lambda^{-1}((\cos\theta - 1)\sigma_3 + \sin\theta\sigma_2)(M_0 + \lambda M_1 + \mathcal{O}(\lambda^2)).\end{aligned}$$

Equating the coefficient of λ^{-1} to 0, we obtain

$$[\sigma_3, M_0] + ((\cos\theta - 1)\sigma_3 + \sin\theta\sigma_2)M_0 = 0,$$

which can be solved for $((\cos\theta - 1)\sigma_3 + \sin\theta\sigma_2)$ through

$$\cos\theta\sigma_3 + \sin\theta\sigma_2 = M_0\sigma_3 M_0^{-1}.$$

Having $\det M_0 = 1$, the equality gives

$$\cos\theta\sigma_3 + \sin\theta\sigma_2 = \begin{pmatrix} M_{11}M_{22} + M_{12}M_{21} & -2M_{11}M_{12} \\ 2M_{21}M_{22} & -M_{11}M_{22} - M_{12}M_{21} \end{pmatrix},$$

which proves the assertion. □

In fact, this can also be retraced using the asymptotic expansions of the modified Jost functions as $|\lambda| \to 0$ in the equation (2.2.10) similar to the asymptotic behavior derived for the modified Jost functions of the NLS equation.

Let us again emphasize the fact that the inverse scattering method for both the NLS and the sG equation is, in general, a powerful framework not solely in order to construct solutions, but also due to the simple time dependence and versatility of the scattering data. Therefore, the three important concepts, we wanted to highlight in this section, are precisely the equivalence of the respective PDE to the compatibility of the Lax system, the scattering data derived from the spectrum and the eigenfunctions of the Lax pair and the correspondence of the scattering data to parameters of solutions, in particular, for the solutions of the NLS equation. In the next chapter, we present different methods in order to find solutions, which have other advantages themselves. In particular, one can observe that two of the three highlighted concepts again emerge in these methods [27]. As for the third concept, the connection of the scattering data to the parameters of the solution, we see that it follows naturally, having the knowledge of the scattering data.

Chapter 3

Solution construction methods

Due to the importance of the PDEs introduced in the last section, the search for solutions has been vividly carried out especially in the last century. As a result, besides the inverse scattering method introduced in the last Sections 2.1 and 2.2, a diverse spectrum of methods has been discovered which enables us to solve integrable nonlinear PDEs, which, in particular, are associated with the AKNS system [1, 38], such as the NLS and sG equation. There are—just to mention a few—the Hirota direct method, the Bäcklund transformation technique and the Dressing method, which is somewhat equivalent to the Darboux transformation. Since, together with the inverse scattering method, each of these methods is applied to the very same integrable nonlinear PDE to construct a solution, one may think that they are all in some way equivalent.

In this chapter, we deal not only with the introduction of a certain subset of these methods, but also with necessary comparisons and resemblances between them. Furthermore, this is the foundation for the next chapter, where we introduce different models of the NLS and sG equation. However, due to the fact that the presentation of more than two methods for the search of solutions is in general not too commonly found in the literature, this chapter takes influences of quite a few sources [5, 11, 14, 21, 27, 10, 32, 33, 34].

3.1 Classical Darboux transformation *vs.* Dressing method *vs.* Bäcklund transformation

In the following, we want to present three additional methods, which can be used to find solutions for the suggested integrable nonlinear PDEs, and show that under the relevant conditions they are indeed equivalent as far as the construction of soliton solutions is concerned.

Therefore, let us first focus on the Bäcklund transformation technique [11, 34], which transforms the problem into two first-order partial matrix differential equations (3.1.2) for a matrix B, which is commonly known as a *Darboux matrix*, with respect to the Lax system.

3.1.1 Bäcklund transformation

Obtaining solutions for nonlinear partial differential equations is usually not as easy as it is illustrated in Section 2.1 for the construction of a one-soliton solution for the NLS equation by the inverse scattering method. In this section, we want to present the Bäcklund transformation technique, which can be used to obtain new solutions from a known solution by solving a system of integrable PDEs. By defining

$$\widetilde{\psi}(t,x,\lambda) = B(t,x,\lambda)\psi(t,x,\lambda), \tag{3.1.1}$$

we consider analogous systems to the ones found in (2.1.2), (2.2.3):

$$\begin{aligned}\widetilde{\psi}_x &= \widetilde{U}\widetilde{\psi},\\ \widetilde{\psi}_t &= \widetilde{V}\widetilde{\psi},\end{aligned}$$

where $\widetilde{U}$, $\widetilde{V}$ are either $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$ with u replaced by $\tilde{u}$ in the case of the NLS equation or $\widetilde{\mathbb{U}}$, $\widetilde{\mathbb{V}}$ with θ replaced by $\tilde{\theta}$ in the case of the sG equation. Simply taking the derivative in equation (3.1.1), we require that the Darboux matrix B satisfies the following partial matrix differential equations for any t and x,

$$\begin{aligned}B_x &= \widetilde{U}B - BU,\\ B_t &= \widetilde{V}B - BV.\end{aligned} \tag{3.1.2}$$

In the case of the NLS equation with $\widetilde{U} = \widetilde{\mathcal{U}}$ and $\widetilde{V} = \widetilde{\mathcal{V}}$, it can be found that if one restricts the matrix $B(t,x,\lambda)$ to be of the form

$$B(t,x,\lambda) = B^{(1)}(t,x) + B^{(0)}(t,x)\lambda^{-1}, \tag{3.1.3}$$

we actually have the following:

Proposition 3.1.1 (Caudrelier, [11])**.** *Under the condition that $\mathcal{B}(t,x,\lambda)$ is of the form* (3.1.3)*, the Darboux matrix in the case of the NLS equation takes the explicit form*

$$\mathcal{B}(t,x,\lambda) = \mathbb{1} + \frac{1}{2\lambda}\begin{pmatrix}\alpha \pm i\sqrt{\beta^2 - |\tilde{u}-u|^2} & -i(\tilde{u}-u)\\ -i(\tilde{u}-u)^* & \alpha \mp i\sqrt{\beta^2-|\tilde{u}-u|^2}\end{pmatrix}, \tag{3.1.4}$$

where $\alpha, \beta \in \mathbb{R}$ are the t and x independent parameters of the transformation.

Similarly, it can be found that in the case of the sG equation with $\widetilde{U} = \widetilde{\mathbb{U}}$ and $\widetilde{V} = \widetilde{\mathbb{V}}$, we have:

Proposition 3.1.2 (Caudrelier, [11])**.** *Under the condition that $\mathbb{B}(t,x,\lambda)$ is of the form* (3.1.3)*, the Darboux matrix in the case of the sG equation takes the explicit form*

$$\mathbb{B}(t,x,\lambda) = \mathbb{1} \pm \frac{i\alpha}{\lambda}\begin{pmatrix}\cos\frac{\tilde{\theta}+\theta}{2} & -i\sin\frac{\tilde{\theta}+\theta}{2}\\ i\sin\frac{\tilde{\theta}+\theta}{2} & -\cos\frac{\tilde{\theta}+\theta}{2}\end{pmatrix}, \tag{3.1.5}$$

where $\alpha \in \mathbb{R}$ is the t and x independent parameter of the transformation.

Effectively, to implement this transformation one has to take two functions u and $\tilde{u}$ or θ and $\tilde{\theta}$. One of these functions, say u or θ, has to be a solution of the underlying PDE. Based on these functions, we construct two Lax pairs as we have demonstrated in the corresponding Sections 2.1 or rather 2.2. Now, if the matrix $B(t,x,\lambda)$ is chosen as in one of the two Propositions 3.1.1 or 3.1.2 and satisfies (3.1.2), then it follows that the second function, $\tilde{u}$ or $\tilde{\theta}$, is also a solution of the underlying PDE.

The proofs for these propositions involve a thorough analysis of the relations (3.1.2) individually equated for each power with respect to λ. We give the proof of Proposition 3.1.1 as a demonstration:

Proof. With the matrix $\mathcal{B}(t,x,\lambda)$ being of the form (3.1.3), we have that the x part of relation (3.1.2) gives the following three equalities

$$\begin{aligned} 0 &= [\mathcal{B}^{(1)}, \sigma_3], && (3.1.6)\\ \mathcal{B}^{(1)}_x &= i[\mathcal{B}^{(0)}, \sigma_3] + \widetilde{\mathcal{Q}}\mathcal{B}^{(1)} - \mathcal{B}^{(1)}\mathcal{Q}, && (3.1.7)\\ \mathcal{B}^{(0)}_x &= \widetilde{\mathcal{Q}}\mathcal{B}^{(0)} - \mathcal{B}^{(0)}\mathcal{Q} && (3.1.8) \end{aligned}$$

as requirements for the powers λ^1, λ^0 and λ^{-1}, respectively. Similarly, we derive from the t part of relation (3.1.2) the following four equalities

$$\begin{aligned} 0 &= [\mathcal{B}^{(1)}, \sigma_3], && (3.1.9)\\ 0 &= 2i[\mathcal{B}^{(0)}, \sigma_3] + 2\widetilde{\mathcal{Q}}\mathcal{B}^{(1)} - 2\mathcal{B}^{(1)}\mathcal{Q}, && (3.1.10)\\ \mathcal{B}^{(1)}_t &= (\widetilde{\mathcal{Q}_1}\big|_{\lambda=0})\mathcal{B}^{(1)} - \mathcal{B}^{(1)}(\mathcal{Q}_1\big|_{\lambda=0}) + 2\widetilde{\mathcal{Q}}\mathcal{B}^{(0)} - 2\mathcal{B}^{(0)}\mathcal{Q}, && (3.1.11)\\ \mathcal{B}^{(0)}_t &= (\widetilde{\mathcal{Q}_1}\big|_{\lambda=0})\mathcal{B}^{(0)} - \mathcal{B}^{(0)}(\mathcal{Q}_1\big|_{\lambda=0}) && (3.1.12) \end{aligned}$$

as requirements for the powers λ^2, λ^1, λ^0 and λ^{-1}, respectively. Equality (3.1.6) immediately implies that the off-diagonal entries of $\mathcal{B}^{(1)}$ are identically zero. Then by (3.1.7), we find that the diagonal entries are independent of x. Further, if we take the limit $|x| \to \infty$ in equation (3.1.11), we obtain that $\mathcal{B}^{(1)}_t \equiv 0$, since the entries are independent of x. Subsequently, normalizing the Darboux matrix $\mathcal{B}(t,x,\lambda)$ via multiplication by $(\mathcal{B}^{(1)})^{-1}$ which is independent of t and x from the left, we only need to determine the entries of the new matrix $(\mathcal{B}^{(1)})^{-1}\mathcal{B}^{(0)}$ which we continue to denote simply as $\mathcal{B}^{(0)}$ and therefore $\mathcal{B}(t,x,\lambda) = \mathbb{1} + \lambda^{-1}\mathcal{B}^{(0)}$. This freedom in normalization stems from the fact that left multiplying of $\mathcal{B}(t,x,\lambda)$ by a matrix G^{-1} independent of t and x is equivalent to transforming the Lax pair $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$ of the NLS equation as $G^{-1}\widetilde{\mathcal{U}}G$, $G^{-1}\widetilde{\mathcal{V}}G$. However, the zero curvature condition is invariant under such transformations.

Therefore, denote

$$\mathcal{B}^{(0)} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

and α_1, α_2 its eigenvalues, which proves helpful in recovering the diagonal entries. First off, we obtain from equation (3.1.7) that the off-diagonal entries can be expressed by $b_2 = -i/2(\tilde{u} - u)$ and $b_3 = -i/2(\tilde{u} - u)^*$. Then, if we compare the coefficients of the characteristic polynomial of $\mathcal{B}^{(0)}$, we find

$$\det \mathcal{B}^{(0)} = \alpha_1\alpha_2, \qquad \operatorname{Tr}\mathcal{B}^{(0)} = \alpha_1 + \alpha_2,$$

from which it is possible to determine b_1 and b_2. Therefore, we have

$$b_1 = \frac{1}{2}\big(\beta_1 \pm i\sqrt{(i\beta_2)^2 - |\tilde{u} - u|^2}\big), \quad b_4 = \frac{1}{2}\big(\beta_1 \mp i\sqrt{(i\beta_2)^2 - |\tilde{u} - u|^2}\big),$$

where $\beta_1 = \alpha_1 + \alpha_2$ and $\beta_2 = \alpha_1 - \alpha_2$. To conclude, we still need to prove that β_1 and β_2 are independent of t and x and particularly $\beta_1 \in \mathbb{R}$ as well as $\beta_2 \in i\mathbb{R}$. For that let us first show that $f(t,x,\lambda) = \det \mathcal{B}(t,x,\lambda)$ and $g(t,x,\lambda) = \operatorname{Tr}\mathcal{B}(t,x,\lambda)$ are independent of t and x:

$$f_x(t,x,\lambda) = \det \mathcal{B}(t,x,\lambda)\operatorname{Tr}\Big(\mathcal{B}^{-1}(t,x,\lambda)\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}x}(t,x,\lambda)\Big)$$

by the Jacobi's formula and with the x part of (3.1.2) and the properties of the trace, we obtain

$$= \det \mathcal{B}(t,x,\lambda)\operatorname{Tr}(\widetilde{\mathcal{U}} - \mathcal{U}) = 0$$

due to $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ being traceless. This can be analogously repeated for $f_t(t,x,\lambda)$, where the tracelessness of $\widetilde{\mathcal{V}}$ and $\mathcal{V}$ provides the independence of t. Since the trace is linear, we find for $g(t,x,\lambda)$ that

$$\begin{aligned} g_x(t,x,\lambda) &= \operatorname{Tr}(\mathcal{B}(t,x,\lambda)(\widetilde{\mathcal{U}}-\mathcal{U})) \\ &= \lambda^{-1}\operatorname{Tr}(\mathcal{B}^{(0)}(t,x)(\widetilde{\mathcal{U}}-\mathcal{U})), \end{aligned}$$

where equation (3.1.7) can be used to derive

$$= i\lambda^{-1}\operatorname{Tr}(\mathcal{B}^{(0)}(t,x)[\sigma_3,\mathcal{B}^{(0)}(t,x)]) = 0.$$

As before, this result can be utilized, taking the limit $|x|\to\infty$, to obtain $g_t(t,x,\lambda) = \operatorname{Tr}(\mathcal{B}(t,x,\lambda)\cdot(\widetilde{\mathcal{V}}-\mathcal{V})) = 0$, since we already know that $g(t,x,\lambda)$ is independent of x and $\widetilde{\mathcal{V}}-\mathcal{V}$ goes to zero as $|x|\to\infty$. Further, we have that the symmetry relation (2.1.4) together with the system (3.1.2) implies that the Darboux matrix satisfies $\mathcal{B}(t,x,\lambda) = \sigma_2(\mathcal{B}(t,x,\lambda^*))^*\sigma_2$ and therefore the eigenvalue problem boils down to either $\alpha_j = \alpha_j^*$, $j = 1,2$, implying $\alpha_j \in \mathbb{R}$ or $\alpha_1 = \alpha_2^*$. In the first case $\alpha := \beta_1 = \alpha_1+\alpha_2 \in \mathbb{R}$ and $\beta := \beta_2 = \alpha_1-\alpha_2 = 0 \in i\mathbb{R}$; in the second case $\alpha = \alpha_1+\alpha_2 = 2\operatorname{Re}\alpha_1 \in \mathbb{R}$ and $\beta = \alpha_1-\alpha_2 = 2i\operatorname{Im}\alpha_1 \in i\mathbb{R}$. □

The proof for Proposition 3.1.2 can be repeated analogously to this proof, especially, considering light-cone coordinates instead of laboratory coordinates. For a large class of integrable PDEs which can be formulated in a specific Lax pair which satisfies similar properties as the Lax pair of the NLS equation, this theory has been proven more general in [11]. In particular, both Darboux matrices for the NLS equation and the sG equation (in laboratory coordinates) can be found therein.

Nevertheless, the point of this method is to construct a solution $\tilde{u}$ of the PDE corresponding to the Lax pair $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$ from the solution u of the PDE corresponding to the Lax pair $\mathcal{U}$, $\mathcal{V}$. At first glance, it is not obvious that the system (3.1.2) with either the Darboux matrix for the NLS equation (3.1.4) or the sG equation (3.1.5) provides the means to achieve this. In [5], it has been shown how to rewrite the relevant equations of the partial matrix differential equations (3.1.2) into a linear system of ordinary differential equations in the case of the NLS equation. Later on, when we mention relations with respect to the solutions u and $\tilde{u}$ or rather θ and $\tilde{\theta}$, which correspond to the relations (3.1.2), we show how to obtain a transformed solution explicitly. For now, however, we have a different goal in mind: We want to present the Bäcklund transformation technique in the course of introducing the classical Darboux transformation or rather the Dressing method. Therefore, we give a simple example on how the Bäcklund transformation can actually be applied through the computation of an ordinary differential equation. On top of that, we indicate similarities in the presented methods as suggested in the introduction of this chapter. Hence, in the next subsection, we mainly deal with one of the other methods to implement new solutions for integrable nonlinear PDEs: the Dressing method. We base this presentation on [14, 32, 33].

3.1.2 (Classical) Darboux transformations *vs.* Dressing method

The common example introducing the classical Darboux transformation discovered by G. Darboux [16] is related to the one-dimensional, time independent Schrödinger equation

$$y_{xx}(x,\lambda) + (\lambda^2 - q(x))y(x,\lambda) = 0, \tag{3.1.13}$$

where the potential q is assumed to be real and vanishing sufficiently fast as $|x|\to\infty$. One should think of q as a substitute for the potential u or θ. Analogous to the direct scattering, we introduce the (time independent) Jost functions

$$y_{\pm}(x,\lambda) \sim e^{\pm i\lambda x}, \quad \text{as } x \to \infty.$$

If we take a real parameter p and an arbitrary constant κ, a solution of equation (3.1.13) is given by

$$f(x) = y_-(x, ip) + \kappa y_+(x, ip), \tag{3.1.14}$$

which is derived from the fundamental solution and is commonly known as the *intermediate wave function.* Then, Darboux showed that there is a mapping from the pair $\{q(x), y(x,\lambda)\}$ to a new pair $\{\tilde{q}(x), \tilde{y}(x,\lambda)\}$ satisfying equation (3.1.13).

Theorem 3.1.3 (Darboux)**.** *Let $f = f(x)$ be a particular solution of equation* (3.1.13) *for the value of the parameter p and $\sigma = f_x f^{-1}$. Consider the Darboux operator*

$$\mathcal{D} = \frac{-i}{\lambda + ip}(\partial_x - \sigma).$$

Given a pair $\{q(x), y(x,\lambda)\}$ satisfying equation (3.1.13)*, a new pair $\{\tilde{q}(x), \tilde{y}(x,\lambda)\}$ also satisfying the equation can be found by the definitions*

$$\begin{aligned} \tilde{q}(x) &= q(x) - 2\sigma_x, \\ \tilde{y}(x,\lambda) &= \mathcal{D}y(x,\lambda). \end{aligned} \tag{3.1.15}$$

Proof. Inserting $\tilde{y}(x,\lambda)$ and $\tilde{q}(x)$ into the left hand side of equation (3.1.13), omitting for the moment the dependencies and the quotient $-i/(\lambda+ip)$ which arises in each term, leads to

$$\tilde{y}_{xx} + (\lambda^2 - \tilde{q})\tilde{y} = y_{xxx} - (\sigma y)_{xx} + (\lambda^2 - q + 2\sigma_x)(y_x - \sigma y)$$

and after using the equation (3.1.13) for y_{xxx} and y_{xx} to cancel out terms

$$= y(q_x - \sigma_{xx} - 2\sigma_x\sigma) = 0,$$

where it can be shown with similar arguments that the bracket is zero, thereby providing a solution $\tilde{y}(x,\lambda)$ to equation (3.1.13) with the potential $\tilde{q}(x)$. □

Based on the definition for the new potential $\tilde{q}(x)$, we can derive the corresponding Bäcklund transformation. Identifying

$$\sigma_x(x) = \left(\frac{f_x(x)}{f(x)}\right)_x = \frac{f_{xx}(x)f(x) - f_x^2(x)}{f^2(x)} = q(x) + p^2 - \sigma^2(x),$$

we have by the definition of the potential (3.1.15) the following

$$\tilde{q}(x) + q(x) + 2p^2 = 2\sigma^2(x).$$

Therefore, $2\sigma(x) = \sqrt{2(\tilde{q}(x) + q(x) + 2p^2)}$ and by differentiating this equality with respect to x inserting again the definition of the new potential as $\sigma_x(x) = \frac{1}{2}(q(x) - \tilde{q}(x))$, we obtain after reordering of the terms

$$\tilde{q}_x(x) + q_x(x) = (q(x) - \tilde{q}(x))\sqrt{2(\tilde{q}(x) + q(x) + 2p^2)}. \tag{3.1.16}$$

Introducing the new fields

$$\tilde{w}(x) = \int_x^\infty \tilde{q}(y)\,\mathrm{d}y, \qquad w(x) = \int_x^\infty q(y)\,\mathrm{d}y \quad \text{and due to } \lim_{x\to\infty} \sigma(x) = p,$$

we obtain the commonly known standard Bäcklund transformation for the Korteweg–de Vries equation, thereby taking the form

$$\tilde{w}_x(x) + w_x(x) = \frac{1}{2}(w(x) - \tilde{w}(x))(\tilde{w}(x) - w(x) + 4p).$$

Further, equation (3.1.13) can be cast in matrix form as

$$Y_x(x,\lambda) = \begin{pmatrix} i\lambda & q(x) \\ 1 & -i\lambda \end{pmatrix} Y(x,\lambda), \tag{3.1.17}$$

where we say that the fundamental solution is given by

$$Y(x,\lambda) = \begin{pmatrix} Y_{11}(x,\lambda) & Y_{12}(x,\lambda) \\ Y_{21}(x,\lambda) & Y_{22}(x,\lambda) \end{pmatrix} = \begin{pmatrix} (y_+)_x(x,\lambda) + i\lambda y_+(x,\lambda) & (y_-)_x(x,\lambda) + i\lambda y_-(x,\lambda) \\ y_+(x,\lambda) & y_-(x,\lambda) \end{pmatrix}. \tag{3.1.18}$$

Therefore, the Darboux transformation described through Theorem 3.1.3 written in terms of matrix solutions $Y(x,\lambda)$ and $\widetilde{Y}(x,\lambda)$ of the spectral problem (3.1.17), where the potentials are given by $q(x)$ and $\tilde{q}(x)$, respectively, becomes

$$\widetilde{Y}(x,\lambda) = DY(x,\lambda),$$

where D is a 2×2 matrix commonly known as a Darboux matrix. In particular, the correspondence to the matrix in Subsection 3.1.1 is not surprising, since the definition $\widetilde{Y}(x,\lambda) = DY(x,\lambda)$ resembles the one we assumed in (3.1.1). Moreover, the derivatives are eliminated and only a pure matrix multiplication is applied with D taking the form

$$D = \frac{-i}{\lambda + ip} \begin{pmatrix} i\lambda - \sigma & \sigma^2 - p^2 \\ 1 & -i\lambda - \sigma \end{pmatrix}.$$

The Darboux matrix D is nonsingular for $\lambda \neq \pm i\lambda_1$ and we note that for our purposes it takes the convenient form

$$D = \left(\mathbb{1} - \frac{2ip}{\lambda + ip} P(x)\right)\sigma_3, \quad P(x) = \frac{1}{2}\begin{pmatrix} 1 - \frac{\sigma}{p} & p - \frac{\sigma^2}{p} \\ \frac{1}{p} & 1 + \frac{\sigma}{p} \end{pmatrix},$$

where P is a 2×2 projection matrix, i.e. $P^2 = P$, which can be calculated by hand with the expressions we already mentioned in this subsection.

With the fundamental solution (3.1.18), we can introduce two particular solutions of equation (3.1.17) at ip and $-ip$ obtained through

$$F(x) = Y^{(1)}(x,ip) + \kappa Y^{(2)}(x,ip), \quad G(x) = Y^{(1)}(x,-ip) + \bar{\kappa} Y^{(2)}(x,-ip).$$

Expressing the entries of $F(x) = (F_1(x), F_2(x))^\intercal$ in terms of the intermediate wave function $f(x)$, we have $F_1(x) = f_x(x) - pf(x)$ and $F_2(x) = f(x)$. Then, to be able to write $G(x) = (G_1(x), G_2(x))^\intercal$ in terms of the intermediate wave function, we need to demand that $\bar{\kappa} = \kappa$ such that the relation

$y_+(x,-ip) = y_-(x,ip)$ gives the appropriate coefficients for the entries of $G(x)$ in order to obtain $G_1(x) = f_x(x) + pf(x)$ and $G_2(x) = f(x)$. Taking the quotients

$$\Delta(x) = -\frac{F_2(x)}{F_1(x)} = -(\sigma - p)^{-1}, \quad \widetilde{\Delta}(x) = \frac{G_1(x)}{G_2(x)} = (\sigma + p),$$

the projector matrix can be written as

$$P(x) = \frac{1}{1 + \Delta(x)\widetilde{\Delta}(x)} \begin{pmatrix} 1 & \widetilde{\Delta}(x) \\ \Delta(x) & \Delta(x)\widetilde{\Delta}(x) \end{pmatrix}.$$

Ultimately, this enables us to obtain the new potential through

$$\tilde{q}(x) = q(x) - 2\widetilde{\Delta}_x(x) \tag{3.1.19}$$

via what is called the Dressing method.

Example 3.1.4. *Given the initial potential $q(x) \equiv 0$ and the parameter $p = 0$, the three equivalent methods* (3.1.15), (3.1.16) *and* (3.1.19) *can be used to derive the new potential $\tilde{q}(x) = 2(x+c)^2$.*

(i) For the Darboux transformation, we find that the intermediate wave function needs to be zero when differentiated twice with respect to x. Hence, $f(x) = c_1 + c_2 x$ and this leads to the result using (3.1.15) *when defining $c = c_1/c_2$.*

(ii) For the Bäcklund transformation inserting the assumptions into (3.1.16)*, we obtain the first-order nonlinear ordinary differential equation $\tilde{q}_x(x) = -\sqrt{2\tilde{q}^3(x)}$, which can be solved to obtain the same new potential, where the constant c comes from integrating.*

(iii) For the Dressing method, the intermediate wave function from (i) also gives the expression of $\Delta(x)$ and therefore with (3.1.19) *the same result.*

Hence, we have shown with this simple application that the two methods, the Darboux transformation and the Dressing method, are actually the same in the context of the one-dimensional, time independent Schrödinger equation (3.1.13), since the Darboux matrix D is just the Darboux operator $\mathcal{D}$ in matrix form. While we established this result, we also connected these methods to the Bäcklund transformation technique presented in the last subsection.

However, this consideration is only a representative introduction into the idea of the equivalence of these methods. The next step is therefore to apply a similar reasoning for the methods applied to the AKNS systems of the NLS and sG equation for which these circumstances are by no means that obvious. The presentation of the Dressing method for Lax systems follows primarily [21, 27], where in the case of the sG equation additional information [10] is necessary in order to give the complete picture in laboratory coordinates.

3.2 Dressing method for the Lax systems of the NLS and sG equation

Based on the last section, it is reasonable to think of the Dressing method as an extension of the Darboux transformation if it can be applied and when it comes to the construction of soliton solutions for any nonlinear PDE the expressions are usually used tantamount. In particular, it is the goal of this section to develop this very theory of constructing soliton solutions with the Darboux transformation for both the NLS and the sG equation.

As we worked out for the linear Schrödinger equation, first off we want to introduce the fundamental solution of the matrix equations (2.1.2). Therefore, the fundamental solution given by the composition of two linearly independent column solutions of the Lax system is

$$Y(t,x,\lambda) = (\psi_-^{(1)}, \psi_+^{(2)}) = \begin{pmatrix} [\psi_-]_{11} & [\psi_+]_{12} \\ [\psi_-]_{21} & [\psi_+]_{22} \end{pmatrix}.$$

Since the vectors $\psi_-^{(1)}$ and $\psi_+^{(2)}$ are initially taken as column vectors from the 2×2-matrices ψ_- and ψ_+, the notation above makes sense. Now, we want to introduce the function related to the intermediate wave function (3.1.14) regarding the Dressing method for the AKNS systems. For that, we take two complex numbers λ_1 and λ_1^* which belong to the upper and lower half-plane of the complex λ-plane, respectively; in particular, $\lambda_1, \lambda_1^* \notin \mathbb{R}$. Further, let u_0, v_0 be arbitrary constants so that a solution at $\lambda = \lambda_1$ is given by

$$\psi_1(t,x) = u_0 \psi_-^{(1)}(t,x,\lambda_1) + v_0 \psi_+^{(2)}(t,x,\lambda_1). \tag{3.2.1}$$

In fact, given this column solution of the initial Lax system at $\lambda = \lambda_1$, we take the intermediate wave function to be $\Delta(t,x) = [\psi_1]_2/[\psi_1]_1$ in order to write $D[1]$, the *one-fold dressing matrix*, in the following form

$$D[1] = \mathbb{1} + \frac{\lambda_1^* - \lambda_1}{\lambda - \lambda_1^*} P[1], \quad P[1] = \frac{1}{1 + |\Delta(t,x)|^2} \begin{pmatrix} 1 & \Delta^*(t,x) \\ \Delta(t,x) & |\Delta(t,x)|^2 \end{pmatrix}, \tag{3.2.2}$$

where $P[1](t,x)$ is a projector matrix depending on $\psi_1(t,x)$. Then as before, assume that we are given a pair of solutions $u[0]$ or $\theta[0]$ and $\psi[0]$ of the so-called *undressed* Lax system of the NLS equation (2.1.2) or the sG equation (2.2.3). This should remind of the given pair q and Y in the context of the time independent Schrödinger equation. However, due to possibilities of iteration in the Dressing method for AKNS systems, we denote the transformation instead of $\{q, Y\}$ to $\{\tilde{q}, \tilde{Y}\}$ by $\{u[0], \psi[0]\}$ to $\{u[1], \psi[1]\}$ or rather $\{\theta[0], \psi[0]\}$ to $\{\theta[1], \psi[1]\}$. Further, note that we reasonably use the Lax pairs $\mathcal{U}[0]$, $\mathcal{V}[0]$ and $\mathcal{U}[1]$, $\mathcal{V}[1]$ instead of $u[0]$ and $u[1]$ or rather $\mathbb{U}[0]$, $\mathbb{V}[0]$ and $\mathbb{U}[1]$, $\mathbb{V}[1]$ instead of $\theta[0]$ and $\theta[1]$. The solutions $u[0]$ and $\theta[0]$ associated with the undressed Lax system are commonly called *seed solutions*. Consequently, the gauge-like transformation

$$\psi[1] = D[1]\psi[0]$$

introduces a new solution of the Lax system

$$\begin{aligned} \psi[1]_x &= U[1]\psi[1], \\ \psi[1]_t &= V[1]\psi[1] \end{aligned}$$

with the corresponding Lax pair $U[1]$, $V[1]$. Moreover, the Lax pairs $U[0]$, $V[0]$ and $U[1]$, $V[1]$ are required to be structurally identical with updated potentials $u[1]$ for the NLS equation and $\theta[1]$ for the sG equation. Particularly, this condition implies that the one-fold dressing matrix $D[1]$, similar to the Darboux matrix in (3.1.2), satisfies

$$\begin{aligned} D[1]_x &= U[1]D[1] - D[1]U[0], \\ D[1]_t &= V[1]D[1] - D[1]V[0]. \end{aligned} \tag{3.2.3}$$

Therefore, we should have all the necessary information to introduce the new pair of solutions from the given pair as in Theorem 3.1.3. Indeed, the last step is to calculate or rather verify the existing reconstruction formulae for the new solutions $u[1]$ and $\theta[1]$. Even though, the idea is that this method can be iterated, we treat at first only the one-fold dressing matrix in the case of the NLS equation:

Proposition 3.2.1. *Let $\psi_1 = \psi_1(t,x)$ be a particular solution of the undressed Lax system* (2.1.2) *corresponding to the seed solution $u[0](t,x)$ for the NLS equation at the spectral parameter $\lambda = \lambda_1$ and $D[1]$ be the one-fold dressing matrix* (3.2.2)*. Now, given a solution $\psi[0](t,x,\lambda)$ to the undressed Lax system, a new pair satisfying the Lax system with updated Lax pair $\mathcal{U}[1]$ and $\mathcal{V}[1]$ associated to the new solution $u[1](t,x)$ can be found by*

$$\begin{aligned} \mathcal{Q}[1](t,x) &= \mathcal{Q}[0](t,x) - i(\lambda_1 - \lambda_1^*)[\sigma_3, P[1]], \\ \psi[1](t,x,\lambda) &= D[1](t,x,\lambda)\psi[0](t,x,\lambda). \end{aligned} \tag{3.2.4}$$

A similar summary as for the Bäcklund transformation can be given. The idea of the Dressing method for the NLS equation consists of the following: Given a seed solution $u[0]$ of the NLS equation, we construct the Lax pair $\mathcal{U}[0]$, $\mathcal{V}[0]$. Furthermore, we take the general solution $\psi[0]$ of the resulting Lax system consisting of $\psi[0]_x = \mathcal{U}[0]\psi[0]$ and $\psi[0]_t = \mathcal{V}[0]\psi[0]$. Additionally, define ψ_1 as in (3.2.1) and subsequently $P[1]$ as well as $D[1]$ as in (3.2.2). Then, let $\psi[1](t,x,\lambda) = D[1](t,x,\lambda)\psi[0](t,x,\lambda)$. With the definition

$$\mathcal{Q}[0](t,x) = \begin{pmatrix} 0 & u[0] \\ -u[0]^* & 0 \end{pmatrix},$$

adapted from Section 2.1, it follows that

$$\begin{pmatrix} 0 & u[1] \\ -u[1]^* & 0 \end{pmatrix} = \mathcal{Q}[1](t,x) = \mathcal{Q}[0](t,x) - i(\lambda_1 - \lambda_1^*)[\sigma_3, P[1]]$$

contains a new solution $u[1]$ to the NLS equation, from which a Lax pair $\mathcal{U}[1]$, $\mathcal{V}[1]$ may be constructed.

To prove this statement, we need to show that the definitions (3.2.4) are indeed enough to prove that the one-fold dressing matrix satisfies relations (3.2.3). Therefore, we give an equivalent expression [27] for the one-fold dressing matrix

$$D[1](t,x,\lambda) = \frac{1}{\lambda - \lambda_1^*}(\lambda\mathbb{1} - S(t,x)), \tag{3.2.5}$$

where it can be shown that the matrix $S(t,x)$ can be written as the product $S(t,x) = H\Lambda H^{-1}$, where Λ is given as a diagonal matrix with entries λ_1, λ_1^* and $H(t,x)$ consists of the column vectors $\psi_1(t,x)$, $\varphi_1(t,x) = -i\sigma_2\psi_1^*(t,x)$. It should be noted that due to the symmetry relation given in Lemma 2.1.2, $-i\sigma_2\psi_1^*(t,x)$ is in fact a solution of the Lax system at the spectral parameter $\lambda = \lambda_1^*$. Expressing $D[1](t,x,\lambda)$ in this manner simplifies the calculations in the proof substantially. However before proving Proposition 3.2.1, let us mention some useful properties we can derive from equality (3.2.5):

Lemma 3.2.2. *Let $\psi_1 = \psi_1(t,x)$ be a particular solution of the undressed Lax system* (2.1.2) *corresponding to the seed solution $u[0](t,x)$ for the NLS equation at the spectral parameter $\lambda = \lambda_1$*

and $D[1]$ *be the one-fold dressing matrix* (3.2.2)*. Then,*

$$D[1]^{-1} = \frac{1}{\lambda - \lambda_1}(\lambda \mathbb{1} - S^\dagger), \tag{3.2.6}$$

$$S_x = -i[\sigma_3, S]S + [\mathcal{Q}[0], S], \tag{3.2.7}$$

$$S_t = -2i[\sigma_3, S]S^2 + 2[\mathcal{Q}[0], S]S + [\mathcal{V}[0]\big|_{\lambda=0}, S], \tag{3.2.8}$$

$$\mathbb{1} = \frac{\lambda^2 \mathbb{1} - \lambda(S + S^\dagger) + SS^\dagger}{(\lambda - \lambda_1^*)(\lambda - \lambda_1)}. \tag{3.2.9}$$

In particular, we have $\det(D[1]) = \frac{\lambda - \lambda_1}{\lambda - \lambda_1^*}$ *independent of* t *and* x *as well as*

$$S^\dagger = H\Lambda^* H^{-1} \quad \textit{and} \quad S^{-1} = H\Lambda^{-1}H^{-1}. \tag{3.2.10}$$

Proof. First off, equations (3.2.10) can be easily calculated

$$S^\dagger = (H\Lambda H^{-1})^\dagger = (H^\dagger)^{-1}\Lambda^\dagger H^\dagger = H\Lambda^* H^{-1},$$

where $H^\dagger = H^{-1}\det(H)$ and $\Lambda^\dagger = \mathrm{diag}(\lambda_1^*, \lambda_1) = \Lambda^*$ as well as

$$S^{-1} = (H\Lambda H^{-1})^{-1} = H\Lambda^{-1}H^{-1}.$$

Then, the identity (3.2.9) is implied, due to

$$\begin{aligned} S + S^\dagger &= H(\Lambda + \Lambda^*)H^{-1} = H(\lambda_1 + \lambda_1^*)\mathbb{1}H^{-1} = (\lambda_1 + \lambda_1^*)\mathbb{1}, \\ SS^\dagger &= H\Lambda H^{-1}H\Lambda^* H^{-1} = H|\lambda_1|^2\mathbb{1}H^{-1} = |\lambda_1|^2\mathbb{1}. \end{aligned}$$

The multiplication of the matrix $D[1]^{-1}$ defined in (3.2.6) with $D[1]$ as defined in (3.2.5) leads exactly to the right hand side of (3.2.9), thereby proving that $D[1]^{-1}$ is indeed the inverse of $D[1]$. For the derivatives of S with respect to t and x it is useful to consider the respective derivative of H first, since

$$S_t = H_t\Lambda H^{-1} - H\Lambda H^{-1}H_tH^{-1} = H_tH^{-1}H\Lambda H^{-1} - H\Lambda H^{-1}H_tH^{-1} = [H_tH^{-1}, S]$$

and the same for the x derivative. In particular, since the column entries of H are solutions of the Lax system (2.1.2) for $\lambda = \lambda_1$ and $\lambda = \lambda_1^*$, we obtain the following

$$\begin{aligned} H_t &= ((\psi_1)_t, (\varphi_1)_t) \\ &= ((-2i\lambda_1^2\sigma_3 + 2\lambda_1\mathcal{Q}[0] + \mathcal{V}[0]\big|_{\lambda=0})\psi_1, (-2i(\lambda_1^*)^2\sigma_3 + 2\lambda_1^*\mathcal{Q}[0] + \mathcal{V}[0]\big|_{\lambda=0})\varphi_1) \\ &= -2i\sigma_3 H\Lambda^2 + 2\mathcal{Q}[0]H\Lambda + \mathcal{V}[0]\big|_{\lambda=0}H. \end{aligned}$$

Therefore, equation (3.2.8) follows with $H_tH^{-1} = -2i\sigma_3 S^2 + 2Q[0]S + \mathcal{V}[0]\big|_{\lambda=0}$. Further, we have

$$\begin{aligned} H_x &= ((\psi_1)_x, (\varphi_1)_x) \\ &= ((-i\lambda_1\sigma_3 + \mathcal{Q}[0])\psi_1, (-i\lambda_1^*\sigma_3 + \mathcal{Q}[0])\varphi_1) \\ &= -i\sigma_3 H\Lambda + 2\mathcal{Q}[0]H \end{aligned}$$

and therefore, we find $H_xH^{-1} = -i\sigma_3 S + \mathcal{Q}[0]$ and also (3.2.7). Finally,

$$\begin{aligned}
\det(D[1]) &= \det\Big(\mathbb{1} + \frac{\lambda_1^* - \lambda_1}{\lambda - \lambda_1^*}\frac{1}{1+|\Delta|^2}\begin{pmatrix} 1 & \Delta^* \\ \Delta & |\Delta|^2 \end{pmatrix}\Big) \\
&= \frac{(1+|\Delta|^2)^{-2}}{(\lambda-\lambda_1^*)^2}\det\begin{pmatrix} (\lambda-\lambda_1)+(\lambda-\lambda_1^*)|\Delta|^2 & (\lambda_1^*-\lambda_1)\Delta^* \\ (\lambda_1^*-\lambda_1)\Delta & (\lambda-\lambda_1^*)+(\lambda-\lambda_1)|\Delta|^2 \end{pmatrix} \\
&= \frac{(1+|\Delta|^2)^{-2}}{(\lambda-\lambda_1^*)^2}(\lambda-\lambda_1^*)(\lambda-\lambda_1)(1+|\Delta|^2)^2 \\
&= \frac{\lambda-\lambda_1}{\lambda-\lambda_1^*},
\end{aligned}$$

where we only give the important steps. □

Note that most properties shown in Lemma 3.2.2 hold in theory not only for the NLS equation, but also among others for the sG equation, except for the equalities (3.2.7) and (3.2.8), where the Lax system of the specific equation needs to be utilized.

With that, we can give a comprehensible proof to Proposition 3.2.1:

Proof. By relations (3.2.3), we find

$$\mathcal{U}[1] = D[1]_x D[1]^{-1} + D[1]\mathcal{U}[0]D[1]^{-1}$$

and by Lemma 3.2.2, this is equal to

$$\begin{aligned}
&= \frac{1}{(\lambda-\lambda_1^*)(\lambda-\lambda_1)}(-i\lambda^3\sigma_3 + i\lambda^2\sigma_3 S^\dagger + \lambda^2\mathcal{Q}[0] + i\lambda^2 S\sigma_3 \\
&\quad - \lambda S_x - \lambda\mathcal{Q}[0]S^\dagger - i\lambda S\sigma_3 S^\dagger - \lambda S\mathcal{Q}[0] + S_x S^\dagger + SQ[0]S^\dagger).
\end{aligned} \tag{3.2.11}$$

Here, the trick to structure the terms in the brackets of (3.2.11) is to sort them by powers of λ and to identify the important terms utilizing identity (3.2.9) so that we can be write the terms as

$$(-i\lambda\sigma_3 + \mathcal{Q}[0] + i[S,\sigma_3])(\lambda^2\mathbb{1} - \lambda(S+S^\dagger) + SS^\dagger).$$

Hence, by the identity and noting that $S = \lambda_1^*\mathbb{1} - (\lambda_1^* - \lambda_1)P[1]$, the definition (3.2.4) gives $\mathcal{U}[1] = -i\lambda\sigma_3 + \mathcal{Q}[1]$.

Further, we have for the t part of the relations (3.2.3) the following

$$\mathcal{V}[1] = D[1]_t D[1]^{-1} + D[1]\mathcal{V}[0]D[1]^{-1}$$

and by Lemma 3.2.2, this is equal to

$$\begin{aligned}
&= \frac{1}{(\lambda-\lambda_1^*)(\lambda-\lambda_1)}(-2i\lambda^4\sigma_3 + 2i\lambda^3\sigma_3 S^\dagger + 2\lambda^3\mathcal{Q}[0] + 2i\lambda^3 S\sigma_3 \\
&\quad - 2\lambda^2\mathcal{Q}[0]S^\dagger + \lambda^2\mathcal{V}[0]\big|_{\lambda=0} - 2i\lambda^2 S\sigma_3 S^\dagger - 2\lambda^2 S\mathcal{Q}[0] \\
&\quad - \lambda S_t - \lambda\mathcal{V}[0]\big|_{\lambda=0}S^\dagger + 2\lambda S\mathcal{Q}[0]S^\dagger - \lambda S\mathcal{V}[0]\big|_{\lambda=0} + S_t S^\dagger + S\mathcal{V}[0]\big|_{\lambda=0}S^\dagger).
\end{aligned}$$

This leads after a lengthy calculation with the same ideas as for the x part to

$$\mathcal{V}[1] = -2i\sigma_3\lambda^2 + 2\lambda(\mathcal{Q}[0] + i[S, \sigma_3]) + (\mathcal{V}[0]\big|_{\lambda=0} - 2[S, \mathcal{Q}[0]] + 2i[S, \sigma_3]S).$$

Due to the definition of $\mathcal{Q}[1]$, we see that the coefficient of λ is indeed $2\mathcal{Q}[1]$. The coefficient of zero-th power needs to be verified explicitly. With the definition (3.2.4), we obtain

$$\begin{aligned}\mathcal{V}[1]\big|_{\lambda=0} &= i\sigma_3(\mathcal{Q}[1]_x - (\mathcal{Q}[1])^2)\\ &= \sigma_3(i\mathcal{Q}[0]_x - [S_x, \sigma_3] - i(\mathcal{Q}[0])^2 + \mathcal{Q}[0][S, \sigma_3] + [S, \sigma_3]\mathcal{Q}[0] + i[S, \sigma_3][S, \sigma_3]).\end{aligned}$$

Noting that $\mathcal{V}[0]\big|_{\lambda=0} = i\sigma_3(\mathcal{Q}[0]_x - (\mathcal{Q}[0])^2)$, we need to prove that the remaining terms of $\mathcal{V}[1]\big|_{\lambda=0}$ are equal to the remaining terms of the 0-th power coefficient: Combining the first term of $-\sigma_3[S_x, \sigma_3]$, when we insert S_x, see (3.2.7), with $i[S, \sigma_3][S, \sigma_3]$, we obtain

$$i\sigma_3([[\sigma_3, S]S, \sigma_3] + [S, \sigma_3][S, \sigma_3]) = 2i\sigma_3(\sigma_3 S\sigma_3 S - SS) = 2i[S, \sigma_3]S$$

and combining the remaining term of $-\sigma_3[S_x, \sigma_3]$ with $\mathcal{Q}[0][S, \sigma_3] + [S, \sigma_3]\mathcal{Q}[0]$, we derive

$$\sigma_3(-[[\mathcal{Q}[0], S], \sigma_3] + \mathcal{Q}[0][S, \sigma_3] + [S, \sigma_3]\mathcal{Q}[0]) = 2(\mathcal{Q}[0]S - S\mathcal{Q}[0]) = -2[S, \mathcal{Q}[0]],$$

where we use the fact that for the off-diagonal matrix $\mathcal{Q}[0]$ the following equality $\sigma_3\mathcal{Q}[0] = -\mathcal{Q}[0]\sigma_3$ holds. □

It is worth noting that the method of constructing a new pair of solutions by Proposition 3.2.1 is indeed—analogous to Theorem 3.1.3—only relying on the intermediate wave function and not as relations (3.2.3) might suggest on the solutions $\mathcal{U}[1]$, $\mathcal{V}[1]$ and therefore on $u[1]$ which is not known in the beginning.

Since Proposition 3.2.1 holds for an arbitrary seed solution $u[0](t, x)$, the method can be iterated with distinct spectral parameters $\lambda = \lambda_j$, $j = 1 \dots N$, such that $\lambda_j \neq \lambda_k^*$, $j, k = 1, \dots, N$. As a consequence, the N column solutions $\psi_j(t, x)$ of the undressed Lax system (2.1.2) corresponding to $\lambda = \lambda_j$ are linearly independent. After the first iteration the updated particular solution which is necessary to apply Proposition 3.2.1 is given by $\psi_2[1] = D[1]\big|_{\lambda=\lambda_2}\psi_2$ and so on. Therefore, the *N-fold dressing matrix* $D[N]$ is given by the iteration of $D[1]$ in the following sense

$$D[N] = \Big(\mathbb{1} + \frac{\lambda_N^* - \lambda_N}{\lambda - \lambda_N^*}P[N]\Big)\cdots\Big(\mathbb{1} + \frac{\lambda_1^* - \lambda_1}{\lambda - \lambda_1^*}P[1]\Big), \tag{3.2.12}$$

where $P[j]$ are projector matrices defined by

$$P[j] = \frac{\psi_j[j-1]\psi_j^\dagger[j-1]}{\psi_j^\dagger[j-1]\psi_j[j-1]}, \quad \psi_j[j-1] = D[j-1]\big|_{\lambda=\lambda_j}\psi_j. \tag{3.2.13}$$

To summarize, we then have

Proposition 3.2.3 (Gu, Hu & Zhou, [27]). *Let $\psi_j = \psi_j(t, x)$, $j = 1, \dots, N$, be particular solutions of the undressed Lax system* (2.1.2) *corresponding to the seed solution $u[0](t, x)$ for the NLS equation at pairwise distinct spectral parameters $\lambda = \lambda_j$ and $D[N]$ be the dressing matrix* (3.2.12). *Now, given a solution $\psi[0](t, x, \lambda)$ to the undressed Lax system* (2.1.2), *a new pair satisfying the Lax system with updated Lax pair $\mathcal{U}[N]$ and $\mathcal{V}[N]$ associated to the new solution $u[N](t, x)$ can be found by*

$$\begin{aligned}\mathcal{Q}[N](t, x) &= \mathcal{Q}[0](t, x) - i\sum_{j=1}^{N}(\lambda_j - \lambda_j^*)[\sigma_3, P[j]],\\ \psi[N](t, x, \lambda) &= D[N](t, x, \lambda)\psi[0](t, x, \lambda).\end{aligned} \tag{3.2.14}$$

In particular, the pairwise distinct spectral parameters λ_j and therefore the linear independence of the solutions ψ_j ensuress that $\psi_j[j-1] = D[j-1]\big|_{\lambda=\lambda_j}\psi_j$ is not zero, $j = 1, \dots, N$. As before, the spectral analogue of (3.2.14) is given by

$$\begin{aligned} D[N]_x &= \mathcal{U}[N]D[N] - D[N]\mathcal{U}[0], \\ D[N]_t &= \mathcal{V}[N]D[N] - D[N]\mathcal{V}[0] \end{aligned} \tag{3.2.15}$$

and connects the undressed Lax system with the following Lax system

$$\begin{aligned} \psi[N]_x &= \mathcal{U}[N]\psi[N], \\ \psi[N]_t &= \mathcal{V}[N]\psi[N]. \end{aligned}$$

Once again, this method can also be applied to the sG equation, see [10, 21, 43], where we waive the application of the one-fold dressing matrix and immediately state the proposition in terms of an N-fold dressing matrix.

Proposition 3.2.4. *Let $\psi_j = \psi_j(t,x)$, $j = 1, \dots, N$, be particular solutions of the undressed Lax system* (2.2.3) *corresponding to the seed solution $\theta[0](t,x) \equiv 0$ of the sG equation at spectral parameters $\lambda = \lambda_j$ and $D[N]$ be the corresponding N-fold dressing matrix* (3.2.12)*. Now, given a solution $\psi[0](t,x,\lambda)$ of the undressed Lax system* (2.2.3)*, a new pair satisfying the Lax system with updated Lax pair $\mathbb{U}[N]$ and $\mathbb{V}[N]$ associated to the new solution $\theta[N]$ can be found by*

$$\begin{aligned} e^{i\frac{\theta[N]}{2}\sigma_1} &= D[N]\big|_{\lambda=0}\sigma_3^{N_s}, \\ \psi[N](t,x,\lambda) &= D[N](t,x,\lambda)\psi[0](t,x,\lambda). \end{aligned} \tag{3.2.16}$$

Proof. We follow the ideas given in the proof in [10]. For $\theta[0](t,x) \equiv 0$, we have $\mathbb{U}[0] = -i(\lambda - \lambda^{-1})/4\sigma_3$ and $\mathbb{V}[0] = i(\lambda + \lambda^{-1})/4\sigma_3$. By (3.2.16), we therefore obtain

$$\begin{aligned} (\psi[N])_x(\psi[N])^{-1} &= D[N]_x D[N]^{-1} - \frac{i(\lambda - \lambda^{-1})}{4} D[N]\sigma_3 D[N]^{-1}, \\ (\psi[N])_t(\psi[N])^{-1} &= D[N]_t D[N]^{-1} + \frac{i(\lambda + \lambda^{-1})}{4} D[N]\sigma_3 D[N]^{-1}. \end{aligned}$$

Now, if we expand $D[N]$ in the limit of $|\lambda| \to \infty$:

$$D[N](t,x,\lambda) = \mathbb{1} + \lambda^{-1}\Sigma(t,x) + \mathcal{O}(\lambda^{-2}),$$

we can derive the limit behavior of $(\psi[N])_x(\psi[N])^{-1}$. We have

$$(\psi[N])_x(\psi[N])^{-1}\big|_{\lambda=0} = \frac{i}{4\lambda} D[N]\big|_{\lambda=0}\sigma_3\big(D[N]\big|_{\lambda=0}\big)^{-1} + D[N]_x\big|_{\lambda=0}\big(D[N]\big|_{\lambda=0}\big)^{-1} \tag{3.2.17}$$

as $|\lambda| \to 0$ and

$$(\psi[N])_x(\psi[N])^{-1} = -\frac{i\lambda}{4}\sigma_3 - \frac{i}{4}[\Sigma(t,x), \sigma_3] + \mathcal{O}(\lambda^{-1}) \quad \text{as } |\lambda| \to \infty.$$

Therefore, the difference

$$(\psi[N])_x(\psi[N])^{-1} - \Big(-\frac{i\lambda}{4}\sigma_3 - \frac{i}{4}[\Sigma(t,x), \sigma_3] + \frac{i}{4\lambda} D[N]\big|_{\lambda=0}\sigma_3\big(D[N]\big|_{\lambda=0}\big)^{-1}\Big)$$

is constructed to be analytic in the whole complex λ-plane. Further taking the limit $|\lambda| \to \infty$, we have that the difference is zero and therefore by Liouville's theorem it is equal to zero for all λ. Repeating the same steps for the t part, we obtain

$$(\psi[N])_x(\psi[N])^{-1} = -\frac{i\lambda}{4}\sigma_3 - \frac{i}{4}[\Sigma(t,x),\sigma_3] + \frac{i}{4\lambda}D[N]\big|_{\lambda=0}\sigma_3\big(D[N]\big|_{\lambda=0}\big)^{-1},$$
$$(\psi[N])_t(\psi[N])^{-1} = \frac{i\lambda}{4}\sigma_3 + \frac{i}{4}[\Sigma(t,x),\sigma_3] + \frac{i}{4\lambda}D[N]\big|_{\lambda=0}\sigma_3\big(D[N]\big|_{\lambda=0}\big)^{-1}$$

so that subtracting the first line from the second results in

$$(\psi[N])_t(\psi[N])^{-1} - (\psi[N])_x(\psi[N])^{-1} = \frac{i\lambda}{2}\sigma_3 + \frac{i}{2}[\Sigma(t,x),\sigma_3]. \tag{3.2.18}$$

Again, taking the limit $|\lambda| \to 0$ and using equation (3.2.17) and the corresponding equation of the t part, we ultimately derive

$$\big(D[N]_t\big|_{\lambda=0} - D[N]_x\big|_{\lambda=0}\big)\big(D[N]\big|_{\lambda=0}\big)^{-1} = \frac{i}{2}[\Sigma(t,x),\sigma_3].$$

Therefore, this enables us to give an expression of $\mathbb{U}[N]$ and $\mathbb{V}[N]$ only relying on the dressing matrix evaluated at $\lambda = 0$:

$$\begin{aligned}\mathbb{U}[N] = (\psi[N])_x(\psi[N])^{-1} &= -\frac{i\lambda}{4}\sigma_3 - \frac{1}{2}\big(D[N]_t\big|_{\lambda=0} - D[N]_x\big|_{\lambda=0}\big)\big(D[N]\big|_{\lambda=0}\big)^{-1} \\ &\quad + \frac{i}{4\lambda}D[N]\big|_{\lambda=0}\sigma_3\big(D[N]\big|_{\lambda=0}\big)^{-1}, \\ \mathbb{V}[N] = (\psi[N])_t(\psi[N])^{-1} &= \frac{i\lambda}{4}\sigma_3 + \frac{1}{2}\big(D[N]_t\big|_{\lambda=0} - D[N]_x\big|_{\lambda=0}\big)\big(D[N]\big|_{\lambda=0}\big)^{-1} \\ &\quad + \frac{i}{4\lambda}D[N]\big|_{\lambda=0}\sigma_3\big(D[N]\big|_{\lambda=0}\big)^{-1}.\end{aligned}$$

Assuming that $\mathbb{U}[N]$ and $\mathbb{V}[N]$ are of the same form as the Lax pair (2.2.2) for the sG equation, we have that the following equalities should hold

$$\frac{i}{2}(\theta[N]_t - \theta[N]_x)\sigma_1 = \big(D[N]_t\big|_{\lambda=0} - D[N]_x\big|_{\lambda=0}\big)\big(D[N]\big|_{\lambda=0}\big)^{-1}, \tag{3.2.19}$$
$$e^{i\frac{\theta[N]}{2}\sigma_1}\sigma_3 e^{-i\frac{\theta[N]}{2}\sigma_1} = D[N]\big|_{\lambda=0}\sigma_3\big(D[N]\big|_{\lambda=0}\big)^{-1}, \tag{3.2.20}$$

where we use that the matrix coefficient of λ^{-1} in $\mathbb{U}[N]$ can be written in the form $e^{i\frac{\theta[N]}{2}\sigma_1}\sigma_3 e^{-i\frac{\theta[N]}{2}\sigma_1}$. Equation (3.2.20) implies, while keeping track of the determinants of both sides, that necessarily

$$e^{i\frac{\theta[N]}{2}\sigma_1} = D[N]\big|_{\lambda=0}\sigma_3^{N_s}, \tag{3.2.21}$$

where $\det(D[N]) = (-1)^N = (-1)^{N_s}$. Then, inserting this into equation (3.2.19), we see that (3.2.21) is, in fact, sufficient to uphold both equalities, which concludes the proof. $\square$

There are several important properties of the dressing matrix. Since it has been studied thoroughly, the literature on the method is extensive. In the following, we want to mention some of these results in order to convey a better understanding of the transformation. First, let us comment once more on why we imposed that the spectral parameters $\lambda_1, \ldots, \lambda_N$, $\lambda_j \neq \lambda_k^*$, $j, k = 1, \ldots, N$, have to be pairwise distinct.

Proposition 3.2.5. *Under the condition that the spectral parameters $\lambda_1, \ldots, \lambda_N$ and $\lambda_1^*, \ldots, \lambda_N^*$ are distinct as well as in the complex plane without the real line, i.e. $\mathbb{C} \setminus \mathbb{R}$, we have that the corresponding solutions of the Lax system* (2.1.2) *of the NLS equation are linearly independent.*

Proof. We prove this by contradiction. Therefore, assume that the solutions $\psi_1, \ldots, \psi_N$ and $\varphi_1, \ldots, \varphi_N$ corresponding to the spectral parameters $\lambda_1, \ldots, \lambda_N$ and $\lambda_1^*, \ldots, \lambda_N^*$ are linearly dependent. Hence, we have that there exist $2N-1$ constants $c_1, \ldots, c_{2N-1}$ so that ψ_1 can be written as a linear combination

$$\psi_1 = \sum_{j=1}^{N} c_{2j-1}\varphi_j + \sum_{j=2}^{N} c_{2j-2}\psi_j. \tag{3.2.22}$$

By the x part of the Lax system (2.1.2), we can write the x derivative of ψ_1 in two ways:

$$(-i\lambda_1\sigma_3 + \mathcal{Q})\psi_1 = \sum_{j=1}^{N} c_{2j-1}(-i\lambda_j^*\sigma_3 + \mathcal{Q})\varphi_j + \sum_{j=2}^{N} c_{2j-2}(-i\lambda_j\sigma_3 + \mathcal{Q})\psi_j.$$

For the term involving the matrix $\mathcal{Q}$ multiplied with the solutions, we obtain equality on both sides and inserting an additive zero, we find that for the remaining terms

$$-i\lambda_1\sigma_3\psi_1 = -i\lambda_N^*\sigma_3\Big(\sum_{j=1}^{N} c_{2j-1}\varphi_j + \sum_{j=2}^{N} c_{2j-2}\psi_j\Big) + i\sum_{j=1}^{N-1}(\lambda_N^* - \lambda_j^*)c_{2j-1}\varphi_j + \sum_{j=2}^{N} c_{2j-2}(\lambda_N^* - \lambda_j)\psi_j.$$

In particular, the linear combination (3.2.22) implies that

$$i\sigma_3\Big((\lambda_1 - \lambda_N^*)\psi_1 + \sum_{j=1}^{N-1}(\lambda_N^* - \lambda_j^*)c_{2j-1}\varphi_j + \sum_{j=2}^{N} c_{2j-2}(\lambda_N^* - \lambda_j)\psi_j\Big) = 0.$$

Then, by assumption $\lambda_1, \ldots, \lambda_N, \lambda_1^*, \ldots, \lambda_N^*$ are distinct and because of that it is possible to write ψ_1 as a linear combination of only $\psi_2, \ldots, \psi_N, \varphi_1, \ldots, \varphi_{N-1}$. Repeating this step additional $2N-3$ times to also eliminate $\psi_2, \ldots, \psi_N, \varphi_2, \ldots, \varphi_{N-1}$, we derive that ψ_1 and φ_1 are linearly dependent which gives a contradiction.

Moreover, the $2N \times 2N$-matrix

$$\begin{pmatrix} H_1 & H_2 & \cdots & H_N \\ H_1\Lambda_1 & H_2\Lambda_2 & \cdots & H_N\Lambda_N \\ \vdots & \vdots & & \vdots \\ H_1\Lambda_1^{N-1} & H_2\Lambda_2^{N-1} & \cdots & H_N\Lambda_N^{N-1} \end{pmatrix}$$

is non-degenerate with the assumptions declared in Proposition 3.2.5, due to the linear independence of the column vectors of $H_1, \ldots, H_N$. □

Remark 3.2.6. Proposition 3.2.5 also applies to the sG equation. For the proof one would follow the same steps while considering the difference of the x and t part of the Lax system (2.2.3) such that one essentially deals with the Lax system of the sG equation in light-cone coordinates. Regarding these coordinates the x part of the Lax pair of the sG equation is closely related to the x part of the Lax pair of the NLS equation.

The second result, we want to mention, is bound to the application of higher order dressing matrices. It answers the question whether it makes a difference changing the order of the solutions ψ_j and corresponding spectral parameter $\lambda = \lambda_j$, $j = 1, \dots, N$, in the process of determining the dressing matrix. Therefore, it is sufficient to investigate what happens if we apply a two-fold dressing matrix

$$D[2] = D_2' D_1 = \frac{1}{(\lambda - \lambda_1^*)(\lambda - \lambda_2^*)}(\lambda \mathbb{1} - S_2')(\lambda \mathbb{1} - S_1) = \frac{1}{(\lambda - \lambda_1^*)(\lambda - \lambda_2^*)}(\lambda \mathbb{1} - S_1')(\lambda \mathbb{1} - S_2) = D_1' D_2$$

from a pair of solutions ψ_1 and ψ_2 in connection with the corresponding spectral parameters λ_1 and λ_2, see Figure 3.1. As before, we find $S_j = H_j \Lambda_j H_j^{-1}$ and H_j consisting of the column vectors ψ_j, φ_j and Λ_j as a diagonal matrix $\operatorname{diag}(\lambda_j, \lambda_j^*)$ for $j = 1, 2$.

Theorem 3.2.7 (Theorem of permutability, Gu, Hu & Zhou, [27]). *Suppose*

$$\det \begin{pmatrix} H_1 & H_2 \\ H_1 \Lambda_1 & H_2 \Lambda_2 \end{pmatrix} \neq 0, \tag{3.2.23}$$

then the two-fold dressing matrix is symmetric to S_1 and S_2.

Proof. With the process of updating the solution $\psi_2[1] = D[1]\big|_{\lambda=\lambda_2} \psi_2$ in mind, it follows that ψ_2, φ_2 are transformed to $(\lambda_2 \mathbb{1} - S_1)\psi_2$, $(\lambda_2 \mathbb{1} - S_1)\varphi_2$ implying the transformation $H_2' = (S_2 - S_1)H_2$. Then again, $S_2' = (S_2 - S_1)S_2(S_2 - S_1)^{-1}$, where $S_2 - S_1$ is non-degenerate due to (3.2.23). Hence, changing the order corresponds to first applying the unmodified S_2 and subsequently $S_1' = (S_1 - S_2)S_1(S_1 - S_2)^{-1}$ in the dressing matrix and the symmetry stands for

$$(\lambda \mathbb{1} - S_2')(\lambda \mathbb{1} - S_1) = (\lambda \mathbb{1} - S_1')(\lambda \mathbb{1} - S_2), \tag{3.2.24}$$

where the left and right hand side can be calculated as

$$\lambda^2 - \lambda(S_2^2 - S_1^2)(S_2 - S_1)^{-1} + (S_2 - S_1)S_2(S_2 - S_1)^{-1}S_1$$

and the same with S_1 and S_2 interchanged. Then, the coefficients of the two resulting polynomials with respect to λ can be compared. The equality for the first order coefficient is straightforward and for the zero-th order coefficient, one uses basic matrix multiplication rules $S_2(S_2 - S_1)^{-1}S_1 = (S_1^{-1} - S_2^{-1})^{-1}$ and $S_1(S_1 - S_2)^{-1}S_2 = (S_2^{-1} - S_1^{-1})^{-1}$. □

This important property of permutability in the Dressing method can be summarized in a Bianchi diagram, see Figure 3.1. Further, the symmetries of the Lax systems imply another interesting property.

Remark 3.2.8. In the context of the NLS and sG equation, the dressing matrix admits the inverse

$$D[N]^{-1}(t, x, \lambda) = D[N]^\dagger(t, x, \lambda^*).$$

For $N = 1$, the result is already stated in Lemma 3.2.2 and written in terms of (3.2.2), we note that $P[1]^\dagger = P[1]$. Thus, this idea can easily be generalized to each factor of the N-fold dressing matrix $D[N]$ which means

$$\left(\mathbb{1} + \frac{\lambda_j^* - \lambda_j}{\lambda^* - \lambda_j^*} P[j]\right)^\dagger \left(\mathbb{1} + \frac{\lambda_j^* - \lambda_j}{\lambda - \lambda_j^*} P[j]\right) = \mathbb{1}, \quad \text{for } j = 1, \dots, N$$

and therefore $D[N](t, x, \lambda) D[N]^{-1}(t, x, \lambda) = D[N](t, x, \lambda) D[N]^\dagger(t, x, \lambda^*) = \mathbb{1}$. Then, the determinant of the N-fold dressing matrix can similarly be generalized, since for each factor of the product an analogous calculation as for $\det D[1] = \frac{\lambda - \lambda_1}{\lambda - \lambda_1^*}$ in Lemma 3.2.2 can be applied.

$$\begin{tikzcd}
& (u[1],\psi[1])_1 \arrow[dr, "D_2'"] & \\
(u[0],\psi[0]) \arrow[ur, "D_1"] \arrow[dr, "D_2"'] & & (u[2],\psi[2])_{12} = (u[2],\psi[2])_{21} \\
& (u[1],\psi[1])_2 \arrow[ur, "D_1'"'] &
\end{tikzcd}$$

Fig. 3.1. Permutability $D_2'D_1 = D_1'D_2$ defined as in (3.2.24). Here, the prime and the indices correspond to the order in which the dressing matrix is applied.

Remark 3.2.9. The determinant of the one-fold dressing matrix can be generalized to

$$D[N] = \prod_{k=1}^{N} \frac{\lambda - \lambda_k}{\lambda - \lambda_k^*},$$

where $\lambda_1, \dots, \lambda_N$ are the spectral parameters used in the dressing method.

Consequently, we have seen that the Dressing method is a powerful method to introduce specific solutions into the framework of AKNS systems for the NLS and sG equation. So now, let us take a closer look at the specificity of these solutions. As for the inverse scattering method, distinct spectral parameters $\lambda_1, \dots, \lambda_N$ are introduced representing simple eigenvalues of the function $u[N]$. In the context of the Dressing method, these simple eigenvalues arise in the algebraic construction of the N-fold dressing matrix $D[N]$, providing zeros and associated kernel vectors of $\prod_{j=1}^{N}(\lambda - \lambda_j^*)D[N](t,x,\lambda)$, which is of importance later on. Nevertheless, since it seems to be the case that part of the scattering data arise in the context of the Dressing method, we want to investigate further if it is possible to give a complete description.

3.3 Change of scattering data under the Dressing method

With scattering data $(\rho, \{\lambda_j, C_j\}_{j=1}^N)$, $\lambda_j \in \mathbb{C}_+$ for all $j = 1, \dots, N$, we want to give the relevant information needed to retrace the change of scattering data under the Dressing method. As in the inverse scattering method, it is of importance that the solution and its derivative with respect to x is sufficiently fast decaying as $|x| \to \infty$. Then, assume we are given a spectral parameter $\lambda_0 \in \mathbb{C}_+ \setminus \{\lambda_1, \dots, \lambda_N\}$ and a column solution of the undressed Lax system

$$\begin{aligned}
\psi_0(t,x) &= u_0 \psi_-^{(1)}(t,x,\lambda_0) + v_0 \psi_+^{(2)}(t,x,\lambda_0) \\
&= u_0 \widehat{\psi}_-^{(1)}(t,x,\lambda_0) e^{-i\Theta(t,x,\lambda_0)} + v_0 \widehat{\psi}_+^{(2)}(t,x,\lambda_0) e^{i\Theta(t,x,\lambda_0)},
\end{aligned}$$

which is given in both cases for the NLS equation and the sG equation. As before, the intermediate wave function, the quotient of the second and first entry of this solution, is given by

$$\Delta(t,x) = \frac{[\widehat{\psi}_-]_{21}(t,x,\lambda_0) + \frac{v_0}{u_0}[\widehat{\psi}_+]_{22}(t,x,\lambda_0) e^{2i\Theta(t,x,\lambda_0)}}{[\widehat{\psi}_-]_{11}(t,x,\lambda_0) + \frac{v_0}{u_0}[\widehat{\psi}_+]_{12}(t,x,\lambda_0) e^{2i\Theta(t,x,\lambda_0)}}.$$

Then, in turn, we obtain an expression for the ratio of $\frac{v_0}{u_0}$, i.e.

$$\frac{v_0}{u_0} = -\frac{[\widehat{\psi}_-]_{21}(t,x,\lambda_0) - \Delta(t,x)[\widehat{\psi}_-]_{11}(t,x,\lambda_0)}{[\widehat{\psi}_+]_{22}(t,x,\lambda_0) - \Delta(t,x)[\widehat{\psi}_+]_{12}(t,x,\lambda_0)} e^{-2i\Theta(t,x,\lambda_0)}. \tag{3.3.1}$$

Also, the one-fold Darboux transformation corresponding to λ_0 and ψ_0 or rather $\Delta(t,x)$ takes the form

$$D[1] = \frac{1}{\lambda - \lambda_0^*}\left(\lambda \mathbb{1} + \frac{1}{1+|\Delta(t,x)|^2}\begin{pmatrix} -\lambda_0 - \lambda_0^*|\Delta(t,x)|^2 & (\lambda_0^* - \lambda_0)\Delta^*(t,x) \\ (\lambda_0^* - \lambda_0)\Delta(t,x) & -\lambda_0^* - \lambda_0|\Delta(t,x)|^2 \end{pmatrix}\right). \tag{3.3.2}$$

The properties of the Jost functions imply

$$\lim_{x\to-\infty}\Delta(t,x) = \infty, \quad \lim_{x\to+\infty}\Delta(t,x) = 0. \tag{3.3.3}$$

Therefore, adding a simple eigenvalue or pole or zero of the one-fold dressing matrix to the scattering data (2.1.14) or (2.2.14) under Dressing method can be explained by the following:

Theorem 3.3.1 (Gu, Hu & Zhou, [27])**.** *Let the scattering data $(\rho, \{\lambda_j, C_j\}_{j=1}^N)$ be given. Applying the Dressing method with $\lambda_0 \in \mathbb{C}_+ \setminus \{\lambda_1, \ldots, \lambda_N\}$ and $\psi_0(t,x) = u_0\psi_-^{(1)}(t,x,\lambda_0) + v_0\psi_+^{(2)}(t,x,\lambda_0)$, where $u_0 \in \mathbb{C}$, $v_0 \in \mathbb{C}\setminus\{0\}$, we add an eigenvalue to the scattering data leaving the original eigenvalues unchanged. In particular, denoting the transformed spectral functions and parameters with a prime, we have*

$$\begin{aligned}
a_{11}'(\lambda) &= \frac{\lambda-\lambda_0}{\lambda-\lambda_0^*}a_{11}(\lambda), \quad \lambda\in\mathbb{C}_+\cup\mathbb{R}, & \rho'(\lambda) &= \frac{\lambda-\lambda_0^*}{\lambda-\lambda_0}\rho(\lambda), \quad \lambda\in\mathbb{R},\\
a_{21}'(\lambda) &= a_{21}(\lambda), \quad \lambda\in\mathbb{R}, & &\\
b_j' &= b_j, \quad j=1,\ldots,N, & C_j' &= \frac{\lambda_j-\lambda_0^*}{\lambda_j-\lambda_0}C_j, \quad j=1,\ldots,N,\\
b_0' &= -\frac{v_0}{u_0}, & C_0' &= -\frac{v_0}{u_0}\frac{\lambda_0-\lambda_0^*}{a_{11}(\lambda_0)}.
\end{aligned}$$

Proof. The scattering data rely heavily on the Jost functions. That is why, the first step is to find the behavior of the Jost functions in the transformed system. Therefore, we need to see what the limit values of the one-fold dressing matrix are. By the observations (3.3.2) and (3.3.3), we derive

$$\begin{aligned}
\lim_{x\to-\infty} D[1](t,x,\lambda) &= \frac{1}{\lambda-\lambda_0^*}\operatorname{diag}(\lambda-\lambda_0^*, \lambda-\lambda_0),\\
\lim_{x\to+\infty} D[1](t,x,\lambda) &= \frac{1}{\lambda-\lambda_0^*}\operatorname{diag}(\lambda-\lambda_0, \lambda-\lambda_0^*).
\end{aligned}$$

Then, we can deduce that the transformed Jost functions can be expressed through

$$(\psi_-^{(1)})'(t,x,\lambda) = D[1](t,x,\lambda)\psi_-^{(1)}(t,x,\lambda), \quad (\psi_+^{(2)})'(t,x,\lambda) = D[1](t,x,\lambda)\psi_+^{(2)}(t,x,\lambda),$$

which is also passed onto $(\widehat{\psi}_-^{(1)})'$ and $(\widehat{\psi}_+^{(2)})'$. As already mentioned in Section 2.1, $a_{11}(\lambda) = \det[\psi_-^{(1)}|\psi_+^{(2)}]$. It follows that for $\lambda\in\mathbb{C}_+\cup\mathbb{R}$, the limit values of $[\widehat{\psi}_-]_{11}$ and $[\widehat{\psi}_+]_{22}$ are $a_{11}(\lambda)$ as x goes to $+\infty$ and $-\infty$, respectively. So that we have

$$a_{11}'(\lambda) = \lim_{x\to\infty}([\widehat{\psi}_-]_{11})' = \frac{\lambda-\lambda_0}{\lambda-\lambda_0^*}a_{11}(\lambda).$$

Analogously, we find for $a_{21}(\lambda)$ that

$$a_{21}(\lambda) = -[\psi_-]_{11}[\psi_+]_{21} + [\psi_-]_{21}[\psi_+]_{11} = (-[\widehat{\psi}_-]_{11}[\widehat{\psi}_+]_{21} + [\widehat{\psi}_-]_{21}[\widehat{\psi}_+]_{11})e^{-2i\Theta(t,x,\lambda)},$$

and therefore the limit values of $[\widehat{\psi}_-]_{21}$ and $-[\widehat{\psi}_+]_{21}$ behave as $a_{21}(\lambda)e^{2i\Theta(t,x,\lambda)}$ as x goes to $+\infty$ and $-\infty$, respectively. Consequently,

$$a'_{21}(\lambda) = \lim_{x\to\infty}([\widehat{\psi}_-]_{21})' = a_{21}(\lambda).$$

Also resulting in $\rho'(\lambda) = \frac{\lambda-\lambda_0^*}{\lambda-\lambda_0}\rho(\lambda)$. Since the Jost functions we relate in order to obtain b_j are changed identically by multiplication with $D[1](t,x,\lambda)$, the parameters b_j remain unchanged, i.e. $b'_j = b_j$ for $j = 1,\dots,N$. Then, by the definition of C_j, we can calculate

$$C'_j = b'_j\Big(\frac{\mathrm{d}a'_{11}(\lambda_j)}{\mathrm{d}\lambda}\Big)^{-1} = \frac{\lambda_j-\lambda_0^*}{\lambda_j-\lambda_0}C_j, \quad j = 1,\dots,N.$$

At the new eigenvalue $\lambda = \lambda_0$, we have that the transformed Jost function are also identically changed by

$$D[1](t,x,\lambda_0) = \frac{1}{1+|\Delta(t,x)|^2}\begin{pmatrix}|\Delta(t,x)|^2 & -\Delta^*(t,x)\\ -\Delta(t,x) & 1\end{pmatrix}.$$

Hence, as we calculated already in (3.3.1), we obtain

$$b'_0 = \frac{([\psi_-]_{21})'(t,x,\lambda_0)}{([\psi_+]_{22})'(t,x,\lambda_0)} = \frac{[\psi_-]_{21}(t,x,\lambda_0)-\Delta(t,x)[\psi_-]_{11}(t,x,\lambda_0)}{[\psi_+]_{22}(t,x,\lambda_0)-\Delta(t,x)[\psi_+]_{12}(t,x,\lambda_0)} = -\frac{v_0}{u_0}.$$

Subsequently, the weight for the added eigenvalue is readily obtained by

$$C'_0 = b'_0\Big(\frac{\mathrm{d}a'_{11}(\lambda_0)}{\mathrm{d}\lambda}\Big)^{-1} = -\frac{v_0}{u_0}\frac{\lambda_0-\lambda_0^*}{a_{11}(\lambda_0)},$$

thereby concluding the proof. □

Remark 3.3.2. A particular example is dressing a pure soliton solution into the NLS equation (or the sG equation) from the zero seed solution $u[0](t,x) = u_{sol}(t,x;\{\}) \equiv 0$ for which $a_{11}(\lambda) = 1$, $a_{21}(\lambda) = 0$ such that $\rho(\lambda) = 0$. Then, successively inserting simple eigenvalues $\lambda_1,\dots,\lambda_N \in \mathbb{C}_+$ with corresponding $\frac{v_j}{u_j} \in \mathbb{C}\setminus\{0\}$, $j = 1,\dots,N$, results in the transformed spectral functions

$$a_{11}^{(N)}(\lambda) = \prod_{j=1}^{N}\frac{\lambda-\lambda_j}{\lambda-\lambda_j^*}, \quad a_{12}^{(N)}(\lambda) = 0,$$

which then also gives the transformed scattering data as

$$S(u[N]) = (0, \{\lambda_j, C_j^{(N)}\}_{j=1}^N),$$

where

$$C_j^{(N)} = -\frac{v_j}{u_j}\prod_{k=1}^{N}(\lambda_j-\lambda_k^*)\Big(\prod_{k=1}^{N}{}'(\lambda_j-\lambda_k)\Big)^{-1}.$$

Here, the prime indicates that the term with $k = j$ is omitted from the product.

Now, after establishing the connection between the Dressing method and the scattering data, we want to continue elaborating on the equivalence between the methods of constructing certain solutions for the integrable nonlinear PDEs. Moreover, we incorporate this into the presentation of the proper models of the NLS and sG equation, which are the main focus of this thesis.

Chapter 4

Models

4.1 Implementing defect conditions

With the preliminary consideration of the scattering method in mind, we introduce the integrable models which are the main focus of this thesis. Therefore, we need to follow up on the idea of the Bäcklund transformation presented in Subsection 3.1.1. In particular, the context, in which the Bäcklund matrices (3.1.4) and (3.1.5) have been analyzed in [11], is to generate the so-called *defect conditions*, which essentially corresponds to considering the Bäcklund transformation as frozen at a specific point x_f and for all $t \in \mathbb{R}$. And since we are talking about conditions for the solutions, the application completely changes. So instead of the transformation of solutions into different solutions satisfying the same system via the Bäcklund transformation, the *frozen Bäcklund transformation* should here be understood as a condition connecting two existing solutions at a specific point x_f. To ease notation, we only work with $x_f = 0$. However, it should be noted that all of the upcoming arguments still hold even with an arbitrary $x_f \in \mathbb{R}$ and further that the arguments also hold with more than just one point where defect conditions are present.

4.1.1 General setting

First off, we make this idea more precise in the context of the first important concept worked out in Chapter 2, the Lax systems. Based on the transformation (3.1.1), we view the two Lax systems with U, V and $\widetilde{U}$, $\widetilde{V}$ not as two systems corresponding to the same integrable PDE on the whole line $x \in \mathbb{R}$. Rather, we restrict the potentials—u, $\tilde{u}$ and θ, $\tilde{\theta}$ for the NLS and sG equation, respectively, to either side of the defect. Here, since $x_f = 0$, we restrict u and θ or rather U and V to the positive half-line $x \in \mathbb{R}_+$ as well as $\tilde{u}$ and $\tilde{\theta}$ or rather $\widetilde{U}$ and $\widetilde{V}$ to the negative half-line $x \in \mathbb{R}_-$. Thus, the potentials are still solutions of the same PDE as before, which is, in particular, equivalent to the zero curvature condition, with the difference that they satisfy the PDE on different domains. Consequently, given solutions of the PDE on the respective domain, we also have knowledge of the complete Lax pairs and with that the partial matrix differential equations (3.1.2), which are reduced to a single point in space $x_f = 0$, connect these Lax pairs given a matrix B. Hence, instead of serving as a transformation of one solution to another of the same PDE on the whole line, the relations

$$\begin{aligned} B_x\big|_{x=0} &= (\widetilde{U}B - BU)\big|_{x=0}, \\ B_t\big|_{x=0} &= (\widetilde{V}B - BV)\big|_{x=0} \end{aligned} \tag{4.1.1}$$

for $t \in \mathbb{R}$ now constitute the spectral equivalent to some conditions. However, as we have seen for the Bäcklund transformation itself, not just any matrix suffices in order to obtain a system which is indeed solvable or by the ideas mentioned in the Introduction integrable. If we however choose B to be for example the identity matrix, then (4.1.1) implies $U = \widetilde{U}$ and $V = \widetilde{V}$ at $x = 0$ which therefore corresponds together with the inherent Lax systems to the ordinary PDE on the whole line. In the following subsection, we elaborate on the more relevant examples for matrices B which leave the NLS and sG equation integrable. Note that integrability for these models is meant in the sense explained in the Introduction. Further, by noticing a symmetry in the phase $\Theta(t, x, \lambda)$, one is able to adapt this idea to also include *boundary conditions* for only one potential on one half-line.

4.1.2 Models of the NLS and sG equation

As indicated before, the Darboux matrices (3.1.4) and (3.1.5) are derived with the idea of the frozen Bäcklund transformation in mind. Hence, it is straightforward to give the defect conditions, which are equivalent to relations (4.1.1), for the NLS and sG equation with respect to their solutions. In this subsection, we only give the essentials of the algebraic calculations in order not to disturb the flow of reading, since it is indeed quite lengthy to perform them accurately. Nevertheless, due to the fact that there is no uniqueness when it comes to Lax pairs and thus to the corresponding Darboux matrix and the importance of these computations to be exact, we can not just simply cite the existing literature and therefore we give the complete algebraic calculations in Appendix A. For the NLS equation, we find:

Proposition 4.1.1. *Inserting the Lax pairs $\mathcal{U}$, $\mathcal{V}$ and $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$ of the NLS equation* (2.1.3) *corresponding to the solutions u and $\tilde{u}$ on the positive and negative half-line, respectively, together with the Darboux matrix* (3.1.4) *into the frozen Bäcklund transformation* (4.1.1) *is equivalent to the defect conditions*

$$\begin{aligned} (\tilde{u} - u)_x &= i\alpha(\tilde{u} - u) \pm \Omega(\tilde{u} + u), \\ (\tilde{u} - u)_t &= -\alpha(\tilde{u} - u)_x \pm i\Omega(\tilde{u} + u)_x + i(\tilde{u} - u)(|u|^2 + |\tilde{u}|^2) \end{aligned} \tag{4.1.2}$$

at $x = 0$ with $\Omega = \sqrt{\beta^2 - |\tilde{u} - u|^2}$ and defect parameters $\alpha, \beta \in \mathbb{R}$.

Proof. Inserting the Lax pairs $\mathcal{U}$, $\mathcal{V}$ and $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$ of the NLS equation (2.1.3) and the Darboux matrix $\mathcal{B}$ into (4.1.1), we obtain for the left hand side of the x part

$$i \begin{pmatrix} \pm\Omega_x & -(\tilde{u} - u)_x \\ -(\tilde{u} - u)_x^* & \mp\Omega_x \end{pmatrix}$$

and calculating the right hand side of the x part, we have

$$i \begin{pmatrix} -(|\tilde{u}|^2 - |u|^2) & -i\alpha(\tilde{u} - u) \mp \Omega(\tilde{u} + u) \\ i\alpha(\tilde{u} - u)^* \mp \Omega(\tilde{u} + u)^* & (|\tilde{u}|^2 - |u|^2) \end{pmatrix}$$

and for the t-part $\mathcal{B}_t = \widetilde{\mathcal{V}}\mathcal{B} - \mathcal{B}\mathcal{V}$, we find the (11)- as well as the (12)-entry after a similar calculation to be

$$\begin{aligned} \pm i\Omega_t &= (i\alpha \mp \Omega)(|\tilde{u}|^2 - |u|^2) + \tilde{u}_x(\tilde{u} - u)^* - u_x^*(\tilde{u} - u) \\ -i(\tilde{u} - u)_t &= (\tilde{u} - u)(|\tilde{u}|^2 + |u|^2) + 2\lambda\tilde{u}(\alpha \mp i\Omega) + i\tilde{u}_x(2\lambda + \alpha \mp i\Omega) \\ &\quad - 2\lambda u(\alpha \pm i\Omega) - iu_x(2\lambda + \alpha \pm i\Omega), \end{aligned} \tag{4.1.3}$$

respectively. Using the (12)-entry of the x part, we see that the term of first order in λ in the (12)-entry of the t part is zero. Hence, the left hand side

$$i\begin{pmatrix} \pm\Omega_t & -(\tilde{u}-u)_t \\ -(\tilde{u}-u)_t^* & \mp\Omega_t \end{pmatrix}$$

is equal to the right hand side of the t part (4.1.3) at $x=0$ if and only if the defect conditions (4.1.2) hold. With the definition of Ω as in the proposition, it can further be verified that the (11)-entries of the x part and of the t part of these relations are satisfied for all $t \in \mathbb{R}$ and $x=0$. □

Proposition 4.1.2. *Inserting the Lax pairs $\mathbb{U}$, $\mathbb{V}$ and $\widetilde{\mathbb{U}}$, $\widetilde{\mathbb{V}}$ of the sG equation* (2.2.2) *corresponding to the solutions θ and $\tilde{\theta}$ on the positive and negative half-line, respectively, together with the Darboux matrix* (3.1.5) *into the frozen Bäcklund transformation* (4.1.1) *is equivalent to the defect conditions*

$$\begin{aligned} \tilde{\theta}_x + \theta_t &= \pm\Big(\alpha \sin\frac{\tilde{\theta}+\theta}{2} + \frac{1}{\alpha}\sin\frac{\tilde{\theta}-\theta}{2}\Big), \\ \tilde{\theta}_t + \theta_x &= \mp\Big(\alpha \sin\frac{\tilde{\theta}+\theta}{2} - \frac{1}{\alpha}\sin\frac{\tilde{\theta}-\theta}{2}\Big), \end{aligned} \tag{4.1.4}$$

at $x=0$ and with the defect parameter $\alpha \in \mathbb{R}$.

Proof. Inserting the Lax pairs $\mathbb{U}$, $\mathbb{V}$ and $\widetilde{\mathbb{U}}$, $\widetilde{\mathbb{V}}$ of the sG equation (2.2.2) and the Darboux matrix $\mathbb{B}$ into (4.1.1), we obtain for the left hand sides

$$\begin{aligned} \mathbb{B}_x &= \pm\frac{i\alpha}{\lambda}\frac{(\tilde{\theta}+\theta)_x}{2}\Big(\sigma_2\cos\frac{\tilde{\theta}+\theta}{2} - \sigma_3\sin\frac{\tilde{\theta}+\theta}{2}\Big), \\ \mathbb{B}_t &= \pm\frac{i\alpha}{\lambda}\frac{(\tilde{\theta}+\theta)_t}{2}\Big(\sigma_2\cos\frac{\tilde{\theta}+\theta}{2} - \sigma_3\sin\frac{\tilde{\theta}+\theta}{2}\Big). \end{aligned}$$

On the right hand sides of $\widetilde{\mathbb{U}}\mathbb{B} - \mathbb{B}\mathbb{U}$ and $\widetilde{\mathbb{V}}\mathbb{B} - \mathbb{B}\mathbb{V}$, we obtain

$$\begin{aligned} &\frac{i}{4\lambda}\Big[\pm\alpha(\tilde{\theta}_t + \theta_t - \tilde{\theta}_x - \theta_x) - 2\sin\frac{\tilde{\theta}-\theta}{2}\Big]\sigma_3\sin\frac{\tilde{\theta}+\theta}{2} \\ &\frac{i}{4\lambda}\Big[\pm\alpha(\tilde{\theta}_x + \theta_x - \tilde{\theta}_t - \theta_t) + 2\sin\frac{\tilde{\theta}-\theta}{2}\Big]\sigma_3\sin\frac{\tilde{\theta}+\theta}{2} \end{aligned}$$

for the diagonal entries of order λ^{-1} and

$$\begin{aligned} &\frac{i}{4\lambda}\Big[\pm\alpha(\tilde{\theta}_x + \theta_x - \tilde{\theta}_t - \theta_t) + 2\sin\frac{\tilde{\theta}-\theta}{2}\Big]\sigma_2\cos\frac{\tilde{\theta}+\theta}{2} \\ &\frac{i}{4\lambda}\Big[\pm\alpha(\tilde{\theta}_t + \theta_t - \tilde{\theta}_x - \theta_x) - 2\sin\frac{\tilde{\theta}-\theta}{2}\Big]\sigma_2\cos\frac{\tilde{\theta}+\theta}{2} \end{aligned}$$

for the off-diagonal entries of order λ^{-1}, so that for both sides to be equal,

$$\tilde{\theta}_t + \theta_t + \tilde{\theta}_x + \theta_x = \pm\frac{2}{\alpha}\sin\frac{\tilde{\theta}-\theta}{2} \tag{4.1.5}$$

needs to hold at $x=0$. Further, the off-diagonal entries (which correspond to the expressions multiplied with the first Pauli matrix σ_1) of zero-th order in λ give

$$-\tilde{\theta}_t + \theta_t + \tilde{\theta}_x - \theta_x = \pm 2\alpha\sin\frac{\tilde{\theta}+\theta}{2}. \tag{4.1.6}$$

Adding and subtracting these two equalities (4.1.5) and (4.1.6), the defect conditions for the sG equation are readily obtained. □

Remark 4.1.3. In [11], the defect conditions were initially introduced in the context of light-cone coordinates

$$\xi = \frac{x-t}{2}, \qquad \eta = \frac{x+t}{2},$$

for the sG equation, which considering the transformation

$$v(\eta,\xi) = \theta(\eta-\xi, \eta+\xi), \qquad \theta(t,x) = v\Big(\frac{x+t}{2}, \frac{x-t}{2}\Big)$$

takes the form $v_{\xi\eta} = \sin v$. Therefore, the two equalities (4.1.5) and (4.1.6) can be written as

$$\begin{aligned}(\tilde{v}-v)_\xi &= \pm 2\alpha \sin\frac{\tilde{v}+v}{2},\\ (\tilde{v}+v)_\eta &= \pm\frac{2}{\alpha}\sin\frac{\tilde{v}-v}{2}.\end{aligned} \tag{4.1.7}$$

In the literature the relations (4.1.2) and (4.1.4) are commonly found with regard to the usual Bäcklund transformation holding for all $t, x \in \mathbb{R}$. The significance in using the very same Bäcklund transformation frozen at a specific point $x = x_f$ lies in the fact that the generating function for the integral of motion can be adjusted to include the defect conditions. Hence, in the sense of integrability as the presence of an infinite set of conservation laws, the systems including a defect presented above can be shown to be integrable [11]. In fact,

Remark 4.1.4. The connection of the defect conditions to a frozen Bäcklund transformation (4.1.1) has been discussed, among other publications, in [11, 15]. The authors additionally prove for both models that there exists an infinite set of modified conservation laws, which means that the defect conditions are integrable in the aforementioned sense.

Resuming the elaboration of equivalence in the solution construction methods, we want to briefly address this idea in the context of the sG equation in light-cone coordinates. The derived conditions are when viewed as transformation holding for all $\eta, \xi \in \mathbb{R}$ the usual Bäcklund transformation and therefore, if we consider the Bäcklund transformation (4.1.7) of the vacuum solution $v \equiv 0$, we find that:

Remark 4.1.5. Equations (4.1.7) can be integrated to give the solution

$$\tilde{v}(\eta,\xi) = 4\arctan e^{\alpha\xi + \frac{1}{\alpha}\eta + c}$$

under the assumption of $v \equiv 0$ and the plus sign, see [19]. Transforming the coordinates back to laboratory coordinates, we find the single one-soliton solution which we associate to the sG equation in this thesis as

$$\tilde{\theta}(t,x) = 4\arctan e^{(\alpha+\frac{1}{\alpha})x - (\alpha-\frac{1}{\alpha})t + c}.$$

Note that, as indicated before, even though this is in combination with the Theorem of permutability 3.2.7 a powerful method to explicitly construct solutions, the scattering data are outside the scope of this method. Nonetheless, we continue this consideration of equivalence between the solution construction methods in the next section.

4.2 Bäcklund transformation vs. Dressing method for Lax systems

Another important property, we want to highlight, is that, similar to Subsection 3.1.2, the one-fold Dressing matrices for the Lax system of the NLS and sG equation are strongly related to the respective Bäcklund transformation presented in Subsection 3.1.1. Among the literature we already mentioned in this context in Chapter 3, this section is particularly inspired by the ideas mentioned in [11, 36]. In fact, with the right spectral parameters, one matrix can be transformed into the other. Note that these results hold for t and x in the respective domains and not necessarily $x = 0$. With respect to the model introduced in Sections 2.1 and 2.2, one has $t, x \in \mathbb{R}$.

Proposition 4.2.1.

(i) The one-fold dressing matrix (3.2.2)*, constructed by* $\mathbb{C}\setminus\mathbb{R} \ni \lambda_1 = \xi_1 + i\eta_1$ *and* $\psi_1(t,x)$*, satisfies* (3.1.2) *with* $\widetilde{U} = U[1]$ *and* $U = U[0]$*.*

- *For the NLS equation, we have that, up to a function of λ, the dressing matrix can be written as* $\frac{\lambda-\lambda_1^*}{\lambda}D[1] = \mathcal{B}$*, see* (3.1.4)*, where* $\alpha = -2\xi_1$*,* $\beta^2 = (2\eta_1)^2 \neq 0$ *and the $\pm$ sign in front of the square root is determined by the sign of η_1 and by the condition that the absolute value of the intermediate wave function is either greater or equal, or less or equal than 1.*
- *For the sG equation, we find that, up to a function of λ, the dressing matrix can be written as* $\frac{\lambda-\lambda_1^*}{\lambda}D[1] = \mathbb{B}$*, see* (3.1.5)*, where* $\xi_1 = 0$*,* $\alpha = \eta_1 \neq 0$ *and the $\pm$ sign is determined by* $\operatorname{sign}\eta_1$*.*

(ii) (Caudrelier, [11]) *The Bäcklund transformation*

- *for the NLS equation $\mathcal{B}$ with $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}\setminus\{0\}$ admits a projector matrix*

$$\mathcal{P} = \frac{1}{\lambda_1^* - \lambda_1}(\mathcal{B}^{(0)}/2 + \lambda_1^* \mathbb{1}), \tag{4.2.1}$$

  where $\lambda_1 = -\frac{\alpha}{2} + \frac{i\beta}{2}$ *and* $\mathcal{B}^{(0)}$ *is the matrix coefficient of the Bäcklund transformation of* λ^{-1}*. In particular, there exists a t and x dependent kernel vector of $\mathcal{B}$ at λ_1.*

- *for the sG equation $\mathbb{B}$ with $\alpha \in \mathbb{R}\setminus\{0\}$ admits a projector matrix*

$$\mathbb{P} = \frac{1}{\lambda_1^* - \lambda_1}(\mathbb{B}^{(0)}/2 + \lambda_1^* \mathbb{1}), \tag{4.2.2}$$

  where $\lambda_1 = \frac{i\alpha}{2}$ *and* $\mathbb{B}^{(0)}$ *is the matrix coefficient of the Bäcklund transformation of* λ^{-1}*. In particular, there exists a t and x dependent kernel vector of $\mathbb{B}$ at λ_1.*

Proof. We begin with the proof for the NLS equation. By the reconstruction formula for the one-fold dressing matrix (3.2.4), we have

$$u[1] - u[0] = 4\eta_1 \frac{\Delta^*}{1 + |\Delta|^2} \tag{4.2.3}$$

and therefore a simple calculation leads to

$$\sqrt{(2\eta_1)^2 - |u[1] - u[0]|^2} = 2|\eta_1| \frac{|1 - |\Delta|^2|}{1 + |\Delta|^2}. \tag{4.2.4}$$

Then, we want to utilize the definition of the Bäcklund transformation as in (4.1.2) and therefore we calculate

$$\Delta_x = -u[0]^* + 2i\lambda_1\Delta - u[0]\Delta^2,$$
$$\Delta_t = (-2\lambda_1 u[0]^* + iu[0]^*_x) + 2i(2\lambda_1^2 - |u[0]|^2)\Delta - (2\lambda_1 u[0] + iu[0]_x)\Delta^2.$$

Therefore, we can derive the t and x derivatives of the difference of the dressed solution and the seed solution

$$\begin{aligned}(u[1]-u[0])_x &= 4\eta_1\Big(\frac{\Delta^*}{1+|\Delta|^2}\Big)_x\\ &= 4\eta_1\frac{\Delta^*_x - \Delta_x(\Delta^*)^2}{(1+|\Delta|^2)^2}\\ &= -2i\xi_1\Big(4\eta_1\frac{\Delta^*}{1+|\Delta|^2}\Big) + 4\eta_1\frac{1-|\Delta|^2}{1+|\Delta|^2}\Big(-u[0] - 2\eta_1\frac{\Delta^*}{1+|\Delta|^2}\Big).\end{aligned}$$

By equality (4.2.3), we find the first bracket to be $u[1]-u[0]$ and the second bracket to be $-(u[1]+u[0])/2$ and using the equality for the square root (4.2.4), we obtain

$$(u[1]-u[0])_x = i(-2\xi_1)(u[1]-u[0]) - \mathrm{sign}(\eta_1(1-|\Delta|^2))\sqrt{(2\eta_1)^2 - |u[1]-u[0]|^2}(u[1]+u[0]).$$

Comparing this result with (4.1.2), we confirm that $-2\xi_1 = \alpha$, $(2\eta_1)^2 = \beta^2$ and $\mathrm{sign}(\eta_1)$ determines the sign in front of the square root under the assumption made for the absolute value of the intermediate wave function Δ. The same can be done for $(u[1]-u[0])_t$ using the expression for Δ_t.

On the other hand, given the Bäcklund transformation (3.1.4), we define $\lambda_1 = -\frac{\alpha}{2} + \frac{i\beta}{2}$ as well as the matrix $\mathcal{P}$ as in (4.2.1). Hence, we derive

$$\mathcal{D}(t,x,\lambda) = \frac{\lambda}{\lambda-\lambda_1^*}\mathcal{B}(t,x,\lambda) = \mathbb{1} + \frac{\lambda_1^* - \lambda_1}{\lambda-\lambda_1^*}\mathcal{P}(t,x).$$

For $\mathcal{D}$ to be a one-fold dressing matrix, we need that $\mathcal{P}$ is indeed a projection matrix, i.e. $\mathcal{P}^2 = \mathcal{P}$ and that there exists a kernel vector of $\mathcal{D}$ at the chosen spectral parameter λ_1. Therefore, we calculate

$$\begin{aligned}\mathcal{P}^2 &= \left(\frac{1}{2\beta}\begin{pmatrix}\beta\mp\Omega & \tilde u - u\\ (\tilde u - u)^* & \beta\pm\Omega\end{pmatrix}\right)^2\\ &= \frac{1}{4\beta^2}\begin{pmatrix}(\beta\mp\Omega)^2 + |\tilde u - u|^2 & (\tilde u - u)(\beta\mp\Omega+\beta\pm\Omega)\\ (\tilde u - u)^*(\beta\mp\Omega+\beta\pm\Omega) & (\beta\pm\Omega)^2 + |\tilde u - u|^2\end{pmatrix}\\ &= \frac{1}{2\beta}\begin{pmatrix}\beta\mp\Omega & \tilde u - u\\ (\tilde u - u)^* & \beta\pm\Omega\end{pmatrix} = \mathcal{P}.\end{aligned}$$

Thus, $\mathcal{P}$ is a projector matrix and particularly it can be easily seen that the determinant and the trace of $\mathcal{P}$ are $\det\mathcal{P} = 0$ and $\mathrm{Tr}\,\mathcal{P} = 1$. Therefore, $\mathcal{P}$ has the eigenvalues 0 and 1. In particular, the vector

$$\upsilon = c_1\begin{pmatrix}(\beta\mp\Omega)\\ (\tilde u - u)^*\end{pmatrix} + c_2\begin{pmatrix}(\tilde u - u)\\ (\beta\pm\Omega)\end{pmatrix}$$

satisfies $\mathcal{P}\upsilon = \upsilon$ such that $\mathcal{D}\upsilon = 0$ at $\lambda = \lambda_1$.

Now, we prove the result for the sG equation. Note that in this case, we have $\theta = \theta[0] \equiv 0$, $\tilde{\theta} = \theta[1]$ and $\lambda_1 = i\eta_1$. Again, by the reconstruction formula for the one-fold dressing matrix (3.2.21), we have that

$$\cos\frac{\theta[1]}{2} = -\frac{1-|\Delta|^2}{1+|\Delta|^2}, \qquad i\sin\frac{\theta[1]}{2} = \frac{2\Delta^*}{1+|\Delta|^2}.$$

Using the definition of Δ, we also find

$$\Delta_x = -\frac{1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big)\Delta, \qquad \Delta_t = \frac{1}{2}\Big(\eta_1 - \frac{1}{\eta_1}\Big)\Delta.$$

Calculating the x and t derivative of

$$\theta[1] = -2\arcsin\Big(\frac{2i\Delta^*}{1+|\Delta|^2}\Big)$$

and using the equalities for Δ_x and Δ_t, we end up with

$$\theta[1]_x = \Big(\eta_1 + \frac{1}{\eta_1}\Big)\sin\frac{\theta[1]}{2}, \qquad \theta[1]_t = -\Big(\eta_1 - \frac{1}{\eta_1}\Big)\sin\frac{\theta[1]}{2},$$

which matches relation (3.1.5) if $\eta_1 = \alpha$ and $\operatorname{sign}\eta_1$ determines the sign in each relation.

Given the Bäcklund transformation (3.1.5) for the sG equation corresponding to $\theta \equiv 0$, we define $\lambda_1 = \frac{i\alpha}{2}$ and the matrix $\mathbb{P}$ as in (4.2.2). We calculate

$$\mathbb{P}^2 = \left(\frac{1}{2}\begin{pmatrix} 1 \mp \cos\frac{\theta[1]}{2} & i\sin\frac{\theta[1]}{2} \\ -i\sin\frac{\theta[1]}{2} & 1 \pm \cos\frac{\theta[1]}{2}\end{pmatrix}\right)^2 = \mathbb{P}.$$

Again, $\det\mathbb{P} = 0$ and $\operatorname{Tr}\mathbb{P} = 1$ so that there exists a vector

$$\upsilon = c_1\begin{pmatrix} -1 \pm \cos\frac{\theta[1]}{2} \\ i\sin\frac{\theta[1]}{2}\end{pmatrix} + c_2\begin{pmatrix} -i\sin\frac{\theta[1]}{2} \\ -1 \mp \cos\frac{\theta[1]}{2}\end{pmatrix}$$

for which $\mathbb{P}\upsilon = \upsilon$ and where c_1 and c_2 may depend on t and x, but not on λ and therefore $\mathbb{D} = \frac{\lambda}{\lambda-\lambda_1^*}\mathbb{B}\upsilon = 0$ at $\lambda = \lambda_1$. □

So, let us stress again that this is an important observation: In general, there is a way to interpret the Bäcklund transformation, specifically, from Subsection 3.1.1 for the NLS or sG equation as one-fold dressing matrix which we introduced in Section 3.2. Note that, starting from a Bäcklund transformation, we also identified the kernel vector corresponding to the spectral parameter $\lambda = \lambda_1$ from which the dressing matrices in Propositions 3.1.1 and 3.1.2 are constructed. Further, we take from this proposition that it is possible that a solution constructed through the Dressing method satisfies the equalities (4.1.2) with the plus sign on a specific domain $x \in E \subset \mathbb{R}$ and the minus sign on the complement $x \in \mathbb{R} \setminus E$ with respect to the whole line. On the other hand, a solution constructed with the Bäcklund transformation can always be expressed in terms of the Dressing method.

Remark 4.2.2. As a special case of Proposition 4.2.1 it is also possible to apply it to a frozen Bäcklund transformation or one-fold dressing matrix at $x = 0$.

4.3 Implementing boundary conditions

In Section 4.1, we presented the model of the NLS equation or sG equation on two half-lines which are connected through the defect conditions at $x = 0$ while conserving integrability. Taking this idea further, we want to look at these PDEs in the quarter plane $x \in \mathbb{R}_+$, $t \in \mathbb{R}_+$ in connection with boundary conditions which correspond to integrable models. In fact, one way to approach boundary conditions is by the unified transform method, initially invented to serve as a generalization to the inverse scattering method for half-line problems. Therefore, a class of boundary conditions has been filtered out for which it is possible to solve the generalized problem with the same level of efficiency as the one for the problem on the full line. These boundary conditions are called *linearizable boundary conditions.*

4.3.1 General setting

For the NLS and sG equation this means that we assume that there exists an t dependent, x independent, nonsingular matrix $K(t, 0, \lambda)$ such that

$$K_t(t, 0, \lambda) = V(t, 0, r(\lambda))K(t, 0, \lambda) - K(t, 0, \lambda)V(t, 0, \lambda), \tag{4.3.1}$$

where $r(\lambda)$ reflects a certain symmetry inherent to the respective equation.

Note that this relation (4.3.1) has structural differences to the relations (4.1.1). Instead of relating one side of the defect with the other, in this case, we exploit a symmetry of the system. Moreover, it is in fact just a condition on the t part in contrast to conditions on both the t and x part of the Lax pair. Again, we give brief calculations to ensure the flow of reading and refer to Appendix A for more details, since these results may be found in a similar but not exactly the same manner in the literature.

4.3.2 Models of NLS and sG equation

For the NLS equation, we have that the symmetry yields that $r(\lambda) = -\lambda$. With that in mind, we find two matrices which satisfy the relation (4.3.1), thereby correlating to certain boundary conditions for the NLS equation.

Proposition 4.3.1. *Boundary matrices for the Lax pair $\mathcal{U}$, $\mathcal{V}$ of the NLS equation* (2.1.3) *corresponding to the Robin boundary condition, see* [28],

$$u_x(t, 0) = \alpha u(t, 0) \tag{4.3.2}$$

with $\alpha \in \mathbb{R}$ and the new boundary condition, see [41],

$$u_x(t, 0) = \frac{iu_t(t, 0)}{2\Omega(t, 0)} - \frac{u(t, 0)\Omega(t, 0)}{2} + \frac{u(t, 0)|u(t, 0)|^2}{2\Omega(t, 0)} - \frac{u(t, 0)\alpha^2}{2\Omega(t, 0)} \tag{4.3.3}$$

with $\Omega(t, 0) = \sqrt{\beta^2 - |u(t, 0)|^2}$, $\alpha, \beta \in \mathbb{R}$ are given by

$$\mathcal{K}(\lambda) = \frac{1}{i\alpha + 2\lambda} \begin{pmatrix} i\alpha - 2\lambda & 0 \\ 0 & i\alpha + 2\lambda \end{pmatrix}, \tag{4.3.4}$$

$$\mathcal{K}(t, 0, \lambda) = \frac{1}{(2\lambda - i|\beta|)^2 - \alpha^2} \begin{pmatrix} 4\lambda^2 + 4i\lambda\Omega(t, 0) - (\alpha^2 + \beta^2) & 4i\lambda u(t, 0) \\ 4i\lambda u^*(t, 0) & 4\lambda^2 - 4i\lambda\Omega(t, 0) - (\alpha^2 + \beta^2) \end{pmatrix}, \tag{4.3.5}$$

respectively. Moreover, for both boundary matrices, $\mathcal{K}^{-1}(t, 0, \lambda) = \mathcal{K}(t, 0, -\lambda)$ holds.

Proof. We omit the denominator of the boundary matrices in the following calculations, since it is time independent in both cases and thus present in every term of the equalities. For the boundary matrix of the Robin boundary condition, we immediately have that the diagonal entries of the relation (4.3.1) are zero. For the off-diagonal entries, after cancellation we obtain

$$\mathcal{V}(t,0,-\lambda)\mathcal{K}(\lambda)-\mathcal{K}(\lambda)\mathcal{V}(t,0,\lambda)=\begin{pmatrix}0 & 4i\lambda(u_x-\alpha u)\\ -4i\lambda(u_x-\alpha u)^* & 0\end{pmatrix},$$

which is at $(t,x=0)$ equivalent to the Robin boundary condition.

For the boundary matrix of the new boundary conditions, the left hand side of (4.3.1) amounts to

$$4i\lambda\begin{pmatrix}\Omega_t & u_t\\ u_t^* & -\Omega_t\end{pmatrix}$$

and after some calculation, the right hand side can be written as

$$4i\lambda\begin{pmatrix}i(u^*u_x-uu_x^*) & -iu(\alpha^2+\beta^2)-2i\Omega u_x+2i\lambda|u|^2u\\ iu^*(\alpha^2+\beta^2)+2i\Omega u_x^*-2i\lambda|u|^2u^* & -i(u^*u_x-uu_x^*)\end{pmatrix}.$$

Hence with the identification $\Omega^2=\beta^2-|u|^2$, the off-diagonal at $(t,x=0)$ is equivalent to the new boundary condition and with this condition it can be confirmed that the equality $\Omega_t=i(u^*u_x-uu_x^*)$ for the diagonal entries holds.

The property that the inverse boundary matrix is equal to the boundary matrix with λ changed to $-\lambda$ relies on the fact that the denominator normalizes the determinant of the boundary matrix. For the Robin boundary condition, we have

$$\mathcal{K}^{-1}(\lambda)=\frac{\det(\mathcal{K}(\lambda))^{-1}}{i\alpha+2\lambda}\begin{pmatrix}i\alpha+2\lambda & 0\\ 0 & i\alpha-2\lambda\end{pmatrix}=\frac{1}{(i\alpha+2(-\lambda))}\begin{pmatrix}i\alpha-2(-\lambda) & 0\\ 0 & i\alpha+2(-\lambda)\end{pmatrix},$$

which is $\mathcal{K}(-\lambda)$ with $\det(\mathcal{K}(\lambda))=\frac{i\alpha-2\lambda}{i\alpha+2\lambda}$. Further, for the new boundary condition, the equality of

$$\mathcal{K}^{-1}(t,0,\lambda)=\frac{\det(\mathcal{K}(t,0,\lambda))^{-1}}{(2\lambda-i|\beta|)^2-\alpha^2}\begin{pmatrix}4\lambda^2-4i\lambda\Omega-(\alpha^2+\beta^2) & -4i\lambda u\\ -4i\lambda u^* & 4\lambda^2+4i\lambda\Omega-(\alpha^2+\beta^2)\end{pmatrix}$$

to $\mathcal{K}(t,0,-\lambda)$ holds, since we can derive the following equality

$$\begin{aligned}\det(\mathcal{K}(t,0,\lambda))&=\frac{(4\lambda^2-\alpha^2-\beta^2+4i\lambda\Omega)\cdot(4\lambda^2-\alpha^2-\beta^2-4i\lambda\Omega)+16\lambda^2|u|^2}{((2\lambda-i|\beta|)^2-\alpha^2)^2}\\ &=\frac{(4\lambda^2-\alpha^2-\beta^2)^2+16\lambda^2(\Omega^2+|u|^2)}{((2\lambda-i|\beta|)^2-\alpha^2)^2}\\ &=\frac{((2\lambda-i|\beta|)^2-\alpha^2)\cdot((2\lambda+i|\beta|)^2-\alpha^2)}{((2\lambda-i|\beta|)^2-\alpha^2)^2}\\ &=\frac{(2\lambda+i|\beta|)^2-\alpha^2}{(2\lambda-i|\beta|)^2-\alpha^2}\end{aligned}$$

for the determinant, thereby concluding the proof of the assertions. □

For the sG equation, we have that the symmetry yields that $r(\lambda)=\lambda^{-1}$. With that in mind, we find three matrices which satisfy the relation (4.3.1), thereby correlating to certain boundary conditions for the sG equation.

Proposition 4.3.2. *Boundary matrices for the Lax pair* $\mathbb{U}$, $\mathbb{V}$ *of the sG equation* (2.2.2) *corresponding to a Dirichlet boundary condition*

$$\theta(t,0)=\alpha \tag{4.3.6}$$

with $\alpha\in\mathbb{R}$ *and a* sin*-boundary condition, see* [43],

$$\theta_x(t,0)=\alpha\sin\frac{\theta(t,0)}{2} \tag{4.3.7}$$

with $\alpha\in\mathbb{R}$ *and a* cos*-boundary condition*

$$\theta_x(t,0)=\alpha\cos\frac{\theta(t,0)}{2} \tag{4.3.8}$$

with $\alpha\in\mathbb{R}$ *are given by*

$$\mathbb{K}(\lambda)=\frac{1}{\sqrt{\lambda^2+\frac{1}{\lambda^2}+2\cos\alpha}}\Big[\Big(\lambda+\frac{1}{\lambda}\Big)\mathbb{1}\cos\frac{\alpha}{2}+i\Big(\lambda-\frac{1}{\lambda}\Big)\sigma_1\sin\frac{\alpha}{2}\Big], \tag{4.3.9}$$

$$\mathbb{K}(t,0,\lambda)=\frac{1}{\sqrt{(\lambda-\frac{1}{\lambda})^2+\alpha^2}}\Big[-\alpha\mathbb{1}-i\Big(\lambda-\frac{1}{\lambda}\Big)\Big(\sigma_3\cos\frac{\theta(t,0)}{2}+\sigma_2\sin\frac{\theta(t,0)}{2}\Big)\Big], \tag{4.3.10}$$

$$\mathbb{K}(t,0,\lambda)=\frac{1}{\sqrt{(\lambda+\frac{1}{\lambda})^2+\alpha^2}}\Big[i\alpha\sigma_1-i\Big(\lambda+\frac{1}{\lambda}\Big)\Big(\sigma_3\cos\frac{\theta(t,0)}{2}+\sigma_2\sin\frac{\theta(t,0)}{2}\Big)\Big], \tag{4.3.11}$$

respectively. Furthermore, for the Dirichlet boundary and sin*-boundary matrices* $\mathbb{K}^{-1}(t,0,\lambda)=\mathbb{K}(t,0,\lambda^{-1})$ *and for the* cos*-boundary matrix* $\mathbb{K}^{-1}(t,0,\lambda)=-\mathbb{K}(t,0,\lambda^{-1})$ *holds.*

Proof. Similarly to the calculation for the boundary matrices of the NLS equation, we omit the denominators due to their time independence. Starting with the Dirichlet boundary condition, we see that the left hand side of (4.3.1) is zero. On the other hand, the right hand side results after cancellation and under the use of appropriate trigonometric identities in

$$\frac{i\lambda}{2}\Big(\lambda-\frac{1}{\lambda}\Big)\Big(\lambda+\frac{1}{\lambda}\Big)\Big(\sigma_3\sin\frac{\theta}{2}-\sigma_2\cos\frac{\theta}{2}\Big)\Big[\sin\frac{\alpha}{2}\cos\frac{\theta}{2}-\cos\frac{\alpha}{2}\sin\frac{\theta}{2}\Big]\Big),$$

which is zero for all $t\in\mathbb{R}_+$ if and only if $\theta(t,0)\equiv\alpha$. Then, we have for the left hand side of (4.3.1) for the sin- and cos-boundary condition

$$\begin{gathered}i\Big(\lambda-\frac{1}{\lambda}\Big)\Big(\sigma_3\sin\frac{\theta}{2}-\sigma_2\cos\frac{\theta}{2}\Big)\frac{\theta_t}{2},\\ i\Big(\lambda+\frac{1}{\lambda}\Big)\Big(\sigma_3\sin\frac{\theta}{2}-\sigma_2\cos\frac{\theta}{2}\Big)\frac{\theta_t}{2},\end{gathered}$$

respectively. After some calculation, we obtain for the right hand side of (4.3.1) for the sin- and cos-boundary condition

$$\begin{gathered}\frac{i}{2}\Big(\lambda-\frac{1}{\lambda}\Big)\Big(\sigma_3\Big[\frac{\alpha}{2}(1-\cos\theta)+(\theta_t-\theta_x)\sin\frac{\theta}{2}\Big]-\sigma_2\Big[\frac{\alpha}{2}\sin\theta+(\theta_t-\theta_x)\cos\frac{\theta}{2}\Big]\Big),\\ \frac{i}{2}\Big(\lambda+\frac{1}{\lambda}\Big)\Big(\sigma_3\Big[\frac{\alpha}{2}\sin\theta+(\theta_t-\theta_x)\sin\frac{\theta}{2}\Big]-\sigma_2\Big[\frac{\alpha}{2}(\cos\theta+1)+(\theta_t-\theta_x)\cos\frac{\theta}{2}\Big]\Big),\end{gathered}$$

respectively. Comparing the two sides in each case, we see that the time derivatives θ_t cancel and the remaining terms

$$\frac{\alpha}{2}(1-\cos\theta)-\theta_x\sin\frac{\theta}{2}, \qquad \frac{\alpha}{2}\sin\theta-\theta_x\cos\frac{\theta}{2},$$
$$\frac{\alpha}{2}\sin\theta-\theta_x\sin\frac{\theta}{2}, \qquad \frac{\alpha}{2}(\cos\theta+1)-\theta_x\cos\frac{\theta}{2}$$

are, after dividing by either $\sin\frac{\theta}{2}$ or $\cos\frac{\theta}{2}$ and using the appropriate trigonometric identities

$$\alpha\frac{(1-\cos\theta)}{2\sin\frac{\theta}{2}}-\theta_x=\alpha\sin\frac{\theta}{2}-\theta_x=\alpha\frac{\sin\theta}{2\cos\frac{\theta}{2}}-\theta_x,$$
$$\alpha\frac{\sin\theta}{2\sin\frac{\theta}{2}}-\theta_x=\alpha\cos\frac{\theta}{2}-\theta_x=\alpha\frac{(\cos\theta+1)}{2\cos\frac{\theta}{2}}-\theta_x,$$

equivalent to the sin-boundary condition (4.3.7) and the cos-boundary condition (4.3.8), respectively.

The property that the inverse of the boundary matrix is equal to the boundary matrix with λ changed to λ^{-1} relies on the fact that the denominator normalizes the determinant of the boundary matrix. The boundary matrix of the Dirichlet boundary condition satisfies

$$\mathbb{K}^{-1}(\lambda)=\frac{\det(\mathbb{K}(\lambda))^{-1}}{\sqrt{\lambda^2+\frac{1}{\lambda^2}+2\cos\alpha}}\Big[(\lambda+\frac{1}{\lambda})\mathbb{1}\cos\frac{\alpha}{2}-(\lambda-\frac{1}{\lambda})\sigma_1\sin\frac{\alpha}{2}\Big]=\mathbb{K}(\lambda^{-1}),$$

since $\det(\mathbb{K}(\lambda))=1$. For the sin-boundary condition, we have

$$\mathbb{K}^{-1}(t,0,\lambda)=\frac{\det(\mathbb{K}(t,0,\lambda))^{-1}}{\sqrt{(\lambda-\frac{1}{\lambda})^2+\alpha^2}}\Big[-\alpha\mathbb{1}+i\Big(\lambda-\frac{1}{\lambda}\Big)\Big(\sigma_3\cos\frac{\theta}{2}+\sigma_2\sin\frac{\theta}{2}\Big)\Big]=\mathbb{K}(t,0,\lambda^{-1}),$$

and for the cos-boundary condition, we have

$$\mathbb{K}^{-1}(t,0,\lambda)=\frac{\det(\mathbb{K}(t,0,\lambda))^{-1}}{\sqrt{(\lambda+\frac{1}{\lambda})^2+\alpha^2}}\Big[-i\alpha\sigma_1+i\Big(\lambda+\frac{1}{\lambda}\Big)\Big(\sigma_3\cos\frac{\theta}{2}+\sigma_2\sin\frac{\theta}{2}\Big)\Big]=-\mathbb{K}(t,0,\lambda^{-1}),$$

where in each case the determinant of the boundary matrix is normalized as $\det(\mathbb{K}(t,0,\lambda))=1$. □

4.4 Preliminary considerations

Before we turn to apply the Dressing method to the presented models of the NLS and sG equation with either defect or boundary conditions, we preliminarily consider some aspects which prove to be useful in this endeavor. In the case of the NLS or sG equation on the whole line $x\in\mathbb{R}$, we have seen that it is possible to construct soliton solutions or breather solutions using distinct spectral parameters which are taken from $\mathbb{C}\setminus\mathbb{R}$. Now, it is a priori not obvious which spectral parameters need to be paired on each side of the defect for it to be preserved under the Dressing method. Therefore, the goal of the following subsection is to clarify under which conditions for the spectral parameters—that we know of—the defect conditions and further the boundary conditions are preserved. As shown in detail in Section 4.2, the frozen Bäcklund transformation can in theory be connected to a frozen one-fold dressing matrix. Hence, we can also view the defect conditions as a single soliton which is bound to $x = 0$. Therefore, one of the solutions, which we ignore for the time being, is to construct a one-soliton on one side of the defect interacting destructively with the 'frozen' soliton at $x = 0$; Especially, since this is a very specific scenario which can not be iterated. Hence, we consider the Dressing method using spectral parameters which are not on the real line and differ from the spectral parameters mentioned in Proposition 4.2.1 being $\lambda_0 = -\frac{\alpha}{2} + \frac{i\beta}{2}$ and λ_0^* for the NLS and $\lambda_0 = \frac{i\alpha}{2}$ and λ_0^* for the sG equation.

4.4.1 A 'space-evolution' interpretation

We want to discuss time direct scattering for the t part of the respective Lax pair U, V at $x = 0$ under the simplifying assumption that we have the zero seed solution. This subsection is inspired by the analysis given in [42] combined with the direct scattering process [2], we presented in Chapter 2. Assuming that, the function and its derivatives with regard to x vanish faster than any exponential as $|t|$ goes to infinity, similar to the case for the (space) direct scattering, one obtains Jost functions

$$\phi_\pm(t, 0, \lambda) \sim e^{-i\Theta(t,0,\lambda)\sigma_3}, \quad \text{as } t \to \pm\infty. \tag{4.4.1}$$

Regarding the space scattering process, the x part of the phase in case of the NLS equation is multiplied by λ and in the case of the sG equation by $(\lambda - \lambda^{-1})$ and we have seen in Sections 2.1 and 2.2 that the Jost functions can be continued analytically in either the upper or lower half-plane. With regard to the time scattering process, the phase in the case of the NLS equation is multiplied by λ^2 and in the case of the sG equation by $(\lambda + \lambda^{-1})$ so that the domains in the λ-plane in which the Jost functions (4.4.1) can in general be continued analytically are split into four quadrants in the case of the NLS equation and into four distinct domains in the case of the sG equation, see Figure 4.1. The proof is essentially the same as for the (space) direct scattering, see Theorems 2.1.1 and 2.2.3 or [2]. Therefore, we have that the first column of ϕ_- as well as the second column of ϕ_+, i.e. $\phi_-^{(1)}$ and $\phi_+^{(2)}$, and the second column of ϕ_- as well as the first column of ϕ_+, i.e. $\phi_-^{(2)}$ and $\phi_+^{(1)}$, can be continued analytically into the gray and the white domain, respectively. However, note that due to the zero seed solution, all four Jost functions are entire functions of λ, since Volterra integral equations on a finite interval always have absolutely convergent Neumann series solutions [2].

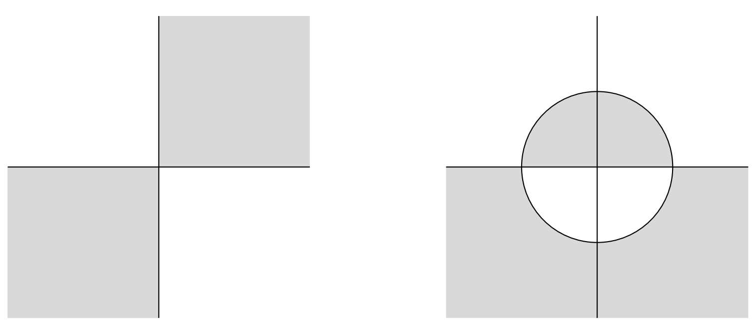

Fig. 4.1. Analyticity domains of the Jost functions for the time direct scattering

Hence, the same reasoning as in the (space) direct scattering implies that there exists a t independent matrix $A(\lambda)$ such that

$$\phi_-(t, 0, \lambda) = \phi_+(t, 0, \lambda)A(\lambda), \quad \lambda \in \mathbb{R},$$

where all the scattering coefficients can be analytically extended, since we assume to have the zero seed solution. In particular, we have

$$A(\lambda) = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{pmatrix}.$$

As indicated in Chapter 2, where the inverse scattering method is presented, normalizing the Jost functions as above leads to a linear evolution of the entries of the space scattering matrix $A(\lambda)$:

$$\frac{\partial A(\lambda)}{\partial x} = (\phi_-\phi_+^{-1})_x = U\phi_-\phi_+^{-1} - \phi_-\phi_+^{-1}U = [U, A(\lambda)],$$

where $U(t,x,\lambda)$ is in the case of the zero seed solution equal to $\mathcal{U}(\lambda) = -i\lambda\sigma_3$ and $\mathbb{U}(\lambda) = -\frac{i}{4}(\lambda - \frac{1}{\lambda})\sigma_3$ in the case of the NLS equation and the sG equation, respectively.

Following the analysis in Sections 2.1 as well as 2.2, one could derive a Riemann–Hilbert problem, where soliton solutions of the NLS equation and sG equation correspond to zeros of $a_{11}(\lambda)$, which in general come from the gray domains of Figure 4.1.

Now, with the consideration of defect conditions (4.1.2) or (4.1.4) at $x = 0$, we examine the Jost functions $\widetilde{\phi}_\pm(t,0,\lambda)$ and $\phi_\pm(t,0,\lambda)$ which are related through the frozen Bäcklund transformation according to $\widetilde{\phi}(t,0,\lambda) = B(t,0,\lambda)\phi(t,0,\lambda)$. Therefore, we derive

$$\begin{aligned}
\widetilde{A}(\lambda) &= \widetilde{\phi}_-(t,0,\lambda)\widetilde{\phi}_+^{-1}(t,0,\lambda) \\
&= B(t,0,\lambda)\phi_-(t,0,\lambda)\big(B(t,0,\lambda)\phi_+(t,0,\lambda)\big)^{-1} \\
&= B(t,0,\lambda)A(\lambda)B^{-1}(t,0,\lambda).
\end{aligned}$$

Similarly for the boundary conditions (4.3.2), (4.3.3), (4.3.8) and (4.3.7), we obtain with the relation $\phi(t,0,r(\lambda)) = K(t,0,\lambda)\phi(t,\lambda)$ the constraint on $A(\lambda)$ of the form

$$A(r(\lambda)) = \phi_-(t,0,r(\lambda))\phi_+^{-1}(t,0,r(\lambda)) = K(t,0,\lambda)A(\lambda)K^{-1}(t,0,\lambda).$$

Since we assume to have zero seed solutions, the frozen Bäcklund transformations are of the form $\mathcal{B} = \mathrm{diag}(1 + (\alpha \pm i|\beta|)/(2\lambda), 1 + (\alpha \mp i|\beta|)/(2\lambda))$ and $\mathbb{B} = \mathrm{diag}(1 \pm i\alpha/\lambda, 1 \mp i\alpha/\lambda)$ for the NLS and sG equation, respectively. Therefore,

$$\tilde{a}_{11}(\lambda) = a_{11}(\lambda), \qquad \tilde{a}_{21}(\lambda) = \frac{2\lambda + \alpha \mp i|\beta|}{2\lambda + \alpha \pm i|\beta|}a_{21}(\lambda),$$

$$\tilde{a}_{11}(\lambda) = a_{11}(\lambda), \qquad \tilde{a}_{21}(\lambda) = \frac{\lambda \mp i\alpha}{\lambda \pm i\alpha}a_{21}(\lambda)$$

holds for the NLS and the sG equation, respectively. The same concept applied to the boundary matrices $\mathcal{K}(\lambda)$ for the Robin boundary and $\mathcal{K}(t,0,\lambda)$ for the new boundary condition as well as $\mathbb{K}(t,0,\lambda)$ for the sin-boundary condition results in the relations

$$a_{11}(-\lambda) = a_{11}(\lambda), \qquad a_{21}(-\lambda) = \frac{i\alpha + 2\lambda}{i\alpha - 2\lambda}a_{21}(\lambda),$$

$$a_{11}(-\lambda) = a_{11}(\lambda), \qquad a_{21}(-\lambda) = \frac{(2\lambda - i|\beta|)^2 - \alpha^2}{(2\lambda + i|\beta|)^2 - \alpha^2}a_{21}(\lambda)$$

as well as

$$a_{11}(\lambda^{-1}) = a_{11}(\lambda), \qquad a_{21}(\lambda^{-1}) = \frac{\alpha - i(\lambda - \lambda^{-1})}{\alpha + i(\lambda - \lambda^{-1})}a_{21}(\lambda),$$

respectively. These relations give the fundamental idea on how to choose the zeros of $a_{11}(\lambda)$ and $\tilde{a}_{11}(\lambda)$ on each side of the defect or $a_{11}(\lambda)$ for the boundary condition for the respective condition to be preserved under the Dressing method. That is if $\lambda_1, \dots, \lambda_N$ and $\tilde{\lambda}_1, \dots, \tilde{\lambda}_N$ are the zeros of $a_{11}(\lambda)$ and $\tilde{a}_{11}(\lambda)$ on the right and the left side of the defect, respectively, then $\tilde{a}_{11}(\lambda) = a_{11}(\lambda)$ implies that one way to construct a solution corresponds to choosing the set of zeros to be the same not necessarily in the right order. However by the Theorem of permutability 3.2.7 for the Dressing method, the order is not of primary importance and therefore we can take $\tilde{\lambda}_1 = \lambda_1, \dots, \tilde{\lambda}_N = \lambda_N$ without loss of generality. Further, the relation of $\tilde{a}_{21}(\lambda)$ to $a_{21}(\lambda)$ provides an insight into the relation of the norming constants. On the other hand, the relations $a_{11}(-\lambda) = a_{11}(\lambda)$ and $a_{11}(\lambda^{-1}) = a_{11}(\lambda)$ for the NLS equation and the sG equation with a boundary condition imply that if λ_1 is used in the Dressing method to introduce new zeros, then $-\lambda_1$ and λ_1^{-1} or rather $r(\lambda_1)$ should also emerge as a zero. Beyond that, the relations of $a_{21}(r(\lambda))$ to $a_{21}(\lambda)$ again foreshadow the relation of the norming constants. Nonetheless, this is only an idea and the effort of this thesis is to make it precise. That being said, it is still instrumental to see that the choice of relations of zeros has an origin. We refer to Figure 4.2 for an exemplary distribution of such zeros.

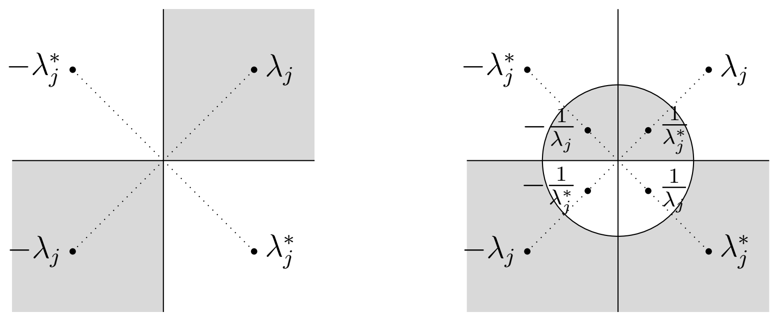

Fig. 4.2. Distribution of zeros in the presence of boundary conditions for the NLS equation (left) and for the sG equation (right).

Remark 4.4.1. The zero seed solution $\theta \equiv 0$ for the sG equation only satisfies the sin-boundary condition out of the three boundary conditions given in Proposition 4.3.2 and therefore this viewpoint only makes sense for this boundary condition.

Now assume we are given seed solutions which satisfy the defect conditions (4.1.2) or (4.1.4). Then, Propositions 3.1.1 and 3.1.2 imply that this is equivalent to the respective frozen Bäcklund transformation, say B_0, satisfying (4.1.1) and thus connecting the Lax pairs of the respective solution, which shifts the problem to the spectral side. Subsequently, we utilize the Dressing method to construct new solutions, as illustrated in this subsection. In theory, the last step would be to verify that the constructed solutions again satisfy the defect conditions. In order to be able to check this assertion, we want to show that there exists a frozen Bäcklund transformation represented through the matrix B_N which satisfies (3.1.2) at $x = 0$ for the Lax pairs corresponding to the by the Dressing method constructed solutions, see Figure 4.3.

In particular, the frozen Bäcklund transformation represented by the matrix B_0 is initially treated as a (frozen) one-fold dressing matrix as in Proposition 4.2.1. In turn, the frozen Bäcklund transformation represented by the matrix B_N is then at first also introduced as a (frozen) one-fold dressing matrix connecting the new solutions. So the last step mentioned above is to show that the constructed dressing matrix can be written as a matrix representing the frozen Bäcklund transformation containing the right parameters which are needed in order for the defect conditions to be preserved. In the case of a nonzero seed solution for the NLS equation, determining the

$$\begin{tikzcd}
\theta[0]=0 \arrow[r, "D[N]"'] \arrow[d, leftrightarrow, "\mathbb{B}_0"'] & \theta[N] \arrow[d, leftrightarrow, "\mathbb{B}_N"] \\
\tilde{\theta}[0]=0 \arrow[r, "\widetilde{D}[N]"'] & \tilde{\theta}[N]
\end{tikzcd}
\qquad
\begin{tikzcd}
u[0] \arrow[r, "D[N]"'] \arrow[d, leftrightarrow, "\mathcal{B}_0"'] & u[N] \arrow[d, leftrightarrow, "\mathcal{B}_N"] \\
\tilde{u}[0] \arrow[r, "\widetilde{D}[N]"'] & \tilde{u}[N]
\end{tikzcd}$$

Fig. 4.3. Schematic plan of the preservation of the frozen Bäcklund transformation under the Dressing method for the sG (left) and NLS equation (right).

$\pm$ sign turns out to be more convoluted. Indeed a similar requirement as in Proposition 4.2.1 regarding the value of the intermediate wave function is necessary in order to make sure that the sign stays the same. On top of that, the crux of the matter is that we need to determine a value at which we can verify that the sign is preserved in the first place which is structurally different from Proposition 4.2.1. In that regard, it turns out to be purposeful to analyze the (frozen) dressing matrix in more detail beforehand.

4.4.2 Frozen one-fold dressing matrix

So, the goal of this subsection is to establish conditions under which we can determine the sign of a matrix representing the frozen Bäcklund transformation of two solutions for which we applied the Dressing method. In that regard, important properties of a Bäcklund transformation with respect to x have been in detail discussed in detail in [17]. In particular, it is shown that the transformation $\mathcal{B}_{\operatorname{Im}(\lambda_1),\psi_1}\colon u \mapsto \tilde{u} = \mathcal{B}_{\operatorname{Im}(\lambda_1),\psi_1} u$, the Bäcklund transformation of $u(t,\cdot)$ with respect to $\{\operatorname{Im}(\lambda_1),\psi_1\}$ on $\mathbb{R}$, is a bijection from $H^{1,1}(\mathbb{R})$ onto $H^{1,1}(\mathbb{R})$. Similarly, we want to analyze the iteration of N one-fold dressing matrices as Bäcklund transformations with respect to t at $x=0$. Thus, for functions $f(\cdot,0,\lambda)$, we introduce the function spaces

$$H_t^{0,1}(\mathbb{R}) = \{f \in L^2(\mathbb{R})\colon tf \in L^2(\mathbb{R})\}, \qquad H_t^{1,1}(\mathbb{R}) = \{f \in L^2(\mathbb{R})\colon \partial_t f, tf \in L^2(\mathbb{R})\}$$

and state the following lemma, which is essential in the proof.

Lemma 4.4.2. *Let $f(\cdot,0,\lambda) \in H_t^{0,1}(\mathbb{R})$, $g(\cdot,0,\lambda) \in H_t^{1,1}(\mathbb{R})$ and $\operatorname{Im}(\lambda^2) < 0$. Then,*

$$\Big\| \int_{\langle t\rangle}^{\infty} f(\tau,0,\lambda) g(\tau,0,\lambda)\,\mathrm{d}\tau \Big\|_{H_t^{1,1}(\mathbb{R}_+)} \le c \|f(\cdot,0,\lambda)\|_{H_t^{0,1}(\mathbb{R}_+)} \|g(\cdot,0,\lambda)\|_{H_t^{1,1}(\mathbb{R}_+)},$$
$$\Big\| \int_{\langle t\rangle}^{\infty} f(\tau,0,\lambda) e^{-4\operatorname{Im}(\lambda^2)(\langle t\rangle-\tau)}\,\mathrm{d}\tau \Big\|_{H_t^{1,1}(\mathbb{R})} \le c \|f(\cdot,0,\lambda)\|_{H_t^{0,1}(\mathbb{R})},$$

where c depends on λ.

Proof. Analogously to the proof in [17], we take $t>0$ and show

$$\begin{aligned}
\Big| \int_t^{\infty} f(\tau,0,\lambda) g(\tau,0,\lambda)\,\mathrm{d}\tau \Big| &\le \int_t^{\infty} \frac{\tau^2+1}{t^2+1} |f(\tau,0,\lambda)|\, |g(\tau,0,\lambda)|\,\mathrm{d}\tau \\
&\le \frac{1}{t^2+1} \|f(\cdot,0,\lambda)\|_{H_t^{0,1}(\mathbb{R}_+)} \|g(\cdot,0,\lambda)\|_{H_t^{0,1}(\mathbb{R}_+)}.
\end{aligned}$$

Thus, the integral is in $H_t^{0,1}(\mathbb{R}_+)$. For the first derivative of the integral, we have the following equality $\frac{d}{dt}\int_t^\infty f(\tau,0,\lambda)g(\tau,0,\lambda)\,\mathrm{d}\tau = -f(t,0,\lambda)g(t,0,\lambda)$ and therefore the first inequality follows. Observing that

$$\int_t^\infty f(\tau,0,\lambda)e^{-4\operatorname{Im}(\lambda^2)(t-\tau)}\,\mathrm{d}\tau = \int_0^\infty f(t+\tau,0,\lambda)e^{4\operatorname{Im}(\lambda^2)\tau}\,\mathrm{d}\tau,$$

we have

$$\begin{aligned}&\left\|(1+|\langle t\rangle|)\int_{\langle t\rangle}^\infty f(\tau,0,\lambda)e^{-4\operatorname{Im}(\lambda^2)(\langle t\rangle-\tau)}\,\mathrm{d}\tau\right\|_{L^2(\mathbb{R})}\\ &\le \int_0^\infty \|(1+|\langle t\rangle|)f(\langle t\rangle+\tau,0,\lambda)\|_{L^2(\mathbb{R})}e^{4\operatorname{Im}(\lambda^2)\tau}\,\mathrm{d}\tau\end{aligned}$$

adding a zero with $\tau-\tau$ in the bracket, we can use the Minkowski inequality to obtain

$$\le c\|f(\cdot,0,\lambda)\|_{H_t^{0,1}(\mathbb{R})}\int_0^\infty (1+\tau)e^{4\operatorname{Im}(\lambda^2)\tau}\,\mathrm{d}\tau \le c\frac{1-\operatorname{Im}(\lambda^2)}{\operatorname{Im}(\lambda^2)^2}\|f(\cdot,0,\lambda)\|_{H_t^{0,1}(\mathbb{R}_+)}$$

Since the derivative of the integral is

$$\frac{d}{dt}\int_t^\infty f(\tau,0,\lambda)e^{-4\operatorname{Im}(\lambda^2)(t-\tau)}\,\mathrm{d}\tau = -f(t,0,\lambda) - 4\operatorname{Im}(\lambda^2)\int_t^\infty f(\tau,0,\lambda)e^{-4\operatorname{Im}(\lambda^2)(t-\tau)}\,\mathrm{d}\tau,$$

we use the same steps to show that it is in $L^2(\mathbb{R})$ and can conclude the proof. □

Given the spectral parameter $\lambda_1 = \xi_1 + i\eta_1$, we can denote the one-fold dressing matrix (3.2.2) as

$$D[1](t,x,\lambda) = \frac{1}{\lambda-\lambda_1^*}\begin{pmatrix}\lambda-\xi_1-i\eta_1\frac{1-|\Delta(t,x)|^2}{1+|\Delta(t,x)|^2} & -2i\eta_1\frac{\Delta^*(t,x)}{1+|\Delta(t,x)|^2}\\ -2i\eta_1\frac{\Delta(t,x)}{1+|\Delta(t,x)|^2} & \lambda-\xi_1+i\eta_1\frac{1-|\Delta(t,x)|^2}{1+|\Delta(t,x)|^2}\end{pmatrix}.$$

Therefore, the reconstruction formula (3.2.4) implies

$$\begin{aligned}u[1](t,x) &= u(t,x) + 4\eta_1\frac{\Delta^*(t,x)}{1+|\Delta(t,x)|^2},\\ u[1]_x(t,x) &= u_x(t,x) - 2i\xi_1(u[1](t,x)-u(t,x)) - 2\eta_1\frac{1-|\Delta(t,x)|^2}{1+|\Delta(t,x)|^2}(u[1](t,x)+u(t,x)).\end{aligned} \tag{4.4.2}$$

In particular, we assume that in the following $u(t,0)$ is given and the one-fold dressing matrix is used to determine $u[1](t,0)$. So, we have a well defined transformation $\mathcal{B}^t_{\lambda_1,\psi_1}: u \mapsto u[1] = \mathcal{B}^t_{\lambda_1,\psi_1}u$ mapping $u(\cdot,0)\in L^1_{\text{loc}}(\mathbb{R}) \to L^1_{\text{loc}}(\mathbb{R}) \ni u[1](\cdot,0)$. The denominator $1+|\Delta(t,0)|^2$ can not be zero, since ψ_1 is a solution of $\psi_t = (-2i\lambda^2\sigma_3 + \mathcal{Q}_1)\psi$ at $\lambda=\lambda_1$. If there exists a $t_0\in\mathbb{R}$ such that $\psi_1(t_0,0)=0$, then $(\psi_1)_t(t_0,0)=0$ and therefore $\psi_1(t,0)=0$ for every $t\in\mathbb{R}$. The assumption of a nonzero asymptotic limit of ψ_1 gives the contradiction. In particular, we want to have that if $u(t,0)$ vanishes for $|t|\to\infty$, then the same is true for the transformed function $u[1](t,0)$. Therefore, we work with functions $u(\cdot,0), u_x(\cdot,0)\in H_t^{1,1}(\mathbb{R})$. In particular, for $u(\cdot,0), u_x(\cdot,0)\in H_t^{0,1}(\mathbb{R})$, we can show that

$$\begin{aligned}\|\mathcal{Q}_1(\cdot,0,\lambda)\|_{L^1(\mathbb{R})} &\le \begin{pmatrix}\|u^2\|_{L^1(\mathbb{R})} & \|2\lambda u + iu_x\|_{L^1(\mathbb{R})}\\ \|2\lambda u^* + iu_x^*\|_{L^1(\mathbb{R})} & \|u^2\|_{L^1(\mathbb{R})}\end{pmatrix}\\ &\le \begin{pmatrix}\|u\|^2_{L^2(\mathbb{R})} & 2|\lambda|\|u\|_{H_t^{0,1}(\mathbb{R})} + \|u_x\|_{H_t^{0,1}(\mathbb{R})}\\ 2|\lambda|\|u\|_{H_t^{0,1}(\mathbb{R})} + \|u_x\|_{H_t^{0,1}(\mathbb{R})} & \|u\|^2_{L^2(\mathbb{R})}\end{pmatrix}.\end{aligned}$$

Thus, we can prove the following result.

Proposition 4.4.3. *$\mathcal{B}^t_{\lambda_1,\psi_1}$, where $\lambda_1 \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, maps functions $u(\cdot,0), u_x(\cdot,0) \in H^{1,1}_t(\mathbb{R})$ onto $u[1](\cdot,0), u[1]_x(\cdot,0) \in H^{1,1}_t(\mathbb{R})$.*

Proof. Following the proof for the Bäcklund transformation with respect to x, see [17, Prop. 4.7], we want to introduce a t dependent (Jost) function. In that regard, we freeze the space variable x, particularly, at $x = 0$. Then, given the limit behaviors $|u(t,0)| \to 0$ and $|u_x(t,0)| \to 0$ as $|t| \to \infty$, it is reasonable to assume that there exists a 2×1-vector-valued solution m to the spectral problem

$$\psi_t = (-2i\lambda^2\sigma_3 + \mathcal{Q}_1)\psi$$

admitting the asymptotic behavior $m(t,0,\lambda) \sim e_1 e^{-2i\lambda^2 t}$ as $t \to \infty$. Then, we also define the normalized t dependent (Jost) function by

$$\hat{m}(t,0,\lambda) = m(t,0,\lambda)e^{2i\lambda^2 t},$$

which admits the normalization $\lim_{t\to\infty} \hat{m}(t,0,\lambda) = e_1$. The solution $m(t,0,\lambda) = \hat{m}(t,0,\lambda)e^{-2i\lambda^2 t}$ is uniquely specified by the asymptotic behavior $\hat{m}(t,0,\lambda) \to e_1$ as $t \to \infty$. As in the usual scattering process, see [2], the normalized (Jost) function can be constructed by solving the following Volterra integral equation

$$\hat{m}(t,0,\lambda) = e_1 - \int_t^\infty \begin{pmatrix} 1 & 0 \\ 0 & e^{4i\lambda^2(t-\tau)} \end{pmatrix} \mathcal{Q}_1(\tau,0,\lambda)\, \hat{m}(\tau,0,\lambda)\, \mathrm{d}\tau. \tag{4.4.3}$$

This, we show by defining the operator

$$\mathcal{T}[\hat{m}](t,0,\lambda) = -\int_t^\infty \begin{pmatrix} 1 & 0 \\ 0 & e^{4i\lambda^2(t-\tau)} \end{pmatrix} \mathcal{Q}_1(\tau,0,\lambda)\, \hat{m}(\tau,0,\lambda)\, \mathrm{d}\tau,$$

which is a bounded operator mapping from $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$ for any fixed λ such that $\mathrm{Im}(\lambda^2) < 0$, since $t - \tau \leq 0$. Also, we define

$$\mathcal{T}_j[\hat{m}](t,0,\lambda) = -\int_t^{t_{j-1}} \begin{pmatrix} 1 & 0 \\ 0 & e^{4i\lambda^2(t-\tau)} \end{pmatrix} \mathcal{Q}_1(\tau,0,\lambda)\, \hat{m}(\tau,0,\lambda)\, \mathrm{d}\tau,$$

where we fix λ such that $\mathrm{Im}(\lambda^2) = 0$. For an arbitrary interval $(t_{j-1}, t_j) \subset \mathbb{R}$, we obtain the estimate

$$||\mathcal{T}_j[\hat{m}](\cdot,0,\lambda)||_{L^\infty(t_{j-1},t_j)} \leq ||\mathcal{Q}_1(\cdot,0,\lambda)||_{L^1(t_{j-1},t_j)} ||\hat{m}(\cdot,0,\lambda)||_{L^\infty(t_{j-1},t_j)}.$$

Then, we can choose t_j in such a way that the operator $\mathcal{T}_j$ is a contraction from $L^\infty(t_{j-1},t_j)$ to $L^\infty(t_{j-1},t_j)$. Repeating this argument starting from $t_0 = -\infty$ and appropriately chosen t_1, $\ldots$, to $t_{\ell-1}$ and $t_\ell = \infty$, we can obtain finitely many intervals so that $\mathcal{T}_j$ is contraction from $L^\infty(t_{j-1},t_j)$ to $L^\infty(t_{j-1},t_j)$, $j = 1,\ldots,\ell$. Setting $\hat{m}_0(t,0,\lambda) \equiv e_1$ on (t_0,t_1), we can find a function $\hat{m}_j(\cdot,0,\lambda) \in L^\infty(t_{j-1},t_j)$ by the Banach Fixed Point Theorem such that it solves the equation

$$\hat{m}_j(t,0,\lambda) = \hat{m}_{j-1}(t_j,0,\lambda) + \mathcal{T}_j[\hat{m}_j](t,0,\lambda), \qquad t \in (t_{j-1},t_j)$$

for every $j = 2,\ldots,\ell$. Combining these functions, we find a continuous function in $L^\infty(\mathbb{R})$ satisfying the Volterra integral equation (4.4.3), which covers the existence.

Now, for the claims regarding the continuation of $\hat{m}(t,0,\lambda)$ to $\mathrm{Im}(\lambda^2) \leq 0$. Analogously to the x dependent Jost solution $\widehat{\psi}^{(1)}_-(t,x,\lambda)$, we introduce for $\hat{m}(t,0,\lambda)$ the Neumann series $\sum_{j=0}^\infty \mathcal{T}^j[m_0](t,0,\lambda)$, where $m_0(t,0,\lambda) \equiv e_1$, which is formally a solution of the Volterra integral equation (4.4.3). Then, it is possible to derive a bound of the iterated operator $\mathcal{T}$. We define $h(t,\lambda)$ by

$$h(t,\lambda) = \int_t^\infty |\mathcal{Q}_1(\tau,0,\lambda)| \,\mathrm{d}\tau \leq \int_0^\infty |\mathcal{Q}_1(\tau,0,\lambda)| \,\mathrm{d}\tau \leq \|\mathcal{Q}_1(\cdot,0,\lambda)\|_{L^1(\mathbb{R})}.$$

By induction, we have

$$\begin{aligned}
|\mathcal{T}^{j+1}[\hat{m}](t,0,\lambda)| &\leq c\frac{\|\hat{m}(\cdot,0,\lambda)\|_{L^\infty(\mathbb{R})}}{j!}\int_t^\infty |\mathcal{Q}_1(\tau,0,\lambda)|\,(h(\tau,\lambda))^j \,\mathrm{d}\tau \\
&\leq c\frac{\|\hat{m}(\cdot,0,\lambda)\|_{L^\infty(\mathbb{R})}}{j!}\int_0^{h(t,\lambda)} s^j \,\mathrm{d}s \\
&= c\|\hat{m}(\cdot,0,\lambda)\|_{L^\infty(\mathbb{R})}\frac{(h(t,\lambda))^{j+1}}{(j+1)!},
\end{aligned}$$

where we put $s = h(\tau,\lambda)$. Thus, we have that $\sum_{j=0}^\infty \mathcal{T}^j[m_0](t,0,\lambda)$ is majorized in norm by a uniformly convergent power series and is therefore itself uniformly convergent for $\mathrm{Im}(\lambda) \leq 0$. The analyticity and continuity continuation for $\hat{m}(t,0,\lambda)$ in $\{\lambda \in \mathbb{C}\setminus\{0\}\colon \mathrm{Im}(\lambda^2) \leq 0\}$ and in $\{\lambda \in \mathbb{C}\setminus\{0\}\colon \mathrm{Im}(\lambda^2) < 0\}$, respectively, holds for the function $m(t,0,\lambda)$, which can be proven similarly as in the proof of Theorem 2.1.1. It is left, to show that the entries of $\hat{m}(\cdot,0,\lambda) - e_1$ are in $H_t^{1,1}(\mathbb{R})$. Since $\mathcal{T}$ maps $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$ and writing $\hat{m}(t,0,\lambda) = (\hat{m}_1, \hat{m}_2)$, we can estimate using Lemma 4.4.2,

$$\begin{aligned}
\|\hat{m}_2(\cdot,0,\lambda)\|_{H_t^{1,1}(\mathbb{R})} &\leq c\|([\mathcal{Q}_1]_{21}\hat{m}_1)(\cdot,0,\lambda)\|_{H_t^{0,1}(\mathbb{R})} + c\|([\mathcal{Q}_1]_{22}\hat{m}_2)(\cdot,0,\lambda)\|_{H_t^{0,1}(\mathbb{R})} \\
&\leq \|\hat{m}_1(\cdot,0,\lambda)\|_{L^\infty(\mathbb{R})}\|[\mathcal{Q}_1(\cdot,0,\lambda)]_{21}\|_{H_t^{0,1}(\mathbb{R})} \\
&\quad + \|\hat{m}_2(\cdot,0,\lambda)\|_{L^\infty(\mathbb{R})}\|[\mathcal{Q}_1(\cdot,0,\lambda)]_{22}\|_{H_t^{0,1}(\mathbb{R})}
\end{aligned}$$

and

$$\begin{aligned}
\|\hat{m}_1(\cdot,0,\lambda) - 1\|_{H_t^{1,1}(\mathbb{R})} &\leq c\|([\mathcal{Q}_1]_{11}\hat{m}_1)(\cdot,0,\lambda)\|_{H_t^{1,1}(\mathbb{R})} + c\|([\mathcal{Q}_1]_{12}\hat{m}_2)(\cdot,0,\lambda)\|_{H_t^{1,1}(\mathbb{R})} \\
&\leq \|\hat{m}_1(\cdot,0,\lambda)\|_{L^\infty(\mathbb{R})}\|[\mathcal{Q}_1(\cdot,0,\lambda)]_{11}\|_{H_t^{1,1}(\mathbb{R})} \\
&\quad + \|\hat{m}_2(\cdot,0,\lambda)\|_{L^\infty(\mathbb{R})}\|[\mathcal{Q}_1(\cdot,0,\lambda)]_{12}\|_{H_t^{1,1}(\mathbb{R})}.
\end{aligned}$$

And for the entries of $\mathcal{Q}_1(t,0,\lambda)$, we find

$$\begin{aligned}
\|[\mathcal{Q}_1(\cdot,0,\lambda)]_{11}\|_{H_t^{1,1}(\mathbb{R})} &\leq \|u(\cdot,0)\|_{L^\infty(\mathbb{R})}\|u(\cdot,0)\|_{H_t^{1,1}(\mathbb{R})}, \\
\|[\mathcal{Q}_1(\cdot,0,\lambda)]_{12}\|_{H_t^{1,1}(\mathbb{R})} &\leq 2|\lambda|\|u(\cdot,0)\|_{H_t^{1,1}(\mathbb{R})} + \|u_x(\cdot,0)\|_{H_t^{1,1}(\mathbb{R})}, \\
\|[\mathcal{Q}_1(\cdot,0,\lambda)]_{21}\|_{H_t^{0,1}(\mathbb{R})} &\leq 2|\lambda|\|u(\cdot,0)\|_{H_t^{0,1}(\mathbb{R})} + \|u_x(\cdot,0)\|_{H_t^{0,1}(\mathbb{R})}, \\
\|[\mathcal{Q}_1(\cdot,0,\lambda)]_{22}\|_{H_t^{0,1}(\mathbb{R})} &\leq \|u(\cdot,0)\|_{L^\infty(\mathbb{R})}\|u(\cdot,0)\|_{H_t^{0,1}(\mathbb{R})}.
\end{aligned} \tag{4.4.4}$$

Thus, if $u(\cdot,0), u_x(\cdot,0) \in H_t^{1,1}(\mathbb{R})$, then $\hat{m}(\cdot,0,\lambda) - e_1 \in H_t^{1,1}(\mathbb{R})$. For the uniqueness, we refer to the proof of Theorem 2.1.1. Next, we consider a solution $n(t,0,\lambda)$ of the t part of the Lax pair defined on $\mathrm{Im}(\lambda^2) \leq 0$ and $t \in \mathbb{R}$ with the property

$$n(t,0,\lambda) = (e_2 + r_1(t))e^{2i\lambda^2 t}, \quad r_1 \in H_t^{1,1}(\mathbb{R}).$$

Here, $e_2 = (0,1)^\intercal$ and $n(t,0,\lambda)$ is a non-unique solution of the differential equation defined on the same domain as $m(t,0,\lambda)$ and these vectors are linearly independent for all $t \in \mathbb{R}$:

For given $u(\cdot,0), u_x(\cdot,0) \in H_t^{1,1}(\mathbb{R})$ and $\lambda \in \{\lambda \in \mathbb{C}\setminus\{0\} \colon \operatorname{Im}(\lambda^2) \le 0\}$, we fix $t_0 > -\infty$ such that each entry of $\int_{t_0}^\infty |\mathcal{Q}_1(\tau,0,\lambda)|\,\mathrm{d}\tau$ is bounded by a constant $c_{t_0} < 1$ and by the arbitrary choice of t_0, the non-uniqueness is apparent. For $\operatorname{Im}(\lambda^2) \le 0$, we consider the following integral equation for $n(t,0,\lambda)$,

$$\begin{aligned} n(t,0,\lambda) = e^{2i\lambda^2 t} e_2 &+ \int_{t_0}^t \begin{pmatrix} e^{-2i\lambda^2(t-\tau)} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{Q}_1(\tau,0,\lambda)\, n(\tau,0,\lambda)\,\mathrm{d}\tau \\ &- \int_t^\infty \begin{pmatrix} 0 & 0 \\ 0 & e^{2i\lambda^2(t-\tau)} \end{pmatrix} \mathcal{Q}_1(\tau,0,\lambda)\, n(\tau,0,\lambda)\,\mathrm{d}\tau, \quad t \ge t_0. \end{aligned}$$

Set $\hat{n}(t,0,\lambda) = n(t,0,\lambda)e^{-2i\lambda^2 t}$, then the integral equation becomes

$$\hat{n}(t,0,\lambda) = e_2 + (\mathcal{N}\hat{n})(t,0,\lambda), \quad t \ge t_0, \tag{4.4.5}$$

where $\mathcal{N}$ is an integral operator defined by

$$\begin{aligned} (\mathcal{N}\hat{n})(t,0,\lambda) = &\int_{t_0}^t \begin{pmatrix} e^{-4i\lambda^2(t-\tau)} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{Q}_1(\tau,0,\lambda)\, \hat{n}(\tau,0,\lambda)\,\mathrm{d}\tau \\ &- \int_t^\infty \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{Q}_1(\tau,0,\lambda)\, \hat{n}(\tau,0,\lambda)\,\mathrm{d}\tau, \quad \hat{n}(\cdot,0,\lambda) \in L^\infty[t_0,\infty). \end{aligned}$$

By the same argument as for $\hat{m}(t,0,\lambda)$, we have existence of $\hat{n}(t,0,\lambda)$ for $t \in (t_0,\infty)$. As $\operatorname{Im}(\lambda^2) \le 0$ and each entry of $\mathcal{Q}_1(\cdot,0,\lambda)$ being in $L^1[t_0,\infty)$, $\mathcal{N}$ is a bounded operator from $L^\infty[t_0,\infty)$ to $L^\infty[t_0,\infty)$. Similar to before, put $\hat{n}_0(t,0,\lambda) = e_2$ and define $\hat{n}_{j+1}(t,0,\lambda) = e_2 + (\mathcal{N}\hat{n}_j)(t,0,\lambda)$, inductively. Then,

$$\|(\hat{n}_{j+1} - \hat{n}_j)(\cdot,0,\lambda)\|_{L^\infty[t_0,\infty)} \le c_{t_0}^j, \quad j \ge 0.$$

Indeed $\|\hat{n}_1(\cdot,0,\lambda) - \hat{n}_0(\cdot,0,\lambda)\|_{L^\infty[t_0,\infty)} \le c_{t_0}$ and for $j \ge 1$,

$$\begin{aligned} \|(\hat{n}_{j+1} - \hat{n}_j)(\cdot,0,\lambda)\|_{L^\infty[t_0,\infty)} &= \|(\mathcal{N}(\hat{n}_j - \hat{n}_{j-1}))(\cdot,0,\lambda)\|_{L^\infty[t_0,\infty)} \\ &\le \|(\hat{n}_j - \hat{n}_{j-1})(\cdot,0,\lambda)\|_{L^\infty[t_0,\infty)} \int_{t_0}^\infty |\mathcal{Q}_1(\tau,0,\lambda)|\,\mathrm{d}\tau \\ &= c_{t_0} \|(\hat{n}_j - \hat{n}_{j-1})(\cdot,0,\lambda)\|_{L^\infty[t_0,\infty)}. \end{aligned}$$

Therefore, $\hat{n}(t,0,\lambda) = \hat{n}_0(t,0,\lambda) + \sum_{j=1}^\infty \hat{n}_j(t,0,\lambda) - \hat{n}_{j-1}(t,0,\lambda)$ converges in $L^\infty[t_0,\infty)$ and solves the integral equation (4.4.5). Writing $\hat{n}(t,0,\lambda) = (\hat{n}_1, \hat{n}_2)^\intercal$, (4.4.5) becomes

$$\begin{aligned} \hat{n}_1(t,0,\lambda) &= \int_{t_0}^t e^{-4i\lambda^2(t-\tau)} ([\mathcal{Q}_1(\tau,0,\lambda)]_{11}\hat{n}_1(\tau,0,\lambda) + [\mathcal{Q}_1(\tau,0,\lambda)]_{12}\hat{n}_2(\tau,0,\lambda))\,\mathrm{d}\tau, \\ \hat{n}_2(t,0,\lambda) &= 1 - \int_t^\infty [\mathcal{Q}_1(\tau,0,\lambda)]_{21}\hat{n}_1(\tau,0,\lambda) + [\mathcal{Q}_1(\tau,0,\lambda)]_{22}\hat{n}_2(\tau,0,\lambda)\,\mathrm{d}\tau. \end{aligned}$$

As for $\hat{m}(t,0,\lambda)$, we can prove that if $u(\cdot,0), u_x(\cdot,0) \in H_t^{1,1}(\mathbb{R})$, then $\hat{n}_1(\cdot,0,\lambda) \in H_t^{1,1}[t_0,\infty)$. Therefore, we consider with Lemma 4.4.2 the estimate

$$\begin{aligned} \|\hat{n}_1(\cdot,0,\lambda)\|_{H_t^{1,1}[t_0,\infty)} &\le c\|([\mathcal{Q}_1]_{11}\hat{n}_1)(\cdot,0,\lambda)\|_{H_t^{0,1}[t_0,\infty)} + c\|([\mathcal{Q}_1]_{12}\hat{n}_2)(\cdot,0,\lambda)\|_{H_t^{0,1}[t_0,\infty)} \\ &\le \|\hat{n}_1(\cdot,0,\lambda)\|_{L^\infty[t_0,\infty)} \|[\mathcal{Q}_1(\cdot,0,\lambda)]_{11}\|_{H_t^{0,1}(\mathbb{R})} \\ &\quad + \|\hat{n}_2(\cdot,0,\lambda)\|_{L^\infty[t_0,\infty)} \|[\mathcal{Q}_1(\cdot,0,\lambda)]_{12}\|_{H_t^{0,1}(\mathbb{R})}. \end{aligned}$$

A similar reasoning involving Lemma 4.4.2 implies that $\hat{n}_2(\cdot,0,\lambda)-1\in H_t^{1,1}[t_0,\infty)$. We have

$$\begin{aligned}\|\hat{n}_2(\cdot,0,\lambda)-1\|_{H_t^{1,1}[t_0,\infty)} &\le c\|([\mathcal{Q}_1]_{21}\hat{n}_1)(\cdot,0,\lambda)\|_{H_t^{1,1}[t_0,\infty)}+c\|([\mathcal{Q}_1]_{22}\hat{n}_2)(\cdot,0,\lambda)\|_{H_t^{1,1}[t_0,\infty)}\\ &\le \|\hat{n}_1(\cdot,0,\lambda)\|_{L^\infty[t_0,\infty)}\|[\mathcal{Q}_1(\cdot,0,\lambda)]_{21}\|_{H_t^{1,1}(\mathbb{R})}\\ &\quad+\|\hat{n}_2(\cdot,0,\lambda)\|_{L^\infty[t_0,\infty)}\|[\mathcal{Q}_1(\cdot,0,\lambda)]_{22}\|_{H_t^{1,1}(\mathbb{R})}.\end{aligned}$$

Except for

$$\begin{aligned}\|[\mathcal{Q}_1(\cdot,0,\lambda)]_{21}\|_{H_t^{1,1}(\mathbb{R})} &\le 2|\lambda|\|u(\cdot,0)\|_{H_t^{1,1}(\mathbb{R})}+\|u_x(\cdot,0)\|_{H_t^{1,1}(\mathbb{R})},\\ \|[\mathcal{Q}_1(\cdot,0,\lambda)]_{22}\|_{H_t^{1,1}(\mathbb{R})} &\le \|u(\cdot,0)\|_{L^\infty(\mathbb{R})}\|u(\cdot,0)\|_{H_t^{1,1}(\mathbb{R})},\end{aligned}$$

all estimates on the entries of $\mathcal{Q}_1(t,0,\lambda)$ are already done in (4.4.4). Therefore, we indeed have that $\hat{n}(\cdot,0,\lambda)-e_2\in H_t^{1,1}[t_0,\infty)$ if $u(\cdot,0),u_x(\cdot,0)\in H_t^{1,1}(\mathbb{R})$. We know that $n(t,0,\lambda)$ defined through $\hat{n}(t,0,\lambda)$ solves the integral equation (4.4.3) for $t\in\mathbb{R}$ and we have its existence in $t\ge t_0$, it follows that, given t_0, $n(t,0,\lambda)$ can be uniquely extended to a solution of the t part of the Lax system for $\operatorname{Im}(\lambda^2)<0$.

The linear independence of $m(t,0,\lambda_1)$ and $n(t,0,\lambda_1)$, $\lambda_1\in\{\lambda\in\mathbb{C}\colon\ \operatorname{Im}(\lambda^2)<0\}$, can be shown by

$$\lim_{t\to\infty}\det(m(t,0,\lambda_1),n(t,0,\lambda_1))=1.$$

Since $\mathcal{V}$ has zero trace, we conclude that

$$\det(m(t,0,\lambda_1),n(t,0,\lambda_1))=1,\quad t\in\mathbb{R}.$$

Then, for $t\in\mathbb{R}$, we can write $\psi_1(t,0)$ as a linear combination of $m(t,0,\lambda_1)$ and $n(t,0,\lambda_1)$ so that

$$\psi_1(t,0)=c_1m(t,0,\lambda_1)+c_2n(t,0,\lambda_1)$$

for some constants c_1, c_2. If $c_2=0$, then as $t\to\infty$,

$$\psi_1(t,0)=c_1e^{-2i\lambda_1^2t}\begin{pmatrix}1+r_2(t)\\ r_3(t)\end{pmatrix},\quad r_j\in H_t^{1,1}(\mathbb{R}),\quad j=2,3.$$

Hence,

$$[P[1](t,0)]_{12}=\frac{(1+r_2(t))r_3(t)^*}{|1+r_2(t)|^2+|r_3(t)|^2}\in H_t^{1,1}(\mathbb{R}).$$

As in the argumentation for the Darboux matrix being a map from $u(\cdot,0)\in L^1_{\mathrm{loc}}(\mathbb{R})\to L^1_{\mathrm{loc}}(\mathbb{R})\ni u[1](\cdot,0)$, the denominator $|1+r_2(t)|^2+|r_3(t)|^2$ can not be zero, due to $m(t,0,\lambda)$ being a solution of the spectral problem $\psi_t=(-2i\lambda^2\sigma_3+\mathcal{Q}_1)\psi$ and given its asymptotic behavior as t goes to infinity. If $c_2\neq0$, then as $t\to\infty$,

$$\psi_1(t,0)=c_2e^{2i\lambda_1^2t}\begin{pmatrix}r_4(t)\\ 1+r_5(t)\end{pmatrix},\quad r_j\in H_t^{1,1}(\mathbb{R}),\quad j=4,5.$$

The same reasoning makes sure that the denominator can not be zero and hence,

$$[P[1](t,0)]_{12}=\frac{(1+r_5(t))^*r_4(t)}{|1+r_4(t)|^2+|r_5(t)|^2}\in H_t^{1,1}(\mathbb{R}).$$

Thus,

$$u[1](t,0)=u(t,0)+4\eta_1[P[1](t,0)]_{12}\in H_t^{1,1}(\mathbb{R}).$$

By the second line of equation (4.4.2), it can also be shown that $u[1]_x(\cdot,0)\in H_t^{1,1}(\mathbb{R})$ in both cases. For $\operatorname{Im}(\lambda^2)\ge0$, the choice of the normalization of $m(t,0,\lambda)$ and $n(t,0,\lambda)$ is reversed. □

If we extend this reasoning to the dressing matrix $D[1](t, x, \lambda)$, we can infer a helpful property in the determination of a value for which the sign of the matrix corresponding to the frozen Bäcklund transformation is preserved.

Lemma 4.4.4 (Deift & Park, [17]). *Let $u(\cdot, 0), u_x(\cdot, 0) \in H_t^{1,1}(\mathbb{R})$, and $D[1](t, x, \lambda)$ be a dressing matrix constructed by λ_1 and $\psi_1(t, x) = ([\psi_1]_1, [\psi_1]_2)^\intercal$. Then, $(\lambda - \lambda_1^*)D[1](t, 0, \lambda)$ goes to either $\mathrm{diag}(\lambda - \lambda_1^*, \lambda - \lambda_1)$ or $\mathrm{diag}(\lambda - \lambda_1, \lambda - \lambda_1^*)$ as $t \to \infty$, depending on the limit behavior of $\psi_1(t, 0)$.*

Proof. At $t = 0$ and $x = 0$, ψ_1 is either being produced by $(1, c)^\intercal$, $c \in \mathbb{C}$, or $(0, 1)^\intercal$. In the first case, $\psi_1(t, 0) = c_1 m(t, 0, \lambda_1) + c_2 n(t, 0, \lambda_1)$ for some constants c_1, c_2, where $m(t, 0, \lambda)$ and $n(t, 0, \lambda)$ are the linearly independent solutions of the t part of the Lax system as constructed in the proof of Proposition 4.4.3. If ψ_1 is proportional to $m(t, 0, \lambda_1)$, then necessarily $c_2 = 0$. As a consequence $\frac{[\psi_1]_2}{[\psi_1]_1} = \frac{m_2(t,0,\lambda_1)}{m_1(t,0,\lambda_1)} \to 0$ as $t \to \infty$ and so $(\lambda - \lambda_1^*)D[1](t, 0, \lambda)$ goes to $\mathrm{diag}(\lambda - \lambda_1, \lambda - \lambda_1^*)$ as $t \to \infty$. If $c_2 \neq 0$, then, as $t \to \infty$, $\psi_1(t, 0) = c_2 e^{2i\lambda_1^2 t} \begin{pmatrix} r_4(t) \\ 1 + r_5(t) \end{pmatrix}$, where $r_4, r_5 \in H_t^{1,1}(\mathbb{R}_+)$ as before. Therefore, $\frac{[\psi_1]_1}{[\psi_1]_2} \to 0$ as $t \to \infty$ and so $(\lambda - \lambda_1^*)D[1](t, 0, \lambda)$ goes to $\mathrm{diag}(\lambda - \lambda_1^*, \lambda - \lambda_1)$ as $t \to \infty$. In the second case, we necessarily have $c_1 = 0$ and again by $n(t, 0, \lambda_1)$, we have that $(\lambda - \lambda_1^*)D[1]\big|_{x=0}$ goes to $\mathrm{diag}(\lambda - \lambda_1^*, \lambda - \lambda_1)$ as $t \to \infty$. $\square$

In particular, this property can be restated in the following sense:

Remark 4.4.5. Let $u(\cdot, 0), u_x(\cdot, 0) \in H_t^{1,1}(\mathbb{R})$, then consecutively using one-fold dressing matrices corresponding to distinct spectral parameters $\lambda_1, \ldots, \lambda_N$ maps the functions u, u_x onto a function $u[N](\cdot, 0), u[N]_x(\cdot, 0) \in H_t^{1,1}(\mathbb{R})$. Moreover, the N-fold dressing matrix defined as in (3.2.12) multiplied by $\prod_{j=1}^{N}(\lambda - \lambda_j^*)$ goes to a product of diagonal matrices of the form $\mathrm{diag}(\lambda - \lambda_j^*, \lambda - \lambda_j)$ or $\mathrm{diag}(\lambda - \lambda_j, \lambda - \lambda_j^*)$ depending on the limit behavior of $\psi_j(t, 0)$ for $j = 1, \ldots, N$.

Chapter 5

Dressing

In the first section of this chapter, we want to apply a method, which we call *dressing the defect*, based on the Dressing method for Lax systems to prove that it is possible to explicitly construct soliton and breather solutions for the models of the sG and NLS equation on two half-lines which are connected via defect conditions as presented in Subsection 4.1.2. Initially, this method has been introduced as an alternative and more natural approach to the mirror image technique [5, 6] in order to construct solutions regarding the sG and NLS equation on the half-line with a sin-boundary condition [43] and a Robin boundary condition [42], respectively, where the authors called the method in the second more detailed paper *dressing the boundary.* As it turns out, it is by no means sufficient to take the method presented in [42] and to just apply it to other models as is, since it seems to be more or less specifically tailored to the NLS equation on the half-line with a Robin boundary condition or rather boundary conditions which are structurally similar to the Robin boundary condition. Thus, even though the methods of dressing the defect or dressing the boundary are called the same in this thesis, they should be understood as generalizations of the method initially developed.

5.1 Initial value problems with defect conditions

In general, the method of dressing the defect, as we generalized it, can be divided into three steps:

(a) We show that the functions derived by the Dressing method indeed satisfy the respective PDE on the appropriate domain which is more or less a formality. In order to verify this, Proposition 3.2.5 and Remark 3.2.6 are of importance.

(b) Afterwards, we construct a matrix, which is on the one hand not determined in terms of the solutions from (a) and on the other hand it is chosen so that the spectral equivalent of the defect conditions at $x = 0$ is satisfied. This is due to the fact that the spectral condition itself can be transformed into an equivalent, more handy expression which can, in turn, be proven by a comparison of two polynomial matrices with respect to the spectral parameter λ.

(c) Then, since the matrices are, up to a function of λ, polynomial matrices of degree one with respect to λ, we know by Propositions 3.1.1 and 3.1.2 the explicit forms of the (frozen) Darboux matrices. Thus, the goal is to determine the sign and the defect parameters and to verify that they match the sign and the defect parameters of the frozen Bäcklund transformation associated with the seed solutions.

In that regard, it appears that in every application, whether it is in the case of the sG or NLS equation, there are slightly different conditions which, in turn, lead to subtle changes in the proof. For example, in the case of the sG equation, it turns out to be advantageous that the seed solution is naturally assumed to be zero as utilized in Proposition 3.2.4 and also that for the reconstruction formula one evaluates the zero-th order matrix coefficient of the dressing matrix. Thus, we first apply dressing the defect to construct solutions in the case of the sG equation, since the proof has the most basic specifications.

5.1.1 The sG equation

For the convenience of the reader, we invoke the model we want to solve explicitly. Therefore, consider the sG equation on two half-lines

$$\theta_{tt} - \theta_{xx} + \sin\theta = 0 \tag{5.1.1}$$

for $\theta(t,x)\colon \mathbb{R}\times\mathbb{R}_+ \mapsto \mathbb{C}$ and initial conditions $\theta(0,x) = \theta_0(x)$ and $\theta_t(0,x) = \theta_1(x)$ for $x \in \mathbb{R}_+$ together with

$$\tilde{\theta}_{tt} - \tilde{\theta}_{xx} + \sin\tilde{\theta} = 0 \tag{5.1.2}$$

for $\theta(t,x)\colon \mathbb{R}\times\mathbb{R}_- \mapsto \mathbb{C}$ and initial condition $\tilde{\theta}(0,x) = \tilde{\theta}_0(x)$ and $\tilde{\theta}_t(0,x) = \tilde{\theta}_1(x)$ for $x \in \mathbb{R}_-$. Further in accordance to the defect conditions, $\theta(t,0)$ and $\tilde{\theta}(t,0)$ satisfy (4.1.4).

Similar to the method applied in [26], we want to utilize dressing the defect to insert soliton and breather solutions. As worked out in Subsection 4.4.1, it is sufficient when constructing a soliton and/or breather on one side of the defect to also construct a soliton and/or breather on the other side of the defect in order for the defect conditions to be preserved. Structurally, one way to achieve this preservation is to consider the same spectral parameter λ_j and appropriately chosen quotients of v_j and u_j used in the Dressing method on each of the half-lines for $j = 1,\dots,N$. This is worked out in the following statement.

Proposition 5.1.1. *Consider zero seed solutions $\theta[0] = 0$ and $\tilde{\theta}[0] = 0$ to the sG equation* (5.1.1) *and* (5.1.2)*, which at $x = 0$ satisfy the defect conditions* (4.1.4) *with $\alpha \in \mathbb{R}\setminus\{0\}$. Further, take solutions ψ_j, $j = 1,\dots,N$, of the Lax system* (2.2.3) *corresponding to $\theta[0]$ for distinct $\lambda = \lambda_j \in \mathbb{C}\setminus\big(\mathbb{R}\cup\{-i\alpha, i\alpha\}\big)$. Assume that there exist paired solutions $\widetilde{\psi}_j$, $j = 1,\dots,N$, of the Lax system* (2.2.3) *corresponding to $\tilde{\theta}[0]$ for the same spectral parameter $\lambda = \lambda_j$ and that they satisfy*

$$\widetilde{\psi}_j\big|_{x=0} = \mathbb{B}_0(t,0,\lambda_j)\psi_j\big|_{x=0}, \qquad j = 1,\dots,N, \tag{5.1.3}$$

where the matrix $\mathbb{B}_0$ is associated to the frozen Bäcklund transformation (3.1.5) *representing the defect conditions with either a plus or a minus sign. Then, two N-fold dressing matrices $D[N]$, $\widetilde{D}[N]$ using the corresponding solutions and spectral parameters lead to solutions $\theta[N]$ and $\tilde{\theta}[N]$ to the sG equation on the respective half-line, for which the defect conditions* (4.1.4) *are preserved.*

To this end, we shall show that the functions $\theta[N]$ and $\tilde{\theta}[N]$ constructed with the N-fold dressing matrices (a) satisfy the sG equation on the positive and negative half-line, respectively, (b) are regarding the Lax systems subject to defect conditions with a matrix $\mathbb{B}_N$, which is for the time being unspecified with respect to the solutions, and further, that (c) $\mathbb{B}_N$ can be written as a matrix corresponding to the frozen Bäcklund transformation (3.1.5) for the two solutions $\theta[N]$ and $\tilde{\theta}[N]$ with the sign and spectral parameter being preserved.

Proof. (a) Due to the analysis in Section 2.2, it is clear that there are two cases for spectral parameters λ_j, $j \in \{1, \dots, N\}$. The first case is represented by paired spectral parameters $\lambda_j \in i\mathbb{R}$ and $\lambda_j^* = -\lambda_j$ corresponding to single solitons for which there is another solution $\varphi_j(t,x) = -i\sigma_2\psi_j^*(t,x)$ or $\varphi_j(t,x) = \sigma_1\psi_j(t,x)$ according to the symmetry of the Lax pair, see (2.2.4). Since the norming constant is purely imaginary in that case, the choices for φ_j are not necessarily equal to ψ_j, but ψ_j and φ_j are linearly dependent. In the second case, the spectral parameters come in quadruples $\lambda_j \in \mathbb{C} \setminus \big(\mathbb{R} \cup i\mathbb{R}\big)$, $\lambda_k = -\lambda_j$, λ_j^* and $-\lambda_j^* = \lambda_k^*$, $k \in \{1, \dots, N\}$ and $k \neq j$, corresponding to breathers for which there are two additional solutions of the Lax system $\varphi_j(t,x) = -i\sigma_2\psi_j^*(t,x)$ for $\lambda = \lambda_j^*$ and $\varphi_k(t,x) = -i\sigma_2\psi_k^*(t,x)$ for $\lambda = \lambda_k^*$. Furthermore, there is also a connection between the solutions corresponding to λ_j and $\lambda_k = -\lambda_j$ given by $\psi_k(t,x) = \sigma_1\psi_j(t,x)$, which follows again from the symmetry of the Lax pair (2.2.4). Here, this distinction turns out to be not as important as later on in the case of the sG equation on the half-line with boundary conditions, however, one should be aware of this fact.

From Remark 3.2.6, we then derive that, since the spectral parameters $\lambda_1, \dots, \lambda_N$ and their complex conjugates are distinct, all solutions mentioned are linearly independent. Moreover, constructing the dressing matrix $D[N]$ from the vectors ψ_j and corresponding spectral parameters λ_j, we know that the dressing matrix multiplied by $\prod_{k=1}^N(\lambda - \lambda_k^*)$ has the following zeros and associated kernel vectors

$$\Big(\prod_{k=1}^N(\lambda - \lambda_k^*)D[N]\Big)\big|_{\lambda=\lambda_j}\psi_j = 0, \quad \Big(\prod_{k=1}^N(\lambda - \lambda_k^*)D[N]\Big)\big|_{\lambda=\lambda_j^*}\varphi_j = 0 \tag{5.1.4}$$

for $j = 1, \dots, N$. Moreover, the same reasoning for $\widetilde{\psi}_j$, $j = 1, \dots, N$, and $\widetilde{\varphi}_j$ chosen accordingly leads to

$$\Big(\prod_{k=1}^N(\lambda - \lambda_k^*)\widetilde{D}[N]\Big)\big|_{\lambda=\lambda_j}\widetilde{\psi}_j = 0, \quad \Big(\prod_{k=1}^N(\lambda - \lambda_k^*)\widetilde{D}[N]\Big)\big|_{\lambda=\lambda_j^*}\widetilde{\varphi}_j = 0. \tag{5.1.5}$$

In particular, if we arrange the systems (5.1.4) and (5.1.5) separately as sets of algebraic equations with the dressing matrices written in terms of a polynomial matrix in λ:

$$\prod_{k=1}^N(\lambda - \lambda_k^*)D[N] = \lambda^N\mathbb{1} + \sum_{k=1}^N \lambda^{N-k}\Sigma_k, \qquad \prod_{k=1}^N(\lambda - \lambda_k^*)\widetilde{D}[N] = \lambda^N\mathbb{1} + \sum_{k=1}^N \lambda^{N-k}\widetilde{\Sigma}_k,$$

it is possible to determine each matrix coefficient $\Sigma_1, \dots, \Sigma_N$ and $\widetilde{\Sigma}_1, \dots, \widetilde{\Sigma}_N$ explicitly if the vectors are linearly independent. We have

$$(\Sigma_1, \cdots, \Sigma_N)\begin{pmatrix} \lambda_1^{N-1}\psi_1 & (\lambda_1^*)^{N-1}\varphi_1 & \cdots & \lambda_N^{N-1}\psi_N & (\lambda_N^*)^{N-1}\varphi_N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_1 & \varphi_1 & \cdots & \psi_N & \varphi_N \end{pmatrix} = \begin{pmatrix} -\lambda_1^N\psi_1 \\ \vdots \\ -\lambda_N^N\varphi_N \end{pmatrix}.$$

The $2N \times 2N$-matrix

$$\begin{pmatrix} \lambda_1^{N-1}\psi_1 & (\lambda_1^*)^{N-1}\varphi_1 & \cdots & \lambda_N^{N-1}\psi_N & (\lambda_N^*)^{N-1}\varphi_N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_1 & \varphi_1 & \cdots & \psi_N & \varphi_N \end{pmatrix}$$

filled with the kernel vectors is invertible. Assuming it is not invertible, it follows that the columns of the matrix are linearly dependent vectors. Therefore, there exist constants $c_1, \dots, c_{2N} \in \mathbb{C}$ so that

$$c_1\psi_1 + c_2\varphi_1 + \cdots + c_{2N-1}\psi_N + c_{2N}\varphi_N = 0,$$

which is a contradiction to the linear independence of the kernel vectors derived in Remark 3.2.6 due to the corresponding spectral parameters being distinct. Applying the very same reasoning to $\prod_{k=1}^{N}(\lambda-\lambda_k^*)\widetilde{D}[N]$, we have that every matrix coefficient can be determined explicitly and therefore the solutions derived via Proposition 3.2.4 are in fact solutions of the sG equation on the respective half-line.

(b) Without loss of generality, we assume that the plus sign is used in the frozen Bäcklund transformation $\mathbb{B}_0$ which is then given by

$$\mathbb{B}_0(t,0,\lambda)=\mathbb{1}+\frac{i\alpha}{\lambda}\sigma_3,$$

not only taken at $x=0$, but also t independent. The goal of this step is to construct a matrix, which we note as $\mathbb{B}_N$ and which particularly satisfies $\mathbb{B}_N=\widetilde{D}[N]\mathbb{B}_0D[N]^{-1}$ at $x=0$. Equivalent to this equality, we have at $x=0$ the following

$$\Big(\prod_{k=1}^{N}(\lambda-\lambda_k^*)\widetilde{D}[N]\Big)(\lambda\mathbb{B}_0)\big|_{x=0}=(\lambda\mathbb{B}_N)\Big(\prod_{k=1}^{N}(\lambda-\lambda_k^*)D[N]\Big)\big|_{x=0}, \tag{5.1.6}$$

where both sides are multiplied by $\lambda\prod_{k=1}^{N}(\lambda-\lambda_k^*)$ to have polynomials with positive powers in λ which is be used later on. This equality functions as a way to prove the spectral equivalent (4.1.1) of the defect conditions (4.1.4) for the sG equation. Therefore, let us show that if we have a matrix $\mathbb{B}_N$ satisfying (5.1.6), then $\mathbb{B}_N$ satisfies the relations (4.1.1) at $x=0$ connecting $\widetilde{\mathbb{U}}[N]$, $\widetilde{\mathbb{V}}[N]$ and $\mathbb{U}[N]$, $\mathbb{V}[N]$. To demonstrate this, we exemplary take the equality $\mathbb{B}_N=\widetilde{D}[N]\mathbb{B}_0D[N]^{-1}$ and differentiate with respect to x (evaluated at $x=0$) in order to obtain

$$\begin{aligned}(\mathbb{B}_N)_x&=(\widetilde{D}[N]\mathbb{B}_0(D[N])^{-1})_x\\&=(\widetilde{D}[N])_x\mathbb{B}_0(D[N])^{-1}+\widetilde{D}[N](\mathbb{B}_0)_x(D[N])^{-1}+\widetilde{D}[N]\mathbb{B}_0((D[N])^{-1})_x.\end{aligned}$$

Then, the first two summands can be simplified using (3.2.15) and the x part of the spectral version of the frozen Bäcklund transformation (4.1.1) so that

$$\begin{aligned}(\widetilde{D}[N])_x\mathbb{B}_0+\widetilde{D}[N](\mathbb{B}_0)_x&=(\widetilde{\mathbb{U}}[N]\widetilde{D}[N]-\widetilde{D}[N]\widetilde{\mathbb{U}}[0])\mathbb{B}_0+\widetilde{D}[N](\widetilde{\mathbb{U}}[0]\mathbb{B}_0-\mathbb{B}_0\mathbb{U}[0])\\&=\widetilde{\mathbb{U}}[N]\widetilde{D}[N]\mathbb{B}_0-\widetilde{D}[N]\mathbb{B}_0\mathbb{U}[0].\end{aligned}$$

In addition, it can be shown with (3.2.15) that $((D[N])^{-1})_x$ from the third summand satisfies

$$\begin{aligned}((D[N])^{-1})_x&=-(D[N])^{-1}(D[N])_x(D[N])^{-1}\\&=-(D[N])^{-1}\mathbb{U}[N]+\mathbb{U}[0](D[N])^{-1}.\end{aligned}$$

If we put these results together and notice that the expressions $\widetilde{D}[N]\mathbb{B}_0(D[N])^{-1}$ are in fact again $\mathbb{B}_N$, we obtain

$$\begin{aligned}(\mathbb{B}_N)_x&=\widetilde{\mathbb{U}}[N]\widetilde{D}[N]\mathbb{B}_0(D[N])^{-1}-\widetilde{D}[N]\mathbb{B}_0(D[N])^{-1}\mathbb{U}[N]\\&=\widetilde{\mathbb{U}}[N]\mathbb{B}_N-\mathbb{B}_N\mathbb{U}[N].\end{aligned}$$

Similarly, the t part of (4.1.1) is implied effectively using the t part of the relations (3.2.15) and the t part of the frozen Bäcklund transformation (4.1.1) for $\mathbb{B}_0$, which is indeed simplified to

$\widetilde{\mathbb{V}}[0]\mathbb{B}_0 - \mathbb{B}_0\mathbb{V}[0] = 0$ due to the t independence of $\mathbb{B}_0$. Note that the terms in this calculation are always evaluated at $x = 0$ which is not written out in every term to ensure readability.

Now for the construction of the matrix $\mathbb{B}_N$, we define $\lambda_0 = i\alpha$ and take a closer look at the matrix multiplication $\widetilde{D}[N]\mathbb{B}_0 D[N]^{-1}$. By Remark 3.2.8, $D[N]^{-1}(t,x,\lambda) = D[N]^\dagger(t,x,\lambda^*)$ and therefore multiplying both sides with $\lambda\prod_{k=1}^N((\lambda-\lambda_k^*)(\lambda-\lambda_k))$ and relating the factors λ to $\mathbb{B}_0$, $\prod_{k=1}^N(\lambda-\lambda_k^*)$ to $\widetilde{D}[N]$ as well as $\prod_{k=1}^N(\lambda-\lambda_k)$ to $D[N]^{-1}$, we obtain a polynomial matrix of degree $2N+1$ in λ:

$$\Big(\sum_{k=1}^{2N+1}\lambda^k m_k\Big)\mathbb{B}_N = \sum_{k=0}^{2N+1}\lambda^k M_k,$$

where the exact expressions of the constant coefficients m_k and constant matrix coefficients M_k with respect to λ are not relevant in their entirety. One observation, we want to note, is, that the highest order coefficients are $m_{2N+1} = 1$ and $M_{2N+1} = \mathbb{1}$. Therefore, it makes sense to assume that $\mathbb{B}_N$ is of the form $\mathbb{B}_N = \mathbb{1} + \frac{1}{\lambda}\mathbb{B}^{(0)}$ at $x = 0$. However, for now we want to think of the matrices $\mathbb{B}_0$ and $\mathbb{B}_N$ being, up to a function of λ, one-fold dressing matrices. On the one hand, Proposition 4.2.1 implies that $\mathbb{B}_0$ admits at $x = 0$ a kernel vector $v(t) = -2c_2(t)e_2$, since $\theta[0] = 0$ and $\tilde{\theta}[0] = 0$, corresponding to the spectral parameter λ_0. On the other hand, we have that a solution of the Lax system at $\lambda = \lambda_0$ is given by $\psi_0(t,x) = u_0\psi_-^{(1)}(t,x,\lambda_0) + v_0\psi_+^{(2)}(t,x,\lambda_0)$, where u_0, v_0 are complex constants. Since we consider zero seed solutions $\theta[0] \equiv 0$ and $\tilde{\theta}[0] \equiv 0$, the Jost functions adopt the explicit forms $\psi_-^{(1)}(t,x,\lambda_0) = e_1 e^{-i\Theta(t,x,\lambda_0)}$ and $\psi_+^{(2)}(t,x,\lambda_0) = e_2 e^{i\Theta(t,x,\lambda_0)}$. Hence, choosing $u_0 = 0$ and $c_2(t)$ appropriately, we can infer that $\psi_0(t,0) = v(t)$ holds. Then, in particular, $\mathbb{B}_0\psi_0\big|_{x=0} = 0$. With these preliminary considerations, it turns out to be advantageous to construct the one-fold dressing matrix $\mathbb{B}_N$ with the vector $\psi_0' = D[N]\big|_{\lambda=\lambda_0}\psi_0$ corresponding to the spectral parameter $\lambda = \lambda_0 \in i\mathbb{R}\setminus\{0\}$ so that $\mathbb{B}_N\big|_{\lambda=\lambda_0}\psi_0' = 0$. Note that the constructed matrix is initially known for $x \in \mathbb{R}$ underlining the free choice of the point of the defect.

Now, we write the left and right hand side of the equality (5.1.6) at $x = 0$ as matrix polynomials $L(\lambda)$ and $R(\lambda)$, respectively. Hence,

$$L(\lambda) = \Big(\prod_{k=1}^N(\lambda-\lambda_k^*)\widetilde{D}[N]\Big)(\lambda\mathbb{B}_0)\big|_{x=0} = \lambda^{N+1}L_{N+1} + \lambda^N L_N + \cdots + L_0,$$
$$R(\lambda) = (\lambda\mathbb{B}_N)\Big(\prod_{k=1}^N(\lambda-\lambda_k^*)D[N]\Big)\big|_{x=0} = \lambda^{N+1}R_{N+1} + \lambda^N R_N + \cdots + R_0.$$

Further, we have made sure that the highest order matrix coefficient for both polynomials are equal $L_{N+1} = \mathbb{1} = R_{N+1}$, since the highest order coefficients of each individual matrix is the identity matrix. In the following, we want to show similarly to step (a) that the remaining matrix coefficients $L_N, R_N, \ldots, L_0, R_0$ are equal. However, since we only need them to be equal and we have that the highest order coefficients already satisfy this assertion, it is advantageous to consider the difference of the two matrix polynomials and the corresponding zeros and associated kernel vectors. First off, it is obvious that the zeros and kernel vectors for the dressing matrix $D[N]$ also function as zeros and associated kernel vectors for $R(\lambda)$ at $x = 0$. Moreover, due to assumption (5.1.3), at $x = 0$ the same is true for $L(\lambda)$ yielding

$$L(\lambda)\big|_{\lambda=\lambda_j}\psi_j = 0, \qquad R(\lambda)\big|_{\lambda=\lambda_j}\psi_j = 0, \qquad j = 1,\ldots,N.$$

Then, having cancelled out the singularities λ_j^* of the dressing matrices, this equality is transferred to the solutions φ_j of the Lax system at $\lambda = \lambda_j^*$ which can be expressed in terms of ψ_j. In particular, at $x = 0$ we obtain

$$L(\lambda)\big|_{\lambda=\lambda_j^*}\varphi_j = 0, \qquad R(\lambda)\big|_{\lambda=\lambda_j^*}\varphi_j = 0, \qquad j = 1, \dots, N.$$

For $R(\lambda)$, this is again true due to the properties of the dressing matrix and for $L(\lambda)$, we need to consider the assumption (5.1.3) in combination with the fact that the matrix $\mathbb{B}_0$ representing the frozen Bäcklund transformation admits similar symmetry relations to (2.2.4), i.e. $\mathbb{B}_0(\lambda) = \sigma_2\mathbb{B}_0^*(\lambda^*)\sigma_2$ or $\mathbb{B}_0(\lambda) = \sigma_1\mathbb{B}_0(-\lambda)\sigma_1$. If we compare the prerequisites in (b) with the ones in (a), it can be noticed that for a polynomial matrix of degree N with $N+1$ unknown matrix coefficients there is a need for $2(N+1)$ zeros and associated kernel vectors in order to determine them completely. Not counting the highest order coefficients for which we already have equality, it is therefore necessary to have $2(N+1)$ zeros and associated kernel vectors which is two more than the ones we obtain from the dressing matrix and assumption (5.1.3). This is exactly the reasoning for the interpretation of the matrices representing the frozen Bäcklund transformations as one-fold dressing matrices. Therefore, we have at $x = 0$ and for $j = 0$ that

$$L(\lambda)\big|_{\lambda=\lambda_0}\psi_0 = 0, \qquad L(\lambda)\big|_{\lambda=\lambda_0^*}\varphi_0 = 0, \qquad R(\lambda)\big|_{\lambda=\lambda_0}\psi_0 = 0, \qquad R(\lambda)\big|_{\lambda=\lambda_0^*}\varphi_0 = 0$$

from the properties we stated for $\mathbb{B}_0$ and $\mathbb{B}_N$. Then, to deduce the matrix coefficients of the difference $C(\lambda) = L(\lambda) - R(\lambda) = \lambda^N C_N + \cdots + C_0$, we arrange this system as a set of algebraic equations

$$\begin{aligned} \Big(\lambda_0^N C_N + \cdots + \lambda_0 C_1 + C_0\Big)\psi_0 &= 0, & \Big((\lambda_0^*)^N C_N + \cdots + \lambda_0^* C_1 + C_0\Big)\varphi_0 &= 0,\\ &\vdots & &\vdots\\ \Big(\lambda_N^N C_N + \cdots + \lambda_N C_1 + C_0\Big)\psi_N &= 0, & \Big((\lambda_N^*)^N C_N + \cdots + \lambda_N^* C_1 + C_0\Big)\varphi_N &= 0. \end{aligned}$$

This set of algebraic equations can further be written in matrix form resulting in

$$(C_N, \cdots, C_0)\begin{pmatrix} \lambda_0^N\psi_0 & (\lambda_0^*)^N\varphi_0 & \cdots & \lambda_N^N\psi_N & (\lambda_N^*)^N\varphi_N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_0 & \varphi_0 & \cdots & \psi_N & \varphi_N \end{pmatrix} = 0. \tag{5.1.7}$$

We can determine the matrix coefficients $C_0, \dots, C_N$, since the $2(N+1) \times 2(N+1)$-matrix filled with the kernel vectors is invertible. This again holds, due to the same argumentation we have given in step (a) and the two additional kernel vectors coming from distinct spectral parameters. Hence, it follows that all matrix coefficients $C_0, \dots, C_N$ are zero which comes from multiplying the equality (5.1.7) with the inverse $2(N+1) \times 2(N+1)$-matrix filled with the kernel vectors from the right. Since every matrix coefficient of $C(\lambda)$ is zero, the matrix polynomial $C(\lambda)$ is identically zero and consequently the matrix coefficient of $L(\lambda)$ and $R(\lambda)$ of the same power with respect to λ are necessarily equal. In summary, we present a way to construct a one-fold dressing matrix $\mathbb{B}_N$ which admits kernel vectors at the same spectral parameters as the initial frozen Bäcklund transformation represented through $\mathbb{B}_0$. Further by choosing $\mathbb{B}_N$ as proposed, equality (5.1.6) is satisfied so that $\mathbb{B}_N$ can at $x = 0$ also be expressed as $\widetilde{D}[N]\mathbb{B}_0(D[N])^{-1}$. Therefore, this concludes the task of step (b).

(c) By the reconstruction formula (3.2.16), we have two possibilities: First, an even or an odd number N_s of single solitons corresponding to the reconstruction formula $e^{i\frac{\theta[N]}{2}\sigma_1} = D[N]\big|_{\lambda=0}$ or $e^{i\frac{\theta[N]}{2}\sigma_1} = D[N]\big|_{\lambda=0}\sigma_3$, respectively, and similarly for $\tilde{\theta}[N]$ and $\widetilde{D}[N]$ as the multi-soliton on the two half-lines only differ regarding the norming constants represented through the different coefficients of the vectors ψ_j and $\widetilde{\psi}_j$, $j = 1, \dots, N$. In both cases we use the respective reconstruction formula to obtain the expression of the matrix $\mathbb{B}_N = \mathbb{1} + \frac{1}{\lambda}\mathbb{B}^{(0)}$ at $x = 0$ representing the frozen Bäcklund transformation of $\theta[N]$ and $\widetilde{\theta}[N]$ in terms of the solution space. In theory, taking into account Proposition 3.1.2, we know that $\mathbb{B}^{(0)}$ needs to be of the form

$$\mathbb{B}^{(0)} = \pm\frac{i\gamma}{\lambda}\begin{pmatrix} \cos\frac{\tilde{\theta}[N]+\theta[N]}{2} & -i\sin\frac{\tilde{\theta}[N]+\theta[N]}{2} \\ i\sin\frac{\tilde{\theta}[N]+\theta[N]}{2} & -\cos\frac{\tilde{\theta}[N]+\theta[N]}{2} \end{pmatrix}$$

with the parameter $\gamma \in \mathbb{R}$ and the sign $\pm$ to be determined. Assuming an

- even number of single solitons, we consider the equality of the zero-th order matrix coefficients in λ of $L(\lambda)$ and $R(\lambda)$ which is given by

$$L(\lambda)\big|_{\lambda=0} = \Big(\Big(\prod_{k=1}^{N}(\lambda - \lambda_k^*)\widetilde{D}[N]\Big)(\lambda\mathbb{B}_0)\big|_{x=0}\Big)\big|_{\lambda=0} = \prod_{k=1}^{N}(-\lambda_k^*)e^{i\frac{\tilde{\theta}[N]}{2}\sigma_1}\cdot(i\alpha)\sigma_3$$

as well as

$$R(\lambda)\big|_{\lambda=0} = \Big((\lambda\mathbb{B}_N)\Big(\prod_{k=1}^{N}(\lambda - \lambda_k^*)D[N]\Big)\big|_{x=0}\Big)\big|_{\lambda=0} = \prod_{k=1}^{N}(-\lambda_k^*)\mathbb{B}^{(0)}(t)\cdot e^{i\frac{\theta[N]}{2}\sigma_1}.$$

Evaluating for example the (11)-entry of $L(0) = R(0)$, we have

$$\pm i\gamma\left(\cos\frac{\tilde{\theta}[N]+\theta[N]}{2}\cos\frac{\theta[N]}{2} + \sin\frac{\tilde{\theta}[N]+\theta[N]}{2}\sin\frac{\theta[N]}{2}\right) = i\alpha\cos\frac{\tilde{\theta}[N]}{2}.$$

Under the trigonometric identities

$$\begin{aligned} \cos\frac{\tilde{\theta}[N]+\theta[N]}{2} &= \cos\frac{\tilde{\theta}[N]}{2}\cos\frac{\theta[N]}{2} - \sin\frac{\tilde{\theta}[N]}{2}\sin\frac{\theta[N]}{2}, \\ \sin\frac{\tilde{\theta}[N]+\theta[N]}{2} &= \sin\frac{\tilde{\theta}[N]}{2}\cos\frac{\theta[N]}{2} + \cos\frac{\tilde{\theta}[N]}{2}\sin\frac{\theta[N]}{2}, \end{aligned}$$

see (A.0.1), we can conclude that $\pm\gamma = \alpha$. In particular, the other entries consolidate this result and therefore $\mathbb{B}_N$ is at $x = 0$ determined in terms of the solutions $\theta[N]$ and $\widetilde{\theta}[N]$.

- odd number of single solitons, consider the equality of the zero-th order matrix coefficients

$$L(\lambda)\big|_{\lambda=0} = \prod_{k=1}^{N}(-\lambda_k^*)e^{i\frac{\tilde{\theta}[N]}{2}\sigma_1}\cdot(i\alpha) = \prod_{k=1}^{N}(-\lambda_k^*)\mathbb{B}^{(0)}(t)\cdot e^{i\frac{\theta[N]}{2}\sigma_1}\sigma_3 = R(\lambda)\big|_{\lambda=0},$$

which motivates the same calculation: Evaluating the (11)-entry of $L(0) = R(0)$, we obtain

$$\pm i\gamma\left(\cos\frac{\tilde{\theta}[N]+\theta[N]}{2}\cos\frac{\theta[N]}{2} + \sin\frac{\tilde{\theta}[N]+\theta[N]}{2}\sin\frac{\theta[N]}{2}\right) = i\alpha\cos\frac{\tilde{\theta}[N]}{2},$$

yielding the same result $\pm\gamma = \alpha$ for the one-fold dressing matrix $\mathbb{B}_N$ at $x = 0$ as in the case of an even number of single solitons. Checking this with the other entries may have different expressions than in the other case at some point, but eventually confirms the result.

Therefore at $x = 0$, both cases lead to the same matrix

$$\mathbb{B}_N(t,0,\lambda) = \mathbb{1} + \frac{i\alpha}{\lambda}\begin{pmatrix} \cos\frac{\tilde{\theta}[N]+\theta[N]}{2} & -i\sin\frac{\theta[\tilde{N}]+\theta[N]}{2} \\ i\sin\frac{\theta[\tilde{N}]+\theta[N]}{2} & -\cos\frac{\theta[\tilde{N}]+\theta[N]}{2}\end{pmatrix},$$

representing the frozen Bäcklund transformation connecting $\tilde{\theta}[N]$ and $\theta[N]$. Further, the spectral parameter α and the sign of the frozen Bäcklund transformation represented by $\mathbb{B}_0$ are preserved. The proof can be adapted to a minus sign of the initial frozen Bäcklund transformation.

□

With Proposition 5.1.1, we have given a first impression on the utility of the dressing the defect method. In particular, we have seen that the method can be applied to the sG equation on two half-lines connected via defect conditions (4.1.4). The application has been simplified due to the inherent assumption of zero seed solutions in the case of the sG equation. At this point we want to give some insight on how to take this further. First off, we mention step (a) in the upcoming propositions. However, since the argumentation is straightforward, the parts in the proof, which are close to this argumentation we have already seen, can be omitted. Then, step (b) and (c) become in the case of the NLS equation more intricate, mainly due to the consideration of nonzero seed solutions, where the defect conditions need to be satisfied initially. However, the main steps of the proofs are the same for dressing the defect in the case of the NLS equation.

In general, we deal with the dressing the defect and the dressing the boundary method in this chapter. Particularly, this means that we only give propositions which build the foundation to construct solutions via the Dressing method, while the construction itself is treated separately. In that regard, we have shown for the model of the sG equation on two half-lines (5.1.1) and (5.1.2) connected via defect conditions (4.1.4) that there is a way to explicitly construct soliton and/or breather solutions, where the spectral parameters on each side need to be equal and the norming constants need to satisfy some kind of relation which we inspect later on. As suggested in other works, the defect conditions with defect parameter α seem to behave as though they are 'half' a soliton [15]. Structurally, this explains the fact that the spectral parameters used for the Dressing method need to be distinct of the spectral parameter $\pm i\alpha$. If we use such a parameter for the construction of the dressing matrix, the defect conditions interact with the soliton in a such way that Proposition 5.1.1 would not be applicable. Since, in this case, the soliton is infinitely delayed—or swallowed—by the defect conditions, the need to construct a paired soliton solution on the other half-line becomes no longer necessary. Nonetheless, it can still be proven that such a swallowed solution exists. Later on, we elaborate on this particular case.

5.1.2 The NLS equation

As a generalization of the NLS equation (2.1.1), we present the model of the NLS equation on two half-lines

$$iu_t + u_{xx} + 2|u|^2u = 0 \tag{5.1.8}$$

for $u(t,x)\colon \mathbb{R}\times\mathbb{R}_+ \mapsto \mathbb{C}$ and initial condition $u(0,x) = u_0(x)$ for $x \in \mathbb{R}_+$ and

$$i\tilde{u}_t + \tilde{u}_{xx} + 2|\tilde{u}|^2\tilde{u} = 0 \tag{5.1.9}$$

for $\tilde{u}(t,x)\colon \mathbb{R}\times\mathbb{R}_- \mapsto \mathbb{C}$ and initial condition $\tilde{u}(0,x) = \tilde{u}_0(x)$ for $x \in \mathbb{R}_-$. In that context, taking for example $u(t,0) = \tilde{u}(t,0)$ and $u_x(t,0) = \tilde{u}_x(t,0)$ as boundary conditions, the two half-lines are connected so that there is no reflection and trivial transmission and by redefining the initial condition accordingly, we end up with the NLS equation as in (2.1.1). As suggested before, this idea corresponds to the choice of the identity matrix representing the frozen Bäcklund transformation (4.1.1). However, the model we are interested in arises with defect conditions (4.1.2) at $x = 0$. Therefore, assume we are given seed solutions $u[0](t,x)$ and $\tilde{u}[0](t,x)$ to the NLS equations on the respective half-line and two defect parameters $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$ together with the plus or minus sign for the defect so that these solutions satisfy the defect conditions (4.1.2). Then, this is equivalent to the fact that the matrix

$$\mathcal{B}_0(t,0,\lambda) = \mathbb{1} + \frac{1}{2\lambda}\begin{pmatrix} \alpha \pm i\sqrt{\beta^2 - |\tilde{u}[0]-u[0]|^2} & -i(\tilde{u}[0]-u[0]) \\ -i(\tilde{u}[0]-u[0])^* & \alpha \mp i\sqrt{\beta^2 - |\tilde{u}[0]-u[0]|^2} \end{pmatrix} \tag{5.1.10}$$

represents the frozen Bäcklund transformation (4.1.1) connecting the Lax pairs associated to $u[0](t,x)$ and $\tilde{u}[0](t,x)$. In contrast to the model of the sG equation with defect conditions, in the model of the NLS equation with defect conditions it is possible to start with more general seed solutions $u[0](t,x)$ and $\tilde{u}[0](t,x)$, since the reconstruction formula is not reliant on the seed solutions to be zero. In that regard, it is not surprising that this generalization is accompanied by a more intricate proof to show that dressing the defect can be applied. Especially, we use the results of Subsection 4.4.2 to ensure that the signs of the frozen Bäcklund transformations match, which was in the case of the sG equation not a difficulty.

Proposition 5.1.2. *Consider seed solutions $u[0]$ and $\tilde{u}[0]$ to the NLS equation* (5.1.8) *and* (5.1.9)*, which at $x = 0$ both satisfy the defect conditions* (4.1.2) *with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}\setminus\{0\}$ and are together with their first x-derivatives in the function space $H_t^{1,1}(\mathbb{R})$. Further, take solutions ψ_j, $j = 1,\ldots,N$, of the Lax system* (2.1.2) *corresponding to $u[0]$ for distinct $\lambda = \lambda_j \in \mathbb{C}\setminus\big(\mathbb{R}\cup i\mathbb{R}\cup\{-\frac{\alpha}{2}+i\frac{\beta}{2}, -\frac{\alpha}{2}-i\frac{\beta}{2}\}\big)$, $j = 1,\ldots,N$. Assume that there exist paired solutions $\widetilde{\psi}_j$, $j = 1,\ldots,N$, of the Lax system* (2.1.2) *corresponding to $\tilde{u}[0]$ for the same spectral parameter $\lambda = \lambda_j$ and that they satisfy*

$$\widetilde{\psi}_j\big|_{x=0} = \mathcal{B}_0(t,0,\lambda_j)\psi_j\big|_{x=0}, \quad j = 1,\ldots,N, \tag{5.1.11}$$

where the matrix $\mathcal{B}_0$ is associated to the frozen Bäcklund transformation (5.1.10) *representing the defect conditions with either a plus or a minus sign. Then, two N-fold dressing matrices $D[N]$, $\widetilde{D}[N]$ using the corresponding solutions and spectral parameters lead to solutions $u[N]$ and $\tilde{u}[N]$ to the NLS equation on the respective half-line, for which the defect conditions* (4.1.2) *are preserved under $\mathcal{B}_N$ of form* (3.1.4) *if*

$$\operatorname{Im}\Big(\lim_{\lambda\to 0}[2\lambda(\mathcal{B}_N(t,0,\lambda)-\mathbb{1})]_{11}\Big)$$

is greater than or equal to or rather less than or equal to 0 for all $t \in \mathbb{R}$ depending on its limit as $|t| \to \infty$.

Proof. (a) The N-fold dressing matrices $D[N]$, $\widetilde{D}[N]$ construct, as presented in Section 3.2, solutions $u[N]$, $\tilde{u}[N]$ from seed solutions $u[0]$, $\tilde{u}[0]$, which satisfy the same partial differential equations. In contrast to the sG equation, we have that each solution $\psi_j(t,x)$ to the Lax system corresponding to the spectral parameter $\lambda = \lambda_j$ comes as a pair, where $\varphi_j(t,x) = -i\sigma_2\psi_j^*(t,x)$ is the paired solution to the spectral parameter $\lambda = \lambda_j^*$, $j = 1,\ldots,N$. The requirement that the $2N$ solution vectors $\psi_j(t,x)$ and $\varphi_j(t,x)$, $j = 1,\ldots,N$, are linearly independent relies on the fact that the spectral parameter $\lambda_1,\ldots,\lambda_N$ and their complex conjugates $\lambda_1^*,\ldots,\lambda_N^*$ are distinct which is ensured by assumption.

(b) We define $\lambda_0 = -\frac{\alpha}{2} + i\frac{\beta}{2}$ and assume that the $\pm$ sign in the matrix $\mathcal{B}_0$, see (5.1.10), is a plus. Again, the idea is to construct a one-fold dressing matrix we note by $\mathcal{B}_N$ which at $x = 0$ satisfies the equality

$$\Big(\prod_{k=1}^{N}(\lambda - \lambda_k^*)\widetilde{D}[N]\Big)(\lambda\mathcal{B}_0)\big|_{x=0} = (\lambda\mathcal{B}_N)\Big(\prod_{k=1}^{N}(\lambda - \lambda_k^*)D[N]\Big)\big|_{x=0}. \tag{5.1.12}$$

The argumentation for the form of the matrix $\mathcal{B}_N = \mathbb{1} + \frac{1}{\lambda}\mathcal{B}^{(0)}$ as well as that equality (5.1.12) implies that $\mathcal{B}_N$ satisfies the relations of the frozen Bäcklund transformation for the dressed solutions $u[N]$, $\tilde{u}[N]$ is close to the one given in Proposition 5.1.1. However, due to the seed solutions not being zero, we have that the initial matrix representing the frozen Bäcklund transformation is time dependent and therefore the derivative of $\mathcal{B}_0$ is in this case not equal to zero. Therefore, we need to include at $x = 0$ the following equality $(\mathcal{B}_0)_t = \widetilde{\mathcal{V}}[0]\mathcal{B}_0 - \mathcal{B}_0\mathcal{V}[0]$ in the derivation of $(\mathcal{B}_N)_t$ resulting in

$$\begin{aligned}
(\mathcal{B}_N)_t &= (\widetilde{D}[N]\mathcal{B}_0(D[N])^{-1})_t \\
&= \widetilde{D}_t[N]\mathcal{B}_0(D[N])^{-1} + \widetilde{D}[N](\mathcal{B}_0)_t(D[N])^{-1} + \widetilde{D}[N]\mathcal{B}_0((D[N])^{-1})_t. \\
&= \widetilde{\mathcal{V}}[N]\widetilde{D}[N]\mathcal{B}_0(D[N])^{-1} - \widetilde{D}[N]\mathcal{B}_0(D[N])^{-1}\mathcal{V}[N] \\
&= \widetilde{\mathcal{V}}[N]\mathcal{B}_N - \mathcal{B}_N\mathcal{V}[N].
\end{aligned}$$

Moreover, due to $\mathcal{B}_0$ having off-diagonal entries, we can not simply identify the compatible kernel vector for the spectral parameter λ_0 immediately as for the sG equation. Rather, we need to assume in a more general fashion that there exists a kernel vector v_0 of the defect matrix $\mathcal{B}_0$ by means of Proposition 4.2.1 in addition to the vector ψ_0 chosen as the usual solution of the Lax system (2.1.2) corresponding to $u[0]$ for the spectral parameter λ_0. In theory, we would want to use the zero λ_0 and associated kernel vector v_0 for $\mathcal{B}_0$ to introduce a new kernel vector to the same zero in order to construct a one-fold dressing matrix as in the case of the sG equation so that we obtain two additional zeros and associated kernel vectors for the equality of the matrix polynomials. However, at this point it is not clear that the vector v_0 and $\psi_1, \dots, \psi_N, \varphi_1, \dots, \varphi_N$ are linearly independent. With that in mind, we differentiate two cases: The two vectors v_0 and ψ_0 are

1. linearly dependent at $x = 0$. This is the case we covered in the proof of Proposition 5.1.1. Since by Proposition 3.2.5 ψ_0 can not be expressed as a linear combination of $\psi_1, \dots, \psi_N$, we define

$$\psi_0' = D[N](t, x, \lambda_0)\psi_0 \neq 0.$$

Then, constructing, up to a function of λ, a one-fold dressing matrix, which we denote by $\mathcal{B}_N$ with the vector ψ_0' and the corresponding spectral parameter λ_0, at $x = 0$ we have

$$\begin{aligned}
\widetilde{D}[N](t, x, \lambda_0)(\lambda\mathcal{B}_0)(t, x, \lambda_0)\psi_0 &= (\lambda\mathcal{B}_N)(t, x, \lambda_0)D[N](t, x, \lambda_0)\psi_0 = 0, \\
\widetilde{D}[N](t, x, -\lambda_0^*)(\lambda\mathcal{B}_0)(t, x, \lambda_0^*)\varphi_0 &= (\lambda\mathcal{B}_N)(t, x, \lambda_0^*)D[N](t, x, \lambda_0^*)\varphi_0 = 0,
\end{aligned} \tag{5.1.13}$$

where $\varphi_0 = -i\sigma_2\psi_0^*$ is orthogonal to ψ_0.

2. linearly independent at $x = 0$. Then, the diagram of Figure 5.1 holds. In other words, if we have that ψ_0 is a solution of the Lax system (2.1.2) corresponding to $u[0]$, where $\psi_0, \dots, \psi_N$ are by assumption linearly independent, then the N-fold dressing matrix $D[N]$ is nonsingular at $\lambda = \lambda_0$. Therefore, we can transform ψ_0 to $\psi_0' = D[N]\big|_{\lambda=\lambda_0}\psi_0$ and by this transformation and the fact that

the dressing matrix satisfies relations (3.2.15), we can infer that

$$\begin{aligned}
(\psi_0')_x &= \left(D[N]\big|_{\lambda=\lambda_0}\right)_x\psi_0 + D[N]\big|_{\lambda=\lambda_0}(\psi_0)_x \\
&= \mathcal{U}[N]\big|_{\lambda=\lambda_0} D[N]\big|_{\lambda=\lambda_0}\psi_0 \\
&= \mathcal{U}[N]\big|_{\lambda=\lambda_0}\psi_0'
\end{aligned}$$

and the same for the t part so that ψ_0' is a solution of the Lax system corresponding to $u[N]$ at $\lambda = \lambda_0$. Now, since the kernel vector and ψ_0 are linearly independent, we can at $x = 0$ follow the same argumentation to derive a transformed solution $\widetilde{\psi}_0 = \mathcal{B}_0(t, 0, \lambda)\psi_0$, which is not the zero vector and satisfies the Lax system corresponding to $\tilde{u}[0]$ at $x = 0$ and $\lambda = \lambda_0$ due to the matrix $\mathcal{B}_0(t, 0, \lambda)$ representing the frozen Bäcklund transformation satisfying (4.1.1). Then again, this can be expanded by the application of $\widetilde{D}[N]$ to a solution $\widetilde{\psi}_0' = \widetilde{D}[N]\widetilde{\psi}_0$ of the Lax system corresponding to $\tilde{u}[N]$ also at $x = 0$ and $\lambda = \lambda_0$. Then, the connection of ψ_0' to $\widetilde{\psi}_0'$ implies that the product of matrices $\widetilde{D}[N]\mathcal{B}_0(D[N])^{-1}$ satisfies the relations (4.1.1) with $\widetilde{\mathcal{U}}[N]$, $\widetilde{\mathcal{V}}[N]$ and $\mathcal{U}[N]$, $\mathcal{V}[N]$ at $x = 0$ and $\lambda = \lambda_0$. In other words, there exists a matrix, we call $\mathcal{B}_N = \widetilde{D}[N]\mathcal{B}_0(D[N])^{-1}$ satisfying

$$\begin{aligned}
((\mathcal{B}_N)_x - \widetilde{U}[N]\mathcal{B}_N + \mathcal{B}_N U[N])\psi_0' = 0, \\
((\mathcal{B}_N)_t - \widetilde{V}[N]\mathcal{B}_N + \mathcal{B}_N V[N])\psi_0' = 0
\end{aligned}$$

at $\lambda = \lambda_0$ and $x = 0$. In that regard, as we have already shown this is equivalent to

$$\begin{aligned}
&\widetilde{D}[N](t, x, \lambda_0)(\lambda\mathcal{B}_0)(t, x, \lambda_0)\psi_0 = (\lambda\mathcal{B}_N)(t, x, \lambda_0)D[N](t, x, \lambda_0)\psi_0 \neq 0, \\
&\widetilde{D}[N](t, x, -\lambda_0^*)(\lambda\mathcal{B}_0)(t, x, \lambda_0^*)\varphi_0 = (\lambda\mathcal{B}_N)(t, x, \lambda_0^*)D[N](t, x, \lambda_0^*)\varphi_0 \neq 0
\end{aligned} \tag{5.1.14}$$

at $x = 0$. Note that we have not exactly constructed the matrix $\mathcal{B}_N$, but merely given a reasoning for the existence of a polynomial matrix $\lambda\mathcal{B}_N$ of degree one which satisfies (5.1.14).

$$\begin{tikzcd}
\psi_0 \text{ solves } \begin{cases}\psi_x = \mathcal{U}[0]\psi \\ \psi_t = \mathcal{V}[0]\psi\end{cases} \arrow[r, "D{[N]}"'] \arrow[d, "\mathcal{B}_0"'] & \psi_0' = D[N]\psi_0 \text{ solves } \begin{cases}\psi_x = \mathcal{U}[N]\psi \\ \psi_t = \mathcal{V}[N]\psi\end{cases} \arrow[d, dashed, "\mathcal{B}_N"] \\
\widetilde{\psi}_0 = \mathcal{B}_0\psi_0 \text{ solves } \begin{cases}\psi_x = \widetilde{\mathcal{U}}[0]\psi \\ \psi_t = \widetilde{\mathcal{V}}[0]\psi\end{cases} \arrow[r, "\widetilde{D}{[N]}"'] & \widetilde{\psi}_0' = \widetilde{D}[N]\widetilde{\psi}_0 \text{ solves } \begin{cases}\psi_x = \widetilde{\mathcal{U}}[N]\psi \\ \psi_t = \widetilde{\mathcal{V}}[N]\psi\end{cases}
\end{tikzcd}$$

Fig. 5.1. Properties of ψ_0 at $\lambda = \lambda_0$ and $x = 0$ if $\mathcal{B}_0(t, 0, \lambda_0)\psi_0(t, 0) \neq 0$ for all $t \in \mathbb{R}$.

Given $\mathcal{B}_N$ as in one of the two cases leads to commuting matrices at the point $x = 0$ of the defect conditions. To prove (5.1.12), we write each side as a matrix polynomial. Denoting the left and right hand side as $L(\lambda)$ and $R(\lambda)$, respectively, we obtain in both cases the following

$$\begin{aligned}
L(\lambda) &= \Big(\prod_{k=1}^{N}(\lambda - \lambda_k^*)\widetilde{D}[N]\Big)(\lambda\mathcal{B}_0)\big|_{x=0} = \lambda^{N+1}L_{N+1} + \lambda^N L_N + \cdots + \lambda L_1 + L_0, \\
R(\lambda) &= (\lambda\mathcal{B}_N)\Big(\prod_{k=1}^{N}(\lambda - \lambda_k^*)D[N]\Big)\big|_{x=0} = \lambda^{N+1}R_{N+1} + \lambda^N R_N + \cdots + \lambda R_1 + R_0.
\end{aligned}$$

Since again $L_{N+1} = \mathbb{1} = R_{N+1}$, only L_N, R_N, …, L_1, R_1, L_0 and R_0 need to be determined. In that regard, we consider the zeros and associated kernel vectors of $L(\lambda)$ and $R(\lambda)$. By construction of the dressing matrices $D[N]$, $\widetilde{D}[N]$, we have that $D[N](t,x,\lambda_j)\psi_j = 0$ and $\widetilde{D}[N](t,x,\lambda_j)\widetilde{\psi}_j = 0$, $j = 1,\dots,N$, which we combine with the assumed relation between ψ_j and $\widetilde{\psi}_j$. Thus, for the $2N$ linearly independent vectors $\psi_1,\dots,\psi_N$ and $\varphi_1,\dots,\varphi_N$, at $x = 0$ we obtain

$$\begin{aligned} L(\lambda)\big|_{\lambda=\lambda_j}\psi_j &= 0, & R(\lambda)\big|_{\lambda=\lambda_j}\psi_j &= 0,\\ L(\lambda)\big|_{\lambda=\lambda_j^*}\varphi_j &= 0, & R(\lambda)\big|_{\lambda=\lambda_j^*}\varphi_j &= 0 \end{aligned}$$

for $j = 1,\dots,N$. For the matrix $\lambda\mathcal{B}_N$ of order one, this is not enough to ensure equality in (5.1.12). However, we derived additional conditions for $\mathcal{B}_N$, i.e. (5.1.13) and (5.1.14), so that there is an additional vector pair for which the two sides are equal, but not necessarily zero. Hence in both cases, at $x = 0$ we have $L(\lambda)\big|_{\lambda=\lambda_0}\psi_0 = R(\lambda)\big|_{\lambda=\lambda_0}\psi_0$, where this equality is either nonzero in the case of linear independence or zero in the case of linear dependence of v_0 and ψ_0. As before, the symmetry of the Lax pair provides another vector φ_0 for which at $x = 0$ the equality $L(\lambda)\big|_{\lambda=\lambda_0^*}\varphi_0 = R(\lambda)\big|_{\lambda=\lambda_0^*}\varphi_0$ holds. Let us stress again that it is not important having additional zeros and associated kernel vectors of $L(\lambda)$ and $R(\lambda)$, but rather linearly independent vectors for which an equality as above holds, since, in the end, we consider the difference of the polynomial matrices rather than the polynomial matrices themselves. Thus, this *additional* pair of vectors is sufficient to determine the difference $C(\lambda) = L(\lambda) - R(\lambda) = \lambda^N C_N + \cdots + \lambda C_1 + C_0$. Together with the zeros and associated kernel vectors of the dressing matrices $D[N]$, $\widetilde{D}[N]$, it can be written as a set of algebraic equations. In matrix form, we have

$$(C_N,\cdots,C_0)\begin{pmatrix} \lambda_0^N\psi_0 & (\lambda_0^*)^N\varphi_0 & \cdots & \lambda_N^N\psi_N & (\lambda_N^*)^N\varphi_N\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \psi_0 & \varphi_0 & \cdots & \psi_N & \varphi_N \end{pmatrix} = 0.$$

As for the sG equation, the $(2N+2)\times(2N+2)$ matrix filled with $\psi_0,\varphi_0,\dots,\psi_N,\varphi_N$ is invertible. If the determinant is zero, we could find coefficients in $\mathbb{C}$ such that a linear combination of $\psi_0,\varphi_0,\dots,\psi_N,\varphi_N$ would be zero, which is a contradiction to their linear independence, which we justified in Proposition 3.2.5. Therefore, $L(\lambda) = R(\lambda)$ holds in both cases either linear dependence or linear independence of the kernel vector v_0 and ψ_0, which, in turn, implies that we actually have found a matrix $\mathcal{B}_N$ satisfying equality (5.1.12). Moreover, since we now know that in both cases $\mathcal{B}_N$ can be written as $\widetilde{D}[N]\mathcal{B}_0(D[N])^{-1}$ at $x = 0$, we can infer that the matrix $\lambda\mathcal{B}_N$ becomes singular at $\lambda = \lambda_0$ and $\lambda = \lambda_0^*$. Moreover, note that the determinant of each factor is t and x independent to begin with. In fact,

$$\begin{aligned} \det(\lambda\mathcal{B}_N) &= \det\Big(\prod_{k=1}^{N}(\lambda-\lambda_k^*)\widetilde{D}[N]\Big)\det(\lambda\mathcal{B}_0)\det\Big(\prod_{k=1}^{N}(\lambda-\lambda_k^*)D[N]\Big)^{-1}\\ &= \det(\lambda\mathcal{B}_0)\\ &= \Big(\lambda+\frac{\alpha}{2}\Big)^2 + \frac{\beta^2 - |\tilde{u}[0]-u[0]|^2}{4} + \frac{1}{4}|\tilde{u}[0]-u[0]|^2\\ &= (\lambda-\lambda_0)(\lambda-\lambda_0^*)\\ &= \lambda^2 + \alpha\lambda + \frac{\alpha^2+\beta^2}{4}, \end{aligned}$$

where the determinant of the dressing matrices is given by

$$\det\Big(\prod_{k=1}^{N}(\lambda-\lambda_k^*)D[N]\Big)=\det\Big(\prod_{k=1}^{N}(\lambda-\lambda_k^*)\widetilde{D}[N]\Big)=\prod_{k=1}^{N}((\lambda-\lambda_k)(\lambda-\lambda_k^*)),$$

see Remark 3.2.9. Consequently, the matrix $\lambda\mathcal{B}_N$ admits kernel vectors at these spectral parameters. Due to the fact that in the case of linear dependence we already identified a kernel vector $\psi_0'=D[N]\big|_{\lambda=\lambda_0}\psi_0$, this kernel vector is the foundation for both constructions of a one-fold dressing matrix $\mathcal{B}_N$ satisfying (5.1.12). Particularly, the dressing matrix $\mathcal{B}_N$ is initially defined for $t\in\mathbb{R}$, $x\in\mathbb{R}$ and for the equality (5.1.12) restricted to $x=0$.

(c) Consequently, part (b) implies that there exists a matrix $\mathcal{B}_1^{(0)}$ which is t and x dependent such that $\mathcal{B}_N(t,x,\lambda)=\mathbb{1}+\frac{1}{\lambda}\mathcal{B}_1^{(0)}(t,x)$ satisfies the frozen Bäcklund transformation (4.1.1) with $\widetilde{U}=\widetilde{\mathcal{U}}[N]$, $U=\mathcal{U}[N]$ and with $\widetilde{V}=\widetilde{\mathcal{V}}[N]$, $V=\mathcal{V}[N]$. By Proposition 3.1.1, there exist spectral parameters $\gamma,\delta\in\mathbb{R}$ and a plus or minus sign such that $\mathcal{B}_N(t,x,\lambda)$ can be at $x=0$ expressed as

$$\mathcal{B}_N(t,0,\lambda)=\mathbb{1}+\frac{1}{2\lambda}\begin{pmatrix}\gamma\pm i\sqrt{\delta^2-|\tilde{u}[N]-u[N]|^2} & -i(\tilde{u}[N]-u[N])\\ -i(\tilde{u}[N]-u[N])^* & \gamma\mp i\sqrt{\delta^2-|\tilde{u}[N]-u[N]|^2}\end{pmatrix}.$$

Comparing the determinant of this matrix multiplied by λ to the determinant of $\lambda\mathcal{B}_N$ we already calculated by the definition as a matrix multiplication, we obtain two conditions on the spectral parameters:

$$\gamma=\alpha,\qquad \frac{\gamma^2+\delta^2}{4}=\frac{\alpha^2+\beta^2}{4}.$$

Hence, the spectral parameter of $\mathcal{B}_N(t,0,\lambda)$ can effectively be determined to be $\gamma=\alpha$ and $\delta^2=\beta^2$. However, this observation carries no information on the $\pm$ sign. In that regard, we know that from solutions $u[0]$, $\tilde{u}[0]$ to the defect conditions with a selected sign, we can construct solutions $u[N]$, $\tilde{u}[N]$ which satisfy the defect conditions with either the plus or the minus sign. A particular case can be determined for which we are able to prove that the sign stays the same, ultimately restricting the solution space. Therefore, two additional assertions are necessary in order to ensure that

(i) the plus sign is preserved at least at a specific time;

(ii) the sign can not change under time evolution.

For the first point, the assertion

$$u[0](\cdot,0),\tilde{u}[0](\cdot,0),u[0]_x(\cdot,0),\tilde{u}[0]_x(\cdot,0)\in H_t^{1,1}(\mathbb{R})\tag{5.1.15}$$

is sufficient. Then, we have by Proposition 4.4.3 that $u[N](\cdot,0),\tilde{u}[N](\cdot,0),u[N]_x(\cdot,0),\tilde{u}[N]_x(\cdot,0)\in H_t^{1,1}(\mathbb{R})$, since $D[N]$, $\widetilde{D}[N]$ are N transformations of the form $\mathcal{B}^t_{\lambda_j,\psi_j[j-1]}$ for $j=1,\dots,N$. In this class of solutions, we can derive that the $\pm$ sign of the frozen Bäcklund transformation, or rather their matrix representation being either $\mathcal{B}_0(t,0,\lambda)$ or $\mathcal{B}_N(t,0,\lambda)$, is closely related to the kernel vector at the corresponding spectral parameter of their respective form as dressing matrix. In relation to that, we have worked out in (b) that ψ_0 is the kernel vector of $\mathcal{B}_0$ at $\lambda=\lambda_0$ and by construction, we have that $\psi_0'=D[N]\big|_{\lambda=\lambda_0}\psi_0$ is the kernel vector of $\mathcal{B}_N$ at $\lambda=\lambda_0$ and $x=0$. On the other hand, if one interprets $\mathcal{B}_0(t,0,\lambda)$ as a dressing matrix transforming $\psi_j(t,0)$ to $\widetilde{\psi}_j(t,0)$, by Lemma 4.4.4, we have that as $|t|$ goes to infinity $\mathcal{B}_0(t,0,\lambda)$ becomes a diagonal matrix and as a consequence, the limit behaviors of $\psi_j(t,0)$, $\widetilde{\psi}_j(t,0)$ are the same for $j = 1,\dots,N$ due to assumption (5.1.11). Consequently, by Lemma 4.4.4, the dressing matrices $\left(\prod_{k=1}^{N}(\lambda-\lambda_k^*)\widetilde{D}[N]\right)$ and $\left(\prod_{k=1}^{N}(\lambda-\lambda_k^*)D[N]\right)$ have at $x = 0$ the same distribution of $\lambda-\lambda_j$ and $\lambda-\lambda_j^*$ in their diagonal form as $|t| \to \infty$. Thus,

$$\begin{aligned}\lim_{|t|\to\infty} \mathcal{B}_N(t,0,\lambda) &= \lim_{|t|\to\infty} \widetilde{D}[N](t,0,\lambda)\mathcal{B}_0(t,0,\lambda)(D[N](t,0,\lambda))^{-1}\\ &= \lim_{|t|\to\infty} \mathcal{B}_0(t,0,\lambda).\end{aligned} \tag{5.1.16}$$

Alternatively, one could look at the vectors ψ_0 and ψ_0', which are at $\lambda = \lambda_0$ and $x = 0$ the kernel vectors of $\mathcal{B}_0$ and $\mathcal{B}_N$, respectively. Since they are connected by $D[N]\big|_{\lambda=\lambda_0}$ which admits a diagonal structure as $|t| \to \infty$, the kernel vectors also have the same limit behavior as $|t| \to \infty$. Both reasonings imply that given a plus sign in the (11)-entry of the matrix $\mathcal{B}_0$ corresponding to the frozen Bäcklund transformation, we can then conclude by the limit behavior of ψ_0 and ψ_0' or (5.1.16) that the sign in the (11)-entry of the matrix $\mathcal{B}_N(t,0,\lambda)$ also needs to be a plus sign at least for $|t|$ big enough.

Then, for the second point, the assertion regarding the imaginary part of a particular entry of the matrix $\mathcal{B}_N$, which is exactly the term influenced by the sign, is sufficient to ensure that the sign stays the same. By the definition of the matrix $\mathcal{B}_N$ representing the frozen Bäcklund transformation, we have on the one hand that $\mathrm{Im}(2\lambda\mathcal{B}_1^{(0)}(t,0)) = \pm\sqrt{\beta^2 - |\tilde{u}[N](t,0) - u[N](t,0)|^2}$. On the other hand, by the construction in (b) we find that it can also be expressed as

$$\mathrm{Im}(2\lambda\mathcal{B}_1^{(0)}(t,0)) = -\beta\frac{1-|\Delta_0(t,0)|^2}{1+|\Delta_0(t,0)|^2}, \qquad \Delta_0(t,0) = \frac{[D[N](t,0,\lambda_0)\psi_0(t,0)]_2}{[D[N](t,0,\lambda_0)\psi_0(t,0)]_1}.$$

Now, by the first point, the sign of the imaginary part is fixed as $+|\beta|$ for $|t|$ big enough. Consequently, the limit behavior of $|\Delta_0(t,0)|$ is either zero or infinity as $|t|$ goes to infinity depending on the sign of β, so that $\mathrm{Im}(2\lambda\mathcal{B}_1^{(0)}(t,0))$ goes to $-\beta$ or β in order to match $|\beta|$. Then, the assertion, which is equivalent to either $|\Delta_0(t,0)| \geq 1$ or $|\Delta_0(t,0)| \leq 1$, ensures that the sign stays the same for all $t \in \mathbb{R}$.

Therefore, the solutions $\widetilde{u}[N]$ and $u[N]$ satisfy defect conditions based on the same defect parameters and sign as the defect conditions for $\widetilde{u}[0]$ and $u[0]$ inferring the result under the assertions of Proposition 5.1.2. □

Here, the frozen Bäcklund transformation of Proposition 5.1.2 actually admits a t dependence accompanied by a matrix representation which not only has entries on the diagonal in contrast to the one applied in Proposition 5.1.1. This leads to a necessary development in order for the method of dressing the defect to still be applicable. Superficially, the same steps (a), (b) and (c) which we have worked out earlier need to be employed in order to prove the result for the model of the NLS equation. However, no particular changes are needed to adapt step (a) apart from the different analysis regarding the spectral parameters. The main difference in step (b) is that due to the seed solution not necessarily being zero the connection of the kernel vector of $\mathcal{B}_0(t,0,\lambda)$ at $\lambda = \lambda_0$ to the solution ψ_0 of the Lax system corresponding to $u[0]$ at $\lambda = \lambda_0$ is not as evident as for the sG equation with defect conditions. That being said, if we use zero seed solutions for the NLS model as in Proposition 5.1.1 for the sG model, the t dependence of the frozen Bäcklund transformation would disappear and further we would be left with a diagonal matrix for which the steps are basically indistinguishable. Continuing, in step (c) the consequences of this more general frozen Bäcklund transformation ultimately go back to the arbitrariness of the kernel vector that is the inability to exactly determine the parameters and the sign of the matrix $\mathcal{B}_N(t, 0, \lambda)$ in the form of a matrix representing the frozen Bäcklund transformation of $\widetilde{u}[N]$ and $u[N]$. At the same time, the kernel vector in this case can be used to resolve this additional complication by considering the limits as the time goes to infinity and assuming that the sign stays that way under time evolution. Again, starting with zero seed solutions, this step would be the same as in Proposition 5.1.1 for the sG model. Moreover, this calls for the necessity to restrict the solution space in order to be able to analyze the limits of the kernel vectors as time goes to infinity.

The important feature of the two proofs of Proposition 5.1.1 and 5.1.2 is that the matrix $\mathcal{B}_0(t, 0, \lambda)$ (or $\mathbb{B}_0(t, 0, \lambda)$) is *interchangeable* with the dressing matrices $D[N]$ and $\widetilde{D}[N]$ in the sense of Figure 4.3. In turn, this is realized through the transition of the matrix $\mathcal{B}_0$ to a dressing transformation and vice versa the introduced one-fold dressing matrix $\mathcal{B}_N(t, x, \lambda)$ (or $\mathbb{B}_N(t, x, \lambda)$) to a matrix which at $x = 0$ represents a frozen Bäcklund transformation.

Initially, the method of dressing the defect has been developed for models of partial differential equations on the half-line subject to boundary conditions at $x = 0$, for which it is known as dressing the boundary. In this context, it has been successfully applied to the NLS equation with Robin boundary conditions [42] and the sG equation with sin-boundary conditions [43]. Both of these models can be written with a corresponding Lax system for which the boundary matrix is diagonal and particularly t independent. As we have seen in this section, inserting solitons with N-fold dressing matrices under these circumstances requires slightly less effort due to correspondence of the kernel vector and the solution of the Lax system corresponding to the seed solution both at the same spectral parameter $\lambda = \lambda_0$. In fact, the consideration of the kernel vector can be omitted entirely if it is e.g. possible to identify an *additional* equality for the coefficients of the matrix polynomials $L(\lambda)$ and $R(\lambda)$ instead. In the next section, we want to show that the method we have developed for dressing the defect can without difficulty be adapted to most of the models for the NLS and sG equation on the half-line with boundary conditions presented in Section 4.3. In that regard, we have suggested in Subsection 4.4.1 on how to choose *pairs* of spectral parameters for the boundary conditions, which can be expressed with the corresponding boundary matrix by relation (4.3.1) to be preserved.

5.2 Initial-boundary value problems

Not surprisingly, there are plenty of differences between dressing the defect and dressing the boundary. The first one coming to mind is the spectral side equivalent to the defect or boundary conditions, which is in the case of the defect conditions given by two matrix relations (4.1.1) representing the t and x part and in the case of the boundary conditions by a matrix relation (4.3.1) representing a symmetry with respect to λ inherent only to the t part. Further due to these relations, sets of spectral parameters, which are sufficient for the defect or boundary conditions to be preserved under the Dressing method, could be deduced. For the defect conditions, we have identified the possibility where on both half-lines the same spectral parameter is used to construct a solution preserving the defect conditions. On the other hand for the boundary condition, the same process has led us to the choice of a *pair* of spectral parameters which underlies the inherent symmetry with respect to λ of the t part of the matrix relation [42]. It also turns out that in the cases where the boundary matrix for the seed solution is not a diagonal matrix, i.e. the Dirichlet and cos-boundary condition for the sG equation, this process of finding sufficient pairings of spectral parameters is not adequate. Furthermore, having the distribution of zeros under the symmetry from Figure 4.2 in mind as well as the proofs for the method of dressing the defect, where the existence of sufficiently many kernel vectors with associated spectral parameters is ensured, it is predictable that the cases in which zeros may coincide need to be treated separately. For the NLS equation, this case occurs with boundary-bound soliton solutions if for example $\lambda_1 = i\eta_1$, since then $-\lambda_1^* = i\eta_1$; for the sG equation, this case occurs with boundary-bound single soliton solutions or boundary-bound breather solutions if for example $N_s = 1$ and $|\lambda_1| = 1$, since then $\lambda_1 = \pm i$ and $1/\lambda_1^* = \pm i$. In particular, $\lambda \mapsto -\lambda$ and $\lambda \mapsto 1/\lambda$ are the aforementioned symmetries of the t part for the NLS and the sG equation, respectively.

5.2.1 The sG equation with boundary conditions

Consider the sG equation on the (positive) half-line

$$\theta_{tt} - \theta_{xx} + \sin\theta = 0 \tag{5.2.1}$$

for $\theta(t,x)\colon \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{C}$ and initial conditions $\theta(0,x) = \theta_0(x)$ and $\theta_t(0,x) = \theta_1(x)$ for $x \in \mathbb{R}_+$ together with the sin-boundary condition

$$\theta_x(t,0) = \alpha \sin\frac{\theta(t,0)}{2},$$

where $\alpha \in \mathbb{R}$. Then, it is obvious that the zero seed solution $\theta[0](t,x) \equiv 0$ satisfies this model with zero initial conditions. Note that in the case of cos- or even Dirichlet boundary conditions this is in general not true for arbitrary $\alpha \in \mathbb{R}$ and for the case that it is true ($\alpha = 0$), we have that these are already special cases of the sin-boundary condition. For this result, we split the simple eigenvalues or zeros of the Dressing method into $N = N_s + 2N_b + 2N_{bbb}$, N_s the number of single solitons for which we have two linearly independent solutions $\psi_j(t,x)$ and $\varphi_j(t,x) = \sigma_1\psi_j(t,x)$, $j = 1,\dots,N_s$, of the Lax system (2.2.3) corresponding to $\theta[0]$ for the spectral parameter $\lambda = \lambda_j$ and $\lambda = \lambda_j^* = -\lambda_j$, respectively, N_b the number of breathers for which we have four linearly independent solutions $\psi_j(t,x)$, $\varphi_j(t,x) = \sigma_1\psi_j(t,x)$, $\psi_{j+N_b}(t,x) = \sigma_3\psi_j^*(t,x)$ and $\varphi_{j+N_b}(t,x) = \sigma_1\psi_{j+N_b}(t,x)$, $j = N_s+1,\dots,N_s+N_b$ to the spectral parameter λ_j, $-\lambda_j$, $\lambda_{j+N_b} = -\lambda_j^*$ and $\lambda_{j+N_b}^* = \lambda_j^*$, respectively, and N_{bbb} the number of boundary-bound breathers for which these four spectral parameters additionally lie on $\{\lambda \in \mathbb{C}\colon |\lambda| = 1\}$. Further, we assume that the selected spectral parameter are always sorted so that for λ_j, $j = 1,\dots,N$, we have that $\lambda_1,\dots,\lambda_{N_s} \in i\mathbb{R} \setminus \{-i,0,i\}$, $\lambda_{N_s+1},\dots,\lambda_{N_s+2N_b} \in \mathbb{C} \setminus \big(\mathbb{R} \cup i\mathbb{R} \cup \{\lambda \in \mathbb{C}\colon |\lambda| = 1\}\big)$ and $\lambda_{N_s+2N_b+1},\dots,\lambda_N \in \{\lambda \in \mathbb{C}\colon |\lambda| = 1\} \setminus \{-i,-1,1,i\}$. Moreover, we define $N_d = 2N_s + 4N_b + 2N_{bbb}$.

Proposition 5.2.1. *Consider the zero seed solution $\theta[0] = 0$ to the sG equation on the half-line (5.2.1), which at $x = 0$ satisfies the* sin*-boundary condition with $\alpha \in \mathbb{R} \setminus \{0\}$. Further, take solutions ψ_j, $j = 1,\dots,N$, of the Lax system (2.2.3) corresponding to $\theta[0]$ for distinct $\lambda = \lambda_j \in \mathbb{C} \setminus \big(\mathbb{R} \cup \{-i,i\}\big)$. If $|\alpha| < 2$, the spectral parameters λ_j, $j = 1,\dots,N$, further need to be different from the four points $\pm\frac{i\alpha}{2} \pm \sqrt{1 - \frac{\alpha^2}{4}}$, whereas if $|\alpha| > 2$, the spectral parameters λ_j need to be different from the four purely imaginary points $i\Big(\pm\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - 1}\Big)$. Assume that there exist paired solutions*

(i) $\widehat{\psi}_j$, $j = 1,\dots,N_s$, of the same Lax system for the spectral parameter $\lambda = \lambda_j^{-1}$ and that they satisfy

$$\widehat{\psi}_j\big|_{x=0} = \mathbb{K}_0(\lambda_j^{(-1)^{N_{bbb}}})\psi_j\big|_{x=0}; \tag{5.2.2}$$

(ii) $\widehat{\psi}_j$, $\widehat{\psi}_{j+N_b}$, $j = N_s+1,\dots,N_s+N_b$, *of the same Lax system for the spectral parameter* $\lambda = \lambda_j^{-1}$ *and* $\lambda = \lambda_{j+N_b}^{-1}$, *respectively, and that they satisfy*

$$\widehat{\psi}_j\big|_{x=0} = \mathbb{K}_0(\lambda_j^{(-1)^{N_{bbb}}})\psi_j\big|_{x=0}, \qquad \widehat{\psi}_{j+N_b}\big|_{x=0} = \mathbb{K}_0((-\lambda_j^*)^{(-1)^{N_{bbb}}})\psi_{j+N_b}\big|_{x=0}. \tag{5.2.3}$$

- *Further, for* $j = N_s+2N_b+1,\dots,N_s+2N_b+N_{bbb}$, *assume that the solutions of the Lax system* ψ_j, φ_j, $\psi_{j+N_{bbb}}$ *and* $\varphi_{j+N_{bbb}}$ *satisfy the following relations*

$$\varphi_{j+N_{bbb}}\big|_{x=0} = \mathbb{K}_0(\lambda_j^{(-1)^{N_{bbb}}})\psi_j\big|_{x=0}, \qquad \psi_{j+N_{bbb}}\big|_{x=0} = \mathbb{K}_0((-\lambda_j)^{(-1)^{N_{bbb}}})\varphi_j\big|_{x=0}. \tag{5.2.4}$$

The matrix $\mathbb{K}_0(\lambda)$ *is associated to the boundary matrix* (4.3.10) *representing the* sin*-boundary condition. Then, an* N_d*-fold dressing matrix* $D[N_d]$ *using the corresponding solutions and spectral parameters leads to the solution* $\theta[N_d]$ *to the sG equation on the half-line, for which the* sin*-boundary condition* (4.3.7) *with either* α *or* $-\alpha$ *as boundary parameter is preserved.*

As for dressing the defect, we shall show that the function $\theta[N_d]$ constructed with the N_d-fold dressing matrix (a) satisfies the sG equation on the half-line, (b) is regarding to the Lax system subject to the boundary constraint with a matrix $\mathbb{K}_N$, which is not specified in terms of the solution, and in conclusion, that (c) $\mathbb{K}_N$ inherits the parameter α or $-\alpha$ from $\mathbb{K}_0$. Therefore, it is important to note that the zero seed solution satisfies the sin-boundary condition with both boundary parameters α and $-\alpha$.

Proof. (a) For $j = 1,\dots,N_s$, we take the distinct spectral parameters λ_j and λ_j^{-1} and corresponding linearly independent solutions of the Lax system. Further, for $j = N_s+1,\dots,N_s+N_b$, we take the distinct spectral parameters λ_j, λ_{j+N_b}, λ_j^{-1} and $\lambda_{j+N_b}^{-1}$ and corresponding linearly independent solutions of the Lax system. Finally for $j = N_s+2N_b+1,\dots,N_s+2N_b+N_{bbb}$, we take the distinct spectral parameters λ_j and $\lambda_{j+N_{bbb}}$ and corresponding linearly independent solutions of the Lax system. Altogether, we then construct the N_d-fold dressing matrix with these N_d linearly independent solutions, see Remark 3.2.6, of the Lax system (2.2.3) corresponding to $\theta[0]$. This ensures that $\theta[N_d]$ is, in fact, a solution of the sG equation on the half-line (5.2.1). Clarifying which particular spectral parameter we use in order to construct the dressing matrix comprehensibly provides us with the means to obtain the equality of the polynomial matrices more easily. Now, for the dressing matrix $D[N_d](t,x,\lambda)$ and the dressing matrix $D[N_d](t,x,\lambda^{-1})$ where we inverse the spectral parameter, we suggest multiplying the matrices with polynomials of λ in order to remove the singularities: For single solitons, we multiply $D[N_d](t,x,\lambda)$ with $\Pi_1 = \prod_{k=1}^{N_s}((\lambda-\lambda_k^*)(\lambda-1/\lambda_k^*))$ which is the same for $D[N_d](t,x,\lambda^{-1})$ since $\prod_{k=1}^{N_s}((\lambda-\lambda_k^*)(\lambda-1/\lambda_k^*)) = \lambda^2\prod_{k=1}^{N_s}((\lambda^{-1}-1/\lambda_k^*)(\lambda^{-1}-\lambda_k^*))$. Similarly for breathers, we denote the following term

$$\prod_{k=N_s+1}^{N_s+N_b}\Big(\Big(\lambda-\lambda_k^*\Big)\Big(\lambda+\lambda_k\Big)\Big(\lambda-\frac{1}{\lambda_k^*}\Big)\Big(\lambda+\frac{1}{\lambda_k}\Big)\Big) = \lambda^4\prod_{k=N_s+1}^{N_s+N_b}\Big(\Big(\frac{1}{\lambda}-\frac{1}{\lambda_k^*}\Big)\Big(\frac{1}{\lambda}+\frac{1}{\lambda_k}\Big)\Big(\frac{1}{\lambda}-\lambda_k^*\Big)\Big(\frac{1}{\lambda}+\lambda_k\Big)\Big)$$

by Π_2 and multiply $D[N_d](t,x,\lambda)$ and $D[N_d](t,x,\lambda^{-1})$ with it. Merely, the factors for the boundary bound breathers differ, i.e. $\Pi_3 = \prod_{k=N_s+2N_b+1}^{N_s+2N_b+N_{bbb}}((\lambda-\lambda_k^*)(\lambda+\lambda_k))$ for $D[N_d](t,x,\lambda)$ and $\Pi_3' = \lambda^2\prod_{k=N_s+2N_b+1}^{N_s+2N_b+N_{bbb}}((\lambda^{-1}-\lambda_k^*)(\lambda^{-1}+\lambda_k))$ for $D[N_d](t,x,\lambda^{-1})$. Therefore, we prove step (b) with the dressing matrices multiplied with their respective three products denoting each one again as $D[N_d](t,x,\lambda)$ and $D[N_d](t,x,\lambda^{-1})$. Further, given these modifications, the new dressing matrices are polynomial matrices of degree N_d and can be written as

$$D[N_d](t,x,\lambda) = \lambda^{N_d}\mathbb{1} + \cdots + \Sigma_{N_d}(t,x) \quad\text{and}\quad D[N_d](t,x,\lambda^{-1}) = \lambda^{N_d}\Sigma_{N_d}(t,x) + \cdots + \mathbb{1}.$$

(b) The sin-boundary matrix is given by

$$\mathbb{K}_0(t,\lambda^{(-1)^{N_{bbb}}}) = \frac{1}{\sqrt{(\lambda-\frac{1}{\lambda})^2+\alpha^2}}\Big(-(-1)^{N_{bbb}}i\Big(\lambda-\frac{1}{\lambda}\Big)\sigma_3-\alpha\mathbb{1}\Big).$$

which is, in particular, t independent, and therefore

$$\mathbb{V}[0](t,0,\lambda^{-1})\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}}) - \mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}})\mathbb{V}[0](t,0,\lambda) = (\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}}))_t = 0. \tag{5.2.5}$$

The minus sign for α can also be chosen to be plus due to the zero seed solution satisfying the sin-boundary condition with α and $-\alpha$. Therefore, the goal is to construct a matrix, which we note as $\mathbb{K}_N$ and which particularly satisfies $\mathbb{K}_N\big|_{x=0} = D[N_d](t,0,\lambda^{-1})\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}})D[N_d]^{-1}(t,0,\lambda)$. Similarly to equality (5.1.6), we have

$$(D[N_d](t,x,\lambda^{-1})\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}}))\big|_{x=0} = (\mathbb{K}_N(t,x,\lambda)D[N_d](t,x,\lambda))\big|_{x=0}. \tag{5.2.6}$$

Moreover, it is important to note that equality (5.2.6) is sufficient for $\mathbb{K}_N$ to satisfy (4.3.1) at $x=0$ except for the zeros of the Lax system. To show this explicitly, we multiply the equation with $(D[N_d](t,0,\lambda))^{-1}$ from the right and differentiate the resulting equation with respect to t to obtain

$$\begin{aligned}(\mathbb{K}_N)_t\big|_{x=0} &= (D[N_d](t,0,\lambda^{-1})\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}})(D[N_d](t,0,\lambda))^{-1})_t \\ &= D_t[N_d](t,0,\lambda^{-1})\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}})(D[N_d](t,0,\lambda))^{-1} \\ &\quad + D[N_d](t,0,\lambda^{-1})\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}})((D[N_d](t,0,\lambda))^{-1})_t.\end{aligned}$$

Utilizing the t part of (3.2.15) for a N_d-fold dressing matrix, we have

$$\begin{aligned}D_t[N_d](t,0,\lambda^{-1}) &= \mathbb{V}[N_d](t,0,\lambda^{-1})D[N_d](t,0,\lambda^{-1}) - D[N_d](t,0,\lambda^{-1})\mathbb{V}[0](t,0,\lambda^{-1}),\\ (D[N_d](t,0,\lambda))_t^{-1} &= -(D[N_d](t,0,\lambda))^{-1}\mathbb{V}[N_d](t,0,\lambda) + \mathbb{V}[0](t,0,\lambda)(D[N_d](t,0,\lambda))^{-1}.\end{aligned}$$

Hence, since equality (5.2.5) holds, by identifying every product of the matrices $D[N_d](t,0,\lambda^{-1})$, $\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}})$ and $D[N_d]^{-1}(t,0,\lambda)$ with $\mathbb{K}_N\big|_{x=0}$ we can derive that

$$(\mathbb{K}_N)_t\big|_{x=0} = \mathbb{V}[N_d](t,0,\lambda^{-1})\mathbb{K}_N(t,0,\lambda) - \mathbb{K}_N(t,0,\lambda)\mathbb{V}[N_d](t,0,\lambda). \tag{5.2.7}$$

The multiplication of the dressing matrices with a product only depending on λ has no impact on this calculation. Now for the construction of the matrix $\mathbb{K}_N$, we take a closer look at the matrix multiplication. Due to the polynomial expressions of $D[N_d](t,x,\lambda)$ and $D[N_d](t,x,\lambda^{-1})$, it is reasonable to assume that as in the proof of Proposition 5.1.1 the matrix $\mathbb{K}_N$ can be written as

$$\left(\sqrt{\Big(\lambda-\frac{1}{\lambda}\Big)^2+\alpha^2}\right)\cdot\mathbb{K}_N\big|_{x=0} = \mathbb{K}^{(1)}\lambda + \mathbb{K}^{(0)} + \frac{1}{\lambda}\mathbb{K}^{(-1)},$$

where $\mathbb{K}^{(1)}$, $\mathbb{K}^{(0)}$ and $\mathbb{K}^{(-1)}$ are t dependent matrix coefficients which need to be determined. Therefore multiplying both sides of equality (5.2.6) with $\lambda\sqrt{(\lambda-\frac{1}{\lambda})^2+\alpha^2}$, we obtain polynomials $L(\lambda)$ and $R(\lambda)$ with matrix coefficients on each side. In particular,

$$\begin{aligned}L(\lambda) &= \lambda^{N_d+2}L_{N_d+2} + \lambda^{N_d+1}L_{N_d+1} + \cdots + L_0,\\ R(\lambda) &= \lambda^{N_d+2}R_{N_d+2} + \lambda^{N_d+1}R_{N_d+1} + \cdots + R_0.\end{aligned}$$

Considering the highest order matrix coefficient of each factor, we have $L_{N_d+2} = -(-1)^{N_{bbb}} i\Sigma_{N_d}\sigma_3$. Furthermore, similarly to the proof of Proposition 5.1.1, by assumptions (5.2.2), (5.2.3) and (5.2.4) and the construction of the dressing matrix we have $2N_d$ zeros and associated kernel vectors at $x = 0$ for the right and left side given by

$$
\begin{aligned}
&R(\lambda)\big|_{\lambda=\lambda_j}\psi_j = 0, \quad && R(\lambda)\big|_{\lambda=-\lambda_j^*}\psi_{j+N_b} = 0, \quad && R(\lambda)\big|_{\lambda=\frac{1}{\lambda_j}}\widehat{\psi}_j = 0, \quad && R(\lambda)\big|_{\lambda=\frac{-1}{\lambda_j^*}}\widehat{\psi}_{j+N_b} = 0,\\
&R(\lambda)\big|_{\lambda=-\lambda_j}\varphi_j = 0, && R(\lambda)\big|_{\lambda=\lambda_j^*}\varphi_{j+N_b} = 0, && R(\lambda)\big|_{\lambda=\frac{-1}{\lambda_j}}\widehat{\varphi}_j = 0, && R(\lambda)\big|_{\lambda=\frac{1}{\lambda_j^*}}\widehat{\varphi}_{j+N_b} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&L(\lambda)\big|_{\lambda=\lambda_j}\psi_j = 0, \quad && L(\lambda)\big|_{\lambda=-\lambda_j^*}\psi_{j+N_b} = 0, \quad && L(\lambda)\big|_{\lambda=\frac{1}{\lambda_j}}\widehat{\psi}_j = 0, \quad && L(\lambda)\big|_{\lambda=\frac{-1}{\lambda_j^*}}\widehat{\psi}_{j+N_b} = 0,\\
&L(\lambda)\big|_{\lambda=-\lambda_j}\varphi_j = 0, && L(\lambda)\big|_{\lambda=\lambda_j^*}\varphi_{j+N_b} = 0, && L(\lambda)\big|_{\lambda=\frac{-1}{\lambda_j}}\widehat{\varphi}_j = 0, && L(\lambda)\big|_{\lambda=\frac{1}{\lambda_j^*}}\widehat{\varphi}_{j+N_b} = 0,
\end{aligned}
$$

respectively. Therefore, the whole set of eight zeros and associated kernel vectors is only provided in the case of breathers, i.e. for $j = N_s+1, \dots, N_s+N_b$. In the case of single solitons, $j = 1, \dots, N_s$, we have $\lambda_j^* = -\lambda_j$ so that there are essentially four zeros and associated kernel vectors after matching the ones which are replicates. In particular, the second and the fourth column are repetitions of the first and third column, respectively. Also in the case of boundary-bound breathers, $j = N_s + 2N_b + 1, \dots, N_s + 2N_b + N_{bbb}$, we have $\lambda_j^* = \lambda_j^{-1}$ so that again there are only four zeros and associated kernel vectors after matching the ones which are identical under the assumption (5.2.4). In fact, the third and the fourth column are repetitions of the second and the first column, respectively. Hence, there are $4N_s + 8N_b + 4N_{bbb} = 2N_d$ equalities. Note that the property $\mathbb{K}_0^{-1}(\lambda^{\pm 1}) = \mathbb{K}_0(\lambda^{\mp 1})$ for the sin-boundary matrix proven in Proposition 4.3.2 is used here in order to derive $\mathbb{K}_0(\lambda_j^{\mp 1})\widehat{\psi}_j = \psi_j$ at $x = 0$ justified by assumption (5.2.2), etc. Further, the symmetry of $\mathbb{K}_0$ given by $\mathbb{K}_0(\lambda) = \sigma_1\mathbb{K}_0(-\lambda)\sigma_1$ is needed to identify the zeros for the left hand side. As we have explicitly seen in the proof of Proposition 5.1.1, we need to have double the amount of zeros and associated kernel vectors if we want to determine the unknown matrix coefficients. Therefore, the $2N_d$ zeros are not enough to prove equality of $L(\lambda)$ and $R(\lambda)$.

Consequently, we devote special attention to the boundary matrix. Interpreting the boundary matrix $\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}})$ as a two-fold dressing matrix, we again have that the zeros are actually the zeros of the diagonal entries of $\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}})$. As a result, we calculate the zeros of $\lambda^2 - (-1)^{N_{bbb}} i\alpha\lambda - 1$ which are in the case $|\alpha| < 2$ given by

$$\lambda_\pm = (-1)^{N_{bbb}}\frac{i\alpha}{2} \pm \sqrt{1 - \frac{\alpha^2}{4}} \in \{\lambda \in \mathbb{C}\colon |\lambda| = 1\} \setminus \{-i, i, -1, 1\},$$

in the case of $|\alpha| = 2$, there is a double zero $\lambda_\pm = (-1)^{N_{bbb}} i$, and in the case $|\alpha| > 2$ by

$$\lambda_\pm = i\Big((-1)^{N_{bbb}}\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - 1}\Big) \in i\mathbb{R} \setminus \{-i, 0, i\}.$$

At the same time, we have that $\lambda_\pm^*$ are the zeros of $i\lambda$ times the (22)-entry of $\mathbb{K}_0(\lambda)$ yielding

$$\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}}) = \frac{(-1)^{N_{bbb}}}{i\lambda\sqrt{(\lambda - \frac{1}{\lambda})^2 + \alpha^2}}\begin{pmatrix}(\lambda - \lambda_+)(\lambda - \lambda_-) & 0\\ 0 & (\lambda - \lambda_+^*)(\lambda - \lambda_-^*)\end{pmatrix}.$$

Then, as before due to the zero seed solution $\theta[0] = 0$, we take the solutions $\psi_{\pm,0}$ of the Lax system corresponding to the zero seed solution at $\lambda = \lambda_\pm$ with $u_0 \neq 0$ and $v_0 = 0$ in (3.2.1) for which we then have $\mathbb{K}_0(\lambda_\pm^{(-1)^{N_{bbb}}})\psi_{\pm,0}\big|_{x=0} = 0$. Therefore, constructing $\mathbb{K}_N(t,x,\lambda)$ as a multiplication of λL_{N_d+2} with a two-fold dressing matrix with the kernel vectors $D[N_d]\big|_{\lambda=\lambda_\pm}\psi_{\pm,0}$ corresponding to the zeros $\lambda_\pm$, we have at least two *additional* zeros and associated kernel vectors of both matrix polynomials $L(\lambda)$ and $R(\lambda)$, which, in particular, are together with the vectors used for the dressing matrix linearly independent. In other words, at $x = 0$ we obtain

$$R(\lambda)\big|_{\lambda=\lambda_\pm}\psi_{\pm,0} = 0, \qquad L(\lambda)\big|_{\lambda=\lambda_\pm}\psi_{\pm,0} = 0,$$
$$R(\lambda)\big|_{\lambda=\lambda_\pm^*}\varphi_{\pm,0} = 0, \qquad L(\lambda)\big|_{\lambda=\lambda_\pm^*}\varphi_{\pm,0} = 0.$$

In the case of $|\alpha| = 2$, we technically only have a one-fold dressing matrix which we just multiply by itself to obtain a matrix of sufficient order. If we arrange these $2N_d + 2$ equalities of the difference $C(\lambda) = L(\lambda) - R(\lambda)$ which is a polynomial matrix of degree $N_d + 1$ in λ as a system of zeros and associated kernel vectors in matrix form similar to (5.1.7), it can be concluded that the difference is in fact zero. In case $|\alpha| \neq 2$, the additional observation that the highest order coefficients are equal is unnecessary. Hence, we have found a matrix $\mathbb{K}_N$ for which at $x = 0$ equality (5.2.6) holds.

(c) By the reconstruction formula (3.2.16), we have that the dressing matrix multiplied with the products can be expressed as

$$\big(\Pi_1\Pi_2\Pi_3 D[N_d]\big)\big|_{\lambda=0} = (-1)^{N_{bbb}} D[N_d]\big|_{\lambda=0} = (-1)^{N_{bbb}} e^{i\frac{\theta[N_d]}{2}\sigma_1},$$

since N_d is always even. On the other hand, we have written $D[N_d]$ multiplied by the products at $x = 0$ also as polynomial matrix so that $\big(\Pi_1\Pi_2\Pi_3 D[N_d]\big)\big|_{\lambda=0} = \Sigma_{N_d}(t,0)$. Thus, if we compare at $x = 0$ the zero-th and the $(2N_d + 2)$-th order matrix coefficients, we obtain the equality of

$$L_0 = (-1)^{N_{bbb}} i\sigma_3 = \mathbb{K}^{(-1)}\Sigma_{N_d}(t,0) = R_0$$

and the equality of

$$L_{N_d+2} = -(-1)^{N_{bbb}} i\Sigma_{N_d}(t,0)\sigma_3 = \mathbb{K}^{(1)} = R_{N_d+2},$$

respectively. Consequently, the two t dependent matrix coefficients of $\mathbb{K}_N$ are at $x = 0$ given by

$$\mathbb{K}^{(1)} = -\mathbb{K}^{(-1)} = -i\left(\sigma_3\cos\frac{\theta[N_d]}{2} + \sigma_2\sin\frac{\theta[N_d]}{2}\right)\Bigg|_{x=0}.$$

Moreover, the symmetries of $\mathbb{V}(t,x,\lambda)$ given in Section 2.2 imply that

$$\mathbb{K}_N(t,x,\lambda) = \sigma_1\big(\mathbb{K}_N(t,x,-\lambda)\big)\sigma_1,$$
$$\mathbb{K}_N(t,x,\lambda) = \sigma_2\big(\mathbb{K}_N(t,x,\lambda^*)\big)^*\sigma_2,$$
$$\mathbb{K}_N(t,x,\lambda) = \sigma_3\big(\mathbb{K}_N(t,x,-\lambda^*)\big)^*\sigma_3.$$

From the first symmetry, we deduce that $\mathbb{K}^{(0)}(t,0) = \sigma_1\mathbb{K}^{(0)}(t,0)\sigma_1$ so that

$$\mathbb{K}^{(0)}(t,0) = \begin{pmatrix} \mathbb{K}_{11}^{(0)}(t,0) & \mathbb{K}_{12}^{(0)}(t,0) \\ \mathbb{K}_{12}^{(0)}(t,0) & \mathbb{K}_{11}^{(0)}(t,0) \end{pmatrix},$$

where the entries $\mathbb{K}_{11}^{(0)}(t,0)$ and $\mathbb{K}_{12}^{(0)}(t,0)$ still need to be determined. By the second and third symmetry, we obtain $\mathbb{K}^{(0)}(t,0) = \sigma_2\mathbb{K}^{(0)}(t,0)^*\sigma_2 = \sigma_3\mathbb{K}^{(0)}(t,0)^*\sigma_3$ and as a consequence

$$\mathrm{Im}(\mathbb{K}_{11}^{(0)}(t,0)) = \mathrm{Re}(\mathbb{K}_{12}^{(0)}(t,0)) = 0.$$

Finally, by the particular choice of the products Π_1, Π_2, Π_3 and Π_3', we have

$$\det\Big(\Pi_1\Pi_2\Pi_3 D[N_d](t,x,\lambda)\Big) = \det\Big(\Pi_1\Pi_2\Pi_3' D[N_d](t,x,\lambda^{-1})\Big)$$

so that calculating the determinant of $\mathbb{K}_N$ at $x=0$ via the matrix product, we derive

$$\det\mathbb{K}_N\Big|_{x=0} = \det(\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}})) = 1,$$

where we already calculated the determinant of $\mathbb{K}_0(\lambda^{\pm1})$ in the proof of Proposition 4.3.2. Then, comparing this determinant of the matrix $\mathbb{K}_N$ at $x=0$ to simply calculating it with the information we have, we see that $(\mathbb{K}_{11}^{(0)}(t,0))^2 - (\mathbb{K}_{12}^{(0)}(t,0))^2 = (\mathrm{Re}(\mathbb{K}_{11}^{(0)}(t,0)))^2 + (\mathrm{Im}(\mathbb{K}_{12}^{(0)}(t,0)))^2 = \alpha^2$. With regards to the expressions of the boundary matrices for the sG equation, see Proposition 4.3.2, it seems that $\mathbb{K}_N$ could be consisting of a mixture of $\alpha\mathbb{1}$ and $i\alpha\sigma_1$ representing the sin- and cos-boundary matrix, respectively. However, as suggested in the proof of Proposition 4.3.2, there is a way to distinguish both cases, since they differ in the condition regarding the inverse. Correspondingly, let us show that $\mathbb{K}_N^{-1}(t,0,\lambda) = \mathbb{K}_N(t,0,\lambda^{-1})$, where we know that this property holds for $\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}})$. On one hand, we have

$$\mathbb{K}_N^{-1}(t,0,\lambda) = D[N_d](t,0,\lambda)\mathbb{K}_0^{-1}(\lambda^{(-1)^{N_{bbb}}})D[N_d](t,0,\lambda^{-1})$$

and on the other hand, we have

$$\mathbb{K}_N(t,0,\lambda^{-1}) = D[N_d](t,0,\lambda)\mathbb{K}_0((\lambda^{-1})^{(-1)^{N_{bbb}}})D[N_d](t,0,\lambda^{-1}),$$

which is thus equal. Hence, if we write out this property for the matrix $\mathbb{K}_N$ as devised so far, $\mathbb{K}_{12}^{(0)}(t,0) = 0$ is implied immediately and furthermore $\mathbb{K}_{11}^{(0)}(t,0) = \pm\alpha\mathbb{1}$.

Therefore, we have found a matrix $\mathbb{K}_N$ which at $x=0$ is of the form of the sin-boundary matrix and also satisfies equality (5.2.7) so that the sin-boundary condition with either α or $-\alpha$ or $\theta[N_d]$ is satisfied. □

As for dressing the defect, applying the method of dressing the boundary to the sG equation is simplified considerably due to the zero seed solution. By the process of determining the sign of the frozen Bäcklund transformation we have developed for the NLS equation, it is possible to take the limit t to infinity in order to match the sign in front of the spectral parameter in the proof we have just worked out. However, this is only feasible in the case where there are no boundary-bound breathers, since we could then show similarly to the NLS equation that $\theta[N_d]$ goes to a multiple of 2π as t goes to infinity. Nonetheless, we omit this analysis here, since the seed solution of the sG equation satisfies the boundary condition with both α and $-\alpha$ and therefore 'preserving' this condition is a given either way.

Remark 5.2.2. In the case of a boundary-bound single soliton solution which corresponds to the choice $\lambda_1 = \pm i$, we can explicitly calculate that the sin-boundary condition is not satisfied for arbitrary $\alpha \in \mathbb{R}$.

Nevertheless, Proposition 5.2.1 gives the means to implement an arbitrary combination of a single soliton, breather and boundary-bound breather solutions into the sG equation on one half-line subject to the sin-boundary condition. Therefore, the partition of the spectral parameters plays a crucial role. Comparing the initial number N of spectral parameters which are meant to be used in the Dressing method and the actual number N_d which is used in Proposition 5.2.1, it is noticeable that single solitons and breathers come in pairs and boundary-bound breathers need to satisfy a particular relation. Moreover, this distribution is also used in the upcoming models of the NLS equation. An attentive reader might as well have noticed that the exception of the boundary-bound single solitons is not limited to the application of the N_d-fold dressing matrix, but also influences the proof where this is equivalent to $|\alpha| = 2$. Particularly, the following observation comes in handy.

Remark 5.2.3. The sin-boundary matrix $\mathbb{K}(t, 0, \lambda)$ with the boundary parameter α, where $|\alpha| > 2$, can be viewed with $\kappa_{\alpha,j} = \left(\frac{\alpha}{2} + (-1)^j\sqrt{\frac{\alpha^2}{4} - 1}\right)$ as the product of two Darboux matrices and a rotation matrix. Let

$$\mathbb{B}_j(t, x, \lambda) = \mathbb{1} \pm \frac{i\kappa_{\alpha,j}}{\lambda}\begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix}$$

where both matrices have the same sign, then the sin-boundary condition (4.3.10) admits the factorization

$$\mathbb{K}(t, 0, \lambda) = \frac{-i\lambda}{\sqrt{(\lambda - \frac{1}{\lambda})^2 + \alpha^2}}\begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix}\mathbb{B}_1(t, x, \lambda)\mathbb{B}_2(t, x, \lambda)\big|_{x=0}.$$

Note the following instrumental equalities $\kappa_{\alpha,1} + \kappa_{\alpha,2} = \alpha$, $\kappa_{\alpha,1}\kappa_{\alpha,2} = 1$ as well as $(\sigma_3\cos\frac{\theta}{2} + \sigma_2\sin\frac{\theta}{2})^2 = \mathbb{1}$.

Taking this into consideration, the factorization of $\mathbb{K}_0(\lambda^{(-1)^{N_{bbb}}})$ in the proof of Proposition 5.2.1 in order to identify the zeros and associated kernel vectors becomes more transparent. Effectively for $|\alpha| > 2$, the boundary condition is represented by two single solitons which are present at the boundary. In that regard, it is comprehensible that in the case of $|\alpha| < 2$, where the zeros are represented by a boundary-bound breather at the boundary, the expression in Remark 5.2.3 is not sufficient due to the defect matrix $\mathbb{B}$ only corresponding to single solitons.

5.2.2 The NLS equation with boundary conditions

Now, we consider the NLS equation on the (positive) half-line

$$iu_t + u_{xx} + 2|u|^2u = 0 \tag{5.2.8}$$

for $u(t, x)\colon \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{C}$ and initial condition $u(0, x) = u_0(x)$ for $x \in \mathbb{R}_+$ and complement it with a Robin boundary condition

$$u_x(t, 0) = \alpha u(t, 0), \tag{5.2.9}$$

where $\alpha \in \mathbb{R}$. The Robin boundary condition is one of the conditions for which the boundary matrix has entries only on the diagonal and is time independent. Therefore, it is not necessary to restrict the solution space. In contrast to the sin-boundary condition regarding the sG equation, for the Robin boundary condition regarding the NLS equation it is possible to construct single boundary-bound solitons. In a similar fashion as for Proposition 5.2.1, we divide the number $N = N_s + N_{bbs}$ of spectral parameters $\lambda_j \in \mathbb{C} \setminus \mathbb{R}$ and order them accordingly so that the spectral parameters

$\lambda_j \in \mathbb{C} \setminus \big(\mathbb{R} \cup i\mathbb{R}\big)$, $j = 1, \ldots, N_s$, correspond to solitons and $\lambda_j \in i\mathbb{R} \setminus \{0\}$, $j = N_s + 1, \ldots, N$, correspond to boundary-bound solitons. Moreover, we define $N_d = 2N_s + N_{bbs}$ so that only the number of solitons is doubled. Thus, we can state the following

Proposition 5.2.4. *Consider the seed solution $u[0]$ to the NLS equation on the half-line* (5.2.8), *which at $x = 0$ satisfies the Robin boundary condition with $\alpha \in \mathbb{R} \setminus \{0\}$. Further, take solutions ψ_j, $j = 1, \ldots, N$, of the Lax system* (2.1.2) *corresponding to $u[0]$ for distinct $\lambda = \lambda_j \in \mathbb{C} \setminus \big(\mathbb{R} \cup \{-i\alpha, i\alpha\}\big)$. Assume that there exist paired solutions $\widehat{\psi}_j$, $j = 1, \ldots, N_s$, of the same Lax system for the spectral parameter $\lambda = \widehat{\lambda}_j = -\lambda_j$ and that they satisfy*

$$\widehat{\psi}_j\big|_{x=0} = \mathcal{K}_0((-1)^{N_{bbs}}\lambda_j)\psi_j\big|_{x=0}, \quad \widehat{\lambda}_k \neq \lambda_j. \tag{5.2.10}$$

Further for $j = N_s + 1, \ldots, N$, assume that the solutions of the Lax system ψ_j, $\varphi_j = -i\sigma_2\psi_j^*$ corresponding to the spectral parameters λ_j and λ_j^* satisfy the following relation

$$\varphi_j\big|_{x=0} = \mathcal{K}_0((-1)^{N_{bbs}}\lambda_j)\psi_j\big|_{x=0}. \tag{5.2.11}$$

The matrix $\mathcal{K}_0(\lambda)$ is associated to the boundary matrix (4.3.4) *representing the Robin boundary condition. Then, an N_d-fold dressing matrix $D[N_d]$ using the corresponding solutions and spectral parameters leads to the solution $u[N_d]$ to the NLS equation on the half-line, for which the Robin boundary condition* (5.2.9) *is preserved.*

Proof. (a) The spectral parameters are divided into $\lambda_j \in \mathbb{C} \setminus \big(\mathbb{R} \cup i\mathbb{R}\big)$, $j = 1, \ldots, N_s$, and $\lambda_j \in i\mathbb{R} \setminus \{0\}$, $j = N_s + 1, \ldots, N$, for the sole purpose of having distinct N_d spectral parameters $\lambda_1, \ldots, \lambda_{N_s}, -\lambda_1, \ldots, -\lambda_{N_s}, \lambda_{N_s+1}, \ldots, \lambda_N$. Therefore, constructing an N_d-fold dressing matrix $D[N_d]$ with these exact spectral parameters and their associated solutions of the Lax system (2.1.2), which are by Proposition 3.2.5 linearly independent, we can derive a solution $u[N_d]$ to the NLS equation on the half-line (5.2.8). Note that it is again helpful to multiply the dressing matrices with $\Pi_1 = \prod_{k=1}^{N_s}((\lambda + \lambda_j^*)(\lambda - \lambda_j^*)) = \prod_{k=1}^{N_s}(((-\lambda) - \lambda_j^*)((-\lambda) + \lambda_j^*))$ and $\Pi_2 = \prod_{k=N_s+1}^{N}(\lambda - \lambda_j^*)$ or $\Pi_2' = \prod_{k=N_s+1}^{N}((-\lambda) - \lambda_j^*)$ so that

$$\begin{aligned} \Pi_1\Pi_2 D[N_d](t,x,\lambda) &= \lambda^{N_d}\mathbb{1} + \cdots + \Sigma_{N_d}, \\ \Pi_1\Pi_2' D[N_d](t,x,-\lambda) &= (-1)^{N_{bbs}}\lambda^{N_d}\mathbb{1} + \cdots + \Sigma_{N_d}. \end{aligned} \tag{5.2.12}$$

For notational purposes, we redefine the dressing matrix $D[N_d](t,x,\lambda)$ and the dressing matrix with the negative spectral parameter $D[N_d](t,x,-\lambda)$ as the first and second row of (5.2.12) for the upcoming steps (b) and (c).

(b) By assumption, we have

$$\mathcal{K}_0((-1)^{N_{bbs}}\lambda) = \frac{1}{i\alpha + 2(-1)^{N_{bbs}}\lambda}(i\alpha\mathbb{1} - 2(-1)^{N_{bbs}}\lambda\sigma_3).$$

Therefore, analogously to the proof of Proposition 5.2.1, we need to show that there exists a matrix $\mathcal{K}_N$ which satisfies

$$(D[N_d](t,x,-\lambda)\mathcal{K}_0((-1)^{N_{bbs}}\lambda))\big|_{x=0} = (\mathcal{K}_N(t,x,\lambda)D[N_d](t,x,\lambda))\big|_{x=0}. \tag{5.2.13}$$

Additionally to the zeros and kernel vectors of the N_d-fold dressing matrix, it is straightforward to see that there are two zeros $\lambda_0 = (-1)^{N_{bbs}}i\alpha/2$ and the parameter with opposite sign of the boundary matrix $(i\alpha + 2(-1)^{N_{bbs}}\lambda)\mathcal{K}_0(\lambda)$ for which $v_0 = e_1$ and $-i\sigma_2 v_0^* = e_2$ are the kernel vectors. In theory, by the definition of the dressing matrix we have given in Section 3.2, the matrix $\mathcal{K}_N$ we want to construct is, in fact, a dressing matrix multiplied by σ_3, which means that the highest order matrix coefficients of the usual polynomial matrices $L(\lambda)$ and $R(\lambda)$ agree. Consequently, considering the solution ψ_0 of the Lax system (2.1.2) corresponding to $u[0]$ at $\lambda = \lambda_0$, there are two scenarios:

1. The kernel vector v_0 of $\mathcal{K}_0(\lambda_0)$ and ψ_0 are linearly dependent at $x = 0$. As before, we can then define with ψ_0, since $\psi_0, \dots, \psi_N$ are linearly independent, the following

$$\psi_0' = D[N_d](t, x, \lambda_0)\psi_0,$$

which serves as the kernel vector for the dressing matrix $\mathcal{K}_N(t, x, \lambda)$. It is important to note that constructing $\mathcal{K}_N(t, x, \lambda)$ in this manner results in the following relations for ψ_0 and the orthogonal vector $\varphi_0 = -i\sigma_2\psi_0^*$ at $x = 0$:

$$\begin{aligned} D[N_d](t, x, -\lambda_0)\mathcal{K}_0(\lambda_0)\psi_0 &= \mathcal{K}_N(t, x, \lambda_0)D[N_d](t, x, \lambda_0)\psi_0 = 0, \\ D[N_d](t, x, -\lambda_0^*)\mathcal{K}_0(\lambda_0^*)\varphi_0 &= \mathcal{K}_N(t, x, \lambda_0^*)D[N_d](t, x, \lambda_0^*)\varphi_0 = 0. \end{aligned} \tag{5.2.14}$$

2. The kernel vector v_0 of $\mathcal{K}_0(\lambda_0)$ and ψ_0 are linearly independent at $x = 0$. Then, a similar diagram to Figure 5.1 holds with $\mathcal{B}_0$ and $\mathcal{B}_N$ replaced by $\mathcal{K}_0$ and $\mathcal{K}_N$, respectively, so that there exists a matrix $\mathcal{K}_N$ which is at $x = 0$ the product of the three matrices $D[N_d](t, x, -\lambda) \cdot \mathcal{K}_0((-1)^{N_{bbs}}\lambda) \cdot (D[N_d](t, x, \lambda))^{-1}$ and for which at $x = 0$ the following holds

$$\begin{aligned} (D[N_d](t, x, -\lambda_0)\mathcal{K}_0(\lambda_0)\psi_0) &= (\mathcal{K}_N(t, x, \lambda_0)D[N_d](t, x, \lambda_0)\psi_0) \neq 0, \\ (D[N_d](t, x, -\lambda_0^*)\mathcal{K}_0(\lambda_0^*)\varphi_0) &= (\mathcal{K}_N(t, x, \lambda_0^*)D[N_d](t, x, \lambda_0^*)\varphi_0) \neq 0. \end{aligned} \tag{5.2.15}$$

Further, if we evaluate the determinant of $\mathcal{K}_N|_{x=0}$ at the spectral parameter λ_0 or λ_0^*, we obtain in accordance with the matrix product that both $\det(\mathcal{K}_N(t, 0, \lambda_0))$ and $\det(\mathcal{K}_N(t, 0, \lambda_0^*))$ are zero. Thus, there exists a kernel vector at the specific parameter $\lambda = \lambda_0$ which we use to construct the dressing matrix $\mathcal{K}_N$ subject to the relations (5.2.15).

By the same argumentation as before, it is reasonable to assume that the following product $(i\alpha + 2(-1)^{N_{bbs}}\lambda)\mathcal{K}_N(t, x, \lambda)$ is a matrix polynomial of degree one. We construct this matrix as in one of the two cases multiplied by an arbitrary t dependent, λ independent matrix. Then, the left and right hand side of equality (5.2.13) multiplied by $(i\alpha + 2(-1)^{N_{bbs}}\lambda))$ is given by

$$\begin{aligned} L(\lambda) &= (D[N_d](t, x, -\lambda)\mathcal{K}_0((-1)^{N_{bbs}}\lambda))\big|_{x=0} = \lambda^{N_d+1}L_{N_d+1} + \lambda^{N_d}L_{N_d} + \dots + L_0, \\ R(\lambda) &= (\mathcal{K}_N(t, x, \lambda)D[N_d](t, x, \lambda))\big|_{x=0} = \lambda^{N_d+1}R_{N_d+1} + \lambda^{N_d}R_{N_d} + \dots + R_0, \end{aligned}$$

respectively, admitting at $x = 0$ the following zeros and associated kernel vectors, $j = 1, \dots, N$,

$$\begin{aligned} &R(\lambda)\big|_{\lambda=\lambda_j}\psi_j = 0, \qquad R(\lambda)\big|_{\lambda=-\lambda_j}\widehat{\psi}_j = 0, \qquad R(\lambda)\big|_{\lambda=\lambda_j^*}\varphi_j = 0, \qquad R(\lambda)\big|_{\lambda=-\lambda_j^*}\widehat{\varphi}_j = 0, \\ &L(\lambda)\big|_{\lambda=\lambda_j}\psi_j = 0, \qquad L(\lambda)\big|_{\lambda=-\lambda_j}\widehat{\psi}_j = 0, \qquad L(\lambda)\big|_{\lambda=\lambda_j^*}\varphi_j = 0, \qquad L(\lambda)\big|_{\lambda=-\lambda_j^*}\widehat{\varphi}_j = 0. \end{aligned}$$

The whole set of four zeros and associated kernel vectors is only provided in the case of solitons, i.e. for $j = 1, \dots, N_s$. In the case of boundary-bound solitons, $j = N_s, \dots, N$, we have $\lambda_j^* = -\lambda_j$ so that there are essentially two zeros and associated kernel vectors after matching the ones which are replicates. In particular, the second and the fourth column are repetitions of the third and first column, respectively. For $R(\lambda)$, these relations follow simply by the construction of the dressing matrix $D[N_d]$. On the other hand, for $L(\lambda)$, we additionally need assumptions (5.2.10) and (5.2.11) as well as the properties $\mathcal{K}_0^{-1}(\lambda) = \mathcal{K}_0(-\lambda)$ given in Proposition 4.3.1 and $\mathcal{K}_0(\lambda) = \sigma_2 \mathcal{K}_0(\lambda^*)^* \sigma_2$ which follows from the symmetry of the Lax pair (2.1.4). Moreover, due to the construction of $\mathcal{K}_N$, we have (for $j = 0$) two values and associated vectors which are equal by equalities (5.2.14) and (5.2.15), but not necessarily zero for $L(\lambda)$ and $R(\lambda)$, i.e.

$$R(\lambda)\big|_{\lambda=\lambda_0}\psi_0 = L(\lambda)\big|_{\lambda=\lambda_0}\psi_0, \qquad R(\lambda)\big|_{\lambda=-\lambda_0}\varphi_0 = L(\lambda)\big|_{\lambda=-\lambda_0}\varphi_0.$$

However, for the difference $C(\lambda) = L(\lambda) - R(\lambda)$ which is a matrix polynomial of degree $2N+1$ in λ these values and vectors function as zeros and associated kernel vectors. Arranging these $4N+2$ equalities of the difference $C(\lambda)$ once again as a system of zeros and associated kernel vectors in matrix form, it follows that each matrix coefficient is zero and therefore the constructed $\mathcal{K}_N$ satisfies equality (5.2.13) in both cases.

(c) To reconstruct $(i\alpha + (-1)^{N_{bbs}} 2\lambda)\mathcal{K}_N = \lambda\mathcal{K}^{(1)} + \mathcal{K}^{(0)}$ at $x = 0$ as boundary matrix, we analyze the equality (5.2.13). In particular for the equality of the matrix coefficients of order $N_d + 1$ in λ, we have

$$L_{N_d+1} = (-2\sigma_3)(-1)^{2N_{bbs}} = -2\sigma_3 = \mathcal{K}^{(1)} = R_{N_d+1}, \tag{5.2.16}$$

confirming the suspected form of $\mathcal{K}^{(1)}$ as σ_3 times the highest order matrix coefficient of the dressing matrix which is $\mathbb{1}$, up to a function of λ. For the equality of the matrix coefficients of order N_d, we obtain with (5.2.16) that

$$2(\Sigma_1(t,0)\sigma_3 + \sigma_3\Sigma_1(t,0)) + (-1)^{2N_{bbs}} i\alpha\mathbb{1} = \mathcal{K}^{(0)},$$

where Σ_1 is the $(N_d - 1)$-th order matrix coefficient of $D[N_d]$. Thus, the off-diagonal entries of $\mathcal{K}^{(0)}$ are zero and to determine the diagonal entries, we consider the determinant of $\mathcal{K}_N$ at $x = 0$ in two ways. First, note that the dressing matrices multiplied by the products defined in (a) satisfy

$$\det D[N_d](t,x,\lambda) = \prod_{k=1}^{N_s}((\lambda-\lambda_k)(\lambda-\lambda_k^*)(\lambda+\lambda_k)(\lambda+\lambda_k^*)) \prod_{k=N_s+1}^{N}((\lambda-\lambda_k)(\lambda-\lambda_k^*)),$$
$$\det D[N_d](t,x,-\lambda) = \prod_{k=1}^{N_s}((\lambda+\lambda_k)(\lambda+\lambda_k^*)(\lambda-\lambda_k)(\lambda-\lambda_k^*)) \prod_{k=N_s+1}^{N}((\lambda+\lambda_k)(\lambda+\lambda_k^*)),$$

where the equality of the first products can be seen right away and the equality of the second products is justified by $-\lambda_j = \lambda_j^*$ for $j = N_s+1,\ldots,N$ and therefore the determinants are equal. Thus, by the matrix product, we have

$$\det((i\alpha + 2(-1))^{2N_{bbs}}\lambda)\mathcal{K}_N\big|_{x=0}) = \det((i\alpha + 2(-1))^{2N_{bbs}}\lambda)\mathcal{K}_0((-1))^{2N_{bbs}}\lambda)) = -4\lambda^2 - \alpha^2.$$

Subsequently, due to what we have found for $\mathcal{K}_N$ already, we can calculate

$$\det((i\alpha - 2(-1)^N\lambda)\mathcal{K}_N\big|_{x=0}) = (-2\lambda + \mathcal{K}_{11}^{(0)})(2\lambda + \mathcal{K}_{22}^{(0)}) = -4\lambda^2 + 2\lambda(\mathcal{K}_{11}^{(0)} - \mathcal{K}_{22}^{(0)}) + \mathcal{K}_{11}^{(0)}\mathcal{K}_{22}^{(0)}$$

so that the following two equalities need to hold

$$\mathcal{K}_{11}^{(0)} - \mathcal{K}_{22}^{(0)} = 0, \qquad \mathcal{K}_{11}^{(0)}\mathcal{K}_{22}^{(0)} = -\alpha^2.$$

This system can be solved and we obtain $\mathcal{K}^{(0)} = \pm i\alpha\mathbb{1}$. However, by the zero-th order equality of (5.2.13), we can verify that the sign of the boundary parameter is preserved, since we need to have $\mathcal{K}^{(0)} = i\alpha\mathbb{1}$ in order for

$$L_0 = i\alpha\Sigma_N(t,0) = \mathcal{K}^{(0)}\Sigma_N(t,0) = R_0$$

to hold.

□

This concludes the theoretical application of the dressing the boundary method to the NLS equation on the half-line subject to the Robin boundary condition (5.2.9). Now, a combination of the framework in all propositions regarding the Dressing method applied in the presence of defect or boundary conditions is necessary in order to expand the results to the new boundary condition

$$u_x(t,0) = \frac{iu_t(t,0)}{2\Omega(t,0)} - \frac{u(t,0)\Omega(t,0)}{2} + \frac{u(t,0)|u(t,0)|^2}{2\Omega(t,0)} - \frac{u(t,0)\alpha^2}{2\Omega(t,0)} \tag{5.2.17}$$

for the NLS equation on the half-line (5.2.8), see [25]. First off, a similar observation as in Remark 5.2.3 can be made, where, in particular, every boundary matrix representing the new boundary condition can be expressed by a multiplication of two matrices (3.1.4) representing frozen Bäcklund transformations.

Proposition 5.2.5. *The boundary matrix $\mathcal{K}(t,0,\lambda)$ for the new boundary condition* (4.3.5) *can be viewed, up to a function of λ, as combination of two Darboux matrices*

$$\mathcal{B}_{0,\alpha}(t,x,\lambda) = \mathbb{1} + \frac{1}{2\lambda}\begin{pmatrix} \alpha \pm i\sqrt{\beta^2 - |u|^2} & iu \\ iu^* & \alpha \mp i\sqrt{\beta^2 - |u|^2} \end{pmatrix},$$

sharing the same sign $\pm$ and where $\alpha, \beta \in \mathbb{R}\setminus\{0\}$ as well as the potential $\tilde{u}$ is assumed to be zero. Then,

$$((2\lambda - i|\beta|)^2 - \alpha^2)\mathcal{K}(t,0,\lambda) = 4\lambda^2\mathcal{B}_{0,\alpha}(t,x,\lambda)\mathcal{B}_{0,-\alpha}(t,x,\lambda)\big|_{x=0}.$$

In particular, it is important that the product of the two matrices $\mathcal{B}_{0,\alpha}$ and $\mathcal{B}_{0,-\alpha}$ is commutative. Therefore, it is comprehensible that a kernel vector for each of the matrices $\mathcal{B}_{0,\pm\alpha}$ at particular, different λ_1, λ_2 introduce the same kernel vectors for the product $\mathcal{K}(t,0,\lambda)$ at these values of λ.

Remark 5.2.6. In the case of a boundary-bound soliton solution which corresponds to the choice $\lambda_1 \in i\mathbb{R}\setminus\{0\}$, we can calculate explicitly that the new boundary conditions are not satisfied.

This exclusion of boundary-bound solitons solidifies the choice we have to make in order to determine whether the new boundary conditions are preserved. Namely as in Proposition 5.1.2, it is sufficient to assume that the seed solution—and therefore by Proposition 4.4.3 also the solution constructed by the Dressing method—and its first x-derivative are in the function space $H_t^{1,1}(\mathbb{R})$ at $x = 0$. And since, by that assumption, the choice of spectral parameters is restricted to $\mathbb{C}\setminus\big(\mathbb{R}\cup i\mathbb{R}\big)$, spectral parameters corresponding to boundary-bound solitons can not be considered. Given these assumptions, we can state the following:

Proposition 5.2.7. *Consider a seed solution $u[0](t,x)$ of the NLS equation on the half-line* (5.2.8), *which at $x = 0$ both satisfies the new boundary conditions* (4.3.3) *with $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}\setminus\{0\}$ and is together with its first x-derivative in the function space $H_t^{1,1}(\mathbb{R})$. Further, take solutions ψ_j, $j = 1,\ldots,N$, of the Lax system* (2.1.2) *corresponding to $u[0]$ for distinct $\lambda = \lambda_j \in \mathbb{C}\setminus\big(\mathbb{R}\cup i\mathbb{R}\cup$

$\{-\frac{\alpha}{2}+i\frac{\beta}{2}, -\frac{\alpha}{2}-i\frac{\beta}{2}, \frac{\alpha}{2}+i\frac{\beta}{2}, \frac{\alpha}{2}-i\frac{\beta}{2}\}$). *Assume that there exist paired solutions* $\widehat{\psi}_j$, $j = 1, \ldots, N$, *of the same Lax system for the spectral parameter* $\lambda = \widehat{\lambda}_j = -\lambda_j$ *and that they satisfy*

$$\widehat{\psi}_j\big|_{x=0} = \mathcal{K}_0(t, 0, \lambda_j)\psi_j\big|_{x=0}, \quad \widehat{\lambda}_k \neq \lambda_j, \tag{5.2.18}$$

where the matrix $\mathcal{K}_0(t, 0, \lambda)$ *is associated to the boundary matrix* (4.3.5) *representing the new boundary condition. Then, a* $2N$*-fold dressing matrix* $D[2N]$ *using the corresponding solutions and spectral parameters leads to the solution* $u[2N]$, *denoted by* $\widehat{u}[N]$, *of the NLS equation on the half-line, for which the new boundary condition* (4.3.3) *is preserved under* $\mathcal{K}_N$ *of form* (4.3.5) *if*

$$\operatorname{Im}\left(\lim_{\lambda\to 0}\frac{1}{\lambda}\left[\frac{(2\lambda - i|\beta|)^2 - \alpha^2}{4}\mathcal{K}_N(t, 0, \lambda) + \frac{\alpha^2+\beta^2}{4}\mathbb{1}\right]_{11}\right)$$

is greater than or equal to or rather less than or equal to 0 for all $t \in \mathbb{R}_+$ *depending on its limit as* $t \to \infty$.

Proof. (a) As discussed in step (a) of Proposition 5.2.4, the condition $\widehat{\lambda}_k \neq \lambda_j$ implies that all spectral parameters λ_j and $-\lambda_j$, $j = 1, \ldots, N$, used in the construction of the dressing matrix are distinct. Therefore, the dressing matrix is uniquely determined and the constructed solution $\widehat{u}[N]$ satisfies the NLS equation on the half-line (5.2.8). Multiplying the dressing matrices with $\Pi_1 = \prod_{k=1}^{N}((\lambda + \lambda_j^*)(\lambda - \lambda_j^*)) = \prod_{k=1}^{N}(((-\lambda) - \lambda_j^*)((-\lambda) + \lambda_j^*))$, we obtain

$$\begin{aligned} \Pi_1 D[2N](t, x, \lambda) &= \lambda^{2N}\mathbb{1} + \cdots + \Sigma_{2N}, \\ \Pi_1 D[2N](t, x, -\lambda) &= \lambda^{2N}\mathbb{1} - \cdots + \Sigma_{2N}, \end{aligned} \tag{5.2.19}$$

which we then again redefine as the actual dressing matrices for steps (b) and (c).

(b) As before, in order to prove that there is a matrix $\mathcal{K}_N(t, x, \lambda)$ satisfying

$$(\mathcal{K}_N)_t(t, x, \lambda)\big|_{x=0} = (V[2N](t, x, -\lambda)\mathcal{K}_N(t, x, \lambda) - \mathcal{K}_N(t, x, \lambda)V[2N](t, x, \lambda))\big|_{x=0},$$

it is of advantage to consider the equivalent equality

$$(D[2N](t, x, -\lambda)\mathcal{K}_0(t, 0, \lambda))\big|_{x=0} = (\mathcal{K}_N(t, x, \lambda)D[2N](t, x, \lambda))\big|_{x=0}, \tag{5.2.20}$$

where on both sides the matrices $\mathcal{K}_0(t, 0, \lambda)$ and $\mathcal{K}_N(t, x, \lambda)\big|_{x=0}$ are multiplied by $((2\lambda - i|\beta|)^2 - \alpha^2)/4$. Further, we define $\lambda_0 = -\frac{\alpha}{2} - i\frac{|\beta|}{2}$. In view of this equation, it is plausible to assume that the matrix, we wish to find, is of second order in λ, i.e. $\mathcal{K}_N\big|_{x=0} = \lambda^2\mathcal{K}^{(2)}(t, 0) + \lambda\mathcal{K}^{(1)}(t, 0) + \mathcal{K}^{(0)}(t, 0)$. Due to Proposition 5.2.5, $\mathcal{K}_0(t, 0, \lambda) = 4\lambda^2\mathcal{B}_{0,\alpha}(t, x, \lambda)\mathcal{B}_{0,-\alpha}(t, x, \lambda)\big|_{x=0}$ and we can deduce that there exist two kernel vectors v_0 and $\widehat{v}_0$ at two distinct spectral parameters λ_0 and $\widehat{\lambda}_0 = -\lambda_0$, respectively, for which

$$\mathcal{B}_{0,\alpha}(t, x, \lambda_0)v_0 = 0, \quad \mathcal{B}_{0,-\alpha}(t, x, \widehat{\lambda}_0)\widehat{v}_0 = 0.$$

Therefore, $\mathcal{K}_0(t, 0, \lambda)$ can be seen as (frozen) two-fold dressing matrix with the inherited kernel vectors of $\mathcal{B}_{0,\pm\alpha}$ at λ_0 and $\widehat{\lambda}_0$, so that

$$\mathcal{K}_0(t, 0, \lambda_0)v_0\big|_{x=0} = 0, \quad \mathcal{K}_0(t, 0, \widehat{\lambda}_0)\widehat{v}_0\big|_{x=0} = 0.$$

As before, these kernel vectors are introduced in order to ensure that the vectors with which we construct the two-fold dressing matrix $\mathcal{K}_N(t, x, \lambda)$ are linearly independent. Therefore, consider the solutions of the Lax system (2.1.2) corresponding to $u[0]$ at $\lambda = \lambda_0$ and $\lambda = \widehat{\lambda}_0$ given by ψ_0 and $\widehat{\psi}_0$. Then, we distinguish two cases for each of these vectors:

1. The kernel vector υ_0 of $\mathcal{K}_0(t,0,\lambda_0)$ and ψ_0 are linearly dependent at $x=0$. As before, define $\psi_0' = D[2N](t,x,\lambda_0)\psi_0$ serving as one of the kernel vectors for the dressing matrix $\mathcal{K}_N(t,x,\lambda)$. Hence, we obtain at $x=0$ the following relations

$$\begin{aligned} D[2N](t,x,-\lambda_0)\mathcal{K}_0(t,0,\lambda_0)\psi_0 &= \mathcal{K}_N(t,x,\lambda_0)D[2N](t,x,\lambda_0)\psi_0 = 0, \\ D[2N](t,x,-\lambda_0^*)\mathcal{K}_0(t,0,\lambda_0^*)\varphi_0 &= \mathcal{K}_N(t,x,\lambda_0^*)D[2N](t,x,\lambda_0^*)\varphi_0 = 0. \end{aligned} \tag{5.2.21}$$

2. The kernel vector υ_0 of $\mathcal{K}_0(t,0,\lambda_0)$ and ψ_0 are linearly independent at $x=0$. Then, a similar diagram to Figure 5.1 holds implying the existence of a kernel vector by which the dressing matrix $\mathcal{K}_N(t,x,\lambda)$ can be constructed so that at $x=0$ the following relations can be given

$$\begin{aligned} (D[2N](t,x,-\lambda_0)\mathcal{K}_0(t,0,\lambda_0)\psi_0) &= (\mathcal{K}_N(t,x,\lambda_0)D[2N](t,x,\lambda_0)\psi_0) \neq 0, \\ (D[2N](t,x,-\lambda_0^*)\mathcal{K}_0(t,0,\lambda_0^*)\varphi_0) &= (\mathcal{K}_N(t,x,\lambda_0^*)D[2N](t,x,\lambda_0^*)\varphi_0) \neq 0. \end{aligned} \tag{5.2.22}$$

This idea of deriving the kernel vector in order to construct the two-fold dressing matrix $\mathcal{K}_N(t,x,\lambda)$ can be repeated for the second, distinct parameter $\widehat{\lambda}_0$. As a result, we can use the two kernel vectors corresponding to the spectral parameters $\lambda=\lambda_0$ and $\lambda=\widehat{\lambda}_0$ to construct a two-fold dressing matrix multiplied by a function of λ in order to adjust the highest matrix coefficient, call it $\mathcal{K}_N$, which may be only given at $x=0$ and which satisfies a combination of relations (5.2.21) and (5.2.22) at these spectral parameters. Thus, we use this constructed matrix to prove that equalition (5.2.20) holds. First, we write the equality as matrix polynomials of degree $2N+2$ in λ and denote them as $L(\lambda)$ and $R(\lambda)$ so that

$$\begin{aligned} L(\lambda) &= (D[2N](t,x,-\lambda)\mathcal{K}_0(t,0,\lambda))\big|_{x=0} = \lambda^{2N+2}L_{2N+2} + \lambda^{2N+1}L_{2N+1} + \cdots + \lambda L_1 + L_0, \\ R(\lambda) &= (\mathcal{K}_N(t,x,\lambda)D[2N](t,x,\lambda))\big|_{x=0} = \lambda^{2N+2}R_{2N+2} + \lambda^{2N+1}R_{2N+1} + \cdots + \lambda R_1 + R_0. \end{aligned}$$

Since every factor on the left and right hand side has, after adapting the one for $\mathcal{K}_N$, the identity matrix times a constant as matrix coefficient of the highest order in λ, we find $L_{2N+2} = \mathbb{1} = R_{2N+2}$. If $\alpha \neq 0$, this property is obsolete. With respect to the property $\mathcal{K}^{-1}(t,0,\lambda) = \mathcal{K}(t,0,-\lambda)$ for $\mathcal{K}_0$ proven in Proposition 4.3.1, the spectral parameters and corresponding solutions of the Lax system provide $4N$ zeros and associated kernel vectors

$$\begin{aligned} &R(\lambda)\big|_{\lambda=\lambda_j}\psi_j = 0, \quad R(\lambda)\big|_{\lambda=\widehat{\lambda}_j}\widehat{\psi}_j = 0, \quad R(\lambda)\big|_{\lambda=\lambda_j^*}\varphi_j = 0, \quad R(\lambda)\big|_{\lambda=\widehat{\lambda}_j^*}\widehat{\varphi}_j = 0, \\ &L(\lambda)\big|_{\lambda=\lambda_j}\psi_j = 0, \quad L(\lambda)\big|_{\lambda=\widehat{\lambda}_j}\widehat{\psi}_j = 0 \quad L(\lambda)\big|_{\lambda=\lambda_j^*}\varphi_j = 0, \quad L(\lambda)\big|_{\lambda=\widehat{\lambda}_j^*}\widehat{\varphi}_j = 0 \end{aligned} \tag{5.2.23}$$

at $x=0$ for $j=1,\ldots,N$. For $R(\lambda)$, the equalities are clear from the definition of the dressing matrix $D[2N]$ and with the assumption (5.2.18), the equalities for $L(\lambda)$ follow immediately. The choice of $\mathcal{K}_N$ further implies that the relations (5.2.23) can be extended to $j=0$ in the following sense

$$\begin{aligned} R(\lambda)\big|_{\lambda=\lambda_0}\psi_0 &= L(\lambda)\big|_{\lambda=\lambda_0}\psi_0, & R(\lambda)\big|_{\lambda=\widehat{\lambda}_0}\widehat{\psi}_0 &= L(\lambda)\big|_{\lambda=\widehat{\lambda}_0}\widehat{\psi}_0, \\ R(\lambda)\big|_{\lambda=\lambda_0^*}\varphi_0 &= L(\lambda)\big|_{\lambda=\lambda_0^*}\varphi_0, & R(\lambda)\big|_{\lambda=\widehat{\lambda}_0^*}\widehat{\varphi}_0 &= L(\lambda)\big|_{\lambda=\widehat{\lambda}_0^*}\widehat{\varphi}_0. \end{aligned}$$

In the case $\alpha = 0$, λ_0 and $\widehat{\lambda}_0^*$ coincide and consequently we only have two spectral parameters where the equality holds. At this point, it is important that all vectors are linearly independent. In view of the *additional* vectors from the construction of $\mathcal{K}_N$, we see that these equalities are at this point not necessarily zero. Nevertheless, arranging the zeros and associated kernel vectors for the difference $C(\lambda) = L(\lambda) - R(\lambda)$ in matrix form, we can conclude that the matrix coefficients of $L(\lambda)$ and $R(\lambda)$ are the same. In particular, this gives us that independently of the construction of $\mathcal{K}_N$, the kernel vectors are indeed as described in the first case, namely the dressing matrix $D[2N]$ evaluated at $\lambda = \lambda_0$ and $\lambda = \widehat{\lambda}_0$ multiplied with the solution of the Lax system corresponding to the seed solution $u[0]$ for $\lambda = \lambda_0$ and $\lambda = \widehat{\lambda}_0$ and thus the matrix $\mathcal{K}_N$ is given for $x \in \mathbb{R}_+$.

(c) Given $\mathcal{K}_N(t,x,\lambda)$ of the form $\lambda^2\mathbb{1} + \lambda K^{(1)}(t,0) + K^{(0)}(t,0)$ at $x = 0$, we want to determine the matrix coefficients to confirm that the boundary conditions are preserved. Therefore, the symmetry of the t part of the Lax pair $\mathcal{V}$ given in (2.1.4) implies $\mathcal{K}_N(t,x,\lambda) = \sigma_2(\mathcal{K}_N(t,x,\lambda^*))^*\sigma_2$ resulting in

$$\mathcal{K}_N(t,0,\lambda) = \lambda^2\mathbb{1} + \lambda\begin{pmatrix} \mathcal{K}_{11}^{(1)}(t,0) & \mathcal{K}_{12}^{(1)}(t,0) \\ -\big(\mathcal{K}_{12}^{(1)}(t,0)\big)^* & \big(\mathcal{K}_{11}^{(1)}(t,0)\big)^* \end{pmatrix} + \begin{pmatrix} \mathcal{K}_{11}^{(0)}(t,0) & \mathcal{K}_{12}^{(0)}(t,0) \\ -\big(\mathcal{K}_{12}^{(0)}(t,0)\big)^* & \big(\mathcal{K}_{11}^{(0)}(t,0)\big)^* \end{pmatrix}.$$

The equality $L_{2N+1} = R_{2N+1}$ gives for the off-diagonal entries of $\mathcal{K}^{(1)}(t,0)$ that $\mathcal{K}_{12}^{(1)}(t,0) = iu[2N](t,0)$ and $\mathcal{K}_{21}^{(1)}(t,0) = -\big(\mathcal{K}_{12}^{(1)}(t,0)\big)^* = iu^*[2N](t,0)$. For the entries on the diagonal of $\mathcal{K}^{(1)}(t,0)$, we obtain from the same equality

$$\begin{aligned} \mathcal{K}_{11}^{(1)}(t,0) &= i\sqrt{\beta^2 - |u[0](t,0)|^2} - 2(\Sigma_1(t,0))_{11}, \\ \big(\mathcal{K}_{11}^{(1)}(t,0)\big)^* &= -i\sqrt{\beta^2 - |u[0](t,0)|^2} - 2(\Sigma_1^*(t,0))_{11}, \end{aligned} \tag{5.2.24}$$

where Σ_1 is the matrix coefficient of λ^{2N-1} of the matrix $D[2N](t,x,\lambda)$. To determine the remaining entries of the matrix coefficients, we need to extract information from the determinant of $\mathcal{K}_N(t,x,\lambda)$. Again, we have that

$$\det\Big(\prod_{k=1}^{N}(\lambda-\lambda_k^*)(\lambda+\lambda_k^*)D[2N](t,x,\lambda)\Big) = \det\Big(\prod_{k=1}^{N}(\lambda-\lambda_k^*)(\lambda+\lambda_k^*)D[2N](t,x,-\lambda)\Big),$$

which implies for the determinant of $\mathcal{K}_N$ that

$$\begin{aligned} \det\Big(\frac{(2\lambda - i|\beta|)^2 - \alpha^2}{4}\mathcal{K}_N\big|_{x=0}\Big) &= \det\Big(\frac{(2\lambda - i|\beta|)^2 - \alpha^2}{4}\mathcal{K}_0\Big) \\ &= \lambda^4 - \frac{\alpha^2-\beta^2}{2}\lambda^2 + \frac{(\alpha^2+\beta^2)^2}{16}. \end{aligned}$$

Formally, calculating the determinant of the matrix $\mathcal{K}_N(t,0,\lambda)$ in polynomial form as given above, we can match the coefficients yielding

$$\begin{aligned} \mathrm{Tr}(\mathcal{K}^{(1)}(t,0)) &= 0, \\ \mathrm{Tr}(\mathcal{K}^{(0)}(t,0)) + \det(\mathcal{K}^{(1)}(t,0)) &= -\frac{\alpha^2-\beta^2}{2}, \\ 2\,\mathrm{Re}\big(\mathcal{K}_{11}^{(1)}(t,0)\big(\mathcal{K}_{11}^{(0)}(t,0)\big)^*\big) - 2\big(\mathcal{K}_{12}^{(0)}(t,0)\big)^*\,\mathrm{Im}(u[2N](t,0)) &= 0, \\ \det(\mathcal{K}^{(0)}(t,0)) &= \frac{(\alpha^2+\beta^2)^2}{16}. \end{aligned} \tag{5.2.25}$$

Combining the first line in (5.2.25) with the expressions we have for $\mathcal{K}_{11}^{(1)}(t,0)$ and its complex conjugate, see (5.2.24), we can deduce that $\mathrm{Re}(\mathcal{K}_{11}^{(1)}(t,0)) = 0$. Further, evaluating the equality of (5.2.20) of order $2N$ in λ, i.e. $L_{2N} = R_{2N}$, we obtain

$$-i\Sigma_1\begin{pmatrix} \sqrt{\beta^2 - |u[0](t,0)|^2} & u[0](t,0) \\ u[0]^*(t,0) & -\sqrt{\beta^2 - |u[0](t,0)|^2} \end{pmatrix} - \frac{\alpha^2+\beta^2}{4}\mathbb{1} = \mathcal{K}^{(1)}(t,0)\Sigma_1 + \mathcal{K}^{(0)}(t,0)$$

at $x=0$. Matching the (12)-entry of this equality, we derive

$$(u[2N](t,0)-u[0](t,0))\frac{\sqrt{\beta^2-|u[0](t,0)|^2}}{2}-iu[0](t,0)(\Sigma_1(t,0))_{11}=$$
$$-\frac{i}{2}\mathcal{K}_{11}^{(1)}(t,0)(u[2N](t,0)-u[0](t,0))+iu[2N](t,0)(\Sigma_1^*(t,0))_{11}+\mathcal{K}_{12}^{(0)}(t,0)$$

and using the expressions in (5.2.24) we have for $(\Sigma_1)_{11}$ and $(\Sigma_1^*)_{11}$, we obtain after cancellation that

$$\mathcal{K}_{12}^{(0)}(t,0)-iu[2N]\,\mathrm{Re}(\mathcal{K}_{11}^{(1)}(t,0))=0.$$

However, we already calculated that $\mathrm{Re}(\mathcal{K}_{11}^{(1)}(t,0))$ needs to be zero in order for the determinants to be equal. Hence, also $\mathcal{K}_{12}^{(0)}(t,0)$ and thus the off-diagonal of $\mathcal{K}^{(0)}(t,0)$ vanishes. It follows by the third equation of (5.2.25) that $\mathrm{Im}(\mathcal{K}_{11}^{(0)}(t,0))=0$ and then, by the fourth equation we have $\mathcal{K}^{(0)}(t,0)=\pm\frac{\alpha^2+\beta^2}{4}\mathbb{1}$. To verify that it is indeed minus as for $\mathcal{K}_0(t,0,\lambda)$, we confirm with the equality of $L_0=R_0$, which gives

$$-\frac{\alpha^2+\beta^2}{4}\Sigma_{2N}(t,0)=\mathcal{K}^{(0)}(t,0)\Sigma_{2N}(t,0),$$

where Σ_{2N} is the zero-th order matrix coefficient of the dressing matrix $D[2N](t,x,\lambda)$. For this to be satisfied for all $t\in\mathbb{R}_+$, we need to have $\mathcal{K}^{(0)}=-\frac{\alpha^2+\beta^2}{4}\mathbb{1}$. Therefore, we obtain $\mathrm{Tr}(\mathcal{K}^{(0)}(t,0))=-\frac{\alpha^2+\beta^2}{2}$. Thus, the second equation of (5.2.25) implies that

$$\mathcal{K}_{11}^{(1)}(t,0)=\pm i\sqrt{\beta^2-|u[2N](t,0)|^2},$$
$$\left(\mathcal{K}_{11}^{(1)}(t,0)\right)^*=\mp i\sqrt{\beta^2-|u[2N](t,0)|^2}.$$

Now, we need to determine the sign of the diagonal entries of $\mathcal{K}^{(1)}(t,0)$ to ensure that $\mathcal{K}_N(t,x,\lambda)$ preserves the boundary constraint at $x=0$, i.e. we need to show that the signs coincide with the signs in the same entry of $\mathcal{K}_0(t,0,\lambda)$ in front of the square root.

Therefore, a similar analysis as in Proposition 5.1.2 is needed, where we use the fact that under the Dressing method functions $u[0](\cdot,0),u[0]_x(\cdot,0)$ in the function space $H_t^{1,1}(\mathbb{R})$ are mapped onto functions, here $u[2N](\cdot,0),u[2N]_x(\cdot,0)$, which lie in the function space $H_t^{1,1}(\mathbb{R})$. Further, assume that $\mathcal{K}_0(t,0,\lambda)$ has a positive sign in the (11)-entry in front of the square root. We have identified the kernel vectors ψ_0 and $\widehat{\psi}_0$ of $\mathcal{K}_0(t,0,\lambda)$ at $x=0$ and $\lambda=\lambda_0$ and $\lambda=\widehat{\lambda}_0$, respectively. Then, for $\mathcal{K}_0(t,0,\lambda)$ multiplied by $((2\lambda-i|\beta|)^2-\alpha^2)/4$ as t goes to infinity, we have that

$$\frac{(2\lambda-i|\beta|)^2-\alpha^2}{4}\lim_{t\to\infty}\mathcal{K}_0(t,0,\lambda)=\mathrm{diag}\Big(\lambda^2+i|\beta|\lambda-\frac{(\alpha^2+\beta^2)}{4},\lambda^2-i|\beta|\lambda-\frac{(\alpha^2+\beta^2)}{4}\Big)$$
$$=\mathrm{diag}((\lambda-\lambda_0)(\lambda-\widehat{\lambda}_0^*),(\lambda-\lambda_0^*)(\lambda-\widehat{\lambda}_0)).$$

In turn, this implies that the kernel vectors of $\mathcal{K}_0(t,0,\lambda)$ necessarily admit the limit behavior $\psi_0\sim e_1$ and $\widehat{\psi}_0\sim e_2$ as t goes to infinity. Since the dressing matrix $D[2N](t,x,\lambda)$ also becomes diagonal as t goes to infinity, see Lemma 4.4.4, the kernel vectors $\psi_0'=D[2N]\big|_{\lambda=\lambda_0}\psi_0$, $\widehat{\psi}_0'=D[2N]\big|_{\lambda=\widehat{\lambda}_0}\widehat{\psi}_0$ of $\mathcal{K}_N$ inherit the long time behavior of their corresponding vector. Therefore, the signs can be determined to be positive in the (11)-entry and negative in the (22)-entry in front of the square root.

Secondly, the assertion regarding the imaginary part corresponding to the sign of $\mathcal{K}_{11}^{(1)}(t,0)$ for $t \in \mathbb{R}_+$ makes sure that the sign can not simply change over time.

Hence, the boundary condition for the solution $\widehat{u}[N]$ of the NLS equation on the half-line is preserved. □

Remark 5.2.8. Similar to the analysis of the long time behavior of the kernel vectors, one could look at the long time behavior of the dressing matrix $D[2N](t,x,\lambda)$ to deduce the same result through the equality of $\mathcal{K}_N\big|_{x=0}$ with the product of the three matrices $D[2N](t,x,-\lambda)\cdot\mathcal{K}_0(t,0,\lambda)\cdot(D[2N](t,x,\lambda))^{-1}$ at $x=0$. Nevertheless, these behaviors are closely related to one another, since the limit behavior of the kernel vectors of $D[2N](t,0,\lambda)$ determines the distribution of factors $\lambda-\lambda_j$, $\lambda-\widehat{\lambda}_j$, $\lambda-\lambda_j^*$ and $\lambda-\widehat{\lambda}_j^*$ for $j=1,\dots,N$ in the diagonal entries as t goes to infinity.

Remark 5.2.9. As for the NLS equation with defect conditions, it is possible to express the assertion about the sign of $\mathcal{K}_{11}^{(1)}(t,0)$ in terms of the kernel vectors the boundary matrix $\mathcal{K}_N$ is constructed from. However, this condition becomes very situational due to the fact that in theory we deal with a two-fold dressing matrix.

We have shown that the method of dressing the boundary can as well be applied to the new boundary conditions constituted as in [30]. In order to achieve this, it is necessary to apply the techniques developed for the other defect and boundary conditions, most importantly, the determination of a dressing matrix which satisfies a particular equality, see (5.2.21) and (5.2.22) as well as the identification procedure for the $\pm$ sign inside the boundary matrix $\mathcal{K}_N$. Integrability for this boundary condition has been established recently in [40] together with a simplified application of the Dressing method.

In conclusion, we have in theory established the method of dressing the defect for the defect conditions (4.1.2) regarding the NLS equation and (4.1.4) regarding the sG equation on two half-lines as presented in Section 5.1 as well as the method of dressing the boundary for the sin-boundary condition (4.3.7) regarding the sG equation and the Robin (4.3.2) as well as the new boundary condition (4.3.3) regarding the NLS equation on the half-line in Section 5.2. As emphasized before, the Dressing method stands, in particular, for a straightforward application to practically obtain soliton or breather solutions in connection with the Lax systems of the NLS and sG equation. Therefore, we dedicate the next chapter to the application of the presented propositions to derive explicit solutions of the respective model and further to graphically present the results. Moreover, we use the theory introduced in Section 3.3 for the change of scattering data under the Dressing method to explicitly describe the complete scattering data which is needed in order to utilize the propositions and prove in the case of the defect conditions for the NLS equation that each soliton is transmitted through the defect independently as conjectured in [15].

Chapter 6

Soliton solutions

In this chapter, we want to apply the main results worked out in Chapter 5 in order to construct and visualize explicit solutions of the presented models using Matlab. Thus, we are interested in those solutions, which can be constructed by the Dressing method. Nonetheless, let us first elucidate the notion of *solitons* especially for the NLS and sG equation more accurately. Before we give mathematically rigorous definitions of solitons in these two cases, in general solitons can be described as solutions of a nonlinear equation which admit three properties:

1. They are of permanent form;

2. They are localized within a region;

3. They can interact with other solitons, and emerge from the collision unchanged, except for a phase shift.

This definition has been given in [20] and it should be mentioned that, due to the broad spectrum of where these solutions can be found, it is by no means the only definition. In particular, solitons arise when the properties of a nonlinear equation are such that the dispersion and nonlinear effects precisely counteract each other, see Figure 6.1 for a sketch of this idea. The NLS and sG equation

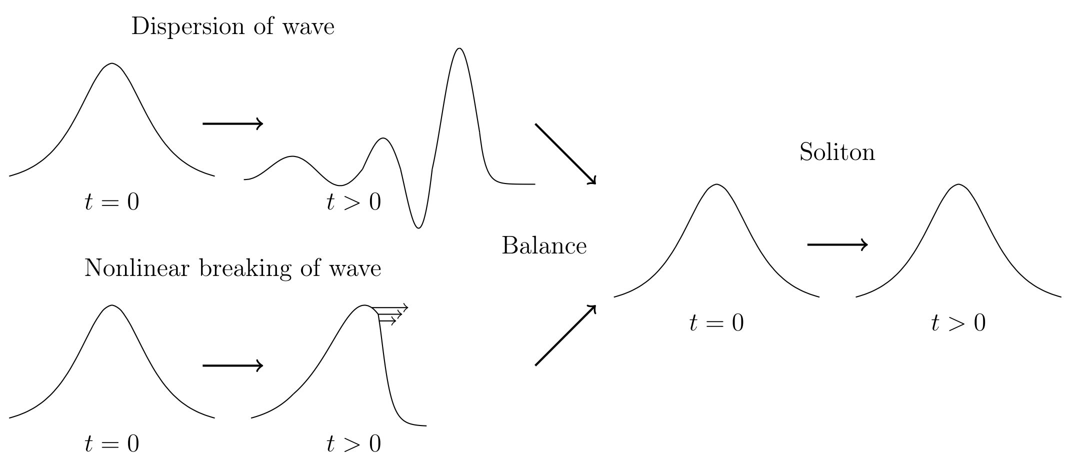

Fig. 6.1. Balance effects of dispersion and breaking in a soliton, see also [20].

both meet this criteria and the inverse scattering transformation is typically utilized to obtain these solutions, as worked out in Sections 2.1 and 2.2. Moreover, with Definitions 2.1.4 and 2.2.6 in mind, we can give rigorous definitions of N-soliton solutions with respect to the scattering data

$\mathcal{S}(u) = (\rho(\lambda), \{\lambda_j, C_j\}_{j=1}^N)$ or $\mathcal{S}(\theta) = (\rho(\lambda), \{\lambda_j, C_j\}_{j=1}^N)$ associated to the initial data $u \in \mathcal{G}_N$ or $\theta \in \mathcal{G}_N$.

Definition 6.0.1. Given the initial data generates pairwise distinct simple eigenvalues $\lambda_1, \dots, \lambda_N$ and moreover $\rho(\lambda) = 0$ for all $\lambda \in \mathbb{R}$, the corresponding solution of (2.1.1) or (2.2.1) is called an *N-soliton solution.* For $\mathcal{D}_N = \{\lambda_j, C_j\}_{j=1}^N \subset (\mathbb{C}_+)^N \times (\mathbb{C} \setminus \{0\})^N$, we adapt the notation

$$u_{sol}(t, x; \mathcal{D}_N), \qquad \theta_{sol}(t, x; \mathcal{D}_N).$$

Note that the one-soliton solution given in (2.1.21) is based on this definition. Moreover, the zero solutions $u = 0$ and $\theta = 0$ are covered by this definition as zero-soliton solutions. In both equations, special solitonic structures, the so-called *breathers*, can be found. In addition to the properties of a soliton, the breather also admits a periodicity as can be seen later on. As indicated before, breathers in the case of the NLS equation at least require that two simple eigenvalues share the same real part, whereas in the case of the sG equation they correspond to a pair of simple eigenvalues λ_1 and $\lambda_2 = -\lambda_1^*$ not lying on the imaginary axis. Beyond that, all these solutions can appear as *boundary-bound* soliton solutions. Again, differentiating the cases of the NLS equation and the sG equation, for which boundary-bound solitons correspond to simple eigenvalues lying on the imaginary axis and on the unit circle of the complex plane, respectively. In Figure 6.2, we give examples associated to the distributions of simple eigenvalues for both equations.

Now, in the forthcoming sections, we especially aspire to construct new solutions and keep information which can be obtained through other literature to a sensible minimum. Dealing with the reconstruction formulae from the Dressing method in both cases, the sG equation (3.2.16) and NLS equation (3.2.14), one may notice that the solutions arising in the case of the sG equation are more complicated than the ones in the case of the NLS equation. Thus, connecting the expressions of the scattering data with relevant parameters on the solution side is less feasible. Nevertheless, we shall provide some insights by explicitly constructing the single one-soliton and a breather solution under the Dressing method in the case of the sG equation. For the NLS equation, however, we utilize these relations of the scattering data to parameters of the solution in order to prove that each soliton is transmitted through the defect independently in the model with defect conditions and that the parameters for the suggested pairs of solitons have specific relations in the model with boundary conditions.

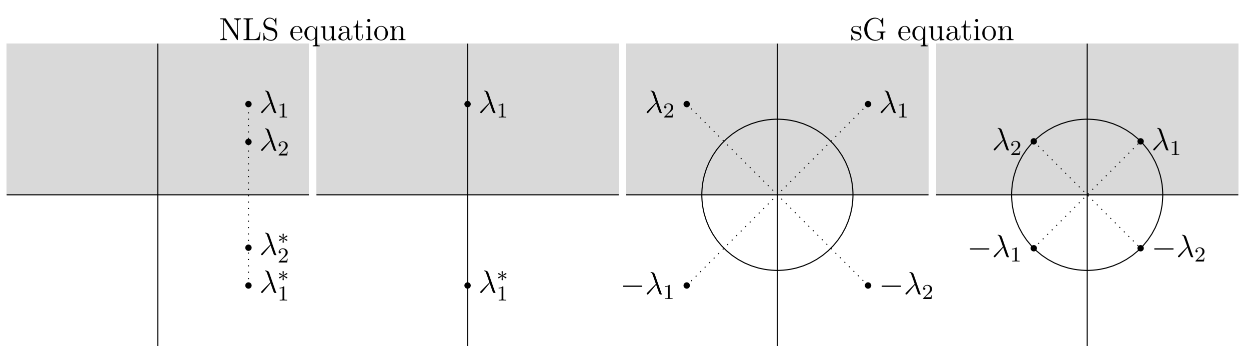

Fig. 6.2. Exemplary distribution of simple eigenvalues for breathers (in each case left) and boundary-bound solitons (in each case right).

6.1 Soliton solutions for models of the sG equation

As we have seen, the zero seed solution $\theta_{sol}(t,x;\{\}) = 0$ corresponds to the scattering data $(\rho(\lambda) = 0)$. In fact, one can derive $a_{11}(\lambda) = 1$ and $a_{21}(\lambda) = 0$. Therefore, applying a one-fold dressing matrix to construct a one-soliton solution with the new simple eigenvalue $\lambda_1 = i\eta_1 \in i\mathbb{R}$ and constants u_1, v_1 such that the quotient $v_1/u_1 \in i\mathbb{R} \setminus \{0\}$, by Theorem 3.3.1 we obtain the following new data

$$\begin{aligned} a_{11}(\lambda) &= \frac{\lambda-\lambda_1}{\lambda-\lambda_1^*}, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R}, & \rho(\lambda) &= 0, & \lambda \in \mathbb{R}, \\ a_{21}(\lambda) &= 0, \quad \lambda \in \mathbb{R}, \\ b_1 &= -\frac{v_1}{u_1}, & C_1 &= -\frac{v_1}{u_1}(\lambda_1 - \lambda_1^*). \end{aligned}$$

The relevant scattering data $(\rho = 0, \{\lambda_1, C_1\})$ corresponds to a one-soliton solution which we construct in the following:

Lemma 6.1.1. *Given the scattering data $(0, \{\lambda_1 = i\eta_1, C_1 = 2i\eta_1 b_1\})$, the one-soliton solution of the sG equation is given by*

$$\theta_{sol}(t,x;\{\lambda_1, C_1\}) = 4\,\mathrm{sign}\,\mathrm{Im}(b_1)\arctan e^{\frac{1}{2}(\eta_1+\frac{1}{\eta_1})x - \frac{1}{2}(\eta_1-\frac{1}{\eta_1})t - \log|b_1|}. \tag{6.1.1}$$

Proof. The proof structurally follows the ideas given in [27]. For the Dressing method, we first state the fundamental solution

$$\Phi(t,x,\lambda) = (\psi_-^{(1)}, \psi_+^{(2)}) = \begin{pmatrix} e^{\frac{\vartheta(\lambda)-i\zeta(\lambda)}{2}} & 0 \\ 0 & e^{-\frac{\vartheta(\lambda)-i\zeta(\lambda)}{2}} \end{pmatrix}$$

for the Lax system (2.2.3) of the sG equation corresponding to the zero seed solution, where

$$\begin{aligned} -\vartheta(\lambda) &= \mathrm{Re}(2i\Theta(t,x,\lambda)) = \frac{\mathrm{Im}(\lambda)}{2}\Big[\Big(1+\frac{1}{|\lambda|^2}\Big)x - \Big(1-\frac{1}{|\lambda|^2}\Big)t\Big], \\ \zeta(\lambda) &= \mathrm{Im}(2i\Theta(t,x,\lambda)) = \frac{\mathrm{Re}(\lambda)}{2}\Big[\Big(1-\frac{1}{|\lambda|^2}\Big)x - \Big(1+\frac{1}{|\lambda|^2}\Big)t\Big]. \end{aligned}$$

Therefore, we can give an explicit solution of the Lax system (2.2.3) at the spectral parameter $\lambda = \lambda_1$ by

$$\psi_1(t,x) = u_1\psi_-^{(1)}(t,x,\lambda_1) + v_1\psi_+^{(2)}(t,x,\lambda_1) = e^{-i\Theta(t,x,\lambda_1)\sigma_3}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \tag{6.1.2}$$

where $(u_1, v_1)^\intercal$ is connected to C_1 as noted beforehand and the phase $\Theta(t,x,\lambda)$ is the phase function from the scattering process for the sG equation. Effectively, only the quotient of v_1 and u_1 is relevant and as in Section 3.2, we take the quotient of the second entry and the first entry of ψ_1 to obtain

$$\Delta_1 = -b_1 e^{2i\Theta(t,x,\lambda_1)} = -b_1 e^{-\vartheta(\lambda_1)+i\zeta(\lambda_1)},$$

where $b_1 = -\frac{v_1}{u_1}$. Then, we derive the one-fold dressing matrix $D[1]$ for an arbitrary $\mathbb{C}\setminus\mathbb{R} \ni \lambda = \lambda_1$, for which we take on the usual notation $\lambda_1 = \xi_1 + i\eta_1$, and $b_1 = -\frac{v_1}{u_1}$. By adopting the notation $\vartheta = \vartheta(\lambda_1) - \log|b_1|$ and $\zeta = \zeta(\lambda_1) + \arg b_1$, we obtain

$$D[1] = \mathbb{1} - \frac{i\eta_1}{\lambda - \lambda_1^*}\begin{pmatrix} 1+\tanh(\vartheta) & -\mathrm{sech}(\vartheta)e^{-i\zeta} \\ -\mathrm{sech}(\vartheta)e^{i\zeta} & 1-\tanh(\vartheta) \end{pmatrix}. \tag{6.1.3}$$

Consequently for $N_s = 1$, we can calculate the one-soliton solution of the sG equation corresponding to a purely imaginary spectral parameter, $\xi_1 = 0$. The reconstruction formula (3.2.16) implies that $\sin(\theta[1]/2)$ can be constructed by evaluating the (12)-entry of $-iD[1]\big|_{\lambda=0}\sigma_3$, which ultimately gives

$$\theta[1] = 2\,\mathrm{sign}\,\mathrm{Im}(b_1)\arcsin(\mathrm{sech}\,\vartheta). \tag{6.1.4}$$

Further, this can be transformed into the more commonly known version of the one-soliton solution

$$\theta[1] = 4\,\mathrm{sign}\,\mathrm{Im}(b_1)\arctan e^{\frac{1}{2}(\eta_1+\frac{1}{\eta_1})x-\frac{1}{2}(\eta_1-\frac{1}{\eta_1})t-\log|b_1|}.$$

□

Note that the general one-fold dressing matrix (6.1.3) can be used to obtain the breather solution, since the Dressing method is an iterative method. Hence, we have:

Lemma 6.1.2. *Given the scattering data* $(0, \{\lambda_1 = \xi_1 + i\eta_1, -\lambda_1^*, C_1 = 2i\lambda_1\frac{\eta_1}{\xi_1}b_1, -C_1^*\})$, *where* $\lambda_j \notin i\mathbb{R}$, *the breather solution of the sG equation is given by*

$$\theta[2] = 4\arctan\left(-\frac{\eta_1\cos\left(\frac{\xi_1}{2}\left[\left(1-\frac{1}{|\lambda_1|^2}\right)x-\left(1+\frac{1}{|\lambda_1|^2}\right)t\right]+\arg(b_1)\right)}{\xi_1\cosh\left(-\frac{\eta_1}{2}\left[\left(1+\frac{1}{|\lambda_1|^2}\right)x-\left(1-\frac{1}{|\lambda_1|^2}\right)t\right]-\log|b_1|\right)}\right). \tag{6.1.5}$$

Proof. The calculations to obtain the one-fold dressing transformation come in handy for the derivation of a breather solution, where the scattering data consist of $\lambda_1 \in \mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$ and $\lambda_2 = -\lambda_1^*$ with respective norming constants $b_1 = -\frac{v_1}{u_1}$ and $b_2 = -b_1^* = \frac{v_1^*}{u_1^*}$. By the one-fold dressing transformation $D[1]$, we can work out the fundamental solution for the Lax system of the Lax pair $\mathbb{U}[1]$ and $\mathbb{V}[1]$ as

$$\Phi[1](t,x,\lambda) = \begin{pmatrix} \left(1-\frac{i\eta_1}{\lambda-\lambda_1^*}(1+\tanh(\vartheta)\right)e^{\frac{\vartheta(\lambda)-i\zeta(\lambda)}{2}} & \frac{i\eta_1}{\lambda-\lambda_1^*}\,\mathrm{sech}(\vartheta)e^{-i\zeta}e^{-\frac{\vartheta(\lambda)-i\zeta(\lambda)}{2}} \\ \frac{i\eta_1}{\lambda-\lambda_1^*}\,\mathrm{sech}(\vartheta)e^{i\zeta}e^{\frac{\vartheta(\lambda)-i\zeta(\lambda)}{2}} & \left(1-\frac{i\eta_1}{\lambda-\lambda_1^*}(1-\tanh(\vartheta)\right)e^{-\frac{\vartheta(\lambda)-i\zeta(\lambda)}{2}} \end{pmatrix},$$

where we continue to utilize the notation we have introduced in the proof of Lemma 6.1.1. From this we obtain at $\lambda_2 = -\lambda_1^*$ and $b_2 = -b_1^*$ that the quotient yields

$$\Delta_2[1] = \frac{-i\eta_1\cos\zeta + \xi_1 e^{-\vartheta-i\zeta}\cosh\vartheta}{\xi_1\cosh\vartheta - i\eta_1 e^{-\vartheta-i\zeta}\cos\zeta}.$$

Hence by the reconstruction formula (3.2.16), evaluating the (12)-entry of the two-fold dressing matrix

$$D[2] = \left(\mathbb{1} - \frac{2i\eta_1}{\lambda-\lambda_1}\frac{1}{1+|\Delta_2[1]|^2}\begin{pmatrix}1 & \Delta_2^*[1]\\ \Delta_2[1] & |\Delta_2[1]|^2\end{pmatrix}\right)\left(\mathbb{1} - \frac{2i\eta_1}{\lambda-\lambda_1^*}\frac{1}{1+|\Delta_1|^2}\begin{pmatrix}1 & \Delta_1^*\\ \Delta_1 & |\Delta_1|^2\end{pmatrix}\right)$$

at $\lambda = 0$ and multiplying the result by $-i$, gives an expression for the solution in terms of $\sin(\theta[2]/2)$. Then, we have

$$\theta[2] = 2\arcsin\left(-\frac{2\xi_1\eta_1\cos\zeta\cosh\vartheta}{\eta_1^2\cos^2\zeta+\xi_1^2\cosh^2\vartheta}\right) \tag{6.1.6}$$

or in the more commonly known form of the breather solution

$$\theta[2] = 4\arctan\left(-\frac{\eta_1\cos\zeta}{\xi_1\cosh\vartheta}\right).$$

□

With these two explicitly constructed soliton solutions, we want to give examples of the solutions of the models of the sG equation on two half-lines connected by defect conditions and on one half-line with the sin-boundary condition. In general, if one merely wants to calculate expressions of soliton solutions for the sG equation, it seems that other methods for instance the Bäcklund transformation or variable transformations are more efficient when it comes to the direct derivation of expressions with the arctan, based on the available literature. Nevertheless, we want to address two pivotal advantages of the Dressing method, which aligns the resulting soliton solutions with the structural analysis we want to apply. Firstly, let us emphasize that in the Dressing method the parameters of the resulting soliton solutions are completely determined by the scattering data, which we have defined during the process of the direct scattering. Not only that, but in regards to the propositions we have worked out in Chapter 5, which are stated with the Dressing method in mind, the solutions depend on relations of the scattering data, making it extremely important to have a direct connection of the scattering data to parameters in the N-soliton solution. Secondly, particularly for the NLS equation, the Dressing method provides the means to let the computer handle the lengthy algebraic computations. Therefore, even without having the explicit expressions of the solutions at hand, it is still possible to visualize exact N-soliton solutions until the processing capacity is reached.

6.1.1 sG equation on two half-lines connected by a defect condition

Now, given the explicit soliton solutions derived in Section 6.1 and the assumption (5.1.3), we can give the explicit expressions of a one-solion and a breather solution for the sG equation on two half-lines connected via defect conditions. Taking paired solutions $\widetilde{\psi}_j$, $j = 1$ or $j = 1, 2$, to the Lax system corresponding to the sG equation on the negative half-line (5.1.2), it follows that each indiviual quotient $\tilde{v}_j/\tilde{u}_j$ needs to satisfy

$$\frac{\tilde{v}_j}{\tilde{u}_j} = \frac{\lambda_j - i\alpha}{\lambda_j + i\alpha}\frac{v_j}{u_j} \tag{6.1.7}$$

by assumption (5.1.3), where we choose the + sign in the matrix $\mathbb{B}_0(\lambda) = \mathbb{1} + i\alpha\sigma_3$ representing the frozen Bäcklund transformation, which is used in Proposition 5.1.1. The quotient representing the defect conditions in the spectral data is transferred to the norming constants $\widetilde{C}_j$, $j = 1$ or $j = 1, 2$, of the scattering data yielding

$$\widetilde{C}_j = \frac{\lambda_j - i\alpha}{\lambda_j + i\alpha} C_j.$$

Hence, a one-soliton solution of the sG equation on two half-lines connected by the defect conditions is given by the combination of the solutions

$$\tilde{\theta}_{sol}(t, x; \{\lambda_1, \widetilde{C}_1\}) = 4\,\mathrm{sign}\Big(\frac{\widetilde{C}_1}{\eta_1}\Big)\arctan e^{\frac{1}{2}(\eta_1 + \frac{1}{\eta_1})x - \frac{1}{2}(\eta_1 - \frac{1}{\eta_1})t - \log\left|\frac{\tilde{C}_1}{2\eta_1}\right|},$$
$$\theta_{sol}(t, x; \{\lambda_1, C_1\}) = 4\,\mathrm{sign}\Big(\frac{C_1}{\eta_1}\Big)\arctan e^{\frac{1}{2}(\eta_1 + \frac{1}{\eta_1})x - \frac{1}{2}(\eta_1 - \frac{1}{\eta_1})t - \log\left|\frac{C_1}{2\eta_1}\right|}$$

on the negative and positive half-line, respectively. Therefore, similar observations based on the quotient $\frac{\lambda_j - i\alpha}{\lambda_j + i\alpha}$ as in [8] hold, see also Figure 6.3.

1. If $\eta_1 > \alpha$, then the soliton is simply delayed, since the quotient is less than or equal to one. For bigger η_1 the delay is getting less, since the quotient gets closer to one.

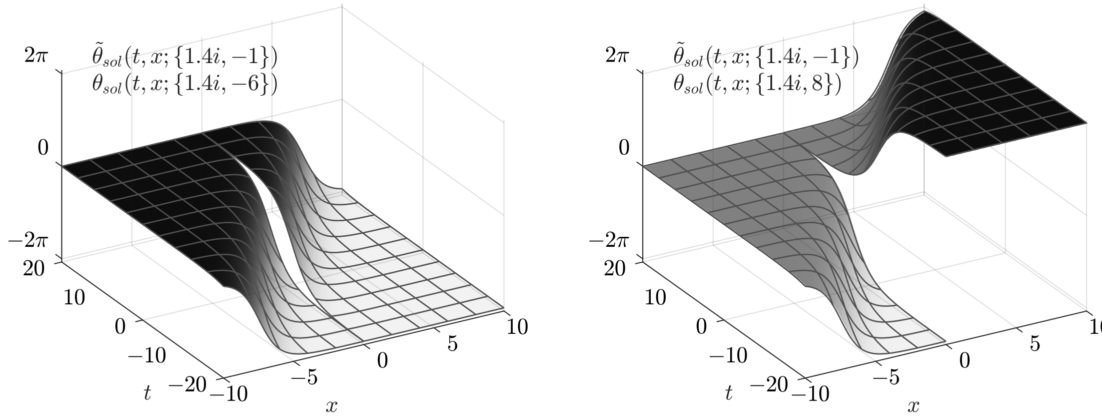

Fig. 6.3. One-soliton interacting with the defect (left: $\alpha = 1$, right: $\alpha = 1.8$).

2. If $\eta_1 = \alpha$, then the soliton can not be described by Proposition 5.1.1, since it is swallowed by the defect.

3. If $\eta_1 < \alpha$, then the constant $\widetilde{C}_1$ has the opposite sign of C_1. Therefore, an incoming soliton is transmitted through the defect as an anti-soliton, or vice-versa.

The same steps can be applied to the breather solution which then amounts to

$$\tilde{\theta}_{sol}(t,x;\{\lambda_1,-\lambda_1^*,\widetilde{C}_1,-\widetilde{C}_1^*\}) = 4\arctan\left(-\frac{\eta_1\cos\Big(\frac{\xi_1}{2}\Big[\big(1-\frac{1}{|\lambda_1|^2}\big)x-\big(1+\frac{1}{|\lambda_1|^2}\big)t\Big]+\arg(\tilde{b}_1)\Big)}{\xi_1\cosh\Big(-\frac{\eta_1}{2}\Big[\big(1+\frac{1}{|\lambda_1|^2}\big)x-\big(1-\frac{1}{|\lambda_1|^2}\big)t\Big]-\log|\tilde{b}_1|\Big)}\right),$$

$$\theta_{sol}(t,x;\{\lambda_1,-\lambda_1^*,C_1,-C_1^*\}) = 4\arctan\left(-\frac{\eta_1\cos\Big(\frac{\xi_1}{2}\Big[\big(1-\frac{1}{|\lambda_1|^2}\big)x-\big(1+\frac{1}{|\lambda_1|^2}\big)t\Big]+\arg(b_1)\Big)}{\xi_1\cosh\Big(-\frac{\eta_1}{2}\Big[\big(1+\frac{1}{|\lambda_1|^2}\big)x-\big(1-\frac{1}{|\lambda_1|^2}\big)t\Big]-\log|b_1|\Big)}\right)$$

on the negative an the positive half-line, respectively. The connections between b_1, $\tilde{b}_1$ and C_1, $\widetilde{C}_1$ are given by

$$b_1 = \frac{C_1}{2i\lambda_1}\frac{\xi_1}{\eta_1}, \qquad \tilde{b}_1 = \frac{\widetilde{C}_1}{2i\lambda_1}\frac{\xi_1}{\eta_1},$$

respectively, where $\lambda_1 = \xi_1 + i\eta_1 \in \mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$ is the spectral parameter. Further, the parameters b_1, b_2 and $\tilde{b}_1$, $\tilde{b}_2$ indeed satisfy the following relations

$$\begin{aligned} b_1 &= -\frac{v_1}{u_1}, & b_2 &= -\frac{v_2}{u_2} = \frac{v_1^*}{u_1^*} = -b_1^*, \\ \tilde{b}_1 &= -\frac{v_1}{u_1}\frac{\lambda_1 - i\alpha}{\lambda_1 + i\alpha}, & \tilde{b}_2 &= -\frac{v_2}{u_2}\frac{\lambda_2 - i\alpha}{\lambda_2 + i\alpha} = \frac{v_1^*}{u_1^*}\frac{\lambda_1^* + i\alpha}{\lambda_1^* - i\alpha} = -\tilde{b}_1^* \end{aligned}$$

due to assumption (5.1.3). In both cases, of the single soliton and the breather solution, these relations amount to

$$\begin{aligned} \tilde{x}_1 - x_1 &= -\log\left|\frac{\lambda_1 - i\alpha}{\lambda_1 + i\alpha}\right|, \\ \tilde{\varphi}_1 - \varphi_1 &= \arg\Big(\frac{\lambda_1 - i\alpha}{\lambda_1 + i\alpha}\Big), \end{aligned} \tag{6.1.8}$$

where $x_1 = -\log|b_1|$ and $\varphi_1 = \arg b_1$ and similar notation for the other two constants. In that regard, we have shown that each soliton in the one-soliton solution and the specific two-soliton solution experience the defect independently, which seems to be not as easily generalizable as for the NLS equation. We give examples of the breather (two-soliton) solution in Figures 6.4 and 6.5.

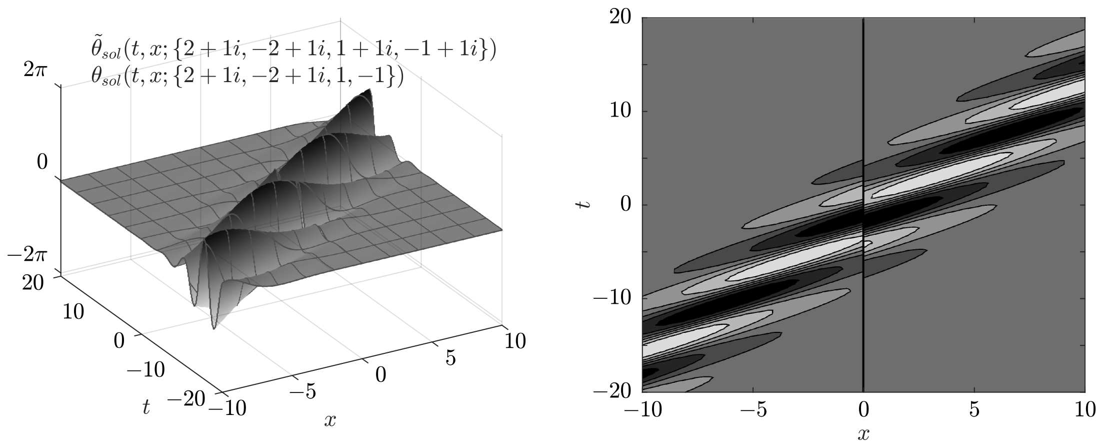

Fig. 6.4. Breather interacting with the defect ($\alpha = -1$) and its contour.

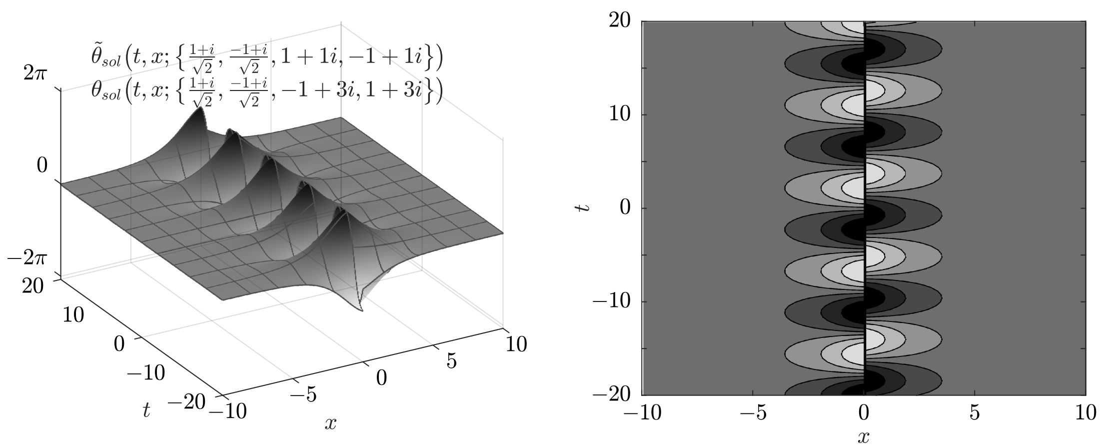

Fig. 6.5. Boundary-bound breather interacting with the defect ($\alpha = \frac{1}{\sqrt{2}}$) and its contour.

A special solution can be identified [7], which interacts destructively with the defect condition. Therefore, a specific choice of the simple eigenvalue λ_1 is necessary which is $\lambda_1 = i\alpha$, $\alpha \in \mathbb{R}\setminus\{0\}$. This results for example in the following formula

$$\begin{aligned}
\tilde{\theta}_{sol}(t,x;\{\lambda_1,\widetilde{C}_1\}) &= 4\,\mathrm{sign}(\mathrm{Im}(\tilde{b}_1))\arctan e^{\frac{1}{2}(\alpha+\frac{1}{\alpha})x-\frac{1}{2}(\alpha-\frac{1}{\alpha})t-\log|\tilde{b}_1|},\\
\theta_{sol}(t,x;\{\}) &= 0
\end{aligned}$$

and very similarly for the other half-line which we visualize in Figure 6.6. Inserting this solution into the defect condition (4.1.4) of the sG equation, we find

$$\begin{aligned}
(\tilde{\theta}_{sol}(t,x;\{\lambda_1,\widetilde{C}_1\}))_x\big|_{x=0} &= 4\,\mathrm{sign}(\mathrm{Im}(\tilde{b}_1))\frac{1}{4}\Big(\alpha+\frac{1}{\alpha}\Big)\,\mathrm{sech}\Big(\frac{1}{2}(\alpha-\frac{1}{\alpha})t+\log|\tilde{b}_1|\Big)\\
&= \Big(\alpha+\frac{1}{\alpha}\Big)\sin\frac{\tilde{\theta}_{sol}(t,0;\{\lambda_1,\widetilde{C}_1\})}{2}
\end{aligned} \tag{6.1.9}$$

in combination with the expression (6.1.4) of $\sin(\theta/2)$ from the proof of Lemma 6.1.1. It then also follows that

$$\begin{aligned}(\tilde{\theta}_{sol}(t,x;\{\lambda_1,\widetilde{C}_1\}))_t\big|_{x=0} &= -\operatorname{sign}(\operatorname{Im}(\tilde{b}_1))\Big(\alpha-\frac{1}{\alpha}\Big)\operatorname{sech}\Big(\frac{1}{2}(\alpha-\frac{1}{\alpha})t+\log|\tilde{b}_1|\Big)\\ &= -\Big(\alpha-\frac{1}{\alpha}\Big)\sin\frac{\tilde{\theta}_{sol}(t,0;\{\lambda_1,\widetilde{C}_1\})}{2}.\end{aligned}$$

This analysis can be repeated with $\lambda_1=-i\alpha$ where the solution then satisfies the defect condition with the other sign. In fact, the existence of these solutions, which are not fitting in the description of solutions by Proposition 5.1.1, provides no added value to the construction of N-soliton solutions due to the requirement of a zero seed solution in the propositions for the sG equation. Nevertheless, since in some special cases, the construction of N-soliton solutions combined with destructively interacting solutions could turn out useful, it is at least worth mentioning them for the sG equation.

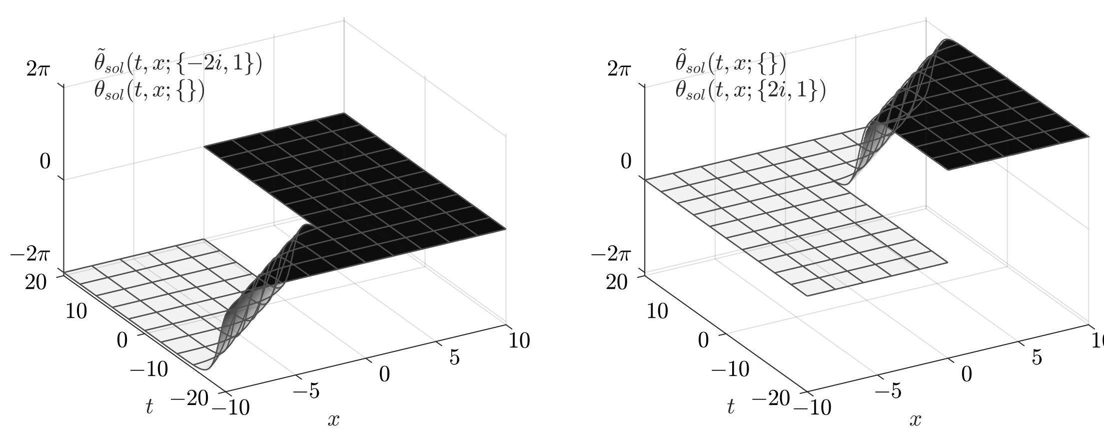

Fig. 6.6. Single soliton swallowed by (left) or emerging from (right) the defect ($\alpha=-2$).

6.1.2 sG equation on the half-line with sin-boundary condition

Subsequently, we consider the sG equation on the half-line (5.2.1) with the sin-boundary condition (4.3.7). As it is imminent from Proposition 5.2.1, single solitons and breathers come in pairs and boundary-bound breathers are subject to a specific condition represented by assumption (5.2.4). In that regard, results for the paired single soliton solution have been discussed in [43], where it can be seen that similarly to the sG equation with defect condition, the boundary can effectively combine a soliton with a soliton or an anti-soliton depending on the boundary parameter α. By the exact formulae for the single one-soliton (6.1.1) and breather solution (6.1.5), we are satisfied by giving the solution formula of a boundary-bound breather solution

$$\theta_{sol}(t,x;\{\lambda_1,-\lambda_1^*,C_1,-C_1^*\}) = 4\arctan\left(-\frac{\eta_1\cos(\xi_1 t)}{\xi_1\cosh(\eta_1 x+\log|b_1|)}\right)$$

with $|\lambda_1|=1$ and the usual relations of the parameters. Then, we can visualize the solution, see Figure 6.7 of a boundary-bound breather solution subject to the sin-boundary condition. Assumption (5.2.4) implies that $b_1=-\frac{v_1}{u_1}=-\sqrt{\frac{\alpha-2\eta_1}{\alpha+2\eta_1}}$, where the quotient inside the root needs to be positive.

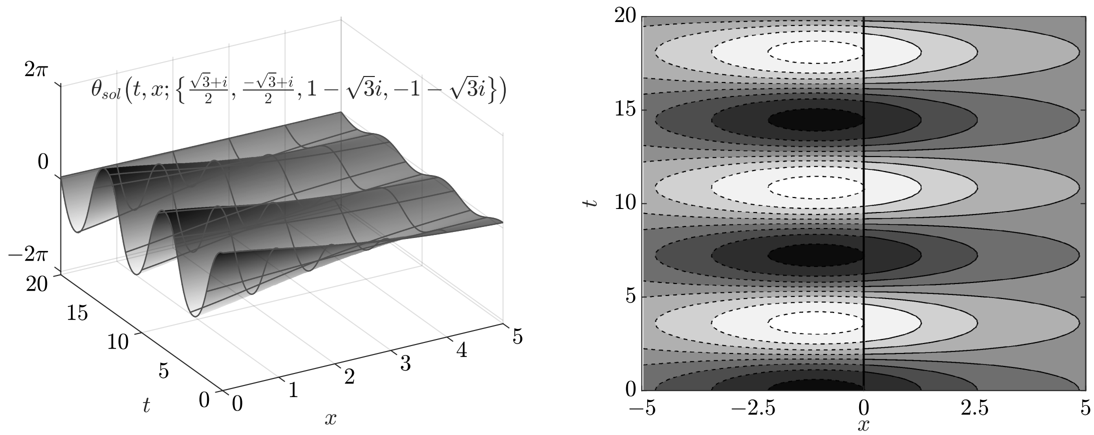

Fig. 6.7. Boundary-bound breather solution interacting with the sin-boundary ($\alpha = 2$).

As it can be seen in equation (6.1.9), if we take a spectral parameter $\lambda_1 = i\eta_1$ with the property $\eta_1 + 1/\eta_1 = \alpha$, then the constructed single soliton solution satisfies the sin-boundary condition. Again, this solution is not compatible with Proposition 5.2.1. It may be natural to think that, since for the defect condition the destructively interacting one-soliton solution is constructed with the zero as simple eigenvalue, which is associated to the defect parameter, the zeros, which are associated to the boundary parameter, can also be used to construct destructively interacting solutions. However, given the boundary parameter α, the boundary-bound breather

$$\theta_{breather}(t,x) := \theta_{sol}(t,x;\{\lambda_1, -\lambda_1^*, C_1, -C_1^*\}) = 4\arctan\left(-\frac{\eta_1 \cos\big(\xi_1 t - \arg b_1\big)}{\xi_1 \cosh\big(\eta_1 x + \log|b_1|\big)}\right),$$

where λ_1 is such that $|\lambda_1| = 1$, satisfies the sin-boundary condition

$$\begin{aligned}(\theta_{breather}(t,x))_x\big|_{x=0} &= -2\eta_1 \tanh(\log|b_1|)\Big(-\frac{2\xi_1\eta_1\cos\big(\xi_1 t - \arg b_1\big)\cosh(\log|b_1|)}{\eta_1^2\cos^2\big(\xi_1 t - \arg b_1\big) + \xi_1^2\cosh^2(\log|b_1|)}\Big) \\ &= \alpha \sin\frac{\theta_{sol}(t,0;\{\lambda_1, -\lambda_1^*, C_1, -C_1^*\})}{2}\end{aligned}$$

if $-2\eta_1 \tanh(\log|b_1|) = \alpha$ by the expression (6.1.6) of the breather from the proof of Lemma 6.1.2. And as it turns out, this equation is equivalent to $b_1 = -\frac{v_1}{u_1} = -\sqrt{\frac{\alpha-2\eta_1}{\alpha+2\eta_1}}$ stated generally for breather solutions.

6.2 Soliton solutions for the NLS equation

The Dressing method presented in Section 3.2 gives the algebraic means to derive, in the case of the NLS equation, N-soliton solutions simply by calculating the (12)-entry of the projector matrices $(P[j])_{12}$ for $j = 1, \dots, N$ recursively and then sum them up or by the direct calculation of the quotient of two $2N \times 2N$ matrices, which represents the (12)-entry of the sum of projector matrices, i.e. $(\Sigma_1)_{12}$, as presented in [42]. In that sense, it is in theory feasible to give an explicit expression for an arbitrary soliton solution, however, as indicated by the recursiveness or rather the dimension of the matrix, the expressions rapidly become unhandy. On the other hand, since we have algebraic expressions, it is reasonable to use programming in order to calculate these solutions and thus to visualize them.

As for the sG equation, we concentrate on the construction of soliton solutions originating from the zero seed solutions for the NLS equation. With that said, it is clear that the zero seed solutions are in the appropriate function spaces for all three propositions to be applicable, which translates into $u[0](\cdot,0), \tilde{u}[0](\cdot,0), u[0]_x(\cdot,0), \tilde{u}[0]_x(\cdot,0) \in H_t^{1,1}(\mathbb{R})$. Due to the seed solution being constant, this property holds particularly for an arbitrary fixed value x_f which corresponds to the point where the defect conditions are imposed. For the defect condition, the generality of the point of the defect gives some further insight. In fact, this circumstance implies that there exists a matrix corresponding to a frozen Bäcklund transformation for every $x_f \in \mathbb{R}$ connecting the Lax pairs corresponding to the zero seed solutions for $x < x_f$ and $x > x_f$. Consequently, applying Proposition 5.1.2 without the additional assumption on the imaginary part of the (11)-entry of the matrix corresponding to the frozen Bäcklund transformation for $\tilde{u}[N](t,x_f)$ and $u[N](t,x_f)$, $x_f \neq 0$, for different values of x_f, we obtain matrices corresponding to frozen Bäcklund transformations for every $x_f \in \mathbb{R}$ connecting the Lax pairs corresponding to the N-soliton solutions constructed through the method of dressing the defect. With that in mind, let us construct solutions in the model of the NLS equation subject to a defect condition.

6.2.1 NLS equation on two half-lines connected by a defect condition

In addition to the zero seed solutions $u_{sol}(t,x;\{\}) = 0$ and $\tilde{u}_{sol}(t,x;\{\}) = 0$, we take $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}\setminus\{0\}$ and define $\lambda_0 = -\frac{\alpha}{2} + i\frac{\beta}{2} \in \mathbb{C}\setminus\mathbb{R}$. Hence, the matrix $\mathcal{B}_0(\lambda)$, corresponding to the frozen Bäcklund transformation connecting the Lax pairs of the zero seed solutions and representing the defect conditions, can be written as

$$\mathcal{B}_0(\lambda) = \mathbb{1} + \frac{1}{2\lambda}\begin{pmatrix} \alpha - i\beta & 0 \\ 0 & \alpha + i\beta \end{pmatrix} = \mathbb{1} + \frac{1}{\lambda}\begin{pmatrix} -\lambda_0 & 0 \\ 0 & -\lambda_0^* \end{pmatrix},$$

where the $\pm$ sign is chosen so that $\pm|\beta| = -\beta$. Then, for the N-fold dressing matrix, we take solutions ψ_j to the Lax system (2.1.2) corresponding to $u[0]$ for distinct $\lambda = \lambda_j \in \mathbb{C}\setminus\big(\mathbb{R}\cup\{\lambda_0,\lambda_0^*\}\big)$, $j = 1,\ldots,N$, which are given by

$$\psi_j(t,x) = u_j\psi_-^{(1)}(t,x,\lambda_j) + v_j\psi_+^{(2)}(t,x,\lambda_j) = e^{(-i\lambda_j x - 2i\lambda_j^2 t)\sigma_3}\begin{pmatrix} u_j \\ v_j \end{pmatrix} \tag{6.2.1}$$

with constants u_j and v_j. Further, since the relation

$$\widetilde{\psi}_j\big|_{x=0} = \mathcal{B}_0(\lambda_j)\psi_j\big|_{x=0}$$

is assumed to hold for $j = 1,\ldots,N$ and solutions defined by $\widetilde{\psi}_j(t,x) = e^{(-i\lambda_j x - 2i\lambda_j^2 t)\sigma_3}(\tilde{u}_j,\tilde{v}_j)^\intercal$ of the Lax system corresponding to $\tilde{u}[0]$ for $\lambda = \lambda_j$ with constants $\tilde{u}_j$ and $\tilde{v}_j$, we obtain the following relation for the constants $\tilde{u}_j$, $\tilde{v}_j$, u_j and v_j:

$$\frac{\tilde{u}_j}{\tilde{v}_j} = \frac{2\lambda_j + \alpha - i\beta}{2\lambda_j + \alpha + i\beta}\frac{u_j}{v_j}, \quad j = 1,\ldots,N. \tag{6.2.2}$$

Note that effectively only the quotient of u_j and v_j, $j = 1,\ldots,N$, is relevant for the scattering data and that changing the sign of β is, via the relation $\pm|\beta| = -\beta$, the same as changing the $\pm$ sign in the defect condition. The exact expressions for C_j and $\widetilde{C}_j$ for $j = 1,\ldots,N$ can be derived straightforwardly and we state them in Remark 6.2.2. Therefore, we can apply Proposition 5.1.2 in order to construct N-soliton solutions $u_{sol}(t,x;\{\lambda_j,C_j\}_{j=1}^N)$ and $\tilde{u}_{sol}(t,x;\{\lambda_j,\widetilde{C}_j\}_{j=1}^N)$ which at

$x = x_f = 0$ are in the function space $H_t^{1,1}(\mathbb{R})$, see Proposition 4.4.3, and with respect to x, we have $u_{sol}(t, \cdot; \{\lambda_j, C_j\}_{j=1}^N) \in H^{1,1}(\mathbb{R}_+)$ and $\tilde{u}_{sol}(t, \cdot; \{\lambda_j, \widetilde{C}_j\}_{j=1}^N) \in H^{1,1}(\mathbb{R}_-)$ by [17, Prop. 4.7]. As in the proof of Proposition 5.1.2, we can use this fact to make sure that, after finding the matrices $B_N(t, x_f, \lambda)$, $x_f \in \mathbb{R}$, corresponding to the frozen Bäcklund transformation for the N-soliton solutions, the sign in front of the root in the (11)-entry is consistent with the sign of the initial matrix $\mathcal{B}_0(\lambda)$. Ultimately, we can use this extension to show that each soliton of the N-soliton solution interacts with the defect individually.

Taking the same dressing matrices, applying them to zero seed solutions $u_{sol}(t, x; \{\}) = 0$ and $\tilde{u}_{sol}(t, x; \{\}) = 0$ on the whole line $x \in \mathbb{R}$, we obtain two N-soliton solutions for the NLS equation for $x \in \mathbb{R}$, which we denote by $u_N(t, x)$ and $\tilde{u}_N(t, x)$ for now to distinguish them from the half-line solutions. By the aforementioned argumentation in Section 4.1, we have that at each point $x_f \in \mathbb{R}$, we can give by Proposition 5.1.2 a matrix $\mathcal{B}_N(t, x_f, \lambda)$ corresponding to a frozen Bäcklund transformation. Hence, it makes sense to assume that the solutions of the Lax systems for $u_N(t, x)$ and $\tilde{u}_N(t, x)$ are connected by a matrix of degree one in λ or in other words

$$\widetilde{\psi}(t, x, \lambda) = \mathcal{B}_N(t, x, \lambda)\psi(t, x, \lambda).$$

Hence, the matrix $\mathcal{B}_N(t, x, \lambda)$ solves the system (4.1.1). Assuming this matrix is linear in λ, it can only be of the form described in Proposition 3.1.1, which means there exist real parameters δ, $\gamma \in \mathbb{R}$ and a $\pm$ sign to be determined and

$$\mathcal{B}_N(t, x, \lambda) = \mathbb{1} + \frac{1}{2\lambda}\begin{pmatrix} \delta \pm i\sqrt{\gamma^2 - |\tilde{u}_N(t, x) - u_N(t, x)|^2} & -i(\tilde{u}_N(t, x) - u_N(t, x)) \\ -i(\tilde{u}_N(t, x) - u_N(t, x))^* & \delta \mp i\sqrt{\gamma^2 - |\tilde{u}_N(t, x) - u_N(t, x)|^2} \end{pmatrix}.$$

However, precisely at $x = 0$, we have by Proposition 5.1.2 that

$$\mathcal{B}_N(t, 0, \lambda) = \mathbb{1} + \frac{1}{2\lambda}\begin{pmatrix} \alpha - i\sqrt{\beta^2 - |\tilde{u}_N(t, 0) - u_N(t, 0)|^2} & -i(\tilde{u}_N(t, 0) - u_N(t, 0)) \\ -i(\tilde{u}_N(t, 0) - u_N(t, 0))^* & \alpha + i\sqrt{\beta^2 - |\tilde{u}_N(t, 0) - u_N(t, 0)|^2} \end{pmatrix},$$

where the full line solutions $u_N(t, 0)$ and $\tilde{u}_N(t, 0)$ can effectively be reduced to their half-line counterpart $u_{sol}(t, 0; \{\lambda_j, C_j\}_{j=1}^N)$ and $\tilde{u}_{sol}(t, 0; \{\lambda_j, \widetilde{C}_j\}_{j=1}^N)$, respectively. Therefore, we can deduce that $\delta = \alpha$, $\gamma^2 = \beta^2$ and the sign complies with $\mathcal{B}_0$ for $x = 0$ and $t \in \mathbb{R}$. Additionally, we fix the sign to be minus and, for $-|\beta| = -\beta$ to hold, thus $\beta > 0$. Moreover, starting with zero seed solutions, we can assume that the matrix $\mathcal{B}_N(t, x, \lambda)$ is constructed by the vector $\psi_0'(t, x) = D[N](t, x, \lambda_0)u_0\psi_-^{(1)}(t, x, \lambda_0)$, $u_0 \in \mathbb{C} \setminus \{0\}$, as suggested in the proof of Proposition 5.1.2. Hence, for the quotient $\Delta_0(t, x) = [\psi_0'(t, x)]_2/[\psi_0'(t, x)]_1$, we have

$$\lim_{|t|\to\infty} |\Delta_0(t, 0)| = \lim_{|t|\to\infty}\left|\frac{[\psi_0']_2(t, 0)}{[\psi_0']_1(t, 0)}\right| = \lim_{|t|\to\infty}\left|\frac{[D[N]]_{21}(t, 0, \lambda_0)}{[D[N]]_{11}(t, 0, \lambda_0)}\right| = 0$$

by Remark 4.4.5. Furthermore, by the construction of dressing matrices, we find for $\mathcal{B}_N(t, x, \lambda)$, up to a polynomial in λ, that

$$\begin{aligned} \mathcal{B}_N(t, x, \lambda) &= \frac{\lambda - \lambda_0^*}{\lambda}\left(\mathbb{1} + \frac{\lambda_0^* - \lambda_0}{\lambda - \lambda_0^*}\frac{1}{1 + |\Delta_0(t, x)|^2}\begin{pmatrix} 1 & \Delta_0^*(t, x) \\ \Delta_0(t, x) & |\Delta_0(t, x)|^2 \end{pmatrix}\right) \\ &= \mathbb{1} + \frac{1}{\lambda}\frac{1}{1 + |\Delta_0(t, x)|^2}\begin{pmatrix} -(\lambda_0 + \lambda_0^*|\Delta_0(t, x)|^2) & (\lambda_0^* - \lambda_0)\Delta_0^*(t, x) \\ (\lambda_0^* - \lambda_0)\Delta_0(t, x) & -(\lambda_0^* + \lambda_0|\Delta_0(t, x)|^2). \end{pmatrix} \end{aligned} \tag{6.2.3}$$

Then, to put the additional property of Proposition 5.1.2 into perspective, we identify

$$\mathrm{Im}\Big(2\lambda([\mathcal{B}_N(t, 0, \lambda)]_{11} - 1)\Big) = -\sqrt{\beta^2 - |\tilde{u}_N(t, 0) - u_N(t, 0)|^2} \leq 0.$$

On the other hand, by the construction (6.2.3) of $\mathcal{B}_N$, we find

$$\operatorname{Im}\Big(2\lambda([\mathcal{B}_N(t,x,\lambda)]_{11}-1)\Big) = \operatorname{Im}\Big(\alpha + i\beta\frac{|\Delta_0(t,x)|^2-1}{|\Delta_0(t,x)|^2+1}\Big) = \beta\frac{|\Delta_0(t,x)|^2-1}{|\Delta_0(t,x)|^2+1}$$

so that the assumed property is equivalent to

$$\beta\frac{|\Delta_0(t,0)|^2-1}{|\Delta_0(t,0)|^2+1} \leq 0 \tag{6.2.4}$$

and indeed, since $\lim_{t\to\infty}|\Delta_0(t,0)| = 0$, we find $-\beta < 0$, which is true by the choice of the sign in the matrix representing the frozen Bäcklund transformation. By the same reasoning as for t to infinity, we can deduce that the quotient $|\Delta_0(t,x)|$ has the following limits

$$\lim_{|x|\to\infty}|\Delta_0(t,x)| = \lim_{|x|\to\infty}\left|\frac{[\psi_0'(t,x)]_2}{[\psi_0'(t,x)]_1}\right| = \lim_{|x|\to\infty}\left|\frac{[D[N](t,x,\lambda_0)]_{21}}{[D[N](t,x,\lambda_0)]_{11}}\right| = 0.$$

Consequently, there exists an $R \in \mathbb{R}$ for which

$$\mathcal{B}_N(t,x,\lambda) = \mathbb{1} + \frac{1}{2\lambda}\begin{pmatrix} \alpha - i\sqrt{\beta^2 - |\tilde{u}_N(t,x)-u_N(t,x)|^2} & -i(\tilde{u}_N(t,x)-u_N(t,x)) \\ -i(\tilde{u}_N(t,x)-u_N(t,x))^* & \alpha + i\sqrt{\beta^2 - |\tilde{u}_N(t,x)-u_N(t,x)|^2} \end{pmatrix},$$

where $|x| > R$. This means that the matrix $\mathcal{B}_N(t,x,\lambda)$, constructed in the proof in order to show that the defect condition is preserved, has in fact a continuation $\mathcal{B}_N(t,x,\lambda)$ for $x \in \mathbb{R}$ and even though the square root in the diagonal entries may become zero at a point $(t,x) \in \mathbb{R}\times(\mathbb{R}\setminus\{0\})$ and even change sign, ultimately, the sign changes back for big enough x and consequently, we have

$$\mathcal{B}_\infty(\lambda) = \lim_{|x|\to\infty}\mathcal{B}_N(t,x,\lambda) = \lim_{|x|\to\infty}\mathcal{B}_0(\lambda) = \mathbb{1} + \frac{1}{\lambda}\begin{pmatrix} -\lambda_0 & 0 \\ 0 & -\lambda_0^* \end{pmatrix}. \tag{6.2.5}$$

Then again, this can be generalized for arbitrary seed solutions satisfying the assumptions of Proposition 5.1.2. Due to the dressing matrix always admitting a diagonal form for the limit of $|x|$ or t going to infinity, the limit behavior of $|\Delta_0(t,x)|$ strongly depends on the limit behavior of the kernel vector ψ_0 of the matrix $\mathcal{B}_0$ representing the frozen Bäcklund transformation associated to the parameter λ_0. Knowing that, we see that the Jost functions have relations induced by the Bäcklund transformation and the same normalization factor $\mathcal{B}_\infty^{-1}(\lambda)$,

$$\widetilde{\psi}_\pm(t,x,\lambda) = \mathcal{B}_N(t,x,\lambda)\psi_\pm(t,x,\lambda)\mathcal{B}_\infty^{-1}(\lambda). \tag{6.2.6}$$

In turn, this relation implies the following relation for the corresponding scattering matrices (2.1.6):

$$\widetilde{\mathcal{A}}(\lambda) = \mathcal{B}_\infty(\lambda)\mathcal{A}(\lambda)\mathcal{B}_\infty^{-1}(\lambda), \quad \lambda \in \mathbb{R}. \tag{6.2.7}$$

This observation is, in fact, similar to the 'space-evolution' interpretation given in Subsection 4.4.1. However, this time the relation is with respect to the usual functions emerging in the scattering process. With these insights, we can give a similar formula for the relations of the soliton parameters in the solution space as (6.1.8) in the case of the sG equation. Yet, due to the well-known correspondence of the scattering data to soliton parameters in the solution space in the case of the NLS equation, we can state more generally the following:

Corollary 6.2.1. *Let $u(t,x)$ and $\tilde{u}(t,x)$ be two N-soliton solutions of the NLS equation on $\mathbb{R}$ constructed by the corresponding vectors used in and satisfying the assumptions of Proposition 5.1.2 and let their restrictions to the positive and negative half-line, respectively, be subject to the defect conditions (4.1.2) at $x=0$. Then for $\lambda_j = \xi_j + i\eta_j \in \mathbb{C}_+ \setminus \big(i\mathbb{R} \cup \{-\frac{\alpha}{2} + i\frac{\beta}{2}\}\big)$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$, it follows that solitons are transmitted through the defect independently of one another, i.e. for all $j = 1,\dots,N$ the following holds*

$$\tilde{x}_j - x_j = \frac{1}{2\eta_j}\log\Big(\Big|\frac{2\lambda_j + \alpha + i\beta}{2\lambda_j + \alpha - i\beta}\Big|\Big),$$
$$\tilde{\phi}_j - \phi_j = \arg\Big(\frac{2\lambda_j + \alpha + i\beta}{2\lambda_j + \alpha - i\beta}\Big).$$

Proof. By the definition $\lambda_0 = -\frac{\alpha}{2} + i\frac{\beta}{2}$ and the analysis above, we know that in the assumed scenario

$$\mathcal{B}_\infty(\lambda) = \lim_{|x|\to\infty} \mathcal{B}_N(t,x,\lambda) = \mathbb{1} - \frac{1}{\lambda}\operatorname{diag}(\lambda_0, \lambda_0^*).$$

The relation of the Jost functions (6.2.6) gives

$$\widetilde{\psi}_-^{(1)} = \mathcal{B}_N(t,x,\lambda)\psi_-^{(1)}\Big(1 - \frac{\lambda_0}{\lambda}\Big)^{-1}, \qquad \widetilde{\psi}_+^{(2)} = \mathcal{B}_N(t,x,\lambda)\psi_+^{(2)}\Big(1 - \frac{\lambda_0^*}{\lambda}\Big)^{-1}. \tag{6.2.8}$$

Using (6.2.8), we can deduce for the respective relation (2.1.10) regarding $u(t,x)$ and $\tilde{u}(t,x)$ that, $j = 1,\dots,N$,

$$\widetilde{\psi}_-^{(1)}(t,x,\lambda_j) = \frac{\lambda_j b_j}{\lambda_j - \lambda_0}\mathcal{B}_N(t,x,\lambda_j)\psi_+^{(2)}(t,x,\lambda_j) = \frac{\lambda_j - \lambda_0^*}{\lambda_j - \lambda_0}\frac{b_j}{\tilde{b}_j}\widetilde{\psi}_-^{(1)}(t,x,\lambda_j).$$

Therefore, the constants $\tilde{b}_j$ and b_j can be related by

$$\frac{\tilde{b}_j}{b_j} = \frac{\lambda_j - \lambda_0^*}{\lambda_j - \lambda_0}, \quad j = 1,\dots,N. \tag{6.2.9}$$

Moreover, the relation (6.2.7) for the scattering matrices implies

$$\begin{aligned} \tilde{a}_{11}(\lambda) &= a_{11}(\lambda), \\ \tilde{a}_{21}(\lambda) &= \frac{\lambda - \lambda_0^*}{\lambda - \lambda_0}a_{21}(\lambda). \end{aligned} \tag{6.2.10}$$

These two relations (6.2.9) and (6.2.10) can be combined to relate the norming constants $\widetilde{C}_j$ and C_j, $j = 1,\dots,N$, of the respective scattering data in the following way

$$\frac{\widetilde{C}_j}{C_j} = \frac{\tilde{b}_j}{b_j}\frac{\frac{\mathrm{d}a_{11}}{\mathrm{d}\lambda}}{\frac{\mathrm{d}\tilde{a}_{11}}{\mathrm{d}\lambda}}\Big|_{\lambda=\lambda_j} = \frac{\lambda_j - \lambda_0^*}{\lambda_j - \lambda_0},$$

from where we can see the influence on the N-soliton solution. Therefore, writing the norming constants as

$$C_j = 2\eta_j e^{2\eta_j x_j + i\phi_j}, \quad \widetilde{C}_j = 2\eta_j e^{2\eta_j \tilde{x}_j + i\tilde{\phi}_j}$$

for $j = 1,\dots,N$ as motivated for the one-soliton solution in Section 2.1, we obtain for the spatial shift $\tilde{x}_j - x_j$ and the phase shift $\tilde{\phi}_j - \phi_j$ the following

$$\tilde{x}_j - x_j = \frac{1}{2\eta_j}\log\Big(\Big|\frac{\lambda_j - \lambda_0^*}{\lambda_j - \lambda_0}\Big|\Big), \qquad \tilde{\phi}_j - \phi_j = \arg\Big(\frac{\lambda_j - \lambda_0^*}{\lambda_j - \lambda_0}\Big),$$

which indicates that the solitons experience the defect independently of one another. $\square$

Remark 6.2.2. Another way to prove the assertion of Corollary 6.2.1 is by the ideas given in Theorem 3.3.1. Therefore, after iterative application of the Dressing method to the zero seed solutions, we obtain

$$\begin{aligned}
&a_{11}^{(N)}(\lambda) = \prod_{j=1}^{N} \frac{\lambda - \lambda_j}{\lambda - \lambda_j^*}, && \tilde{a}_{11}^{(N)}(\lambda) = \prod_{j=1}^{N} \frac{\lambda - \lambda_j}{\lambda - \lambda_j^*}, && \lambda \in \mathbb{C}_+ \cup \mathbb{R},\\
&a_{21}^{(N)}(\lambda) = 0, && \tilde{a}_{21}^{(N)}(\lambda) = 0, && \lambda \in \mathbb{R},\\
&C_j^{(N)} = -\frac{v_j}{u_j} 2i\,\mathrm{Im}(\lambda_j) \prod_{k=1}^{N}{}' \frac{\lambda_j - \lambda_k^*}{\lambda_j - \lambda_k}, && \widetilde{C}_j^{(N)} = -\frac{\tilde{v}_j}{\tilde{u}_j} 2i\,\mathrm{Im}(\lambda_j) \prod_{k=1}^{N}{}' \frac{\lambda_j - \lambda_k^*}{\lambda_j - \lambda_k}, && j = 1, \dots, N.
\end{aligned}$$

The relevant equations appearing in the proof of Corollary 6.2.1 hold and particularly, by equation (6.2.2) we have

$$\frac{\widetilde{C}_j^{(N)}}{C_j^{(N)}} = \frac{\tilde{v}_j}{\tilde{u}_j} \frac{u_j}{v_j} = \frac{2\lambda_j + \alpha + i\beta}{2\lambda_j + \alpha - i\beta}, \quad j = 1, \dots, N.$$

From the first approach, we can deduce two helpful properties. First off, we can give an equivalent expression for the assumption on $\mathrm{Im}\Big(\lim_{\lambda\to 0}[2\lambda(\mathcal{B}_N(t,0,\lambda) - \mathbb{1})]_{11}\Big)$ from Proposition 5.1.2 on the spectral side, see (6.2.4), thereby underlining the necessity of this assumption. With the choice $\beta > 0$ and the minus sign in the defect condition, the assumption is equivalent to

$$|\Delta_0(t,0)| \leq 1, \quad t \in \mathbb{R}. \tag{6.2.11}$$

And by a similar reasoning $|\Delta_0'(t,0)| \geq 1$ for $t \in \mathbb{R}$ and $\Delta_0'(t,x) = \frac{[D[N](t,x,\lambda_0^*)]_{22}}{[D[N](t,x,\lambda_0^*)]_{12}}$ which is equivalent to (6.2.11). Now, assuming that there exists a $t_0 \in \mathbb{R}$ such that $|\Delta_0(t_0,0)| = 1$, for the one-fold dressing matrix $D[1](t_0,0,\lambda_0)$, see (3.2.2), we find that this is equivalent to

$$\begin{aligned}
|[D[1](t_0,0,\lambda_0)]_{11}|^2 - |[D[1](t_0,0,\lambda_0)]_{21}|^2 = 0\\
|(\lambda_0 - \lambda_1) + (\lambda_0 - \lambda_1^*)|\Delta(t_0,0)||^2 - |\lambda_1^* - \lambda_1|^2 |\Delta(t_0,0)|^2 = 0,
\end{aligned}$$

where both entries of the one-fold dressing matrix are multiplied by $(\lambda_0 - \lambda_1^*)(1 + |\Delta(t_0,0)|^2)$. Then, given the usual identification $\lambda_0 = -\frac{\alpha}{2} + i\frac{\beta}{2}$ and $\lambda_1 = \xi_1 + i\eta_1$, we have

$$\begin{aligned}
\Big(\big(\frac{\alpha}{2} + \xi_1\big)^2 + \big(\frac{\beta}{2} + \eta_1\big)^2\Big)|\Delta(t_0,0)|^4 + \Big(2\big(\frac{\alpha}{2} + \xi_1\big)^2 + \frac{\beta^2}{2} - 6\eta_1^2\Big)|\Delta(t_0,0)|^2&\\
+ \Big(\big(\frac{\alpha}{2} + \xi_1\big)^2 + \big(\frac{\beta}{2} - \eta_1\big)^2\Big) &= 0.
\end{aligned}$$

Thus, we want to show that there exists a combination of solutions of the NLS equation on the respective half-line for which the defect condition is satisfied with either plus or minus sign depending on the time t, which underlines the necessity of assumption (6.2.11). Writing the equation for $|\Delta(t_0,0)|$ as a polynomial $p(y) = y^4 + c_1 y^2 + c_2$ in $y \geq 0$, we have

$$p(y) = y^4 + \frac{2\big(\frac{\alpha}{2} + \xi_1\big)^2 + \frac{\beta^2}{2} - 6\eta_1^2}{\big(\frac{\alpha}{2} + \xi_1\big)^2 + \big(\frac{\beta}{2} + \eta_1\big)^2} y^2 + \frac{\big(\frac{\alpha}{2} + \xi_1\big)^2 + \big(\frac{\beta}{2} - \eta_1\big)^2}{\big(\frac{\alpha}{2} + \xi_1\big)^2 + \big(\frac{\beta}{2} + \eta_1\big)^2} = 0.$$

Since $c_2 > 0$, the effective behavior of the polynomial only depends on c_1, see Figure 6.8. If η_1 is big enough in order for $p(y)$ to have real zeros, then there indeed exists at least one $t_0 \in \mathbb{R}$ such

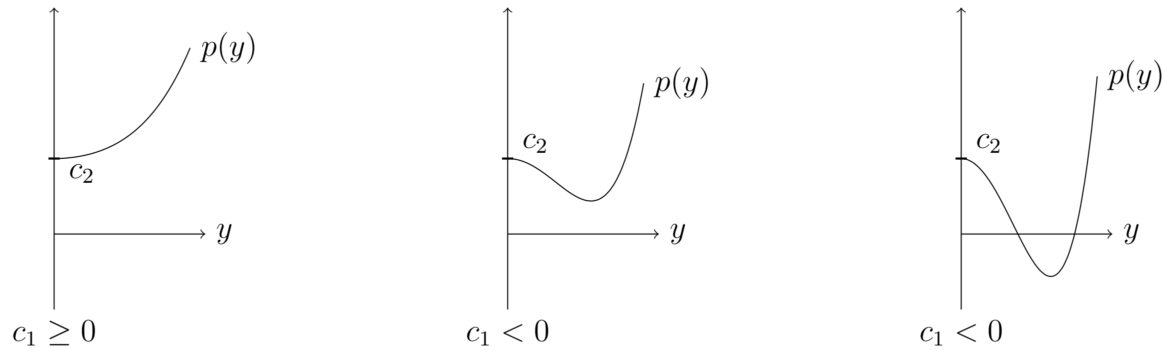

Fig. 6.8. Behavior of the polynomial $p(y)$ for different values of c_1.

that $|\Delta_0(t_0, 0)| = 1$.
Let us elaborate on this occurrence with the following selection of parameters: $\alpha = 0$, $\beta = 2$, $\lambda_1 = \xi_1 + i\eta_1 = 1 + 2i$, $C_1 = -4i$, which further implies $b_1 = -1$, $\widetilde{C}_1 = 4 - 8i$ and $\tilde{b}_1 = -2 - i$ and particularly $c_1 = -2$ as well as $c_2 = 0.2$. Correspondingly, we obtain by Proposition 5.1.2 the following solution formulae for u_{sol} and $\tilde{u}_{sol}$:

$$\begin{aligned} u_{sol}(t, x; \{1 + 2i, -4i\}) &= 4e^{-i(2x-12t)} \operatorname{sech}(4(x + 4t)), \\ \tilde{u}_{sol}(t, x; \{1 + 2i, 4 - 8i\}) &= 4e^{-i(2x-12t+\pi/2-\arctan(2))} \operatorname{sech}(4(x + 4t) - \log(5)/2). \end{aligned}$$

By the constants of the polynomial $p(y)$, we can derive that the third plot in Figure 6.8 represents the polynomial with the positive zeros $\sqrt{1 \pm \sqrt{0.8}}$. If we want to connect these zeros to the times t they occur at, we need to look at the time evolution of $|\Delta_0(t, 0)|$, which is given by $|\Delta_0(t, 0)| = |b_1|e^{-16t}$. Therefore, we find that between the critical values $t_0^+ = -\log(\sqrt{1 + \sqrt{0.8}})/16$ and $t_0^- = -\log(\sqrt{1 - \sqrt{0.8}})/16$, the sign of the defect condition is not as desired. Now, for the first equation of the defect condition (4.1.2), we obtain that

$$(\tilde{u}_{sol} - u_{sol})_x = -2i(\tilde{u}_{sol} - u_{sol}) - 4\tanh(4(x + 4t) - \log(5)/2)\tilde{u}_{sol} + 4\tanh(4(x + 4t))u_{sol}$$

is supposed to be equal to

$$\pm\sqrt{4 - |\tilde{u}_{sol} - u_{sol}|^2}(\tilde{u}_{sol} + u_{sol})$$

for $x = 0$ and $t \in \mathbb{R}$. If we approximate the critical interval as $[t_0^+, t_0^-] \approx [-0.02, 0.07]$, we have for $t \notin [t_0^+, t_0^-]$, i.e. $t = -0.25$ and $t = 1$, that the defect condition in the first equality hold with a minus sign

$$\begin{aligned} (\tilde{u}_{sol} - u_{sol})_x\big|_{(t,x)=(-0.25,0)} &\approx 0.4139 - 0.0001i \approx -\sqrt{4 - |\tilde{u}_{sol} - u_{sol}|^2}(\tilde{u}_{sol} + u_{sol})\big|_{(t,x)=(-0.25,0)}, \\ (\tilde{u}_{sol} - u_{sol})_x\big|_{(t,x)=(1,0)} &\approx (-3.5921 + 4.4178i) \cdot 10^{-6} \approx -\sqrt{4 - |\tilde{u}_{sol} - u_{sol}|^2}(\tilde{u}_{sol} + u_{sol})\big|_{(t,x)=(1,0)}. \end{aligned}$$

However, if we check inside the interval $t \in (t_0^+, t_0^-)$, we have that the defect condition in the first equality is satisfied with a plus sign

$$(\tilde{u}_{sol} - u_{sol})_x\big|_{(t,x)=(0,0)} \approx 4.4444 - 0.8889i \approx +\sqrt{4 - |\tilde{u}_{sol} - u_{sol}|^2}(\tilde{u}_{sol} + u_{sol})\big|_{(t,x)=(0,0)}.$$

Swapping the roles of $D[N]$ and $\widetilde{D}[N]$ in Proposition 5.1.2, while retaining the norming constants $\widetilde{C}_1 = 4 - 8i$ and adjusting the norming constant C_1 to $C_1 = (4 - 8i)\frac{1+2i+i}{1+2i-i} = 16 - 12i$, leads to a different critical interval, now depending on the values of t for which $\sqrt{1 + \sqrt{0.8}} < |\widetilde{\Delta}(t, 0)| <$

$\sqrt{1-\sqrt{0.8}}$, where $\widetilde{\Delta}(t,x)$ is the quotient of the solution (3.2.1) used in the one-fold dressing matrix $\widetilde{D}[1]$ as in (3.2.2). In this particular example, the associated simplified interval is then given by $[t_0^+, t_0^-] = [\log(5/(1+\sqrt{0.8}))/32, \log(5/(1-\sqrt{0.8}))/32]$. In that regard, it may seem that the determination of solutions for the NLS equation with defect conditions is more complicated than just applying the Dressing method to construct them. The actual aim of the preceding analysis is however just to give an easy example for solutions of the NLS equation on the respective half-lines which are not subject to the defect condition with consistent sign for $t \in \mathbb{R}$. In general, the inequality of (6.2.11) can be verified easily with the expressions for the N-fold dressing matrix $D[N]$ and given the kernel vector ψ_0 of the matrix $\mathcal{B}_0$ corresponding to the frozen Bäcklund transformation for the seed solutions $u[0]$ and $\tilde{u}[0]$. Now, since we have complete knowledge of the kernel vector ψ_0 in the case of zero seed solutions, the verification is a matter of adding a few lines in the code which lets us calculate the N-soliton solution.

The advantage of the first approach in determining the difference in the initial position and phase is that it is easy to imagine making a comparison between a soliton under trivial transmission, a soliton subject to the defect condition and a soliton-soliton interaction with the defect considered to be half a soliton. For that comparison, let us first discuss what is known [2, 38] about the asymptotic states as $t \to \pm\infty$ of an N-soliton solution for the NLS equation on the full line. If $\mathrm{Re}(\lambda_j) = \xi_j \neq \xi_k = \mathrm{Re}(\lambda_k)$ for $j \neq k$, then for $t \to \pm\infty$ the potential $u \in \mathcal{G}_N$ breaks up into individual solitons of the form of a one-soliton (2.1.21) so that

$$u_{sol}^{\pm}(t,x;\{\lambda_j, C_j\}_{j=1}^N) \sim \sum_{j=1}^{N} u_{sol}(t,x;\{\lambda_j, C_j^{\pm}\}), \quad \text{as } t \to \pm\infty$$

with

$$u_{sol}(t,x;\{\lambda_j, C_j^{\pm}\}) = 2\eta_j e^{-i(2\xi_j x + 4(\xi_j^2-\eta_j^2)t + (\phi_j^{\pm}+\pi/2))} \operatorname{sech}(2\eta_j(x+4\xi_j t - x_j^{\pm})),$$

where $\lambda_j = \xi_j + i\eta_j$, $j = 1,\dots,N$. Let us assume that the soliton parameters related to the velocity $-4\xi_j$ of the soliton are arranged in such a way that $\xi_1 < \xi_2 < \cdots < \xi_N$. Then, for a large enough negative time or rather $t \to -\infty$, the solitons are distributed along the x-axis in order of decreasing velocities, thus, $\xi_1, \xi_2, \dots, \xi_N$, while this order is reversed as $t \to \infty$. Based on this circumstance, we want to discuss the consequence of the interaction between solitons by tracing the influence this development has on the respective eigenfunctions. Further, let $x_j(t)$ denote the soliton coordinates at the instant of time t, where $|t|$ is assumed to be large enough so that it makes sense to talk about individual solitons. By the above argumentation if $t \to -\infty$, then $x_1 \ll x_2 \ll \cdots \ll x_N$. The function $\psi_-^{(1)}(t,x,\lambda_j)$ admits the form $\psi_-^{(1)}(t,x,\lambda_j) \sim e_1 e^{-i(\lambda_j x + 2\lambda_j^2 t)}$ in the region $x \ll x_1$. If there is any interaction between the soliton corresponding to λ_1 and a soliton corresponding to λ_j, according to equation (2.1.6) and Remark 3.3.2 the form of $\psi_-^{(1)}(t,x,\lambda_1)$ would change by the factor of the coefficient $a_1(\lambda_j)$ for $x \gg x_1$, where $a_1(\lambda) = \frac{\lambda-\lambda_1}{\lambda-\lambda_1^*}$ is the scattering coefficient $a_{11}(\lambda)$ relative to the first soliton. After repeated application of this argument, we obtain

$$\psi_-^{(1)}(t,x,\lambda_j) \sim \prod_{k=1}^{j-1} a_k(\lambda_j) \begin{pmatrix} e^{-i(\lambda_j x + 2\lambda_j^2 t)} \\ 0 \end{pmatrix}, \qquad x_{j-1} \ll x \ll x_j.$$

Then, through the j-th soliton interaction corresponding to a bound state, we find

$$\psi_-^{(1)}(t,x,\lambda_j) \sim 2\eta_j e^{2\eta_j x_j^- + i\phi_j^-} \prod_{k=1}^{j-1} a_k(\lambda_j) \begin{pmatrix} 0 \\ e^{i(\lambda_j x + 2\lambda_j^2 t)} \end{pmatrix}, \qquad x_j \ll x \ll x_{j+1}.$$

Coming from the other side $x \gg x_N$ and repeating the argument, we have

$$\psi_+^{(2)}(t,x,\lambda_j) \sim \prod_{k=j+1}^{N} a_k(\lambda_j) \begin{pmatrix} 0 \\ e^{i(\lambda_j x + 2\lambda_j^2 t)} \end{pmatrix}, \qquad x_j \ll x \ll x_{j+1}.$$

Therefore, relation (2.1.10) implies, using the expressions of the norming constants, that

$$C_j \sim 2\eta_j e^{2\eta_j x_j^- + i\phi_j^-} \frac{1}{\frac{\mathrm{d}a_{11}}{\mathrm{d}\lambda}\big|_{\lambda=\lambda_j}} \prod_{k=1}^{j-1} a_k(\lambda_j) \prod_{k=j+1}^{N} a_k(\lambda_j)^{-1}, \quad t \to -\infty,$$

where x_j^- and ϕ_j^- describe the asymptotics of the functions x_j, ϕ_j as $t \to -\infty$.

Analogously, for $t \to +\infty$, we find

$$C_j \sim 2\eta_j e^{2\eta_j x_j^+ + i\phi_j^+} \frac{1}{\frac{\mathrm{d}a_{11}}{\mathrm{d}\lambda}\big|_{\lambda=\lambda_j}} \prod_{k=1}^{j-1} a_k(\lambda_j)^{-1} \prod_{k=j+1}^{N} a_k(\lambda_j), \quad t \to +\infty.$$

Combining these two results, we can deduce

$$e^{2\eta_j(x_j^+ - x_j^-) + i(\phi_j^+ - \phi_j^-)} = \prod_{k=1}^{j-1} a_k(\lambda_j)^2 \prod_{k=j+1}^{N} a_k(\lambda_j)^{-2} = \prod_{k=1}^{j-1} \left(\frac{\lambda_j - \lambda_k}{\lambda_j - \lambda_k^*} \right)^2 \prod_{k=j+1}^{N} \left(\frac{\lambda_j - \lambda_k^*}{\lambda_j - \lambda_k} \right)^{-2}.$$

In general, this means that under the condition stated in the beginning, $\mathrm{Re}(\lambda_j) = \xi_j \neq \xi_k = \mathrm{Re}(\lambda_k)$ for $j \neq k$, we have that an N-soliton is actually a combination of N single solitons and that the N-soliton solution characterizes the interaction of these individual solitons. Consequently, the two-soliton interaction with $\mathrm{Re}(\lambda_1) < \mathrm{Re}(\lambda_2)$ corresponds to the following spatial and phase shift:

$$\begin{aligned} x_1^+ - x_1^- &= \frac{1}{\eta_j} \log\left|\frac{\lambda_1 - \lambda_2^*}{\lambda_1 - \lambda_2}\right|, \quad x_2^+ - x_2^- = \frac{1}{\eta_j} \log\left|\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2^*}\right| \\ \phi_1^+ - \phi_1^- &= 2\arg\left(\frac{\lambda_1 - \lambda_2^*}{\lambda_1 - \lambda_2}\right), \quad \phi_2^+ - \phi_2^- = 2\arg\left(\frac{\lambda_1 - \lambda_2}{\lambda_1^* - \lambda_2}\right). \end{aligned} \tag{6.2.12}$$

By these explicit expressions, we can capture three cases of interaction. If we furthermore assume $\mathrm{Im}(\lambda_j) > 0$, $j = 1, 2$, so that

$$\left|\frac{\lambda_1 - \lambda_2^*}{\lambda_1 - \lambda_2}\right| > 1,$$

then we see that the spatial shift for the first and second soliton is $x_1^+ - x_1^- > 0$ corresponding to a shift into the positive x-direction and $x_2^+ - x_2^- < 0$ corresponding to a shift into the negative x-direction, respectively. In addition, we can differentiate the three cases by the velocities $\nu_j = -4\,\mathrm{Re}(\lambda_j)$, $j = 1, 2$, and find for

(i) $\nu_2 < \nu_1 < 0$ that the second soliton is on the right of the first soliton as $t \to -\infty$ and that they both travel from the right to the left so that as $t \to +\infty$ the places are switched. Consequently, the second soliton overtakes the first one due to it being faster $|\nu_2| > |\nu_1|$. The faster soliton is shifted forward and the slower soliton backward with respect to the direction of travel;

(ii) $\nu_1 < 0 < \nu_2$ that the first soliton is on the right of the second soliton as $t \to -\infty$ and travels from the right to the left and the the second soliton travels from the left to the right so that again as $t \to +\infty$ the places are switched. Consequently, due to them travelling towards each other, they interact. Both soltions are shifted forward with respect to their direction of travel;

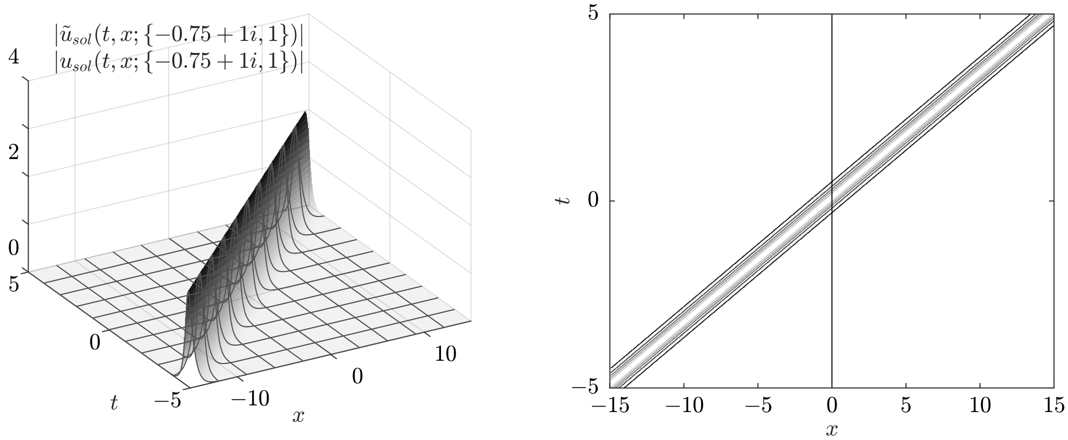

Fig. 6.9. One-soliton purely transmitted and its contour.

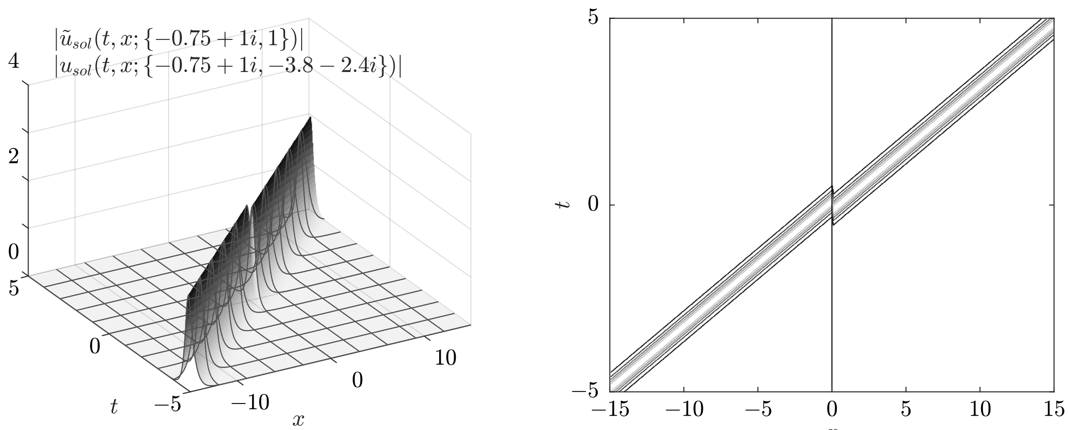

Fig. 6.10. One-soliton interacting with the defect ($\alpha = 1$, $\beta = 3$) and its contour.

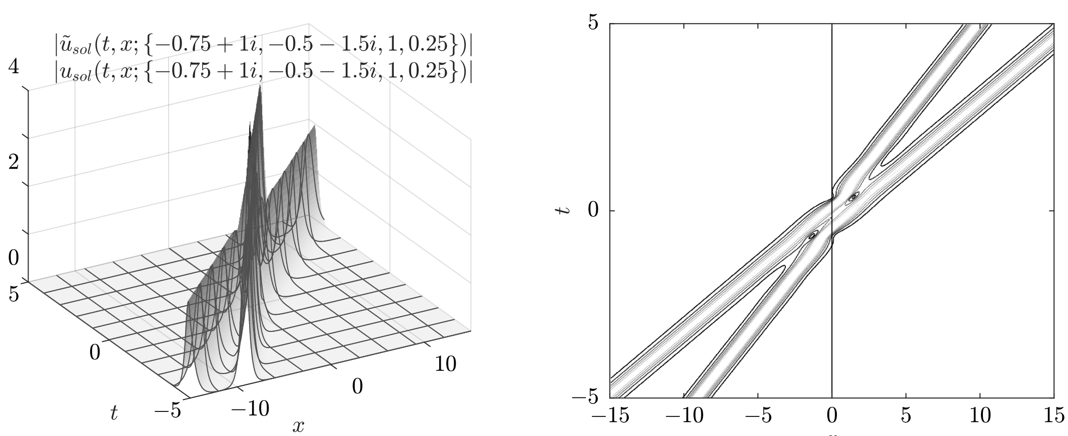

Fig. 6.11. Two-soliton purely transmitted with the defect interpreted as soliton and its contour.

(iii) $0 < \nu_1 < \nu_2$ that the second soliton is on the left of the first soliton as $t \to -\infty$ and that they both travel from the left to the right so that as $t \to +\infty$ the places are switched. Consequently, the second soliton overtakes the first one due to it being faster $|\nu_2| > |\nu_1|$ as in (i). Analogous to the first case, the faster soliton is shifted forward and the slower soliton backward with respect to the direction of travel.

If we compare these expressions with the spatial and phase shift each soliton experiences independently after interacting with the defect, it becomes evident that the defect can in theory be seen as 'half' a soliton, see [8]. Since, as we have seen before, the defect parameters are used as half the real and half the imaginary part of a spectral parameter from which we construct a frozen one-fold dressing matrix in the proof of Proposition 5.1.2. On the other hand, by comparison of the spatial and phase shift (6.2.12) with the result of Corollary 6.2.1, there is again a factor of one half. To summarize this idea, we refer to Figures 6.9, 6.10 and 6.11, where we compare a one-soliton solution being purely transmitted, satisfying the defect condition and interacting with the defect condition interpreted as half a soliton. In the transition from the second to the third plot, one can observe that the phase shift one of the solitons experiences is doubled when the defect is interpreted as a soliton itself.

The expression $\frac{2\lambda_j+\alpha+i\beta}{2\lambda_j+\alpha-i\beta}$ lets us, similarly to the argumentation for the sG equation, state some facts about the behavior of the spatial and phase shift of the N-soliton after interacting with the defect. For the general idea of the spatial shift and the influence of the relation of $\mathrm{Re}(\lambda_j)$ to the defect parameter α, we refer to the explanation regarding the two-soliton interaction, we just have given. Additionally, letting β go to zero, the quotient goes to 1, which indicates that the discontinuity at $x = 0$ disappears, suggesting that α by itself can not maintain the defect condition. On the other hand, letting $|\beta|$ go to infinity, the quotient goes to -1, which means no considerable spatial shift as $\tilde{x}_j - x_j$ goes to zero and essentially a shape inversion as $\tilde{\phi}_j - \phi_j$ goes to π for all $j = 1, \dots, N$. The effect of the spatial shift growing and decreasing can be observed in Figure 6.12, where for $\alpha = 0$ the maximal space shift is given at $\beta = 2.5$. However, if we take $\beta \in \mathbb{R} \setminus \{0\}$ and let $|\alpha|$ go to infinity, the effect of the discontinuity also disappears, i.e. $\tilde{x}_j - x_j$ and $\tilde{\phi}_j - \phi_j$ both go to zero for all $j = 1, \dots, N$. Hence, the second defect parameter may be understood as a means to smooth out the discontinuity in the presence of the defect condition ($\beta \neq 0$).

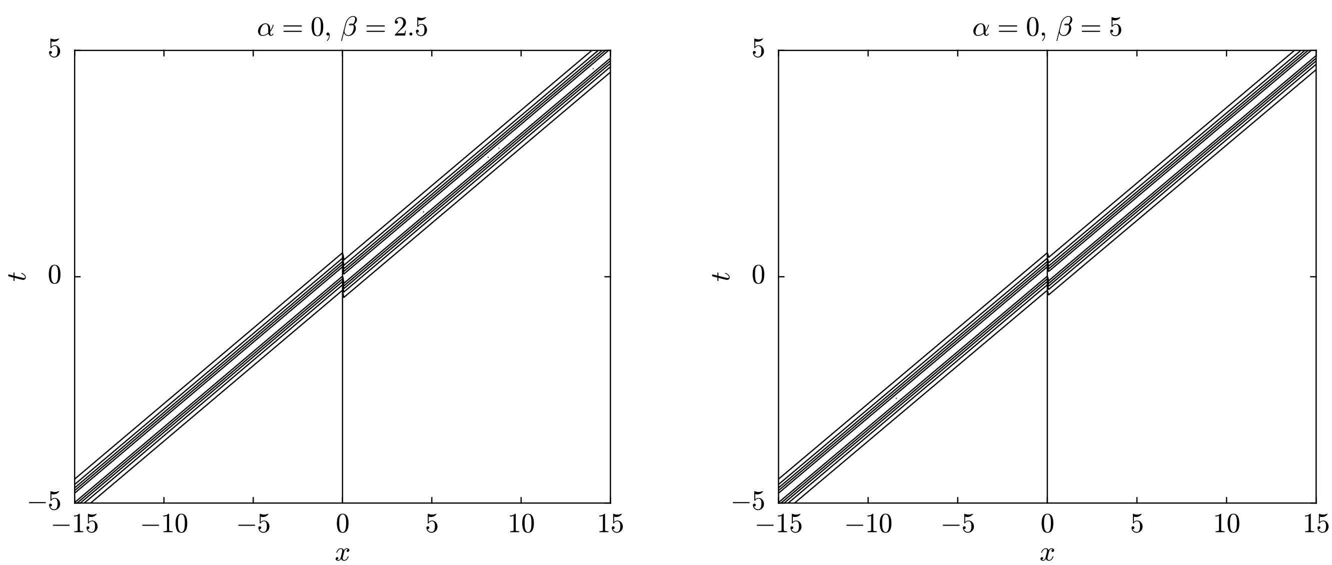

Fig. 6.12. One-soliton $|\tilde{u}_{sol}(t, x, \{-0.75 + i, 1\})|$ interacting with the defect resulting in $|u_{sol}(t, x, \{-0.75 + i, -3i\})|$ on the left and in $|u_{sol}(t, x, \{-0.75 + i, -5/3 - 4/3i\})|$ on the right.

Remark 6.2.3. To translate the expression into the notation used in [15], where the authors confirmed by direct calculation the one- and two-soliton satisfying the defect condition with switched defect parameter notation and $\alpha = 0$, first off, we need to forget about the defect parameter we call β and additionally take $\Omega = \sqrt{\alpha^2 - |\tilde{u} - u|^2}$. Then, for the one-soliton solution consider $\frac{v_1}{u_1} = 1$, $a = 2\eta_1$, $c = -2\xi$, $p = e^{-2\eta_1 \tilde{x}_1}$ and finally $q = e^{-i\tilde{\phi}_1}$ to recover the same result.

In the method we have presented, there are no limitations on the amount of solitons one can construct. Even though, it may not be obvious that the defect conditions are satisfied, we give a four-soliton solution which satisfies the defect conditions in Figure 6.13.

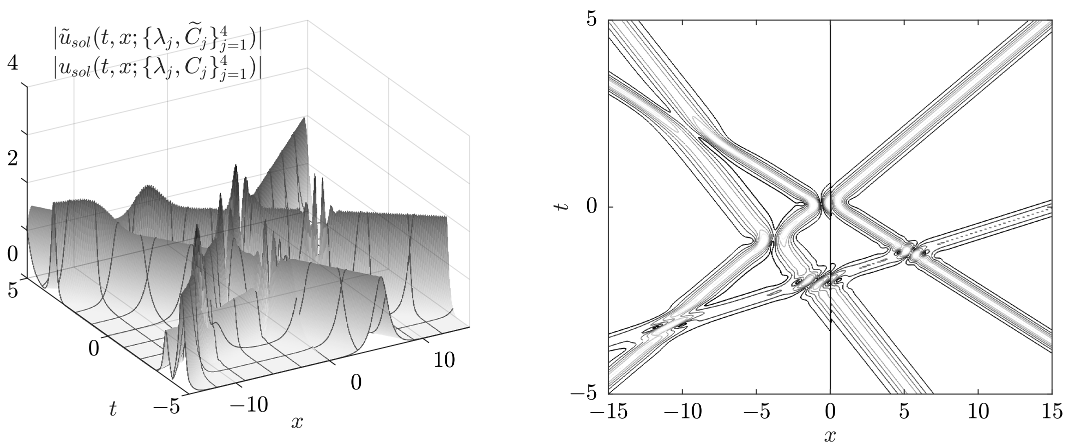

Fig. 6.13. Four-soliton $|\tilde{u}_{sol}(t, x, \{-0.75 + i, 1 + i, 0.5 + 0.75i, -2 + 0.5i, 1, 2, e^{-5}, e^{15}\})|$ interacting with the defect ($\alpha = -1$, $\beta = 3$) and its contour.

There is also a particular solution known which is not covered by Proposition 5.1.2. As in the case of the sG equation, this solution interacts destructively with the defect. Firstly, let us give the resulting solution

$$\tilde{u}_{dsol}(t, x) = \tilde{u}_{sol}\left(t, x; \left\{-\frac{\alpha}{2} + i\frac{\beta}{2}, C_1\right\}\right) = \beta e^{-i(-\alpha x + (\alpha^2 - \beta^2)t + (\phi_1 + \pi/2))} \operatorname{sech}(\beta(x - 2\alpha t - x_1)),$$
$$u_{sol}(t, x; \{\}) = 0,$$

where α, β and x_1 again need to be chosen in such a way that the argument $\beta(2\alpha t + x_1)$ is either positive or negative for all $t \in \mathbb{R}$. Therefore, it can be derived that α needs to be zero and subsequently, the $\pm$ sign of the defect condition needs to be chosen in order to compensate the resulting sign of βx_1, where it is imminent that $x_1 \neq 0$. This property is equivalent to the assertion on $\operatorname{Im}\Big(\lim_{\lambda \to 0} [2\lambda(\mathcal{B}_N(t, 0, \lambda) - \mathbb{1})]_{11}\Big)$ in Proposition 5.1.2, since in this case

$$\pm \operatorname{Im}\Big(\lim_{\lambda \to 0} [2\lambda(\mathcal{B}_N(t, 0, \lambda) - \mathbb{1})]_{11}\Big) = \Omega(t, 0) = \sqrt{\beta^2 - |\tilde{u}_{dsol}(t, 0)|^2} = |\beta \tanh(\beta x_1)|.$$

Moreover, we can calculate that this solution satisfies the defect condition with $\alpha = 0$, thus we need

$$(\tilde{u}_{dsol})_x(t, 0) = \pm \Omega(t, 0) \tilde{u}_{dsol}(t, 0),$$
$$(\tilde{u}_{dsol})_t(t, 0) = \pm i\Omega(t, 0)(\tilde{u}_{dsol})_x(t, 0) + i\tilde{u}_{dsol}(t, 0)|\tilde{u}_{dsol}(t, 0)|^2.$$

The derivatives of $\tilde{u}_{dsol}(t, x)$ evaluated at $x = 0$ are

$$(\tilde{u}_{dsol})_x(t, 0) = \beta \tanh(\beta x_1))\tilde{u}_{dsol}(t, 0),$$
$$(\tilde{u}_{dsol})_t(t, 0) = i\beta^2 \tilde{u}_{dsol}(t, 0).$$

Due to $\Omega(t, 0)$ being equal to $|\beta \tanh(\beta x_1)|$, both these equations hold if $\operatorname{sign}(x_1) = \pm 1$.

Remark 6.2.4. The destructive soliton solution with $\alpha = 0$ is in fact a boundary-bound soliton solution, which is not covered by Proposition 5.1.2, but mentioned in [15].

6.2.2 NLS equation on the half-line with boundary conditions

By the argumentation of the proofs in Chapter 5, it follows that in the case of zero seed solutions, it is not necessary to distinguish cases when introducing the frozen dressing matrix $\mathcal{K}_N(t, 0, \lambda)$. Due to the complete knowledge of the solutions of the Lax system (2.1.2) and the fact that the boundary matrices are diagonal, we can immediately identify the linear dependence of the two vectors υ_0 and ψ_0 by a particular choice of constants as in the proofs of the propositions for the sG equation. On the other hand, given for example a non-zero seed solution $u[0] = \rho e^{2i\rho^2 t}$ with constant background $\rho > 0$ which satisfies the Neumann boundary condition $u_x(t, 0) = 0$, this criterion can not be applied.

As for the defect condition connecting two half-lines, the zero seed solution $u[0] = 0$ can be taken as a foundation to construct soliton solutions for the NLS equation on the half-line with both the Robin and the new boundary condition. In compliance with Proposition 5.2.4, we thus take pairs of solutions $\psi_j(t, x) = e^{-i(\lambda_j x + 2\lambda_j^2 t)\sigma_3}(u_j, v_j)^\intercal$ and $\widehat{\psi}_j(t, x) = e^{-i(-\lambda_j x + 2\lambda_j^2 t)\sigma_3}(\hat{u}_j, \hat{v}_j)^\intercal$ of the Lax system (2.1.2) at the spectral parameters $\lambda = \lambda_j$ and $\lambda = -\lambda_j$, $j = 1, \ldots, N_s$, respectively, making sure that they are distinct $-\lambda_k \neq \lambda_j$ for all $1 \leq j \leq k \leq N_s$. Further, with regards to assumption (5.2.10), we impose

$$\widehat{\psi}_j\Big|_{x=0} = \begin{pmatrix} \hat{u}_j e^{-2i\lambda_j^2 t} \\ \hat{v}_j e^{2i\lambda_j^2 t} \end{pmatrix} = \begin{pmatrix} i\alpha - (-1)^{N_{bbs}} 2\lambda_j & 0 \\ 0 & i\alpha + (-1)^{N_{bbs}} 2\lambda_j \end{pmatrix} \begin{pmatrix} u_j e^{-2i\lambda_j^2 t} \\ v_j e^{2i\lambda_j^2 t} \end{pmatrix},$$

which is $\mathcal{K}_0((-1)^{N_{bbs}}\lambda_j)\psi_j\big|_{x=0}$ and easily translates into the following equality for the quotients

$$\frac{\hat{u}_j}{\hat{v}_j} = \frac{i\alpha - (-1)^{N_{bbs}} 2\lambda_j}{i\alpha + (-1)^{N_{bbs}} 2\lambda_j} \frac{u_j}{v_j}, \quad j = 1, \ldots, N_s. \tag{6.2.13}$$

As in the proposition, N_s and N_{bbs} are the numbers of the solitons for which the spectral parameter satisfies $\lambda_j \in \mathbb{C}\backslash(\mathbb{R} \cup i\mathbb{R})$, $j = 1, \ldots, N_s$, corresponding to solitons or $\lambda_j \in i\mathbb{R}\backslash\{0\}$, $j = N_s + 1, \ldots, N$ corresponding to boundary-bound solitons, respectively. On the other hand, the assumption (5.2.11) prescribes the choice of the norming constant for the included boundary-bound solitons. We have

$$\left|\frac{v_j}{u_j}\right| = \sqrt{-\frac{\alpha - (-1)^{N_{bbs}} 2\,\mathrm{Im}(\lambda_j)}{\alpha + (-1)^{N_{bbs}} 2\,\mathrm{Im}(\lambda_j)}}, \quad j = N_s + 1, \ldots, N.$$

The argument under the square root needs to be positive. Since we are not able to freely construct boundary-bound solitons for the new boundary condition due to the restriction in Proposition 5.2.7, we take this opportunity to give an example of boundary-bound solitons on the half-line with the Robin boundary condition in Figure 6.14.

Note that if $\lambda_j \in \mathbb{C}_+$, then $\widehat{\lambda}_j = -\lambda_j \in \mathbb{C}_-$ which, in turn, implies that $\widehat{\psi}_j$ has 'opposite' limit behavior as ψ_j for $x \to \pm\infty$. In order to apply Theorem 3.3.1 to the Dressing method corresponding to $\widehat{\lambda}_j$ and $\widehat{\psi}_j$, we instead use the counterpart $\widehat{\lambda}_j^*$ and $\widehat{\varphi}_j$. Since with $\widehat{\lambda}_j^* \in \mathbb{C}_+$, the vector $\widehat{\varphi}_j = e^{-i(\widehat{\lambda}_j^* x + 2(\widehat{\lambda}_j^*)^2 t)\sigma_3}(-\hat{v}_j^*, \hat{u}_j^*)^\intercal$, admits the same limit behavior as ψ_j for $x \to \pm\infty$. Similar to Remark 3.3.2 following Theorem 3.3.1, we can deduce for a two-fold dressing matrix consisting of the spectral parameters $\lambda_1 \in \mathbb{C}_+ \setminus i\mathbb{R}$ and $-\lambda_1^* \in \mathbb{C}_+ \setminus i\mathbb{R}$ with the corresponding solutions ψ_1 and $\widehat{\varphi}_1$ of the Lax system (2.1.2) that the weights of the scattering data can be calculated as

$$C_1^{(2)} = -\frac{v_1}{u_1}\frac{(\lambda_1 - \lambda_1^*)(\lambda_1 - \widehat{\lambda}_1)}{\lambda_1 - \widehat{\lambda}_1^*}, \qquad C_2^{(2)} = -\frac{\hat{u}_1^*}{-\hat{v}_1^*}\frac{(\widehat{\lambda}_1^* - \lambda_1^*)(\widehat{\lambda}_1^* - \widehat{\lambda}_1)}{\widehat{\lambda}_1^* - \lambda_1}.$$

This results in the following relation for the norming constants of the scattering data under the Robin boundary condition

$$C_1^{(2)}(C_2^{(2)})^* = -4\lambda_1^2 \cdot \frac{i\alpha - 2\lambda_1}{i\alpha + 2\lambda_1} \cdot \frac{\mathrm{Im}(\lambda_1)^2}{\mathrm{Re}(\lambda_1)^2}, \tag{6.2.14}$$

which is up to notation the same as in [6]. To align the notation, one would need to complex conjugate (6.2.14) and then it would be compliant with equation (2.36) in their paper after replacing $k_1 = -\lambda_1^*$. This is due to the differently defined potential $\mathcal{Q}_1$ of the matrix $\mathcal{V}$, which as a consequence gives the existence of Jost functions with different asymptotic behavior and continuations into different parts of the complex plane. Further, the following relations between the initial positions and phases of a $2N_s$-soliton solution are valid.

Remark 6.2.5 (Biondini & Hwang, [6])**.** In general, we can construct a $2N_s$-fold dressing matrix using the information given by the distinct spectral parameters $\lambda_1, \dots, \lambda_{N_s}$ and $-\lambda_1^*, \dots, -\lambda_{N_s}^*$ in $\mathbb{C}_+ \setminus i\mathbb{R}$ as well as their respective solutions of the Lax system (2.1.2) corresponding to the zero seed solution $\psi_1, \dots, \psi_{N_s}$ and $\widehat{\varphi}_1, \dots, \widehat{\varphi}_{N_s}$. In particular, we have $\lambda_j = \xi_j + i\eta_j$ and consequently $-\lambda_j^* = -\xi_j + i\eta_j$ for $j = 1, \dots, N_s$. Then, for $j = 1, \dots, N$ the relation for a pair of initial positions x_j and $\hat{x}_j = x_{N_s+j}$ as well as phases ϕ_j and $\hat{\phi}_j = \phi_{N_s+j}$ amounts to

$$\begin{aligned} x_j + \hat{x}_j &= \frac{1}{2\eta_j}\log\Big(1 + \frac{\eta_j^2}{\xi_j^2}\Big) + \frac{1}{4\eta_j}\log\Big(\frac{(2\xi_j)^2 + (\alpha - 2\eta_j)^2}{(2\xi_j)^2 + (\alpha + 2\eta_j)^2}\Big) \\ &\quad - \frac{1}{2\eta_j}\sum_{k=1}^{N_s}{}' \log\frac{[(\xi_j - \xi_k)^2 + (\eta_j - \eta_k)^2][(\xi_j + \xi_k)^2 + (\eta_j - \eta_k)^2]}{[(\xi_j + \xi_k)^2 + (\eta_j + \eta_k)^2][(\xi_j - \xi_k)^2 + (\eta_j + \eta_k)^2]}, \\ \phi_j - \hat{\phi}_j &= 2\arg(\lambda_j) + \arg\Big(\frac{2\xi_j + i(2\eta_j - \alpha)}{2\xi_j + i(2\eta_j + \alpha)}\Big) \\ &\quad - \sum_{k=1}^{N_s}{}' \arg\Big(\frac{[(\xi_j - \xi_k) + i(\eta_j - \eta_k)][(\xi_j + \xi_k) + i(\eta_j - \eta_k)]}{[(\xi_j + \xi_k) + i(\eta_j + \eta_k)][(\xi_j - \xi_k) + i(\eta_j + \eta_k)]}\Big), \end{aligned}$$

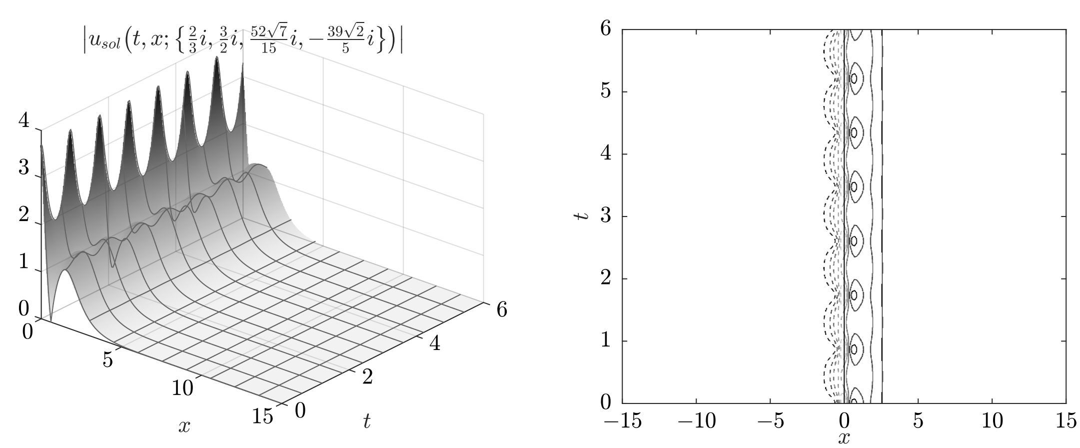

Fig. 6.14. Boundary-bound two-soliton subject to (5.2.9) with $\alpha = -1$ and its contour.

while the product of a pair of weights C_j, $\widehat{C}_j = C_{N_s+j}$ is

$$C_j\widehat{C}_j^* = -4\lambda_j^2 \frac{i\alpha - 2\lambda_j}{i\alpha + 2\lambda_j} \frac{(2\eta_j)^2}{(2\xi_j)^2} \Big[\prod_{k=1}^{N_s}{}' \frac{(\lambda_j - \lambda_k^*)(\lambda_j + \lambda_k)}{(\lambda_j - \lambda_k)(\lambda_j + \lambda_k^*)} \Big]^2.$$

Before graphically presenting the results, we want to give a similar argumentation for the new boundary condition corresponding, in the case of the zero seed solution, to the boundary matrix

$$\mathcal{K}_0(\lambda) = \frac{1}{(2\lambda - i|\beta|)^2 - \alpha^2} \begin{pmatrix} 4\lambda^2 + 4i\lambda|\beta| - (\alpha^2 + \beta^2) & 0 \\ 0 & 4\lambda^2 - 4i\lambda|\beta| - (\alpha^2 + \beta^2) \end{pmatrix}.$$

Taking the spectral parameters $\lambda = \lambda_j$ and $\lambda = -\lambda_j$, $j = 1, \ldots, N$, as well as their corresponding solutions ψ_j and $\widehat{\psi}_j$ of the Lax system (2.1.2) as in the case of Robin boundary conditions with the difference that the paired solutions need to satisfy assumption (5.2.18), we obtain the relevant relation for the quotients

$$\frac{\hat{u}_j}{\hat{v}_j} = \frac{(2\lambda_j + i|\beta|)^2 - \alpha^2}{(2\lambda_j - i|\beta|)^2 - \alpha^2} \frac{u_j}{v_j}, \quad j = 1, \ldots, N,$$

which is the counterpart to relation (6.2.13). For the two-soliton solution, this yields the relation

$$C_1^{(2)}(C_2^{(2)})^* = -4\lambda_1^2 \cdot \frac{(2\lambda_1 + i|\beta|)^2 - \alpha^2}{(2\lambda_1 - i|\beta|)^2 - \alpha^2} \cdot \frac{\mathrm{Im}(\lambda_1)^2}{\mathrm{Re}(\lambda_1)^2}$$

regarding the norming constants, where it is obvious that the factor $\frac{(2\lambda_1+i|\beta|)^2-\alpha^2}{(2\lambda_1-i|\beta|)^2-\alpha^2}$ is the only difference to the same result regarding the Robin boundary condition, where one has $\frac{i\alpha-2\lambda_1}{i\alpha+2\lambda_1}$. Moreover, by defining $\lambda_1 = \xi_1 + i\eta_1$ and $\widehat{\lambda}_1^* = -\xi_1 + i\eta_1$ as well as the corresponding weights $C_1 = 2\eta_1 e^{2\eta_1 x_1 + i\phi_1} = C_1^{(2)}$ and $\widehat{C}_1 = 2\eta_1 e^{2\eta_1 \hat{x}_1 + i\hat{\phi}_1} = C_2^{(2)}$, we obtain a relation between the initial positions and phases of the two-soliton

$$\begin{aligned} x_1 + \hat{x}_1 &= \frac{1}{2\eta_1} \log\Big(1 + \frac{\eta_1^2}{\xi_1^2}\Big) + \frac{1}{4\eta_1} \log\Big(\frac{(4\xi_1^2 - \alpha^2 - (2\eta_1 + |\beta|)^2)^2 + (4\xi_1(2\eta_1 + |\beta|))^2}{(4\xi_1^2 - \alpha^2 - (2\eta_1 - |\beta|)^2)^2 + (4\xi_1(2\eta_1 - |\beta|))^2} \Big), \\ \phi_1 - \hat{\phi}_1 &= 2\arg(\lambda_1) + \arg\Big(\frac{4\xi_1^2 - \alpha^2 - (2\eta_1 + |\beta|)^2 + i4\xi_1(2\eta_1 + |\beta|)}{4\xi_1^2 - \alpha^2 - (2\eta_1 - |\beta|)^2 + i4\xi_1(2\eta_1 - |\beta|)} \Big) + \pi. \end{aligned}$$

Remark 6.2.6. In general, we can construct a $2N$-fold dressing matrix using the information given by the distinct spectral parameters $\lambda_1, \ldots, \lambda_N$ and $-\lambda_1^*, \ldots, -\lambda_N^*$ in $\mathbb{C}_+ \setminus i\mathbb{R}$ as well as their respective solutions of the Lax system (2.1.2) corresponding to the zero seed solution $\psi_1, \ldots, \psi_N$ and $\widehat{\varphi}_1, \ldots, \widehat{\varphi}_N$. In particular, we have $\lambda_j = \xi_j + i\eta_j$ and consequently $-\lambda_j^* = -\xi_j + i\eta_j$ for $j = 1, \ldots, N$. Then, for $j = 1, \ldots, N$ the relation for a pair of initial positions x_j and $\hat{x}_j = x_{N+j}$ as well as phases ϕ_j and $\hat{\phi}_j = \phi_{N+j}$ amounts to

$$\begin{aligned} x_j + \hat{x}_j &= \frac{1}{2\eta_j} \log\Big(1 + \frac{\eta_j^2}{\xi_j^2}\Big) + \frac{1}{4\eta_j} \log\Big(\frac{(4\xi_j^2 - \alpha^2 - (2\eta_j + |\beta|)^2)^2 + (4\xi_j(2\eta_j + |\beta|))^2}{(4\xi_j^2 - \alpha^2 - (2\eta_j - |\beta|)^2)^2 + (4\xi_j(2\eta_j - |\beta|))^2} \Big) \\ &\quad - \frac{1}{2\eta_j} \sum_{k=1}^{N}{}' \log \frac{[(\xi_j - \xi_k)^2 + (\eta_j - \eta_k)^2][(\xi_j + \xi_k)^2 + (\eta_j - \eta_k)^2]}{[(\xi_j + \xi_k)^2 + (\eta_j + \eta_k)^2][(\xi_j - \xi_k)^2 + (\eta_j + \eta_k)^2]}, \\ \phi_j - \hat{\phi}_j &= 2\arg(\lambda_j) + \arg\Big(\frac{4\xi_j^2 - \alpha^2 - (2\eta_j + |\beta|)^2 + i4\xi_j(2\eta_j + |\beta|)}{4\xi_j^2 - \alpha^2 - (2\eta_j - |\beta|)^2 + i4\xi_j(2\eta_j - |\beta|)} \Big) + \pi \\ &\quad - \sum_{k=1}^{N}{}' \arg\Big(\frac{[(\xi_j - \xi_k) + i(\eta_j - \eta_k)][(\xi_j + \xi_k) + i(\eta_j - \eta_k)]}{[(\xi_j + \xi_k) + i(\eta_j + \eta_k)][(\xi_j - \xi_k) + i(\eta_j + \eta_k)]} \Big), \end{aligned}$$

whereas the product of a pair of weights C_j and $\widehat{C}_j = C_{N+j}$ is

$$C_j\widehat{C}_j^* = -4\lambda_j^2 \frac{(2\lambda_j + i|\beta|)^2 - \alpha^2}{(2\lambda_j - i|\beta|)^2 - \alpha^2} \frac{(2\eta_j)^2}{(2\xi_j)^2} \Big[\prod_{k=1}^{N}{}' \frac{(\lambda_j - \lambda_k^*)(\lambda_j + \lambda_k)}{(\lambda_j - \lambda_k)(\lambda_j + \lambda_k^*)} \Big]^2,$$

where the prime indicates that the term with $k = j$ is omitted from the sum and product.

Let us now focus on the visualization of the solutions in the case of boundary conditions for the NLS equation. For $N_s = 1$, consider the spectral parameter $\lambda_1 = \xi_1 + i\eta_1$, where it is comprehensible with regard to (2.1.21) that ξ_1 and η_1 describe the velocity and the amplitude of the physical one-soliton, respectively, as indicated in the discussion for the defect conditions. Further, the quotient of the constants u_1 and v_1 is highly related to the initial position x_1 and phase ϕ_1 of the soliton. Consequently, the mirror soliton corresponding to $\widehat{\lambda}_1^* = -\xi_1 + i\eta_1$ has opposite velocity to and the same amplitude as the physical soliton. Particularly, we have visualized the behavior in Figures 6.15 and 6.16 for the Robin and the new boundary condition.

On the other hand, the Dirichlet boundary condition $u(t, 0) = 0$ occurs as a special case of the Robin boundary condition, when $\alpha \to \infty$, or of the new boundary condition (4.3.3), when for example $|\alpha| \to \infty$, $|\beta| \to \infty$ or $\beta \to 0$; the Neumann boundary condition $u_x(t, 0) = 0$ only occurs when $\alpha = 0$ in the Robin boundary condition. Indeed, structurally these cases correspond to the boundary matrix $\mathcal{K}_0(\lambda) = \mathbb{1}$ or $\mathcal{K}_0(\lambda) = -\sigma_3$. Therefore, we plotted in Figure 6.17 on the left and right the reflection of a one-soliton solution $|u_{sol}(t, x; \{\lambda_j, C_j\}_{j=1}^2)|$ subject to the Dirichlet and Neumann boundary condition, respectively.

Then, in Figure 6.18, we choose particular parameters for the new boundary condition to plot an example of a physical three-soliton solution $|u_{sol}(t, x; \{\lambda_j, C_j\}_{j=1}^6)|$, which is reflected at the boundary, in three dimensions on the left and as a contour plot together with the mirror soliton on the right. Then, in Figure 6.19, we repeat this idea with the scattering data of a breather

$$\Big\{ \frac{3+3i}{4}, \frac{3+1i}{4}, -\frac{3+3i}{4}, -\frac{3+1i}{4}, 1.6e^{28.8}, 0.6e^{10.8}, 1.6\frac{-541+12i}{1105}e^{28.8}, 0.6\frac{-77-84i}{265}e^{10.8} \Big\}$$

for the NLS equation subject to the new boundary condition which is characterised by two spectral parameters having the same real part or rather velocity and overlapping spatial positions. It is observable that in these cases the physical soliton and the mirror soliton change roles, after the usual soliton interaction, with the physical soliton visible before and the mirror soliton visible after the interaction with the boundary. Additionally, in the case of the Dirichlet boundary condition the interaction of the pair of solitons results in the whole solution being zero at the boundary.

Picking up, the one-soliton solution swallowed by the defect is given by

$$u_{dsol}(t, x) = u_{sol}\Big(t, x; \Big\{ -\frac{\alpha}{2} + i\frac{\beta}{2}, C_1 \Big\} \Big) = \beta e^{-i(-\alpha x + (\alpha^2 - \beta^2)t + (\phi_1 + \pi/2))} \operatorname{sech}(\beta(x - 2\alpha t - x_1)),$$

and hence, we can infer that it satisfies

$$\begin{aligned} (u_{dsol})_x(t, 0) &= \beta \tanh(\beta x_1) u_{dsol}(t, 0) = -\sqrt{\beta^2 - |u_{dsol}(t, 0)|^2} u_{dsol}(t, 0), \\ (u_{dsol})_t(t, 0) &= i\beta^2 u_{dsol}(t, 0) = -i\sqrt{\beta^2 - |u_{dsol}(t, 0)|^2} (u_{dsol})_x(t, 0) + i|u_{dsol}(t, 0)|^2 u_{dsol}(t, 0), \end{aligned}$$

under the condition that $\alpha = 0$ and if $\operatorname{sign}(x_1) = -1$. Therefore, multiplying the first equality with $\sqrt{\beta^2 - |u_{dsol}(t, 0)|^2}$ and adding the second one multiplied with $-i$, we obtain

$$2\sqrt{\beta^2 - |u_{dsol}(t, 0)|^2} (u_{dsol})_x(t, 0) = i(u_{dsol})_t(t, 0) + 2|u_{dsol}(t, 0)|^2 u_{dsol}(t, 0) - \beta^2 u_{dsol}(t, 0),$$

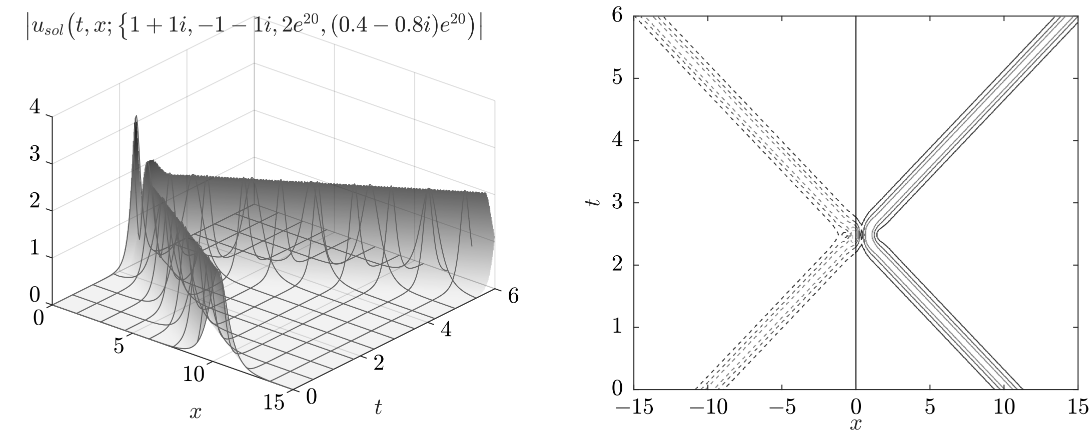

Fig. 6.15. One-soliton interacting with the Robin boundary ($\alpha = -2$) and its contour.

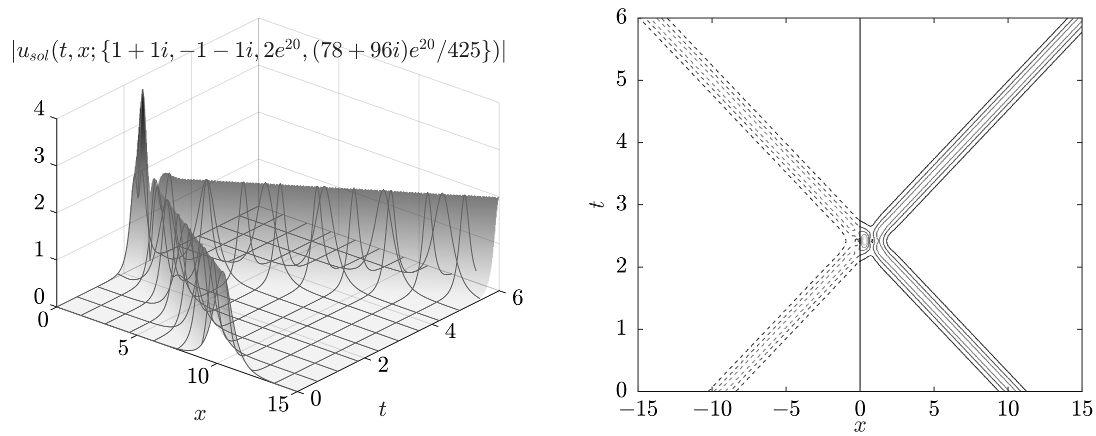

Fig. 6.16. One-soliton interacting with the new boundary ($\alpha = -1$, $\beta = 2$) and its contour.

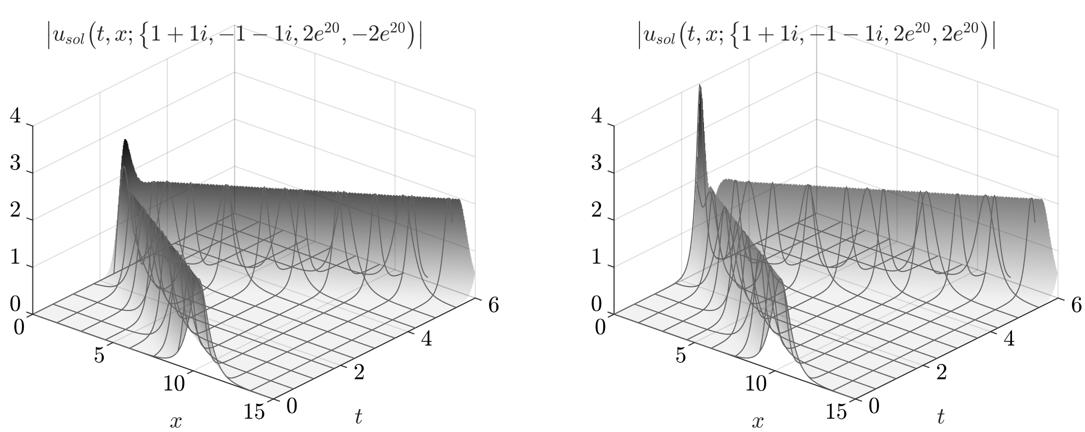

Fig. 6.17. One-soliton interacting with the Dirichlet (left) and the Neumann (right) boundary.

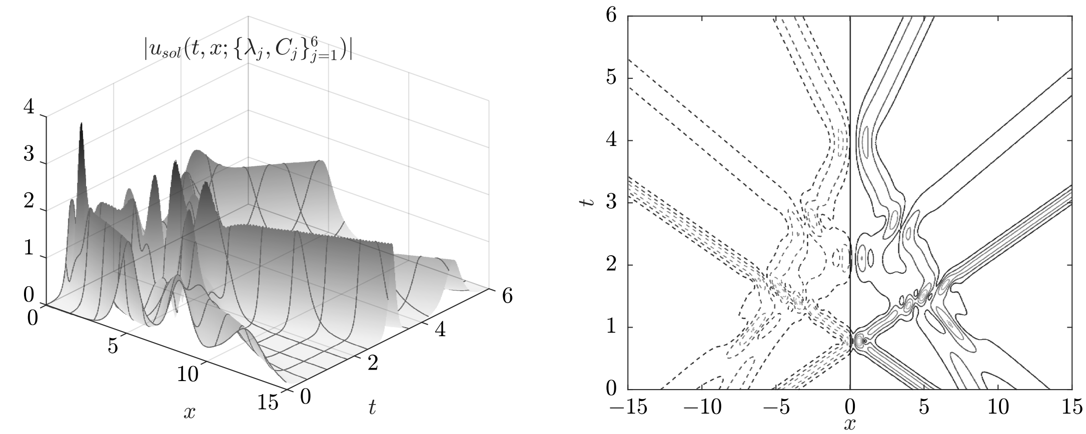

Fig. 6.18. Three-soliton interacting with the new boundary ($\alpha = 4$, $\beta = 2$) and its contour.

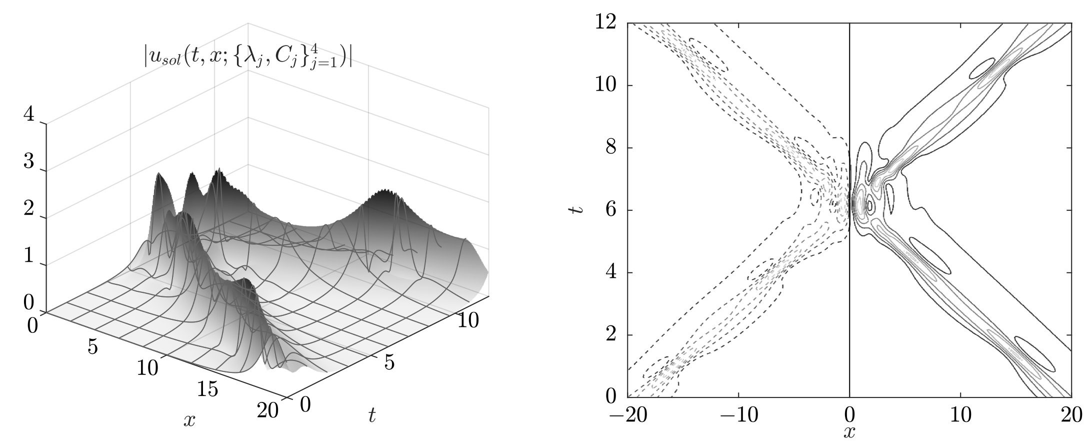

Fig. 6.19. Breather interacting with the new boundary ($\alpha = 2$, $\beta = 0.5$) and its contour.

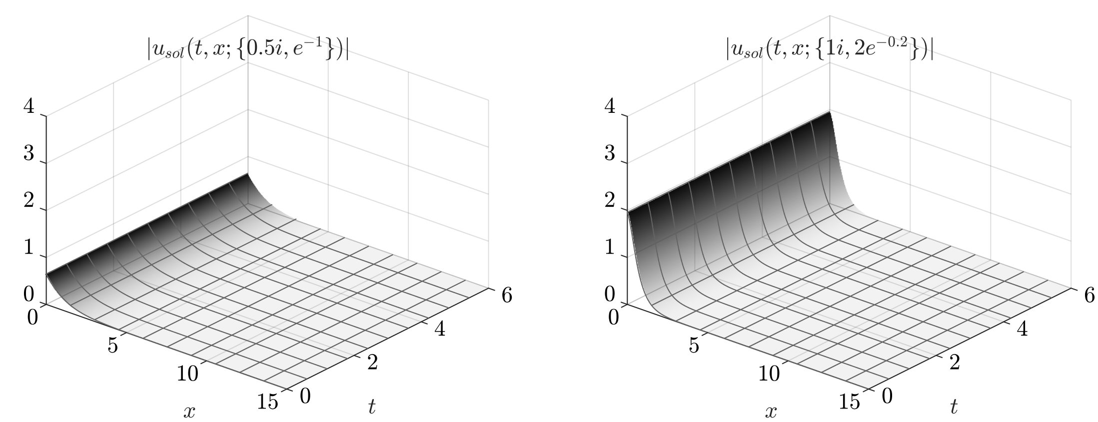

Fig. 6.20. Boundary-bound soliton interacting destructively with the new boundary with $\alpha = 0$ as well as $\beta = 1$ on the right and $\beta = 2$ on the left.

which corresponds exactly to the new boundary condition with $\alpha = 0$. Hence, this boundary-bound soliton is a solution of the new boundary condition interacting destructively with the boundary which eliminates the need to construct a paired soliton. This viewpoint is based on the idea that we already know what we want to achieve. In theory, we could assume more generally that $\alpha \in \mathbb{R}$. Indeed, it yields the same result which in hindsight one might connect to the assertion concerning the imaginary part of the (11)-entry of $\mathcal{K}_N$ or rather $|u_{dsol}(t,0)| = |\beta \operatorname{sech}(-\beta(\alpha t + x_1))| < |\beta|$ if $\beta \neq 0$.

Chapter 7

Conclusion

The main result of this work is the application of the Dressing method to different integrable models on the half-line and two half-lines connected through defect conditions. Explicitly, starting from zero seed solutions, we have constructed pure N-soliton solutions subject to the Robin and new boundary condition for the NLS equation on the half-line and the sin-boundary condition for the sG equation on the half-line. Furthermore, for the NLS equation, we have taken a closer look at the corresponding relations of the norming constants and put forth their explicit relations in terms of parameters of the solution. Again, given zero seed solutions, we have also constructed pure N-soliton solutions subject to defect conditions connecting the NLS or sG equation on two half-lines. Particularly, for the NLS equation, we have shown that each soliton is transmitted through the defect independently, which proves the statement conjectured in [15].

Different extensions of the presented method could probably be realized. Among them the application to integrable models on a star-graph with more than two half-lines, to integrable models of other PDEs associated to the AKNS system and to nonzero seed solutions for the NLS equation satisfying the corresponding boundary or defect conditions. Unsurprisingly, for each of these extensions the application is made more intricate. To begin with, for the consideration of three or more half-lines the concept of integrability needs to be generalized and the preliminary considerations with respect to the distribution of simple eigenvalues on each half-line needs to adapted. If one is interested in a different equation other than the NLS or sG equation particularly on the half-line, the associated symmetry with respect to the spectral parameter of the term including the time t of the phase needs to be considered, e.g. λ^2 for the NLS and $\lambda + \lambda^{-1}$ for the sG equation. In that regard, it is not always straightforward to obtain this very symmetry as pointed out to us by C. Zhang. Finally, the topic of a nonzero seed solution at least for the Neumann boundary condition has been treated in [42] and it is interesting to see if it is possible to give other examples, which then may also be related to physical phenomena.

Since the nonlinear method of images serves as an alternative method to find exact solutions for integrable models on the half-line, it should be possible to formulate Proposition 5.2.7 in terms of this method. This may lead to further insights regarding the boundary conditions.

Appendices

Appendix A

Calculations

For the proofs, we want to reiterate on, the following trigonometric identities are instrumental.

Lemma A.0.1 (trigonometric identities, [3])**.** *The Product-to-sum formulae give*

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y. \tag{A.0.1}$$

Conversely, the Sum-to-product formulae are

$$2 \sin \frac{x-y}{2} \cos \frac{x+y}{2} = \sin x - \sin y, \qquad -2 \sin \frac{x-y}{2} \sin \frac{x+y}{2} = \cos x - \cos y. \tag{A.0.2}$$

$$2 \cos \frac{x-y}{2} \sin \frac{x+y}{2} = \sin x + \sin y, \qquad 2 \cos \frac{x-y}{2} \cos \frac{x+y}{2} = \cos x + \cos y. \tag{A.0.3}$$

The double-angle formulae yield

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}, \tag{A.0.4}$$

$$\cos x + 1 = 2 \cos^2 \frac{x}{2}, \tag{A.0.5}$$

$$\cos x - 1 = -2 \sin^2 \frac{x}{2}. \tag{A.0.6}$$

Proof. By Euler's formula, we know that $\sin x = i(e^{-ix} - e^{ix})/2$ and $\cos x = (e^{-ix} + e^{ix})/2$. Therefore, the left hand sides of equalities (A.0.1) are, in fact,

$$\sin(x \pm y) = \frac{i}{2}(e^{-i(x \pm y)} - e^{i(x \pm y)}), \qquad \cos(x \pm y) = \frac{1}{2}(e^{-i(x \pm y)} + e^{i(x \pm y)}).$$

On the other hand, we find for the first line

$$\begin{aligned} \sin x \cos y \pm \cos x \sin y &= \frac{i}{4}\Big[(e^{-ix} - e^{ix})(e^{-iy} + e^{iy}) \pm (e^{-ix} + e^{ix})(e^{-iy} - e^{iy})\Big] \\ &= \frac{i}{4}\Big[(e^{-i(x+y)} - e^{i(x+y)})(1 \pm 1) + (e^{-i(x-y)} - e^{i(x-y)})(1 \mp 1)\Big], \end{aligned}$$

which is the same as the expression for $\sin(x \pm y)$ and for the second line

$$\begin{aligned} \cos x \cos y \mp \sin x \sin y &= \frac{1}{4}\Big[(e^{-ix} + e^{ix})(e^{-iy} + e^{iy}) \pm (e^{-ix} - e^{ix})(e^{-iy} - e^{iy})\Big] \\ &= \frac{i}{4}\Big[(e^{-i(x+y)} + e^{i(x+y)})(1 \pm 1) + (e^{-i(x-y)} + e^{i(x-y)})(1 \mp 1)\Big], \end{aligned}$$

which is the same as the expression for $\cos(x \pm y)$. Further, we have the simple equality

$$\sin^2 x + \cos^2 x = -\frac{1}{4}(e^{-2ix} - 2 + e^{2ix}) + \frac{1}{4}(e^{-2ix} + 2 + e^{2ix}) = 1.$$

Using the identities (A.0.1), we have

$$\begin{aligned}
2\sin\frac{x \pm y}{2}\cos\frac{x \mp y}{2} &= 2\Big(\sin\frac{x}{2}\cos\frac{y}{2} \pm \cos\frac{x}{2}\sin\frac{y}{2}\Big)\Big(\cos\frac{x}{2}\cos\frac{y}{2} \pm \sin\frac{x}{2}\sin\frac{y}{2}\Big)\\
&= 2\sin\frac{x}{2}\cos\frac{x}{2}\Big[\cos^2\frac{y}{2} + \sin^2\frac{y}{2}\Big] \pm 2\sin\frac{y}{2}\cos\frac{y}{2}\Big[\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\Big]\\
&= \sin x \pm \sin y,
\end{aligned}$$

where incidentally, $\sin x = \sin\frac{x+x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2}$ is a special case of (A.0.1). We also have

$$\begin{aligned}
-2\sin\frac{x-y}{2}\sin\frac{x+y}{2} &= -2\Big(\sin\frac{x}{2}\cos\frac{y}{2} - \cos\frac{x}{2}\sin\frac{y}{2}\Big)\Big(\sin\frac{x}{2}\cos\frac{y}{2} + \cos\frac{x}{2}\sin\frac{y}{2}\Big)\\
&= -\Big(\sin^2\frac{x}{2}\Big(1 - \sin^2\frac{y}{2} + \cos^2\frac{y}{2}\Big) - \cos^2\frac{x}{2}\Big(1 - \cos^2\frac{y}{2} + \sin^2\frac{y}{2}\Big)\Big)\\
&= \Big(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\Big) - \Big(\cos^2\frac{y}{2} - \sin^2\frac{y}{2}\Big)\Big(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\Big),\\
&= \cos x - \cos y,
\end{aligned}$$

where we have used that $2\cos^2\frac{y}{2} = 1 - \sin^2\frac{y}{2} + \cos^2\frac{y}{2}$ and $2\sin^2\frac{y}{2} = 1 - \cos^2\frac{y}{2} + \sin^2\frac{y}{2}$ in the first line and the special case of the second line of (A.0.1), which is $\cos x = \cos\frac{x+x}{2} = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$, in the third line. Lastly, we have

$$\begin{aligned}
2\cos\frac{x-y}{2}\cos\frac{x+y}{2} &= 2\Big(\cos\frac{x}{2}\cos\frac{y}{2} + \sin\frac{x}{2}\sin\frac{y}{2}\Big)\Big(\cos\frac{x}{2}\cos\frac{y}{2} - \sin\frac{x}{2}\sin\frac{y}{2}\Big)\\
&= \Big(\cos^2\frac{x}{2}\Big(1 - \sin^2\frac{y}{2} + \cos^2\frac{y}{2}\Big) - \sin^2\frac{x}{2}\Big(1 - \cos^2\frac{y}{2} + \sin^2\frac{y}{2}\Big)\Big)\\
&= \Big(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\Big) + \Big(\cos^2\frac{y}{2} - \sin^2\frac{y}{2}\Big)\Big(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\Big),\\
&= \cos x + \cos y.
\end{aligned}$$

And since these two equalities hold, we find with $y = 0$ that

$$\begin{aligned}
\cos x + 1 &= \cos x + \cos 0 = 2\cos^2\frac{x}{2},\\
\cos x - 1 &= \cos x - \cos 0 = -2\sin^2\frac{x}{2},
\end{aligned}$$

which implies the last two trigonometric identities stated in Lemma A.0.1. $\square$

A.1 Proof of Proposition 4.1.1

Proposition A.1.1. *The matrix*

$$\mathcal{B}(t,x,\lambda) = \mathbb{1} + \frac{1}{2\lambda}\begin{pmatrix} \alpha \pm i\sqrt{\beta^2 - |\tilde{u}-u|^2} & -i(\tilde{u}-u) \\ -i(\tilde{u}-u)^* & \alpha \mp i\sqrt{\beta^2 - |\tilde{u}-u|^2} \end{pmatrix},$$

	σ_3	σ_+	σ_-
σ_3	$\mathbb{1}$	$-\sigma_+$	σ_-
σ_+	σ_+	0	$\operatorname{diag}(0,1)$
σ_-	$-\sigma_-$	$\operatorname{diag}(1,0)$	0

Table A.1. Row entry times column entry. Elementary matrix multiplications (NLS equation).

	σ_1	σ_2	σ_3
σ_1	$\mathbb{1}$	$-i\sigma_3$	$i\sigma_2$
σ_2	$i\sigma_3$	$\mathbb{1}$	$-i\sigma_1$
σ_3	$-i\sigma_2$	$i\sigma_1$	$\mathbb{1}$

Table A.2. Row entry times column entry. Pauli matrix multiplications (sG equation).

representing the frozen Bäcklund transformation (4.1.1) *for the Lax pairs*

$$\mathcal{U} = \begin{pmatrix} -i\lambda & u \\ -u^* & i\lambda \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} -2i\lambda^2 + i|u|^2 & 2\lambda u + iu_x \\ -2\lambda u^* + iu_x^* & 2i\lambda^2 - i|u|^2 \end{pmatrix},$$
$$\widetilde{\mathcal{U}} = \begin{pmatrix} -i\lambda & \tilde{u} \\ -\tilde{u}^* & i\lambda \end{pmatrix}, \qquad \widetilde{\mathcal{V}} = \begin{pmatrix} -2i\lambda^2 + i|\tilde{u}|^2 & 2\lambda \tilde{u} + i\tilde{u}_x \\ -2\lambda \tilde{u}^* + i\tilde{u}_x^* & 2i\lambda^2 - i|\tilde{u}|^2 \end{pmatrix},$$

of the NLS equation corresponds to the defect conditions

$$\begin{aligned} (\tilde{u}-u)_x &= i\alpha(\tilde{u}-u) \pm \sqrt{\beta^2 - |\tilde{u}-u|^2}(\tilde{u}+u), \\ (\tilde{u}-u)_t &= -\alpha(\tilde{u}-u)_x \pm i\sqrt{\beta^2 - |\tilde{u}-u|^2}(\tilde{u}+u)_x + i(\tilde{u}-u)(|u|^2+|\tilde{u}|^2) \end{aligned} \tag{A.1.1}$$

at $x = 0$ *and* $\alpha, \beta \in \mathbb{R}$.

Proof. Writing the relevant matrices in terms of $\mathbb{1}$, σ_3, σ_+ and σ_-, we obtain

$$\begin{aligned} 2\lambda\mathcal{B}(t,x,\lambda) &= (2\lambda+\alpha)\mathbb{1} \pm i\sqrt{\beta^2 - |\tilde{u}-u|^2}\sigma_3 - i(\tilde{u}-u)\sigma_+ - i(\tilde{u}-u)^*\sigma_-, \\ \mathcal{U}(t,x,\lambda) &= -i\lambda\sigma_3 + u\sigma_+ - u^*\sigma_-, \qquad \widetilde{\mathcal{U}}(t,x,\lambda) = -i\lambda\sigma_3 + \tilde{u}\sigma_+ - \tilde{u}^*\sigma_-, \\ \mathcal{V}(t,x,\lambda) &= (-2i\lambda^2 + i|u|^2)\sigma_3 + (2\lambda u + iu_x)\sigma_+ + (-2\lambda u^* + iu_x^*)\sigma_-, \\ \widetilde{\mathcal{V}}(t,x,\lambda) &= (-2i\lambda^2 + i|\tilde{u}|^2)\sigma_3 + (2\lambda \tilde{u} + i\tilde{u}_x)\sigma_+ + (-2\lambda \tilde{u}^* + i\tilde{u}_x^*)\sigma_-. \end{aligned}$$

Therefore, on the left hand side of equality $(2\lambda\mathcal{B})_x = \widetilde{\mathcal{U}}(2\lambda\mathcal{B}) - (2\lambda\mathcal{B})\mathcal{U}$, we have at $x = 0$ that

$$2\lambda\mathcal{B}_x(t,0,\lambda) = \mp i \frac{(\tilde{u}-u)_x(\tilde{u}-u)^* + (\tilde{u}-u)(\tilde{u}-u)_x^*}{2\sqrt{\beta^2 - |\tilde{u}-u|^2}}\sigma_3 - i(\tilde{u}-u)_x\sigma_+ - i(\tilde{u}-u)_x^*\sigma_- \tag{A.1.2}$$

and on the right hand side, due to the elementary matrix multiplications given in Tabular A.1, that

$$\begin{aligned} \widetilde{\mathcal{U}}(2\lambda\mathcal{B}) = &- i\lambda\big[(2\lambda+\alpha)\sigma_3 \pm i\sqrt{\beta^2 - |\tilde{u}-u|^2}\cdot\mathbb{1} - i(\tilde{u}-u)\sigma_+ - i(\tilde{u}-u)^*(-\sigma_-)\big] \\ &+ \tilde{u}\big[(2\lambda+\alpha)\sigma_+ \pm i\sqrt{\beta^2 - |\tilde{u}-u|^2}\cdot(-\sigma_+) - i(\tilde{u}-u)\cdot 0 - i(\tilde{u}-u)^*\operatorname{diag}(1,0)\big] \\ &- \tilde{u}^*\big[(2\lambda+\alpha)\sigma_- \pm i\sqrt{\beta^2 - |\tilde{u}-u|^2}\cdot\sigma_- - i(\tilde{u}-u)\operatorname{diag}(0,1) - i(\tilde{u}-u)^*\cdot 0\big], \\ -(2\lambda\mathcal{B})\mathcal{U} = &+ i\lambda\big[(2\lambda+\alpha)\sigma_3 \pm i\sqrt{\beta^2 - |\tilde{u}-u|^2}\cdot\mathbb{1} - i(\tilde{u}-u)(-\sigma_+) - i(\tilde{u}-u)^*\sigma_-\big] \\ &- u\big[(2\lambda+\alpha)\sigma_+ \pm i\sqrt{\beta^2 - |\tilde{u}-u|^2}\cdot\sigma_+ - i(\tilde{u}-u)\cdot 0 - i(\tilde{u}-u)^*\operatorname{diag}(0,1)\big] \\ &+ u^*\big[(2\lambda+\alpha)\sigma_- \pm i\sqrt{\beta^2 - |\tilde{u}-u|^2}\cdot(-\sigma_-) - i(\tilde{u}-u)\operatorname{diag}(1,0) - i(\tilde{u}-u)^*\cdot 0\big]. \end{aligned}$$

Thus, we can make out the terms corresponding to the matrices σ_3, σ_+ and σ_-. For σ_+, we have

$$-2\lambda(\tilde{u}-u)+\tilde{u}(2\lambda+\alpha\mp i\sqrt{\beta^2-|\tilde{u}-u|^2})-u(2\lambda+\alpha\pm i\sqrt{\beta^2-|\tilde{u}-u|^2})$$

and noticing that the expression of order λ cancel, we are left with

$$\alpha(\tilde{u}-u)\mp i\sqrt{\beta^2-|\tilde{u}-u|^2}(\tilde{u}+u), \tag{A.1.3}$$

which is with respect to the equality (A.1.2) equivalent to the first equality of the defect condition (A.1.1). Analogously, for σ_-, we find

$$2\lambda(\tilde{u}-u)^*-\tilde{u}^*(2\lambda+\alpha\pm i\sqrt{\beta^2-|\tilde{u}-u|^2})+u^*(2\lambda+\alpha\mp i\sqrt{\beta^2-|\tilde{u}-u|^2})$$

so that, similar to the expression multiplied by σ_- in (A.1.2), it can be written as the negative complex conjugate of the expression (A.1.3) multiplied by σ_+:

$$-\alpha(\tilde{u}-u)^*\mp i\sqrt{\beta^2-|\tilde{u}-u|^2}(\tilde{u}+u)^*.$$

After cancellation, the remaining terms on the diagonal are

$$-i(\tilde{u}(\tilde{u}-u)^*+u^*(\tilde{u}-u))\operatorname{diag}(1,0)+i(\tilde{u}^*(\tilde{u}-u)+u(\tilde{u}-u)^*)\operatorname{diag}(0,1),$$

which can be simplified to

$$-i(|\tilde{u}|^2-|u|^2)\sigma_3.$$

Hence, it suffices to check with the first equality of the defect condition (A.1.1) that

$$(\tilde{u}-u)_x(\tilde{u}-u)^*+(\tilde{u}-u)(\tilde{u}-u)^*_x=\pm\sqrt{\beta^2-|\tilde{u}-u|^2}2(|\tilde{u}|^2-|u|^2), \tag{A.1.4}$$

which then, in turn, gives the equality of $(2\lambda\mathcal{B})_x=\widetilde{\mathcal{U}}(2\lambda\mathcal{B})-(2\lambda\mathcal{B})\mathcal{U}$ at $x=0$ on the diagonal.

On the other hand, on the left hand side of equality $(2\lambda\mathcal{B})_t=\widetilde{\mathcal{V}}(2\lambda\mathcal{B})-(2\lambda\mathcal{B})\mathcal{V}$, we have at $x=0$ that

$$2\lambda\mathcal{B}_t(t,0,\lambda)=\mp i\frac{(\tilde{u}-u)_t(\tilde{u}-u)^*+(\tilde{u}-u)(\tilde{u}-u)^*_t}{2\sqrt{\beta^2-|\tilde{u}-u|^2}}\sigma_3-i(\tilde{u}-u)_t\sigma_+-i(\tilde{u}-u)^*_t\sigma_- \tag{A.1.5}$$

and on the right hand side, due to the elementary matrix multiplications given in Tabular A.1, that

$$\begin{aligned}
\widetilde{\mathcal{V}}(2\lambda\mathcal{B})=&+(-2i\lambda^2+i|\tilde{u}|^2)\big[(2\lambda+\alpha)\sigma_3\pm i\sqrt{\beta^2-|\tilde{u}-u|^2}\cdot\mathbb{1}-i(\tilde{u}-u)\sigma_+-i(\tilde{u}-u)^*(-\sigma_-)\big]\\
&+(2\lambda\tilde{u}+i\tilde{u}_x)\big[(2\lambda+\alpha)\sigma_+\pm i\sqrt{\beta^2-|\tilde{u}-u|^2}\cdot(-\sigma_+)-i(\tilde{u}-u)^*\operatorname{diag}(1,0)\big]\\
&+(-2\lambda\tilde{u}^*+i\tilde{u}^*_x)\big[(2\lambda+\alpha)\sigma_-\pm i\sqrt{\beta^2-|\tilde{u}-u|^2}\cdot\sigma_--i(\tilde{u}-u)\operatorname{diag}(0,1)\big],\\
-(2\lambda\mathcal{B})\mathcal{V}=&-(-2i\lambda^2+i|u|^2)\big[(2\lambda+\alpha)\sigma_3\pm i\sqrt{\beta^2-|\tilde{u}-u|^2}\cdot\mathbb{1}-i(\tilde{u}-u)(-\sigma_+)-i(\tilde{u}-u)^*\sigma_-\big]\\
&-(2\lambda u+iu_x)\big[(2\lambda+\alpha)\sigma_+\pm i\sqrt{\beta^2-|\tilde{u}-u|^2}\cdot\sigma_+-i(\tilde{u}-u)^*\operatorname{diag}(0,1)\big]\\
&-(-2\lambda u^*+iu^*_x)\big[(2\lambda+\alpha)\sigma_-\pm i\sqrt{\beta^2-|\tilde{u}-u|^2}\cdot(-\sigma_-)-i(\tilde{u}-u)\operatorname{diag}(1,0)\big].
\end{aligned}$$

This time, for σ_+, we find

$$\begin{aligned}
(-4\lambda^2+(|\tilde{u}|^2+|u|^2))(\tilde{u}-u)+(2\lambda\tilde{u}+i\tilde{u}_x)(2\lambda+\alpha\mp i\sqrt{\beta^2-|\tilde{u}-u|^2})&\\
-(2\lambda u+iu_x)(2\lambda+\alpha\pm i\sqrt{\beta^2-|\tilde{u}-u|^2}),&
\end{aligned}$$

which equates after cancellation of the second and first order terms of λ to

$$(|\tilde{u}|^2 + |u|^2)(\tilde{u} - u) + i\alpha(\tilde{u} - u)_x \pm \sqrt{\beta^2 - |\tilde{u} - u|^2}(\tilde{u} + u)_x. \tag{A.1.6}$$

Equating (A.1.6) to $-i(\tilde{u} - u)_t$ from (A.1.5) is equivalent to the second equality of the defect condition (A.1.1). And again, we find for σ_- that

$$\begin{aligned}(4\lambda^2 - (|\tilde{u}|^2 + |u|^2))(\tilde{u} - u)^* + (i\tilde{u}_x^* - 2\lambda\tilde{u}^*)((2\lambda + \alpha \pm i\sqrt{\beta^2 - |\tilde{u} - u|^2})\\ + (2\lambda u^* - iu_x^*)(2\lambda + \alpha \mp i\sqrt{\beta^2 - |\tilde{u} - u|^2}),\end{aligned}$$

which equates also after cancellation of the second and first order terms of λ to

$$-(|\tilde{u}|^2 + |u|^2)(\tilde{u} - u) + i\alpha(\tilde{u} - u)_x^* \mp \sqrt{\beta^2 - |\tilde{u} - u|^2}(\tilde{u} + u)_x^*,$$

the negative complex conjugate of (A.1.6), and therefore to $-i(\tilde{u} - u)_t^*$ from (A.1.5). However, for the diagonal entries, we have

$$\begin{aligned}(|\tilde{u}|^2 - 2\lambda^2)[(2\lambda + \alpha)i\sigma_3 \mp \sqrt{\beta^2 - |\tilde{u} - u|^2}\mathbb{1}] + (2\lambda^2 - |u|^2)[(2\lambda + \alpha)i\sigma_3 \mp \sqrt{\beta^2 - |\tilde{u} - u|^2}\mathbb{1}]\\ + (\tilde{u}_x - 2i\lambda\tilde{u})(\tilde{u} - u)^* \operatorname{diag}(1,0) - (2i\lambda u^* + u_x^*)(\tilde{u} - u) \operatorname{diag}(1,0)\\ + (\tilde{u}_x^* + 2i\lambda\tilde{u}^*)(\tilde{u} - u) \operatorname{diag}(0,1) + (2i\lambda u - u_x)(\tilde{u} - u)^* \operatorname{diag}(0,1).\end{aligned}$$

After eliminating $-2\lambda^2$ from the first line with $2\lambda^2$ from the second line, we find for the coefficient of order one in λ that

$$2i(|\tilde{u}|^2 - |u|^2)\sigma_3 + (-2i(\tilde{u}(\tilde{u} - u)^* + u^*(\tilde{u} - u))) \operatorname{diag}(1,0) + (2i(\tilde{u}^*(\tilde{u} - u) + u(\tilde{u} - u)^*)) \operatorname{diag}(0,1),$$

which equates to

$$2i(|\tilde{u}|^2 - |u|^2)\sigma_3 + (-2i(|\tilde{u}|^2 - |u|^2)) \operatorname{diag}(1,0) + (2i(|\tilde{u}|^2 - |u|^2)) \operatorname{diag}(0,1) = 0.$$

Hence, the remaining expression is independent of λ. Particularly,

$$\begin{aligned}(|\tilde{u}|^2 - |u|^2)[i\alpha\sigma_3 \mp \sqrt{\beta^2 - |\tilde{u} - u|^2}\mathbb{1}] + (\tilde{u}_x(\tilde{u} - u)^* - u_x^*(\tilde{u} - u)) \operatorname{diag}(1,0)\\ + (\tilde{u}_x^*(\tilde{u} - u) - u_x(\tilde{u} - u)^*) \operatorname{diag}(0,1).\end{aligned} \tag{A.1.7}$$

Using the first equality of the defect condition (A.1.1) to express $\tilde{u}_x^*$ and u_x in terms of $\tilde{u}$, u, u_x^* and $\tilde{u}$, u, $\tilde{u}_x$, respectively, we obtain for the last expression in (A.1.7) the following

$$\begin{aligned}\tilde{u}_x^*(\tilde{u} - u) - u_x(\tilde{u} - u)^* &= (u_x^* - i\alpha(\tilde{u} - u)^* \pm \sqrt{\beta^2 - |\tilde{u} - u|^2}(\tilde{u} + u)^*)(\tilde{u} - u)\\ &\quad - (\tilde{u}_x - i\alpha(\tilde{u} - u) \mp \sqrt{\beta^2 - |\tilde{u} - u|^2}(\tilde{u} + u))(\tilde{u} - u)^*\\ &= u_x^*(\tilde{u} - u) - \tilde{u}_x(\tilde{u} - u)^* \pm 2\sqrt{\beta^2 - |\tilde{u} - u|^2}(|\tilde{u}|^2 - |u|^2)\end{aligned} \tag{A.1.8}$$

so that (A.1.7) can be written as

$$[\tilde{u}_x(\tilde{u} - u)^* - u_x^*(\tilde{u} - u) + (i\alpha \mp \sqrt{\beta^2 - |\tilde{u} - u|^2})(|\tilde{u}|^2 - |u|^2)]\sigma_3. \tag{A.1.9}$$

Hence, we check with the second equality of the defect condition (A.1.1) that

$$\begin{aligned}(|\tilde{u} - u|^2)_t = (-\alpha(\tilde{u} - u)_x \pm i\sqrt{\beta^2 - |\tilde{u} - u|^2}(\tilde{u} + u)_x + i(\tilde{u} - u)(|u|^2 + |\tilde{u}|^2))(\tilde{u} - u)^*\\ - (\tilde{u} - u)(\alpha(\tilde{u} - u)_x^* \pm i\sqrt{\beta^2 - |\tilde{u} - u|^2}(\tilde{u} + u)_x^* + i(\tilde{u} - u)^*(|u|^2 + |\tilde{u}|^2)).\end{aligned}$$

We already calculated the term $((\tilde{u} - u)_x(\tilde{u} - u)^* + (\tilde{u} - u)(\tilde{u} - u)_x^*)$ which is multiplied by α in (A.1.4) and with regards to (A.1.9), substituting the term $\tilde{u}_x^*(\tilde{u} - u) - u_x(\tilde{u} - u)^*$, which we also calculated already in (A.1.8), we obtain

$$(|\tilde{u} - u|^2)_t = \pm 2i\sqrt{\beta^2 - |\tilde{u} - u|^2}[\tilde{u}_x(\tilde{u} - u)^* - u_x^*(\tilde{u} - u) + (i\alpha \mp \sqrt{\beta^2 - |\tilde{u} - u|^2})(|\tilde{u}|^2 - |u|^2)]$$

from which we can ultimately confirm the equality $(2\lambda\mathcal{B})_t = \widetilde{\mathcal{V}}(2\lambda\mathcal{B}) - (2\lambda\mathcal{B})\mathcal{V}$, thereby concluding the proof. $\square$

A.2 Proof of Proposition 4.1.2

Proposition A.2.1. *The matrix*

$$\mathbb{B}(t,x,\lambda) = \mathbb{1} \pm \frac{i\alpha}{\lambda}\Big(\cos\frac{\tilde{\theta}+\theta}{2}\sigma_3 + \sin\frac{\tilde{\theta}+\theta}{2}\sigma_2\Big),$$

representing the frozen Bäcklund transformation (4.1.1) *for the Lax pairs*

$$\begin{aligned}
\mathbb{U} &= \frac{i}{4}\Big[(\theta_x-\theta_t)\sigma_1 + \frac{1}{\lambda}\sin\theta\sigma_2 + \Big(\frac{1}{\lambda}\cos\theta-\lambda\Big)\sigma_3\Big],\\
\mathbb{V} &= \frac{i}{4}\Big[(\theta_t-\theta_x)\sigma_1 + \frac{1}{\lambda}\sin\theta\sigma_2 + \Big(\frac{1}{\lambda}\cos\theta+\lambda\Big)\sigma_3\Big],\\
\widetilde{\mathbb{U}} &= \frac{i}{4}\Big[(\tilde{\theta}_x-\tilde{\theta}_t)\sigma_1 + \frac{1}{\lambda}\sin\tilde{\theta}\sigma_2 + \Big(\frac{1}{\lambda}\cos\tilde{\theta}-\lambda\Big)\sigma_3\Big],\\
\widetilde{\mathbb{V}} &= \frac{i}{4}\Big[(\tilde{\theta}_t-\tilde{\theta}_x)\sigma_1 + \frac{1}{\lambda}\sin\tilde{\theta}\sigma_2 + \Big(\frac{1}{\lambda}\cos\tilde{\theta}+\lambda\Big)\sigma_3\Big],
\end{aligned}$$

of the sG equation corresponds to the defect conditions

$$\begin{aligned}
\tilde{\theta}_x + \theta_t &= \pm\Big(\alpha\sin\frac{\tilde{\theta}+\theta}{2} + \frac{1}{\alpha}\sin\frac{\tilde{\theta}-\theta}{2}\Big),\\
\tilde{\theta}_t + \theta_x &= \mp\Big(\alpha\sin\frac{\tilde{\theta}+\theta}{2} - \frac{1}{\alpha}\sin\frac{\tilde{\theta}-\theta}{2}\Big)
\end{aligned} \tag{A.2.1}$$

at $x=0$ with $\alpha\in\mathbb{R}$.

Proof. As for the defect condition for the NLS equation, only elementary matrix multiplications, see Table A.2, are necessary in order to prove the claim. For the left hand side of the equality $\mathbb{B}_x = \widetilde{\mathbb{U}}\mathbb{B} - \mathbb{B}\mathbb{U}$, we have at $x=0$ the following

$$\mathbb{B}_x(t,0,\lambda) = \pm\frac{i\alpha}{\lambda}\frac{(\tilde{\theta}+\theta)_x}{2}\Big(-\sin\frac{\tilde{\theta}+\theta}{2}\sigma_3 + \cos\frac{\tilde{\theta}+\theta}{2}\sigma_2\Big). \tag{A.2.2}$$

For the right hand side, we calculate

$$\begin{aligned}
\widetilde{\mathbb{U}}(t,0,\lambda)\mathbb{B}(t,0,\lambda) =& \frac{i}{4}\Big[(\tilde{\theta}_x-\tilde{\theta}_t)\sigma_1 + \frac{1}{\lambda}\sin\tilde{\theta}\sigma_2 + \Big(\frac{1}{\lambda}\cos\tilde{\theta}-\lambda\Big)\sigma_3\Big]\\
&\mp \frac{\alpha}{4\lambda}\cos\frac{\tilde{\theta}+\theta}{2}\Big[(\tilde{\theta}_x-\tilde{\theta}_t)(-i\sigma_2) + \frac{1}{\lambda}\sin\tilde{\theta}(i\sigma_1) + \Big(\frac{1}{\lambda}\cos\tilde{\theta}-\lambda\Big)\mathbb{1}\Big]\\
&\mp \frac{\alpha}{4\lambda}\sin\frac{\tilde{\theta}+\theta}{2}\Big[(\tilde{\theta}_x-\tilde{\theta}_t)(i\sigma_3) + \frac{1}{\lambda}\sin\tilde{\theta}\cdot\mathbb{1} + \Big(\frac{1}{\lambda}\cos\tilde{\theta}-\lambda\Big)(-i\sigma_1)\Big],\\
-\mathbb{B}(t,0,\lambda)\mathbb{U}(t,0,\lambda) =& -\frac{i}{4}\Big[(\theta_x-\theta_t)\sigma_1 + \frac{1}{\lambda}\sin\theta\sigma_2 + \Big(\frac{1}{\lambda}\cos\theta-\lambda\Big)\sigma_3\Big]\\
&\pm \frac{\alpha}{4\lambda}\cos\frac{\tilde{\theta}+\theta}{2}\Big[(\theta_x-\theta_t)(i\sigma_2) + \frac{1}{\lambda}\sin\theta(-i\sigma_1) + \Big(\frac{1}{\lambda}\cos\theta-\lambda\Big)\mathbb{1}\Big]\\
&\pm \frac{\alpha}{4\lambda}\sin\frac{\tilde{\theta}+\theta}{2}\Big[(\theta_x-\theta_t)(-i\sigma_3) + \frac{1}{\lambda}\sin\theta\cdot\mathbb{1} + \Big(\frac{1}{\lambda}\cos\theta-\lambda\Big)(i\sigma_1)\Big].
\end{aligned}$$

Picking out the expressions corresponding to the identity matrix $\mathbb{1}$, we have

$$\mp\frac{\alpha}{4\lambda}\cos\frac{\tilde{\theta}+\theta}{2}\Big[\Big(\frac{1}{\lambda}\cos\tilde{\theta}-\lambda\Big) - \Big(\frac{1}{\lambda}\cos\theta-\lambda\Big)\Big] \mp \frac{\alpha}{4\lambda^2}\sin\frac{\tilde{\theta}+\theta}{2}\Big[\sin\tilde{\theta}-\sin\theta\Big],$$

which, using the trigonometric identities (A.0.2), equates to

$$\mp\frac{\alpha}{4\lambda^2}\Big[\cos\frac{\tilde\theta+\theta}{2}\Big(-2\sin\frac{\tilde\theta-\theta}{2}\sin\frac{\tilde\theta+\theta}{2}\Big)+\sin\frac{\tilde\theta+\theta}{2}\Big(2\sin\frac{\tilde\theta-\theta}{2}\cos\frac{\tilde\theta+\theta}{2}\Big)\Big]=0.$$

Then, proceeding similarly with the expressions corresponding to the first Pauli matrix σ_1, we find

$$\frac{i}{4}[(\tilde\theta-\theta)_x-(\tilde\theta-\theta)_t]\mp\frac{i\alpha}{4\lambda^2}\cos\frac{\tilde\theta+\theta}{2}(\sin\tilde\theta+\sin\theta)\pm\frac{i\alpha}{4\lambda}\sin\frac{\tilde\theta+\theta}{2}\Big(\frac{1}{\lambda}(\cos\tilde\theta+\cos\theta)-2\lambda\Big)$$

so that the term corresponding to the negative second order in λ again using the trigonometric identities (A.0.3) equates to

$$\mp\frac{i\alpha}{4\lambda^2}\Big[\cos\frac{\tilde\theta+\theta}{2}\Big(2\sin\frac{\tilde\theta+\theta}{2}\cos\frac{\tilde\theta-\theta}{2}\Big)-\sin\frac{\tilde\theta+\theta}{2}\Big(2\cos\frac{\tilde\theta-\theta}{2}\cos\frac{\tilde\theta+\theta}{2}\Big)\Big]=0.$$

Assuming that the remaining term multiplied by $-4i$ is zero:

$$(\tilde\theta-\theta)_x-(\tilde\theta-\theta)_t\mp 2\alpha\sin\frac{\tilde\theta+\theta}{2}=0, \tag{A.2.3}$$

we can derive that this is equivalent to the subtraction of the second from the first equality of the defect condition (A.2.1). Now, for the expressions corresponding to the second Pauli matrix σ_2, we have

$$\frac{i}{4\lambda}\Big[\sin\tilde\theta-\sin\theta\Big]\pm\frac{i\alpha}{4\lambda}\cos\frac{\tilde\theta+\theta}{2}\Big[\tilde\theta_x-\tilde\theta_t+\theta_x-\theta_t\Big].$$

Utilizing the trigonometric identity (A.0.2) for $\sin(x)-\sin(y)$, we thus obtain

$$\frac{i}{4\lambda}\cos\frac{\tilde\theta+\theta}{2}\Big[2\sin\frac{\tilde\theta-\theta}{2}\pm\alpha(\tilde\theta_x-\tilde\theta_t+\theta_x-\theta_t)\Big].$$

Equating this to the expression corresponding to σ_2 in (A.2.2) gives

$$(\tilde\theta+\theta)_x+(\tilde\theta_t+\theta_t)=\pm\frac{2}{\alpha}\sin\frac{\tilde\theta-\theta}{2}, \tag{A.2.4}$$

which is equivalent to the addition of the two equalities of the defect condition (A.2.1). This leaves the examination of the expressions corresponding to the third Pauli matrix σ_3, for which we obtain

$$\frac{i}{4}\Big[\Big(\frac{1}{\lambda}\cos\tilde\theta-\lambda\Big)-\Big(\frac{1}{\lambda}\cos\theta-\lambda\Big)\Big]\mp\frac{i\alpha}{4\lambda}\sin\frac{\tilde\theta+\theta}{2}\Big[\tilde\theta_x-\tilde\theta_t+\theta_x-\theta_t\Big].$$

By means of the trigonometric identities (A.0.2) for $\cos(x)-\cos(y)$, we then find in combination with the equality to the expression corresponding to the third Pauli matrix σ_3 in (A.2.2) that

$$\mp\frac{i\alpha}{\lambda}\frac{(\tilde\theta+\theta)_x}{2}\sin\frac{\tilde\theta+\theta}{2}=\frac{i}{4\lambda}\Big[-2\sin\frac{\tilde\theta+\theta}{2}\sin\frac{\tilde\theta-\theta}{2}\Big]\mp\frac{i\alpha}{4\lambda}\sin\frac{\tilde\theta+\theta}{2}\Big[\tilde\theta_x-\tilde\theta_t+\theta_x-\theta_t\Big].$$

Multiplying this equality with $\mp\frac{4\lambda}{i\alpha}(\sin\frac{\tilde\theta+\theta}{2})^{-1}$ and transferring all derivatives to the left hand side, we obtain the same equality (A.2.4) as the one corresponding to the second Pauli matrix. Therefore, we have shown that the defect condition imply both derived equalities (A.2.3) and (A.2.4). On the other hand, assuming the equalities (A.2.3) and (A.2.4), derived from the frozen Bäcklund transformation, hold, then adding them up and subtracting (A.2.3) from (A.2.4) is equivalent to two times the first and second equality of the defect condition (A.2.1), respectively.

For the left hand side of the equality $\mathbb{B}_t = \widetilde{\mathbb{V}}\mathbb{B} - \mathbb{B}\mathbb{V}$, we have at $x = 0$ the following

$$\mathbb{B}_t(t,0,\lambda) = \pm\frac{i\alpha}{\lambda}\frac{(\tilde{\theta}+\theta)_t}{2}\Big(-\sin\frac{\tilde{\theta}+\theta}{2}\sigma_3 + \cos\frac{\tilde{\theta}+\theta}{2}\sigma_2\Big). \tag{A.2.5}$$

For the right hand side, we calculate

$$\begin{aligned}
\widetilde{\mathbb{V}}(t,0,\lambda)\mathbb{B}(t,0,\lambda) =& \frac{i}{4}\Big[(\tilde{\theta}_t - \tilde{\theta}_x)\sigma_1 + \frac{1}{\lambda}\sin\tilde{\theta}\sigma_2 + \Big(\frac{1}{\lambda}\cos\tilde{\theta} + \lambda\Big)\sigma_3\Big] \\
&\mp \frac{\alpha}{4\lambda}\cos\frac{\tilde{\theta}+\theta}{2}\Big[(\tilde{\theta}_t - \tilde{\theta}_x)(-i\sigma_2) + \frac{1}{\lambda}\sin\tilde{\theta}(i\sigma_1) + \Big(\frac{1}{\lambda}\cos\tilde{\theta} + \lambda\Big)\mathbb{1}\Big] \\
&\mp \frac{\alpha}{4\lambda}\sin\frac{\tilde{\theta}+\theta}{2}\Big[(\tilde{\theta}_t - \tilde{\theta}_x)(i\sigma_3) + \frac{1}{\lambda}\sin\tilde{\theta}\cdot\mathbb{1} + \Big(\frac{1}{\lambda}\cos\tilde{\theta} + \lambda\Big)(-i\sigma_1)\Big], \\
-\mathbb{B}(t,0,\lambda)\mathbb{V}(t,0,\lambda) =& -\frac{i}{4}\Big[(\theta_t - \theta_x)\sigma_1 + \frac{1}{\lambda}\sin\theta\sigma_2 + \Big(\frac{1}{\lambda}\cos\theta + \lambda\Big)\sigma_3\Big] \\
&\pm \frac{\alpha}{4\lambda}\cos\frac{\tilde{\theta}+\theta}{2}\Big[(\theta_t - \theta_x)(i\sigma_2) + \frac{1}{\lambda}\sin\theta(-i\sigma_1) + \Big(\frac{1}{\lambda}\cos\theta + \lambda\Big)\mathbb{1}\Big] \\
&\pm \frac{\alpha}{4\lambda}\sin\frac{\tilde{\theta}+\theta}{2}\Big[(\theta_t - \theta_x)(-i\sigma_3) + \frac{1}{\lambda}\sin\theta\cdot\mathbb{1} + \Big(\frac{1}{\lambda}\cos\theta + \lambda\Big)(i\sigma_1)\Big].
\end{aligned}$$

Picking out the expressions corresponding to the identity matrix $\mathbb{1}$, we have

$$\mp\frac{\alpha}{4\lambda}\cos\frac{\tilde{\theta}+\theta}{2}\Big[\Big(\frac{1}{\lambda}\cos\tilde{\theta} + \lambda\Big) - \Big(\frac{1}{\lambda}\cos\theta + \lambda\Big)\Big] \mp \frac{\alpha}{4\lambda^2}\sin\frac{\tilde{\theta}+\theta}{2}\Big[\sin\tilde{\theta} - \sin\theta\Big],$$

which is the same as for the x part after eliminating $\lambda - \lambda = 0$ in the first bracket. Then, proceeding similarly with the expressions corresponding to the first Pauli matrix σ_1, we find

$$\frac{i}{4}[(\tilde{\theta}-\theta)_t - (\tilde{\theta}-\theta)_x] \mp \frac{i\alpha}{4\lambda^2}\cos\frac{\tilde{\theta}+\theta}{2}(\sin\tilde{\theta} + \sin\theta) \pm \frac{i\alpha}{4\lambda}\sin\frac{\tilde{\theta}+\theta}{2}\Big(\frac{1}{\lambda}(\cos\tilde{\theta} + \cos\theta) + 2\lambda\Big),$$

which results, up to an insignificant minus sign, in the same as (A.2.3). For the expressions corresponding to the second Pauli matrix σ_2, we have

$$\frac{i}{4\lambda}\Big[\sin\tilde{\theta} - \sin\theta\Big] \pm \frac{i\alpha}{4\lambda}\cos\frac{\tilde{\theta}+\theta}{2}\Big[\tilde{\theta}_t - \tilde{\theta}_x + \theta_t - \theta_x\Big].$$

Utilizing the trigonometric identity (A.0.2) for $\sin(x) - \sin(y)$ and equating the term to the expression corresponding to σ_2 in (A.2.5), we thus obtain

$$(\tilde{\theta}+\theta)_x + (\tilde{\theta}_t + \theta_t) = \pm\frac{2}{\alpha}\sin\frac{\tilde{\theta}-\theta}{2},$$

where we multiplied with $\pm\frac{4\lambda}{i\alpha}(\cos\frac{\tilde{\theta}+\theta}{2})^{-1}$ and transferred the derivatives on the left hand side. Repeating this for the expressions corresponding to the third Pauli matrix σ_3, we find

$$\frac{i}{4\lambda}\Big[\cos\tilde{\theta} - \cos\theta\Big] \mp \frac{i\alpha}{4\lambda}\sin\frac{\tilde{\theta}+\theta}{2}\Big[\tilde{\theta}_t - \tilde{\theta}_x + \theta_t - \theta_x\Big].$$

Therefore, applying the trigonometric identity (A.0.2) for $\cos(x) - \cos(y)$, equating the result to the expression corresponding to the third Pauli matrix from (A.2.5), then multiplying with $\mp\frac{4\lambda}{i\alpha}(\sin\frac{\tilde{\theta}+\theta}{2})^{-1}$ and finally transferring all derivatives to the left hand side, again leads to the same equality as for the second Pauli matrix and thus to (A.1.6). Hence, the t part for the frozen Bäcklund transformation is merely a repetition of the results we derived for the x part. □

A.3 Proof of Proposition 4.3.1

Proposition A.3.1. *The boundary matrices*

$$\mathcal{K}(\lambda) = \frac{1}{i\alpha + 2\lambda}(i\alpha\mathbb{1} - 2\lambda\sigma_3),$$

$$\mathcal{K}(t,0,\lambda) = \frac{1}{(2\lambda - i|\beta|)^2 - \alpha^2}((4\lambda^2 - (\alpha^2+\beta^2))\mathbb{1} + 4i\lambda\Omega(t,0)\sigma_3 + 4i\lambda u(t,0)\sigma_+ + 4i\lambda u^*(t,0)\sigma_-),$$

representing the symmetry relation (4.3.1) *for the Lax pair*

$$\mathcal{U} = -i\lambda\sigma_3 + u\sigma_+ - u^*\sigma_-, \qquad \mathcal{V} = (-2i\lambda^2 + i|u|^2)\sigma_3 + (2\lambda u + iu_x)\sigma_+ + (-2\lambda u^* + iu_x^*)\sigma_-,$$

of the NLS equation correspond to the Robin boundary condition

$$u_x(t,0) = \alpha u(t,0)$$

with $\alpha \in \mathbb{R}$ and the new boundary condition

$$u_x(t,0) = \frac{iu_t(t,0)}{2\Omega(t,0)} - \frac{u(t,0)\Omega(t,0)}{2} + \frac{u(t,0)|u(t,0)|^2}{2\Omega(t,0)} - \frac{u(t,0)\alpha^2}{2\Omega(t,0)}$$

with $\Omega(t,0) = \sqrt{\beta^2 - |u(t,0)|^2}$, $\alpha, \beta \in \mathbb{R}$, respectively.

Proof. For the Robin boundary condition, we need to verify the symmetry relation $0 = \mathcal{V}(t,0,-\lambda)\cdot((i\alpha+2\lambda)\mathcal{K}(\lambda)) - ((i\alpha+2\lambda)\mathcal{K}(\lambda))\cdot\mathcal{V}(t,0,\lambda)$. Hence, by Table A.1 the multiplications yield

$$\begin{aligned}
\mathcal{V}(t,0,-\lambda)((i\alpha+2\lambda)\mathcal{K}(\lambda)) =& + i\alpha\big[(-2i\lambda^2 + i|u|^2)\sigma_3 + (-2\lambda u + iu_x)\sigma_+ + (2\lambda u^* + iu_x^*)\sigma_-\big] \\
& - 2\lambda\big[(-2i\lambda^2 + i|u|^2)\mathbb{1} + (-2\lambda u + iu_x)(-\sigma_+) + (2\lambda u^* + iu_x^*)\sigma_-\big], \\
-((i\alpha+2\lambda)\mathcal{K}(\lambda))\mathcal{V}(t,0,\lambda) =& - i\alpha\big[(-2i\lambda^2 + i|u|^2)\sigma_3 + (2\lambda u + iu_x)\sigma_+ + (-2\lambda u^* + iu_x^*)\sigma_-\big] \\
& + 2\lambda\big[(-2i\lambda^2 + i|u|^2)\mathbb{1} + (2\lambda u + iu_x)\sigma_+ + (-2\lambda u^* + iu_x^*)(-\sigma_-)\big].
\end{aligned}$$

It can easily be seen that the expressions corresponding to the identity $\mathbb{1}$ and third Pauli matrix σ_3 are of opposite sign and therefore vanishing. For the expressions corresponding to σ_+, we obtain

$$i\alpha[(-2\lambda u + iu_x) - (2\lambda u + iu_x)] + 2\lambda[(-2\lambda u + iu_x) + (2\lambda u + iu_x)] = 4i\lambda[u_x - \alpha u];$$

for the expressions corresponding to σ_-, we find

$$i\alpha[(2\lambda u^* + iu_x^*) - (-2\lambda u^* + iu_x^*)] - 2\lambda[(2\lambda u^* + iu_x^*) + (-2\lambda u^* + iu_x^*)] = -4i\lambda[u_x^* - \alpha u^*],$$

thereby confirming the equivalence of the the symmetry relation to the Robin boundary condition.

Now, for the new boundary condition, we need to verify the symmetry relation

$$\mathcal{K}_t(t,0,\lambda) = \mathcal{V}(t,0,-\lambda)\mathcal{K}(t,0,\lambda) - \mathcal{K}(t,0,\lambda)\mathcal{V}(t,0,\lambda).$$

As should be clear and as we have seen in the other cases, the multiplication of a polynomial in λ with the boundary matrix is not affecting this relation. Thus, the boundary matrix for the new boundary condition is to be taken without its denominator $(2\lambda - i|\beta|)^2 - \alpha^2$. For the left hand side, we find that

$$\mathcal{K}_t(t,0,\lambda) = 4i\lambda\Omega_t(t,0)\sigma_3 + 4i\lambda u_t(t,0)\sigma_+ + 4i\lambda u_t^*(t,0)\sigma_-$$

and for the right hand side, we find with Table A.1 that

$$\begin{aligned}\mathcal{V}(t,0,-\lambda)\mathcal{K}(t,0,\lambda) =& + (4\lambda^2-(\alpha^2+\beta^2))\big[(-2i\lambda^2+i|u|^2)\sigma_3 + (-2\lambda u + iu_x)\sigma_+ + (2\lambda u^* + iu_x^*)\sigma_-\big]\\ &+ 4i\lambda\Omega\big[(-2i\lambda^2+i|u|^2)\mathbb{1} + (-2\lambda u + iu_x)(-\sigma_+) + (2\lambda u^* + iu_x^*)\sigma_-\big]\\ &+ 4i\lambda u\big[(-2i\lambda^2+i|u|^2)\sigma_+ + (-2\lambda u + iu_x)\cdot 0 + (2\lambda u^* + iu_x^*)\,\mathrm{diag}(0,1)\big]\\ &+ 4i\lambda u^*\big[(-2i\lambda^2+i|u|^2)(-\sigma_-) + (-2\lambda u + iu_x)\,\mathrm{diag}(1,0) + (2\lambda u^* + iu_x^*)\cdot 0\big],\\ -\mathcal{K}(t,0,\lambda)\mathcal{V}(t,0,\lambda) =& - (4\lambda^2-(\alpha^2+\beta^2))\big[(-2i\lambda^2+i|u|^2)\sigma_3 + (2\lambda u + iu_x)\sigma_+ + (-2\lambda u^* + iu_x^*)\sigma_-\big]\\ &- 4i\lambda\Omega\big[(-2i\lambda^2+i|u|^2)\mathbb{1} + (2\lambda u + iu_x)\sigma_+ + (-2\lambda u^* + iu_x^*)(-\sigma_-)\big]\\ &- 4i\lambda u\big[(-2i\lambda^2+i|u|^2)(-\sigma_+) + (2\lambda u + iu_x)\cdot 0 + (-2\lambda u^* + iu_x^*)\,\mathrm{diag}(1,0)\big]\\ &- 4i\lambda u^*\big[(-2i\lambda^2+i|u|^2)\sigma_- + (2\lambda u + iu_x)\,\mathrm{diag}(0,1) + (-2\lambda u^* + iu_x^*)\cdot 0\big].\end{aligned}$$

Then, picking the expressions corresponding to the matrix σ_+, we obtain

$$\begin{aligned}(4\lambda^2-(\alpha^2+\beta^2))[(-2\lambda u + iu_x) - (2\lambda u + iu_x)] - 4i\lambda\Omega[(-2\lambda u + iu_x) + (2\lambda u + iu_x)]\\ + 4i\lambda u[(-2i\lambda^2+i|u|^2) + (-2i\lambda^2+i|u|^2)].\end{aligned}$$

If we compare the coefficients with respect to a polynomial in λ, we find that the third order coefficient $-16u + 16u$ is zero as well as the second and zero-th order coefficient so that the only contribution in the equality to $\mathcal{K}_t(t,0,\lambda)$ comes from the first order coefficient. Therefore, the equality with respect to σ_+ divided by 4λ amounts to

$$iu_t(t,0) = u(t,0)(\alpha^2+\beta^2) + 2\Omega(t,0)u_x(t,0) - 2u(t,0)|u(t,0)|^2, \tag{A.3.1}$$

which is under simply conversion equivalent to the new boundary condition. Similarly, the expressions corresponding to the matrix σ_- give

$$(4\lambda^2-(\alpha^2+\beta^2))[4\lambda u^*] + 4i\lambda\Omega[2iu_x^*] - 4i\lambda u^*[-4i\lambda^2 + 2i|u|^2].$$

Hence, the resulting contribution to the equality to $\mathcal{K}_t(t,0,\lambda)$ is again limited to the coefficient with respect to the first order in λ and for this equality, we can write after dividing by 4λ:

$$iu_t^*(t,0) = -u^*(t,0)(\alpha^2+\beta^2) - 2\Omega(t,0)u_x^*(t,0) + 2u^*(t,0)|u(t,0)|^2, \tag{A.3.2}$$

which is equivalent to the complex conjugate of the new boundary condition. Now, after simply cancellation of the expressions corresponding to the identity $\mathbb{1}$ and third Pauli matrix σ_3, we are left with the equalities

$$\begin{aligned}4i\lambda\Omega_t &= 4i\lambda u^*(-2\lambda u + iu_x) - 4i\lambda u(-2\lambda u^* + iu_x^*)\\ -4i\lambda\Omega_t &= 4i\lambda u(2\lambda u^* + iu_x^*) - 4i\lambda u^*(2\lambda u + iu_x)\end{aligned}$$

on the diagonal coming from the expressions corresponding to $\mathrm{diag}(1,0)$ and $\mathrm{diag}(0,1)$. First, by simplifying these equalities, one can notice that they are redundant and in particular, dividing by 4λ, one has

$$i\Omega_t(t,0) = u(t,0)u_x^*(t,0) - u^*(t,0)u_x(t,0).$$

On the other hand, by the definition of $\Omega(t,0)$, we can calculate

$$i\Omega_t(t,0) = -i\frac{u(t,0)u_t^*(t,0) + u^*(t,0)u_t(t,0)}{2\Omega(t,0)}$$

for which we can use the to the new boundary condition equivalent expressions (A.3.1) and (A.3.2) in order to find after cancellation the following

$$\begin{aligned}
&= -\frac{u(t,0)(-2\Omega(t,0)u_x^*(t,0)) + u^*(t,0)(2\Omega(t,0)u_x(t,0))}{2\Omega(t,0)} \\
&= u(t,0)u_x^*(t,0) - u^*(t,0)u_x(t,0).
\end{aligned}$$

In particular, this calculation confirms the equivalence of the new boundary condition to the symmetry relation with regards to the chosen boundary matrix $\mathcal{K}(t,0,\lambda)$. $\square$

A.4 Proof of Proposition 4.3.2

Proposition A.4.1. *The boundary matrices*

$$\mathbb{K}(\lambda) = \frac{1}{\sqrt{\lambda^2 + \frac{1}{\lambda^2} + 2\cos\alpha}}\Big[\Big(\lambda + \frac{1}{\lambda}\Big)\mathbb{1}\cos\frac{\alpha}{2} + i\Big(\lambda - \frac{1}{\lambda}\Big)\sigma_1 \sin\frac{\alpha}{2}\Big],$$

$$\mathbb{K}(t,0,\lambda) = \frac{1}{\sqrt{(\lambda - \frac{1}{\lambda})^2 + \alpha^2}}\Big[-\alpha\mathbb{1} - i\Big(\lambda - \frac{1}{\lambda}\Big)\Big(\sigma_3\cos\frac{\theta(t,0)}{2} + \sigma_2\sin\frac{\theta(t,0)}{2}\Big)\Big],$$

$$\mathbb{K}(t,0,\lambda) = \frac{1}{\sqrt{(\lambda + \frac{1}{\lambda})^2 + \alpha^2}}\Big[i\alpha\sigma_1 - i\Big(\lambda + \frac{1}{\lambda}\Big)\Big(\sigma_3\cos\frac{\theta(t,0)}{2} + \sigma_2\sin\frac{\theta(t,0)}{2}\Big)\Big].$$

for the Lax pair of the sG equation

$$\begin{aligned}
\mathbb{U} &= \frac{i}{4}\Big[(\theta_x - \theta_t)\sigma_1 + \frac{1}{\lambda}\sin\theta\sigma_2 + \Big(\frac{1}{\lambda}\cos\theta - \lambda\Big)\sigma_3\Big], \\
\mathbb{V} &= \frac{i}{4}\Big[(\theta_t - \theta_x)\sigma_1 + \frac{1}{\lambda}\sin\theta\sigma_2 + \Big(\frac{1}{\lambda}\cos\theta + \lambda\Big)\sigma_3\Big],
\end{aligned}$$

correspond to a Dirichlet boundary condition

$$\theta(t,0) = \alpha$$

with $\alpha \in \mathbb{R}$, a sin*-boundary condition*

$$\theta_x(t,0) = \alpha\sin\frac{\theta(t,0)}{2}$$

with $\alpha \in \mathbb{R}$ and a cos*-boundary condition*

$$\theta_x(t,0) = \alpha\cos\frac{\theta(t,0)}{2}$$

with $\alpha \in \mathbb{R}$, respectively.

Proof. For all three cases, we need to check the symmetry relation

$$\mathbb{K}_t(t,0,\lambda) = \mathbb{V}(t,0,\lambda^{-1})\mathbb{K}(t,0,\lambda) - \mathbb{K}(t,0,\lambda)\mathbb{V}(t,0,\lambda),$$

where we multiply this equality by the denominator of $\mathbb{K}$ in each case. Therefore, the left hand sides of the Dirichlet, the sin- and the cos-boundary condition are given by

$$\begin{aligned}\mathbb{K}_t(\lambda) &= 0,\\ \mathbb{K}_t(t,0,\lambda) &= i\Big(\lambda - \frac{1}{\lambda}\Big)\frac{\theta_t(t,0)}{2}\Big(\sigma_3 \sin\frac{\theta(t,0)}{2} - \sigma_2\cos\frac{\theta(t,0)}{2}\Big),\\ \mathbb{K}_t(t,0,\lambda) &= i\Big(\lambda + \frac{1}{\lambda}\Big)\frac{\theta_t(t,0)}{2}\Big(\sigma_3 \sin\frac{\theta(t,0)}{2} - \sigma_2\cos\frac{\theta(t,0)}{2}\Big),\end{aligned}$$

respectively. Then, only elementary matrix multiplications are necessary in order to obtain

$$\begin{aligned}\mathbb{V}(t,0,\lambda^{-1})\mathbb{K}(\lambda) = &+\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\cos\frac{\alpha}{2}\Big[(\theta_t-\theta_x)\sigma_1 + \lambda\sin\theta\sigma_2 + \Big(\lambda\cos\theta + \frac{1}{\lambda}\Big)\sigma_3\Big]\\ &-\frac{1}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\sin\frac{\alpha}{2}\Big[(\theta_t-\theta_x)\mathbb{1} + \lambda\sin\theta(-i\sigma_3) + \Big(\lambda\cos\theta + \frac{1}{\lambda}\Big)i\sigma_2\Big],\\ -\mathbb{K}(\lambda)\mathbb{V}(t,0,\lambda) = &-\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\cos\frac{\alpha}{2}\Big[(\theta_t-\theta_x)\sigma_1 + \frac{1}{\lambda}\sin\theta\sigma_2 + \Big(\frac{1}{\lambda}\cos\theta + \lambda\Big)\sigma_3\Big]\\ &+\frac{1}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\sin\frac{\alpha}{2}\Big[(\theta_t-\theta_x)\mathbb{1} + \frac{1}{\lambda}\sin\theta i\sigma_3 + \Big(\frac{1}{\lambda}\cos\theta + \lambda\Big)(-i\sigma_2)\Big]\end{aligned}$$

for the Dirichlet boundary condition. Since the expressions for the identity $\mathbb{1}$ and the first Pauli matrix σ_1 show up in pairs with opposite sign, it is clear that they cancel out. Leaving the expressions corresponding to the second Pauli matrix σ_2 which can be summarized as

$$\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\cos\frac{\alpha}{2}\Big[\lambda\sin\theta - \frac{1}{\lambda}\sin\theta\Big] - \frac{i}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\sin\frac{\alpha}{2}\Big[\lambda\cos\theta + \frac{1}{\lambda} + \frac{1}{\lambda}\cos\theta + \lambda\Big].$$

This, however, can be simplified to

$$\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\Big(\lambda-\frac{1}{\lambda}\Big)\Big[\cos\frac{\alpha}{2}\sin\theta - \sin\frac{\alpha}{2}(\cos\theta+1)\Big]. \tag{A.4.1}$$

From the trigonometric identities (A.0.4), $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, and (A.0.5), $\cos\theta+1 = 2\cos^2\frac{\theta}{2}$, we have that equation (A.4.1) can be written as

$$\frac{i}{2}\Big(\lambda+\frac{1}{\lambda}\Big)\Big(\lambda-\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}\Big[\cos\frac{\alpha}{2}\sin\frac{\theta}{2} - \sin\frac{\alpha}{2}\cos\frac{\theta}{2}\Big] = \frac{i}{2}\Big(\lambda+\frac{1}{\lambda}\Big)\Big(\lambda-\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}\sin\frac{\theta-\alpha}{2},$$

which is zero for all $t \in \mathbb{R}_+$ if either $\theta(t,0) - \alpha = 2\pi C$ or $\theta(t,0) = \pi C - \pi/2$ for $C \in \mathbb{Z}$. However, since one identifies the solutions of the sG equation up to a multiple of 2π, the first condition is essentially the Dirichlet boundary condition. Moreover, the expressions corresponding to the third Pauli matrix σ_3 are

$$\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\cos\frac{\alpha}{2}\Big[\Big(\lambda\cos\theta+\frac{1}{\lambda}\Big) - \Big(\frac{1}{\lambda}\cos\theta+\lambda\Big)\Big] + \frac{i}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\sin\frac{\alpha}{2}\Big[\lambda\sin\theta + \frac{1}{\lambda}\sin\theta\Big],$$

where we can use similar means, in particular the trigonometric identities (A.0.4) and (A.0.6), $\cos\theta - 1 = -2\sin^2\frac{\theta}{2}$, in order to obtain

$$\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\Big(\lambda-\frac{1}{\lambda}\Big)\Big[\cos\frac{\alpha}{2}(\cos\theta-1) + \sin\frac{\alpha}{2}\sin\theta\Big] = -\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\Big(\lambda-\frac{1}{\lambda}\Big)\sin\frac{\theta}{2}\sin\frac{\theta-\alpha}{2}.$$

Consequently, the expressions corresponding to the third Pauli matrix σ_3 are zero if either $\theta(t,0) - \alpha = 2\pi C$ or $\theta(t,0) = \pi C$ for $C \in \mathbb{Z}$. Combining the results for the second and third Pauli matrix, the only possibility for both expressions to be zero is given by the Dirichlet boundary condition. And vice versa, the Dirichlet boundary condition is sufficient for the symmetry relation with the respective boundary matrix to hold.

In the case of the sin-boundary condition, the right hand side of the symmetry relation yields

$$\begin{aligned}
\mathbb{V}(t,0,\lambda^{-1})\mathbb{K}(t,0,\lambda) = &-\frac{i\alpha}{4}\Big[(\theta_t-\theta_x)\sigma_1+\lambda\sin\theta\sigma_2+\Big(\lambda\cos\theta+\frac{1}{\lambda}\Big)\sigma_3\Big]\\
&+\frac{1}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}\Big[(\theta_t-\theta_x)(-i\sigma_2)+\lambda\sin\theta i\sigma_1+\Big(\lambda\cos\theta+\frac{1}{\lambda}\Big)\mathbb{1}\Big]\\
&+\frac{1}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\sin\frac{\theta}{2}\Big[(\theta_t-\theta_x)i\sigma_3+\lambda\sin\theta\mathbb{1}+\Big(\lambda\cos\theta+\frac{1}{\lambda}\Big)(-i\sigma_1)\Big],\\
-\mathbb{K}(t,0,\lambda)\mathbb{V}(t,0,\lambda) = &+\frac{i\alpha}{4}\Big[(\theta_t-\theta_x)\sigma_1+\frac{1}{\lambda}\sin\theta\sigma_2+\Big(\frac{1}{\lambda}\cos\theta+\lambda\Big)\sigma_3\Big]\\
&-\frac{1}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}\Big[(\theta_t-\theta_x)i\sigma_2+\frac{1}{\lambda}\sin\theta(-i\sigma_1)+\Big(\frac{1}{\lambda}\cos\theta+\lambda\Big)\mathbb{1}\Big]\\
&-\frac{1}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\sin\frac{\theta}{2}\Big[(\theta_t-\theta_x)(-i\sigma_3)+\frac{1}{\lambda}\sin\theta\mathbb{1}+\Big(\frac{1}{\lambda}\cos\theta+\lambda\Big)i\sigma_1\Big].
\end{aligned}$$

Following the same strategy as before, we filter the expressions corresponding to the identity matrix

$$\frac{1}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}\Big[\Big(\lambda\cos\theta+\frac{1}{\lambda}\Big)-\Big(\frac{1}{\lambda}\cos\theta+\lambda\Big)\Big]+\frac{1}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\sin\frac{\theta}{2}\Big[\lambda\sin\theta-\frac{1}{\lambda}\sin\theta\Big],$$

which can be written as

$$\frac{1}{4}\Big(\lambda-\frac{1}{\lambda}\Big)^2\Big[\cos\frac{\theta}{2}(\cos\theta-1)+\sin\frac{\theta}{2}\sin\theta\Big].$$

With the trigonometric identities (A.0.4) and (A.0.6), we find

$$\frac{1}{4}\Big(\lambda-\frac{1}{\lambda}\Big)^2\Big[-2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\Big]=0.$$

First, note that the derivatives of θ in the expressions corresponding to the first Pauli matrix σ_1 cancel so that we effectively obtain the following

$$\frac{i}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\Big(\lambda+\frac{1}{\lambda}\Big)\Big[\cos\frac{\theta}{2}\sin\theta-\sin\frac{\theta}{2}(\cos\theta+1)\Big]$$

for the expressions corresponding to the first Pauli matrix σ_1. Then, the trigonometric identities (A.0.4) and (A.0.5) yield

$$\frac{i}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\Big(\lambda+\frac{1}{\lambda}\Big)\Big[2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\Big]=0.$$

Further, the expressions corresponding to the second Pauli matrix σ_2 are given by

$$-\frac{i\alpha}{4}\Big[\lambda\sin\theta-\frac{1}{\lambda}\sin\theta\Big]-\frac{i}{4}\Big(\lambda-\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}\Big[(\theta_t-\theta_x)+(\theta_t-\theta_x)\Big].$$

If we utilize the trigonometric identity (A.0.4), we obtain

$$-\frac{i}{2}\Big(\lambda-\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}\Big[\alpha\sin\frac{\theta}{2}+(\theta_t-\theta_x)\Big].$$

By the expression corresponding the second Pauli matrix of the derivative with respect to t of the boundary matrix, we see that the term $-\frac{i\theta_t}{2}\left(\lambda-\frac{1}{\lambda}\right)\cos\frac{\theta}{2}$ is the same on each side of the equality and therefore, we are left with

$$-\frac{i}{2}\left(\lambda-\frac{1}{\lambda}\right)\cos\frac{\theta}{2}\left[\alpha\sin\frac{\theta}{2}-\theta_x\right]=0. \tag{A.4.2}$$

On the other hand, analyzing the expressions corresponding to the third Pauli matrix σ_3 yields

$$-\frac{i\alpha}{4}\left[\left(\lambda\cos\theta+\frac{1}{\lambda}\right)-\left(\frac{1}{\lambda}\cos\theta+\lambda\right)\right]+\frac{i}{4}\left(\lambda-\frac{1}{\lambda}\right)\sin\frac{\theta}{2}\left[(\theta_t-\theta_x)+(\theta_t-\theta_x)\right].$$

Again, the application of the trigonometric identity (A.0.6) then implies

$$-\frac{i}{2}\left(\lambda-\frac{1}{\lambda}\right)\sin\frac{\theta}{2}\left[-\alpha\sin\frac{\theta}{2}-\theta_t+\theta_x\right].$$

Together with the expression corresponding to the same Pauli matrix of the t derivative of the boundary matrix, we see that the terms $\frac{i\theta_t}{2}\left(\lambda-\frac{1}{\lambda}\right)\sin\frac{\theta}{2}$ on both sides cancel. Therefore, we obtain

$$-\frac{i}{2}\left(\lambda-\frac{1}{\lambda}\right)\sin\frac{\theta}{2}\left[-\alpha\sin\frac{\theta}{2}+\theta_x\right]=0. \tag{A.4.3}$$

As for the Dirichlet boundary condition, the combination of the two equalities (A.4.2) and (A.4.3) is equivalent to the sin-boundary condition, even though the the first and second equality are also satisfied if $\cos\frac{\theta}{2}=0$ and $\sin\frac{\theta}{2}=0$ hold, respectively.

Lastly, in the case of the cos-boundary condition, the right hand side of the symmetry relation can be calculated as

$$\begin{aligned}
\mathbb{V}(t,0,\lambda^{-1})\mathbb{K}(t,0,\lambda)=&-\frac{\alpha}{4}\left[(\theta_t-\theta_x)\mathbb{1}+\lambda\sin\theta(-i\sigma_3)+\left(\lambda\cos\theta+\frac{1}{\lambda}\right)i\sigma_2\right]\\
&+\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)\cos\frac{\theta}{2}\left[(\theta_t-\theta_x)(-i\sigma_2)+\lambda\sin\theta i\sigma_1+\left(\lambda\cos\theta+\frac{1}{\lambda}\right)\mathbb{1}\right]\\
&+\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)\sin\frac{\theta}{2}\left[(\theta_t-\theta_x)i\sigma_3+\lambda\sin\theta\mathbb{1}+\left(\lambda\cos\theta+\frac{1}{\lambda}\right)(-i\sigma_1)\right],\\
-\mathbb{K}(t,0,\lambda)\mathbb{V}(t,0,\lambda)=&+\frac{\alpha}{4}\left[(\theta_t-\theta_x)\mathbb{1}+\frac{1}{\lambda}\sin\theta i\sigma_3+\left(\frac{1}{\lambda}\cos\theta+\lambda\right)(-i\sigma_2)\right]\\
&-\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)\cos\frac{\theta}{2}\left[(\theta_t-\theta_x)i\sigma_2+\frac{1}{\lambda}\sin\theta(-i\sigma_1)+\left(\frac{1}{\lambda}\cos\theta+\lambda\right)\mathbb{1}\right]\\
&-\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)\sin\frac{\theta}{2}\left[(\theta_t-\theta_x)(-i\sigma_3)+\frac{1}{\lambda}\sin\theta\mathbb{1}+\left(\frac{1}{\lambda}\cos\theta+\lambda\right)i\sigma_1\right].
\end{aligned}$$

For the expressions corresponding to the identity matrix $\mathbb{1}$, we have, after the obvious cancellation of the derivatives of θ, the following

$$\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)\cos\frac{\theta}{2}\left[\left(\lambda\cos\theta+\frac{1}{\lambda}\right)-\left(\frac{1}{\lambda}\cos\theta+\lambda\right)\right]+\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)\sin\frac{\theta}{2}\left[\lambda\sin\theta-\frac{1}{\lambda}\sin\theta\right].$$

Using the trigonometric identities (A.0.4) and (A.0.6) for $\sin\theta$ and $\cos\theta-1$ yields

$$\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)\left(\lambda-\frac{1}{\lambda}\right)\left[-2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]=0.$$

Then, for the expressions corresponding to the first Pauli matrix σ_1, we find

$$\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}\Big[\lambda\sin\theta-\frac{1}{\lambda}\sin\theta\Big]-\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\sin\frac{\theta}{2}\Big[\Big(\lambda\cos\theta+\frac{1}{\lambda}\Big)+\Big(\frac{1}{\lambda}\cos\theta+\lambda\Big)\Big],$$

which under the trigonometric identities (A.0.4) and (A.0.5) equates to

$$\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\Big(\lambda-\frac{1}{\lambda}\Big)\Big[2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\Big]=0.$$

Subsequently, the expressions corresponding to the second Pauli matrix σ_2 are given by

$$-\frac{i\alpha}{4}\Big[\Big(\lambda\cos\theta+\frac{1}{\lambda}\Big)+\Big(\frac{1}{\lambda}\cos\theta+\lambda\Big)\Big]-\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}\Big[(\theta_t-\theta_x)+(\theta_t-\theta_x)\Big].$$

By the trigonometric identity (A.0.5) for $\cos\theta+1$, we derive

$$-\frac{i}{2}\Big(\lambda+\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}\Big[\alpha\cos\frac{\theta}{2}+\theta_t-\theta_x\Big].$$

After cancelling the term $-\frac{i\theta_t}{2}\Big(\lambda+\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}$ involving the factor θ_t with the derivative of the boundary matrix on the left hand side of the equality, we are left with

$$-\frac{i}{2}\Big(\lambda+\frac{1}{\lambda}\Big)\cos\frac{\theta}{2}\Big[\alpha\cos\frac{\theta}{2}-\theta_x\Big]=0. \tag{A.4.4}$$

Finally, we mention the expressions corresponding to the third Pauli matrix σ_3:

$$\frac{i\alpha}{4}\Big[\lambda\sin\theta+\frac{1}{\lambda}\sin\theta\Big]+\frac{i}{4}\Big(\lambda+\frac{1}{\lambda}\Big)\sin\frac{\theta}{2}\Big[(\theta_t-\theta_x)+(\theta_t-\theta_x)\Big],$$

which can be written as

$$\frac{i}{2}\Big(\lambda+\frac{1}{\lambda}\Big)\sin\frac{\theta}{2}\Big[\alpha\cos\frac{\theta}{2}+\theta_t-\theta_x\Big]$$

with the trigonometric identity (A.0.4) for $\sin\theta$. Noticing that $\frac{i\theta_t}{2}\Big(\lambda+\frac{1}{\lambda}\Big)\sin\frac{\theta}{2}$ is cancelled with the same term on the left hand side of the equality with the time derivative of the boundary matrix, this can be reduced to

$$\frac{i}{2}\Big(\lambda+\frac{1}{\lambda}\Big)\sin\frac{\theta}{2}\Big[\alpha\cos\frac{\theta}{2}-\theta_x\Big]=0. \tag{A.4.5}$$

As for the other two boundary conditions, the combination of equalities (A.4.4) and (A.4.5) leads to the equivalence of the cos-boundary condition to the symmetry relation with the respective boundary matrix $\mathbb{K}(t,0,\lambda)$. □

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