One-skeleton galleries and cell combinatorics in type A

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Abstract

We describe the image of a cell in the Bialynicki-Birula decomposition of Gaussent and Littelmann's Bott-Samelson type variety, which is a desingularization of an affine Schubert variety, in type A for two particular cases. In the first case, we take a one-skeleton gallery, which is completely included in the dominant Weyl chamber. We can show, that the closure of the image of the cells associated to galleries, that belong to certain parts of the associated crystal, are in fact MV-cycles. In the second case we take a gallery of type $N\omega_1$, which is completely included in the dominant Weyl chamber. We generalize the notion of a Young tableau and use the 1:1 correspondance between tableaux and one-skeleton galleries to analyze, how the associated cells behave under the bumping algorithm. Finally, we can show that if and only if we take two words, equivalent under the Knuth relations, then the closure of the images of the cells associated to the galleries coming from these word, are the same. In addition, this closure is an MV-cycle. This allows a geometric interpretation of the combinatoric Knuth relations in the plactic monoid.

Kurzzusammenfassung

Wir analysieren die Bilder der Zellen in der Bialynicki-Birula Zerlegung von Gaussent und Littelmanns Bott-Samelson Auflösung von affinen Schubert Varietäten im Typ A für zwei spezielle Fälle. Im ersten Fall nehmen wir eine Eins-Skelett Galerie, die vollständig in der dominanten Weyl Kammer liegt. Wir zeigen, dass die Abschlüsse der Bilder von Zellen von Galerien aus Teilen des dazugehörigen Kristallgraphen MV-Zykel sind. Im zweiten Fall nehmen wir eine Galerie vom Typ $N\omega_1$, die vollständig in der dominanten Weyl Kammer liegt. Wir verallgemeinern den Begriff eines Young Tableau und nutzen die 1:1 Beziehung zwischen Tableaux und Eins-Skelett Galerien um das Verhalten der Zellen unter dem Bumping Algorithmus zu analysieren. Letztendlich können wir zeigen, dass wenn man zwei unter den Knuth Relationen äquivalente Wörter nimmt, dann und nur dann die Abschlüsse der Bilder der Zellen, die von den Galerien zu diesen Wörtern kommen, dieselben sind. Zusätzlich handelt es sich bei diesem Abschluss um einen MV-Zykel. Dies erlaubt eine geometrische Interpretation der kombinatorischen Knuth Relationen im plaktischen Monoid.

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Contents

Introduction

In their paper "One skeleton galleries, the path model and a generalization of Macdonald's formula for Hall-Littlewood polynomials" ([GL11]), Gaussent and Littelmann introduced the notion of one-skeleton galleries to give a direct geometric interpretation of the path model. This interpretation allows to compute the coefficients of the expansion of the Hall-Littlewood polynomials in the monomial basis. In their article, they also describe a Bott-Samelson type variety of galleries to give a desingularization of affine Schubert varieties. In this thesis, we will only consider the case G being of type A. Let λ be a dominant coweight and γ_λ a combinatorial one-skeleton gallery joining $\mathfrak{o}$ and λ. Let $\Sigma(\gamma_\lambda)$ be the associated Bott-Samelson type variety and $\pi : \Sigma(\gamma_\lambda) \to X_\lambda$ the desingularization map. By fixing a generic anti-dominant coweight $\eta : \mathbb{C}^* \to T$, we can consider a decomposition of $\Sigma(\gamma_\lambda)$ in Bialynicki-Birula cells:

$$C_\delta = \left\{ x \in \Sigma(\gamma_\lambda) \mid \lim_{t \to 0} \eta(t).x = \delta \right\},$$

where γ is a combinatorial gallery of the same type es γ_λ. But since the map π is $SL_n(\mathcal{O})$-equivariant, we can observe

$$\pi(C_\delta) \subset \left\{ y \in \mathcal{G} \mid \lim_{t \to 0} \eta(t).y = \mu \right\} = U^-(\mathcal{K}).\nu,$$

where μ is the target of δ. And since $\pi(C_\delta)$ is included in the Schubert variety X_λ, the set $\pi(C_\delta)$ has to be included in an MV-cycle. In this thesis, we want to determine particular cases in type A, in which the set $\overline{\pi(C_\delta)}$ really is an MV-cycle, but for a possibly smaller Schubert variety. In the first chapter, we will introduce the basic notions for our group G. In addition, we will define affine Schubert varieties and MV-cycles. In the second chapter, we characterizing the affine and the spherical building. Then we will define one-skeleton galleries. We introduce the notion of minimality and of positive foldings. We also describe a 1:1 correspondence between (positively folded) one-skeleton galleries of the same type as $\gamma_{\underline{\lambda}}$ and (semi-standard) Young tableaux of shape λ. By defining root operators we obtain a crystal structure on the set $\Sigma(\gamma_\lambda)$ of combinatorial galleries of type γ_λ starting in $\mathfrak{o}$. At the end, we will cite results by John R. Stembridge, describing simply-laced crystals in general. In the third chapter, we define the already mentioned Bott-Samelson type variety and we will describe a cellular decomposition of this variety. We will quote a result by Gaussent and Littelmann, giving an upper bound for the dimension of these cells.

In the forth chapter, we obtain a result about the dimension of these cells in type A, giving the exact dimension of a cell C_δ. We will consider a gallery δ, which is completely included in the dominant Weyl chamber. Then we will look at the crystal associated to this gallery using the root operators defined by Gaussent and Littelmann ([GL05]). By using a result from Stembridge ([Ste03]), we achieve:

Corollary 4.2.3 If δ is a gallery of type γ_λ, such that for every $j \in \{1, ..., n\}$ we have $\tilde{f}_{\alpha_j}\delta = 0$ and if $\tilde{e}^k_{\alpha_i}\delta$ is defined, then $\overline{\pi(C_{\tilde{e}^k_{\alpha_i}\delta})}$ is a MV-cycle of coweight (λ, μ).

In the last chapter, we will consider a dominant coweight λ and an LS-gallery of type γ_λ. By reading the associated Young tableau box by box, we obtain a new gallery, which is not necessarily an LS-gallery. The cells associated to these galleries will be in different Bott-Samelson type varieties, we want to relate the image of these cells in the affine Grassmannian. This can be done inductively, by studying how these images relate under the application of the bumping algorithm ([Ful97]). To be more precise, we will generalize the notion of a tableau and allow columns, which are not necessarily decreasing in length. Then we will consider the tableaux

$$T_1 = T' * \begin{array}{|c|c|} \hline j & i_1 \\ \hline \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & i_{l-1} \\ \cline{2-2} \multicolumn{1}{c|}{} & i_l \\ \cline{2-2} \multicolumn{1}{c|}{} & i_{l+1} \\ \cline{2-2} \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & i_r \\ \cline{2-2} \end{array} * T \text{ and } T_2 = T' * \begin{array}{|c|c|} \hline i_1 & i_l \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} i_{l-1} & \multicolumn{1}{c}{} \\ \cline{1-1} j & \multicolumn{1}{c}{} \\ \cline{1-1} i_{l+1} & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} i_r & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} * T,$$

where T_2 is achieved by bumping $\boxed{j}$ into the first column. If we denote δ_1 the gallery obtained by reading T_1 box by box, δ_2 the gallery obtained by reading T_2 columnwise and π, π' the associated desingularizations, we obtain:

Lemma 5.2.5 We have the following inclusion

$$\pi(C_{\delta_2}) \subset \pi'(C_{\delta_1}).$$

If we consider a dominant gallery γ of type $N\omega_1$, δ a gallery in the crystal associated to γ and T the Young tableaux associated to δ, we can divide T into a second part, which is semi-standard and a first part, which is not. Let μ be the target of δ. By gradually bumping the single boxes of the first part into the second part, we will obtain a semi-standard Young tableaux δ' at the end and relate the image of the cells of δ and δ'. We obtain:

Theorem 5.2.1 If and only if two words are equivalent under the Knuth relations, then the closure of the images of the cells associated to two words are the same. They form an MV-cycle of coweight (λ, μ), where μ is their target and λ is the shape of the semi-standard Young tableau resulting from the bumping algorithm.

1 Affine Schubert varieties

We want to fix the notations for our group G, introduce affine Schubert varieties and MV-cycle. For a reference see [Kum02] and [MV07].

1.1 Notations for the group

Let G be a connected complex semisimple algebraic group. Let us fix a Borel subgroup $B \subset G$ and a maximal torus T. We denote by B^- the opposite Borel group. We denote the unipotent radicals of B and B^- by U and U^-. Let $N_G(T)$ be the normalizer of T in G and the *Weyl group* of G and T is given by $W = N_G(T)/T$.

Example: Let $G = SL_n(\mathbb{C})$. The Borel subgroup B consists of the upper triangular matrices in $SL_n(\mathbb{C})$, while the set of diagonal matrices in $SL_n(\mathbb{C})$ will form a maximal torus in G. The Weyl group in this case is isomorphic to S_n.

The *character group* (respectively *cocharacter group*) associated to T is defined by $X = X^*(T) = Mor(T, \mathbb{C}^*)$ respectively $X^\vee = X_*(T) = Mor(\mathbb{C}^*, T)$. We use the notation $\mu : \mathbb{C}^* \to T, s \mapsto s^\mu$. We write Φ and $\Phi^\vee$ for the *root* and *coroot*-system. According to our choice of B we denote by Φ^+ and Φ^- the set of positive and negative roots of G, and use the notation $\Phi_+^\vee$ and $\Phi_-^\vee$ for the corresponding subsets of the coroots. We denote the set

$$X_+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0, \ \forall \alpha^\vee \in \Phi_+^\vee\}$$

the *dominant weights* and the set

$$X_+^\vee = \{\lambda^\vee \in X^\vee \mid \langle \alpha, \lambda^\vee \rangle \geq 0, \ \forall \alpha \in \Phi_+\}$$

the *dominant coweights*. Let $\alpha_1, ..., \alpha_n$ be the simple roots and let ρ be half the sum of the positive roots. We denote by $\omega_i \in X^\vee$ the *fundamental coweight* corresponding to α_i. By $l(w)$ we denote the length of a Weyl group element, i.e. the length of a reduced expression $w = s_{i_1}...s_{i_r}$.

1.2 Schubert varieties

We denote $\mathcal{O} = \mathbb{C}[[t]]$ the ring of complex formal power series and $\mathcal{K} = \mathbb{C}((t))$ its quotient field. Let $v : \mathcal{K}^* \to \mathbb{Z}$ be the standard valuation such that $\mathcal{O} = \{f \in \mathcal{K} \mid v(f) \geq 0\}$. We call the quotient

$$\mathcal{G} = G(\mathcal{K})/G(\mathcal{O})$$

the *affine Grassmannian*. It can be seen that $G(\mathcal{K})$ and $\mathcal{G}$ are *ind*-schemes and $G(\mathcal{O})$ is a group scheme (see [Kum02]).

Example: Considering the example $G = SL_n(\mathbb{C})$, we have $G(\mathcal{K}) = SL_n(\mathcal{K})$, matrices with entries in $\mathcal{K}$ and determinant equal to 1.

The $G(\mathcal{O})$-orbits in $\mathcal{G}$ are finite dimensional quasi-projective varieties. Given a $\lambda \in X^\vee$, we can view λ as an element of $G(\mathcal{K})$ and, when we do, we will use the notation t^λ. By abuse of notation, we will also write t^λ for the corresponding class in $\mathcal{G}$. Let $ev : G(\mathcal{O}) \to G$ be the evaluation map at $t = 0$ and let $\mathcal{B} = ev^{-1}(B)$ the corresponding Iwahori subgroup. Then we have the following decompositions:

$$\mathcal{G} = \bigcup_{\lambda \in X^\vee} \mathcal{B}t^\lambda = \bigcup_{\lambda \in X^\vee_+} G(\mathcal{O})t^\lambda.$$

The closure of an orbit $\mathcal{B}t^\lambda$ is the *Schubert variety* $X(\lambda) = \overline{\mathcal{B}t^\lambda}$. Note that for $\lambda \in X^\vee_+$ we have

$$\overline{G(\mathcal{O})t^\lambda} = X(w_0(\lambda)),$$

with w_0 being the longest element in the Weyl group W. We will write X_λ for $X(w_0(\lambda))$.

1.3 Affine Kac-Moody groups

We can naturally equip the field $\mathcal{K}$ with the *rotation operation* $\gamma : \mathbb{C}^* \to Aut(\mathcal{K})$, that "rotates" the indeterminante: $\gamma(z)(f(t)) = f(zt)$. This action lifts to an operation on the group $G(\mathcal{K})$, $\gamma_G : \mathbb{C}^* \to Aut(G(\mathcal{K}))$. Let us denote the semidirect product $\mathbb{C}^* \ltimes G(\mathcal{K})$ by $\mathcal{L}(G(\mathcal{K}))$, the *loop group* corresponding to G. Since the action restricts to an operation of $\mathcal{O}$, we also get $\mathcal{L}(G(\mathcal{O})) = \mathbb{C}^* \ltimes G(\mathcal{O})$.

Let $\hat{\mathcal{L}}(G)$ be the affine Kac-Moody group associated to the affine Kac-Moody algebra

$$\hat{\mathcal{L}}(\mathfrak{g}) = \mathfrak{g} \otimes \mathcal{K} \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $0 \to \mathbb{C}c \to \mathfrak{g} \otimes \mathcal{K} \oplus \mathbb{C}c \to \mathfrak{g} \otimes \mathcal{K} \to 0$ is the universal central extension of the *loop algebra* $\mathfrak{g} \otimes \mathcal{K}$ and d denotes the scaling element. Let $\mathcal{P}_\mathcal{O} \subset \hat{\mathcal{L}}(g)$ be the "parabolic" subgroup $\pi^{-1}(\mathcal{L}(G(\mathcal{O})))$, then

$$\mathcal{G} = G(\mathcal{K})/G(\mathcal{O}) = \mathcal{L}(G(\mathcal{K}))/\mathcal{L}(G(\mathcal{O})) = \hat{\mathcal{L}}(G)/\mathcal{P}_\mathcal{O}.$$

We denote by $N_\mathcal{K}$ the subgroup of $G(\mathcal{K})$ generated by N and $T(\mathcal{K})$ and let $\mathcal{T} \subset \hat{\mathcal{L}}(G)$ be the corresponding standard maximal torus. By $\mathcal{N}$ we denote its stabilizer in $\hat{\mathcal{L}}(G)$. The extended affine Weyl group can be described the following ways:

$$W^\mathfrak{a} = N_\mathcal{K}/T \simeq \mathcal{N}/\mathcal{T}.$$

Remark 1 The affine Weyl group can be realized as the semidirect product of the classical Weyl group of G with the coroot lattice on which the Weyl group acts naturally.

1.4 MV-cycles

Inside the affine Grassmannian, we are interested in two types of orbits. It is clear that $G(\mathcal{O})$ acts on $\mathcal{G}$ by left multiplication. As already mentioned in section 1.2, these orbits are finite dimensional and can be indexed by the set of dominant coweights:

$$\mathcal{G} = \bigcup_{\lambda \in X_+^\vee} G(\mathcal{O})t^\lambda.$$

We will denote the $G(\mathcal{O})$-orbit by $\mathcal{G}_\lambda = G(\mathcal{O})t^\lambda$. The second type of orbits we are interested in are the semi-infinite orbits S_ν^w for $\nu \in X^\vee$ and $w \in W$. They are defined as

$$S_\nu^w = wU^-(\mathcal{K})w^{-1}t^\lambda$$

and we denote S_ν^{Id} by S_ν. We have well known closure relations for both types of orbits, see [MV07].

Definition 1.4.1 ([And03],§5.3, Def. and [Kam10],2.2) Let $\lambda \in X_+^\vee$ and $\mu \in X^\vee$. If the intersection $\mathcal{G}_\lambda \cap S_\mu$ is not empty we call the irreducible components of $\overline{\mathcal{G}_\lambda \cap S_\mu}$ the *MV-cycles of coweight* (λ, μ).

Mirkovic and Vilonen showed, that the collection of all MV-cycles of coweights (λ, ν) for $\nu \in X^\vee$ form a natural basis of the irreducible representation $V(\lambda)$ for $G^\vee$ of highest weight λ, where $G^\vee$ is the Langlands dual group of G. In addition, the MV-cycles of coweight (λ, ν) span the weight space of the coweight ν. The geometric Satake correspondence identifies the underlying space of the irreducible rational representation $V(\lambda)$ with highest weight λ with the intersection cohomology of X_λ and the MV-cycles afford a basis of this intersection cohomology.

2 Buildings and galleries

In this chapter, we will introduce the affine and spherical building. Then we can define our main object, a one-skeleton gallery in the affine building. We will introduce the notion of a minimal and positively folded one-skeleton gallery. Finally we define root operators on the set of combinatorial galleries of a given type, that start in $\mathfrak{o}$, similar as in Littelmann's path model ([Lit95]). At the end we will state some of Stembridge's results ([Ste03]) about simply-laced crystals.

2.1 Buildings

Given a root and coroot datum, we associate to it the real vector space $\mathbb{A} = X^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ together with the hyperplane arrangement defined by the set $\{(\alpha, n) \mid \alpha \in \Phi, n \in \mathbb{Z}\}$ of affine roots. The couple (α, n) corresponds to the real affine root $\alpha + n\delta$ with δ being the smallest positive imaginary root. To an affine root (α, n) we associate the affine reflection $s_{\alpha,n} : x \mapsto x - (\langle \alpha, x^\vee \rangle)\alpha^\vee$ and the affine hyperplane $H_{\alpha,n} = \{x \in \mathbb{A} \mid \langle \alpha, x^\vee \rangle + n = 0\}$. We write

$$H^+_{\alpha,n} = \{x \in \mathbb{A} \mid \langle \alpha, x^\vee \rangle + n \geq 0\}$$

for the corresponding closed half-space and analogously

$$H^-_{\alpha,n} = \{x \in \mathbb{A} \mid \langle \alpha, x^\vee \rangle + n \leq 0\}$$

for the negative half space.

Definition 2.1.1 We call the irreducible components of

$$\mathbb{A} - \bigcup_{\alpha \in \Phi^+} H_{\alpha,0}$$

open chambers and their closures *closed chambers*, *Weyl chambers* or just *chambers*. An *open alcove* is an irreducible component of

$$\mathbb{A} - \bigcup_{(\alpha,n) \in \Phi^+ \times \mathbb{Z}} H_{\alpha,n},$$

its closure is denoted as a *closed alcove* or just *alcove*.

It is well known, that the Weyl group W can also be described as the finite subgroup of $GL(\mathbb{A})$ generated by the reflections $s_{\alpha,0}, \alpha \in \Phi$, the affine Weyl group $W^{\mathfrak{a}}$ can be described as the group of affine transformations of $\mathbb{A}$ generated by the affine reflections $s_{\alpha,n}, (\alpha, n) \in \Phi \times \mathbb{Z}$. A fundamental domain for the action of W on $\mathbb{A}$ ist given by

$$C^+ = \{x \in \mathbb{A} \mid \langle \alpha, x^\vee \rangle \geq 0,\ \forall \alpha \in \Phi^+\} = \bigcap_{\alpha \in \Phi^+} H^+_{\alpha,0},$$

which we denote by the dominant Weyl chamber. Similarly the *fundamental alcove*

$$\Delta_f = \{x \in \mathbb{A} \mid 0 \leq \langle \alpha, x^\vee \rangle \leq 1,\ \forall \alpha \in \Phi^+\} = \bigcap_{\alpha \in \Phi^+} H^+_{\alpha,0} \cap \bigcap_{\alpha \in \Phi, n>0} H^+_{\alpha,n}$$

is a fundamental domain for the action of $W^{\mathfrak{a}}$ on $\mathbb{A}$.

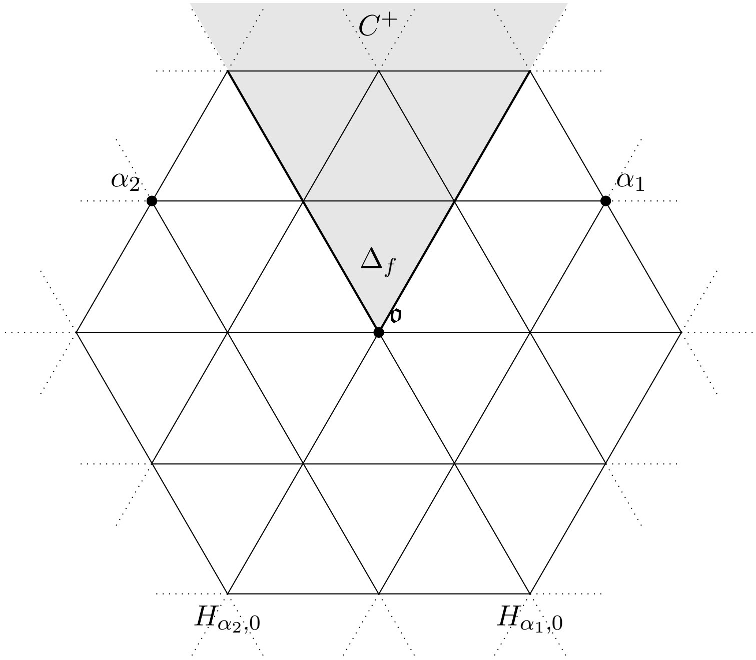

Figure 2.1: Example in the case A_2

Definition 2.1.2 A subset F of $\mathbb{A}$, which can be written as

$$F = \bigcap_{(\beta,m)} H^\bullet_{\beta,n},$$

where we choose $H^\bullet_{\beta,n}$ for every pair $(\beta, n), \beta \in \Phi^+, n \in \mathbb{Z}$ as either the hyperplane, the positive or the negative halfspace, will be called a *face*. The corresponding *open face* F^o will be the subset of F obtained by replacing the closed affine halfspaces by the corresponding open affine halfspaces.

The *support* of a face F is the affine span $\langle F^o \rangle_{\text{aff}} = \langle F \rangle_{\text{aff}}$, the *dimension* of a face is the dimension of its support. We call the support of a codimension one face a *wall of an alcove*. A face of dimension one in $\mathbb{A}$ is denoted an *edge* and a face of dimension zero a *vertex*. For any vertex v let $\Phi_v \subset \Phi$ be the subrootsystem of all roots α such that $v \in H_{\alpha,n}$ for some integer n. We call a vertex v special if $\Phi_v = \Phi$.

Example: In the case of G being of type A, every vertex is special.

A *sector* with vertex $\nu \in \mathbb{A}$ is a closed chamber translated by ν, i.e. there exists a closed chamber C such that

$$\mathfrak{s} := \{\lambda \in \mathbb{A} \mid \lambda = \nu + z \text{ for some } z \in C\}.$$

We will write $-\mathfrak{s}$ for the sector

$$-\mathfrak{s} = \nu - C = \{\mu \in \mathbb{A} \mid \mu = \nu - x \text{ for some } x \in C\}.$$

We can translate a sector $\mathfrak{s}$ with vertex V by $\mu - \nu$, where $\mu \in \mathbb{A}$ and get:

$$\begin{aligned} \mathfrak{s}(\mu) &= \{\lambda \in \mathbb{A} \mid \lambda = (\mu - \nu) + z \text{ for some } z \in \mathfrak{s}\} \\ &= \{\lambda \in \mathbb{A} \mid \lambda = \mu + z \text{ for some } z \in C\}. \end{aligned}$$

Let $N = N_G(T)$ be the normalizer in G of the fixed maximal torus $T \subset G$. The Weyl group W of G is isomorphic to N/T. For any real number $r \in \mathbb{R}$ we set

$$U_{\alpha,r} = \{1\} \cup \{x_\alpha(f) \mid f \in \mathcal{K}^*, v(f) \geq r\} \subset U_\alpha(\mathcal{K}).$$

For any non-empty subset $\Omega \subset \mathbb{A}$ let $l_\alpha(\Omega) = -\inf_{x\in\Omega} \langle \alpha, x \rangle$ and set

$$U_\Omega := \left\langle U_{\alpha,l_\alpha(\Omega)} \mid \alpha \in \Phi \right\rangle \subset G(\mathcal{K}).$$

Let $N(\mathcal{K})$ be the subgroup of $G(\mathcal{K})$ generated by N and $T(\mathcal{K})$ and we define the affine building $\mathcal{J}^{\mathfrak{a}}$ as

$$G(\mathcal{K}) \times \mathbb{A}/\sim,$$

where

$$(g, x) \sim (h, y) \text{ if } \exists n \in N(\mathcal{K}) \text{ such that } nx = y \text{ and } g^{-1}hn \in U_x := U_{\{x\}}.$$

Using the injective and $N(\mathcal{K})$-equivariant map $\mathbb{A} \to \mathcal{J}^{\mathfrak{a}}, x \mapsto (1, x)$ we can identity $\mathbb{A}$ with its image in $\mathcal{J}^{\mathfrak{a}}$. Any subset A of $\mathcal{J}^{\mathfrak{a}}$ of the form $g\mathbb{A}$ for some $g \in G(\mathcal{K})$ is called an *apartment*. Similarly, we extend the notion of a face F and a chamber C.

We also want to look at another building, the *residue building*. Let V be a vertex of $\mathcal{J}^{\mathfrak{a}}$ and $\mathcal{J}^{\mathfrak{a}}_V$ be the set of all faces F, such that $V \subset F$. The simplicial structure is given by the relation $F \subset F'$, for two faces F, F' such that $V \subset F$ and $V \subset F'$. If H_V is the connected reductive subgroup of G with root system Φ_V, then the structure of the spherical building on the set of all parabolic subgroups of H_V is isomorphic to the one on $\mathcal{J}^{\mathfrak{a}}_V$. For any face F of $\mathcal{J}^{\mathfrak{a}}$ containing V, we denote the associated face in $\mathcal{J}^{\mathfrak{a}}_V$ by F_V. Given a sector $\mathfrak{s} = V + C$ in $\mathbb{A}$, one associates the chamber $\mathfrak{s}_V$ of $\mathbb{A}_V$ in the following way: Let $\Delta \supset V$ be the unique alcove in $\mathbb{A}$ such that $\Delta^o \cap \mathfrak{s}^o \neq \emptyset$, then $\mathfrak{s}_V := \Delta_V$. Let $C^{\pm}_V$ be the positive (resp. negative) chamber in $\mathbb{A}_V$ associated to $V + C^{\pm}$.

2.2 One-skeleton galleries

Compared to a gallery of alcoves, a one-skeleton gallery will be a series of edges in $\mathcal{J}^{\mathfrak{a}}$, where two subsequent ones will share a vertex. We will denote combinatorial galleries, those, that stay in the apartment $\mathbb{A}$.

Definition 2.2.1 ([GL11], Def. 5) A sequence $\gamma = (V_0 \subset E_0 \supset V_1 \subset E_1 \supset .... \supset V_r \subset E_r \supset V_{r+1})$ of faces in $\mathbb{A}$ is called a *combinatorial one-skeleton gallery* if

- the faces V_i are vertices in $\mathbb{A}$;

- the faces E_i are edges in $\mathbb{A}$;
- the vertex V_0 (the *source* of the gallery) and the vertex V_{r+1} (the *target* of the gallery) are special vertices.

Let $\gamma' = V_0' \subset E_0' \supset ... \subset E_t' \supset V_{t+1}')$ be another gallery such that $V_0' = V_{r+1}$, then we can concatenate these galleries to get

$$\gamma * \gamma' = (V_0 \subset E_0 \supset ... \supset V_r \subset E_r \supset V_{r+1} = V_0' \subset E_0' \supset ... \subset E_t' \supset V_{t+1}').$$

We will also concatenate galleries, if $V_0' \neq V_{r+1}$, in this case we take $\gamma * \gamma'$ to be the concatenation of γ and the translated gallery $\gamma' + (V_{r+1} - V_0')$.

We want to associate a combinatorial one-skeleton gallery to a fundamental coweight ω. Let $\mathbb{R}_{\geq 0}\omega \subset \mathbb{A}$ be the extremal ray of the dominant Weyl chamber C^+ spanned by ω. We set $V_0 = 0$ and let E_0 be the unique edge in the intersection of Δ_f and $\mathbb{R}_{\geq 0}\omega$. If the first vertex V_1 is not ω, then we let E_1 be the unique dimension one face in $\mathbb{R}_{\geq 0}\omega$ different vom E_0 having V_1 as a common vertex with E_0. By iterating we obtain a gallery

$$\gamma_\omega = (V_0 = \mathfrak{o} \subset E_0 \supset V_1 \subset ... \supset V_r \subset E_r \supset \omega = V_{r+1})$$

which joins $\mathfrak{o}$ and ω. We call these galleries *fundamental galleries* and their faces *fundamental faces*.

If we have a fixed enumeration $\omega_1, ..., \omega_n$ of the fundamental coweights and $\lambda = \sum a_i \omega_i$ for an arbitrary dominant coweight λ, we set

$$\gamma_\lambda = \gamma_{a_1\omega_1} * ... * \gamma_{a_n\omega_n},$$

where $\gamma_{a_i\omega_i} = \underbrace{\gamma_{\omega_i} * ... * \gamma_{\omega_i}}_{a_i \text{ times}}$.

Definition 2.2.2 ([GL11], Def. 6) We define

$$S^{\mathfrak{a}} = \{(\alpha, n) \mid \Delta_f \cap H_{\alpha,n} \text{ is a face of codimension one}\}.$$

The *type* of a face F of the fundamental alcove is $S^{\mathfrak{a}}(F) = \{(\alpha, n) \mid F \subset H_{\alpha,n}\}$. For an arbitrary face F we set $S^{\mathfrak{a}}(F) = S^{\mathfrak{a}}(F^f)$, where F^f is the unique face of the fundamental alcove such that there exists a $w \in W^{\mathfrak{a}}$ with $w(F) = F^f$.

Definition 2.2.3 ([GL11], Def. 7) Let $\gamma = (V_0 \subset E_0 \supset V_1 \subset ... \supset V_r \subset E_r \supset V_{r+1})$ be a combinatorial one-skeleton gallery. The *type* t_γ of γ is defined as

$$t_\gamma := (S^{\mathfrak{a}}(V_0) \subset S^{\mathfrak{a}}(E_0) \supset S^{\mathfrak{a}}(V_1) \subset ... \supset S^{\mathfrak{a}}(V_r) \subset S^{\mathfrak{a}}(E_r) \supset S^{\mathfrak{a}}(V_{r+1})),$$

which is also denoted the gallery of types. We denote by $\Gamma(t_\gamma, V_0)$ the set of all combinatorial galleries starting in V_0 of type t_γ.

For any dominant coweight λ, a gallery of type λ is a gallery of type γ_λ. The set of all combinatorial galleries of type γ_λ starting in $\mathfrak{o}$ will be denoted by $\Sigma(\gamma_\lambda)$. For an example see Figure 2.2.

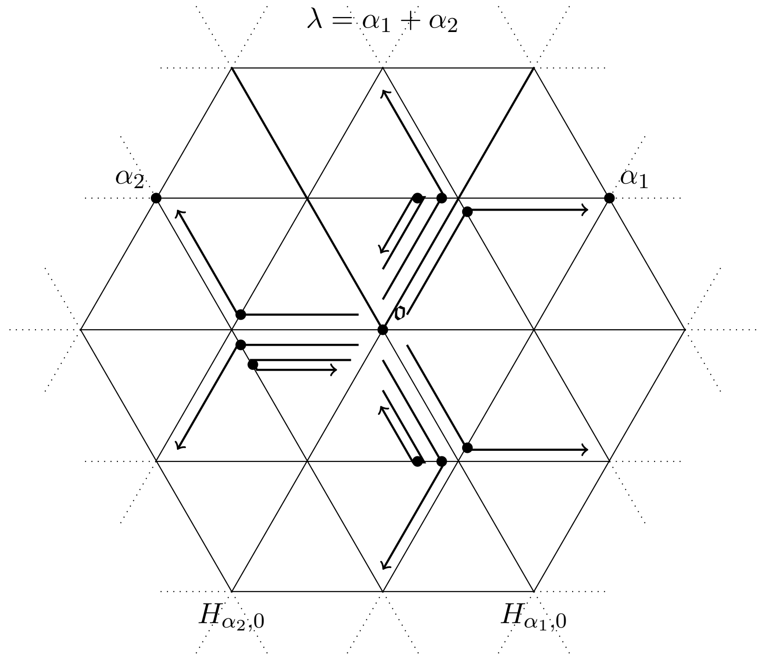

Figure 2.2: Example in the case A_2, galleries of type $\lambda = \alpha_1 + \alpha_2$

2.3 Minimal one-skeleton galleries

We know, that for two given sectors $\mathfrak{s}_1, \mathfrak{s}_2$, there exists an apartment A and subsectors $\mathfrak{s}_1' \subset \mathfrak{s}_1, \mathfrak{s}_2' \subset \mathfrak{s}_2$ such that $\mathfrak{s}_1', \mathfrak{s}_2' \subset A$. There is an bijection between the equivalence classes of sectors and the set of Weyl chambers in $\mathbb{A}$. These equivalence classe will be denoted by $\underline{\mathfrak{s}}$.

Definition 2.3.1 ([GL11], Def. 9) Let $\delta = (V_0 \subset E_0 \supset ... \subset E_r \supset V_{r+1})$ be a one-skeleton gallery. It is called minimal, if every step $(V_i \subset E_i)$ is included in the same equivalence class of one sector, i.e. there exists an equivalence class of sectors $\underline{\mathfrak{s}}_\delta$ and representatives $\mathfrak{s}_0, ..., \mathfrak{s}_r \in \underline{s}_\delta$ such that V_i is the vertex for the sector $\mathfrak{s}_i$ and $V_i \subset E_i \subset \mathfrak{s}_i$. The equivalence class $\underline{\mathfrak{s}}_i$ is not necessarily uniquely determined by δ.

We call the sequence $\underline{\mathfrak{s}}(\delta) = (\mathfrak{s}_0, ..., \mathfrak{s}_r)$ a *chain of sectors* associated to δ.

Example: Again, we consider the apartment of type A_2. For any element w of the Weyl group, we set $\gamma_{w(\omega_i)} := w(\gamma_{\omega_i})$. It is easy to see, that the galleries $\gamma_1 = \gamma_{s_1(\omega_1)} * \gamma_{s_1 s_2(\omega_2)}$ and $\gamma_2 = \gamma_{s_1 s_2(\omega_2)} * \gamma_{s_2 s_1(\omega_1)}$ are minimal with $\underline{\mathfrak{s}}_{\gamma_1} = \underline{s_1 s_2(C^+)}$ and $\underline{\mathfrak{s}}_{\gamma_2} = \underline{s_1 s_2 s_1(C^+)}$. The gallery $\gamma := \gamma_{s_1(\omega_1)} * \gamma_{s_1 s_2(\omega_2)} * \gamma_{s_2 s_1 \omega_1}$ is not minimal in the sense above (see Figure 2.3).

We have a natural action of $G(\mathcal{K})$ on $\mathcal{J}^{\mathfrak{a}}$ that induces a natural action on one-skeleton galleries. For δ any one-skeleton gallery and $\mathbf{g} \in G(\mathcal{K})$ we set:

$$\mathbf{g}.\delta = (\mathbf{g}.V_0 \subset \mathbf{g}.E_0 \supset .... \subset \mathbf{g}.E_r \supset \mathbf{g}.V_{r+1}).$$

Proposition 2.3.1 ([GL11],Prop. 8) *Let δ be a minimal one-skeleton gallery in $\mathcal{J}^{\mathfrak{a}}$ starting in $\mathfrak{o}$ and ending in $\lambda = V_{r+1}$, where $\lambda \in C^+$. Since $V_{r+1} = \lambda$ is special, it* *is a coweight, which we can identify with an element in $G(\mathcal{K})/G(\mathcal{O})$. The following natural map between the $G(\mathcal{O})$-orbit of δ and the $G(\mathcal{O})$-orbit of λ in $\mathcal{G}$ is bijective:*

$$G(\mathcal{O}).\delta \to G(\mathcal{O})t^{\lambda} \subset G(\mathcal{K})/G(\mathcal{O}), \mathbf{g}.\delta \mapsto \mathbf{g}t^{\lambda}.$$

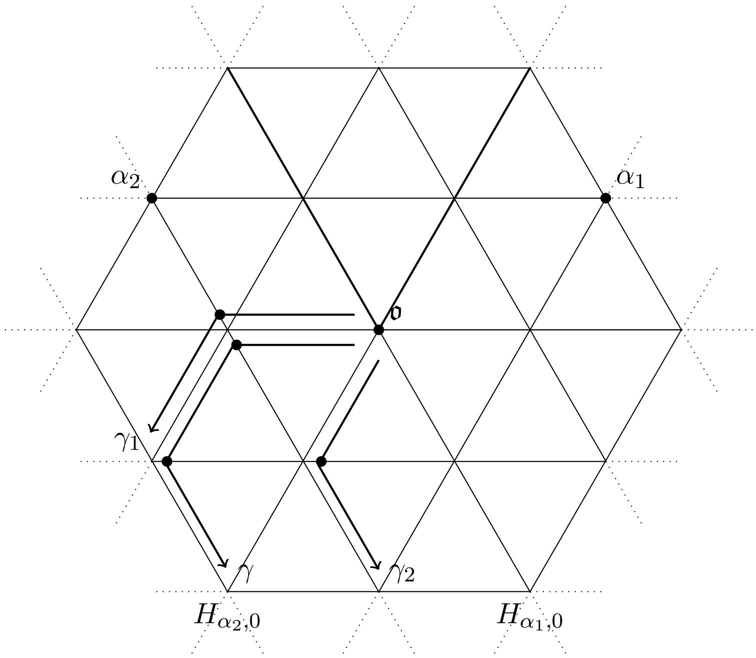

Figure 2.3: Example of minimal galleries in the case A_2

2.4 Positively folded galleries

We call a sequence $(V_0 \subset E \supset V \subset F \supset V_1)$ a *two step gallery*, but we often omit V_0 and V_1. Such a gallery is called *minimal* if there exists a sector $\mathfrak{s}$ with Vertex V_0 such that $E \subset \mathfrak{s}$ and $F \subset \mathfrak{s}(V)$.

Definition 2.4.1 ([GL11], Def. 10) A two step gallery $(E \supset V \subset F') \subset \mathbb{A}$ is obtained from $(E \supset V \subset F) \subset \mathbb{A}$ by a *positive folding* if there exists an affine root (α, n) such that

$$V \in H_{\alpha,n}, F' = s_{\beta,n}(F) \text{ and } H_{\alpha,n} \text{ separates } F \text{ and } C^{-}(V) \text{ from } F'.$$

A two step gallery $(E \supset V \subset F)$ in $\mathbb{A}$ is called *positively folded* if either the gallery is minimal, or if there exist faces $F_0, ..., F_s$ containing V such that:

- $(E \supset V \subset F_0)$ is minimal and $F_s = F$,
- $\forall j = 1, ..., s : (E \supset V \subset F_j)$ is obtained from $(E \supset V \subset F_{j-1})$ by a positive folding.

If we look at the residue building at a vertex V, we say that (E_V, F_V) is a *minimal pair* if there exists two opposite sectors $\mathfrak{s}$ and $-\mathfrak{s}$ with vertex V such that $E \subset \mathfrak{s}$ and $F \subset -\mathfrak{s}$. Using this language, we can define a positively folded two-step gallery the following way:

Definition 2.4.2 ([GL11], Def. 11) A two-step gallery $(E \supset V \subset F)$ in $\mathbb{A}$ is called *positively folded* if

- there exist faces $F_{0,V}, ..., F_{s,V}$ such that $(E_V, F_{0,V})$ is a minimal pair, and $F_{s,V} = F_V$,
- for all $j = 1, ..., s$ there exists an affine root (β_j, n_j) such that $\beta_j \in \Phi_V, V \in H_{\beta_j,n_j}, s_{\beta_j,n_j}(F_{j-1,V}) = F_{j,V}$ and H_{β_j,n_j} separates C_V^- and $F_{j-1,V}$ from $F_{j,V}$.

We have seen, that the equivalence classes of sectors are in bijection with the Weyl chambers, therefore we can endow the set of equivalence classes with the Bruhat order: $\underline{\mathfrak{s}} \geq \underline{\mathfrak{s}}'$ iff $\underline{\mathfrak{s}} = \tau(C^+), \underline{\mathfrak{s}}' = \kappa(C^+)$ and $\tau \geq \kappa$. For a minimal gallery δ and its associated chain of sectors $\underline{\mathfrak{s}}(\delta) = (\mathfrak{s}_0, ..., \mathfrak{s}_r)$ we hav $\underline{\mathfrak{s}}_0 = ... = \underline{\mathfrak{s}}_r$.

Definition 2.4.3 ([GL11], Def. 12) Let λ be a dominant coweight and γ_λ a minimal one-skeleton gallery contained in C^+ with start in $\mathfrak{o}$ and target λ. A combinatorial one-skeleton gallery of type t_{γ_λ}

$$\delta = (V_0 = \mathfrak{o} \subset E_0 \supset ... \subset E_r \supset V_{r+1}) \subset \mathbb{A}$$

is called *globally positively folded* or just *positively folded* if

i) the gallery is locally positively folded, i.e. every two step gallery $(E_{i-1} \supset V_i \subset E_i)$ is positively folded for $i = 1, ..., r$,

ii) there is a chain of sectors $\underline{\mathfrak{s}}(\delta) = (\mathfrak{s}_0, ..., \mathfrak{s}_r)$ such that for all $i = 0, ..., r$, V_i is the vertex of $\mathfrak{s}_i$, $E_i \subset \mathfrak{s}_i$ and $\underline{\mathfrak{s}}_0 \geq ... \geq \underline{\mathfrak{s}}_r$.

Let us look at the special case of G in type A_n. The fundamental weights can be written $\omega_i = \varepsilon_1 + ... + \varepsilon_i$. We have $\gamma_{\omega_i} = (\mathfrak{o} \subset E \supset \omega_i)$, where $E = \{t\omega_i \mid t \in [0, 1]\}$. It is easy to see, that the galleries of the same type as γ_{ω_i} are exactly the galleries $\gamma_{\sigma(\omega_i)}$ with $\sigma \in W/W_{\omega_i}$. Let λ be a dominant weight. Remember, if we have $\lambda = \sum a_i\omega_i$, we set

$$\gamma_\lambda = \gamma_{a_1\omega_1} * ... * \gamma_{a_n\omega_n}.$$

To a gallery $\delta = \delta_1 * ... * \delta_r, \sum a_i = r$, of the same type as $\gamma_{\underline{\lambda}}$, we can associate a Young tableau of shape λ in the following way: Let $\varepsilon_{k_1} + ... + \varepsilon_{k_j}$ be the target of δ_k. We associate to δ_k the column C_{δ_k} of j boxes filled with the numbers $k_1, ..., k_j$ (decreasing from top to bottom). We get the Young tableau $Y_\delta = (C_{\delta_r}, ..., C_{\delta_1})$, by putting the columns next to each other in reverse order. In this way, the galleries of the same type as $\gamma_{\underline{\lambda}}$ can be identified with Young tableaux of shape λ with entries strictly increasing in the columns, and the positively folded galleries are identified with the semi-standard tableaux ([GL11]).

Let us look at an example: Consider the case A_2 and the dominant weight $\lambda = 2\omega_1 + \omega_2$, the associated gallery $\gamma_{\underline{\lambda}}$ is $\gamma_{\varepsilon_1} * \gamma_{\varepsilon_1} * \gamma_{\varepsilon_1+\varepsilon_2}$ and its Young tableau is

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\quad .$$

Another tableau of shape λ and entries strictly increasing in the columns would be

$$\begin{array}{|c|c|c|}\hline 2 & 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

to which we associate the gallery $\delta = \gamma_{\varepsilon_2} * \gamma_{\varepsilon_1} * \gamma_{\varepsilon_2+\varepsilon_3}$, which is of the same type as $\gamma_{\underline{\lambda}}$. If we have a semi-standard Young tableau of shape λ, like

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

we get the gallery $\delta = \gamma_{\varepsilon_2} * \gamma_{\varepsilon_2} * \gamma_{\varepsilon_1+\varepsilon_3}$, which is of the same type as $\gamma_{\underline{\lambda}}$ and positively folded.

Let us look at the example above in the case of A_2 and $\lambda = \alpha_1 + \alpha_2$. We had $\gamma_\lambda = \gamma_{\varepsilon_1} * \gamma_{\varepsilon_1+\varepsilon_2}$, the associated Young tableau is

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\;.$$

We get seven additional semi-standard Young tableaux of shape λ:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\;,\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\;,\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\;,\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\;,\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\;,\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\;,\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\;.$$

These are associated to seven positively folded galleries of type λ (see Figure 2.4):

$$\gamma_{\varepsilon_2} * \gamma_{\varepsilon_1+\varepsilon_2}, \gamma_{\varepsilon_3} * \gamma_{\varepsilon_1+\varepsilon_2}, \gamma_{\varepsilon_1} * \gamma_{\varepsilon_1+\varepsilon_3}, \gamma_{\varepsilon_2} * \gamma_{\varepsilon_1+\varepsilon_3}, \gamma_{\varepsilon_3} * \gamma_{\varepsilon_1+\varepsilon_3}, \gamma_{\varepsilon_2} * \gamma_{\varepsilon_2+\varepsilon_3}, \gamma_{\varepsilon_2} * \gamma_{\varepsilon_2+\varepsilon_3}.$$

2.5 Root operators

We want to define root operators for one-skeleton galleries, which are basically the root operators defined by Littelmann in his path model (see [Lit95]). They put a crystal structure on the set $\Gamma(\gamma_\lambda)$.

Let $\delta = (\mathfrak{o} \subset E_0 \supset V_1 \subset \ldots \supset V_r \subset E_r \supset \mu)$ be any one-skeleton gallery. Fix a simple root α and let m the biggest integer, such that there exists a $V_i \in H_{\alpha,m}$. Let k be minimal, such that $V_k \in H_{\alpha,m}$, i.e. let V_k be the first face of δ, that touches $H_{\alpha,m}$. The root operator $\tilde{e}_\alpha$ is defined the following way:

If $m = 0$, we set $\tilde{e}_\alpha\delta = 0$. Otherwise, we set:

$$\tilde{e}_\alpha\delta = (\mathfrak{o} \subset \ldots \supset E_{k-2} \supset V_{k-1} \subset s_{\alpha,m}E_{k-1} \supset t_\alpha V_k \subset \ldots \supset t_\alpha V_r \subset t_\alpha E_r \supset \mu + \alpha).$$

Here t_α is the translation by α.

Now let k be maximal, such that $V_k \in H_{\alpha,m}$, i.e. let V_k be the last face of δ, that touches $H_{\alpha,m}$. Then we definde:

If $m = -\langle \mu, \alpha^\vee \rangle$, we set $\tilde{f}_\alpha\delta = 0$. Otherwise we set:

$$\tilde{f}_\alpha\delta = (\mathfrak{o} \subset \ldots \supset E_{k-1} \supset V_k \subset s_{\alpha,m}E_k \supset t_{-\alpha}V_k \subset \ldots \supset t_{-\alpha}V_r \subset t_{-\alpha}E_r \supset \mu - \alpha).$$

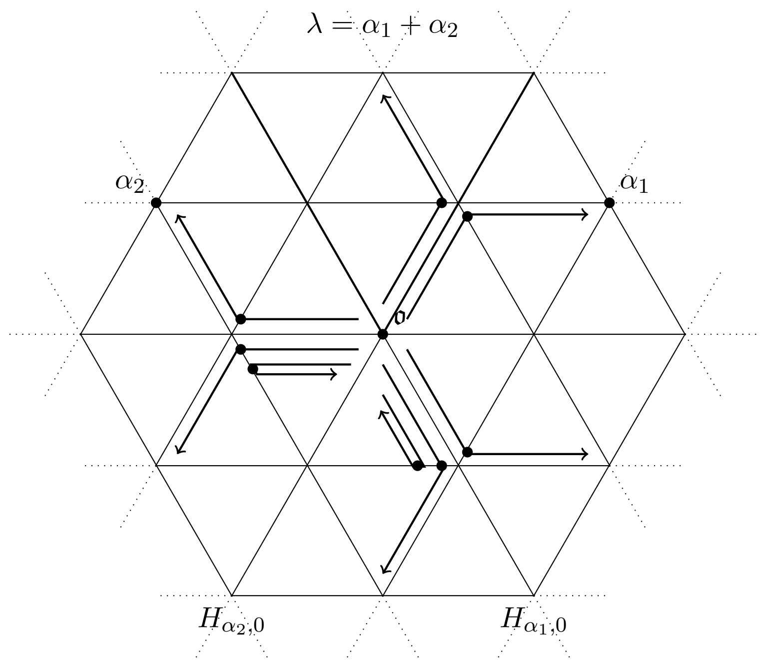

Figure 2.4: Example in the case A_2, LS-galleries of type $\alpha_1 + \alpha_2$

For an easy example see Figure 2.5. Considering the functions

$$wt : \Gamma(\gamma_\lambda) \to X^\vee$$
$$\delta \mapsto target(\delta),$$

$$\varepsilon_\alpha(\delta) = m,$$

and

$$\phi_\alpha(\delta) = m + \langle \mu, \alpha^\vee \rangle ,$$

it can be seen, that we have a crystal structure on $\Gamma(\gamma_\lambda)$.

2.6 Characterization of simply-laced crystals

In the following chapter, we will need a result from an article of John R. Stembride ([Ste03]), in which he describes simply-laced crystals.

We denote I a finite index set and $A = [a_{ij}]_{i,j\in I}$ the Cartan matrix of a simply-laced Kac-Moody algebra. In this case we have $a_{ii} = 2$ and $a_{ij} = a_{ji} \in \{0, -1\}$ for $i \neq j$. Stembridge associates to A a class of (possibly infinite) directed graphs, that he calls *A-regular*. He asks the graphs to be edge-colored, i.e. the edges will be labelled by elements from I.

Let $y = E_i(x)$ if there is an i-colored edge $x \leftarrow y$, and dually $z = F_i(x)$ if there is an i-colored edge $x \to z$. We define the *i-string through* x to be the maximal path of the form

$$F_i^{-d}(x) \to ... \to F_i^{-1}(x) \to x \to F_i(x) \to ... \to F_i^r(x) \ (r, d \geq 0).$$

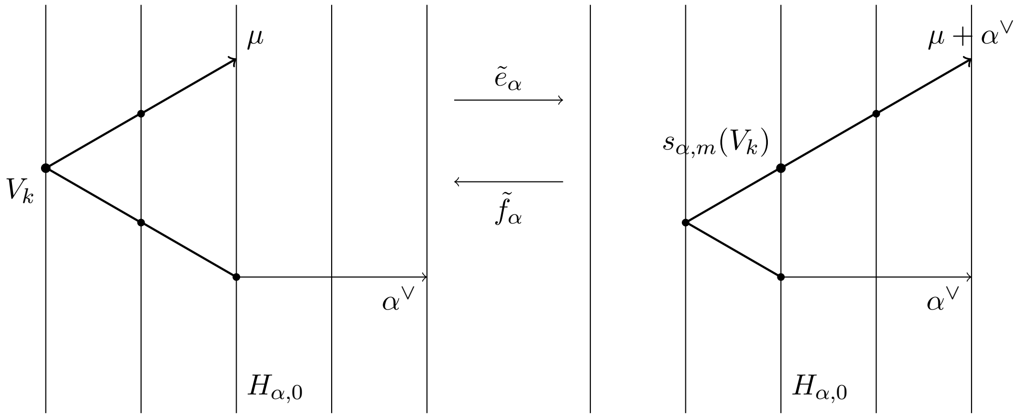

Figure 2.5: Example in the case A_2

We define the *i-rise* and i-depth to be $\varepsilon(x,i) := r$ and $\delta(x,i) := -d$. In addition, he defines difference operators Δ_i:

$$\Delta_i\delta(x,j) = \delta(E_ix,j) - \delta(x,j), \Delta_i\varepsilon(x,j) = \varepsilon(E_ix,j) - \varepsilon(x,j)$$

whenever E_ix ist defined (i.e. $\delta(x,i) < 0$), and analogously

$$\nabla_i\delta(x,j) = \delta(x,j) - \delta(F_ix,j), \nabla_i\varepsilon(x,j) = \varepsilon(x,j) - \varepsilon(F_ix,j).$$

Definition 2.6.1 ([Ste03], Def. 1.1) Let A be a simply-laced Cartan matrix. An edge-colored directed graph is A-regular if it satisfies:

(P1) All monochromatic directed paths in X have finite length. In particular, X has no monochromatic circuits.

(P2) For every vertex x and every $i \in I$, there is at most one edge $x \leftarrow y$ with color i, and dually, at most one edge $x \rightarrow y$ with color i.

(P3) $\Delta_i\delta(x,j) + \Delta_i\varepsilon(x,j) = a_{ij}$.

(P4) $\Delta_i\delta(x,j) \leq 0, \Delta_i\varepsilon(x,j) \leq 0$.

(P5) If E_ix and E_jx are defined: $\Delta_i\delta(x,j) = 0$ implies $E_iE_jx = E_jE_ix$ and $\nabla_j\varepsilon(y,i) = 0$, where $y = E_iE_jx = E_jE_ix$.

(P6) If E_ix and E_jx are defined: $\Delta_i\delta(x,j) = \Delta_j\delta(x,i) = -1$ implies $E_iE_j^2E_ix = E_jE_i^2E_jx$ and $\nabla_i\varepsilon(y,j) = \nabla_j\varepsilon(y,i) = -1$, where $y = E_iE_j^2E_ix = E_jE_i^2E_jx$.

(P5') If F_ix and F_j are defined: $\nabla_i\varepsilon(x,j) = 0$ implies $F_iF_jx = F_jF_ix$ and $\Delta_j\delta(y,i) = 0$, where $y = F_iF_jx = F_jF_ix$.

(P6') $\nabla_i\varepsilon(x,j) = \nabla_j\varepsilon(x,i) = -1$ implies $F_iF_j^2F_ix = F_jF_i^2F_jx$ and $\Delta_i\delta(x,j) = \Delta_j\delta(y,i) = -1$, where $y = F_iF_j^2F_ix = F_jF_i^2F_jx$.

Let us look at **(P3)** and **(P4)**. For $i \neq j$, we have $a_{ij} \in \{0,1\}$, therefore **(P3)** and **(P4)** only allow three possibilities:

$$(a_{ij}, \Delta_i \delta(x,j), \Delta_i \varepsilon(x,j)) = (0,0,0), (-1,-1,0), \text{ or } (-1,0,-1).$$

Let us assume for a vertex x: $\varepsilon(x,i) = 0 \; \forall j \in I, i \neq j$. Because of the observation above, we have

$$\varepsilon(E_i x, j) = \varepsilon(E_i x, j) - \varepsilon(E, j) = \Delta_i \varepsilon(x,j) \in \{0,-1\} \text{ for all } i \neq j.$$

But since $\varepsilon(E_i x, j) > 0$, we can only have

$$\varepsilon(E_i x, j) = 0 \text{ for all } i \neq j.$$

Now it is easy to observe, that if we look at the crystal obtained by applying the root operators to a gallery δ, which is completely contained in C^+, the associated crystal graph is A-regular, where A is a Cartan matrix of type A. Using the observation above, we proved:

Lemma 2.6.1 *If δ is a gallery, such that $\tilde{f}_{\alpha_j}\delta = 0$ for all $i \neq j$, we have $\tilde{f}_{\alpha_j}(\tilde{e}_{\alpha_i}\delta) = 0$ for all $i \neq j$.*

3 Bott-Samelson varieties

In this chapter we will provide a desingularization of an affine Schubert variety in form of a Bott-Samelson type variety consisting of all galleries of a given type. By fixing a generic anti-dominant coweight, we obtain a decomposition of these varieties in Bialynicki-Birula cells. We give a description of these cells and an upper bound for their dimensions ([GL11]).

3.1 A Bott-Samelson type variety

To every face in $\mathbb{A}$ we can associate a unique parabolic subgroup of the affine Kac-Moody group $\hat{\mathcal{L}}(G)$, which contains $\mathcal{T}$ and a unique parahoric subgroup in $G(\mathcal{K})$ which contains T. To any root vector $X_\alpha \in Lie(G)$ we can associate the one-parameter subgroup $U_\alpha = \{x_\alpha(f) = \exp(X_\alpha \otimes f) \mid f \in \mathcal{K}\}$ of $G(\mathcal{K})$ (resp. of $\hat{\mathcal{L}}(G)$). Analogously, to any real affine root $\alpha + n\delta$ we associate the one-parameter subgroup $U_{\alpha+n\delta} = \{x_\alpha(at^n) \mid a \in \mathbb{C}\}$. We will use both notations for the affine root morphism:

$$x_\alpha(at^n) = x_{(\alpha,n)}(a).$$

The following computation rules are used later in our proofs:

i) For all $\lambda^\vee \in X^\vee$, α a root, $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$ we have

$$a^\lambda x_\alpha(a) = x_\alpha(a^{\langle \alpha, \lambda^\vee \rangle} b) a^\lambda.$$

ii) For any root α and $a, b \in \mathbb{C}$ such that $1 + ab \neq 0$,

$$x_\alpha(a) x_{-\alpha}(b) = x_{-\alpha}(\frac{b}{1+ab})(1+ab)^{\alpha^\vee} x_\alpha(\frac{a}{1+ab}).$$

iii) For any positive root α and any $b \in \mathbb{C}^*$

$$x_\alpha(b) x_{-\alpha}(-b^{-1}) x_\alpha(b) = x_{-\alpha}(-b^{-1}) x_\alpha(b) x_{-\alpha}(-b^{-1}) = b^{\alpha^\vee} \overline{s_\alpha} = \overline{s_\alpha} b^{-\alpha^\vee}.$$

iv) For any roots $\alpha, \beta, \alpha \neq -\beta$ and $a, b \in \mathbb{C}$ we have:

$$x_\alpha(a) x_\beta(b) = \begin{cases} x_\beta(b) x_\alpha(a) & \text{if } \alpha + \beta \notin \Phi \\ x_{\alpha+\beta}(ab) x_\beta(b) x_\alpha(a) & \text{if } \alpha + \beta \in \Phi. \end{cases}$$

Note that an easy calculation shows, that iv) also holds in the affine case:

$$x_{(\alpha,n)}(a) x_{(\beta,m)}(b) = \begin{cases} x_{(\beta,m)}(b) x_{(\alpha,n)}(a) & \text{if } \alpha + \beta \notin \Phi \\ x_{(\alpha+\beta,n+m)}(ab) x_{(\beta,m)}(b) x_{(\alpha,n)}(a) & \text{if } \alpha + \beta \in \Phi \end{cases}$$

Definition 3.1.1 ([GL11], Def. 3) For a face F we define $\hat{P}_F$ as the *unique parabolic subgroup* of $\hat{\mathcal{L}}(G)$ containing $\mathcal{T}$ and all root subgroups $U_{\alpha+n\delta}$ such that $F \subset H^+_{\alpha,n}$. We let U_F be the subgroup of $G(\mathcal{K})$ generated by all $x_\alpha(f)$ such that $f \in \mathcal{K}^*$,$v(f) \geq n$ and $F \subset H^+_{\alpha,n}$ and define P_F to be the *unique parahoric subgroup* of $G(\mathcal{K})$ containing T and U_F.

As we did in chapter 1, we extend the notion of a parahoric subgroup P_F associated to a face F to the affine building $\mathcal{J}^{\mathfrak{a}}$. The action of $G(\mathcal{K})$ is such that $U_{\alpha+n\delta}$ fixes the halfspace $H^+_{\alpha,n}$. It is easy to see, that $x_\alpha(at^n)$ belongs to U_x, therefore we have $(x_\alpha(at^n), x) \sim (1, x)$.

The map $r_{-\infty} : \mathcal{J}^{\mathfrak{a}} \to \mathbb{A}$ denotes the retraction centered at $-\infty$. It is a chamber complex map and its fibers are exactly the $U^-(\mathcal{K})$-orbits in $\mathcal{J}^{\mathfrak{a}}$ (see [GL05] for further information).

To any combinatorial one-skeleton gallery $\gamma = (V_0 = \mathfrak{o} \subset E_0 \supset V_1 \subset ... \supset V_r \subset E_r \supset V_{r+1})$ we can associate a sequence of parahoric subgroups

$$G(\mathcal{O}) \supset P_{E_0^f} \subset P_{V_1^f} \supset ... \subset P_{V_r^f} \supset P_{E_r^f} \subset P_{V_{r+1}^f}.$$

Using this correspondence we can identify one-skeleton galleries with points in Bott-Samelson varieties.

Definition 3.1.2 ([GL11], Def. 8) Let $\Sigma(t_\gamma)$ be the closed subvariety of

$$G(\mathcal{K})/G(\mathcal{O}) \times G(\mathcal{K})/P_{E_0^f} \times ... \times G(\mathcal{K})/P_{E_r^f} \times G(\mathcal{K})/P_{V_{r+1}^f}$$

given by sequences of parahoric subgroups of the form

$$G(\mathcal{O}) \supset Q_0 \subset R_1 \supset Q_1 \subset ... \supset Q_r \subset R_{r+1},$$

where R_i is conjugate to $P_{V_i^f}$ for $i \in \{1, ..., r+1\}$ and Q_i is conjugate to $P_{E_i^f}$ for $i = 0, ..., r$. We call it the variety of galleries of type t_γ starting in $V_0 = \mathfrak{o}$.

We can naturally extend the action of $G(\mathcal{K})$ on $\mathcal{J}^{\mathfrak{a}}$ to an action on the set of galleries. This action is type preserving and $\Sigma(t_\gamma)$ is stable under the action of $G(\mathcal{O})$. Since there is a bijection between faces of $\mathcal{J}^{\mathfrak{a}}$ and parahoric subgroups, we have a bijection between the points in $\Sigma(t_\gamma)$ and the one-skeleton galleries in $\mathcal{J}^{\mathfrak{a}}$ of type t_λ. The combinatorial galleries, which are those galleries in $\mathcal{J}^{\mathfrak{a}}$ included in $\mathbb{A}$, correspond to sequences of subgroups who are conjugated to the $P_{E_i^f}$'s and $P_{V_i^f}$'s by elements in the affine Weyl group $W^{\mathfrak{a}}$. These are precisely the T-fixed points in $\Sigma(t_\gamma)$. To a sequence of parahoric subgroups

$$G(\mathcal{O}) \supset P_{E_0^f} \subset P_{V_1^f} \supset P_{E_1^f} \subset ... \supset P_{E_r^f} \subset P_{V_{r+1}^f}$$

we associate the fibred product

$$G(\mathcal{O}) \times_{P_{E_0^f}} P_{V_1^f} \times_{P_{E_1^f}} ... \times_{P_{E_{r-1}^f}} P_{V_r^f}/P_{E_r^f},$$

which is defined as the quotient of $P_{V_0^f} \times ... \times P_{V_r^f}$ by $P_{E_0^f} \times ... \times P_{E_r^f}$ given by the action

$$(p_0, ..., p_r) \cdot (q_0, ..., q_r) = (q_0 p_0, p_0^{-1} q_1 p_1, ..., p_{r-1}^{-1} q_r p_r).$$

Proposition 3.1.1 ([GL11], Prop. 2) *As a variety, $\Sigma(t_\gamma)$ is isomorphic to the fibred product via the map*

$$(g_0, ..., g_r) \mapsto$$

$$(P_{V_0} \supset g_0 P_{E_0} g_0^{-1} \subset g_0 P_{V_1} g_0^{-1} \supset g_0 g_1 P_{E_1}(g_0 g_1)^{-1} \subset ... \subset g_0 ... g_r P_{V_{r+1}}(g_0 ... g_r)^{-1}).$$

To a dominant coweight λ we associate the gallery $\gamma_\lambda = (\mathfrak{o} \subset E_0 \supset ... \subset E_r \supset \lambda)$ described above. The variety of galleries of type t_{γ_λ} starting in $\mathfrak{o}$ is called the Bott-Samelson variety associated to the gallery γ_λ and we denote it by:

$$\Sigma(\gamma_\lambda) := G(\mathcal{O}) \times_{P_{E_0^f}} P_{V_1^f} \times_{P_{E_1^f}} ... \times_{P_{E_{r-2}^f}} P_{V_{r-1}^f} \times_{P_{E_{r-1}^f}} P_{V_r^f}/P_{E_r^f}.$$

The combinatorial galleries in $\Sigma(\gamma_\lambda)$ will be denoted by $\Gamma(\gamma_\lambda)$.

Proposition 3.1.2 ([GL11],Prop. 3) *We denote by λ_f the point in $\mathcal{G}$ corresponding to the vertex of the fundamental alcove of the same type as λ. The canonical product map*

$$\pi : \Sigma(\gamma_\lambda) := G(\mathcal{O}) \times_{P_{E_0^f}} P_{V_1} \times_{P_{E_1^f}} ... \times_{P_{E_{r-1}^f}} P_{V_r^f}/P_{E_r^f} \to \mathcal{G}$$

$$[g_0, ..., g_r] \mapsto g_0 ... g_r t^{\lambda^f} G(\mathcal{O})$$

has as image the Schubert variety X_λ. The induced map $\pi : \Sigma(\gamma_\lambda) \to X_\lambda$ defines a desingularization of the variety X_λ.

3.2 Cells

Let us fix a generic anti-dominant coweight $\eta : \mathbb{C}^* \to T$. As already mentioned, the set of η-fixed points in $\Sigma(\gamma_\lambda)$ is finite and in bijection with the set of all combinatorial galleries of the same type as γ_λ. For such a gallery γ we denote by C_γ the corresponding Bialynicki-Birula cell, i.e.

$$C_\gamma = \left\{ x \in \Sigma(\gamma_\lambda) \mid \lim_{t \to 0} \eta(t).x = \gamma \right\}.$$

Bruhat and Tits associate to a face F of the Coxeter complex the function

$$f_F : \alpha \mapsto \inf_{k \in \mathbb{Z}} \{\alpha(F) + k \geq 0\}.$$

For any $\alpha \in \Phi$ is $f_F(\alpha)$ the smallest integer such that $F \subset H^+_{\alpha,n}$. For two faces F, V in $\mathbb{A}$, such that $V \subset F$, we denote by $\Phi^{\mathfrak{a}}_-(V, F)$ the set of all affine roots $(\alpha, n), \alpha \in \Phi^-, n \in \mathbb{Z}$, such that $V \in H_{\alpha,n}$ and $F \not\subset H^+_{\alpha,n}$. Let $Stab_-(V, F)$ be the subgroup of $U^-(\mathcal{K})$ generated by the elements of the form $x_\alpha(at^n)$ with $(\alpha, n) \in \Phi^{\mathfrak{a}}_-(V, F)$ and $a \in \mathbb{C}$.

Proposition 3.2.1 ([GL11],Prop. 4) *1. The stabilizer $Stab_-(F)$ of a face of the Coxeter complex is generated by the elements $x_\alpha(p)$ where $\alpha \in \Phi^-$ and $p \in \mathcal{K}$ such that $val(p) \geq f_F(\alpha)$.*

2. *Let F and V be two faces of the Coxeter complex such that $V \subset F$. Then $Stab_-(V,F)$ is a set of representatives for the right cosets of $Stab_-(F)$ in $Stab_-(V)$. For any total order on the set $\Phi^{\mathfrak{a}}_-(V,F)$, the map*

$$(a_\beta)_{\beta\in\Phi^{\mathfrak{a}}_-(V,F)} \mapsto \prod_{\beta\in\Phi^{\mathfrak{a}}_-(V,F)} x_\beta(a_\beta)$$

is a bijection from $C^{\Phi^{\mathfrak{a}}_-(V,F)}$ onto $Stab_-(V,F)$, where $C^{\Phi^{\mathfrak{a}}_-(V,F)}$ is the set of all mappings from $\Phi^{\mathfrak{a}}_-(V,F)$ to $\mathbb{C}$.

Let $\delta = [\delta_0, ..., \delta_r] = (0 = V_0 \subset E_0 \supset V_1 \subset ... \supset V_r \subset E_r \supset V_{r+1}) \in \Gamma(\gamma_\lambda)$, we set

$$Stab_-(\delta) = Stab_-(V_0, E_0) \times ... \times Stab_-(V_r, E_r).$$

Proposition 3.2.2 ([GL11],Prop.6) *The map*

$$f : (v_0, ..., v_r) \mapsto [v_0\overline{\delta_0}, \overline{\delta_0}^{-1} v_1\overline{\delta_0\delta_1}, [\overline{\delta_0\delta_1}^{-1} v_2\overline{\delta_0\delta_1\delta_2}, ..., \overline{\delta_0...\delta_{r-1}}^{-1} v_r\overline{\delta_0...\delta_r}]$$

from $Stab_-(\delta)$ to $\Sigma(\gamma_\lambda)$ is injective and its image is C_δ (by $\overline{x}$ we denote a coset representative of x in $G(\mathcal{K})$). Therefore, C_δ is isomorphic to $\mathbb{C}^{\Phi^{\mathfrak{a}}_-(V_0,E_0)} \times ... \times \mathbb{C}^{\Phi^{\mathfrak{a}}_-(V_r,E_r)}$.

3.3 Dimension of C_δ

We remember $\Phi^{\mathfrak{a}}_-(V,E)$ being defined as the set of affine roots $(\alpha, n) \in \Phi_- \times \mathbb{Z}$, such that $V \in H_{\alpha,n}$ and $E \not\subset H^+_{\alpha,n}$. For an affine root $(\alpha, n) \in \Phi^- \times \mathbb{N}$ we will say that (V,E) crosses the wall $H_{\alpha,n}$ in the positive (negative) direction if $F \not\subset H^-_{\alpha,n}$ (respectively $F \not\subset H^+_{\alpha,n}$), i.e. a wall crossing is positive if $(-\alpha, -n) \in \Phi^{\mathfrak{a}}_-(V,F)$.

Let us denote for a gallery δ the number of positive wall crossings appearing in that gallery by $\sharp^+\delta$, analogously the number of negative wall crossings by $\sharp^-\delta$ and the sum of both by $\sharp^\pm\delta$:

$$\sharp^+\delta = \sum_{i=0}^{r} (\sharp \text{ positive wall crossings of } (V_i, E_i))$$
$$\sharp^-\delta = \sum_{i=0}^{r} (\sharp \text{ negative wall crossings of } (V_i, E_i))$$
$$\sharp^\pm\delta = \sharp^+\delta + \sharp^-\delta.$$

It is easy to see, that $\sharp^\pm\delta = \langle\lambda, 2\rho\rangle$.

Lemma 3.3.1 ([GL11], Lemma 9) $\sharp^+\delta = \dim(C_\delta)$.

There is an upper bound for the dimension of a cell C_δ:

Proposition 3.3.1 ([GL11],Prop. 16) *Let μ be the target of δ, we have $\sharp^+\delta \leq \langle\lambda + \mu, \rho\rangle$.*

Important for us is the following ([GL11]): Let

$$Z_{\lambda,\mu} = G(\mathcal{O})t^\lambda \cap U^-(\mathcal{K})t^\mu$$

for a dominant coweight λ and an arbitrary coweight μ. If δ is an LS-gallery of type γ_λ with target μ, then $\overline{Z_{\lambda,\mu} \cap C_\delta}$ is an MV-cycle of coweight (λ, μ).

4 Cells in type A

Gaussent and Littelmann determine an upper bound for the dimension of the cells C_δ. In the first part of this chapter, we provide the exact dimension of these cells in type A. In the second part, we will give a description of the image of these cells by the desingularization map π in type A. In a particular case, we can prove, that the closure of this image is an MV-cycle.

4.1 Cell dimension

Let G be of type A_n. We have seen in chapter 3, that there is an upper bound for the dimension of the cell C_δ associated to a gallery δ. In this section we want to show, that in the case of G being of type A_n, we even have equality, i.e. $\dim(C_\delta) = \langle \lambda + \mu, \rho \rangle$.

The fundamental weights are of the form $\omega_i = \varepsilon_1 + ... + \varepsilon_i$, we denote the simple roots $\alpha_1, ..., \alpha_n$. By $F_{\varepsilon_{i_1}+...\varepsilon_{i_k}}$ resp. $(0 \subset F_{\varepsilon_{i_1}+...\varepsilon_{i_k}})$ we will denote the face going from 0 to $\varepsilon_{i_1} + ... + \varepsilon_{i_k}$. By abuse of notation, we will denote by $F_{\varepsilon_{i_1}+...\varepsilon_{i_k}}$ or $(V \subset F_{\varepsilon_{i_1}+...+\varepsilon_{i_k}})$ the face going from the vertex V to $V + \varepsilon_{i_1} + ... + \varepsilon_{i_k}$. The context will make it clear, which face we mean. We want to determine the number of positive wall crossings of a gallery of the form $V_0 \subset E_0 \supset V_1$. In the case of $V_0 = 0$, these galleries are all of the form:

$$(0 \subset F_{\varepsilon_{i_1}+...+\varepsilon_{i_k}} \supset \varepsilon_{i_1} + ... + \varepsilon_{i_k}).$$

Lemma 4.1.1 *We have*

$$\left|\Phi^{\mathfrak{a}}_{-}(0, F_{\varepsilon_k})\right| = n - k + 1.$$

Proof. To determine the number of positive wall crossings for $(0 \subset F_{\varepsilon_k})$ it is enough to determine the number of negative roots α, such that $\langle \alpha, \varepsilon_k \rangle < 0$. All the negative roots in type A_n are given by

$$-\alpha_i, -\alpha_i - \alpha_{i+1}, ..., -\alpha_i - ... - \alpha_n \text{ for } i = 1, ..., n.$$

Since $\varepsilon_i = \omega_i - \omega_{i-1}$, $\langle \alpha, \varepsilon_k \rangle$ is negative for exactly the following negative roots:

$$\left\{-\alpha_k, -\alpha_k - \alpha_{k+1}, ..., -\alpha_k - ... - \alpha_n\right\},$$

we get $\left|\Phi^{\mathfrak{a}}_{-}(0, F_{\varepsilon_k})\right| = n - k + 1.$ $\square$

Let us calculate the positive wall crossings of $(0 \subset F_{\varepsilon_{i_1}+...+\varepsilon_{i_k}}), i_1 < ... < i_k$. We want to compare the number of positive wall crossings of $(0 \subset F_{\varepsilon_{i_1}}), ..., (0 \subset F_{\varepsilon_{i_k}})$ to the number of positive wall crossings of $(0 \subset F_{\varepsilon_{i_1}+...+\varepsilon_{i_k}})$.

Lemma 4.1.2 *We have*

$$\left|\Phi_-^{\mathfrak{a}}(0, F_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_k}})\right| = \sum_{j=1}^{k}\left|\Phi_-^{\mathfrak{a}}(0, F_{\varepsilon_{i_j}})\right| - \frac{k(k-1)}{2}.$$

Proof. As above, we count the number of negative roots α, such that

$$\langle \alpha, \varepsilon_{i_1} + \ldots + \varepsilon_{i_k} \rangle < 0.$$

It is easy to see, that these are exactly the following roots:

- $-\alpha_{i_1}, \alpha_{i_1} - \alpha_{i_1+1}, \ldots, -\alpha_{i_1} - \ldots - \alpha_{i_2-2}$ (if $i_2 - 2 \geq i_1$),
  $-\alpha_{i_1} - \ldots - \alpha_{i_2}, \ldots, -\alpha_{i_1} - \ldots - \alpha_{i_3-2}$ (if $i_3 - 2 \geq i_2$),
  $\ldots,$
  $-\alpha_{i_1} - \ldots - \alpha_{i_k}, \ldots, -\alpha_{i_1} - \ldots - \alpha_n$ (if $n - i_k \geq 0$)
- $\ldots$
- $-\alpha_{i_{k-1}}, -\alpha_{i_{k-1}} - \alpha_{i_{k-1}+1}, \ldots, -\alpha_{i_{k-1}} - \ldots - \alpha_{i_k-2}$ (if $i_k - 2 \geq i_{k-1}$),
  $\ldots,$
  $-\alpha_{i_{k-1}} - \ldots - \alpha_{i_k}, \ldots, -\alpha_{i_{k-1}} - \ldots - \alpha_n$ (if $n - i_k \geq 0$)
- $-\alpha_{i_k}, -\alpha_{i_k} - \alpha_{i_{k+1}}, \ldots, -\alpha_{i_k} - \ldots - \alpha_n$ (if $n - i_k \geq 0$).

If we have $i_j - 2 \ngeq i_{j-1}$, then $\Phi_-^{\mathfrak{a}}(0, F_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_k}})$ will not contain any root of the form $(-\alpha_{i_j} - \ldots, 0)$. An easy calculation gives us:

$$\begin{aligned}
&\Phi_-^{\mathfrak{a}}(0, F_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_k}}) \\
&= n - i_k + 1 + \sum_{j=1}^{k-1}((i_{j+1} - i_j - 1) + \ldots + (i_k - i_{k-1} - 1) + (n - i_k + 1)) \\
&= -\sum_{j=1}^{k} i_j - \frac{k(k-1)}{2} + kn + k.
\end{aligned}$$

By Lemma 4.1.1, we get

$$\sum_{j=1}^{k} \Phi_-^{\mathfrak{a}}(0, F_{\varepsilon_{i_j}}) = -\sum_{j=1}^{k} i_j + kn + k.$$

□

Lemma 4.1.3 *The faces $(0 \subset F_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_k}})$ and $(V \subset F_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_k}})$, for any vertex V in $\mathbb{A}$, have the same number of positive wall crossings. More precisely, the negative roots α appearing in $(\alpha, 0) \in \Phi_-^{\mathfrak{a}}(0, F_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_k}})$ are the same as in $\Phi_-^{\mathfrak{a}}(V, F_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_k}})$.*

Proof. For an affine root $(\alpha, 0) \in \Phi_-^{\mathfrak{a}}(0, F)$, we have

$$\begin{aligned}
F \subset H_{\alpha,n}^- &\Rightarrow \alpha(F) \leq 0 \\
&\Rightarrow \alpha(F) + \alpha(V) + (-\alpha(V)) \leq 0 \\
&\Rightarrow \alpha(V + F) + (-\alpha(V)) \leq 0.
\end{aligned}$$

But this means $F \subset H_{\alpha,-\alpha(V)}^- \Rightarrow (\alpha, -\alpha(V)) \subset \Phi_-^{\mathfrak{a}}(V, F)$, since in type A_n every vertex is special. □

Let λ be a dominant weight. Let δ be a gallery of the same type as $\gamma_{\underline{\lambda}}$ and μ be the target of δ. We have the following Lemma:

Lemma 4.1.4 *We have*

$$\dim(C_\delta) = \sharp^+\delta = \langle \lambda + \mu, \rho^\vee \rangle .$$

Proof. We have $\varepsilon_i = \omega_i - \omega_{i-1}, i = 1, ..., n$. Since $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$, we get

$$\langle \varepsilon_i, \rho \rangle = \langle \omega_i - \omega_{i-1}, \rho \rangle = \frac{n - 2i + 2}{2}.$$

Write

$$\lambda = \sum_{i=1}^{n} a_i \omega_i = \sum_{i=1}^{n} (a_i \sum_{j=1}^{i} \varepsilon_j).$$

The target μ of δ can be written as

$$\mu = \sum_{i=1}^{n} \sum_{j=1}^{a_i} (\varepsilon_{l_{i,j,1}} + ... + \varepsilon_{l_{i,j,i}}),$$

where $l_{i,j,1} < ... < l_{i,j,i}$ and a_i is the number of columns of length i. An easy calculation gives us:

$$\begin{aligned}
\langle \lambda + \mu, \rho \rangle &= \left\langle \sum_{i=1}^{n} (a_i \sum_{k=1}^{i} \varepsilon_j), \rho \right\rangle + \left\langle \sum_{i=1}^{n} \sum_{j=1}^{a_i} \varepsilon_{l_{i,j,1}} + ... + \varepsilon_{l_{i,j,i}}, \rho \right\rangle \\
&= \sum_{i=1}^{n} \left(a_i (\sum_{k=1}^{i} \frac{n - 2k + 2}{2}) + \sum_{j=1}^{a_i} (\frac{n - 2l_{i,j,1} + 2}{2} + ... + \frac{n - 2l_{i,j,i} + 2}{2}) \right) \\
&= \sum_{i=1}^{n} \left(a_i (ni + 2i - \frac{i(i+1)}{2}) - \sum_{j=1}^{a_i} (l_{i,j,1} + ... + l_{i,j,i}) \right) .
\end{aligned}$$

By using Lemma 4.1.1 and Lemma 4.1.2, we obtain the following:

$$\begin{aligned}
\sharp^+\delta &= \sum_{i=1}^{n} \sum_{j=1}^{a_i} \Phi^{\mathfrak{a}}_{-}(0, F_{\varepsilon_{l_{i,j,1}} + ... + \varepsilon_{l_{i,j,i}}}) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{a_i} (\Phi^{\mathfrak{a}}_{-}(0, F_{\varepsilon_{l_{i,j,1}}}) + ... + \Phi^{\mathfrak{a}}_{-}(0, F_{\varepsilon_{l_{i,j,i}}}) - \frac{(i-1)i}{2}) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{a_i} ((n - l_{i,j,1} + 1) + ... + (n - l_{i,j,i} + 1) - \frac{(i-1)i}{2}) \\
&= \sum_{i=1}^{n} \left(a_i (ni + 2i - \frac{i(i+1)}{2}) - \sum_{j=1}^{a_i} (l_{i,j,1} + ... + l_{i,j,i}) \right) \\
&= \langle \lambda + \mu, \rho \rangle .
\end{aligned}$$

□

4.2 Cells and MV-cycles I

Let $\pi : \Sigma_\gamma \to X_\lambda$ be the map from the Bott-Samelson variety onto the affine Schubert variety. We are now interested in the image of a cell $C_\delta \subset \Sigma_\gamma$ under this map. We want to give a partial answer to the question, in which cases and for which coweights the image $\overline{\pi(C_\delta)}$ is an MV-cycle in a possibly smaller Schubert variety.

Let

$$\delta = (V_0 \subset E_0 \supset V_1 \subset E_1 \supset ... \subset E_r \supset \mu)$$

be a gallery of the same type as γ_λ with target μ.

Lemma 4.2.1 *Let $\alpha \in \Phi^-$, such that $\mu \in H^-_{\alpha,0}$. For every $0 \leq i < -\langle \mu, \alpha \rangle$ there exist a $j \in \{0, ..., r\}$, such that $(\alpha, i) \in \Phi^{\mathfrak{a}}_-(V_j, E_j)$. In other words, if we consider a negative root α and if the target of the gallery is on the negative side of $H_{\alpha,0}$, the gallery will cross every wall between $H_{\alpha,0}$ and $H_{\alpha,-\langle\mu,\alpha\rangle}$ positively.*

Proof. Let $\alpha = -\alpha_k - ... - \alpha_l$. Let us assume the gallery δ has no positive wall crossing at the wall $H_{\alpha,i}$ for any $0 \leq i < -\langle \mu, \alpha \rangle$. In this case, every edge in δ, up to translation, must be of the form

$$F_{...+\varepsilon_k+...+\varepsilon_{l+1}+...}$$

or

$$F_{...+\widehat{\varepsilon_k}+...}$$

Let $\mu = \sum_{i=1}^{n+1} a_i\varepsilon_i$. Since $\mu \in H^-_{\alpha,0}$, we must have $a_{l+1} \leq a_k$, in contradiction to the above form of δ. ∎

The first case, in which we can give an answer to our question, if $\overline{\pi(C_\delta)}$ is an MV-cycle, is the case of δ being completely included in the dominant Weyl chamber:

Lemma 4.2.2 *If we have $\delta \subset C^+$, we get*

$$\pi(C_\delta) = U^-(\mathcal{K})t^\mu \cap G(\mathcal{O})t^\mu.$$

Proof. The image of the cell C_δ can be described as:

$$\pi(C_\delta) = \left\{ \prod_{i=0}^{r} \left(\prod_{(\alpha,n)\in\Phi^{\mathfrak{a}}_-(V_i,E_i)} x_{(\alpha,n)}(a(i,\alpha,n)) \right) t^\mu G(\mathcal{O}) \mid a(i,\alpha,n) \in \mathbb{C} \right\}$$
$$\subset U^-(\mathcal{K})t^\mu$$

For every $(\alpha, n) \in \Phi^{\mathfrak{a}}_-(V_i, E_i)$ we have $E_i \not\subset H^+_{(\alpha,n)}$ and since $\delta \subset C^+$, n must be non negative. Therefore we have $\pi(C_\delta) \subset G(\mathcal{O})t^\mu \cap U^-(\mathcal{K})t^\mu$. Let $x \in G(\mathcal{O})t^\mu \cap U^-(\mathcal{K})t^\mu = U^-(\mathcal{O})t^\mu$. Then x will be of the form:

$$x = \prod_{\alpha\in\Phi^-} x_\alpha(f_\alpha)t^\mu$$

with $f_\alpha \in \mathcal{O}$. Using the relation

$$x_\alpha(f_\alpha)t^\mu G(\mathcal{O}) = t^\mu x_\alpha(t^{\langle\mu,\alpha\rangle} f_\alpha)G(\mathcal{O})$$

we get $x_\alpha(f_\alpha)t^\mu G(\mathcal{O}) = t^\mu G(\mathcal{O})$ if $\deg(f_\alpha) \geq -\langle \mu, \alpha \rangle$. Using Lemma 4.2.1 and since the maximal n, such that $(\alpha, n) \in \Phi^{\mathfrak{a}}_-(E_j, V_j)$ for any j is at least $-\langle \mu, \alpha \rangle$, we get

$$G(\mathcal{O})t^\mu \cap U^-(\mathcal{K})t^\mu \subset \pi(C_\delta).$$

□

Obviously, $\mathcal{G}_\mu \cap S_\mu \cap \overline{\pi(C_\delta)}$ is dense in $\overline{\pi(C_\delta)}$ and we have the following corollary:

Corollary 4.2.1 *If we have $\delta \subset C^+$ and μ is the target of δ then $\pi(\overline{C_\delta})$ is a MV-Cycle of coweight (μ, μ).*

We will consider the crystal of galleries of type γ_λ and start at the "bottom" of the graph. Let δ be a gallery of type γ_λ, such that for every $i \in \{1, ..., n\}$ we have $\tilde{f}_{\alpha_i}\delta = 0$. We want to determine $\pi(C_\delta)$.

Lemma 4.2.3 *In this case, for a wall $H_{\alpha,n}$ with α a negative root, which is positively crossed by δ, we have $\mu \in H^+_{\alpha,n}$.*

Proof. Let $m_\alpha = \max\{m \mid \exists V_i : V_i \in H_{\alpha,m}\}$. Since we have $\tilde{f}_{\alpha_i}\delta = 0 \Leftrightarrow m_\alpha = -\langle \alpha_i, \mu \rangle$, the statement is clear for α being a simple root. Now let the wall

$$H_{-\alpha_i - ... - \alpha_j, n}$$

be positively crossed by δ, with $i < j$. In this case, there exists $V_i \subset E_i$ in δ such that $V_i \in H_{-\alpha_i-...-\alpha_j,n}$ and $E_i \subset H^-_{-\alpha_i-...-\alpha_j,n}$.

Let us assume $\mu \in H^-_{-\alpha_i-...-\alpha_j,n}$, i.e. $(-\alpha_i - ... - \alpha_j)(\mu) + n \leq 0$. In this case, either $-\alpha_i(\mu) \leq 0$ or $(-\alpha_{i+1} - ... - \alpha_j)(\mu) + n \leq 0$. In the case of $\mu \in H^-_{-\alpha_i,0}$, by Lemma 4.2.1, we would have a wall $H_{-\alpha_i,k}$, which is positively crossed, but with $\mu \in H^-_{-\alpha_i,k}$, in contradiction to our above statement. Iteratively, we can prove our statement in the general case. □

Lemma 4.2.4 *In the special case of δ being such that $\tilde{f}_{\alpha_i}\delta = 0$ for every $i \in \{i, ..., n\}$, we have $\pi(C_\delta) = \{t^\mu G(\mathcal{O})\}$, where $\pi(\delta) = \mu$.*

Proof. According to the lemma above, for every wall $H_{\alpha,m}$, which is positively crossed by δ, we have $\mu \in H^+_{\alpha,m}$, i.e. $\alpha(\mu) + m \geq 0$. If we look at the associated root subgroup $x_\alpha(at^m), a \in \mathbb{C}$. We have

$$x_\alpha(at^m)t^\mu G(\mathcal{O}) = t^\mu x_\alpha(at^{\langle \alpha,\mu \rangle + m})G(\mathcal{O}) = t^\mu G(\mathcal{O}).$$

Therefore we have

$$\begin{aligned}\pi(C_\delta) &= \left\{ \left(\prod_{i=1}^{r} (\prod_{(\alpha,n) \in \Phi^{\mathfrak{a}}_-(V_i,E_i)} x_\alpha(a_{\alpha,i}t^n)) \right) t^\mu G(\mathcal{O}) \mid a_{\alpha,i} \in \mathbb{C} \right\} \\ &= \{t^\mu G(\mathcal{O})\}.\end{aligned}$$

□

Starting at δ with $\tilde{f}_{\alpha_i}\delta = 0$ for $i = 1, ..., n$, we want to look at $\tilde{e}^k_{\alpha_i}\delta$ for $k \geq 0$. We want to give an explicit description of the set $\pi(C_\delta)$.

Lemma 4.2.5 *If $\tilde{e}^k_{\alpha_i}\delta$ is defined, we have:*

$$\pi(C_{\tilde{e}^k_{\alpha_i}\delta}) =$$
$$\Bigg\{\left(\prod_{i_1=1}^{r_1} x_{\beta_{i_1}}(b_{i_1,1}t^{m_{i_1,1}})\right) x_{-\alpha_i}(a_1 t^{m_1})\cdot \ldots \cdot \left(\prod_{i_k=1}^{r_k} x_{\beta_{i_k}}(b_{i_k,k}t^{m_{i_k,k}})\right)$$
$$x_{-\alpha_i}(a_i t^{m_k}) t^{\mu+k\alpha_i} G(\mathcal{O}) \mid b_{i_j,j}, a_j \in \mathbb{C}, r_j \in \mathbb{N}\Bigg\},$$

and we have $x_{\beta_{i_j}}(b_{i_j,j}t^{m_{i_j,j}}) \in Stab_-(t^{\mu+k\alpha_i})$.

Proof. Remember that $Stab_-(F)$ for a face F of the Coxeter complex is generated by the elements $x_\alpha(p)$, where $\alpha \in \Phi_-$ and $p \in \mathcal{K}$ satisfying $val(p) \geq f_F(\alpha)$. Since $f_\mu(\alpha) = -\langle \mu, \alpha\rangle$, $Stab_-(\mu)$ is generated by $x_\alpha(at^m)$ with $m \geq -\langle \mu, \alpha\rangle$.

Let us first look at the application of one root operator. The image of the cell C_δ is

$$\pi(C_\delta) = \left\{\prod_{i=1}^{r} \prod_{(\alpha,n)\in\Phi^{\mathfrak{a}}_-(V_i,E_i)} x_\alpha(a_{i,\alpha,n}t^n)t^\mu G(\mathcal{O}) \mid a_{i,\alpha,n} \in \mathbb{C}\right\},$$

where every $x_\alpha(a_{i,\alpha,n})$ is in the stabilizer of $t^\mu G(\mathcal{O})$,

$$Stab_-(t^\mu) = \langle x_\beta(at^n) \mid \beta \prec 0, n \geq -\langle \mu, \beta\rangle, a \in \mathbb{C}\rangle,$$

as we have just seen in Lemma 4.2.4. Let us see what happens, if we apply the root operator $\tilde{e}_{\alpha_i}$ just once. Let $\tilde{e}_{\alpha_i}\delta \neq 0$, m and k be as in the definition of the root operator. We get

$$\pi(C_{\tilde{e}_{\alpha_i}\delta}) = \Bigg\{\left(\prod_{i=1}^{k-2} \prod_{(\alpha,n)\in\Phi^{\mathfrak{a}}_-(V_i,E_i)} x_\alpha(a_{i,\alpha,n}t^n)\right) x_{-\alpha_i}(ct^{m-1})$$
$$\prod_{(\alpha,n)\in s_{\alpha_i,m-1}(\Phi^{\mathfrak{a}}_-(V_{k-1},E_{k-1}))} x_\alpha(a_{k-1,\alpha,n}t^n)$$
$$\left(\prod_{i=k}^{r} \prod_{(\alpha,n)\in\tau_{\alpha_i^\vee}(\Phi^{\mathfrak{a}}_-(V_i,E_i))} x_\alpha(a_{i,\alpha,n}t^n)\right) t^{\mu+\alpha_i}G(\mathcal{O}) \mid a_{i,\alpha,n}, c \in \mathbb{C}\Bigg\},$$

where

$$\tau_{\alpha_i^\vee}(\Phi^{\mathfrak{a}}_-(V_i,E_i)) = \left\{(\alpha, n + \langle \alpha, \alpha_i^\vee\rangle) \mid (\alpha,n) \in \Phi^{\mathfrak{a}}_-(V_i,E_i)\right\}.$$

Note that we have $m \geq 1$, since $\tilde{e}_{\alpha_i}$ is applicable at least once to δ. Since we have

$$x_\alpha(a_{i,\alpha,n}t^n) \in Stab_-(t^\mu) \text{ for every } (\alpha,n) \in \Phi^{\mathfrak{a}}_-(V_i,E_i), i = 1,\ldots,r,$$

it is easy to see that

$$x_\alpha(a_{i,\alpha,n}t^n) \in Stab_-(t^{\mu+\alpha_i}) \text{ for every } (\alpha,n) \in \tau_{\alpha_i^\vee}(\Phi^{\mathfrak{a}}_-(V_i,E_i)), i = k,\ldots,r$$

and

$$x_\alpha(a_{i,\alpha,n}t^n) \in Stab_-(t^{\mu+\alpha_i}) \text{ for every } (\alpha,n) \in s_{\alpha_i,m-1}(\Phi^{\mathfrak{a}}_-(V_{k-1},E_{k-1})).$$

Let us consider $x_\beta(at^n) \in Stab_-(t^\mu)$. We have

$$n \geq -\langle \mu, \beta^\vee \rangle .$$

Assume now, that $\beta - \alpha_i \in \Phi^-$, which is only possible if

$$\beta \in \{-\alpha_j - ... - \alpha_{i-1}, -\alpha_{i+1} - ... - \alpha_{j'} \mid j < i-1, j' > i+1\},$$

therefore $\langle \beta, \alpha_i^\vee \rangle = 1$. We have the following commutation relation:

$$x_\beta(at^n)x_{-\alpha_i}(ct^{m-1}) = x_{-\alpha_i}(ct^{m-1})x_\beta(at^m)x_{\beta-\alpha_i}(act^{n+m-1}).$$

But since $m - 1 \geq 0$ and $-\langle \mu, \beta^\vee \rangle \geq -\langle \mu + \alpha_i, \beta^\vee \rangle$, we have $n + m - 1 \geq -\langle \mu + \alpha_i, \beta^\vee \rangle$ and therefore

$$x_{\beta-\alpha_i}(act^{n-m-1}) \in Stab_-(t^{\mu+\alpha_i}).$$

This basically shows, that for every root subgroup appearing in the product before $x_{-\alpha_i}(ct^{m-1})$, if it does not commute with $x_{-\alpha_i}(ct^{m-1})$, the new root subgroup coming from the commutation relation will stabilize $t^{\mu+\alpha_i}$. Now it is left to show, that the root subgroups appearing before $x_{-\alpha_i}(ct^{m-1})$ in the product, do not only stabilize t^μ, but also $t^{\mu+\alpha_i}$. For every $x_\alpha(a_{i,\alpha,n}t^n)$ appearing in

$$\prod_{i=1}^{k-2} \prod_{(\alpha,n)\in\Phi_-^{\mathfrak{a}}(V_i,E_i)} x_\alpha(a_{i,\alpha,n}t^n),$$

we have $x_\alpha(a_{i,\alpha,n}t^n) \in Stab_-(t^\mu) \Leftrightarrow n \geq -\langle \mu, \alpha^\vee \rangle$. But since we have

$$-\langle \mu, \alpha^\vee \rangle \geq -\langle \mu + \alpha_i, \alpha^\vee \rangle \Leftrightarrow \alpha \notin \{-\alpha_k - ... - \alpha_i, -\alpha_i - ... - \alpha_k\},$$

we have

$$x_\alpha(a_{i,\alpha,n}t^n) \in Stab_-(t^{\mu+\alpha_i}) \text{ for every } \alpha \notin \{-\alpha_k - ... - \alpha_i, -\alpha_i - ... - \alpha_k\}.$$

Now let $\beta = -\alpha_k - ... - \alpha_i$ and let δ cross the wall $H_{\beta,m'}$ positively. The critical case would be $m' = -\langle \mu, \beta^\vee \rangle$, because then we would have

$$m' = -\langle \mu, \beta^\vee \rangle < -\langle \mu + \alpha_i, \beta^\vee \rangle ,$$

and if the reflection from the operator $\tilde{e}_{\alpha_i}$ is applied after the gallery has crossed $H_{\beta,m'}$ positively, we would have $x_\beta(ct^m) \notin Stab_-(t^{\mu+\alpha_i})$.

Let $V_l \subset E_l$ be the vertex, at which the gallery crosses the wall $H_{\beta,m'}$ positively. Let us assume without loss of generality $V_l = 0$, therefore $m' = 0$. We will just look at the last part of the gallery beginning at V_l, we denote this gallery $\delta_{\geq l}$. Let us assume the operator $\tilde{e}_{\alpha_i}$ is applicable to $\delta_{\geq l}$, in which case $x_\beta(c) \notin Stab_-(t^{\mu+\alpha_i})$. Since we have $\tilde{f}_{\alpha_i}\delta = 0$ for every $i = 1, ..., n$, the target μ of δ has to be antidominant:

$$\langle \mu, \alpha_1^\vee \rangle \leq 0, ..., \langle \mu, \alpha_n^\vee \rangle \leq 0.$$

Now it is easy to see

$$0 = \langle \mu, \beta^\vee \rangle = \langle \mu, -\alpha_k^\vee \rangle + ... + \langle \mu, -\alpha_i^\vee \rangle$$

$$\Rightarrow 0 = \langle \mu, -\alpha_k^\vee \rangle = ... = \langle \mu, -\alpha_i^\vee \rangle .$$

But since we have $\tilde{f}_{\alpha_i}\delta = 0$, by definition, we have $m = -\langle \mu, \alpha_i^\vee \rangle = 0$ and $\tilde{e}_{\alpha_i}\delta_{\geq l} = 0$, in contradiction to our assumption. Therefore, in our original gallery δ, if $\tilde{e}_{\alpha_i}$ is applicable, the reflection will be applied before the vertex V_l and our critical case does not appear. If we have $\beta = -\alpha_i - ... - \alpha_k$, the proof is completely analogous.

The general case can now be proven inductively. We apply $\tilde{e}_{\alpha_i}$ to the gallery $\tilde{e}^s_{\alpha_i}\delta$. Every argument given above can be applied analogously by applying the lemma to $\tilde{e}^s_{\alpha_i}\delta$ itself, we only have to consider the critical case. Again, let $\beta = -\alpha_i - ... - \alpha_k$ and let $\tilde{e}^s_{\alpha_i}\delta$ cross the wall $H_{\beta,m'}$ positively. Let

$$m' = -\langle \mu + s\alpha_i, \beta^\vee \rangle .$$

Let $V_l \in H_{\beta,m'}$ be the vertex, at which $\tilde{e}^s_{\alpha_i}\delta$ crosses the wall positively and let us consider $(\tilde{e}^s_{\alpha_i}\delta)_{\geq l} = \tilde{e}^s_{\alpha_i}(\delta_{\geq l})$, therefore we assume $m' = 0$. We have

$$0 = \langle \mu + s\alpha_i, -\beta^\vee \rangle = \langle \mu, \alpha_i^\vee \rangle + ... + \langle \mu, \alpha_k^\vee \rangle + s.$$

But since $\tilde{e}^s_{\alpha_i}\delta \neq 0$, we have $\langle \mu, \alpha_i^\vee \rangle \leq -s$. Using Lemma 2.6.1, we have

$$\langle \mu, \alpha_{i+1}^\vee \rangle , ..., \langle \mu, \alpha_k^\vee \rangle \leq 0,$$

and therefore we must have

$$m = -\langle \mu, \alpha_i^\vee \rangle = s.$$

But then we have $\tilde{e}_{\alpha_i}\tilde{e}^s_{\alpha_i}\delta = 0$. □

Corollary 4.2.2 *If $\tilde{e}^k_{\alpha_i}\delta$ is defined, we have:*

$$\pi(C_{\tilde{e}^k_{\alpha_i}\delta}) = \left\{ x_{-\alpha_i}(a_1 t^{m_1}) \cdot ... \cdot x_{-\alpha_i}(a_k t^{m_k}) t^{\mu + k\alpha_i} G(\mathcal{O}) \mid a_j \in \mathbb{C} \right\},$$

and $m_j \geq 0$ for all $j = 1, ..., k$.

Proof. As we have seen in Lemma 4.2.5, every other root subgroup $x_\beta(ct^n)$ besides $x_{-\alpha_j}(a_j t^{m_j})$ is in $Stab_-(t^{\mu+k\alpha_i})$. If the $x_\beta(ct^n)$ and $x_{-\alpha_j}(a_j t^{m_j})$ do not commute, we can use the computation rules given in the chapter before and we get:

$$x_\beta(ct^n) x_{-\alpha_j}(a_j t^{m_j}) = x_{-\alpha_j}(a_j t^{m_j}) x_{\beta-\alpha_j}(a_j c t^{n+m_j}) x_\beta(ct^n).$$

But by the definition of the root operator $\tilde{e}_{\alpha_i}$, we must have $m_j \geq 0$ and we have

$$x_{\beta-\alpha_j}(a_j c t^{n+m_j}) \in Stab_-(t^{\mu+k\alpha_i}).$$

Now remember our argument about the additional subgroups coming from commutation relations in the proof of Lemma 4.2.5, which proves the corollary. □

If δ is as in the corollary above, we obtain directly:

$$\pi(C_{\tilde{e}^k_{\alpha_i}\delta}) = U^-(\mathcal{K})t^\mu \cap G(\mathcal{O})t^\mu.$$

Now, similarly as in Corollary 4.2.1, we have

Corollary 4.2.3 *If δ is a gallery of type γ_λ, such that for every $j \in \{1, ..., n\}$ we have $\tilde{f}_{\alpha_j}\delta = 0$ and if $\tilde{e}^k_{\alpha_i}\delta$ is defined, $\overline{\pi(C_{\tilde{e}^k_{\alpha_i}\delta})}$ is a MV-cycle of coweight (λ, μ).*

5 Cell combinatorics in type A

In chapter 4, we have shown for a certain part of the crystal graph associated to a gallery, that the closure of the image of the cells associated to these galleries is not only contained in an MV-cycle, but actually is an MV-cycle. In this chapter we will take a gallery δ of type $N\omega_1$, which is completely included in the dominant Weyl chamber. By applying the root operators, we get a crystal containing the gallery δ. We will use the bumping algorithm to show, that the cells associated to these galleries include an MV-cycle.

5.1 The plactic monoid and bumping

We will introduce column bumping according to [Ful97]. Row bumping is defined analogously. Column insertion or bumping is a construction, where a positive integer x is added to a semi-standard Young tableau to obtain a new semi-standard Young tableau. The integer x is added at the bottom of the first column, if it is strictly bigger than every entry in it. If not, it replaces the smallest entry in the column, that is larger than or equal to x. The replaced entry is bumped the same way into the next column. The process stops if one entry can go to the bottom of the next column or until it becomes the entry of a new column. Let us look at an easy example, let us bump 3 into the tableau

$$\underset{3\to}{\begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline 2&3&5&5\\\hline 4&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array}} \to \underset{4\to}{\begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline 2&3&5&5\\\hline 3&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array}} \to \underset{4\to}{\begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline 2&3&5&5\\\hline 3&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array}}$$

$$\to \underset{5\to}{\begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline 2&3&4&5\\\hline 3&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array}} \to \begin{array}{|c|c|c|c|c|}\hline 1&2&2&3&5\\\hline 2&3&4&5\\\cline{1-4} 3&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array}$$

Now we want to introduce the plactic monoid. We will write words as a sequence of letters (positive integers) and write ww' for the juxtaposition of the two words w and w'. Let T be a tableau, we define the *column word* $w_{\mathrm{col}}(T)$ of T by listing the entries from bottom to top in each column, starting in the left column and moving to right. The *row word* $w_{\mathrm{row}}(T)$ of T will be obtained by reading the entries from left right, starting at the bottom row going to the top. This section is quoted from Fulton's book "Young Tableaux" ([Ful97]) and we use his notation. But in the rest of this thesis, we will always read a tableau columnwise from top to bottom, going from right to left. The bumping algorithm can now be broken down into smaller steps, we call the *elementary Knuth transformations*:

$$\begin{array}{ll} (K') & yzx \mapsto yxz \ \text{ if } \ x < y \leq z, \\ (K'') & xzy \mapsto zxy \ \text{ if } \ x \leq y < z. \end{array}$$

It is easier to see this transformations as simple row-bumpings:

$$\begin{array}{|c|c|}\hline y & z \\ \hline \end{array} \bullet \begin{array}{|c|}\hline x \\ \hline \end{array} = \begin{array}{|c|c|}\hline x & z \\ \hline y \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|}\hline x & z \\ \hline \end{array} \bullet \begin{array}{|c|}\hline y \\ \hline \end{array} = \begin{array}{|c|c|}\hline x & y \\ \hline z \\ \cline{1-1} \end{array}$$

Now two words w and w' will be called Knuth equivalent, if they can be changed into each other by a sequence of elementary Knuth transformations, and we write $w \equiv w'$.

Lemma 5.1.1 ([Ful97], p.27) *We have*

$$w_{col}(T) \equiv w_{row}(T).$$

5.2 Cells and MV-cycles II

Let λ be a dominant coweight and let γ_λ be a combinatorial gallery joining $\mathfrak{o}$ and λ. Consider the map $\pi : \Sigma(\gamma_\lambda) \to \mathcal{G}$, which is $SL_n(\mathcal{O})$-equivariant. For a generic anti-dominant coweight $\eta : \mathbb{C}^* \to T$, we have

$$\eta(t)\pi(x) = \pi(\eta(t).x).$$

For any gallery $\delta \in \Sigma(\gamma_\lambda)$ with $\pi(\delta) = \mu$ we have

$$\pi(C_\delta) \subset \left\{ y \in \mathcal{G} \mid \lim_{t \to 0} \eta(t).y = \mu \right\} = U^-(\mathcal{K}).\nu.$$

But by construction we also have $\pi(C_\delta) \subset X_\lambda$ and therefore $\pi(C_\delta)$ must be contained in an MV-cycle of a Schubert variety contained in X_λ. The aim of this chapter is to show in a particular case, that $\overline{\pi(C_\delta)}$ in fact is a MV-cycle for a possibly smaller Schubert variety.

Let us start by showing a Lemma, which we will need in the proofs of this chapter. Set $U^-(\mathcal{O}) = U^-(\mathcal{K}) \cap SL_n(\mathcal{O})$. Our Bott-Samelson variety $\Sigma(\gamma_\lambda)$ inherits naturally a $SL_n(\mathcal{O})$-action and the map π is equivariant with respect to this action. Remember, that

$$C_\delta = \left\{ z \in \Sigma(\gamma_\lambda) \mid \lim_{t \to 0} \eta(t).z = \delta \right\}$$

for a fixed anti-dominant one-parameter subgroup $\eta : \mathbb{C}^* \to T$. Using

$$\lim_{t \to 0} \eta(t) u \eta(t)^{-1} = 1,$$

we get $u.z \in C_\delta$ for all $z \in C_\delta$ and $u \in U^-(\mathcal{O})$. Therefore it follows:

Lemma 5.2.1 *The image of the cell C_δ under π is $U^-(\mathcal{O})$-stable, in other words*

$$x_{-\alpha,m}(a).\pi(C_\delta) = \pi(C_\delta) \ \forall a \in \mathbb{C}, \alpha \succ 0 \text{ and } m \in \mathbb{N}.$$

Now, for $k \geq 0$, consider

$$\pi(C_\delta)^{\geq k} = \left\{ \prod_{j=k}^{r} \prod_{(\alpha,n)\in\Phi_-^{\mathfrak{a}}(V_j,E_j)} x_{\alpha,m}(a_{j,\alpha,m})t^\mu \mid a_{j,\alpha,m} \in \mathbb{C} \right\},$$

where μ is the target of δ and the gallery $\delta^{\geq k}$, which is obtained by δ by just considering it after the vertex V_k. We replace the group $U^-(\mathcal{O})$ by

$$U_k^- = \left\{ \prod x_{(-\alpha,m)} \mid \alpha \succ 0, m \in \mathbb{Z}, V_k \in H^+_{(\alpha,m)} \right\}.$$

Now let φ_k be the coweight associated to V_k.

Lemma 5.2.2 *The set $\pi(C_\delta)^{\geq k}$ is U_k^--stable, in other words*

$$x_{(-\alpha,m)}.\pi(C_\delta)^{\geq k} = \pi(C_\delta)^{\geq k} \ \forall \alpha \succ 0, m \geq -\langle \varphi_k, \alpha^\vee \rangle .$$

Proof. We can view $\varphi_k : \mathbb{C}^* \to T$ as a one-parameter subgroup as well as a $\mathcal{K}$ rational point in G. Let t_{φ_k} and $t_{-\varphi_k}$ be the translations by φ_k, respectively $-\varphi_k$. We have $t_{\varphi_k}(0) = V_k$ and $t_{-\varphi_k}(V_k) = 0$ and $\delta_0^{\geq k} := t_{-\varphi_k}(\delta^{\geq k})$. The translations act on the roots by shifts: $t_{\varphi_k} : (\alpha, l) \mapsto (\alpha, l + \langle\varphi_k, \alpha^\vee\rangle)$ and $t_{-\varphi_k} : (\alpha, l) \mapsto (\alpha, l - \langle\varphi_k, \alpha^\vee\rangle)$. Note that

$$\varphi_k x_{(\alpha,l)}(a)\varphi_k^{-1} = x_{(\alpha,l+\langle\varphi_k,\alpha^\vee\rangle)}(a).$$

We have

$$\varphi_k.\pi(C_{\delta_0^{\geq k}}) = \pi(C_\delta)^{\geq k}.$$

By the lemma above, $U^-(\mathcal{O})$ stabilizes $C_{\delta_0^{\geq k}}$, therefore $\varphi_k U^-(\mathcal{O})\varphi_k^{-1}$ stabilizes $\pi(C_\delta)^{\geq k}$. If $\alpha \succ 0$ and $m \geq 0$, we have

$$\varphi_k x_{(-\alpha,m)}\varphi_k^{-1} = x_{(-\alpha,m+\langle\varphi_k,\alpha^\vee\rangle)} = x_{(-\alpha,l)} \text{ with } l \geq -\langle \varphi_k, \alpha^\vee \rangle .$$

□

Now let us consider an LS-gallery δ of type γ_λ with λ being a dominant coweight. As we did at the beginning of chapter 4, we will write the target μ of δ as

$$\mu = \sum_{i=1}^{n}\sum_{j=1}^{a_i}(\varepsilon_{l_{i,j,1}} + ... + \varepsilon_{l_{i,j,i}})$$

according to the filling of the associated (not necessarily semi-standard) generalized tableau and $\varepsilon_{l_{i,j,1}} + ... + \varepsilon_{l_{i,j,i}}$ belonging to the column

$$\begin{array}{|c|} \hline l_{i,j,1} \\ \hline \vdots \\ \hline l_{i,j,i} \\ \hline \end{array},$$

where $l_{i,j,1} > ... > l_{i,k,i}$. To this Young tableau we can also associate the following gallery, obtained by reading the tableau box by box:

$$\begin{aligned}\delta' =&\gamma_{(\varepsilon_{l_{1,1,1}}}) * ... * (\gamma_{\varepsilon_{l_{1,a_1,1}}}) * (\gamma_{\varepsilon_{l_{2,1,1}}} * \gamma_{\varepsilon_{l_{2,1,2}}}) * ...\\ &... * (\gamma_{\varepsilon_{l_{2,a_2,1}}} * \gamma_{\varepsilon_{l_{2,a_2,2}}}) * ... * (\gamma_{\varepsilon_{l_{n,a_n,1}}} * ... * \gamma_{\varepsilon_{l_{n,a_n,n}}}),\end{aligned}$$

which, in most of the cases, will not be an LS-gallery. We will now consider two different Bott-Samelson varieties. We have

$$C_\delta \subset \Sigma(\gamma_\lambda),$$

where δ is of type γ_λ and

$$C_{\delta'} \subset \Sigma(\gamma_{N\omega_1}),$$

where N is the number of boxes in our Young tableau. Let $\pi : \Sigma(\gamma_\lambda) \to \mathcal{G}$ and $\pi' : \Sigma(\gamma_{N\omega_1}) \to \mathcal{G}$ the associated desingularizations.

Lemma 5.2.3 *We have*

$$\pi(C_\delta) = \pi'(C_{\delta'}).$$

Proof. Let us first proof the statement for a single column. We have the following two galleries:

$$\delta = \gamma_{\varepsilon_{i_1}+...+\varepsilon_{i_r}}$$

and

$$\delta' = \gamma_{\varepsilon_{i_1}} * ... * \gamma_{\varepsilon_{i_r}}.$$

As we have seen in Lemma 4.1.1, we have the following negative roots, associated to a wall, which is positively crossed by δ':

$$-\alpha_{i_j}, -\alpha_{i_j} - \alpha_{i_j+1}, ..., -\alpha_{i_j} - ... - \alpha_n \text{ for } j = 1, ..., r.$$

More precisely, let F_{ε_i} as defined in section 4.1, then:

- $\Phi^{\mathfrak{a}}_-(0, F_{\varepsilon_{i_1}}) = \{(-\alpha_{i_1}, 0), ..., (-\alpha_{i_1} - ... - \alpha_n, 0)\}$
- $\Phi^{\mathfrak{a}}_-(\varepsilon_{i_1}, \varepsilon_{i_1} + F_{\varepsilon_{i_2}})$
  $= \left\{(-\alpha_{i_2}, \left\langle\varepsilon_{i_1}, \alpha^\vee_{i_2}\right\rangle), ..., (-\alpha_{i_2} - ... - \alpha_n, \left\langle\varepsilon_{i_1}, \alpha^\vee_{i_2} + ... + \alpha^\vee_n\right\rangle)\right\}$
- ...
- $\Phi^{\mathfrak{a}}_-(\varepsilon_{i_1} + ... + \varepsilon_{i_{r-1}}, \varepsilon_{i_1} + ... + \varepsilon_{i_{r-1}} + F_{\varepsilon_{i_r}}) =$
  $\{(-\alpha_{i_r}, \langle\varepsilon_{i_1} + ... + \varepsilon_{i_{r-1}}, \alpha^\vee_{i_r}\rangle), ...,$
  $(-\alpha_{i_r} - ... - \alpha_n, \langle\varepsilon_{i_1} + ... + \varepsilon_{i_{r-1}}, \alpha^\vee_{i_r} + ... + \alpha^\vee_n\rangle)\}$.

But since $i_1 < i_2 < ... < i_r$, we have

$$\left\langle\varepsilon_{i_1} + ...\varepsilon_{i_{j-1}}, -\alpha^\vee_{i_j} - ... - \alpha^\vee_s\right\rangle = 0$$

and the description of the associated root subgroups simplifies:

- $\Phi^{\mathfrak{a}}_-(0, F_{\varepsilon_{i_1}}) = \{(-\alpha_{i_1}, 0), ..., (-\alpha_{i_1} - ... - \alpha_n, 0)\}$

- $\Phi^{\mathfrak{a}}_{-}(\varepsilon_{i_1}, \varepsilon_{i_1} + F_{\varepsilon_{i_2}}) = \{(-\alpha_{i_2}, 0), ..., (-\alpha_{i_2} - ... - \alpha_n, 0)\}$
- ...
- $\Phi^{\mathfrak{a}}_{-}(\varepsilon_{i_1} + ... + \varepsilon_{i_{r-1}}, \varepsilon_{i_1} + ... + \varepsilon_{i_{r-1}} + F_{\varepsilon_{i_r}}) = \{(-\alpha_{i_r}, 0), ..., (-\alpha_{i_r} - ... - \alpha_n, 0)\}$.

Remember the proof of Lemma 4.1.2, if $i_{j+1} - 2 \ngeq i_j$, the set $\Phi^{\mathfrak{a}}_{-}(0, F_{\varepsilon_{i_1}+...+\varepsilon_{i_r}})$ will not contain any root of the form $(-\alpha_{i_j} - ..., 0)$. Therefore $\Phi^{\mathfrak{a}}_{-}(0, F_{\varepsilon_{i_1}+...+\varepsilon_{i_r}})$ consists of the roots:

- $(-\alpha_{i_1}, 0), (-\alpha_{i_1} - \alpha_{i_1+1}, 0), ..., (-\alpha_{i_1} - ... - \alpha_{i_2-2}, 0)$ (if $i_2 - 2 \geq i_1$),
  $(-\alpha_{i_1} - ... - \alpha_{i_2}, 0), ..., (-\alpha_{i_1} - ... - \alpha_{i_3-2}, 0)$ (if $i_3 - 2 \geq i_2$),
  ...,
  $(-\alpha_{i_1} - ... - \alpha_{i_k}, 0), ..., (-\alpha_{i_1} - ... - \alpha_n, 0)$ (if $n - i_k \geq 0$)
- ...
- $(-\alpha_{i_{k-1}}, 0), (-\alpha_{i_{k-1}} - \alpha_{i_{k-1}+1}, 0), ..., (-\alpha_{i_{k-1}} - ... - \alpha_{i_k-2}, 0)$ (if $i_k - 2 \geq i_{k-1}$),
  ...,
  $(-\alpha_{i_{k-1}} - ... - \alpha_{i_k}, 0), ..., (-\alpha_{i_{k-1}} - ... - \alpha_n, 0)$ (if $n - i_k \geq 0$)
- $(-\alpha_{i_k}, 0), (-\alpha_{i_k} - \alpha_{i_{k+1}}, 0), ..., (-\alpha_{i_k} - ... - \alpha_n, 0)$ (if $n - i_k \geq 0$),

and we see right away

$$\Phi^{\mathfrak{a}}_{-}(0, F_{\varepsilon_{i_1}+...+\varepsilon_{i_r}}) \subset \Phi^{\mathfrak{a}}_{-}(0, F_{\varepsilon_{i_1}}) \cup ... \cup \Phi^{\mathfrak{a}}_{-}(\varepsilon_{i_1} + ... + \varepsilon_{i_{r-1}}, \varepsilon_{i_1} + ... + \varepsilon_{i_{r-1}} + F_{\varepsilon_{i_r}}),$$

and hence

$$\pi(C_\delta) \subset \pi'(C_{\delta'}) \subset U^-(\mathcal{O}) t^{\varepsilon_{i_1}+...+\varepsilon_{i_r}}.$$

The last inclusion follows from the observation, that all the root subgroups appearing in $\pi'(C_{\delta'})$ are in $U^-(\mathcal{O})$. But by Lemma 5.2.1, we have

$$U^-(\mathcal{O}) t^{\varepsilon_{i_1}+...+\varepsilon_{i_r}} = \pi(C_\delta),$$

and we conclude

$$\pi(C_\delta) = \pi'(C_{\delta'}).$$

Now we want to add another column

$$C = \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_r \\ \hline \end{array}$$

from the left to a tableau T, and consider

$$T * C.$$

By induction, assume that our statement is already true for T, which means that the image of the cell is the same, if we read T column- or boxwise. Let δ_T be the gallery associated to T and μ be the target of δ_T. We have already seen

$$\Phi^{\mathfrak{a}}_{-}(0, F_{\varepsilon_{i_1}+...+\varepsilon_{i_r}}) \subset \Phi^{\mathfrak{a}}_{-}(0, F_{\varepsilon_{i_1}}) \cup ... \cup \Phi^{\mathfrak{a}}_{-}(\varepsilon_{i_1} + ... + \varepsilon_{i_{r-1}}, \varepsilon_{i_1} + ... + \varepsilon_{i_{r-1}} + F_{\varepsilon_{i_r}}),$$

therefore we have

$$\begin{aligned}&\Phi^{\mathfrak{a}}_{-}(\mu, \mu + F_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_r}})\\ \subset&\Phi^{\mathfrak{a}}_{-}(\mu, \mu + F_{\varepsilon_{i_1}}) \cup \ldots \cup \Phi^{\mathfrak{a}}_{-}(\mu + \varepsilon_{i_1} + \ldots + \varepsilon_{i_{r-1}}, \mu + \varepsilon_{i_1} + \ldots + \varepsilon_{i_{r-1}} + F_{\varepsilon_{i_r}})\end{aligned}$$

and it follows

$$\pi(C_{\delta_{C*T}}) \subset \pi'(C_{\delta'_{C*T}}).$$

But similar, as we have seen before, the root subgroups appearing in $\Phi^{\mathfrak{a}}_{-}(\mu, \mu + F_{\varepsilon_{i_l}+\ldots+\varepsilon_{i_k}}), 1 \leq i < k \leq r-1$ are in

$$U^{-}(\mu) = \left\{ \prod x_{(-\alpha,m)} \mid \alpha \succ 0, m \in \mathbb{Z}, \mu \in H^{+}_{(\alpha,m)} \right\}.$$

And we have

$$\pi'(C_{\delta'_{C*T}}) \subset U^{-}(\mu)U(T)t^{\mu+\varepsilon_{i_1}+\ldots\varepsilon_{i_r}},$$

and $U(T)$ is the product of root subgroups coming from T, where we know by induction, that it is irrelevant, if we read T boxwise or columnwise. But again, similar to the case of a single column, we can use Lemma 5.2.2 to obtain

$$U^{-}(\mu)U(T)t^{\mu+\varepsilon_{i_1}+\ldots+\varepsilon_{i_r}} = \pi(C_\delta),$$

and we have proven the lemma. □

Now we want to look at the case of bumping a single box into a column and relate this to the geometry of cells in the Bott-Samelson varieties. We generalize the notion of a Young tableau. By a tableau we mean an arrangement of boxes, compared to a Young tableau, the numbers of boxes in the columns do not have to be decreasing from left to right. We will still ask the entries in the columns of such tableaux to be strictly increasing. Consider the tableau:

$$T_1 = \begin{array}{|c|c|} \hline j & i_1 \\ \hline & \vdots \\ \cline{2-2} & i_{l-1} \\ \cline{2-2} & i_l \\ \cline{2-2} & i_{l+1} \\ \cline{2-2} & \vdots \\ \cline{2-2} & i_r \\ \cline{2-2} \end{array}$$

and bump the single box $\boxed{j}$ into the column to get a Young tableau

$$T_2 = \begin{array}{|c|c|} \hline i_1 & i_l \\ \hline \vdots & \\ \cline{1-1} i_{l-1} & \\ \cline{1-1} j & \\ \cline{1-1} i_{l+1} & \\ \cline{1-1} \vdots & \\ \cline{1-1} i_r & \\ \cline{1-1} \end{array},$$

where $i_l \geq j > i_{l-1}$. The case $j > i_r$ will be discussed later. To T_1, we will associate the gallery δ_1, which we get by reading the tableau box by box:

$$\delta_1 = \gamma_{\varepsilon_{i_1}} * ... * \gamma_{\varepsilon_{i_r}} * \gamma_{\varepsilon_j}.$$

We will read T_2 columnwise to get

$$\delta_2 = \gamma_{\varepsilon_{i_l}} * \gamma_{\varepsilon_{i_1}+...+\varepsilon_{i_{l-1}}+\varepsilon_j+\varepsilon_{i_{l+1}}+...+\varepsilon_{i_r}}.$$

As we did before, we will consider two different Bott-Samelson varieties and corresponding cells, $C_{\delta_1} \subset \Sigma(\gamma_{N\omega_1})$ and $C_{\delta_2} \subset \Sigma(\gamma_\lambda)$, where N is the number of boxes in our tableau T_1 and λ is the shape of T_2. As before, we will denote $\pi' : \Sigma(\gamma_{N\omega_1}) \to \mathcal{G}$ and $\pi : \Sigma(\gamma_\lambda) \to \mathcal{G}$ the desingularizations.

Let us first fix some notations to facilitate the following proofs. If

$$T = \boxed{\begin{array}{c} i_1 \\ \vdots \\ i_r \end{array}},$$

we set

$$u_T = \prod_{j=1}^{r} \prod_{k>i_j} x_{(-\varepsilon_k-\varepsilon_{i_j},0)}.$$

We have a shifted version for a weight μ:

$$u_T(\mu) = \prod_{j=1}^{r} \prod_{k>i_j} x_{(-\varepsilon_k-\varepsilon_{i_j},-\langle\mu,-\varepsilon_k-\varepsilon_{i_j}\rangle)}.$$

Whenever there is no confusion possible, we will write $(\alpha, \overline{\mu})$ for the affine root $(\alpha, -\langle\mu, \alpha\rangle)$. If we have a tableau T consisting of the columns $K_1, ..., K_q$ from right to left. Let ν_j be the weight of the tableau consisting of the columns $K_1, ..., K_{j-1}$. If

$$K_j = \boxed{\begin{array}{c} i_{1j} \\ \vdots \\ i_{r_j j} \end{array}},$$

we set

$$u_{K_j} = \prod_{j=1}^{r} \prod_{\substack{k>i_j \\ k\neq i_{(r+1)j},...,i_{r_j j}}} x_{(-\varepsilon_k-\varepsilon_{i_j},\overline{\nu_j})}.$$

Then we have

$$u_K = \prod_{j=1}^{q} u_{K_1} u_{K_2} ... u_{K_q}.$$

There is also a shifted version

$$u_K(\mu) = \prod_{j=1}^{q} u_{K_j}(\mu),$$

where

$$u_{K_j}(\mu) = \prod_{r=1} t \prod_{\substack{k>i_j \\ k\neq i_{(r+1)j},...,i_{r_j j}}} x_{(-\varepsilon_k-\varepsilon_{i_j},\overline{\mu+\nu_j})}.$$

Lemma 5.2.4 *For any weight μ and*

$$T_1 = \begin{array}{|c|c|}\hline c & a_1 \\ \hline \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & a_s \\ \cline{2-2}\end{array} \text{ and } T_2 = \begin{array}{|c|c|}\hline a_1 & c \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_s & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

we have

$$u_{T_1}(\mu) = u_{T_2}(\mu).$$

Proof. We have

$$u_{T_1}(\mu) = (\prod_{j=1}^{s} \prod_{\substack{l>a_j \\ l\neq a_{j+1},\dots,a_s}} u_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})}) \prod_{m>c} u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})}.$$

Since we have $m > c > a_j$ for all j, the root subgroups $u_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})}$ and $u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})}$ commute, unless $l = c$, in which case we have

$$u_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})}(b)u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})}(a) = u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})}(a)u_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})}u_{(\varepsilon_m-\varepsilon_{a_j},\overline{\mu})}(ab).$$

But since $m > c > a_s$, we have $m > a_j$ and $m \neq a_{j+1}, ..., a_s$. Therefore the rootsubgroups

$$u_{(\varepsilon_m-\varepsilon_{a_j},\overline{\mu})}(ab)$$

appear in the product

$$\prod_{j=1}^{s} \prod_{\substack{l>a_j \\ l\neq a_{j+1},\dots,a_s}} u_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})}.$$

We can reorder our product and get

$$(\prod_{j=1}^{s} \prod_{\substack{l>a_j \\ l\neq a_{j+1},\dots,a_s}} u_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})})u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})} = u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})}(\prod_{j=1}^{s} \prod_{\substack{l>a_j \\ l\neq a_{j+1},\dots,a_s}} u_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})}),$$

if we neglect the parameters. Let us look at:

$$\begin{aligned}&(\prod_{j=1}^{s} \prod_{\substack{l>a_j \\ l\neq a_{j+1},\dots,a_s}} u_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})}(x_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})}))u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})}(x_{(\varepsilon_m-\varepsilon_c,\overline{\mu})}) \\ &= u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})}(x_{(\varepsilon_m-\varepsilon_c,\overline{\mu})})(\prod_{j=1}^{s} \prod_{\substack{l>a_j \\ l\neq a_{j+1},\dots,a_s}} u_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})}(x'_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})})).\end{aligned}$$

The parameter $x'_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})}$ is of the form

$$x'_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})} = x_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})} + p,$$

where p ist a polynomial depending on $x_{(\beta,\overline{\mu})}$ with $\beta < \varepsilon_{a_j} - \varepsilon_l$. Therefore we can find a reparametrization such that we have an equality of varieties

$$(\prod_{j=1}^{s} \prod_{\substack{l>a_j \\ l\neq a_{j+1},\dots,a_s}} u_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})})u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})} \text{ "} = \text{" } u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})}(\prod_{j=1}^{s} \prod_{\substack{l>a_j \\ l\neq a_{j+1},\dots,a_s}} u_{(\varepsilon_l-\varepsilon_{a_j},\overline{\mu})}).$$

□

Corollary 5.2.1 *Let K, L be tableaux and T_1, T_2 as above, with*

$$K = K' * T_1 * K'', L = K' * T_2 * K''.$$

Then we have

$$u_K = u_L.$$

Proof. If μ_1 is the weight of K'', we set $\mu_2 = \mu_1 + \varepsilon_c + \varepsilon_{a_1} + ... + \varepsilon_{a_s}$. By Lemma 5.2.4 we have

$$u_K = u_{K''} u_{T_1}(\mu_1) u_{K'}(\mu_2) = u_{K''} u_{T_2}(\mu_1) u_{K'}(\mu_2) = u_L.$$

□

Now let

$$T_1 = \begin{array}{|c|c|} \hline b & c \\ \hline d_1 & \\ \cline{1-1} \vdots & \\ \cline{1-1} d_T & \\ \cline{1-1} \end{array} \quad \text{and } T_2 = \begin{array}{|c|c|} \hline b & c \\ \hline & d_1 \\ \cline{2-2} & \vdots \\ \cline{2-2} & d_t \\ \cline{2-2} \end{array}.$$

Lemma 5.2.5 *For any weight μ, there exists a dense subset of the set of parameters for the root subgroups, such that*

$$u_{T_1}(\mu) = u_{T_2}(\mu) \prod_{j=1}^{t} u_{(\varepsilon_{d_j} - \varepsilon_c, \overline{\mu})}(f(\Delta)),$$

where $f(\Delta)$ indicates, that these parameters can not be chosen freely and depend on the entries $d_1, ..., d_t$ of the tableaux.

Proof. We write

$$u_{\underline{d}}(\mu) = (\prod_{j=1}^{s} \prod_{\substack{m>d_j \\ m \neq d_{j+1},...,d_t}}^{n} u_{(\varepsilon_m - \varepsilon_{d_j}, \overline{\mu})}).$$

We get

$$u_{T_1}(\mu) = (\prod_{l>c} u_{(\varepsilon_l - \varepsilon_c, \overline{\mu})}) u_{\underline{d}}(\mu) (u_{(\varepsilon_c \varepsilon_b, \overline{\mu}-1)} \prod_{\substack{k>b \\ k \neq c, d_1,...,d_t}} u_{(\varepsilon_k - \varepsilon_b, \overline{\mu})})$$

and

$$u_{T_2}(\mu) = (\prod_{\substack{l>c \\ l \neq d_1,...,d_t}} u_{(\varepsilon_l - \varepsilon_c, \overline{\mu})}) u_{\underline{d}}(\mu) (u_{(\varepsilon_c - \varepsilon_b, \overline{\mu}-1)} \prod_{j=1}^{t} u_{(\varepsilon_{d_j} - \varepsilon_b, \overline{\mu}-1)} \prod_{\substack{k>b \\ k \neq c, d_1,...,d_t}} u_{(\varepsilon_k - \varepsilon_b, \overline{\mu})}).$$

Let us first look at $u_{T_1}(\mu)$:

$$u_{T_1}(\mu) = (\prod_{l>c} u_{(\varepsilon_l - \varepsilon_c, \overline{\mu})})(\prod_{j=1}^{t} u_{(\varepsilon_{d_j} - \varepsilon_c, \overline{\mu})}) u_{\underline{d}}(\mu)(u_{(\varepsilon_c - \varepsilon_b, \overline{\mu}-1)}) \prod_{\substack{k>b \\ k \neq c, d_1,...,d_t}} u_{(\varepsilon_k - \varepsilon_b, \overline{\mu})},$$

where we split the first product. The root subgroups in $u_{\underline{d}}(\mu)$ (which are of the form $u_{(\varepsilon_m - \varepsilon_{d_l}, \overline{\mu})}, m > d_l > b, m \neq d_p$ for all p) and in $\prod_{j=1}^{t} u_{(\varepsilon_{d_j} - \varepsilon_c, \overline{\mu})}$ do not necessarily commute. But root subgroups appearing by using the commutation relation are of the form $u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})}$ and therefore commute with the roots in $\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})}$. Again, by neglecting the parameters we have

$$\begin{aligned}u_{T_1}(\mu) = (\prod_{l>c} u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})})u_{\underline{d}}(\mu)(\prod_{l>c} u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})}(x_{\underline{d},\underline{j}}))\\ (\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})})u_{(\varepsilon_c-\varepsilon_b,\overline{\mu}-1)} \prod_{\substack{k>b\\ k\neq c,d_1,\ldots,d_t}} u_{(\varepsilon_k-\varepsilon_b,\overline{\mu})}.\end{aligned}$$

The parameter $x_{\underline{d},\underline{j}}$ depends on data in $u_{\underline{d}}(\mu)$ and $\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})}$. The root subgroups $u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})}, l > c, l \neq d_1, \ldots, d_t$ commute with the root subgroups $u_{(\varepsilon_m-\varepsilon_{d_l},\overline{\mu})}, m > d_l > b, m \neq d_p$ for all p, which occur in $u_{\underline{d}}$. By using the commutation relation, we obtain root subgroups that correspond to roots occuring in the product

$$\prod_{l>c} u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})}.$$

After reparametrization, we get

$$\begin{aligned}u_{T_1}(\mu) = &(\prod_{l>c} u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})})u_{\underline{d}}(\mu)(\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})})\\ &u_{(\varepsilon_c-\varepsilon_b,\overline{\mu}-1)} \prod_{\substack{k>b\\ k\neq c,d_1,\ldots,d_t}} u_{(\varepsilon_k-\varepsilon_b,\overline{\mu})}\\ =&(\prod_{l>c} u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})})u_{\underline{d}}(\mu)(\prod_{\substack{k>b\\ k\neq c,d_1,\ldots,d_t}} u_{(\varepsilon_k-\varepsilon_b,\overline{\mu})})\\ &(\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})})u_{(\varepsilon_c-\varepsilon_b,\overline{\mu}-1)},\end{aligned}$$

since the root subgroups $u_{(\varepsilon_k-\varepsilon_b,\overline{\mu})}$ commute with the root subgroups $u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})}$ and $u_{(\varepsilon_c-\varepsilon_b,\overline{\mu}-1)}$. We rewrite the expression:

$$u_{\begin{array}{|c|c|}\hline b & c \\ \hline d_1 \\ \cline{1-1} \vdots \\ \cline{1-1} d_T \\ \cline{1-1}\end{array}}(\mu) = u_{\begin{array}{|c|}\hline c \\ \hline d_1 \\ \hline \vdots \\ \hline d_T \\ \hline\end{array}}(\mu)(\prod_{\substack{k>b\\ k\neq c,d_1,\ldots,d_t}} u_{(\varepsilon_k-\varepsilon_b,\overline{\mu})})(\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})})u_{(\varepsilon_c-\varepsilon_b,\overline{\mu}-1)}.$$

We commute

$$\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})}$$

and $u_{(\varepsilon_c-\varepsilon_b,\overline{\mu}-1)}$ using the commutating relation, and we get root subgroups of the form $u_{(\varepsilon_{d_j}-\varepsilon_b,\overline{\mu}-1)}$. Since $u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})}$ commutes with the $u_{(\varepsilon_k-\varepsilon_b,\overline{\mu})}$, we get

$$u_{\begin{array}{|c|c|}\hline b & c \\ \hline d_1 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} d_T & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}(\mu) \text{ " } = \text{ " } u_{\begin{array}{|c|c|}\hline b & c \\ \hline \multicolumn{1}{c|}{} & d_1 \\ \cline{2-2} \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & d_t \\ \cline{2-2}\end{array}}(\mu) \prod_{j=1}^{t} \left(u_{(\varepsilon_{d_j}-\varepsilon_b,\overline{\mu})}(f(\Delta))\right).$$

This is not an equality, since commuting the root subgroups in the last step leads to the two parameters x and y of two roots appearing as xy in the sum of the two roots and the parameter of this root being equal to zero only if one of the parameters equals zero. So this the equality does hold only on an open dense subset. $\square$

Lemma 5.2.6 *For any weight μ there exists a dense subset of the set of parameters for the root subgroups such that*

$$u_{\begin{array}{|c|c|}\hline b & b \\ \hline d_1 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} d_t & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}(\mu) = u_{\begin{array}{|c|c|}\hline b & b \\ \hline \multicolumn{1}{c|}{} & d_1 \\ \cline{2-2} \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & d_t \\ \cline{2-2}\end{array}}(\mu) \left(\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_b,\overline{\mu})}(f(\Delta))\right),$$

where $f(\Delta)$ indicates that these parameters can not be chosen freely and depend on the entries $d_1, ..., d_t$.

Proof. We abbreviate

$$u_{\underline{d}}(\mu) = \left(\prod_{j=1}^{s} \prod_{\substack{m>d_j \\ m\neq d_{j+1},\dots,d_t}} u_{(\varepsilon_m-\varepsilon_{d_j},\overline{\mu})}\right).$$

We get

$$u_{\begin{array}{|c|c|}\hline b & c \\ \hline d_1 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} d_t & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}(\mu) = \prod_{l>c} u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})} u_{\underline{d}}(\mu) \left(u_{(\varepsilon_c-\varepsilon_b,\overline{\mu}-1)} \prod_{\substack{k>b \\ k\neq c,d_1,\dots,d_t}} u_{(\varepsilon_k-\varepsilon_b,\overline{\mu})}\right).$$

We also have

$$u_{\begin{array}{|c|c|}\hline b & c \\ \hline \multicolumn{1}{c|}{} & d_1 \\ \cline{2-2} \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & d_t \\ \cline{2-2}\end{array}}(\mu) = (\prod_{\substack{l>c \\ l\neq d_1,\dots,d_t}} u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})}) u_{\underline{d}}(\mu)$$

$$(u_{(\varepsilon_c-\varepsilon_b,\overline{\mu}-1)}\prod_{j=1}^{t}u_{(\varepsilon_{d_j}-\varepsilon_b,\overline{\mu}-1)}\prod_{\substack{k>b\\ k\neq c,d_1,\dots,d_t}}u_{(\varepsilon_k-\varepsilon_b,\overline{\mu})})$$

Let us now rearrange the first product:

$$u_{\begin{array}{|c|c|}\hline b & c \\ \hline d_1 \\ \cline{1-1} \vdots \\ \cline{1-1} d_t \\ \cline{1-1}\end{array}}(\mu)=(\prod_{\substack{l>c\\ l\neq d_1,\dots,d_t}}u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})})(\prod_{j=1}^{t}u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})})u_{\underline{d}}(\mu)$$

$$\left(u_{(\varepsilon_c-\varepsilon_b,\overline{\mu}-1)}\prod_{\substack{k>b\\ k\neq c,d_1,\dots,d_t}}u_{(\varepsilon_k-\varepsilon_b,\overline{\mu})}\right).$$

By commuting the $u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})}$ with $u_{\underline{d}}(\mu)$, we obtain new root subgroups of the form $u_{(\varepsilon_m-\varepsilon_c,\overline{\mu})}$, which commute with

$$\prod_{j=1}^{t}u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})}.$$

So again, if we neglect the parameters, we get

$$u_{\begin{array}{|c|c|}\hline b & c \\ \hline d_1 \\ \cline{1-1} \vdots \\ \cline{1-1} d_t \\ \cline{1-1}\end{array}}(\mu)=(\prod_{\substack{l>c\\ l\neq d_1,\dots,d_t}}u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})})u_{\underline{d}}(\mu)(\prod_{\substack{l>c\\ l\neq d_1,\dots,d_t}}u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})}(x_{\underline{d},\underline{j}}))$$

$$(\prod_{j=1}^{t}u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu},\overline{\mu})})\left(u_{(\varepsilon_c-\varepsilon_b,\overline{\mu}-1)}\prod_{\substack{k>b\\ k\neq c,d_1,\dots,d_t}}u_{(\varepsilon_k-\varepsilon_b,\overline{\mu})}\right).$$

The parameter $x_{\underline{d},\underline{j}}$ depends on data hidden in $u_{\underline{d}}(\mu)$ and

$$\prod_{j=1}^{t}u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu})}.$$

The root subgroups $u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})}, l>c, l\neq d_1,\dots,d_t$ commute with the root subgroups $u_{(\varepsilon_m-\varepsilon_{d_l},\overline{\mu})}, m>d_l>b, m\neq d_p$ for all p occuring in $u_{\underline{d}}$. The root subgroups appearing when we commute the two products occur in the product

$$\prod_{\substack{l>c\\ l\neq d_1,\dots,d_t}}u_{(\varepsilon_l-\varepsilon_c,\overline{\mu})},$$

and after a reparametrization, we obtain:

$$u_{\begin{array}{|c|c|}\hline b & b \\ \hline d_1 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} d_t & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}(\mu) = \Big(\prod_{\substack{l>b \\ l\neq d_1,\dots,d_t}} u_{(\varepsilon_l-\varepsilon_b,\overline{\mu})}\Big) u_{\underline{d}}(\overline{\mu})\Big(\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_b,\overline{\mu})}\Big) \prod_{\substack{k>b \\ k\neq d_1,\dots,d_t}} u_{(\varepsilon_k-\varepsilon_b,\overline{\mu}+1)}\Big)$$

and therefore

$$u_{\begin{array}{|c|c|}\hline b & b \\ \hline d_1 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} d_t & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}(\mu) = u_{\begin{array}{|c|c|}\hline b & c \\ \hline \multicolumn{1}{c|}{} & d_1 \\ \cline{2-2} \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & d_t \\ \cline{2-2}\end{array}}(\mu).$$

$\square$

Let us suppose now, we have generalized tableaux K, L of the form

$$K = K' * \begin{array}{|c|c|}\hline b & c \\ \hline d_1 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} d_t & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} * K'', L = K' * \begin{array}{|c|c|}\hline b & c \\ \hline \multicolumn{1}{c|}{} & d_1 \\ \cline{2-2} \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & d_t \\ \cline{2-2}\end{array} * K''.$$

so both have the same weight μ. In the following, we will also allow the case of $b = c$. Let μ_1 be the weight of K'', let $\mu_2 = \mu_1 + \varepsilon_c + \varepsilon_{d_1} + \dots + \varepsilon_{d_t}$ and $\mu_3 = \mu_2 + \varepsilon_b$.

Proposition 5.2.1 *The sets $\{u_K t^\mu\}$ and $\{u_L t^\mu\}$ have a common dense subset, in particular, $\overline{\{u_K t^\mu\}} = \overline{\{u_L t^\mu\}}$ in the affine Grassmannian.*

Proof. Let us first assume $b > c$. We have $\left\langle \mu_1, \varepsilon_{d_j} - \varepsilon_c \right\rangle = \left\langle \mu_s, \varepsilon_{d_j} - \varepsilon_c \right\rangle = \left\langle \mu_3, \varepsilon_{d_j} - \varepsilon_c \right\rangle$, and therefore

$$u_K t^\mu = u_{K''} u_{\begin{array}{|c|c|}\hline b & c \\ \hline \multicolumn{1}{c|}{} & d_1 \\ \cline{2-2} \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & d_t \\ \cline{2-2}\end{array}}(\mu_1)\Big(\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu_1})}(f(\Delta))\Big) u_{K'}(\mu_3) t^\mu$$

$$= u_{K''}(\mu_1) u_{\begin{array}{|c|}\hline c \\ \hline d_1 \\ \hline \vdots \\ \hline d_t \\ \hline\end{array}}(\mu_1) u_{\begin{array}{|c|}\hline c \\ \hline\end{array}}(\mu_2)\Big(\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu_1})}(f(\Delta))\Big) u_{K'}(\mu_3) t^\mu.$$

Now the products of the form

$$\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu_2})}$$

are elements ofn $U^-(\mathcal{O}_{\mu_s})$, which is the intersction of $U^-(\mathcal{K})$ and the stabilizer $G(\mathcal{O}_{\mu_2})$ in $G(\mathcal{K})$ of the vertex μ_2. All the root subgroups $u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu_2})}$ lie in the stabilizer of the edge joining μ_2 with μ_3. Hence we know by Lemma 5.2.2 that

$$\begin{aligned} & u_{\boxed{c}}(\mu_2)\Big(\prod_{j=1}^{t} u_{(\varepsilon_{d_j}-\varepsilon_c,\overline{\mu_1})}(f(\Delta))\Big)u_{K'}(\mu_3)t^\mu \\ &= u_{\boxed{c}}(\mu_2)U^-(\mathcal{O}_{\mu_2})u_{K'}(\mu_3)t^\mu \\ &= u_{\boxed{c}}(\mu_2)u_{K'}(\mu_3)t^\mu \end{aligned}$$

and as a direct consequence, we obtain

$$u_K t^\mu = u_L t^\mu$$

on a dense subset. Suppose now $b = c$, then by Lemma 5.2.6, we have

$$u_K t^\mu = u_{K''} u_{\begin{array}{|c|c|}\hline b & b \\ \hline d_1 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} d_T & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} u_{K'}(\mu_3)t^\mu = u_{K''} u_{\begin{array}{|c|c|}\hline b & b \\ \hline \multicolumn{1}{c|}{} & d_1 \\ \cline{2-2} \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & d_T \\ \cline{2-2}\end{array}} u_{K'}(\mu_3)t^\mu = u_L t^\mu.$$

□

Theorem 5.2.1 *Given a generalized tableau K, let T_K be the unique semi-standard Young tableaux obtained from K by the bumping algorithm and denote by λ_K the shape of T_K and by μ_k the weight of K. Then*

$$\overline{\pi(C_K)} = \overline{\pi(C_{T_K})}$$

is an MV-cycle in the Schubert variety X_{λ_K} of weight μ_K.

Proof. Given a generalized tableaux $K = K_1 * K_2 * K_3$, such that K_2 is just one column and K_3 is a semi-standard Young tableaux and maximal with this property. Let μ be the weight of K. The proof will be by induction on the number of boxes of K_3. Remember, that K_3 is never empty, but it may be just one column or even just one box. We will show, that there exists another generalized tableau, which can be obtained by the bumping algorithm from K and its semi-standard part has at least one box more. If we have

$$K_2 = \begin{array}{|c|}\hline b \\ \hline p_2 \\ \vdots \\ \hline p_s \\ \hline\end{array} \text{ then let } L = K_1 * \begin{array}{|c|c|}\hline p_2 & b \\ \hline p_3 & \multicolumn{1}{c}{} \\ \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} p_2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} * K_3.$$

Now $u_K t^\mu = u_L t^\mu$ since read box-wise K and L result in the same gallery and are related by the bumping algorithm. So without loss of generality, we may assume

$$K = K_1 * \boxed{b} * \underbrace{\begin{array}{|c|}\hline q_1 \\ \hline q_2 \\ \hline \vdots \\ \hline q_s \\ \hline \end{array} * K_3'}_{K_3}.$$

If we have $p > q_s$, then we set

$$L = K_1 * \begin{array}{|c|}\hline q_1 \\ \hline q_2 \\ \hline \vdots \\ \hline q_s \\ \hline b \\ \hline \end{array} * K_3'.$$

Again, we have $u_K t^\mu = u_L t^\mu$ since K and L result in the same gallery read box-wise, but they are related by bumping and in L the semi-standard part has one box more than in K. Let K be of the form

$$K = K_1 * \underbrace{\begin{array}{|c|c|}\hline b & a_1 \\ \hline \multicolumn{1}{c|}{} & a_2 \\ \cline{2-2} \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & a_s \\ \cline{2-2} \multicolumn{1}{c|}{} & c \\ \cline{2-2} \multicolumn{1}{c|}{} & d_1 \\ \cline{2-2} \multicolumn{1}{c|}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & d_t \\ \cline{2-2} \end{array}}_{:=K_3} * K_3',$$

where $a_1 < a_2 < ... < a_s < b \leq c < d_1 < ... < d_t \leq n+1$. By the above, we have $u_K t^\mu = u_L t^\mu$ since read box-wise, K and L give the same gallery, where

$$L = K_1 * \begin{array}{|c|c|c|}\hline b & c & a_1 \\ \hline \multicolumn{1}{c|}{} & d_1 & a_2 \\ \cline{2-3} \multicolumn{1}{c|}{} & \vdots & \vdots \\ \cline{2-3} \multicolumn{1}{c|}{} & \cdot & a_s \\ \cline{2-3} \multicolumn{1}{c|}{} & \cdot & \multicolumn{1}{c}{} \\ \cline{2-2} \multicolumn{1}{c|}{} & d_t & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} * K_3'.$$

Again, K and L are related by the bumping algorithm. By Proposition 5.2.1, $u_M t^\mu$

and $u_L t^\mu$ have a common open subset, where

$$M = K_1 * \begin{array}{|c|c|c|}
\hline
b & c & a_1 \\
\hline
d_1 & \multicolumn{1}{c|}{} & a_2 \\
\cline{1-1}\cline{3-3}
\vdots & \multicolumn{1}{c|}{} & \vdots \\
\cline{1-1}\cline{3-3}
\cdot & \multicolumn{1}{c|}{} & a_s \\
\cline{1-1}\cline{3-3}
d_t & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array} * K_3'.$$

Note, that M and L are related by the bumping algorithm. By Corollary 5.2.1, $u_M t^\mu$ and $u_L t^\mu$ have a common open subset, where

$$M = K_1 * \begin{array}{|c|c|c|}
\hline
b & a_1 & c \\
\hline
d_1 & a_2 & \multicolumn{1}{c}{} \\
\cline{1-2}
\vdots & \vdots & \multicolumn{1}{c}{} \\
\cline{1-2}
\cdot & a_s & \multicolumn{1}{c}{} \\
\cline{1-2}
d_t & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array} * K_3',$$

and the associated generalized tableaux are related by the bumping algorithm. We set

$$N = K_1 * \begin{array}{|c|c|}
\hline
a_1 & c \\
\hline
a_2 & \multicolumn{1}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{1}{c}{} \\
\cline{1-1}
a_2 & \multicolumn{1}{c}{} \\
\cline{1-1}
b & \multicolumn{1}{c}{} \\
\cline{1-1}
d_1 & \multicolumn{1}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{1}{c}{} \\
\cline{1-1}
d_t & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array} * K_3'$$

and $u_M t^\mu = u_N t^\mu$, since read box-wise M and N result in the same gallery and are related by the bumping algorithm. By iterating this process, after a finite number of steps we obtain a generalized tableau Q, such that its semi-standard part hast at least one box more and $u_K t^\mu$ and $u_Q t^\mu$ coincide on a dense subset. $\square$

Now consider two words equivalent under the Knuth relations. We regard them as tableau of type $N\omega_1$. Since the equivalence means, that they result in the same semi-standard Young tableau using the bumping algorithm and since the MV-cycle associated to the cell of an LS-gallery is unique, we obtain the following theorem:

Theorem 5.2.2 *If and only if two words are equivalent under the Knuth relations, then the closure of the images of the cells associated to two words are the same. They form an MV-cycle of coweight (λ, μ), where μ is their target and λ is the shape of the semi-standard Young tableau resulting from the bumping algorithm.*

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Erklärung

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Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Herrn Professor Doktor Peter Littelmann betreut worden.

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