The front of the randomized Fisher-KPP equation and the parabolic Anderson model

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Kurzzusammenfassung

In dieser Arbeit untersuchen wir Phänomene der randomisierten *Fisher-KPP Gleichung*

$$\begin{aligned} w_t(t,x) &= \frac{1}{2} w_{xx}(t,x) + \xi(x,\omega) \cdot F\big(w(t,x)\big), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ w(0,x) &= w_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{F-KPP}$$

sowie ihrer Linearisierung, dem *parabolischen Anderson Modell*,

$$\begin{aligned} u_t(t,x) &= \frac{1}{2} u_{xx}(t,x) + \xi(x,\omega) \cdot u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{PAM}$$

Hierbei ist der Koeffizient $\xi = (\xi(x,\omega))_{x \in \mathbb{R}}$, $\omega \in \Omega$, ein stochastischer Prozess auf einem Wahrscheinlichkeitsraum $(\Omega, \mathcal{F}, \mathbb{P})$, welcher als Modellierung eines nicht einsehbaren und daher als zufällig angenommenen Mediums fungiert. ξ ist außerdem beschränkt, stationär und mischend und besitzt hinreichend reguläre Pfade. Der nichtlineare Term F wird in der Literatur üblicherweise als „Fisher-KPP Nichtlinearität“ geführt.

Im ersten Kapitel nehmen wir eine Einführung in das Thema durch die Diskussion klassischer und aktueller Resultate vor und geben eine Übersicht der Arbeit. Im zweiten Kapitel leiten wir das nötige mathematische Rüstzeug her. Im dritten Kapitel untersuchen wir dann zunächst die Lösung u zu (PAM), welche aufgrund der Lineariät von (PAM) eine explizite sogenannte *Feynman-Kac Darstellung* zulässt. Für diese (zufällige) Lösung wird ein Invarianzprinzip hergeleitet. Dieses Ergebnis wird dann auf die Front der Lösung $\overline{m}(t) = \sup\{x \in \mathbb{R} : u(t,x) \geq M\}$, $M > 0$, übertragen, indem wir ausnutzen, dass die Lösung u stabil bezüglich Störungen in Zeit und Raum ist. Als Hauptresultat zeigen wir dann, dass die Fronten der Lösung zu (F-KPP) und der Lösung zu (PAM) für große Zeiten t höchstens $C \ln t$ Raumeinheiten voneinander entfernt sind, wobei $C > 0$ eine deterministische Konstante ist. Dies wird mithilfe einer Darstellung beider Lösungen durch ein Funktional eines Verzweigungsprozesses erreicht. Als Anwendung des Hauptresultates erhalten wir dann ein Invarianzprinzip für die Lösung zu (F-KPP). Im vierten Kapitel zeigen wir, dass sich die Umgebung der Front, in der die Lösung von (F-KPP) den Übergang der Werte 0 und 1 aufweist, die sogenannte *Transition front* der Lösung, unbeschränkt ausdehnen kann. Als Anwendung unserer Beweismethoden zeigen wir weiter, dass die Lösung zu (F-KPP) – selbst für große Zeiten t – nicht monoton im Raum sein muss. Dies steht im Gegensatz zum klassischen Resultat der Lösung zu (F-KPP) mit konstantem Potential $\xi \equiv \text{const}$, nach dem für großes t in der Umgebung der Front die Lösung einen scharfen Übergang der Werte 0 und 1 aufweist und die Lösung monoton ist. Für die Lösung zu (PAM) zeigen wir, dass die Transition front beschränkt bleibt. Anschließend werden noch die Modellannahmen diskutiert. Die Arbeit endet mit einem Ausblick und Fragestellungen für zukünftige Forschung.

Abstract

In this work we investigate phenomena of the randomized *Fisher-KPP equation*

$$
\begin{aligned}
&w_t(t,x) = \frac{1}{2} w_{xx}(t,x) + \xi(x,\omega)\cdot F\big(w(t,x)\big), \quad (t,x)\in(0,\infty)\times\mathbb{R},\\
&w(0,x) = w_0(x), \quad x\in\mathbb{R},
\end{aligned}
\tag{F-KPP}
$$

and its linearization, the *parabolic Anderson model*

$$
\begin{aligned}
&u_t(t,x) = \frac{1}{2} u_{xx}(t,x) + \xi(x,\omega)\cdot u(t,x), \quad (t,x)\in(0,\infty)\times\mathbb{R},\\
&u(0,x) = u_0(x), \quad x\in\mathbb{R}.
\end{aligned}
\tag{PAM}
$$

The coefficient $\xi = (\xi(x,\omega))_{x\in\mathbb{R}}$, $\omega\in\Omega$, is a stochastic process on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, which models an uncertain and therefore as random assumed medium. Furthermore, ξ is bounded, stationary and mixing and has sufficiently regular paths. The nonlinear term F is usually listed in the literature as "Fisher-KPP nonlinearity".

The first chapter serves as an introduction to the topic through the discussion of classic and current results and gives an overview of the thesis. In the second chapter we provide the necessary mathematical tools. In the third chapter we first examine the solution u to (PAM), which allows an explicit so-called *Feyman-Kac representation* due to the linearity of (PAM). For this (random) solution we derive an invariance principle. This result is then transferred to the front of the solution $\overline{m}(t) = \sup\{x\in\mathbb{R} : u(t,x)\geq M\}$, $M>0$, by exploiting the fact that the solution u is stable with respect to perturbations in time and space. As the main result we then show that for long times t the fronts of the solution to (F-KPP) and the solution to (PAM) are at most $C\ln t$ spatial units apart, where $C>0$ is a deterministic constant. This is achieved by using a representation of both solutions by a functional of a branching process. As an application of the main result, we then get an invariance principle for the solution to (F-KPP). In the fourth chapter we show that the so-called *transition front* of the solution to (F-KPP), i.e. the area around the front in which the solution has the transition of the values 0 and 1, can have unbounded expansion. As an application of our proof methods we further show that the solution to (F-KPP) – even for large times t – does not have to be monotone in space. This is in contrast to the classic result of the solution to (F-KPP) with constant potential $\xi\equiv$ const, according to which for large times t the solution has a sharp, monotone transition of the values 0 and 1 and the solution is monotone in space. However, for the solution to (PAM) we show that the transition front remains bounded. Subsequently, the model assumptions are discussed. The work ends with an outlook and questions for future research.

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"Probability is the most important concept in modern science,
especially as nobody has the slightest notion of what it means."

– Bertrand Russell

In Lecture (1929), as quoted in E.T. Bell (ed.), Development of Mathematics (1940).

Contents

Chapter One

Introduction

1.1 A classic model in population dynamics

The actual starting point of this thesis – the randomized Fisher-KPP (F-KPP) equation – is motivated by a classic question from biological population genetics. We consider a population of diploid individuals (multicellular organisms with two sets of chromosomes) living in a linear habitat, such as a shore line or a river bank. This population occupies its habitat with uniform density, see [23, p. 69]. It is assumed that for a given gene in the population an advantageous gene mutation occurs. One expects this mutant gene to spread within the population at the expense of the gene variant (allele) which previously occupied the same locus. This requires the production of offspring by two randomly meeting individuals, of whom at least one carries the mutated gene. This is possible as the population is assumed to be non-static, meaning that every individual moves with the same dynamics, independently of whether it carries a mutant gene or not. The process of advance of the mutant gene is first completed in the neighborhood of the occurrence of the mutation and later, after the mutant gene has diffused into the surrounding population, in the adjacent portions of its range.

One is now interested in the distribution of the mutated gene among the whole population, more precisely, how the proportion $w(t,x)$ of the population carrying the mutant gene evolves in time t and space x. As time goes by, this proportion changes from the *unstable state* $w = 0$ (in the absence of any individual carrying a mutant gene) to *stable state* $w = 1$ (where the mutant gene is present in the entire surrounding population). As described above, there are two major influences on the change of w in time and space:

- the growth rate caused by reproduction of the individuals and thus propagation of the mutant gene and
- the flux or diffusion of the individuals.

To be precise, the proportion of the mutant gene at time t in some domain $[x, x + \triangle x]$ can be described as $w(t,x)\triangle x$ and its rate of change is equal to

$$\frac{\partial}{\partial t}\big(w(t,x)\triangle x\big) = \begin{cases} +\text{ growth rate in } [x, x+\triangle x], \\ +\text{ entry rate at } x, \\ -\text{ departure rate at } x+\triangle x, \end{cases}$$

see [45, (15.3)], which can be written as

$$\frac{\partial}{\partial t} w(t,x) \triangle x = f(t,x,w(t,x)) \triangle x + J(t,x) - J(t,x+\triangle x).$$

Here, f denotes a "suitable" growth rate (see below) and J is the left-to-right flux (or diffusion) of the mutant-gene population. Dividing by $\triangle x$ and passing to the limit $\triangle x \to 0$, we arrive at

$$\frac{\partial}{\partial t} w(t,x) = f(t,x,w(t,x)) - \frac{\partial}{\partial x} J(t,x),$$

which is usually called the *conservation law*. A common assumption is (see [23]) that the flux obeys Fick's law, i.e. it is proportional to the negative gradient of the proportion of the population carrying the mutant gene,

$$J(t,x) = -D \frac{\partial}{\partial x} w(t,x)$$

for some constant $D > 0$. This leads to the classical reaction-diffusion equation

$$\frac{\partial}{\partial t} w(t,x) = D \frac{\partial^2}{\partial x^2} w(t,x) + f(t,x,w(t,x)).$$

The question remains which choices for the growth rate f are "suitable". In population dynamics, as in the model for gene spreading above, a usual assumption is that the rate function f depends on the solution w and, furthermore, follows a logistic law. This results in *Fisher's equation*

$$\frac{\partial}{\partial t} w(t,x) = D \frac{\partial^2}{\partial x^2} w(t,x) + \kappa w(t,x)\big(1 - w(t,x)\big), \tag{1.1.1}$$

where $\kappa > 0$ denotes some constant, representing the intensity of selection in favor of the mutant gene. The logistic term $w(1-w)$ is the simplest form of a density-dependent regulation, which is usually postulated whenever there is a certain growth restriction in terms of capacity. In the example of a spreading gene, growth is exponential in the beginning, but the gradual mutation of the genotypes of the surrounding population finally leads to a "saturation" and the growth rate tends to zero.

1.1.1 Related PDEs

Although it seems reasonable to make use of the quadratic function $w(1-w)$, there are other choices possible. In fact, the equation (1.1.1) is only appropriate for a sexually reproducing species, if there is a certain constellation of phenotypes ([15, p. 1]). If this condition is not fulfilled, the assumptions on the dynamics result in a (cubic) growth term of the form $w^2(1-w)$, see [68, 14]. However, equation (1.1.1) remains the model of choice in many situations. For example, in addition to the population dynamics presented above (see also [63, (4)]), another field of application is biological cellular tissue growth, in which w represents the cell population density divided by the steady-state tissue density (see e.g. [67, (20)]). To give an overview of equations with density-dependent growth term, we provide a short list. This list (see [29, p. 1 ff.]) is by no means complete and we put emphasis on equations which are relevant with respect to results in this thesis. For simplicity, we set $D = \frac{1}{2}$ and $\kappa = 1$. This, of course, can be achieved using change of variables $t \to t/\kappa$ and $x \to \sqrt{2D/\kappa} \cdot x$. The choice of the diffusion constant $\frac{1}{2}$ simplifies some formulas in the thesis.

- *Fisher's equation*, [23],
$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} + w(1-w). \tag{1.1.2}$$
Its motivation is given above. It is the archetypal model to describe the spread of an advantageous allele in a population with uniform density living in a one-dimensional habitat.

- The *Newell-Whitehead*, *amplitude-* or *Ginsburg-Landau equation*, [53, 57],
$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} + w(1-w^q),$$
for some $q \in \mathbb{N}$. This equation describes the nonlinear distribution of temperature in an infinitely thin and long rod or as the flow velocity of a fluid in an infinitely long pipe with small diameter. It has wide applicability in mechanical and chemical engineering and bio-engineering, see [57, (1)]. The choice $q = 1$ yields Fisher's equation.

- The *Zeldovich equation*, [18, p. 3],
$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} + w^2(1-w).$$
This question naturally arises in combustion theory, where w represents temperature, while the shape of the growth term corresponds to the generation of heat by combustion.

- The *Kolmogorov-Petrowski-Piskunov equation* (or *KPP equation*), [44],
$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} + F(w), \tag{1.1.3}$$
where F is differentiable on $[0,1]$, $(0,1] \ni w \mapsto F(w)/w$ can be continuously extended to $w = 0$ and we have
$$\begin{aligned} F(0) = F(1) = 0, \quad F(w) > 0 \ \forall w \in (0,1), \quad F(w) < 0 \ \forall w > 1, \\ F'(0) = \lim_{w \downarrow 0} F(w)/w = \sup_{w \in (0,1)} F(w)/w = 1 \geq \sup_{w \in (0,1)} F'(w). \end{aligned} \tag{1.1.4}$$
This equation covers the Newell-Whitehead (and thus Fisher's) equation, but not Zeldovich's equation.

- The *Nagumo*, or *bistable equation*, [51],
$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} + w(1-w)(w-\alpha), \tag{1.1.5}$$
for some $\alpha \in (0,1)$. This equation is used to model the transmission of electrical pulses in a nerve axon. The term "bistable" is explained by the fact that the nonlinearity has three roots. The root α is "repelling" and the two others are "attracting". This terminology is described in more detail in the next subsection.

- The *ignition-type equation* in combustion theory, [5, 6],

$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} + F^\theta(w)$$

with

$$F^\theta(w) = \begin{cases} 0, & w \in [0,\theta], \\ g(u), & w \in (\theta, 1]. \end{cases} \tag{1.1.6}$$

Here, $\theta \in (0,1)$ is a threshold or "ignition temperature" parameter and g is a differentiable function fulfilling $g(w) > 0$ for all $w \in (\theta, 1]$, $g'(1) < 0$ and $g(1) = 0$. Note that the influence of the growth function vanishes for small values of w. This equation describes deflagration, e.g. for compressible reacting gases with one reactant in a single step chemical reaction.

We further mention another very important equation, that is used throughout Chapter 3. Especially due to the upper unboundedness of its (linear) growth function, we don't put it in line with the other equations enumerated above, but stress its importance for our later investigations.

- The linearized KPP-equation, by ecologists usually referred to as *KISS (Kierstead, Slobodkin, Skellam) model*, [45, p. 272],

$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} + w. \tag{1.1.7}$$

This is a *linearization* of equation (1.1.2) (for small values of w) and has its own interpretation. It has been used to describe the propagation of a population into an unbounded habitat, e.g. red tide outbreaks, see [43].

For more references to the corresponding applications of above equations, we refer to [29, Section 1].

1.1.2 The solution

Let us now take a look at the KPP equation (1.1.3), of which Fisher's equation (1.1.2) is a special case. How can we actually solve it? Let us make a few observations. We first notice that $w \equiv 0$ and $w \equiv 1$ are *stationary solutions* to (1.1.3). They are usually called *unstable* ($w \equiv 0$) and *stable* ($w \equiv 1$) stationary solutions. In the neighborhood of these solutions, the growth term F from (1.1.3) fulfills $F(w) \approx 0$. Due to the property of the term $F(w)$ being positive for $0 < w < 1$, a solution w such that $0 < w < 1$ is *repelled* away from zero and *attracted* to one. We, thus, expect a solution to evolve (as time goes on, locally in space) from unstable to stable state. Consequently, regions where the solution is near to one tend to grow and occupy adjacent sites where the solution is small.

Pioneering work concerning this topic has been published by the Briton Ronald A. Fisher [23] in 1937, already mentioned above, and also by three Russians, Andrey N. Kolmogorov, Ivan G. Petrowski and Nikolaj S. Piskunov [44], publishing almost at the same time. Let us briefly outline the analysis given in [44]. The ansatz is to write $w(t,x) = p(x - vt)$ for some real function p and some *velocity* v and exploit the fact that if w is a solution to (1.1.3), p solves the ordinary differential equation (ODE)

$$\frac{1}{2}p'' = -vp' - F(p), \tag{1.1.8}$$

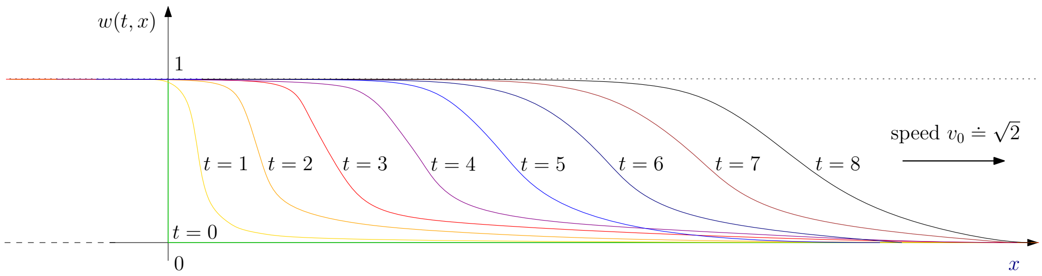

Figure 1.1: Illustration of the "traveling wave" phenomenon. Here we have $w(x, 0) = 1$ for all $x < 0$ and $w(0, x) = 0$ for all $x \geq 0$.

where p' denotes the first and p'' the second derivative of p. Using phase-plane analysis, i.e. defining $q := p'$ the system of ODEs

$$\begin{aligned} p' &= q, \\ q' &= -2F(p) - 2vq, \end{aligned}$$

gives rise to the critical points $(p, q) = (0, 0)$, which is a saddle point, and $(1, 0)$, which is a stable node for $v \geq \sqrt{2}$ and a spiral for $v \in (0, \sqrt{2})$. It is shown in [44, §2], using the above ansatz and assuming the initial condition to be of *Heaviside* type, i.e. $w(0, x) = \mathbb{1}_{(-\infty,0]}(x)$, that a solution to (1.1.8) for $v = \sqrt{2}$, fulfilling $p(-\infty) = 1$ and $p(+\infty) = 0$ exists. Furthermore, this solution is shown to be unique up to spatial translation, i.e. if p is a solution then for every $y \in \mathbb{R}$ the function $\widetilde{p}(\cdot) = p(\cdot + y)$ is also a solution. In turns out, see [44, §3], that

- there exists a function m such that

$$w(t, m(t) + x) \underset{t \to \infty}{\longrightarrow} p(x) \quad \text{uniformly in } x \in \mathbb{R}, \tag{1.1.9}$$

  where p is a solution to (1.1.8) fulfilling $p(-\infty) = 1$ and $p(+\infty) = 0$, see [44, Theorem 14],

- for all $t > 0$ and $x \in \mathbb{R}$ we have $\frac{\partial}{\partial x} w(t, x) \leq p'(x)$, i.e. the shape of the solution "flattens out", see [44, Theorem 12],

- the function m fulfills

$$\frac{m(t)}{t} \underset{t \to \infty}{\longrightarrow} v_0 = \sqrt{2}, \tag{1.1.10}$$

  see [44, Theorem 17], but

- for any $x > 0$ we have $w(t, x + \sqrt{2}t) \underset{t \to \infty}{\longrightarrow} 0$, i.e. we only have a first asymptotic for m and the "correct" speed is smaller than $v_0 = \sqrt{2}$, see [44, Theorem 10].

Thus, as $t \to \infty$, the part of the solution $w(t, x)$ (as a function of the spatial variable x) corresponding to the *transition front*, where the solution drops from one to zero, moves to the right with the critical speed $v_0 = \sqrt{2}$. The shape of the transition front approaches the graph of the solution to (1.1.8). This behavior of the solution resembles a so-called *traveling wave*. An illustration of this phenomenon is given in Figure 1.1.

1.2 Bramson's result for the classical F-KPP equation

Since the pioneering works [23, 44] the investigation of equations from Section 1.1.1 has developed considerably. In [40] and [21], more general nonlinearities f than of KPP-type (1.1.4) are considered and it shown, in the case of Heaviside initial condition, that the solution converges to a traveling wave. In [2, 3], these results are extended to more general initial conditions and higher dimensions.

However, especially the question of the "correct" speed of the *wave front* (or just *front*) m apart from (1.1.10) for the KPP equation (1.1.3) had been unanswered for a long time. In 1978, Maury Bramson published his famous seminal paper [13] in which he improves the results from Kolmogorov et al. [44] considerably. He shows for the same initial conditions as used in [44], that for every $\varepsilon \in (0,1)$, the function $m^\varepsilon : [0,\infty) \to \mathbb{R}$,

$$m^\varepsilon(t) = \sup\{x \in \mathbb{R} : w(t,x) = \varepsilon\} \tag{1.2.1}$$

fulfills the asymptotics

$$m^\varepsilon(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t + \mathcal{O}_\varepsilon(1) \quad \text{as } t \to \infty, \tag{1.2.2}$$

and $w(t, x + m^\varepsilon(t)) \underset{t\to\infty}{\longrightarrow} p(x)$ uniformly in $x \in \mathbb{R}$, where p is a solution to (1.1.8) fulfilling $p(-\infty) = 1$ and $p(+\infty) = 0$ (as above). Bramson takes advantage of two things. On the one hand, the solution to (1.1.3) can be represented as a functional of a *branching Brownian motion* (BBM). Let us give a short probabilistic description of its dynamics:

- We start with one initial particle at the origin.
- This particle moves as a standard Brownian motion in $\mathbb{R}$.
- After a mean 1-exponential time, which is independent of the motion of the particle, the particle (ancestor) dies and gives birth to a random number of new particles, its *offspring*.
- Each of the offspring particles is an independent copy of its ancestor, i.e. starts where the ancestor has died, moves as a standard Brownian motion and dies with exponential rate 1, independently of all other particles.

Then, if we assume Heaviside initial condition $\mathbb{1}_{(-\infty,0]}$ and the nonlinearity F to be generated by the offspring distribution of the BBM, the solution to (1.1.3) is the tail of the distribution function of the right-most particle of the BBM. This is known at least since the papers by Ikeda, Nagasawa and Watanabe [36, 37, 38], and Bramson cites a result from McKean [52, Section 2]. The above mentioned representation in terms of a BBM is commonly known as the *McKean representation* of the solution to (1.1.3).

On the other hand, Bramson uses that the solution to the linearized equation (1.1.7) at time t and site x equals the expected number of particles of a BBM which are to the right of x at time t. This allows one to analyze the solution to the KPP equation (and the solution to the linearized equation) by probabilistic arguments. For the linearized equation, which is much easier to handle, we also get precise asymptotics for the corresponding front. This can be achieved without too much effort, using standard Gaussian estimates (this is explained in Chapter 2). That is, for $\varepsilon > 0$ the front $\overline{m}^\varepsilon : [0,\infty) \to \mathbb{R}$

$$\overline{m}^\varepsilon(t) = \sup\{x \in \mathbb{R} : u(t,x) = \varepsilon\}$$

of the solution u to (1.1.7) fulfills

$$\overline{m}^\varepsilon(t) = \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t + \mathcal{O}_\varepsilon(1) \quad \text{as } t \to \infty.$$

This leads to the interesting fact that the difference of the fronts m^ε and $\overline{m}^\varepsilon$ of the solutions to (1.1.3) and (1.1.7) is of logarithmic order, i.e.

$$0 \leq \overline{m}^\varepsilon(t) - m^\varepsilon(t) \leq \frac{1}{\sqrt{2}} \ln t + \mathcal{O}_\varepsilon(1) \quad \text{as } t \to \infty. \tag{1.2.3}$$

We call this *Log-distance* of the fronts. The main theorem of the thesis deals with a similar result as in (1.2.3).

Later on, in [12] Bramson shows his results for a large class of initial conditions. Heaviside can be replaced by initial conditions w_0 not vanishing as $x \to -\infty$ and having a sufficiently fast decaying tail as $x \to \infty$. It is shown, see [12, Theorem A], that the first order speed $v \geq \sqrt{2}$ is determined by the (tail-)behavior of the function $w_0(x)$ as $x \to \infty$. A lighter tail results in a larger value of the wave speed v of the traveling wave solution. The initial condition $\mathbb{1}_{(-\infty,0]}$ results in the *critical* case $v_0 = \sqrt{2}$, whereas $v > \sqrt{2}$ is usually referred to as *supercritical* speed.

1.3 Results for periodic and random potential

It has already been observed in Fisher's paper [23] that a more realistic model would be obtained by considering spatial heterogeneous reproduction rates. The propagation of the mutant gene is then influenced e.g. by local weather or temperature influences. Having pointed out several results for the homogeneous equation, where the reaction term depends only on the solution itself, the natural question arises whether similar results as in the last two sections hold true for more general reaction terms as well. However, it makes sense if we limit ourselves to certain nonlinearities, which are important in the context of this thesis. We assume that the reaction term can be represented as the product of a time-independent and space-dependent *potential* ξ and a nonlinearity F:

$$\begin{aligned} &w_t(t,x) = \frac{1}{2} w_{xx}(t,x) + \xi(x) \cdot F(w(t,x)), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ &w(0,x) = w_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1.3.1}$$

We focus on results for *periodic potential* and then turn attention to *random potential.*

1.3.1 Periodic potential

Let the function ξ be a periodic function on $\mathbb{R}$. In this case, a recent breakthrough in generalizing Bramson's results has been achieved in [32]. The authors assume a bounded, strictly positive and 1-periodic potential ξ, i.e. $\xi(x) = \xi(x+1)$ for all $x \in \mathbb{R}$, KPP-type nonlinearity F (see (1.1.4)) and initial conditions w_0 fulfilling $w_0(x) \in [0,1]$ for all $x \in \mathbb{R}$ and having compact support. Under these assumptions it is shown that there exist v_0, λ_0 such that for any $\varepsilon \in (0,1)$ and some $s(\varepsilon) \in (0,\infty)$ the solution w to (1.3.1) fulfills

$$\begin{aligned} &w(t,x) > 1-\varepsilon \ \text{ for all } \ t > s(\varepsilon) \ \text{ and all } \ x \in [0, m(t) - \mathcal{O}_\varepsilon(1)] \quad \text{and} \\ &w(t,x) < \varepsilon \ \text{ for all } \ t > s(\varepsilon) \ \text{ and all } \ x \in [m(t) + \mathcal{O}_\varepsilon(1), \infty). \end{aligned} \tag{1.3.2}$$

The function m is given by

$$m(t) = v_0 t - \frac{3}{2\lambda_0} \ln t, \tag{1.3.3}$$

see [32, Theorem 1.1]. This, on the one hand, gives an asymptotic of the front of the solution w up to logarithmic order, similar to Bramson's result (1.2.2). On the other hand, it shows that the area where the solution drops from the value one to zero, the so-called *transition front*, is uniformly bounded by a constant. To get (1.3.3), the authors in [32] use that the solution to (1.3.1) behaves similarly to the front of the solution to the linearized equation

$$\begin{aligned} u_t(t,x) &= \frac{1}{2} u_{xx}(t,x) + \xi(x) \cdot u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R} \end{aligned} \tag{1.3.4}$$

with a Dirichlet boundary condition $u(t, v_0 t) = 0$. For instance, the parameters v_0 and λ_0 arise in variational problems including the eigenvalue of a 1-periodic eigenvalue problem in terms of a linear second order ODE operator, see [32, (4)]. Then, to get the upper bound in (1.3.2), one uses that the solution to the nonlinear problem is upper bounded by the solution to the linearized PDE (1.3.4). The (more complicated) proof of the lower bound requires a detailed treatment of a "subsolution" of (1.3.1), which is a rescaled version of the solution to (1.3.4), see [32, (11)]. Instead of Bramson, who uses a probabilistic argument to show the results in [13], the techniques in [32] are purely analytic.

In this context it should be mentioned that a probabilistic interpretation of the solution to (1.3.1) for constant ξ, as we explained it in Section 1.2, does *not* hold for non-constant ξ. In this case, the solution to (1.3.1) is *not* given by the distribution of the maximum of a BBM (with space-dependent branching rates). However, there is another interpretation in terms of a BBM, which is explained in Section 2.2.3 and we take advantage of this Chapter 3 and 4.

1.3.2 Random potential

Another option is to take the potential $\xi = (\xi(x,\omega))_{x\in\mathbb{R}}$, $\omega \in \Omega$, in (1.3.1) as random. Here, ξ is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Of course, one may need the paths of ξ to be "nice" and ξ to fulfill a certain stochastic regularity structure like stationarity or ergodicity with respect to spatial shifts of the paths. The object of interest is thus the solution to the *randomized Fisher-KPP (F-KPP) equation*

$$\begin{aligned} w_t(t,x) &= \frac{1}{2} w_{xx}(t,x) + \xi(x,\omega) \cdot F\big(w(t,x)\big), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ w(0,x) &= w_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{F-KPP}$$

As in the periodic case, it turns out useful to consider its linearization, the so-called *parabolic Anderson model* (PAM),

$$\begin{aligned} u_t(t,x) &= \frac{1}{2} u_{xx}(t,x) + \xi(x,\omega) \cdot u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{PAM}$$

Similar topics as in the homogeneous setting have been treated for random potential as well. For example, Shen [65] examines a "random traveling wave" phenomenon under suitable regularity assumptions on ξ and F. This thesis, however, is aimed towards the more specific question on the behaviour of the solution in the vicinity of the wave *front*. To be precise, we focus on the behavior of the corresponding (left- and right-) fronts

$$m^{\varepsilon}(t) := \sup\{x \in \mathbb{R} : w(t,x) \geq \varepsilon\} \quad \text{and}$$
$$m^{\varepsilon,-}(t) := \inf\{x \geq 0 : w(t,x) \leq \varepsilon\}, \quad \varepsilon \in (0,1),$$

for the solution w to (F-KPP), consider the (left- and right-) fronts

$$\overline{m}^{M}(t) := \sup\{x \in \mathbb{R} : u(t,x) \geq M\} \quad \text{and}$$
$$\overline{m}^{M,-}(t) := \inf\{x \geq 0 : u(t,x) \leq M\}, \quad M > 0,$$

for the solution u to (PAM) and examine the connection between these quantities.

Let us give a short overview of previous results. To begin with, under fairly general assumptions and using large deviation principles, Freĭdlin and Gärtner [28] have derived the existence and characterization of the propagation speed, i.e. the linear order $\lim_{t\to\infty} m^{\varepsilon}(t)/t$ of the front. That is, there exists $v_0 > 0$, such that $\mathbb{P}$-a.s., the solution w to (F-KPP) with a KPP-type nonlinearity F converges to 0 (resp. 1), uniformly for all $x \geq vt$ with $v > v_0$ (resp. for all $x \leq vt$ with $v < v_0$), as t tends to infinity. In a similar way, this result is also valid for the solution to (PAM) with the same speed v_0, i.e. $\overline{m}^{M}(t)/t \xrightarrow[t\to\infty]{} v_0$ $\mathbb{P}$-a.s. This shows that the speeds of both fronts m and $\overline{m}$ coincide in the first order. Consequently, as in the homogeneous case, the question of second order corrections arises naturally.

Unfortunately, sharp asymptotics up to logarithmic order as in the periodic case (1.3.3) are not generally known for nontrivial random potential. However, as the front is random, it seems reasonable to ask for a central limit theorem of m (resp. $\overline{m}$) around v_0. This has been partially investigated for supercritical initial conditions by Nolen [54]. Let us take a closer look at these assumptions. He examines (F-KPP) by assuming F to be Fisher-type, i.e. $F(w) = w(1-w)$, and ξ to be bounded, positive, stationary with respect to spatial shifts and to fulfill the following mixing condition: For all $j \leq k$ and $X \in L^2(\Omega, \mathcal{F}_j, \mathbb{P})$ and $Y \in L^2(\Omega, \mathcal{F}^k, \mathbb{P})$, where $\mathcal{F}_j := \sigma\{\xi(x,\cdot) : x \leq j\}$ and $\mathcal{F}^k := \sigma\{\xi(x,\cdot) : x \geq k\}$, Nolen assumes

$$\big|\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\big| \leq \big(\rho(k-j)\big)^{1/2}\big(\mathbb{E}[X^2]\mathbb{E}[Y^2]\big)^{1/2}, \tag{1.3.5}$$

for some function ρ fulfilling $\sum_k \rho(k)^{1/2} < \infty$. This is usually called *ρ-mixing*. Furthermore, Nolen requires initial conditions satisfying

$$\mathcal{C}_1(\omega) g^{\omega,\gamma}(x) \leq w_0(x,\omega) \leq \mathcal{C}_2(\xi) g^{\omega,\gamma}(x) \quad \forall\, x > 0, \tag{1.3.6}$$

where $\mathcal{C}_1(\omega), \mathcal{C}_2(\omega) > 0$ and $g = g^{\omega,\gamma}$ is a solution to the ODE

$$g''(x) + (\xi(x) - \gamma)g(x) = 0, \ x > 0,$$

for $\gamma > \overline{\gamma}$ and some suitable $\overline{\gamma} > 0$. The rate $\mu(\gamma) := \lim_{x\to\infty} -\frac{1}{x} \ln g^{\xi,\gamma}(x)$ is shown to exist $\mathbb{P}$-a.s and to be increasing in γ. For technical reasons, Nolen additionally requires $\gamma < \gamma^*$, i.e. the initial condition $w_0(x)$ must not decay too fast as x tends to infinity. As explained at the end of Section 1.2, initial conditions fulfilling (1.3.6) can be categorized to be "supercritical". As his main result, Nolen obtains a central limit theorem for m in this case, see [54, Theorem 1.4].

These results are obtained also for other types of nonlinearities. For ignition-type (see (1.1.6)) or bistable (see (1.1.5)) nonlinearities, Nolen shows in another paper [55] that the front of the solution to (3.3.14) obeys a central limit theorem and, for non-vanishing variance, an invariance principle, see [55, Corollary 1.2].

Another question is whether the transition front is *uniformly bounded* or "sharp", as has been shown for the periodic case, see (1.3.2). For ignition-type nonlinearities, the answer is yes. More precisely, in [56] the authors assume ξ to be stationary, ergodic with respect to the spatial shift, positive, bounded and Lipschitz continuous. Then, for nonnegative initial conditions having compact support, the authors show that for every $\varepsilon \in (0,1)$ there exists a constant $C_\varepsilon \in (0,\infty)$ such that

$$0 \leq m^{\varepsilon}(t) - m^{1-\varepsilon,-}(t) \leq C_\varepsilon, \tag{1.3.7}$$

see [56, Proposition 2.3]. This yields a so-called *uniformly bounded transition front.*

We have already mentioned that exact asymptotics of the front of the solution to (F-KPP) up to logarithmic order are generally unknown for random potentials. A different approach would then be to compare the KPP-front m to the front $\overline{m}$ of the corresponding linearized equation. This has already been done in (1.2.3) for the homogeneous case: For large times t, m lags logarithmically in t behind $\overline{m}$, i.e. the difference $\overline{m}(t) - m(t)$ is $\frac{1}{\sqrt{2}} \ln t$ up to some constant as $t \to \infty$. The question arises whether we can find a similar result in the case of random potential.

In the context of branching processes, this question can be answered in the affirmative. To explain this, we recall from Section 1.2 that in the homogeneous setting $\xi \equiv 1$, the solution to (F-KPP) can be represented in terms of the distribution function of the right-most particle of a BBM. Furthermore, it can be shown that the solution to the linearized equation (PAM) for $\xi \equiv 1$ can be represented in terms of the expected site of the right-most particle. Now for a discrete-space analogon of the latter process with inhomogeneous branching rates, in the recently published paper by Jiří Černý and Alexander Drewitz [16], it is shown that the median of the right-most particle at time t is at most $C \ln t$ space units behind the expected site.

Although for non-constant ξ, there is another representation of a branching process, which is described in Section 2.2, the interpretation of the solutions to (F-KPP) and (PAM) in terms of the right-most particle fails to hold for non-constant ξ. It is not straight-forward to adapt the results from [16] to the continuous-time setting and deduce a statement about the Log-distance of the fronts of equations (F-KPP) and (PAM). Fortunately, at least the proof technics in [16] turn out to be applicable in our setup.

The next section gives a guideline and a short summary of every chapter of the thesis.

1.4 Organization of the thesis

Recall the *randomized Fisher-KPP equation* (F-KPP equation)

$$\begin{aligned} w_t(t,x) &= \frac{1}{2} w_{xx}(t,x) + \xi(x,\omega) \cdot F\big(w(t,x)\big), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ w(0,x) &= w_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{F-KPP}$$

and its linearization, the *parabolic Anderson model*,

$$\begin{aligned} u_t(t,x) &= \frac{1}{2} u_{xx}(t,x) + \xi(x,\omega) \cdot u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{PAM}$$

where ξ is a stochastic process fulfilling suitable boundedness, stationarity and mixing conditions, and F is a standard F-KPP-nonlinearity, see Section 3.1.1 for precise assumptions. We start with recalling several technical tools in Chapter 2 which are necessary to treat the topics in the subsequent Chapters 3 and 4.

1.4.1 Chapter 2: Technical tools

Although in Sections 1.1 and 1.2 we have mostly spoken about solutions to certain partial differential equations, this thesis mainly belongs to the field of stochastics. This is explained by the fact that we are able to take advantage of crucial connections between the theory of PDEs and the theories of diffusion and branching processes. It turns out that the solution to (F-KPP) and (PAM) can be represented as a functional of a Brownian motion, also known as *Feynman-Kac representation.* For (PAM) the solution is given explicitly and for (F-KPP), with nonlinear function F, implicitly. While the implicit expression is used at some point in this thesis, we frequently take advantage of another, explicit expression for the solution to (F-KPP) in terms of a branching process.

Chapter 2 ends with a summary of the results from Sections 2.1 and 2.2, serving as a preparation of Chapters 3 and 4.

1.4.2 Chapter 3: Log-distance and invariance principles of the F-KPP- and PAM-front

Motivated by the results from Sections 1.2 and 1.3, Chapter 3 deals with the fronts of the respective solutions to (F-KPP) and (PAM). Our assumptions on ξ are stronger than those from Nolen's paper [54], given in (1.3.5), i.e. we assume ψ- instead of ρ-mixing. However, our nonlinearity F fulfills general KPP-conditions and is not of Fisher-type. The main difference to Nolen's setup is that we consider a class of "critical" initial conditions $w_0(x)$, vanishing for $x > 0$ and non-vanishing for $x \to -\infty$. First, we investigate the solution u to (PAM) with the help of a change-of-measure technique obtained from the Feynman-Kac representation of u. This makes it possible to derive an exact large deviation principle for the solution u and also provides an invariance principle for $\ln u$. These methods are then used to control perturbations of the solution u in time and space and to obtain an invariance principle for the front $\overline{m}$ of the solution to (PAM). In the main part of the chapter we show the so-called *Log-distance* of both fronts, i.e. for all t large enough, the front m of the solution to (F-KPP) lags at most $C \ln t$ space units behind the front $\overline{m}$, where $C \in (0, \infty)$ is a deterministic constant. Here, the representation of the solutions to (F-KPP) and (PAM) in terms of a branching process plays a crucial role. As a corollary, we then get an invariance principle of m.

Chapter 3 is taken from the article [19].

1.4.3 Chapter 4: (Un)-bounded transition fronts, (non)-monotonicity and model assumptions

Due to the Log-distance proved in Chapter 3, we get that the difference of the fronts m and $\overline{m}$ is bounded by a logarithmically increasing function. In Section 4.1, however, it becomes apparent that their corresponding transition fronts behave differently. On the one hand, with the methods already established in Chapter 3, one can easily show that for every $\varepsilon, M > 0$, $\varepsilon \leq M$, the transition front of the solution to (PAM) remains bounded, i.e. for t large enough we have

$$0 \leq \overline{m}^{\varepsilon}(t) - \overline{m}^{M,-}(t) \leq C_{\varepsilon,M},$$

where $C_{\varepsilon,M} \in (0,\infty)$ is some constant, not depending on the initial condition. As has been mentioned in (1.3.7) that this behavior is also observed for the KPP-front in the case of bistable and ignition-type nonlinearity F. On the other hand, we show a different behavior of the front for KPP-type nonlinearities. More precisely, for a suitable subsequence of times $(t_n)_{n\in\mathbb{N}}$ tending to infinity, we prove that the transition front of the solution to (F-KPP) tends to infinity, i.e. $m^{\varepsilon}_{t_n} - m^{1-\varepsilon,-}_{t_n} \underset{n\to\infty}{\longrightarrow} \infty$. Here we use once more that the solution to (F-KPP) can be represented in terms of a branching process. The idea of the proof is to apply a coupling argument for two branching processes, one starting in a site surrounded by high values of ξ and the other one starting in a site surrounded by low values of ξ. In Section 4.2, parts of the latter proof can then be used to show that the solution to (F-KPP) does not need to be monotone in space, even for large times. This is in contrast to the case of constant ξ and Heaviside initial condition, where we have seen that the graph of the solution flattens out and the solution is monotone in space.

The remaining part of the chapter, Section 4.3, is dedicated to our model assumptions. In addition to the usual regularity requirements, we frequently demand a purely technical assumption (VEL). We show that (VEL) is not trivial in the sense that there are potentials which satisfy all assumptions except (VEL). However, we then show that all results from Chapters 3 and 4 apply simultaneously for a rich family of potentials.

Sections 4.1 and 4.2 are taken from the article [20] and Section 4.3 contains results from [19] and [20].

1.4.4 Chapter 5: Outlook

In Chapter 5 we discuss several consequences of our results and treat questions that remain unanswered from Chapters 3 and 4.

Chapter Two

Technical tools

This chapter gives a short introduction into several concepts from the theory of partial differential equations (PDEs) and branching processes. Since we will consider PDEs with non-smooth initial conditions, it is desirable to expand the space of classical solutions by introducing the concept of so-called *generalized solutions*. This is also known under the name *Feynman-Kac representation* and connects the field of PDEs to the theory of diffusion processes. The second section we introduce a branching process, the *branching Brownian motion in random environment* (BBMRE), and we show that the (generalized) solution to certain PDEs can be represented by a functional of a BBMRE. In Section 2.3, we apply the results of Sections 2.1 and 2.2 to a certain PDE, which is examined in the main part of the thesis.

Those readers already familiar with the theory of PDEs (especially with the concept of *generalized solutions of parabolic PDEs*) and the theory of branching processes can move directly to Section 2.3.

2.1 Partial differential equations and diffusion processes

In the following we need some notation regarding the theory of PDEs. In addition to the notation of partial derivatives, we also have to consider function spaces, which are important in the context of generalized solutions.

2.1.1 Notation

We start with introducing some function spaces and notation for partial derivatives. For $A \subset \mathbb{R}^n$, $n \in \{1,2\}$, and $B \subset \mathbb{R}$, we write $C(A,B)$ for the space of continuous functions, mapping from A to B. Furthermore, for open $A \subset \mathbb{R}^2$, $B \subset \mathbb{R}$ and $p,q \in \mathbb{N}$, $C^{p,q}(A,B)$ denotes the space of functions $f : A \to B$ such that the partial derivatives $A \ni (t,x) \mapsto \frac{\partial^{k+l}}{\partial t^k \partial x^l} f(t,x)$ exist and are continuous for all $k \in \{1,\dots,p\}$ and $l \in \{1,\dots,q\}$. Then $C^\infty(A,B) := \bigcap_{p,q\in\mathbb{N}} C^{p,q}(A,B)$ is the space of smooth functions. For $A \subset \mathbb{R}^n$, $n \in \{1,2\}$, and $B \subset \mathbb{R}$, we denote by $C_c(A,B)$ the space of continuous functions with compact support, and by $C_b(A,B)$ the space of bounded continuous functions.

Furthermore, to abbreviate notation and because we mostly deal with functions w in two variables, we write $w_t(t,x) := \frac{\partial}{\partial t} w(t,x)$, $w_x(t,x) := \frac{\partial}{\partial x} w(t,x)$ and $w_{xx}(t,x) := \frac{\partial^2}{\partial x^2} w(t,x)$, if these partial derivatives exist. We sometimes omit the arguments of w, but nonetheless write the subscripts t,x to indicate the corresponding partial derivatives w_t, w_x and w_{xx}; e.g. $w_x(s,y) = \frac{\partial}{\partial x} w(t,x)\big|_{(t,x)=(s,y)}$.

2.1.2 The classical reaction-diffusion equation

Let us consider the PDE

$$
\begin{aligned}
w_t(t,x) &= \frac{1}{2} w_{xx}(t,x) + f(t,x,w(t,x)), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\
w(0,x) &= w_0(x), \quad x \in \mathbb{R}.
\end{aligned}
\tag{2.1.1}
$$

Equation (2.1.1) is called (time and space inhomogeneous) *reaction-diffusion equation.* The term f is usually referred to as the *reaction term* or *growth term.* In Section 1.1.1 we discussed examples of different reaction terms and applications of the corresponding PDEs.

In the main part of this thesis we focus on specific choices of the function f. That is, in Chapters 3 and 4 we examine (2.1.1) for the cases of function f being either linear or quasi-linear in w. As will be shown in the remainder of Section 2.1, in these cases it is possible to represent the solution to (2.1.1) in terms of a functional of Brownian paths, establishing a connection to probability theory.

2.1.3 Linear reaction term

General concepts and statements for linear PDEs

This section follows [27, Chapter 6.4]. We assume that the reaction term f in (2.1.1) depends linearly on the solution. Therefore, we consider

$$
\begin{aligned}
w_t(t,x) &= \frac{1}{2} w_{xx}(t,x) + c(t,x) \cdot w(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\
w(0,x) &= w_0(x), \quad x \in \mathbb{R}.
\end{aligned}
\tag{2.1.2}
$$

We want to show that under suitable conditions for the function c, the solutions to (2.1.2) admit a representation in terms of a *fundamental solution*:

Definition 2.1. *Let $T > 0$. A fundamental solution to* (2.1.2) *in* $[0,T] \times \mathbb{R}$ *is a function* $\Gamma(t,x;s,y)$*, defined for all* $(t,x) \in [0,T] \times \mathbb{R}$ *and all* $(s,y) \in [0,T] \times \mathbb{R}$ *such that* $s < t$*, which fulfills the following condition:*

For any function $w_0 \in C_c(\mathbb{R},\mathbb{R})$ *and any* $s \in [0,T)$*, the function* $w : (s,T] \times \mathbb{R} \to \mathbb{R}$*,*

$$
w(t,x) = \int_{\mathbb{R}} \Gamma(t,x;s,y) w_0(y) \, \mathrm{d}y
$$

satisfies

$$
\begin{aligned}
w_t(t,x) &= \frac{1}{2} w_{xx}(t,x) + c(t,x) w(t,x), \quad (t,x) \in (s,T] \times \mathbb{R}, \\
\lim_{t \downarrow s} w(t,x) &= w_0(x), \quad x \in \mathbb{R}.
\end{aligned}
\tag{2.1.3}
$$

To guarantee the existence of such a function, we have to introduce the following condition for the coefficient c in (2.1.2).

Assumption 2.2. For every $T > 0$ the function c fulfills $c \in C_b([0,T] \times \mathbb{R}, \mathbb{R})$ and is Hölder continuous (with exponent α) in x, uniformly with respect to (t,x) in compact subsets of $[0,T] \times \mathbb{R}$, i.e. for every compact subset $I \subset [0,T] \times \mathbb{R}$ there exists a constant $C(I) < \infty$, such that

$$\sup_{\substack{t\in[0,T],\ x,y\in\mathbb{R}:\\ (t,x),(t,y)\in I}} \frac{|c(t,x)-c(t,y)|}{|x-y|^\alpha} \le C(I).$$

Note that every function in $\mathcal{C}_T^\alpha(C)$ for some $C>0$ fulfills Assumption 2.2. Assumption 2.2 guarantees the fundamental solution to have a sufficiently fast decaying derivative, which is stated in the next proposition. This is needed later in the proof of Proposition 2.6.

Proposition 2.3 ([27, Theorem 6.4.5]). *Let Assumption 2.2 be fulfilled. Then there exists a fundamental solution* Γ *to* (2.1.2) *satisfying*

$$\Big|\frac{\partial^k}{\partial x^k}\Gamma(t,x;s,y)\Big| \le \frac{A}{(t-s)^{(1+k)/2}} e^{-a\frac{(x-y)^2}{t-s}} \tag{2.1.4}$$

for $k=0,1$*, where* $a, A>0$ *are positive constants.*

The proof of this result as well as the construction of the function Γ can be found in [26, Chapter 9]. Then we represent the solution to (2.1.2) in terms of the corresponding fundamental solution. This is stated in the next proposition.

Proposition 2.4 ([26, Theorem 9.4.3]). *Let* $T>0$*, Assumption 2.2 be fulfilled and let* $w_0\in C(\mathbb{R},\mathbb{R})$*. Further assume that there exist positive constants* $\widetilde{a}, \widetilde{A}>0$ *such that*

$$|w_0(x)| \le \widetilde{A}e^{\widetilde{a}x^2}, \quad x\in\mathbb{R}.$$

Then there exists a solution w *to* (2.1.2) *in* $[0,T^*]\times\mathbb{R}$*, i.e.* $w\in C^{1,2}((0,T^*]\times\mathbb{R},\mathbb{R})\cap C([0,T^*]\times\mathbb{R},\mathbb{R})$*, where* $T^*:=T\wedge(b/\widetilde{a})$ *and* $b>0$ *is some constant depending on the function* c*, and* w *fulfills*

$$|w(t,x)| \le \widehat{A}e^{\widehat{a}x^2}, \quad (t,x)\in[0,T^*]\times\mathbb{R},$$

for some positive constants $\widehat{a},\widehat{A}>0$*. The solution* w *is given by* $w(0,x)=w_0(x)$ *and*

$$w(t,x) = \int_{\mathbb{R}} \Gamma(t,x;0,y)w_0(y)\,\mathrm{d}y, \quad (t,x)\in(0,T^*]\times\mathbb{R} \tag{2.1.5}$$

Stochastic representation of the solution

Although by Proposition 2.4, the representation (2.1.5) of the solution w to (2.1.2) is given explicitly and w can be investigated using analytical tools, it is useful to have another, *stochastic*, interpretation of the solution. A first example can be given for the easiest case $c\equiv 0$, which is usual referred to as the *heat equation*. In this case the fundamental solution Γ is given by $\Gamma(t,x;s,y)=\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{(x-y)^2}{2(t-s)}}$ and fulfills $\Gamma(\cdot,\cdot,s,y)\in C^\infty((s,\infty)\times\mathbb{R},\mathbb{R})$, $s>0$, $y\in\mathbb{R}$. Indeed, for every $s>0$ and $w_0\in C_c(\mathbb{R},\mathbb{R})$, using dominated convergence, the function $w(t,x)=\int_{\mathbb{R}}\Gamma(t,x;s,y)w_0(y)\,\mathrm{d}y$ solves (2.1.3). Then, due to Proposition 2.4, w solves (2.1.2) (for $c\equiv 0$). A trivial, but nevertheless interesting, observation from a probabilistic point of view is that for the heat equation, Γ is the density at site x of the probability distribution of a standard Brownian motion $(B_r)_{r\ge 0}$ at time $t-s$, starting in y. The function w defined above is usually called a *mild solution* to (2.1.2) and can be considered as a functional of a Brownian path, weighted by the initial condition w_0. This is known by the name *Feynman-Kac formula* or *Feynman-Kac representation*, which we now deduce for the equation (2.1.2).

To this end, we need the initial condition w_0 to be continuous and to fulfill a certain growth condition:

$$w_0 \in C(\mathbb{R}, \mathbb{R}) \quad \text{and there exist constants } a, A > 0, \text{ such that} \\ |w_0(x)| \le A(1 + |x|^a), \quad x \in \mathbb{R}. \tag{2.1.6}$$

Lemma 2.5. *Let $T > 0$, Assumption 2.2 be fulfilled and w_0 satisfy* (2.1.6)*. Then there exists a unique solution w to* (2.1.2) *satisfying*

$$\Big|\frac{\partial^k}{\partial x^k} w(t,x)\Big| \le B(1 + |x|^a), \quad (t,x) \in [0,T] \times \mathbb{R}, \tag{2.1.7}$$

for $k = 0, 1$, where $a > 0$ is given in (2.1.6) *and $B > 0$ is some positive constant.*

Proof. First note that all assumptions of Proposition 2.4 are fulfilled. Thus a solution exists and it can be represented as in (2.1.5). Uniqueness follows from [27, Corollary 6.4.4]. Estimate (2.1.7) for $k = 0$ follows jointly from (2.1.4) (for the case $k = 0$), (2.1.5) and (2.1.6). By (2.1.4) (for $k = 1$) and (2.1.6), using dominated convergence we obtain (2.1.7) for $k = 1$. □

Now we can prove the Feynman-Kac formula for the linear equation.

Proposition 2.6. *Let Assumption 2.2 be fulfilled and w_0 satisfy* (2.1.6)*. Then the unique solution w to* (2.1.2) *is given by*

$$w(t,x) = E_x\big[e^{\int_0^t c(t-s,B_s)\,\mathrm{d}s} w_0(B_t)\big], \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Here and for the remainder of the thesis, we denote by E_x the expectation operator of the probability measure P_x under which the process $(B_t)_{t\ge 0}$ is a standard Brownian motion starting in $x \in \mathbb{R}$.

Proof of Proposition 2.6. Fix $T > 0$ and define

$$Y_t := w(T-t, B_t) e^{\int_0^t c(T-s,B_s)\,\mathrm{d}s}, \quad t \in [0,T].$$

Note that by Proposition 2.4 we have $w \in C^{1,2}((0,\infty) \times \mathbb{R}, \mathbb{R}) \cap C([0,\infty) \times \mathbb{R}, \mathbb{R})$. Then Itô's formula (see e.g. [8, I.III.1.10, page 43]) yields for all $t \in [0,T)$

$$\begin{aligned} Y_t = Y_0 + \int_0^t \Big(& c(T-s,B_s) w(T-s,B_s) - w_t(T-s,B_s) \\ & + \frac{1}{2} w_{xx}(T-s,B_s)\Big) e^{\int_0^s c(T-r,B_r)\,\mathrm{d}r}\,\mathrm{d}s \\ & + \int_0^t w_x(T-s,B_s) e^{\int_0^s c(T-r,B_r)\,\mathrm{d}r}\,\mathrm{d}B_t \\ = Y_0 + \int_0^t & w_x(T-s,B_s) e^{\int_0^s c(T-r,B_r)\,\mathrm{d}r}\,\mathrm{d}B_t, \end{aligned}$$

where we used that w solves (2.1.2). Furthermore the assumptions of Lemma 2.5 are fulfilled.

Using (2.1.7) (for $k = 1$) and boundedness of c, we have

$$\sup_{0\le t\le T}\int_0^t E_x\big[\big(w_x(T-s,B_s)e^{\int_0^s c(T-r,B_r)}\big)^2\big]\,\mathrm{d}s < \infty.$$

Therefore (see [64, Proposition A.5] or [41, Proposition 3.2.10]), the stochastic integral is a martingale with respect to the filtration generated by $(B_t)_{t\ge0}$ and thus has zero expectation, implying

$$E_x\big[w(T-t,B_t)e^{\int_0^t c(T-s,B_s)\,\mathrm{d}s}\big] = E_x\big[Y_t\big] = E_x\big[Y_0\big] = w(T,x).$$

Additionally, Proposition 2.4 provides

$$\lim_{t\uparrow T} w(T-t,B_t)e^{\int_0^t c(T-s,B_s)\,\mathrm{d}s} = w(0,B_T)e^{\int_0^T c(T-s,B_s)\,\mathrm{d}s} \quad P_x\text{-a.s.},$$

where we used that $[0,T]\times\mathbb{R}\ni(t,x)\mapsto w(t,x)$ is continuous. Furthermore, by (2.1.7) (for $k = 0$) we have

$$E_x\Big[\sup_{0\le t\le T}|w(T-t,B_t)|\Big] \le \text{const}\cdot E_x\Big[\sup_{0\le t\le T}(1+|B_t|^a)\Big] < \infty, \tag{2.1.8}$$

where the latter inequality is due to $E_x[\sup_{0\le t\le T}|B_t|^a] < \infty$ by Doob's maximal inequality [8, I.I.1.20 (a), p. 10]. Applying dominated convergence, we finally get

$$E_x[w_0(B_T)e^{\int_0^T c(T-s,B_s)\,\mathrm{d}s}] = \lim_{t\uparrow T} E_x\big[w(T-t,B_t)e^{\int_0^t c(T-s,B_s)\,\mathrm{d}s}\big] = w(T,x).$$

Since $T > 0$ is chosen arbitrarily, we can conclude. □

It is worth noting that if the function c does not depend on t, we get the following corollary. It is of special interest in the main part of the thesis.

Corollary 2.7. *Assume w_0 fulfills* (2.1.6) *and $c:\mathbb{R}\to\mathbb{R}$ to be bounded and uniformly Hölder continuous on compact subsets of $\mathbb{R}$. Then the unique solution to*

$$\begin{aligned} w_t(t,x) &= \frac{1}{2}w_{xx}(t,x) + c(x)\cdot w(t,x), \quad (t,x)\in(0,\infty)\times\mathbb{R},\\ w(0,x) &= w_0(x), \quad x\in\mathbb{R}, \end{aligned}$$

is given by

$$w(t,x) = E_x\big[e^{\int_0^t c(B_s)\,\mathrm{d}s}w_0(B_t)\big], \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

2.1.4 Quasilinear reaction term

The interlude in the last subsection reveals that the solution to the reaction-diffusion equation may admit a representation in terms of a functional of Brownian paths. The proof relies heavily on the fact that the reaction term is linear in w. To preserve this structure, we restrict to the case where the reaction term

$$f(t,x,w(t,x)) = c(t,x,w(t,x))\cdot w(t,x)$$

admits an (implicit) linearity in w. Thus let us consider the *quasilinear PDE*

$$\begin{aligned} w_t(t,x) &= \frac{1}{2} w_{xx}(t,x) + c(t,x,w(t,x)) \cdot w(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ w(0,x) &= w_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{2.1.9}$$

In contrast to the previous section, the term $c(t,x,w(t,x))$ can depend on the solution w itself. Therefore, one may expect that the solution to (2.1.1) is given only implicitly. However, it is shown in this section that w still admits a representation in terms of a functional of Brownian paths. We abbreviate

$$c(t,x;w) := c(t,x,w(t,x)).$$

The results in this subsection have first been published as a remark without a proof in the paper [24] and then (also without a complete proof) republished in [25, Section 5.3] for a more general PDE. We follow the representation from [39, §1, §2], where the author considers a slightly more general model than [24] and covers the model in the main part of this thesis in the Chapters 3 and 4.

We fix $T > 0$ for the remainder of Section 2.1.4.

Generalized solution

For non-smooth initial conditions it is not clear whether there exists a classical solution to (2.1.9). However, it is possible to expand the space of the solutions in such a way that we also get a solution concept for such initial conditions. These solutions are called *generalized solutions*.

Let us introduce a function space that is needed in the context of generalized solutions. For $t > 0$ let $\mathcal{U}_t$ be the Banach space of bounded Borel measurable functions $w : [0,t] \times \mathbb{R} \to \mathbb{R}$, equipped with the norm $\|w\|_t := \sup\{|w(s,x)| : (s,x) \in [0,t] \times \mathbb{R}\}$, $w \in \mathcal{U}_t$.

We further assume that c from (2.1.9) is defined for all $w \in \mathcal{U}_T$ and all $(t,x) \in [0,T] \times \mathbb{R}$ and that $[0,T] \times \mathbb{R} \ni (t,x) \mapsto c(t,x;w)$ is Borel measurable for every $w \in \mathcal{U}_T$. We need some conditions for c. In essence, we demand c to be suitably bounded and $w \mapsto c(\cdot,\cdot;w)$ to be Lipschitz continuous.

Assumption 2.8. (i) For all $K > 0$ there exist $K_1, K_2 > 0$ such that for all $u, v \in \mathcal{U}_T$ fulfilling $\|u\|_T \le K$ and $\|v\|_T \le K$ we have

$$|c(t,x;u)| \le K_1, \quad (t,x) \in [0,T] \times \mathbb{R},$$

and

$$|c(t,x;u) - c(t,x;v)| \le K_2 \|u - v\|_t, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

(ii) There exists a constant $K_3 > 0$ such that $c(t,x;w) \le K_3$ for all $(t,x) \in [0,T] \times \mathbb{R}$ and for all $w \in \mathcal{U}_T$ with $w \ge 0$.

As is shown below, under the assumptions on c given above, there exists a unique solution to (2.1.9) in an implicit way. Solutions of such a type are defined to fulfill a certain (implicit) Feynman-Kac formula and are called *generalized solutions*. This is made precise in the following definition.

Definition 2.9. *A function $w : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is called* generalized solution *to* (2.1.9)*, if it fulfills the equation*

$$w(t,x) = E_x\big[e^{\int_0^t c(t-s,B_s;w))\,\mathrm{d}s} w_0(B_t)\big], \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{2.1.10}$$

The next proposition ensures the existence and uniqueness of the generalized solution to (2.1.9).

Proposition 2.10 ([39, Theorem 1])**.** *Assume that the coefficient c satisfies Assumption 2.8. Then* (2.1.10) *has a unique solution in $\mathcal{U}_T$ for any bounded nonnegative measurable function w_0.*

Proof. We define the operator $\Phi : \mathcal{U}_T \to \mathcal{U}_T$,

$$(\Phi \circ w)(t,x) = E_x\Big[e^{\int_0^t c(t-s,B_s;w)\,\mathrm{d}s} w_0(B_t)\Big], \quad (t,x) \in [0,T] \times \mathbb{R}. \tag{2.1.11}$$

Let K_3 be such that Assumption 2.8 (ii) is satisfied. For $K := \|w_0\|_\infty e^{K_3 T}$ let K_1, K_2 be as in Assumption 2.8 (i). Further, for $t \in [0,T]$ let

$$\mathcal{U}_t(K) := \{w \in \mathcal{U}_t : \|w\|_t \leq K \text{ and } w \geq 0\},$$

which is a complete subset of $\mathcal{U}_t$. Then by definition of K, Φ maps $\mathcal{U}_t(K)$ to itself. We now show that it is contractive on $\mathcal{U}_{t_1}(K)$ for some suitable $t_0 > 0$. To this end, for $(t,x) \in [0,T] \times \mathbb{R}$ and $u, w \in \mathcal{U}_t(K)$ we get

$$\begin{aligned}
\big|(\Phi \circ u)(t,x) - (\Phi \circ w)(t,x)\big| &\leq \|w_0\|_\infty E_x\Big[\Big|e^{\int_0^t c(t-s,B_s;u)\,\mathrm{d}s} - e^{\int_0^t c(t-s,B_s;w)\,\mathrm{d}s}\Big|\Big] \\
&\leq \|w_0\|_\infty \Big(E_x\big[e^{\int_0^t 2c(t-s,B_s;w)\,\mathrm{d}s}\big]\Big)^{1/2} \Big(E_x\Big[\big|e^{\int_0^t c(t-s,B_s;u)-c(t-s,B_s;w)\,\mathrm{d}s} - 1\big|^2\Big]\Big)^{1/2} \\
&\leq \|w_0\|_\infty e^{K_3 t}\Big(E_x\Big[\Big|\int_0^t \big(c(t-s,B_s;u) - c(t-s,B_s;w)\big) \\
&\qquad\qquad \times e^{\int_0^s \left(c(t-r,B_r;u)-c(t-r,B_r;w)\right)\,\mathrm{d}r}\,\mathrm{d}s\Big|^2\Big]\Big)^{1/2} \\
&\leq \|w_0\|_\infty e^{(2K_1+K_3)T} t K_2 \|u-w\|_t,
\end{aligned}$$

where we used Cauchy-Schwarz in the second, the identity $e^{\int_0^t f(s)\,\mathrm{d}s} - 1 = \int_0^t f(s) e^{\int_0^s f(r)\,\mathrm{d}r}\,\mathrm{d}s$ in the third and Assumption 2.8 (i) in the last inequality. Thus for $t_0 := e^{-(2K_1+K_3)T}\frac{1}{2K_2\|w_0\|_\infty}$ the operator Φ is contractive on $\mathcal{U}_{t_0}(K)$ and by the Banach fixed-point theorem (cf. e.g. [1, Theorem 1.1]), there exists a unique u_0 on $\mathcal{U}_{t_0}(K)$ such that $(\Phi \circ u_0)(t,x) = u_0(t,x)$ for all $(t,x) \in [0,t_0] \times \mathbb{R}$. To show that this function can be uniquely continued for times $t > t_0$, let $w \in \mathcal{U}_T$ be an extension of u_0, i.e. $w(t,x) = u_0(t,x)$ for all $(t,x) \in [0,t_0] \times \mathbb{R}$. Then we have $(\Phi \circ w)(t_0,\cdot) = (\Phi \circ u_0)(t_0,\cdot) = u_0(t_0,\cdot)$ and the Markov-property of $(B_t)_{t\geq 0}$ yields that for all $(t,x) \in [0,T-t_0] \times \mathbb{R}$ we get

$$\begin{aligned}
(\Phi \circ w)(t_0+t,x) &= E_x\Big[e^{\int_0^t c(t+t_0-s,B_s;w)\,\mathrm{d}s}(\Phi \circ w)(t_0,B_t)\Big] \\
&= E_x\Big[e^{\int_0^t c'(t-s,B_s;w)\,\mathrm{d}s} u_0(t_0,B_t)\Big],
\end{aligned} \tag{2.1.12}$$

where we set

$$\begin{aligned} c'(t,x;w) &:= c(t+t_0,x;w^*), \\ w^*(t,x) &:= u_0(t,x)\mathbb{1}_{[0,t_0]}(t) + w(t-t_0,x)\mathbb{1}_{(t_0,T]}(t). \end{aligned} \tag{2.1.13}$$

Then we can solve a fixed-point problem for the operator $\Phi' : \mathcal{U}'_{T-t_0} \to \mathcal{U}'_{T-t_0}$, where

$$\begin{aligned} (\Phi' \circ w)(t,x) &= E_x\Big[e^{\int_0^t c'(t-s,B_s;w)\,\mathrm{d}s} u_0(t_0,B_t)\Big], \quad (t,x) \in [0,T-t_0]\times\mathbb{R}, \\ \mathcal{U}'_t &:= \{w \in \mathcal{U}_t : w(0,\cdot) = u_0(t_0,\cdot)\}, \quad 0 \le t \le T-t_0. \end{aligned}$$

Indeed, because by Assumption 2.8 and also using

$$\|u_0(t_0,\cdot)\|_\infty e^{K_3(T-t_0)} \le \|w_0\|_\infty e^{K_3 T} = K,$$

Φ' is contractive on $\mathcal{U}'_{t_0}(K) := \left\{w \in \mathcal{U}'_{t_0} : \|w\|_t \le K \text{ and } w \ge 0\right\}$ and the Banach fixed-point theorem yields a unique function v_0 on $\mathcal{U}'_{t_0}(K)$ such that $(\Phi' \circ v_0)(t,x) = v_0(t,x)$ for all $(t,x) \in [0,t_0]\times\mathbb{R}$. But in view of (2.1.12), the function v_0^* as defined in (2.1.13) is the unique function on $\mathcal{U}_{2t_0}$ such that $(\Phi \circ v_0^*)(t,x) = v_0^*(t,x)$ for all $(t,x) \in [0,2t_0]\times\mathbb{R}$. We can repeat this procedure until we finally get a unique function on $\mathcal{U}_T$ fulfilling (2.1.10). □

Classical solution

The proof of Proposition 2.10 reveals that a generalized solution solves a certain fixed-point equation and it is not clear whether it is a solution to (2.1.1) in the classical sense. In this subsection we show that under certain additional assumptions, the solution to (2.1.10) also solves (2.1.1).

To this end, we introduce some additional function space. For $t > 0$ let $\mathcal{C}_t^\alpha$ be the space of bounded continuous functions $w : [0,t]\times\mathbb{R} \to \mathbb{R}$, which are uniformly Hölder continuous in x with Hölder exponent $\alpha > 0$, i.e. for $w \in \mathcal{C}_t^\alpha$ there exists some Hölder coefficient $C = C(w,t) > 0$ such that

$$\sup_{x,y\in\mathbb{R}:\,x\neq y,\ s\in[0,t]} \frac{|w(s,x)-w(s,y)|}{|x-y|^\alpha} \le C.$$

For $L > 0$ we abbreviate $\mathcal{C}_t^\alpha(L) := \{w \in \mathcal{C}_t^\alpha : C(w,t) \le L\}$ to be the set of Hölder continuous functions whose Hölder coefficient is at most L.

We take advantage of the results from Section 2.1.3. Thus, additionally to Assumption 2.8, we have to impose further conditions on the function c.

Assumption 2.11. (i) For all $t \in [0,T]$ we have that $w \in \mathcal{C}_t^\alpha$ implies $[0,t]\times\mathbb{R} \ni (s,x) \mapsto c(s,x;w) \in \mathcal{C}_t^\alpha$;

(ii) For all $K > 0$ there exist constants $K_4, K_5 > 0$ such that for all $w \in \mathcal{C}_T^\alpha$ with Hölder coefficient L and $\|w\|_T \le K$ we have

$$|c(t,x;w) - c(t,y;w)| \le (K_4 + K_5 L)|x-y|^\alpha, \quad t \in [0,T],\ x,y \in \mathbb{R}.$$

The next example treats a special case for the nonlinearity which is used in the main part of the thesis. For this case, Assumptions 2.8 and 2.11 are fulfilled.

Example 2.12. Assume we have $c(t,x;w) = \xi(x)\cdot k(w(t,x))$ such that

- $\xi : \mathbb{R} \to \mathbb{R}$ is bounded and Hölder continuous with exponent α^ξ and coefficient L^ξ and
- $k : \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz continuous with Lipschitz constant C.

We observe that c fulfills Assumption 2.8. Additionally, c fulfills Assumption 2.11 for all $\alpha \leq \alpha^\xi$. Indeed, because ξ is bounded, by [22, (1.81)] we have

$$\sup_{x \neq y} \frac{|\xi(x) - \xi(y)|}{|x-y|^\alpha} \leq 2(L^\xi)^{\alpha/\alpha^\xi} \cdot \big(\|\xi\|_\infty\big)^{\frac{\alpha^\xi - \alpha}{\alpha^\xi}} \quad \forall \alpha \in [0, \alpha^\xi).$$

Thus we get that ξ is Hölder continuous also for all $\alpha \in [0, \alpha^\xi)$ with some uniform Hölder coefficient $\widetilde{C}$. Then, also using that k is Lipschitz continuous and bounded, for $w \in \mathcal{C}_T^\alpha(L)$ we get

$$\begin{aligned} |c(t,x;w) - c(t,y;w)| &\leq |\xi(x) - \xi(y)| \cdot |k(w(t,x))| + |\xi(y)| \cdot |k(w(t,x)) - k(w(t,y))| \\ &\leq (\|k\|_\infty \widetilde{C} + \|\xi\|_\infty C \cdot L)|x-y|^\alpha, \quad t \in [0,T],\ x, y \in \mathbb{R}. \end{aligned}$$

Assuming additional regularity for the initial function, we get existence and uniqueness of the classical solution to (2.1.9), which is stated in the next proposition.

Proposition 2.13. *Let Assumptions 2.8 and 2.11 be fulfilled. If an initial condition w_0 is nonnegative, bounded and satisfies $|w_0(x) - w_0(y)| \leq L_0|x-y|^{\alpha_0}$, $x, y \in \mathbb{R}$, for some $\alpha_0, L_0 > 0$, then there exists a unique solution $w \in \mathcal{C}_T^{\alpha_0}$ of* (2.1.10) *which is the unique solution to* (2.1.9)*.*

The proof is based on the following lemma, which we prove by adapting the proof of [39, Lemma 2]. However, we replace the condition [39, (IV)] therein by Assumption 2.11 (ii), and we obtain a slightly stronger statement. We abbreviate by

$$\mathcal{C}_t^\alpha(L, K) := \big\{ w \in \mathcal{C}_t^\alpha(L) :\ w \geq 0,\ \|w\|_t \leq K \big\},$$

the set of nonnegative functions $w \in \mathcal{C}_t^\alpha(L)$ which satisfy $\|w\|_t \leq K$.

Lemma 2.14. *Let Assumptions 2.8 and 2.11 be fulfilled and w_0 be nonnegative, bounded and satisfying $|w_0(x) - w_0(y)| \leq L_0|x-y|^{\alpha_0}$, $x, y \in \mathbb{R}$, for some $\alpha_0, L_0 > 0$. Further, let $K = \|w_0\|_\infty e^{K_3 T}$ with K_3 from Assumption 2.8* (ii)*. Then the mapping Φ, defined in* (2.1.11)*, maps $\mathcal{C}_t^\alpha(L, K)$ to $\mathcal{C}_t^\alpha(J_1 L_0 + (J_2 + J_3 L)t, K)$ for each $t \leq T$. J_1, J_2 and J_3 are positive constants, depending only on K and T.*

Proof. For $K = \|w_0\|_\infty e^{K_3 T}$ let K_1, K_2 be the constants from Assumption 2.8 (i), and K_4, K_5 be the constants from Assumption 2.11 (ii). For $w \in \mathcal{C}_T^{\alpha_0}(L, K)$, $s, t \in [0, T]$ such that $s \leq t$ and $x, y \in \mathbb{R}$, we have

$$\begin{aligned} &\big|(\Phi \circ w)(t,x) - (\Phi \circ w)(s,y)\big| \\ &\quad = \Big| E_0\Big[e^{\int_0^t c(t-r, B_r + x; w)\,\mathrm{d}r} w_0(B_t + x)\Big] - E_0\Big[e^{\int_0^s c(s-r, B_r + y; w)\,\mathrm{d}r} w_0(B_s + y)\Big]\Big| \\ &\quad \leq E_0\Big[|w_0(B_t + x) - w_0(B_s + y)| e^{\int_0^t c(t-r, B_r + x; w)\,\mathrm{d}r}\Big] \\ &\qquad + E_0\Big[|w_0(B_s + y)| \cdot \big| e^{\int_0^t c(t-r, B_r + x; w)\,\mathrm{d}r} - e^{\int_0^s c(s-r, B_r + y; w)\,\mathrm{d}r}\big|\Big] \\ &\quad =: I_1 + I_2. \end{aligned}$$

Then we have by Cauchy-Schwarz

$$I_1 \le L_0 \big(E_0\big[|B_t - B_s + x - y|^{2\alpha_0}\big]\big)^{1/2} \cdot e^{K_3 t} \le 2L_0\big(|x-y|^{\alpha_0} + |t-s|^{\alpha_0/2}\cdot \kappa\big)e^{K_3 t},$$

where $\kappa := \big(\int_{\mathbb{R}} |z|^{2\alpha_0}(2\pi)^{-1/2}e^{-z^2/2}\,\mathrm{d}z\big)^{1/2}$. Furthermore,

$$\begin{aligned} I_2 &\le \|w_0\|_\infty e^{K_3 s} \\ &\quad\times \left(E_0\Big[\Big|\exp\Big\{\int_0^t c(t-r, B_r + x; w)\,\mathrm{d}r - \int_0^s c(s-r, B_r + y; w)\,\mathrm{d}r\Big\} - 1\Big|^2\Big]\right)^{1/2} \end{aligned}$$

and because of the identity $e^{\int_0^t f(s)\,\mathrm{d}s} - 1 = \int_0^t f(s)e^{\int_0^s f(r)\,\mathrm{d}r}\,\mathrm{d}s$ for some integrable function f, the assumption $\|c\|_t \le K_1$ and $|a+b|^2 \le 2|a|^2 + 2|b|^2$, $a, b \in \mathbb{R}$, we get

$$\begin{aligned} &E_0\Big[\Big|\exp\Big\{\int_0^t c(t-r, B_r + x; w)\,\mathrm{d}r - \int_0^s c(s-r, B_r + y; w)\,\mathrm{d}r\Big\} - 1\Big|^2\Big] \\ &\quad = E_0\Big[\Big|\int_0^t \big(c(t-r, B_r + x; w) - \underline{c}(s-r, B_r + y; w)\big) \\ &\qquad\quad \times \exp\Big\{\int_0^r \big(c(t-z, B_z + x; w) - \underline{c}(s-z, B_z + y; w)\big)\,\mathrm{d}z\Big\}\,\mathrm{d}r\Big|^2\Big] \\ &\quad \le e^{4K_1 t}E_0\Big[2\Big|\int_0^s \big(c(t-r, B_r + x; w) - \underline{c}(s-r, B_r + y; w)\big)\,\mathrm{d}r\Big|^2 + 2K_1|t-s|^2\Big] \\ &\quad \le e^{4K_1 t}\left(2sE_0\Big[\int_0^s \big(c(t-r, B_r + x; w) - c(s-r, B_r + y; w)\big)^2\,\mathrm{d}r\Big] + 2K_1|t-s|^2\right), \end{aligned}$$

where we set $\underline{c}(s,\cdot;\cdot) := c(s,\cdot;\cdot)$ for $s > 0$ and $\underline{c}(s,\cdot;\cdot) \equiv 0$ for $s \le 0$. Thus by the estimates for I_1 and I_2, using continuity and boundedness of c, we observe that $[0,T] \times \mathbb{R} \ni (t,x) \mapsto (\Phi \circ w)(t,x)$ is continuous for all $w \in \mathcal{C}_T^{\alpha_0}(L,K)$. By the above inequalities we get for $x, y \in \mathbb{R}$ and all $t \in [0,T]$

$$\begin{aligned} &\big|(\Phi \circ w)(t,x) - (\Phi \circ w)(t,y)\big| \le 2L_0 e^{K_3 T}|x-y|^{\alpha_0} \\ &\quad + \|w_0\|_\infty e^{K_3 T + 2K_1 T}(2t)^{1/2} \times \left(E_0\Big[\int_0^t \big(c(t-r, B_r + x; w) - c(t-r, B_r + y; w)\big)^2\,\mathrm{d}r\Big]\right)^{1/2} \\ &\quad \le \Big(2L_0 e^{K_3 T} + \|w_0\|_\infty e^{K_3 T + 2K_1 T}2^{1/2}(K_4 + K_5 L)t\Big)\,|x-y|^{\alpha_0}, \end{aligned}$$

i.e. Φ maps $\mathcal{C}_t^{\alpha}(L,K)$ to $\mathcal{C}_t^{\alpha}(J_1 L_0 + (J_2 + J_3 L)t, K)$ for each $t \le T$, where

$$J_1 := 2e^{K_3 T}, \quad J_2 := Ke^{2K_1 T}2^{1/2}K_4, \quad \text{and} \quad J_3 := Ke^{2K_1 T}2^{1/2}K_5.$$

□

Subsequently, we can now prove Proposition 2.13 by following the argument in [39, Proof of Theorem 2].

Proof of Proposition 2.13. Let Φ be defined in (2.1.11) and let us first show that the equation

$$w = \Phi \circ w$$

has a unique solution in $\mathcal{C}_T^{\alpha_0}$. We put $\Phi^{(0)} := \mathrm{Id}$, $\Phi^{(n)} := \Phi \circ \Phi^{(n-1)}$, $n \in \mathbb{N}$, and set

$K := \|w_0\|_\infty e^{K_3 T}$. Using Lemma 2.14 n times, we get that $\Phi^{(n)}$ maps $\mathcal{C}_t^{\alpha_0}(L_0, K)$ to $\mathcal{C}_t^{\alpha_0}(L_n, K)$ with

$$L_n := (J_3 t)^n L_0 + (J_1 L_0 + J_2 t) \sum_{k=0}^{n-1} (J_3 t)^k.$$

Let $t_1 > 0$ be as in the proof of Proposition 2.10 (i.e. such that Φ is a contraction on $\mathcal{U}_{t_1}$) and set $t_1' := t_1 \wedge \frac{1}{2(J_2 \vee J_3)}$, where $a \wedge b$ denotes the minimum of $a, b \in \mathbb{R}$. Then Φ is contractive and by Banach's fixed-point theorem [1, Theorem 1.1] we get that there exists a unique fixed point $u_0 = \lim_{n\to\infty} \Phi^{(n)} \circ w_0$ in $\mathcal{C}_{t_1'}^{\alpha_0}(1 + (1 + 2J_1)L_0, K)$. Then, just as we did at the end of the proof of Proposition 2.10, we can uniquely prolong the solution u_0 to $w \in \mathcal{C}_T^{\alpha_0}$.

To show that w is a solution to (2.1.9), we mention that by Assumption 2.11 (i) and $w \in \mathcal{C}_T^{\alpha_0}$ we have $c(\cdot, \cdot; w) \in \mathcal{C}_T^{\alpha_0}$. Thus we can apply Proposition 2.4, which yields that there exists a unique solution v to the PDE

$$\begin{aligned} v_t(t, x) &= \frac{1}{2} v_{xx}(t, x) + c(t, x; w) \cdot v(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\ v(0, x) &= w_0(x), \quad x \in \mathbb{R}. \end{aligned}$$

Let us show $v(t, x) = w(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}$. To do so, fix $t \in (0, T]$ and define

$$Y_s := v(t - s, B_s) e^{\int_0^s c(t-r, B_r; w)\,\mathrm{d}r}, \quad s \in [0, t].$$

Then applying Itô's formula from [8, I.III.1.10, page 43], we get for all $s < t$

$$Y_s = Y_0 + \int_0^s v_x(t - r, B_r) e^{\int_0^r c(t-u, B_u; w)\,\mathrm{d}u}\,\mathrm{d}B_r.$$

Since w_0 is bounded, Lemma 2.5 provides that the stochastic integral is a martingale and has zero expectation with respect to P_x. Taking expectation on both sides of the latter equality and putting $s = t$ (this is possible by the same arguments as in (2.1.8)), we get

$$w(t, x) = E_x\big[w_0(B_t) e^{\int_0^t c(t-r, B_r; w)\,\mathrm{d}r}\big] = E_x\big[Y_t\big] = E_x\big[Y_0\big] = v(t, x).$$

□

2.2 Branching processes

The second major tool for the investigations in the main part of the thesis stems from the theory of branching processes. We provide a construction of a certain branching process and then derive useful applications.

2.2.1 Construction of BBMRE

The basic construction is based on the model from [66, Section 2.1]. While the author of [66] models the randomness of a certain medium by a Poisson point process hit by a Wiener sausage, our medium is modeled by a continuous nonnegative random-function ξ. With regard to the next subsection, we also use notation from [34] and [33], and the main theorem of the next subsection is a corollary of the results and ideas in these papers.

Before we provide a construction of a branching Brownian motion in random environment

(BBMRE), we supply a guideline of its dynamics. We follow the representations from [34, Section 2], where a more general model is described.

A BBMRE can be characterize as follows. Let $\xi = (\xi(x))_{x\in\mathbb{R}} = (\xi(x,\omega))_{x\in\mathbb{R}}$, $\omega \in \Omega$, be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with continuous and nonnegative paths $x \mapsto \xi(x)$. Given ξ, we start the branching process with one particle at x. This particle moves as a standard Brownian motion. When at position y, the particles *branches* at rate $\xi(y)$, i.e. it dies and gives birth to a random number of new particles, which start at the site where their parent has died. We further assume that at every branching event, at least one particle is born and the process will never die out. Each of the new born particles then independently repeats the stochastic behavior of its parent.

Construction

Let $\xi : \mathbb{R} \to \mathbb{R}$ be a nonnegative, bounded and continuous function.

Furthermore, let $\mathscr{M} := (\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\geq 0}, (B_t)_{t\geq 0}, (P_x)_{x\in\mathbb{R}}, (\theta_t)_{t\geq 0})$ be a standard Brownian motion on $\mathbb{R}$. Here, $(\theta_t)_{t\geq 0}$ denotes the time shift operator of paths, i.e. for each $\hat{\omega} \in \hat{\Omega}$, $B_t(\theta_s\hat{\omega}) = B_{t+s}(\hat{\omega})$ identically for any $s, t \geq 0$, and P_x is the probability measure under which we have $P_x(B_0 = x) = 1$. Furthermore, let τ be a nonnegative random variable on $(\hat{\Omega}, \hat{\mathscr{F}}, P_x)$, which is independent of the Brownian motion and exponentially distributed with mean 1, i.e. $P_x(\tau > t) = e^{-t}$ for every $t \geq 0$. Furthermore, let

$$S = S(\hat{\omega}, \xi) := \inf\Big\{t \geq 0 : \int_0^t \xi(B_s(\hat{\omega}))\,\mathrm{d}s \geq \tau(\hat{\omega})\Big\}, \quad \hat{\omega} \in \hat{\Omega}. \tag{2.2.1}$$

Then we have

$$\begin{aligned} P_x(S(\cdot,\xi) > t) &= P_x\big(\tau > \int_0^t \xi(B_s)\,\mathrm{d}s\big) = E_x\big[P_x\big(\tau > \int_0^t \xi(B_s)\,\mathrm{d}s \,\big|\, (B_s)_{s\geq 0}\big)\big] \\ &= E_x\big[e^{-\int_0^t \xi(B_s)\,\mathrm{d}s}\big], \quad t \geq 0, \end{aligned}$$

where in the third equality we used independence of τ and $(B_t)_{t\geq 0}$. Thus under P_x, $S(\cdot, \xi)$ can be regarded as an exponentially distributed random variable with inhomogeneous rate $\xi(B_s(\cdot))$.

Let $(p_n)_{n\in\mathbb{N}}$ be a probability distribution supported on $\mathbb{N}$, i.e. $p_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^\infty p_n = 1$. We further assume $p_1 < 1$, so that we exclude the trivial case where there is no actual branching. Further, let I be an $\mathbb{N}$-valued random variable on $(\hat{\Omega}, \hat{\mathscr{F}}, P_x)$, which is independent of τ and the Brownian motion $(B_t)_{t\geq 0}$ and has distribution $(p_n)_{n\in\mathbb{N}}$, i.e. $P_x(I = n) = p_n$, $n \in \mathbb{N}$.

To describe the genealogy of the particles of the branching process, we use the following labeling system. The set

$$\mathbf{K} := \{\emptyset\} \cup \bigcup_{n\in\mathbb{N}} K^n,$$

where

$$K^1 := \{(1)\}, \quad K^n := \{(1, k_2, \dots, k_n) : \; k_2, \dots, k_n \in \mathbb{N}\}, \; n \geq 2,$$

is called the set of *particles*; particle $\emptyset$ is referred to as the *initial ancestor*. We use the convention that the initial ancestor has exactly one child (1). As an example, $(1, 4, 9) \in \mathbf{K}$ denotes the ninth child of the fourth child of the child of the initial ancestor. For $k \in \mathbb{N}$ and $\mathbf{u} = (1)$ we set $\mathbf{v} := \mathbf{u}k = (1)k := (1, k) \in K^2$ and if $\mathbf{u} = (1, k_2, \dots, k_n) \in K^n$, $n \geq 2$, we define $\mathbf{v} := \mathbf{u}k := (1, k_2, \dots, k_n, k) \in K^{n+1}$; we say that $\mathbf{v}$ is the k-th *child* of $\mathbf{u}$. For convention, we set $\emptyset(1) := (1) \in K^1$ and allow the notation $\emptyset(k) := (k)$, $k \in \mathbb{N} \setminus \{1\}$; recall that $(k) \notin \mathbf{K}$ for $k \in \mathbb{N} \setminus \{1\}$ and this notation is introduced only to shorten some notation below.

Since we wish to have a certain view of particles, as a system evolving in time and space, we associate to every particle certain random objects such as motion, life- and death-time as well as the number of its children. To be precise, let $(B_t^{\mathbf{u}})_{t\geq 0}$ and $\tau^{\mathbf{u}}$, $\mathbf{u} \in \mathbf{K}$, be independent copies of the random objects $(B_t)_{t\geq 0}$ and τ, respectively, and denote by $S^{\mathbf{u}}$ the corresponding branching time with $(B_t)_{t\geq 0}$ and τ in (2.2.1) replaced by $(B_t^{\mathbf{u}})_{t\geq 0}$ and $\tau^{\mathbf{u}}$, $\mathbf{u} \in \mathbf{K}$. Furthermore, set $I^{\emptyset} := 1$ and let $I^{\mathbf{u}}$, $\mathbf{u} \in \mathbf{K} \setminus \{\emptyset\}$, be independent copies of τ. We extend $\mathscr{M}$ such that all objects are assumed to be defined on the same probability space.

Then define the family of random variables $T^{\mathbf{u}}$ and $(X_t^{\mathbf{u}})_{t\geq 0}$, indexed by $\mathbf{u} \in \mathbf{K}$, on $(\hat{\Omega}, \hat{\mathscr{F}})$ as follows; for each $\hat{\omega} \in \hat{\Omega}$ define $T^{\emptyset}(\hat{\omega}) := T^{\emptyset}(\hat{\omega}, \xi) := 0$ and $X_t^{\emptyset}(\hat{\omega}) := X_t^{\emptyset}(\hat{\omega}, \xi) := B_t^{\emptyset}(\hat{\omega})$ for all $t \geq 0$. Then for all $\mathbf{u} \in \mathbf{K}$, $k \in \mathbb{N}$ and $\hat{\omega} \in \hat{\Omega}$ we set

$$T^{\mathbf{u}k}(\hat{\omega}) := T^{\mathbf{u}k}(\hat{\omega}, \xi) := \begin{cases} T^{\mathbf{u}}(\hat{\omega}, \xi) + S^{\mathbf{u}k}(\theta_{T^{\mathbf{u}}(\hat{\omega},\xi)}\hat{\omega}, \xi), & \text{if } k \leq I^{\mathbf{u}}(\hat{\omega}), \\ \infty, & \text{if } k \geq I^{\mathbf{u}}(\hat{\omega}) + 1, \end{cases}$$

and

$$X_t^{\mathbf{u}k}(\hat{\omega}) := X_t^{\mathbf{u}k}(\hat{\omega}, \xi) := \begin{cases} X^{\mathbf{u}}_{T^{\mathbf{u}}(\hat{\omega},\xi)}(\hat{\omega}, \xi) + B_t^{\mathbf{u}k}(\hat{\omega}) - B^{\mathbf{u}k}_{T^{\mathbf{u}}(\hat{\omega},\xi)}(\hat{\omega}), \\ \qquad \text{if } T^{\mathbf{u}}(\hat{\omega}, \xi) \leq t < T^{\mathbf{u}k}(\hat{\omega}, \xi) \ \text{ and } \ k \leq I^{\mathbf{u}}(\hat{\omega}), \\ \Delta, \qquad \text{otherwise}, \end{cases}$$

where $\Delta \notin \mathbb{R}$ is some cemetary state. We use $X_t^{\mathbf{u}}$ and $T^{\mathbf{u}}$ to denote, respectively, the *position* and *branching time* of the particle $\mathbf{u}$. More precisely, we can describe the branching Brownian motion as follows:

- At time 0, one initial particle with index $\emptyset$ dies at site $X_0^{\emptyset} = B_0^{\emptyset}$ and gives birth to exactly one Brownian particle (1).
- A Brownian particle with index $\mathbf{u} \in \mathbf{K} \setminus \{\emptyset\}$ dies at time $T^{\mathbf{u}}$ and site $X^{\mathbf{u}}_{T^{\mathbf{u}}}$ and produces k new Brownian particles with probability $P_x(I^{\mathbf{u}} = k)$, $k \in \mathbb{N}$.
- These Brownian particles, indexed by $\mathbf{u}1, \ldots, \mathbf{u}I^{\mathbf{u}}$, respectively, start from $X^{\mathbf{u}}_{T^{\mathbf{u}}}$, and independently repeat the stochastic behaviour of their parent.

Note that we included the cemetary state Δ only for the sake of completeness. One could think of that at a branching event there are infinitely many particles born, but those with index $k \geq I^{\mathbf{u}} + 1$ are dead-born (or "never born") and taken to the cemetary. These particles are irrelevant in the course of the thesis, since we only consider particles $\mathbf{u}k$ up to time t such that $T^{\mathbf{u}} \leq t$, which is impossible if $k \geq I^{\mathbf{u}} + 1$.

Above construction of the branching process is, due to the relatively simple branching mechanism, quite short and self-contained. However, we can consult general results ensuring existence of branching processes, described e.g. in the monographs [36, 37, 38]. There, the authors use semigroup theory to construct general branching Markov processes. As a special case, the process constructed above is given in [37, Example 3.4 (A) Branching Brownian motions].

For a given function ξ, we denote by P_x^{ξ} the probability measure under which the above constructed process $(X_t^{\mathbf{u}})_{t\geq 0, \mathbf{u}\in\mathbf{K}}$ starts in $x \in \mathbb{R}$.

The next step is to take ξ random. More precisely, let $\xi = (\xi(x, \omega))_{x\in\mathbb{R}}$, $\omega \in \Omega$, be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}$-a.s. continuous, nonnegative and bounded paths. Conditioned on ξ, we let $\mathtt{P}_x^\xi$ be the probability measure of a the above branching process, under which the process starts at site $x \in \mathbb{R}$. Thus $\mathtt{P}_x^\xi$ is the regular conditional probability of the above branching process. The existence (also for more general branching processes) of the regular conditional probability $\mathtt{P}_x^\xi$ is proven in [62, Theorem 4.2] using the construction [37, Theorem 3.5]. It usual called *quenched* law of the branching process.

We call the above described process the *branching Brownian motion in random environment* (*BBMRE*).

Finally, let us introduce some notation. For $t \geq 0$ we let

$$\begin{aligned} N(t,A) &:= \Big\{ \mathbf{u}k : \ T^{\mathbf{u}} \leq t < T^{\mathbf{u}k}, \ X_t^{\mathbf{u}} \in A, \ \mathbf{u} \in \mathbf{K}, k \in \mathbb{N} \Big\}, \quad A \in \mathcal{B}(\mathbb{R}), \\ N(t) &:= N(t,\mathbb{R}), \\ N^{\leq}(t,x) &:= |N(t,(-\infty,x])|, \quad x \in \mathbb{R}, \\ N^{\geq}(t,x) &:= |N(t,[x,\infty))|, \quad x \in \mathbb{R}, \end{aligned}$$

where $\mathcal{B}(\mathbb{R})$ is the Borel-σ-algebra on $\mathbb{R}$. Further, $(X_s^{\mathbf{u}})_{0 \leq s \leq t}$ denotes the path of the unique ancestor of $\mathbf{u}$ up to time t. Finally, we let $\mathtt{E}_x^\xi$ be the corresponding expectation associated to $\mathtt{P}_x^\xi$.

2.2.2 The many-to-few lemmata

As the construction of the previous subsection reveals, a BBMRE is a priori complicated to analyze because there is a large number of ingredients governing the dynamics. Nevertheless, if one wants to calculate the moments of the number of particles, there exist useful, if not even indispensable tools: the *many-to-few lemmata*. These lemmata simplify the calculation of cumbersome expressions, including *many* particles of the BBMRE, by a functional of only *few* Brownian particles. What is more, in the case of the first moment, the *many-to-one formula* and the Feynman-Kac formula for a suitable PDE actually coincide. This forms a bridge between the theory of branching processes and the theory of PDEs and this observation is crucial for the main part of the thesis.

The main ingredients in this subsection are the papers [34] and [30]. As [34] treats the many-to-few lemmata in their most general form, [30] gives an application of certain results from [34] for so-called branching random walks in random environment, which we can easily adapt to our setting. Furthermore, for our purposes, it suffices to state the many-to-few lemmata up to the second moments (i.e. *many-to-one* and *many-to-two formula*).

Let $\xi = (\xi(x))_{x\in\mathbb{R}} = (\xi(x,\omega))$, $\omega \in \Omega$, be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}$-a.s. continuous, nonnegative and bounded paths. Further, let $\mathtt{P}_x^\xi$, be the probability measure of a BBMRE with offspring distribution $(p_n)_{n\in\mathbb{N}}$, starting in $x \in \mathbb{R}$. For the offspring distribution, we assume the following assumptions on the moments:

$$m_1 = \sum_{k=1}^{\infty} k p_k > 1, \quad m_2 := \sum_{k=1}^{\infty} k^2 p_k \in (1,\infty). \tag{2.2.2}$$

Proposition 2.15. *Let φ_1, $\varphi_2 : [0,\infty) \to [-\infty,\infty]$ be càdlàg functions satisfying $\varphi_1 \leq \varphi_2$. Then $\mathbb{P}$-a.s., the first and second moments of the number of particles in $N(t)$ whose genealogy* *stays between* φ_1 *and* φ_2 *in the time interval* $[0,t]$ *are given by*

$$\begin{aligned}\mathtt{E}_x^{\xi}\Big[\big|\{\mathbf{u}\in N(t):\varphi_1(s)\le X_s^{\mathbf{u}}\le\varphi_2(s)\ \forall s\in[0,t]\}\big|\Big]\\ = E_x\Big[e^{(m_1-1)\int_0^t\xi(B_r)\,\mathrm{d}r};\varphi_1(s)\le B_s\le\varphi_2(s)\ \forall s\in[0,t]\Big]\end{aligned}\tag{Mom1}$$

and

$$\begin{aligned}&\mathtt{E}_x^{\xi}\Big[\big|\mathbf{u}\in N(t):\varphi_1(s)\le X_s^{\mathbf{u}}\le\varphi_2(s)\ \forall s\in[0,t]\big|^2\Big]\\ &= E_x\Big[e^{(m_1-1)\int_0^t\xi(B_r)\,\mathrm{d}r};\varphi_1(s)\le B_s\le\varphi_2(s)\ \forall s\in[0,t]\Big]\\ &+(m_2-m_1)\int_0^t E_x\Big[e^{(m_1-1)\int_0^s\xi(B_r)\,\mathrm{d}r}\,\xi(B_s)\mathbb{1}_{\{\varphi_1(r)\le B_r\le\varphi_2(r)\ \forall 0\le r\le s\}}\\ &\times\Big(E_y\big[e^{(m_1-1)\int_0^{t-s}\xi(B_r)\,\mathrm{d}r}\,\mathbb{1}_{\{\varphi_1(r+s)\le B_r\le\varphi_2(r+s)\ \forall 0\le r\le t-s\}}\big]\Big)^2_{|y=B_s}\Big]\,\mathrm{d}s,\end{aligned}\tag{Mom2}$$

respectively.

Here and for the remainder of the thesis we use the notation $E_x[f;\cdot]=E_x[f\mathbb{1}_{\{\cdot\}}]$ and $\mathtt{E}_x^{\xi}[g;\cdot]=\mathtt{E}_x^{\xi}[g\mathbb{1}_{\{\cdot\}}]$ for suitable functions f,g.

Proof of Proposition 2.15. The proof is basically an adaptation of the proof of [30, Theorem 2.1]. For $\mathbf{u}\in N(t)$ we define $B^{\mathbf{u}}$ as the birth time (i.e. the death time of its parent $\mathbf{v}\in\mathbf{K}$ such that $\mathbf{u}=\mathbf{v}k$ for some $k\in\mathbb{N}$) and $T^{\mathbf{u}}$ as the death time of $\mathbf{u}$, and abbreviate $B^{\mathbf{u}}(t):=B^{\mathbf{u}}\wedge t$ and $T^{\mathbf{u}}(t):=T^{\mathbf{u}}\wedge t$. To apply [34, Lemma 1] for the cases $k=1,2$ therein, we have to define new probability measures $\mathtt{Q}_x^{(i)}(\cdot)$, $i=1,2$, as follows:

i) We start with one particle at x, carrying i marks (as well as their positions).

ii) We denote $\psi_t^j\in N(t)$ as the particle carrying mark j, and Z_t^j as its position at time t. The particles carrying marks are called *spines*. For $i=1$ there is one spine for all $t\ge0$ and for $i=2$ there is one spine at time 0 (i.e. $Z_0^1=Z_0^2$) and at most two spines for all times $t>0$.

iii) A particle at position y carrying j marks diffuses as a standard Brownian motion, branches at rate $m_j\cdot\xi(y)$ and is replaced by k offspring particles with probability $p_k^{(j)}:=\frac{k^jp_k}{m_j}$, $j=1,2$, $k\in\mathbb{N}$. The number of offspring particles for each particle is independent of everything else.

iv) At a branching event of a particle carrying j marks, each mark independently and uniformly at random chooses one of the offspring particles to follow.

v) Particles not carrying any marks behave as under $\mathtt{P}_x^{\xi}$.

The existence of the measures $\mathtt{Q}_x^{(i)}(\cdot)$, $i=1,2$, is proven in [34, Section 5]. We write $\mathtt{E}_x^{(i)}[\cdot]$, $i=1,2$, as the corresponding expectation. The set of particles carrying at least one mark up to time t is called *skeleton* at time t. It is denoted by $\mathrm{skel}(t)$. Further we define $D(\mathbf{v})\in\{1,2\}$ to be the number of marks carried by particle $\mathbf{v}\in\mathrm{skel}(t)$. Let us abbreviate

$$A(t):=\exp\Big\{\sum_{\mathbf{v}\in\mathrm{skel}(t)}\int_{B^{\mathbf{v}}(t)}^{T^{\mathbf{v}}(t)}(m_{D(\mathbf{v})}-1)\xi(X_r^{\mathbf{v}})\,\mathrm{d}r\Big\},$$

$$Y(\mathbf{u},t):=\mathbb{1}_{\{\varphi_1(s)\le X_s^{\mathbf{u}}\le\varphi_2(s)\ \forall s\le t\}},\ \mathbf{u}\in N(t),$$

and

$$N^{\varphi_1,\varphi_2}(t) := \big\{\mathbf{u} \in N(t) :\ \varphi_1(s) \le X_s^{\mathbf{u}} \le \varphi_2(s)\, \forall s \in [0,t]\big\}.$$

Then applying the general many-to-few lemma [34, Lemma 1] with $\zeta \equiv 1$ and $k = 1,2$ therein, we obtain

$$\begin{aligned}
\mathtt{E}_x^{\xi}\big[|N^{\varphi_1,\varphi_2}(t)|\big] &= \mathtt{E}_x^{\xi}\Big[\sum_{\mathbf{u}\in N(t)} Y(\mathbf{u},t)\Big] = \mathtt{E}_x^{(1)}\Big[A(t)\sum_{\mathbf{v}\in N(t)} Y(\mathbf{v},t)\mathbb{1}_{\{\mathbf{v}=\psi_t^1\}}\Big],\\
\mathtt{E}_x^{\xi}\big[|N^{\varphi_1,\varphi_2}(t)|^2\big] &= \mathtt{E}_x^{\xi}\Big[\sum_{\mathbf{u}_1,\mathbf{u}_2\in N(t)} Y(\mathbf{u}_1,t)Y(\mathbf{u}_2,t)\Big] \\
&= \mathtt{E}_x^{(2)}\Big[A(t)\sum_{\mathbf{v}_1,\mathbf{v}_2\in N(t)} Y(\mathbf{v}_1,t)Y(\mathbf{v}_2,t)\mathbb{1}_{\{\mathbf{v}_i=\psi_t^i\ \forall i=1,2\}}\Big].
\end{aligned} \tag{2.2.3}$$

We see that in the latter display, the quantities in the expectations $\mathtt{E}_x^{(1)}$ and $\mathtt{E}_x^{(2)}$ only contain particles from the skeleton. In the first equation, the skeleton consists of one single spine $\mathbf{v}$ and under $\mathtt{Q}_x^{(1)}(\cdot)$, its position $(Z_t^1)_{t\ge 0}$ diffuses as a standard Brownian motion. The splitting mechanism is irrelevant in this case. We thus have $B^{\mathbf{v}}(t) = 0$ and $T^{\mathbf{v}}(t) = t$ and we get

$$\begin{aligned}
\mathtt{E}_x^{\xi}\big[|N^{\varphi_1,\varphi_2}(t)|\big] &= \mathtt{E}_x^{(1)}\Big[e^{\int_0^t (m_1-1)\xi(Z_r^1)\,\mathrm{d}r}; \varphi_1(s) \le Z_s^1 \le \varphi_2(s)\ \forall s \le t\Big]\\
&= E_x\Big[e^{\int_0^t (m_1-1)\xi(X_r)\,\mathrm{d}r}; \varphi_1(s) \le B_s \le \varphi_2(s)\ \forall s \le t\Big]
\end{aligned}$$

and hence (Mom1). To show (Mom2), we have to make sure that the right-hand side in (Mom2) equals the very last term in (2.2.3), where there are two marks. We condition on whether there is a *splitting* before time t, i.e. the initial particle, carrying both marks, branches *and* the two marks choose different lines to follow. Let $\mathtt{T}$ be the time of the first splitting event. If at a branching event we branch into k particles, this happens with probability $1/k$. Thus at site y, the rate of $\mathtt{T}$ is equal to $m_2\xi(y)\cdot\sum_k(1-\frac{1}{k})p_k^{(2)} = \xi(y)(m_2 - m_1)$. On $\{\mathtt{T} > t\}$ the skeleton consists only of one single spine $\mathbf{v} \in \mathrm{skel}(t)$ until time t, which fulfills $B^{\mathbf{v}}(t) = 0$, $T^{\mathbf{v}}(t) = t$, $D(\mathbf{v}) = 2$ and $(X_s^{\mathbf{v}})_{s\in[0,t]} = (Z_s^1)_{s\in[0,t]}$. Thus we have

$$\mathtt{Q}_x^{(2)}\big(\mathtt{T} > t\,|\,Z^1\big) = \exp\Big\{-\int_0^t (m_2-m_1)\xi(Z_r^1)\,\mathrm{d}r\Big\}$$

and then

$$\begin{aligned}
&\mathtt{E}_x^{(2)}\Big[A(t)\mathbb{1}_{\{\mathtt{T}>t\}}\sum_{\mathbf{v}_1,\mathbf{v}_2\in N(t)} Y(\mathbf{v}_1,t)Y(\mathbf{v}_2,t)\mathbb{1}_{\{\mathbf{v}_i=\psi_t^i\ \forall i=1,2\}}\,\Big|\,Z^1\Big]\\
&\qquad\qquad = e^{\int_0^t (m_1-1)\xi(Z_r^1)\,\mathrm{d}r}\mathbb{1}_{\{\varphi_1(s)\le Z_s^1\le\varphi_2(s)\ \forall s\le t\}}.
\end{aligned}$$

Integrating this with respect to $\mathtt{Q}_x^{(2)}$, we get

$$\begin{aligned}
&\mathtt{E}_x^{(2)}\Big[A(t)\mathbb{1}_{\{\mathtt{T}>t\}}\sum_{\mathbf{v}_1,\mathbf{v}_2\in N(t)} Y(\mathbf{v}_1,t)Y(\mathbf{v}_2,t)\mathbb{1}_{\{\mathbf{v}_i=\psi_t^i\ \forall i=1,2\}}\Big]\\
&\qquad\qquad = \mathtt{E}_x^{(2)}\big[e^{\int_0^t (m_1-1)\xi(Z_r^1)\,\mathrm{d}r}\mathbb{1}_{\{\varphi_1(s)\le Z_s^1\le\varphi_2(s)\ \forall s\le t\}}\big]\\
&\qquad\qquad = E_x\big[e^{\int_0^t (m_1-1)\xi(B_r)\,\mathrm{d}r}\mathbb{1}_{\{\varphi_1(s)\le B_s\le\varphi_2(s)\ \forall s\le t\}}\big].
\end{aligned} \tag{2.2.4}$$

giving the first summand in (Mom2). For the conditional density, we get

$$\mathtt{Q}_x^{(2)}\big(\mathtt{T}\in \mathrm{d}s\,|\,Z^1\big)=\exp\big\{-\int_0^s (m_2-m_1)\xi(Z_r^1)\,\mathrm{d}r\big\}\xi(Z_s^1)(m_2-m_1)\,\mathrm{d}s,$$

and if $\mathtt{T}\le t$, we have

$$A(t)=\exp\left\{\int_0^{\mathtt{T}}(m_2-1)\xi(Z_r^1)\,\mathrm{d}r\right\}\cdot\prod_{i=1}^2 e^{\int_{\mathtt{T}}^t (m_1-1)\xi(Z_r^i)\,\mathrm{d}r}$$

and

$$\begin{aligned}&\sum_{\mathbf{v}_1,\mathbf{v}_2\in N(t)} Y(\mathbf{v}_1,t)Y(\mathbf{v}_2,t)\mathbb{1}_{\{\mathbf{v}_i=\psi_t^i\ \forall i=1,2\}}\\ &\qquad=\mathbb{1}_{\{\varphi_1(r)\le Z_r^1\le\varphi_2(r)\ \forall r\le\mathtt{T}\}}\prod_{i=1}^2\mathbb{1}_{\{\varphi_1(\mathtt{T}+r)\le Z_{\mathtt{T}+r}^i\le\varphi_2(\mathtt{T}+r)\ \forall r\le t-\mathtt{T}\}}.\end{aligned}$$

Applying the strong Markov property we then get

$$\begin{aligned}&\mathtt{E}_x^{(2)}\Big[A(t)\mathbb{1}_{\{\mathtt{T}\le t\}}\sum_{\mathbf{v}_1,\mathbf{v}_2\in N(t)}Y(\mathbf{v}_1,t)Y(\mathbf{v}_2,t)\mathbb{1}_{\{\mathbf{v}_i=\psi_t^i\ \forall i=1,2\}}\,\Big|\,Z^1\Big]\\ &\quad=\int_0^t e^{\int_0^s (m_1-1)\xi(Z_r^1)\,\mathrm{d}r}(m_2-m_1)\xi(Z_s^{(1)})\mathbb{1}_{\{\varphi_1(r)\le Z_r^1\le\varphi_2(r)\ \forall r\le s\}}\\ &\qquad\times\Big(\mathtt{E}_{Z_s^1}^{\xi}\big[|N^{\varphi_1(\cdot+s),\varphi_2(\cdot+s)}(t-s)|\big]\Big)^2\,\mathrm{d}s.\end{aligned}$$

Integrating the latter expression with respect to $\mathtt{Q}_x^{(2)}$ and using (2.2.4), we see that the very last term in (2.2.3) equals the term on the right-hand side in (Mom2). This completes the proof. □

2.2.3 The McKean representation

Finally, we are in the position to represent the solution to (2.1.1) for a certain class of nonlinearities and initial conditions in terms of a functional of a BBMRE. We also take the chance to get in touch with some model assumptions from Chapters 3 and 4.

To be precise, consider the *randomized Fisher-KPP equation*

$$\begin{aligned}w_t&=\frac{1}{2}w_{xx}+\xi(x,\omega)\cdot F(w),\quad t>0,\ x\in\mathbb{R},\\ w(0,x)&=w_0(x),\quad x\in\mathbb{R},\end{aligned}\tag{F-KPP}$$

where we make the following assumptions:

We assume that $\xi=(\xi(x))_{x\in\mathbb{R}}=(\xi(x,\omega))_{x\in\mathbb{R}}$, $\omega\in\Omega$, is a stochastic process on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, which is bounded away form 0 and infinity as well as Hölder continuous, i.e. $\mathbb{P}$-a.s. we have

$$0<\mathtt{ei}:=\operatorname*{ess\,inf}_{\omega}\xi(x)<\operatorname*{ess\,sup}_{\omega}\xi(x)=:\mathtt{es}<\infty\quad\text{for all } x\in\mathbb{R},,\tag{BDD}$$

where ess inf denotes the essential infimum and ess sup the essential supremum of ξ, and there exist $\alpha = \alpha(\omega) > 0$ and $C = C(\omega) > 0$, such that

$$|\xi(x) - \xi(y)| \leq C \cdot |x - y|^{\alpha} \quad \forall x, y \in \mathbb{R}. \tag{HOEL}$$

Let $(p_k)_{k\in\mathbb{N}}$ be a sequence of real numbers and the function $F = F^{(p_k)_{k\in\mathbb{N}}}$ on $[0,1]$ be as follows:

$$p_k \in [0,1]\ \forall k \in \mathbb{N}, \quad \sum_{k=1}^{\infty} p_k = 1, \quad m_1 = \sum_{k=1}^{\infty} k p_k \equiv 2, \quad m_2 = \sum_{k=1}^{\infty} k^2 p_k < \infty;$$
$$F(u) = 1 - u - \sum_{k=1}^{\infty} p_k (1-u)^k,\ u \in [0,1]. \tag{PROB}$$

Note that $F \in C^1([0,1],[0,1])$ and that F can be extended to a bounded C^1 function on the real line. However, this is not important for our applications, as our choice of initial functions forces the solution to stay in $[0,1]$, see Corollary C.3.

Denoting the solution to (F-KPP) with initial function w_0 by w^{w_0}, let

$$\widetilde{\mathcal{I}}^{\text{smooth}}_{\text{F-KPP}} := \big\{ w_0 \in C(\mathbb{R},[0,1]) : w^{w_0} \in C([0,\infty)\times\mathbb{R},\mathbb{R}) \cap C^{1,2}((0,\infty)\times\mathbb{R},\mathbb{R}) \\ \text{and } w^{w_0} \text{ solves (F-KPP)} \big\}$$

be the class of continuous initial functions such that the corresponding solution to (F-KPP) exists and is classical and let

$$\widetilde{\mathcal{I}}_{\text{F-KPP}} := \big\{ w_0 :\ 0 \leq w_0 \leq 1 \text{ and } \exists (\widetilde{w}_0^{(n)})_{n\in\mathbb{N}} \subset \widetilde{\mathcal{I}}^{\text{smooth}}_{\text{F-KPP}} : \widetilde{w}_0^{(n)} \overset{\text{mon.}}{\to} w_0 \big\}.$$

Here, $\overset{\text{mon.}}{\to}$ denotes pointwise monotone convergence, i.e. $a_n \overset{\text{mon.}}{\to} a$ if $a_n \leq a_{n+1}$ or $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$ and $a_n \underset{n\to\infty}{\longrightarrow} a$. $\widetilde{\mathcal{I}}_{\text{F-KPP}}$ is the class of functions which can be approximated pointwise by a monotonically increasing or decreasing sequence of functions from $\widetilde{\mathcal{I}}^{\text{smooth}}_{\text{F-KPP}}$.

Let us give a few remarks why we use these conditions. Note that by the assumptions in (PROB), F is Lipschitz continuous and bounded on $[0,1]$ with Lipschitz constant 1 (and these properties are fulfilled on the real line by a suitable extension of F). Therefore, Proposition 2.10 gives that there exists a unique bounded generalized solution to (F-KPP), see Definition 2.9 for the notation of generalized solutions. In addition, under the conditions (BDD) and (HOEL), the assumptions from Example 2.12 are fulfilled. Thus, by Proposition 2.13, if the initial function fulfills a suitable Hölder condition, the solution is classical. Consequently, the class of Hölder continuous functions is a subset of $\widetilde{\mathcal{I}}^{\text{smooth}}_{\text{F-KPP}}$ and we have $\widetilde{\mathcal{I}}^{\text{smooth}}_{\text{F-KPP}} \subset \widetilde{\mathcal{I}}_{\text{F-KPP}}$. The definition of $\widetilde{\mathcal{I}}_{\text{F-KPP}}$ is explained in the following remark. Basically, if an initial function w_0 fulfills $w_0 \in \widetilde{\mathcal{I}}_{\text{F-KPP}}$, the corresponding generalized solution w^{w_0} can be approximated by classical solutions. This guarantees the generalized solution to be monotone in its initial function (i.e. $w_0 \leq u_0$ implies $w^{w_0} \leq w^{u_0}$), which is due to the corresponding monotonicity of classical solutions in their initial functions, see Corollary C.2.

Remark 2.16. For initial conditions in $\widetilde{\mathcal{I}}_{\text{F-KPP}}$, generalized solutions can be approximated by classical solutions. Indeed, if $(w_0^{(n)})_{n\in\mathbb{N}} \subset \widetilde{\mathcal{I}}^{\text{smooth}}_{\text{F-KPP}}$ approximates $w_0 \in \widetilde{\mathcal{I}}_{\text{F-KPP}}$ monotonically pointwise, then by Corollary C.2 the corresponding sequence of classical solutions $(w^{(n)})_{n\in\mathbb{N}} = (w^{w_0^{(n)}})_{n\in\mathbb{N}}$ to (F-KPP) is also monotone and thus the limit $w(t,x) := \lim_{n\to\infty} w^{(n)}(t,x)$ exists for all $(t,x) \in [0,\infty)\times\mathbb{R}$. Dominated convergence (note that ξ, w_0 and $w \mapsto \frac{F(w)}{w}$ are bounded) and the fact that $w^{(n)}$ is also a generalized solution (which is due to Proposition 2.10) then imply

$$\begin{aligned} w(t,x) &= \lim_{n\to\infty} w^{(n)}(t,x) = \lim_{n\to\infty} E_x\Big[\exp\Big\{\int_0^t \xi(B_s)\frac{F(w^{(n)}(t-s,B_s))}{w^{(n)}(t-s,B_s)}\,\mathrm{d}s\Big\}\, w_0^{(n)}(B_t)\Big] \\ &= E_x\Big[\exp\Big\{\int_0^t \xi(B_s)\frac{F(w(t-s,B_s))}{w(t-s,B_s)}\,\mathrm{d}s\Big\}\, w_0(B_t)\Big], \end{aligned}$$

i.e. w is the generalized solution to (F-KPP) with initial condition w_0.

As constructed in Section 2.2.1, let $\mathtt{P}_x^\xi$, $x \in \mathbb{R}$, be the probability measure of a BBMRE with offspring distribution $(p_n)_{n\in\mathbb{N}}$, starting in $x \in \mathbb{R}$. Recall further notation for BBMRE from page 26. Then we get the following *McKean representation* of the solution to (F-KPP):

Proposition 2.17. *Let* $w_0 \in \widetilde{\mathcal{I}}_{F\text{-}KPP}$*,* ξ *fulfill* (BDD) *and* (HOEL) *and let* F *fulfill* (PROB)*. Then* $\mathbb{P}$*-a.s., the solution to* (F-KPP) *is given by*

$$w(t,x) = 1 - \mathtt{E}_x^\xi\Big[\prod_{\mathbf{u}\in N(t)}\big(1 - w_0(X_t^{\mathbf{u}})\big)\Big], \quad (t,x)\in[0,\infty)\times\mathbb{R}. \tag{McKean}$$

Proof. We divide the proof into two parts, one in which w_0 is assumed to be continuous and one in which we have $w_0 \in \widetilde{\mathcal{I}}_{\text{F-KPP}}$.

Let w_0 be continuous and $w_0(x) \in [0,1]$ for all $x \in \mathbb{R}$. We abbreviate $\partial_x^2 := \frac{\partial^2}{\partial x^2}$, define $u := 1 - w$ and $u_0 := 1 - w_0$. Then by Corollary 2.7, the function $u^{(1)}(t,x) = E_x\big[e^{-\int_0^t \xi(B_r)\,\mathrm{d}r}u_0(B_t)\big]$ is the unique classical solution to $u_t^{(1)}(t,x) = \frac{1}{2}u_{xx}^{(1)}(t,x) - \xi(x)u^{(1)}(t,x)$ fulfilling $u^{(1)}(0,x) = u_0(x)$. Furthermore, by the same argument, for all $s < t$ and $k \in \mathbb{N}$ fixed, the function

$$u^{(2)}(t,x;s,k) := E_x\Big[\xi(B_{t-s})e^{-\int_0^{t-s}\xi(B_r)\,\mathrm{d}r}u(s,B_{t-s})^k\Big]$$

is the unique classical solution to $u_t^{(2)}(t,x;s,k) = \frac{1}{2}u_{xx}^{(2)}(t,x;s,k) - \xi(x)u^{(2)}(t,x;s,k)$. By dominated convergence, taking advantage of the uniform continuity of the first and second order derivatives, for every fixed $t' \le t$, it is allowed to interchange the limits to obtain the identities

$$\begin{gathered} \sum_k p_k \int_0^{t'} \frac{\partial}{\partial t}u^{(2)}(t,x;s,k)\,\mathrm{d}s = \frac{\partial}{\partial t}\sum_k p_k \int_0^{t'} u^{(2)}(t,x;s,k)\,\mathrm{d}s, \\ \sum_k p_k \int_0^{t'} \big(\frac{1}{2}\partial_x^2 - \xi(x)\big)u^{(2)}(t,x;s,k)\,\mathrm{d}s = \big(\frac{1}{2}\partial_x^2 - \xi(x)\big)\sum_k p_k\int_0^{t'} u^{(2)}(t,x;s,k)\,\mathrm{d}s. \end{gathered} \tag{2.2.5}$$

Conditioning on the first splitting time of the initial particle and on how many children are born, we have

$$\begin{aligned} u(t,x) &= E_x\big[e^{-\int_0^t \xi(B_r)\,\mathrm{d}r}u_0(B_t)\big] + \sum_k p_k \int_0^t E_x\Big[\xi(B_s)e^{-\int_0^s \xi(B_s)}u(t-s,B_s)^k\Big]\,\mathrm{d}s \\ &= u^{(1)}(t,x) + \sum_k p_k \int_0^t E_x\Big[\xi(B_{t-s})e^{-\int_0^{t-s}\xi(B_r)\,\mathrm{d}r}u(s,B_{t-s})^k\Big]\,\mathrm{d}s, \end{aligned}$$

where we used the Markov property of the process $(B_t)_{t\geq 0}$. Then for $h \neq 0$,

$$\begin{aligned}\frac{1}{h}\big(u(t+h,x)-u(t,x)\big) &= \frac{1}{h}\big(u^{(1)}(t+h,x)-u^{(1)}(t,x)\big)\\ &\quad+\sum_k p_k E_x\Big[\frac{1}{h}\int_t^{t+h}\xi(B_{t+h-s})e^{-\int_0^{t+h-s}\xi(B_r)\,\mathrm{d}r}u(s,B_{t+h-s})^k\,\mathrm{d}s\Big]\\ &\quad+\sum_k p_k\int_0^t\frac{1}{h}\big(u^{(2)}(t+h,x;s,k)-u^{(2)}(t,x;s,k)\big)\,\mathrm{d}s.\end{aligned}$$

Invoking once again the arguments from the beginning of the proof, as h tends to zero, the first summand converges to $\frac{1}{2}u^{(1)}_{xx}(t,x)-\xi(x)u^{(1)}(t,x)$ and by dominated convergence, the second summand converges to $\xi(x)\sum_k p_k u(t,x)^k$. For the third term, we observe that the integrand is uniformly bounded since $u^{(2)}_t$ is continuous. Again, due to dominated convergence and (2.2.5), the latter term converges to

$$\big(\frac{1}{2}\partial_x^2-\xi(x)\big)\sum_k p_k\int_0^t u^{(2)}(t,x;s,k)\,\mathrm{d}s.$$

Combining these observations, we arrive at

$$\begin{aligned}u_t(t,x) &= u^{(1)}_t(t,x)+\xi(x)\sum_k p_k u(t,x)^k+\big(\frac{1}{2}\partial_x^2-\xi(x)\big)\sum_k p_k\int_0^t u^{(2)}(t,x;s,k)\,\mathrm{d}s\\ &= \big(\frac{1}{2}\partial_x^2-\xi(x)\big)u(t,x)+\xi(x)\sum_k p_k u(t,x)^k,\end{aligned}$$

which is equivalent to w being a solution to (F-KPP). Thus, we have shown the claim for continuous w_0.

To show that the function w in (McKean) is the unique (probably generalized) solution to (F-KPP) for every $w_0\in\widetilde{\mathcal{I}}_{\text{F-KPP}}$, let $(w_0^{(n)})_{n\in\mathbb{N}}\subset\widetilde{\mathcal{I}}^{\text{smooth}}_{\text{F-KPP}}$ be a monotone sequence approximating w_0. Then by the step 1), the corresponding McKean representations $w^{(n)}=w^{w_0^{(n)}}$, $n\in\mathbb{N}$, are classical solutions to (F-KPP) and converge pointwise to w by dominated convergence (note that $w_0^{(n)}\in[0,1]$ for all $n\in\mathbb{N}$). On the other side, by Remark 2.16, the limit of classical solutions is the unique (generalized) solution. Thus the McKean presentation is the unique (generalized) solution to (F-KPP) for all $w_0\in\mathcal{I}_{\text{F-KPP}}$. □

Let us discuss an application of Proposition 2.17.

Remark 2.18. In Chapters 3 and 4 we frequently use Proposition 2.17 for the function $w_0=\mathbb{1}_{(-\infty,0]}$, resulting in

$$\begin{aligned}w(t,x) &= 1-\mathtt{E}_x^{\xi}\Big[\prod_{\mathbf{u}\in N(t)}\mathbb{1}_{(0,\infty)}(X_t^{\mathbf{u}})\Big]=1-\mathtt{P}_x^{\xi}\big(X_t^{\mathbf{u}}>0\ \forall\mathbf{u}\in N(t)\big)\\ &= \mathtt{P}_x^{\xi}\big(N^{\leq}(t,0)\geq 1\big).\end{aligned}\tag{2.2.6}$$

Indeed, although the function $\mathbb{1}_{(-\infty,0]}$ is not smooth and we can't expect a continuous solution to (F-KPP), the function can be approximated by a monotonically decreasing sequence of functions $(w_0^{(n)})_{n\in\mathbb{N}}$ such as

$$w_0^{(n)}(x) := \begin{cases} 1, & x \le 0, \\ e^{\frac{x}{nx-1}}, & 0 < x < \frac{1}{n}, \\ 0, & x \ge \frac{1}{n}. \end{cases}$$

Because all these functions are smooth, by Proposition 2.13 the corresponding solutions to (F-KPP) are classical. Thus $\mathbb{1}_{(-\infty,0]} \in \widetilde{\mathcal{I}}_{\text{F-KPP}}$.

Remark 2.19. Let us recall what we already mentioned in Section 1.2. Proposition 2.17 slightly differs from the McKean representation in homogeneous branching environment, used by Bramson [13]. More precisely, for $\xi \equiv c$ being a constant function and $w(0,\cdot) = \mathbb{1}_{(-\infty,0]}$, the canonical representation is given by $w(t,x) = \mathtt{P}_0^c(N^{\ge}(t,x) \ge 1)$. This representation follows from Proposition 2.17 using the symmetry $\mathtt{P}_x^c(N^{\le}(t,0) \ge 1) = \mathtt{P}_0^c(N^{\ge}(t,x) \ge 1)$ which is a consequence of the reflection symmetry of the Brownian motion and the homogeneity of the environment. Thus $w(t,\cdot)$ is the tail of the distribution function of the rightmost particle of a BBM at time t with branching rate c, starting in the origin. For non-homogeneous potential, however, this interpretation fails to hold due to non-symmetry of the BBM. In this case, we have to work with the presentation from (2.2.6).

2.3 Summary

Before we turn our attention to the actual content of this thesis in the next two chapters, we give a short summary of the last two sections. As we have already seen, there are several connections between the theory of PDEs, functionals of Brownian motion and those of BBMRE. Let us summarize these links and provide the formulas we frequently use. Of course, as it is in the nature of a summary, there may be overlaps with the previous content. Nevertheless, this section serves well as referencing is made easier.

Let us state the theorem about existence, uniqueness and representation of the corresponding solutions to the randomized Fisher-KPP equation

$$\begin{aligned} &w_t(t,x) = \frac{1}{2}w_{xx}(t,x) + \xi(x,\omega)\cdot F(w(t,x)), \quad (t,x)\in(0,\infty)\times\mathbb{R}, \\ &w(0,x) = w_0(x), \quad x\in\mathbb{R}, \end{aligned} \tag{F-KPP}$$

as well as its linearization, the *parabolic Anderson model*,

$$\begin{aligned} &u_t(t,x) = \frac{1}{2}u_{xx}(t,x) + \xi(x,\omega)\cdot u(t,x), \quad (t,x)\in(0,\infty)\times\mathbb{R}, \\ &u(0,x) = u_0(x), \quad x\in\mathbb{R}, \end{aligned} \tag{PAM}$$

with $\xi = (\xi(x,\omega))_{x\in\mathbb{R}}$, $\omega\in\Omega$, a stochastic process on a probability space $(\Omega,\mathcal{F},\mathbb{P})$.

As before, we work under some boundedness and smoothness conditions for ξ. More precisely, $\mathbb{P}$-a.s. we have

$$0 < \mathtt{ei} := \operatorname*{ess\,inf}_{\omega} \xi(x) < \operatorname*{ess\,sup}_{\omega} \xi(x) =: \mathtt{es} < \infty \quad \text{for all } x\in\mathbb{R}, \tag{BDD}$$

where ess inf denotes the essential infimum and ess sup the essential supremum of ξ, and that there exist $\alpha = \alpha(\omega) > 0$ and $C = C(\omega) > 0$, such that

$$|\xi(x) - \xi(y)| \le C \cdot |x-y|^\alpha \quad \forall x, y \in \mathbb{R}. \tag{HOEL}$$

Further, we recall some special nonlinearity, generated by the offspring distribution of the BBMRE. That is, let $(p_k)_{k\in\mathbb{N}}$ be a sequence of real numbers and the function $F = F^{(p_k)_{k\in\mathbb{N}}}$ on $[0,1]$ be as follows:

$$p_k \in [0,1] \; \forall k \in \mathbb{N}, \quad \sum_{k=1}^\infty p_k = 1, \quad m_1 = \sum_{k=1}^\infty k p_k \equiv 2, \quad m_2 = \sum_{k=1}^\infty k^2 p_k < \infty;$$
$$F(u) = 1 - u - \sum_{k=1}^\infty p_k (1-u)^k, \; u \in [0,1]. \tag{PROB}$$

Note that $F \in C^1([0,1],[0,1])$.

Let us also recall a set of initial conditions for the solution to (F-KPP). Denoting the solution to (F-KPP) with initial function w_0 by w^{w_0}, let

$$\widetilde{\mathcal{I}}_{\text{F-KPP}}^{\text{smooth}} := \big\{ w_0 \in C(\mathbb{R},[0,1]) : w^{w_0} \in C([0,\infty)\times\mathbb{R},\mathbb{R}) \cap C^{1,2}((0,\infty)\times\mathbb{R},\mathbb{R}) \text{ and } w^{w_0} \text{ solves (F-KPP)} \big\}$$

be the class of continuous initial functions such that the corresponding solution to (F-KPP) exists and is classical and let

$$\widetilde{\mathcal{I}}_{\text{F-KPP}} := \big\{ w_0 : \; 0 \le w_0 \le 1 \text{ and } \exists (\widetilde{w}_0^{(n)})_{n\in\mathbb{N}} \subset \widetilde{\mathcal{I}}_{\text{F-KPP}}^{\text{smooth}} : \widetilde{w}_0^{(n)} \overset{\text{mon.}}{\to} w_0 \big\}.$$

We recall the probability measure $\mathtt{P}_x^\xi$ and the corresponding expectation $\mathtt{E}_x^\xi$ of a BBMRE.

Proposition 2.20. $\mathbb{P}$-*a.s., let* ξ *satisfy* (BDD) *and* (HOEL). *Then the following hold* $\mathbb{P}$-*a.s.:*

(a) Let $F \in C^1([0,1],\mathbb{R})$ *be such that* $(0,1] \ni w \mapsto \frac{F(w)}{w}$ *is bounded, Lipschitz continuous and continuously extendable to* $w = 0$. *Then for every bounded, nonnegative and measurable initial condition* w_0 *there exists a unique generalized solution* w *to* (F-KPP), *i.e.* w *fulfills*

$$w(t,x) = E_x\Big[\exp\Big\{ \int_0^t \xi(B_s) \frac{F\big(w(t-s,B_s)\big)}{w(t-s,B_s)} \,\mathrm{d}s \Big\} w_0(B_t)\Big], \; (t,x) \in [0,\infty)\times\mathbb{R}. \tag{2.3.1}$$

If w_0 *is bounded and Hölder continuous, then the generalized solution is a classical one.*

(b) If F *is given by* (PROB), *then for all* $w_0 \in \widetilde{\mathcal{I}}_{\text{F-KPP}}$ *the solution to* (F-KPP) *is given by* (McKean) *and in the case* $w_0 = \mathbb{1}_{(-\infty,0]}$ *we have*

$$w(t,x) = \mathtt{P}_x^\xi\big(N^{\le}(t,0) \ge 1\big), \quad (t,x) \in [0,\infty)\times\mathbb{R}. \tag{2.3.2}$$

(c) If u_0 *is continuous and of at most polynomial growth (as in* (2.1.6)*), the generalized solution* u *(or: Feynman-Kac formula) to* (PAM), *defined by*

$$u(t,x) = E_x\Big[\exp\Big\{ \int_0^t \xi(B_s)\,\mathrm{d}s \Big\} u_0(B_t)\Big], \; (t,x) \in [0,\infty)\times\mathbb{R}. \tag{2.3.3}$$

is the unique classical solution to (PAM)*.*

(d) Let F fulfill (PROB)*. Then for the case $u_0 = \mathbb{1}_{[a,b]}$, $a, b \in \mathbb{R}$, $a < b$, the generalized solution u to* (PAM) *is given by*

$$u(t,x) = \mathtt{E}^{\xi}_{x}\big[|N(t,[a,b])|\big], \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{2.3.4}$$

and for the case $u_0 = \mathbb{1}_{(-\infty,0]}$ the solution u is given by

$$u(t,x) = \mathtt{E}^{\xi}_{x}\big[N^{\leq}(t,0)\big], \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{2.3.5}$$

Proof. (a) If $w \mapsto \frac{F(w)}{w}$ is bounded and Lipschitz continuous, then for every $T > 0$ and $w \in \mathcal{U}_T$ the function c,

$$[0,T] \times \mathbb{R} \ni (t,x) \mapsto c(t,x,w(t,x)) = \xi(x)\frac{F(w(t,x))}{w(t,x)}$$

fulfills Assumption 2.8. Then by Proposition 2.10, there exists a unique generalized solution to (F-KPP), which, by definition, is given by (2.3.1). Further, the function c fulfills Assumptions 2.8 and 2.11. Thus, if the initial condition w_0 is bounded, nonnegative and Hölder continuous, then by Proposition 2.13, the generalized solution is a classical one, i.e. it solves (F-KPP).

(b) The first part of the claim follows from Proposition 2.17. Furthermore, as was explained in Remark 2.18, we have $\mathbb{1}_{(-\infty,0]} \in \widetilde{\mathcal{I}}_{\text{F-KPP}}$ and an application of Proposition 2.17 gives that (2.3.2) is the unique (generalized) solution to (F-KPP) with initial condition $\mathbb{1}_{(-\infty,0]}$.

(c) The claim follows from Corollary 2.7.

(d) As shown in Remark 2.18, we have $\mathbb{1}_{(-\infty,0]} \in \widetilde{\mathcal{I}}_{\text{F-KPP}}$. A similar argument gives $\mathbb{1}_{[a,b]} \in \widetilde{\mathcal{I}}_{\text{F-KPP}}$ for all $a, b \in \mathbb{R}$ such that $a < b$. Applying (2.3.3) and using the many-to-one formula (Mom1) from Proposition 2.15, we can conclude. $\square$

Chapter Three

Log-distance and invariance principles for the fronts of F-KPP and PAM

The rest of the thesis deals with the randomized *Fisher-KPP equation*

$$\begin{aligned} w_t(t,x) &= \frac{1}{2} w_{xx}(t,x) + \xi(x,\omega)\cdot F\big(w(t,x)\big), \quad (t,x)\in(0,\infty)\times\mathbb{R}, \\ w(0,x) &= w_0(x), \quad x\in\mathbb{R}, \end{aligned} \tag{F-KPP}$$

as well as its linearization, the *parabolic Anderson model*,

$$\begin{aligned} u_t(t,x) &= \frac{1}{2} u_{xx}(t,x) + \xi(x,\omega)\cdot u(t,x), \quad (t,x)\in(0,\infty)\times\mathbb{R}, \\ u(0,x) &= u_0(x), \quad x\in\mathbb{R}. \end{aligned} \tag{PAM}$$

In order to be able to summarize our results, we introduce some notation. Let $\varepsilon\in(0,1)$ and $M>0$. Furthermore, write $w=w^{\xi,F,w_0}$ for the solution to (F-KPP) with initial condition w_0 and nonlinearity F and $u=u^{\xi,u_0}$ for the solution to (PAM) with initial condition u_0. We denote the fronts of the respective solutions by

$$\begin{aligned} \overline{m}^{u_0,M}(t) &:= \overline{m}^{\xi,u_0,M}(t) := \sup\big\{x\in\mathbb{R} : u(t,x)\geq a\big\}, \\ m^{F,w_0,\varepsilon}(t) &:= m^{\xi,F,w_0,\varepsilon}(t) := \sup\big\{x\in\mathbb{R} : w(t,x)\geq \varepsilon\big\}, \end{aligned} \tag{3.0.1}$$

and sometimes use the abbreviations

$$\begin{aligned} \overline{m}(t) &:= \overline{m}^{\xi}(t) := \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},1/2}(t), \\ m(t) &:= m^{\xi,F}(t) := m^{\xi,F,\mathbb{1}_{(-\infty,0]},1/2}(t). \end{aligned} \tag{3.0.2}$$

Our findings are, on the one hand, motivated by the respective results from Section 1.2 for the homogeneous case, which provide information about the position of the fronts of the solutions to the respective equations, and thus their respective backlog as well, see (1.2.3). On the other hand, we investigate the fluctuations of the fronts, similar to Nolen's central limit theorem [54, Theorem 1.4], already mentioned in Section 1.3.2.

Under suitable assumptions, our results are summarized in the following statements:

(a) There exist a constant $C\in(0,\infty)$ and a $\mathbb{P}$-a.s. finite random time $\mathcal{T}(\omega)$ such that for all $t\geq\mathcal{T}(\omega)$,

$$\overline{m}(t)-m(t)\leq C\ln t;$$

see Theorem 3.5 below.

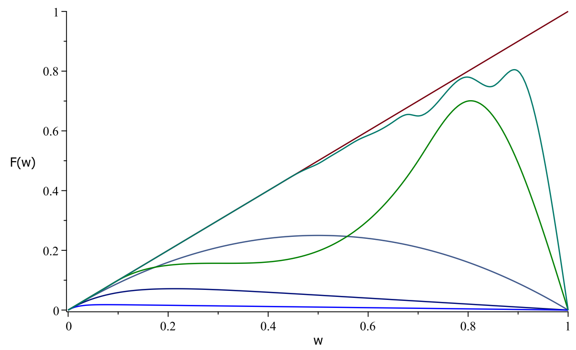

Figure 3.1: Sketches of functions fulfilling (SC), all of which are dominated by the identity function.

(b) After centering and diffusive rescaling, the stochastic processes $[0,\infty) \ni t \mapsto \overline{m}(t)$ and $[0,\infty) \ni t \mapsto m(t)$ fulfill an invariance principle; see Theorem 3.4 and Corollary 3.6 below.

This chapter is taken from the preprint article [19] and for the sake of brevity we refrain from detailed references to the relevant parts of [19].

3.1 Results

In order to be able to precisely formulate the previously summarized results, we have to introduce our model assumptions, some of which we have already seen in Section 2.2.3, where we prove the McKean representation of the solution to (F-KPP).

3.1.1 Precise model assumptions

Let us start with introducing the *standard conditions* for the nonlinearity, i.e., F in (F-KPP) has to fulfill the following:

$$\begin{aligned} &F \in C^1([0,1],[0,1]), \quad F(0)=F(1)=0, \quad F(w)>0 \quad \forall w \in (0,1), \\ &F'(0) = 1 = \sup_{w>0} F(w)w^{-1}, \quad F'(1)<0, \quad \limsup_{w \downarrow 0} \frac{1-F'(w)}{w} < \infty. \end{aligned} \tag{SC}$$

We sometimes need a special kind of nonlinearity, generated by a probability distribution. That is, let $(p_k)_{k\in\mathbb{N}}$ be a sequence of real numbers and the function $F = F^{(p_k)_{k\in\mathbb{N}}}$ on $[0,1]$ be as follows:

$$\begin{aligned} &p_k \in [0,1] \ \forall k \in \mathbb{N}, \quad \sum_{k=1}^{\infty} p_k = 1, \quad m_1 = \sum_{k=1}^{\infty} k p_k \equiv 2, \quad m_2 = \sum_{k=1}^{\infty} k^2 p_k < \infty; \\ &\qquad F(u) = 1 - u - \sum_{k=1}^{\infty} p_k (1-u)^k, \ u \in [0,1]. \end{aligned} \tag{PROB}$$

It is easy to check that every F fulfilling (PROB) also fulfills (SC).

We now specify the classes of initial conditions under consideration for both, (F-KPP) and (PAM). For this purpose, we fix $\delta' \in (0,1)$ and $C' > 1$, and require an initial condition

u_0 of (PAM) to fulfill

$$\delta'\mathbb{1}_{[-\delta',0]} \le u_0 \le C'\mathbb{1}_{(-\infty,0]}. \tag{PAM-INI}$$

In addition, let us introduce a tail condition for the initial condition of (F-KPP), which is the same as the one for the case $\xi \equiv 1$ stated in [12, (1.17)]. For this purpose, additionally to the constant δ' from (PAM-INI), we fix $N, N' > 0$, and require w_0 as in (F-KPP) to fulfill

$$0 \le w_0 \le \mathbb{1}_{(-\infty,0]} \qquad \text{and} \qquad \int_{[x-N',x]} w_0(y)\,\mathrm{d}y \ge \delta' \quad \forall x \le -N. \tag{KPP-INI}$$

As in Section 2.2.3, we first define subclasses for our initial conditions, which guarantee a classical solution, that are

$$\mathcal{I}^{\text{smooth}}_{\text{F-KPP}} := \big\{w_0 : w^{w_0} \in C([0,\infty)\times\mathbb{R},\mathbb{R}) \cap C^{1,2}((0,\infty)\times\mathbb{R},\mathbb{R}) \text{ and } w^{w_0} \text{ solves (F-KPP)}\big\},$$
$$\mathcal{I}^{\text{smooth}}_{\text{PAM}} := \big\{u_0 : u^{u_0} \in C([0,\infty)\times\mathbb{R},\mathbb{R}) \cap C^{1,2}((0,\infty)\times\mathbb{R},\mathbb{R}) \text{ and } u^{u_0} \text{ solves (PAM)}\big\},$$

Then we define the classes of initial conditions as follows.

$$\begin{aligned}\mathcal{I}_{\text{F-KPP}} := \mathcal{I}_{\text{F-KPP}}(\delta') := \big\{w_0 : w_0 &\text{ fulfills (KPP-INI) and} \\ &\exists (w_0^{(n)})_{n\in\mathbb{N}} \subset \mathcal{I}^{\text{smooth}}_{\text{F-KPP}} : w_0^{(n)} \overset{\text{mon.}}{\to} w_0\big\},\\ \mathcal{I}_{\text{PAM}} := \mathcal{I}_{\text{PAM}}(\delta', C') := \big\{u_0 : u_0 &\text{ fulfills (PAM-INI) and} \\ &\exists (u_0^{(n)})_{n\in\mathbb{N}} \subset \mathcal{I}^{\text{smooth}}_{\text{PAM}} : u_0^{(n)} \overset{\text{mon.}}{\to} u_0\big\}.\end{aligned}$$

An emblematic example which is contained in both, $\mathcal{I}_{\text{F-KPP}}$ and $\mathcal{I}_{\text{PAM}}$, is the function $\mathbb{1}_{(-\infty,0]}$ of Heaviside type, as we already saw in Remark 2.18. We also have that the class of nonnegative bounded Hölder continuous functions is a subset of $\mathcal{I}^{\text{smooth}}_{\text{F-KPP}}$ by Proposition 2.13 and that the class of bounded nonnegative continuous functions is a subset of $\mathcal{I}^{\text{smooth}}_{\text{PAM}}$ by Corollary 2.7.

It remains to specify the assumptions on ξ. We assume $\xi = (\xi(x))_{x\in\mathbb{R}} = (\xi(x,\omega))_{x\in\mathbb{R}}$, $\omega \in \Omega$, to be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, having Hölder continuous paths, i.e. $\mathbb{P}$-a.s. there exists $\alpha = \alpha(\omega) > 0$ and $C = C(\omega) > 0$, such that

$$|\xi(x) - \xi(y)| \le C \cdot |x-y|^\alpha \quad \forall x, y \in \mathbb{R}. \tag{HÖL}$$

and such that the following conditions are fulfilled:

- ξ is *uniformly bounded away from* 0 *and* ∞: $\mathbb{P}$-a.s. we have

$$0 < \mathtt{ei} := \operatorname*{ess\,inf}_{\omega} \xi(x) < \operatorname*{ess\,sup}_{\omega} \xi(x) =: \mathtt{es} < \infty \quad \text{ for all } x \in \mathbb{R}; \tag{BDD}$$

- ξ is *stationary*: For every $h \in \mathbb{R}$ we have

$$(\xi(x))_{x\in\mathbb{R}} \overset{d}{=} (\xi(x+h))_{x\in\mathbb{R}}, \tag{STAT}$$

  i.e. the finite dimensional distributions of both processes in (STAT) coincide.

- ξ fulfills a ψ*-mixing* condition: Let $\mathcal{F}_x := \sigma(\xi(z) : z \le x)$ and $\mathcal{F}^y := \sigma(\xi(z) : z \ge y)$,

$x, y \in \mathbb{R}$ and assume that there is a continuous, non-increasing function $\psi : [0,\infty) \to [0,\infty)$, such that for all $j \leq k$ as well as $X \in \mathcal{L}^1(\Omega, \mathcal{F}_j, \mathbb{P})$ and $Y \in \mathcal{L}^1(\Omega, \mathcal{F}^k, \mathbb{P})$ we have $\mathbb{P}$-a.s.

$$\begin{aligned} \big|\mathbb{E}\big[X - \mathbb{E}[X] \,|\, \mathcal{F}^k\big]\big| &\leq \mathbb{E}[|X|] \cdot \psi(k-j), \\ \big|\mathbb{E}\big[Y - \mathbb{E}[Y] \,|\, \mathcal{F}_j\big]\big| &\leq \mathbb{E}[|Y|] \cdot \psi(k-j), \\ \sum_{k=1}^{\infty} \psi(k) &< \infty. \end{aligned} \tag{MIX}$$

Note that (MIX) implies the ergodicity of ξ with respect to the shift operator θ_y acting via $\xi(\cdot) \circ \theta_y = \xi(\cdot + y)$, $y \in \mathbb{R}$.

Let us make a few remarks about our model assumptions. The first remark considers the mixing condition (MIX).

Remark 3.1. ψ-mixing is usually defined in the following manner. Set

$$\widetilde{\psi}(\mathcal{G}_1, \mathcal{G}_2) := \sup_{A \in \mathcal{G}_1^*, B \in \mathcal{G}_2^*} \left| \frac{\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(A)\mathbb{P}(B)} \right|,$$

where $\mathcal{G}_1, \mathcal{G}_2$ are sub-σ-algebras of $\mathcal{F}$ and $\mathcal{G}_i^* := \{A \in \mathcal{G}_i : \ \mathbb{P}(A) > 0\}$, $i = 1, 2$, as well as

$$\sup_j \widetilde{\psi}(\mathcal{F}_j, \mathcal{F}^{j+k}) =: \widetilde{\psi}(k), \quad k \geq 0. \tag{3.1.1}$$

Then ξ is called ψ-mixing if ξ is stationary and $\widetilde{\psi}(k) \to 0$ as $k \to \infty$, which by simple arguments, see Lemma A.1, corresponds to $\psi(k) \to 0$, where ψ is as in (MIX).

In Section 1.3, we mentioned Nolen's paper [54], which demands a ρ-mixing condition (1.3.5) for the potential, similar to (MIX). However, due to Remark 3.1 and [11, (1.12)], Nolen's condition (1.3.5) is weaker than our ψ-mixing condition (MIX).

In the next remark we take a look at our nonlinearity F and our initial conditions.

Remark 3.2. Note that the condition $F'(0) = 1$ in (SC) can be replaced by $F'(0) > 0$. Indeed, in this case we would define $\widehat{F}(w) := F(w)/F'(0)$, $\widehat{\xi}(x) := \xi(x) \cdot F'(0)$, and arrive at the same equation (F-KPP). The statements about (PAM) would still be valid by defining $\widehat{\texttt{es}} := \texttt{es} \cdot F'(0)$, $\widehat{\texttt{ei}} := \texttt{ei} \cdot F'(0)$ and suitable $\widehat{\psi}$, which makes the assumptions (BDD), (STAT) and (MIX) hold for the corresponding quantities. Additionally, note that we assumed in (PAM-INI) and (KPP-INI) our initial conditions to vanish for $x > 0$. This can be weakened in the sense that we can demand the initial condiditons to vanishes for $x > b$, where $b > 0$ is fixed. Indeed, if u^{ξ,u_0} solves (PAM) for $u_0 \in \mathcal{I}_{\text{PAM}}$, then $\widehat{u}(t,x) := u(t, x-b)$ solves (PAM) with data $\widehat{\xi}(x) := \xi(x-b)$ and $\widehat{u}_0(x) := u_0(x-b)$ instead of ξ and u_0. Furthermore, $\widehat{\xi}$ still fulfills (BDD), (STAT) and (MIX), while $\widehat{u}_0(x)$ vanishes for $x > b$. We even expect our results to hold for initial conditions that decay sufficiently fast at infinity, as well as for initial conditions that grow towards minus infinity with sufficiently small exponential rate. However, in order to avoid further technical complications we stick to the above set of initial conditions.

Summarizing, we arrive at the following standing assumptions:

We assume conditions (BDD), (STAT) and (MIX) to be fulfilled from now on, if not explicitly mentioned otherwise. (**Standing assumptions**)

As was summarized in Proposition 2.20, under the above model assumptions we get that both (F-KPP) and (PAM) have a unique generalized solution, which are classical solutions if the corresponding initial conditions are nice.

If the nonlinearity F fulfills (PROB), i.e. is generated by the probability distribution of the number of offspring particles of the BBMRE, then the solution to (F-KPP) is given by the McKean representation (McKean), i.e.

$$w(t,x) = 1 - \mathtt{E}_x^\xi\Big[\prod_{\mathbf{u}\in N(t)} \big(1 - w_0(X_t^{\mathbf{u}})\big)\Big], \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

Recall from Proposition 2.20 (c) that the solution to (PAM) can be represented by the Feynman-Kac formula

$$u(t,x) = E_x\Big[\exp\Big\{\int_0^t \xi(B_s)\,\mathrm{d}s\Big\}\,u_0(B_t)\Big]. \tag{3.1.2}$$

Due to the many-to-one formula (Mom1) (with $m_1 = 2$), for $u_0 = \mathbb{1}_{(-\infty,0]}$, the solution to (PAM) can be represented as the expected number of particles of a BBMRE which are to the left of the origin, i.e.

$$u(t,x) = \mathtt{E}_x^\xi\big[N^{\le}(t,0)\big] = E_x\Big[\exp\Big\{\int_0^t \xi(B_s)\,\mathrm{d}s\Big\}; B_t \le 0\Big].$$

3.1.2 An invariance principle for the PAM front

In Section 1.2 in the case of constant ξ, we saw that the linearized equation exhibits a similar behavior as the corresponding nonlinear equation. We thus expect that investigating the solution to (PAM) might also provide some insight into the solution to (F-KPP).

Let us therefore explain the strategy of the proofs. Starting with the first order of the front as a function of time, it turns out useful to consider the so-called *Lyapunov exponent*

$$\Lambda(v) := \lim_{t\to\infty} \frac{1}{t} \ln u(t,vt). \tag{3.1.3}$$

We will see in Proposition 3.7 and Corollary 3.22 that the Lyapunov exponent exists $\mathbb{P}$-a.s. for all $v \in \mathbb{R}$, is non-random, and does not depend on the initial condition in $\mathcal{I}_{\mathrm{PAM}}$. Furthermore, the function $[0,\infty) \ni v \mapsto \Lambda(v)$ is concave, tends to $-\infty$ as $v \to \infty$ and $\Lambda(0) = \mathtt{es}$, where $\mathtt{es}$ is defined in (BDD). $\Lambda(v)$ describes the exponential growth of the solution in the linear regime with speed v. By Proposition 3.7, there exists a unique $v_0 > 0$, such that

$$\Lambda(v_0) = 0,$$

which we call *velocity* or *speed* of the solution to (PAM). Using the properties of the Lyapunov exponent, we immediately infer the first order asymptotics for $\overline{m}$ to $\mathbb{P}$-a.s. satisfy

$$\frac{\overline{m}(t)}{t} \underset{t\to\infty}{\longrightarrow} v_0.$$

It will turn out that our methods only work if we require v_0 to be strictly larger than some "critical" value v_c, defined in Lemma 3.9 (d). Roughly speaking, the condition $v > v_c$ allows us to find a suitable additive tilting parameter in the exponent of the Feynman-Kac representation, see (2.3.3), which depends on v and makes the solution $u(t,x)$ to (PAM)

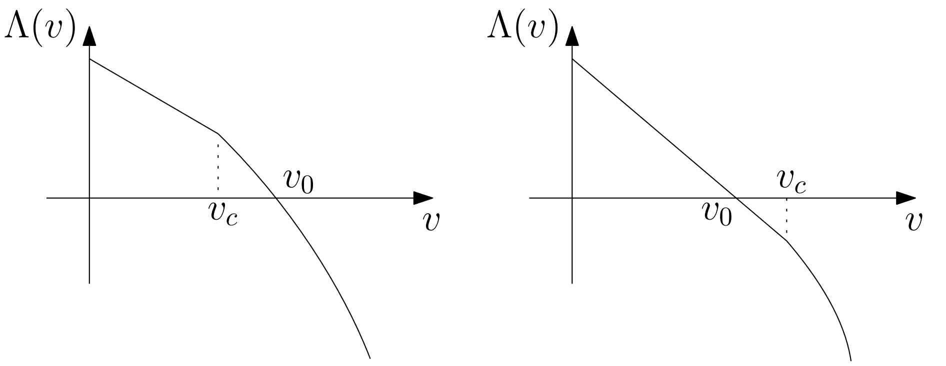

Figure 3.2: Illustration of the Lyapunov exponent from (3.1.3), depending on whether condition (VEL) is fulfilled (left picture) or not (right picture).

amenable to the investigation by standard tools for values $x \approx vt$ and large t. Hence, we work under the assumption

$$v_0 > v_c, \tag{VEL}$$

where v_c is defined in Lemma 3.9 (c). As will be shown in Section 4.3, this assumption is fulfilled for a rich class of potentials ξ.

We start with investigating the fluctuations of $\ln u(t, vt)$ around $t\Lambda(v)$ for values v in a neighborhood of v_0, which are interesting in their own right. To this end, we define a metric ρ on $C([0,\infty))$, the space of continuous functions $f : [0,\infty) \to \mathbb{R}$, making $(C([0,\infty)), \rho)$ a complete separable metric space (see [71, Theorem 2.1] for a proof of the latter claim), that is for $f, g \in C([0,\infty))$ we let $\|f-g\|_j := \sup_{x\in[0,j]} |f(x)-g(x)|$ and

$$\rho(f,g) := \sum_{j=1}^{\infty} 2^{-j} \frac{\|f-g\|_j}{1+\|f-g\|_j}. \tag{3.1.4}$$

Theorem 3.3. *Let $u_0 \in \mathcal{I}_{\mathrm{PAM}}$ and $u = u^{\xi,u_0}$ be the corresponding solution to* (PAM)*. Furthermore, let $V \subset (v_c, \infty)$ be a compact interval such that $v_0 \in \operatorname{int}(V)$. Then for each $v \in V$, as $n \to \infty$ the sequence of random variables $(nv)^{-1/2}\big(\ln u(n, vn) - n\Lambda(v)\big)$, $n \in \mathbb{N}$, converges in $\mathbb{P}$-distribution to a centered Gaussian random variable with variance $\sigma_v^2 \in [0,\infty)$, where σ_v^2 is defined in* (3.3.1)*. If $\sigma_v^2 > 0$, the sequence of processes*

$$[0,\infty) \ni t \mapsto \frac{1}{\sqrt{nv\sigma_v^2}}\big(\ln u(nt, vnt) - nt\Lambda(v)\big), \quad n \in \mathbb{N},$$

converges as $n \to \infty$ in $\mathbb{P}$-distribution to a standard Brownian motion in the sense of weak convergence of measures on $C([0,\infty))$ endowed with the metric ρ from (3.1.4)*.*

In combination with certain perturbation estimates for u, we will use this result in order to infer an invariance principle for the front of the solution to (PAM). Note that since the function $t \mapsto \overline{m}(t)$ may only be càdlàg, we consider convergence in the Skorohod space $D([0,\infty))$ in the following result.

Theorem 3.4. *Let* (VEL) *be fufilled, $u_0 \in \mathcal{I}_{\mathrm{PAM}}$ and $M > 0$. Then for $n \to \infty$, the sequence $n^{-1/2}\big(\overline{m}^{\xi,u_0,M}(n) - v_0 n\big)$, $n \in \mathbb{N}$, converges in $\mathbb{P}$-distribution to a centered Gaussian random variable with variance $\widetilde{\sigma}_{v_0}^2 \in [0,\infty)$, where $\widetilde{\sigma}_{v_0}^2$ is defined in* (3.3.76)*. If $\widetilde{\sigma}_{v_0}^2 > 0$, the sequence* *of processes*

$$[0,\infty) \ni t \mapsto \frac{\overline{m}^{\xi,u_0,M}(nt) - v_0 nt}{\sqrt{n\widetilde{\sigma}_{v_0}^2}}, \quad n \in \mathbb{N},$$

converges as $n \to \infty$ in $\mathbb{P}$-distribution to a standard Brownian motion in the Skorohod space $D([0,\infty))$.

We stress that in the latter theorems the case $\sigma_v^2 = 0$ (resp. $\widetilde{\sigma}_{v_0}^2 = 0$) is allowed and leads to a degenerate limit of the corresponding sequences. This can be excluded, e.g. if the finite-dimensional projections of the stochastic process $\xi = (\xi(x))_{x\in\mathbb{R}}$ are associated (see e.g. [59] or [58] for definitions and results). In this case the covariances in (3.3.1) are nonnegative and $\sigma_v^2 > 0$ follows. [54, Proposition 2.1] provides an example of a potential, which is generated by an i.i.d. sequence of random variables and thus associated.

3.1.3 Log-distance of the F-KPP and PAM front

Coming back to the original equation of interest, it is natural to ask whether we can obtain results for (F-KPP), which are in some sense counterparts to those derived in Section 3.1.2 for (PAM). As already mentioned in (a) on page 37, the next result, which can be considered as the main results of the thesis, states that there is an at most logarithmic distance of the fronts of (F-KPP) and (PAM).

Theorem 3.5. *Let* (VEL) *be fufilled. Then for each F fulfilling* (SC) *there exists a constant $C_1 > 0$ such that the following holds: For all $M > 0$, $\varepsilon \in (0,1)$, $u_0 \in \mathcal{I}_{\text{PAM}}$ and $w_0 \in \mathcal{I}_{\text{F-KPP}}$, there exists a non-random $C = C(\varepsilon, M, u_0, w_0) > 0$ and a $\mathbb{P}$-a.s. finite random time $\mathcal{T} = \mathcal{T}(\xi, \varepsilon, M, u_0, w_0) \geq 0$, such that*

$$-C \leq \overline{m}^{\xi,u_0,M}(t) - m^{\xi,F,w_0,\varepsilon}(t) \leq C_1 \ln t + C \quad \forall t \geq \mathcal{T}. \tag{3.1.5}$$

Moreover for $w_0 = u_0$ and $M \leq \varepsilon$, the left inequality in (3.1.5) is zero for all $t \geq 0$.

Furthermore, combining Theorem 3.4 and Theorem 3.5, we can deduce an invariance principle for the front of (F-KPP) as well.

Corollary 3.6. *Let* (VEL) *be fufilled, F fulfill* (SC)*, $w_0 \in \mathcal{I}_{\text{F-KPP}}$ and $\varepsilon \in (0,1)$. Then as $n \to \infty$, the sequence $n^{-1/2}\big(m^{\xi,F,w_0,\varepsilon}(n) - v_0 n\big)$, $n \in \mathbb{N}$, converges in $\mathbb{P}$-distribution to a centered Gaussian random variable with variance $\widetilde{\sigma}_{v_0}^2 \in [0,\infty)$, where $\widetilde{\sigma}_{v_0}^2$ is defined in* (3.3.76)*. If $\widetilde{\sigma}_{v_0}^2 > 0$, the sequence of processes*

$$[0,\infty) \ni t \mapsto \frac{m^{\xi,F,w_0,\varepsilon}(nt) - v_0 nt}{\sqrt{n\widetilde{\sigma}_{v_0}^2}}, \quad n \in \mathbb{N},$$

converges as $n \to \infty$ in $\mathbb{P}$-distribution to a standard Brownian motion in the Skorohod space $D([0,\infty))$.

3.2 First observations and technical tools

Before we turn our attention to the actual proofs of results from Sections 3.1.2 and 3.1.3, let us first commit ourselves to some notation which we use throughout the remaining part of the thesis.

Notational conventions

We will frequently use sums of real-indexed quantities A_x, $x \in \mathbb{R}$. In this case, we write

$$\sum_{i=1}^{x} A_i := \sum_{i=1}^{\lfloor x \rfloor} A_i + A_x, \quad x \in [0,\infty) \setminus \mathbb{N}_0,$$

where $\sum_{i=1}^{0} := 0$. This notion remains consistent if we also allow for additive constants $b \in \mathbb{R}$, i.e.

$$\sum_{i=1}^{x} (A_i + b) = \sum_{i=1}^{\lfloor x \rfloor} A_i + A_x + b \cdot x = \sum_{i=1}^{x} A_i + \sum_{i=1}^{x} b, \quad x \in [0,\infty).$$

Finally, we set

$$\sum_{i=x+1}^{y} A_i := \begin{cases} \sum_{i=1}^{y} A_i - \sum_{i=1}^{x} A_i, & x \le y, \\ \sum_{i=1}^{x} A_i - \sum_{i=1}^{y} A_i, & x > y, \end{cases} \quad x, y \in [0,\infty).$$

A prime example is the quantity $A_x = \ln E_x\big[e^{\int_0^{H_{\lceil x \rceil - 1}} (\xi(B_s) - \mathtt{es})\,\mathrm{d}s}\big]$, where $H_y := \inf\{t \ge 0 : B_t = y\}$. Indeed, by the strong Markov property we have

$$\ln E_x\big[e^{\int_0^{H_0} (\xi(B_s) - \mathtt{es})\,\mathrm{d}s}\big] = \sum_{i=1}^{\lfloor x \rfloor} A_i + A_x \quad \text{for all } x \in [0,\infty) \setminus \mathbb{N}_0.$$

Furthermore, we will often use positive finite constants $c_1, c_2, \dots$ in the proofs, primarily in large chains of inequalities. This numbering is consistent within any of the proofs, and it is reset after each proof. Furthermore, $C_1, C_2, \dots$ will be used to denote positive finite constants that are fixed throughout the rest of the thesis, and they oftentimes depend on each other. Other constants like $c, C, \varepsilon, \delta$ etc. in the proofs are used to compare certain quantities and are also reset after each proof.

Dependence of an object or a statement G on the environment ξ (and thus on the randomness in $\omega \in \Omega$) is usually abbreviated by the notation $G(\xi)$. The phrase "$\mathbb{P}$-a.s. for all t large enough, ..." abbreviates that "There exists a $\mathbb{P}$-a.s. finite random time $\mathcal{T} = \mathcal{T}(\xi)$, such that for all $t \ge \mathcal{T}$, ...".

This section serves to prepare for the proofs of the main results of the thesis. We will see direct consequences of our model assumptions and introduce important tools that accompany us through the chapter.

3.2.1 The Lyapunov exponent of the solution to PAM

Let us state a first result about the *asymptotic velocity* of the solution to (PAM).

Proposition 3.7. *Let $u = u^{\mathbb{1}_{[x-\delta,x+\delta]}}$ be a solution to* (PAM) *with initial condition $\mathbb{1}_{[x-\delta,x+\delta]}$. Then the limit*

$$\Lambda(v) := \lim_{t\to\infty} \frac{1}{t} \ln u(t, vt), \quad v \in \mathbb{R}, \tag{3.2.1}$$

exists $\mathbb{P}$-a.s., is non-random and independent of $x \in \mathbb{R}$ and $\delta > 0$. The function $[0,\infty) \ni v \mapsto \Lambda(v)$ is concave, $\Lambda(0) = \mathtt{es}$ and $\lim_{v\to\infty} \frac{\Lambda(v)}{v} = -\infty$. In particular, there exists a unique $v_0 > 0$ such that $\Lambda(v_0) = 0$.

Proof. Let $x = 0$, $\lambda \in (0,1)$ and $v_1, v_2 \in \mathbb{R}$, and set $v := \lambda v_1 + (1-\lambda)v_2$ and $A := [(1-\lambda)v_2 t - \delta, (1-\lambda)v_2 t + \delta]$. By (3.1.2), the solution to (PAM) admits the Feynman-Kac representation $u^{\mathbb{1}_{[x-\delta,x+\delta]}}(t, vt) = E_{vt}\big[e^{\int_0^t \xi(B_s)\,\mathrm{d}s}; B_t \in [x-\delta, x+\delta]\big]$. Then by the Markov property, for all $y \in [vt-\delta, vt+\delta]$ we have

$$\begin{aligned} \ln E_y\big[e^{\int_0^t \xi(B_s)\,\mathrm{d}s}; B_t \in [-\delta,\delta]\big] &\geq \ln E_y\big[e^{\int_0^{\lambda t} \xi(B_s)\,\mathrm{d}s}; B_{\lambda t} \in A\big] \\ &\quad + \inf_{z\in A} \ln E_z\big[e^{\int_0^{(1-\lambda)t} \xi(B_s)\,\mathrm{d}s}; B_{(1-\lambda)t} \in [-\delta,\delta]\big]. \end{aligned} \tag{3.2.2}$$

Defining $\overline{\mu}^{\delta}_{s,t}(v) := \inf_{y\in[vt-\delta,vt+\delta]} \ln E_y\big[e^{\int_0^{t-s} \xi(B_r)\,\mathrm{d}r}; B_{t-s} \in [vs-\delta, vs+\delta]\big]$, $s<t$, by the same argument as in the last display one can see that $\overline{\mu}^{\delta}_{s,t}(v) \geq \overline{\mu}^{\delta}_{s,u}(v) + \overline{\mu}^{\delta}_{u,t}(v)$ for all $s<u<t$. Furthermore for the h-lateral shift θ_h on ξ (i.e. $\xi(\cdot)\circ\theta_h = \xi(\cdot + h)$) we have

$$\begin{aligned} \overline{\mu}^{\delta}_{s,t}(v)\circ\theta_h &= \inf_{y\in[vt-\delta,vt+\delta]} \ln E_y\big[e^{\int_0^{t-s} \xi(B_r+h)\,\mathrm{d}r}; B_{t-s} \in [vs-\delta, vs+\delta]\big] \\ &= \inf_{y\in[vt+h-\delta,vt+h+\delta]} \ln E_y\big[e^{\int_0^{t-s} \xi(B_r)\,\mathrm{d}r}; B_{t-s} \in [vs+h-\delta, vs+h+\delta]\big] \\ &= \overline{\mu}^{\delta}_{s+\frac{h}{v},t+\frac{h}{v}}(v) \end{aligned}$$

for every $h \in \mathbb{R}$, $v \neq 0$ and $s<t$. By (BDD) we get

$$\overline{\mu}^{\delta}_{0,t}(v) \geq \mathtt{ei}t + \ln P_0\Big(B_1 \in \big[v\sqrt{t} - 2\delta/\sqrt{t}, v\sqrt{t}\big]\Big)$$

and a Gaussian estimate gives $\inf_{t\geq 1} \overline{\mu}^{\delta}_{0,t}(v)/t \geq -K_{\delta,v}$ for some $K_{\delta,v} \in (0,\infty)$. By (BDD) we also have $\overline{\mu}^{\delta}_{0,t}(v) \leq \mathtt{es}\cdot t$ and thus $\overline{\mu}^{\delta}_{0,t}(v) \in L^1$ for all $t>0$. Thus, Kingman's subadditive ergodic theorem [49, Theorem 1.10] to $(\overline{\mu}^{\delta}_{s,t}(v))_{0<s<t}$ implies that the limit $\Lambda^{\delta}(v) := \lim_{t\to\infty} \frac{1}{t}\overline{\mu}^{\delta}_{0,t}(v)$ exists $\mathbb{P}$-a.s. Furthermore, by (MIX), ξ is mixing and thus ergodic, so the $\Lambda^{\delta}(v)$ is non-random. By standard estimates, the limit is independent of $\delta>0$ and we can exchange $x=0$ by any real number $x \in \mathbb{R}$.

To show the concavity of $v \mapsto \Lambda(v)$, for $v_1, v_2 \geq 0$ and dividing by t, the left-hand side of (3.2.2) converges $\mathbb{P}$-a.s. to $\Lambda(\lambda v_1 + (1-\lambda)v_2)$, while the second term on the right-hand side of (3.2.2) converges $\mathbb{P}$-a.s. to $(1-\lambda)\Lambda(v_2)$. Dividing by t and using (STAT), the first term on the right-hand side of (3.2.2) converges in distribution to the constant $\lambda\Lambda(v_1)$, proving the concavity of $v \mapsto \Lambda(v)$.

By [25, Theorem 7.5.2] we have $\lim_{t\to\infty} \frac{1}{t} \ln u^{\mathbb{1}_{(-\delta,\delta)}}(t,0)\big] = \mathtt{es}$, independent of $\delta>0$, giving $\Lambda(0) = \mathtt{es}$. Due to (BDD) and a standard Gaussian computation, i.e. $\ln P_{vt}(B_t \in [-\delta,\delta]) \leq -\mathrm{const}\cdot v^2 t$ for large t, we have $\lim_{v\to\infty} \frac{\Lambda(v)}{v} = -\infty$. This, together with $\Lambda(0) = \mathtt{es} > 0$ and the concavity of $v \mapsto \Lambda(v)$, implies the existence of a unique $v_0 > 0$, such that $\Lambda(v_0) = 0$. □

We will see that the for $v \geq 0$ the Lyapunov exponent $\Lambda(v)$ exists and is independent of the initial condition $u_0 \in \mathcal{I}_{\mathrm{PAM}}$ (i.e. also for initial conditions with non-compact support as was assumed in Proposition 3.7), see Corollary 3.22.

3.2.2 Change of measure

In this section we introduce a change of measure which turns out to assure the Brownian motion in the Feynman-Kac formula started at tv, some $v \neq 0$, to be close to the origin at time t. For this purpose, let $(\xi(x))_{x\in\mathbb{R}}$ be as in (BDD) and define the shifted potential

$$\zeta := \xi - \texttt{es}.$$

Then $\mathbb{P}$-a.s.,

$$\zeta(x) \in [-(\texttt{es} - \texttt{ei}), 0] \quad \forall x \in \mathbb{R}. \tag{3.2.3}$$

We oftentimes write

$$H_y := \inf\big\{t \geq 0 : B_t = y\big\}, \quad y \in \mathbb{R}, \quad \text{and} \quad \tau_i := H_{i-1} - H_i, \quad i \in \mathbb{Z}, \tag{3.2.4}$$

for the first hitting times and their pairwise differences. Then for $x, y \in \mathbb{R}$ as well as $\eta \leq 0$ define the probability measures $P^{\zeta,\eta}_{x,y}$ via

$$P^{\zeta,\eta}_{x,y}(A) := \frac{1}{Z^{\zeta,\eta}_{x,y}} E_x\Big[\exp\Big\{\int_0^{H_{x-y}} \big(\zeta(B_s)+\eta\big)\,\mathrm{d}s\Big\}; A\Big], \quad A \in \sigma\big(B_{t\wedge H_{x-y}} : t \geq 0\big), \tag{3.2.5}$$

with normalizing constant

$$Z^{\zeta,\eta}_{x,y} := E_x\Big[\exp\Big\{\int_0^{H_{x-y}} \big(\zeta(B_s)+\eta\big)\,\mathrm{d}s\Big\}\Big] \in (0,\infty),$$

For $A \in \sigma\big(B_{t\wedge H_{x-y}} : t \geq 0\big)$, using the strong Markov property at time H_{x-y}, we infer $P^{\zeta,\eta}_{x,y}(A) = P^{\zeta,\eta}_{x,y'}(A)$ for all $y' \geq y$. Thus, by Kolmogorov's extension theorem (see e.g. [70, Theorem 2.4.3]),

$$\big(P^{\zeta,\eta}_{x,y}\big)_{y\geq 0} \text{ can be extended to a unique probability measure } P^{\zeta,\eta}_x \text{ on } \sigma\big(B_t : t \geq 0\big). \tag{3.2.6}$$

We write $E^{\zeta,\eta}_x$ for the corresponding expectation and introduce the logarithmic moment generating functions

$$\begin{aligned} L^\zeta_x(\eta) &:= \ln E_x\Big[\exp\Big\{\int_0^{H_{\lceil x\rceil -1}} \big(\zeta(B_s)+\eta\big)\,\mathrm{d}s\Big\}\Big], \quad x \in \mathbb{R}, \\ \overline{L}^\zeta_x(\eta) &:= \frac{1}{x}\sum_{i=1}^{x} L^\zeta_i(\eta) = \frac{1}{x}\ln E_x\Big[\exp\Big\{\int_0^{H_0} (\zeta(B_s)+\eta)\,\mathrm{d}s\Big\}\Big], \quad x > 0, \end{aligned} \tag{3.2.7}$$

where we recall the notation introduced in Section 3.2, and where the last equality is due to the Markov property. In addition, set

$$L(\eta) := \mathbb{E}\big[L^\zeta_1(\eta)\big]. \tag{3.2.8}$$

Due to (3.2.3), for any $\eta \leq 0$ the quantities above are well-defined, and it is easy to check that in this case and under (BDD), the expressions defined in (3.2.7) – (3.2.8) are finite.

Analytical properties

We have the following useful analytical properties of the latter functions.

Lemma 3.8. *(a) L, L^ζ_x, for $x \in \mathbb{R}$, and $\overline{L}^\zeta_x$, for $x > 0$, are infinitely differentiable on*

$(-\infty, 0)$. *Furthermore, for all* $\eta < 0$ *we have*

$$\big(L_x^\zeta\big)'(\eta) = \frac{E_x\Big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_r)+\eta)\,\mathrm{d}r}H_{\lceil x\rceil-1}\Big]}{E_x\Big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_r)+\eta)\,\mathrm{d}r}\Big]} = E_x^{\zeta,\eta}[\tau_{\lceil x\rceil-1}], \quad x \in \mathbb{R}, \tag{3.2.9}$$

$$\big(\overline{L}_x^\zeta\big)'(\eta) = \frac{1}{x}E_x^{\zeta,\eta}\big[H_0\big], \quad x > 0, \tag{3.2.10}$$

$$L'(\eta) = \mathbb{E}\Big[\frac{E_1\big[e^{\int_0^{H_0}(\zeta(B_r)+\eta)\,\mathrm{d}r}H_0\big]}{E_1\big[e^{\int_0^{H_0}(\zeta(B_r)+\eta)\,\mathrm{d}r}\big]}\Big] = \mathbb{E}\big[E_1^{\zeta,\eta}[H_0]\big], \tag{3.2.11}$$

and

$$\begin{aligned}\big(L_x^\zeta\big)''(\eta) &= \frac{E_x\big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_r)+\eta)\,\mathrm{d}r}H_{\lceil x\rceil-1}^2\big]}{E_x\big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_r)+\eta)\,\mathrm{d}r}\big]} - \left(\frac{E_x\big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_r)+\eta)\,\mathrm{d}r}H_{\lceil x\rceil-1}\big]}{E_x\big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_r)+\eta)\,\mathrm{d}r}\big]}\right)^2\\ &= E_x^{\zeta,\eta}\big[\tau_{\lceil x\rceil-1}^2\big] - \big(E_x^{\zeta,\eta}[\tau_{\lceil x\rceil-1}]\big)^2 = \mathrm{Var}_x^{\zeta,\eta}(\tau_{\lceil x\rceil-1}) > 0, \quad x \in \mathbb{R},\end{aligned} \tag{3.2.12}$$

$$\big(\overline{L}_x^\zeta\big)''(\eta) = \frac{1}{x}\mathrm{Var}_x^{\zeta,\eta}(H_0), \quad x > 0, \tag{3.2.13}$$

$$\begin{aligned}L''(\eta) &= \mathbb{E}\left[\frac{E_1\big[e^{\int_0^{H_0}(\zeta(B_r)+\eta)\,\mathrm{d}r}H_0^2\big]}{E_1\big[e^{\int_0^{H_0}(\zeta(B_r)+\eta)\,\mathrm{d}r}\big]} - \Big(\frac{E_1\big[e^{\int_0^{H_0}(\zeta(B_r)+\eta)\,\mathrm{d}r}H_0\big]}{E_1\big[e^{\int_0^{H_0}(\zeta(B_r)+\eta)\,\mathrm{d}r}\big]}\Big)^2\right]\\ &= \mathbb{E}\Big[E_1^{\zeta,\eta}[H_0^2] - \big(E_1^{\zeta,\eta}[H_0]\big)^2\Big] = \mathbb{E}\big[\mathrm{Var}_1^{\zeta,\eta}(H_0)\big] > 0.\end{aligned} \tag{3.2.14}$$

(b) For each compact interval $\triangle \subset (-\infty, 0)$ *there exists a constant* $C_2 = C_2(\triangle) > 0$, *such that the following inequalities hold* $\mathbb{P}$*-a.s.:*

$$-C_2 \le \inf_{\substack{\eta\in\triangle,\\ x\ge 1}} \big\{L_{\lfloor x\rfloor}^\zeta(\eta), \overline{L}_x^\zeta(\eta), L(\eta)\big\} \le \sup_{\substack{\eta\in\triangle,\\ x\ge 1}} \big\{L_{\lfloor x\rfloor}^\zeta(\eta), \overline{L}_x^\zeta(\eta), L(\eta)\big\} \le -C_2^{-1},$$

$$C_2^{-1} \le \inf_{\substack{\eta\in\triangle,\\ x\ge 1}} \big\{(L_{\lfloor x\rfloor}^\zeta)'(\eta), (\overline{L}_x^\zeta)'(\eta), L'(\eta)\big\} \le \sup_{\substack{\eta\in\triangle,\\ x\ge 1}} \big\{(L_{\lfloor x\rfloor}^\zeta)'(\eta), (\overline{L}_x^\zeta)'(\eta), L'(\eta)\big\} \le C_2,$$

$$C_2^{-1} \le \inf_{\substack{\eta\in\triangle,\\ x\ge 1}} \big\{(L_{\lfloor x\rfloor}^\zeta)''(\eta), (\overline{L}_x^\zeta)''(\eta), L''(\eta)\big\} \le \sup_{\substack{\eta\in\triangle,\\ x\ge 1}} \big\{(L_{\lfloor x\rfloor}^\zeta)''(\eta), (\overline{L}_x^\zeta)''(\eta), L''(\eta)\big\} \le C_2.$$

Proof. (a) Due to the convexity of the exponential function we have $\big|\frac{e^{hx}-1}{h}\big| \le xe^{hx} \vee 1$ for all $x \ge 0$ and $h \in \mathbb{R}$. If we choose $h_0 := \frac{|\eta|}{2}$, then since $\zeta \le 0$, we have that for all $|h| \le h_0$,

$$\Big|\frac{1}{h}e^{\int_0^{H_y}(\zeta(B_r)+\eta)\,\mathrm{d}r}\big(e^{hH_y} - 1\big)\Big| \le e^{\eta H_y/2}(H_y \vee 1). \tag{3.2.15}$$

Due to $H_y e^{\eta H_y} \le \frac{1}{|\eta|}$ for all $H_y \ge 0$ and $\eta < 0$, as well as $\lim_{h\to 0}\frac{e^{hH_y}-1}{h} = H_y e^{hH_y}$ for all $H_y \ge 0$, dominated convergence yields for all $\eta < 0$ that

$$\frac{\mathrm{d}}{\mathrm{d}\eta}E_x\big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_r)+\eta)\,\mathrm{d}r}\big] = E_x\big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_r)+\eta)\,\mathrm{d}r}H_{\lceil x\rceil-1}\big]. \tag{3.2.16}$$

Then (3.2.9) is a consequence of the chain rule and the fact that the expectation on the left-hand side in (3.2.16) is positive. Then (3.2.10) follows from linearity of the derivative.

To show (3.2.11), we have to apply dominated convergence once more. This time, we additionally need that the expectation on the left-hand side in (3.2.16) for $x = 1$ is strictly bounded from below by the constant $E_1[e^{-(\mathtt{es}-\mathtt{ei}-\eta)H_0}] = e^{-\sqrt{2(\mathtt{es}-\mathtt{ei}-\eta)}} > 0$ due to (BDD) and [8, (2.0.1), p. 204]. Using the mean value theorem and (3.2.15) entail the existence of $c_1 > 0$ such that

$$\sup_{|h|\le|\eta|/2}\Big|\frac{1}{h}\big(L_1^\zeta(\eta+h)-L_1^\zeta(\eta)\big)\Big| \le \sup_{|h|\le|\eta|/2}(L_1^\zeta)'(\eta+h) \le c_1E_1\big[e^{\eta H_0/2}(H_0\vee 1)\big] \le c_1\big(\frac{2}{|\eta|}\vee 1\big).$$

Using (3.2.9) for $x = 1$ and dominated convergence, we arrive at (3.2.11).

By induction and similar arguments as above, it follows that for all $n\in\mathbb{N}$, the function $(-\infty,0)\ni\eta\mapsto E_x\big[(H_y)^n e^{\int_0^{H_y}(\zeta(B_r)+\eta)\,\mathrm{d}r}\big]$ is positive $\mathbb{P}$-a.s. and differentiable. Due to (BDD), one can interchange expectation and differentiation in η, yielding (3.2.12) and (3.2.13). The first equality in (3.2.14) is then again a consequence of dominated convergence. The strict inequalities in (3.2.12) and (3.2.14) are due to the fact that under P_1, and thus also $\mathbb{P}$-a.s. under $P_1^{\zeta,\eta}$, the random variable H_0 is non-degenerate.

(b) Observe that the function $\zeta\mapsto E_x\big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_s)+\eta)\,\mathrm{d}s}\big]$ is nondecreasing. Consequently, using the notation $\triangle = [\eta_*,\eta^*]$, we have

$$-\infty < e^{-\sqrt{2(\mathtt{es}-\mathtt{ei}+|\eta_*|)}} \le E_x\big[e^{(\mathtt{ei}-\mathtt{es}+\eta_*)H_{\lceil x\rceil-1}}\big] \le \inf_{\eta\in\triangle}\operatorname*{ess\,inf}_{\zeta} E_x\big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_s)+\eta)\,\mathrm{d}s}\big]$$
$$\le \sup_{\eta\in\triangle}\operatorname*{ess\,sup}_{\zeta} E_x\big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_s)+\eta)\,\mathrm{d}s}\big] \le E_x\big[e^{\eta^* H_{\lceil x\rceil-1}}\big] = e^{(x-\lceil x\rceil+1)\sqrt{2|\eta^*|}},$$

where we used [8, (2.0.1), p. 204] for the second inequality and last equality. Due to the inequality $e^{-xy}x \le \frac{2}{y}e^{-xy/2}$ for all $x\ge 0$ and $y > 0$, these estimates can be used to derive similar bounds for $E_x\big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_r)+\eta)\,\mathrm{d}r}H_{\lceil x\rceil-1}^k\big]$, $k = 1,2$. Thus, estimating the numerator and the denominator of the corresponding expressions in (3.2.9) to (3.2.14), we can conclude. □

Asymptotic properties and Legendre transformation

The next lemma deals with the asymptotic properties of the functions in (3.2.7) and (3.2.8) as well as introduces the Legendre transformation for the function L from (3.2.8) and its properties.

Lemma 3.9. *(a) We have $\mathbb{P}$-a.s. that*

$$\lim_{x\to\infty}\overline{L}_x^\zeta(\eta) = L(\eta) \quad \textit{for all } \eta\le 0. \tag{3.2.17}$$

(b) $L'(\eta)\downarrow 0$ as $\eta\downarrow-\infty$

(c) For every $v > v_c := \frac{1}{L'(0-)}$ (where $\frac{1}{+\infty} := 0$), which we call critical velocity*, there exists a*

$$\textit{unique solution } \overline{\eta}(v) < 0 \textit{ to the equation } L'(\overline{\eta}(v)) = \frac{1}{v}. \tag{3.2.18}$$

$\overline{\eta}(v)$ can be characterized as the unique maximizer to $(-\infty,0] \ni \eta \mapsto \frac{\eta}{v} - L(\eta)$, i.e.

$$\sup_{\eta \le 0} \Big(\frac{\eta}{v} - L(\eta)\Big) = \frac{\overline{\eta}(v)}{v} - L\big(\overline{\eta}(v)\big). \tag{3.2.19}$$

The function $(v_c,\infty) \ni v \mapsto \overline{\eta}(v)$ is continuously differentiable and strictly decreasing.

Proof of Lemma 3.9. (a) By [25, Theorem 7.5.1], for every $\eta \le 0$ we get $\mathbb{P}$-a.s. that $\lim_{x\to\infty} \overline{L}_x^\zeta(\eta) = \mathbb{E}\big[L_1^\zeta(\eta) \,|\, \mathcal{F}_{\text{inv}}^\zeta\big]$, where $\mathcal{F}_{\text{inv}}^\zeta$ is the σ-algebra of all $\mathbb{P}$-invariant sets. Due to our standing assumptions, the family $\zeta(x)$, $x \in \mathbb{R}$, is mixing and thus ergodic. Thus, $\mathcal{F}_{\text{inv}}^\zeta$ is $\mathbb{P}$-trivial, i.e., $\mathbb{E}\big[L_1^\zeta(\eta) \,|\, \mathcal{F}_{\text{inv}}^\zeta\big] = L(\eta)$. By continuity of the functions $\overline{L}_x^\zeta(\cdot)$ and $L(\cdot)$ by Lemma 3.8, the statement follows.

(b) We note that L is strictly increasing and strictly convex on $(-\infty,0)$ by (3.2.11) and (3.2.14) and

$$L(\eta) \ge \mathbb{E}\Big[\ln E_1\big[e^{-(\texttt{es}-\texttt{ei}-\eta)H_0}\big]\Big] = -\sqrt{2(\texttt{es}-\texttt{ei}-\eta)} \quad \text{for all } \eta \le 0,$$

where the equality is due to [8, (2.0.1), p. 204]. Thus, we infer that its derivative $L'(\eta)$ tends to 0 as $\eta \to -\infty$.

(c) Using that $1/v_c > 1/v$ (where $\frac{1}{0} := +\infty$) and the fact that L' is strictly increasing with $L'(\eta) \downarrow 0$ for $\eta \downarrow -\infty$, we can find a unique $\overline{\eta}(v) < 0$ such that $L'(\overline{\eta}(v)) = 1/v$, giving (3.2.18). On the other hand, (3.2.19) is a direct consequence of (3.2.18) and standard properties of the Legendre transformations of strictly convex functions. Because L' is strictly increasing and smooth on $(-\infty,0)$, it has a strictly increasing inverse function $(L')^{-1}$, which is differentiable on $(0,1/v_c)$. By (3.2.18), for $v > v_c$ we have $\overline{\eta}(v) = (L')^{-1}(1/v)$ and thus by the formula for the derivative of the inverse function we get

$$\overline{\eta}'(v) = -\frac{1}{v^2} \cdot \frac{1}{L''(\overline{\eta}(v))}.$$

Because the right-hand side of the latter display is continuous in v and negative, we conclude.

□

We use the standard notation $L^* : \mathbb{R} \to (-\infty,\infty]$ to denote the Legendre transformation

$$v \mapsto \sup_{\eta \le 0} \big(\eta v - L(\eta)\big)$$

of L. Lemma 3.9 entails that

$$L^*(1/v) = \frac{\overline{\eta}(v)}{v} - L\big(\overline{\eta}(v)\big). \tag{3.2.20}$$

In the next part of this section, we are interested in a suitable *tilting parameter* $\eta_x^\zeta(v)$ such that

$$E_x^{\zeta,\eta_x^\zeta(v)}\big[H_0\big] = \frac{x}{v}, \quad x > 0,\ v > 0, \tag{3.2.21}$$

holds true (setting $\eta_x^\zeta(v) := 0$ if no such parameter exists). For $\eta_x^\zeta(v)$ fulfilling (3.2.21) we observe that under $P_x^{\zeta,\eta_x^\zeta(v)}$, the Brownian motion is tilted to have time-averaged velocity

v until it reaches the origin. In Lemma 3.10, we will show that for suitable v and x large enough, a tilting parameter as postulated in (3.2.21) actually exists. Furthermore, we will show that the random parameter $\eta_x^\zeta(v)$ concentrates around the deterministic $\overline{\eta}(v)$ defined in (3.2.18). The last result is a perturbation estimate for $\eta_x^\zeta(v)$ in x, cf. Lemma 3.12.

3.2.3 Concentration inequalities

We have the following result regarding the existence (or, more precisely, negativity) and concentration properties of the postulated parameter $\eta_x^\zeta(v)$.

Lemma 3.10. *(a) For every $v > v_c$ there exists a finite random variable $\mathcal{N} = \mathcal{N}(v)$ such that for all $x \geq \mathcal{N}$ the solution $\eta_x^\zeta(v) < 0$ to* (3.2.21) *exists.*

(b) For each $q \in \mathbb{N}$ and each compact interval $V \subset (v_c, \infty)$, there exists $C_3 := C_3(V, q) \in (0, \infty)$ such that

$$\mathbb{P}\Big(\sup_{v\in V}\sup_{x\in[n,n+1)} |\eta_x^\zeta(v) - \overline{\eta}(v)| \geq C_3\sqrt{\frac{\ln n}{n}}\Big) \leq C_3 n^{-q} \qquad \textit{for all } n \in \mathbb{N}. \tag{3.2.22}$$

Proof. We recall that due to Lemma 3.8, the tilting parameter $\eta_x^\zeta(v)$ can alternatively be characterized as the unique solution $\eta_x^\zeta(v) \in (-\infty, 0)$ to

$$\big(\overline{L}_x^\zeta\big)'(\eta_x^\zeta(v)) = \frac{1}{v}, \tag{3.2.23}$$

if the solution exists, and $\eta_x^\zeta(v) = 0$ otherwise. We start with noting that Part (a) directly follows from Part (b). Indeed, let A_n, $n \in \mathbb{N}$, be the event in the probability on the left-hand side of (3.2.22). Then $\sum_n \mathbb{P}(A_n) < \infty$ for $q \geq 2$. By the first Borel-Cantelli lemma, $\mathbb{P}$-a.s. only finitely many of the A_n occur. In combination with the fact that $\overline{\eta}(v) < 0$, cf. (3.2.18), this implies that $\mathbb{P}$-a.s., the value of $\eta_x^\zeta(v)$ can only vanish for $x > 0$ small enough. In particular, we deduce the existence of a $\mathbb{P}$-a.s. finite random variable $\mathcal{N}$ as postulated.
Hence, it remains to show (3.2.22). For this purpose, in the following lemma we investigate the fluctuations of the functions through which the parameters $\eta_x^\zeta(v)$ and $\overline{\eta}(v)$ are implicitly defined; we will then infer the desired bounds on the fluctuations of the parameters themselves through perturbation estimates for these functions.

Lemma 3.11. *For every compact interval $\triangle \subset (-\infty, 0)$ and each $q \in \mathbb{N}$, there exists a constant $C_4 = C_4(\triangle, q) \in (0, \infty)$ such that*

$$\mathbb{P}\Big(\sup_{\eta\in\triangle}\sup_{x\in[n,n+1)} \Big|\big(\overline{L}_x^\zeta\big)'(\eta) - L'(\eta)\Big| \geq C_4\sqrt{\frac{\ln n}{n}}\Big) \leq C_4 n^{-q} \qquad \textit{for all } n \in \mathbb{N}. \tag{3.2.24}$$

In order not to hinder the flow of reading, we postpone the proof of this auxiliary result to the end of the proof of Lemma 3.10 and finish the proof of Lemma 3.10 (b) first. Let $q \in \mathbb{N}$ and $V \subset (v_c, \infty)$ be a compact interval. By Lemma 3.8, for each compact $\triangle \subset (-\infty, 0)$ we have $\mathbb{P}$-a.s.,

$$\begin{aligned} C_2^{-1} &\leq \inf_{\eta\in\triangle} L''(\eta) \leq \sup_{\eta\in\triangle} L''(\eta) \leq C_2, \\ C_2^{-1} &\leq \inf_{x\geq 1}\inf_{\eta\in\triangle} \big(\overline{L}_x^\zeta\big)''(\eta) \leq \sup_{x\geq 1}\sup_{\eta\in\triangle} \big(\overline{L}_x^\zeta\big)''(\eta) \leq C_2. \end{aligned} \tag{3.2.25}$$

Therefore, and because the function $V \ni v \mapsto \overline{\eta}(v)$ is strictly decreasing by Lemma 3.9, it is possible to find $N = N(V) \in \mathbb{N}$ and a compact interval $\triangle = \triangle(N, V) \subset (-\infty, 0)$, where for notational convenience we write

$$V = [v_*, v^*] \quad \text{and} \quad \triangle = [\eta_*, \eta^*], \tag{3.2.26}$$

such that, using standard calculus for sets,

$$\big(\overline{\eta}(V) - C_4 C_2 \sqrt{\ln n/n}\big) \cup \big(\overline{\eta}(V) + C_4 C_2 \sqrt{\ln n/n}\big) \subset \triangle \quad \text{for all } n \geq N.$$

Let $n \geq N$ and assume that the complement of the event on the left-hand side of (3.2.24),

$$\sup_{\eta \in \triangle} \sup_{x \in [n,n+1)} \big|\big(\overline{L}_x^{\zeta}\big)'(\eta) - L'(\eta)\big| < C_4 \sqrt{\frac{\ln n}{n}}, \tag{3.2.27}$$

occurs. On this event, for all $v \in V$ and all $x \in [n, n+1)$,

$$\big(\overline{L}_x^{\zeta}\big)'\big(\overline{\eta}(v) - C_4 C_2 \sqrt{\ln n/n}\big) \leq \frac{1}{v} \leq \big(\overline{L}_x^{\zeta}\big)'\big(\overline{\eta}(v) + C_4 C_2 \sqrt{\ln n/n}\big) \tag{3.2.28}$$

and thus, due to the strict monotonicity of $(\overline{L}_x^{\zeta})'$ implied by (3.2.25), there exists a unique $\eta_x^{\zeta}(v) \in \triangle$ such that $\big(\overline{L}_x^{\zeta}\big)'(\eta_x^{\zeta}(v)) = 1/v$. Due to (3.2.28), still assuming (3.2.27), we have

$$\sup_{v \in V} \sup_{x \in [n,n+1)} |\overline{\eta}(v) - \eta_x^{\zeta}(v)| \leq C_4 C_2 \sqrt{\frac{\ln n}{n}}.$$

Thus, for $n \geq N$, choosing $C_3 > 2C_4C_2$, the probability in (3.2.22) is bounded by the right-hand side of (3.2.24), which finishes the proof. □

It remains to prove Lemma 3.11.

Proof of Lemma 3.11. Applying the strong Markov property, we get

$$x\big(\overline{L}_x^{\zeta}\big)'(\eta) = E_x^{\zeta,\eta}(H_0) = E_x^{\zeta,\eta}\big[H_{\lfloor x \rfloor}\big] + E_{\lfloor x \rfloor}^{\zeta,\eta}\big[H_0\big] = E_x^{\zeta,\eta}\big[H_{\lfloor x \rfloor}\big] + \lfloor x \rfloor \big(\overline{L}_{\lfloor x \rfloor}^{\zeta}\big)'(\eta).$$

Furthermore, $0 \leq E_x^{\zeta,\eta}\big[H_{\lfloor x \rfloor}\big] = \big(L_x^{\zeta}\big)'(\eta) \leq C_2$ by (3.2.9) and Lemma 3.8 b), and thus also $0 \leq \big(\overline{L}_x^{\zeta}\big)'(\eta) = \frac{1}{x}\sum_{i=1}^{x} \big(L_i^{\zeta}\big)'(\eta) \leq C_2$ for all $x \geq 1$ and all $\eta \in \triangle$, $\mathbb{P}$-a.s. As a consequence, we get that for all $x \geq 1$,

$$\big|\big(\overline{L}_x^{\zeta}\big)'(\eta) - L'(\eta)\big| \leq \big|\big(\overline{L}_{\lfloor x \rfloor}^{\zeta}\big)'(\eta) - L'(\eta)\big| + \frac{2C_2}{\lfloor x \rfloor}.$$

It is therefore enough to prove

$$\mathbb{P}\Big(\sup_{\eta \in \triangle} \Big|\big(\overline{L}_n^{\zeta}\big)'(\eta) - L'(\eta)\Big| \geq C_4 \sqrt{\frac{\ln n}{n}}\Big) \leq C_4 n^{-q} \qquad \text{for all } n \in \mathbb{N}. \tag{3.2.29}$$

For each $\eta \in \triangle$, $((L_i^{\zeta})'(\eta) - L'(\eta))_{i \in \mathbb{Z}}$ is a family of stationary, centered and bounded random variables. Furthermore, they fulfill the exponential mixing condition (A.4) due to Lemma A.2. Since $\sigma((L_i^{\zeta})'(\eta) : i \geq k) \subset \sigma(\xi(x) : x \geq k-1)$ and (MIX), setting $Y_i := (L_i^{\zeta})'(\eta) - L'(\eta)$, the left-hand side in (B.1) is bounded by some constant $c_2 > 0$, uniformly for every i. Then setting $m_i := c_1$, condition (B.1) is fulfilled and we can apply the Hoeffding-type inequality from Corollary B.2 to get $c_2 > 0$ such that

$$\mathbb{P}\Big(\Big|\big(\overline{L}_n^\zeta\big)'(\eta) - L'(\eta)\Big| \geq c_2\sqrt{\frac{\ln n}{n}}\Big) \leq c_2 n^{-q-1} \qquad \text{for all } \eta \in \triangle \text{ and all } n \in \mathbb{N}.$$

Let $\triangle_n := (\triangle \cap \frac{1}{n}\mathbb{Z}) \cup \{\eta_*, \eta^*\}$, recalling the notation of (3.2.26). Because $|\triangle \cap \frac{1}{n}\mathbb{Z}| \leq n \cdot \mathrm{diam}(\triangle) + 1$, taking advantage of the previous display we infer

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\eta\in\triangle_n}\Big|\big(\overline{L}_n^\zeta\big)'(\eta) - L'(\eta)\Big| \geq c_2\sqrt{\frac{\ln n}{n}}\Big)\\
&\leq |\triangle_n| \cdot \sup_{\eta\in\triangle_n} \mathbb{P}\Big(\Big|\big(\overline{L}_n^\zeta\big)'(\eta) - L'(\eta)\Big| \geq c_2\sqrt{\frac{\ln n}{n}}\Big) \leq c_3(\triangle) n^{-q}, \qquad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

By Lemma 3.8 b), we have $\mathbb{P}$-a.s. $\sup_n \sup_{\eta\in\triangle} \big|\big(\overline{L}_n^\zeta(\eta)\big)''\big| \vee \big|L''(\eta)\big| \leq C_2$. Thus, the mean value theorem entails that

$$\sup_{\eta\in\triangle}\big|\big(\overline{L}_n^\zeta\big)'(\eta) - L'(\eta)\big| \leq \sup_{\eta\in\triangle_n}\big|\big(\overline{L}_n^\zeta\big)'(\eta) - L'(\eta)\big| + \frac{2C_2}{n},$$

and thus we find $C_4 > 0$ such that (3.2.29) and hence (3.2.24) hold true. □

In what comes below, many results will implicitly depend on the choice of compact intervals V and $\triangle$, which have already occurred before. Thus, in order to avoid ambiguity and due to assumption (VEL), we now

$$\text{fix arbitrary compact intervals } V \subset (v_c, \infty) \text{ and } \triangle = \triangle(V) \subset (-\infty, 0) \text{ such that } v_0 \in \mathrm{int}(V) \text{ and } \overline{\eta}(V) \subset \mathrm{int}(\triangle). \tag{3.2.30}$$

Furthermore, as a consequence of Lemma 3.10, there exists a $\mathbb{P}$-a.s. finite random variable $\mathcal{N}_1 = \mathcal{N}_1(\xi, C_3(V, 2))$ such that

$$\mathcal{H}_x := \mathcal{H}_x(V) := \big\{\eta_x^\zeta(v) \in \triangle \text{ for all } v \in V\big\} \quad \text{occurs for all } x \geq \mathcal{N}_1. \tag{3.2.31}$$

We write

$$\big(\overline{L}_x^\zeta\big)^*\Big(\frac{1}{v}\Big) = \sup_{\eta<0}\Big(\frac{\eta}{v} - \overline{L}_x^\zeta(\eta)\Big) = \frac{\eta_x^\zeta(v)}{v} - \overline{L}_x^\zeta(\eta_x^\zeta(v)), \quad x \geq 1,$$

for Legendre transformation of the weighted averages. We also recall to set $\eta_x^\zeta(v) = 0$, if there is no solution $\eta_x^\zeta(v) \in \triangle$ to (3.2.23); note that this can only happen on $\mathcal{H}_x^c$.

In order to show an invariance principle for the Legendre transformation $(\overline{L}_x^\zeta)^*$ in the following section, we now derive a perturbation result on the tilting parameter $\eta_x^\zeta(v)$ in x.

Lemma 3.12. *There exists a constant $C_5 > 0$ such that $\mathbb{P}$-a.s., for all $x \in (0, \infty)$ large enough, uniformly in $v \in V$ and $0 \leq h \leq x$,*

$$\big|\eta_x^\zeta(v) - \eta_{x+h}^\zeta(v)\big| \leq C_5\frac{h}{x}. \tag{3.2.32}$$

Proof. By Lemma 3.10 we can choose x large enough such that $\eta_y^\zeta(v) \in \triangle$ for all $y \geq x$ and all $v \in V$. For $h = 0$, the statement is obvious. For $0 < h \leq x$, it suffices to show that there exists $c_1 > 0$ such that

$$\sup_{\eta\in\triangle} \big|\big(\overline{L}_{x+h}^\zeta\big)'(\eta) - \big(\overline{L}_x^\zeta\big)'(\eta)\big| \le c_1\frac{h}{x}. \tag{3.2.33}$$

Indeed, using (3.2.23) we can write

$$\begin{aligned}\big(\overline{L}_{x+h}^\zeta\big)'(\eta_{x+h}^\zeta(v)) - \big(\overline{L}_x^\zeta\big)'(\eta_{x+h}^\zeta(v)) &= \big(\overline{L}_x^\zeta\big)'(\eta_x^\zeta(v)) - \big(\overline{L}_x^\zeta\big)'(\eta_{x+h}^\zeta(v))\\ &= \big(\overline{L}_x^\zeta\big)''(\widetilde{\eta})(\eta_x^\zeta(v) - \eta_{x+h}^\zeta(v))\end{aligned}$$

for some $\widetilde{\eta} \in \triangle$ between $\eta_x^\zeta(v)$ and $\eta_{x+h}^\zeta(v)$. By the second display in (3.2.25) we know that $\mathbb{P}$-a.s. $\inf_{\eta\in\triangle,x\ge1}(\overline{L}_x^\zeta)''(\eta) \ge C_2^{-1}$. Using this, inequality (3.2.32) is a direct consequence of (3.2.33) with $C_5 := c_1C_2$. To prove (3.2.33), recall that for all $\eta \in \triangle$, $x \ge 1$, and $0 < h \le x$, by the strong Markov property applied at time H_x,

$$\begin{aligned}\big(\overline{L}_{x+h}^\zeta\big)'(\eta) - \big(\overline{L}_x^\zeta\big)'(\eta) &= \frac{1}{x+h}\big(E_{x+h}^{\zeta,\eta}[H_x] + E_x^{\zeta,\eta}[H_0]\big) - \frac{1}{x}E_x^{\zeta,\eta}[H_0]\\ &= -\frac{h}{x+h}\big(\overline{L}_x^\zeta\big)'(\eta) + \frac{h}{x+h}\frac{1}{h}E_{x+h}^{\zeta,\eta}[H_x].\end{aligned}$$

Finally, recall that by Lemma 3.8 there exists $C_2 = C_2(\triangle) > 0$ such that $\mathbb{P}$-a.s. we have $\sup_{\eta\in\triangle,x>0}\big|\big(\overline{L}_x^\zeta\big)'(\eta)\big| \le C_2$. By exactly the same argument used for the proof of the latter inequality (see proof of (3.8)), one can show that also $\sup_{\eta\in\triangle,x,h>0}\big|\frac{1}{h}E_{x+h}^{\zeta,\eta}[H_x]\big| \le C_2$ holds $\mathbb{P}$-a.s. with the same constant C_2. (3.2.33) now follows choosing $c_1 := 2C_2$. □

3.3 Large deviations and perturbation results for the PAM

The main objective of this section is to establish certain exact large deviation results for hitting times of Brownian motion under the tilted measures introduced above, and then to apply these in order to obtain perturbation results. For this purpose let

$$\begin{aligned}&V_x^{\zeta,v}(\eta) := \frac{\eta}{v} - L_x^\zeta(\eta), \quad x \in \mathbb{R},\\ &\sigma_v^2 := \mathrm{Var}_{\mathbb{P}}(V_1^{\zeta,v}(\overline{\eta}(v)) + 2\sum_{i=2}^\infty \mathrm{Cov}_{\mathbb{P}}\big(V_1^{\zeta,v}(\overline{\eta}(v)), V_i^{\zeta,v}(\overline{\eta}(v))\big), \quad \sigma_v := \sqrt{\sigma_v^2}, \quad v \in V.\end{aligned} \tag{3.3.1}$$

We start with observing that $\sigma_v^2 \in [0,\infty)$ for all $v \in V$. Indeed, $(\widetilde{L}_i)_{i\in\mathbb{N}}$, where $\widetilde{L}_i := L_i^\zeta(\overline{\eta}(v)) - \mathbb{E}[L_i^\zeta(\overline{\eta}(v))]$, is a sequence of bounded (see Lemma 3.8), centered and mixing (see Lemma A.2) random variables, giving

$$\sum_{i=1}^\infty \big|\mathrm{Cov}_{\mathbb{P}}\big(V_1^{\zeta,v}(\overline{\eta}(v)), V_i^{\zeta,v}(\overline{\eta}(v))\big)\big| = \sum_{i=1}^\infty \big|\mathbb{E}\big[\widetilde{L}_1\widetilde{L}_i\big]\big| = \sum_{i=1}^\infty \big|\mathbb{E}\big[\widetilde{L}_i\mathbb{E}[\widetilde{L}_1 \,|\, \mathcal{F}^{i-1}]\big]\big| < \infty, \tag{3.3.2}$$

where the last inequality is due to uniform boundedness of $\widetilde{L}_i$ in i, (A.2) and the summability criterion in (MIX). Thus $\sigma_v^2 < \infty$. Furthermore, $\sigma_v^2 \ge 0$ is due to (3.3.2) and [61, Lemma 1.1].

We now introduce the process $W_x^v(t)$ of empirical Legendre transformations

$$W_x^v(t) := t\sqrt{x}\Big((\overline{L}_{xt}^{\zeta})^*(1/v) - L^*(1/v)\Big), \quad t, x > 0,\ v \in V, \tag{3.3.3}$$

and set $W_0^v(t) = W_x^v(0) = 0$ for $t, x > 0$, $v \in V$, and obtain a first functional Central limit theorem for it.

Proposition 3.13. *For every $v \in V$, $W_n^v(1)$ converges in $\mathbb{P}$-distribution to a centered Gaussian random variable with variance $\sigma_v^2 \geq 0$. If $\sigma_v^2 > 0$, the sequence of processes*

$$[0,\infty) \ni t \mapsto \frac{1}{\sigma_v} W_n^v(t), \quad n \in \mathbb{N},$$

converges in $\mathbb{P}$-distribution to a standard Brownian motion in the sense of weak convergence of measures on $C([0,\infty))$ with topology induced by the metric ρ from (3.1.4).

Proof. It is sufficient to show the claim if $(W_n^v(t))_{t\in[0,\infty)}$ is replaced by $(W_n^v(t)\cdot \mathbb{1}_{\mathcal{H}_{nt}})_{t\in[0,\infty)}$, $n \in \mathbb{N}$, with $\mathcal{H}_{nt}$ as defined in (3.2.31), since the $\mathbb{P}$-probability of $\mathcal{H}_{nt}$ tends to 1 for $n \to \infty$ by Lemma 3.10. In the notation of (3.3.1), setting

$$S_x^{\zeta,v}(\eta) := \sum_{i=1}^{x} V_i^{\zeta,v}(\eta), \ x \in \mathbb{R}, \tag{3.3.4}$$

on $\mathcal{H}_{nt}$ we have

$$\big(\overline{L}_{nt}^{\zeta}\big)^*\Big(\frac{1}{v}\Big) = \frac{\eta_{nt}^{\zeta}(v)}{v} - \overline{L}_{nt}^{\zeta}(\eta_{nt}^{\zeta}(v)) = \frac{1}{nt}\sum_{i=1}^{nt} V_i^{\zeta,v}(\eta_{nt}^{\zeta}(v)) = \frac{1}{nt} S_{nt}^{\zeta,v}(\eta_{nt}^{\zeta}(v)).$$

Thus, we can rewrite the relevant term as a sum of three differences

$$\begin{aligned} nt\Big((\overline{L}_{nt}^{\zeta})^*(1/v) - L^*(1/v)\Big) &= \Big(S_{nt}^{\zeta,v}(\eta_{nt}^{\zeta}(v)) - S_{nt}^{\zeta,v}(\overline{\eta}(v))\Big) \\ &+ \Big(S_{nt}^{\zeta,v}(\overline{\eta}(v)) - \mathbb{E}\big[S_{nt}^{\zeta,v}(\overline{\eta}(v))\big]\Big) + \Big(\mathbb{E}\big[S_{nt}^{\zeta,v}(\overline{\eta}(v))\big] - ntL^*(1/v)\Big), \end{aligned} \tag{3.3.5}$$

where we note that the third summand vanishes. Indeed, we have

$$\mathbb{E}\big[S_{nt}^{\zeta,v}(\overline{\eta}(v))\big] = nt\Big(\frac{\overline{\eta}(v)}{v} - \mathbb{E}[L_1^{\zeta}(\overline{\eta}(v))]\Big) = ntL^*(1/v),$$

where the last equality is due to (3.2.19) and the definition of the Legendre transform. The proof is completed by the use of Lemmas 3.15 and 3.14 below, which show that the second summand of (3.3.5) exhibits the postulated diffusive behavior whereas the latter summand is negligible in that scaling. □

Lemma 3.14. *For every $v \in V$, $\frac{1}{\sqrt{n}}\Big(S_{nt}^{\zeta,v}(\overline{\eta}(v)) - \mathbb{E}\big[S_{nt}^{\zeta,v}(\overline{\eta}(v))\big]\Big)$ converges in $\mathbb{P}$-distribution to a centered Gaussian random variable with variance $\sigma_v^2 \geq 0$. If $\sigma_v^2 > 0$, the sequence of processes*

$$[0,\infty) \ni t \mapsto \frac{1}{\sigma_v\sqrt{n}}\Big(S_{nt}^{\zeta,v}(\overline{\eta}(v)) - \mathbb{E}\big[S_{nt}^{\zeta,v}(\overline{\eta}(v))\big]\Big), \quad n \in \mathbb{N},$$

converges in $\mathbb{P}$-distribution to a standard Brownian motion in the sense of weak convergence of measures on $C([0,\infty))$ with topology induced by the metric ρ from (3.1.4).

Proof. Let $\widetilde{L}_i := L_i^\zeta(\overline{\eta}(v)) - \mathbb{E}[L_i^\zeta(\overline{\eta}(v))]$, $\widetilde{V}_i := V_i^\zeta(\overline{\eta}(v)) - \mathbb{E}[V_i^\zeta(\overline{\eta}(v))]$ and $M \in \mathbb{N}$. Further set $\widetilde{L}_i^{(M)} := \sum_{j=1+(i-1)M}^{iM} \widetilde{L}_j$. Then $(\widetilde{L}_i^{(M)})_{i\in\mathbb{Z}}$ is a sequence of centered, stationary and (by Lemma 3.8) bounded random variables. To show the central limit theorem on $C([0,M])$, we use the method of martingale approximation from [31], which is summarized as a theorem in [54, Section 2.3], see Theorem A.3, and it turns out to be applicable in our situation. That is, we have to make sure that condition (A.7) is fulfilled. Indeed, replacing in (A.6) $\mathcal{F}^j$ by $\mathcal{F}_k$ and noting that quantity A in (A.6) is $\mathcal{F}_k$-measurable we get

$$\sum_{k=1}^{\infty} |\widetilde{L}_0^{(M)} - \mathbb{E}[\widetilde{L}_0^{(M)} \,|\, \mathcal{F}_k]| \le c_1 \sum_{k=1}^{\infty} e^{-k/c_1} < \infty,$$

giving the convergence of the first series in (A.7). Furthermore, using that $\widetilde{L}_k^{(M)}$ is $\mathcal{F}^{(k-1)M}$-measurable and bounded, also recalling (MIX), we get

$$\sum_{k=1}^{\infty} |\mathbb{E}[\widetilde{L}_k^{(M)} \,|\, \mathcal{F}_0]| \le \sum_{k=1}^{\infty} \psi(k-1)\mathbb{E}[|\widetilde{L}_0^{(M)}|] < \infty.$$

Because the series in (3.3.1) is absolutely convergent, by [61, Lemma 1.1] and Theorem A.3 we have $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\big[\big(\sum_{k=1}^n \widetilde{L}_k^{(M)}\big)^2\big] = M \cdot \lim_{n\to\infty} \frac{1}{Mn}\mathbb{E}\big[\big(\sum_{k=1}^{Mn} \widetilde{L}_k\big)^2\big] = M \cdot \sigma_v^2 \in [0,\infty)$. Furthermore, if $\sigma_v^2 > 0$, Theorem A.3 entails that the sequence of processes

$$[0,1] \ni t \mapsto X_n^{(M)}(t) := \frac{1}{\sigma_v\sqrt{nM}}\Big(\sum_{k=1}^{\lfloor nt\rfloor} \widetilde{L}_k^{(M)} + (nt - \lfloor nt\rfloor)\widetilde{L}_{\lfloor nt\rfloor+1}^{(M)}\Big), \quad n \in \mathbb{N},$$

converges in $\mathbb{P}$-distribution to a standard Brownian motion $(B_t)_{t\in[0,1]}$ in the sense of weak convergence of measures on $C([0,1])$ with the topology induced by the uniform metric. Then by definition, above convergence also holds true for $(\widetilde{V}_i)_{i\ge1}$ instead of $(\widetilde{L}_i)_{i\ge1}$. Furthermore, we have the uniform bound

$$\sup_{t\in[0,M],n\in\mathbb{N}} \Big| S_{nt}^{\zeta,v}(\overline{\eta}(v)) - \Big(\sum_{i=1}^{\lfloor \frac{nt}{M}\rfloor M} V_i^{\zeta,v} + (nt - \lfloor \frac{nt}{M}\rfloor M) \sum_{i=1+\lfloor \frac{nt}{M}\rfloor M}^{M+\lfloor \frac{nt}{M}\rfloor M} V_i^{\zeta,v}\Big)\Big| \le c_2 \quad \mathbb{P}\text{-a.s.}$$

Consequently, the sequence $[0,M] \ni t \mapsto \frac{1}{\sigma_v\sqrt{n}}\Big(S_{nt}^{\zeta,v}(\overline{\eta}(v)) - \mathbb{E}\big[S_{nt}^{\zeta,v}(\overline{\eta}(v))\big]\Big)$ has the same weak limit as $\big(\sqrt{M}\cdot X_n^{(M)}(t/M)\big)_{t\in[0,M]}$, $n \in \mathbb{N}$, which converges to $\big(\sqrt{M}\cdot B(t/M)\big)_{t\in[0,M]}$ and the latter process is a standard Brownian motion on $[0,M]$. Because $M \in \mathbb{N}$ was arbitrary, [71, first Theorem 2.4 on page 15] gives weak convergence on $C([0,\infty))$. □

The next lemma shows that the first summand in (3.3.5) is asymptotically negligible.

Lemma 3.15. *There exists a constant $C_6 \in (0,\infty)$ such that for every $v \in V$ and $M > 0$,*

$$\limsup_{n\to\infty} \frac{1}{\ln n} \sup_{0\le t\le M} \Big| S_{nt}^{\zeta,v}(\eta_{nt}^\zeta(v)) - S_{nt}^{\zeta,v}(\overline{\eta}(v))\Big| \le C_6 \quad \mathbb{P}\text{-a.s.}$$

Proof. There exists a $\mathbb{P}$-a.s. finite time $\mathcal{N}_1(\omega)$, defined before (3.2.31), such that for all $x \ge \mathcal{N}_1$ and all $v \in V$ we have $\eta_x^\zeta(v) \in \triangle$. Furthermore, by Lemma 3.8, $S_x^{\zeta,v}$ is infinitely differentiable on $(-\infty,0)$, so for all $x \ge \mathcal{N}_1$ there exists $\widetilde{\eta}_x^\zeta(v) \in [\overline{\eta}(v) \wedge \eta_x^\zeta(v), \overline{\eta}(v) \vee \eta_x^\zeta(v)]$

such that

$$S_x^{\zeta,v}(\overline{\eta}(v)) = S_x^{\zeta,v}(\eta_x^\zeta(v)) + \big(S_x^{\zeta,v}\big)'(\eta_x^\zeta(v))\big(\overline{\eta}(v) - \eta_x^\zeta(v)\big) + \frac{\big(S_x^{\zeta,v}\big)''(\widetilde{\eta}_x^\zeta(v))}{2}\big(\overline{\eta}(v) - \eta_x^\zeta(v)\big)^2.$$

Due to (3.2.23), $(S_x^{\zeta,v})'(\eta_x^\zeta(v)) = 0$ and by Lemma 3.8 we have

$$\sup_{\eta\in\triangle}\sup_{x\geq 1}\big|(S_x^{\zeta,v})''(\eta)\big|/x \leq c_1.$$

By (3.2.22) and the first Borel-Cantelli lemma, there exists a finite random variable $\mathcal{N}_2 \geq \mathcal{N}_1$ such that for $x \geq \mathcal{N}_2$ the complementary event on the left-hand side of (3.2.22) occurs, hence

$$\sup_{x\geq\mathcal{N}_1}\big(\eta_x^\zeta(v) - \overline{\eta}(v)\big)^2\frac{x}{\ln x} \leq C_3^2(V,2)$$

and thus

$$\sup_{x\geq\mathcal{N}_1}\frac{1}{\ln x}\big|S_x^{\zeta,v}(\eta_x^\zeta(v)) - S_x^{\zeta,v}(\overline{\eta}(v))\big| \leq C_6 \tag{3.3.6}$$

with $C_6 := \frac{c_1 C_3^2(V,2)}{2}$. Finally, we have

$$\begin{aligned}\sup_{0\leq t\leq M}\Big|S_{nt}^{\zeta,v}(\eta_{nt}^\zeta(v)) - S_{nt}^{\zeta,v}(\overline{\eta}(v))\Big| &\leq \sup_{0\leq x\leq\mathcal{N}_1}\Big|S_x^{\zeta,v}(\eta_x^\zeta(v)) - S_x^{\zeta,v}(\overline{\eta}(v))\Big| \\ &\quad + \sup_{\mathcal{N}_1/n\leq t\leq M}\Big|S_{nt}^{\zeta,v}(\eta_{nt}^\zeta(v)) - S_{nt}^{\zeta,v}(\overline{\eta}(v))\Big| \\ &\leq 2\mathcal{N}_1 c_2 + C_6\ln M + C_6\ln n,\end{aligned}$$

where in the last inequality we used that $\mathbb{P}$-a.s., every summand in the definition of $S_n^{\zeta,\eta}$ is uniformly bounded by c_2. The $\mathbb{P}$-a.s. finiteness of $\mathcal{N}_1$ gives the claim. □

As a by-product of the proof above we get an approximation result of W_x^v being a centered stationary sequence.

Corollary 3.16. *For every $v \in V$ and all t such that $vt \geq \mathcal{N}_1$ we have*

$$\Big|\sqrt{vt}W_{vt}^v(1) - \sum_{i=1}^{vt}\big(L(\overline{\eta}(v)) - L_i^\zeta(\overline{\eta}(v))\big)\Big| \leq C_6\ln v + C_6\ln t.$$

Proof. By the definition of $W_x^v(t)$ and $S_x^{\zeta,v}(\eta)$ from (3.3.3) and (3.3.4), as well as the definition in (3.2.20) for the corresponding Legendre transformations, we have

$$\begin{aligned}
\sqrt{vt}W^v_{vt}(1) &= vt\Big(\big(\overline{L}^\zeta_{vt}\big)^*(1/v) - L^*(1/v)\Big)\\
&= vt\Big(\frac{\eta^\zeta_{vt}(v)}{v} - \overline{L}_{vt}\big(\eta^\zeta_{vt}(v)\big) - \frac{\overline{\eta}(v)}{v} + L(\overline{\eta}(v))\Big)\\
&= S^{\zeta,v}_{vt}\big(\eta^\zeta_{vt}(v)\big) - \sum_{i=1}^{vt}\big(\frac{\overline{\eta}(v)}{v} - L^\zeta_i(\overline{\eta}(v))\big) + \sum_{i=1}^{vt}\big(L(\overline{\eta}(v)) - L^\zeta_i(\overline{\eta}(v))\big)\\
&= S^{\zeta,v}_{vt}\big(\eta^\zeta_{vt}(v)\big) - S^{\zeta,v}_{vt}\big(\overline{\eta}(v)\big) + \sum_{i=1}^{vt}\big(L(\overline{\eta}(v)) - L^\zeta_i(\overline{\eta}(v))\big).
\end{aligned}$$

Then we can conclude using (3.3.6). □

3.3.1 An exact large deviation result for auxiliary processes

For $x \geq 0$ and $v > 0$ we introduce

$$\begin{aligned}
Y^{\approx}_v(x) &:= E_x\Big[e^{\int_0^{H_0}\zeta(B_s)\,\mathrm{d}s}; H_0 \in \Big[\frac{x}{v} - K, \frac{x}{v}\Big]\Big],\\
Y^{>}_v(x) &:= E_x\Big[e^{\int_0^{H_0}\zeta(B_s)\,\mathrm{d}s}; H_0 < \frac{x}{v} - K\Big], \quad \text{and}\\
Y_v(x) &:= E_x\Big[e^{\int_0^{H_0}\zeta(B_s)\,\mathrm{d}s}; H_0 \leq \frac{x}{v}\Big] = Y^{\approx}_v(x) + Y^{>}_v(x),
\end{aligned}$$

where $K > 0$ is a constant, defined in (3.3.16) below. For $v \in V$ and $x \geq 1$ we define

$$\sigma^\zeta_x(v) := \begin{cases} \big|\eta^\zeta_x(v)\big|\sqrt{\mathrm{Var}^{\zeta,\eta^\zeta_x(v)}_x(H_0)}, & \text{on } \mathcal{H}_x,\\ \sup_{\eta\in\triangle}|\eta|\sqrt{\mathrm{Var}^{\zeta,\sup_{\eta\in\triangle}}_x(H_0)}, & \text{on } \mathcal{H}^c_x. \end{cases}$$

Furthermore, by Lemma 3.8, there exists some $C > 1$ such that $\mathrm{Var}^{\zeta,\eta}_x(H_0) = x\big(\overline{L}^\zeta_x\big)''(\eta) \in [xC_2^{-1}, xC_2]$ for all $x \geq 1$. Thus, there is some constant $C_7 = C_7(\triangle) > 1$ such that

$$C_7^{-1}\sqrt{x} \leq \sigma^\zeta_x(v) \leq C_7\sqrt{x} \quad \text{for all } v \in V,\ x \geq 1,\ \text{on } \mathcal{H}_x. \tag{3.3.7}$$

We now prove the following result.

Proposition 3.17. *Let V be as in* (3.2.30)*, σ_v defined by* (3.3.1) *and $W^v_x(t)$ as in* (3.3.3)*, $v \in V$, and $K > 0$ be such that* (3.3.16) *holds. Then there exists a constant $C_8 \in (1,\infty)$, such that for all $v \in V$ and all $x \geq 1$, on $\mathcal{H}_x$ we have*

$$\sigma^\zeta_x(v) Y^{>}_v(x)\exp\big\{xL^*(1/v) + \sqrt{x}W^v_x(1)\big\} \in \big[C_8^{-1}, C_8\big]. \tag{3.3.8}$$

Furthermore, for all $v \in V$ and all $x \geq 1$, on $\mathcal{H}_x$ we have

$$\frac{Y^{\approx}_v(x)}{Y^{>}_v(x)} \in [C_8^{-1}, C_8], \tag{3.3.9}$$

and the sequence $n^{-1/2}\big(\ln Y^{\approx}_v(n) - nL^*(1/v)\big)$, $n \in \mathbb{N}$, converges to a centered Gaussian random variable with variance $\sigma^2_v \in [0,\infty)$, where σ^2_v is defined in (3.3.1) *and if $\sigma^2_v > 0$, the* *sequence of processes*

$$[0,\infty) \ni t \mapsto \frac{1}{\sigma_v\sqrt{n}}\big(\ln Y_v^{\approx}(nt) + ntL^*(1/v)\big), \quad n \in \mathbb{N}, \tag{3.3.10}$$

converges in $\mathbb{P}$*-distribution to standard Brownian motion in* $C([0,\infty))$.

Proof. We start with proving (3.3.8) and for this purpose let $x \geq 1$ such that $\mathcal{H}_{\lfloor x \rfloor}$ occurs. Write $\eta := \eta_x^\zeta(v)$ and $\sigma := \sigma_x^\zeta(v)$, and recall the notation introduced in (3.2.4), i.e. $\tau_i = H_i - H_{i-1}$, $i = 1,\dots,\lceil x\rceil - 1$, and set $\tau_x := H_{\lceil x\rceil -1} - H_x$, $x \in \mathbb{R}\backslash\mathbb{Z}$, (which would be consistent with the definition in (3.2.4) for x integer) to define $\widehat{\tau}_i := \widehat{\tau}_i^{(x)} := \tau_i - E_x^{\zeta,\eta}[\tau_i]$. Then $\sum_{i=1}^x E_x^{\zeta,\eta}[\tau_i] = E_x^{\zeta,\eta}[H_0] = \frac{x}{v}$. We now rewrite

$$\begin{aligned} Y_v^{\approx}(x) &= E_x\Big[e^{\int_0^{H_0}(\zeta(B_s)+\eta)\,\mathrm{d}s}\exp\Big\{-\eta\sum_{i=1}^x\widehat{\tau}_i\Big\};\ \sum_{i=1}^x\widehat{\tau}_i \in [-K,0]\Big]\exp\Big\{-x\frac{\eta}{v}\Big\} \\ &= E_x^{\zeta,\eta}\Big[\exp\Big\{-\sigma\frac{\eta}{\sigma}\sum_{i=1}^x\widehat{\tau}_i\Big\};\ \frac{\eta}{\sigma}\sum_{i=1}^x\widehat{\tau}_i \in \Big[0,-\frac{K\eta}{\sigma}\Big]\Big]\exp\Big\{-x\Big(\frac{\eta}{v}-\overline{L}_x^\zeta(\eta)\Big)\Big\}. \end{aligned} \tag{3.3.11}$$

Analogously, we get

$$Y_v^{>}(x) = E_x^{\zeta,\eta}\Big[\exp\Big\{-\sigma\frac{\eta}{\sigma}\sum_{i=1}^x\widehat{\tau}_i\Big\};\ \frac{\eta}{\sigma}\sum_{i=1}^x\widehat{\tau}_i > -\frac{K\eta}{\sigma}\Big]\exp\Big\{-x\Big(\frac{\eta}{v}-\overline{L}_x^\zeta(\eta)\Big)\Big\}.$$

We define $\mu_x^{\zeta,v}$ as the distribution of $\frac{\eta}{\sigma}\sum_{i=1}^x\widehat{\tau}_i$ under $P_x^{\zeta,\eta}$. Then

$$Y_v^{\approx}(x) = e^{-x(\frac{\eta}{v}-\overline{L}_x^\zeta(\eta))}\int_0^{\frac{-K\eta}{\sigma}} e^{-\sigma y}\ \mathrm{d}\mu_x^{\zeta,v}(y) \tag{3.3.12}$$

and

$$Y_v^{>}(x) = e^{-x(\frac{\eta}{v}-\overline{L}_x^\zeta(\eta))}\int_{\frac{-K\eta}{\sigma}}^\infty e^{-\sigma y}\ \mathrm{d}\mu_x^{\zeta,v}(y). \tag{3.3.13}$$

Using Lemma 3.18 below, the integrals on the right-hand side of (3.3.12) and (3.3.13), multiplied by σ, are bounded from below and above by positive constants. Display (3.3.8) now follows by the definition of W_x^v, and (3.3.9) is direct a consequence of (3.3.12)-(3.3.15). The last two statements, are a consequence of (3.3.8), (3.3.9), $W_{nt}^v(1) = \frac{1}{\sqrt{t}}W_n^v(t)$ and Proposition 3.13. □

To complete the previous proof, it remains to prove the following.

Lemma 3.18. *Under the conditions of Proposition 3.17, there exists a constant* $C_9 > 1$ *such that for all* $v \in V$ *and* $x \geq 1$, *on* $\mathcal{H}_x$,

$$\sigma_x^\zeta(v)\int_0^{-K\eta_x^\zeta(v)/\sigma_x^\zeta(v)} e^{-\sigma_x^\zeta(v)y}\,\mathrm{d}\mu_x^{\zeta,v}(y) \in [C_9^{-1}, C_9] \tag{3.3.14}$$

and

$$\sigma_x^\zeta(v)\int_{-K\eta_x^\zeta(v)/\sigma_x^\zeta(v)}^\infty e^{-\sigma_x^\zeta(v)y}\,\mathrm{d}\mu_x^{\zeta,v}(y) \in [C_9^{-1}, C_9], \tag{3.3.15}$$

with $\mu_x^{\zeta,v}$ *as in the proof of Proposition 3.17.*

Proof. We write $n := \lceil x \rceil$ and recall that under $P_x^{\zeta,\eta}$, the sequence $\Big(\sqrt{n}\frac{\eta_x^\zeta(v)}{\sigma_x^\zeta(v)}\widehat{\tau}_i\Big)_{i=1,\dots,\lfloor x\rfloor,x}$ is a sequence of independent, centered random variables. Thus, on $\mathcal{H}_x$ we obtain

$$\frac{1}{n}\sum_{i=1}^{x} \mathrm{Var}_x^{\zeta,\eta}\Big(\sqrt{n}\frac{\eta_x^\zeta(v)}{\sigma_x^\zeta(v)}\widehat{\tau}_i\Big) = \Big(\frac{\eta_x^\zeta(v)}{\sigma_x^\zeta(v)}\Big)^2 \mathrm{Var}_x^{\zeta,\eta}\Big(\sum_{i=1}^{x}\widehat{\tau}_i\Big) = \Big(\frac{\eta_x^\zeta(v)}{\sigma_x^\zeta(v)}\Big)^2 \mathrm{Var}_x^{\zeta,\eta}(H_0) = 1.$$

Additionally, the $\widehat{\tau}_i$'s have uniform exponential moments. Thus, the conditions of [7, Theorem 13.3] are fulfilled and an application of [7, (13.43)] yields

$$\sup_{\mathcal{C}} \big|\mu_x^{\zeta,v}(\mathcal{C}) - \Phi(\mathcal{C})\big| \le c_1 n^{-1/2},$$

where the supremum is taken over all Borel-measurable convex subsets of $\mathbb{R}$, Φ denotes the standard Gaussian measure on $\mathbb{R}$ and c_1 only depends on the uniform bound of the exponential moments of the $\widehat{\tau}_i$'s. Without loss of generality, we assume $c_1 > 4$. Then, due to (3.3.7), by denoting $\mathcal{C} := \big[0, -K\eta_x^\zeta(v)/\sigma_x^\zeta(v)\big]$ we can choose $K > 0$ large enough, so that

$$\Phi(\mathcal{C}) \ge 2c_1^{-1}n^{-1/2} \quad \text{for all } n \in \mathbb{N} \text{ and } v \in V. \tag{3.3.16}$$

We thus get

$$\begin{aligned} c_1^{-1}n^{-1/2} &\le \Phi(\mathcal{C}) - \big|\mu_x^{\zeta,v}(\mathcal{C}) - \Phi(\mathcal{C})\big| \le \mu_x^{\zeta,v}(\mathcal{C}) \\ &\le \big|\mu_x^{\zeta,v}(\mathcal{C}) - \Phi(\mathcal{C})\big| + \Phi(\mathcal{C}) \le c_2(K)\cdot n^{-1/2}. \end{aligned} \tag{3.3.17}$$

Because the integrand in (3.3.14) is bounded away from 0 and infinity on the respective interval of integration (uniformly in $n \in \mathbb{N}$), (3.3.14) is a direct consequence of (3.3.7) and (3.3.17). For (3.3.15), we split the integral into a sum:

$$\begin{aligned} \int_{-K\eta_x^\zeta(v)/\sigma_x^\zeta(v)}^{\infty} e^{-\sigma_x^\zeta(v)y}\,\mathrm{d}\mu_x^{\zeta,v}(y) &= \sum_{k=1}^{\infty}\int_{-kK\eta_x^\zeta(v)/\sigma_x^\zeta(v)}^{-(k+1)K\eta_x^\zeta(v)/\sigma_x^\zeta(v)} e^{-\sigma_x^\zeta(v)y}\,\mathrm{d}\mu_x^{\zeta,v}(y) \\ \le \sum_{k=1}^{\infty}\mu_x^{\zeta,v}\Big(-\frac{K\eta_x^\zeta(v)}{\sigma_x^\zeta(v)}[k,k+1]\Big)e^{-k\cdot K|\eta_x^\zeta(v)|} &\le c_2 n^{-1/2}\sum_{k=1}^{\infty} e^{-k\cdot K|\eta^*|} \le C_9 n^{-1/2}, \end{aligned}$$

where we recall the notation from (3.2.26). The lower bound in (3.3.15) can be obtained by noting that

$$\begin{aligned} \int_{-K\eta_x^\zeta(v)/\sigma_x^\zeta(v)}^{\infty} e^{-\sigma_x^\zeta(v)y}\,\mathrm{d}\mu_x^{\zeta,v}(y) &\ge \int_{-K\eta_x^\zeta(v)/\sigma_x^\zeta(v)}^{-2K\eta_x^\zeta(v)/\sigma_x^\zeta(v)} e^{-\sigma_x^\zeta(v)y}\,\mathrm{d}\mu_x^{\zeta,v}(y) \\ &\ge e^{2K\eta_x^\zeta(v)}\mu_x^{\zeta,v}\big([-K\eta_x^\zeta(v)/\sigma_x^\zeta(v), -2K\eta_x^\zeta(v)/\sigma_x^\zeta(v)]\big). \end{aligned}$$

Analogously to (3.3.17), choosing C_9 large enough, the last expression is bounded from below by $C_9^{-1}n^{-1/2}$. Combining this with (3.3.7), we finally arrive at (3.3.15). □

We are now in the position to prove the following result.

Lemma 3.19. *Under the conditions of Proposition 3.17, for all $\delta > 0$ there exists a constant*

$C_{10} = C_{10}(\delta) \in (1, \infty)$ *such that for all* $v \in V$, $t > 0$, *on* $\mathcal{H}_{vt}$ *we have*

$$C_{10}^{-1} Y_v^{\approx}(vt) \le E_{vt}\Big[e^{\int_0^t \zeta(B_s)\,\mathrm{d}s}; B_t \in [-\delta, 0]\Big] \le E_{vt}\Big[e^{\int_0^t \zeta(B_s)\,\mathrm{d}s}; B_t \le 0\Big] \le C_{10} Y_v^{\approx}(vt). \tag{3.3.18}$$

Proof. The second inequality is obvious. Since $\{B_t \le 0\} \subset \{H_0 \le t\}$ and $\zeta \le 0$, we get $E_{vt}\big[e^{\int_0^t \zeta(B_s)\,\mathrm{d}s}; B_t \le 0\big] \le Y_v(vt) \le (1 + C_8) Y_v^{\approx}(vt)$ by Proposition 3.17 and thus the last inequality in (3.3.18) is obtained. Therefore, it remains to show the first inequality. For this purpose, define the function $p(s) := E_0\big[e^{\int_0^s \zeta(B_r)\,\mathrm{d}r}; B_s \in [-\delta, 0]\big]$ which is bounded from below by $c_1(K, \delta) > 0$ for all $s \in [0, K]$. Using the strong Markov property at H_0, we finally get

$$\begin{aligned} Y_v^{\approx}(vt) &= E_{vt}\left[e^{\int_0^{H_0} \zeta(B_r)\,\mathrm{d}r}; H_0 \in [t - K, t]\right] \\ &\le c_1(K, \delta)^{-1} E_{vt}\left[e^{\int_0^{H_0} \zeta(B_r)\,\mathrm{d}r} p(t - s)_{|s = H_0}; H_0 \in [t - K, t]\right] \\ &\le c_1(K, \delta)^{-1} E_{vt}\left[e^{\int_0^{H_0} \zeta(B_r)\,\mathrm{d}r} p(t - s)_{|s = H_0}\right] \\ &= c_1(K, \delta)^{-1} E_{vt}\left[e^{\int_0^{t} \zeta(B_r)\,\mathrm{d}r}; B_t \in [-\delta, 0]\right]. \end{aligned}$$

and the claim follows by choosing $C_{10} := c_1(K, \delta) \vee (1 + C_8)$. □

Plugging the relation $\xi(x) = \zeta(x) + \mathtt{es}$, $x \in \mathbb{R}$, into Lemma 3.19 immediately supplies us with the following corollary.

Corollary 3.20. *Let* C_{10} *be as in Lemma 3.19. Then for all* $v \in V$, $t > 0$, *on* $\mathcal{H}_{vt}$ *we have*

$$C_{10}^{-1} e^{\mathtt{es} \cdot t} Y_v^{\approx}(vt) \le E_{vt}\left[e^{\int_0^t \xi(B_s)\,\mathrm{d}s}; B_t \le 0\right] \le C_{10} e^{\mathtt{es} \cdot t} Y_v^{\approx}(vt).$$

Using the Feynman-Kac formula (3.1.2) we also get the following result. Recall that u^{u_0} denoted the solution to (PAM) with initial condition $u_0 \in \mathcal{I}_{\mathrm{PAM}}$.

Corollary 3.21. *Let* $C_{10} = C_{10}(\delta)$ *be as in Lemma 3.19. Then for all* $v \in V$, $t > 0$, *on* $\mathcal{H}_{vt}$ *we have*

$$u^{\mathbb{1}_{[-\delta,0]}}(t, vt) \le u^{\mathbb{1}_{(-\infty,0]}}(t, vt) \le C_{10}^2 \cdot u^{\mathbb{1}_{[-\delta,0]}}(t, vt).$$

The previous results are fundamental for proving perturbation statements in the next section, which themselves allow us to analyze path probabilities of the branching process.

3.3.2 Proof of Theorem 3.3

Now we are ready to prove our first main result.

Proof of Theorem 3.3. We first assume $\sigma_v^2 > 0$ and consider the case $u_0 = \mathbb{1}_{(-\infty,0]}$ to show the second part of the claim, i.e. that the sequence of processes

$$[0, \infty) \ni t \mapsto \frac{1}{\sqrt{nv\sigma_v^2}}\big(\ln u(nt, vnt) - nt\Lambda(v)\big), \quad n \in \mathbb{N}, \tag{3.3.19}$$

converges in $\mathbb{P}$-distribution to standard Brownian motion. Because $[0, \infty) \ni t \mapsto \ln u(t, vt)$ might be discontinuous *only* in 0, we have to make clear what we mean by above convergence. In fact, we show the invariance principle for a sequence of auxiliary processes $(X_n^v(t))_{t \ge 0}$,

$n \in \mathbb{N}$, where for every $n \in \mathbb{N}$ and $t \geq \frac{1}{n}$, $X_n^v(t)$ is the same as in (3.3.19), whereas for $t \in [0, 1/n]$ the term $\ln u(nt, vnt)$ in (3.3.19) is replaced by $(1-nt)\ln u(0,0)+nt\ln(1,v)$, making $(X_n^v(t))_{t\geq 0}$ continuous. Because the difference of the processes in (3.3.19) and $(X_n^v(t))_{t\geq 0}$ converges uniformly to zero as $n \to \infty$, convergence of the processes in (3.3.19) to a standard Brownian motion is defined as the convergence of the processes $(X_n^v(t))_{t\geq 0}$, $n \in \mathbb{N}$, to a standard Brownian motion in $C([0,\infty))$ with topology induced by the metric ρ from (3.1.4).

By Proposition 3.17 and Corollary 3.20, on $\mathcal{H}_{nvt}$ (recall the notation from (3.2.31)) we have

$$\begin{aligned} &-\ln C_8 \leq \ln \sigma_{nvt}^{\zeta}(v) + \ln Y_v^{\approx}(nvt) + nvtL^*(1/v) + \sqrt{nvt}W_{vnt}^v(1) \leq \ln C_8, \quad \text{and} \\ &-\ln C_{10} \leq \ln u(nt, vnt) - \mathtt{es}\cdot nt - \ln Y_v^{\approx}(vnt) \leq \ln C_{10}. \end{aligned} \tag{3.3.20}$$

Recall that a sequence of processes $t \mapsto A_n(t)$, $n \in \mathbb{N}$, converges in $\mathbb{P}$-distribution to standard Brownian motion if and only if for all $\sigma > 0$ the sequence $t \mapsto \sigma^{-1}A_n(\sigma^2 t)$, $n \in \mathbb{N}$, converges in $\mathbb{P}$-distribution to a standard Brownian motion. Applying this to (3.3.10), the sequence of processes

$$[0,\infty) \ni t \mapsto \frac{1}{\sqrt{nv\sigma_v^2}}\big(\ln Y_v^{\approx}(vnt) + vntL^*(1/v)\big), \quad n \in \mathbb{N}, \tag{3.3.21}$$

converges in $\mathbb{P}$-distribution to a standard Brownian motion. Further, by the second line in (3.3.20),

$$-\ln C_{10} \leq \big(\ln u(nt, vnt) - nt(\mathtt{es} - vL^*(1/v))\big) - \big(\ln Y_v^{\approx}(vnt) + vntL^*(1/v)\big) \leq \ln C_{10}$$

holds. Consequently, if we can prove that

$$\Lambda(v) = \mathtt{es} - vL^*(1/v) \quad \forall v \in V. \tag{3.3.22}$$

the claim follows from (3.3.21). To prove (3.3.22), we set $n = 1$ in (3.3.20) and note that $\frac{W_{vt}^v(1)}{\sqrt{t}} \underset{t\to\infty}{\longrightarrow} 0$ $\mathbb{P}$-a.s. for all $v \in V$, because $W_n(1)$ converges in $\mathbb{P}$-distribution to a centered normally distributed random variable by Proposition 3.13. Using (3.3.24), (3.3.20) and (3.3.7), we get (3.3.22).

It remains to show the claim for arbitrary $u_0 \in \mathcal{I}_{\text{PAM}}$. Recall that there exist $\delta' \in (0,1)$ and $C' > 1$, such that $\delta'\mathbb{1}_{[-\delta',0]}(x) \leq u_0(x) \leq C'\mathbb{1}_{(-\infty,0]}(x)$ for all $x \in \mathbb{R}$. Therefore, using Lemma 3.19, there exists a constant $c_1 > 0$ such that

$$c_1 E_{vt}[e^{\int_0^t \xi(B_s)\,\mathrm{d}s}; B_t \leq 0] \leq E_{vt}[e^{\int_0^t \xi(B_s)\,\mathrm{d}s}u_0(B_t)] \leq C' E_{vt}[e^{\int_0^t \xi(B_s)\,\mathrm{d}s}; B_t \leq 0]. \tag{3.3.23}$$

Thus, the convergence of (3.3.19) for arbitrary initial condition $u_0 \in \mathcal{I}_{\text{PAM}}$ follows from the one with initial condition $\mathbb{1}_{(-\infty,0]}$. This gives the second part of Theorem 3.3.

It remains to show that $(nv)^{-1/2}\big(\ln u(n, vn) - n\Lambda(v)\big)$ converges in $\mathbb{P}$-distribution to a Gaussian random variable. For $u_0 = \mathbb{1}_{(-\infty,0]}$ this is a direct consequence of (3.3.20) for $t = 1$, (3.3.22) and the second part of Proposition 3.17. For general u_0 the claim follows from (3.3.23). □

In view of Corollary 3.21, Proposition 3.7 and (3.3.22), it is possible to say that the Lyapunov exponent Λ, defined in (3.1.3), determines the exponential decay resp. growth for solutions to (PAM) for *arbitrary* initial conditions $u_0 \in \mathcal{I}_{\text{PAM}}$ (and not only for those with compact support).

Corollary 3.22. *For all $v \geq 0$ and all $u_0 \in \mathcal{I}_{PAM}$ we have*

$$\Lambda(v) = \lim_{t\to\infty} \frac{1}{t} \ln u^{u_0}(t, vt) \quad \mathbb{P}\text{-}a.s. \tag{3.3.24}$$

Λ is linear on $[0, v_c]$ and strictly concave on (v_c, ∞) and the convergence in (3.3.24) holds uniformly on every compact interval $K \subset [0, \infty)$.

Proof. Note that by Proposition 3.7, $\delta'\mathbb{1}_{[-\delta',0]} \leq u_0 \leq C'\mathbb{1}_{(-\infty,0]}$ and Corollary 3.21 we have that (3.3.24) holds for all $v \in V$ and all compact $V \subset (v_c, \infty)$. Thus (3.3.24) is true for all $v > v_c$. The strict concavity of Λ on (v_c, ∞) follows from the strict convexity of $L^*(1/v)$, which in turn follows from the strict convexity of L and usual properties of the Legendre transformation. If $v_c = 0$, the proof is complete due to $\lim_{t\to\infty} \frac{1}{t} \ln u^{\delta'\mathbb{1}_{[-\delta',0]}} = \Lambda(0) = \mathtt{es}$ by Proposition 3.7 and $u^{u_0} \leq e^{\mathtt{est}}$ for all $u_0 \in \mathcal{I}_{\mathrm{PAM}}$. Let us assume $v_c > 0$. First observe that $L^*(1/v)$ tends to $L^*(1/v_c) = -L(0)$ as $v \downarrow v_c$. Indeed, recalling Lemma 3.9 (d), for all $v > v_c$ there exists a unique $\overline{\eta}(v) \in (-\infty, 0)$ defined by $L'\big(\overline{\eta}(v)\big) = \frac{1}{v}$, such that

$$L^*(1/v) = \frac{\overline{\eta}(v)}{v} - L(\overline{\eta}(v)).$$

Furthermore $(v_c, \infty) \ni v \mapsto \overline{\eta}(v)$ is continuously differentiable and strictly decreasing, bounded from above by 0, and $(-\infty, 0) \ni \eta \mapsto L'(\eta)$ is smooth and strictly monotone and tends to $L'(0-)$ as $\eta \uparrow 0$, we get that $\eta(v) \uparrow 0$ as $v \downarrow v_c$ and thus $L^*(1/v) \to L^*(1/v_c)$ as $v \downarrow v_c$. Therefore $\Lambda(v) = \mathtt{es} - vL^*(1/v)$ for all $v \in [v_c, \infty)$. Furthermore, for all $u_0 \in \mathcal{I}_{\mathrm{PAM}}$, due to (BDD)

$$\begin{aligned} u^{u_0}(t, vt) &\leq C' e^{\mathtt{est}} E_{vt}\big[e^{\int_0^t \zeta(B_s)\,\mathrm{d}s}; B_t \leq 0\big] \leq C' e^{\mathtt{est}} E_{vt}\big[e^{\int_0^{H_0} \zeta(B_s)\,\mathrm{d}s}; H_0 \leq t\big] \\ &\leq C' \exp\Big\{t\big(\mathtt{es} + v\overline{L}_{vt}(0)\big)\Big\}. \end{aligned} \tag{3.3.25}$$

Using Lemma 3.9 (b), taking logarithm and deviding by t, we get that the (by Proposition 3.7) concave function Λ is bounded from above by the linear function $[0, \infty) \ni v \mapsto \mathtt{es} - vL(0) = \mathtt{es} + vL^*(1/v_c)$ and coincides with this function at $v = 0$ and $v = v_c$ and thus on the whole interval $[0, v_c]$. Using (3.3.25) once more, we infer (3.3.24) for all $v \geq 0$ and $u_0 \in \mathcal{I}_{\mathrm{PAM}}$.

To show that the convergence is uniform on every compact interval $K \subset [0, \infty)$, for $\varepsilon > 0$ arbitrary we consider $\varepsilon\mathbb{Z} := \{k\varepsilon : k \in \mathbb{Z}\}$, and for $y \in \mathbb{R}$ set $\lfloor y \rfloor_\varepsilon := \sup\{x \in \varepsilon\mathbb{Z} : x \leq y\}$. Then the convergence holds uniformly for all $y \in K \cap \varepsilon\mathbb{Z}$. A fortiori, for t large enough, uniformly in $y \in K$,

$$u(t, t\lfloor y \rfloor_\varepsilon) \geq e^{t(\Lambda(\lfloor y \rfloor_\varepsilon) - \varepsilon)}.$$

Lemma D.4 then entails that

$$\inf_{z \in [t\lfloor y \rfloor_\varepsilon - 1, t\lfloor y \rfloor_\varepsilon + 1]} u(t+1, z) \geq \frac{1}{C_{20}} u(t, t\lfloor y \rfloor_\varepsilon) \geq \frac{1}{C_{20}} e^{t(\Lambda(\lfloor y \rfloor_\varepsilon) - \varepsilon)}. \tag{3.3.26}$$

Furthermore, using $0 \leq y - \lfloor y \rfloor_\varepsilon \leq \varepsilon$, by a Gaussian estimate we get

$$\begin{aligned} P_{yt}(B_{\varepsilon t} \in [t\lfloor y \rfloor_\varepsilon - 1, t\lfloor y \rfloor_\varepsilon + 1]) &\geq \sqrt{\frac{2}{\pi\varepsilon t}} \inf_{x \in [t\lfloor y \rfloor_\varepsilon - 1, t\lfloor y \rfloor_\varepsilon + 1]} e^{-\frac{(x-yt)^2}{2\varepsilon t}} \\ &\geq \sqrt{\frac{2}{\pi}} \cdot \exp\Big\{ -\frac{\varepsilon t}{2} - \frac{\ln(\varepsilon t)}{2} - 1 - \frac{1}{2\varepsilon} \Big\}. \end{aligned} \tag{3.3.27}$$

Using the Feynman-Kac formula in the equality, the Markov property at time εt, and (BDD) in the first inequality, we infer that

$$\begin{aligned} u(t+1+\varepsilon t, yt) &= E_{yt}\big[e^{\int_0^{t+1+\varepsilon t}\xi(B_s)\,\mathrm{d}s}u_0(B_{t+1+\varepsilon t})\big] \\ &\geq e^{\mathtt{ei}\varepsilon t}\cdot P_{yt}(B_{\varepsilon t}\in[t\lfloor y\rfloor_\varepsilon-1, t\lfloor y\rfloor_\varepsilon+1])\cdot \inf_{z\in[t\lfloor y\rfloor_\varepsilon-1,t\lfloor y\rfloor_\varepsilon+1]} u(t+1,z) \\ &\geq c_1\exp\Big\{t\Big(\Lambda(\lfloor y\rfloor_\varepsilon)+(\mathtt{ei}-3/2)\varepsilon-\frac{\ln(\varepsilon t)}{2t}-\frac{1}{t}-\frac{1}{2\varepsilon t}\Big)\Big\}, \end{aligned} \tag{3.3.28}$$

where in the last inequality we used (3.3.26) and (3.3.27). Setting $t' := t+1+\varepsilon t$ and $y' := \frac{t'}{t}y$, we get

$$\frac{1}{t'}\ln u(t',t'y)-\Lambda(y)=\frac{t}{t'}\Big(\frac{1}{t}\ln u(t',ty')-\Lambda(\lfloor y'\rfloor_\varepsilon)\Big)+\frac{t}{t'}\Lambda(\lfloor y'\rfloor_\varepsilon)-\Lambda(y).$$

Since (3.3.28) holds uniformly for all $y\in K$, we infer that

$$\begin{aligned} \inf_{y\in K}\Big(\frac{1}{t'}\ln u(t',t'y)-\Lambda(y)\Big) &\geq \frac{t}{t'}\Big(\frac{\ln c_1}{t}+(\mathtt{ei}-3/2)\varepsilon-\frac{\ln(\varepsilon t)}{2t}-\frac{1}{t}-\frac{1}{2\varepsilon t}\Big) \\ &\quad -\sup_{y\in K}\Big|\frac{t}{t'}\Lambda(\lfloor y'\rfloor_\varepsilon)-\Lambda(y)\Big|. \end{aligned} \tag{3.3.29}$$

Since Λ concave and finite, it is uniformly continuous on compact intervals, hence choosing $\varepsilon>0$ small, we deduce the lower bound

$$\liminf_{t\to\infty}\inf_{y\in K}\Big(\frac{1}{t}\ln u(t,ty)-\Lambda(y)\Big)\geq 0. \tag{3.3.30}$$

To derive the matching upper bound, we assume that the convergence does not hold uniformly on K. Then, due to (3.3.30), there exists $\alpha>0$ and sequences $(t_n)_{n\in\mathbb{N}}\subset[0,\infty)$ and $(y_n)_{n\in\mathbb{N}}\subset K$ such that $t_n\to\infty$ and

$$\frac{1}{t_n}\ln u(t_n,t_ny_n)-\Lambda(y_n)\geq\alpha\quad\forall n\in\mathbb{N}. \tag{3.3.31}$$

Retreating to a suitable subsequence, we can assume $y_n \underset{n\to\infty}{\longrightarrow} y\in K$. We choose $\varepsilon>0$ such that

$$(1+\varepsilon)\Lambda(y)\leq\Lambda(y)+\alpha/3\quad\text{ and }\quad\frac{(1+z)^2}{2}\varepsilon\leq\alpha/3\ \forall z\in K. \tag{3.3.32}$$

For n such that $|y_n-y|\leq\varepsilon$, a Gaussian estimate yields

$$\begin{aligned} P_{y(t_n+1+\varepsilon t_n)}\big(B_{\varepsilon t_n}\in[t_ny_n-1,t_ny_n+1]\big) &\geq \sqrt{\frac{2}{\pi\varepsilon t_n}}\inf_{x\in[t_ny_n-1,t_ny_n+1]} e^{-\frac{(x-y(t_n+1+\varepsilon t_n))^2}{2\varepsilon t_n}} \\ &\geq \sqrt{\frac{2}{\pi\varepsilon t_n}}\exp\Big\{-\frac{(1+y)^2}{2\varepsilon t_n}(t_n\varepsilon+1)^2\Big\}. \end{aligned}$$

An argument as in (3.3.28) yields

$$u(t_n+1+\varepsilon t_n,(t_n+1+\varepsilon t_n)y)\geq c_2\cdot u(t_n,t_ny_n)\cdot\sqrt{\frac{1}{\varepsilon t_n}}\exp\Big\{\varepsilon\mathtt{ei}t_n-\frac{(1+y)^2}{2\varepsilon t_n}(t_n\varepsilon+1)^2\Big\} \tag{3.3.33}$$

for all n such that $|y_n - y| \le \varepsilon$. Now first taking logarithms, dividing by t_n, taking $n \to \infty$ and recalling $y_n \underset{n\to\infty}{\longrightarrow} y$ and continuity of Λ, the left-hand side in (3.3.33) converges to $(1+\varepsilon)\Lambda(y)$, whereas by (3.3.31), the limit of the right-hand side is bounded from below by $\Lambda(y) + \alpha - \frac{(1+y)^2}{2}\varepsilon$. But by (3.3.32), this leads to a contradiction. We thus deduce the uniform convergence on K. □

3.3.3 Time perturbation

In the next step we prove perturbation results, i.e., time and space perturbation Lemmas 3.23 and 3.24. These statements will be useful when comparing the expected number of particles evolving which are slightly "slower" or "faster" (within a certain margin) than the ones with given velocity. In the following, $u = u^{u_0}$ denotes the solution to (PAM) with initial condition $u_0 \in \mathcal{I}_{\mathrm{PAM}}$.

Lemma 3.23. *(a) Let $u_0 \in \mathcal{I}_{PAM}$ and let $\varepsilon : (0,\infty) \to (0,\infty)$ be a function such that $\varepsilon(t) \to 0$ and $t\varepsilon(t) \to \infty$ as $t \to \infty$. Then there exists $C_{11} = C_{11}((\varepsilon(t))_{t\ge 0}, u_0)$ such that $\mathbb{P}$-a.s., for all t large enough,*

$$\sup_{(v,h)\in\mathcal{E}_t} \Big| \ln\Big(\frac{u^{u_0}(t+h, vt)}{u^{u_0}(t, vt)}\Big) - h(\mathsf{es} - \overline{\eta}(v))\Big| \le C_{11} + C_{11}|h|\Big(\sqrt{\frac{\ln t}{t}} + \frac{|h|}{t}\Big), \quad (3.3.34)$$

where $\mathcal{E}_t := \Big\{(v,h) :\ v \in V,\ |h| \le t\varepsilon(t),\ \frac{vt}{t+h} \in V\Big\}$.

(b) For all $\varepsilon > 0$ and $u_0 \in \mathcal{I}_{PAM}$ there exists a constant $C_{12} > 1$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}_1$ such that for all $t \ge \mathcal{T}_1$, uniformly in $0 \le h \le t^{1-\varepsilon}$, $v \in V$ and $\frac{vt}{t+h} \in V$,

$$C_{12}^{-1} e^{h/C_{12}} u^{u_0}(t, vt) \le u^{u_0}(t+h, vt) \le C_{12} e^{C_{12}h} u^{u_0}(t, vt). \quad (3.3.35)$$

Proof. (a) Note that it suffices to show the claim for $u_0 = \mathbb{1}_{(-\infty,0]}$. Indeed, for all $u_0 \in \mathcal{I}_{\mathrm{PAM}}$ we have $\delta'\mathbb{1}_{[-\delta',0]} \le u_0 \le C'\mathbb{1}_{(-\infty,0]}$. Using Corollary 3.21, we infer that for all $u_0 \in \mathcal{I}_{\mathrm{PAM}}$, all $v \in V$, and all t large enough

$$\delta' C_{10}^{-2} u^{\mathbb{1}_{(-\infty,0]}}(t, vt) \le u^{u_0}(t, vt) \le C' u^{\mathbb{1}_{(-\infty,0]}}(t, vt).$$

where u^{u_0} denotes the solution to (PAM) with initial condition u_0. Let t be large enough such that $\mathcal{H}_{vt}$ occurs for all $v \in V$, which is possible by (3.2.31). Letting $(v,h) \in \mathcal{E}_t$ and writing $v' := \frac{vt}{t+h} \in V$, we infer that

$$\frac{u^{\mathbb{1}_{(-\infty,0]}}(t+h, vt)}{u^{\mathbb{1}_{(-\infty,0]}}(t, vt)} = \frac{E_{vt}\Big[e^{\int_0^{t+h} \xi(B_s)\,\mathrm{d}s}; B_{t+h} \le 0\Big]}{E_{vt}\Big[e^{\int_0^{t} \xi(B_s)\,\mathrm{d}s}; B_{t} \le 0\Big]} = e^{\mathsf{es}\cdot h} \frac{E_{vt}\Big[e^{\int_0^{t+h} \zeta(B_s)\,\mathrm{d}s}; B_{t+h} \le 0\Big]}{E_{vt}\Big[e^{\int_0^{t} \zeta(B_s)\,\mathrm{d}s}; B_{t} \le 0\Big]}. \quad (3.3.36)$$

Using Lemma 3.19, on $\mathcal{H}_{vt}$, the last fraction divided by $\frac{Y_{v'}^{\approx}(vt)}{Y_{v}^{\approx}(vt)}$ is bounded away from 0 and infinity for all t large enough. As in the derivation of (3.3.11), the term $\frac{Y_{v'}^{\approx}(vt)}{Y_{v}^{\approx}(vt)}$ can be written as

$$\frac{E_{vt}^{\zeta,\eta_{vt}^{\zeta}(v')}\Big[\exp\Big\{-\eta_{vt}^{\zeta}(v')\sum_{i=1}^{vt}\widehat{\tau}_i\Big\};\sum_{i=1}^{vt}\widehat{\tau}_i\in[-K,0]\Big]}{E_{vt}^{\zeta,\eta_{vt}^{\zeta}(v)}\Big[\exp\Big\{-\eta_{vt}^{\zeta}(v)\sum_{i=1}^{vt}\widetilde{\tau}_i\Big\};\sum_{i=1}^{vt}\widetilde{\tau}_i\in[-K,0]\Big]}\cdot\frac{\exp\Big\{-vt\Big(\frac{\eta_{vt}^{\zeta}(v')}{v'}-\overline{L}_{vt}^{\zeta}(\eta_{vt}^{\zeta}(v'))\Big)\Big\}}{\exp\Big\{-vt\Big(\frac{\eta_{vt}^{\zeta}(v)}{v}-\overline{L}_{vt}^{\zeta}(\eta_{vt}^{\zeta}(v))\Big)\Big\}},\tag{3.3.37}$$

where $\widehat{\tau}_i=\tau_i-E_{vt}^{\zeta,\eta_{vt}^{\zeta}(v')}[\tau_i]$ and $\widetilde{\tau}_i:=\tau_i-E_{vt}^{\zeta,\eta_{vt}^{\zeta}(v)}[\tau_i]$. But now, since $v'\in V$, as in the proof of Proposition 3.17, the first fraction of the previous display is bounded from below and above by positive constants, for all t large enough. Indeed, setting $x=vt$ in (3.3.12), the denominator in (3.3.37) equals the integral in (3.3.12) and, replacing v by v' in (3.3.12), the numerator equals the integral in (3.3.12). The claim then follows due to Lemma 3.18 and using (3.3.7). Therefore, taking logarithms in (3.3.36) and recalling the definition of $S_{vt}^{\zeta,v}(\eta)$ in (3.3.4), according to the previous considerations it suffices to show that the logarithm of the second fraction in (3.3.37) plus $\overline{\eta}(v)\cdot h$, i.e.

$$\big(S_{vt}^{\zeta,v}(\eta_{vt}^{\zeta}(v))-S_{vt}^{\zeta,v}(\eta_{vt}^{\zeta}(v'))\big)+\big(S_{vt}^{\zeta,v}(\eta_{vt}^{\zeta}(v'))-S_{vt}^{\zeta,v'}(\eta_{vt}^{\zeta}(v'))\big)+\overline{\eta}(v)\cdot h,\tag{3.3.38}$$

satisfies bound on the right-hand side of (3.3.34), uniformly in $(v,h)\in\mathcal{E}_t$. Recall that $\frac{1}{v'}=\frac{1}{v}\left(1+\frac{h}{t}\right)$, thus the second summand in (3.3.38) is $-h\cdot\eta_{vt}^{\zeta}(v')$. The triangular inequality entails

$$|\eta_{vt}^{\zeta}(v')-\overline{\eta}(v)|\le|\eta_{vt}^{\zeta}(v')-\overline{\eta}(v')|+|\overline{\eta}(v')-\overline{\eta}(v)|,\tag{3.3.39}$$

and so by Lemma 3.10, uniformly for $v'\in V$ and t large enough, the first term on the right-hand side of (3.3.39) can be upper bounded by $C_3\sqrt{\frac{\ln vt}{vt}}$, $\mathbb{P}$-a.s. Furthermore, by Lemma 3.9 d) we know that $\overline{\eta}$ is continuously differentiable and strictly decreasing, having uniform positive bounds of the derivative on every bounded subinterval of (v_c,∞). Hence, $\overline{\eta}(\cdot)$ is Lipschitz continuous on V and we therefore get that the second summand in (3.3.39) can be upper bounded by $c_1|v-v'|=c_1v\frac{|h|}{|t+h|}=c_1v\frac{|h|}{t}\cdot\frac{t}{|t+h|}\le c_2\frac{|h|}{t}$, uniformly for all $v,v'\in V$ and all t large enough, where the last inequality is due to $|h|/t\le\varepsilon(t)\to 0$. Therefore, the absolute value of the sum of the second and third summand in (3.3.38) is upper bounded by $C_{11}|h|\big(\sqrt{\frac{\ln t}{t}}+\frac{|h|}{t}\big)$ with $C_{11}:=c_2\vee C_3$.

It remains to show that the first summand in (3.3.38) tends to 0 as t tends to ∞. We write

$$\begin{aligned}S_{vt}^{\zeta,v}(\eta_{vt}^{\zeta}(v'))=S_{vt}^{\zeta,v}(\eta_{vt}^{\zeta}(v))&+\big(\eta_{vt}^{\zeta}(v')-\eta_{vt}^{\zeta}(v)\big)\cdot\big(S_{vt}^{\zeta,v}\big)'(\eta_{vt}^{\zeta}(v))\\&+\frac{1}{2}\big(\eta_{vt}^{\zeta}(v')-\eta_{vt}^{\zeta}(v)\big)^2\cdot\big(S_{vt}^{\zeta,v}\big)''(\widetilde{\eta})\end{aligned}$$

for some $\widetilde{\eta}\in[\eta_{vt}^{\zeta}(v')\wedge\eta_{vt}^{\zeta}(v),\eta_{vt}^{\zeta}(v')\vee\eta_{vt}^{\zeta}(v)]$. Recall that $S_{vt}^{\zeta,v}(\eta)=vt\big(\frac{\eta}{v}-\overline{L}_{vt}^{\zeta}(\eta)\big)$ and by definition $\big(\overline{L}_{vt}^{\zeta}\big)'\big(\eta_{vt}^{\zeta}(v)\big)=\frac{1}{v}$. Thus, $\big(S_{vt}^{\zeta,v}\big)'(\eta_{vt}^{\zeta}(v))=0$. Furthermore, $\big(S_{vt}^{\zeta,v}\big)''(\eta)=-vt\big(\overline{L}_{vt}^{\zeta}\big)''(\eta)$ and the function $\big(\overline{L}_{vt}^{\zeta}\big)''$ is uniformly bounded away from 0 and infinity by Lemma 3.8 on V. Thus, by the characterizing equation (3.2.23) and the implicit function theorem, on $\mathcal{H}_{vt}$, the function $\eta_{vt}^{\zeta}(\cdot)$ is differentiable with uniformly bounded first derivative, i.e.

$$\big|\eta_{vt}^{\zeta}(v')-\eta_{vt}^{\zeta}(v)\big|\le c_3|v'-v|\le c_3v\frac{|h|}{t(1-\varepsilon(t))}.\tag{3.3.40}$$

Thus, on $\mathcal{H}_{vt}$, the first summand in (3.3.38) can be bounded by $c_4 \cdot t \cdot \frac{h^2}{t^2} = c_4 \cdot \frac{h^2}{t}$, uniformly in $(v,h) \in \mathcal{E}_t$, for all t large enough. This implies a).

(b) The first part of the proof is similar to that of a); indeed, dividing by $u^{u_0}(t,vt)$ and taking logarithms in (3.3.35), one arrives at (3.3.38) again. The second part then consists of showing that (3.3.38) is lower and upper bounded by two strictly increasing linear functions for all $0 < h \le t^{1-\varepsilon}$. Following the same computations as in the proof of a), for $v \in V$ and $h > 0$, such that $v' \in V$, we have up to some additive constant, which is independent of h, that

$$\ln \frac{u^{u_0}(t+h,vt)}{u^{u_0}(t,vt)} \asymp c_t \frac{vt}{2}(v-v')^2 - \eta^{\zeta}_{vt}(v') \cdot h + \mathtt{es} \cdot h,$$

where c_t is a function which for t large enough is positive and bounded away from 0 and infinity. Because $\eta^{\zeta}_{vt}(v') < 0$, the latter expression is bounded from below by h/C_{12} and bounded from above by $C_{12} \cdot h$ for our choice of parameters, hence we can conclude. □

3.3.4 Space perturbation

While in the previous section we have been investigating the effects of time perturbations of u and related quantities, here we consider space perturbations. As before, let u^{u_0} denotes the solution to (PAM) with initial condition $u_0 \in \mathcal{I}_{\mathrm{PAM}}$.

Lemma 3.24. *Let $\varepsilon(t)$ be a positive function such that $\varepsilon(t) \to 0$ and $\frac{t\varepsilon(t)}{\ln t} \to \infty$ as $t \to \infty$. Then for all $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that $\mathbb{P}$-a.s., for all $u_0 \in \mathcal{I}_{PAM}$ we have*

(a)

$$\limsup_{t\to\infty} \sup \left\{ \left| \frac{1}{h} \ln\left(\frac{u(t,vt+h)}{u(t,vt)}\right) - L\big(\overline{\eta}(v)\big) \right| : (v,h) \in \mathcal{E}_t \right\} \le \varepsilon, \tag{3.3.41}$$

where $\mathcal{E}_t := \big\{(v,h) : \ v, v + \frac{h}{t} \in V, \ C(\varepsilon)\ln t \le |h| \le t\varepsilon(t)\big\}$.

(b) Let $\varepsilon(t)$ be a positive function such that $\varepsilon(t) \to 0$. Then there exists a constant $C_{13} < \infty$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}_2$ such that for all $t \ge \mathcal{T}_2$, uniformly in $0 \le h \le t\varepsilon(t)$, $v \in V$, $v + \frac{h}{t} \in V$ and $u_0 \in \mathcal{I}_{PAM}$ we have

$$C_{13}^{-1} e^{-C_{13}h} \cdot u(t,vt) \le u(t,vt+h) \le C_{13} e^{-h/C_{13}} \cdot u(t,vt). \tag{3.3.42}$$

Proof of Lemma 3.24. (a) For all $u_0 \in \mathcal{I}_{\mathrm{PAM}}$ we have $\delta' \mathbb{1}_{[-\delta',0]} \le u_0 \le C' \mathbb{1}_{(-\infty,0]}$. Using Corollary 3.21, we infer that for all $u_0 \in \mathcal{I}_{\mathrm{PAM}}$, all $v \in V$, and all t large enough

$$\delta' C_{10}^{-2} u^{\mathbb{1}_{(-\infty,0]}}(t,vt) \le u^{u_0}(t,vt) \le C' u^{\mathbb{1}_{(-\infty,0]}}(t,vt). \tag{3.3.43}$$

where u^{u_0} denotes the solution to (PAM) with initial condition u_0.

For this u_0, the solution to (PAM) can be represented by the Feynman-Kac formula (see (3.1.2))

$$u(t,vt) = E_{vt}\Big[e^{\int_0^t \xi(B_s)\,\mathrm{d}s}; B_t \le 0\Big].$$

If follows from Corollary 3.20 and (3.3.9) that if $\mathcal{H}_{vt}$ occurs, then, up to a universal multiplicative constant, this can be approximated by

$$E_{vt}\Big[e^{\int_0^{H_0} \xi(B_s)\,\mathrm{d}s}; H_0 \le t\Big].$$

We now consider t large enough such that $\mathcal{H}_{vt}$ occurs for all $v \in V$. Taking $(v,h) \in \mathcal{E}_t$ and defining $v' := v + \frac{h}{t} \in V$, we see that $\mathcal{H}_{v't}$ occurs as well. Therefore the fraction in (3.3.41), up to a positive multiplicative constant, is equal to

$$\frac{E_{v't}\big[\exp\big\{\int_0^{H_0}\zeta(B_s)\,\mathrm{d}s\big\}; H_0 \le t\big]}{E_{vt}\big[\exp\big\{\int_0^{H_0}\zeta(B_s)\,\mathrm{d}s\big\}; H_0 \le t\big]} \\ = \frac{E_{v't}\big[\exp\big\{\int_0^{H_0}\big(\zeta(B_s)+\eta^\zeta_{v't}(v')\big)\,\mathrm{d}s\big\}e^{-\eta^\zeta_{v't}(v')H_0}; H_0 \le t\big]}{E_{vt}\big[\exp\big\{\int_0^{H_0}\big(\zeta(B_s)+\eta^\zeta_{vt}(v)\big)\,\mathrm{d}s\big\}e^{-\eta^\zeta_{vt}(v)H_0}; H_0 \le t\big]}.$$

Using that $E_{vt}^{\zeta,\eta_{vt}(v)}[H_0] = E_{v't}^{\zeta,\eta_{v't}(v')}[H_0] = t$, recalling (3.2.5) and (3.3.4), the latter fraction can be written as

$$\frac{E_{v't}^{\zeta,\eta^\zeta_{v't}(v')}\big[e^{-\eta^\zeta_{v't}(v')(H_0 - E_{v't}^{\zeta,\eta^\zeta_{v't}(v')}[H_0])}; H_0 - E_{v't}^{\zeta,\eta^\zeta_{v't}(v')}[H_0] \le 0\big]}{E_{vt}^{\zeta,\eta^\zeta_{vt}(v)}\big[e^{-\eta^\zeta_{vt}(v)(H_0 - E_{vt}^{\zeta,\eta^\zeta_{vt}(v)}[H_0])}; H_0 - E_{vt}^{\zeta,\eta^\zeta_{vt}(v)}[H_0] \le 0\big]} \\ \times \exp\Big\{S_{vt}^{\zeta,v}(\eta^\zeta_{vt}(v)) - S_{v't}^{\zeta,v'}(\eta^\zeta_{v't}(v'))\Big\},$$

Since $\mathcal{H}_{vt}$ and $\mathcal{H}_{v't}$ occur, the fraction is bounded from below and above by positive constants (see Lemma 3.18). The logarithm of the second factor divided by h can be written as

$$\frac{1}{h}\big(S_{vt}^{\zeta,v}(\eta^\zeta_{vt}(v)) - S_{v't}^{\zeta,v'}(\eta^\zeta_{vt}(v))\big) + \frac{1}{h}\big(S_{v't}^{\zeta,v'}(\eta^\zeta_{vt}(v)) - S_{v't}^{\zeta,v'}(\eta^\zeta_{v't}(v'))\big). \tag{3.3.44}$$

We claim that the second summand in (3.3.44) tends to 0 uniformly in $(v,h) \in \mathcal{E}_t$ as $t \to \infty$, $\mathbb{P}$-a.s. Indeed, by a Taylor expansion we get

$$\begin{aligned} &S_{v't}^{\zeta,v'}(\eta^\zeta_{vt}(v)) - S_{v't}^{\zeta,v'}(\eta^\zeta_{v't}(v')) \\ &\quad = (S_{v't}^{\zeta,v'})'(\eta^\zeta_{v't}(v'))\cdot\big(\eta^\zeta_{vt}(v) - \eta^\zeta_{v't}(v')\big) + \frac{1}{2}(S_{v't}^{\zeta,v'})''(\widetilde{\eta})\big(\eta^\zeta_{vt}(v) - \eta^\zeta_{v't}(v')\big)^2 \end{aligned} \tag{3.3.45}$$

for some $\widetilde{\eta} \in \triangle$ between $\eta^\zeta_{v't}(v')$ and $\eta^\zeta_{vt}(v)$. Using again $E_{vt}^{\zeta,\eta_{vt}(v)}[H_0] = t$ and (3.2.10) we have $(S_{v't}^{\zeta,v'})'(\eta^\zeta_{v't}(v')) = 0$. Lemma 3.8(b) entails that $(\overline{L}^\zeta_{v't})''(\cdot)$ is uniformly bounded on $\triangle$ and thus

$$(S_{v't}^{\zeta,v'})''(\widetilde{\eta}) = -v't(\overline{L}^\zeta_{v't})''(\widetilde{\eta}) \in [-v'tc_1^{-1}, -v'tc_1]. \tag{3.3.46}$$

Furthermore, by Lemma 3.12 we have

$$\big|\eta^\zeta_{vt}(v) - \eta^\zeta_{v't}(v)\big| \le c_2\frac{|h|}{vt} \le c_3\frac{|h|}{t}, \tag{3.3.47}$$

and by (3.3.40)

$$\big|\eta^\zeta_{v't}(v) - \eta^\zeta_{v't}(v')\big| \le c_4|v - v'| = c_4\frac{|h|}{t}. \tag{3.3.48}$$

Thus, for all t large enough, uniformly in $(v,h) \in \mathcal{E}_t$, we get

$$\Big|\frac{1}{h}\big(S_{v't}^{\zeta,v'}(\eta^\zeta_{vt}(v)) - S_{v't}^{\zeta,v'}(\eta^\zeta_{v't}(v'))\big)\Big| \le c_5\frac{|h|}{t} \le \varepsilon(t), \tag{3.3.49}$$

which tends to zero by assumption.

It remains to show convergence of the first summand in (3.3.44). We first note that

$$\frac{1}{h}\big(S^{\zeta;v}_{vt}(\eta^{\zeta}_{vt}(v)) - S^{\zeta;v'}_{v't}(\eta^{\zeta}_{vt}(v))\big) = \frac{1}{h}\sum_{i=vt+1}^{v't} L^{\zeta}_{i}(\eta^{\zeta}_{vt}(v)). \tag{3.3.50}$$

To finish the proof, we use the following lemma. Recall $\mathcal{N}_1$ from definition (3.2.31) and let $\varepsilon^*(t) := \sup_{s\in[\lfloor t\rfloor,\lceil t\rceil]}\varepsilon(s)$.

Lemma 3.25 (cf. [16, Claim 5.2]). *For every $\delta > 0$ and every $q \in \mathbb{N}$, there exists $C_0 = C_0(q,\delta) > 0$ such that for all $t \geq 1$*

$$\mathbb{P}\Big(\sup_{\substack{C_0\ln\lfloor t\rfloor\leq|h|\leq\lceil t\rceil\cdot\varepsilon^*(t),\\ v\in V}}\Big|L(\overline{\eta}(v)) - \frac{1}{h}\sum_{i=vt+1}^{vt+h} L^{\zeta}_{i}(\eta^{\zeta}_{vt}(v))\Big| > \delta, \big(vt \geq \mathcal{N}_1\ \forall v\in V\big)\Big) \leq ct^{-q}. \tag{3.3.51}$$

To not disturb the flow of reading, we postpone the proof of Lemma 3.25 to the end of the proof of Lemma 3.24. We let A_t be the first event and B_t be the second event on the left-hand side of (3.3.51). By Lemma 3.25 with $q = 2$ and $C_0 = C_0(2,\delta/3)$, $\sum_n \mathbb{P}(A_n, B_n) < \infty$ and thus, by the first Borel-Cantelli lemma, $\mathbb{P}$-a.s. the event $A_n \cap B_n$ occurs only finitely often. Because $\mathcal{N}_1$ is $\mathbb{P}$-a.s. finite, we get

$$\sup_{\substack{C_0\ln t\leq|h|\leq t\varepsilon(t),\\ v\in V}}\Big|L(\overline{\eta}(v)) - \frac{1}{h}\sum_{i=v\lfloor t\rfloor+1}^{v\lfloor t\rfloor+h} L^{\zeta}_{i}(\eta^{\zeta}_{v\lfloor t\rfloor}(v))\Big| \leq \frac{\delta}{3} \tag{3.3.52}$$

$\mathbb{P}$-a.s. for all t large enough.

To bound the right-hand side of (3.3.50), we need to replace $v\lfloor t\rfloor$ in (3.3.52) by vt. First note that due to the strong Markov property at H_y we have $\sum_{i=x+1}^{z} L^{\zeta}_{i}(\eta) = \sum_{i=x+1}^{y} L^{\zeta}_{i}(\eta) + \sum_{i=y+1}^{z} L^{\zeta}_{i}(\eta)$ and thus

$$\sum_{i=v\lfloor t\rfloor+1}^{v\lfloor t\rfloor+h} L^{\zeta}_{i}(\eta) - \sum_{i=vt+1}^{vt+h} L^{\zeta}_{i}(\eta) = \ln E_{vt}\big[e^{\int_0^{H_{v\lfloor t\rfloor}}(\zeta(B_s)+\eta)}\big] - \ln E_{vt+h}\big[e^{\int_0^{H_{v\lfloor t\rfloor+h}}(\zeta(B_s)+\eta)}\big].$$

By [8, (2.0.1), p. 204] we have

$$\ln E_x\big[e^{-cH_y}\big] = \sqrt{2c}|y-x|, \qquad \text{for all } c\geq 0 \text{ and } x,y\in\mathbb{R}. \tag{3.3.53}$$

Therefore, for all t large enough, for every $\eta \in \triangle \subset (-\infty, 0)$ and $0 \geq \zeta(x) \geq -(\mathtt{es}-\mathtt{ei})$,

$$\sup_{\substack{C_0\ln t\leq|h|\leq t\varepsilon(t),\\ v\in V}}\Big|\frac{1}{h}\Big(\sum_{i=v\lfloor t\rfloor+1}^{v\lfloor t\rfloor+h} L^{\zeta}_{i}(\eta) - \sum_{i=vt+1}^{vt+h} L^{\zeta}_{i}(\eta)\Big)\Big| \leq \frac{2v\sqrt{2(|\eta|+(\mathtt{es}-\mathtt{ei}))}}{C_0\ln t} \leq \frac{\delta}{3}. \tag{3.3.54}$$

In particular, since $\eta^{\zeta}_{v\lfloor t\rfloor}(v) \in \triangle \subset (-\infty,0)$ (cf. (3.2.31)), (3.3.54) holds with η replaced by $\eta^{\zeta}_{v\lfloor t\rfloor}(v)$. Moreover, By Lemma 3.12, there exists $C_5 > 0$ such that $\mathbb{P}$-a.s. for all x large enough we have $\sup_{v\in V}|\eta^{\zeta}_{x+h}(v) - \eta^{\zeta}_{x}(v)| \leq C_5\frac{h}{x}$ for all $h\in[0,x]$. Furthermore, as a direct consequence of Lemma 3.8 (b), we have that the family of functions

$$\big\{I \ni \eta \mapsto L_x^\zeta(\eta): \ x \in \mathbb{R}, -(\mathtt{es}-\mathtt{ei}) \le \zeta \le 0\big\}$$

is equicontinuous on every compact interval $I \subset (-\infty, 0)$. All together this implies that $\mathbb{P}$-a.s. for all t large enough we have

$$\sup_{\substack{C_0 \ln t \le |h| \le t\varepsilon(t),\\ v \in V}} \Big| \frac{1}{h} \sum_{i=vt+1}^{vt+h} \big(L_i^\zeta(\eta_{v\lfloor t\rfloor}^\zeta(v)) - L_i^\zeta(\eta_{vt}^\zeta(v))\big)\Big| \le \frac{\delta}{3}. \tag{3.3.55}$$

Applying the triangle inequality to the inequalities (3.3.52)–(3.3.55), the absolute value of the difference of the right-hand side of (3.3.50) and $L(\overline{\eta}(v))$ is bounded from above by δ, uniformly in $(v,h) \in \mathcal{E}_t$ for all t large enough, completing the proof of claim (a).

(b) Analogously to the first steps in the proof of (a), it is enough to consider the case $u_0 = \mathbb{1}_{(-\infty,0]}$, and then to show that the expression in (3.3.44) is bounded from above and below by negative constants, uniformly for all $0 < h \le t\varepsilon(t)$. Performing the same calculations as in the proof of (a), i.e. using equations (3.3.45) to (3.3.48), one can observe that the second summand in (3.3.44) is contained in the interval $[-c_5 \frac{h}{t}, 0]$ for c_5 from (3.3.49) uniformly for all $v \in V$ and $v' := v + \frac{h}{t} \in V$ and all t large enough.

For the first summand in (3.3.44), we mention that due to the strong Markov property at time H_{vt}, we have

$$\begin{aligned} S_{vt}^{\zeta,v}(\eta_{vt}^\zeta(v)) - S_{v't}^{\zeta,v'}(\eta_{vt}^\zeta(v)) &= \ln E_{vt+h}\big[e^{\int_0^{H_0}(\zeta(B_s)+\eta_{vt}^\zeta(v))\,\mathrm{d}s}\big] - \ln E_{vt}\big[e^{\int_0^{H_0}(\zeta(B_s)+\eta_{vt}^\zeta(v))\,\mathrm{d}s}\big] \\ &= \ln E_{vt+h}\big[e^{\int_0^{H_{vt}}(\zeta(B_s)+\eta_{vt}^\zeta(v))\,\mathrm{d}s}\big]. \end{aligned}$$

Using (3.3.53),(BDD) and $\eta_{vt}^\zeta(v) \in \triangle \subset (-\infty, 0)$, for all t large enough, we get

$$-\sqrt{2(|\eta_{vt}^\zeta(v)| + \mathtt{es}-\mathtt{ei})}h \le \ln E_{vt+h}\big[e^{\int_0^{H_{vt}}(\zeta(B_s)+\eta_{vt}^\zeta(v))\,\mathrm{d}s}\big] \le -\sqrt{2|\eta_{vt}^\zeta(v)|}h \tag{3.3.56}$$

and we can conclude. □

To finish the proof of Lemma 3.24, we still have to provide the proof of Lemma 3.25.

Proof of Lemma 3.25. We decompose the difference in (3.3.51) as

$$L(\overline{\eta}(v)) - \sum_{i=vt+1}^{vt+h} L_i^\zeta(\eta_{vt}^\zeta(v)) = L(\overline{\eta}(v)) - \sum_{i=vt+1}^{vt+h} L_i^\zeta(\overline{\eta}(v)) + \sum_{i=vt+1}^{vt+h} \big(L_i^\zeta(\eta_{vt}^\zeta(v)) - L_i^\zeta(\overline{\eta}(v))\big). \tag{3.3.57}$$

To bound the last summand on the right-hand side, we again recall that the family $\big(L_i^\zeta(\cdot) : i \in \mathbb{R}, 0 \ge \zeta(\cdot) \ge \mathtt{ei}-\mathtt{es}\big)$ is bounded and uniformly equicontinuous on $\triangle$. Therefore, by Lemma 3.10, we have

$$\mathbb{P}\Big(\sup_{\substack{\ln\lfloor t\rfloor \le |h| \le \lceil t\rceil \varepsilon^*(t),\\ v \in V}} \Big|\frac{1}{h}\sum_{i=vt+1}^{vt+h} \big(L_i^\zeta(\eta_{vt}^\zeta(v)) - L_i^\zeta(\overline{\eta}(v))\big)\Big| > \frac{\delta}{2},\ vt \ge \mathcal{N}_1\ \forall v \in V\Big) \le ct^{-q}$$

for t large enough. It thus suffices to bound the first summand in (3.3.57), i.e. to show that there exists $C_0 = C_0(\varepsilon, q) > 0$ such that for all t large enough we have

$$\mathbb{P}\Big(\sup_{\substack{C_0 \ln\lfloor t\rfloor \le |h| \le \lceil t\rceil \varepsilon^*(t),\\ v\in V}} \Big|L(\overline{\eta}(v)) - \frac{1}{h}\sum_{i=vt+1}^{vt+h} L_i^\zeta(\overline{\eta}(v))\Big| > \frac{\delta}{2}\Big) \le ct^{-q}. \tag{3.3.58}$$

Hence, for every h we write $hL(\eta) - \sum_{i=vt+1}^{vt+h} L_i^\zeta(\eta) = \sum_{i=1}^{\lfloor h\rfloor+2} \widetilde{L}_i^{\zeta,h,v}(\eta)$, where the sequence of random variables $(\widetilde{L}_i^{\zeta,h,v}(\eta))_{i=1}^{\lfloor |h|\rfloor+2}$ is centered and $\mathbb{P}$-a.s. uniformly bounded in $v \in V$, $h \in \mathbb{R}$, $t \in \mathbb{R}$ and $\eta \in \triangle$, as well as fulfills the mixing condition from Lemma A.2. Thus, we can apply Corollary B.2 to show that there exist constants $C > 0$ and $C_0(\varepsilon, q) > 0$, such that for all $v \in V$ and all h fulfilling $|h| \ge C_0 \ln t$ we have

$$\mathbb{P}\Big(\Big|L(\eta) - \frac{1}{h}\sum_{i=vt+1}^{vt+h} L_i^\zeta(\eta(v))\Big| > \frac{\delta}{2}\Big) \le \sqrt{e}\exp\Big\{-\frac{1}{2C\lfloor h\rfloor}\Big(\frac{|h|\varepsilon}{2}\Big)^2\Big\} \le c_1 t^{-q-3}$$

for all t large enough.

To get the "uniform bound" from (3.3.58), we first show it on the grid $V_n := (\frac{1}{n}\mathbb{Z}) \cap V$ and $C_n^{(t)} := (\frac{1}{n}\mathbb{Z}) \cap [\ln\lfloor t\rfloor, \lceil t\rceil\varepsilon^*(t)]$, $n \in \mathbb{N}$. Indeed, because $|V_n| \le (\mathrm{diam}(V) + 1)n$ and $|C_n^{(t)}| \le nt$, we get

$$\mathbb{P}\Big(\sup_{|h|\in C_{\lfloor t\rfloor}^{(t)},\ v\in V_{\lfloor t\rfloor}} \Big|L(\overline{\eta}(v)) - \frac{1}{h}\sum_{i=vt+1}^{vt+h} L_i^\zeta(\overline{\eta}(v))\Big| > \frac{\delta}{2}\Big) \le \mathrm{diam}(V)c_2 \cdot t^{-q} \tag{3.3.59}$$

for all t large enough. To control all $v \in V$ and $|h| \in [\ln\lfloor t\rfloor, \lceil t\rceil\varepsilon^*(t)]$, we note that for all $s \ge 0$ we have

$$\begin{aligned}
&\ln E_{vt+\frac{k}{n}+s}\big[e^{\int_0^{H_{vt}}(\zeta(B_s)+\eta)\,\mathrm{d}s}\big] - \ln E_{vt+\frac{k}{n}}\big[e^{\int_0^{H_{vt}}(\zeta(B_s)+\eta)\,\mathrm{d}s}\big]\\
&\quad = \ln E_{vt+\frac{k}{n}+s}\big[e^{\int_0^{H_{vt+\frac{k}{n}}}(\zeta(B_s)+\eta)\,\mathrm{d}s}\big] \in \Big[-s\sqrt{2(\mathtt{es}-\mathtt{ei}+|\eta|)}, 0\Big]
\end{aligned}$$

where the last display is again a consequence of (3.3.53). Thus for all h not on the grid the terms in (3.3.59) differ at most by a term of order $1/t$. A similar statement holds for all $v \in V$ not on the grid, because $\eta(\cdot)$ is uniformly Lipschitz continuous on V (see Lemma 3.9 (c)). Thus the uniform bound in (3.3.59) can be extended to be valid for all h such that $C_0 \ln\lfloor t\rfloor \le |h| \le \lceil t\rceil\varepsilon^*(t)$. This completes the proof. □

3.3.5 Approximation results

In this section we mainly show how moment generating functions can be used in order to approximate quantities related to the solution to (PAM) and BBMRE. As usual, for $u_0 \in \mathcal{I}_{\mathrm{PAM}}$ let u^{u_0} be the solution to (PAM) with initial condition u_0.

Lemma 3.26. *There exists a constant $C_{14} > 0$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}_3$ such that for all $u_0 \in \mathcal{I}_{PAM}$ and $t \ge \mathcal{T}_3$,*

$$\Big|\ln u^{u_0}(t, v_0 t) - \sum_{i=1}^{v_0 t}\big(L_i^\zeta(\overline{\eta}(v_0)) - L(\overline{\eta}(v_0))\big)\Big| \le C_{14}\ln t. \tag{3.3.60}$$

Proof. By (3.3.22) and $\Lambda(v_0) = 0$, we have $L^*(1/v_0) = \frac{\mathtt{es}}{v_0}$. Also, by (PAM-INI) and monotonicity of the solution to (PAM) in its initial condition, we have $u^{\delta'\mathbb{1}_{[-\delta',0]}} \le u^{u_0} \le u^{C'\mathbb{1}_{(-\infty,0]}}$. Thus, on the one hand, by the many-to-few lemma (Proposition 2.15) and Lemma 3.19, for all $u_0 \in \mathcal{I}_{\text{PAM}}$ and t such that $v_0 t \ge \mathcal{N}_1$, where $\mathcal{N}_1$ was defined in (3.2.31), we have

$$\big|\ln u^{u_0}(t, v_0 t) - \big(\ln Y^{\approx}_{v_0}(v_0 t) + v_0 t L^*(1/v_0)\big)\big| \le \ln(C_{10}/\delta').$$

On the other hand, due to Proposition 3.17 and Corollary 3.16, there exists a finite random variable $\mathcal{N}$, such that for all $t \ge \mathcal{N}$ we have

$$\Big|\ln Y^{\approx}_{v_0}(v_0 t) + v_0 t L^*(1/v_0) + \ln \sigma^{\zeta}_{v_0 t}(v_0) - \sum_{i=1}^{v_0 t}\big(L_i^{\zeta}(\overline{\eta}(v_0)) - L(\overline{\eta}(v_0))\big)\Big| \\ \le \ln C_8 + C_6 \ln v_0 + C_6 \ln t.$$

Finally, by (3.3.7), $|\ln \sigma^{\zeta}_{v_0 t}(v_0) - \frac{1}{2}\ln t| \le \ln C_7 + \frac{1}{2}|\ln v_0|$ for all t such that $v_0 t \ge \mathcal{N}_1$. Combining this with the previous two display, inequality (3.3.60) follows with $\mathcal{T}_3 := (\mathcal{N} \vee \mathcal{N}_1)/v_0$ and C_{14} suitable. ☐

We introduce the so-called *breakpoint inverse*

$$T_x^{u_0,M} := \inf\big\{t \ge 0 : u^{u_0}(t,x) \ge M\big\}, \quad x \in \mathbb{R},\ M \in [0,\infty),\ u_0 \in \mathcal{I}_{\text{PAM}}, \tag{3.3.61}$$

and abbreviate

$$T_x^{(M)} := T_x^{\mathbb{1}_{(-\infty,0]},M}. \tag{3.3.62}$$

Next, we state an important approximation result for $T_x^{u_0,M}$, $x \ge 0$, in terms of the centered logarithmic moment generating functions.

Lemma 3.27. *There exists a constant $C_{15} < \infty$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{C}_1 = \mathcal{C}_1(M, u_0)$, $M > 0, u_0 \in \mathcal{I}_{PAM}$, such that for all $x \ge 1$,*

$$\Big|T_x^{u_0,M} - \frac{1}{v_0 L(\overline{\eta}(v_0))}\sum_{i=1}^{x} L_i^{\zeta}(\overline{\eta}(v_0))\Big| \le \mathcal{C}_1 + C_{15}\ln x. \tag{3.3.63}$$

Additionally, for each $u_0 \in \mathcal{I}_{PAM}$ and $M > 0$,

$$\lim_{x\to\infty} \frac{T_x^{u_0,M}}{x} = \frac{1}{v_0} \quad \mathbb{P}\text{-}a.s. \tag{3.3.64}$$

Proof. We set $t = x/v_0$ and let

$$h_t := \frac{1}{v_0 L(\overline{\eta}(v_0))}\sum_{i=1}^{v_0 t}\big(L(\overline{\eta}(v_0) - L_i^{\zeta}(\overline{\eta}(v_0))\big).$$

We first note that due to Lemma A.2, the family $(L(\overline{\eta}(v_0)) - L_i^{\zeta}(\overline{\eta}(v_0)))_{i\in\mathbb{Z}}$ satisfies the assumptions of Corollary B.2 with all m_i equal to some large enough finite constant and thus $\sum_n \mathbb{P}(|h_n| \ge C\sqrt{n \ln n}) < \infty$ for some $C > 1$ large enough. The first Borel-Cantelli lemma then readily supplies us with $|h_n| < C\sqrt{n\ln n}$ $\mathbb{P}$-a.s. for all n large enough. To control non-integer t, we recall

$$h_t - h_{\lfloor t \rfloor} = \frac{1}{v_0 L(\overline{\eta}(v_0))}\big(v_0(t-\lfloor t \rfloor)L(\overline{\eta}(v_0)) - \ln E_{v_0 t}\big[e^{\int_0^{H_{v_0\lfloor t \rfloor}}(\zeta(B_s)+\overline{\eta}(v_0))\,\mathrm{d}s}\big]\big)$$

and hence $\mathbb{P}$-a.s. that $|h_t - h_{\lfloor t \rfloor}| \le 1 + \frac{\sqrt{2(\mathtt{es}-\mathtt{ei}-\overline{\eta}(v_0))}}{|L(\overline{\eta}(v_0))|}$ by (BDD) and [8, (2.0.1), p. 204], thus giving

$$|h_t| < c_1\sqrt{t \ln t} \text{ } \mathbb{P}\text{-a.s. for all } t \text{ large enough.} \tag{3.3.65}$$

To show the desired inequality, we note that (3.3.63) is equivalent to

$$\big|T^{u_0,M}_{v_0 t} - (t-h_t)\big| \le \mathcal{C}_1 + C_{15}\ln(v_0 t). \tag{3.3.66}$$

For proving the latter, observe that it is sufficient to show that we can choose $C_{15} > 0$ as well as a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}$, such that

$$\begin{aligned} &u^{\delta'\mathbb{1}_{[-\delta',0]}}(t-h_t+C_{15}\ln t, v_0 t) \ge M \quad \text{ and} \\ &u^{\mathbb{1}_{(-\infty,0]}}(t-h_t-C_{15}\ln t, v_0 t) < \frac{M}{2C'},\ \forall t \ge \mathcal{T}. \end{aligned} \tag{3.3.67}$$

with δ', C' from (PAM-INI). Indeed, due to (PAM-INI), the first inequality in (3.3.67) implies $T^{u_0,M}_{v_0 t} \le T^{\delta'\mathbb{1}_{[-\delta',0]},M}_{v_0 t} \le t-h_t+C_{15}\ln t$ for $t \ge \mathcal{T}$. To use the second inequality, first note that $T^{u_0,M}_{v_0 t} \ge T^{C'\mathbb{1}_{(-\infty,0]},M}_{v_0 t} = T^{\mathbb{1}_{(-\infty,0]},M/C'}_{v_0 t}$. Then, using Lemma D.2, (3.3.67) implies $T^{\mathbb{1}_{(-\infty,0]},M/C'}_{v_0 t} \ge t-h_t-C_{15}\ln t$ for all $t \ge \mathcal{T}$. For $t < \mathcal{T}$ on the other hand, we can use that $\mathbb{P}$-a.s., the family $T^{u_0,M}_{v_0 t}$, $t < \mathcal{T}$, as well as the $L^\zeta_i(\overline{\eta}(v_0))$ are uniformly bounded, allowing us to upper bound the remaining cases of (3.3.66) by some finite random variable $\mathcal{C}_1$.

Thus, in order to show (3.3.67), note that for $\alpha \in \mathbb{R}$ and uniformly in $u_0 \in \mathcal{I}_{\mathrm{PAM}}$ we have that

$$\begin{aligned} &\ln u^{u_0}(t-h_t+\alpha\ln t, v_0 t) \\ &= \ln\Big(\frac{u^{u_0}(t-h_t+\alpha\ln t, v_0 t)}{u^{u_0}(t, v_0 t)}\Big) + \sum_{i=1}^{v_0 t}\big(L^\zeta_i(\overline{\eta}(v_0)) - L(\overline{\eta}(v_0))\big) + a_t \\ &= (-h_t+\alpha\ln t)(\mathtt{es}-\overline{\eta}(v_0)) + v_0 L(\overline{\eta}(v_0))h_t + b(\alpha,t) = \alpha(\mathtt{es}-\overline{\eta}(v_0))\ln t + b(\alpha,t), \end{aligned}$$

for some error terms a_t and $b(\alpha,t)$ fulfilling $|a_t| \le C_{14}\ln t$ and

$$|b(\alpha,t)| \le C_{14}\ln t + C_{11} + C_{11}\cdot|\alpha\ln t - h_t|\cdot\Big(\sqrt{\frac{\ln t}{t}} + \frac{|\alpha\ln t - h_t|}{t}\Big)$$

for all t large enough. Indeed, the first equality is due to Lemma 3.26, the second due to the time perturbation Lemma 3.23, the last one due to the identity $\mathtt{es} - \overline{\eta}(v_0) = v_0 L(\overline{\eta}(v_0))$. Then due to $|h_t| \le c_1\sqrt{t\ln t}$ for large t (cf. (3.3.65)), choosing $C_{15} = \alpha > \frac{2C_{11}\cdot C^2}{\mathtt{es}-\overline{\eta}(v_0)}$ the latter term tends to infinity, supplying us with (3.3.67).

To complete the proof, equation (3.3.64) is a direct consequence of (3.3.63) and (3.2.17). □

Recall the definition $\overline{m}^{u_0,M} = \overline{m}^{\xi,u_0,M}$ from (3.0.1).

Corollary 3.28. *For all $u_0 \in \mathcal{I}_{PAM}$ and $M > 0$ we have*

$$\frac{\overline{m}^{u_0,M}(t)}{t} \underset{t\to\infty}{\longrightarrow} v_0 \quad \mathbb{P}\text{-}a.s. \tag{3.3.68}$$

Proof. For an upper bound, we have

$$\limsup_{t\to\infty} \frac{\overline{m}^{u_0,M}(t)}{t} = \limsup_{t\to\infty} \frac{\overline{m}^{u_0,M}(t)}{T^{u_0,M}_{\overline{m}^{u_0,M}(t)}} \frac{T^{u_0,M}_{\overline{m}^{u_0,M}(t)}}{t} \leq v_0,$$

where the last inequality is due to $T^{u_0,M}_{\overline{m}^{u_0,M}(t)} \leq t$ and (3.3.64). To get a lower bound, we can use the properties of the Lyapunov exponent from Corollary 3.22, giving the lower bound $\liminf_{t\to\infty} \frac{\overline{m}^{u_0,M}(t)}{t} \geq v$ for all $v \in [0, v_0)$, and we can conclude. □

Lemma 3.29. *For every $M > 0$ there exists a constant $C_{16} = C_{16}(M) > 0$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}_4 = \mathcal{T}_4(M)$ such that for all $u_0 \in \mathcal{I}_{PAM}$ and $t \geq \mathcal{T}_4$,*

$$t - C_{16} \leq T^{u_0,M}_{\overline{m}^{u_0,M}(t)} \leq t. \tag{3.3.69}$$

Proof. By definition, the inequality $T^{u_0,M}_{\overline{m}^{u_0,M}(t)} \leq t$ follows directly. To show $t - C_{16} \leq T^{u_0,M}_{\overline{m}^{u_0,M}(t)}$, recall that due to (3.3.68) we can use time perturbation. Indeed, by defining $C_{16} := C_{12} \ln\big(C_{12} \cdot 3C'/M\big)$ with C' from (PAM-INI) and C_{12} from Lemma 3.23 b), for all t large enough

$$u^{\mathbb{1}_{(-\infty,0]}}(t - C_{16}, \overline{m}^{u_0,M}(t)) \leq C_{12} e^{-C_{16}/C_{12}} u^{u_0}(t, \overline{m}^{u_0,M}(t)) < \frac{M}{2C'}$$

and thus, recalling Lemma D.2, we get $T^{u_0,M}_{\overline{m}^{u_0,M}(t)} \geq T^{C'\mathbb{1}_{(-\infty,0]},M}_{\overline{m}^{u_0,M}(t)} = T^{\mathbb{1}_{(-\infty,0]},M/C'}_{\overline{m}^{u_0,M}(t)} \geq t - C_{16}$ for all t large enough and we can conclude. □

Recall definition (3.3.62) for $T^{(M)}_x$.

Corollary 3.30. *There exists $\overline{K} \in (1, \infty)$ such that $\mathbb{P}$-a.s., for all $M > 0$ and for all x large enough*

$$\sup_{|y|\leq 1} T^{(M)}_{x+y} - \overline{K} \leq T^{(M)}_x \leq \inf_{|y|\leq 1} T^{(M)}_{x+y} + \overline{K}. \tag{3.3.70}$$

Proof. We set $\overline{K} := 1 + C_{12}\big(\ln(2C_{12}C_{13}) + C_{13}\big)$. Then due to (3.3.64), $\mathbb{P}$-a.s. for all x large enough we have

$$\frac{x + y'}{T^{(M)}_{x+y''} \pm \overline{K}} \in V \quad \forall\, y', y'' \in [-1, 1].$$

This allows us to apply the inequalities (3.3.35) and (3.3.42) for $u_0 = \mathbb{1}_{(-\infty,0]}$. Indeed, for all $|y| \leq 1$,

$$\begin{aligned} u(T^{(M)}_{x+y} - \overline{K}, x) &\leq C_{13} e^{C_{13}} u(T^{(M)}_{x+y} - \overline{K}, x + y) \leq C_{13} e^{C_{13}} C_{12} e^{-(\overline{K}-1)/C_{12}} u(T^{(M)}_{x+y} - 1, x + y) \\ &< \frac{M}{2}, \end{aligned}$$

where the first inequality is due to (3.3.42), the second one due to (3.3.35) and the last one uses $u(T^{(M)}_{x+y} - 1, x + y) < M$. By Lemma D.2 we get $u(t, x) < M$ for all $t \leq T^{(M)}_{x+y} - \overline{K}$ and thus the left-hand side in (3.3.70). Analogously, first applying (3.3.42) and then (3.3.35), we have

$$\begin{aligned} u(T^{(M)}_{x+y}+\overline{K},x) &\geq C_{13}^{-1}e^{-C_{13}}u(T^{(M)}_{x+y}+\overline{K},x+y) \geq C_{13}^{-1}e^{-C_{13}}C_{12}^{-1}e^{\overline{K}/C_{12}}u(T^{(M)}_{x+y},x+y) \\ &\geq M \end{aligned}$$

for all $|y| \leq 1$, giving the right-hand side of (3.3.70). □

Corollary 3.31. *Let $\overline{m}^M(t) = \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},M}(t)$, $M > 0$, be defined in (3.0.1). Then for all $0 < \varepsilon \leq M$ there exists $C = C(\varepsilon, M)$ such that $\mathbb{P}$-a.s. for all t large enough*

$$0 \leq \overline{m}^{\varepsilon}(t) - \overline{m}^M(t) \leq C.$$

Proof. The first inequality is clear. By Corollary 3.28, we can use the second inequality from Lemma 3.24 b) and get the claim by defining $C := C_{13}\ln\big(C_{13}\cdot M/\varepsilon\big)$ with C_{13} from Lemma 3.24 b) to get $u^{\mathbb{1}_{(-\infty,0]}}(t, \overline{m}^{\varepsilon}(t) - C) \geq M$ and thus $\overline{m}^M(t) \geq \overline{m}^{\varepsilon}(t) - C$ for all t large enough. □

3.3.6 Proof of Theorem 3.4

Using the preparatory results from the previous sections, it is now possible to obtain an invariance principle for the front of the solution to (PAM). Roughly speaking, up to some error which can be controlled by the results from the previous sections, we have $\overline{m}(t) \approx \ln u(t, v_0 t)$ and can then use the invariance principle from Theorem 3.3 to conclude.

Proof of Theorem 3.4. Let $u_0 \in \mathcal{I}_{\text{PAM}}$, $M > 0$ and abbreviate $u = u^{u_0}$ and $\overline{m} := \overline{m}^{u_0,M}$. We first assume $\sigma^2_{v_0} > 0$. Then we have to show that the sequence of processes

$$[0,\infty) \ni t \mapsto \frac{\overline{m}(nt) - v_0 nt}{\sqrt{n\widetilde{\sigma}^2_{v_0}}}, \quad n \in \mathbb{N}, \tag{3.3.71}$$

where $\widetilde{\sigma}^2_{v_0} > 0$ is given in (3.3.76) below, converges in $\mathbb{P}$-distribution to standard Brownian motion in the Skorohod space $D([0,\infty))$. Notice that $[0,\infty) \ni t \mapsto \overline{m}(t)$ might not be càdlàg *only*, but this can happen only in 0. To avoid this issue, above convergence is defined as convergence of the sequence of processes in (3.3.71) where we set $\overline{m}(t) \equiv 0$ for t such that $\overline{m}(t) \leq 0$, making it càdlàg.

Due to the limiting behavior and the continuity of $x \mapsto u(t,x)$ for $t > 0$, the value $r(t) := \overline{m}(t) - v_0 t$ is the largest solution to

$$\ln u(t, v_0 t + r(t)) = -\ln 2.$$

We define

$$\begin{aligned} \mathcal{L}(t,h) &:= \ln \frac{u(t, v_0 t + h)}{u(t, v_0 t)}, \quad t > 0,\ h \in \mathbb{R}, \\ U(t) &:= -\ln u(t, v_0 t) - \ln 2 = \mathcal{L}(t, r(t)), \quad t \geq 0, \end{aligned}$$

let

$$\delta \in \big(0, |L(\overline{\eta}(v_0))|\big)$$

and $\varepsilon(t)$ be a positive function such that $\varepsilon(t) \to 0$ and $\varepsilon(t)t^{1/2} \to \infty$. Then by Lemma 3.24, there is $C(\varepsilon) > 0$ such that for t large enough and all $h \in \mathbb{R}$ fulfilling $C(\varepsilon)\ln t \leq |h| \leq \varepsilon(t)t$

and $v_0 + \frac{h}{t} \in V$ we have

$$-\big(|L(\overline{\eta}(v_0))| + \delta\big)h \le \mathcal{L}(t,h) \le -\big(|L(\overline{\eta}(v_0))| - \delta\big)h. \tag{3.3.72}$$

Now, multiplying (3.3.63) by v_0, replacing x by $\overline{m}(t)$ in (3.3.63) and recalling that $t - C_{16} \le T_{\overline{m}(t)} \le t$ by Lemma 3.29, we get

$$\Big|(\overline{m}(t) - v_0 t) - \frac{1}{L(\overline{\eta}(v_0))}\sum_{i=1}^{\overline{m}(t)}\big(L_i^{\zeta}(\overline{\eta}(v_0)) - L(\overline{\eta}(v_0))\big)\Big| \le v_0\mathcal{C}_1 + C_{16} + v_0 C_{15}\ln(\overline{m}(t)) \tag{3.3.73}$$

for all t large enough. Next, recall that $\frac{\overline{m}(t)}{t} \to v_0$ by Corollary 3.28 and that the standardized sum $\frac{1}{\sqrt{n}}\sum_{i=1}^{nt}\big(L_i^{\zeta}(\overline{\eta}(v_0)) - L(\overline{\eta}(v_0))\big)$ converges in distribution to a non-degenerate Gaussian random variable by Lemma 3.14. As a consequence, in combination with (3.3.73), we infer that $|r(t)| = |\overline{m}(t) - v_0 t| \in [C(\varepsilon)\ln t, \varepsilon(t)t]$ with probability tending to 1 as t tends to infinity. This and (3.3.72) implies

$$r(t) \in \Big[\frac{U(t)}{|L(\overline{\eta}(v_0))| \mp \delta}, \frac{U(t)}{|L(\overline{\eta}(v_0))| \pm \delta}\Big], \tag{3.3.74}$$

with probability tending to 1 as t tends to ∞, where the upper sign is chosen if $U(t) > 0$ and the lower sign if $U(t) < 0$. If $\sigma_{v_0}^2 > 0$, due to $\Lambda(v_0) = 0$ and Theorem 3.3, the sequence of processes

$$[0,\infty) \ni t \mapsto \frac{1}{\sqrt{n v_0 \sigma_{v_0}^2}} U(nt), \quad n \in \mathbb{N}, \tag{3.3.75}$$

converges in $\mathbb{P}$-distribution to standard Brownian motion. Because (3.3.74) holds for all $\delta > 0$ small enough, Theorem 3.4 is a direct consequence of the convergence in distribution of (3.3.75) by choosing

$$\widetilde{\sigma}_{v_0} := \frac{\sqrt{\sigma_{v_0}^2 v_0}}{|L(\overline{\eta}(v_0))|}. \tag{3.3.76}$$

where $\sigma_{v_0}^2$ is defined in (3.3.1). This gives the second part of Theorem 3.4. If $\sigma_{v_0}^2 = 0$, we can proceed analogously and the first part of Theorem 3.4 follows from the first part of Theorem 3.3 and (3.3.74). □

3.4 Log-distance of the fronts of the solutions to PAM and F-KPP

We finally prove our last main result of this chapter, Theorem 3.5. In Sections 3.4.1 and 3.4.2, we will assume that $u_0 = w_0 = \mathbb{1}_{(-\infty,0]}$. Indeed, using a comparison argument in the proof of Theorem 3.5, it will turn out that this is actually sufficient for or purposes. It should also be mentioned here that the tools we employ are inherently probabilistic. As a consequence, and for notational convenience, we will mostly formulate the respective results in terms of the BBMRE in what follows below; the correspondence to the results in PDE terms is immediate from (3.1.2) and Remark 2.18.

In the case $u_0 = w_0 = \mathbb{1}_{(-\infty,0]}$, using Markov's inequality we infer

$$\mathtt{P}_x^{\xi}\big(N^{\le}(t,0) \ge 1\big) \le \mathtt{E}_x^{\xi}\big[N^{\le}(t,0)\big]$$

and thus $\overline{m}(t) \geq m(t)$ for all $t \geq 0$, which establishes the first inequality in (3.1.5). The rest of this section will be dedicated to deriving the second inequality in (3.1.5), i.e., that the front of the randomized F-KPP equations lags behind the front of the solution to the parabolic Anderson model at most logarithmically. We introduce some notation and, recalling the notation $X^{\mathbf{u}}$ introduced from section 2.2.1, start with considering certain "well-behaved" particles

$$\begin{aligned} N_{s,u,t}^{\mathcal{L},M} := \big|\big\{\mathbf{u} \in N(s) :\ & X_s^{\mathbf{u}} \leq 0, \\ & H_k^{\mathbf{u}} \geq u - T_k^{(M)} - 5\chi_1(\overline{m}^M(t))\ \forall k \in \{1, \dots, \lfloor \overline{m}^M(t) \rfloor\}\big\}\big|, \\ & M > 0,\ s, t, u \geq 0; \end{aligned} \tag{3.4.1}$$

(3.4.1) here, $H_k^{\mathbf{u}} := \inf\{t \geq 0 : X_t^{\mathbf{u}} = k\}$, the random variable $T_k^{(M)}$ has been defined in (3.3.62), and

$$\chi_b(x) := \mathcal{C}_1 + b(1 + \overline{K} + C_{16}) + C_{15}(\ln x \vee 1), \quad x \in (0, \infty),\ b \in \mathbb{R}, \tag{3.4.2}$$

where $\mathcal{C}_1$ and C_{15} have been defined in Lemma 3.27, $\overline{K}$ is taken from Corollary 3.30, the constant C_{16} from Lemma 3.29. We abbreviate $N_t^{\mathcal{L},M} := N_{t,t,t}^{\mathcal{L},M}$ and call the particles contributing to $N_t^{\mathcal{L},M}$ *leading particles at time t*. Cauchy-Schwarz immediately gives

$$\mathtt{P}_x^{\xi}\left(N^{\leq}(t,0) \geq 1\right) \geq \mathtt{P}_x^{\xi}\left(N_t^{\mathcal{L},M} \geq 1\right) \geq \frac{\mathtt{E}_x^{\xi}\left[N_t^{\mathcal{L},M}\right]^2}{\mathtt{E}_x^{\xi}\left[(N_t^{\mathcal{L},M})^2\right]}. \tag{3.4.3}$$

The next two sections are dedicated to deriving an upper bound for the denominator and a lower bound for the numerator of the right-hand side, both for x in a neighborhood of $\overline{m}(t)$.

3.4.1 First moment of leading particles

The biggest chunk of this section consists of proving the following first moment bound on the number of leading particles. Recall the notation $\overline{m}^M(t)$ from (3.0.2).

Lemma 3.32. *For all $M > 0$ there exists $\gamma_1 = \gamma_1(M) \in (0, \infty)$ such that $\mathbb{P}$-a.s., for all t large enough*

$$\inf_{x \in [\overline{m}^M(t)-1, \overline{m}^M(t)+1]} \mathtt{E}_x^{\xi}\left[N_t^{\mathcal{L},M}\right] \geq t^{-\gamma_1}.$$

Proof. Let $M > 0$. To simplify notation, we omit the index $M > 0$ in the quantities involved and write $N_{s,u,t}^{\mathcal{L}} := N_{s,u,t}^{\mathcal{L},M}$, $T_x^{(M)} := T_x$, $\overline{m}^M(t) := \overline{m}(t)$ from now on.

Let $A_{u,t} := \{H_k \geq u - T_k - 5\chi_1(\overline{m}(t))\ \forall k \in \{1, \dots, \lfloor m(t) \rfloor\}\}$, let K be such that (3.3.16) holds and set $\bar{t} := T_{\lfloor \overline{m}(t) \rfloor}$. We obtain for all t large enough that

$$\begin{aligned}
\inf_{x\in[\overline{m}(t)-1,\overline{m}(t)+1]} \mathtt{E}_x^\xi \left[N_t^{\mathcal{L}}\right] &\geq \frac{\inf_{x\in[\overline{m}(t)-1,\overline{m}(t)+1]} \mathtt{E}_x^\xi \left[N_t^{\mathcal{L}}\right]}{2\mathtt{E}^\xi_{\lfloor \overline{m}(t)\rfloor}\left[N^{\leq}(\bar{t},0)\right]} \\
&\geq \frac{c}{2}\frac{\mathtt{E}^\xi_{\lfloor m(t)\rfloor}\left[N^{\mathcal{L}}_{t-1,t,t+1}\right]}{\mathtt{E}^\xi_{\lfloor m(t)\rfloor}\left[N^{\leq}\left(\bar{t},0\right)\right]} = \frac{c}{2}\frac{E_{\lfloor m(t)\rfloor}\left[e^{\int_0^{t-1}\xi(B_s)\,\mathrm{d}s}; B_{t-1}\leq 0, A_{t+1,t}\right]}{E_{\lfloor m(t)\rfloor}\left[e^{\int_0^{\bar{t}}\xi(B_s)\,\mathrm{d}s}; B_{\bar{t}}\leq 0\right]} \\
&\geq c_1\frac{E_{\lfloor m(t)\rfloor}\left[e^{\int_0^{t-1}\zeta(B_s)\,\mathrm{d}s}; B_{t-1}\leq 0, A_{t+1,t}\right]}{E_{\lfloor m(t)\rfloor}\left[e^{\int_0^{\bar{t}}\zeta(B_s)\,\mathrm{d}s}; B_{\bar{t}}\leq 0\right]};
\end{aligned} \tag{3.4.4}$$

here, the first inequality follows from the definition of $T_{\lfloor \overline{m}(t)\rfloor}$, the second inequality is due to Lemma D.3, the equality follows using Proposition 2.15 and the last inequality is due to $\xi = \zeta + \mathtt{es}$, as well as (3.3.70) which gives $\bar{t} = T_{\lfloor \overline{m}(t)\rfloor} \leq T_{\overline{m}(t)} + \overline{K} \leq t + \overline{K}$. Now the numerator can be bounded from below by

$$\begin{aligned}
&E_{\lfloor m(t)\rfloor}\left[e^{\int_0^{t-1}\zeta(B_s)\,\mathrm{d}s}; H_0\in[t-3\overline{K}-C_{16},t-1], B_{t-1}\leq 0, A_{t+1,t}\right] \\
&\quad\geq E_{\lfloor \overline{m}(t)\rfloor}\left[e^{\int_0^{H_0}\zeta(B_s)\,\mathrm{d}s}\, E_0\left[e^{\int_0^{r}\zeta(B_s)\,\mathrm{d}s}; B_{t-1-r}\leq 0\right]\Big|_{r=t-1-H_0};\right. \\
&\qquad\qquad\left. H_0\in[t-3\overline{K}-C_{16},t-1], A_{t+1,t}\right] \\
&\quad\geq c_2 E_{\lfloor \overline{m}(t)\rfloor}\left[e^{\int_0^{H_0}\zeta(B_s)\,\mathrm{d}s}; H_0\in[t-3\overline{K}-C_{16},t-1], A_{t+1,t}\right],
\end{aligned}$$

where the second inequality is due to $\zeta \geq -(\mathtt{es}-\mathtt{ei})$ and $P_0(B_s\leq 0)\geq 1/2$ for all $s\geq 0$. Now using the inclusion $\{B_{\bar{t}}\leq 0\}\subset\{H_0\leq \bar{t}\}$ in combination with $\zeta\leq 0$, we infer $E_{\lfloor \overline{m}(t)\rfloor}\left[e^{\int_0^{\bar{t}}\zeta(B_s)\,\mathrm{d}s}; B_{\bar{t}}\leq 0\right] \leq E_{\lfloor \overline{m}(t)\rfloor}\left[e^{\int_0^{H_0}\zeta(B_s)\,\mathrm{d}s}; H_0\leq \bar{t}\right]$. Thus, recalling $\overline{\eta}(v_0)<0$ and (3.2.5), we can continue to lower bound (3.4.4) via

$$\begin{aligned}
\inf_{x\in[\overline{m}(t)-1,\overline{m}(t)+1]} \mathtt{E}_x^\xi \left[N_t^{\mathcal{L}}\right] &\geq c_3\frac{E^{\zeta,\overline{\eta}(v_0)}_{\lfloor \overline{m}(t)\rfloor}\left[e^{-\overline{\eta}(v_0)H_0}; H_0\in[t-3\overline{K}-C_{16},t-1], A_{t+1,t}\right]}{E^{\zeta,\overline{\eta}(v_0)}_{\lfloor \overline{m}(t)\rfloor}\left[e^{-\overline{\eta}(v_0)H_0}; H_0\leq\bar{t}\right]} \\
&\geq c_4\frac{P^{\zeta,\overline{\eta}(v_0)}_{\lfloor \overline{m}(t)\rfloor}\Big(H_0\in[t-3\overline{K}-C_{16},t-1],\ H_k\geq t+1-T_k-5\chi_1(\overline{m}(t)),\ \forall k\leq\lfloor\overline{m}(t)\rfloor\Big)}{P^{\zeta,\overline{\eta}(v_0)}_{\lfloor \overline{m}(t)\rfloor}\left(H_0\leq\bar{t}\right)} \\
&\geq c_4 P^{\zeta,\overline{\eta}(v_0)}_{\lfloor \overline{m}(t)\rfloor}\Big(H_0\in[\bar{t}-2\overline{K},\bar{t}-\overline{K}-1],\ H_k\geq\bar{t}-T_k-5\chi_0(\overline{m}(t)),\ \forall k\leq\lfloor\overline{m}(t)\rfloor\Big),
\end{aligned}$$

where the last inequality is due to $t\geq T_{\overline{m}(t)}\geq \bar{t}-\overline{K}$ and $\bar{t}\geq T_{\overline{m}(t)}-C_{16}-\overline{K}$ (by (3.3.70) and (3.3.69)). Now as we recall that $\frac{\lfloor\overline{m}(t)\rfloor}{t}\to v_0$, abbreviating $\eta=\overline{\eta}(v_0)$, $n:=\lfloor\overline{m}(t)\rfloor$ and thus $\bar{t}=T_n$, we see that in order to finish the proof, it suffices to show that there exists $\gamma\in(0,\infty)$ such that $\mathbb{P}$-a.s., for all $n\in\mathbb{N}$ large enough,

$$P_n^{\zeta,\eta}\big(H_0\in[T_n-2\overline{K},T_n-\overline{K}-1], H_k\geq T_n-T_k-5\chi_0(n)\ \forall k\in\{1,\dots,n\}\big)\geq n^{-\gamma}. \tag{3.4.5}$$

Using the notation

$$\widehat{H}_k^{(n)} := H_k - E_n^{\zeta,\eta}\left[H_k\right] \quad \text{as well as} \quad R_k^{(n)} := T_n - T_k - E_n^{\zeta,\eta}\left[H_k\right], \tag{3.4.6}$$

the probability in (3.4.5) can be rewritten as

$$P_n^{\zeta,\eta}\big(\widehat{H}_0^{(n)} \in [R_0^{(n)} - 2\overline{K}, R_0^{(n)} - \overline{K} - 1], \widehat{H}_k^{(n)} \geq R_k^{(n)} - 5\chi_0(n)\ \forall k \in \{1,\dots,n\}\big). \quad (3.4.7)$$

In order to facilitate computations, we approximate the sequence $(R_k^{(n)})$ by a stationary one, setting

$$\rho_i := \frac{L_i^\zeta(\eta)}{v_0 L(\eta)} - (L_i^\zeta)'(\eta) = \frac{1}{v_0 L(\eta)}\big(L_i^\zeta(\eta) - L(\eta)\big) - \big(E_n^{\zeta,\eta}[\tau_{i-1}] - \mathbb{E}[E_n^{\zeta,\eta}[\tau_{i-1}]]\big) \quad (3.4.8)$$

and

$$\widehat{R}_k^{(n)} := \sum_{i=k+1}^{n} \rho_i, \quad k < n, \quad (3.4.9)$$

where $\tau_{i-1} = H_{i-1} - H_i$, and in the equality we used $E_n^{\zeta,\eta}[\tau_{i-1}] = (L_i^\zeta)'(\eta)$ and $\mathbb{E}[E_n^{\zeta,\eta}[H_k]] = \frac{n-k}{v_0}$. Applying inequality (3.3.63) from Lemma 3.27 and using the identity $\mathbb{E}[E_n^{\zeta,\eta}[H_k]] = \frac{n-k}{v_0}$, we get that $\mathbb{P}$-a.s.,

$$\big|R_k^{(n)} - \widehat{R}_k^{(n)}\big| \leq 2\,(\mathcal{C}_1 + C_{15}\ln n) \quad \text{for all } n \in \mathbb{N} \text{ and each } k \in \{0,\dots,n\}. \quad (3.4.10)$$

From now on we write $\chi := \chi_0$. Then by (3.4.10), the probability in (3.4.7) can be lower bounded by

$$P_n^{\zeta,\eta}\big(\widehat{H}_0^{(n)} \in [R_0^{(n)} - 2\overline{K}, R_0^{(n)} - \overline{K} - 1]; \widehat{H}_k^{(n)} \geq \widehat{R}_k^{(n)} - 3\chi(n)\ \forall k \in \{1,\dots,n\}\big). \quad (3.4.11)$$

Now, for every n, enlarging the underlying probability space if necessary, we introduce two processes $(B_t^{(i,n)})_{t\geq 0}$, $i = 1, 2$, which are independent from everything else and Brownian motions under P_n, starting in n, and, without further formal definition, we tacitly assume in the following that the tilting of the probability measure $P_n^{\zeta,\eta}$ of our original Brownian motion *also applies to* $(B_t^{(i,n)})_{t\geq 0}$, $i = 1, 2$, in the obvious way. For $i = 1, 2$, let $H_k^{(i,n)} := \inf\{t \geq 0 : B_t^{(i,n)} = k\}$, $k \in \mathbb{Z}$, be the corresponding hitting times, $\widehat{H}_k^{(i,n)} := H_k^{(i,n)} - E_n^{\zeta,\eta}[H_k^{(i,n)}]$ and let Σ_n be a random variable which, under P_n, is uniformly distributed on $\{1,\dots,n-1\}$ and independent of everything else. We define

$$\beta_k^{(i,n)} := \widehat{H}_k^{(i,n)} - \widehat{R}_k^{(n)}, \quad k = n-1, n-2, \dots, \quad i = 1, 2,$$
$$\beta_k^{(n)} := \begin{cases} \beta_k^{(1,n)}, & \Sigma_n \leq k < n, \\ \beta_{\Sigma_n}^{(1,n)} + \big(\beta_k^{(2,n)} - \beta_{\Sigma_n}^{(2,n)}\big), & k < \Sigma_n. \end{cases}$$

The ξ-adaptedness of the process $(\widehat{R}_k^{(n)})_{k<n}$ implies that the processes $(\beta_k^{(i,n)})_{k<n}$, $i = 1, 2$, are $P_n^{\zeta,\eta}$-independent and have the same distribution as $(\beta_k^{(n)})_{k<n}$. We can therefore rewrite (3.4.11) as

$$P_n^{\zeta,\eta}\left(\beta_k^{(n)} \geq -3\chi(n)\ \forall k \in \{1,\dots,n\}, \beta_0^{(n)} \in I_n\right), \quad (3.4.12)$$

where $I_n := \big[R_0^{(n)} - \widehat{R}_0^{(n)} - 2\overline{K}, R_0^{(n)} - \widehat{R}_0^{(n)} - \overline{K} - 1\big]$. Due to (3.4.10) we have that $\mathbb{P}$-a.s.

for all n large enough, $R_0^{(n)} - \widehat{R}_0^{(n)} - 2\overline{K} \geq -3\chi(n)$, i.e.

$$I_n \subset [-3\chi(n), \infty). \tag{3.4.13}$$

For each $k \in \{0, \dots, n\}$ we introduce

$$\overline{\beta}_k^{(1,n)} := \beta_{n-1-k}^{(1,n)} - \beta_{n-1}^{(1,n)}, \qquad \overline{\beta}_k^{(2,n)} := \beta_k^{(2,n)} - \beta_0^{(2,n)},$$

and note that

$$\beta_0^{(n)} = \overline{\beta}_{n-1-\Sigma_n}^{(1,n)} - \overline{\beta}_{\Sigma_n}^{(2,n)} + \beta_{n-1}^{(1,n)}. \tag{3.4.14}$$

An illustration of the various processes introduced above is given in figure 3.3 below. Now the key to bound the probability in (3.4.12) is the following lemma.

Lemma 3.33. *(a) There exists $\gamma' < \infty$ such that $\mathbb{P}$-a.s. for all n large enough,*

$$P_n^{\zeta,\eta}\big(\overline{\beta}_k^{(1,n)} \geq 0\ \forall 0 \leq k \leq n,\ \overline{\beta}_n^{(1,n)} \geq n^{1/4}\big) \geq n^{-\gamma'}, \qquad \text{and}$$
$$P_n^{\zeta,\eta}\big(\overline{\beta}_k^{(2,n)} \geq 0\ \forall 0 \leq k \leq n,\ \overline{\beta}_n^{(2,n)} \geq n^{1/4}\big) \geq n^{-\gamma'}.$$

(b) There exists $C(\gamma') > 0$ such that $\mathbb{P}$-a.s. for all n large enough,

$$P_n^{\zeta,\eta}\Big(\max_{1 \leq k \leq n, i \in \{1,2\}} \big|\beta_k^{(i,n)} - \beta_{k-1}^{(i,n)}\big| \leq C(\gamma')\ln n\Big) \geq 1 - n^{-3\gamma'}. \tag{3.4.15}$$

(c) Let $\delta \in (0,1)$. There exists $c > 0$ such that for all $x \geq 1$ and all $n \in \mathbb{Z}$,

$$P_n^{\zeta,\eta}\big(\beta_{n-1}^{(1,n)} \in [x, x+\delta]\big) \geq c\delta e^{-x/c}.$$

Before proving Lemma 3.33, we finish the current proof in order not to interrupt the Reader. To this end let

$$J_n := \sup\big\{k \in \{1, \dots, n-1\} : I_n - \overline{\beta}_{n-k+1}^{(1,n)} + \overline{\beta}_k^{(2,n)} \subset [0, 2C(\gamma')\ln n]\big\},$$

where as always $\sup \emptyset := -\infty$. We have

$$\begin{aligned}
&\Big\{\beta_k^{(n)} \geq -3\chi(n)\ \forall 0 \leq k \leq n-1,\ \beta_0^{(n)} \in I_n\Big\}\\
&\supset \Big(\big\{\overline{\beta}_k^{(1,n)} \geq 0\ \forall 0 \leq k \leq n,\ \overline{\beta}_n^{(1,n)} \geq n^{1/4}\big\} \cap \big\{\max_{1 \leq k \leq n} |\overline{\beta}_k^{(1,n)} - \overline{\beta}_{k-1}^{(1,n)}| \leq C(\gamma')\ln n\big\}\\
&\quad \cap \big\{\overline{\beta}_k^{(2,n)} \geq 0\ \forall 0 \leq k \leq n,\ \overline{\beta}_n^{(2,n)} \geq n^{1/4}\big\} \cap \big\{\max_{1 \leq k \leq n} |\overline{\beta}_k^{(2,n)} - \overline{\beta}_{k-1}^{(2,n)}| \leq C(\gamma')\ln n\big\}\\
&\quad \cap \big\{\beta_{n-1}^{(1,n)} \in I_n - \overline{\beta}_{n-1-\Sigma_n}^{(1,n)} + \overline{\beta}_{\Sigma_n}^{(2,n)}\big\} \cap \{\Sigma_n = J_n\}\Big).
\end{aligned} \tag{3.4.16}$$

Indeed, due to (3.4.14), the fifth event on the right-hand side of (3.4.16) entails that $\beta_0^{(n)} \in I_n$ must hold. On the last two events on the right-hand side of (3.4.16) we have $\beta_{n-1}^{(1,n)} \geq 0$ and thus the first event on the right-hand side of (3.4.16) implies that $\beta_k^{(n)}$ is nonnegative for $k \geq \Sigma_n$. The third event then implies monotonicity at times $k < \Sigma_n$. Since $I_n \subset [-3\chi(n), \infty)$ due to (3.4.13), this gives the first condition on the left-hand side of (3.4.16). Now the first and third event on the right hand-side of (3.4.16) are independent under $P_n^{\zeta,\eta}$ and their probabilities are bounded from below by $n^{-\gamma'}$ due to Lemma 3.33 a). Thus, as a

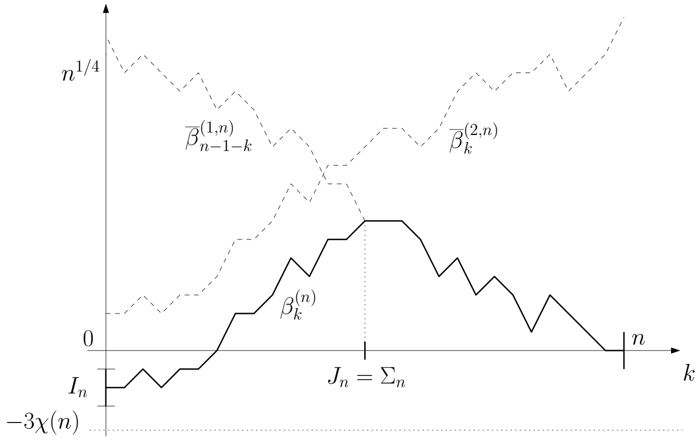

Figure 3.3: Illustration of (3.4.16).

consequence of Lemma 3.33 b), for n large enough the probability of the first four events is bounded from below by $n^{-2\gamma'} - n^{-3\gamma'}$. Furthermore, the first four events imply that $J_n \in \{1, \ldots, n-1\}$. Thus, due to Lemma 3.33, conditionally on the occurrence of the first four events, the probability of the last two events on the right-hand side in (3.4.16) can be bounded from below by $cn^{-1}e^{-C(\gamma')\ln n/c} \geq n^{-\gamma''}$ for n large enough. The proof of (3.4.5) and thus of Lemma 3.32 is completed by the choice $\gamma_1 > 2\gamma' + \gamma''$. □

Proof of Lemma 3.33. (*b*) We have $H_k^{(1,n)} - H_{k-1}^{(1,n)} \stackrel{d}{=} \tau_{k-1}$ under $P_n^{\zeta,\eta}$. Thus, recalling $\mathbb{E}\big[E_n^{\zeta,\eta}[\tau_k]\big] = \frac{1}{n}$ and the definition of $\widehat{R}_k^{(n)}$ from (3.4.8), we have $\beta_k^{(1,n)} - \beta_{k-1}^{(1,n)} = \tau_{k-1} - \frac{L_k^\zeta(\eta)}{v_0 L(\eta)}$. Now $L_k^\zeta(\eta)$ is $\mathbb{P}$-a.s. bounded by Lemma 3.8. Furthermore, for all θ such that $|\theta| \leq |\eta^*|$ (where $\triangle = [\eta_*, \eta^*]$), we have

$$0 \leq E_n^{\zeta,\eta}\big[e^{\theta\tau_{k-1}}\big] = E_k^{\zeta,\eta}\big[e^{\theta\tau_{k-1}}\big] \leq \frac{1}{E_k\big[e^{(-\mathtt{es}+\mathtt{ei}+\eta_*)H_{k-1}}\big]} = e^{\sqrt{2(\mathtt{es}-\mathtt{ei}+|\eta_*|)}} < \infty,$$

where the last equation is due to [8, (2.0.1), p. 204]. I.e., τ_k has uniform exponential moments under $P_n^{\zeta,\eta}$ and thus (3.4.15) follows by a union bound in combination with the exponential Chebyshev inequality.

(*c*) We have $\beta_{n-1}^{(1,n)} = H_{n-1}^{(1,n)} - E_n^{\zeta,\eta}[H_{n-1}^{(1,n)}] - \widehat{R}_{n-1}^{(n)} = H_{n-1}^{(1,n)} - E_n^{\zeta,\eta}[\tau_{n-1}] - \rho_n$, thus recalling definition (3.4.8), the event in (c) is equivalent to $\{H_{n-1}^{(1,n)} \in [x, x+\delta] + \frac{L_n^\zeta(\eta)}{v_0 L(\eta)}\}$. Because $\frac{L_n^\zeta(\eta)}{v_0 L(\eta)}$ is uniformly bounded and nonnegative, it suffices to check that for every $C > 0$, there exists $c > 0$ such that $\inf_{y \in [0,C]} P_n^{\zeta,\eta}(H_{n-1}^{(1,n)} \in [x+y, x+y+\delta]) \geq c\delta e^{-x/c}$ for all $x \geq 1$. Indeed, recalling (3.2.5), we can lower bound

$$\begin{aligned} P_n^{\zeta,\eta}(H_{n-1}^{(1,n)} \in [x+y, x+y+\delta]) &\geq E_n\big[e^{-(\mathtt{es}-\mathtt{ei}-\eta_*)H_{n-1}}; H_{n-1}^{(1,n)} \in [x+y, x+y+\delta]\big] \\ &\geq e^{-(\mathtt{es}-\mathtt{ei}-\eta_*)(x+y+\delta)} P_n\big(H_{n-1}^{(1,n)} \in [x+y, x+y+\delta]\big) \\ &\geq \frac{\delta e^{-(\mathtt{es}-\mathtt{ei}-\eta_*)(x+y+\delta)}}{\sqrt{2\pi(x+y+\delta)^3}} e^{-\frac{1}{2(x+y)}}, \end{aligned}$$

where the last inequality is due to [8, (2.0.2), p. 204]. Now since the latter term can be lower bounded by $c\delta e^{-x/c}$, uniformly in $y \in [0, C]$, the claim follows.

(a) We will prove the second inequality, and explain the modifications that are necessary to show the first one at the end of the proof. For later reference it will serve our purposes to exclude some potentially bad behavior of the process $(\widehat{R}_k^{(n)} - \widehat{R}_{k-1}^{(n)})_k$. To do so, we take advantage of the next claim, the proof of which we provide after concluding the proof of Lemma 3.33.

Claim 3.34. *For each $n \in \mathbb{Z}$, the sequence $(\rho_i)_{i\in\mathbb{Z}}$ consists of $\mathbb{P}$-centered and $\mathbb{P}$-stationary random variables, and the family $(\rho_i)_{i\in\mathbb{Z}}$ is bounded $\mathbb{P}$-a.s. In addition, ρ_i is $\mathcal{F}^{i-1}$-adapted and there exists $C_{17} > 0$ such that $\mathbb{P}$-a.s., for all $k, n \in \mathbb{Z}$, $k < n$, we have*

$$\big|\mathbb{E}\big[\rho_{n-k}|\mathcal{F}^n\big]\big| \leq C_{17} \cdot \big(\psi(k/2) + e^{-k/C_{17}}\big). \tag{3.4.17}$$

Furthermore, there exists $\overline{\sigma} \in [0, \infty)$ such that $n^{-1/2}\sum_{l=1}^n \rho_l$ and $n^{-1/2}\sum_{l=1}^n \rho_{-l}$ converge in $\mathbb{P}$-distribution to $\overline{\sigma}X$ as $n \to \infty$, where $X \sim \mathcal{N}(0,1)$ is a standard Normal random variable.

Now due to (3.4.17), $(\rho_i)_{i\in\mathbb{Z}}$ fulfills the conditions of Corollary B.2. As a consequence we deduce that for $k \in \mathbb{N}$ and $x \geq 0$ we have $\mathbb{P}\big(\sum_{l=1}^k \rho_l \geq x\big) \leq c_1 e^{-\frac{x^2}{c_1 k}}$, which, using stationarity, can be extended to the maximal inequality (e.g. by [42, Theorem 1])

$$\mathbb{P}\Big(\max_{0\leq k\leq y}\big(\widehat{R}_{r+k}^{(n)} - \widehat{R}_r^{(n)}\big) \geq x\Big) = \mathbb{P}\Big(\max_{0\leq k\leq y}\sum_{l=0}^k \rho_l \geq x\Big) \leq c_2 e^{-\frac{x^2}{c_2 y}} \quad \forall r, y \in \mathbb{Z},\ x \geq 0. \tag{3.4.18}$$

Furthermore, recalling (3.4.6), (3.4.8) and (3.4.9), the increments of the process $(\overline{\beta}_k^{(2,n)})_k$ can be written as

$$\begin{aligned}\overline{\beta}_k^{(2,n)} - \overline{\beta}_{k-1}^{(2,n)} &= \big(H_k - H_{k-1} - E_n^{\zeta,\eta}[H_k - H_{k-1}]\big) - \Big(\sum_{i=k+1}^n \rho_i - \sum_{i=k}^n \rho_i\Big)\\ &= \big(-\tau_{k-1} + (L_k^\zeta)'(\eta)\big) - \Big(-\frac{L_k^\zeta(\eta)}{v_0 L(\eta)} + (L_k^\zeta)'(\eta)\Big) = \frac{L_k^\zeta(\eta)}{v_0 L(\eta)} - \tau_{k-1}.\end{aligned}$$

$\mathbb{P}$-a.s., by Lemma 3.8 b), the last fraction in the previous display is positive and uniformly bounded away from zero and infinity, whereas under $P_n^{\zeta,\eta}$, τ_{k-1} is an absolutely continuous random variable with positive density on $(0, \infty)$. Therefore, for the constant

$$a := \frac{1}{4}\sup_{k\in\mathbb{Z}}\operatorname*{ess\,inf}_{\xi}\big(\overline{\beta}_k^{(2,n)} - \overline{\beta}_{k-1}^{(2,n)}\big),$$

we have $\operatorname{ess\,inf}_{k,n\in\mathbb{Z}:\ k\leq n,\ \xi} P_n^{\zeta,\eta}(\overline{\beta}_k^{(n)} - \overline{\beta}_{k-1}^{(n)} \geq 2a) \geq \delta$ for some universal constant $\delta \in (0,1)$. We now split the environment into $\overline{\xi}(j) := (\xi(l))_{l\geq j}$ and $\underline{\xi}(j) := (\xi(l))_{l<j}$ and set $t_0 = t_{-1} := 0$ as well as $t_i := 2^i$ for $i \geq 1$. Furthermore, we introduce two constants: $\overline{c} > 0$, which is defined in (3.4.31) below, and $\overline{C} > 0$, which is independent of $\overline{c}$ and will be chosen large enough such that the sums in (3.4.21) and (3.4.35) below are finite. For $i \geq 1$, we define the random variables

$$Z_i^{(n)} := \operatorname*{ess\,inf}_{\overline{\xi}(t_{i+1})} \inf_{x \geq at_{i-1}^{1/2}} P_n^{\zeta,\eta}\big(\overline{\beta}_{t_i}^{(2,n)} \geq at_i^{1/2}, \overline{\beta}_k^{(2,n)} \geq t_i^{1/4}\ \forall k \in \{t_{i-1}, \dots, t_i\} \mid \overline{\beta}_{t_{i-1}}^{(2,n)} = x\big)$$
$$= \operatorname*{ess\,inf}_{\overline{\xi}(t_{i+1})} P_n^{\zeta,\eta}\big(\overline{\beta}_{t_i}^{(2,n)} \geq at_i^{1/2}, \overline{\beta}_k^{(2,n)} \geq t_i^{1/4}\ \forall k \in \{t_{i-1}, \dots, t_i\} \mid \overline{\beta}_{t_{i-1}}^{(2,n)} = at_{i-1}^{1/2}\big), \tag{3.4.19}$$

where $\operatorname*{ess\,inf}_{\overline{\xi}(x)}$ means taking the essential infimum with respect to $\overline{\xi}(x)$, and where the second equality is due to the monotonicity of the first probability in (3.4.19) as a function in x. Thus, as a random variable, $Z_i^{(n)}$ is measurable with respect to $\mathcal{F}_{t_{i+1}}$. Now since $\overline{\beta}_k^{(2,n)}$ is $\mathcal{F}^k$-measurable, we have that $Z_i^{(n)}$ is $(\mathcal{F}^{t_{i-1}} \cap \mathcal{F}_{t_{i+1}})$-measurable. Setting $i(n) := \log_2\big(\lfloor(\overline{C}\ln n)^2\rfloor\big)$, we further define

$$Y^{(n)} := P_n^{\zeta,\eta}\Big(\overline{\beta}_k^{(2,n)} \geq 0\ \forall \lfloor \overline{C}\ln n\rfloor \leq k \leq t_{i(n)}, \overline{\beta}_{t_{i(n)}}^{(2,n)} \geq a\lfloor \overline{C}\ln n\rfloor \;\Big|\; \overline{\beta}_{\lfloor \overline{C}\ln n\rfloor}^{(2,n)} = 2a\lfloor \overline{C}\ln n\rfloor\Big).$$

Writing $j(n) := \lceil \log_2(n)\rceil$, due to the Markov property of the process $\overline{\beta}^{(2,n)}$ under $P_n^{\zeta,\eta}$, we have $\mathbb{P}$-almost surely that for all n large enough,

$$P_n^{\zeta,\eta}\Big(\overline{\beta}_n^{(2,n)} \geq n^{1/4}, \overline{\beta}_k^{(2,n)} \geq 0\ \forall k \leq n\Big) \geq \prod_{k=1}^{\lfloor \overline{C}\ln n\rfloor} P_n^{\zeta,\eta}\big(\overline{\beta}_k^{(2,n)} - \overline{\beta}_{k-1}^{(2,n)} \geq 2a\big) Y^{(n)} \prod_{i=i(n)+1}^{j(n)} Z_i^{(n)}$$
$$\geq \delta^{\lfloor \overline{C}\ln n\rfloor} \cdot Y^{(n)} \cdot \exp\Big\{\sum_{i=i(n)+1}^{j(n)} \ln Z_i^{(n)} \mathbb{1}_{B_i^{(n)}}\Big\}, \tag{3.4.20}$$

where the event

$$B_i^{(n)} := \Big\{\max_{r \in [t_{i-1}, t_i],\ 0 \leq k \leq \frac{5}{2}at_i^{1/2}/\overline{c}} \big(\widehat{R}_{r+k} - \widehat{R}_r\big) < at_{i-1}^{1/2}/16\Big\}$$

occurs $\mathbb{P}$-almost surely for all $i \in [i(n), j(n)]$ and all n large enough. Indeed, by (3.4.18) we have

$$\sum_n \sum_{i=i(n)}^{j(n)} \mathbb{P}\big((B_i^{(n)})^c\big) \leq \sum_n \sum_{i=i(n)}^{j(n)} t_{i-1}\mathbb{P}\Big(\max_{0 \leq k \leq \frac{5}{2}at_i^{1/2}/\overline{c}} \big(\widehat{R}_k - \widehat{R}_0\big) \geq at_{i-1}^{1/2}/16\Big)$$
$$\leq c_3 \sum_n n \sum_{i=i(n)}^{j(n)} e^{-a^2\overline{c}t_{i-1}^{1/2}/c_3} \leq c_4 \sum_n n\log_2(n) e^{-a^2\overline{C}\ln n/c_4} < \infty, \tag{3.4.21}$$

where the last inequality holds true for $\overline{C}$ large enough. Thus, the Borel-Cantelli lemma implies that $\mathbb{P}$-a.s., for all n large enough the events $B_i^{(n)}$ occur for all $i \in [i(n), j(n)]$. Furthermore, it is possible to show that $\mathbb{P}$-almost surely, for all n large enough we have $Y^{(n)} \geq n^{-\gamma''}$. We postpone a proof of this fact, because it uses the same arguments as the following paragraph and we will describe necessary adaptations afterwards, cf. page 87. Thus, for the time being it remains to show that there exists $\widetilde{c} > 0$ such that $\mathbb{P}$-almost surely, for all n large enough,

$$\sum_{i=i(n)}^{j(n)} \ln(Z_i^{(n)}) \mathbb{1}_{B_i^{(n)}} \geq -\widetilde{c} \cdot j(n). \tag{3.4.22}$$

Then the second inequality in Lemma 3.33 (a) follows from (3.4.20) with $\gamma' > \overline{C}\ln(1/\delta) + \gamma'' + \widetilde{c}/\ln(2)$.

In order to show (3.4.22), we use the following result, whose proof we postpone to the argument that it actually implies (3.4.22).

Lemma 3.35. *There exist $c'', \theta > 0$, independent of $\widetilde{c}$, such that for all i large enough,*

$$\sup_n \mathbb{E}\Big[e^{-\theta \ln(Z_i^{(n)}) \mathbb{1}_{B_i^{(n)}}}\Big] \leq c''. \tag{3.4.23}$$

Indeed, if (3.4.23) holds, setting $\widetilde{Z}_i^{(n)} := \ln(Z_i^{(n)}) \mathbb{1}_{B_i^{(n)}}$, by Markov's inequality we have

$$\begin{aligned}
\mathbb{P}\Big(\sum_{i=i(n)}^{j(n)} \widetilde{Z}_i^{(n)} < -\widetilde{c} \cdot j(n) \Big) &\leq \mathbb{P}\Big(\sum_{k=0}^{3} \sum_{i=\lceil i(n)/4 \rceil}^{\lfloor \frac{j(n)}{4} \rfloor -1} \widetilde{Z}_{4i+k}^{(n)} < -\widetilde{c} \cdot j(n) \Big) \\
\leq \sum_{k=0}^{3} \mathbb{P}\Big(\sum_{i=\lceil i(n)/4 \rceil}^{\lfloor \frac{j(n)}{4} \rfloor -1} \widetilde{Z}_{4i+k}^{(n)} < -\widetilde{c} \cdot j(n)/4 \Big) &\leq 4 e^{-\theta \widetilde{c} \cdot j(n)/4} \max_{k \leq 4} \mathbb{E}\Big[e^{-\theta \sum_{i=\lceil i(n)/4 \rceil}^{\lfloor j(n)/4 \rfloor -1} \widetilde{Z}_{4i+k}^{(n)}} \Big].
\end{aligned}$$

We will only estimate the above expectation for the case $k = 0$; the cases $k \in \{1, 2, 3\}$ can be estimated similarly. Now $\widetilde{Z}_{4i}^{(n)}$ is $\mathcal{F}^{t_{4i-1}}$-measurable, hence, also recalling $t_{4i-1} - t_{4i-2} = 2^{4i-2}$, by (MIX) we have

$$\mathbb{E}\big[e^{-\theta \widetilde{Z}_{4i}^{(n)}} | \mathcal{F}_{t_{4i-2}}\big] \leq (1 + \psi(2^{4i-2})) \mathbb{E}[e^{-\theta \widetilde{Z}_{4i}^{(n)}}].$$

Since furthermore $\widetilde{Z}_{4(i-1)}^{(n)}$ is $\mathcal{F}^{t_{4i-2}}$-measurable, we obtain via iterated conditioning that

$$\begin{aligned}
\mathbb{E}\Big[e^{-\theta \sum_{i=\lceil \frac{i(n)}{4} \rceil}^{\lfloor \frac{j(n)}{4} \rfloor} \widetilde{Z}_{4i}^{(n)}} \Big] &= \mathbb{E}\Big[\mathbb{E}\big[\cdots \mathbb{E}\big[\mathbb{E}\big[e^{-\theta \sum_{i=\lceil \frac{i(n)}{4} \rceil}^{\lfloor \frac{j(n)}{4} \rfloor} \widetilde{Z}_{4i}^{(n)}} \,|\, \mathcal{F}_{t_{4j(n)-2}} \big] \,|\, \mathcal{F}_{t_{4j(n)-6}} \big] \cdots \,|\, \mathcal{F}_{t_2} \big] \Big] \\
&\leq \prod_{i=\lceil i(n)/4 \rceil}^{\lfloor \frac{j(n)}{4} \rfloor} (1 + \psi(2^{4i-2})) \mathbb{E}[e^{-\theta \widetilde{Z}_{4i}^{(n)}}] \leq (c_6 c'')^{j(n)},
\end{aligned}$$

for some $c_6 > 0$ and n large enough. Choosing $\widetilde{c}$ large enough, by a Borel-Cantelli argument similar to the proof of Lemma 3.10, inequality (3.4.22) follows. We thus have to show Lemma 3.35.

Proof of Lemma 3.35. Note that

$$Z_i^{(n)} = Z_i^{(n)}(\xi(\cdot)) = Z_i^{(n-k)}(\xi(\cdot + k)) \overset{d}{=} Z_i^{(n-k)}(\xi(\cdot)) = Z_i^{(n-k)}, \tag{3.4.24}$$

so we can drop the supremum in (3.4.23). In the following, we first choose i large enough (and from then on fixed) such that several estimates in the remaining part of the proof hold, and afterwards we adapt $n = n(i)$ to ensure $0 \le i \le \lceil \log_2(n) \rceil$. For simplicity, we write $Z_i := Z_i^{(n)}$, $\beta_k := \overline{\beta}_k^{(2,n)}$, $\widehat{H}_k := H_k - E_n^{\zeta,\eta}[H_k]$, $\widehat{R}_k := \widehat{R}_k^{(n)}$ and define

$$\overline{\rho}_k^{(j)} := \operatorname*{ess\ sup}_{\overline{\xi}(j)} \rho_k, \quad \overline{R}_k^{(j)} := \sum_{l=0}^{k} \overline{\rho}_l^{(j)}, \quad 0 \le k \le j.$$

Furthermore, Thus, $\overline{\rho}_k^{(j)}$ is $\mathcal{F}_j$-measurable and $\overline{R}_{k+l}^{(j)} - \overline{R}_k^{(j)}$ is $(\mathcal{F}^k \cap \mathcal{F}_j)$-measurable for all $l \ge 0$. Let $M_R := \operatorname{ess\,sup} \rho_0$ and $L := at_i^{1/2}$, and note that the latter choice corresponds to diffusive scaling. Then we define

$$r_0 := t_{i-1},\ m := \frac{L}{16M_R},\ s_0 := \big(\inf\big\{k \ge r_0 :\ \overline{R}_k^{(k+m)} - \overline{R}_{r_0}^{(k+m)} \ge L/8\big\} - 1\big) \wedge t_i, \tag{3.4.25}$$

and for $j \ge 1$ let

$$\begin{aligned} r_j &:= s_{j-1} + \left\lceil \frac{L}{8M_R} \right\rceil, \\ s_j &:= \big(\inf\big\{k \ge r_j : \overline{R}_k^{(k+m)} - \overline{R}_{r_j}^{(k+m)} \ge L/8\big\} - 1\big) \wedge (r_j + (t_i - t_{i-1})). \end{aligned} \tag{3.4.26}$$

Heuristically, s_j is the first time after which the process $\overline{R}$ (and thus $\widehat{R}$) increases at least by the amount $L/8$ after time r_j. Such large increments of $\widehat{R}$ are potentially troublesome, since as a consequence, the process β might decrease too much and cause the event in the definition of Z_i to have too small probability. In order to cater for this inconvenience, we start noting that by definition, $s_j - r_j$ is bounded by $t_i - t_{i-1}$ and $\mathcal{F}_{s_j+m}$-measurable, and $r_{j+1} - (s_j + m) \ge m$. Thus, by condition (MIX), for every nonnegative measurable function f we notice for later reference that

$$\mathbb{E}[f(s_j - r_j) \,|\, \mathcal{F}^{r_{j+1}}] \le \big(1 + \psi(m)\big)\mathbb{E}[f(s_j - r_j)]. \tag{3.4.27}$$

Next, we define

$$\begin{aligned} \mathcal{G}_j &:= \Big\{ \inf_{r_j \le k \le s_j} (\widehat{H}_k - \widehat{H}_{r_j}) \ge -L/8,\ \beta_{s_j} \ge 2L \Big\}, \\ \mathcal{G}_j' &:= \Big\{ \inf_{s_j \le k \le r_{j+1}} (\widehat{H}_k - \widehat{H}_{s_j}) \ge -L/8 \Big\}, \\ J &:= \inf\big\{j : s_j - r_j = t_i - t_{i-1}\big\} \wedge \inf\{j : s_j \ge t_i\}, \quad \text{as well as} \\ \mathcal{G} &:= \bigcap_{j=0}^{J} \mathcal{G}_j \cap \bigcap_{j=0}^{J-1} \mathcal{G}_j', \end{aligned}$$

and claim that

$$Z_i \ge P_n^{\zeta,\eta}\big(\mathcal{G} \mid \beta_{t_{i-1}} = at_{i-1}^{1/2}\big). \tag{3.4.28}$$

Indeed, on $[r_0, s_0]$, the process $\overline{R}$ (and thus also $\widehat{R}$) increases by at most $L/8$, and the process $\widehat{H}$ decreases by at most $L/8$ on $\mathcal{G}_0$. Moreover, for $j \ge 1$, on $[s_{j-1}, r_j]$, the process $\widehat{R}$ increases by at most $L/8$, and $\widehat{H}$ decreases by at most $L/8$ on $\mathcal{G}_j'$. Finally, on $[r_j, s_j]$, the process $\overline{R}$ (and thus $\widehat{R}$) increases by at most $L/8$, and on $\mathcal{G}_j$, $\widehat{H}$ decreases by at most $L/8$ and $\beta_{s_j} \ge 2L$. All in all, conditioning on $\beta_{t_{i-1}} = at_{i-1}^{1/2} = L/\sqrt{2} \ge L/2$, we have

$\beta_k \geq L/4 \geq t_i^{1/4}$ for $k \in [r_0, s_0]$ and $\beta_k \geq L$ for all $k \in [s_0, s_J]$. Since by definition, $s_J \geq t_i$, we get $\beta_{t_i} \geq L = at_i^{1/2}$, implying (3.4.28).

Furthermore, we can continue to lower bound

$$P_n^{\zeta,\eta}(\mathcal{G} \mid \beta_{t_{i-1}} = at_{i-1}^{1/2}) \geq P_n^{\zeta,\eta}\Big(\mathcal{G}_0 \mid \beta_{r_0} = L/\sqrt{2}\Big) \prod_{j=0}^{J-1} P_n^{\zeta,\eta}\left(\mathcal{G}_j'\right) \prod_{j=1}^{J} P_n^{\zeta,\eta}\left(\mathcal{G}_j \mid \beta_{r_j} = 2L\right). \tag{3.4.29}$$

To see this, successively condition on $\beta_{r_j} \geq 2L$, $j = 1, \dots, J$, and use the Markov property of the process $\widehat{H}$ as well as the fact that $x \mapsto P_n^{\zeta,\eta}\left(\mathcal{G}_j \,|\, \beta_{r_j} = x\right)$ is increasing. Then use the fact that under $P_n^{\zeta,\eta}$, the event $\mathcal{G}_j'$ is independent of β_{r_j} by the independence of the increments of $\widehat{H}$, $j = 0, \dots, J-1$.

In order to lower bound (3.4.29), observe that under $P_n^{\zeta,\eta}$, the sequence $(\widehat{H}_{k+1} - \widehat{H}_k)_{k \geq r_j}$ consists of independent and centered random variables, whose $P_n^{\zeta,\eta}$-moment generating function is finite in a neighborhood of zero. Thus, the central limit theorem entails that for i large enough we have $P_n^{\zeta,\eta}(\mathcal{G}_j') \geq 1/2$ for all relevant choices of j. Moreover,

$$P_n^{\zeta,\eta}\left(\mathcal{G}_j \mid \beta_{r_j} = 2L\right) \geq P_n^{\zeta,\eta}\Big(\widehat{H}_{s_j} - \widehat{H}_{r_j} \geq 5L/2, \inf_{r_j \leq k \leq s_j} (\widehat{H}_k - \widehat{H}_{r_j}) \geq -L/8\Big).$$

We see that both events are non-decreasing in the (independent) increments of $\widehat{H}$. By Harris' inequality ([9, Theorem 2.15]) we get

$$\begin{aligned} &P_n^{\zeta,\eta}\Big(\widehat{H}_{s_j} - \widehat{H}_{r_j} \geq 5L/2, \inf_{r_j \leq k \leq s_j} (\widehat{H}_k - \widehat{H}_{r_j}) \geq -L/8\Big) \\ &\quad \geq P_n^{\zeta,\eta}\big(\widehat{H}_{s_j} - \widehat{H}_{r_j} \geq 5L/2\big) \cdot P_n^{\zeta,\eta}\Big(\inf_{r_j \leq k \leq s_j} (\widehat{H}_k - \widehat{H}_{r_j}) \geq -L/8\Big). \end{aligned}$$

Recalling $s_j - r_j \leq t_i - t_{i-1} = \frac{L^2}{2a^2}$, a Gaussian scaling yields $P_n^{\zeta,\eta}\Big(\inf_{r_j \leq k \leq s_j}(\widehat{H}_k - \widehat{H}_{r_j}) \geq -L/8\Big) \geq c_7 > 0$. To bound the first factor, we recall that by (A.3) and (A.5), we have $\mathbb{P}$-a.s.

$$0 \leq \overline{\rho}_{r_j+l}^{(r_j+k+m)} - \rho_{r_j+l} \leq c_8(\psi(m/2) + e^{-m/c_8}) \quad \text{for all } l \leq k \leq t_i/2.$$

Because $m = \frac{L}{16M_R}$ and $t_i/2 = \frac{L^2}{2a^2}$, due to (MIX) we finally get for all i (and thus L) large enough (due to $\psi(x) \cdot x \to 0$ $(x \to \infty)$, which itself is due to summability of $\psi(k)$), that

$$\begin{aligned} 0 &\leq \big(\overline{R}_{r_j+k}^{(r_j+k+m)} - \overline{R}_{r_j}^{(r_j+k+m)}\big) - \big(\widehat{R}_{r_j+k} - \widehat{R}_{r_j}\big) = \sum_{l=1}^{k} \big(\overline{\rho}_{r_j+l}^{(r_j+k+m)} - \rho_{r_j+l}\big) \\ &\leq c_8 L^2(\psi(L/16M_R) + e^{-L/c_8}) \leq L/16 \quad \text{for all } k \in \Big\{0, \dots, \frac{L^2}{2a^2}\Big\}. \end{aligned} \tag{3.4.30}$$

By $s_j - r_j \leq t_i - t_{i-1} = \frac{L^2}{2a^2}$ and (3.4.26), we see that $s_j - r_j \geq L/16$ for all i large enough. Recall that under $P_n^{\zeta,\eta}$, the sequence $\widehat{H}_{s_j} - \widehat{H}_{r_j}$ is a sum of independent centered random variables, whose moment generating function is uniformly bounded in a neighborhood of the origin. Then by [72, (1)], we can apply [72, Theorem 4] in the following manner: Let $c' > 0$ be as in [72, Theorem 4] and in the notation of the latter theorem we choose $k = s_j - r_j$,

$\alpha = \operatorname{ess\,inf}_{\zeta,k<n} E_n^{\zeta,\eta}[(\widehat{H}_k - \widehat{H}_{k-1})^2] > 0$, $M = |\eta|/2$, $u_1 = \ldots = u_k = 1$ and

$$\overline{c} := c' \cdot \frac{|\eta|}{2} \cdot \alpha \tag{3.4.31}$$

Then a lower Bernstein-type inequality from [72, Theorem 4] gives that on $B_i^{(n)}$ we have

$$P_n^{\zeta,\eta}\big(\widehat{H}_{s_j} - \widehat{H}_{r_j} \geq 5L/2\big) \geq c_9 e^{-\frac{L^2}{c_9(s_j - r_j)}}. \tag{3.4.32}$$

Note that the condition in $B_i^{(n)}$ makes [72, Theorem 4] applicable by ensuring 'enough' summands $s_j - r_j$ and is the main reason we have to introduce the sets $B_i^{(n)}$. We will write $c = c_7 \wedge c_9$ from now on. Using (3.4.28) in combination with the lower bounds for the factors of (3.4.29) just derived, the term in (3.4.23) can be bounded from above by

$$\begin{aligned}
\mathbb{E}\big[e^{-\theta \ln(Z_i)}\big] &\leq c_{10}\mathbb{E}\Big[\exp\Big\{\theta J \ln(2/c) + \theta \sum_{j=0}^{J} \frac{L^2}{c(s_j - r_j)}\Big\}\Big] \\
&\leq c_{10} \sum_{k=0}^{\infty} \mathbb{E}\Big[\Big(\frac{2}{c}\Big)^{\theta k} \exp\Big\{\theta \frac{L^2}{c(s_k - r_k)} \mathbb{1}_{s_k - r_k = t_i - t_{i-1}} + \theta \sum_{j=0}^{k-1} \frac{L^2}{c(s_j - r_j)} \mathbb{1}_{s_j - r_j < t_i - t_{i-1}}\Big\}\Big] \\
&\leq c_{10} \sum_{k=0}^{\infty} \Big(\frac{2}{c}\Big)^{\theta k} \big(1 + \psi(m)\big)^k e^{\frac{2\theta L^2}{c t_{i-1}}} \cdot \prod_{j=0}^{k-1} \mathbb{E}\Big[\exp\Big\{\frac{\theta L^2}{c(s_j - r_j)}\Big\} \mathbb{1}_{s_j - r_j < t_i/2}\Big],
\end{aligned} \tag{3.4.33}$$

where we recall m from (3.4.26), and the last inequality is due to (3.4.27) in combination with $t_i - t_{i-1} = t_i/2$. For the latter expectation we have

$$\begin{aligned}
\mathbb{E}\left[\exp\Big\{\frac{\theta L^2}{c(s_j - r_j)}\Big\} \mathbb{1}_{s_j - r_j < t_i/2}\right] &= \int_0^{\infty} \mathbb{P}\Big(e^{\frac{\theta L^2}{c(s_j - r_j)}} \mathbb{1}_{s_j - r_j < t_i/2} \geq x\Big)\, \mathrm{d}x \\
&\leq e^{\frac{2\theta L^2}{c t_i}} \mathbb{P}\Big(s_j - r_j < t_i/2\Big) + \int_{e^{\frac{2\theta L^2}{c t_i}}}^{\infty} \mathbb{P}\Big(e^{\frac{\theta L^2}{c(s_j - r_j)}} \geq x\Big)\, \mathrm{d}x.
\end{aligned} \tag{3.4.34}$$

Substituting $x = e^{\frac{\theta L^2}{cy}}$, the second summand can be written as

$$\int_0^{t_i/2} \frac{\theta L^2}{c y^2} e^{\frac{\theta L^2}{cy}} \mathbb{P}\left(s_j - r_j \leq y\right) \mathrm{d}y.$$

In order to obtain an upper bound, we start with the probability inside the integral and get

$$\begin{aligned}
\mathbb{P}\left(s_j - r_j \leq y\right) &= \mathbb{P}\Big(\max_{1 \leq k \leq y} \sum_{l=1}^{k} \overline{\rho}_{r_j + l}^{(r_j + k + m)} \geq L/8\Big) \leq \mathbb{P}\Big(\max_{1 \leq k \leq y} \sum_{l=1}^{k} \rho_{r_j + l} \geq L/16\Big) \\
&\leq c_{11} e^{-\frac{L^2}{c_{11} y}}, \quad \forall\, y \in [0, t_i/2];
\end{aligned}$$

here the first inequality is due to (3.4.30) and the last inequality due to (3.4.18).

Putting these bounds together, the second summand in (3.4.34) can be bounded from above by

$$\int_0^{t_i/2} \frac{\theta L^2}{cy^2} e^{\frac{\theta L^2}{cy}} c_{10} e^{-\frac{L^2}{yc_{10}}} \, \mathrm{d}y = c_{11} \int_0^{t_i/2} \frac{\theta L^2}{cy^2} e^{\frac{L^2}{cy}(\theta - c_{12})} \, \mathrm{d}y \leq c_{13} \int_0^{1/2a^2} \frac{\theta}{z^2} e^{\frac{1}{cz}(\theta - c_{12})} \, \mathrm{d}z$$
$$\leq c_{14}\theta \int_0^{\infty} e^{\frac{x(\theta - c_{11})}{c}} \, \mathrm{d}x.$$

Now the latter term can be made arbitrarily close to zero by choosing $\theta > 0$ small enough. Furthermore, again choosing $\theta > 0$ small enough, the first term in (3.4.34) is strictly smaller than one by the central limit theorem from Lemma 3.34. Thus, $\theta > 0$ can be chosen small enough such that for all i large enough, the sum on the right-hand side in (3.4.33) converges, with a finite upper bound independent of i. The proof of Lemma 3.35 is complete. □

To complete the proof of Lemma 3.33 (a), there are still two things to show. First we have to show $Y^{(n)} \geq n^{-\gamma''}$. Let us adapt the strategy in the proof of (3.4.22), i.e. set $L := 2a\overline{C}\ln n$, $r_0 = \lfloor \overline{C} \ln n \rfloor$ and $J := \inf\{j : s_j \geq \lfloor (\overline{C}\ln n)^2 \rfloor\}$ and keep the other definitions as in (3.4.25) and (3.4.26). Then by the same argument below display (3.4.23), $Y^{(n)} \geq n^{-\gamma''}$ for some suitable $\gamma'' > 0$ follows if $\mathbb{E}[e^{-\theta Y^{(n)}}] \leq c$ for some constant $c > 0$, some small $\theta > 0$ and all n large enough. But this follows (as in the argument leading to the definition of $B_i^{(n)}$), if the process $(\widehat{R}_k - \widehat{R}_{k-1})_k$ does not decrease too fast, see the Borel-Cantelli argument below display (3.4.32), which itself is a consequence of

$$\sum_n (\overline{C}\ln n)^2 \cdot \mathbb{P}\big(\sup_{0 \leq k \leq L/8\overline{c}} \big(\widehat{R}_k - \widehat{R}_0\big) \geq \frac{L}{8}\big) \leq c_{15} \sum_n (\overline{C}\ln n)^2 e^{-a\overline{C}\ln n / c_{15}} < \infty. \quad (3.4.35)$$

Secondly, we will now explain how to adapt the latter arguments for the proof of the first inequality in (a). We define $\beta_k = \overline{\beta}_k^{(1,n)} - \overline{\beta}_{n-1}^{(1,n)}$. In the definition of $Z_i^{(n)}$, we have to take the essential infimum over $\overline{\xi}(n - t_{i-2})$ and have to replace the subscripts k of β_k by $n-k$, i.e. "running backwards" from n. Thus, $Z_i^{(n)}$ is $(\mathcal{F}^{n-t_i} \cap \mathcal{F}_{n-t_{i-2}})$-measurable. It is then enough to consider the case $n = 0$ due to the argument in (3.4.24). Writing $Z_i := Z_i^{(0)}$, $\beta_k := \beta_k^{(0)}$, $\widehat{H}_k := H_k - E_0^{\zeta,\eta}[H_k]$, $\widehat{R}_k := \widehat{R}_k^{(0)}$ and defining

$$\overline{\rho}_k^{(j)} := \operatorname*{ess\,sup}_{\overline{\xi}(j)} \rho_k, \qquad \overline{R}_k^{(j)} := \sum_{l=k+1}^{0} \overline{\rho}_l^{(j)}, \quad k < 0, \; k \in \mathbb{Z},$$

we have to adapt the definitions of r_j and s_j by the expressions

$$\begin{aligned}
&r_0 := -t_{i-1}, \qquad s_0 := \big(\sup\big\{k \leq r_0 : \; \widehat{R}_k - \widehat{R}_{r_0} \geq L/8\big\} + 1\big) \vee (-t_i), \\
&r_j := s_{j-1} - \Big\lceil \frac{L}{8M_R} \Big\rceil, \quad s_j' := s_{j-1} - \frac{L}{16M_R}, \quad j \geq 1, \\
&s_j := \big(\sup\big\{k \leq r_j : \overline{R}_k^{(s_j')} - \overline{R}_{r_j}^{(s_j')} \geq L/8\big\} + 1\big) \vee \left(r_j - (t_i - t_{i-1})\right), \quad j \geq 1, \\
&J := \inf\{j : s_j - r_j = -t_i\} \vee \sup\{j : s_j \leq -t_i\}.
\end{aligned}$$

The remaining part of the proof essentially follows the same steps as for the second inequality in (a). This completes the proof Lemma 3.33. □

Proof of Claim 3.34. Boundedness, stationarity and adaptedness are direct consequences of the corresponding properties of the sequences $(L_i^{\zeta}(\eta))_{i \in \mathbb{Z}}$ and $((L_i^{\zeta})'(\eta))_{i \in \mathbb{Z}}$ and Lemma 3.8.

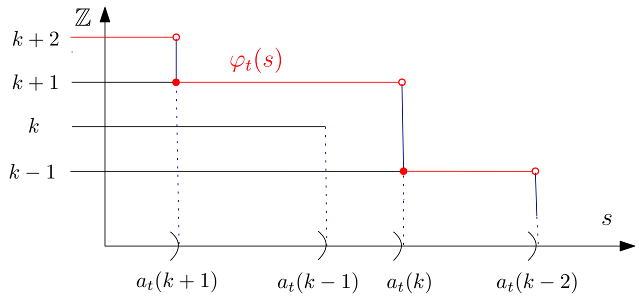

Figure 3.4: Illustration of φ_t, which is the red line. We denote $a_t(k) := t - T_k - 5\chi^\xi(\overline{m}(t))$. Note that the sequence $(a_t(k))_{k\in\mathbb{Z}}$ does not have to be monotone and thus the interval $[a_t(k+1), a_t(k))$ might be empty. In this case the graph of φ_t jumps at least two steps at time $s = a_t(k)$.

Display (3.4.17) is due to Lemma A.2. To show that the central limit theorem, we note that the sequence $(\rho_i)_{i\in\mathbb{Z}}$ fulfills the same conditions as the sequence $(\widetilde{L}_i)_{i\in\mathbb{Z}}$ in the proof of Lemma 3.14 □

3.4.2 Second moment of leading particles

Recall the notation $N_t^{\mathcal{L}}$ from below (3.4.2) and that of $\overline{m}(t)$ from (3.0.2). For the second moment of the leading particles we now prove the following upper bound.

Lemma 3.36. *For every function F fulfilling* (PROB) *and for every $a > 0$, there exists $\gamma_2 = \gamma_2(F, M) < \infty$ such that $\mathbb{P}$-a.s., for all t large enough,*

$$\sup_{x\in[\overline{m}^M(t)-1,\overline{m}^M(t)+1]} \mathrm{E}_x^\xi\Big[\big(N_t^{\mathcal{L},M}\big)^2\Big] \le t^{\gamma_2}. \tag{3.4.36}$$

Proof. We omit the superscript M in the quantities involved and use the same abbreviations as in the beginning of the proof of Lemma 3.32.

We want to show (3.4.36) with the help of the second-moment formula (Mom2). To this end, define the function $\varphi_t^\xi : [0,t] \to \mathbb{Z} \cup \{-\infty\}$,

$$\varphi_t(s) := \lfloor \overline{m}(t) \rfloor \wedge \sup\big\{k \in \mathbb{Z} : s \in \big[t - T_{k+1} - 5\chi_1(\overline{m}(t)), t - T_k - 5\chi_1(\overline{m}(t))\big)\big\},$$

where $\sup\emptyset := -\infty$ and χ_1 has been defined in (3.4.2). Due to $T_k = 0$ for all $k \le 0$ (recall the notation T_k from (3.3.61) and (3.3.62)), we have $1 \le \varphi_t(0) \le \lfloor \overline{m}(t) \rfloor$. Furthermore, $\varphi_t(t) = -\infty$, because $T_k \ge 0$ and $\chi_1^\xi(\overline{m}(t)) \ge 0$. To apply (Mom2), the following upper bound will prove useful.

Claim 3.37. *We have*

$$N_t^{\mathcal{L}} \le |\{\mathbf{u} \in N(t) : X_t^{\mathbf{u}} \le 0, X_s^{\mathbf{u}} > \varphi_t(s)\ \forall s \in [0,t]\}| \tag{3.4.37}$$

and $\mathbb{P}$-a.s. for all t large enough, the function $[0,t] \ni s \mapsto \varphi_t(s)$ is a non-increasing, càdlàg step function.

In order not to hinder the flow of reading, we postpone the proof of the latter claim to the end of the proof of Lemma 3.36. By the Feynman-Kac formula (cf. Proposition 2.15)

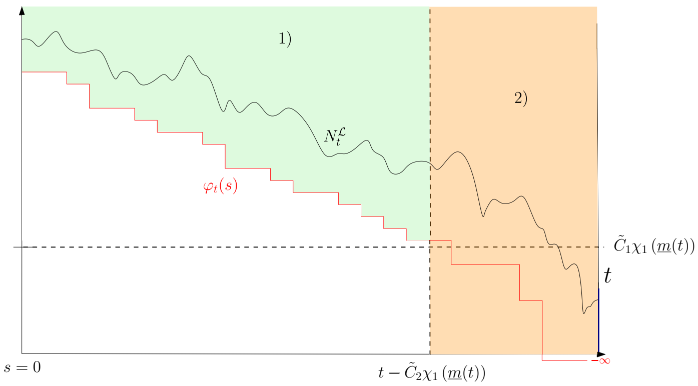

Figure 3.5: Leading particles and the different areas in the proof of Lemma 3.36.

and (3.4.37), we have

$$\begin{aligned}\mathtt{E}_x^\xi\big[(N_t^{\mathcal{L}})^2\big] \leq \mathtt{E}_x^\xi\left[N_t^{\mathcal{L}}\right] + (m_2-2)\int_0^t E_x\Big[e^{\int_0^s \xi(B_r)\,\mathrm{d}r}\xi(B_s)\mathbb{1}_{\{B_r\geq\varphi_t(r)\ \forall r\in[0,s]\}} \\ \times\Big(E_y\Big[e^{\int_0^{t-s}\xi(B_r)\,\mathrm{d}r}\mathbb{1}_{\{B_r\geq\varphi_t(r+s)\ \forall r\in[0,t-s]\},B_{t-s}\leq 0}\Big]\Big)^2_{|y=B_s}\Big]\,\mathrm{d}s.\end{aligned} \tag{3.4.38}$$

For the first summand we have

$$\begin{aligned}\sup_{x\in[\overline{m}(t)-1,\overline{m}(t)+1]}\mathtt{E}_x^\xi\left[N_t^{\mathcal{L}}\right] &\leq \sup_{x\in[\overline{m}(t)-1,\overline{m}(t)+1]}\mathtt{E}_x^\xi\left[N^{\leq}(t,0)\right] \\ &\leq c_1\mathtt{E}_{\overline{m}(t)+1}^\xi\left[N^{\leq}(t,0)\right]\leq\frac{c_1}{2},\end{aligned} \tag{3.4.39}$$

where the first inequality is due to the first inequality in (3.3.42) and the last one due to the definition of $\overline{m}(t)$. Recall that the Markov property provides us with

$$E_x\Big[e^{\int_0^s\xi(B_r)\,\mathrm{d}r}E_y\big[e^{\int_0^{t-s}\xi(B_r)\,\mathrm{d}r}\mathbb{1}_{\{B_{t-s}\leq 0\}}\big]_{|y=B_s}\Big] = \mathtt{E}_x^\xi[N^{\leq}(t,0)].$$

Using $\xi\leq\mathtt{es}$ and the two previous displays, the second summand in (3.4.38) can thus be bounded from above by

$$\begin{aligned}&\mathtt{es}(m_2-2)\sup_{x\in[\overline{m}(t)-1,\overline{m}(t)+1]}\mathtt{E}_x^\xi[N^{\leq}(t,0)]\cdot\int_0^t\sup_{y\geq\varphi_t(s)}E_y\big[e^{\int_0^{t-s}\xi(B_r)\,\mathrm{d}r};B_{t-s}\leq 0\big]\,\mathrm{d}s \\ &\leq\frac{\mathtt{es}(m_2-2)c_1}{2}\int_0^t\sup_{y\geq\varphi_t(s)}\mathtt{E}_y^\xi[N^{\leq}(t-s,0)]\,\mathrm{d}s.\end{aligned} \tag{3.4.40}$$

It thus suffices to upper bound $\sup_{y\geq\varphi_t(s)}\mathtt{E}_y^\xi[N^{\leq}(t-s,0)]$ by a polynomial in t. We treat different areas for s and y separately and we will need an additional claim, the proof of which will be provided after this proof. It guarantees that the assumptions of the time perturbation Lemma 3.23 are satisfied in our setting.

Claim 3.38. *There exists $\widetilde{C}_1\in(0,\infty)$ such that $\mathbb{P}$-a.s. for all t large enough and all $y\geq\widetilde{C}_1\chi_1(\overline{m}(t))$ we have $\frac{y}{T_y-1},\frac{y}{T_y+\overline{K}+5\chi_1(\overline{m}(t))}\in V$, where V is defined in (3.2.30). Furthermore, there exists $\widetilde{C}_2=\widetilde{C}_2(\widetilde{C}_1)\in(0,\infty)$ such that $\mathbb{P}$-a.s. for all t large enough and all $s\in[0,t-\widetilde{C}_2\chi_1(\overline{m}(t)))$ we have $\varphi_t(s)\geq\widetilde{C}_1\chi_1(\overline{m}(t))$.*

We choose $\gamma' > 5C_{12}C_{15}$. Then, recalling the definition of χ_1 from (3.4.2) and that $\mathbb{P}$-a.s. $\frac{\overline{m}(t)}{t} \to v_0$ by Corollary 3.28, for t large enough, the statements from Claim 3.38 hold true, we have $C_{12}e^{C_{12}(\overline{K}+1+5\chi_1(\overline{m}(t)))} \le t^{\gamma'}$ and also $T_{\lfloor y\rfloor+1} \le T_y + \overline{K}$ for all $y \ge \widetilde{C}_1\chi_1(\overline{m}(t))$ by Corollary 3.30.
1) Let $s \in \big[0, t - \widetilde{C}_2\chi_1(\overline{m}(t))\big)$ and $y \ge \varphi_t(s)$. Then by Claim 3.38, $y \ge \widetilde{C}_1\chi_1(\overline{m}(t))$ and thus $\frac{y}{T_y-1}, \frac{y}{T_y+\overline{K}+5\chi_1(\overline{m}(t))} \in V$. By definition of φ_t we have $s \ge t - T_{\lfloor y\rfloor+1} - 5\chi_1(\overline{m}(t))$ and thus $T_y + \overline{K} + 5\chi_1(\overline{m}(t)) \ge T_{\lfloor y\rfloor+1} + 5\chi_1(\overline{m}(t)) \ge t - s$. Thus, by Lemma D.2, we infer that $\mathtt{E}_y^\xi\big[N^{\le}(t-s,0)\big] \le 2\mathtt{E}_y^\xi\big[N^{\le}\big(T_y + \overline{K} + 5\chi_1(\overline{m}(t)), 0\big)\big]$, and then the second inequality in (3.3.35) entails that for all t large enough,

$$\sup_{\substack{0\le s<t-\widetilde{C}_2\chi_1(\overline{m}(t))\\ y\ge\varphi_t(s)}} \mathtt{E}_y^\xi[N^{\le}(t-s,0)] \le 2C_{12}e^{C_{12}(\overline{K}+1+5\chi_1(\overline{m}(t)))} \times \sup_{y\in\mathbb{R}} \mathtt{E}_y^\xi\big[N^{\le}((T_y-1)\vee 0, 0)\big] \le t^{\gamma'}.$$

2) The remaining part of the domain above the graph of φ_t not controlled by 1) is a subset of

$$\big\{(s,y) \in [0,t]\times\mathbb{R} : t - \widetilde{C}_2\chi_1(\overline{m}(t)) \le s \le t\big\}.$$

Recalling the definition of χ_1 from (3.4.2) and that $\mathbb{P}$-a.s., $\frac{\overline{m}(t)}{t} \to v_0$ by Corollary 3.28, choosing $\gamma'' > \mathtt{es}\widetilde{C}_2C_{15}$, on the the above domain we get that $\mathbb{P}$-a.s., for all t large enough,

$$\mathtt{E}_y^\xi\big[N^{\le}(t-s,0)\big] \le 2\mathtt{E}_y^\xi\big[N^{\le}\big(\widetilde{C}_2\chi_1(\overline{m}(t)), 0\big)\big] \le 2e^{\mathtt{es}\widetilde{C}_2\chi_1(\overline{m}(t))} \le t^{\gamma''}. \tag{3.4.41}$$

To conclude the proof, defining $\gamma_2 := 1 \vee \gamma' \vee \gamma'' + 1$, inequalities (3.4.39), (3.4.40) and the estimates (3.4.41) and (3.4.41) for the term $E_y^\xi\big[N^{\le}(t-s,0)\big]$ entail the statement of Lemma 3.36. □

Proof of Claim 3.37. Let $a_t(k) := t - T_k - 5\chi_1^\xi(\overline{m}(t))$ and recall the definition of $N_t^{\mathcal{L}}$:

$$N_t^{\mathcal{L}} = \big|\big\{\mathbf{u} \in N(t) : X_t^{\mathbf{u}} \le 0,\ H_k^{\mathbf{u}} \ge a_t(k)\ \forall\, k \in \{1,\dots,\lfloor\overline{m}(t)\rfloor\}\big\}\big|.$$

To prove (3.4.37), note that $H_k^{\mathbf{u}} \ge a_t(k)$ if and only if $X_s^{\mathbf{u}} > k$ for all $s < a_t(k)$. But the property $X_s^{\mathbf{u}} > k$ for all $s \in [0, a_t(k))$ and all $k \in \{1,\dots,\lfloor\overline{m}(t)\rfloor\}$ implies that $X_s^{\mathbf{u}} > \sup\{k \in \mathbb{Z} : s \in [a_t(k+1), a_t(k))\} \wedge \lfloor\overline{m}(t)\rfloor = \varphi_t(s)$ for all $s \in [0,t]$ and thus (3.4.37) is shown. The property of φ_t being a càdlàg step-function is a direct consequence of the use of left-closed, right-open intervals in the definition of φ_t. It remains to show that $s \mapsto \varphi_t(s)$ is non-increasing. For this purpose, let us first prove by induction in $k = \lfloor\overline{m}(t)\rfloor, \lfloor\overline{m}(t)\rfloor - 1, \dots$ that for all t large enough and all $k \le \overline{m}(t)$,

$$[0, a_t(k-1)) \subset \bigcup_{l=k}^{\lfloor\overline{m}(t)\rfloor} [a_t(l), a_t(l-1)) \tag{3.4.42}$$

holds. By Corollary 3.30 and Lemma 3.29, there exist $\overline{K}, C_{16} > 0$ such that $t - T_{\lfloor\overline{m}(t)\rfloor-1} \le t - T_{\overline{m}(t)} + \overline{K} \le C_{16} + \overline{K}$ and thus $a_t(\lfloor\overline{m}(t)\rfloor) \le 0$ for all t large enough. Assume now that

(3.4.42) holds for some $k \le \overline{m}(t)$. Then

$$[0, a_t(k-2)) \subset [0, a_t(k-1)) \cup [a_t(k-1), a_t(k-2)) \subset \bigcup_{l=k-1}^{\lfloor \overline{m}(t) \rfloor} [a_t(l), a_t(l-1)),$$

where the last inclusion is due to induction hypothesis. Thus, we have shown (3.4.42). Now let $0 \le s_1 \le s_2$. Assume there exists k_2 such that $s_2 \in [a_t(k_2), a_t(k_2-1))$. Then by (3.4.42), there exists $k_1 \in \mathbb{Z}$ with $k_2 \le k_1 \le \lfloor \overline{m}(t) \rfloor$, such that $s_1 \in [a_t(k_1), a_t(k_1-1))$. By definition we get $\varphi_t(s_1) \ge \varphi_t(s_2)$. If no such k_2 exists, then $\varphi_t(s_2) = -\infty \le \varphi_t(s_1)$. □

Proof of Claim 3.38. We write $V = [v_*, v^*]$. Since $\mathbb{P}$-a.s. we have $\frac{y}{T_y} \to v_0 \in \operatorname{int}(V)$ by (3.3.64), it follows that $\frac{y}{T_y - 1}, \frac{y}{T_y} \in V$ for all y large enough. Among others, there exists $\varepsilon = \varepsilon(v_*, v^*, v_0) > 0$ and $\mathcal{N}'(\xi)$ such that $v_*(1+\varepsilon) \le \frac{y}{T_y} \le (1-\varepsilon) v^*$ for all $y \ge \mathcal{N}'$. Choosing $\widetilde{C}_1 > \frac{5v^*}{\varepsilon}$, this implies $1 \le \frac{T_y + \overline{K} + 5\chi_1(\overline{m}(t))}{T_y} \le 1 + \varepsilon$ for all $y \ge \widetilde{C}_1 \cdot \chi_1(\overline{m}(t))$ and all t large enough. Thus, we get

$$v_* \le \frac{y}{T_y} \cdot \frac{T_y}{T_y + \overline{K} + 5\chi_1(\overline{m}(t))} \le v^* \quad \text{for all } y \ge \widetilde{C}_1 \chi_1(\overline{m}(t)) \text{ and all } t \text{ large enough.}$$

This gives the first part of the Claim 3.38. For the second part, recall that $T_y \le \frac{1}{v} y$ for all $y \ge \mathcal{N}'$. Furthermore, by the definition of φ_t we have $\varphi_t(s) \ge \lfloor y \rfloor + 1$ for all $s \in \big[0, t - T_{\lfloor y \rfloor} - 5\chi_1(\overline{m}(t))\big)$. Choosing $y := \lfloor \widetilde{C}_1 5 \chi_1(\overline{m}(t)) \rfloor$ and $\widetilde{C}_2 > \frac{\widetilde{C}_1}{v} + 1$, this implies that for t large enough we get $\varphi_t(s) \ge \widetilde{C}_1 5\chi_1(\overline{m}(t))$ for all $s \in \big[0, t - \widetilde{C}_2 5 \chi_1(\overline{m}(t))\big)$. □

3.4.3 Proof of Theorem 3.5

We start with an amplification result.

Lemma 3.39. *For every $(p_k)_{k\in\mathbb{N}}$ fulfilling* (PROB) *there exists $C_{18} = C_{18}((p_k)) > 1$ and $t_0 > 0$ such that $\mathbb{P}$-a.s., for all $t \ge t_0$,*

$$\sup_{x\in\mathbb{R}} \mathtt{P}_x^{\xi}\left(|N(t, [x-1, x+1])| \le C_{18}^t\right) \le C_{18}^{-t}.$$

Proof. For the proof it is enough to show the claim for binary branching with rate $\xi(x) \equiv \mathtt{ei}' := \mathtt{ei}(1 - p_1)$ (which is the rate of first branching into more than one particle) by a straightforward coupling argument. Due to the spatial homogeneity of $\mathtt{ei}'$ it is enough to show

$$\mathtt{P}_0^{\mathtt{ei}'}\left(|N(t, [-1,1])| \le C_{18}^t\right) \le C_{18}^{-t}$$

for all $t \ge t_0$, where $\mathtt{P}_0^{\mathtt{ei}'}$ is the probability measure under which the branching Brownian motion starts with one particle in 0 and has constant branching rate $\mathtt{ei}' > 0$. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\mathtt{P}_0^{\mathtt{ei}'}\left(|N(t/3, [-\varepsilon, \varepsilon])| \ge \delta t\right) \ge 1 - e^{-\delta t} \tag{3.4.43}$$

for all t large enough. Indeed, the probability that the initial particle does not leave the interval $[-\varepsilon t/2, \varepsilon t/2]$ before time $t/3$ is at least $1 - e^{-c_1 \varepsilon^2 t}$. If this happens, the particle produces more than $t\mathtt{ei}'/4$ offsprings with probability $1 - e^{-c_2 t}$ before time $t/3$, while each of these offsprings does not leave the interval $[-\varepsilon t, \varepsilon t]$ before time $t/3$ with probablity at least

$1-e^{-c_1\varepsilon^2 t/2}$. Combining these observations and choosing $\delta(\varepsilon)>0$ small enough provides us with (3.4.43). For a particle $\mathbf{u}\in N(t/3)$ let $D^{\mathbf{u}}(t/3+s)$ be the set of offsprings of $\mathbf{u}$ in the interval $[X^{\mathbf{u}}_{t/3}-1, X^{\mathbf{u}}_{t/3}+1]$ at time $t/3+s$, $s\geq 0$. We will show the existence of some $p>0$ and $c>1$ such that

$$\mathtt{P}_0^{\mathtt{ei}'}\big(|D^{\mathbf{u}}(t/3+s)|\geq c^s\big)\geq p \qquad (3.4.44)$$

for all s large enough. To obtain (3.4.44), let $r>0$ be such that

$$\inf_{y\in[-1,+1]}\mathtt{E}_y^{\mathtt{ei}'}\big[|N(r,[-1,+1])|\big]=:\mu>1,$$

(the feasibility of such a choice of r is a direct consequence of the Feynman-Kac formula). For $\mathbf{u}\in D_\varepsilon(t/3)$ consider the following process under $\mathtt{P}_0^{\mathtt{ei}'}$, conditionally on $X^{\mathbf{u}}_{t/3}$:

- the process starts with one particle at position $X^{\mathbf{u}}_{t/3}$;
- between times $r(n-1)$ and rn, $n\in\mathbb{N}$, the process evolves as a branching Brownian motion with branching rate $\mathtt{ei}'$;
- at times rn, $n\in\mathbb{N}$, particles outside of the interval $\big[X^{\mathbf{u}}_{t/3}-1, X^{\mathbf{u}}_{t/3}+1\big]$ are killed.

Using the Markov property, one readily observe that the number of particles of the latter process stochastically dominates the number of particles of a Galton-Watson process $(L_n)_{n\in\mathbb{N}}$ which starts with one particle and whose offspring distribution has expectation μ. Then by [4, Theorem 1, section I.5], the Galton-Watson process has positive probability to survive, i.e. $P(L_n>0\ \forall n\in\mathbb{N})=:p_1>0$. Conditioned on surviving, there exists $c>1$ such that

$$P\big(L_k\geq c^k\,|\,L_n>0\ \forall n\in\mathbb{N}\big)\geq\frac{1}{2}\quad\text{for all } k\in\mathbb{N}.$$

One can see that for every $\mathbf{u}\in N(t/3)$, inequality (3.4.44) holds true with the choice $p:=p_1/2$ for all $s\in r\cdot\mathbb{N}$. By a straightforward comparison argument, this extends to all $s\geq 0$. Therefore, we can now apply (3.4.43) and (3.4.44) in order to deduce

$$\mathtt{P}_0^{\mathtt{ei}'}\big(|N(2t/3,[-\varepsilon t,\varepsilon t])|\geq c^t\big)\geq 1-e^{c'(\varepsilon)t}. \qquad (3.4.45)$$

Furthermore, we have

$$P_{\varepsilon t}\big(X_{t/3}\in[-1,1]\big)\geq c_3 t^{-1/2}e^{-3\varepsilon^2 t/2}\geq\Big(\frac{1+c}{2}\Big)^{-t/3},$$

for all t large enough $\varepsilon>0$ small enough and $c>1$ suitable, where for the last inequality we used that ε does not depend on c. The latter inequality and a large deviation statement then gives for all $t\geq t_4\geq t_3$

$$\mathtt{P}_0^{\mathtt{ei}'}\Big(|N(t,[-1,1])|\geq\frac{1}{2}c^t\Big(\frac{1+c}{2}\Big)^{-t/3}\,\big|\,|N(2t/3,[-\varepsilon t,\varepsilon t])|\geq c^t\Big)\geq 1-e^{-c_4 t}. \qquad (3.4.46)$$

Thus, for $t\geq t_0$, where t_0 is chosen large enough, by (3.4.45) and (3.4.46), in combination with $c>1$, we infer the desired result. □

Now recall $\overline{m}^M(t)=\sup\{x\in\mathbb{R}:u(t,x)\geq M\}$ and $m^\varepsilon(t)=\sup\{x\in\mathbb{R}:w(t,x)\geq\varepsilon\}$, where u and w are the solutions to (PAM) and (F-KPP), respectively.

With the help of Lemmas 3.32, 3.36, and 3.39, it is now possible to state a crucial result for the proof of Theorem 3.5.

Proposition 3.40. *For every $q > 0$, F satisfying* (PROB) *and $M > 0$, there exist a constant $C_1 = C_1(q,F) \in (0,\infty)$, a $\mathbb{P}$-a.s. finite $C = C(t) = C(t,q,F,\xi) > 0$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}_5 = \mathcal{T}_5(M,q,F,\xi)$ such that for all $t \geq \mathcal{T}_5$, we have $C(t) \leq C_1$ and*

$$\mathtt{P}^{\xi}_{\overline{m}^M(t)-C\ln t}\left(N^{\leq}(t,0) \geq 1\right) \geq 1 - 2t^{-q}. \tag{3.4.47}$$

Proof. For simplicity, we write $\overline{m}(t) := \overline{m}^M(t)$. Without loss of generality, it is enough to show the claim for all $q > 2(\gamma_1 + \gamma_2)$, where $\gamma_1 = \gamma_1(M)$ and $\gamma_2 = \gamma_2(M)$ are defined in Lemmas 3.32 and 3.36, respectively. Let further C_{18} and t_0 be as in Lemma 3.39, and c_1 be such that for $r := c_1 \ln t$ we have $C_{18}^{-r} = t^{-q}$.

We claim that there exist $C_1, C(t)$ and $\mathcal{T}_5$ as above such that $\overline{m}(t-r) = \overline{m}(t) - C(t)\ln t$, $C(t) \leq C_1$ and the conclusions of Lemmas 3.32 and 3.36 hold for all $t \geq \mathcal{T}_5$. Indeed, writing $u(t,x) = \mathtt{E}^{\xi}_x[N^{\leq}(t,0)]$, by the time and space perturbation Lemmas 3.23 and 3.24, defining $c_2 := C_{12} \vee C_{13}$, we deduce that

$$u(t-r, \overline{m}(t) - C_1 \ln t) \geq c_2^{-1} e^{C_1 \ln t/c_2} u(t-r, \overline{m}(t)) \geq c_2^{-2} e^{C_1 \ln t/c_2 - r/c_2} u(t, \overline{m}(t)) \geq M, \tag{3.4.48}$$

where for the last inequality we choose C_1 and $\mathcal{T}_5 = \mathcal{T}_5(\varepsilon)$ large enough such that the last inequality holds for all $t \geq \mathcal{T}_5$. As a consequence, we infer

$$\overline{m}(t-r) \geq \overline{m}(t) - C_1 \ln t. \tag{3.4.49}$$

By (3.3.35), we also get $u(t, \overline{m}(t-r)) \geq C_{12}^{-1} e^{r/C_{12}} u(t-r, \overline{m}(t-r)) \geq M$ for all t large enough, i.e.

$$\overline{m}(t-r) \leq \overline{m}(t).$$

Thus, combining the two previous displays, we can find $C(t) \leq C_1$ such that $\overline{m}(t-r) = \overline{m}(t) - C(t)\ln t$. Now let $x := \overline{m}(t) - C(t)\ln t = \overline{m}(t-r)$. Conditioning on whether until time r there are more or less than C_{18}^r particles in $[x-1, x+1]$, we get for all $t \geq \mathcal{T}_5$,

$$\begin{aligned}
&\mathtt{P}^{\xi}_x\left(N^{\leq}(t,0) \geq 1\right)\\
&\quad \geq 1 - \mathtt{P}^{\xi}_x\big(|N(r,[x-1,x+1])| \leq C_{18}^r\big) - \sup_{y\in[x-1,x+1]} \Big(\mathtt{P}^{\xi}_y\left(N^{\leq}(t-r,0) = 0\right)\Big)^{C_{18}^r}\\
&\quad \geq 1 - C_{18}^{-r} - \sup_{y\in[x-1,x+1]} \Big(\mathtt{P}^{\xi}_y\big(N^{\mathcal{L},M}_{t-r} = 0\big)\Big)^{C_{18}^r},
\end{aligned}$$

using Lemma 3.39 in the last inequality. Now using Cauchy-Schwarz as in (3.4.3), in combination with Lemmas 3.32 and 3.36, we infer

$$\sup_{y\in[x-1,x+1]} \big(\mathtt{P}^{\xi}_y(N^{\mathcal{L},M}_{t-r} = 0)\big)^{C_{18}^r} \leq \big(1 - t^{-2\gamma_1-\gamma_2}\big)^{t^q} \leq t^{-q},$$

adapting $\mathcal{T}_5 = \mathcal{T}_5(M,q,\xi,q)$ such that the last inequality holds for all $t \geq \mathcal{T}_5$. □

Proof of Theorem 3.5. 1) We first prove the result under the additional assumption that F fulfills (PROB). Let w^{ξ,F,w_0} be the solution to (F-KPP) with initial condition $w_0 \in \mathcal{I}_{\text{F-KPP}}$, so in particular $0 \leq w_0 \leq \mathbb{1}_{(-\infty,0]}$. Because F fulfills (PROB), by (McKean) and the Markov property we infer

$$\begin{aligned} w^{\xi,F,w_0}(t,x) &= \mathtt{E}_x^{\xi}\Big[1-\prod_{\mathbf{u}\in N(t)}\big(1-w_0(X_t^{\mathbf{u}})\big)\Big] \\ &\geq \mathtt{E}_x^{\xi}\Big[1-\prod_{\mathbf{u}\in N(t)}\big(1-w_0(X_t^{\mathbf{u}})\big); N^{\leq}(t-s,0)\geq 1\Big] \\ &\geq \mathtt{P}_x^{\xi}\big(N^{\leq}(t-s,0)\geq 1\big)\cdot \inf_{y\leq 0} w^{\mathtt{ei},F,w_0}(s,y), \end{aligned} \tag{3.4.50}$$

where $w = w^{\mathtt{ei},F,w_0}$ solves the homogenous equation $w_t = \frac{1}{2}w_{xx} + \mathtt{ei}\cdot F(w)$ with initial condition $w(0,\cdot) = w_0$. Then we have $w^{1,F,\widetilde{w}_0} = w^{\mathtt{ei},F,w_0}(\frac{t}{\mathtt{ei}}, \frac{x}{\sqrt{\mathtt{ei}}})$ with $\widetilde{w}_0(x) := w_0(x/\sqrt{\mathtt{ei}})$. Because $w^{\mathtt{ei},F,w_0}(0,x) = 0$ for $x > 0$, conditions [12, (8.1) and (1.17)] are fulfilled. Together with (KPP-INI) and [12, Theorem 3, p. 141], $w^{1,F,\widetilde{w}_0}$ (and thus also $w^{\mathtt{ei},F,w_0}$) is a travelling wave solution, i.e., there exist $m^{\mathtt{ei}}(t) = \sqrt{2\mathtt{ei}}t + o(t)$ and some g fulfilling $\lim_{x\to-\infty} g(x) = 1$ and $\lim_{x\to\infty} g(x) = 0$ such that

$$\sup_y \big|w^{\mathtt{ei},F,w_0}(t, y + m^{\mathtt{ei}}(t)) - g(y)\big| \to 0, \quad t\to\infty. \tag{3.4.51}$$

Now let $\varepsilon \in (0,1)$ and choose $\delta > 0$ such that $\frac{\varepsilon}{1-\delta} \in (0,1)$. Then by (3.4.51) we get

$$\inf_{y\leq 0} w^{\mathtt{ei},F,w_0}(s,y) = \inf_{y\leq -m^{\mathtt{ei}}(s)} w^{\mathtt{ei},F,w_0}(s, y + m^{\mathtt{ei}}(s)) \geq 1-\delta \quad \text{for all } s \geq s_0(F, w_0, \delta, \mathtt{ei}),$$

which, together with (3.4.50), gives

$$m^{\xi,F,w_0,\varepsilon}(t) \geq m^{\xi,F,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{1-\delta}}(t-s_0) \quad \text{for all } t\geq s_0(F,w_0,\delta,\mathtt{ei}). \tag{3.4.52}$$

The inequality

$$m^{\xi,F,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{1-\delta}}(t-s_0) \geq \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{1-\delta}}(t-s_0) - C_1\ln(t), \quad \text{for all } t\geq \mathcal{T}_1(\xi,F,\varepsilon,\delta),$$

follows from Proposition 3.40. By Corollary 3.31, for $C' > 1$ from (PAM-INI) we get

$$\begin{aligned} \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{1-\delta}}(t-s_0) &\geq \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{C'}}(t-s_0) - c_1(\varepsilon,\delta,C') \\ &\geq \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{C}}(t) - c_2(\varepsilon,\delta,C',s_0) \quad \text{for all } t\geq \mathcal{T}_2(\xi,\varepsilon,\delta), \end{aligned}$$

where the second inequality can be obtained similarly to the argument in (3.4.48) and (3.4.49). Combining the above inequalities, we arrive at

$$\begin{aligned} m^{\xi,F,w_0,\varepsilon}(t) &\geq \overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\frac{\varepsilon}{C'}}(t) - C_1\ln(t) - c_3 \\ &= \overline{m}^{\xi,C'\mathbb{1}_{(-\infty,0]},\varepsilon}(t) - C_1\ln(t) - c_3 \quad \text{for all } t\geq \mathcal{T}_3(\xi). \end{aligned}$$

Now by (PAM-INI) every $u_0 \in \mathcal{I}_{\text{PAM}}$ is upper bounded by the function $C'\mathbb{1}_{(-\infty,0]}$, and since we have $\overline{m}^{\xi,\mathbb{1}_{(-\infty,0]},\varepsilon}(t) \geq m^{\xi,F,w_0,\varepsilon}(t)$ for all $\varepsilon\in(0,1)$ and $w_0 \in \mathcal{I}_{\text{F-KPP}}$, this finishes the proof for F fulfilling (PROB).

2) Now let F fulfill (SC) and w_0 be such that $w^{\xi,w_0,F}$ is a classical solution to (F-KPP). By Lemma D.1 there exists some function G fulfilling (PROB), such that $F(w) \geq G(w)$ for all $w\in[0,1]$. We now use a sandwich argument. By Corollary C.2 the solutions

$w^F = w^{\xi,w_0,F}$ and $w^G = w^{\xi,w_0,G}$ to

$$w_t^F - \frac{1}{2}w_{xx}^F - \xi(x)F(w^F) = 0 = w_t^G - \frac{1}{2}w_{xx}^G - \xi(x)G(w^G)$$

(which are classical by Proposition 2.17) fulfill $w^F \geq w^G$. As a consequence, we infer that $m^{\xi,F,w_0,\varepsilon}(t) \geq m^{\xi,G,w_0,\varepsilon}(t) \geq \overline{m}^M(t) - C_1 \ln(t)$, where the second inequality is due to step 1). The claim for arbitrary $w_0 \in \mathcal{I}_{\text{F-KPP}}$ is then true by an approximation argument of w_0 by continuous functions, that is if $F \geq G$, by Remark 2.16 we have $w^{w_0,F} \geq w^{w_0,G}$ and consequently $m^{\xi,w_0,F,\varepsilon}(t) \geq m^{\xi,w_0,G,\varepsilon}(t)$ for all $t \geq 0$ and we can conclude. $\square$

Chapter Four
Complementary results

The main result of Chapter 3 tells us that the upper front m^ε of the solution to (F-KPP) lags at most logarithmically behind the upper front $\overline{m}^M$ of the solution to (PAM), see Theorem 3.5. We could directly transfer asymptotic results for the front of the solution to (PAM), such as the first order asymptotics $\overline{m}^M(t)/t \to v_0$ or the invariance principles in Theorem 3.3, to those for the front of the solution to (F-KPP), see Corollary 3.6.

However, several topics remain untreated. A natural question is whether the logarithmic upper bound in Theorem 3.5 is "sharp", i.e. if there exists a constant $c > 0$ such that the fronts of both models at time t lag *at least* $c \log t$ behind each other, for t large enough. Although we are not able to treat this lower bound as generally as the upper bound in Chapter 3, there is a partially *positive* answer to this question for a subsequence of times, see (4.1.4). This is a consequence of the main result of the first section in this chapter. That is, in Section 4.1 we show a somewhat surprising behavior of the *transition front* of both models, where the solution changes from small to large values. We can show, see Theorem 4.2, that the solution to (PAM) has a uniformly bounded (or "sharp") transition front. Surprisingly, this is not generally true for the solution to (F-KPP) and we give a counterexample (Theorem 4.3). As a by-product of the construction, we show in Section 4.2 that the solution to (F-KPP) need *not* be monotone in space, even for large times, and this is in stark contrast to the observations which have been made for constant potential. We close the chapter with Section 4.3, where we discuss some of our model assumptions from Section 3.1.1. We first show that there are potentials ξ such that all relevant results in this thesis hold *simultaneously*, which ensures that we have not made too restrictive assumptions about our model. Then we show that for unbounded potential we cannot expect a linear order of the front of (PAM). Further, we take a closer look at the technical condition (VEL), which is essential for the proofs of Theorem 3.4 and 3.5. We show that it is fulfilled for a large class of bounded potentials, but, more importantly, that it is non-trivial, which is shown by constructing an example of a potential fulfilling all model assumptions except (VEL).

We use the same notation as in Chapter 3. Recall the PDEs of interest

$$\begin{aligned} &w_t(t,x) = \frac{1}{2} w_{xx}(t,x) + \xi(x,\omega) \cdot F\big(w(t,x)\big), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ &w(0,x) = w_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{F-KPP}$$

as well as its linearization

$$\begin{aligned} u_t(t,x) &= \frac{1}{2}u_{xx}(t,x) + \xi(x,\omega)\cdot u(t,x), \quad (t,x)\in(0,\infty)\times\mathbb{R},\\ u(0,x) &= u_0(x), \quad x\in\mathbb{R}, \end{aligned} \tag{PAM}$$

with (see Section 3.1.1) $\xi = (\xi(x))_{x\in\mathbb{R}} = (\xi(x,\omega))_{x\in\mathbb{R}}$, $\omega\in\Omega$, being a stochastic process on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ fulfilling (HÖL), (BDD), (STAT) and (MIX), F fulfilling (PROB), $u_0\in\mathcal{I}_{\text{PAM}}$ and $w_0\in\mathcal{I}_{\text{F-KPP}}$.

Sections 4.1 and 4.2 are taken from the preprint article [20] and we omit a reference to corresponding parts in the paper. Proposition 4.14 and Claim 4.13 from Section 4.3 are taken from [19, Section 4.4], and we also refrain from references.

We continue the numbering of the constants $C_1, C_2, \dots$ that has been used in Chapter 3.

4.1 Unbounded F-KPP and bounded PAM transition front

In order to investigate the position of the front, we recall already existing and introduce new notation. For $\varepsilon\in(0,1)$, $M>0$ and $t\geq 0$ consider the quantities

$$\begin{aligned} m^{\varepsilon}(t) &:= m^{w_0,F,\varepsilon}(t) := \sup\{x\in\mathbb{R} : w(t,x)\geq\varepsilon\},\\ m^{\varepsilon,-}(t) &:= m^{w_0,F,\varepsilon,-}(t) := \inf\{x\geq 0 : w(t,x)\leq\varepsilon\},\\ \overline{m}^{M}(t) &:= \overline{m}^{u_0,M}(t) := \sup\{x\in\mathbb{R} : u(t,x)\geq M\},\\ \overline{m}^{M,-}(t) &:= \overline{m}^{u_0,M,-}(t) := \inf\{x\geq 0 : u(t,x)\leq M\}. \end{aligned} \tag{4.1.1}$$

In this section we are mainly interested in the transition fronts of the fronts, more precisely whether these areas are bounded or not. Let us explain what we mean by this.

Definition 4.1. The solution to (F-KPP) is said to have a *uniformly bounded transition front* if for each $\varepsilon\in(0,\frac{1}{2})$, there exists a constant $C_\varepsilon\in(0,\infty)$ such that $\mathbb{P}$-a.s., for all t large enough we have

$$m^{\varepsilon}(t) - m^{1-\varepsilon,-}(t) \leq C_\varepsilon.$$

The solution to (PAM) is said to have a *uniformly bounded transition front* if for all $\varepsilon, M\in(0,\infty)$ with $\varepsilon\leq M$, there exists a constant $C_{\varepsilon,M}\in(0,\infty)$ such that $\mathbb{P}$-a.s., for all t large enough,

$$\overline{m}^{\varepsilon}(t) - \overline{m}^{M,-}(t) \leq C_{\varepsilon,M}. \tag{4.1.2}$$

Observe that the transition front of the solution to (PAM) stays bounded uniformly in time. This is stated in the next theorem.

Theorem 4.2. *Let* (HÖL), (BDD), (STAT), (MIX) *and* (VEL) *be fulfilled. Then for all* $u_0\in\mathcal{I}_{PAM}(\delta',C')$, *the solution to* (PAM) *has a uniformly bounded transition front. Furthermore, for* $\delta',C'>0$ *fixed, the corresponding constant* $C_{\varepsilon,M}$ *in* (4.1.2) *is independent of* $u_0\in\mathcal{I}_{PAM}(\delta',C')$.

However, the transition fronts of the solutions to (F-KPP) and (PAM) differ fundamentally. That is, as a second result we get that an analogous statement as in Theorem 4.2 is *not* true for the solution to (F-KPP) in general.

Theorem 4.3. *Let* $w_0=\mathbb{1}_{(-\infty,0]}$ *and* F *fulfill* (PROB). *Then there exist potentials* ξ *fulfilling* (HÖL), (BDD), (STAT) *and* (MIX) *such that the transition front of the solution* *to* (F-KPP) *is not uniformly bounded in time. More precisely, such* ξ *can be chosen so that for any* $\delta \in (0,1)$ *and any* $\varepsilon > 0$ *we find a sequence* $(x_n, t_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R} \times [0, \infty)$ *as well as a function* $\varphi \in \Theta(\ln n)$ *such that*

(a) $x_n, t_n \to \infty$ *as* $n \to \infty$, *and* $(x_n)_{n\in\mathbb{N}} \in \Theta(n)$,

(b) for all $n \in \mathbb{N}$,

$$\delta = w(t_n, x_n) \leq w(t_n, x_n + \varphi(n)) + \varepsilon. \tag{4.1.3}$$

This means that, at least along a subsequence of times, the interval of transition in which the solution changes from being locally unstable ($w \approx 0$) to locally stable ($w \approx 1$), grows at least logarithmically in time as $t \to \infty$. This phenomenon is different to the behavior of the front for certain other reaction terms. For example, if the potential ξ is constant, or, as mentioned in Section 1.3.2, if the potential is ergodic and the nonlinearity F is of ignition-type, the transition front is bounded, see (1.3.7).

As already mentioned in the introduction of this chapter, the question arises whether the logarithmic upper bound from Theorem 3.5 is "sharp". Theorem 4.3 provides the following partial affirmative answer: There exist $c_0 \in (0, \infty)$, an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of times with $t_n \in (0, \infty)$ fulfilling $\lim_{n\to\infty} t_n = \infty$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers such that $\overline{m}^{\frac{1}{2}}(t_n) - x_n \geq c_0 \log t_n$ such that for all $n \in \mathbb{N}$:

$$w(t_n, x_n) < \frac{1}{2} \quad \text{and (by definition)} \quad u(t_n, \overline{m}^{\frac{1}{2}}(t_n)) = \frac{1}{2}. \tag{4.1.4}$$

As has already been mentioned, at first glance, it may seem slightly difficult to reconcile the statement of Theorem 4.2 with the statements of Theorem 4.3. In particular, it might seem surprising given that oftentimes the linearization of a nonlinear PDE is considered to be a good approximation for the original PDE. However, from a probabilistic point of view, the phenomenon of the uniformly bounded transition front of the solution to (PAM) from Theorem 4.2 can be explained as follows: Consider the Feynman-Kac representation

$$u(t,x) = E_x\big[e^{\int_0^t \xi(B_s)\,\mathrm{d}s} \mathbb{1}_{(-\infty,0]}(B_t)\big] \tag{4.1.5}$$

of the solution to (PAM) for x of linear order in time t. Then it is "costly" to start in some value $x + C$ for C large instead, i.e. a Brownian motion starting in x and being to the left of the origin at time t has to make less of an effort in terms of large deviations than a Brownian motion starting in $x + C$ and being to the left of the origin at time t. However, the former can still collect at least as high potential values as the latter, since, typically between x and 0 there are enough locations where the potential is large. This yields $u(t, x) \gg u(t, x + C)$ for C large.

On the other hand, see (2.3.2), the solution w to (F-KPP) has been shown to fulfill the probabilistic representation

$$w(t,x) = \mathtt{P}_x^\xi\big(N^{\leq}(t,0) \geq 1\big), \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{4.1.6}$$

To show Theorem 4.3, we take advantage of (4.1.6) and the fact that the term in (4.1.6) is much more sensible with respect to the starting point x of the BBMRE than the term in (4.1.5). This can be explained as follows. Note that it is likely that a BBMRE starting in some site r surrounded by large potential produces more particles in the beginning than starting in some site l surrounded by small potential. One then expects, if these two sides are "not too far away" from each other, that the offspring of the particle starting in r has higher probability to reach the origin before time t than the offspring of the particle starting in l and thus $w(t,r) > w(t,l)$ by (4.1.6). The leading idea of the proof of Theorem 4.3 is therefore to show that a BBMRE starting in some site r produces more particles than the corresponding BBMRE starting in $l < r$, the former set of particles eventually "overtakes" latter set and $r - l$ can be taken to tend to infinity.

We first show that there is a potential ξ that fulfills (**Standing assumptions**), so that $\mathbb{P}$-a.s. we can find wide islands with small values of the potential and (to the right of these) neighboring wide islands of large values. Then we construct a coupling of two BBMRE, one starting in a site l surrounded by the small potential island and one starting in a site $r > l$ surrounded by the large potential. The coupling aims at controlling the offspring particles of the process started in l through appropriate "mirroring" and "matching" by the offspring particles of the process started in r.

4.1.1 Bounded PAM front

The boundedness of the front of (PAM) essentially follows from the space perturbation result from Section 3.3.4 (b) and the properties of the Lyapunov exponent proved in Corollary 3.22. Let us first recall what need for the proof of Theorem 4.2.

Due to (VEL), we can choose a compact interval $V \subset (v_c, \infty)$ such that v_0 is in the interior of V. By Corollary 3.28 we know that for all $u_0 \in \mathcal{I}_{\text{PAM}}$ and $M > 0$ we have

$$\frac{\overline{m}^{u_0,M}(t)}{t} \underset{t\to\infty}{\longrightarrow} v_0 \quad \mathbb{P}\text{-a.s.}$$

Furthermore we have (trivially) $\overline{m}^{u_0,\varepsilon,-}(t) \leq \overline{m}^{u_0,M}(t)$ for $0 < \varepsilon \leq M$. For the Lyapunov exponent Λ we have $\Lambda(v) > 0$ for all $v \in [0, v_0)$ and by Corollary 3.22 the convergence of the Lyapunov exponent holds uniformly on every compact interval $K \subset [0,\infty)$. Thus we get $\liminf_{t\to\infty} \inf_{x\in[0,v]} \frac{1}{t} \ln u^{u_0}(t, xt) > 0$ for all $v < v_0$. Consequently, for arbitrary initial conditions $u_0 \in \mathcal{I}_{\text{PAM}}$ and every $\varepsilon, M > 0$, $\varepsilon \leq M$, we have $\mathbb{P}$-a.s.

$$\lim_{t\to\infty} \frac{\overline{m}^{u_0,\varepsilon}(t)}{t} = \lim_{t\to\infty} \frac{\overline{m}^{u_0,M,-}(t)}{t} = v_0. \tag{4.1.7}$$

In particular, $\overline{m}^{\varepsilon}(t)/t \in V$ and $\overline{m}^{M,-}(t)/t \in V$ for t large enough, since we assume that v_0 is in the interior of V, and $a_t := \overline{m}^{\varepsilon}(t) - \overline{m}^{M,-}(t) \in o(t)$. By Lemma 3.24 (b), uniformly in $u_0 \in \mathcal{I}_{\text{PAM}}$ and $v \in V$, for all t large enough such that $vt + a_t \in V$, we get

$$\begin{aligned} \frac{u(t, vt + a_t)}{u(t, vt)} &= \prod_{k=1}^{\lfloor\sqrt{t}\rfloor} \frac{u(t, vt + k a_t/\lfloor\sqrt{t}\rfloor)}{u(t, vt + (k-1)a_t/\lfloor\sqrt{t}\rfloor)} \\ &\leq \big(C_{13} e^{-a_t/(C_{13}\lfloor\sqrt{t}\rfloor)}\big)^{\lfloor\sqrt{t}\rfloor} = e^{-a_t/C_{13}(1-\frac{\lfloor\sqrt{t}\rfloor}{a_t}\cdot C_{13}\ln C_{13})}. \end{aligned} \tag{4.1.8}$$

Now we have all we need to prove Theorem 4.2.

Proof of Theorem 4.2. Set $C_{\varepsilon,M} := 2C_{13} \ln\big(\frac{2MC_{13}}{\varepsilon}\big)$ and $\varepsilon(t) := 2t^{-1/2} C_{13} \ln C_{13}$. Assume by contradiction that the claim of the theorem does not hold. Then there exist $0 < \varepsilon \leq M$ and a (random) sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \underset{n\to\infty}{\longrightarrow} \infty$ and $a_{t_n} = \overline{m}^{\varepsilon}(t_n) - \overline{m}^{M,-}(t_n) \geq C_{\varepsilon,M}$ for all $n \in \mathbb{N}$. Recalling that $\overline{m}^{\varepsilon}(t)/t \in V$, we get for all n large enough that

$$\varepsilon = u(t_n, \overline{m}^{\varepsilon}(t_n)) = u\big(t, \overline{m}^{M,-}(t_n) + a_{t_n}\big) \leq u(t_n, \overline{m}^{M,-}(t_n)) \cdot C_{13} e^{-a_{t_n}/2C_{13}} \leq \varepsilon/2,$$

where in the first inequality we used Lemma 3.24 (b) if $a_{t_n} \le t_n\varepsilon(t_n)$ and (4.1.8) if $a_{t_n} > t_n\varepsilon(t_n)$. This is a contradiction. As a consequence, we must have $0 \le \overline{m}^\varepsilon(t) - \overline{m}^M(t) \le C_{\varepsilon,M}$ for all t large enough. Furthermore, this inequality holds uniformly for all $u_0 \in \mathcal{I}_{\mathrm{PAM}}(\delta', C')$, because C_{13} is independent of $u_0 \in \mathcal{I}_{\mathrm{PAM}}(\delta', C')$, proving the claim of the theorem. □

4.1.2 Unbounded F-KPP-front: The potential

We start the proof of Theorem 4.3 by constructing a suitable potential ξ, for which we then show the unboundedness of the transition front of the solution to (F-KPP) with $w_0 = \mathbb{1}_{(-\infty,0]}$. For the constants $\mathtt{es}$ and $\mathtt{ei}$ from (BDD) we assume that

$$\frac{\mathtt{es}}{\mathtt{ei}} > 2. \tag{4.1.9}$$

We further let $\delta_1, \delta_2 \in (0,1)$ be small positive constants, which will be fixed at the end of the proof of Lemma 4.6, see the paragraph below (4.1.34).

It is an interesting open question whether the condition (4.1.9) is necessary for the unboundedness of the front. We could not improve it using the methods of this section, see in particular after (4.1.31) where the condition (4.1.9) is crucially needed.

Let furthermore $\chi : \mathbb{R} \to [0,1]$ be a Lipschitz-continuous non-increasing function with $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$, and let $\omega = (\omega^i)_{i\in\mathbb{Z}}$ be a Poisson point process on $\mathbb{R}$ with intensity 1 constructed on $(\Omega, \mathcal{F}, \mathbb{P})$, see e.g. [60, Chapter 3] for a construction and related results. Among others, $\omega = (\omega^i)_i$ is a mapping from Ω into the set of all locally finite point configurations on $\mathbb{R}$, satisfying the following properties:

- $|\{i : \omega^i \in B\}| < \infty$ for every bounded Borel set,
- $\omega^i \ne \omega^j$ for all $i \ne j$,
- for pairwise disjoint Borel sets $B_1, \dots, B_n$ and $k_1, \dots, k_n \in \mathbb{N}_0$, $n \in \mathbb{N}$, the events $\{|\omega \cap B_1| = k_1\}, \dots, \{|\omega \cap B_n| = k_n\}$ are independent and
- for all $k \in \mathbb{N}_0$ and all Borel sets B we have

$$\mathbb{P}(|\omega \cap B| = k) = \begin{cases} \frac{\lambda(B)^k}{k!} e^{-\lambda(B)}, & \lambda(B) < \infty, \\ 0, & \lambda(B) = \infty. \end{cases}$$

We then define our potential via

$$\xi(x) := \mathtt{ei} + (\mathtt{es} - \mathtt{ei}) \cdot \sup\{\chi(x - \omega^i) : i \in \mathbb{Z}\}. \tag{4.1.10}$$

Observe that the map $x \mapsto \xi(x)$ is a Lipschitz continuous function (i.e. (HÖL) is fulfilled), $\xi(x) \in [\mathtt{ei}, \mathtt{es}]$ for all $x \in \mathbb{R}$ (i.e. ξ fulfills (BDD)), $\xi(x) = \mathtt{ei}$ if $|x - \omega^i| > 2$ for all i, and $\xi(x) = \mathtt{es}$ if there exists ω^i such that $|x - \omega^i| \le 1$. Also, because $(\omega^i)_{i\in\mathbb{Z}} \overset{d}{=} (\omega^i + h)_{i\in\mathbb{Z}}$ by translation invariance of the Lebesgue measure, (STAT) is fulfilled and by the independence property of the Poisson process on disjoint Borel sets, we get that ξ also fulfills (MIX). Thus ξ fulfills all conditions from Section 3.1.1. We refer to Figure 4.1 for an illustration of this potential.

The crucial property of this potential is that it has long stretches where it equals $\mathtt{ei}$ that are adjacent to comparably long stretches where it equals $\mathtt{es}$, as is proved in the next lemma.

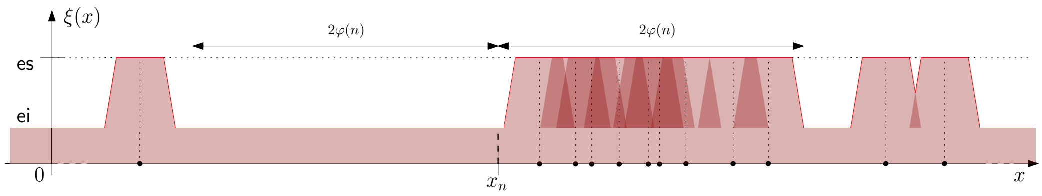

Figure 4.1: Realization of a potential ξ (top red line) fulfilling (4.1.11) with $\varphi(n) = c_0 \ln n$. Here we chose $\chi(x) = ((3-2x) \wedge 1) \vee 0$.

Lemma 4.4. *There is a constant $c_0 > 0$ such that $\mathbb{P}$-a.s. there exists a (random) increasing sequence $(x_n)_{n\in\mathbb{N}}$ of reals tending to infinity, such that*

$$\begin{aligned} \xi(x) &= \texttt{ei} \quad \forall x \in [x_n - 2c_0 \ln n, x_n], \\ \xi(x) &= \texttt{es} \quad \forall x \in [x_n + 2, x_n + 2c_0 \ln n - 2], \end{aligned} \tag{4.1.11}$$

and $\xi(\cdot)$ is non-decreasing on $[x_n - 2c_0 \ln n, x_n + 2c_0 \ln n - 2]$. Moreover, $\mathbb{P}$-a.s.,

$$1 \le \liminf_{n\to\infty} n^{-1} x_n \le \limsup_{n\to\infty} n^{-1} x_n \le 2. \tag{4.1.12}$$

Proof. The proof is an easy application of the Borel-Cantelli lemma. For $k \in \mathbb{N}$, let $A_{k,n}$ be the event

$$A_{k,n} = \left\{ \begin{array}{c} \omega : \omega \cap [n + (4k-2)c_0 \ln n - 2, n + 4kc_0 \ln n + 2) = \emptyset \quad \text{and} \\ \omega \cap [n + 4kc_0 \ln n + \ell, n + 4kc_0 \ln n + \ell + 1) \neq \emptyset \\ \text{for all } \ell = 2, \dots, \lfloor 2c_0 \ln n \rfloor - 3 \end{array} \right\}.$$

Observe that if $A_{k,n}$ occurs, then ξ satisfies (4.1.11) with $x_n = n + 4kc_0 \ln n$, and that $A_{k,n}$ only depends on ω in the interval $[n + (4k-2)\ln n - 2, n + 4k \ln n + \lfloor 2 \ln n \rfloor - 2)$. Therefore, the events $(A_{k,n})_{k\in\mathbb{N}}$ are independent. Moreover,

$$\mathbb{P}(A_{k,n}) = e^{-2c_0 \ln n - 4} \prod_{\ell=2}^{\lfloor 2c_0 \ln n \rfloor - 3} (1 - e^{-1}) \ge \alpha^{-c_0 \ln n} = n^{-c_0 \ln \alpha}$$

for some $\alpha > 1$ independent of c_0. Therefore, using $1 - x \le e^{-x}$,

$$\mathbb{P}\Big(\bigcap_{k=0}^{n/(4c_0 \ln n)-1} A_{k,n}^c \Big) \le (1 - n^{-c_0 \ln \alpha})^{n/(4c_0 \ln n)} \le \exp\{-n^{1-c_0 \ln \alpha} (4c_0 \ln n)^{-1}\}.$$

For $c_0 < 1/\ln \alpha$, the right-hand side is summable and thus by the Borel-Cantelli lemma, almost surely for n large enough, there exists $k \in [0, n/(4c_0 \ln n) - 1]$ such that $A_{k,n}$ occurs. This implies that $\mathbb{P}$-a.s. for n large enough there is $x \in [n, 2n]$ satisfying (4.1.11), completing the proof. □

In the following, if not mentioned otherwise, we will always refer to the sequence (x_n) as the one the existence of which is provided by (4.4).

4.1.3 Unbounded F-KPP-front: The coupling

In the next step towards a proof of Theorem 4.3, we construct a *coupling* of two BBMREs started in the vicinity of the points x_n where the potential satisfies the conditions (4.1.11) of Lemma 4.4. A coupling is a proof technique used to compare random objects X and Y by defining them on the same probability space such that the marginal distribution of (X, Y) correspond to X and Y, respectively. In our case, X will have the distribution of a BBMRE starting in small potential and Y that of a BBMRE starting in high potential to the right of X. The idea is to "couple their motion" such that under the common probability measure it is very likely that the offspring particles of the BBMRE whose ancestor started in high potential will overtake those particles starting in low potential.

Throughout this section, we assume that the constant c_0 and the random sequence x_n are as in Lemma 4.4, and write

$$\varphi(n) = c_0 \ln n. \tag{4.1.13}$$

In order to emphasize the dependence of the BBMRE on the starting point, we write $N_x = (N_x(t))_{t\geq 0}$ for the set of particles of a BBMRE started from x, that is for the process whose distribution is $\mathtt{P}_x^\xi$.

The content of the next proposition is the coupling alluded to above. Its statement is slightly more general than needed to show Theorem 4.3, since we construct couplings for many different starting points. This additional control will be useful in the proof of Theorem 4.9 in Section 4.2. Recall that the (possibly small but) positive parameter δ_1 is fixed below (4.1.34).

As before, for $x, y \in \mathbb{R}$ and $t \geq 0$ we set $N_x^{\leq}(t, y) = |\{\mathbf{u} \in N_x(t) : X_t^{\mathbf{u}} \leq y\}|$.

Proposition 4.5. *For every $\varepsilon > 0$ there exists $C_{19} = C_{19}(\varepsilon) \in (0, \infty)$ such that for all n large enough, $l \in [x_n - 5\delta_1\varphi(n), x_n - 4\delta_1\varphi(n)]$, and $r \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]$, there exists a coupling $\mathtt{Q}_{l,r}^\xi$ of the BBMREs N_l and N_r such that*

$$\mathtt{Q}_{l,r}^\xi\big(N_l(t) \subset N_r(t)\, \forall t \geq C_{19} \ln n\big) \geq 1 - \varepsilon. \tag{4.1.14}$$

For an illustration of the coupling and an explanation of the strategy to show that the event in (4.1.14) occurs with high probability, we refer to Figure 4.2 on page 105.

Before proving Proposition 4.5, let us first show that it implies Theorem 4.3.

Proof of Theorem 4.3. Using the notation from Proposition 4.5 we set

$$t_n := \inf\{t \geq 0 : w(t, x_n - 4\delta_1\varphi(n)) = \delta\}.$$

Note that $t_n \geq C_{19} \ln n$ for all n large enough (using $x_n \geq n$ and the fact that the front moves at least linearly, which is essentially a consequence of $w^\xi \geq w^{\mathtt{ei}}$ and by the results of Bramson [13], $w^{\mathtt{ei}}$ moves linearly in first order $\sqrt{2\mathtt{ei}}$). By (4.1.12) and (4.1.13) we get $\varphi \in \Omega(\ln n)$, $x_n, t_n \to \infty$, $(x_n)_{n\in\mathbb{N}} \in \mathcal{O}(n)$ and it remains to show (4.1.3). Let us abbreviate $l := x_n - 4\delta_1\varphi(n)$ and $r := x_n + 2\delta_1\varphi(n)$. By definition of the coupling $\mathtt{Q}_{l,r}^\xi$ and the representation $w(t, x) = \mathtt{P}_x^\xi\big(N^{\leq}(t, 0) \geq 1\big)$ of the solution to (F-KPP) (see (4.1.6)), we have for all n large enough that

$$\begin{aligned}\delta = w(t_n, x_n - 4\delta_1\varphi(n)) &= \mathtt{P}_l^\xi\big(N^{\le}(t_n, 0) \ge 1\big) = \mathtt{Q}_{l,r}^\xi\big(N_l^{\le}(t_n, 0) \ge 1\big)\\ &\le \mathtt{Q}_{l,r}^\xi\big(N_l^{\le}(t_n, 0) \ge 1, N_l(t) \subset N_r(t)\, \forall t \ge C_{19}\ln n\big) + \varepsilon\\ &\le \mathtt{Q}_{l,r}^\xi\big(N_r^{\le}(t_n, 0) \ge 1\big) + \varepsilon = \mathtt{P}_r^\xi\big(N^{\le}(t_n, 0) \ge 1\big) + \varepsilon\\ &= w(t_n, x_n + 2\delta_1\varphi(n)) + \varepsilon,\end{aligned}$$

where we used (4.1.14) in the first inequality. Adapting the notation to that of the statement, we can conclude. □

Proof of Proposition 4.5. To construct the coupling, we endow every particle in N_l and N_r at every time with a type. The type of the particle does not influence its dynamics within N_l or N_r, but rather helps to encode the dependence between N_l and N_r under $\mathtt{Q}_{l,r}^\xi$. At any given time, every particle in N_l can have either of the types *l-mirrored*, *l-coupled*, or *bad*. Similarly, every particle in N_r can have either of the types *r-mirrored*, *r-coupled*, or *free*. We denote $\mathtt{LM}(t), \mathtt{LC}(t), \mathtt{B}(t)$ and $\mathtt{RM}(t)$, $\mathtt{RC}(t)$ and $\mathtt{F}(t)$ the sets of particles with those respective types at time t. A particle is given a type when it is created, and its type can change only if it branches, meets another particle or hits some special point in space, as we will describe later. The assignment of the type is a right-continuous function in times, in the sense that if, e.g., a particle $\mathbf{u}$ changes its type from l-mirrored to bad at time t, then $\in \mathtt{B}(t)$ and $\mathbf{u} \in \mathtt{LM}(t-)$.

In addition, under the coupling, at every time $t \ge 0$, there are bijections $\mu_t : \mathtt{LM}(t) \to \mathtt{RM}(t)$ and $\gamma_t : \mathtt{LC}(t) \to \mathtt{RC}(t)$. The bijections μ_t "mirror" the positions of the particles:

$$\text{If } \mathbf{u} \in \mathtt{LM}(t) \text{ and } \mathbf{u}' = \mu_t(\mathbf{u}) \in \mathtt{RM}(t), \text{ then } m - X_t^{\mathbf{u}} = X_t^{\mathbf{u}'} - m, \tag{4.1.15}$$

where m is the midpoint of the segment (l, r),

$$m := \frac{1}{2}(l + r) \in [x_n - 2\delta_1\varphi(n), x_n - \delta_1\varphi(n)].$$

On the other hand, coupled particles are at the same position:

$$\text{If } \mathbf{u} \in \mathtt{LC}(t) \text{ and } \mathbf{u}' = \gamma_t(\mathbf{u}) \in \mathtt{RC}(t), \text{ then } X_t^{\mathbf{u}} = X_t^{\mathbf{u}'}. \tag{4.1.16}$$

As time evolves, the bijections μ_t and γ_t naturally follow the particles. That is, for the mirrored particles, if $\mathbf{u} \in \mathtt{LM}(t) \cap \mathtt{LM}(t')$, $\mathbf{u}' \in \mathtt{RM}(t) \cap \mathtt{RM}(t')$ and $\mathbf{u}' = \mu_t(\mathbf{u})$, then also $\mathbf{u}' = \mu_{t'}(\mathbf{u})$, and similarly for the coupled particles.

We set

$$\mathrm{L} := x_n - \varphi(n) \qquad \text{and} \qquad \mathrm{R} := 2m - \mathrm{L}. \tag{4.1.17}$$

It will turn out that under the coupling constructed below, the l-mirrored particles will always be in the interval (L, m), that is $\{X_t^{\mathbf{u}} : \mathbf{u} \in \mathtt{LM}(t)\} \subset (\mathrm{L}, m)$, see (A) and (C) below. As a consequence of (4.1.15) and (4.1.17), we then have $\{X_t^{\mathbf{u}} : \mathbf{u} \in \mathtt{RM}(t)\} \subset (m, \mathrm{R})$. In particular, in combination with (4.1.11), we infer that the potential is always larger at the position of an r-mirrored particle than at the position of the corresponding l-mirrored particle:

$$\text{If } \mathbf{u} \in \mathtt{LM}(t) \text{ and } \mathbf{u}' = \mu_t(\mathbf{u}), \text{ then } \xi(X_t^{\mathbf{u}}) \le \xi(X_t^{\mathbf{u}'}). \tag{4.1.18}$$

We can now describe the dynamics of N_l, N_r and of the types under the coupling $\mathtt{Q}_{l,r}^\xi$. At time 0, there is one (l-mirrored) particle at position l in N_l and one (r-mirrored) particle at

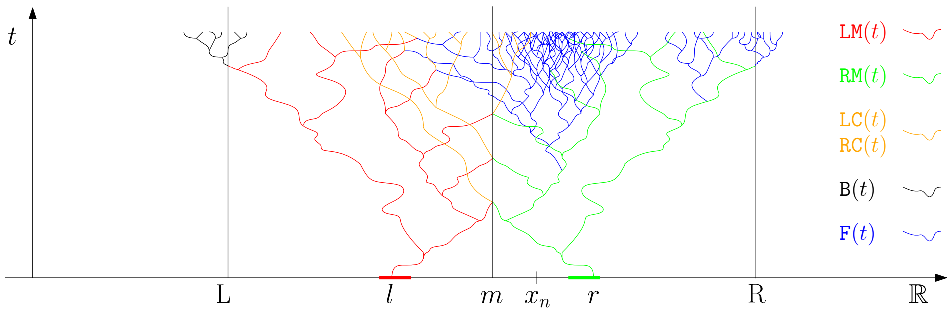

Figure 4.2: An illustration of the coupling mechanism. l-mirrored particles are illustrated in red, r-mirrored particles in green, while l- and r-coupled particles are illustrated in orange. Free particles are blue and bad particles are black. The fat red (resp. green) line on the $\mathbb{R}$-axis denotes the set $[x_n - 5\delta_1\varphi(n), x_n - 4\delta_1\varphi(n)]$ (resp. $[x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]$). Note that x_n is nearer to the green domain, forcing a particle $\mathbf{u} \in N_l$ to go a long way to reach high branching-potential. The event in (4.1.14) occurs, if at time $t = c\ln n$, all l-mirrored particles (red) are already turned into l-coupled ones (orange) and no l-mirrored particles have crossed L yet. But then there will be no bad particles (black) either, which already implies the event in (4.1.14).

position r in N_r; this determines the bijection μ_0 uniquely. Every particle in N_l (resp. N_r) performs Brownian motion, independently of the other particles in N_l (resp. N_r). The corresponding mirrored and coupled particles are required to satisfy (4.1.15) and (4.1.16) respectively, which is possible, since the law of Brownian motion is invariant by reflection; besides these two conditions the motion of particles in N_l is independent of the motion of particles in N_r.

The branching events occur according to the following rules.

(a) At time t, every $\mathbf{u} \in N_l$ branches with rate $\xi(X_t^{\mathbf{u}})$. It is replaced by k new particles, with probability p_k, independently of remaining randomness. The type of the new particles is the same as of $\mathbf{u}$.

If a particle $\mathbf{u}$ is l-mirrored (resp. l-coupled), $\mathbf{u} \in \mathtt{LM}(t-)$ (resp. $\mathbf{u} \in \mathtt{LC}(t-)$) before time t, then the corresponding r-mirrored particle $\mathbf{u}' = \mu_{t-}(\mathbf{u})$ (resp. r-coupled particle, $\mathbf{u}' = \gamma_{t-}(\mathbf{u})$) branches as well. It is replaced by the same number k of particles. The newly created particles are set to be r-mirrored (resp. r-coupled) and the bijection μ_t (resp. γ_t) is a natural extension of μ_{t-} (resp. γ_{t-}) to the newly created particles.

(b) At time t, every r-mirrored particle $\mathbf{u}' \in \mathtt{RM}(t-)$ (mirrored with $\mathbf{u} = \mu_{t-}^{-1}(\mathbf{u}')$) branches with rate $\xi(X_t^{\mathbf{u}'}) - \xi(2m - X_t^{\mathbf{u}'}) = \xi(X_t^{\mathbf{u}'}) - \xi(X_t^{\mathbf{u}})$, in addition to the branching occurring in (a). This rate is nonnegative due to (4.1.15) and (4.1.18). It is replaced by k new particles, with probability p_k, independently of everything else. One of the newly created particles, say $\mathbf{v}'$, is set to be r-mirrored, and we set $\mu_t(\mathbf{u}) := \mathbf{v}'$. The type of the remaining newly created particles is free.

(c) At time t, every free particle $\mathbf{u}' \in \mathtt{F}(t)$ branches with rate $\xi(X_t^{\mathbf{u}'})$. It is replaced by k new particles, with probability p_k, independently of everything else. The type of the new particles is free.

As a result of the rules (a)–(c), every $\mathbf{u}' \in N_r$ branches with rate $\xi(X_t^{\mathbf{u}'})$ at time t, as it should.

Finally, the particles can change their type if one of the following events occur:

(A) If an l-mirrored particle hits m, that is $\mathbf{u} \in \mathtt{LM}(t-)$ and $X_t^{\mathbf{u}} = m$, then, by consequence of (4.1.15), the corresponding particle $\mathbf{u}' = \mu_{t-}(\mathbf{u})$ satisfies $X_t^{\mathbf{u}'} = m$ as well. We thus change the types of $\mathbf{u}$ and $\mathbf{u}'$ to l-coupled and r-coupled, respectively, and define $\gamma_t(\mathbf{u}) := \mathbf{u}'$.

(B) If an l-mirrored particle $\mathbf{u} \in \mathtt{LM}(t-)$ meets a free particle at time t, that is there is $\mathbf{v}' \in \mathtt{F}(t-)$ with $X_t^{\mathbf{v}'} = X_t^{\mathbf{u}}$, then we change the types of $\mathbf{u}$ and $\mathbf{v}'$ to l-coupled and r-coupled, respectively, and define with $\gamma_t(\mathbf{u}) := \mathbf{v}'$. The type of the r-mirrored particle $\mathbf{u}' = \mu_{t-}(\mathbf{u})$ that was mirrored with $\mathbf{u}$ is changed to free.

(C) If an l-mirrored particle hits L, that is $\mathbf{u} \in \mathtt{LM}(t-)$ and $X_t^{\mathbf{u}} = \mathrm{L}$, then the type of $\mathbf{u}$ is changed to bad, and the type of the corresponding r-mirrored particle $\mathbf{u}' = \mu_{t-}(\mathbf{u})$ is changed to free.

To show that the coupling succeeds, i.e. that (4.1.14) holds, it is sufficient to show that with probability at least $1-\varepsilon$, there are no l-mirrored and bad particles after time $C_{19} \log n$. In this vein, we define two good events:

$$\mathcal{G}_1(t) := \{N_l^{\leq}(s, \mathrm{L}) = 0 \, \forall s \leq t\}, \tag{4.1.19}$$

i.e., on $\mathcal{G}_1(t)$ no particle from N_l enters $(-\infty, \mathrm{L})$ before time t, and

$$\mathcal{G}_2(t) := \{N_r^{\leq}(t, \mathrm{L}) \geq 1\}; \tag{4.1.20}$$

i.e., there is a (necessarily free, if $\mathcal{G}_1(t)$ occurs as well) particle to the left of L at time t. We now need the following lemma which ensures that we can find t such that those events are typical.

Lemma 4.6. *For any $\varepsilon > 0$ there exists $t' < 1$ such that for all n large enough, with $t = t'\varphi(n)/\sqrt{2\mathtt{ei}}$,*

$$\mathtt{Q}_{l,r}^{\xi}\big(\mathcal{G}_1(t) \cap \mathcal{G}_2(t)\big) \geq 1 - \varepsilon. \tag{4.1.21}$$

We postpone the proof of this lemma and complete the proof of Proposition 4.5 first. Let t be as in Lemma 4.6. We claim that

$$\{N_l(t) \subset N_r(t)\} \supset \mathcal{G}_1(t) \cap \mathcal{G}_2(t). \tag{4.1.22}$$

If we show this, then the claim of Proposition 4.5 follows with $C_{19} = t/\ln n = t'c_0/\sqrt{2\mathtt{ei}}$.

To prove (4.1.22), recall first that bad particles can only be created if an l-mirrored particle hits L. As a consequence,

$$\text{on } \mathcal{G}_1(t) \text{ there cannot be any bad particles at time } t. \tag{4.1.23}$$

Next, we show that

$$\text{on } \mathcal{G}_1(t) \cap \mathcal{G}_2(t) \text{ there are no l-mirrored particles at time } t \tag{4.1.24}$$

either. To this end define $\mathcal{R}(t) = \inf\{X_t^{\mathbf{u}'} : \mathbf{u}' \in \mathtt{F}(t)\}$ to be the position of the leftmost free particle, and $\mathcal{L}(t) = \sup\{X_t^{\mathbf{u}} : \mathbf{u} \in \mathtt{LM}(t)\}$ to be the position of the rightmost l-mirrored particle, with the convention $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$; in the remaining cases, a.s., the infimum and supremum are attained, since $\mathtt{F}(t)$ and $\mathtt{LM}(t)$ are a.s. finite sets). Let

$$\tau := \inf\{t \geq 0 : \mathcal{L}(t) > \mathcal{R}(t)\}.$$

We claim that $\tau = \infty$, $\mathtt{Q}_{l,r}^{\xi}$-a.s. Indeed, we first note that $\mathcal{L}$ and $\mathcal{R}$ are right-continuous. In addition, the only jumps that $\mathcal{L}$ has are downward jumps. They occur a.s. iff the rightmost l-mirrored particle changes its type due to (A)–(C). (If one of (A)–(C) occurs, then a.s. there is only one l-mirrored particle at position $\mathcal{L}(t)$. At branching events, $\mathcal{L}$ is unchanged, as l-mirrored particles are created only at positions where l-mirrored particles are already present, see (a)). Similarly, with the exception of the first jump from $+\infty$, the only jumps that the function $\mathcal{R}$ has are upwards jumps, occurring a.s. iff the leftmost free particle becomes r-coupled due to (B). Therefore, it follows that a.s. $\tau \geq \inf\{t \geq 0 : \mathcal{L}(t) = \mathcal{R}(t)\}$. However, the event $\{\exists t \in [0,\infty) : \mathcal{L}(t) = \mathcal{R}(t)\}$ cannot occur by the construction of the coupling, since if an l-mirrored and a free particle meet, then at this instant they become l-/r-coupled immediately. Hence, $\tau = \infty$ almost surely, as claimed.

Assume now that $\mathcal{G}_1(t) \cap \mathcal{G}_2(t)$ occurs. At time t, there is thus a particle from N_r and no particle from N_l to the left of L. From the construction, this particle is neither r-coupled (since on $\mathcal{G}_1(t)$ there is no corresponding l-coupled particle there), nor r-mirrored (as all r-mirrored particles are always in (m, R)). Therefore, it must be free and thus $\mathcal{R}(t) < \mathrm{L}$. Since $\tau = \infty$ a.s., $\mathcal{L}(t) < \mathrm{L}$ as well. However, by construction, l-mirrored particles are always located in (L, m), and thus $\mathcal{L}(t) < \mathrm{L}$ implies $\mathcal{L}(t) = -\infty$, that is $\mathtt{LM}(t) = \emptyset$, establishing (4.1.24).

All in all, from the above it follows that on $\mathcal{G}_1(t) \cap \mathcal{G}_2(t)$, (4.1.23) as well as (4.1.24) hold true, i.e., there do not exist any l-mirrored or bad particles at time t. Hence, all particles in $N_l(t)$ are necessarily l-coupled, which proves (4.1.22). This completes the proof of Proposition 4.5. □

It remains to show Lemma 4.6.

Proof of Lemma 4.6. Let us first estimate the probability of $\mathcal{G}_1(t)$ as a function of $t \in [0, \varphi(n)/\sqrt{2\mathtt{ei}}]$. To this end we write $\mathcal{N}(t)$ for the number of particles from N_l that hit L before t; here, we only count the first hit of L by any particle. That is, we disregard possible successive hits of L by the same particle, and also the fact that this particle could branch between the hitting of L and the time t, and thus produce more particles at time t that hit L. The expectation of $\mathcal{N}(t)$ can be written as

$$\mathtt{E}_l^{\xi}[\mathcal{N}(t)] = E_l\Big[e^{\int_0^{H_{\mathrm{L}}} \xi(X_s)\mathrm{d}s}; H_{\mathrm{L}} < t\Big] \leq E_l\Big[e^{\int_0^{H_{\mathrm{L}}} \tilde{\xi}(X_s)\mathrm{d}s}; H_{\mathrm{L}} < t\Big], \tag{4.1.25}$$

where the potential $\tilde{\xi}$ is given by $\tilde{\xi}(x) = \mathtt{es}$ if $x \geq x_n$, and $\tilde{\xi}(x) = \mathtt{ei}$ if $x < x_n$. To estimate the right-hand side, note that there are two possible scenarios for a particle to hit L. Either, it stays all the time in the interval (L, x_n) where the potential equals $\mathtt{ei}$ and hits L (i.e., it displaces by altogether at least $l - \mathrm{L} \geq (1 - 5\delta_1)\varphi(n)$). Or, it spends some s units of time in the interval $[x_n, \infty)$, where the potential is $\mathtt{es}$, but then it should displace by at least $(x_n - l) + (x_n - \mathrm{L}) \geq (1 + 4\delta_1)\varphi(n)$ in $t - s$ units of time. Ignoring prefactors which are sub-exponential in $\varphi(n)$ and using standard Gaussian tail bounds, we thus arrive at the following upper bound:

$$\begin{aligned}\mathtt{E}_l^{\xi}[\mathcal{N}(t)] &\lesssim \exp\Big\{t\mathtt{ei} - \frac{(1-5\delta_1)^2\varphi(n)^2}{2t}\Big\} + \sup_{s \leq t} \exp\Big\{(t-s)\mathtt{ei} + s\mathtt{es} - \frac{(1+4\delta_1)^2\varphi(n)^2}{2(t-s)}\Big\} \\ &= \exp\Big\{\sigma(n)\Big(t' - \frac{(1-5\delta_1)^2}{t'}\Big)\Big\} + \sup_{s' < t'} \exp\Big\{\sigma(n)\Big(t' + s'\,\frac{\mathtt{es}-\mathtt{ei}}{\mathtt{ei}} - \frac{(1+4\delta_1)^2}{t'-s'}\Big)\Big\},\end{aligned} \tag{4.1.26}$$

where we introduced

$$\sigma(n) = \varphi(n)\sqrt{\frac{\texttt{ei}}{2}} \qquad \text{and} \qquad t' = \frac{t\,\texttt{ei}}{\sigma(n)} \tag{4.1.27}$$

in order to put the various terms on the same scale. Using Markov's inequality, to show that $\mathtt{P}_l^\xi(\mathcal{G}_1(t)^c) \to 0$, it is sufficient to show that both summands on the right-hand side of (4.1.26) tend to 0. For this to be the case for the first one, it is sufficient to require

$$t' < (1 - 5\delta_1). \tag{4.1.28}$$

Before dealing with the second term (which we will do below (4.1.30)), we turn our attention to the event $\mathcal{G}_2(t)$.

To control the probability of the event $\mathcal{G}_2$, we need two claims.

Claim 4.7. *For every $\varepsilon > 0$ there exists $t_0 < \infty$ such that for all n large enough,*

$$\mathtt{P}_r^\xi\Big(\big|\big\{\mathbf{u} \in N_r(t) : X_t^{\mathbf{u}} \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]\big\}\big| \geq e^{(1-\delta_2)\texttt{es}\,t}\Big) \geq 1 - \varepsilon/2, \ \forall\, t \geq t_0.$$

In order not to hinder the flow of reading, we postpone the proof of Claim 4.7 to the end of the proof of Lemma 4.6.

Claim 4.8. *Let $t = t'\varphi(n)/\sqrt{2\texttt{ei}}$ with $t' < 1$ and $\eta > 0$. Then for every $y \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]$ and all n large enough*

$$\mathtt{P}_y^\xi(N^{\leq}(t,\mathrm{L}) \geq 1) \geq \exp\Big\{\sigma(n)\Big(t' - \frac{(1+2\delta_1)^2}{t'} - \eta\Big)\Big\}. \tag{4.1.29}$$

Proof. Obviously $\mathtt{P}_y^\xi(N^{\leq}(t,\mathrm{L}) \geq 1) \geq \mathtt{P}_y^{\texttt{ei}}(N^{\leq}(t,\mathrm{L}) \geq 1) \geq \mathtt{P}_{x_n+2\delta_1\varphi(n)}^{\texttt{ei}}(N^{\leq}(t,\mathrm{L}) \geq 1)$. Moreover, by the large deviation lower bound from [17, Theorem 1], for every $v > \sqrt{2\texttt{ei}}$ and $\eta > 0$, if t is sufficiently large, then

$$\mathtt{P}_0^{\texttt{ei}}(N^{\leq}(t, -vt) \geq 1) \geq \exp\{t(\texttt{ei} - v^2/2 - \eta)\}.$$

Using this estimate with $v = (x_n + 2\delta_1\varphi(n) - \mathrm{L})/t = (1+2\delta_1)\varphi(n)/t = (1+2\delta_1)\sqrt{2\texttt{ei}}/t' > \sqrt{2\texttt{ei}}$, and by rewriting it using the notation introduced in (4.1.27), the claim follows. □

Using these two claims, we have that for any $0 < s' < t' < 1$ as well as for $t = t'\varphi(n)/\sqrt{2\texttt{ei}}$ and $s = s'\varphi(n)/\sqrt{2\texttt{ei}}$, that

$$\begin{aligned}
\mathtt{P}_r^\xi(\mathcal{G}_2(t)^c) &\leq \mathtt{P}_r^\xi\Big(|\{\mathbf{u} \in N_r(s) : X_s^{\mathbf{u}} \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]\}| \leq e^{(1-\delta_2)\texttt{es}\,s}\Big) \\
&\quad + \mathtt{P}_r^\xi\Big(\mathcal{G}_2(t)^c \,\Big|\, |\{\mathbf{u} \in N_r(s) : X_s^{\mathbf{u}} \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]\}| \geq e^{(1-\delta_2)\texttt{es}\,s}\Big) \\
&\leq \frac{\varepsilon}{2} + \Big(1 - \exp\Big\{\sigma(n)\Big(t' - s' - \frac{(1+2\delta_1)^2}{t'-s'} - \eta\Big)\Big\}\Big)^{\exp\{(1-\delta_2)\texttt{es}\,s\}}.
\end{aligned}$$

The second summand on the right-hand side converges to 0 as $n \to \infty$ if

$$\begin{aligned}
&\exp\Big\{\sigma(n)\Big(t' - s' - \frac{(1+2\delta_1)^2}{t'-s'} - \eta\Big)\Big\} \cdot \exp\{(1-\delta_2)\texttt{es}\,s\} \\
&= \exp\Big\{\sigma(n)\Big(t' + s'\frac{\texttt{es}(1-\delta_2) - \texttt{ei}}{\texttt{ei}} - \frac{(1+2\delta_1)^2}{t'-s'} - \eta\Big)\Big\} \to \infty.
\end{aligned} \tag{4.1.30}$$

The factors in the exponents of (4.1.26) and (4.1.30) are both of the form $t' + As' - B/(t' - s')$ such that (for $\delta_2 > 0$ small) $A > 0$ and $B > 1$. For A, B and t', fixed, this function is maximized for $s \in [0, t']$ by

$$s' = \begin{cases} t' - \sqrt{B/A}, \\ 0, \end{cases} \text{with a maximum value of } \begin{cases} (1+A)t' - 2\sqrt{AB}, & \text{if } t' > \sqrt{B/A}, \\ t' - B/t', & \text{otherwise.} \end{cases} \tag{4.1.31}$$

Ignoring for a moment the constants δ_2 and η, we write $A = (\mathtt{es} - \mathtt{ei})/\mathtt{ei}$, $B_1 = (1 + 4\delta_1)^2$, and $B_2 = (1 + 2\delta_1)^2$. Observe that $A > 1$ by (4.1.9). In order to satisfy (4.1.30) and let (4.1.26) tend to 0, we must fix t' and δ_1 so that (4.1.28) holds, and at the same time

$$\sup_{0<s'<t'} t' + s'A - B_1/(t' - s') < 0, \tag{4.1.32}$$

$$\sup_{0<s'<t'} t' + s'A - B_2/(t' - s') > 0. \tag{4.1.33}$$

Since $B_2 > 1$ and $t' < 1$, the analysis in (4.1.31) implies that the supremum in (4.1.33) can be positive only if

$$t' > \max\left(\sqrt{\frac{B_2}{A}}, \frac{2\sqrt{AB_2}}{1+A}\right) = \frac{2\sqrt{AB_2}}{1+A}, \tag{4.1.34}$$

where to obtain the equality we used the fact that $A > 1$. We thus fix $\delta_1 > 0$ small enough so that $1 - 5\delta_1 > 2\sqrt{AB_2}/(1 + A)$ and (4.1.28) as well as (4.1.34) can be both satisfied; this is possible only if $A > 1$ which is true by assumption. We then fix t' satisfying (4.1.28) and (4.1.34), so that the supremum in (4.1.33) is positive (this is by construction), but small enough, so that the supremum in (4.1.32) is negative; this is possible since $B_1 > B_2$. Finally, we fix $\delta_2 > 0$, $\eta > 0$ so that the validity of the established inequalities is not modified. With this choice of constants, (4.1.30) holds and the right-hand side of (4.1.26) tends to 0, as required. Hence, for $t = t'\varphi(n)/\sqrt{2\mathtt{ei}}$ we have $\mathtt{Q}^{\xi}_{r,l}(\mathcal{G}_1(t)^c \cup \mathcal{G}_2(t)^c) \le \mathtt{P}^{\xi}_l(\mathcal{G}_1(t)^c) + \mathtt{P}^{\xi}_r(\mathcal{G}_2(t)^c) \to 0$ as $n \to \infty$. This completes the proof. □

It remains to prove Claim 4.7.

Proof of Claim 4.7. The proof follows by a comparison with branching processes split into two phases. For the first phase we recall that by Lemma 3.39 there exist $\kappa > 1$ and $t_1 < \infty$ such that, $\mathbb{P}$-a.s.,

$$\sup_{x\in\mathbb{R}} \mathtt{P}^{\xi}_x\big(|\{\mathbf{u} \in N(t) : X^{\mathbf{u}}_t \in [x-1, x+1]\}| \le \kappa^t\big) \le \kappa^{-t} \qquad \text{for all } t \ge t_1. \tag{4.1.35}$$

For the second phase we need few preparatory steps. We fix $T > 0$ such that

$$e^{(1-\frac{\delta_2}{2})\mathtt{es}\, T} \le \frac{1}{4} e^{\mathtt{es}\, T} \quad \text{and} \quad P_0(B_T > 1) \ge \frac{7}{16}. \tag{4.1.36}$$

We further fix $K_1 > 1$ large enough so that

$$\inf_{x\in[-K_1-1, K_1+1]} P_x\big(B_T \in [-K_1, K_1]\big) \ge \frac{3}{8}, \tag{4.1.37}$$

which is possible due to the second part of (4.1.36). Finally, we fix $K_2 > K_1$ large enough so that

$$\sup_{x\in[-K_1-1,K_1+1]} P_x\big(B_s \notin [-K_2,K_2] \text{ some } s\in[0,T]\big) \leq \frac{1}{16}, \tag{4.1.38}$$

so (4.1.37) in combination with (4.1.38) entail that

$$\inf_{x\in[-K_1-1,K_1+1]} P_x\big(B_T \in [-K_1,K_1], B_s \in [-K_2,K_2]\,\forall s\leq T\big) \geq \frac{5}{16}. \tag{4.1.39}$$

Next, assume that n is large enough, so that $\delta_1\varphi(n) > K_2/2$, and in particular ξ equals $\texttt{es}$ on $[x_n+\delta_1\varphi(n)-K_2, x_n+\delta_1\varphi(n)+K_2]$. For $x\in[x_n+\delta_1\varphi(n)-1, x_n+\delta_1\varphi(n)+1]$, define

$$x' = \begin{cases} x_n+\delta_1\varphi(n)+K_1, & \text{if } x < x_n+\delta_1\varphi(n)+K_1,\\ x_n+2\delta_1\varphi(n)-K_1, & \text{if } x > x_n+2\delta_1\varphi(n)-K_1,\\ x, & \text{otherwise,}\end{cases} \tag{4.1.40}$$

and set $I_i = [x'-K_i, x'-K_i]$, $i=1,2$, so that $I_1\subset I_2$. We now consider the BBMRE started at x and for $k\geq 1$ we define

$$Z_k = |\{\mathbf{u}\in N(kT) : X^{\mathbf{u}}_{lT}\in I_1\,\forall 1\leq l\leq K, X^{\mathbf{u}}_s\in I_2\,\forall s<kT\}|. \tag{4.1.41}$$

Z_k can be interpreted as the number of particles in the k-th generation of a multi-type branching process; here, the type corresponds to the position of the particle in I_1 at which it is born (with exception of the initial particle which is at most at distance 1 from I_1), and where the number of offspring of a particle of type y is distributed as $|\{\mathbf{u}\in N(T) : X^{\mathbf{u}}_T\in I_1, X^{\mathbf{u}}_s\in I_2\,\forall s\leq T\}|$ under $\mathtt{P}^{\texttt{es}}_y$. In particular, using the Feynman-Kac formula as well as (4.1.39) and then (4.1.36), the expected offspring number of a particle of type y satisfies

$$\begin{aligned} &\mathtt{E}^{\texttt{es}}_y[|\{\mathbf{u}\in N(T) : X^{\mathbf{u}}_T\in I_1, X^{\mathbf{u}}_s\in I_2\,\forall s\leq T\}|]\\ &\quad = e^{\texttt{es}\,T}P_y(B_T\in I_1, B_s\in I_2\,\forall s\leq T) \geq \frac{5}{16}e^{\texttt{es}\,T} \geq e^{(1-\frac{\delta_2}{2})\texttt{es}\,T}, \end{aligned} \tag{4.1.42}$$

uniformly over all admissible types y. In addition, the second moment of the same quantity is finite, again uniformly over all admissible types, by comparison with branching process with branching rate $\texttt{es}$. It thus follows by the standard results on multi-type branching processes that for some $\rho\geq e^{(1-\frac{\delta_2}{2})\texttt{es}\,T}$ finite, Z_k/ρ^k converges in distribution to a nonnegative random variable W with $P(W>0)>0$ (see e.g. [35, Theorem 14.1], where ρ is the principal eigenvalue of the expectation operator of the multi-type branching process; observe also that Condition 10.1 of this theorem is easily checked for V being the Lebesgue measure). In particular, one can find $\varepsilon_2>0$ and k_0 large such that

$$\mathtt{P}^{\texttt{es}}_x\big(Z_k\geq \varepsilon_2 e^{(1-\frac{\delta_2}{2})\texttt{es}\,kT}\big) \geq \mathtt{P}^{\texttt{es}}_x(Z_k\geq\varepsilon_2\rho^k) \geq \varepsilon_2 \qquad \text{for all } k\geq k_0, \tag{4.1.43}$$

uniformly in $x\in[x_n+\delta_1\varphi(n)-1, x_n+\delta_1\varphi(n)+1]$. This terminates the investigation of the second phase of comparison with a branching process, and we may now proceed to the proof of Claim 4.7.

To this end, fix K such that $(1-\varepsilon_2)^K<\varepsilon/4$ and set (for κ and t_1 from (4.1.35))

$$t' = \inf\{s\in[t_1,t], \kappa^s > K\vee(4/\varepsilon), t-s = kT \text{ for some } k\in\mathbb{N}\}. \tag{4.1.44}$$

Observe that there is $c<\infty$ such that $t'<c$ for all $t\geq c$. Setting $\mathcal{N} = \{\mathbf{u}\in N(t') : X^{\mathbf{u}}_{t'}\in$

$[r-1, r+1]\}$, we have, using (4.1.35) and (4.1.44) for the last inequality, that

$$\begin{aligned} &\mathtt{P}_r^\xi\Big(\big|\{\mathbf{u} \in N_r(t) : X_t^{\mathbf{u}} \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]\}\big| \le e^{(1-\delta_2)\mathtt{es}\, t}\Big) \\ &\quad \le \mathtt{P}_r^\xi(|\mathcal{N}| < \kappa^{t'}) + \mathtt{P}_r^\xi\big(\{|\mathcal{N}| \ge \kappa^{t'}\} \cap \mathcal{A}\big) \le \frac{\varepsilon}{4} + \mathtt{P}_r^\xi\big(\{|\mathcal{N}| \ge \kappa^{t'}\} \cap \mathcal{A}\big), \end{aligned} \tag{4.1.45}$$

where $\mathcal{A}$ denotes the event that each particle in $\mathcal{N}$ produces less than $e^{(1-\delta_2)\mathtt{es}\, t}$ particles in $[x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]$ at time t. For a particle at position $x \in [r-1, r+1]$, we then fix the intervals I_1, I_2 as above, and observe that the number of its children in $[x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]$ at time $t - t' =: k_t T$ dominates Z_{k_t} under $\mathtt{P}_x^{\mathtt{es}}$. Since the offspring of different particles are independent, for t large enough such that $e^{(1-\delta_2)\mathtt{es}\, t} \le \varepsilon_2 e^{(1-\frac{\delta_2}{2})\mathtt{es}\, k_t T}$, we obtain

$$\begin{aligned} \mathtt{P}_r^\xi\big(\{|\mathcal{N}| \ge \kappa^{t'}\} \cap \mathcal{A}\big) &\le \mathtt{E}_r^\xi\Big[\prod_{\mathbf{u}\in\mathcal{N}} \mathtt{P}_{X_{t'}^{\mathbf{u}}}\big(Z_{k_t} \le e^{(1-\delta_2)\mathtt{es}\, t}\big);\, |\mathcal{N}| \ge \kappa^{t'}\Big] \\ &\le \mathtt{E}_r^\xi\Big[\prod_{\mathbf{u}\in\mathcal{N}} \mathtt{P}_{X_{t'}^{\mathbf{u}}}\big(Z_{k_t} \le \varepsilon_2 e^{(1-\frac{\delta_2}{2})\mathtt{es}\, k_t T}\big);\, |\mathcal{N}| \ge \kappa^{t'}\Big] \\ &\le \mathtt{E}_r^\xi\Big[(1-\varepsilon_2)^{|\mathcal{N}|}; |\mathcal{N}| \ge \kappa^{t'}\Big] \le (1-\varepsilon_2)^{\kappa^{t'}} \le (1-\varepsilon_2)^K \le \frac{\varepsilon}{4}, \end{aligned} \tag{4.1.46}$$

where for the third inequality we used (4.1.43) and for the last two inequalities we applied (4.1.44). Combining (4.1.45) with the last display completes the proof of the claim. □

4.2 Spatial non-monotonicity of the solution to F-KPP

While the previous result has been derived using probabilistic techniques, we will enhance it employing analytic techniques to show that the statement of Theorem 4.3 is true even for some negative ε. In particular, this entails the non-monotonicity of the solution w to (F-KPP) in space.

Theorem 4.9. *Let* $w_0 = \mathbb{1}_{(-\infty,0]}$. *There exist potentials* ξ *fulfilling* (HÖL), (BDD), (STAT) *and* (MIX), *some* $\varepsilon > 0$ *small enough, and sequences* $(t_n')_{n\in\mathbb{N}} \subset [0,\infty)$, $(l_n')_{n\in\mathbb{N}} \subset \mathbb{R}$ *and* $(r_n')_{n\in\mathbb{N}} \subset \mathbb{R}$ *such that* $t_n' \to \infty$, $l_n' < r_n'$ *for all* n, $r_n - l_n \in \Theta(\ln n)$ *and for all* $n \in \mathbb{N}$,

$$w(t_n', l_n') \le w(t_n', r_n') - \varepsilon.$$

The proof of Theorem 4.9 is based on the simple idea that if there are two adjacent long stretches, the left one with potential `ei` and the right one with `es`, where the values of w are comparable at some time t_n, as proved in Theorem 4.3, then at some later time $t_n + s$ the function w must be non-monotone, since it grows faster on the right stretch.

For the first time we use the analytical expression for the solution w to (F-KPP), i.e. w fulfills

$$w(t,x) = E_x\Big[\exp\Big\{\int_0^t \xi(B_s)c(w(t-s,B_s))\,\mathrm{d}s\Big\}w_0(B_t)\Big],\ (t,x) \in [0,\infty)\times\mathbb{R}, \tag{4.2.1}$$

where we abbreviated $c(w) := \frac{F(w)}{w}$, with some $F : [0,1] \to [0,1]$ fulfilling (PROB), see Proposition 2.20 (a). It is easy to see that c is strictly decreasing, can be extended continuously to $w = 0$, i.e. $c(0) = \lim_{w\downarrow 0} c(w) = \sup_{w\in(0,1]} c(w) = 1$, $c(1) = 0$ and the function $c : [0,1] \to [0,1]$ is Lipschitz continuous with Lipschitz constant $H \in (0,\infty)$.

Proof of Theorem 4.9. For every $\varepsilon > 0$ we choose $K = K(\varepsilon)$ such that

$$f(K) := e^{\texttt{es}} P_0\Big(\sup_{0 \le u \le 1} |B_u| > K \Big) \le \varepsilon. \tag{4.2.2}$$

Recall that by Proposition 4.5, the definition of the coupling $\mathtt{Q}^{\xi}_{l,r}$ and the representation $w(t,x) = \mathtt{P}^{\xi}_x\big(N^{\le}(t,0) \ge 1\big)$ of the solution to (F-KPP) (see (4.1.6)), for $\delta \in (0,1)$ there exist l_n, r_n, t_n such that $t_n \to \infty$, $w(t_n, l_n) = \delta$, $r_n - l_n \underset{n\to\infty}{\longrightarrow} \infty$ and such that for all n large enough

$$\sup_{l \in [l_n - K, l_n + K]} w(t_n, l) \le \inf_{r \in [r_n - K, r_n + K]} w(t_n, r) + \varepsilon \tag{4.2.3}$$

holds. We will prove the result by contradiction and therefore assume for the time being that the claim of the theorem does not hold. Then, for all $\varepsilon > 0$, all n large enough and all $s \in [0,1]$, we have

$$\inf_{l \in [l_n - K, l_n + K]} w(t_n + s, l) \ge \sup_{r \in [r_n - K, r_n + K]} w(t_n + s, r) - \varepsilon. \tag{4.2.4}$$

Let us choose $\varepsilon \in (0, \delta)$, $s' \in (0,1]$ small enough and $b \in (0,1)$ such that for all $s \in [0, s']$,

$$e^{\texttt{es}\, s}(\delta + 3\varepsilon) \le b. \tag{4.2.5}$$

Due to (4.2.4) and $w \in [0,1]$, for all $s \in [0,1]$ we have

$$\sup_{l \in [l_n - K, l_n + K]} c\big(w(t_n + s, l)\big) \le \inf_{r \in [r_n - K, r_n + K]} c\big(w(t_n + s, r)\big) + H\varepsilon. \tag{4.2.6}$$

Furthermore, by the Feynman-Kac formula (4.2.1) and the Markov property, for all $s \ge 0$ we have

$$w(t_n + s, l_n) = E_{l_n}\Big[\exp\Big\{ \int_0^s \xi(B_u) c\big(w(t_n + s - u, B_u)\big)\, \mathrm{d}u \Big\} w(t_n, B_s) \Big].$$

Then due to $\xi \le \texttt{es}$, $w \in [0,1]$, $c \le 1$, (4.2.3), (4.2.4), (4.2.2), and (4.2.5), for all n large enough we have for all $s \in [0, s']$ that

$$w(t_n + s, l_n) \le e^{\texttt{es}\, s}\Big(P_{l_n}\big(\sup_{0 \le u \le 1} |B_u - l_n| > K \big) + \sup_{l \in [l_n - K, l_n + K]} w(t_n, l) \Big) \le b. \tag{4.2.7}$$

Furthermore, using $\xi \le \texttt{es}$, $w \in [0,1]$ and $c(w) \in [0,1]$ for $w \in [0,1]$ we get that for all $s \in [0,1]$ we have

$$\begin{aligned} w(t_n + s, l_n) \le\ & E_{l_n}\Big[\exp\Big\{ \int_0^s \xi(B_u) c\big(w(t_n + s - u, B_u)\big)\, \mathrm{d}u \Big\} w(t_n, B_s); \sup_{0 \le u \le 1} |B_u - l_n| \le K \Big] \\ & + e^{\texttt{es}} P_0\Big(\sup_{0 \le u \le 1} |B_u| > K \Big). \end{aligned}$$

To bound the first summand, we recall (by definition of l_n, r_n) that $\xi(l) = \texttt{ei}$ for all $l \in [l_n - K, l_n + K]$ and $\xi(r) = \texttt{es}$ for all $r \in [r_n - K, r_n + K]$. Using (4.2.3) and (4.2.6), we see that the first summand can be bounded from above by

$$\begin{aligned}
&E_{l_n}\Big[\exp\Big\{\frac{\mathtt{ei}}{\mathtt{es}}\int_0^s \xi(B_u - l_n + r_n)\big(c(w(t_n+s-u, B_u - l_n + r_n)) + H\varepsilon\big)\,\mathrm{d}u\Big\}\\
&\qquad\times\big(w(t_n, B_s - l_n + r_n) + \varepsilon\big);\ \sup_{0\le u\le 1}|B_u - l_n| \le K\Big]\\
&= e^{\mathtt{ei}H\varepsilon s}E_{r_n}\Big[\exp\Big\{\frac{\mathtt{ei}}{\mathtt{es}}\int_0^s \xi(B_u)c(w(t_n+s-u, B_u))\,\mathrm{d}u\Big\}\\
&\qquad\times (w(t_n, B_s) + \varepsilon);\ \sup_{0\le u\le 1}|B_u - r_n| \le K\Big].
\end{aligned}$$

Recall the inequality $e^{ax} \le e^x - (1-a)x$ for all $a \in [0,1]$ and $x \ge 0$. Then, since $\frac{\mathtt{ei}}{\mathtt{es}} \in (0,1)$, we get

$$\begin{aligned}
&w(t_n+s, l_n)\\
&\le f(K) + e^{\mathtt{ei}H\varepsilon s}\bigg(\varepsilon e^{\mathtt{ei}s} + E_{r_n}\Big[\exp\Big\{\int_0^s \xi(B_u)c(w(t_n+s-u,B_u))\,\mathrm{d}u\Big\}w(t_n,B_s)\Big]\\
&\quad - \big(1-\frac{\mathtt{ei}}{\mathtt{es}}\big)E_{r_n}\Big[\int_0^s \xi(B_u)c(w(t_n+s-u,B_u))\,\mathrm{d}u\ w(t_n,B_s);\ \sup_{0\le u\le 1}|B_u - r_n| \le K\Big]\bigg).
\end{aligned} \tag{4.2.8}$$

Recalling (4.2.3), we also have $\inf_{r\in[r_n-K,r_n+K]} w(t_n,r) \ge \delta - \varepsilon$. Furthermore, using the properties of c, for ε small enough such that $\varepsilon + b < 1$, we have that $\underline{c} = \underline{c}(\varepsilon, b) := \inf_{v\in[0,b+\varepsilon]} c(v) > 0$. Using (4.2.2), $\xi \ge \mathtt{ei}$, (4.2.4), (4.2.7), the inequality $e^x \le 1 + 2x$ for $x \ge 0$ small enough, and $w \in [0,1]$, we get, choosing $s = s'$ from (4.2.5) and continuing the bound from (4.2.8),

$$\begin{aligned}
w(t_n+s', l_n) &\le \varepsilon(1 + e^{(1+H\varepsilon)\mathtt{ei}s'}) + (1 + 2H\varepsilon\mathtt{ei}s')w(t_n+s', r_n)\\
&\qquad - (1 - \mathtt{ei}/\mathtt{es})\,\mathtt{ei}\,\underline{c}\,(\delta-\varepsilon)(1-\varepsilon)s'\\
&\le w(t_n+s', r_n) + \varepsilon(1 + 2\mathtt{ei}(1+2H\varepsilon))\\
&\qquad - (1 - \mathtt{ei}/\mathtt{es})\,\mathtt{ei}\,\underline{c}\,(\delta-\varepsilon)(1-\varepsilon)s'
\end{aligned}$$

and the right-hand side can made smaller than $w(t_n+s', r_n) - 2\varepsilon$ if we choose s' (say) of order $\sqrt{\varepsilon}$ and ε small enough. But this is a contradiction to (4.2.4), which hence proves Theorem 4.9. □

4.3 On boundedness of the potential and the condition $v_c < v_0$

We close this chapter with some complementary results about our model assumptions. While (HÖL) is needed for the existence of the solution and (STAT) and (MIX) are needed to get "enough randomness" for the central limit theorems in Sections 3.1.2 and 3.1.3, the questions arise whether the assumption (BDD) is necessary and, more importantly, the assumption

$$v_c < v_0, \tag{VEL}$$

is always fulfilled. What is more, we needed the condition

$$\mathtt{es} > 2\mathtt{ei} \tag{4.3.1}$$

for Theorems 4.3 and 4.9 to hold, see (4.1.9). The first result in this chapter shows that all these condition can be fulfilled and thus all relevant results from Chapter 3 and (4) (so far) can hold *simultaneously.*

Proposition 4.10. *There exists ξ, such that* (HÖL), (BDD), (STAT), (MIX), (VEL) *and* (4.3.1) *are fulfilled.*

To show Proposition 4.10, we will take advantage of the following lemma. Recall the definition of L from (3.2.8), its derivative (3.2.11) and the defintion of $v_c = \frac{1}{L'(0-)}$ from Lemma 3.9 (c).

Lemma 4.11. *Let ξ be the potential constructed in* (4.1.10) *for real numbers* $\mathtt{es}$ *and* $\mathtt{ei}$ *satisfying* $0 < \mathtt{ei} < \mathtt{es}$ *(with* (4.1.9) *not necessarily fulfilled). Then, making the dependence of L explicit in writing $L = L_\xi$, we have that the family of real numbers $\frac{1}{L'_{C\xi}(0-)}$, $C \in [1, \infty)$, is upper bounded away from infinity.*

Proof. Equation (3.2.11) and monotone convergence entail that for all $C \in [1, \infty)$ we have

$$L'_{C\xi}(0-) = \mathbb{E}\left[\frac{E_1\left[e^{C\int_0^{H_0}(\xi(B_r)-\mathtt{es})\,\mathrm{d}r}H_0\right]}{E_1\left[e^{C\int_0^{H_0}(\xi(B_r)-\mathtt{es})\,\mathrm{d}r}\right]}\right].$$

Since the expectation in the denominator on the right-hand side of the previous display is $\mathbb{P}$-a.s. upper bounded by 1, we can continue the above to infer that for some positive constant $c > 0$ and all $C \in [1, \infty)$ we have

$$\begin{aligned} L'_{C\xi}(0-) &\geq \mathbb{E}\Big[E_1\big[e^{C\int_0^{H_0}(\xi(B_r)-\mathtt{es})\,\mathrm{d}r}H_0\big] \cdot \mathbb{1}_{\{\xi(x)=\mathtt{es}\ \forall x\in[0,2]\}}\Big] \\ &\geq \mathbb{E}\Big[E_1[H_0 \cdot \mathbb{1}_{\{B_r\in[0,2]\ \forall r\in[0,H_0]\}}] \cdot \mathbb{1}_{\{\xi(x)=\mathtt{es}\ \forall x\in[0,2]\}}\Big] \geq c > 0, \end{aligned}$$

which finishes the proof of the lemma. □

Now we have everything to prove Proposition 4.10.

Proof of Proposition 4.10. Let ξ be as in (4.1.10) with $\mathtt{ei}, \mathtt{es}$ such that (4.3.1) is fulfilled. We had seen that this ξ fulfills (HÖL), (BDD), (STAT) and (MIX) as well. Furthermore, it is clear that all these conditions are fulfilled for the potential $C\xi$ for any $C \in [1, \infty)$. It is thus sufficient to choose $C \in [1, \infty)$ such that for the potential $C\xi$ condition (VEL) holds true. For this purpose, note that by Lemma 4.11 $\frac{1}{L'_{C\xi}(0-)}$ and thus $v_c(C\xi)$ is upper bounded away from infinity $C \to \infty$. Regarding v_0, a comparison with the constant potentials $C\mathtt{ei}$ yields that $v_0(C\xi) \to \infty$ as $C \to \infty$, so (VEL) holds true for all C large enough. □

Furthermore, for unbounded potential, we have superlinear speed of the front. This is made more precise in the next lemma. It essentially follows the argument from [48, Remark 5.4] and *cannot* be considered as a new result.

Lemma 4.12 ([48, Remark 5.4])**.** *If* $\mathtt{es} = \infty$*, then* $v_0 = \infty$.

In the proof of this lemma as well as in the proof of the next Proposition 4.14, we will take advantage of the following alternative representation of v_0.

Claim 4.13.

$$v_0 = \inf_{\eta \leq 0} \frac{\eta - \mathtt{es}}{L(\eta)}. \tag{4.3.2}$$

Proof. As shown in [25, p. 514ff.], the function

$$I : (0, \infty) \ni y \mapsto \sup_{\eta \le -\texttt{es}} \big(y\eta - L(\eta + \texttt{es})\big)$$

is strictly decreasing, finite, convex, fulfills $\lim_{y\downarrow 0} I(y) = +\infty$ and $\lim_{y\uparrow\infty} I(y) = -\infty$, there exists a unique $v^* > 0$ such that $I(1/v^*) = 0$, and one has

$$v^* = \inf_{\eta \le 0} \frac{\eta - \texttt{es}}{L(\eta)}.$$

We now show $v^* = v_0$. For this purpose, let w and u be solutions to (F-KPP) and (PAM), respectively, both with initial condition $\mathbb{1}_{[-1,0]}$. Then $w \le u$ and thus by [25, Lemma 7.6.3] and Proposition 3.7 we have for all $v > v^*$ that $\mathbb{P}$-a.s.,

$$\Lambda(v) = \lim_{t\to\infty} \frac{1}{t} \ln u(t, vt) \ge \liminf_{t\to\infty} \frac{1}{t} \ln w(t, vt) \ge -vI(1/v).$$

Furthermore, the function Λ is concave and the function I is convex. Since both are finite, they are continuous and by passing to the limit $v \downarrow v^*$, we deduce that $\Lambda(v^*) \ge -v^* I(1/v^*) = 0$. Furthermore, $\Lambda(v) < 0$ for all $v > v_0$ and thus we infer $v_0 \ge v^*$.

To get the converse inequality, we use that for all $v > 0$ and all $\eta \le 0$ we have

$$\begin{aligned} u(t, vt) &= E_{vt}\big[e^{\int_0^t (\zeta(B_s) + \texttt{es})\,\mathrm{d}s}; B_t \in [-1, 0]\big] \\ &= E_{vt}\big[e^{\int_0^t (\zeta(B_s) + \texttt{es})\,\mathrm{d}s}; B_t \in [-1, 0], H_0 \le t\big] \\ &\le e^{(\texttt{es} - \eta)t} E_{vt}\big[e^{\int_0^{H_0} (\zeta(B_s) + \eta)\,\mathrm{d}s}\big]. \end{aligned}$$

In combination with (3.2.17) this yields that for all $v > 0$,

$$\begin{aligned} \Lambda(v) = \lim_{t\to\infty} \frac{1}{t} \ln u(t, vt) &\le \inf_{\eta \le 0} \big((\texttt{es} - \eta) + vL(\eta)\big) = -v \sup_{\eta \le -\texttt{es}} \big(\frac{\eta}{v} - L(\eta + \texttt{es})\big) \\ &= -vI(1/v). \end{aligned}$$

But then $\Lambda(v^*) \le -v^* I(1/v^*) = 0$ and thus we must have $v^* \ge v_0$. □

Now we can prove Lemma 4.12.

Proof of Lemma 4.12 . Let ξ be a potential fulfilling (HÖL), (STAT) and (MIX), bounded from below by $\texttt{ei} \in (0, 1)$, but unbounded from above. ξ can be constructed using the method from Section 4.1.2. Indeed, instead of having height 1 of the function χ, the height could be determined by an unbounded random variable, independently of the Poisson point process, and we would get $\texttt{es} = \infty$. Now for every $k \in \mathbb{N}$ let $\underline{\xi}^{(k)} := \xi \wedge k$ be the truncation of ξ. Then we have $\operatorname{ess\,sup} \underline{\xi}^{(k)} = k$ and we can define the quantities $\underline{L}^{(k)} = L^{\underline{\xi}^{(k)}}$ as in (3.2.8) and $\underline{v}^{(k)}$ by the right-hand side of (4.3.2) (again replacing L by $\underline{L}^{(k)}$ and $\texttt{es}$ by k) associated to the potential $\underline{\xi}^{(k)}$. Then $v_0 \ge \underline{v}^{(k)}$ and both $\underline{v}^{(k)}$ and $\underline{L}^{(k)}$ are nondecreasing in k. Thus Claim 4.13 gives

$$v_0 \ge \underline{v}^{(k)} = \inf_{\eta \le 0} \frac{\eta - k}{\underline{L}^{(k)}(\eta)} \ge \inf_{\eta \le -k} \frac{\eta}{\underline{L}^{(0)}(\eta)}.$$

Furthermore $\underline{L}^{(0)}(\eta) = \mathbb{E}\big[\ln E_1\big[e^{\eta H_0}\big]\big] = -\sqrt{2|\eta|}$ by [8, (2.0.1), p. 204]. Thus $\underline{v}^{(k)} \xrightarrow[k\to\infty]{} \infty$

and we get $v_0 = \infty$. □

On the other side, the answer to the question whether (VEL) is *always* fulfilled is generally no, and we will construct a counterexample where the all other model assumptions are satisfied, but (VEL) fails to hold. The next statement can be considered as the main result of this section.

Proposition 4.14. *There exist potentials ξ fulfilling* (HÖL), (BDD), (STAT) *and* (MIX), *and such that $v_c > v_0$; i.e., condition* (VEL) *is violated.*

Proof. Recalling (4.3.2), definition $v_c := \frac{1}{L'(0-)}$ from Lemma 3.9 (c), it is sufficient to show $L(0) + \mathtt{es} \cdot L'(0-) < 0$, which means

$$\mathbb{E}\Big[\ln E_1\big[e^{\int_0^{H_0} \zeta(B_s)\,\mathrm{d}s}\big]\Big] + \mathtt{es} \cdot \mathbb{E}\Big[\frac{E_1\big[H_0 e^{\int_0^{H_0} \zeta(B_s)\,\mathrm{d}s}\big]}{E_1\big[e^{\int_0^{H_0} \zeta(B_s)\,\mathrm{d}s}\big]}\Big] < 0. \tag{4.3.3}$$

To establish the latter, let $\widetilde{\omega}$ be a one-dimensional Poisson point process with intensity one. In a slight abuse of notation, $\widetilde{\omega} = (\widetilde{\omega}^i)_i$ can be seen as a random mapping into the set of all locally finite point configurations; i.e., $\widetilde{\omega} = (\widetilde{\omega}^i)_i$ can be interpreted as a random set of countably many points in $\mathbb{R}$, satisfying $|\{i : \widetilde{\omega}^i \in B\}| < \infty$ for every bounded Borel set and $\widetilde{\omega}^i \neq \widetilde{\omega}^j$ for all $i \neq j$. See [60] for further details. Now denote by $\omega = (\omega^i)_i$ be the point process that is obtained from $\widetilde{\omega}$ by deleting *simultaneously* all points in $\widetilde{\omega}$ which have distance 1 or less to their nearest neighbor in $\widetilde{\omega}$. (see [50, p. 47] for details). Let $\varphi(x)$ be a mollifier with support $[-1/2, 1/2]$, non-decreasing for $x \le 0$, non-increasing for $x \ge 0$ with $\varphi(0) = 1$ and let $\varphi^{(\varepsilon)}(x) := \varphi(x/\varepsilon)$, $\varepsilon > 0$. Finally, for $\varepsilon \in (0,1)$ and $a > 0$, define the potential $\zeta(x) = \zeta^{(\varepsilon,a)}(x) := -a + a\sum_i \varphi^{(\varepsilon)}(x - \omega^i)$. One can easily check that ζ is the corresponding shifted potential as in (3.2.3) of some ξ fulfilling (BDD), (STAT) and (MIX), i.e. $a = \mathtt{es} - \mathtt{ei}$. We will choose $a > \ln 2$, $\mathtt{ei} \in (0, \frac{a}{8})$ and $\varepsilon(a) > 0$ suitably at the end of the proof. Let us now consider both summands in (4.3.3) separately.

1) We observe that $\mathbb{P}$-a.s., $\zeta^{(\varepsilon,a)} \downarrow -a$ as $\varepsilon \downarrow 0$ for all $x \in \mathbb{R}$, as well as by [8, (2.0.1), p. 204]

$$-\sqrt{2a} = \ln E_1\big[e^{-aH_0}\big] \le \ln E_1\big[e^{\int_0^{H_0} \zeta^{(\varepsilon,a)}(B_s)\,\mathrm{d}s}\big] \le 0$$

for all $\varepsilon \in (0,1)$. Thus, by dominated convergence, for all $a > 0$ there exists $\varepsilon_1 = \varepsilon_1(a) > 0$, such that

$$\mathbb{E}\Big[\ln E_1\big[e^{\int_0^{H_0} \zeta^{(\varepsilon_1,a)}(B_s)\,\mathrm{d}s}\big]\Big] \le -\frac{3}{4}\sqrt{2a}. \tag{4.3.4}$$

2) To bound the second summand in (4.3.3), we lower bound its denominator by

$$E_1\big[e^{\int_0^{H_0} \zeta(B_s)\,\mathrm{d}s}\big] \ge E_1\big[e^{-aH_0}\big] = e^{-\sqrt{2a}}. \tag{4.3.5}$$

For the numerator, define $\mathbb{J}$ to be the set of possible point configurations of the process $(\omega^i)_i$. Let us first check that for all $a > 0$ there exists $\varepsilon = \varepsilon(a) > 0$, such that

$$\sup_{(\omega^i)_i \in \mathbb{J}} E_1\big[H_0 e^{\int_0^{H_0}(-a + a\sum_i \varphi^{(\varepsilon)}(B_s - \omega^i))\,\mathrm{d}s}\big] < \infty. \tag{4.3.6}$$

Indeed, letting $g^{\varepsilon,(\omega^i)_i}(x) := \sum_i \varphi^{(\varepsilon)}(x-\omega^i)$, we have

$$
\begin{aligned}
E_1\big[H_0 e^{\int_0^{H_0}(-a+a\cdot g^{\varepsilon,(\omega^i)_i}(B_s))\,\mathrm{d}s}\big] &= \sum_{n=0}^{\infty} E_1\big[H_0 e^{\int_0^{H_0}(-a+a\cdot g^{\varepsilon,(\omega^i)_i}(B_s))\,\mathrm{d}s}; H_0\in[n,n+1)\big]\\
&\le \sum_{n=0}^{\infty}(n+1)E_1\big[e^{\int_0^{n}(-a+a\cdot g^{\varepsilon,(\omega^i)_i}(B_s))\,\mathrm{d}s}\big].
\end{aligned}
\tag{4.3.7}
$$

Note that by the property of all point configurations in $\mathbb{J}$ to have points with mutual distance at least one, we have

$$
\sup_{(\omega^i)_i\in\mathbb{J}}\sup_{x\in\mathbb{R}} E_x\Big[a\int_0^1 g^{\varepsilon,(\omega^i)_i}(B_s)\,\mathrm{d}s\Big] \le a\int_0^1 E_0\big[\mathbb{1}_{A_\varepsilon}(B_s)\big]\,\mathrm{d}s \le \frac{1}{2}
$$

for all $\varepsilon(a) > 0$ small enough, where $A_\varepsilon := \bigcup_{i\in\mathbb{Z}}[-\varepsilon/2+i, \varepsilon/2+i]$. Using Kasminskii's lemma (cf. e.g. [69, Lemma 1.2.1]) we infer $\sup_{x\in\mathbb{R}} E_x\big[e^{a\int_0^1 g^{\varepsilon,(\omega^i)_i}(B_s)\,\mathrm{d}s}\big] \le 2$. An $(n-1)$-fold application of the Markov property at times $1,\dots,n-1$ supplies us with $\sup_{x\in\mathbb{R}} E_x\big[e^{a\int_0^n g^{\varepsilon,(\omega^i)_i}(B_s)\,\mathrm{d}s}\big] \le 2^n$ for all $n\in\mathbb{N}$ and all $(\omega^i)_i\in\mathbb{J}$. Plugging this into (4.3.7) we infer

$$
\sup_{(\omega^i)_i\in\mathbb{J}} E_1\Big[H_0 e^{\int_0^{H_0}(-a+a\cdot g^{\varepsilon,(\omega^i)_i}(B_s))\,\mathrm{d}s}\Big] \le \sum_{n=0}^{\infty}(n+1)e^{-na}2^n,
$$

so the right-hand side in (4.3.7) is finite, and (4.3.6) holds true for all $a > \ln 2$ and $\varepsilon(a)$ small enough as well. Since $g^{\varepsilon,(\omega^i)_i}$ decreases $\mathbb{P}$-a.s. to 0 monotonically as $\varepsilon\downarrow 0$, we infer

$$
\begin{aligned}
\lim_{\varepsilon\downarrow 0}\mathbb{E}\Big[E_1\big[H_0 e^{\int_0^{H_0}\zeta^{(\varepsilon,a)}(B_s)\,\mathrm{d}s}\big]\Big] &= E_1\big[H_0 e^{-aH_0}\big] = -\frac{\mathrm{d}}{\mathrm{d}a}E_1\big[e^{-aH_0}\big]\\
&= -\frac{\mathrm{d}}{\mathrm{d}a}(e^{-\sqrt{2a}}) = \frac{1}{\sqrt{2a}}e^{-\sqrt{2a}},
\end{aligned}
\tag{4.3.8}
$$

using [8, (2.0.1), p. 204] in the third equality. Thus, combining (4.3.5) and (4.3.8) we infer that there exists $\varepsilon_2(a) > 0$ such that the second summand on the left-hand side of (4.3.3) is upper bounded by $\texttt{es}\cdot\frac{4/3}{\sqrt{2a}} = (a+\texttt{ei})\cdot\frac{4/3}{\sqrt{2a}}$. Using this in combination with (4.3.4), we infer that for all $a > \ln 2$ (which is sufficient for (4.3.6)) and $\varepsilon\in(0,\varepsilon_1(a)\wedge\varepsilon_2(a))$, choosing $\texttt{ei}\in(0,\frac{a}{8})$, we get that the left-hand side in (4.3.3) is upper bounded by $-\frac{3}{4}\sqrt{2a}+(a+\texttt{ei})\cdot\frac{4/3}{\sqrt{2a}} < 0$. □

Chapter Five

Outlook and open questions

In this chapter we list several open questions that arise in the course of this thesis.

Discrete-space PDEs

Besides equations (F-KPP) and (PAM), see Chapter 3, we can take a look at the discrete-space counterparts. More precisely, we have the *discrete-space randomized F-KPP equation*

$$\begin{aligned} w_t(t,x) &= \frac{1}{2}\triangle_d w(t,x) + \xi(x,\omega)\cdot F(w(t,x)), \quad (t,x)\in(0,\infty)\times\mathbb{Z}, \\ w(0,x) &= w_0(x), \quad x\in\mathbb{Z}, \end{aligned} \tag{5.0.1}$$

where

$$\triangle_d f(t,x) = \sum_{y\in\mathbb{Z}:|y-x|=1} (f(t,y)-f(t,x)), \quad (t,x)\in(0,\infty)\times\mathbb{Z}, \quad f:\mathbb{Z}\to\mathbb{R},$$

is the generator of a simple random walk, $\xi = (\xi(x))_{x\in\mathbb{Z}} = (\xi(x,\omega))_{x\in\mathbb{Z}}$, $\omega\in\Omega$, is a random field being stationary and fulfilling suitable boundedness and mixing conditions, similar to its real-indexed counterpart $(\xi(x))_{x\in\mathbb{R}}$, and F fulfills the F-KPP standard conditions (SC). The corresponding linearized equation is

$$\begin{aligned} u_t(t,x) &= \frac{1}{2}\triangle_d u(t,x) + \xi(x,\omega)\cdot u(t,x), \quad (t,x)\in(0,\infty)\times\mathbb{Z}, \\ u(0,x) &= u_0(x), \quad x\in\mathbb{Z}. \end{aligned} \tag{5.0.2}$$

Equation (5.0.2) is commonly known by the name *parabolic Anderson model*, which is the reason why we choose this name for the model in continuous space. The equations (5.0.1) and (5.0.2) play a role in the investigation of branching random walks in random environment (BRWRE), see [16]. We believe that the results of this thesis can readily be transferred to statements about the discrete-space counterparts (5.0.1) and (5.0.2). Furthermore, as this work is initially motivated by the paper [16], we believe that the assumptions in [16] on $\xi(x)$, $x\in\mathbb{Z}$, being a sequence of i.i.d. random variables, can be weakened so that $\xi(x)$, $x\in\mathbb{Z}$, fulfills a mixing condition as in (MIX).

Subsequential bounded transition front

As a consequence of Theorem 4.3 we get that the transition front can diverge, i.e. for every $\varepsilon\in(0,1/2)$ there exists a subsequence of time-points $(t_n)_{n\in\mathbb{N}}$ tending to infinity, such that $m^{\varepsilon}_{t_n} - m^{1-\varepsilon,-}_{t_n} \underset{n\to\infty}{\longrightarrow} \infty$. We refer to this as *subsequential unbounded transition front*. However, the question arises whether there is also a *subsequential bounded transition front*, i.e. whether for all $\varepsilon \in (0,1/2)$, with $\mathbb{P}$-probability one, we can find a sequence $(\widetilde{t}_n)_{n\in\mathbb{N}} = (\widetilde{t}_n(\xi,\varepsilon))_{n\in\mathbb{N}}$, tending to infinity, such that $m^{\varepsilon}_{\widetilde{t}_n} - m^{1-\varepsilon,-}_{\widetilde{t}_n} \leq C_\varepsilon$. Although we are not able to show or disprove this statement, there is another interesting result in the recently published article [46] in the context of BRWRE. To explain what the author proves in [46], let us first forge a link to his setup. Recall that it is possible to represent the solution to (F-KPP) in terms of the position M_t of the right-most particle of a BBMRE. More precisely, the function

$$w(t,x) = \mathtt{P}^{\xi}_x\big(M_t \geq 0\big) \tag{5.0.3}$$

solves (F-KPP) with initial condition $w_0 = \mathbb{1}_{[0,\infty)}$. We have seen in Section 1.2 in the continuous-space setting, that for constant potential $\xi \equiv$ const, using the symmetry of the branching Brownian motion, we get

$$\mathtt{P}^{\text{const}}_x\big(M_t \geq 0\big) = \mathtt{P}^{\text{const}}_0\big(M_t \leq x\big),$$

and thus, for $\xi \equiv$ const, it is possible to consider the solution to (F-KPP) as the distribution function of the maximal particle of a BBMRE starting in 0. Unfortunately, this representation is not valid anymore for non-constant potential ξ. In the context of BRWRE, again denoting the position of the right-most particle at time t by M_t, the methods in the proof of [46, Theorem 1] reveal that there exists a sequence of deterministic time-points $(\widehat{t}_n)_{n\in\mathbb{N}}$ such that $(M_{\widehat{t}_n} - \mathtt{E}^{\xi}_0[M_{\widehat{t}_n}])_{n\in\mathbb{N}}$ is tight with respect to the measure $\mathtt{P}^{\xi}_0$ and thus the transition front of the front of the function $x \mapsto \mathtt{P}^{\xi}_0\big(M_t \leq x\big)$ is bounded. Although a direct transfer to the case of BBMRE is not possible, it is an interesting question whether the methods in [46] can also be used to show subsequential tightness for the transition front of the solution to (F-KPP) with the help of (5.0.3).

Lower transition front

In our main result, Theorem 3.5, we show that the upper front m^{ε} of (F-KPP) lags at most logarithmically behind the upper front $\overline{m}^M$ of (PAM). Furthermore, in Theorem 4.2 we show that the transition front of the solution to (PAM) is uniformly bounded, i.e. $\overline{m}^{\varepsilon}(t) - \overline{m}^{M,-} \leq C_{\varepsilon,M}$ for all t large enough. Unfortunately, an analogous statement like Theorem 4.2 for the transition front of the solution to (F-KPP) is not true in general, see Theorem 4.3. Therefore, it is *not* straight-forward to show – and we were not able to prove – that one has an analogous statement as in Theorem 3.5 for the *lower transition fronts*, i.e.

$$|m^{\varepsilon,-}(t) - \overline{m}^{M,-}(t)| \leq C_{\varepsilon,M} \log t \tag{5.0.4}$$

for some constant $C_{\varepsilon,M} \in (0,\infty)$ and all t large enough. Using the methods in this thesis, (5.0.4) follows if we were able to show that inequality (3.4.47) from Proposition 3.40 holds uniformly for all subscripts $x \in [0, \overline{m}^M(t) - C\ln t]$, i.e.

$$\inf_{x\in[0,\overline{m}^M(t)-C\ln t]} \mathtt{P}^{\xi}_x\left(N^{\leq}(t,0) \geq 1\right) \geq 1 - 2t^{-q},$$

and not only for $x = \overline{m}^M(t) - C\ln t$. However, the proof of Proposition 3.40 relied heavily on the fact that the subscript $\overline{m}^M(t) - C\ln t$ is moving linear in time with some velocity in the neighborhood V of v_0. Unfortunately, V must fulfill $V \subset (v_c,\infty)$, see (3.2.30) and (VEL), and we showed in Proposition 4.14 that condition (VEL) *cannot* be dropped. The proof of (5.0.4) thus requires a significant amount of extra work and we address the claim (5.0.4) to future research.

Phase transition from bounded to unbounded transition front for F-KPP

In the notation of this thesis, homogeneous potential means $\mathtt{ei} = \mathtt{es}$ with $\mathtt{ei}, \mathtt{es}$ as in (BDD). In this setting, the transition front of the solution to (F-KPP) is bounded. However, Theorem 4.3 tells us that the condition $\mathtt{es} > 2\mathtt{ei}$ leads to an (at least subsequential) unbounded transition front. We thus expect that the front of the solution to (F-KPP) shifts from exhibiting unbounded transition fronts (essentially when $\mathtt{es}/\mathtt{ei}$ large, and maybe further conditions) to exhibiting bounded transition fronts (essentially if $\mathtt{es}/\mathtt{ei}$ small, and maybe further conditions). While it is not clear if "small" means "vanishes" in this context, let us point out here that – while periodic media are oftentimes taken to be a simple instance for heterogeneous or random media, cf. also [25, 32] – it is clear from our proofs that the phenomenon of long stretches of areas of high and low potential, which is crucial in our proof, is not observed for periodic media.

The condition (VEL)

Recall that directly before Theorem 3.3 we have introduced our assumption (VEL), which reads $v_0 > v_c$; from a technical point of view, this is necessary for our change of measure argument to work. We show in Proposition 4.10 that for a rich class of potentials this condition is satisfied. What is more, however, according to Proposition 4.14 there exist potentials ξ fulfilling (**Standing assumptions**), but such that $v_0 < v_c$ holds true. In this regime our methods do not apply, and it is an interesting and open question to obtain a more profound understanding of this situation as well.

Chapter Six

Appendix

A ψ-mixing

For a stochastic process $\xi = (\xi(x,\omega))_{x\in\mathbb{R}}$, $\omega \in \Omega$, recall the definitions $\mathcal{F}_j = \sigma(\xi(x,\cdot) : x \leq j)$ and $\mathcal{F}^k = \sigma(\xi(x,\cdot) : x \geq k)$.

Lemma A.1. *Let ξ fulfill* (STAT)*. Further, let ψ be some function fulfilling* (MIX) *and $\widetilde{\psi}$ be defined in* (3.1.1)*. Then we have $\psi(t) \underset{t\to\infty}{\longrightarrow} 0$ if and only if $\widetilde{\psi}(t) \underset{t\to\infty}{\longrightarrow} 0$.*

Proof. Let $\psi(t) \underset{t\to\infty}{\longrightarrow} 0$, fix $j \leq k$ and let $A \in \mathcal{F}_j, B \in \mathcal{F}^k$. Then by (MIX)

$$\begin{aligned} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| &\leq \mathbb{E}\big[\big|\mathbb{E}[\mathbb{1}_A - \mathbb{P}(A) \,|\, \mathcal{F}^k]\mathbb{1}_B\big|\big] \\ &\leq \mathbb{P}(A)\mathbb{P}(B)\psi(k-j) \end{aligned}$$

and because our choices of the quantities involved was arbitrary, also using (STAT), we get $\widetilde{\psi}(t) \leq \psi(t) \to 0$.

Now assume $\widetilde{\psi}(t) \underset{t\to\infty}{\longrightarrow} 0$. We use algebraic induction to show that the choice $\psi := \widetilde{\psi}$ is sufficient. We will only show the first inequality in (MIX). The second inequality can be shown by exactly the same arguments. Now $\widetilde{\psi}(t) \to 0$ implies that for all $j \leq k$, $A \in \mathcal{F}_j$ and $B \in \mathcal{F}^k$ with $\mathbb{P}(A), \mathbb{P}(B) > 0$ we have

$$|\mathbb{E}\left[\mathbb{1}_A - \mathbb{P}(A)|B\right]| \leq \mathbb{P}(A) \cdot \widetilde{\psi}(k-j).$$

If the set $\left\{ \left|\mathbb{E}\left[\mathbb{1}_A - \mathbb{P}(A)|\mathcal{F}^k\right]\right| > \mathbb{P}(A) \cdot \widetilde{\psi}(k-j)\right\} =: B \in \mathcal{F}^k$ had positive probability, this would contradict the above inequality and our claim is true for indicator functions. Let $X \in \mathcal{L}^1(\mathcal{F}_j)$ be a simple function, i.e. $X = \sum_{l=1}^n \alpha_l \mathbb{1}_{A_l}$, where $\alpha_l \geq 0$ for all l as well as $A_l \in \mathcal{F}_j$ for all l and the A_l, $l = 1, \ldots, n$, are pairwise disjoint. Then $\mathbb{P}$-a.s.

$$\begin{aligned} \left|\mathbb{E}\left[X - \mathbb{E}[X] \,|\, \mathcal{F}^{k-j}\right] \right| &\leq \sum_{l=1}^n \alpha_l \left|\mathbb{E}\left[\mathbb{1}_{A_l} - \mathbb{P}(A_l) \,|\, \mathcal{F}^k\right]\right| \leq \sum_{l=1}^n \alpha_l \mathbb{P}(A_l) \cdot \widetilde{\psi}(k-j) \\ &= \mathbb{E}[X] \cdot \widetilde{\psi}(k-j) = \mathbb{E}[|X|] \cdot \widetilde{\psi}(k-j). \end{aligned} \tag{A.1}$$

As usual, we can approximate nonnegative X by a non-decreasing sequence of simple functions, e.g. $X = \sup_n Y^{(n)}$ with $Y^{(n)}$ as above and the statement follows by monotone convergence. The step for integrable $X = X^+ - X^-$, $X^\pm \geq 0$, is obvious. Thus (A.1) is true for all $X \in \mathcal{L}^1(\mathcal{F}_j)$ and we can conclude. □

Lemma A.2. *Let $\triangle \subset (-\infty, 0)$ be a compact interval. Then there exists a constant $C_\triangle > 0$ such that $\mathbb{P}$-a.s., for all $i, j \in \mathbb{Z}$ with $i < j$, and all $\eta \in \triangle$,*

$$\big|\mathbb{E}\big[L_i^\zeta(\eta)|\mathcal{F}^j\big] - L(\eta)\big| \le C_\triangle \cdot \Big(\psi\big(\frac{j-i}{2}\big) + e^{-(j-i)/C_\triangle}\Big), \tag{A.2}$$

$$0 \le \Big(\operatorname*{ess\,sup}_{\xi(k):k\ge j} L_i^\zeta(\eta)\Big) - L_i^\zeta(\eta) \le C_\triangle \cdot \Big(\psi\big(\frac{j-i}{2}\big) + e^{-(j-i)/C_\triangle}\Big), \tag{A.3}$$

as well as

$$\Big|\mathbb{E}\big[(L_i^\zeta)'(\eta)|\mathcal{F}^j\big] - L'(\eta)\Big| \le C_\triangle \cdot \Big(\psi\big(\frac{j-i}{2}\big) + e^{-(j-i)/C_\triangle}\Big), \tag{A.4}$$

$$0 \le \Big(\operatorname*{ess\,sup}_{\xi(k):k\ge j} \big(L_i^\zeta\big)'(\eta)\Big) - \big(L_i^\zeta\big)'(\eta) \le C_\triangle \cdot \Big(\psi\big(\frac{j-i}{2}\big) + e^{-(j-i)/C_\triangle}\Big). \tag{A.5}$$

Proof. By translation invariance, it is enough to prove (A.2) for $i = 0$ and $j \ge 2$ (the case $j = 1$ follows immediately from the uniform boundedness of L_0^ζ and L on $\triangle$ due to Lemma 3.8). To show (A.2), let $\eta \in \triangle$ and write $L_0^\zeta(\eta) = \ln(A + B)$ with

$$A = A(\eta) := E_0\Big[e^{\int_0^{H_{-1}}(\zeta(B_s)+\eta)\,\mathrm{d}s};\ \sup_{0\le s\le H_{-1}} B_s < \lceil j/2\rceil\Big]$$

and

$$B = B(\eta) := E_0\Big[e^{\int_0^{H_{-1}}(\zeta(B_s)+\eta)\,\mathrm{d}s};\ \sup_{0\le s\le H_{-1}} B_s \ge \lceil j/2\rceil\Big].$$

Then $A \le E_0[e^{H_{-1}\eta}] \le c_{\triangle,1} < 1$, and, using (3.2.3), at the same time we have

$$A \ge E_0\Big[e^{-(\texttt{es}-\texttt{ei}+|\eta|)H_{-1}};\ \sup_{0\le s\le H_{-1}} B_s < \lceil j/2\rceil\Big] \ge c_{\triangle,2} > 0 \quad \text{for all } j \ge 2.$$

To bound B, we condition on H_{-1} to happen before or after time j and use the reflection principle for Brownian motion and the tail estimate from [10, Lemma 1.1] to infer for all $j \ge 2$

$$0 \le B \le P_0\Big(\sup_{0\le s\le j} B_s \ge \lceil j/2\rceil\Big) + e^{\eta j} = 2P_0(B_j \ge \lceil j/2\rceil) + e^{\eta j} \le 2e^{-j(1/8\wedge|\eta|)}.$$

As $\ln(1+x) \le x$, the above implies that for all $j \ge 2$

$$\ln(A) \le L_0^\zeta(\eta) = \ln(A) + \ln\Big(1 + \frac{B}{A}\Big) \le \ln(A) + c_{\triangle,3}e^{-j/c_{\triangle,3}}.$$

Since L_0^ζ is continuous on $\triangle$, the latter display $\mathbb{P}$-a.s. holds uniformly for all $\eta \in \triangle$. Now $\ln(A)$ is $\mathcal{F}_{\lceil j/2\rceil}$-measurable and bounded, so by (MIX)

$$\begin{aligned}\sup_{\eta\in\triangle}\Big|\mathbb{E}[L_0^\zeta(\eta)|\mathcal{F}^j] - L(\eta)\Big| &\le \sup_{\eta\in\triangle}\big|\mathbb{E}[\ln(A) - \mathbb{E}[\ln(A)]|\mathcal{F}^j]\big| + 2c_{\triangle,3}e^{-j/c_{\triangle,3}}\\ &\le C_\triangle \cdot \big(\psi(j/2) + e^{-j/C_\triangle}\big).\end{aligned} \tag{A.6}$$

The proof of (A.4) is similar. Indeed, using the same notation we have $(L_0^\zeta)' = \frac{A'}{A+B} + \frac{B'}{A+B}$.

Then by $A+B \geq c_{\triangle,2}$ and $e^{\eta H_{-1}}H_{-1} \leq \frac{2}{|\eta|}e^{\eta H_{-1}/2}$, we can use above calculation to conclude that $\frac{B'}{A+B}$ decays exponentially to 0 as $j \to \infty$. Further, $\frac{A'}{A} - \frac{BA'}{A(A+B)} = \frac{A'}{A+B} \leq \frac{A'}{A}$, $F_{\lceil j/2 \rceil}$-measurability of $\frac{A'}{A}$ and above estimates give a similar bound as in (A.6) for $(L_0^\zeta)'(\eta)$ and $L'(\eta)$ instead of $(L_0^\zeta)(\eta)$ and $L(\eta)$. Finally, (A.3) and (A.5) follow by the same arguments as (A.2) and (A.4). □

Theorem A.3 ([54, Theorem 2.1], also cf. [31, Section 5.4]). *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of square-integrable and centered random variables, such that the two series*

$$\sum_{k=1}^{\infty} \big(X_0 - \mathbb{E}[X_0 \,|\, \widetilde{\mathcal{F}}_k]\big) \quad \textit{and} \quad \sum_{k=1}^{\infty} \mathbb{E}[X_k \,|\, \widetilde{\mathcal{F}}_0] \tag{A.7}$$

converge in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, where $\widetilde{\mathcal{F}}_i := \sigma(X_j : j \leq i)$. Then the limit

$$\sigma^2 := \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\Big[\Big|\sum_{k=0}^{N-1} X_k\Big|^2\Big]$$

exists and is finite. If $\sigma^2 > 0$ and $S_n := \sum_{k=0}^{n-1} X_0$, then as $n \to \infty$ the family of processes

$$Z_n(t) = \frac{1}{\sqrt{n\sigma^2}}\big(S_k + (nt-k)X_k\big), \quad k \leq nt \leq k+1, \quad k = 0, 1, \ldots, n-1,$$

converges in $\mathbb{P}$-distribution to a standard Brownian motion on $[0,1]$, in the sense of $C([0,1])$ with topology of uniform convergence.

B Concentration inequalities

Lemma B.1 ([61, Theorem 2.4]). *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of real-valued bounded random variables, $\widetilde{\mathcal{F}}_i := \sigma(X_j : j \leq i)$, $S_n := \sum_{i=1}^n X_i$, and let $(m_1, \ldots, m_n)$ be an n-tuple of positive reals such that for all $i \in \{1, \ldots, n\}$,*

$$\sup_{j\in\{i,i+1,\ldots,n\}} \Big(\|X_i^2\|_\infty + 2\|X_i \sum_{k=i+1}^{j} \mathbb{E}[X_k|\widetilde{\mathcal{F}}_i]\|_\infty\Big) \leq m_i,$$

with the convention $\sum_{k=i+1}^{i} \mathbb{E}[X_k|\widetilde{\mathcal{F}}_i] = 0$. Then for every $x > 0$,

$$\mathbb{P}\left(|S_n| \geq x\right) \leq \sqrt{e}\exp\left\{-x^2/(2m_1 + \cdots + 2m_n)\right\}.$$

Rearranging the quantities in the above result, we arrive at the following corollary which we primarily pronounce explicitly since its formulation tailor-made for our purposes.

Corollary B.2. *Let $(Y_i)_{i\in\mathbb{Z}}$ be a sequence of real-valued bounded random variables, $\widetilde{\mathcal{F}}^k := \sigma(Y_j : j \geq k)$, and let $(m_1, \ldots, m_n)$ be an n-tuple of positive real numbers such that for all $i \in \{1, \ldots, n\}$,*

$$\sup_{j\in\{1,\ldots,i\}} \Big(\|Y_i^2\|_\infty + 2\Big\|Y_i \sum_{k=j}^{i-1} \mathbb{E}[Y_k|\widetilde{\mathcal{F}}^i]\Big\|_\infty\Big) \leq m_i, \tag{B.1}$$

with the convention $\sum_{k=i}^{i-1} \mathbb{E}[Y_k|\widetilde{\mathcal{F}}^i] = 0$. *Then for every* $x > 0$,

$$\mathbb{P}\Big(\Big|\sum_{i=1}^{n} Y_i\Big| \geq x\Big) \leq \sqrt{e}\exp\left\{-x^2/(2m_1 + \cdots + 2m_n)\right\}.$$

Proof. For $i = 1, \dots, n$ we define $\widetilde{m}_i := m_{n+1-i}$, and let $\widetilde{Y}_i := Y_{n+1-i}$ and

$$\begin{aligned}\widetilde{\mathcal{F}}_i &:= \sigma(\widetilde{Y}_j : j \leq i) = \sigma(Y_{n+1-j} : j \leq i) \\ &= \sigma(Y_j : j \geq n+1-i) = \widetilde{\mathcal{F}}^{n+1-i}.\end{aligned}$$

Then rearranging the above quantities and indices we get

$$\sup_{i\leq j\leq n}\Big(||\widetilde{Y}_i^2||_\infty + 2||\widetilde{Y}_i \sum_{k=i+1}^{j} \mathbb{E}[\widetilde{Y}_k|\widetilde{\mathcal{F}}_i]||_\infty\Big) \leq \widetilde{m}_i \quad \forall\, i \in \{1,\dots,n\}$$

if and only if

$$\begin{aligned}&\sup_{i\leq j\leq n}\left(||Y_{n+1-i}^2||_\infty + 2||Y_{n+1-i} \sum_{k=i+1}^{j} \mathbb{E}[Y_{n+1-k}|\widetilde{\mathcal{F}}^{n+1-i}]||_\infty\right) \leq m_{n+1-i} \quad \forall\, i \in \{1,\dots,n\} \\ \Leftrightarrow &\sup_{n+1-i\leq j\leq n}\left(||Y_i^2||_\infty + 2||Y_i \sum_{k=n+2-i}^{j} \mathbb{E}[Y_{n+1-k}|\widetilde{\mathcal{F}}^i]||_\infty\right) \leq m_i \quad \forall\, i \in \{1,\dots,n\} \\ \Leftrightarrow &\sup_{n+1-i\leq j\leq n}\left(||Y_i^2||_\infty + 2||Y_i \sum_{k=n+1-j}^{i-1} \mathbb{E}[Y_k|\widetilde{\mathcal{F}}^i]||_\infty\right) \leq m_i \quad \forall\, i \in \{1,\dots,n\} \\ \Leftrightarrow &\sup_{1\leq j\leq i}\Big(||Y_i^2||_\infty + 2||Y_i \sum_{k=j}^{i-1} \mathbb{E}[Y_k|\widetilde{\mathcal{F}}^i]||_\infty\Big) \leq m_i \quad \forall\, i \in \{1,\dots,n\}.\end{aligned}$$

We see that the conditions in Lemma B.1 are satisfied for the sequence $(\widetilde{Y}_i)_{i\in\mathbb{Z}}$, $(\widetilde{\mathcal{F}}_i)_{i\in\mathbb{Z}}$ and $(\widetilde{m}_1, \dots, \widetilde{m}_n)$. This gives

$$\begin{aligned}\mathbb{P}\Big(\Big|\sum_{i=1}^{n} Y_i\Big| \geq x\Big) = \mathbb{P}\Big(\Big|\sum_{i=1}^{n} \widetilde{Y}_i\Big| \geq x\Big) &\leq \sqrt{e}\exp\left\{-x^2/(2\widetilde{m}_1 + \cdots + 2\widetilde{m}_n)\right\} \\ &= \sqrt{e}\exp\left\{-x^2/(2m_1 + \cdots + 2m_n)\right\}.\end{aligned}$$

□

C PDEs

For the next lemma, we introduce the differential operator

$$(\mathcal{L}_G w)(t,x) := w_t(t,x) - \frac{1}{2}w_{xx}(t,x) - G(x, w(t,x)),$$

where G is uniformly Lipschitz-continuous in w, i.e., there exists a constant $\alpha > 0$, such that

$$|G(x,u) - G(x,v)| \leq \alpha|u - v| \quad \forall\, x,u,v \in \mathbb{R}. \tag{C.1}$$

The next lemma is in the spirit of [2, Proposition 2.1].

Lemma C.1. *Let $T > 0$, $Q := (0,T) \times \mathbb{R}$ and G be such that* (C.1) *holds. Let $w^{(1)}$ and $w^{(2)}$ be nonnegative and bounded functions on $\overline{Q}$, such that for $i \in \{1,2\}$, $w_x^{(i)}$ and $w_{xx}^{(i)}$ are continuous on Q, and such that $w_t^{(i)}$ exists in Q. If $\mathcal{L}_G w^{(1)} \leq \mathcal{L}_G w^{(2)}$ on Q and $0 \leq w^{(1)}(0,x) \leq w^{(2)}(0,x)$ for all $x \in \mathbb{R}$, then also $w^{(1)} \leq w^{(2)}$ on Q.*

Proof.

$$w_t^{(2)}(t,x) - w_t^{(1)}(t,x) \geq \frac{1}{2}(w_{xx}^{(2)}(t,x) - w_{xx}^{(1)}(t,x)) + G(x, w^{(2)}(t,x)) - G(x, w^{(1)}(t,x)).$$

Then, recalling (C.1) and letting

$$v(t,x) := e^{-2\alpha t}\big(w^{(2)}(t,x) - w^{(1)}(t,x)\big),$$

we get

$$\begin{aligned} v_t(t,x) - \frac{1}{2}v_{xx}(t,x) &\geq -2\alpha v(t,x) + e^{-2\alpha t}\big(G(x,w^{(2)}(t,x)) - G(x,w^{(1)}(t,x))\big) \\ &\geq \big(-2 - \operatorname{sgn}(v(t,x))\big)\alpha \cdot v(t,x). \end{aligned}$$

Now the the first factor on the right-hand side is negative and bounded. Applying the maximum principle [47, Theorem 8.1.4] to $-v$ then implies that $v \geq 0$ on Q and we can conclude. □

As an important application we get that the solution w to $\mathcal{L}_G w = 0$ is monotone in G and in the initial condition.

Corollary C.2. *Let $T > 0$, $Q := (0,T)\times\mathbb{R}$, and let G_1 and G_2 fulfill $G_1 \leq G_2$ on $\mathbb{R}\times[0,\infty)$. Furthermore, assume that G_2 satisfies* (C.1)*. In addition, let $w^{(1)}$ and $w^{(2)}$ be nonnegative and bounded functions on $\overline{Q}$, such that for $i \in \{1,2\}$, w_x^i and w_{xx}^i are continuous on Q and w_t^i exist on Q. If $\mathcal{L}_{G_1} w^{(1)} = \mathcal{L}_{G_2} w^{(2)}$ and $w^{(1)}(0,\cdot) \leq w^{(2)}(0,\cdot)$ on $x \in \mathbb{R}$, then we have $w^{(1)} \leq w^{(2)}$ on Q.*

Proof. Since the function $w^{(2)}$ is nonnegative, we have $\mathcal{L}_{G_2} w^{(2)} = \mathcal{L}_{G_1} w^{(1)} \geq \mathcal{L}_{G_2} w^{(1)}$. Then by Lemma C.1, we have $w^{(1)}(t,x) \leq w^{(2)}(t,x)$ for all $(t,x) \in Q$. □

Corollary C.3. *Let G fulfill* (C.1) *and $G(x,0) = G(x,1) = 0$ for all $x \in \mathbb{R}$. Let w be a solution to $\mathcal{L}_G w = 0$ with $0 \leq w(0,x) \leq 1$. Then $0 \leq w(t,x) \leq 1$ for all $(t,x) \in [0,\infty) \times \mathbb{R}$.*

Proof. The functions $w^{(1)}(t,x) = 0$ and $w^{(2)}(t,x) = 1$ are solutions to $\mathcal{L}w^{(1)} = \mathcal{L}w^{(2)} = 0$ and $w^{(1)}(0,x) \leq w(0,x) \leq w^{(2)}(0,x)$. The claim then follows from Lemma C.1. □

D Auxiliary results

Lemma D.1. *For all functions F fulfilling* (SC) *there exists a sequence $(p_k)_{k\in\mathbb{N}}$ fulfilling* (PROB) *such that for the function $G = G^{(p_k)_{k\in\mathbb{N}}} : [0,1] \to [0,\infty), G(u) = 1-u+\sum_{k=1}^{\infty} p_k(1-u)^k$ we have*

$$F(u) \geq G(u) \quad \forall u \in [0,1]$$

Proof. Recall that (SC) implies that there exists $M \in \mathbb{N}$ such that

$$1 - F'(x) \le \frac{M}{2} x \quad \text{and} \quad F(1-x) \ge x M^{-1} \quad \text{for all } x \in [0, M^{-1}]. \tag{D.1}$$

Define $G_n(x) := \frac{1-x}{n}\big(1-(1-x)^n\big)$, $x \in [0,1]$, $n \in \mathbb{N}$. Then each G_n, $n \in \mathbb{N}$, satisfies (PROB) with $p_1 = 1 - n^{-1}$ and $p_{n+1} = n^{-1}$, and our goal is to show that $G_n \le F$ for all $x \in [0,1]$ and all n large enough.

We start with noting that $G_{n+1} \le G_n$ as functions on $x \in [0,1]$, for all $n \in \mathbb{N}$, and that $G_n \downarrow 0$ uniformly as n tends to infinity. Thus, since F is continuous and $F > 0$ on $(0,1)$ due to (SC), we only have to take care of the neighborhoods of 0 and 1. From (D.1) we immediately get $G_M(x) \le (1-x)M^{-1} \le F(x)$ for all $x \in [1-M^{-1}, 1]$. To infer the desired inequality for $x \in [0, 1-M^{-1}]$, Taylor expansion yields

$$(1-x)^M \ge 1 - Mx + \binom{M}{2} x^2 - \binom{M}{3} x^3.$$

Then for M large enough and for all $x \in [0, M^{-1}]$ we get

$$\begin{aligned} G_M(x) &\le (1-x)\Big(x - \frac{M-1}{2}x^2 + \frac{(M-1)(M-2)}{6}x^3\Big) \le x - \frac{M}{3}x^2 \\ &\le x - \frac{M}{4}x^2 = \int_0^x (1 - Mt/2)\,\mathrm{d}t \le \int_0^x F'(t)\,\mathrm{d}t = F(x), \end{aligned}$$

where the last inequality is due to (D.1) again. □

Recall that u^{u_0} denotes the solution to (PAM) with initial condition u_0.

Lemma D.2. *For all $x \in \mathbb{R}$ and $0 \le s \le t$ we have $u^{\mathbb{1}_{(-\infty,0]}}(s,x) \le 2u^{\mathbb{1}_{(-\infty,0]}}(t,x)$.*

Proof. By the Feynman-Kac formula, $\xi \ge 0$ and the Markov property at time s we have

$$\begin{aligned} u^{\mathbb{1}_{(-\infty,0]}}(t,x) &= E_x\Big[e^{\int_0^t \xi(B_r)\,\mathrm{d}r}; B_t \le 0\Big] \ge E_x\Big[e^{\int_0^s \xi(B_r)\,\mathrm{d}r}; B_s \le 0, B_t \le 0\Big] \\ &\ge \mathtt{E}_x^\xi\big[N^{\le}(s,0)\big]\, P_0(B_{t-s} \le 0) = \frac{1}{2} u^{\mathbb{1}_{(-\infty,0]}}(s,x). \end{aligned}$$

□

Lemma D.3. *Let $N_{s,u,t}^{\mathcal{L},\mathcal{M}}$ be as in (3.4.1). Then there exists $c > 0$ such that $\mathbb{P}$-a.s.*

$$\mathtt{E}_x^\xi\big[N_{t,t,t}^{\mathcal{L}}\big] \ge c\mathtt{E}_{\lfloor x \rfloor}^\xi\big[N_{t-1,t+1,t}^{\mathcal{L}}\big] \quad \textit{for all } t \ge 1 \textit{ and } x \ge 1.$$

Proof. Write $A_{y,z,t} := \big\{H_k + y \ge z + t - T_k^{(M)} - 5\chi_1(\overline{m}(t)) \ \forall k \in \{1, \dots, \lfloor \overline{m}(t) \rfloor\}\big\}$ and $p(r) := \inf_{y \le 0} P_y(B_r \le 0) = \frac{1}{2}$. Then

$$\begin{aligned} \mathtt{E}_x^\xi\big[N_{t,t,t}^{\mathcal{L}}\big] &= E_x\big[e^{\int_0^t \xi(B_s)\,\mathrm{d}s}; B_t \le 0, A_{0,0,t}\big] \\ &\ge E_x\Big[e^{\int_{H_{\lfloor x \rfloor}}^{t-1+H_{\lfloor x \rfloor}} \xi(B_s)\,\mathrm{d}s}; B_t \le 0, B_{t-1+H_{\lfloor x \rfloor}} \le 0, A_{H_{\lfloor x \rfloor},1,t}, H_{\lfloor x \rfloor} \le 1\Big] \\ &\ge E_x\Big[E_{\lfloor x \rfloor}\big[e^{\int_0^{t-1} \xi(B_s)\,\mathrm{d}s} \times \inf_{r \in [0,1]} p(r); B_{t-1} \le 0, A_{0,1,t}\big]; H_{\lfloor x \rfloor} \le 1\Big] \\ &\ge \frac{1}{2} P_x(H_{\lfloor x \rfloor} \le 1) E_{\lfloor x \rfloor}\big[e^{\int_0^{t-1} \xi(B_s)\,\mathrm{d}s}; B_{t-1} \le 0, A_{0,1,t}\big] \ge c\mathtt{E}_{\lfloor x \rfloor}^\xi\big[N_{t-1,t+1,t}^{\mathcal{L}}\big], \end{aligned}$$

with $c = \frac{1}{2}P_1(H_0 \le 1)$, where the last inequality can be obtained analogously to the proof of Lemma 3.29. □

The next result is a Harnack inequality for the solution to (PAM).

Lemma D.4. *There exists a constant $C_{20} \in (0,\infty)$ such that $\mathbb{P}$-a.s. for all $y \in \mathbb{R}$, $t \ge 1$ and all $u_0 \in \mathcal{I}_{PAM}$ we have*

$$u^{u_0}(t,y) \le C_{20} \inf_{x\in[y-1,y+1]} u^{u_0}(t+1,x). \tag{D.2}$$

Proof. For $x \in \mathbb{R}$ and $t > 0$ let $f_{t,x}$ be the probability density of a Brownian motion at time t, starting in x. Let us first show that for all $y, z \in \mathbb{R}$ we have

$$f_{1,y}(z) \le \sqrt{2/e} \inf_{x\in[y-1,y+1]} f_{2,x}(z). \tag{D.3}$$

Indeed, using $f_{t,y}(z) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(z-y)^2}{2t}}$, (D.3) follows from

$$\begin{aligned}\inf_{x\in[y-1,y+1]} \big\{2(z-y)^2 - (z-x)^2\big\} &= \begin{cases} 2(z-y)^2 - (z-y+1)^2 = (z-y-1)^2 - 2, & y < z, \\ 2(z-y)^2 - (z-y-1)^2 = (z-y+1)^2 - 2, & y \ge z, \end{cases} \\ &\ge -2.\end{aligned}$$

Now by the Feynman-Kac formula and the Markov property, we have

$$\begin{aligned} u^{u_0}(t,y) = E_y\big[e^{\int_0^t \xi(B_r)\,\mathrm{d}r} u_0(B_t)\big] &\le e^{\mathsf{es}} \int_{\mathbb{R}} f_{1,y}(z) E_z\big[e^{\int_0^{t-1} \xi(B_r)\,\mathrm{d}r} u_0(B_{t-1})\big]\,\mathrm{d}z \\ &\le e^{\mathsf{es}} \sqrt{2/e} \inf_{x\in[y-1,y+1]} \int_{\mathbb{R}} f_{2,x}(z) E_z\big[e^{\int_0^{t-1} \xi(B_r)\,\mathrm{d}r} u_0(B_{t-1})\big]\,\mathrm{d}z \\ &\le e^{\mathsf{es}} \sqrt{2/e} \inf_{x\in[y-1,y+1]} u^{u_0}(t+1,x), \end{aligned}$$

where in the second inequality we used (D.3). Then (D.2) follows with $C_{20} = e^{\mathsf{es}}\sqrt{2/e}$. □

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Notation

Erklärung zur Dissertation

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist, sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Alexander Drewitz betreut worden.

Lars Schmitz, 14. Februar 2021

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- *Invariance principles and Log-distance of F-KPP fronts in a random medium.*
  Alexander Drewitz and Lars Schmitz.
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- *(Un-)bounded transition fronts for the parabolic Anderson model and the randomized F-KPP equation.*
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