On the Lifetime of a Conditioned Brownian Motion

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vorgelegt von
Matthias Erven
aus München

Berichterstatter:	Prof. Dr. Guido Sweers
	Prof. Dr. Bernd Kawohl
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Abstract

Let $\mathbb{E}_x^y(\tau_\Omega)$ denote the lifetime of a Brownian motion starting at $x \in \Omega$, conditioned to be killed at the boundary and to go to $y \in \Omega$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain. It has been conjectured that the lifetime gets maximal for x and y both being boundary points. In this thesis, we give a counterexample and show that there is a multiply connected domain, where the maximal lifetime occurs for interior points but not for boundary points. This domain, which we consider in Part II, consists of several subdomains that are connected through small gaps.

In Part I, we show how to estimate the lifetime on domains like the one considered in Part II. Let $\Omega_l \subset \mathbb{R}^2$ be a domain which is divided into two subdomains A and B by a path with several gaps of size l. We show the asymptotic behaviour of $\mathbb{E}_x^y(\tau_{\Omega_l})$ for $l \to 0$, where the limits obtained are functions of certain lifetimes on A and B.

Kurzzusammenfassung

Sei $\Omega \subset \mathbb{R}^2$ ein beschränktes Gebiet. $\mathbb{E}_x^y(\tau_\Omega)$ bezeichne die Lebensdauer einer Brownschen Bewegung, die in $x \in \Omega$ startet und konditioniert ist, am Rande zu sterben und nach $y \in \Omega$ zu laufen. Es wurde vermutet, dass maximale Lebensdauer für Randpunkte x und y auftritt. In dieser Doktorarbeit geben wir ein Gegenbeispiel und zeigen, dass es eine mehrfach zusammenhängende Menge gibt, wo die maximale Lebensdauer für innere Punkte, aber nicht für Randpunkte auftritt. Dieses Gebiet, das in Teil II betrachtet wird, besteht aus mehreren Teilgebieten, die durch schmale Öffnungen verbunden sind.

In Teil I zeigen wir, wie die Lebensdauer auf Gebieten wie dem aus Teil II abgeschätzt werden kann. Sei $\Omega_l \subset \mathbb{R}^2$ ein Gebiet, das durch einen Pfad mit mehreren Öffnungen der Größe l in zwei Teilgebiete A und B unterteilt wird. Wir zeigen das asymptotische Verhalten von $\mathbb{E}_x^y(\tau_{\Omega_l})$ für $l \to 0$, wobei die erhaltenen Grenzwerte Funktionen von Lebensdauern auf A und B sind.

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Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a domain and let $x, y \in \Omega$. The average lifetime of a Brownian motion that starts at x and is conditioned to go to y and to be killed at the boundary is given by

$$\mathbb{E}_x^y\left(\tau_\Omega\right) = \int_\Omega \frac{G_\Omega\left(x,z\right) G_\Omega\left(z,y\right)}{G_\Omega\left(x,y\right)}\, dz, \tag{1.1}$$

where G_Ω stands for the Green function of the Laplace operator with Dirichlet boundary conditions.

Conditioned Brownian motion originates in the concept of h-conditional Brownian motion, which was introduced by Doob in 1957, see [12]. In this concept, $h : \Omega \to \mathbb{R}$ stands for a positive harmonic function. It is a kind of weight put to the transition density of standard Brownian motion, so the average time that h-conditional Brownian motion starting in x spends in Ω is given by

$$\mathbb{E}_x^h\left(\tau_\Omega\right) = \int_\Omega G_\Omega\left(x,z\right) \frac{h\left(z\right)}{h\left(x\right)}\, dz,$$

whereas the expected lifetime of standard Brownian motion is simply given by

$$\mathbb{E}_x\left(\tau_\Omega\right) = \int_\Omega G_\Omega\left(x,z\right)\, dz.$$

In order to get to (1.1), we replace Ω by $\Omega \setminus B_\varepsilon\left(y\right)$ for small $\varepsilon > 0$, where $B_\varepsilon\left(y\right)$ stands for the ball with centre y and radius ε. We consider special $h_\varepsilon : \Omega \setminus B_\varepsilon\left(y\right) \to \mathbb{R}$ which satisfy

$$\begin{cases} -\Delta h_\varepsilon = 0 & \text{in } \Omega \setminus B_\varepsilon\left(y\right), \\ h_\varepsilon = 0 & \text{on } \partial\Omega, \\ h_\varepsilon = 1 & \text{on } \partial B_\varepsilon\left(y\right). \end{cases}$$

One can show that

$$\lim_{\varepsilon \to 0} \mathbb{E}_x^{h_\varepsilon}\left(\tau_{\Omega \setminus B_\varepsilon(y)}\right) = \int_\Omega \frac{G_\Omega\left(x,z\right) G_\Omega\left(z,y\right)}{G_\Omega\left(x,y\right)}\, dz. \tag{1.2}$$

We call this limit the average lifetime of a Brownian motion starting at x, conditioned to go to y and to be killed at the boundary and write

$$\mathbb{E}_x^y\left(\tau_\Omega\right) := \lim_{\varepsilon \to 0} \mathbb{E}_x^{h_\varepsilon}\left(\tau_{\Omega \setminus B_\varepsilon(y)}\right). \tag{1.3}$$

In 1983, Cranston and McConnell [8] proved that there is a universal constant such that

$$\mathbb{E}_x^h\left(\tau_\Omega\right) \leq C\left|\Omega\right|, \tag{1.4}$$

where $|\Omega|$ stands for the Lebesgue measure of Ω. Combining (1.2) and (1.4), we see that the same upper bound holds for $\mathbb{E}_x^y(\tau_\Omega)$. Hence the lifetime of our conditioned Brownian motion in a bounded domain is finite, and there is an upper bound independent of the position of x and y.

As $\mathbb{E}_x^y(\tau_\Omega)$ depends continuously on x and y for smooth domains, even up to the boundary, one could ask, where the points x and y have to be situated such that $\mathbb{E}_x^y(\tau_\Omega)$ gets maximal. Intuitively, at least in simply connected domains, one would think that the lifetime gets larger the farther the 'distance' between x and y, so one would expect that the maximal lifetime occurs at boundary points. This is an open conjecture, which has been formulated among others in [9], where the authors claim that for simply connected planar domains

$$\sup_{x,y\in\overline{\Omega}} \mathbb{E}_x^y(\tau_\Omega) = \sup_{x,y\in\partial\Omega} \mathbb{E}_x^y(\tau_\Omega)$$

holds. So far, the best result for general domains known to us has been obtained by Griffin, McConnell, and Verchota in their 1993 paper [17], where they showed that

$$\sup_{x\in\partial\Omega,y\in\Omega} \mathbb{E}_x^y(\tau_\Omega) = \sup_{x,y\in\partial\Omega} \mathbb{E}_x^y(\tau_\Omega).$$

For disks, the conjecture is true, which was shown by Dall'Acqua, Grunau and Sweers [9] in 2004. The proof of the conjecture for more general domains seems to be rather involved. In fact, it is shown in [14] that for a two-dimensional surface which resembles a fish bowl with small aperture, the maximal lifetime occurs for interior points.

The present work can be seen as a contribution to testing the limits of the conjecture, as we show that there is a multiply connected domain where the conjecture is not true, i.e., we show that this domain has two interior points such that the lifetime of our Brownian motion between those two points is larger than between all boundary points. This is done in Part II.

The domain we consider consists, roughly speaking, of two disks which are connected by a system of thin tubes, see Figure 1.1. A difficulty that occurs if one wants to compute

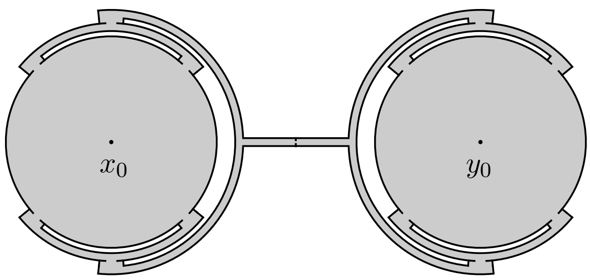


Figure 1.1.: A domain for which the lifetime of our Brownian motion between two interior points, namely x_0 and y_0, is larger than between boundary points.

the lifetime for some given domain Ω is that G_Ω is seldom known explicitly. However, there exists an explicit formula for disks. In Part I, we show how the lifetime of a domain consisting of two subdomains A and B can be computed, if we know the lifetimes on the subdomains. The subdomains are separated by a line with k small gaps of width l, which are centred around the points w_j for $j = 1, \ldots, k$. We are able to show that, for quite general domains Ω_l,

$$\lim_{l\to 0} \mathbb{E}_x^y \left(\tau_{\Omega_l}\right) = \mathbb{E}_x^y \left(\tau_A\right) \tag{1.5}$$

holds if both $x, y \in A$ and that

$$\lim_{l\to 0} \mathbb{E}_x^y \left(\tau_{\Omega_l}\right) = \sum_{j=1}^{k} \left(\mathbb{E}_x^{w_j} \left(\tau_A\right) + \mathbb{E}_{w_j}^y \left(\tau_B\right)\right) \frac{K_A\left(x, w_j\right) K_B\left(y, w_j\right)}{\sum_{m=1}^{k} K_A\left(x, w_m\right) K_B\left(y, w_m\right)}, \tag{1.6}$$

if $x \in A$ and $y \in B$. Here, K_A stands for the Poisson kernel on A. Both limits are uniform in x and y, as long as x and y keep a distance of $\rho > 0$ from the equal size gaps (and points where the boundary is not sufficiently smooth).

This work is structured as follows. Part I is the more theoretical one, where we study the lifetime on domains of type Ω_l. The main theorems there are Theorem 4.7, which gives (1.5), and Theorem 4.10, which gives (1.6). In order to be able to prove the theorems in Chapter 4, we need some preparation. In Chapter 2, we define the domain Ω_l more precisely and give some intuitive explanation for the convergence behaviour of $\mathbb{E}_x^y\left(\tau_{\Omega_l}\right)$. Chapter 3 is dedicated to the presentation of the concepts and results we will use later on. In this spirit, various definitions of boundary smoothness are reviewed in Section 3.1. Conformal mappings, the Riemann mapping theorem and some of its consequences are dealt with in Section 3.2. In Section 3.3, Green functions and their properties are presented thoroughly. In Section 3.4, we give a more detailed introduction to the concept of Brownian motion, though it is still far from being rigorous from the stochastic point of view. Finally, in Section 3.5, further results related to our studies are presented, trying to give a general context to the present work. Theorem 4.7 and Theorem 4.10 in Chapter 4 are formulated under the condition that x and y stay away from the gaps and boundary singularities at a distance of $\rho > 0$. What happens if they go closer? The results of Chapter 5 tell us that we only make a mistake of $C\rho^2$ if we do not consider the points close to the gaps and boundary singularities.

Being equipped with the tools of Part I, we are able to compute the lifetime for our special domain in Part II. We present the domain and properties of the lifetime there in Chapter 6, before we put everything together in Chapter 7, Theorem 7.1.

In the appendices, some supplemental material related to this work can be found. In Appendix A we prove that the limit in (1.2) holds and hence (1.3) is well-defined. A sub– and a supersolution of a special boundary value problem that comes up in the proof of Theorem 4.10 are presented in Appendix B.

Part I and Part II can be read more or less independently if one is willing to believe the results of Part I which are used in Part II. All the results in this work are obtained analytically, although the probabilistic view gives a good intuition.

It would be interesting to see if there are analogous results for three dimensions and higher. Theorem 4.7 and Theorem 4.10 are shown for quite general domains concerning boundary smoothness, transforming them to smoother domains with the help of conformal mappings. However, in higher dimensions, there are far less conformal mappings:

According to Liouville's theorem, a conformal mapping in dimension 3 or greater is either a translation, a similarity, an orthogonal transformation, an inversion, or a composition of those. Hence a proof of Theorem 4.7 and Theorem 4.10 for general domains would need other techniques. Moreover, the counterexample of Part II uses that the lifetime of our conditioned Brownian motion is small on domains with small area, as a consequence of (1.4). This result holds also only for planar domains, a counterexample for dimension 3 can be found in [8, Section 3].

Part I.

On the lifetime of a conditioned Brownian motion on a domain divided into two subdomains by a path with small gaps

2. The setting

2.1. The domain

We consider a planar domain bounded by a finite number of disjoint Jordan curves[1]. Let Γ be a Jordan arc which cuts the domain into two subdomains A and B. We assume that both domains are also bounded by a finite number of disjoint Jordan curves. Moreover, let w_j, $j = 1, \dots, k$, be some points of Γ within the domain. By starting at w_j and following Γ along an arc of length $\frac{l}{2}$ (with $l > 0$) in each direction, we move along the section of Γ which we denote $\Gamma_{j,l}$ and call a *gap*. We assume l to be so small such that the gaps $\Gamma_{j,l}$ do not intersect. We set

$$\Omega_l := A \cup B \cup \left(\bigcup_{j=1}^{k} \Gamma_{j,l} \right).$$

To sum up, Ω_l can be regarded as a domain which is cut into two pieces by a path which has gaps of width l. See Figure 2.1.

Finally, we make some smoothness assumptions. We assume that ∂A and ∂B are Dini smooth except for a set $S \subset (\partial A \cup \partial B)$. This set has to be neither finite nor of measure zero. Moreover, near the points w_j, $j = 1, \dots, k$, we ask Γ to be analytic.[2]

2.2. The results

The domain Ω_l is roughly speaking divided into two subdomains A and B by a path Γ that has gaps of width l.

Now, let $x, y \in A$. We imagine particles that move according to the rules of Brownian motion, starting at x and conditioned to be killed at the boundary of Ω_l and to go to y. If the gaps get smaller, it is harder for the particles to leave A without being killed at the boundary of Ω_l. It is nearly impossible for them to leave A, spend some time in B and then find a way back to A without touching the boundary. That is why, intuitively, we expect that

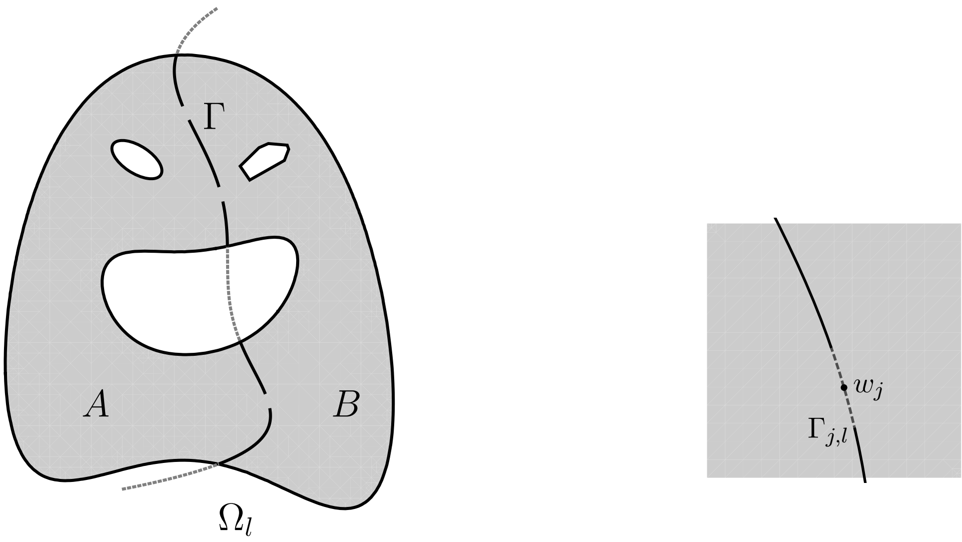
$$\mathbb{E}_x^y\left(\tau_{\Omega_l}\right) \to \mathbb{E}_x^y\left(\tau_A\right) \text{ for } l \to 0.$$[3]

What do we expect if x and y are in different subdomains, say $x \in A$ and $y \in B$? If there is only one gap, the particles have to get through the gap at least once. Again,

[1]For the definitions concerning boundary properties, see Section 3.1.

[2]That implies that for small $l > 0$, $S \subset \partial\Omega_l$ independent of l, as there is no boundary singularity near the gaps.

[3]For an interpretation of $\mathbb{E}_x^y\left(\tau_{\Omega_l}\right)$, see Section 3.4.2.



Ω_l, its subdomains A and B, and the path Γ. A close-up of a gap $\Gamma_{j,l}$.

Figure 2.1.: An example of a domain in our setting.

for a small gap width it seems improbable that a particle which once left A manages to reenter A and then leave it again in order to get to $y \in B$ without being killed. We expect that

$$\mathbb{E}_x^y\left(\tau_{\Omega_l}\right) \to \mathbb{E}_x^{w_1}\left(\tau_A\right) + \mathbb{E}_{w_1}^y\left(\tau_B\right) \text{ for } l \to 0. \tag{2.1}$$

And if there are several gaps of equal size? As each particle has to exit A through one of the gaps, the lifetime should be some average over the lifetimes obtained for the single-gap situation. However, there should be some weight which takes into account that a particle is more likely to leave through a closer gap. It turns out that, for the gap width going to zero, this weight is given by

$$\frac{K_A\left(x, w_j\right) K_B\left(y, w_j\right)}{\sum_{m=1}^k K_A\left(x, w_m\right) K_B\left(y, w_m\right)}.$$

Hence

$$\mathbb{E}_x^y\left(\tau_{\Omega_l}\right) \to \sum_{j=1}^k \left(\mathbb{E}_x^{w_j}\left(\tau_A\right) + \mathbb{E}_{w_j}^y\left(\tau_B\right)\right) \frac{K_A\left(x, w_j\right) K_B\left(y, w_j\right)}{\sum_{m=1}^k K_A\left(x, w_m\right) K_B\left(y, w_m\right)} \text{ for } l \to 0. \tag{2.2}$$

After the preparatory work of Chapter 3, we prove the limits of (2.1) and (2.2) in Chapter 4.

3. Preliminaries

In this chapter, we present basic facts and definitions that will be used when formulating and proving the results. Section 3.1 deals with definitions of boundary smoothness, in Section 3.2, the Riemann mapping theorem and corollaries for A and B of our setting are presented. Section 3.3 is about Green functions and Poisson kernels, which appear in the formula for the lifetime of our conditioned Brownian motion, the concept of which is introduced in Section 3.4. In Section 3.5, we present further results on conditioned Brownian motion in order to put this work into a greater context.

Whenever the facts presented in this section hold in a general context, we will use the letters G or Ω for the domains. If we draw conclusions in our special setting, we will formulate the results for the domains A (and B) directly.

3.1. Definitions of boundary smoothness

By considering $\mathbf{z} = z_1 + \mathbf{i}z_2$ instead of $z = (z_1, z_2)$, points, sets and functions in $\mathbb{R}^2$ can be viewed as points, sets and functions in $\mathbb{C}$. It will turn out to be very useful sometimes to switch from the real point of view to the complex point of view and back. We will use boldface for complex points, sets and functions and normal type face for their real 2-D counterparts.

The boundary regularity and existence theorems of the following sections depend on the smoothness of the boundary. The definitions for boundary smoothness from the complex point of view (presented, e.g., in [26]) differ from those made in $\mathbb{R}^n$ (as they can be found in [15], [16], and [30], for instance). However, they are equivalent. For the sake of completeness, we present the definitions for both contexts in this section.

We start with the complex point of view. According to the definitions of [26, Section 1.1.3], a *Jordan arc* is a continuous curve which is injective (except for the starting point and endpoint, maybe). That is, a Jordan arc is the image of a continuous function $\boldsymbol{\varphi} : [a, b] \to \mathbb{C}$ which is injective on $[a, b)$ and on $(a, b]$. If $\boldsymbol{\varphi}(a) = \boldsymbol{\varphi}(b)$, i.e., if the arc is closed, the Jordan arc is called *Jordan curve.* According to the Jordan curve theorem ([26, p.2]), a Jordan curve divides the complex plane into two components, one of them being bounded. A *Jordan domain* is a domain which is bounded by a Jordan curve.

We say that a domain $\boldsymbol{\Omega} \subset \mathbb{C}$ that is bounded by a finite number of disjoint Jordan curves has a $C^{n,\alpha}$ *boundary near* $\mathbf{z} \in \partial\boldsymbol{\Omega}$, if, locally, there is a boundary parametrisation $[t_0 - \varepsilon, t_0 + \varepsilon] \ni t \mapsto \boldsymbol{\varphi}(t)$ with $\boldsymbol{\varphi}(t_0) = \mathbf{z}$ which belongs to the Hölder space $C^{n,\alpha}([t_0 - \varepsilon, t_0 + \varepsilon]; \mathbb{C})$ and satisfies $\boldsymbol{\varphi}'(t) \neq 0$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. Moreover, we say that the boundary is *analytic near* $\mathbf{z} \in \partial\boldsymbol{\Omega}$ if $\boldsymbol{\varphi}$ is analytic, i.e., it has a power series expansion on $[t_0 - \varepsilon, t_0 + \varepsilon]$ and satisfies $\boldsymbol{\varphi}'(t) \neq 0$.

From the real point of view, we say that a domain $\Omega \subset \mathbb{R}^2$ has a $C^{n,\alpha}$ *boundary near* $z \in \partial\Omega$, if there is some $r > 0$ and a function $\varphi \in C^{n,\alpha}([a,b])$, such that, perhaps after relabeling and reorienting the coordinate axes, $\Omega \cap B_r(z) = \{x = (x_1, x_2) \in B_r(z); x_2 > \varphi(x_1)\}$. It is called *analytic near* $z \in \partial\Omega$, if φ is analytic. We refer to [15, Appendix C].

Both definitions are equivalent in the following way. Let $\boldsymbol{\varphi}$ be a boundary parametrisation in the complex sense with $\boldsymbol{\varphi}(t_0) = \mathbf{z}$. As $\boldsymbol{\varphi}'(t_0) \neq 0$, either $\mathrm{Re}\boldsymbol{\varphi}(t_0)$ or $\mathrm{Im}\boldsymbol{\varphi}(t_0)$ is locally invertible by the inverse function theorem. If, for instance, the real part is invertible, then $x_1 \mapsto \mathrm{Im}\boldsymbol{\varphi}\left((\mathrm{Re}\boldsymbol{\varphi})^{inv}(x_1)\right)$ satisfies the real point of view definition and has the same smoothness. As the boundary is given by Jordan curves, the set Ω only lies on one side of the boundary. Conversely, if the real counterpart Ω of a domain $\boldsymbol{\Omega}$ that is bounded by Jordan curves has a $C^{n,\alpha}$ or analytic boundary near $z \in \partial\Omega$ in the real sense, the function $t \mapsto t + \mathbf{i}\varphi(t)$ (or, maybe, $t \mapsto \varphi(t) + \mathbf{i}t$) parametrises the complex boundary in an equally smooth way, with the first derivative being nonzero.

Now we turn to the concept of Dini smoothness. The Poisson kernel exists at boundary parts that are Dini smooth, see Section 3.3.3. The definitions given here can be found in a more general form in [26, Section 3.3]. Let $\boldsymbol{\varphi} : [a,b] \to \mathbb{C}$ be a continuous function. We define its *modulus of continuity* by

$$\omega(\delta) := \sup\{|\boldsymbol{\varphi}(t_1) - \boldsymbol{\varphi}(t_2)|; t_1, t_2 \in [a,b], |t_1 - t_2| < \delta\}.$$

The function $\boldsymbol{\varphi}$ is called *Dini continuous* if

$$\int_0^\pi t^{-1}\omega(t)\, dt < \infty.$$

Here, the upper bound π is an arbitrary choice, it could be any other positive number. If a function is Hölder continuous, the integrand is bounded. Hence Hölder continuity implies Dini continuity. Finally, a boundary of a complex domain $\boldsymbol{\Omega}$ is said to be *Dini smooth near* $\mathbf{z} \in \partial\boldsymbol{\Omega}$, if it has a parametrisation $t \mapsto \boldsymbol{\varphi}(t)$ around $\mathbf{z}$ such that $\boldsymbol{\varphi}'$ exists, is Dini continuous and nowhere zero. Summing up, one can say that Dini smoothness is more than C^1, but less than $C^{1,\alpha}$ for all $\alpha > 0$.

3.2. Riemann mapping theorem and conformal maps

An important tool in the analysis of harmonic functions and Green functions in two dimensions are *conformal maps*. These are injective meromorphic functions from one complex domain onto another. For basic properties, see [26, Section 1.2, Subsection 1], where the author further refers to [1, Chapter 3, Section 2]. We recall the

Theorem 3.1 (Riemann mapping theorem, cf. [26, Section 1.2])**.** *Let* $\mathbf{G} \subsetneqq \mathbb{C}$ *be a simply connected domain and let* $\mathbf{w}_0 \in \mathbf{G}$, $0 \leq a < 2\pi$. *Then there is a unique conformal map* $\mathbf{f}$ *of the open unit disk* $\mathbb{D}$ *onto* $\mathbf{G}$ *such that* $\mathbf{f}(0) = \mathbf{w}_0$ *and* $\arg \mathbf{f}'(0) = a$.

Every conformal map $\mathbf{f} : \mathbb{D} \to \mathbb{D}$ is of the form

$$\mathbf{f}(\mathbf{z}) = e^{\mathbf{i}a}\frac{\mathbf{z} - \mathbf{z}_0}{1 - \overline{\mathbf{z}_0}\mathbf{z}}$$

with some $\mathbf{z}_0 \in \mathbb{D}$, $0 \leq a < 2\pi$. Such a map is uniquely determined, for instance, if the image of three boundary points is given (note that the image points have to have the same cyclic order). Moreover, there is

Theorem 3.2 (Carathéodory theorem, cf. [26, Theorem 2.6]). *Let* $\mathbf{f}$ *map* $\mathbb{D}$ *conformally onto the bounded domain* $\mathbf{G}$. *Then the following two conditions are equivalent:*

1. $\mathbf{f}$ *has a continuous injective extension to* $\overline{\mathbb{D}}$.
2. $\partial\mathbf{G}$ *is a Jordan curve.*

By first mapping Jordan domains conformally onto the unit disk and then constructing the right conformal mapping of the disk onto itself, one can prove the following corollary.

Corollary 3.3 (cf. [26, Corollary 2.7]). *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be Jordan domains (i.e., bounded by a Jordan curve) and let the points* $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \in \partial\mathbf{G}$ *and* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \partial\mathbf{H}$ *have the same cyclic order. Then there is a unique conformal map* $\mathbf{f}$ *of* $\mathbf{G}$ *onto* $\mathbf{H}$ *that satisfies*

$$\mathbf{f}(\mathbf{z}_j) = \mathbf{v}_j \qquad \textit{for } j = 1, 2, 3.$$

What happens if the domain is multiply connected? In general, it is not possible to map a multiply connected domain conformally onto another given domain with the same number of holes. In fact, it is not even possible to map the annulus $\mathbb{A}_{R_1,1}(0) := \{\mathbf{z} \in \mathbb{C}; R_1 < |\mathbf{z}| < 1\}$ conformally onto the annulus $\mathbb{A}_{R_2,1}(0)$ with $R_1 \neq R_2$. There are several ways to show this. In [25, Section VII.1] Nehari gives a proof by contradiction, assuming that there exists a conformal map $\mathbf{f} : \mathbb{A}_{R_1,1} \to \mathbb{A}_{R_2,1}$. He considers the analytic function $\mathbf{g}(\mathbf{z}) = \log R_1 \log \mathbf{f}(\mathbf{z}) - \log R_2 \log \mathbf{z}$ and finally concludes that $R_1 = R_2$.

Another way to show that there is no conformal map $\mathbf{f} : \mathbb{A}_{R_1,1} \to \mathbb{A}_{R_2,1}$ is making use of the extremal length concept. This concept allows us to compute the so-called extremal distance between the two boundary circles of $\mathbb{A}_{R_1,1}(0)$, which is $-\frac{1}{2\pi}\log R_1$. For the second annulus, we obtain $-\frac{1}{2\pi}\log R_2$. However, the extremal distance is invariant under conformal mappings, i.e., the extremal distance between the boundary circles in both annuli has to be the same if there exists a conformal mapping between the two annuli. Hence there cannot be a conformal mapping between the annuli. For details on the concept of extremal length, see [2, Section 4]. The extremal distance for the annulus can be found there in [2, Section 4.2].

To sum up, it is not possible to map the domain $\mathbf{A}$ of our setting conformally onto an arbitrary given domain of the same connectivity.[1] However, it will be enough to map

[1] For further reading: Is there a possibility to see if two given domains of connectivity $n > 2$ can be mapped conformally onto one another? In [25, Section VII], Nehari shows that for such a domain one needs $3n - 6$ real parameters, which he calls *Riemann moduli*, to determine the *conformal type* of the domain. Two domains of the same conformal type can be mapped conformally onto one another. Furthermore, is there any chance to write down such a conformal map explicitly? Concerning simply connected domains, the Schwarz-Christoffel formula is a tool to describe the mapping of a unit disk onto a polygonal domain with given angles. In recent years, some methods have been developed to construct a conformal mapping of a disk with n circular holes onto a polygonal domain with n polygonal holes and prescribed angles. For a survey, see [5].

the domain $\mathbf{A}$ (or $\mathbf{B}$, respectively) onto *some* other domain with equally many holes and a smooth boundary. In the following lemma, let $\mathbf{s}$ and $\mathbf{t}$ stand for two points of $\mathbf{\Gamma}$ such that all points $\mathbf{w}_j$ lie between $\mathbf{s}$ and $\mathbf{t}$.

Lemma 3.4. *There is a domain $\tilde{\mathbf{A}}$ and a conformal map $\mathbf{h} : \mathbf{A} \to \tilde{\mathbf{A}}$ such that the domain $\tilde{\mathbf{A}}$ has the following properties:*

- *$\tilde{\mathbf{A}}$ is a bounded Jordan domain and $\partial\tilde{\mathbf{A}}$ is C^{10}.*[2]
- *The part of $\mathbf{\Gamma}$ between $\mathbf{s}$ and $\mathbf{t}$ which contains all the $\mathbf{w}_j$ is mapped onto a straight line segment (which we choose to lie on the imaginary axis).*[3]

Moreover, the map $\mathbf{h} : \mathbf{A} \to \tilde{\mathbf{A}}$ has a continuous injective extension to $\overline{\mathbf{A}}$.

Proof. We follow the ideas of the proof of [7, Lemma 6.17] and sketch the construction here. It is illustrated by Figure 3.1.

The boundary of $\mathbf{A}$ is a finite union of Jordan curves. We start with one curve which does not contain the points $\mathbf{w}_j$, $j = 1, \dots, n$. If the curve is already smooth enough, we do nothing. If the curve is not smooth enough, we proceed as follows. According to the Jordan curve theorem ([26, p.2]), the curve splits the complex plain into two components, namely a bounded and an unbounded one. If $\mathbf{A}$ is contained in the bounded component, we can map the whole bounded component onto the unit disk according to the Riemann mapping theorem. In this way, we have achieved that the first part of the boundary is smooth. We observe that the other Jordan curves remain Jordan curves under the transformation, and that the images of the points $\mathbf{w}_j$ still lie between the images of $\mathbf{s}$ and $\mathbf{t}$. In case that $\mathbf{A}$ is contained in the unbounded component, we first apply the conformal map $\mathbf{z} \mapsto (\mathbf{z} - \mathbf{z}_0)^{-1}$, where $\mathbf{z}_0$ is an interior point of the bounded component. Then we can proceed as in the first case, as Jordan curves are still Jordan curves and the images of the points $\mathbf{w}_j$ still lie between the images of $\mathbf{s}$ and $\mathbf{t}$ (but in reverse order), and the image of $\mathbf{A}$ now lies in the bounded component.

Having smoothed out in this way the first Jordan curve of the boundary of $\mathbf{A}$, we proceed successively in the same way with the other boundary parts. We observe that each step does not decrease the smoothness of the boundary parts treated before. For the last curve we choose the Jordan curve which contains $\mathbf{s}$, $\mathbf{t}$, and $\mathbf{w}_j$, $j = 1, \dots, n$.

In the last step, we map the unit disk conformally onto a domain which has a C^{10} boundary and whose boundary is partly a straight line. This time we choose the mapping in such a way that the following requests are fulfilled: The images of the points $\mathbf{s}$ and $\mathbf{t}$ (obtained so far by the preceding steps) are mapped onto the first and the last point of the straight line part of the boundary, and the image of one (arbitrarily chosen) point $\mathbf{w}_{j_0}$ is mapped onto some point of the straight line.

We write $\mathbf{h}$ for the composition of all the mappings applied so far and observe that we have obtained a conformal map onto a domain $\tilde{\mathbf{A}}$ with the properties claimed in the lemma.

[2] We need this boundary regularity when applying the estimates on Green functions and Poisson kernels of Section 3.3.6.

[3] This can always be obtained by a rotation combined with a translation.

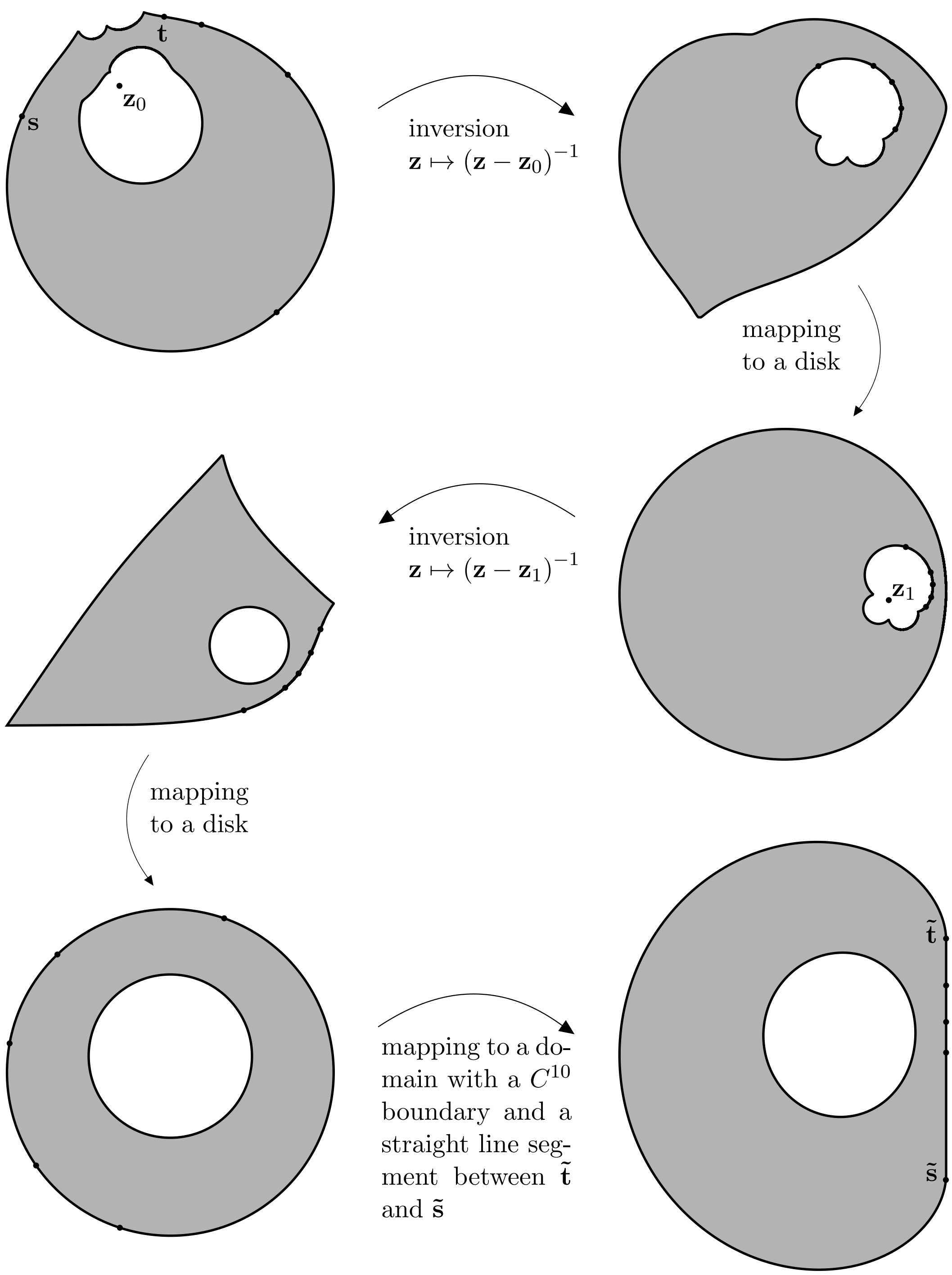


Figure 3.1.: Idea of the proof of Lemma 3.4: A conformal map of the top left domain onto the bottom right domain with the required boundary is obtained as a composition of inversions and mappings of Jordan domains onto circles.

Concerning the possibility to extend the map $\mathbf{h} : \mathbf{A} \to \tilde{\mathbf{A}}$, we make the following observation: In each step of the construction of $\mathbf{h}$, we have picked out one Jordan curve and then made use of a conformal map that maps the whole interior domain onto the unit disk. The other Jordan arcs lie in this interior domain, so if we restrict the map onto the domain bounded by all the Jordan arcs, it is clear that the restricted map can at least be extended to the interior Jordan arcs. It remains to study the behaviour of the map on the outer boundary. The Carathéodory theorem (see Theorem 3.2) states that the inverse of the whole map, namely the map of the unit disk onto the bounded Jordan domain, has a continuous injective extension up to the boundary. Hence also the original map can be extended. We finally observe that mappings of the form $\mathbf{z} \mapsto (\mathbf{z} - \mathbf{z}_0)^{-1}$ are continuous and injective on $\mathbb{C} \setminus \{\mathbf{z}_0\}$. We conclude that $\mathbf{h}$ as a composition of the maps used in the steps of the construction also has a continuous injective extension up to the boundary. □

If the domain $\mathbf{G}$ is bounded by Jordan curves, a conformal map $\mathbf{f} : \mathbf{G} \to \mathbf{H}$ to a domain with smooth boundary is continuous on the whole of $\overline{\mathbf{G}}$. If the boundary of $\mathbf{G}$ is smoother than just a Jordan curve, then also the derivatives of $\mathbf{f}$ have continuous extensions to the whole of $\overline{\mathbf{G}}$. In his book, Pommerenke presents two results.

Theorem 3.5 (cf. [26, Theorem 3.5]; shortened version)**.** *Let* $\mathbf{f}$ *map* $\mathbb{D}$ *conformally onto the interior domain of the Dini smooth Jordan curve* $\mathbf{C}$*. Then* $\mathbf{f}'$ *has a continuous extension to* $\overline{\mathbb{D}}$ *and*

$$\frac{\mathbf{f}(\boldsymbol{\xi}) - \mathbf{f}(\mathbf{z})}{\boldsymbol{\xi} - \mathbf{z}} \to \mathbf{f}'(\mathbf{z}) \neq 0 \text{ for } \boldsymbol{\xi} \to \mathbf{z},\ \boldsymbol{\xi}, \mathbf{z} \in \overline{\mathbb{D}}.$$

Remark 3.6. As a side note, we point out that it does not suffice that the boundary parametrisation $t \mapsto \boldsymbol{\varphi}(t)$ is continuously differentiable with $\boldsymbol{\varphi}'(t) \neq 0$. A counterexample is given in [26, Section 3.2, p.45].

Theorem 3.7 (Kellogg-Warschawski theorem, cf. [26, Theorem 3.6])**.** *Let* $\mathbf{f}$ *map* $\mathbb{D}$ *conformally onto the interior domain of the Jordan curve* $\mathbf{C}$ *of class* $C^{n,\alpha}$ *where* $n = 1, 2, \ldots$ *and* $0 < \alpha < 1$*. Then* $\mathbf{f}^{(n)}$ *has a continuous extension to* $\overline{\mathbb{D}}$ *and*

$$\left|\mathbf{f}^{(n)}(\mathbf{z}_1) - \mathbf{f}^{(n)}(\mathbf{z}_2)\right| \leq M \left|\mathbf{z}_1 - \mathbf{z}_2\right|^{\alpha}.$$

Both results are global in the sense that regularity of the boundary *everywhere* is assumed. Pommerenke refers to works of Kellogg [21] and Warschawski [29], [28]. However, in [29], the existence of a first derivative extension is proven locally under the assumption that the boundary is sufficiently smooth *near a boundary point.* Moreover, it is not important either that the domain is simply connected, being the image of $\mathbb{D}$. We refer to the proof of Lemma 3.4: there is a conformal map that maps a multiply connected domain to an equally connected subdomain of the unit disk. This map is a composition of inversions of the form $(\mathbf{z} - \mathbf{z}_0)^{-1}$ and conformal mappings of Jordan domains (bounded by *one* Jordan curve) to $\mathbb{D}$. The latter mappings are then restricted to subsets defined by the remaining Jordan curves. *Only the boundary behaviour* of those latter mappings is important, as conformal maps are analytic in the interior of the domain. This is described by the two theorems above. Finally, the results also hold for a conformal map $\mathbf{f}$ between two domains $\mathbf{G}$ and $\mathbf{H}$ with equally smooth boundary. In fact, this map can be regarded as the composition of a conformal map $\mathbf{f}_1^{inv}$ of $\mathbf{G}$ to (a subset of) $\mathbb{D}$ and another map $\mathbf{f}_2 = \mathbf{f} \circ \mathbf{f}_1$ from (the same subset of) $\mathbb{D}$ to $\mathbf{H}$. Smoothness of $\mathbf{f}$ is a consequence of the above theorems and $\mathbf{f}_1'$ not getting zero at the boundary.

Corollary 3.8. *Let $\mathbf{f} : \mathbf{G} \to \mathbf{H}$ be a conformal map of a domain that is bounded by a finite number of Jordan curves to another such domain. Let $\mathbf{z}_0 \in \partial\mathbf{G}$. If $\partial\mathbf{G}$ is Dini smooth in a neighbourhood of $\mathbf{z}_0$ and $\partial\mathbf{H}$ is Dini smooth in a neighbourhood of $\mathbf{f}(\mathbf{z}_0)$, then*

$$\frac{\mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{z}_0)}{\mathbf{z} - \mathbf{z}_0} \to \mathbf{f}'(\mathbf{z}_0) \neq 0 \text{ for } \mathbf{z} \to \mathbf{z}_0,\ \mathbf{z} \in \overline{\mathbf{G}}.$$

So far we have presented results that state conditions under which a conformal map can be extended to the boundary. Now we look at the possibility of extending the map $\mathbf{h}$ from Lemma 3.4 even beyond the boundary at certain points. In [22, Section D.9], Koebe shows the following theorem using the Schwarz reflection principle.

Theorem 3.9. *Let $\mathbf{f}$ be a conformal map of a (multiply connected) complex domain $\mathbf{G}$ onto (a multiply connected subset of) the unit disk. If $\partial\mathbf{G}$ is analytic near some point $\mathbf{z}_0 \in \partial\mathbf{G}$, then there is a ball $B_R(\mathbf{z}_0)$ such that $\mathbf{f}$ is biholomorphic on $\mathbf{G} \cup B_R(\mathbf{z}_0)$.*

We can regard the conformal map of Lemma 3.4 as a composition of a conformal map of $\mathbf{A}$ onto a subset of the unit disk and then the inverse of a conformal map of $\tilde{\mathbf{A}}$ onto the same subset. We have assumed that $\partial\mathbf{A}$ is analytic near $\mathbf{w}_j$. Near the corresponding points $\tilde{\mathbf{w}}_\mathbf{j} = \mathbf{h}(\mathbf{w}_j)$, the boundary of $\tilde{\mathbf{A}}$ is straight and therefore analytic, too. This yields the following corollary.

Corollary 3.10. *There is some $R > 0$ such that $\mathbf{h}$ of Lemma 3.4 can be extended to a conformal map defined on $\mathbf{A} \cup \left(\bigcup_{j=1}^k B_R(\mathbf{w}_j)\right)$.*

Let $\rho > 0$ and $\mathbf{w} \in \overline{\mathbf{A} \cup \left(\bigcup_{j=1}^k B_{\frac{R}{2}}(\mathbf{w}_j)\right)} \setminus \bigcup_{\mathbf{s}\in\mathbf{S}} B_\rho(\mathbf{s})$, i.e., we assume that $\mathbf{w}$ is somewhere in the extended set $\overline{\mathbf{A} \cup \left(\bigcup_{j=1}^k B_{\frac{R}{2}}(\mathbf{w}_j)\right)}$ but stays away from the boundary singularities $\mathbf{S}$. We call this set $\overline{\mathbf{A}_{+\frac{R}{2},-\rho}}$ for a moment. Additionally, let $\mathbf{z} \in \overline{\mathbf{A}}$, $\mathbf{z} \neq \mathbf{w}$. We write $\tilde{\mathbf{z}} = \mathbf{h}(\mathbf{z})$ and $\tilde{\mathbf{w}} = \mathbf{h}(\mathbf{w})$. Then

$$|\tilde{\mathbf{z}} - \tilde{\mathbf{w}}| = \frac{|\tilde{\mathbf{z}} - \tilde{\mathbf{w}}|}{|\mathbf{z} - \mathbf{w}|} \cdot |\mathbf{z} - \mathbf{w}| \leq \sup_{\mathbf{v}_1 \in \overline{\mathbf{A}}, \mathbf{v}_2 \in \overline{\mathbf{A}_{+\frac{R}{2},-\rho}}} \frac{|\mathbf{h}(\mathbf{v}_1) - \mathbf{h}(\mathbf{v}_2)|}{|\mathbf{v}_1 - \mathbf{v}_2|} \cdot |\mathbf{z} - \mathbf{w}|\,.$$

We claim that the supremum is finite and that it just depends on R and ρ. Indeed, if $\mathbf{v}_1$ is close to $\mathbf{v}_2$, the quotient can be replaced by $|\mathbf{h}'(\mathbf{v}_3)|$ for $\mathbf{v}_3$ close to $\mathbf{v}_2$. As $\mathbf{h}$ is holomorphic in $\overline{\mathbf{A}_{+\frac{R}{2},-\rho}}$, this is bounded from above.[4] On the contrary, if $\mathbf{v}_1$ is not

[4] This is the reason why we assume $\mathbf{z}$ to stay away from the boundary singularities in $\mathbf{S}$. Near such a singularity, $|\mathbf{h}'|$ could get arbitrarily large.

close to $\mathbf{v}_2$ the quotient is bounded from above anyway. Hence our claim is true. As $\mathbf{h}$ is one-to-one, the supremum is greater than zero. To sum up, there exists some $C > 0$ such that

$$|\tilde{\mathbf{z}} - \tilde{\mathbf{w}}| \leq C \cdot |\mathbf{z} - \mathbf{w}| . \tag{3.1}$$

The same can be done for the inverse mapping, which yields a lower bound for $|\tilde{\mathbf{z}} - \tilde{\mathbf{w}}|$.

The gap width of $\mathbf{\Gamma}_{j,l}$ is l for each j. After the domain transformation under $\mathbf{h}$ the gap width of $\tilde{\mathbf{\Gamma}}_{j,l} = \mathbf{h}(\mathbf{\Gamma}_{j,l})$ may depend on j. However, computing the arclength and using the fact that $\mathbf{h}'$ is bounded and non-zero near $\mathbf{w}_j$, we obtain a similar estimate as in (3.1) for the gap width. We sum up the last results in a corollary.[5]

Corollary 3.11. *Let $\rho > 0$ and let $\mathbf{h} : \mathbf{A} \to \tilde{\mathbf{A}}$ be the conformal map of Lemma 3.4 and Corollary 3.10. There exists $C \geq 1$ such that*

$$\frac{1}{C} \cdot |\mathbf{z} - \mathbf{w}| \leq |\tilde{\mathbf{z}} - \tilde{\mathbf{w}}| \leq C \cdot |\mathbf{z} - \mathbf{w}| .$$

holds for all $\tilde{\mathbf{z}} = \mathbf{h}(\mathbf{z})$ with $\mathbf{z} \in \overline{A}$ and all $\tilde{\mathbf{w}} = \mathbf{h}(\mathbf{w})$ with $\mathbf{w} \in \overline{\mathbf{A} \cup \left(\bigcup_{j=1}^{k} B_{\frac{R}{2}}(\mathbf{w}_j)\right)} \setminus \bigcup_{\mathbf{s} \in \mathbf{S}} B_\rho(\mathbf{s})$.

Let $\tilde{l}_j$ be the width of the gap $\tilde{\mathbf{\Gamma}}_{j,l} = \mathbf{h}(\mathbf{\Gamma}_{j,l})$. There is some other $C \geq 1$ such that for all $j = 1, \ldots, k$

$$\frac{1}{C} \cdot l \leq \tilde{l}_j \leq C \cdot l. \tag{3.2}$$

To end this section, we derive a formula which describes the relation between l, $\tilde{l}_j$ and $\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|$, where $\tilde{\mathbf{h}}$ denotes the inverse of $\mathbf{h}$. For $j = 1, \ldots, k$ we get, using the transformation theorem and then Taylor expansion,

$$l = \int_{\tilde{\Gamma}_{j,l}} \left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}})\right| d\sigma(\tilde{w}) = \int_{\tilde{\Gamma}_{j,l}} \left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j) + \tilde{\mathbf{h}}''\left(\tilde{\xi}_{\tilde{\mathbf{w}}}\right)(\tilde{\mathbf{w}} - \tilde{\mathbf{w}}_j)\right| d\sigma(\tilde{w}),$$

where $\tilde{\xi}_{\tilde{w}}$ lies on the straight line between $\tilde{w}$ and $\tilde{w}_j$. As $\tilde{\mathbf{h}}''$ is bounded near $\tilde{\mathbf{w}}_j$ independently of l, with the help of (3.2) we get

$$l - C_1 l^2 \leq \left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right| \tilde{l}_j \leq l + C_1 l^2$$

for some $C_1 > 0$ independent of l (and j). We have obtained the following lemma.

Lemma 3.12. *Let $\mathbf{h} : \mathbf{A} \to \tilde{\mathbf{A}}$ be the conformal map of Lemma 3.4 and Corollary 3.10. There exist $C > 0$ and $l_1 > 0$, such that*

$$l - Cl^2 \leq \left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right| \tilde{l}_j \leq l + Cl^2$$

for all l with $0 < l < l_1$ and for all $j \in \{1, \ldots, k\}$.

[5] We claim the existence of a constant $C \geq 1$, as $0 < C < 1$ would not make sense in the way we formulate the inequalities.

3.3. Properties of Green functions and Poisson kernels

The expression for the lifetime of our conditioned Brownian motion contains the 2-D Green function for the Laplace equation. For the sake of completeness, we recall the definition and properties of Green functions. In Section 3.3.1 we present the general idea of the construction of the Green function for the Laplace operator on a bounded domain $\Omega \subset \mathbb{R}^2$, as it can be found in [15] or [30]. We present adapted versions for $\mathbb{R}^2$ here. The existence of such a Green function and its regularity depends on the boundary of Ω. In Section 3.3.2 we show that in our setting the Green functions G_A, G_B and G_{Ω_l} exist as continuous functions up to the boundary. Also, the Poisson kernel exists where it is needed, which we show in Section 3.3.3. In Section 3.3.4 basic properties of the Green function are presented, whereas Sections 3.3.6 to 3.3.8 deal with estimates on the shape of Green functions and their derivatives.

3.3.1. Definition

The function $\Phi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, defined by

$$\Phi(x) = -\frac{1}{2\pi} \log |x| \,,$$

is called a *fundamental solution*.[6] It is *harmonic* away from 0, i.e., $\Delta\Phi(x) = 0$ if $x \neq 0$.

If the boundary of a domain $\Omega \subset \mathbb{R}^2$ is sufficiently smooth (see Section 3.3.2), then for fixed $x \in \Omega$ there exists, by Perron's method, a unique solution Φ^x of

$$\begin{cases} -\Delta\Phi^x(y) = 0 & \text{for } y \in \Omega, \\ \Phi^x(y) = \Phi(y - x) & \text{for } y \in \partial\Omega. \end{cases} \tag{3.3}$$

We define the *Green function*[7]

$$G_\Omega(x, y) := \Phi(y - x) - \Phi^x(y).$$

Let $y \in \partial\Omega$. Assume that the outer normal derivative of the Green function exists at y. We call

$$K_\Omega(x, y) := -\frac{\partial}{\partial n_y} G_\Omega(x, y) \tag{3.4}$$

the *Poisson kernel*. By means of the Green function and the Poisson kernel, the unique solution of the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

[6]In three or more dimensions, $-\Delta\Phi(x) = \delta_0(x)$ (where δ_0 is the Dirac measure) together with the condition $\lim_{|x|\to\infty} \Phi(x) = 0$ defines a unique fundamental solution. In two dimensions, this condition cannot be fulfilled, as $\lim_{|x|\to\infty} (-\log |x|) = -\infty$. Hence there is no unique fundamental solution, one could always add a constant to Φ.

[7]If we had added a constant to Φ, this would cancel out here, see [6].

is found by

$$u(x) = \int_\Omega G_\Omega(x,y)f(y)\,dy + \int_{\partial\Omega} K_\Omega(x,y)g(y)\,d\sigma(y). \tag{3.6}$$

If $\Omega = B_R(x_0)$ is a ball, Green function and Poisson kernel are explicitly known:

$$G_{B_R(x_0)}(x,y) = \frac{1}{4\pi}\log\left(1 + \frac{\left(R^2 - |x-x_0|^2\right)\left(R^2 - |y-x_0|^2\right)}{R^2|x-y|^2}\right), \tag{3.7}$$

$$K_{B_R(x_0)}(x,y) = \frac{1}{2\pi R}\frac{R^2 - |x-x_0|^2}{|x-y|^2}. \tag{3.8}$$

3.3.2. Existence of the Green function in our setting

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. The Green function exists if (3.3) has a solution. That is why we look for an answer to the following more general question: When does the *Dirichlet problem for the Laplace equation*,

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases} \tag{3.9}$$

have a solution? Before answering that question, we introduce the concept of superharmonic functions and regular boundary points and present Perron's method. The definition of superharmonic functions is motivated by the following theorem.

Theorem 3.13 (cf. [30, Satz 3.2.3])**.** *Let $u : \Omega \to \mathbb{R}$ be harmonic, i.e., $-\Delta u = 0$, and let $B_R(x_0) \subset\subset \Omega$. Then for all $x \in B_R(x_0)$, it holds*

$$u(x) = \int_{\partial B_R(x_0)} K_{B_R(x_0)}(x,y)\,u(y)\,d\sigma(y).$$

Definition 3.14. A function $u : \Omega \to \mathbb{R}$ is called *superharmonic*, if

$$u(x) \geq \int_{\partial B_R(x_0)} K_{B_R(x_0)}(x,y)\,u(y)\,d\sigma(y)$$

holds for all $B_R(x_0) \subset\subset \Omega$ and all $x \in B_R(x_0)$.

There are equivalent characterisations for superharmonic functions. In particular, if $u \in C^2(\Omega)$, one can make use of the Laplace operator.

Lemma 3.15. *For a function $u \in C^2(\Omega)$, the following assertions are equivalent.*

1. *u is superharmonic.*
2. *$u(x_0) \geq \int_{\partial B_R(x_0)} K_{B_R(x_0)}(x_0,y)\,u(y)\,d\sigma(y)$ for all $B_R(x_0) \subset\subset \Omega$.*
3. *$-\Delta u \geq 0$ in Ω.*

Proof. In [30, Satz 3.3.13], one finds that 1 is equivalent to 2. Theorem 2.1 in [16] states that 2 is a consequence of 3. Moreover, in the proof of this theorem, one finds that if $-\Delta u < 0$ in some $B_R(x_0) \subset \Omega$, then $u(x_0) < \int_{\partial B_{\frac{R}{2}}(x_0)} K_{B_{\frac{R}{2}}(x_0)}(x_0, y)\, u(y)\, d\sigma(y)$. Hence 2 implies 3. □

Perron's method uses superharmonic functions to construct a harmonic function that at least 'tries' to satisfy the boundary condition of (3.9).

Theorem 3.16 (cf. [30, Satz 3.3.6])**.** *Let $\Omega \subset\subset \mathbb{R}^2$ be nonempty, $g : \partial\Omega \to \mathbb{R}$ be bounded and $P(g) := \left\{v \in C^0\left(\overline{\Omega}\right); v \text{ is superharmonic on } \Omega, v \leq \sup g, v|_{\partial\Omega} \geq g\right\}$. Then the function $u : \Omega \to \mathbb{R}$ given by*

$$u(x) := \inf\{v(x); v \in P(g)\}$$

has the following properties:

1. $\inf g \leq u \leq \sup g$.
2. *u is harmonic in Ω.*

It turns out that the function obtained by Perron's method satisfies the boundary condition at regular boundary points.

Definition 3.17 (cf. [30, Definition 3.3.8])**.** Let $\Omega \subset \mathbb{R}^2$ be open and $x_0 \in \partial\Omega$. A function $b \in C^0\left(\overline{\Omega}\right)$ is called a *barrier* for Ω at x_0, if it is superharmonic in Ω and it has the properties

$$b(x) > 0 \text{ for } x \in \overline{\Omega} \setminus \{x_0\}, \ b(x_0) = 0. \tag{3.10}$$

The point x_0 is called *regular*[8], if there is a barrier for Ω at x_0.

Actually, regularity is a local property, as it is stated in [30, Satz 3.3.13].

Theorem 3.18. *Let $\Omega \subset\subset \mathbb{R}^2$, $x_0 \in \partial\Omega$, $r > 0$. There exists a barrier for Ω at x_0 if and only if there exists a barrier for $\Omega \cap B_r(x_0)$ at x_0.*

At regular points, the boundary condition can be satisfied.

Theorem 3.19 (cf. [30, Satz 3.3.9])**.** *Let $\Omega \subset\subset \mathbb{R}^2$ be nonempty, $g : \partial\Omega \to \mathbb{R}$ be bounded and u the harmonic function defined in Theorem 3.16. If $x_0 \in \partial\Omega$ is regular and g continuous at x_0, then*

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = g(x_0).$$

The first assertion of the following theorem is a direct consequence.

[8] We use the term 'regular' here, as it is the standard name in literature. It will only play a role in this section, when we prove the existence of the Green functions in our setting. We hope not to confuse the reader by also talking about a set of boundary 'singularities' S in our setting, see Section 2.1. These are points where the boundary is not Dini smooth. Thus they are regular in the sense of Definition 3.17 but singular in the sense of Section 2.1.

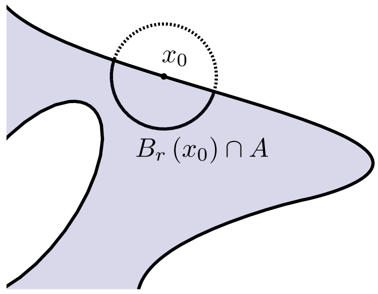


Figure 3.2.: Boundary points of A are regular.

Theorem 3.20 (cf. [30, Satz 3.3.10])**.** *Let $\Omega \subset\subset \mathbb{R}^2$ be nonempty.*

1. *If every boundary point of Ω is regular, (3.9) has a unique solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ for every continuous $g : \partial\Omega \to \mathbb{R}$.*
2. *If (3.9) has a solution for every continuous $g : \partial\Omega \to \mathbb{R}$, every boundary point of Ω is regular.*

Using the results presented above, we are able to show boundary regularity for A and B in our setting.

Proposition 3.21. *Every boundary point of A (and B, respectively) is regular.*

Proof. Let $x_0 \in \partial A$. As A is bounded by a finite number of Jordan curves, there is some $r > 0$ such that $\partial B_r(x_0) \cap \partial A$ consists of only two points and $B_r(x_0) \cap A$ is simply connected, see Figure 3.2. We apply Theorem 3.20.2 and show that there is a solution of

$$\begin{cases} -\Delta u = 0 & \text{in } B_r(x_0) \cap A, \\ u = g & \text{on } \partial B_r(x_0) \cap A \end{cases} \tag{3.11}$$

for every continuous g. By our construction, $B_r(x_0) \cap A$ is a Jordan domain, and according to the Riemann mapping theorem (see Theorem 3.1), its complex counterpart is the image of the unit disk $\mathbb{D} = B_1(0)$ under a conformal map $\mathbf{f}$. According to Theorem 3.2, the function $\mathbf{f}$ – and its inverse – have continuous extensions to the boundary. The composition of a harmonic function and a conformal map is harmonic again.[9] Hence if there is a solution of

$$\begin{cases} -\Delta \tilde{u} = 0 & \text{in } B_1(0), \\ \tilde{u} = g \circ f^{inv} & \text{on } \partial B_1(0), \end{cases} \tag{3.12}$$

then $u := \tilde{u} \circ f$ is a solution of (3.11). A solution of (3.12) is found by (3.6), as the Poisson kernel for a ball exists and is known explicitly, see (3.8). ☐

The boundary of Ω_l is not a union of finitely many Jordan curves. Indeed, if one wants to parametrise the boundary part of Ω_l that belongs to Γ and if one wants the

[9] This follows from (3.15), shown below in Section 3.3.5.

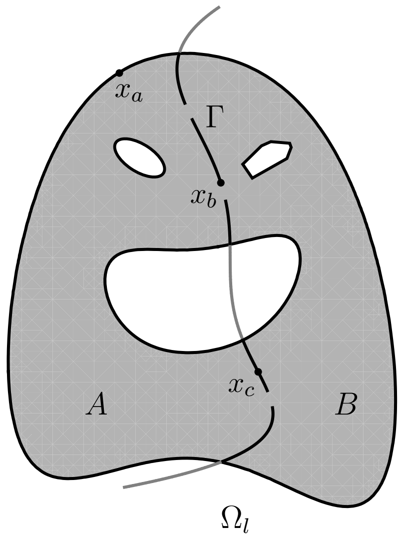


Figure 3.3.: Example of boundary points for three different cases: x_a belongs to $\partial\Omega_l \setminus \Gamma$, x_b is an endpoint of a gap, x_c belongs to Γ but is not an endpoint of a gap.

parametrisation to have the same starting point and endpoint, one has to run through the Γ part twice, which means that the parametrisation is no longer injective. Nevertheless, Ω_l has a regular boundary. This is a consequence of the following proposition and Theorem 3.20.2.

Proposition 3.22. *There exists some $l_1 > 0$, such that for all $0 < l < l_1$, the Dirichlet problem (3.9) has a solution for $\Omega = \Omega_l$ and every continuous $g : \partial\Omega_l \to \mathbb{R}$.*

Proof. Let first $l > 0$ and $g : \partial\Omega_l \to \mathbb{R}$ be a continuous function. We look at the harmonic function $u : \Omega_l \to \mathbb{R}$ obtained by Theorem 3.16 and show that it has a continuous extension up to the boundary, which equals g there.

Let $x_0 \in \partial\Omega_l$. We consider three different cases as illustrated in Figure 3.3. In the first case, if x_0 does not belong to Γ, then it is a regular boundary point, as it has been shown in Proposition 3.21.

In the second case, x_0 belongs to Γ and is the endpoint of a gap. We construct a barrier. See Figure 3.4 for an illustration of the construction. To start, let $f_1 : [0,1] \to \mathbb{R}^2$ be a parametrisation of $\Gamma \cap \partial\Omega_l$ near x_0 with $f_1(0) = x_0$. We switch to complex notation now. We have assumed that $\mathbf{\Gamma}$ is analytic near the $\mathbf{w}_j$'s, and if $l_1 > 0$ is small enough and $0 < l < l_1$, $\mathbf{x}_0$ is close enough to one such $\mathbf{w}_j$, such that $\mathbf{\Gamma}$ is analytic near $\mathbf{x}_0$. Hence we can assume that $\mathbf{f}_1$ is analytic with $\mathbf{f}_1'(0) \neq 0$. This means that $\mathbf{f}_1^{inv}$ maps a neighbourhood of $\mathbf{x}_0$ to a neighbourhood of 0, such that $\partial\mathbf{\Omega}_l$ is mapped to the positive real axis. The function $\mathbf{f}_2 : \mathbf{z} \mapsto \mathbf{z}^2$ maps the upper half plane to the entire plane without the positive real axis. We look at the function $\mathbf{g}(\mathbf{z}) = 1 - e^{\mathbf{iz}}$. It is holomorphic, which implies that the real part is harmonic and thus superharmonic. If $\mathbf{z} = z_1 + \mathbf{i}z_2$, then $\operatorname{Re}(\mathbf{g}(\mathbf{z})) = 1 - e^{-z_2}\cos(z_1)$. This is strictly positive if $z_2 > 0$, and it is zero if $\mathbf{z} = 0$.

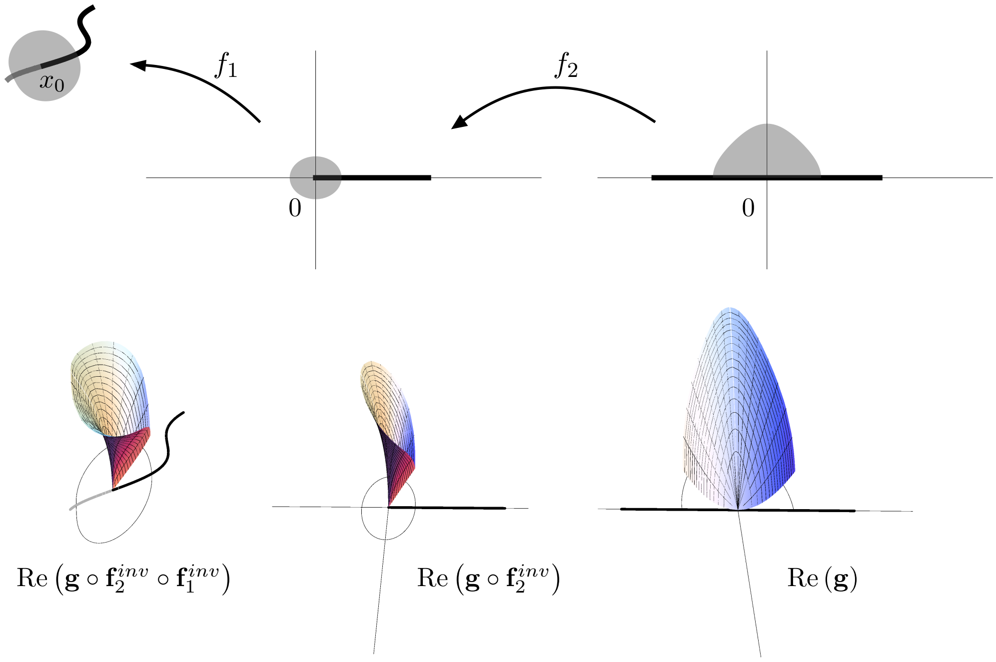


Figure 3.4.: The construction of a barrier if x_0 is an endpoint of a gap. Top: neighbourhoods of the corresponding endpoints under the transformations. Bottom: the barrier on the various neighbourhoods.

We define $b := \operatorname{Re}\left(\mathbf{g} \circ \mathbf{f}_2^{inv} \circ \mathbf{f}_1^{inv}\right)$. We show that b is a local barrier for Ω_l at x_0 in the sense of Theorem 3.18. Due to our construction, (3.10) is satisfied and b is even harmonic in Ω_l. Continuity in a neighbourhood of x_0 is a consequence of the fact that $\operatorname{Re}(\mathbf{g}(z_1 + \mathbf{i}z_2)) = \operatorname{Re}(\mathbf{g}(-z_1 + \mathbf{i}z_2))$, which implies that $\operatorname{Re}\left(\mathbf{g} \circ \mathbf{f}_2^{inv}\right)$ is continuous in a neighbourhood of 0.

Now we turn to the third case, where $x_0 \in \Gamma$ is not the endpoint of a gap. Then it belongs to both ∂A and ∂B. We show that

$$\lim_{\substack{x \to x_0 \\ x \in A}} u(x) = g(x_0) = \lim_{\substack{x \to x_0 \\ x \in B}} u(x), \tag{3.13}$$

which implies continuity of u at x_0. The functions $u|_A$ and $u|_B$ are harmonic and satisfy the respective boundary conditions

$$u|_A(x) = \begin{cases} g(x) \text{ for } x \in \partial A \setminus \left(\bigcup_{j=1}^k \overline{\Gamma_{j,l}}\right), \\ u(x) \text{ for } x \in \bigcup_{j=1}^k \overline{\Gamma_{j,l}}, \end{cases}$$

and

$$u|_B(x) = \begin{cases} g(x) \text{ for } x \in \partial B \setminus \left(\bigcup_{j=1}^k \overline{\Gamma_{j,l}}\right), \\ u(x) \text{ for } x \in \bigcup_{j=1}^k \overline{\Gamma_{j,l}}. \end{cases}$$

The endpoints of the gaps are regular, hence u is continuous on $\bigcup_{j=1}^k \overline{\Gamma_{j,l}}$ and therefore bounded. As A and B have regular boundaries according to Proposition 3.21, u is continuous up to the boundary at x_0 from both sides and (3.13) holds as a consequence of Theorem 3.19.

As u is continuous up to the boundary everywhere, every boundary point of Ω_l is regular according to Theorem 3.20.2. □

As all A, B, and Ω_l have regular boundaries, (3.3) has a solution on each of these domains.

Corollary 3.23. *The Green functions G_A and G_B exist. Moreover, there is some $l_1 > 0$, such that G_{Ω_l} exists for $0 < l < l_1$.*

3.3.3. Existence of the Poisson kernel

Looking at the definition (3.4) of the Poisson kernel, we see that its existence depends on the regularity of the Green function and hence on the smoothness of the boundary. Wherever it is possible to compute the outer normal derivative of G_Ω, the Poisson kernel exists. A rather general result on boundary regularity of solutions of elliptic second order equations in n dimensions is presented in [16, Lemma 6.18]. We present it here in a version adapted to the Laplace operator.

Lemma 3.24. *Let Ω be a domain with a $C^{2,\alpha}$ boundary portion T, and let $f \in C^\alpha\left(\overline{\Omega}\right)$, $g \in C^{2,\alpha}\left(\overline{\Omega}\right)$. Suppose that u is a $C^0\left(\overline{\Omega}\right) \cap C^2(\Omega)$ function satisfying $-\Delta u = f$ in Ω, $u = g$ on T. Then $u \in C^{2,\alpha}(\Omega \cup T)$.*

Moreover, in two dimensions sharper results can be obtained by making use of conformal mappings. According to Lemma 3.4, a domain $\boldsymbol{\Omega}$ that is bounded by finitely many Jordan curves can be mapped to a domain $\tilde{\boldsymbol{\Omega}}$ with smooth C^{10} boundary via a conformal map $\mathbf{f} : \boldsymbol{\Omega} \to \tilde{\boldsymbol{\Omega}}$. $K_{\tilde{\Omega}}$ exists according to the preceding lemma. What about K_Ω? According to Corollary 3.8, $\mathbf{f}'$ has a continuous extension to the boundary at those parts where the original boundary already was Dini smooth. In Section 3.3.5 we look at the transformation behaviour of Green functions and Poisson kernels under conformal mappings. Equation (3.19) states that

$$K_\Omega(x, y) = K_{\tilde{\Omega}}(\tilde{x}, \tilde{y}) \left|\mathbf{f}'(\mathbf{y})\right|,$$

which implies that K_Ω exists at those boundary parts of Ω that are Dini smooth.

Can we solve problem (3.5) even if the boundary is not smooth enough everywhere? Instead of solving it directly, we can look at the equivalent problem on $\tilde{\Omega}$. It is possible to solve the latter with the help of the representation formula (3.6). Transforming the integrals back to Ω, we see that (3.6) is also valid for Ω whenever the boundary data g is zero at those boundary parts where $K_{\tilde{\Omega}}$ is not defined.

To sum up, questions concerning existence and regularity for Ω are answered completely by the boundary behaviour of the conformal map that transforms Ω to the smoother domain $\tilde{\Omega}$.

3.3.4. Basic properties of Green functions

In this section, we list basic properties of Green functions.

- The Green function of a domain Ω is symmetric, i.e., $G_\Omega(x, y) = G_\Omega(y, x)$ for all $x, y \in \Omega$. This statement can be found in [30, Satz 4.5.2].
- For fixed $y \in \Omega$, $x \mapsto G_\Omega(x, y)$ is harmonic if $x \neq y$. This is a direct consequence of the construction of G_Ω.
- Another consequence of the construction is the following. If the boundary of Ω is regular (as it is the case for $\Omega = A, B$, or Ω_l with small l), then $\lim_{x\to x_0} G_\Omega(x, y) = 0$ for $y \in \Omega$, $x_0 \in \partial\Omega$. This means in particular that G_Ω has a continuous extension to the boundary.[10]
- The maximum principle implies that Green functions are nonnegative – otherwise, there would be a superharmonic function that is zero on the boundary and negative somewhere inside the domain.
- Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be two domains with $\Omega_1 \subset \Omega_2$. Then for fixed $x \in \Omega_1$ the difference $G_{\Omega_2}(x, .) - G_{\Omega_1}(x, .)$ is a harmonic function on Ω_1 satisfying nonnegative boundary conditions. Hence, again by the maximum principle, for all $x, y \in \Omega_1$ it holds

$$G_{\Omega_1}(x, y) \leq G_{\Omega_2}(x, y) \tag{3.14}$$

[10]Actually, in [30, Definition 4.1.1], the Green function is *defined* to be continuous up to the boundary, as it is demanded that the solution of (3.3) be $C\left(\overline{\Omega}\right)$. By this definition, a Green function only exists if the boundary is regular.

3.3.5. Green function, Poisson kernel and conformal mappings

Let $\Omega \subset \mathbb{R}^2$ be a domain and let $f : \Omega \to \tilde{\Omega}$ be a map such that its complex counterpart $\mathbf{f} : \boldsymbol{\Omega} \to \tilde{\boldsymbol{\Omega}}$ is conformal. We write $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) = f(x_1, x_2) = f(x)$. Let $u : \Omega \to \mathbb{R}$ be twice differentiable and set

$$\tilde{u}(\tilde{x}) := u(x).$$

In other words $u = \tilde{u} \circ f$. The chain rule combined with the Cauchy–Riemann differential equations yields

$$(\Delta_x u)(x) = (\Delta_{\tilde{x}} \tilde{u})(\tilde{x}) \cdot \left|\mathbf{f}'(\mathbf{x})\right|^2. \tag{3.15}$$

Hence solving the system

$$\begin{cases} -\Delta \tilde{u} = F & \text{in } \tilde{\Omega}, \\ \tilde{u} = G & \text{on } \partial\tilde{\Omega} \end{cases}$$

is equivalent to solving

$$\begin{cases} -\Delta u = (F \circ f) \cdot |\mathbf{f}'|^2 & \text{in } \Omega, \\ u = G \circ f & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

Conversely, the transformation theorem states that

$$\int_{\tilde{\Omega}} w(\tilde{x})\, d\tilde{x} = \int_{\Omega} (w \circ f)(x) \left|\det(\nabla f(x))\right| dx = \int_{\Omega} (w \circ f)(x) \left|\mathbf{f}'(\mathbf{x})\right|^2 dx. \tag{3.17}$$

The correction term $|\mathbf{f}'|^2$ in (3.16) is the same as in (3.17), which yields that

$$G_\Omega(x, y) = G_{\tilde{\Omega}}(\tilde{x}, \tilde{y}). \tag{3.18}$$

As the tangential derivative of $G_\Omega(x, .)$ equals zero on the boundary and $K_\Omega(x, y) \geq 0$, it holds

$$K_\Omega(x, y) = \left|\nabla_y G_\Omega(x, y)\right|.$$

Application of the chain rule then yields

$$K_\Omega(x, y) = K_{\tilde{\Omega}}(\tilde{x}, \tilde{y}) \left|\mathbf{f}'(\mathbf{y})\right|, \tag{3.19}$$

wherever $\mathbf{f}'(\mathbf{y})$ is defined.

3.3.6. Green function and Poisson kernel estimates

Later on, we will need estimates on the shape of the Green function and the Poisson kernel. If the boundary is smooth enough, such results exist. The domain $\tilde{A}$ obtained in Lemma 3.4 of Section 3.2 fulfills these smoothness assumptions, so we will first present the results for $\tilde{A}$ and then give corollaries for A. As before, we will mark the image points in $\tilde{A}$ by a tilde, i.e., we set $\tilde{x} := h(x)$. The estimates presented contain the distance of a point $\tilde{x}$ from the boundary

$$d_{\tilde{A}}(\tilde{x}) := \min\left\{|\tilde{x} - \tilde{z}|\,; \tilde{z} \in \partial\tilde{A}\right\}.$$

In [27, Lemma 4.2], Sweers gives estimates for the Green function on a 2-D domain with a boundary that is at least $C^{1,\gamma}$.

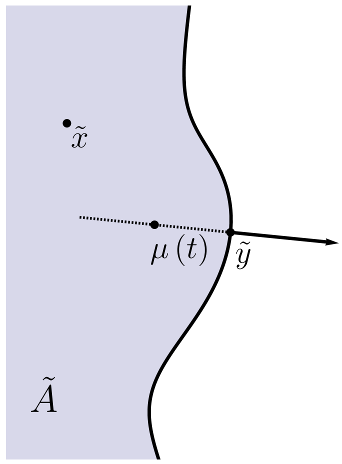


Figure 3.5.: The path μ in the proof of Proposition 3.27.

Proposition 3.25. *There are $c, C > 0$, which only depend on $\tilde{A}$ (from Lemma 3.4), such that*

$$(4\pi)^{-1} \log\left(1 + c\frac{d_{\tilde{A}}(\tilde{x})\, d_{\tilde{A}}(\tilde{y})}{|\tilde{x}-\tilde{y}|^2}\right) \leq G_{\tilde{A}}(\tilde{x},\tilde{y}) \leq (4\pi)^{-1} \log\left(1 + C\frac{d_{\tilde{A}}(\tilde{x})\, d_{\tilde{A}}(\tilde{y})}{|\tilde{x}-\tilde{y}|^2}\right)$$

for all $\tilde{x}, \tilde{y} \in \tilde{A}$.

In [10, Theorem 4], Dall'Acqua and Sweers present an estimate for the Poisson kernel if the domain has at least a C^{10} boundary.

Proposition 3.26. *There exists $C > 0$, which only depends on $\tilde{A}$ (from Lemma 3.4), such that*

$$K_{\tilde{A}}(\tilde{x},\tilde{y}) \leq C\frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{y}|^2}$$

for all $\tilde{x} \in \tilde{A}$, $\tilde{y} \in \partial\tilde{A}$.

Later, we will also need an estimate from below for the Poisson kernel. As we will derive it from the estimate on the Green function from Proposition 3.25, it would be enough to have a $C^{1,\gamma}$ boundary here. Anyway, the boundary of $\tilde{A}$ is smooth enough.

Proposition 3.27. *There is a $c > 0$, which only depends on $\tilde{A}$ (from Lemma 3.4), such that*

$$K_{\tilde{A}}(\tilde{x},\tilde{y}) \geq c\frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{y}|^2}$$

for all $\tilde{x} \in \tilde{A}$, $\tilde{y} \in \partial\tilde{A}$.

Proof. Fix $\tilde{x} \in \tilde{A}$, $\tilde{y} \in \partial\tilde{A}$. We define a path $\mu(t) := \tilde{y} - tn_{\tilde{y}}$, where $n_{\tilde{y}}$ stands for the outer unit normal to $\partial\tilde{A}$ at $\tilde{y}$, see Figure 3.5. For small $t \geq 0$, $\mu(t)$ will stay away from

$\tilde{x}$, hence

$$m_1(t) := G_{\tilde{A}}\left(\tilde{x}, \mu(t)\right)$$

is well-defined. For small $t > 0$, m_1 is continuously differentiable as $\tilde{z} \mapsto G_{\tilde{A}}\left(\tilde{x}, \tilde{z}\right)$ is harmonic for $\tilde{z} \notin \partial\tilde{A} \cup \{\tilde{x}\}$. Moreover, as $\partial\tilde{A}$ is C^{10}, the derivative of the Green function exists even on the boundary and is continuous. To sum up, this implies that m_1 is continuously differentiable for small $t \geq 0$, and

$$m_1'(0) = \left(\nabla_{\tilde{y}} G_{\tilde{A}}\left(\tilde{x}, \mu(t)\right)\right) \cdot \mu'(t)\big|_{t=0} = -\left(\nabla_{\tilde{y}} G_{\tilde{A}}\left(\tilde{x}, \tilde{y}\right)\right) \cdot n_{\tilde{y}} = K_{\tilde{A}}\left(\tilde{x}, \tilde{y}\right). \tag{3.20}$$

Now we define for small $t \geq 0$

$$m_2(t) := (4\pi)^{-1} \log\left(1 + c\frac{d_{\tilde{A}}\left(\tilde{x}\right) d_{\tilde{A}}\left(\mu(t)\right)}{\left|\tilde{x} - \mu(t)\right|^2}\right),$$

where $c > 0$ is the constant in Proposition 3.25. The proposition implies $m_1(t) \geq m_2(t)$ for small $t > 0$.

Furthermore, as $d_{\tilde{A}}\left(\mu(0)\right) = d_{\tilde{A}}\left(\tilde{y}\right) = 0$, we have $m_2(0) = 0 = G_{\tilde{A}}\left(\tilde{x}, \tilde{y}\right) = m_1(0)$. Hence

$$m_1'(0) = \lim_{t\downarrow 0} \frac{m_1(t) - m_1(0)}{t} \geq \lim_{t\downarrow 0} \frac{m_2(t) - m_2(0)}{t} = m_2'(0).$$

We now compute $m_2'(0)$ in order to obtain a lower bound for $K_{\tilde{A}}$ by (3.20). We start with $m_2'(t)$, using that $\frac{d}{dt} d_{\tilde{A}}\left(\mu(t)\right) = 1$ for small $t \geq 0$,

$$m_2'(t) = (4\pi)^{-1} \frac{1}{1 + c\frac{d_{\tilde{A}}(\tilde{x}) d_{\tilde{A}}(\mu(t))}{|\tilde{x} - \mu(t)|^2}} \left(c\frac{d_{\tilde{A}}\left(\tilde{x}\right)}{\left|\tilde{x} - \mu(t)\right|^2} + 2c\frac{d_{\tilde{A}}\left(\tilde{x}\right) d_{\tilde{A}}\left(\mu(t)\right)}{\left|\tilde{x} - \mu(t)\right|^4} \left(\left(\tilde{x} - \mu(t)\right) \cdot n_{\tilde{y}}\right)\right)$$

We plug in $t = 0$ and use $\mu(0) = \tilde{y}$ to get

$$m_2'(0) = (4\pi)^{-1} c\frac{d_{\tilde{A}}\left(\tilde{x}\right)}{\left|\tilde{x} - \tilde{y}\right|^2},$$

which completes the proof. □

The transformation results of (3.18) and (3.19) yield the following corollary.

Corollary 3.28. *Let* $\mathbf{h} : \mathbf{A} \to \tilde{\mathbf{A}}$ *be the conformal map from Lemma 3.4. There exist* $c, C > 0$*, depending only on* A *(and* $\tilde{A}$*), such that*

$$(4\pi)^{-1} \log\left(1 + c\frac{d_{\tilde{A}}\left(\tilde{x}\right) d_{\tilde{A}}\left(\tilde{y}\right)}{\left|\tilde{x} - \tilde{y}\right|^2}\right) \leq G_A\left(x, y\right) \leq (4\pi)^{-1} \log\left(1 + C\frac{d_{\tilde{A}}\left(\tilde{x}\right) d_{\tilde{A}}\left(\tilde{y}\right)}{\left|\tilde{x} - \tilde{y}\right|^2}\right)$$

for all $x, y \in A$*. Moreover, let* $l \leq l_0$ *be so small such that* $\mathbf{h}'\left(\mathbf{y}\right)$ *exists for all* $\mathbf{y} \in \mathbf{\Gamma}_{j,l}$ *(see Corollary 3.10). Then there exist (new)* $c, C > 0$*, depending only on* A*, such that*

$$c\frac{d_{\tilde{A}}\left(\tilde{x}\right)}{\left|\tilde{x} - \tilde{y}\right|^2} \left|\mathbf{h}'\left(\mathbf{y}\right)\right| \leq K_A\left(x, y\right) \leq C\frac{d_{\tilde{A}}\left(\tilde{x}\right)}{\left|\tilde{x} - \tilde{y}\right|^2} \left|\mathbf{h}'\left(\mathbf{y}\right)\right|$$

for all $x \in A$*,* $y \in \bigcup_{j=1}^{k} \Gamma_{j,l}$*.*

3.3.7. Estimates on higher order Green function derivatives

In [10], Dall'Acqua and Sweers present estimates on higher order derivatives of Green functions, whenever the domain has at least a C^{10} boundary. This is the case for $\tilde{A}$ from Lemma 3.4. We cite Theorem 12 (1) (a) in a version adapted to our case ($m = 1$, $n = 2$).

Proposition 3.29. *Let $\tilde{A}$ be the domain in Lemma 3.4. Then for each multiindex $k = (k_1, k_2) \neq (0, 0)$, there exists $C > 0$ such that*

$$\left| D_z^k G_{\tilde{A}}(\tilde{x}, \tilde{z}) \right| \leq C \, |\tilde{x} - \tilde{z}|^{-|k|} \min \left\{ 1, \frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x} - \tilde{z}|} \right\},$$

for all $\tilde{x}, \tilde{z} \in \tilde{A}$.

In the special case $k = (1, 1)$ and writing $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$, we get

$$\left| \frac{\partial}{\partial \tilde{z}_2} \frac{\partial}{\partial \tilde{z}_1} G_{\tilde{A}}(\tilde{x}, \tilde{z}) \right| \leq C \, |\tilde{x} - \tilde{z}|^{-2} \min \left\{ 1, \frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x} - \tilde{z}|} \right\}. \tag{3.21}$$

As the boundary is a straight line in a neighbourhood of $\tilde{w}_j$ and thus C^∞, the Green function $\tilde{z} \mapsto G_{\tilde{A}}(\tilde{x}, \tilde{z})$ and its derivatives have continuous extensions up to the boundary $\partial \tilde{A}$ near $\tilde{w}_j$. Hence, by continuity, the estimate of (3.21) also holds for $\tilde{z}$ being a boundary point near $\tilde{w}_j$. The outer normal at these boundary points is pointing in the $\tilde{z}_1$ direction, hence $K_{\tilde{A}}(\tilde{x}, \tilde{z}) = -\frac{\partial}{\partial \tilde{z}_1} G_{\tilde{A}}(\tilde{x}, \tilde{z})$ there.

We parametrise the boundary $\partial \tilde{A}$ near $\tilde{w}_j$ by $\tilde{z}(t) := \tilde{w}_j + t\,(0, 1)$. By the mean value theorem, there exists a θ_t between 0 and t such that

$$\begin{aligned} K_{\tilde{A}}(\tilde{x}, \tilde{z}(t)) &= K_{\tilde{A}}(\tilde{x}, \tilde{w}_j) + t \, \frac{d}{ds} \left(K_{\tilde{A}}(\tilde{x}, \tilde{z}(s)) \right) \Big|_{s=\theta_t} \\ &= K_{\tilde{A}}(\tilde{x}, \tilde{w}_j) - t \frac{\partial}{\partial \tilde{z}_2} \frac{\partial}{\partial \tilde{z}_1} G_{\tilde{A}}(\tilde{x}, \tilde{z}(\theta_t)). \end{aligned} \tag{3.22}$$

If we assume that $\tilde{x}$ stays sufficiently far away from the gap $\tilde{\Gamma}_{j,l}$, this leads us to the following corollary about Lipschitz regularity of the Poisson kernel.

Corollary 3.30. *Let $\tilde{A}$ be the domain in Lemma 3.4. Let $\tilde{x} \in \tilde{A}$ with $|\tilde{x} - \tilde{w}_j| \geq \tilde{\rho}$ for some $j \in \{1, \ldots, k\}$ and $\tilde{\rho} > 0$. Further, let l be so small that $\tilde{l}_j < \tilde{\rho}$,[11] and let $\tilde{z} \in \tilde{\Gamma}_{j,l}$. Then there is a $C > 0$, depending only on $\tilde{A}$, such that*

$$\left| K_{\tilde{A}}(\tilde{x}, \tilde{z}) - K_{\tilde{A}}(\tilde{x}, \tilde{w}_j) \right| \leq C \min \left\{ \left(\tilde{\rho} - \tilde{l}_j \right)^{-2}, d_{\tilde{A}}(\tilde{x}) \left(\tilde{\rho} - \tilde{l}_j \right)^{-3} \right\} |\tilde{z} - \tilde{w}_j|.$$

Proof. We use equation (3.22) together with (3.21) and the facts that $|\tilde{z} - \tilde{w}_j| = t$ and $|\tilde{x} - \tilde{z}(\theta_t)| \geq |\tilde{x} - \tilde{w}_j| - |\tilde{w}_j - \tilde{z}(\theta_t)| \geq \tilde{\rho} - \tilde{l}_j$. □

[11] As before, $\tilde{l}_j$ denotes the width of the gap $\tilde{\Gamma}_{j,l}$.

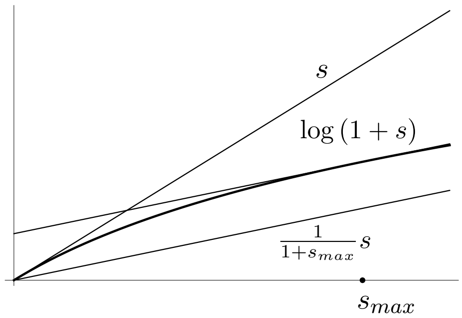


Figure 3.6.: Bounds for the function $s \mapsto \log(1+s)$.

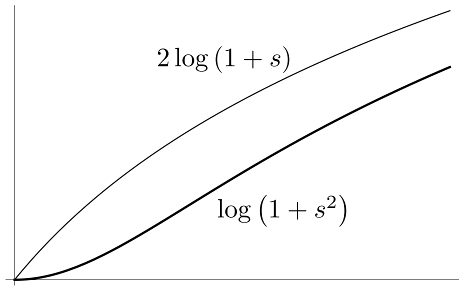


Figure 3.7.: Bounds for the function $s \mapsto \log\left(1+s^2\right)$.

3.3.8. Green function estimates – further estimates on the logarithm

Later it will be helpful to replace the estimates for the Green function in Corollary 3.28 by some (less sharp) estimates that contain linear terms instead of logarithms. For future reference, we list them here.

- For $s \in [0, s_{max}]$, it holds that $\frac{1}{1+s_{max}}s \leq \log(1+s)$, see Figure 3.6. As $\tilde{A}$ is bounded, $c\frac{d_{\tilde{A}}(\tilde{x})d_{\tilde{A}}(\tilde{z})}{|\tilde{x}-\tilde{z}|^2} \leq c\frac{\tilde{M}^2}{|\tilde{x}-\tilde{z}|^2}$ for some $\tilde{M} > 0$. Hence for the estimate from below for the Green function in Corollary 3.28 we get

$$\begin{aligned} G_A(x,z) &\geq (4\pi)^{-1}\log\left(1+c\frac{d_{\tilde{A}}(\tilde{x})\,d_{\tilde{A}}(\tilde{z})}{|\tilde{x}-\tilde{z}|^2}\right) \\ &\geq (4\pi)^{-1}\frac{1}{1+c\frac{\tilde{M}^2}{|\tilde{x}-\tilde{z}|^2}}c\frac{d_{\tilde{A}}(\tilde{x})\,d_{\tilde{A}}(\tilde{z})}{|\tilde{x}-\tilde{z}|^2} \\ &\geq (4\pi)^{-1}\frac{c}{(1+c)\,\tilde{M}^2}d_{\tilde{A}}(\tilde{x})\,d_{\tilde{A}}(\tilde{z}). \end{aligned} \tag{3.23}$$

- For an estimate from above, it does not suffice to use $\log(1+s) \leq s$ for $s \geq 0$ directly (see Figure 3.6). This estimate together with Corollary 3.28 would yield $G_A(x,z) \leq C\frac{d_{\tilde{A}}(\tilde{x})d_{\tilde{A}}(\tilde{z})}{|\tilde{x}-\tilde{z}|^2}$, where the right hand side is no longer integrable in z over A. That is why we have to proceed a bit more carefully, as it is done in [18, Lemma 3.4]. For the sake of completeness, we present the considerations here. Let $\tilde{x}, \tilde{z} \in \tilde{A}$. We distinguish two cases.

Case 1: $|\tilde{x}-\tilde{z}| \leq \frac{1}{2}\max\left\{d_{\tilde{A}}(\tilde{x}), d_{\tilde{A}}(\tilde{z})\right\}$
We first show that $d_{\tilde{A}}(\tilde{z}) \leq 2d_{\tilde{A}}(\tilde{x})$. This holds clearly if $d_{\tilde{A}}(\tilde{z}) \leq d_{\tilde{A}}(\tilde{x})$. If $d_{\tilde{A}}(\tilde{z}) > d_{\tilde{A}}(\tilde{x})$, let $\tilde{w}_{\tilde{x}} \in \partial\tilde{A}$ be a point such that $d_{\tilde{A}}(\tilde{x}) = |\tilde{x}-\tilde{w}_{\tilde{x}}|$. Then $d_{\tilde{A}}(\tilde{z}) \leq |\tilde{z}-\tilde{w}_{\tilde{x}}| \leq |\tilde{z}-\tilde{x}| + d_{\tilde{A}}(\tilde{x}) \leq \frac{1}{2}d_{\tilde{A}}(\tilde{z}) + d_{\tilde{A}}(\tilde{x})$ and thus $\frac{1}{2}d_{\tilde{A}}(\tilde{z}) \leq d_{\tilde{A}}(\tilde{x})$.
For $s \geq 0$, it holds that $\log(1+s^2) \leq \log(1+2s+s^2) \leq 2\log(1+s)$ (see Figure 3.7),

which implies

$$\log\left(1+C\frac{d_{\tilde{A}}(\tilde{x})\,d_{\tilde{A}}(\tilde{z})}{|\tilde{x}-\tilde{z}|^2}\right)\le 2\log\left(1+\sqrt{C\frac{d_{\tilde{A}}(\tilde{x})\,d_{\tilde{A}}(\tilde{z})}{|\tilde{x}-\tilde{z}|^2}}\right)$$
$$\le 2\log\left(1+\sqrt{2C}\frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{z}|}\right).$$

Case 2: $|\tilde{x}-\tilde{z}|>\frac{1}{2}\max\left\{d_{\tilde{A}}(\tilde{x}),d_{\tilde{A}}(\tilde{z})\right\}$

Then $\frac{d_{\tilde{A}}(\tilde{z})}{|\tilde{x}-\tilde{z}|}<2$, and we get

$$\log\left(1+C\frac{d_{\tilde{A}}(\tilde{x})\,d_{\tilde{A}}(\tilde{z})}{|\tilde{x}-\tilde{z}|^2}\right)\le\log\left(1+2C\frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{z}|}\right).$$

Summing up both cases and now making use of $\log(1+s)\le s$, we conclude that there exists $C>0$ such that

$$G_A(x,z)\le C\frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{z}|}. \tag{3.24}$$

3.4. Some facts on Brownian motion

In this section we give a heuristic introduction to the concept of Brownian motion and the expected lifetime of a Brownian motion starting at x, conditioned to be killed at the boundary and to end at y, which we denote by $\mathbb{E}_x^y(\tau_\Omega)$. This introduction is far from being rigorous, but it will explain why it is plausible that the lifetime is given by

$$\mathbb{E}_x^y(\tau_\Omega)=\int_\Omega\frac{G_\Omega(x,z)\,G_\Omega(z,y)}{G_\Omega(x,y)}\,dz,$$

or, put the other way round, we try to give a stochastic interpretation of the $3G$ integral in the equation above.

For the mathematical correctness of the rest of the paper this stochastic view is not necessary, as all the results are obtained by analytical methods. However, the stochastic point of view provides some intuition for tackling the problem and helps to understand the results. That is why we sketch it in this section. We remark that in stochastics, $\frac{1}{2}\Delta$ is often used instead of Δ. Since the factor $\frac{1}{2}$ only changes the scaling but not the model, we will omit it here.

Presenting some general results in this section, we do not use the notation introduced for our special setting but write Ω for an arbitrary bounded domain in $\mathbb{R}^2$. We assume merely that Ω is smooth enough to possess a Green function G_Ω and a heat kernel p_Ω.

3.4.1. Brownian motion

We start with the concept of Brownian motion. For details we refer to [7, Chapter 1], [13, Part 2, Chapter VII], and [23, Chapter 2].

Let $\Omega \subset \mathbb{R}^2$ and let $x \in \Omega$. Imagine a particle that starts moving according to the rules of Brownian motion at x. This means, roughly speaking, that all directions of movement have the same probability, there is no preferred direction. Moreover, at every time and point of the movement, the 'choice of direction' made by the particle does not depend on the path that has led to the actual position.

Eventually, the particle may leave Ω. If we write X_t for the random variable that describes the position at time t, we define the stopping time τ_Ω to be

$$\tau_\Omega := \inf \{t \geq 0; X_t \notin \Omega\} .$$

We are especially interested in Brownian motion that stays within Ω. It can be described by some kind of probability density $p_\Omega(x, z, t)$ in the sense that

$$\mathbb{P}\left(X_t \in \tilde{\Omega}, t < \tau_\Omega \,|X_0 = x\right) = \int_{\tilde{\Omega}} p_\Omega(x, z, t)\, dz \tag{3.25}$$

gives the probability that the particle, having started its motion at x, is in the subset $\tilde{\Omega} \subset \Omega$ at time t and *has not (yet) left* Ω.

It turns out that this probability density is nothing else but the heat kernel with zero Dirichlet boundary conditions. The heat kernel is such a function that

$$\begin{cases} \frac{\partial}{\partial t} u - \Delta u = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u(t, .) = 0 & \text{on } \partial\Omega \text{ for } t \in \mathbb{R}^+, \\ u(0, .) = u_0 & \text{on } \Omega \end{cases} \tag{3.26}$$

is solved by

$$u(t, x) = \int_\Omega p_\Omega(x, z, t)\, u_0(z)\, dz.$$

Why the heat kernel? In [23], Lawler gives an intuitive explanation. Each path of a Brownian motion that starts in x and ends in $\tilde{\Omega}$ corresponds to a path that starts in $\tilde{\Omega}$ and ends in x. Hence we can understand the probability in (3.25) as the relative frequency of particles at point x and time t that started in $\tilde{\Omega}$, where we assume that the particles move according to the rules of Brownian motion and that the particles were equally distributed over Ω at time zero. Let us call these particles 'heat particles' and imagine that the temperature at (x, t) is proportional to the relative frequency of particles at this point and time. Then it is reasonable that the probability in (3.25) is described with the help of the heat kernel of (3.26). The fact that only those particles are taken into account that have not yet left Ω is reflected by the zero Dirichlet boundary condition. It is sometimes said that the particles that reach the boundary are 'killed'.

Now let us turn back to (3.25). The probability that a particle that started its motion at $x \in \Omega$ has not yet left Ω at time t is given by

$$\mathbb{P}(X_t \in \Omega, t < \tau_\Omega \,|X_0 = x) = \int_\Omega p_\Omega(x, z, t)\, dz.$$

We are interested in the expected value of the stopping time τ_Ω, denoted by $\mathbb{E}_x(\tau_\Omega)$ – it is also called the (expected) lifetime. It can be computed by integrating t times the probability that the particle leaves Ω at time t. Hence

$$\begin{aligned}\mathbb{E}_x(\tau_\Omega) &= \int_0^\infty t\left(-\frac{d}{dt}\mathbb{P}\left(X_t \in \Omega, t < \tau_\Omega \,|X_0 = x\right)\right) dt \\ &= -\int_0^\infty t\frac{d}{dt}\int_\Omega p_\Omega(x,z,t)\; dz\, dt = -\int_\Omega\int_0^\infty t\frac{\partial}{\partial t}p_\Omega(x,z,t)\; dt\, dz \\ &= \int_\Omega\int_0^\infty p_\Omega(x,z,t)\; dt\, dz = \int_\Omega G_\Omega(x,z)\; dz.\end{aligned}$$

In the last step, we used that $\int_0^\infty p_\Omega(x,z,t)\; dt = G_\Omega(x,z)$. This can be checked using the facts that, formally, $p_\Omega(x,.,t)$ solves the heat equation (3.26) with $u_0 = \delta_x$ and that, also formally, $G_\Omega(x,.)$ solves $-\Delta G_\Omega(x,.) = \delta_x$, both with zero Dirichlet boundary conditions. Indeed,

$$\begin{aligned}-\Delta\int_0^\infty p_\Omega(x,.,t)\; dt &= -\int_0^\infty \Delta p_\Omega(x,.,t)\; dt \\ &= -\int_0^\infty \frac{\partial}{\partial t}p_\Omega(x,.,t)\; dt = p_\Omega(x,.,0) = \delta_x.\end{aligned}$$

3.4.2. Conditioned Brownian motion

In 1957, Doob introduced the so-called (h-) conditional Brownian motion, see [12]. It is defined as follows. Let $h : \Omega \to \mathbb{R}$ be a strictly positive harmonic function. We replace the probability density $p_\Omega(x,z,t)$ in (3.25) by

$$p_\Omega(x,z,t)\,\frac{h(z)}{h(x)}.$$

The expected lifetime $\mathbb{E}_x^h(\tau_\Omega)$ is then given by

$$\mathbb{E}_x^h(\tau_\Omega) = \int_\Omega G_\Omega(x,z)\,\frac{h(z)}{h(x)}\,dz.$$

In 1983, Cranston and McConnell showed that the lifetime has an upper bound that only depends on the area of the domain, see [8].[12] Later, a shorter proof was presented by Chung in [6]. We present the theorem here as it can be found in [7, Theorem 5.7], adapted to our notation.

Theorem 3.31. *If Ω is a domain in $\mathbb{R}^2$ and $h > 0$ is harmonic in Ω, then for all $x \in \Omega$*

$$\mathbb{E}_x^h(\tau_\Omega) \le C\,|\Omega|\,,$$

where C is an absolute constant and $|\Omega|$ stands for the Lebesgue measure of Ω.

[12] A side note: There seems to have occurred a shift of name from the original ‘conditional’ Brownian motion to the nowadays often used ‘conditioned’ Brownian motion.

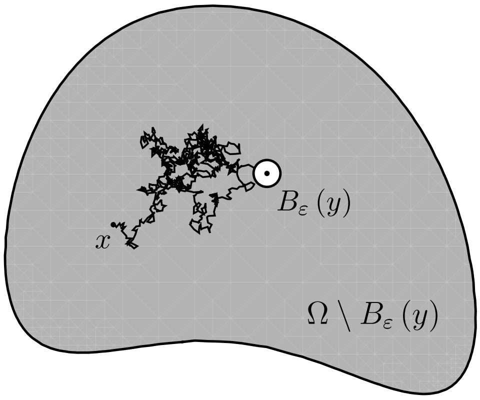


Figure 3.8.: Possible path of a Brownian motion on $\Omega \setminus B_\varepsilon(y)$ starting at x and conditioned to exit through $\partial B_\varepsilon(y)$.

Next, let $y \in \Omega$ and $\varepsilon > 0$ be so small that $B_\varepsilon(y) \subset \Omega$. We replace Ω by $\Omega \setminus B_\varepsilon(y)$. For h, we take the solution of

$$\begin{cases} -\Delta h_\varepsilon = 0 & \text{in } \Omega \setminus B_\varepsilon(y), \\ h_\varepsilon = 0 & \text{on } \partial\Omega, \\ h_\varepsilon = 1 & \text{on } \partial B_\varepsilon(y). \end{cases}$$

Then $\mathbb{E}_x^{h_\varepsilon}\left(\tau_{\Omega\setminus B_\varepsilon(y)}\right)$ can be understood as the expected time that a particle starting at x spends in $\Omega \setminus B_\varepsilon(y)$ before it leaves $\Omega \setminus B_\varepsilon(y)$ through the boundary part where $h_\varepsilon = 1$. That means, by imposing the above boundary values on h_ε, we have conditioned the particle to exit through $\partial B_\varepsilon(y)$. All other paths, which exit trough $\partial\Omega$, are ignored. For an illustration, see Figure 3.8.

What happens if $\varepsilon \to 0$? Intuitively, we will get the average time that it takes a particle to get from x (close) to y, where only those paths that stay inside Ω are taken into account. Corollary A.6 in Appendix A states that the limit of $\mathbb{E}_x^{h_\varepsilon}\left(\tau_{\Omega\setminus B_\varepsilon(y)}\right)$ exists and that

$$\lim_{\varepsilon\to 0} \mathbb{E}_x^{h_\varepsilon}\left(\tau_{\Omega\setminus B_\varepsilon(y)}\right) = \int_\Omega \frac{G_\Omega(x,z)\, G_\Omega(z,y)}{G_\Omega(x,y)}\, dz =: \mathbb{E}_x^y(\tau_\Omega). \tag{3.27}$$

In full length, we call the term on the right hand side *the expected lifetime of Brownian motion starting at x, conditioned to be killed at the boundary and to go to y.* We sum up the model in a kind of algorithm for home experiments:

1. Choose a fixed starting point $x \in \Omega$ and a fixed endpoint $y \in \Omega$.

2. Put a particle on x and let it move randomly within Ω.

3. a) If the particle reaches $\partial\Omega$, remove it.

b) If the particle reaches y, write down the time it has taken the particle to get from x to y.

4. Repeat steps 2 and 3 infinitely many times and average over all times obtained in 3b.

As Theorem 3.31 holds for all harmonic $h > 0$ and hence also for the limit in (3.27), we get the following corollary for $\mathbb{E}_x^y(\tau_\Omega)$.

Corollary 3.32. *There exists an absolute constant $C > 0$, such that*

$$\mathbb{E}_x^y(\tau_\Omega) \leq C\,|\Omega|$$

for all $\Omega \subset \mathbb{R}^2$ (that possess a Green function) and all $x, y \in \Omega$.

3.4.3. Continuity of $\mathbb{E}_x^y(\tau_\Omega)$

Let again $\Omega \subset \mathbb{R}^2$ be a bounded domain and $x \in \Omega$. The function $y \mapsto \mathbb{E}_x^y(\tau_\Omega)$ is continuous in $\Omega \setminus \{x\}$, which can be seen the following way. The function $u^x : \Omega \to \mathbb{R}$ defined by

$$u^x(y) = \int_\Omega G_\Omega(x,z)\, G_\Omega(z,y)\; dz \tag{3.28}$$

lies in $H^1(\Omega)$ and solves in a weak sense

$$-\Delta u^x = G_\Omega(x,.) \text{ in } \Omega.$$

Interior elliptic regularity, see, e.g., [15, Section 6.3.1, Theorem 1], gives $u^x \in H^2_{loc}(\Omega)$. The Sobolev inequalities, see [15, Section 5.6.3, Theorem 6 (ii)], imply further that $u^x \in C^{0,\gamma}\left(\tilde{\Omega}\right)$ for some $0 < \gamma < 1$ and some neighbourhood $\tilde{\Omega} \subset\subset \Omega$ of y. As, moreover, $y \mapsto G_\Omega(x,y)$ is continuous in $y \neq x$, also $y \mapsto \mathbb{E}_x^y(\tau_\Omega)$ is continuous in $\Omega \setminus \{x\}$.[13]

Green functions are symmetric and hence the definition of $\mathbb{E}_x^y(\tau_\Omega)$ in (3.27) implies that

$$\mathbb{E}_x^y(\tau_\Omega) = \mathbb{E}_y^x(\tau_\Omega)\,.$$

Consequently, also $x \mapsto \mathbb{E}_x^y(\tau_\Omega)$ is continuous in $\Omega \setminus \{y\}$.

What happens if $y \to x$? The function u^x in (3.28) is locally continuous near x and therefore bounded. As $\lim_{y\to x} G_\Omega(x,y) = \infty$,

$$\lim_{y\to x} \mathbb{E}_x^y(\tau_\Omega) = \lim_{y\to x} \frac{u^x(y)}{G_\Omega(x,y)} = 0,$$

[13] To be correct, we have only shown that $y \mapsto \mathbb{E}_x^y(\tau_\Omega)$ is equal *almost everywhere* to a continuous function, as weak solutions are L^p functions and hence only unique up to a set of measure zero. Alternatively, one could argue that continuity is a consequence of the continuity of G_Ω and Lebesgue's dominated convergence theorem. A function dominating the integrands can be found with the help of the estimate $G_\Omega(x,z) \leq G_{B_M(x)}(x,z) = -\frac{1}{2\pi}\log\left(\frac{|x-z|}{M}\right)$ for some large M.

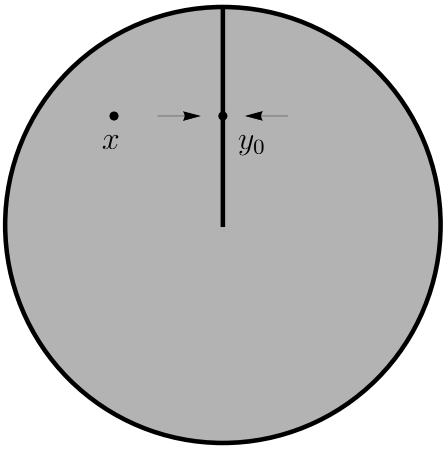


Figure 3.9.: Example with a non-smooth boundary: the limit of $\mathbb{E}_x^y(\tau_\Omega)$ for $y \to y_0$ will depend on whether y approaches y_0 from the left or from the right.

which makes sense as one would expect a motion from x to a very close y to take nearly no time in average.[14] More detailed results on the behaviour of $\mathbb{E}_x^y(\tau_\Omega)$ are presented in [27, Section 4.4] and [4].

What happens if we approach the boundary? If y_0 is a boundary point and if the boundary part near y_0 is sufficiently smooth such that the Poisson kernel exists there (see Section 3.3.3), we can approach y_0 from inside Ω by y. A use of l'Hôpital's rule gives[15]

$$\lim_{\substack{y\to y_0\\ y\in\Omega}} \mathbb{E}_x^y(\tau_\Omega) = \lim_{\substack{y\to y_0\\ y\in\Omega}} \int_\Omega \frac{G_\Omega(x,z)\,G_\Omega(z,y)}{G_\Omega(x,y)}\,dz = \int_\Omega \frac{G_\Omega(x,z)\,K_\Omega(z,y_0)}{K_\Omega(x,y_0)}\,dz =: \mathbb{E}_x^{y_0}(\tau_\Omega).$$

If x_0 belongs to a smooth part of $\partial\Omega$, too, a second application of l'Hôpital's rule enables us to define a continuous extension of the lifetime by

$$\mathbb{E}_{x_0}^{y_0}(\tau_\Omega) := \int_\Omega \frac{K_\Omega(z,x_0)\,K_\Omega(z,y_0)}{-\frac{\partial}{\partial n_x}K_\Omega(x_0,y_0)}\,dz. \tag{3.29}$$

What if the boundary is less smooth? To give an example, we set $\Omega := B_1(0,0) \setminus (\{0\}\times[0,1))$ and $x := \left(-\frac{1}{2},\frac{1}{2}\right)$, see Figure 3.9. What happens to $\mathbb{E}_x^y(\tau_\Omega)$ if y approaches $y_0 =: \left(0,\frac{1}{2}\right) \in \partial\Omega$? Intuitively, this limit should be lower if we approach y_0 from the left of the boundary than it would be if approaching it from the right, as in the latter case, the particle has to take a path round the 'wall' given by $\{0\}\times[0,1)$. As almost always in two dimensions, conformal maps help to solve this problem. Let $\tilde{\mathbf{f}} : B_1(0) \to \mathbf{\Omega}$ be a conformal map of the unit disk onto $\mathbf{\Omega}$ given by the Riemann mapping theorem, see

[14] Again, if one is not satisfied with 'almost everywhere' arguments, one could use $G_\Omega(x,z) \le G_{B_M(w)}(w,z)$ to get an upper bound for u^x.

[15] Again, we also need to find a dominating function for the integrand by $G_\Omega(z,y) \le G_{B_M(y)}(z,y)$.

Theorem 3.1. Then, writing $\tilde{x} := f(x) := \tilde{f}^{inv}(x)$ as usual, we get by (3.17) and (3.18) that

$$\mathbb{E}_x^y(\tau_\Omega) = \int_{B_1(0,0)} \frac{G_{B_1(0,0)}(\tilde{x},\tilde{z})\, G_{B_1(0,0)}(\tilde{z},\tilde{y})}{G_{B_1(0,0)}(\tilde{x},\tilde{y})} \left|\tilde{\mathbf{f}}'(\tilde{\mathbf{z}})\right|^2 d\tilde{z}. \tag{3.30}$$

The point y_0 is 'split into two points' by f in the following sense. If we approach y_0 from the left by a sequence of points $y_n^l \in \Omega \cap (\mathbb{R}^- \times \mathbb{R})$, then the image points $\tilde{y}_n^l$ will converge to some $\tilde{y}_0^l \in \partial B_1(0,0)$. If we approach y_0 from the right hand side by some other sequence, this will converge to another boundary point $\tilde{y}_0^r \neq \tilde{y}_0^l$. Likewise, the integral in (3.30) will converge to different values if we let $\tilde{y}$ tend to $\tilde{y}_0^r$ or $\tilde{y}_0^l$. Summing up, in our example, it will not be possible do define *one* value for $\mathbb{E}_x^{y_0}(\tau_\Omega)$, but there are *two different limits* depending on how we approach y_0.

3.5. Further related results concerning conditioned Brownian motion

3.5.1. Connection to elliptic systems and the positivity preserving property

The integral expressing the lifetime also appears in the study of coupled elliptic systems of the type

$$\begin{cases} L_1 u = f - \lambda g(., v, \nabla v) & \text{in } \Omega, \\ L_2 v = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.31}$$

where L_1 and L_2 are second order elliptic operators. One could ask whether there is some λ_{max} such that for $0 \leq \lambda < \lambda_{max}$, the positivity of f implies the positivity of u (for v it is a consequence of the maximum principle).

To motivate the connection to the lifetime we set $L_1 = L_2 = -\Delta$ and $g(., v, \nabla v) = v$. Then system (3.31) reads

$$\begin{cases} -\Delta u = f - \lambda v & \text{in } \Omega, \\ -\Delta v = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.32}$$

With the help of the Green representation formula (3.6) a solution of (3.32) can be written as

$$u(x) = \int_\Omega G_\Omega(x,z) \left(f(z) - \lambda \int_\Omega G_\Omega(w,z) f(w)\, dw \right) dz.$$

Formally, if $f = \delta_y$ for some $y \in \Omega$, this turns to

$$u(x) = G_\Omega(x,y) - \lambda \int_\Omega G_\Omega(x,z)\, G_\Omega(y,z)\, dz.$$

Hence, if $x \in \Omega$, the positivity of $u(x)$ is equivalent to

$$\lambda < \left(\frac{\int_\Omega G_\Omega(x,z)\, G_\Omega(y,z)}{G_\Omega(x,y)}\, dz \right)^{-1}$$

for all $y \in \Omega$. For results on the positivity preserving property for coupled elliptic systems we refer to [27] and [24].

3.5.2. Maximal lifetime and a conjecture

Corollary 3.32 says that $\sup_{x,y\in\Omega} \mathbb{E}_x^y(\tau_\Omega)$ is finite for bounded two dimensional domains. Where do the points x and y have to be situated such that the lifetime gets maximal? In a simply connected domain (which is sufficiently smooth such that $\mathbb{E}_x^y(\tau_\Omega)$ can be extended to the boundary, see Section 3.4.3), one would expect that one could always increase the lifetime by moving the starting point and the endpoint towards the boundary in 'opposite directions' (whatever that means). This has been conjectured by several authors, we reproduce [9, Conjecture 3].

Conjecture 3.33. *If Ω is a (simply connected) planar domain, then*

$$\sup_{x,y\in\overline{\Omega}} \mathbb{E}_x^y(\tau_\Omega) = \sup_{x,y\in\partial\Omega} \mathbb{E}_x^y(\tau_\Omega).$$

Remark 3.34. Strictly speaking, we abuse notation by writing $\sup_{x,y\in\overline{\Omega}}$ and $\sup_{x,y\in\partial\Omega}$ since $\mathbb{E}_x^y(\tau_\Omega)$ does not have to exist at every boundary point (see the example at the end of Section 3.4.3). We understand the supremum as being taken over all existing values of $\mathbb{E}_x^y(\tau_\Omega)$ and all the possibly existing different limits if approaching the boundary from different sides. Corollary 3.32 makes sure that the supremum is finite.

To our knowledge, this conjecture has not been proven so far. In 1993, Griffin, McConnell, and Verchota (see [17, Corollary 2.4]) were able to show that

$$\sup_{x\in\partial\Omega, y\in\Omega} \mathbb{E}_x^y(\tau_\Omega) = \sup_{x,y\in\partial\Omega} \mathbb{E}_x^y(\tau_\Omega) \tag{3.33}$$

holds for simply connected domains by showing that if the starting point x is situated at the boundary, the lifetime does not decrease, if the endpoint y moves towards the boundary along a hyperbolic geodesic.

If Ω is a disk, Conjecture 3.33 is true. This was shown by Dall'Acqua, Grunau and Sweers [9] in 2004. Later, in 2008, Dittmar [11] gave a proof by elementary conformal mapping techniques.

The assumption that Ω is planar and simply connected seems to be crucial. On the one hand, in [14], it is shown that there is a two-dimensional simply connected surface $S \subset \mathbb{R}^3$ (see Figure 3.10) where

$$\sup_{x,y\in\overline{S}} \mathbb{E}_x^y(\tau_S) > \sup_{x,y\in\partial S} \mathbb{E}_x^y(\tau_S).$$

On the other hand, we show in Theorem 7.1 of Part II that there is a multiply connected domain $\Omega \subset \mathbb{R}^2$ where

$$\left(\sup_{x,y\in\overline{\Omega}} \mathbb{E}_x^y(\tau_\Omega) \geq\right) \mathbb{E}_{x_0}^{y_0}(\tau_\Omega) > \sup_{x,y\in\partial\Omega} \mathbb{E}_x^y(\tau_\Omega)$$

for some interior points x_0, y_0 of Ω. This implies that some kind of maximum principle argument is not enough to show Conjecture 3.33.

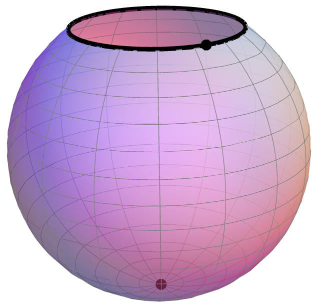


Figure 3.10.: The surface treated in [14]. It resembles a fish bowl with a small aperture. The lifetime of our conditioned Brownian motion between the south pole and boundary points can be made arbitrarily large by choosing the aperture sufficiently small.

3.5.3. Maximal lifetime and domain shape

Let us for a moment write

$$s(\Omega) := \sup_{x,y\in\overline{\Omega}} \mathbb{E}_x^y(\tau_\Omega), \quad s_b(\Omega) := \sup_{x\in\partial\Omega, y\in\overline{\Omega}} \mathbb{E}_x^y(\tau_\Omega), \quad \text{and } s_{bb}(\Omega) := \sup_{x,y\in\partial\Omega} \mathbb{E}_x^y(\tau_\Omega).$$

By the definition of the supremum, it always holds that

$$s_{bb}(\Omega) \le s_b(\Omega) \le s(\Omega),$$

and (3.33) means that for simply connected domains, the first inequality is an equality. Conjecture 3.33 says that for simply connected domains, equality holds everywhere,

$$s_{bb}(\Omega) = s_b(\Omega) = s(\Omega).$$

Furthermore, in this notation, Corollary 3.32 can be reformulated by stating that there is some absolute $C > 0$ such that

$$\frac{s(\Omega)}{|\Omega|} \le C \tag{3.34}$$

holds for all $\Omega \subset \mathbb{R}^2$. Is there an explicit value for C? In [17], it is shown that if one restricts oneself to convex domains, the lowest constant C such that (3.34) holds is $\frac{1}{2\pi}$ (remember we use Δ instead of $\frac{1}{2}\Delta$).

Is there also a lower bound for the quotient in (3.34)? In 1991, Xu [31] gave answers to this question. A consequence of [31, Theorem 2] is that there is some absolute constant $\gamma > 0$, such that for convex domains,

$$\gamma \le \frac{s_b(\Omega)}{|\Omega|} \left(\le \frac{s(\Omega)}{|\Omega|} \right). \tag{3.35}$$

Moreover, Xu shows in [31, Theorem 3] that there is a simply connected (non-convex) domain of *infinite* area where $s(\Omega) < \infty$, which implies that such a γ in general does not exist. Later, in 2002, Kawohl and Sweers went a step further and showed in [20, Corollary 2] that for given $M > 0$, even a starshaped domain with *finite* area, in fact $|\Omega| = 1$, exists such that

$$\frac{s(\Omega)}{|\Omega|} \leq \frac{1}{M}.$$

Hence an absolute constant $\gamma > 0$ such that (3.35) holds at least for bounded domains does not exist. Two years later, Bass, Horák and McKenna (see [3, Theorem 1]) gave another example for a simply connected domain where $\frac{s(\Omega)}{|\Omega|}$ can be chosen arbitrarily small.

Now let us restrict ourselves to convex domains. What is the optimal γ for (3.35) to hold? Or, put the other way round, is there some domain Ω such that

$$\frac{s(\Omega)}{|\Omega|} \leq \frac{s(\Omega^*)}{|\Omega^*|}$$

holds for all convex domains Ω^* with $|\Omega^*| = |\Omega|$? For the unit disk $B_1(0,0)$, we know explicitly, see [9] and (6.13), that

$$\frac{s_{bb}(B_1(0,0))}{|B_1(0,0)|} = \frac{s_b(B_1(0,0))}{|B_1(0,0)|} = \frac{s(B_1(0,0))}{|B_1(0,0)|} = \frac{2\log(2) - 1}{\pi} \approx 0.122961.$$

One could think that the disk is a good candidate for giving the optimal γ, as it maximizes the area for a given diameter, and one might be tempted to think that there is some relation between the diameter and the maximal lifetime. However, this is wrong. In 2002, Kawohl and Sweers (see [19, Theorem 1]) showed that for a sector S of the unit disk with angle $\frac{1}{3}\pi$, it holds that

$$\frac{s_b(S)}{|S|} = \frac{3}{8\pi} \approx 0.119366 < 0.122961 \approx \frac{s_b(B_1(0,0))}{|B_1(0,0)|}.$$

Hence the disk is not optimal for minimizing $\frac{s_b(\Omega)}{|\Omega|}$, and, if Conjecture 3.33 holds, then it neither is a minimizer for $\frac{s(\Omega)}{|\Omega|}$. To our knowledge it is still open to find a minimizer amongst all convex sets.

By the way, knowing the example of the sector, one can understand why the disk maybe was not such a good guess at all. Surely $s_{bb}(\Omega)$ is related to the diameter of the set, as larger distances take more time to be travelled through. However, there should also be a relation between $s_{bb}(\Omega)$ and the width of the domain: if the domain is narrow, then a particle that wanders around too much gets killed at the boundary, which decreases the average lifetime.

4. Convergence of $\mathbb{E}_x^y \left(\tau_{\Omega_l}\right)$

In this chapter, we look at the behaviour of $\mathbb{E}_x^y (\tau_{\Omega_l})$ as $l \to 0$. We distinguish the cases that both x and y lie on the same side of Γ (treated in Section 4.2) and that x and y lie on different sides of Γ (treated in Section 4.3). In both cases, we need some convergence results for G_{Ω_l} and K_{Ω_l}, which will be provided in Section 4.1.

In order to be able to prove the results properly, we introduce some variable $l_0 > 0$ and consider only gap widths l with $0 < l < l_0$, which implies $\Omega_l \subset \Omega_{l_0}$ for all l that are considered.

4.1. Convergence of G_{Ω_l}

As before, we make use of the conformal map $\mathbf{h} : \mathbf{A} \to \tilde{\mathbf{A}}$ of Lemma 3.4 and the corresponding 2-D map $h : A \to \tilde{A}$ and set $\tilde{x} = h(x)$. For the inverse mapping, we write $\tilde{h} = h^{inv}$.

Theorem 4.1. *Let $\rho > 0$. There are $C > 0$ and $l_1 < l_0$, such that the following holds: If $x \in A$ with $|x - w_j| > \rho$ for $j = 1, \dots, k$ and if $z \in A$ somewhere, $z \neq x$, then*

$$0 \leq G_{\Omega_l}(x, z) - G_A(x, z) \leq C d_{\tilde{A}}(\tilde{x})\, l\, |\log l|$$

for $0 < l < l_1$.

Remark 4.2. The estimate in Theorem 4.1 and the following results are uniform in the sense that C neither depends on l nor on x and z.

Proof. The first inequality is clear by (3.14) as $A \subset \Omega_l$. In order to show the second one, we define

$$u_l(x, z) := G_{\Omega_l}(x, z) - G_A(x, z) \tag{4.1}$$

and work in the smoother domain $\tilde{A}$ by setting

$$\tilde{u}_l(\tilde{x}, \tilde{z}) := u_l(x, z) = \left(u_l \circ \left(\tilde{h} \times \tilde{h}\right)\right)(\tilde{x}, \tilde{z}). \tag{4.2}$$

We fix $z \in A$. The function $x \mapsto u_l(x, z)$ is harmonic on A. The transformation property of the Laplace operator (3.15) yields that $\tilde{x} \mapsto \tilde{u}_l(\tilde{x}, \tilde{z})$ is harmonic on $\tilde{A}$. The Green functions G_A and G_{Ω_l} are both zero on the common boundary parts, and that is why $\tilde{u}_l(., \tilde{z})$ satisfies the following boundary condition.

$$\tilde{u}_l(\tilde{x}, \tilde{z}) = \begin{cases} 0 & \text{if } \tilde{x} \in \partial\tilde{A} \setminus \left(\bigcup_{j=1}^k \tilde{\Gamma}_{j,l}\right) \\ G_{\Omega_l}(x, z) & \text{if } \tilde{x} \in \bigcup_{j=1}^k \tilde{\Gamma}_{j,l} \end{cases}$$

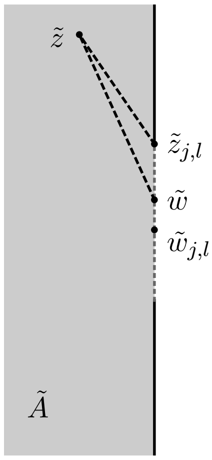


Figure 4.1.: A possible configuration of $\tilde{z}$ and $\tilde{z}_{j,l}$.

Consequently, by means of the Poisson kernel, $\tilde{u}_l(., \tilde{z})$ can be written as

$$\tilde{u}_l(\tilde{x}, \tilde{z}) = \int_{\bigcup_{j=1}^k \tilde{\Gamma}_{j,l}} K_{\tilde{A}}(\tilde{x}, \tilde{w})\, G_{\Omega_l}(w, z)\, d\sigma(\tilde{w}).$$

As Ω_{l_0} is bounded, there is some $M > 0$ such that $\Omega_l \subset \Omega_{l_0} \subset B_M(w)$ for all $l \leq l_0$ and all $w \in \Omega_l$. This implies (see (3.14) again) $G_{\Omega_l}(w, z) \leq G_{B_M(w)}(w, z) = \frac{1}{2\pi} \log\left(\frac{M}{|w-z|}\right)$ (which is positive in Ω_l).

Let $l_1 < \min\left\{\frac{1}{2}R, l_0\right\}$, where R is the radius in Corollary 3.10 such that $\mathbf{h} : \mathbf{A} \to \tilde{\mathbf{A}}$ has an analytic extension on $B_R(\mathbf{w}_j)$. Then the gap width of Γ_{j,l_1} is sufficiently small that we can compare distances in A and $\tilde{A}$ according to Corollary 3.11. Moreover, let $l_1 < \frac{1}{2}\rho$. For $l < l_1$, we apply the estimate from Proposition 3.26 and conclude that there is some $C_1 > 0$ such that

$$\begin{aligned}
\tilde{u}_l(\tilde{x}, \tilde{z}) &\leq C_1 \sum_{j=1}^k \int_{\tilde{\Gamma}_{j,l}} \frac{d_{\tilde{A}}(\tilde{x})}{|x-w|^2} \log\left(\frac{M}{|w-z|}\right) d\sigma(\tilde{w}) \\
&\leq C_1 \left(\frac{4}{3\rho}\right)^2 d_{\tilde{A}}(\tilde{x}) \sum_{j=1}^k \int_{\tilde{\Gamma}_{j,l}} \log\left(\frac{M}{|w-z|}\right) d\sigma(\tilde{w}).
\end{aligned}$$

For each $\tilde{z} \in \tilde{A}$ and each $j \in \{1, \ldots, k\}$, there is some $\tilde{z}_{j,l} \in \overline{\tilde{\Gamma}_{j,l}}$ such that $|\tilde{z} - \tilde{z}_{j,l}| = \min\left\{|\tilde{z} - \tilde{v}|\, ; \tilde{v} \in \overline{\tilde{\Gamma}_{j,l}}\right\}$, see Figure 4.1. Hence $|\tilde{w} - \tilde{z}_{j,l}| \leq |\tilde{w} - \tilde{z}| + |\tilde{z} - \tilde{z}_{j,l}| \leq 2\,|\tilde{w} - \tilde{z}|$. This implies

$$\begin{aligned}
\int_{\tilde{\Gamma}_{j,l}} \log\left(\frac{M}{|w-z|}\right) d\sigma(\tilde{w}) &\leq \int_{\tilde{\Gamma}_{j,l}} \log\left(\frac{2C_2M}{|\tilde{w} - \tilde{z}_{j,l}|}\right) d\sigma(\tilde{w}) \\
&\leq 2\int_0^{C_3 l} \log\left(\frac{2C_2M}{s}\right) ds = 2C_3\left(-l\log l + l\log\left(\frac{2C_2M}{C_3}\right) + l\right).
\end{aligned}$$

The constants C_2 and C_3 come from comparing distances and gap widths in A and $\tilde{A}$ according to Corollary 3.11. Integration along $\tilde{\Gamma}_{j,l}$ was simple as $\tilde{\Gamma}_{j,l}$ is a straight line. If l gets arbitrarily small, $l\,|\log l|$ dominates the sum on the right hand side. Thus, after maybe restricting the size of l_1 and hence l one more time, the lemma is shown. □

In the preceding theorem, we assumed that x stays away from the gaps, whereas z was allowed to be anywhere in A. Later, we will look at cases where z stays away from the gaps, too, at a distance of at least $C_1 l^\alpha$. Then the order of the estimate gets better.

Corollary 4.3. *Let $C_1, \rho > 0$ and $0 \le \alpha < 1$. There are $C > 0$ and $l_1 < l_0$ such that the following holds: If $x, z \in A$, $z \ne x$, with $|x - w_j| > \rho$ and $|z - w_j| > C_1 l^\alpha$ for $j = 1, \dots, k$, then*

$$0 \le G_{\Omega_l}(x,z) - G_A(x,z) \le C d_{\tilde{A}}(\tilde{x}) \min\left\{ d_{\tilde{A}}(\tilde{z})\, l^{2-2\alpha}, l^{2-\alpha} \right\} |\log l|$$

for $0 < l < l_1$.

Remark 4.4. The smaller α, the higher is the order of convergence. This is plausible as $G_A(x,z) = 0$ for $z \in \Gamma_{j,l}$, whereas $G_{\Omega_l}(x,z)$ is strictly positive there. The influence of this difference gets smaller, the farther z stays away from the gap (which is equal to smaller α). On the contrary, if $\alpha \to 1$, z gets too close to the gap, so the order of the estimate approaches the one in Theorem 4.1.

Proof. Again, we look at $\tilde{u}_l$, defined in (4.2) and (4.1). As it is also harmonic in the second component and nonzero only on the gaps, we can write

$$\tilde{u}_l(\tilde{x}, \tilde{z}) = \sum_{j=1}^{k} \int_{\tilde{\Gamma}_{j,l}} K_{\tilde{A}}(\tilde{z}, \tilde{w})\, \tilde{u}_l(\tilde{x}, \tilde{w})\, d\sigma(\tilde{w}).$$

By continuity, the estimate for $\tilde{u}_l(\tilde{x}, \tilde{w})$ of Theorem 4.1 holds also for $\tilde{w} \in \partial \tilde{A}$ if $l < l_1$. Again, we use the estimate for $K_{\tilde{A}}$ from Proposition 3.26 to obtain

$$\tilde{u}_l(\tilde{x}, \tilde{z}) \le C_2 d_{\tilde{A}}(\tilde{x})\, l\, |\log l| \sum_{j=1}^{k} \int_{\tilde{\Gamma}_{j,l}} \frac{d_{\tilde{A}}(\tilde{z})}{|\tilde{z} - \tilde{w}|^2}\, d\sigma(\tilde{w}).$$

for some $C_2 > 0$. We have $d_{\tilde{A}}(\tilde{z}) \le |\tilde{z} - \tilde{w}|$, hence $\frac{d_{\tilde{A}}(\tilde{z})}{|\tilde{z} - \tilde{w}|} \le 1$. Moreover, according to Corollary 3.11, there is a $C > 0$ such that

$$|\tilde{z} - \tilde{w}| \ge |\tilde{z} - \tilde{w}_j| - |\tilde{w}_j - \tilde{w}| \ge \frac{1}{C} |z - w_j| - \tilde{l}_j \ge \frac{C_1}{C} l^\alpha - Cl \ge \left(\frac{C_1}{C} - l^{1-\alpha} C \right) l^\alpha$$

for $\tilde{w} \in \tilde{\Gamma}_{j,l}$. If $l_1 > 0$ is chosen small enough, $\left(\frac{C_1}{C} - l^{1-\alpha} C \right) \ge \frac{C_1}{2C}$ for $0 < l < l_1$. Consequently,

$$\frac{d_{\tilde{A}}(\tilde{z})}{|\tilde{z} - \tilde{w}|^2} \le \min\left\{ d_{\tilde{A}}(\tilde{z}) \left(\frac{2C}{C_1} \right)^2 l^{-2\alpha}, \frac{2C}{C_1} l^{-\alpha} \right\}$$

and

$$\tilde{u}_l(\tilde{x},\tilde{z}) \le C_3 d_{\tilde{A}}(\tilde{x})\, l\,|\log l| \min\left\{d_{\tilde{A}}(\tilde{z})\, l^{-2\alpha}, l^{-\alpha}\right\} \sum_{j=1}^{k} \int_{\tilde{\Gamma}_{j,l}} 1\, d\sigma(\tilde{w})$$
$$\le C_4 d_{\tilde{A}}(\tilde{x}) \min\left\{d_{\tilde{A}}(\tilde{z})\, l^{2-2\alpha}, l^{2-\alpha}\right\} |\log l|$$

for some $C_3, C_4 > 0$, which proves the corollary. □

Until now we have only considered the case that both x and z lie in A. What happens if the points are situated on different sides of Γ? If $x \in A$, $z \in B$, then the function $z \mapsto G_{\Omega_l}(x,z)$ is harmonic on B. Consequently, it attains its maximum at some $z \in \partial B$. As $G_{\Omega_l}(x,.)$ is zero on the boundary parts $\partial B \cap \partial\Omega_l$, $G_{\Omega_l}(x,.)$ is bounded by the values on the $\Gamma_{j,l}$ parts. By continuity, the estimate of Theorem 4.1 holds also for $z \in \Gamma_{j,l}$, and as $G_A(x,z) = 0$ for $z \in \Gamma_{j,l}$, we have obtained an upper bound. We sum up this in the following corollary.

Corollary 4.5. *Let $\rho > 0$. There are $C > 0$ and $l_1 < l_0$ such that the following holds: If $x \in A$ with $|x - w_j| > \rho$ for $j = 1,\dots,k$ and $z \in B$, then*

$$0 \le G_{\Omega_l}(x,z) \le C d_{\tilde{A}}(\tilde{x})\, l\,|\log l|$$

for $0 < l < l_1$.

As in Corollary 4.3, we can go even one step further. Let $z \in B$ stay away from the boundary, too, let's say $|z - w_j| > C_1 l^\alpha$ for some $C_1 > 0$, $0 \le \alpha < 1$ and all $j = 1,\dots,k$. Let g map B to a smoother domain $\hat{B}$ according to Lemma 3.4. We write $\hat{z} = g(z)$. Similar to above, we make use of the Poisson kernel representation in $\hat{B}$ to get

$$G_{\Omega_l}(x,z) = \sum_{j=1}^{k} \int_{\hat{\Gamma}_{j,l}} K_{\hat{B}}(\hat{z},\hat{w})\, G_{\Omega_l}(x,w)\, d\sigma(\hat{w})$$
$$\le C_2 \sum_{j=1}^{k} \int_{\hat{\Gamma}_{j,l}} \frac{d_{\hat{B}}(\hat{z})}{|\hat{z} - \hat{w}_j|^2} d_{\tilde{A}}(\tilde{x})\, l\,|\log l|\, d\sigma(\hat{w})$$
$$\le C_3 \sum_{j=1}^{k} d_{\tilde{A}}(\tilde{x})\, l\,|\log l| \int_{\hat{\Gamma}_{j,l}} \min\left\{\frac{d_{\hat{B}}(\hat{z})}{l^{2\alpha}}, \frac{1}{l^\alpha}\right\} d\sigma(\hat{w})$$
$$\le C_4 d_{\tilde{A}}(\tilde{x}) \min\left\{d_{\hat{B}}(\hat{z})\, l^{2-2\alpha}, l^{2-\alpha}\right\} |\log l|$$

for some constants $C_2, C_3, C_4 > 0$ and sufficiently small $0 < l < l_1$. We made use of the estimate on the Poisson kernel of Proposition 3.26 and the fact that both $d_{\hat{B}}(\hat{z})$ and $|\hat{z} - \hat{w}_j|^{-2}$ are bounded. We formulate the following corollary.

Corollary 4.6. *Let $C_1, \rho > 0$ and $0 \le \alpha < 1$. There are $C > 0$ and $l_1 < l_0$ such that the following holds: If $x \in A$ with $|x - w_j| > \rho$ for $j = 1,\dots,k$ and $z \in B$ with $|z - w_j| > C_1 l^\alpha$ for $j = 1,\dots,k$, then*

$$0 \le G_{\Omega_l}(x,z) \le C d_{\tilde{A}}(\tilde{x}) \min\left\{d_{\hat{B}}(\hat{z})\, l^{2-2\alpha}, l^{2-\alpha}\right\} |\log l|$$

for $0 < l < l_1$.

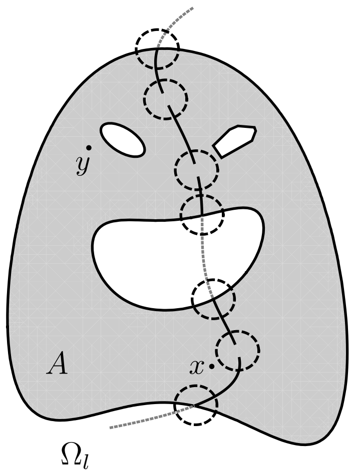


Figure 4.2.: The setting in Theorem 4.7: both points x and y are in A and stay away from the gaps and the boundary singularities.

4.2. Convergence of $\mathbb{E}_x^y(\tau_{\Omega_l})$ – both points in A

Theorem 4.7. *Let $\rho > 0$. There are $C > 0$ and $l_1 \in (0, l_0)$ such that the following holds: If*

- *$x \in A$ with $|x - w_j| > \rho$ for $j = 1, \ldots, k$ and $|x - s| > \rho$ for all $s \in S$, and if*
- *$y \in A$ with $|y - w_j| > \rho$ for $j = 1, \ldots, k$ and $|y - s| > \rho$ for all $s \in S$,*

then

$$|\mathbb{E}_x^y(\tau_{\Omega_l}) - \mathbb{E}_x^y(\tau_A)| \leq Cl\,|\log l|$$

for $0 < l < l_1$.

Remark 4.8. The convergence is uniform in x and y as long as both points stay away from the gaps and the boundary singularities.[1]

Proof. The difference of lifetimes is given by

$$\begin{aligned} |\mathbb{E}_x^y(\tau_{\Omega_l}) - \mathbb{E}_x^y(\tau_A)| &= \left| \int_{\Omega_l} \frac{G_{\Omega_l}(x,z)G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\,dz - \int_A \frac{G_A(x,z)G_A(z,y)}{G_A(x,y)}\,dz \right| \\ &\leq \int_A \left| \frac{G_{\Omega_l}(x,z)G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)} - \frac{G_A(x,z)G_A(z,y)}{G_A(x,y)} \right| dz + \int_B \frac{G_{\Omega_l}(x,z)G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\,dz \end{aligned} \tag{4.3}$$

First we look at the second integral. For the numerator, we use the estimate on G_{Ω_l} from Corollary 4.5. The denominator is greater than G_A, for which we find an estimate

[1] Concerning the boundary singularities, see Remark 4.9 below the proof.

from below in Corollary 3.28 and then a further estimate in inequality (3.23). Everything put together implies that there is a $C_1 < 0$ and some $l_1 < l_0$ such that for all $0 < l < l_1$, we have

$$\left| \int_B \frac{G_{\Omega_l}(x,z) G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)} \, dz \right| \leq C_1 \int_B \frac{d_{\tilde{A}}(\tilde{x})\, l \, |\log l| \cdot d_{\tilde{A}}(\tilde{y})\, l\, |\log l|}{d_{\tilde{A}}(\tilde{x})\, d_{\tilde{A}}(\tilde{y})} \, dz$$
$$= C_1 \int_B dz \cdot l^2 \, |\log l|^2 ,$$

where $\tilde{x} = h(x)$ for $h : A \to \tilde{A}$ as always.

Now we turn to the first integral on the right hand side of (4.3) and rewrite the integrand, replacing G_{Ω_l} by $(G_{\Omega_l} - G_A) + G_A$ in the third step.

$$\begin{aligned}
&\left| \frac{G_{\Omega_l}(x,z) G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)} - \frac{G_A(x,z) G_A(z,y)}{G_A(x,y)} \right| \\
&\quad = \left| \frac{G_{\Omega_l}(x,z) G_{\Omega_l}(z,y) G_A(x,y) - G_A(x,z) G_A(z,y) G_{\Omega_l}(x,y)}{G_{\Omega_l}(x,y) G_A(x,y)} \right| \\
&\quad \leq \frac{|G_{\Omega_l}(x,z) G_{\Omega_l}(z,y) G_A(x,y) - G_A(x,z) G_A(z,y) G_{\Omega_l}(x,y)|}{G_A(x,y) G_A(x,y)} \\
&\quad \leq \frac{|G_{\Omega_l}(x,z) - G_A(x,z)|\, G_A(z,y)}{G_A(x,y)} + \frac{G_A(x,z)\, |G_{\Omega_l}(z,y) - G_A(z,y)|}{G_A(x,y)} \\
&\qquad + \frac{|G_{\Omega_l}(x,z) - G_A(x,z)|\, |G_{\Omega_l}(z,y) - G_A(z,y)|}{G_A(x,y)} \\
&\qquad + \frac{G_A(x,z) G_A(z,y)\, |G_{\Omega_l}(x,y) - G_A(x,y)|}{G_A(x,y) G_A(x,y)}
\end{aligned} \tag{4.4}$$

Each term of this sum can be estimated from above. We start with the first one: An upper bound for $|G_{\Omega_l} - G_A|$ is given by Theorem 4.1. The Green function G_A can be estimated from above and below by Corollary 3.28 together with (3.24) and (3.23) of Section 3.3.8, respectively. The next two terms of (4.4) work in a similar way. In the last term, we leave the expression $G_A(x,z) G_A(z,y) \,(G_A(x,y))^{-1}$ as it is and use only estimates on the other two factors. To sum up, we get that there is a $C_2 > 0$ such that the right hand side of (4.4) is bounded from above by

$$\begin{aligned}
&C_2 \frac{d_{\tilde{A}}(\tilde{x})\, d_{\tilde{A}}(\tilde{y})}{d_{\tilde{A}}(\tilde{x})\, d_{\tilde{A}}(\tilde{y})\, |\tilde{z} - \tilde{y}|} l\, |\log l| + C_2 \frac{d_{\tilde{A}}(\tilde{x})\, d_{\tilde{A}}(\tilde{y})}{d_{\tilde{A}}(\tilde{x})\, d_{\tilde{A}}(\tilde{y})\, |\tilde{z} - \tilde{x}|} l\, |\log l| \\
&\qquad + C_2 \frac{d_{\tilde{A}}(\tilde{x})\, d_{\tilde{A}}(\tilde{y})}{d_{\tilde{A}}(\tilde{x})\, d_{\tilde{A}}(\tilde{y})} l^2\, |\log l|^2 + C_2 \frac{G_A(x,z) G_A(z,y)}{G_A(x,y)} \frac{d_{\tilde{A}}(\tilde{x})\, d_{\tilde{A}}(\tilde{y})}{d_{\tilde{A}}(\tilde{x})\, d_{\tilde{A}}(\tilde{y})} l\, |\log l| \\
&\quad = C_2 \frac{1}{|\tilde{z} - \tilde{y}|} l\, |\log l| + C_2 \frac{1}{|\tilde{z} - \tilde{x}|} l\, |\log l| + C_2 l^2\, |\log l|^2 + C_2 \frac{G_A(x,z) G_A(z,y)}{G_A(x,y)} l\, |\log l| .
\end{aligned} \tag{4.5}$$

Now we integrate over $z \in A$. As y stays away from the boundary singularities, $|\tilde{z} - \tilde{y}| \geq C^{-1} |z - y|$ for some $C \geq 1$ according to Corollary 3.11. Moreover, A is bounded, so there exists some $M > 0$ such that $A \subset B_M(y)$ for all $y \in A$. Hence

$$\int_A \frac{1}{|\tilde{z} - \tilde{y}|}\, dz \le C \int_A \frac{1}{|z-y|}\, dz \le C \int_{B_M(y)} \frac{1}{|z-y|}\, dz = C \int_{B_M(0)} \frac{1}{z}\, dz. \tag{4.6}$$

We conclude that the first two terms on the right hand side of (4.5) have an upper bound that does neither depend on x nor on y. The third term does not depend on z at all, so integration of the third term gives $C_2 l^2 |\log l|^2$ times the area of A. For small $l > 0$, $l^2 |\log l|^2$ is dominated by $l |\log l|$, so the order of convergence will be $l |\log l|$. Integration of the $3G$ expression in the fourth term gives $\mathbb{E}_x^y(\tau_A)$, which is bounded by a constant times the area of A, independent of x and y, see Corollary 3.32. Hence there exist $C > 0$ and $l_1 < l_0$, independent of x and y such that

$$\int_A \left| \frac{G_{\Omega_l}(x,z) G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)} - \frac{G_A(x,z) G_A(z,y)}{G_A(x,y)} \right| dz \le Cl\, |\log l|$$

for all $0 < l < l_1$, which completes the proof. □

Remark 4.9. We conjecture that the convergence of $\mathbb{E}_x^y(\Omega_l)$ remains uniform in x and y even if the points are close to the boundary singularities in S. In fact, the integration in (4.6) is the only step when we need the assumption that x and y stay away from the boundary singularities. We use the estimate $|\tilde{z} - \tilde{y}|^{-1} \le C\, |z-y|^{-1}$ there. We could also proceed as follows, making use of (3.17).

$$\begin{aligned} \int_A \frac{1}{|\tilde{z} - \tilde{y}|}\, dz &= \int_A \frac{1}{|h(z) - \tilde{y}|}\, dz \\ &= \int_{\tilde{A}} \frac{1}{\left| \left(h \circ \tilde{h}\right)(\tilde{z}) - \tilde{y} \right|} \left| \tilde{\mathbf{h}}'(\tilde{\mathbf{z}}) \right|^2 d\tilde{z} \\ &= \int_{\tilde{A}} \frac{1}{|\tilde{z} - \tilde{y}|} \left| \tilde{\mathbf{h}}'(\tilde{\mathbf{z}}) \right|^2 d\tilde{z} \end{aligned}$$

We see that $\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{z}})\right|$ (with $\tilde{h} = h^{inv}$) could cause some troubles in the integration. Now let us assume that ∂A has a singularity in form of a cone with interior angle $\alpha\pi$ and $\alpha \in (0,2)$, see Figure 4.3. We move it to the origin. The corner is straightened by the mapping $\mathbf{z} \mapsto \tilde{\mathbf{z}} = \mathbf{z}^{\frac{1}{\alpha}}$, hence $\tilde{\mathbf{h}}(\tilde{\mathbf{z}}) \approx \tilde{\mathbf{z}}^{\alpha}$ near the singularity. This gives an order of $2\alpha - 2$ for $\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{z}})\right|^2$, and in the 'worst case' for $\tilde{y} = 0$, and after switching to polar coordinates, we obtain an integral of the kind

$$\int_0^1 \frac{r^{2\alpha-2}}{r}\, r dr = \int_0^1 r^{2\alpha-2}\, dr.$$

If the opening angle is larger than $\frac{1}{2}\pi$, this is still integrable and gives an upper bound independent of y.

What if the boundary singularity is even worse? We conjecture that the order of convergence of $\mathbb{E}_x^y(\Omega_l)$ does not depend on x and y, even if they are close to a singularity,

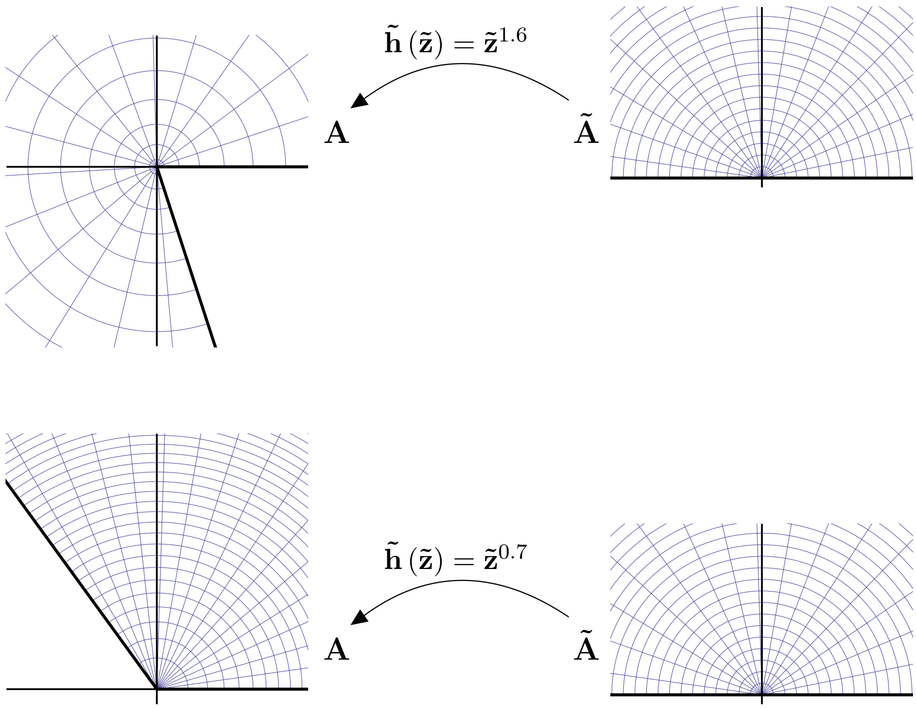


Figure 4.3.: Possible corners for different interior angles $\alpha\pi$ and the respective conformal maps from a domain with straight boundary.

as long as they stay away from the gaps. The factors of the type $|\tilde{y} - \tilde{z}|$ only appear in (4.5) because we need an estimate from above for $G_A(z, y) = G_{\tilde{A}}(\tilde{z}, \tilde{y})$. We could use the estimate of Corollary 3.28 directly, which leads to a logarithmic singularity. But in our proof, we need to be able to divide by the factor $d_{\tilde{A}}(\tilde{y})$, so we use the estimate of (3.24) instead and obtain the singularity of $|\tilde{y} - \tilde{z}|$. This seems to be the price we have to pay. Maybe some more careful estimates lead to a result allowing to approach the boundary singularities.

We can also go one step further in another direction. In our setting, see Section 2.1, we assume that the subdomains A and B are both bounded by a union of finitely many Jordan curves. In Part II, we look at a domain that consists of *several* subdomains and *several* paths with gaps. We apply Theorem 4.7 (and also Theorem 4.10) several times. For this, we first divide the large domain into two subdomains by closing the gap in the middle. Second, we divide these subdomains into subsubdomains and so on. To be precise, the subdomains of the first step are *not* bounded by Jordan curves, as the boundary near the gaps closed in the second step has domain on both sides (see Figure 4.4). However, the statement of Theorem 4.7 (and also Theorem 4.10) holds for such kind of domains, too. The proof works exactly the same way, we just have to put the boundary with domain on both sides to the set of boundary singularities S. The mapping to a smoother tilde domain will then be less regular. Points with domain on both sides will be mapped to two different boundary points of the smoother domain, see also the example at the end of Section 3.4.3. As long as we stay away from these points,

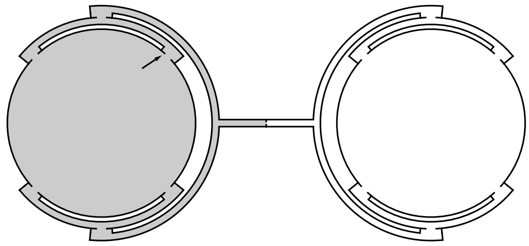


Figure 4.4.: The domain considered in Part II. The gray coloured subdomain is not bounded by a finite union of Jordan curves. The arrow marks a boundary part which has domain on both sides. Nevertheless, the statement of Theorem 4.7 (and also Theorem 4.10) holds if these boundary parts are put to the set of boundary singularities.

nothing bad can happen.

4.3. Convergence of $\mathbb{E}_x^y(\tau_{\Omega_l})$ – one point in A, one in B

Theorem 4.10. *Let $\rho > 0$. There exist $C > 0$ and $l_1 < l_0$ such that the following holds: If*

- *$x \in A$ with $|x - w_j| > \rho$ for $j = 1, \ldots, k$ and $|x - s| > \rho$ for all $s \in S$, and if*
- *$y \in B$ with $|y - w_j| > \rho$ for $j = 1, \ldots, k$ and $|y - s| > \rho$ for all $s \in S$,*

then

$$\left| \mathbb{E}_x^y(\tau_{\Omega_l}) - \sum_{j=1}^{k} \left(\mathbb{E}_x^{w_j}(\tau_A) + \mathbb{E}_{w_j}^y(\tau_B) \right) \frac{K_A(x, w_j)\, K_B(y, w_j)}{\sum_{m=1}^{k} K_A(x, w_m)\, K_B(y, w_m)} \right| \le C l^{\frac{1}{3}} |\log l|$$

for $0 < l < l_1$.

Remark 4.11. For an interpretation of the limit, we refer to Section 2.2.

Remark 4.12. As before in Theorem 4.7, we do not think that x and y really have to stay away from the boundary singularities in order to obtain uniform convergence. It is the same thing as before: At the very end of the proof, we have to integrate over singularities of the form $|\tilde{x} - \tilde{z}|^{-1}$, see (4.36). It is only there we need that $\tilde{x}$ is not close to a boundary singularity. For further comments on this, we refer to Remark 4.9.

Proof. In the proof, we will map both A and B to the smoother domains $\tilde{A}$ and $\hat{B}$ with the help of the transformations h and g, which are given by Lemma 3.4. Around each centre w_j of a gap $\Gamma_{j,l}$, within some ball $B_R(w_j)$, h can be extended even to the outside of A according to Corollary 3.10. The same holds for g, if R is chosen small enough. Moreover, we assume that R is so small that the the balls with radius R around the

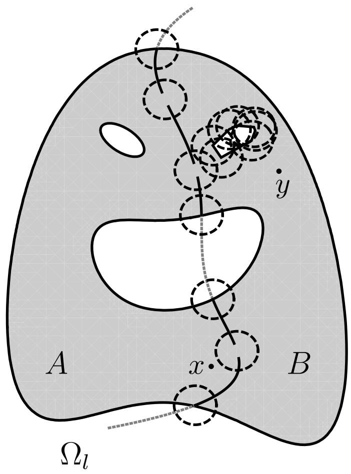


Figure 4.5.: The setting in Theorem 4.10: $x \in A$ and $y \in B$ lie in different subdomains. They both stay away from the gaps and the boundary singularities (with quite many of them in ∂B).

points w_j stay more or less within Ω_l, i.e., to be more precise, $B_R(w_j) \subset (\Omega_l \cup \Gamma)$ for all j. In the proof, we will work with neighbourhoods of the gaps whose sizes depend on l. Without mentioning it there again, we always assume that $l_1 > 0$ is so small that those neighbourhoods are contained in the corresponding balls $B_R(w_j)$, so that the transformations h and g are well–defined there.

After the domain transformation, the width $\tilde{l}_j$ of the gap $\tilde{\Gamma}_{j,l}$ depends both on l and j. Moreover, the width $\hat{l}_j$ of $\hat{\Gamma}_{j,l}$ does not necessarily equal $\tilde{l}_j$. Nevertheless, according to Corollary 3.11, there is a $C \geq 1$ such that

$$\frac{1}{C} \cdot l \leq \tilde{l}_j \leq C \cdot l \qquad \text{and} \qquad \frac{1}{C} \cdot l \leq \hat{l}_j \leq C \cdot l$$

holds for all $j = 1, \ldots, k$. Moreover, if C is chosen large enough, then

$$\frac{1}{C} \cdot |z - w| \leq |\tilde{z} - \tilde{w}| \leq C \cdot |z - w| \qquad \text{and} \qquad \frac{1}{C} \cdot |z - w| \leq |\hat{z} - \hat{w}| \leq C \cdot |z - w|$$

holds for $z \in \overline{A}$ (or $\overline{B}$, respectively) and $w \in \overline{\Gamma_{j,l}}$ for $l < l_1$. Throughout the proof, we will use C for this constant coming from the domain transformation and C_1, C_2, C_A, C_T,... for other constants that appear. For an illustration, see Figure 4.6.

At this point we make another remark on the gap $\tilde{\Gamma}_{j,l}$. We have defined $\Gamma_{j,l}$ to be a gap centred around w_j in the sense that the arc length of $\Gamma_{j,l}$ is $\frac{l}{2}$ in each direction starting from w_j. We have chosen $\tilde{A}$ in a way that after the domain transformation, $\tilde{\Gamma}_{j,l}$ is a subset of the second coordinate axis. However, it is not said that $\tilde{\Gamma}_{j,l}$ is centred around $\tilde{w}_j$. We define $\tilde{w}_j^*$ to be the centre of the gap $\tilde{\Gamma}_{j,l}$. It holds that $\left|\tilde{w}_j - \tilde{w}_j^*\right| \leq \frac{1}{2}\tilde{l}_j$.

Lemma 3.12 implies that for small $l_1 > 0$,

$$\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|^{-1} - C_1 \left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|^{-1} l \leq \frac{\tilde{l}_j}{l} \leq \left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|^{-1} + C_1 \left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|^{-1} l$$

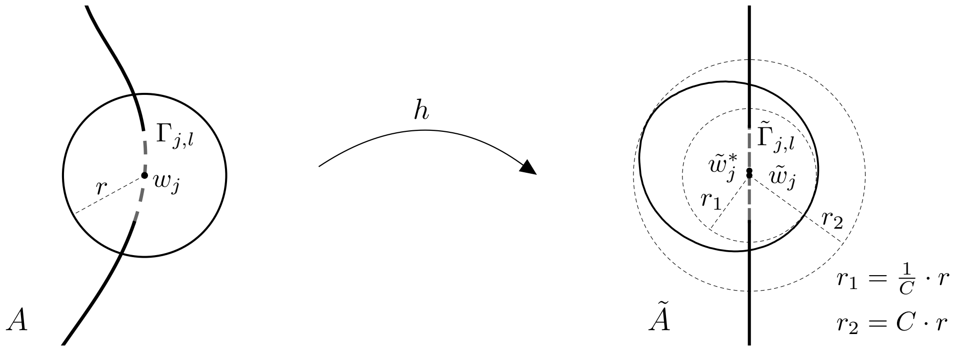


Figure 4.6.: The image of the circle $|z - w_j| = r$ under the domain transformation $h : A \to \tilde{A}$ (plus its extension beyond the gap) does not have to be a circle any more, but it holds that $\frac{1}{C} \cdot r \leq |\tilde{z} - \tilde{w}_j| \leq C \cdot r$. Moreover, observe that $\tilde{w}_j^*$, i.e., the center of the gap $\tilde{\Gamma}_{j,l}$, does not have to coincide with $\tilde{w}_j = h(w_j)$.

for $0 < l < l_1$ and for all $j \in \{1, \dots, k\}$. For $l \to 0$, everything tends to $\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|^{-1} \neq 0$. Hence we can assume l_1 to be so small such that

$$\frac{1}{2}\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|^{-1} \leq \frac{\tilde{l}_j}{l} \leq \frac{3}{2}\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|^{-1} \tag{4.7}$$

holds for all $j \in \{1, \dots, k\}$ and $0 < l < l_1$.[2]

In the proof, we will apply the convergence results for G_{Ω_l} of Section 4.1 several times. Without mentioning it then, we assume that $l_1 > 0$ is so small that the assertions of the estimates given in Section 4.1 hold. Moreover, at several points of the proof, we will state that something holds 'for l with $0 < l < l_1$ if $l_1 > 0$ is small enough'. We do not think that it increases readability to explicitly list all the thresholds here. Hence we simply suppose now that l_1 is small enough.

The proof is divided into several steps. To begin, let $\rho > 0$ and $x \in A$, $y \in B$ with $|x - w_j| > \rho$ and $|y - w_j| > \rho$ for $j = 1, \dots, k$, and let $0 < l < l_1$.

- The lifetime is given by

$$\begin{aligned}\mathbb{E}_x^y(\tau_{\Omega_l}) &= \int_{\Omega_l} \frac{G_{\Omega_l}(x,z)\, G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\, dz \\ &= \int_A \frac{G_{\Omega_l}(x,z)\, G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\, dz + \int_B \frac{G_{\Omega_l}(x,z)\, G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\, dz.\end{aligned}$$

[2] On first reading, all the tilde and hat signs might be a bit confusing. To get an idea about the proof, one could assume that A and B already are smooth enough, such that the estimates on Green functions and Poisson kernels hold. Then h and g are the identity and $\tilde{w}_j = \tilde{w}_j^* = w_j = \hat{w}_j$, all gap widths are equal, and so on. Hence one can simply ignore the tilde and hat signs and the step when we scale to a gap width independent of j.

We will only show that

$$\left| \int_A \frac{G_{\Omega_l}(x,z)\, G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\, dz - \sum_{j=1}^{k} \mathbb{E}_x^{w_j}(\tau_A) \frac{K_A(x,w_j)\, K_B(y,w_j)}{\sum_{m=1}^{k} K_A(x,w_m)\, K_B(y,w_m)} \right| \leq C_1 l^{\frac{1}{3}} |\log l|, \tag{4.8}$$

the other integral can be treated analogously.

- The function $z \mapsto G_{\Omega_l}(z,y)$ is harmonic in A. Hence representation with the help of the Poisson kernel gives

$$G_{\Omega_l}(z,y) = \sum_{j=1}^{k} \int_{\tilde{\Gamma}_{j,l}} K_{\tilde{A}}(\tilde{z},\tilde{w})\, G_{\Omega_l}(w,y)\, d\sigma(\tilde{w}) \tag{4.9}$$

for $z \in A$.

- Our considerations below only work if z (the variable of integration) stays away from the gaps. However, the contribution of those z which are close to the gap is not too big, which we will show in this step and the following one. We split the integral over the $3G$ expression into two integrals by

$$\begin{aligned}
&\int_A \frac{G_{\Omega_l}(x,z)\, G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\, dz \\
&= \int_{A \setminus \left(\bigcup_{j=1}^{k} B_{l^\alpha}(w_j)\right)} \frac{G_{\Omega_l}(x,z)\, G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\, dz + \sum_{j=1}^{k} \int_{A \cap B_{l^\alpha}(w_j)} \frac{G_{\Omega_l}(x,z)\, G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\, dz,
\end{aligned}$$

where α is a fixed exponent with $0 < \alpha < 1$. At the end of the proof, it will turn out that $\alpha = \frac{1}{3}$ is a good choice, but we work with general α now in order to illustrate how the choice of α has an influence on the order of convergence in (4.8). We set

$$I_j := \int_{A \cap B_{l^\alpha}(w_j)} \frac{G_{\Omega_l}(x,z)\, G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\, dz$$

and rewrite the terms $G_{\Omega_l}(.,y)$ with the help of (4.9) to get

$$I_j = \int_{A \cap B_{l^\alpha}(w_j)} \frac{G_{\Omega_l}(x,z) \sum_{m=1}^{k} \int_{\tilde{\Gamma}_{m,l}} K_{\tilde{A}}(\tilde{z},\tilde{w})\, G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})}{\sum_{m=1}^{k} \int_{\tilde{\Gamma}_{m,l}} K_{\tilde{A}}(\tilde{x},\tilde{w})\, G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})}\, dz.$$

The estimates on $K_{\tilde{A}}$ from Propositions 3.26 and 3.27 imply that there is a $C_1 > 0$ such that

$$I_j \leq C_1 \int_{A \cap B_{l^\alpha}(w_j)} \frac{G_{\Omega_l}(x,z) \sum_{m=1}^{k} \int_{\tilde{\Gamma}_{m,l}} \frac{d_{\tilde{A}}(\tilde{z})}{|\tilde{z}-\tilde{w}|^2} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})}{\sum_{m=1}^{k} \int_{\tilde{\Gamma}_{m,l}} \frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{w}|^2} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})}\, dz.$$

As Ω_l is bounded, $|\tilde{x} - \tilde{w}| \leq C\,|x - w|$ is bounded from above by a constant independent of $\tilde{x}$, $\tilde{w}$ and l. Moreover, by the definition of $d_{\tilde{A}}$, $d_{\tilde{A}}(\tilde{z}) \leq |\tilde{z} - \tilde{w}|$ for all $w \in \Gamma_{m,l}$.

This implies for the integral that there is a $C_2 > 0$ such that

$$I_j \le C_2 \frac{\int_{A\cap B_{l^\alpha}(w_j)} G_{\Omega_l}(x,z) \sum_{m=1}^k \int_{\tilde{\Gamma}_{m,l}} |\tilde{z}-\tilde{w}|^{-1} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})\, dz}{d_{\tilde{A}}(\tilde{x}) \sum_{m=1}^k \int_{\tilde{\Gamma}_{m,l}} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})}. \tag{4.10}$$

We choose $l_1 > 0$ to be small enough such that

$$|x-z| \ge |x-w_j| - |w_j - z| > \rho - l^\alpha > \rho - \frac{1}{2}\rho = \frac{1}{2}\rho$$

for $0 < l < l_1$ and that $B_{l^\alpha}(w_j) \subset B_{\frac{3}{2}l^\alpha}(w)$ for all $w \in \Gamma_{j,l}$.

This implies the following for the term $\int_{A\cap B_{l^\alpha}(w_j)} |\tilde{z}-\tilde{w}|^{-1}\, dz$: If $w \in \Gamma_{j,l}$, then

$$\int_{A\cap B_{l^\alpha}(w_j)} |\tilde{z}-\tilde{w}|^{-1}\, dz \le C \int_{A\cap B_{\frac{3}{2}l^\alpha}(w)} |z-w|^{-1}\, dz \le 3\pi C l^\alpha.$$

If $w \in \Gamma_{m,l}$ with $m \ne j$, then there is no singularity in the integrand. That is why, for small $l_1 > 0$, we get that

$$\int_{A\cap B_{l^\alpha}(w_j)} |\tilde{z}-\tilde{w}|^{-1}\, dz \le C_3 l^{2\alpha},$$

where the constant $C_3 > 0$ only depends on the distance between the gaps.

Summing up both cases and taking the lower rate of convergence, we have thus shown that there is a constant $C_4 > 0$ such that

$$\int_{A\cap B_{l^\alpha}(w_j)} |\tilde{z}-\tilde{w}|^{-1}\, dz \le C_4 l^\alpha$$

holds for all $w \in \Gamma_{m,l}$, $m = 1, \dots, k$.

We return to (4.10) and change the order of integration on the right hand side. Moreover, we make use of the upper bound on G_{Ω_l} implied by Theorem 4.1 combined with (3.24) together with the estimates above.

$$\begin{aligned}
I_j &\le C_2 \frac{\sum_{m=1}^k \int_{\tilde{\Gamma}_{m,l}} \int_{A\cap B_{l^\alpha}(w_j)} \left(\frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{z}|} + d_{\tilde{A}}(\tilde{x})\, l\, |\log l| \right) |\tilde{z}-\tilde{w}|^{-1} G_{\Omega_l}(w,y)\, dz d\sigma(\tilde{w})}{d_{\tilde{A}}(\tilde{x}) \sum_{m=1}^k \int_{\tilde{\Gamma}_{m,l}} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})} \\
&\le C_5 \left(2\rho^{-1} + l\,|\log l|\right) \frac{\sum_{m=1}^k \int_{\tilde{\Gamma}_{m,l}} \left(\int_{A\cap B_{l^\alpha}(w_j)} |\tilde{z}-\tilde{w}|^{-1}\, dz \right) G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})}{\sum_{m=1}^k \int_{\tilde{\Gamma}_{m,l}} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})} \\
&\le C_6 \left(2\rho^{-1} + l\,|\log l|\right) \frac{\sum_{m=1}^k \int_{\tilde{\Gamma}_{m,l}} l^\alpha G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})}{\sum_{m=1}^k \int_{\tilde{\Gamma}_{m,l}} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})} \\
&= C_6 \left(2\rho^{-1} + l\,|\log l|\right) l^\alpha \\
&\le C_7 l^\alpha
\end{aligned} \tag{4.11}$$

- The main part of the proof is to show that

$$\left| \int_{A\setminus\left(\bigcup_{n=1}^k B_{l^\alpha}(w_n)\right)} \frac{G_{\Omega_l}(x,z)\,G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\,dz \right.$$
$$\left. - \sum_{j=1}^k \int_{A\setminus\left(\bigcup_{n=1}^k B_{l^\alpha}(w_n)\right)} \frac{G_A(x,z)\,K_A(z,w_j)}{K_A(x,w_j)}\,dz \cdot \frac{K_A(x,w_j)\,K_B(y,w_j)}{\sum\limits_{m=1}^k K_A(x,w_m)\,K_B(y,w_m)} \right| \tag{4.12}$$

gets small for small l. We have already shown that the value of the first integral is not too far away from the value of the integral on the whole domain A. Using analogous estimates, we get

$$\begin{aligned}
&\int\limits_{A\cap\left(\bigcup_{n=1}^k B_{l^\alpha}(w_n)\right)} \frac{G_A(x,z)\,K_A(z,w_j)}{K_A(x,w_j)}\,dz \cdot \frac{K_A(x,w_j)\,K_B(y,w_j)}{\sum_{m=1}^k K_A(x,w_m)\,K_B(y,w_m)} \\
&\quad\le \int_{A\cap\left(\bigcup_{n=1}^k B_{l^\alpha}(w_n)\right)} \frac{G_A(x,z)\,K_{\tilde{A}}(\tilde{z},\tilde{w}_j)\,|\mathbf{h}'(\mathbf{w}_j)|}{K_{\tilde{A}}(\tilde{x},\tilde{w}_j)\,|\mathbf{h}'(\mathbf{w}_j)|}\,dz\cdot 1 \\
&\quad\le C_1 \int_{A\cap\left(\bigcup_{n=1}^k B_{l^\alpha}(w_n)\right)} \frac{\frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{z}|}\frac{d_{\tilde{A}}(\tilde{z})}{|\tilde{z}-\tilde{w}_j|^2}}{\frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{w}_j|^2}}\,dz \\
&\quad\le C_2 \int_{A\cap\left(\bigcup_{n=1}^k B_{l^\alpha}(w_n)\right)} \frac{1}{|\tilde{z}-\tilde{w}_j|}\,dz \le C_3 l^\alpha.
\end{aligned} \tag{4.13}$$

Hence it suffices to show convergence for (4.12) instead of (4.8).

- From now on we assume $z \in A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)$. The key idea of the proof is to rewrite $G_{\Omega_l}(z,y)$ and $G_{\Omega_l}(x,y)$ and then give approximations for it. The steps are the same for both expressions, but the order of convergence will differ as we have assumed that $|x-w_j|$ is bounded from below by a constant, whereas for $|z-w_j|$, we only have a lower bound of the form l^α. We treat both cases simultaneously, always looking at the expression with z first and then stating what happens if z is replaced by x. Figure 4.7 illustrates where the points we use are situated.

To start, we go back to (4.9). We want to replace $K_{\tilde{A}}(\tilde{z},\tilde{w})$ there by $K_{\tilde{A}}(\tilde{z},\tilde{w}_j)$. For this purpose, we make use of the Lipschitz regularity of $K_{\tilde{A}}$ formulated in Corollary 3.30. The term $G_{\Omega_l}(w,y)$ can be estimated by the inequality of Theorem 4.1 applied to B.[3]

[3] It also holds for $w \in \partial B$. For those w, $G_B(w,y) = 0$.

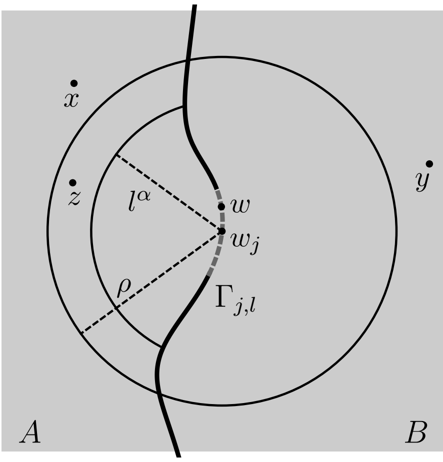


Figure 4.7.: Possible positions of the points used in the proof of Theorem 4.10 in relation to the gap $\Gamma_{j,l}$.

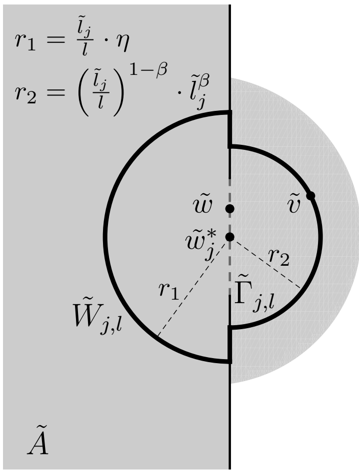


Figure 4.8.: $\tilde{W}_{j,l}$ and possible positions of used points in the transformed $\tilde{A}$ (including the h-extension beyond the gap).

We get that

$$\begin{aligned}
&\left|G_{\Omega_l}(z,y) - \sum_{j=1}^{k} \int_{\tilde{\Gamma}_{j,l}} K_{\tilde{A}}(\tilde{z},\tilde{w}_j)\, G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})\right| \\
&\qquad = \left|\sum_{j=1}^{k} \int_{\tilde{\Gamma}_{j,l}} \left(K_{\tilde{A}}(\tilde{z},\tilde{w}) - K_{\tilde{A}}(\tilde{z},\tilde{w}_j)\right) G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})\right| \\
&\qquad\qquad \le C_1 d_{\hat{B}}(\hat{y})\, l\, |\log l| \sum_{j=1}^{k} \left(C^{-1} l^\alpha - \tilde{l}_j\right)^{-2} \int_{\tilde{\Gamma}_{j,l}} |\tilde{w} - \tilde{w}_j|\, d\sigma(\tilde{w})
\end{aligned}$$

for some constant $C_1 > 0$. We have used here that $|\tilde{z} - \tilde{w}_j| \ge C^{-1} l^\alpha$ holds for $j = 1, \dots, k$ and taken this value for $\tilde{\rho}$ in Corollary 3.30. If $l_1 < (2C^2)^{-\frac{1}{1-\alpha}}$, we get that

$$\tilde{l}_j \le Cl < C \cdot l_1^{1-\alpha} \cdot l^\alpha \le C \cdot \frac{1}{2C^2} \cdot l^\alpha = \frac{1}{2} C^{-1} l^\alpha, \tag{4.14}$$

and hence $\left(C^{-1} l^\alpha - \tilde{l}_j\right)^{-2} \le \left(\frac{1}{2} C^{-1} l^\alpha\right)^{-2}$. The integral is bounded from above by $C^2 l^2$, so

$$\left|G_{\Omega_l}(z,y) - \sum_{j=1}^{k} K_{\tilde{A}}(\tilde{z},\tilde{w}_j) \int_{\tilde{\Gamma}_{j,l}} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})\right| \le C_L d_{\hat{B}}(\hat{y})\, l^{3-2\alpha} |\log l| \tag{4.15}$$

with some constant $C_L > 0$ independent of l.

We do the same for $x \in A$ with $|x - w_j| > \rho$ for $j = 1, \ldots, k$ instead of z. We replace the factor obtained by the Lipschitz type estimate on $K_{\tilde{A}}$ by $d_{\tilde{A}}(\tilde{x}) \left(C^{-1}\rho - \tilde{l}_j\right)^{-3}$, which is bounded from above by $d_{\tilde{A}}(\tilde{x}) \left(\frac{1}{2}C^{-1}\rho\right)^{-3}$, if l_1 is chosen small enough and $0 < l < l_1$. Thus (4.15) has to be replaced by

$$\left| G_{\Omega_l}(x,y) - \sum_{j=1}^{k} K_{\tilde{A}}(\tilde{x}, \tilde{w}_j) \int_{\tilde{\Gamma}_{j,l}} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w}) \right| \leq C_L d_{\tilde{A}}(\tilde{x})\, d_{\hat{B}}(\hat{y})\, l^3 |\log l|, \quad (4.16)$$

where $C_L > 0$ again is independent of l.

• Now fix $j \in \{1, \ldots, k\}$. We look at $G_{\Omega_l}(w,y)$ with $w \in \Gamma_{j,l}$, which appears in both (4.15) and (4.16). As $y \in B$ stays away from the gap $\Gamma_{j,l}$, the function $w \mapsto G_{\Omega_l}(w,y)$ is harmonic near the gap on both sides of Γ. As $h : A \to \tilde{A}$ and its inverse $\tilde{h}$ can be defined on both sides of the gap, even the function $\tilde{w} \mapsto G_{\Omega_l}\left(\tilde{h}(\tilde{w}), y\right) = G_{\Omega_l}(w,y)$ is harmonic in a neighbourhood of $\tilde{\Gamma}_{j,l}$. The idea is to define a set $\tilde{W}_{j,l}$ and write

$$G_{\Omega_l}(w,y) = \int_{\partial \tilde{W}_{j,l}} K_{\tilde{W}_{j,l}}(\tilde{w}, \tilde{v})\, G_{\Omega_l}(v,y)\, d\sigma(\tilde{v}). \quad (4.17)$$

How do we define $\tilde{W}_{j,l}$? We want the set to have two properties: First, the sets $\tilde{W}_{j_1,l}$ and $\tilde{W}_{j_2,l}$ should have the same shape and be just a rescaled version of one another. Second, the boundary should be chosen in such a way that we can approximate $G_{\Omega_l}(v,y)$ by other expressions of a good order of l. Here is how it is done – for an illustration see Figure 4.8.

For $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)$ with negative $\tilde{v}_1$, $\tilde{W}_{j,l}$ is a half disk around $\tilde{w}_j^*$ of radius $\frac{\tilde{l}_j}{l} \cdot \eta$, where η (independent of j) is chosen to be small enough that the half disk is a subset of $\tilde{A}$. For $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)$ with positive $\tilde{v}_1$, it is a half circle around $\tilde{w}_j^*$ of radius $\left(\frac{\tilde{l}_j}{l}\right)^{1-\beta} \cdot \tilde{l}_j^{\beta}$ with $0 < \beta < 1$. Later, we will give the exact value for β. On the one hand, the radius has to be chosen large enough such that G_{Ω_l} can be approximated by G_B at a good order of l (see Corollary 4.3), on the other hand, it has to be chosen small enough such that G_B can be approximated by a Taylor expansion in a second step. As we think that it gives a little bit more insight into the proof, we work with an unspecified β for now and find out later that $\beta = \frac{2}{3}$ is a good choice. To sum up, we set

$$\tilde{W}_{j,l} := \left\{ \tilde{v} = (\tilde{v}_1, \tilde{v}_2); v_1 < 0 \text{ and } \left|\tilde{v} - \tilde{w}_j^*\right| < \frac{\tilde{l}_j}{l} \cdot \eta \right\} \cup \tilde{\Gamma}_{j,l}$$
$$\cup \left\{ \tilde{v} = (\tilde{v}_1, \tilde{v}_2); v_1 > 0 \text{ and } \left|\tilde{v} - \tilde{w}_j^*\right| < \left(\frac{\tilde{l}_j}{l}\right)^{1-\beta} \cdot \tilde{l}_j^{\beta} \right\}.$$

As stated in (4.7), the quotient $\frac{\tilde{l}_j}{l}$ is bounded from below and above by some constants for small $0 < l < l_1$. As we want the sets $W_{j,l}$ for all j to have the same shape, we assume moreover l_1 to be small enough such that $\frac{\tilde{l}_j}{2}$ (half the gap width) is smaller than $\left(\frac{\tilde{l}_j}{l}\right)^{1-\beta} \cdot \tilde{l}_j^\beta$ (the radius of the half disk with positive v_1) and the latter is smaller than $\frac{\tilde{l}_j}{l} \cdot \eta$ for $0 < l < l_1$.

We make a remark on the distinction between $\tilde{w}_j$ and $\tilde{w}_j^*$. Later on, we will apply the convergence results for G_{Ω_l} from Section 4.1. They are formulated under conditions like $|v - w_j| \geq C_1 l^\beta$. Our construction of $\tilde{W}_{j,l}$ leads to bounds of the form $\left|\tilde{v} - \tilde{w}_j^*\right| \geq C_1 l^\beta$ for $\tilde{v} \in \partial\tilde{W}_{j,l}$ with positive $\tilde{v}_1$. Nevertheless, it holds that

$$\begin{aligned}|v - w_j| &\geq C^{-1} |\tilde{v} - \tilde{w}_j| \geq C^{-1} \left(\left|\tilde{v} - \tilde{w}_j^*\right| - \left|\tilde{w}_j^* - \tilde{w}_j\right|\right)\\ &\geq C^{-1} \left(C_1 l^\beta - \frac{1}{2}\tilde{l}_j\right) \geq C^{-1} \left(C_1 l^\beta - \frac{1}{2} C l\right) = C^{-1} \left(C_1 - \frac{1}{2} C l^{1-\beta}\right) l^\beta.\end{aligned}$$

If l_1 and hence $0 < l < l_1$ is chosen small enough, we get an estimate of the form $|v - w_j| \geq C_2 l^\beta$ even for $\tilde{v} \in \partial\tilde{W}_{j,l} \cap (\mathbb{R}^+ \times \mathbb{R})$, so we can apply the convergence results with the same order β. Similarly, $\left|\tilde{v} - \tilde{w}_j^*\right| \leq C_1 l^\beta$ gives $|v - w_j| \leq C_2 l^\beta$ for sufficiently small l.

Now we return to (4.17) and divide the boundary of $\tilde{W}_{j,l}$ into three parts (see Figure 4.8): The half circle which lies within $\tilde{A}$, the other half circle outside $\tilde{A}$, and the straight line part between $\tilde{w}_j^* - \eta\frac{\tilde{l}_j}{l}(0,1)$ and $\tilde{w}_j^* + \eta\frac{\tilde{l}_j}{l}(0,1)$ without the gap $\tilde{\Gamma}_{j,l}$. On the last one, $G_{\Omega_l}(.,y)$ equals zero. The half circle parts of the boundary are sufficiently smooth so that it is correct to write[4]

$$\begin{aligned}G_{\Omega_l}(w,y) = &\int_{\partial\tilde{W}_{j,l}\cap\tilde{A}} K_{\tilde{W}_{j,l}}(\tilde{w},\tilde{v})\, G_{\Omega_l}(v,y)\, d\sigma(\tilde{v})\\ &+ \int_{\partial\tilde{W}_{j,l}\setminus\overline{\tilde{A}}} K_{\tilde{W}_{j,l}}(\tilde{w},\tilde{v})\, G_{\Omega_l}(v,y)\, d\sigma(\tilde{v}) \quad (+0)\,.\end{aligned}$$

According to Corollary 4.6,[5] $G_{\Omega_l}(v,y)$ in the first integrand is bounded by some constant times $d_{\hat{B}}(\hat{y})\, l^2 |\log l|$. The maximum principle[6] implies that also the integral is bounded independent of $\tilde{w}$ by the same upper bound. Hence there is a C_A independent of l such that

$$\left| G_{\Omega_l}(w,y) - \int_{\partial\tilde{W}_{j,l}\setminus\overline{\tilde{A}}} K_{\tilde{W}_{j,l}}(\tilde{w},\tilde{v})\, G_{\Omega_l}(v,y)\, d\sigma(\tilde{v}) \right| \leq C_A d_{\hat{B}}(\hat{y})\, l^2 |\log l|\,.$$

G_{Ω_l} in the remaining integral can be approximated by G_B according to Corollary 4.3.

[4] see Section 3.3.3.

[5] We apply it with $x \in A$ being replaced by $y \in B$ and $z \in B$ replaced by $v \in A$. The distance of v from the gap is bounded from below by a constant, so we apply the corollary for $\alpha = 0$.

[6] We recall that $\int_{\partial\tilde{W}_{j,l}\cap\tilde{A}} K_{\tilde{W}_{j,l}}(\tilde{w},\tilde{v})\, u(\tilde{v})\, d\sigma(\tilde{v})$ gives a harmonic function on $\tilde{W}_{j,l}$ which equals u on $\tilde{W}_{j,l} \cap \tilde{A}$ and zero elsewhere.

There is a constant $C_{\Omega_l \to B}$, which is independent of l, such that

$$\left| G_{\Omega_l}(w,y) - \int_{\partial \tilde{W}_{j,l} \setminus \overline{\tilde{A}}} K_{\tilde{W}_{j,l}}(\tilde{w}, \tilde{v})\, G_B(v,y)\; d\sigma(\tilde{v}) \right| \leq C_A d_{\hat{B}}(\hat{y})\, l^2 \left|\log l\right| + C_{\Omega_l \to B} d_{\hat{B}}(\hat{y})\, l^{2-\beta} \left|\log l\right|.$$

For small l, the second term is dominating, so we replace $C_A l_1^\beta + C_{\Omega_l \to B}$ by $C_{\Omega_l \to B}$ and state that there is a $C_{\Omega_l \to B}$, which is independent of l, such that

$$\left| G_{\Omega_l}(w,y) - \int_{\partial \tilde{W}_{j,l} \setminus \overline{\tilde{A}}} K_{\tilde{W}_{j,l}}(\tilde{w}, \tilde{v})\, G_B(v,y)\; d\sigma(\tilde{v}) \right| \leq C_{\Omega_l \to B} d_{\hat{B}}(\hat{y})\, l^{2-\beta} \left|\log l\right|. \quad (4.18)$$

- At this point we have to emphasise how the integrand has to be understood. $G_B(v,y)$ stands for $G_B\left(\tilde{h}(\tilde{v}), y\right)$. As the boundary is assumed to be analytic near w_j, the Green function is sufficiently smooth to allow a Taylor expansion of the integrand in $\tilde{v}$ around $\tilde{w}_j$ (we do not expand around $\tilde{w}_j^*$!). Hence for each $\tilde{v} \in \partial \tilde{W}_{j,l} \setminus \overline{\tilde{A}} = \partial \tilde{W}_{j,l} \cap (\mathbb{R}^+ \times \mathbb{R})$, there is a $\tilde{\xi}_{\tilde{v}}$ on the straight line between $\tilde{w}_j$ and $\tilde{v}$ such that

$$G_B\left(\tilde{h}(\tilde{v}), y\right) = G_B\left(\tilde{h}(\tilde{w}_j), y\right) + \nabla_{\tilde{v}}\left(G_B\left(\tilde{h}(\tilde{v}), y\right)\right)\Big|_{\tilde{v}=\tilde{w}_j} . (\tilde{v} - \tilde{w}_j) + \frac{1}{2}(\tilde{v} - \tilde{w}_j)\, H_{G_B(\tilde{h}(.),y)}\left(\tilde{\xi}_{\tilde{v}}\right)(\tilde{v} - \tilde{w}_j)^T, \quad (4.19)$$

where $H_{G_B(\tilde{h}(.),y)}$ stands for the Hessian. As $\tilde{h}(\tilde{w}_j) = w_j$, the constant term of the expansion is zero. We have to be a bit careful about the linear term, as G_B is the Green function on B, whereas $\tilde{h}$ is the inverse of the (extended) transformation of the other domain, A. By the chain rule, we get

$$\nabla_{\tilde{v}}\left(G_B\left(\tilde{h}(\tilde{v}), y\right)\right)\Big|_{\tilde{v}=\tilde{w}_j} = \nabla_v\left(G_B(v,y)\right)\big|_{v=w_j} \nabla \tilde{h}(\tilde{v}).$$

As $G_B(.,y)$ equals zero on the boundary, the gradient at the point w_j is pointing in the opposite direction of the outer unit normal and is of the length $K_B(y, w_j)$. Analogous to (3.19), we get

$$\left| \nabla_{\tilde{v}}\left(G_B\left(\tilde{h}(\tilde{v}), y\right)\right)\Big|_{\tilde{v}=\tilde{w}_j} \right| = K_B(y, w_j) \left| \tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j) \right|.$$

In which direction is $\nabla_{\tilde{v}}$ pointing at $\tilde{v} = \tilde{w}_j$? As $t \mapsto \tilde{h}(\tilde{w}_j + t(0,1))$ follows the boundary of B for small $|t|$ and as $G_B(.,y)$ is zero there, it is pointing in the $(1,0)$ direction. Hence, if $p_1 : \mathbb{R}^2 \to \mathbb{R}$ stands for the projection on the first variable, the linear term of the Taylor expansion (4.19) can be rewritten as[7]

$$\nabla_{\tilde{v}}\left(G_B\left(\tilde{h}(\tilde{v}), y\right)\right)\Big|_{\tilde{v}=\tilde{w}_j} . (\tilde{v} - \tilde{w}_j) = K_B(y, w_j) \left| \tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j) \right| p_1(\tilde{v}). \quad (4.20)$$

[7] This is the reason why we could expand $G_B\left(\tilde{h}(.), y\right)$ around $\tilde{w}_j$ instead of $\tilde{w}_j^*$, although the geometry of $\tilde{W}_{j,l}$ would propose the latter: Only the linear term of the Taylor expansion will be of importance, and $p_1(\tilde{v} - \tilde{w}_j) = p_1(\tilde{v}) = p_1\left(\tilde{v} - \tilde{w}_j^*\right)$.

The remainder term of the Taylor expansion in (4.19) contains the Hessian matrix of $\tilde{v} \mapsto G_B\left(\tilde{h}(\tilde{v}), y\right)$. By the chain rule and the product rule, it is a combination of first and second order derivatives of $G_B(., y)$ and derivatives of $\tilde{h}$. As $\tilde{\mathbf{h}}$ is holomorphic in $\overline{\tilde{\mathbf{W}}_{j,l}} \subset \overline{\tilde{\mathbf{W}}_{j,l_1}} \subset \mathbf{h}(B_R(\mathbf{w}_j))$, the derivatives of $\tilde{h}$ are bounded by a constant independent of $l < l_1$. The derivatives of $G_B(., y)$ can be estimated with the help of the estimates on $G_{\hat{B}}$ given by Proposition 3.29 and the fact that the derivatives of g are bounded on $W_{j,l}$, too. The estimates on $G_{\hat{B}}$ give upper bounds of the form

$$|\hat{y} - \hat{v}|^{-|k|} \min\left\{1, \frac{d_{\hat{B}}(\hat{y})}{|\hat{y} - \hat{v}|}\right\}, \tag{4.21}$$

with $|k|$ being either 1 or 2. However, we have that

$$|\hat{y} - \hat{v}| \geq C^{-1}|y - w| \geq C^{-1}|y - w_j| - C^{-1}C|\tilde{v} - \tilde{w}_j| \geq C^{-1}\rho - C_1\tilde{l}_j^{\beta}$$

for some $C_1 > 0$. If l_1 and thus $\tilde{l}_j \leq Cl < Cl_1$ is small enough, $|\hat{y} - \hat{v}|$ is bounded from below by a constant, so (4.21) has an upper bound of the form $d_{\hat{B}}(\hat{y})$ times a constant. Moreover, $|\tilde{v} - \tilde{w}_j| \leq C_2\tilde{l}_j^{\beta}$ for some $C_2 > 0$. That implies together with (4.19) and (4.20)

$$\left|G_B\left(\tilde{h}(\tilde{v}), y\right) - K_B(y, w_j)\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right| p_1(\tilde{v})\right| \leq C_T d_{\hat{B}}(\hat{y})\, l^{2\beta}$$

for some constant $C_T > 0$ independent of l. Combining this result with (4.18), we get

$$\left|G_{\Omega_l}(w, y) - K_B(y, w_j)\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right| \int_{\partial\tilde{W}_{j,l}\setminus\overline{\tilde{A}}} K_{\tilde{W}_{j,l}}(\tilde{w}, \tilde{v})\, p_1(\tilde{v})\, d\sigma(\tilde{v})\right|$$
$$\leq d_{\hat{B}}(\hat{y})\left(C_{\Omega_l \to B} l^{2-\beta}|\log l| + C_T l^{2\beta}\right) \tag{4.22}$$

• Combining (4.15) with (4.22) and the estimate on the Poisson kernel stated in Proposition 3.26 gives

$$\Bigg|G_{\Omega_l}(z, y) - \sum_{j=1}^{k} K_{\tilde{A}}(\tilde{z}, \tilde{w}_j)\, K_B(y, w_j)\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|$$
$$\times \int_{\tilde{\Gamma}_{j,l}} \int_{\partial\tilde{W}_{j,l}\setminus\overline{\tilde{A}}} K_{\tilde{W}_{j,l}}(\tilde{w}, \tilde{v})\, p_1(\tilde{v})\, d\sigma(\tilde{v})\, d\sigma(\tilde{w})\Bigg|$$
$$\leq \left|G_{\Omega_l}(z, y) - \sum_{j=1}^{k} K_{\tilde{A}}(\tilde{z}, \tilde{w}_j) \int_{\tilde{\Gamma}_{j,l}} G_{\Omega_l}(w, y)\, d\sigma(\tilde{w})\right| + \sum_{j=1}^{k} K_{\tilde{A}}(\tilde{z}, \tilde{w}_j)$$
$$\times \int_{\tilde{\Gamma}_{j,l}} \left|G_{\Omega_l}(w, y) - \sum_{j=1}^{k} K_B(y, w_j)\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right| \int_{\partial\tilde{W}_{j,l}\setminus\overline{\tilde{A}}} K_{\tilde{W}_{j,l}}(\tilde{w}, \tilde{v})\, p_1(\tilde{v})\, d\sigma(\tilde{v})\right| d\sigma(\tilde{w})$$

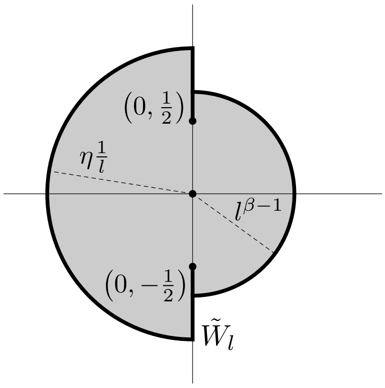


Figure 4.9.: The rescaled domain $\tilde{W}_l$.

$$\leq C_L d_{\hat{B}}(\hat{y}) l^{3-2\alpha} |\log l| + \left(C_{\Omega_l\to B} l^{2-\beta} |\log l| + C_T l^{2\beta}\right) d_{\hat{B}}(\hat{y}) \sum_{j=1}^{k} d_{\tilde{A}}(\tilde{z}) |\tilde{z} - \tilde{w}_j|^{-2} l$$

$$= d_{\hat{B}}(\hat{y}) \left(C_L l^{3-2\alpha} |\log l| + \left(C_{\Omega_l\to B} l^{3-\beta} |\log l| + C_T l^{1+2\beta}\right) \sum_{j=1}^{k} |\tilde{z} - \tilde{w}_j|^{-1}\right). \quad (4.23)$$

In the penultimate line, we have replaced the the original constants $C_{\Omega_l\to B}$ and C_T by larger ones that include the constant coming from the Poisson kernel estimate. Note that we could even get rid of the term $\sum_{j=1}^{k} |\tilde{z} - \tilde{w}_j|^{-1}$ by the estimate $|\tilde{z} - \tilde{w}_j|^{-1} \leq C |z - w_j| \leq C l^{-\alpha}$, but this would lower the order of convergence. As the singularity is integrable in $2D$, the term will give a constant anyways after the z–integration later. That is why we keep it here. Doing analogous estimates with (4.16) instead of (4.15) we get

$$\Bigg| G_{\Omega_l}(x, y) - \sum_{j=1}^{k} K_{\tilde{A}}(\tilde{x}, \tilde{w}_j) K_B(y, w_j) \left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|$$
$$\times \int_{\tilde{\Gamma}_{j,l}} \int_{\partial \tilde{W}_{j,l} \setminus \overline{\tilde{A}}} K_{\tilde{W}_{j,l}}(\tilde{w}, \tilde{v}) p_1(\tilde{v})\, d\sigma(\tilde{v})\, d\sigma(\tilde{w}) \Bigg|$$
$$\leq d_{\tilde{A}}(\tilde{x}) d_{\hat{B}}(\hat{y}) \left(C_L l^3 |\log l| + C_{\Omega_l\to B} l^{3-\beta} |\log l| + C_T l^{1+2\beta}\right). \quad (4.24)$$

• The sets $\tilde{W}_{j,l}$ all have the same shape but are of different size, depending on j. We scale them to a unit size with gap width 1 centred around the origin and set, see also Figure 4.9,

$$\tilde{W}_l := \left\{\tilde{v} = (\tilde{v}_1, \tilde{v}_2); v_1 < 0 \text{ and } |\tilde{v}| < \eta\frac{1}{l}\right\} \cup \left(\{0\} \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$$
$$\cup \left\{\tilde{v} = (\tilde{v}_1, \tilde{v}_2); v_1 > 0 \text{ and } |\tilde{v}| < \left(\frac{1}{l}\right)^{1-\beta}\right\}.$$

The mapping $\mathbf{w} \mapsto \tilde{\mathbf{w}}_{j,l}^* + \tilde{l}_j \mathbf{w}$ maps $\tilde{\mathbf{W}}_l$ conformally onto $\tilde{\mathbf{W}}_{j,l}$. We write $e_\varphi := (\cos(\varphi), \sin(\varphi))$ for a moment and get by the transformation formula (3.19) for the Poisson kernel that

$$K_{\tilde{W}_{j,l}}\left(\tilde{w}_{j,l}^* + \tau \tilde{l}_j (0,1), \tilde{w}_{j,l}^* + \left(\frac{\tilde{l}_j}{l}\right)^{1-\beta} \tilde{l}_j^\beta e_\varphi\right)$$
$$= K_{\tilde{W}_{j,l}}\left(\tilde{w}_{j,l}^* + \tau \tilde{l}_j (0,1), \tilde{w}_{j,l}^* + \frac{\tilde{l}_j}{l^{1-\beta}} e_\varphi\right) = K_{\tilde{W}_l}\left(\tau (0,1), l^{\beta-1} e_\varphi\right) \tilde{l}_j^{-1}.$$

Hence

$$\int_{\tilde{\Gamma}_{j,l}} \int_{\partial \tilde{W}_{j,l} \setminus \overline{\tilde{A}}} K_{\tilde{W}_{j,l}}(\tilde{w}, \tilde{v})\, p_1(\tilde{v})\, d\sigma(\tilde{v})\, d\sigma(\tilde{w})$$
$$= \int_{\tau=-\frac{1}{2}}^{\frac{1}{2}} \int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}} K_{\tilde{W}_{j,l}}\left(\tilde{w}_{j,l}^* + \tau \tilde{l}_j (0,1), \tilde{w}_{j,l}^* + \frac{\tilde{l}_j}{l^{1-\beta}} e_\varphi\right) \frac{\tilde{l}_j}{l^{1-\beta}} \cos(\varphi) \frac{\tilde{l}_j}{l^{1-\beta}}\, d\varphi\, \tilde{l}_j d\tau$$
$$= \tilde{l}_j^2 \int_{\tau=-\frac{1}{2}}^{\frac{1}{2}} \int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}} K_{\tilde{W}_l}\left(\tau(0,1), l^{\beta-1} e_\varphi\right) l^{\beta-1} \cos(\varphi)\, l^{\beta-1} d\varphi d\tau =: \tilde{l}_j^2 I_l.$$

We apply (3.19) to $K_{\tilde{A}}$ and rewrite (4.23)

$$\left| G_{\Omega_l}(z,y) - \sum_{j=1}^k K_A(z, w_j) K_B(y, w_j) \left| \tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j) \right|^2 \tilde{l}_j^2 I_l \right|$$
$$\leq d_{\tilde{B}}(\hat{y}) \left(C_L l^{3-2\alpha} |\log l| + \left(C_{\Omega_l \to B} l^{3-\beta} |\log l| + C_T l^{1+2\beta} \right) \sum_{j=1}^k |\tilde{z} - \tilde{w}_j|^{-1} \right) \quad (4.25)$$

and (4.24)

$$\left| G_{\Omega_l}(x,y) - \sum_{j=1}^k K_A(x, w_j) K_B(y, w_j) \left| \tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j) \right|^2 \tilde{l}_j^2 I_l \right|$$
$$\leq C_L d_{\tilde{A}}(\tilde{x})\, d_{\hat{B}}(\hat{y}) \left(l^3 |\log l| + C_{\Omega_l \to B} l^{3-\beta} |\log l| + C_T l^{1+2\beta} \right). \quad (4.26)$$

- How does I_l depend on l? We define $\tilde{\Psi}_l : \tilde{W}_l \to \mathbb{R}$ by

$$\tilde{\Psi}_l(\tilde{w}) := \int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}} K_{\tilde{W}_l}\left(\tilde{w}, l^{\beta-1} (\cos(\varphi), \sin(\varphi))\right) l^{\beta-1} \cos(\varphi)\, l^{\beta-1} d\varphi$$
$$= \int_{\partial \tilde{W}_l \cap (\mathbb{R}^+ \times \mathbb{R})} K_{\tilde{W}_l}(\tilde{w}, \tilde{v})\, p_1(\tilde{v})\, d\sigma(\tilde{v}).$$

Then

$$I_l = \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{\Psi}_l\left((0,\tau)\right)\, d\tau,$$

and $\tilde{\Psi}_l$ is a harmonic function on $\tilde{W}_l$ (see Figure 4.9) that satisfies the following boundary conditions for $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)$.

$$\tilde{\Psi}(\tilde{v}) = \begin{cases} 0 & \text{if } \tilde{v}_1 < 0 \\ 0 & \text{if } \tilde{v}_1 = 0 \\ \tilde{v}_1 & \text{if } \tilde{v}_1 > 0 \end{cases}$$

We scale $\tilde{W}_l$ by a factor 2 and obtain the set W_{R_1,R_2} considered in Proposition B.1 of Appendix B with $R_1 = 2\eta l^{-1}$ and $R_2 = 2l^{\beta-1}$. The proposition implies that for small $l > 0$, $\tilde{\Psi}(0,\tilde{\tau})$ is bounded from below and above by (nonzero) functions that do not depend on l. Hence there are $C_{sub}, C_{sup} > 0$ such that

$$C_{sub} \leq I_l \leq C_{sup}. \tag{4.27}$$

- Lemma 3.12 states

$$l - C_1 l^2 \leq \left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right| \tilde{l}_j \leq l + C_1 l^2$$

for some $C_1 > 0$, i.e., $\left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right| \tilde{l}_j$ is not too far away from l. Squaring everything gives that there exists some $C_S > 0$ such that

$$l^2 - C_S l^3 \leq \left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|^2 \tilde{l}_j^{\,2} \leq l^2 + C_S l^3.$$

Together with the estimates on K_A and K_B from Proposition 3.26 and the upper bound on I_l from (4.27), we get from (4.25) that

$$\begin{aligned} &\left| G_{\Omega_l}(z,y) - \sum_{j=1}^{k} K_A(z,w_j) K_B(y,w_j) l^2 I_l \right| \\ &\quad \leq d_{\hat{B}}(\hat{y}) \left(C_L l^{3-2\alpha} |\log l| + \left(C_{\Omega_l \to B} l^{3-\beta} |\log l| + C_T l^{1+2\beta} \right) \sum_{j=1}^{k} |\tilde{z} - \tilde{w}_j|^{-1} \right) \\ &\qquad + \sum_{j=1}^{k} K_A(z,w_j) K_B(y,w_j) \left| \left|\tilde{\mathbf{h}}'(\tilde{\mathbf{w}}_j)\right|^2 \tilde{l}_j^{\,2} - l^2 \right| I_l \\ &\quad \leq d_{\hat{B}}(\hat{y}) \left(C_L l^{3-2\alpha} |\log l| + \left(C_{\Omega_l \to B} l^{3-\beta} |\log l| + C_T l^{1+2\beta} + C_S l^3 \right) \sum_{j=1}^{k} |\tilde{z} - \tilde{w}_j|^{-1} \right) \end{aligned} \tag{4.28}$$

for some constant $C_S > 0$ independent of l. Similarly, from (4.26), we get

$$\left| G_{\Omega_l}(x,y) - \sum_{j=1}^{k} K_A(x,w_j) K_B(y,w_j) l^2 I_l \right| \\ \leq d_{\tilde{A}}(\tilde{x}) d_{\hat{B}}(\hat{y}) \left(C_L l^3 |\log l| + C_{\Omega_l \to B} l^{3-\beta} |\log l| + C_T l^{1+2\beta} + C_S l^3 \right). \quad (4.29)$$

• We return to (4.12). According to Corollary 4.3, $G_{\Omega_l}(x,z)$ is close to $G_A(x,z)$ for small l and z with $|z-w_j| \geq l^\alpha$. (4.9) and the estimates of Propositions 3.26 and 3.27 give

$$\left| \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{G_{\Omega_l}(x,z) G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)} dz - \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{G_A(x,z) G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)} dz \right| \\ \leq C_1 \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{d_{\tilde{A}}(\tilde{x}) l^{2-\alpha} |\log l| \sum_{j=1}^k \int_{\tilde{\Gamma}_{j,l}} \frac{d_{\tilde{A}}(\tilde{z})}{|\tilde{z}-\tilde{w}|^2} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})}{\sum_{j=1}^k \int_{\tilde{\Gamma}_{j,l}} \frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{w}|^2} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})} dz \\ \leq C_2 \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{l^{2-\alpha} |\log l| \sum_{j=1}^k l^{-\alpha} \int_{\tilde{\Gamma}_{j,l}} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})}{\sum_{j=1}^k \int_{\tilde{\Gamma}_{j,l}} G_{\Omega_l}(w,y)\, d\sigma(\tilde{w})} dz \\ \leq C_{\Omega_l \to A} l^{2-2\alpha} |\log l|. \quad (4.30)$$

We have used here that $|\tilde{z}-\tilde{w}| \geq |\tilde{z}-\tilde{w}_j| - |\tilde{w}_j - \tilde{w}| \geq C^{-1} l^\alpha - \tilde{l}_j \geq \frac{1}{2} C^{-1} l^\alpha$ according to (4.14).

• In order to increase readability, we set for a moment

$$H(z,y) := \sum_{j=1}^{k} K_A(z,w_j) K_B(y,w_j) l^2 I_l.$$

Our aim is to approximate the second integral on the left hand side of (4.30).

$$\left| \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{G_A(x,z) G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)} dz - \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{G_A(x,z) H(z,y)}{H(x,y)} dz \right| \\ = \left| \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{G_A(x,z) \left(G_{\Omega_l}(z,y) H(x,y) - H(z,y) G_{\Omega_l}(x,y)\right)}{G_{\Omega_l}(x,y) H(x,y)} dz \right| \\ \leq \left| \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{G_A(x,z) H(x,y) \left(G_{\Omega_l}(z,y) - H(z,y)\right)}{G_{\Omega_l}(x,y) H(x,y)} dz \right| \\ + \left| \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{G_A(x,z) H(z,y) \left(H(x,y) - G_{\Omega_l}(x,y)\right)}{G_{\Omega_l}(x,y) H(x,y)} dz \right| =: I_1 + I_2$$

Before we show that I_1 and I_2 tend to zero for $l \to 0$, we list the estimates we will use (with some constants $C_1, C_2, \ldots > 0$).

– By Proposition 3.26 and (4.27),

$$H(z,y) \le C_1 \sum_{j=1}^{k} \frac{d_{\tilde{A}}(\tilde{z})}{|\tilde{z}-\tilde{w}_j|^2} \frac{d_{\hat{B}}(\hat{y})}{|\hat{y}-\hat{w}_j|^2} l^2 I_l \le C_2 d_{\hat{B}}(\hat{y}) \left(\sum_{j=1}^{k} |z-w_j|^{-1} \right) l^2. \quad (4.31)$$

– By Proposition 3.27 and (4.27),

$$H(x,y) \ge C_3 \sum_{j=1}^{k} \frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{w}_j|^2} \frac{d_{\hat{B}}(\hat{y})}{|\hat{y}-\hat{w}_j|^2} l^2 I_l \ge C_4 d_{\tilde{A}}(\tilde{x}) d_{\hat{B}}(\hat{y}) l^2. \quad (4.32)$$

– By (3.24),

$$G_A(x,z) \le C_5 \frac{d_{\tilde{A}}(\tilde{x})}{|\tilde{x}-\tilde{z}|}. \quad (4.33)$$

– We find a lower bound for $G_{\Omega_l}(x,y)$. By (4.29) and (4.32),

$$\begin{aligned} G_{\Omega_l}(x,y) &\ge H(x,y) - |G_{\Omega_l}(x,y) - H(x,y)| \\ &\ge d_{\tilde{A}}(\tilde{x}) d_{\hat{B}}(\hat{y}) l^2 \left(C_4 - C_L l^1 |\log l| - C_{\Omega_l \to B} l^{1-\beta} |\log l| - C_T l^{2\beta-1} - C_S l \right). \end{aligned} \quad (4.34)$$

If $\frac{1}{2} < \beta < 1$, the term in brackets tends to $C_4 > 0$. Consequently, if $l_1 > 0$ is small enough, the right hand side of (4.34) is positive and even greater than $C_6 := \frac{C_4}{2}$. Hence for small $0 < l < l_1$, we have

$$G_{\Omega_l}(x,y) \ge C_6 d_{\tilde{A}}(\tilde{x}) d_{\hat{B}}(\hat{y}) l^2. \quad (4.35)$$

– As x stays away from the boundary singularities, we have that both $|\tilde{x}-\tilde{z}|^{-1} \le C|x-z|$ and $|\tilde{w}_j - \tilde{z}|^{-1} \le C|w_j - z|$ for all $j = 1, \ldots, k$. As x and the w_j's stay away from each other (and as the region of integration avoids neighbourhoods of the gaps anyways), there exists $C_7 > 0$ such that

$$\int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j) \right)} \sum_{j=1}^{k} \frac{1}{|\tilde{x}-\tilde{z}| \, |\tilde{z}-\tilde{w}_j|} \, dz < C_7. \quad (4.36)$$

The estimates (4.28), (4.33), (4.35), and (4.36) give an upper bound for I_1.

$$\begin{aligned} I_1 &= \left| \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j) \right)} \frac{G_A(x,z)\left(G_{\Omega_l}(z,y) - H(z,y)\right)}{G_{\Omega_l}(x,y)} \, dz \right| \\ &\le \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j) \right)} \frac{C_5 d_{\tilde{A}}(\tilde{x}) d_{\hat{B}}(\hat{y})}{|\tilde{x}-\tilde{z}| C_6 d_{\tilde{A}}(\tilde{x}) d_{\hat{B}}(\hat{y}) l^2} \\ &\quad \times \left(C_L l^{3-2\alpha} |\log l| + \left(C_{\Omega_l \to B} l^{3-\beta} |\log l| + C_T l^{1+2\beta} + C_S l^3 \right) \sum_{j=1}^{k} |\tilde{z}-\tilde{w}_j|^{-1} \right) dz \\ &\le \frac{C_5 C_7}{C_6} \left(C_L l^{1-2\alpha} |\log l| + C_{\Omega_l \to B} l^{1-\beta} |\log l| + C_T l^{2\beta-1} + C_S l \right) \end{aligned} \quad (4.37)$$

Similarly, an upper bound for I_2 is given by (4.29), (4.33), (4.31), (4.32), (4.35), and (4.36).

$$\begin{aligned}
I_2 &= \left| \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{G_A(x,z)\,H(z,y)\,(H(x,y)-G_{\Omega_l}(x,y))}{G_{\Omega_l}(x,y)\,H(x,y)}\,dz \right| \\
&\le \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{C_5 d_{\tilde{A}}(\tilde{x})\,C_2 d_{\hat{B}}(\hat{y}) \left(\sum_{j=1}^k |z-w_j|^{-1}\right) l^2 d_{\tilde{A}}(\tilde{x})\,d_{\hat{B}}(\hat{y})}{|\tilde{x}-\tilde{z}|\,C_6 d_{\tilde{A}}(\tilde{x})\,d_{\hat{B}}(\hat{y})\,l^2 C_4 d_{\tilde{A}}(\tilde{x})\,d_{\hat{B}}(\hat{y})\,l^2} \\
&\quad \times \left(C_L l^3 |\log l| + C_{\Omega_l\to B} l^{3-\beta} |\log l| + C_T l^{1+2\beta} + C_S l^3\right) dz \\
&\le \frac{C_2 C_5 C_7}{C_6 C_4} \left(C_L l |\log l| + C_{\Omega_l \to B} l^{1-\beta} |\log l| + C_T l^{2\beta-1} + C_S l\right) \qquad (4.38)
\end{aligned}$$

- It holds that

$$\begin{aligned}
\frac{G_A(x,z)\,H(z,y)}{H(x,y)} &= \frac{G_A(x,z)\sum_{j=1}^k K_A(z,w_j)\,K_B(y,w_j)\,l^2 I_l}{\sum_{j=1}^k K_A(z,w_j)\,K_B(y,w_j)\,l^2 I_l} \\
&= \sum_{j=1}^k \frac{G_A(x,z)\,K_A(z,w_j)}{K_A(x,w_j)} \cdot \frac{K_A(x,w_j)\,K_B(y,w_j)}{\sum_{m=1}^k K_A(z,w_m)\,K_B(y,w_m)}.
\end{aligned}$$

We sum up (4.30), (4.37), and (4.38).

$$\begin{aligned}
&\Bigg| \int_{A \setminus \left(\bigcup_{j=1}^k B_{l^\alpha}(w_j)\right)} \frac{G_{\Omega_l}(x,z)\,G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\,dz \\
&\quad - \sum_{j=1}^k \int_{A \setminus \left(\bigcup_{n=1}^k B_{l^\alpha}(w_n)\right)} \frac{G_A(x,z)\,K_A(z,w_j)}{K_A(x,w_j)}\,dz \cdot \frac{K_A(x,w_j)\,K_B(y,w_j)}{\sum_{m=1}^k K_A(z,w_m)\,K_B(y,w_m)} \Bigg| \\
&\le C_{\Omega_l\to A} l^{2-2\alpha} |\log l| + C_1 \left(C_L l^{1-2\alpha} |\log l| + C_{\Omega_l\to B} l^{1-\beta} |\log l| + C_T l^{2\beta-1} + C_S l\right) \\
&\qquad + C_2 \left(C_L l |\log l| + C_{\Omega_l\to B} l^{1-\beta} |\log l| + C_T l^{2\beta-1} + C_S l\right) \\
&\qquad\qquad \le C_3 l^{1-2\alpha} |\log l| + C_4 l^{1-\beta} |\log l| + C_5 l^{2\beta-1}
\end{aligned}$$

In order to increase readability, in the last line we have omitted the terms, where the l order is obviously higher than the order of other terms appearing. We add the estimates of (4.11) and (4.13) and get

$$\begin{aligned}
&\left| \int_A \frac{G_{\Omega_l}(x,z)\,G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\,dz - \sum_{j=1}^k \mathbb{E}_x^{w_j}(\tau_A) \frac{K_A(x,w_j)\,K_B(y,w_j)}{\sum_{m=1}^k K_A(x,w_m)\,K_B(y,w_m)} \right| \\
&\qquad \le C_3 l^{1-2\alpha} |\log l| + C_4 l^{1-\beta} |\log l| + C_5 l^{2\beta-1} + C_6 l^\alpha. \qquad (4.39)
\end{aligned}$$

How do we choose α and β? So long, we have only assumed that $0 < \alpha < 1$ and $\frac{1}{2} < \beta < 1$. (4.39) suggests that the choices $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$ are best, which implies

$$\left| \int_A \frac{G_{\Omega_l}(x,z)\, G_{\Omega_l}(z,y)}{G_{\Omega_l}(x,y)}\, dz - \sum_{j=1}^{k} \mathbb{E}_x^{w_j}(\tau_A) \frac{K_A(x,w_j)\, K_B(y,w_j)}{\sum_{m=1}^{k} K_A(x,w_m)\, K_B(y,w_m)} \right| \leq C_7 l^{\frac{1}{3}} \left|\log l\right|.$$

□

5. Moving x and y away from the gaps

Theorems 4.7 and 4.10 of the preceding chapter explain how the lifetime of our conditioned Brownian motion in Ω_l can be computed (plus an error depending on the gap width l) if the lifetimes on the subdomains A and B are known. Both theorems are formulated under the assumption that both the starting point and the endpoint stay away from the gaps and the boundary singularities. If we are looking for points of maximal lifetime, however, no great surprises can be expected if we ignore small neighbourhoods of those exceptions. This is a consequence of the lemmas presented in this section. As they do not only hold in a domain like Ω_l and near a gap, we formulate them for a general bounded domain Ω (that is sufficiently smooth such that $\mathbb{E}_x^y(\tau_\Omega)$ exists for all $x, y \in \Omega$) and balls $B_\rho(m)$ with $m \in \Omega$ and do not talk about gaps and boundary singularities at all. We just point out that it is not assumed that $B_\rho(m) \subset \Omega$, hence the results can also be applied if m is the centre of a gap.

Lemma 5.1. *There is an absolute constant $C > 0$ such that the following holds. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $m \in \Omega$. Let $\rho > 0$ such that $B_\rho(m) \cap \Omega$ is a domain, and let $x, y \in \Omega$ with $|y - m| < \rho$. Then*

$$\mathbb{E}_x^y(\tau_\Omega) \leq \sup_{w \in \partial B_\rho(m) \cap \overline{\Omega}} \mathbb{E}_x^w(\tau_\Omega) + C\rho^2.$$

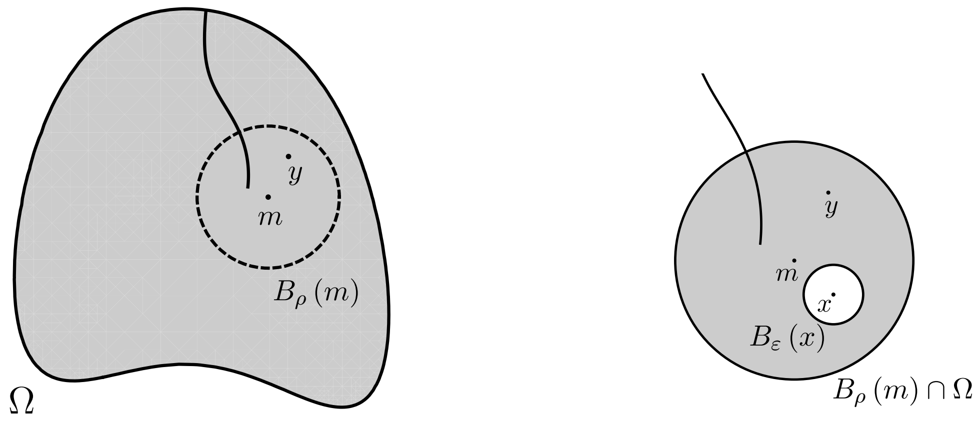
Proof. Let $\rho > 0$ and $x, y \in \Omega$ with $|y - m| < \rho$, see Figure 5.1. We assume that $x \neq y$, as $\mathbb{E}_x^x(\tau_\Omega) = 0$ and thus the assertion holds for $x = y$ anyways. We split the integral that expresses the lifetime into two parts,

$$\begin{aligned} \mathbb{E}_x^y(\tau_\Omega) &= \int_{\Omega \setminus B_\rho(m)} \frac{G_\Omega(x,z)\, G_\Omega(z,y)}{G_\Omega(x,y)}\, dz + \int_{\Omega \cap B_\rho(m)} \frac{G_\Omega(x,z)\, G_\Omega(z,y)}{G_\Omega(x,y)}\, dz \\ &=: I_1 + I_2, \end{aligned} \tag{5.1}$$

and look at I_1 first. As $z \in \Omega \setminus B_\rho(m)$, the mapping $y \mapsto G_\Omega(z,y)$ is harmonic in the domain $\Omega \cap B_\rho(m)$ and can hence be written as

$$G_\Omega(z,y) = \int_{\partial(\Omega \cap B_\rho(m))} K_{\Omega \cap B_\rho(m)}(y,w)\, G_\Omega(z,w)\, d\sigma(w).$$ [1]

[1] To be correct, $K_{\Omega \cap B_\rho(m)}$ cannot be defined in the classical sense here in case of a boundary part $\partial\Omega \cap B_\rho(m)$, as we approach the boundary from both sides. However, $G_\Omega(z,w) = 0$ there, so the formula holds nevertheless.



The domain Ω and $B_\rho(m)$.

If $x \in B_\rho(m) \cap \Omega$, we construct a harmonic function h_ε on $(\Omega \cap B_\rho(m)) \setminus B_\varepsilon(x)$.

Figure 5.1.: Illustration to the proof of Lemma 5.1.

We get

$$\begin{aligned}
I_1 &= \int_{\Omega \setminus B_\rho(m)} \frac{G_\Omega(x,z)\, G_\Omega(z,y)}{G_\Omega(x,y)}\, dz \\
&= \int_{\Omega \setminus B_\rho(m)} \frac{G_\Omega(x,z) \int_{\partial(\Omega \cap B_\rho(m))} K_{\Omega \cap B_\rho(m)}(y,w)\, G_\Omega(z,w)\, d\sigma(w)}{G_\Omega(x,y)}\, dz \\
&= \int_{\Omega} \frac{G_\Omega(x,z) \int_{\partial(\Omega \cap B_\rho(m))} K_{\Omega \cap B_\rho(m)}(y,w)\, G_\Omega(z,w)\, d\sigma(w)}{G_\Omega(x,y)}\, dz \\
&\quad \underbrace{- \int_{\Omega \cap B_\rho(m)} \frac{G_\Omega(x,z) \int_{\partial(\Omega \cap B_\rho(m))} K_{\Omega \cap B_\rho(m)}(y,w)\, G_\Omega(z,w)\, d\sigma(w)}{G_\Omega(x,y)}\, dz}_{:=I_3} \\
&= \int_{\Omega} \frac{G_\Omega(x,z) \int_{\partial(\Omega \cap B_\rho(m)) \setminus \partial\Omega} K_{\Omega \cap B_\rho(m)}(y,w)\, G_\Omega(z,w)\, d\sigma(w)}{G_\Omega(x,y)}\, dz - I_3 \\
&= \int_{\partial B_\rho(m) \cap \overline{\Omega}} K_{\Omega \cap B_\rho(m)}(y,w) \frac{G_\Omega(x,w)}{G_\Omega(x,y)} \int_\Omega \frac{G_\Omega(x,z)\, G_\Omega(z,w)}{G_\Omega(x,w)}\, dz d\sigma(w) - I_3 \\
&= \int_{\partial B_\rho(m) \cap \overline{\Omega}} K_{\Omega \cap B_\rho(m)}(y,w) \frac{G_\Omega(x,w)}{G_\Omega(x,y)} \mathbb{E}_x^w(\tau_\Omega)\, d\sigma(w) - I_3 && (5.2) \\
&\leq \sup_{w \in \partial B_\rho(m) \cap \overline{\Omega}} \mathbb{E}_x^w(\tau_\Omega) \int_{\partial B_\rho(m) \cap \overline{\Omega}} K_{\Omega \cap B_\rho(m)}(y,w) \frac{G_\Omega(x,w)}{G_\Omega(x,y)} d\sigma(w) - I_3 && (5.3)
\end{aligned}$$

$$= \sup_{w \in \partial B_\rho(m) \cap \overline{\Omega}} \mathbb{E}_x^w (\tau_\Omega) \frac{\int_{\partial(\Omega \cap B_\rho(m))} K_{\Omega \cap B_\rho(m)} (y,w) \, G_\Omega (x,w) \; d\sigma (w)}{G_\Omega (x,y)} - I_3 \tag{5.4}$$

$$\leq \sup_{w \in \partial B_\rho(m) \cap \overline{\Omega}} \mathbb{E}_x^w (\tau_\Omega) \frac{G_\Omega (x,y)}{G_\Omega (x,y)} - I_3 \tag{5.5}$$

$$= \sup_{w \in \partial B_\rho(m) \cap \overline{\Omega}} \mathbb{E}_x^w (\tau_\Omega) - I_3 \tag{5.6}$$

The step from (5.4) to (5.5) might need some explanation: If $x \in \Omega \setminus B_\rho (m)$, then $y \mapsto \int_{\partial(\Omega \cap B_\rho(m))} K_{\Omega \cap B_\rho(m)} (y,w) \, G_\Omega (x,w) \; d\sigma (w)$ is a function that is harmonic and equals $G_\Omega (x,.)$ on $\partial (\Omega \cap B_\rho (m))$. It hence is equal to $G_\Omega (x,.)$.[2] In the other case, namely if $x \in \Omega \cap B_\rho (m)$, then $y \mapsto \int_{\partial(\Omega \cap B_\rho(m))} K_{\Omega \cap B_\rho(m)} (y,w) \, G_\Omega (x,w) \; d\sigma (w)$ again is a harmonic function that equals $G_\Omega (x,.)$ on $\partial (\Omega \cap B_\rho (m))$. As the singularity of $G_\Omega (x,.)$, namely x, lies in $\Omega \cap B_\rho (m)$ this time, the integral does not equal $G_\Omega (x,y)$, but still it is smaller than $G_\Omega (x,y)$. The '$\leq$' sign is justified.

Second, we look at the remaining terms and search for an upper bound of $I_2 - I_3$, which can be written as

$$\int_{\Omega \cap B_\rho(m)} \frac{G_\Omega (x,z) \left(G_\Omega (z,y) - \int_{\partial(\Omega \cap B_\rho(m))} K_{\Omega \cap B_\rho(m)} (y,w) \, G_\Omega (z,w) \; d\sigma (w) \right)}{G_\Omega (x,y)} \, dz.$$

The integral $\int_{\partial(\Omega \cap B_\rho(m))} K_{\Omega \cap B_\rho(m)} (y,w) \, G_\Omega (z,w) \; d\sigma (w) := u^z (y)$ gives a harmonic function in y that equals $G_\Omega (z,y)$ on $\partial (\Omega \cap B_\rho (m))$. Hence the difference $G_\Omega (z,y) - u^z (y)$ equals zero at the boundary of $\Omega \cap B_\rho (m)$, is harmonic in y in $\Omega \cap B_\rho (m)$ for $z \neq y$ and has a singularity of the type $-\frac{1}{2\pi} \log |y - z|$. This means that it is nothing but the Green function for $\Omega \cap B_\rho (m)$. Summing up, we have that

$$I_2 - I_3 = \int_{\Omega \cap B_\rho(m)} \frac{G_\Omega (x,z) \, G_{\Omega \cap B_\rho(m)} (z,y)}{G_\Omega (x,y)} \, dz.$$

Again we consider two cases. If $x \in \Omega \setminus B_\rho (m)$, the function $z \mapsto G_\Omega (x,z) =: h(z)$ is harmonic in $\Omega \cap B_\rho (m)$, hence $I_2 - I_3$ can be written in the form

$$I_2 - I_3 = \int_{\Omega \cap B_\rho(m)} \frac{h (z) \, G_{\Omega \cap B_\rho(m)} (z,y)}{h (y)} \, dz,$$

which is bounded from above by a constant times the area of $\Omega \cap B_\rho (m)$ according to Theorem 3.31, where the constant C_1 is independent of the domain and h, i.e.,

$$I_2 - I_3 \leq C_1 \pi \rho^2. \tag{5.7}$$

[2]If $x \in \Omega \cap \partial B_\rho (m)$, one might doubt if this is still true, as the integration on $\partial (\Omega \cap B_\rho (m))$ hits the singularity of $G_\Omega (x,.)$. However, by approaching x by a sequence of points $x_n \in \Omega \setminus \overline{B_\rho (m)}$ and the dominated convergence theorem, one can show that the identity still holds.

If the singularity $x \in \Omega \cap B_\rho(m)$, we use the same strategy as in the definition of $\mathbb{E}_x^y(\tau_\Omega)$, see (3.27) and Corollary A.6 of Appendix A. Let $\varepsilon > 0$ be so small that $B_\varepsilon(x) \subset \Omega \cap B_\rho(m)$, see Figure 5.1, and define h_ε to be the solution of

$$\begin{cases} -\Delta h_\varepsilon = 0 & \text{in } \Omega, \\ h\varepsilon = G_{\Omega \cap B_\rho(m)}(x,.) & \text{on } \partial(B_\rho(m) \cap \Omega) \\ h\varepsilon = G_{\Omega \cap B_\rho(m)}(x,.) & \text{on } \partial B_\varepsilon(x). \end{cases}$$

On $(\Omega \cap B_\rho(m)) \setminus B_\varepsilon(x)$, h_ε equals $G_\Omega(x,.)$. As $G_{(\Omega \cap B_\rho(m)) \setminus B_\varepsilon(x)} \to G_{\Omega \cap B_\rho(m)}$ for $\varepsilon \to 0$, see Lemma A.1, it holds that

$$\lim_{\varepsilon \to 0} \int_{(\Omega \cap B_\rho(m)) \setminus B_\varepsilon(x)} \frac{h_\varepsilon(z)\, G_{(\Omega \cap B_\rho(m)) \setminus B_\varepsilon(x)}(z,y)}{h_\varepsilon(y)}\, dz \\ = \int_{\Omega \cap B_\rho(m)} \frac{G_\Omega(x,z)\, G_{\Omega \cap B_\rho(m)}(z,y)}{G_\Omega(x,y)}\, dz = I_2 - I_3,$$

and as

$$\int_{(\Omega \cap B_\rho(m)) \setminus B_\varepsilon(x)} \frac{h_\varepsilon(z)\, G_{(\Omega \cap B_\rho(m)) \setminus B_\varepsilon(x)}(z,y)}{h_\varepsilon(y)}\, dz \le C_1 \left|(\Omega \cap B_\rho(m)) \setminus B_\varepsilon(x)\right| \\ \le C_1 \pi \rho^2,$$

the estimate of (5.7) also holds for $x \in \Omega \cap B_\rho(m)$. After combining this with (5.1) and (5.6), we have shown that there is some constant $C > 0$ such that

$$\mathbb{E}_x^y(\tau_\Omega) \le \sup_{w \in \partial B_\rho(m) \cap \overline{\Omega}} \mathbb{E}_x^w(\tau_\Omega) + C\rho^2.$$

□

The preceding lemma says that if we are looking for an upper bound for the lifetime of Brownian motion, we can ignore points y in a ρ-neighbourhood of some m_2 at the cost of $C\rho^2$. If also x lies in a neigbourhood of some point m_1 which we want to ignore, we can apply the lemma a second time. The following corollary is obtained.

Corollary 5.2. *There is an absolute constant $C > 0$ such that the following holds. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $m_1, m_2 \in \Omega$. Let $\rho > 0$ such that $B_\rho(m_1) \cap \Omega$ and $B_\rho(m_2) \cap \Omega$ are a domains, and let $x, y \in \Omega$ with $|x - m_1| < \rho$ and $|y - m_2| < \rho$. Then*

$$\mathbb{E}_x^y(\tau_\Omega) \le \sup_{v \in \partial B_\rho(m_1), w \in \partial B_\rho(m_2)} \mathbb{E}_v^w(\tau_\Omega) + 2C\rho^2.$$

Remark 5.3. Going through the proof of Lemma 5.1 once again, one can even find a lower bound for $\mathbb{E}_x^y(\tau_\Omega)$ in the case that $|y - m| < \rho$ and $|x - m| > \rho$. Indeed, in the step from (5.2) to (5.3), we get an estimate from below by replacing 'sup' by 'inf'. With

$x \notin \overline{\Omega \cap B_\rho(m)}$, the '$\leq$' sign in the step from (5.4) to (5.5) can be replaced by a '='. Finally, $I_2 - I_3 > 0$, so

$$\mathbb{E}_x^y(\tau_\Omega) > \inf_{w \in \partial B_\rho(m)} \mathbb{E}_x^w(\tau_\Omega).$$

This shows that for an interior point y, we can always find a point $\tilde{y}$ such that $\mathbb{E}_x^{\tilde{y}}(\tau_\Omega) < \mathbb{E}_x^y(\tau_\Omega)$ (by moving 'closer to' x in an appropriate sense). In general domains, the converse is not true, i.e., it is not possible to find a point $\tilde{y}$ with greater lifetime close to y, as we show in Theorem 7.1. We interpret this philosophically: shortening a lifetime is always easier than extending it.

Part II.

An example for a multiply connected domain where maximal lifetime of conditioned Brownian motion occurs for a pair of interior points

6. A multiply connected domain

In this part, we prove Theorem 7.1, which says that Conjecture 3.33 does not hold for multiply connected domains by presenting a counterexample. In Chapter 6, we present the domain, especially in Section 6.1. The domain consists of subdomains where lifetimes can be computed explicitly. Section 6.2 is about showing how the lifetimes on the subdomains can be added to get (upper bounds for) the lifetimes on the whole domain, whereas in Section 6.3, the required lifetimes on the subdomains are computed explicitly. Finally, everything is put together in Chapter 7 to obtain Theorem 7.1.

6.1. The domain

We consider a domain as it is illustrated in Figure 6.1. It consists, roughly speaking, of two unit disks D^l and D^r, centred at $(-2, 0)$ and $(2, 0)$, respectively. Those disks are connected via a system of tubes of thickness s. The tubes are connected to the disks by four gaps of width d_2. The centres of these gaps are uniformly distributed along the boundary of the disks. Moreover, the system of tubes consists of subdomains which again are linked by gaps. One gap of width d_3 is centred around the origin $0 = (0, 0)$, the other gaps of width d_1 are situated where the tubes branch out.

The nomenclature is illustrated in Figure 6.2, we have tried to make it as suggestive as possible. For instance, the domain $AB^l_{d_1,s}$ stands for the tube system consisting of the subdomains of A and of B type, i.e., A^l_s and $B^{1,l}_s$, $B^{2,l}_s$. The superscript l implies that we consider the left part. The domain $AB^l_{d_1,s}$ depends on the gap width d_1 and the tube thickness s, hence the subscripts d_1 and s.

The centres of the gaps are named after the subdomain whose letter appears earlier

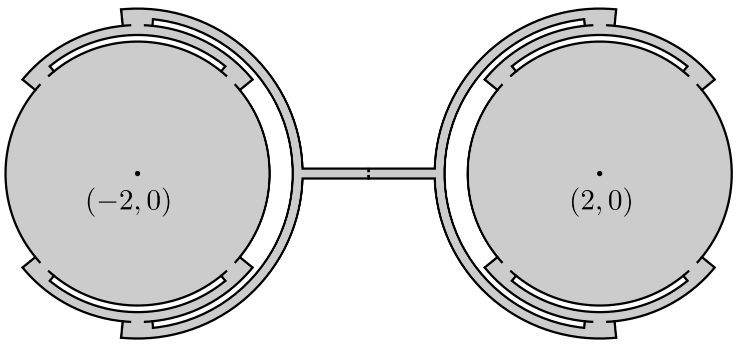
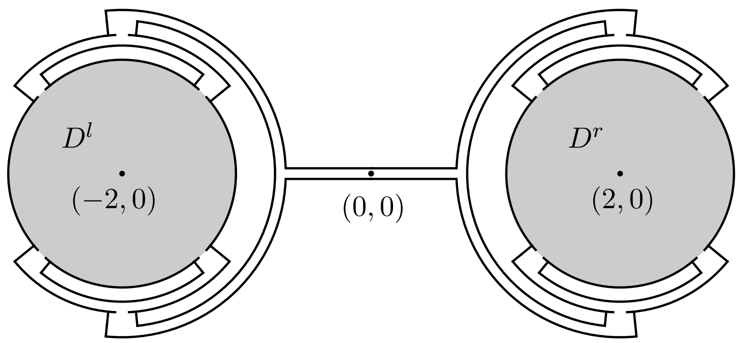
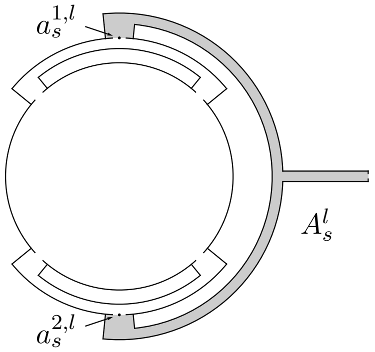


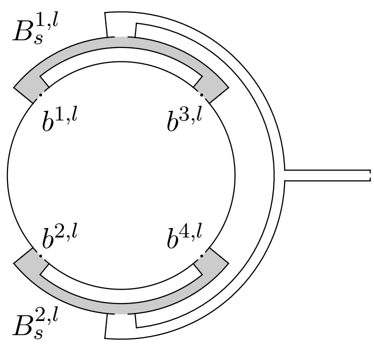
Figure 6.1.: The domain $M_{d_1,d_2,d_3,s}$.



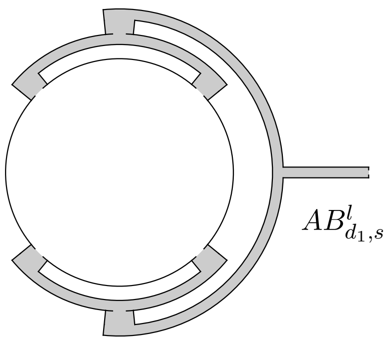
The subdomains D^l and D^r.



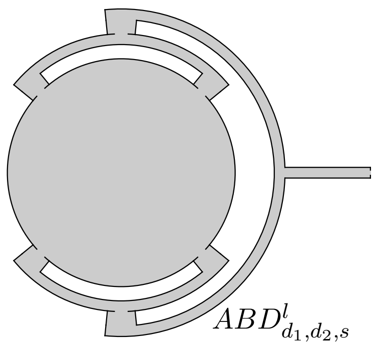
An A type subdomain.



Subdomains of B type.



An AB type subdomain.



An ABD type subdomain.

Figure 6.2.: Points, gaps and subdomains of $M_{d_1,d_2,d_3,s}$ (with oversized s for a better figure resolution).

in the alphabet, for instance the gaps between D^l and $B_s^{1,l}$ are centred around $b^{1,l}$ and $b^{3,l}$, respectively. The gaps are then denoted by the corresponding Greek letter, e.g., the gap around $b^{1,l}$ is named $\beta_{d_2}^{1,l}$.

We call the whole domain $M_{d_1,d_2,d_3,s}$. It is symmetric with respect to the x– and the y-axis, and moreover, the subdomains of B type (that is, $B_s^{1,l}$, $B_s^{2,l}$, and $B_s^{1,r}$, $B_s^{2,r}$) are symmetric with respect to the $(x = -2)$– and the $(x = 2)$-axis, respectively.

The domain has an analytic boundary except for the corners of the tubes and the boundary parts close to a gap, where the set is on both sides of the boundary. We call all these points boundary singularities and write $S'_{d_1,d_2,d_3,s}$ for the set of boundary singularities. In fact, only the set of corners and centres, which we denote by $S_{d_1,d_2,d_3,s}$, will play a role.

Some words about the intuition: We will show that the lifetime of our conditioned Brownian motion between the two centres of the disks is larger than the lifetime between boundary points – in fact, we think it is maximal. Why is that so? First, by the results of Part I, we can compute the lifetime by adding averaged lifetimes on the subdomains. Second, the thin tubes do not contribute much to the total lifetime, as in thin tubes, particles have to move fast in order not to get killed at the boundary. Therefore, it is enough to know the averaged lifetimes on the disks. Nevertheless, thirdly, the number as well as the placement of the tubes connecting the disks is important: The more tubes and hence the more possible disk exit points are given, the easier it is for the particles which started at the boundary to get into a tube and leave the disk, which decreases lifetime. In contrast to this, let us look at particles that start at the centre of a disk. They always have to find their way to the disk boundary if conditioned to move to the other disk. Hence an increasing number of exit points does not influence the time spent within the disk.[1] Summing up, it turns out that four exit points are enough such that the lifetime of our Brownian motion between boundary points is smaller than the lifetime between the disk centres.

6.2. Lifetimes for small gap widths

The following lemma is the key ingredient in the proof of Theorem 7.1. We show how (upper bounds for) $\mathbb{E}_x^y\left(\tau_{M_{d_1,d_2,d_3,s}}\right)$ can be computed for small values of the parameters. For this purpose, we make use of Theorems 4.7 and 4.10 of Part I.

Lemma 6.1. *Let $\rho > 0$. For each $\varepsilon > 0$, there exist $d_1, d_2, d_3, s > 0$ such that the following holds: Let $x, y \in M_{d_1,d_2,d_3,s}$ with $|x - z| > \rho$, $|y - z| > \rho$ for all $z \in S_{d_1,d_2,d_3,s}$. Then the following estimates for the lifetime hold depending on which subdomain the points belong to.*

1. *$x, y \in D^l$: $\mathbb{E}_x^y\left(\tau_{M_{d_1,d_2,d_3,s}}\right) \leq \sup_{p,q\in\overline{D^l}} \mathbb{E}_p^q\left(\tau_{D^l}\right) + \varepsilon$.*
2. *$x \in D^l$, $y \in AB_{d_1,s}^l$: $\mathbb{E}_x^y\left(\tau_{M_{d_1,d_2,d_3,s}}\right) \leq \sup_{p,q\in\overline{D^l}} \mathbb{E}_p^q\left(\tau_{D^l}\right) + \varepsilon$.*

[1] This can be seen in (6.14) later.

3. $x \in D^l$, $y \in AB^r_{d_1,s}$: $\mathbb{E}^y_x\left(\tau_{M_{d_1,d_2,d_3,s}}\right) \leq \sup_{p,q\in\overline{D^l}} \mathbb{E}^q_p\left(\tau_{D^l}\right) + \varepsilon$.

4. $x \in D^l$, $y \in D^r$: $\left|\mathbb{E}^y_x\left(\tau_{M_{d_1,d_2,d_3,s}}\right) - \left(\mathbb{H}_x\left(\tau_{D^l}\right) + \mathbb{H}_y\left(\tau_{D^r}\right)\right)\right| \leq \varepsilon$, *with*

$$\mathbb{H}_x\left(\tau_{D^l}\right) := \sum_{j=1}^{4} \mathbb{E}^{b^{j,l}}_x\left(\tau_{D^l}\right) \frac{K_{D^l}\left(x, b^{j,l}\right)}{\sum_{k=1}^4 K_{D^l}\left(x, b^{k,l}\right)}$$

for $x \in D^l$; $\mathbb{H}_y\left(\tau_{D^r}\right)$ *is defined analogously for* D^r.

5. $x, y \in AB^l_{d_1,s}$: $\mathbb{E}^y_x\left(\tau_{M_{d_1,d_2,d_3,s}}\right) \leq \varepsilon$.

6. $x \in AB^l_{d_1,s}$, $y \in AB^r_{d_1,s}$: $\mathbb{E}^y_x\left(\tau_{M_{d_1,d_2,d_3,s}}\right) \leq \varepsilon$.

Proof. Let $\rho, \varepsilon > 0$. Before going through the different cases, we determine the values of s, d_1, d_2, and d_3.

- According to Corollary 3.32, the lifetime of conditioned Brownian motion between two points of a domain is bounded from above by a constant times the area of the domain. As the tubes are of thickness s but of constant 'length', this area tends to zero for all domains of A, B, and AB type. We choose s to be so small that firstly,

$$\sup_{p,q\in\overline{AB^l_{d_1,s}}} \mathbb{E}^q_p\left(\tau_{AB^l_{d_1,s}}\right) \leq \frac{1}{7}\varepsilon. \tag{6.1}$$

Secondly, let s be so small that $B_{\frac{\rho}{2}}(z)$ covers all the boundary singularities close to a gap with centre z.[2] Let s be fixed from now on.

- We look at the term

$$\frac{\frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}\left(0, b^{k,l}\right)}{\frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}\left(0, b^{j,l}\right)} \tag{6.2}$$

with $k, j = 1, \ldots, 4$. Here, $\frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}\left(0, b^{k,l}\right)$ stands for

$$\lim_{\substack{z_1\to 0\\ z_1<0}} \frac{K_{AB^l_{d_1,s}}\left(\left(z_1, 0\right), b^{k,l}\right) - K_{AB^l_{d_1,s}}\left(\left(0, 0\right), b^{k,l}\right)}{z_1}.$$

We claim that (6.2) tends to 1 if $d_1 \to 0$ and sketch the proof here. Let $k \in \{1, \ldots, 4\}$ and let $\tilde{k}$ be the index of the subdomain $B^{\tilde{k},l}_s$ which $b^{k,l}$ is a boundary point of. As the

[2]This way, we have obtained that all the boundary parts near a gap that are not smooth because there is domain on both sides of them can be covered by a ball centred around the gap centre.

boundary of $AB^l_{d_1,s}$ is sufficiently smooth near the gaps, the derivatives commute, and hence

$$\begin{aligned}\frac{\partial}{\partial z_1}K_{AB^l_{d_1,s}}\left(0,b^{k,l}\right) &= -\frac{\partial}{\partial z_1}\frac{\partial}{\partial n^{k,l}}G_{AB^l_{d_1,s}}\left(0,b^{k,l}\right)\\ &= -\frac{\partial}{\partial n^{k,l}}\frac{\partial}{\partial z_1}G_{AB^l_{d_1,s}}\left(0,b^{k,l}\right)\\ &= \frac{\partial}{\partial n^{k,l}}K_{AB^l_{d_1,s}}\left(b^{k,l},0\right),\end{aligned}$$

where $\frac{\partial}{\partial n^{k,l}}$ stands for the derivative in the direction of the outer normal $n^{k,l}$ at $b^{k,l}$. We approach $b^{k,l}$ by a path $b^k(t)$ which leads orthogonally to the boundary. It is given by

$$b^k(t) = b^{k,l} - tn^{k,l}.$$

The mapping $w \mapsto K_{AB^l_{d_1,s}}(w,0)$ is harmonic in $B^{\tilde{k},l}_s$, hence it can be written as

$$K_{AB^l_{d_1,s}}\left(b^k(t),0\right) = \int_{\alpha^{\tilde{k},l}_{d_1,s}} K_{B^{\tilde{k},l}_s}\left(b^k(t),w\right)K_{AB^l_{d_1,s}}(w,0)\,d\sigma(w).$$

As the boundary of $B^{\tilde{k}}_s$ is sufficiently smooth close to gap $\alpha^{\tilde{k},l}_{d_1 s}$, the function $w \mapsto K_{B^{\tilde{k},l}_s}\left(b^k(t),w\right)$ is Lipschitz continuous along the gap.[3] Moreover, $w \mapsto K_{AB^l_{d_1,s}}(w,0)$ is at least continuous there. Hence, if $d_1 \to 0$,

$$K_{AB^l_{d_1,s}}\left(b^k(t),0\right) = K_{B^{\tilde{k},l}_s}\left(b^k(t),a^{\tilde{k}}_s\right)\int_{\alpha^{\tilde{k},l}_{d_1,s}} K_{AB^l_{d_1,s}}(w,0)\,d\sigma(w) + t\mathcal{O}\left(d_1^2\right), \quad (6.3)$$

where we used that $t = d_{B^{\tilde{k},l}_s}\left(b^k(t)\right)$ for small t. The term $\mathcal{O}\left(d_1^2\right)$ is independent of t, as $b^k(t)$ stays sufficiently far away from the gap $\alpha^{\tilde{k},l}_{d_1 s}$.

Because of the symmetries of the B type domains and $AB^l_{d_1,s}$, the expression on the right hand side of (6.3) is independent of k, so

$$\begin{aligned}&K_{B^{\tilde{j},l}_s}\left(b^j(t),a^{\tilde{j}}_s\right)\int_{\alpha^{\tilde{j},l}_{d_1,s}} K_{AB^l_{d_1,s}}(w,0)\,d\sigma(w)\\ &\qquad = K_{B^{\tilde{k},l}_s}\left(b^k(t),a^{\tilde{k}}_s\right)\int_{\alpha^{\tilde{k},l}_{d_1,s}} K_{AB^l_{d_1,s}}(w,0)\,d\sigma(w)\end{aligned}$$

[3]For the proof, we refer to Corollary 3.30.

for each $j \in \{1, \dots, 4\}$. Consequently,

$$\begin{aligned}
&\frac{\frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}\left(0, b^{k,l}\right)}{\frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}\left(0, b^{j,l}\right)} \\
&\quad = \lim_{t\to 0} \frac{\frac{1}{t}\left(K_{AB^l_{d_1,s}}\left(b^k(t), 0\right) - K_{AB^l_{d_1,s}}\left(b^{k,l}, 0\right)\right)}{\frac{1}{t}\left(K_{AB^l_{d_1,s}}\left(b^j(t), 0\right) - K_{AB^l_{d_1,s}}\left(b^{j,l}, 0\right)\right)} \\
&\quad = \lim_{t\to 0} \frac{\frac{1}{t} K_{AB^l_{d_1,s}}\left(b^k(t), 0\right)}{\frac{1}{t} K_{AB^l_{d_1,s}}\left(b^j(t), 0\right)} \\
&\quad = \lim_{t\to 0} \frac{\frac{1}{t}\left(K_{B_s^{\tilde{k},l}}\left(b^k(t), a_s^{\tilde{k}}\right)\int_{\alpha^{\tilde{k},l}_{d_1,s}} K_{AB^l_{d_1,s}}(w,0)\, d\sigma(w) + t\mathcal{O}\left(d_1^2\right)\right)}{\frac{1}{t}\left(K_{B_s^{\tilde{j},l}}\left(b^j(t), a_s^{\tilde{j}}\right)\int_{\alpha^{\tilde{j},l}_{d_1,s}} K_{AB^l_{d_1,s}}(w,0)\, d\sigma(w) + t\mathcal{O}\left(d_1^2\right)\right)} \\
&\quad = 1 + \mathcal{O}\left(d_1^2\right).
\end{aligned}$$

We chose $d_1 > 0$ to be so small that

$$\left|\frac{\frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}\left(0, b^{k,l}\right)}{\frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}\left(0, b^{j,l}\right)} - 1\right| \leq \frac{1}{4}\left(\sup_{p,q\in\overline{D^l}} \mathbb{E}_p^q\left(\tau_{D^l}\right) + \sup_{p,q\in\overline{AB^l_{d_1,s}}} \mathbb{E}_p^q\left(\tau_{AB^l_{d_1,s}}\right)\right)^{-1} \frac{1}{7}\varepsilon. \quad (6.4)$$

Let d_1 be fixed from now on.

• Let $x \in D^l$, $z \in AB_{d_1,s}$ with $\left|x - b^{j,l}\right| > \rho$ and $\left|z - b^{j,l}\right| > \rho$ for all $j \in \{1, \dots, 4\}$. Moreover, assume that z keeps a distance greater than ρ from the tube corners. According to Theorem 4.10,[4] d_2 can be chosen so small that

$$\begin{aligned}
&\left|\mathbb{E}_x^z\left(\tau_{ABD^l_{d_1,d_2,s}}\right) - \sum_{j=1}^{4}\left(\mathbb{E}_x^{b^{j,l}}\left(\tau_{D^l}\right) + \mathbb{E}^z_{b^{j,l}}\left(\tau_{AB^l_{d_1,s}}\right)\right)\right. \\
&\qquad \left. \times \frac{K_{D^l}\left(x, b^{j,l}\right) K_{AB^l_{d_1,s}}\left(z, b^{j,l}\right)}{\sum_{k=1}^4 K_{D^l}\left(x, b^{k,l}\right) K_{AB^l_{d_1,s}}\left(z, b^{k,l}\right)}\right| \leq \frac{1}{7}\varepsilon \quad (6.5)
\end{aligned}$$

independent of x and z. By continuity, this holds also for $z \to 0$, and by l'Hôpital's rule,

[4] To be precise, Theorem 4.10 (and Theorem 4.7) are formulated under the condition that both subdomains are bounded by a disjoint union of Jordan curves. This is not the case for $AB_{d_1,s}$ for instance, as there is domain on both sides of the boundary near the α type gaps. Nevertheless, the theorems can be applied, as we stay away from these boundary parts. We refer to Remark 4.9.

we get that

$$\Bigg| \mathbb{E}_x^0 \left(\tau_{ABD^l_{d_1,d_2,s}} \right) - \sum_{j=1}^{4} \left(\mathbb{E}_x^{b^{j,l}} \left(\tau_{D^l} \right) + \mathbb{E}_{b^{j,l}}^0 \left(\tau_{AB^l_{d_1,s}} \right) \right) \times \frac{K_{D^l}\left(x, b^{j,l}\right) \frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}\left(0, b^{j,l}\right)}{\sum_{k=1}^{4} K_{D^l}\left(x, b^{k,l}\right) \frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}\left(0, b^{k,l}\right)} \Bigg| \leq \frac{1}{7}\varepsilon.$$

The $\frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}$ terms cancel out as it was shown in (6.4).

$$\begin{aligned} &\left| \mathbb{E}_x^0 \left(\tau_{ABD^l_{d_1,d_2,s}} \right) - \sum_{j=1}^{4} \left(\mathbb{E}_x^{b^{j,l}} \left(\tau_{D^l} \right) + \mathbb{E}_{b^{j,l}}^0 \left(\tau_{AB^l_{d_1,s}} \right) \right) \frac{K_{D^l}\left(x, b^{j,l}\right)}{\sum_{k=1}^{4} K_{D^l}\left(x, b^{j,l}\right)} \right| \\ &\leq \frac{1}{7}\varepsilon + \sum_{j=1}^{4} \left(\mathbb{E}_x^{b^{j,l}} \left(\tau_{D^l} \right) + \mathbb{E}_{b^{j,l}}^0 \left(\tau_{AB^l_{d_1,s}} \right) \right) \\ &\quad \times \left| \frac{K_{D^l}\left(x, b^{j,l}\right) \frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}\left(0, b^{k,l}\right)}{\sum_{k=1}^{4} K_{D^l}\left(x, b^{k,l}\right) \frac{\partial}{\partial z_1} K_{AB^l_{d_1,s}}\left(0, b^{k,l}\right)} - \frac{K_{D^l}\left(x, b^{j,l}\right)}{\sum_{k=1}^{4} K_{D^l}\left(x, b^{k,l}\right)} \right| \\ &\leq \frac{2}{7}\varepsilon \end{aligned}$$

Finally, as $\mathbb{E}_{b^{j,l}}^0 \left(\tau_{AB^l_{d_1,s}} \right)$ is bounded from above by $\frac{1}{7}\varepsilon$, see (6.1),

$$\left| \mathbb{E}_x^0 \left(\tau_{ABD^l_{d_1,d_2,s}} \right) - \sum_{j=1}^{4} \mathbb{E}_x^{b^{j,l}} \left(\tau_{D^l} \right) \frac{K_{D^l}\left(x, b^{j,l}\right)}{\sum_{k=1}^{4} K_{D^l}\left(x, b^{j,l}\right)} \right| \leq \frac{3}{7}\varepsilon. \tag{6.6}$$

- Moreover, according to Theorem 4.7, $d_2 > 0$ can be chosen so small that

$$\left| \mathbb{E}_x^y \left(\tau_{D^l} \right) - \mathbb{E}_x^y \left(\tau_{ABD^l_{d_1,d_2,s}} \right) \right| \leq \frac{1}{7}\varepsilon \tag{6.7}$$

and

$$\left| \mathbb{E}_x^y \left(\tau_{AB^l_{d_1,s}} \right) - \mathbb{E}_x^y \left(\tau_{ABD^l_{d_1,d_2,s}} \right) \right| \leq \frac{1}{7}\varepsilon, \tag{6.8}$$

where both x and y lie in the same respective subdomain and stay away from the gaps and tube corners. Let d_2 be fixed from now on.
- For reasons of symmetry, all the estimates for domains 'on the left hand side' (indicated by l) also hold for the corresponding domains 'on the right hand side' (indicated by r).
- Finally, we choose $d_3 > 0$ to be so small that

$$\left| \mathbb{E}_x^y \left(\tau_{M_{d_1,d_2,d_3,s}} \right) - \left(\mathbb{E}_x^0 \left(\tau_{ABD^l_{d_1,d_2,s}} \right) + \mathbb{E}_0^y \left(\tau_{ABD^r_{d_1,d_2,s}} \right) \right) \right| \leq \frac{1}{7}\varepsilon \tag{6.9}$$

for $x \in ABD^l_{d_1,d_2,s}$ and $y \in ABD^r_{d_1,d_2,s}$ and

$$\left|\mathbb{E}^y_x\left(\tau_{M_{d_1,d_2,d_3,s}}\right) - \mathbb{E}^y_x\left(\tau_{ABD^l_{d_1,d_2,s}}\right)\right| \leq \frac{1}{7}\varepsilon \tag{6.10}$$

for $x, y \in ABD^l_{d_1,d_2,s}$, with x and y staying away from the gaps and corners again in both cases.

So far we have chosen the values of s, d_1, d_2, and d_3. Now we go through the different cases and show the assertions. We assume that x, y stay away from the elements of $S_{d_1,d_2,d_3,s}$ at a distance of at least ρ.

1. Let $x, y \in D^l$. According to (6.10) and (6.7) it holds that

$$\mathbb{E}^y_x\left(\tau_{M_{d_1,d_2,d_3,s}}\right) \leq \mathbb{E}^y_x\left(\tau_{ABD^l_{d_1,d_2,s}}\right) + \frac{1}{7}\varepsilon \leq \mathbb{E}^y_x\left(\tau_{D^l}\right) + \frac{2}{7}\varepsilon \leq \sup_{p,q\in\overline{D^l}} \mathbb{E}^q_p\left(\tau_{D^l}\right) + \varepsilon.$$

2. Let $x \in D^l$, $y \in AB^l_{d_1,s}$. The inequalities (6.10), (6.5) and (6.1) yield

$$\begin{aligned}
&\mathbb{E}^y_x\left(\tau_{M_{d_1,d_2,d_3,s}}\right) \leq \mathbb{E}^y_x\left(\tau_{ABD^l_{d_1,d_2,s}}\right) + \frac{1}{7}\varepsilon \\
&\leq \sum_{j=1}^4 \left(\mathbb{E}^{b^{j,l}}_x\left(\tau_{D^l}\right) + \mathbb{E}^y_{b^{j,l}}\left(\tau_{AB^l_{d_1,s}}\right)\right) \frac{K_{D^l}\left(x, b^{j,l}\right) K_{AB^l_{d_1,s}}\left(y, b^{j,l}\right)}{\sum_{k=1}^4 K_{D^l}\left(x, b^{k,l}\right) K_{AB^l_{d_1,s}}\left(y, b^{k,l}\right)} + \frac{2}{7}\varepsilon \\
&\leq \left(\sup_{p,q\in\overline{D^l}} \mathbb{E}^q_p\left(\tau_{D^l}\right) + \frac{1}{7}\varepsilon\right) \sum_{j=1}^4 \frac{K_{D^l}\left(x, b^{j,l}\right) K_{AB^l_{d_1,s}}\left(y, b^{j,l}\right)}{\sum_{k=1}^4 K_{D^l}\left(x, b^{k,l}\right) K_{AB^l_{d_1,s}}\left(y, b^{k,l}\right)} + \frac{2}{7}\varepsilon \\
&\leq \sup_{p,q\in\overline{D^l}} \mathbb{E}^q_p\left(\tau_{D^l}\right) + \varepsilon.
\end{aligned}$$

3. Let $x \in D^l$, $y \in AB^r_{d_1,s}$. This time, we use (6.9), (6.6), (6.8), and (6.1).

$$\begin{aligned}
\mathbb{E}^y_x\left(\tau_{M_{d_1,d_2,d_3,s}}\right) &\leq \mathbb{E}^0_x\left(\tau_{ABD^l_{d_1,d_2,s}}\right) + \mathbb{E}^y_0\left(\tau_{ABD^r_{d_1,d_2,s}}\right) + \frac{1}{7}\varepsilon \\
&\leq \sum_{j=1}^4 \mathbb{E}^{b^{j,l}}_x\left(\tau_{D^l}\right) \frac{K_{D^l}\left(x, b^{j,l}\right)}{\sum_{k=1}^4 K_{D^l}\left(x, b^{j,l}\right)} + \mathbb{E}^y_0\left(\tau_{AB^r_{d_1,s}}\right) + \frac{5}{7}\varepsilon \\
&\leq \sup_{p,q\in\overline{D^l}} \mathbb{E}^q_p\left(\tau_{D^l}\right) + \varepsilon.
\end{aligned}$$

4. Now we turn to the most important case, $x \in D^l$ and $y \in D^r$. The inequalities (6.9) and (6.6) yield

$$\begin{aligned}
&\mathbb{E}^y_x\left(\tau_{M_{d_1,d_2,d_3,s}}\right) \leq \mathbb{E}^0_x\left(\tau_{ABD^l_{d_1,d_2,s}}\right) + \mathbb{E}^y_0\left(\tau_{ABD^r_{d_1,d_2,s}}\right) + \frac{1}{7}\varepsilon \\
&\leq \sum_{j=1}^4 \mathbb{E}^{b^{j,l}}_x\left(\tau_{D^l}\right) \frac{K_{D^l}\left(x, b^{j,l}\right)}{\sum_{k=1}^4 K_{D^l}\left(x, b^{k,l}\right)} + \sum_{j=1}^4 \mathbb{E}^{b^{j,r}}_y\left(\tau_{D^r}\right) \frac{K_{D^r}\left(x, b^{j,r}\right)}{\sum_{k=1}^4 K_{D^r}\left(x, b^{k,r}\right)} + \frac{7}{7}\varepsilon \\
&\leq \mathbb{H}_x\left(\tau_{D^l}\right) + \mathbb{H}_y\left(\tau_{D^r}\right) + \varepsilon,
\end{aligned}$$

and the reverse inequality holds analogously.

5. Let $x, y \in AB^l_{d_1,s}$. We apply (6.10), (6.8), and (6.1).

$$\mathbb{E}^y_x\left(\tau_{M_{d_1,d_2,d_3,s}}\right) \leq \mathbb{E}^y_x\left(\tau_{ABD^l_{d_1,d_2,s}}\right) + \frac{1}{7}\varepsilon \leq \mathbb{E}^y_x\left(\tau_{AB^l_{d_1,s}}\right) + \frac{2}{7}\varepsilon \leq \varepsilon$$

6. The case $x \in AB^l_{d_1,s}$, $y \in AB^r_{d_1,s}$ is covered by (6.9), (6.8), and (6.1).

$$\begin{aligned}\mathbb{E}^y_x\left(\tau_{M_{d_1,d_2,d_3,s}}\right) &\leq \mathbb{E}^0_x\left(\tau_{ABD^l_{d_1,d_2,s}}\right) + \mathbb{E}^y_0\left(\tau_{ABD^r_{d_1,d_2,s}}\right) + \frac{1}{7}\varepsilon \\ &\leq \mathbb{E}^0_x\left(\tau_{AB^l_{d_1,s}}\right) + \mathbb{E}^y_0\left(\tau_{AB^r_{d_1,s}}\right) + \frac{3}{7}\varepsilon \\ &\leq \varepsilon\end{aligned}$$

□

6.3. Computation of some explicit values

In Lemma 6.1, the only terms that do not get arbitrarily small are $\sup_{p,q\in\overline{D^l}} \mathbb{E}^q_p\left(\tau_{D^l}\right)$ and $\mathbb{H}_x\left(\tau_{D^l}\right)$. As only Green function and Poisson kernels for the unit disk are involved, they can be computed explicitly. We write $\mathbb{D} = B_1(0)$. It holds that

$$\begin{aligned}G_{\mathbb{D}}(x,y) &= \frac{1}{4\pi}\log\left(1 + \frac{\left(1-|x|^2\right)\left(1-|y|^2\right)}{|x-y|^2}\right), \\ K_{\mathbb{D}}(x,y) &= -\frac{\partial}{\partial n_y}G_{\mathbb{D}}(x,y) = \frac{1}{2\pi}\frac{1-|x|^2}{|x-y|^2}, \\ -\frac{\partial}{\partial n_x}K_{\mathbb{D}}(x,y) &= \frac{1}{\pi}\frac{1}{|x-y|^2}.\end{aligned} \tag{6.11}$$

- We first look at $\sup_{p,q\in\overline{D^l}} \mathbb{E}^q_p\left(\tau_{D^l}\right)$. In [9], the authors show that the lifetime in a disk gets maximal if p and q are opposite boundary points. Hence it suffices to compute $\mathbb{E}^{(1,0)}_{(-1,0)}\left(\tau_{\mathbb{D}}\right)$. The lifetime for two boundary points is given by (3.29). We plug in the explicit formulas of (6.11) and make use of complex polar coordinates, $z = re^{i\varphi}$.[5]

$$\begin{aligned}\mathbb{E}^{(1,0)}_{(-1,0)}\left(\tau_{\mathbb{D}}\right) &= \int_{\mathbb{D}} \frac{K_{\mathbb{D}}(z,(1,0))\,K_{\mathbb{D}}(z,(-1,0))}{-\frac{\partial}{\partial n_x}K_{\mathbb{D}}(x,(-1,0))\Big|_{x=(1,0)}}\,dz \\ &= \int_{\mathbb{D}} \frac{\frac{1}{2\pi}\frac{1-|z|^2}{|z-(1,0)|^2}\frac{1}{2\pi}\frac{1-|z|^2}{|z-(-1,0)|^2}}{\frac{1}{\pi}\frac{1}{|(1,0)-(-1,0)|^2}}\,dz\end{aligned}$$

[5]In Part I, we made a sharp distinction in notation between real two-dimensional z and complex $\mathbf{z}$. We skip this here.

$$
\begin{aligned}
&= \frac{1}{\pi} \int_{r=0}^{1} \int_{\varphi=0}^{2\pi} \frac{1-r^2}{(re^{i\varphi}-1)(re^{-i\varphi}-1)} \frac{1-r^2}{(re^{i\varphi}+1)(re^{-i\varphi}+1)} d\varphi r dr \\
&= \frac{1}{\pi} \int_{r=0}^{1} \left(1-r^2\right)^2 r \int_{\varphi=0}^{2\pi} \frac{(-i)e^{i\varphi} i e^{i\varphi}}{(1-re^{i\varphi})(e^{i\varphi}-r)(1+re^{i\varphi})(e^{i\varphi}+r)} d\varphi dr
\end{aligned} \tag{6.12}
$$

The inner integral can be regarded as a complex line integral along the path $\gamma(\varphi) = e^{i\varphi}$ and computed with the help of the residue theorem.

$$
\begin{aligned}
&\int_{\varphi=0}^{2\pi} \frac{(-i)e^{i\varphi} i e^{i\varphi}}{(1-re^{i\varphi})(e^{i\varphi}-r)(1+re^{i\varphi})(e^{i\varphi}+r)} d\varphi \\
&\quad = \int_{\gamma} \frac{(-i)z}{(1-rz)(z-r)(1+rz)(z+r)} dz \\
&\quad = 2\pi i \left(\mathrm{Res}_{z=r} + \mathrm{Res}_{z=-r}\right) \left(\frac{(-i)z}{(1-rz)(z-r)(1+rz)(z+r)} \right) \\
&\quad = 2\pi \left(\frac{r}{(1-r^2)(1+r^2)2r} + \frac{-r}{(1+r^2)(-2r)(1-r^2)} \right) \\
&\quad = \frac{2\pi}{(1-r^2)(1+r^2)}
\end{aligned}
$$

We plug this into (6.12).

$$
\begin{aligned}
\mathbb{E}_{(-1,0)}^{(1,0)}(\tau_{\mathbb{D}}) &= 2 \int_{r=0}^{1} \frac{\left(1-r^2\right)^2 r}{(1-r^2)(1+r^2)} dr \\
&= 2 \int_{r=0}^{1} \frac{\left(1-r^2\right) r}{(1+r^2)} dr \\
&= 2\log\left(1+r^2\right) - r^2 \Big|_{r=0}^{1} \\
&= 2\log(2) - 1 \\
&\approx 0.386294
\end{aligned} \tag{6.13}
$$

- Now we compute $\mathbb{H}_0(\tau_{\mathbb{D}})$, replacing the points $b^{j,l}$ by $b^j := \left(\cos\left(\frac{1}{2}j\pi\right), \sin\left(\frac{1}{2}j\pi\right)\right)$ for $j = 1, \ldots, 4$. We make use of the rotational invariance of $\mathbb{E}_0^z(\tau_{\mathbb{D}})$ for $z \in \partial\mathbb{D}$.

$$
\begin{aligned}
\mathbb{H}_0(\tau_{\mathbb{D}}) &= \sum_{j=1}^{4} \mathbb{E}_0^{b^j}(\tau_{\mathbb{D}}) \frac{K_{\mathbb{D}}\left(0, b^j\right)}{\sum_{k=1}^{4} K_{\mathbb{D}}\left(0, b^k\right)} \\
&= \mathbb{E}_0^{b_4}(\tau_{\mathbb{D}}) \sum_{j=1}^{4} \frac{K_{\mathbb{D}}\left(0, b^j\right)}{\sum_{k=1}^{4} K_{\mathbb{D}}\left(0, b^k\right)} \\
&= \mathbb{E}_0^{(1,0)}(\tau_{\mathbb{D}})
\end{aligned} \tag{6.14}
$$

Plugging in the formulas of (6.11) and writing $z = re^{i\varphi}$ again gives

$$\begin{aligned}\mathbb{H}_0(\tau_{\mathbb{D}}) &= \int_{\mathbb{D}} \frac{G_{\mathbb{D}}(0,z)\, K_{\mathbb{D}}(z,(1,0))}{K_{\mathbb{D}}(0,(1,0))}\, dz \\ &= \int_{\mathbb{D}} \frac{\frac{1}{4\pi}\log\left(1+\frac{(1-|0|^2)(1-|z|^2)}{|0-z|^2}\right)\frac{1}{2\pi}\frac{1-|z|^2}{|(1,0)-z|^2}\, dz}{\frac{1}{2\pi}\frac{1-|0|^2}{|0-(1,0)|^2}} \\ &= \frac{1}{2\pi}\int_{r=0}^{1}\int_{\varphi=0}^{2\pi} \frac{-\log r\left(1-r^2\right)}{\left(1-re^{i\varphi}\right)\left(1-re^{-i\varphi}\right)}\, d\varphi r dr \\ &= \frac{1}{2\pi}\int_{r=0}^{1} -\log r\left(1-r^2\right) r \int_{\varphi=0}^{2\pi} \frac{(-i)\, ie^{i\varphi}}{\left(1-re^{i\varphi}\right)\left(e^{i\varphi}-r\right)}\, d\varphi dr. \end{aligned} \tag{6.15}$$

Again, we compute the inner integral with the help of the residue theorem for integration along the path $\gamma(\varphi) = e^{i\varphi}$.

$$\begin{aligned}\int_{\varphi=0}^{2\pi} \frac{(-i)\, ie^{i\varphi}}{\left(1-re^{i\varphi}\right)\left(e^{i\varphi}-r\right)}\, d\varphi &= \int_{\gamma} \frac{-i}{(1-rz)(z-r)}\, dz \\ &= 2\pi i \mathrm{Res}_{z=r}\left(\frac{-i}{(1-rz)(z-r)}\right) \\ &= 2\pi \left.\frac{1}{(1-rz)}\right|_{z=r} \\ &= \frac{2\pi}{1-r^2}\end{aligned}$$

We plug this back into (6.15).

$$\begin{aligned}\mathbb{H}_0(\tau_{\mathbb{D}}) &= \frac{1}{2\pi}\int_{r=0}^{1} -\log r\left(1-r^2\right) r \frac{2\pi}{1-r^2}\, dr = \int_{r=0}^{1} -r\log r\, dr \\ &= \left.\frac{1}{4}r^2 - \frac{1}{2}r^2\log r\right|_{r=0}^{1} = \frac{1}{4}\end{aligned} \tag{6.16}$$

- Finally, we are interested in $\sup_{x\in\partial\mathbb{D}} \mathbb{H}_x(\tau_{\mathbb{D}})$, where $b^j := \left(\cos\left(\frac{1}{2}j\pi\right), \sin\left(\frac{1}{2}j\pi\right)\right)$ for $j = 1, \ldots, 4$ as above. This term can only be understood as a limit, we hence use an approximation argument.

$$\begin{aligned}\mathbb{H}_x(\tau_{\mathbb{D}}) &= \lim_{t\to 0} \mathbb{H}_{(1-t)x}(\tau_{\mathbb{D}}) \\ &= \lim_{t\to 0} \sum_{j=1}^{4} \mathbb{E}^{b^j}_{(1-t)x}(\tau_{\mathbb{D}}) \frac{K_{\mathbb{D}}\left((1-t)x, b^j\right)}{\sum_{k=1}^{4} K_{\mathbb{D}}\left((1-t)x, b^k\right)} \\ &= \lim_{t\to 0} \sum_{j=1}^{4} \int_{\mathbb{D}} \frac{G_{\mathbb{D}}((1-t)x, z)\, K_{\mathbb{D}}\left(z, b^j\right)}{K_{\mathbb{D}}\left((1-t)x, b^j\right)}\, dz \frac{K_{\mathbb{D}}\left((1-t)x, b^j\right)}{\sum_{k=1}^{4} K_{\mathbb{D}}\left((1-t)x, b^k\right)}\end{aligned}$$

$$= \sum_{j=1}^{4} \int_{\mathbb{D}} \frac{K_{\mathbb{D}}(z,x)\, K_{\mathbb{D}}(z,b^j)}{\sum_{k=1}^{4} -\frac{\partial}{\partial n_x} K_{\mathbb{D}}(x,b^k)}\, dz$$

$$= \left(\sum_{k=1}^{4} \frac{1}{\pi} \frac{1}{|x-b^k|^2}\right)^{-1} \sum_{j=1}^{4} \int_{\mathbb{D}} \frac{1}{2\pi} \frac{1-|z|^2}{|z-x|^2} \frac{1}{2\pi} \frac{1-|z|^2}{|z-b^j|^2}\, dz \tag{6.17}$$

We make use of complex notation and replace x by $e^{i\omega}$ and b^j by $e^{i\frac{1}{2}j\pi}$. We start with the first factor.

$$\sum_{k=1}^{4} \frac{1}{\pi} \frac{1}{|x-b^k|^2} = \frac{1}{\pi} \sum_{k=1}^{4} \frac{1}{\left(e^{i\omega} - e^{i\frac{1}{2}j\pi}\right)\left(e^{-i\omega} - e^{-i\frac{1}{2}j\pi}\right)} = \frac{1}{\pi} \sum_{k=1}^{4} \frac{1}{2 - 2\mathrm{Re} e^{i\left(\omega - \frac{1}{2}j\pi\right)}}$$

$$= \frac{1}{2\pi} \left(\frac{1}{1-\cos\left(\omega - \frac{1}{2}\pi\right)} + \frac{1}{1-\cos(\omega-\pi)} + \frac{1}{1-\cos\left(\omega - \frac{3}{2}\pi\right)} + \frac{1}{1-\cos(\omega-2\pi)} \right)$$

$$= \frac{1}{2\pi} \left(\frac{1}{1-\sin\omega} + \frac{1}{1+\cos\omega} + \frac{1}{1+\sin\omega} + \frac{1}{1-\cos\omega} \right)$$

$$= \frac{1}{\pi} \left(\frac{1}{1-\sin^2\omega} + \frac{1}{1-\cos^2\omega} \right) = \frac{1}{\pi} \frac{1}{\sin^2\omega \cos^2\omega} \tag{6.18}$$

We use polar coordinates for the integral,

$$\int_{\mathbb{D}} \frac{1}{2\pi} \frac{1-|z|^2}{|z-x|^2} \frac{1}{2\pi} \frac{1-|z|^2}{|z-b^j|^2}\, dz$$
$$= \frac{1}{4\pi^2} \int_{r=0}^{1} \left(1-r^2\right)^2 \int_{\varphi=0}^{2\pi} \frac{1}{|re^{i\varphi} - e^{i\omega}|^2 \left|re^{i\varphi} - e^{i\frac{1}{2}j\pi}\right|^2}\, d\varphi r dr, \tag{6.19}$$

and compute the inner integral by means of complex analysis as above.

$$\int_{\varphi=0}^{2\pi} \frac{1}{|re^{i\varphi} - e^{i\omega}|^2 \left|re^{i\varphi} - e^{i\frac{1}{2}j\pi}\right|^2}\, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \frac{1}{\left(re^{i\varphi} - e^{i\omega}\right)\left(re^{-i\varphi} - e^{-i\omega}\right)\left(re^{i\varphi} - e^{i\frac{1}{2}j\pi}\right)\left(re^{-i\varphi} - e^{-i\frac{1}{2}j\pi}\right)}\, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \frac{e^{i\omega} e^{i\frac{1}{2}j\pi} (-i)\, e^{i\varphi} i e^{i\varphi}}{\left(re^{i\varphi} - e^{i\omega}\right)\left(re^{i\omega} - e^{i\varphi}\right)\left(re^{i\varphi} - e^{i\frac{1}{2}j\pi}\right)\left(re^{i\frac{1}{2}j\pi} - e^{i\varphi}\right)}\, d\varphi$$

$$= \int_{\gamma} \frac{e^{i\omega} e^{i\frac{1}{2}j\pi} (-i)\, z}{\left(rz - e^{i\omega}\right)\left(z - re^{i\omega}\right)\left(rz - e^{i\frac{1}{2}j\pi}\right)\left(z - re^{i\frac{1}{2}j\pi}\right)}\, d\varphi$$

$$= 2\pi i \left(\mathrm{Res}_{z=re^{i\omega}} + \mathrm{Res}_{z=re^{i\frac{1}{2}j\pi}}\right) \left(\frac{e^{i\omega} e^{i\frac{1}{2}j\pi} (-i)\, z}{\left(rz - e^{i\omega}\right)\left(z - re^{i\omega}\right)\left(rz - e^{i\frac{1}{2}j\pi}\right)\left(z - re^{i\frac{1}{2}j\pi}\right)} \right)$$

$$
\begin{aligned}
&= 2\pi \left(\frac{e^{i\omega} e^{i\frac{1}{2}j\pi} r e^{i\omega}}{\left(r^2 e^{i\omega} - e^{i\omega}\right)\left(r^2 e^{i\omega} - e^{i\frac{1}{2}j\pi}\right)\left(r e^{i\omega} - r e^{i\frac{1}{2}j\pi}\right)} \right. \\
&\qquad \left. + \frac{e^{i\omega} e^{i\frac{1}{2}j\pi} r e^{i\frac{1}{2}j\pi}}{\left(r^2 e^{i\frac{1}{2}j\pi} - e^{i\omega}\right)\left(r e^{i\frac{1}{2}j\pi} - r e^{i\omega}\right)\left(r^2 e^{i\frac{1}{2}j\pi} - e^{i\frac{1}{2}j\pi}\right)} \right) \\
&= 2\pi \left(\frac{e^{i\omega} e^{i\frac{1}{2}j\pi}}{\left(r^2 - 1\right)\left(r^2 e^{i\omega} - e^{i\frac{1}{2}j\pi}\right)\left(e^{i\omega} - e^{i\frac{1}{2}j\pi}\right)} \right. \\
&\qquad \left. + \frac{e^{i\omega} e^{i\frac{1}{2}j\pi}}{\left(r^2 e^{i\frac{1}{2}j\pi} - e^{i\omega}\right)\left(e^{i\frac{1}{2}j\pi} - e^{i\omega}\right)\left(r^2 - 1\right)} \right) \\
&= 2\pi \frac{e^{i\omega} e^{i\frac{1}{2}j\pi}}{\left(r^2 - 1\right)\left(e^{i\omega} - e^{i\frac{1}{2}j\pi}\right)} \cdot \frac{\left(r^2 e^{i\frac{1}{2}j\pi} - e^{i\omega}\right) - \left(r^2 e^{i\omega} - e^{i\frac{1}{2}j\pi}\right)}{\left(r^2 e^{i\omega} - e^{i\frac{1}{2}j\pi}\right)\left(r^2 e^{i\frac{1}{2}j\pi} - e^{i\omega}\right)} \\
&= 2\pi \frac{1}{\left(r^2 - 1\right)\left(e^{i\omega} - e^{i\frac{1}{2}j\pi}\right)} \cdot \frac{r^2\left(e^{i\frac{1}{2}j\pi} - e^{i\omega}\right) + \left(e^{i\frac{1}{2}j\pi} - e^{i\omega}\right)}{\left(r^2 - e^{i\left(\frac{1}{2}j\pi - \omega\right)}\right)\left(r^2 - e^{-i\left(\frac{1}{2}j\pi - \omega\right)}\right)} \\
&= 2\pi \frac{r^2 + 1}{\left(1 - r^2\right)\left(r^4 - 2r^2 \cos\left(\frac{1}{2}j\pi - \omega\right) + 1\right)}
\end{aligned}
$$

We plug this into (6.19) and get

$$
\begin{aligned}
\int_{\mathbb{D}} \frac{1}{2\pi} \frac{1 - |z|^2}{|z - x|^2} \frac{1}{2\pi} \frac{1 - |z|^2}{|z - b^j|^2} \, dz &= \frac{1}{2\pi} \int_{r=0}^{1} \frac{\left(1 - r^2\right)\left(r^2 + 1\right) r}{r^4 - 2r^2 \cos\left(\frac{1}{2}j\pi - \omega\right) + 1} \, dr \\
&= \frac{1}{4\pi} \int_{s=0}^{1} \frac{1 - s^2}{s^2 - 2s \cos\left(\frac{1}{2}j\pi - \omega\right) + 1} \, ds.
\end{aligned}
$$

Put together with (6.18) into (6.17), this yields

$$
\mathbb{H}_x\left(\tau_{\mathbb{D}}\right) = \frac{1}{4} \sin^2 \omega \, \cos^2 \omega \sum_{j=1}^{4} \int_{s=0}^{1} \frac{1 - s^2}{s^2 - 2s \cos\left(\frac{1}{2}j\pi - \omega\right) + 1} \, ds.
$$

As in (6.18), we expand the sum explicitly,

$$
\begin{aligned}
&\sum_{j=1}^{4} \frac{1}{s^2 - 2s \cos\left(\frac{1}{2}j\pi - \omega\right) + 1} \\
&= \frac{2s^2 + 2}{\left(s^2 + 1 - 2s \cos \omega\right)\left(s^2 + 1 + 2s \cos \omega\right)} + \frac{2s^2 + 2}{\left(s^2 + 1 - 2s \sin \omega\right)\left(s^2 + 1 + 2s \sin \omega\right)}
\end{aligned}
$$

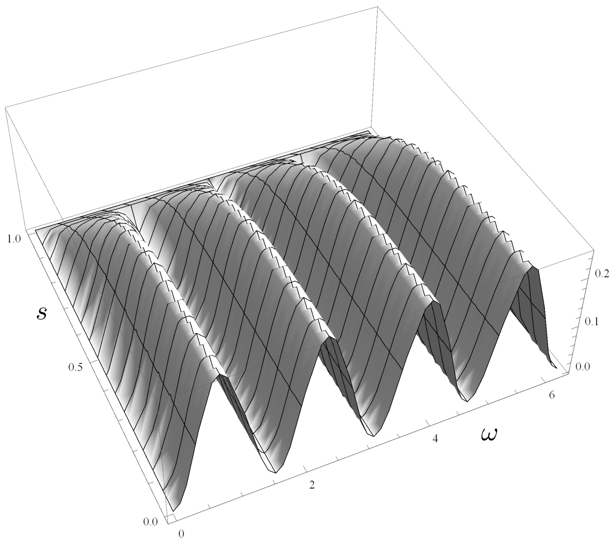


Figure 6.3.: A plot of the function $(s,\omega) \mapsto \frac{\sin^2\omega\cos^2\omega\left(1-s^4\right)\left(1+s^4\right)}{\left(s^2+1\right)^4-4s^2\left(s^2+1\right)^2+16s^4\sin^2\omega\cos^2\omega}$. For fixed s, the maximum is always attained at $\omega = \frac{1}{4}\pi + \frac{1}{2}j\pi$, $j = 1,\dots,4$.

$$= 2\left(s^2+1\right)\frac{2\left(s^2+1\right)^2-4s^2}{\left(\left(s^2+1\right)^2-4s^2\cos^2\omega\right)\left(\left(s^2+1\right)^2-4s^2\sin^2\omega\right)}$$
$$= \frac{4\left(s^2+1\right)\left(s^4+1\right)}{\left(s^2+1\right)^4-4s^2\left(s^2+1\right)^2+16s^4\sin^2\omega\cos^2\omega},$$

and hence get

$$\mathbb{H}_x\left(\tau_{\mathbb{D}}\right) = \int_{s=0}^{1} \frac{\sin^2\omega\cos^2\omega\left(1-s^4\right)\left(1+s^4\right)}{\left(s^2+1\right)^4-4s^2\left(s^2+1\right)^2+16s^4\sin^2\omega\cos^2\omega}\,ds. \tag{6.20}$$

For which $x = (\cos\omega, \sin\omega)$ does $\mathbb{H}_x\left(\tau_{\mathbb{D}}\right)$ attain its maximum? A plot of the integrand is shown in Figure 6.3. We see that for fixed s, the maximum of the integrand is always attained at $\omega = \frac{1}{4}\pi + \frac{1}{2}j\pi$, $j = 1,\dots,4$, independent of s. Analytically, this follows from the fact that for fixed s, the function $(s,t) \mapsto \frac{t\left(1-s^4\right)\left(1+s^4\right)}{\left(s^2+1\right)^4-4s^2\left(s^2+1\right)^2+16s^4t}$ is increasing in $t \in [0, \frac{1}{4}]$, thus it attains its maximum at $t = \frac{1}{4} = \max\left\{\sin^2\omega\cos^2\omega; \omega \in [0,2\pi]\right\}$. Hence the maximal value of $\mathbb{H}_x\left(\tau_{\mathbb{D}}\right)$ is attained at $x = \left(\cos\left(\frac{1}{4}\pi\right), \sin\left(\frac{1}{4}\pi\right)\right)$ or after rotations of x by $\frac{1}{2}\pi$. This also suits the intuition that the lifetime should be longer the farther away the starting point stays from the gaps. The gaps are situated at the angles of 0, $\frac{1}{2}\pi$, π and $\frac{3}{2}\pi$, so we get maximal lifetime if the (boundary) starting point lies just in the middle between two gaps.

We set $x = \left(\cos\left(\frac{1}{4}\pi\right), \sin\left(\frac{1}{4}\pi\right)\right)$ in (6.20) and get

$$\begin{aligned}
\sup_{x\in\partial\mathbb{D}} \mathbb{H}_x\left(\tau_{\mathbb{D}}\right) &= \int_{s=0}^{1} \frac{\frac{1}{4}\left(1-s^4\right)\left(1+s^4\right)}{\left(s^2+1\right)^4 - 4s^2\left(s^2+1\right)^2 + 4s^4}\,ds = \frac{1}{4}\int_{s=0}^{1} \frac{1-s^4}{1+s^4}\,ds \\
&= -\frac{1}{4}s - \frac{\sqrt{2}}{8}\arctan\left(1-\sqrt{2}s\right) + \frac{\sqrt{2}}{8}\arctan\left(1+\sqrt{2}s\right) \\
&\quad -\frac{\sqrt{2}}{16}\log\left(1-\sqrt{2}s+s^2\right) + \frac{\sqrt{2}}{16}\log\left(1+\sqrt{2}s+s^2\right)\Bigg|_{s=0}^{1} \\
&= -\frac{1}{4} + \frac{\sqrt{2}}{16}\pi + \frac{\sqrt{2}}{32}\log\left(17+12\sqrt{2}\right) \approx 0.183486. \qquad (6.21)
\end{aligned}$$

7. Maximal lifetime for interior points

After the preparatory work, it is now easy to proof the following theorem.

Theorem 7.1. *There exists a bounded (multiply connected) domain $\Omega \subset \mathbb{R}^2$ which has two interior points x_0 and y_0 such that*

$$\mathbb{E}_{x_0}^{y_0}(\tau_\Omega) \geq \sup_{x,y \in \partial\Omega} \mathbb{E}_x^y(\tau_\Omega).$$

Proof. We consider $\Omega = M_{d_1,d_2,d_3,s}$ and explain first how we choose the values of d_1, d_2, d_3, and s with the help of Lemma 6.1 and the computations of Section 6.3.

To start, let ρ be so small that it satisfies the following two conditions:

- $2C(2\rho)^2 < 0.02$ with C from Lemma 5.1 and Corollary 5.2.
- The function $x \mapsto \mathbb{H}_x(\tau_{D^l})$ is continuous on $\overline{D^l}$. Choose ρ to be so small that

$$|\tilde{x} - x| \leq 2\rho \text{ implies } |\mathbb{H}_{\tilde{x}}(\tau_{D^l}) - \mathbb{H}_x(\tau_{D^l})| < 0.01. \tag{7.1}$$

Then set $\varepsilon := 0.01$. Finally choose the values of d_1, d_2, d_3, and s such that the assertions of Lemma 6.1 hold for this ε.

Now set $x_0 := (-2, 0)$ and $y_0 := (2, 0)$. Case 4 of Lemma 6.1 together with (6.16) gives

$$\mathbb{E}_{x_0}^{y_0}\left(\tau_{M_{d_1,d_2,d_3,s}}\right) \geq \mathbb{H}_x(\tau_{D^l}) + \mathbb{H}_y(\tau_{D^r}) - \varepsilon = \frac{1}{4} + \frac{1}{4} - 0.01 = 0.49.$$

We turn to $\sup_{x,y\in\partial M_{d_1,d_2,d_3,s}} \mathbb{E}_x^y\left(\tau_{M_{d_1,d_2,d_3,s}}\right)$. By continuity, the assertions of Lemma 6.1 also hold for boundary points. If x or/and y are closer to some $z \in S_{d_1,d_2,d_3,s}$ than 2ρ, we make use of Lemma 5.1/Corollary 5.2. This allows us to replace x (and/or y) by some $\tilde{x} \in \partial B_{2\rho}(z) \cap \overline{M_{d_1,d_2,d_3,s}}$ (or/and eventually y by $\tilde{y}$), for which the lifetimes get maximal, at the cost of $2C\rho^2 < 0.02$.[1] Consequently,

$$\begin{aligned}
&\sup\left\{\mathbb{E}_x^y\left(\tau_{M_{d_1,d_2,d_3,s}}\right); x, y \in \partial M_{d_1,d_2,d_3,s}\right\} \\
&\quad \leq \sup\left\{\mathbb{E}_x^y\left(\tau_{M_{d_1,d_2,d_3,s}}\right); x, y \in \partial\left(M_{d_1,d_2,d_3,s} \setminus \bigcup_{z\in S_{d_1,d_2,d_3,s}} B_{2\rho}(z)\right)\right\} + 0.02.
\end{aligned} \tag{7.2}$$

[1] Actually, Lemma 5.1 and Corollary 5.2 are formulated only for interior, not for boundary points. Nevertheless, we can approach a boundary point by a sequence of interior points and see that the same upper limit of the lifetime holds for all points of the approximating sequence. Hence it also holds for the boundary point.

All the remaining points to look at have a larger distance to the points of $S_{d_1,d_2,d_3,s}$ than ρ, so Lemma 6.1 can be applied. It gives three different upper bounds depending on which (boundary of a) subdomain x and y are part of. These three upper bounds are

- $\varepsilon = 0.01$,
- $\sup_{p,q\in\overline{D^l}} \mathbb{E}_p^q(\tau_{D^l})$, which equals $2\log(2) - 1 \leq 0.39$ according to (6.13),
- and, finally, the sum of

$$\sup\left\{\mathbb{H}_p(\tau_{D^l})\,;\,p \in \partial\left(D^l \setminus \bigcup_{z\in S_{d_1,d_2,d_3,s}} B_{2\rho}(z)\right)\right\}$$

and

$$\sup\left\{\mathbb{H}_q(\tau_{D^r})\,;\,q \in \partial\left(D^r \setminus \bigcup_{z\in S_{d_1,d_2,d_3,s}} B_{2\rho}(z)\right)\right\}.$$

By continuity, see (7.1), and after translation to the unit circle, an upper bound for this sum is given by

$$2\left(\sup_{p\in\partial\mathbb{D}} \mathbb{H}_p(\tau_{\mathbb{D}}) + 0.01\right) < 0.4$$

according to (6.21).

The latter has the largest value for the three upper bounds. We plug it into (7.2) and sum everything up to obtain

$$\sup_{x,y\in\partial M_{d_1,d_2,d_3,s}} \mathbb{E}_x^y\left(\tau_{M_{d_1,d_2,d_3,s}}\right) < 0.4 + 0.02 < 0.49 = \mathbb{E}_{x_0}^{y_0}\left(\tau_{M_{d_1,d_2,d_3,s}}\right).$$

□

Appendices

A. Convergence of $\mathbb{E}_x^{h_\varepsilon}\left(\tau_{\Omega\setminus B_\varepsilon(y)}\right)$

Lemma A.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain for which a Green function G_Ω exists. Let $y \in \Omega$. Then, if $B_\varepsilon(y) \subset\subset \Omega$, the Green function $G_{\Omega\setminus B_\varepsilon(y)}$ exists, too.*

Let $\delta > 0$ be such that $B_\delta(y) \subset\subset \Omega$. If $\varepsilon \in \left(0, \frac{1}{4}\delta\right)$, then

$$\left|G_{\Omega\setminus B_\varepsilon(y)}(x,z) - G_\Omega(x,z)\right| \leq \frac{8\pi}{\log\left(1+\frac{15}{16}\frac{\delta^2}{\varepsilon^2}\right)} G_\Omega(z,y)\, G_\Omega(x,y) \tag{A.1}$$

for all $x, z \in \Omega \setminus B_\delta(y)$ with $x \neq z$.

Remark A.2. As for fixed x and z (with $x \neq y \neq z$), the right hand side of (A.1) tends to zero for $\varepsilon \to 0$, this especially implies that $G_{\Omega\setminus B_\varepsilon(y)} \to G_\Omega$ pointwise. If, moreover, $\partial\Omega$ is smooth enough to allow the Green function estimates of Proposition 3.25, one gets an upper bound of the form

$$\frac{C d_\Omega(x)\, d_\Omega(z)}{\log\left(1+\frac{15}{16}\frac{\delta^2}{\varepsilon^2}\right)},$$

where C depends on Ω and y.

Proof. If G_Ω exists and if $B_\varepsilon(y) \subset\subset \Omega$, then $G_{\Omega\setminus B_\varepsilon(y)}$ exists, too, as all points of $\partial B_\varepsilon(y)$ are regular.

Let $\delta > 0$ with $B_\delta(y) \subset\subset \Omega$ and $x, z \in \Omega \setminus B_\delta(y)$ with $x \neq z$. Because of (3.14), it holds that

$$G_{\Omega\setminus B_\varepsilon(y)}(x,z) \leq G_\Omega(x,z)$$

for all $\varepsilon \in (0, \delta)$.

We now show that for $\varepsilon \in \left(0, \frac{1}{4}\delta\right)$,

$$G_\Omega(x,z) \leq G_{\Omega\setminus B_\varepsilon(y)}(x,z) + \frac{8\pi}{\log\left(1+\frac{15}{16}\frac{\delta^2}{\varepsilon^2}\right)} G_\Omega(z,y)\, G_\Omega(x,y). \tag{A.2}$$

For this purpose, we fix x and define the auxiliary function u_ϵ on $\Omega \setminus B_\varepsilon(y)$ by

$$u_\varepsilon(w) := G_\Omega(x,w) - G_{\Omega\setminus B_\varepsilon(y)}(x,w) - 2\frac{G_\Omega(w,y)\, G_\Omega(x,y)}{\inf_{\tilde{w}\in\partial B_\varepsilon(y)} G_\Omega(\tilde{w},y)}.$$

As both $w \mapsto G_\Omega(x,w) - G_{\Omega\setminus B_\varepsilon(y)}(x,w)$ and $w \mapsto G_\Omega(w,y)$ are harmonic on $\Omega\setminus B_\varepsilon(y)$, u_ε solves the following boundary value problem.

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega \setminus B_\varepsilon(y) \\ u_\varepsilon = 0 & \text{on } \partial\Omega \\ u_\varepsilon = G_\Omega(x,.) - 2\frac{G_\Omega(.,y) G_\Omega(x,y)}{\inf_{\tilde{w}\in\partial B_\varepsilon(y)} G_\Omega(\tilde{w},y)} & \text{on } \partial B_\varepsilon(y) \end{cases}$$

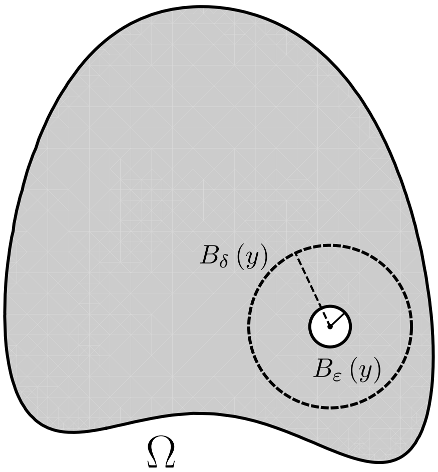


Figure A.1.: The domain $\Omega \setminus B_\varepsilon(y)$ of Lemma A.1.

We show that $u_\varepsilon \le 0$ on $\partial B_\varepsilon(y)$ for small ε. Let $w \in \partial B_\varepsilon(y)$. Then

$$\begin{aligned} u_\varepsilon(w) &= G_\Omega(x,w) - 2\frac{G_\Omega(w,y)\,G_\Omega(x,y)}{\inf_{\tilde{w}\in\partial B_\varepsilon(y)} G_\Omega(\tilde{w},y)} \\ &\le G_\Omega(x,w) - 2G_\Omega(x,y)\,\frac{\inf_{\tilde{w}\in\partial B_\varepsilon(y)} G_\Omega(\tilde{w},y)}{\inf_{\tilde{w}\in\partial B_\varepsilon(y)} G_\Omega(\tilde{w},y)} \\ &= G_\Omega(x,y)\left(\frac{G_\Omega(x,w)}{G_\Omega(x,y)} - 2\right). \end{aligned} \tag{A.3}$$

The function $w \mapsto G_\Omega(x,w)$ is harmonic and positive in $B_\delta(y)$, so the Harnack type inequality stated in Theorem A.3 below gives that

$$G_\Omega(x,w) \le \left(1 + \frac{\varepsilon}{\delta-\varepsilon}\right)^2 G_\Omega(x,y) \tag{A.4}$$

for $w \in \partial B_\varepsilon(y)$. If $\varepsilon \le \frac{1}{4}\delta$, then the right hand side of (A.4) is bounded by $\frac{16}{9}G_\Omega(x,y)$, which implies that the right hand side of (A.3) and, as a consequence, $u_\varepsilon(w)$ is negative for $w \in \partial B_\varepsilon(y)$. Hence, by the maximum principle, $u_\varepsilon \le 0$ on $\Omega \setminus B_\varepsilon(y)$. It holds especially for z, so

$$G_\Omega(x,z) \le G_{\Omega\setminus B_\varepsilon(y)}(x,z) + 2\frac{G_\Omega(z,y)\,G_\Omega(x,y)}{\inf_{\tilde{z}\in\partial B_\varepsilon(y)} G_\Omega(\tilde{z},y)}. \tag{A.5}$$

A lower bound for $\inf_{\tilde{z}\in\partial B_\varepsilon(y)} G_\Omega(\tilde{z},y)$ can be found in the following way. By (3.14) again and the explicit formula for the Green function on a ball, (3.7), we have that

$$G_\Omega(\tilde{z},y) \ge G_{B_\delta(y)}(\tilde{z},y) = \frac{1}{4\pi}\log\left(1 + \frac{\left(\delta^2-\varepsilon^2\right)\delta^2}{\delta^2\varepsilon^2}\right) \ge \frac{1}{4\pi}\log\left(1 + \frac{15}{16}\frac{\delta^2}{\varepsilon^2}\right)$$

for every $\tilde{z} \in B_\varepsilon(y)$ and $\varepsilon \in \left(0, \frac{1}{4}\delta\right)$. Plugging this into (A.5), we get (A.2). □

The following theorem which has been used in the proof above is a direct consequence of the mean value property for harmonic functions. It can be found in [30, Satz 2.2.1], for example. We present a version adapted for dimension 2.

Theorem A.3. *Let $\Omega \subset \mathbb{R}^2$ be open and u be harmonic and nonnegative on Ω. Let $x_1, x_2 \in \Omega$ and set $d := |x_1 - x_2|$. Then*

$$u(x_1) \le \left(1 + \frac{d}{r}\right)^2 u(x_2)$$

for all $r > 0$ with $B_{d+r}(x_2) \subset\subset \Omega$.

The next lemma gives some approximation of $G_\Omega(x,y)$ within $\Omega \setminus B_\delta(y)$.

Lemma A.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain for which a Green function G_Ω exists. Let $y \in \Omega$ and let $\delta > 0$ be such that $B_\delta(y) \subset\subset \Omega$. For $\varepsilon \in (0, \delta)$, let h_ε be the unique solution of*

$$\begin{cases} -\Delta h_\varepsilon = 0 & \text{on } \Omega \setminus B_\varepsilon(y), \\ h_\varepsilon = 0 & \text{on } \partial\Omega, \\ h_\varepsilon = 1 & \text{on } \partial B_\varepsilon(y). \end{cases} \tag{A.6}$$

Let $x \in \Omega \setminus B_\delta(y)$. If $\varepsilon \in \left(0, \frac{1}{4}\delta\right)$, then

$$\left| G_\Omega(x,y) - \left(-\frac{1}{2\pi}\log\varepsilon\right) h_\varepsilon(x) \right| \le \frac{8\pi}{\log\left(1 + \frac{15}{16}\frac{\delta^2}{\varepsilon^2}\right)} G_\Omega(x,y)\, \Phi^y(y)$$

with Φ^y being the function that is used in the construction of G_Ω, see (3.3) in Section 3.3.1.

Remark A.5. As a consequence, $\lim_{\varepsilon \to 0}\left(-\frac{1}{2\pi}\log\varepsilon\right) h_\varepsilon(x) = G_\Omega(x,y)$ pointwise for all $x \in \Omega$, $x \neq y$.

Proof. We set $u_\varepsilon := \left(-\frac{1}{2\pi}\log\varepsilon\right) h_\varepsilon(x) - G_\Omega(x,y)$. The function u_ε is harmonic on $\Omega \setminus B_\varepsilon(y)$ and equals 0 on $\partial\Omega$. Which boundary value is attained at $z \in \partial B_\varepsilon(y)$? As $-\frac{1}{2\pi}\log\varepsilon h_\varepsilon(z) = -\frac{1}{2\pi}\log\varepsilon = \Phi(z-y)$, where Φ is the fundamental solution, see Section 3.3.1, $u_\varepsilon(z) = \Phi^y(z)$ with Φ^y from (3.3). Summing up, u_ε solves

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega \setminus B_\varepsilon(y), \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \\ u_\varepsilon = \Phi^y & \text{on } \partial B_\varepsilon(y). \end{cases} \tag{A.7}$$

If χ_δ stands for a C^∞ function that is 1 on $B_{\frac{1}{4}\delta}(y)$, 0 on $\Omega \setminus B_\delta(y)$, and nonnegative in between, then the solution of (A.7) can be written as

$$u_\varepsilon(x) = \chi_\delta(x)\,\Phi^y(x) - \int_{\Omega \setminus B_\varepsilon(y)} G_{\Omega \setminus B_\varepsilon(y)}(x,z)\left(-\Delta\left(\chi_\delta \Phi^y\right)(z)\right) dz.$$

As $\chi_\delta \Phi^y$ is zero on $\partial\Omega$, $\chi_\delta(x)\Phi^y(x) = \int_\Omega G_\Omega(x,z)\left(-\Delta\left(\chi_\delta\Phi^y\right)(z)\right)dz$. Moreover, with Φ^y being harmonic on Ω and χ_δ being 1 on $B_{\frac{1}{4}\delta}(y)$, $-\Delta\left(\chi_\delta\Phi^y\right) = 0$ on $B_{\frac{1}{4}\delta}(y)$. We make use of Lemma A.1.

$$\begin{aligned}
u_\varepsilon(x) &= \int_\Omega G_\Omega(x,z)\left(-\Delta\left(\chi_\delta\Phi^y\right)(z)\right)dz - \int_{\Omega\setminus B_\varepsilon(y)} G_{\Omega\setminus B_\varepsilon(y)}(x,z)\left(-\Delta\left(\chi_\delta\Phi^y\right)(z)\right)dz \\
&= \int_{\Omega\setminus B_{\frac{1}{4}\delta}(y)} \left(G_\Omega(x,z) - G_{\Omega\setminus B_\varepsilon(y)}(x,z)\right)\left(-\Delta\left(\chi_\delta\Phi^y\right)(z)\right)dz \\
&\le \frac{8\pi}{\log\left(1+\frac{15}{16}\frac{\delta^2}{\varepsilon^2}\right)} G_\Omega(x,y)\int_{\Omega\setminus B_{\frac{1}{4}\delta}(y)} G_\Omega(z,y)\left(-\Delta\left(\chi_\delta\Phi^y\right)(z)\right)dz \\
&\le \frac{8\pi}{\log\left(1+\frac{15}{16}\frac{\delta^2}{\varepsilon^2}\right)} G_\Omega(x,y)\left(\chi_\delta\Phi^y\right)(y) \\
&= \frac{8\pi}{\log\left(1+\frac{15}{16}\frac{\delta^2}{\varepsilon^2}\right)} G_\Omega(x,y)\,\Phi^y(y)
\end{aligned}$$

□

As a consequence of the preceding lemmas, the limit in (3.27) holds true.

Corollary A.6. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain for which a Green function* G_Ω *exists and let* $x, y \in \Omega$ *with* $x \neq y$. *Let* h_ε *be the solution of (A.6). Then*

$$\lim_{\varepsilon\to 0}\int_{\Omega\setminus B_\varepsilon(y)} G_{\Omega\setminus B_\varepsilon(y)}(x,z)\frac{h_\varepsilon(z)}{h_\varepsilon(x)}\,dz = \int_\Omega \frac{G_\Omega(x,z)\,G_\Omega(z,y)}{G_\Omega(x,y)}\,dz.$$

Proof. Let $x, y \in \Omega$. We extend $G_{\Omega\setminus B_\varepsilon(y)}(x,.)$ and h_ε by 0 on $B_\varepsilon(y)$ and look at

$$\int_\Omega G_{\Omega\setminus B_\varepsilon(y)}(x,z)\frac{h_\varepsilon(z)}{h_\varepsilon(x)}\,dz = \int_\Omega G_{\Omega\setminus B_\varepsilon(y)}(x,z)\frac{-\frac{1}{2\pi}\log\varepsilon\cdot h_\varepsilon(z)}{-\frac{1}{2\pi}\log\varepsilon\cdot h_\varepsilon(x)}\,dz.$$

Lemma A.1 and Lemma A.4 give that the integrand on the right hand side converges pointwise (in z for x and y fixed) to the $3G$ expression. It remains to find some dominating function to show convergence of the integral. As $h_\varepsilon \le 1$ according to the maximum principle, it holds that

$$G_{\Omega\setminus B_\varepsilon(y)}(x,z)\frac{-\frac{1}{2\pi}\log\varepsilon\cdot h_\varepsilon(z)}{-\frac{1}{2\pi}\log\varepsilon\cdot h_\varepsilon(x)} \le G_\Omega(x,z)\frac{1}{h_\varepsilon(x)},$$

which is integrable in z. □

B. Sub- and supersolution of a special Dirichlet problem

In the proof of Theorem 4.10, amongst others we deal with a domain W_{R_1,R_2} that consists of two half disks of different radii R_1 and R_2 with $R_2 < R_1$. The half disks are separated by a straight line which has an opening of gap width 2, see Figure B.1. In other words, we set

$$\begin{aligned} W_{R_1,R_2} := &\{v = (v_1, v_2)\,;\, v_1 < 0 \text{ and } |v| < R_1\} \cup (\{0\} \times (-1, 1)) \\ &\cup \{v = (v_1, v_2)\,;\, v_1 > 0 \text{ and } |v| < R_2\}\,. \end{aligned}$$

In the course of the proof, we look at a harmonic function u that satisfies the following boundary conditions for $v = (v_1, v_2) \in \partial W_{R_1,R_2}$.

$$u(v) = \begin{cases} 0 & \text{if } v_1 < 0, \\ 0 & \text{if } v_1 = 0, \\ v_1 & \text{if } v_1 > 0; \end{cases} \tag{B.1}$$

How does the solution behave in the interior at $v_1 = 0$ for large R_1, R_2? We will give a sub– and a supersolution of the boundary value problem, which are independent of large R_1 and R_2. We find the supersolution by mapping W_{R_1,R_2} conformally to some domain which is nearly a circle. As the analysis of the boundary value problem can be done without the context of Theorem 4.10, we present it here in the Appendix and prove the following proposition.

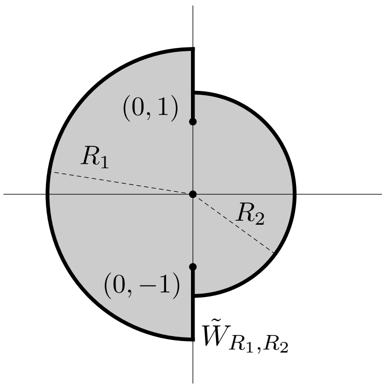


Figure B.1.: The domain $\tilde{W}_{R_1,R_2}$.

Proposition B.1. *Let u be a harmonic function on W_{R_1,R_2} that satisfies the boundary conditions of (B.1). There are some $R > 0$ and functions $u_{sub}, u_{sup} : \mathbb{R}^2 \to \mathbb{R}_0^+$ (with $u_{sub} \neq 0$) which are independent of R_1 and R_2 such that*

$$u_{sub} \leq u \leq u_{sup}$$

on W_{R_1,R_2} if $R_1 > R_2 > R$. Especially for $v_2 \in (-1, +1)$, it holds that

$$0 < u\,(0, v_2) < C_{sup},$$

where C_{sup} is independent of R_1 and R_2.

Proof. To get a subsolution, let u_3 be the harmonic function on $B_1\,(0)$ that fulfills the following boundary conditions for $v = (v_1, v_2) \in \partial B_1\,(0)$.

$$u_3(v) = \begin{cases} 0 & \text{if } v_1 \leq 0 \\ v_1 & \text{if } v_1 > 0 \end{cases}$$

Comparing u_3 with the harmonic functions $u_1 : \mathbb{R}^2 \to \mathbb{R}$, $u_1(v) = 0$ and $u_2 : \mathbb{R}^2 \to \mathbb{R}$, $u_2(v) = v_1$ gives $u_1 < u_3$ and $u_2 < u_3$ in $B_1\,(0)$ according to the maximum principle. We define

$$u_{sub}(v) := \begin{cases} u_1(v) & \text{if } v_1 \leq 0 \text{ and } v \notin B_1\,(0)\,, \\ u_2(v) & \text{if } v_1 > 0 \text{ and } v \notin B_1\,(0)\,, \\ u_3(v) & \text{if } v \in B_1\,(0)\,. \end{cases}$$

It satisfies $u_{sub} = \max\{u_1, u_2, u_3\}$ (and $u_{sub} = \max\{u_1, u_2\}$ outside $B_1\,(0)$). Hence it is subharmonic[1]. Moreover, u_{sub} satisfies the boundary conditions (B.1) if $R_1, R_2 > 1$. In other words, u_{sub} is a subsolution of the problem, so $u_{sub} \leq u$ by the maximum principle. It holds that $u\,(0, v_2) \geq u_{sub}\,(0, v_2) = u_3\,(0, v_2) > 0$ if $v_2 \in (-1, 1)$. The graph of u_{sub} is sketched in Figure B.3.

Now we look for a supersolution. The idea is to switch to complex notation and analyse the conformal mapping $\mathbf{f} : \mathbf{H}_1 \to \mathbf{H}_2$ that maps $\mathbf{H}_1 := \{\mathbf{z} \in \mathbb{C}; \mathrm{Re}\,(\mathbf{z}) > 0\}$ onto $\mathbf{H}_2 := \mathbb{C} \setminus ((-\infty, -1]\,\mathbf{i} \cup [1, \infty)\,\mathbf{i})$. We define a holomorphic function $\mathbf{u}_4$ on $\mathbf{H}_1$ whose real part satisfies the corresponding boundary conditions on $\mathbf{H}_1$ with a $\geq$ sign. As real parts of holomorphic functions are harmonic, we have thus found a supersolution of the original problem by taking $u^{sup}(v_1, v_2) := \mathrm{Re}\left(\mathbf{u}_4\left(\mathbf{f}^{inv}\,(v_1 + \mathbf{i}v_2)\right)\right)$.

The function $\mathbf{f}$ is given by

$$\mathbf{f}\,(\mathbf{z}) = \frac{1}{2}\left(\mathbf{z} - \frac{1}{\mathbf{z}}\right),$$

which is sketched in Figure B.2. To see that this function is the right choice, we look at the image of $\partial \mathbf{H}_1 = \mathbf{i}\mathbb{R}$ under $\mathbf{f}$.

$$\mathbf{f}\,(\mathbf{i}t) = \frac{1}{2}\mathbf{i}\left(t + \frac{1}{t}\right)$$

[1] Subharmonic functions are defined as an analogue to superharmonic functions, with the '$\geq$' sign in Definition 3.14 replaced by $\leq$'. Analogous equivalences to those in Lemma 3.15 hold.

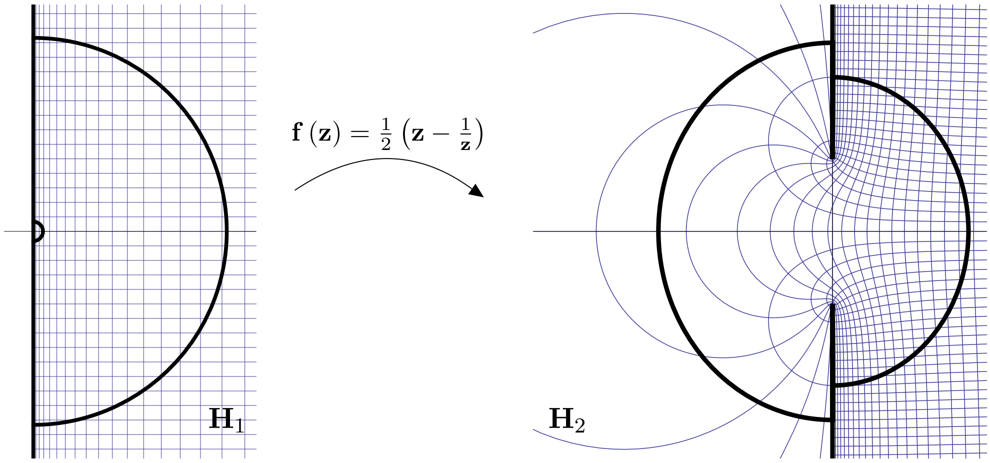


Figure B.2.: The conformal map $\mathbf{f}$. The half circles with radii 0.2 and 4 and their images are indicated by thick lines.

For $t > 0$, $\mathbf{f}(\mathbf{i}t) \in \mathbf{i}[1, \infty)$, because $t \mapsto t + \frac{1}{t}$ has its minimum at $t = 1$ for $t > 0$. For $t < 0$, $\mathbf{f}(\mathbf{i}t) \in \mathbf{i}(-\infty, -1]$. What is the image of the half circle of radius $r > 0$ parametrised by $re^{\mathbf{i}t}$ with $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$?

$$\mathbf{f}\left(re^{\mathbf{i}t}\right) = \frac{1}{2}\left(re^{\mathbf{i}t} - \frac{1}{r}e^{-\mathbf{i}t}\right) = \frac{1}{2}\left(r - \frac{1}{r}\right)\cos t + \mathbf{i}\frac{1}{2}\left(r + \frac{1}{r}\right)\sin t$$

This curve is a half ellipse with the semiaxes $\frac{1}{2}\left(r - \frac{1}{r}\right)$ and $\frac{1}{2}\left(r + \frac{1}{r}\right)$. For large $r > 1$, it is nearly a half circle of radius $\frac{1}{2}r$ with positive real part. For $r = 1$, it is the straight line between $-\mathbf{i}$ and $\mathbf{i}$, for small $0 < r < 1$, it is again close to a half circle, this time of radius $\frac{1}{2r}$ and with negative real part.

We define the function $\mathbf{u}_4 : \mathbf{H}_1 \to \mathbb{C}$ by $\mathbf{u}_4(\mathbf{z}) := \frac{1}{2}\left(\mathbf{z} + \frac{1}{\mathbf{z}}\right)$. For $x, y \in \mathbb{R}$ with $x \geq 0$,

$$\operatorname{Re}\left(\mathbf{u}_4(x + \mathbf{i}y)\right) = \operatorname{Re}\left(\frac{1}{2}\left(x + \mathbf{i}y + \frac{x - \mathbf{i}y}{x^2 + y^2}\right)\right) = \frac{1}{2}\left(x + \frac{x}{x^2 + y^2}\right) \geq \frac{1}{2}x \geq 0. \quad \text{(B.2)}$$

This implies the following. If $v \in \partial W_{R_1,R_2}$ with $v_1 = 0$, the real part of $\mathbf{u}_4 \circ \mathbf{f}^{inv}(v)$ equals zero and hence satisfies the boundary condition in (B.1). For $v_1 < 0$, the real part of $\mathbf{u}_4 \circ \mathbf{f}^{inv}$ is nonnegative, as $\mathbf{f}^{inv}$ maps to $\mathbf{H}_1$, so we have '$\geq$' instead of '$=$' in (B.1).[2] What happens to the third boundary condition? The preimage of the half circle of radius R_2 and positive real part under f lies between the half circles of radius $R_2\left(1 + \sqrt{1 - \frac{1}{R_2^2}}\right)$ and $R_2\left(1 + \sqrt{1 + \frac{1}{R_2^2}}\right)$, with both radii approximating $2R_2$ if R_2 gets large. This gives a hint why, after maybe replacing $\mathbf{u}_4$ by $C_1\mathbf{u}_4 + C_2$ with $C_1 > 1$ and $C_2 > 0$ and calling this $\mathbf{u}_4$ again, $u^{sup}(v_1, v_2) := \operatorname{Re}\left(\mathbf{u}_4\left(\mathbf{f}^{inv}(v_1 + \mathbf{i}v_2)\right)\right)$ also satisfies the third boundary condition for $R_2 > R$ with some large $R > 0$. See Figure B.4. □

[2] Actually, we will have $u_{sup}(v_1, v_2) \approx -v_1$ for $v_1 < 0$.

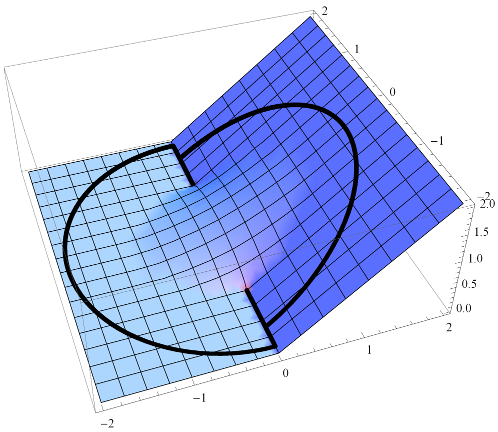


Figure B.3.: The subsolution u_{sub}. The black line marks boundary values attained for one possible domain W_{R_1,R_2}.

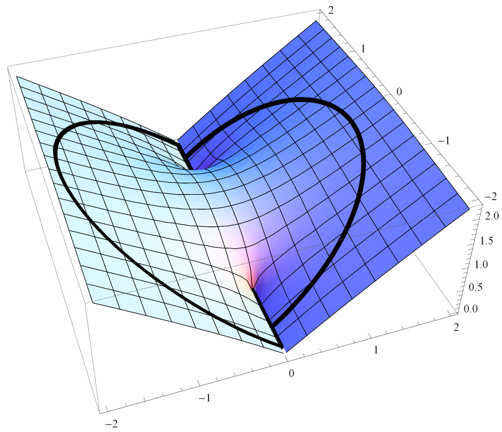


Figure B.4.: The supersolution u_{sup}. The black line marks boundary values attained for one possible domain W_{R_1,R_2}.

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Köln, den 13.10.2011