Weyl modules for equivariant map algebras and Kirillov-Reshetikhin crystals

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Zusammenfassung

Die Darstellungstheorie von (getwisteten) Schleifen-und Stromalgebren hat in den letzen Jahrzenten stark an Attraktivität gewonnen, z.B. wurden lokale Weylmoduln, Demazuremoduln und Kirillov-Reshetikhin-Moduln intensiv untersucht. Die äquivarianten Funktionenalgebren stellen eine umfangreiche Klasse von Algebren dar, welche die (getwisteten) Schleifen-und Stromalgebren verallgemeinern. Wir haben die Definition von lokalen Weylmoduln für äquivariante Funktionenalgebren erweitert, wo $\mathfrak{g}$ halbeinfach, X affin vom endlichen Typ und die Gruppe Γ eine abelsche Gruppe ist, die frei auf X operiert. Wir haben eine Verbindung, genauer einen Isomorphismus, zwischen einer Unterkategorie von Darstellungen von äquivarianten Funktionenalgebren und deren ungetwisteten Analoga erzielt. Wir haben ebenfalls gezeigt, dass weitere Eigenschaften von lokalen Weylmoduln (z.B. deren Charakterisierung durch homologische Eigenschaften und eine Tensorprodukt-Eigenschaft) auch für äquivariante Funktionenalgebren gelten. In dem Fall wo die Operation nicht frei ist, haben wir lokale Weylmoduln für getwistete Stromalgebren untersucht. Wir haben diese mit affinen Demazuremoduln identifiziert und eine explizite Konstruktion dieser ausgehend von ungetwisteten Weylmoduln angegeben, die das Fusionsprodukt verallgemeinert. Mit Hilfe dieser Resultate haben wir somit eine Dimensions und Charakterformel erhalten. Auf der Seite der Kombinatorischen Darstellungstheorie haben wir eine explizite Realisierung von Kirillov-Reshetikhin-Kristallen über Polytope für den affinen Typ A bewiesen. Der Vorteil dieser Realisierung besteht darin, dass alle Formeln explizit angegeben sind. Diese Realisierung erlaubt es die Kombinatorik von kristallinen Basen von Kirillov-Reshetikhin-Moduln zu beschreiben.

Abstract

The representation theory of (twisted) loop and current algebras has gained a lot of attraction during the last decades, e.g. local Weyl modules, Demazure modules and Kirillov-Reshetikhin modules were investigated intensively. The equivariant map algebras are a large class of algebras that are generalizations of (twisted) loop and current algebras. We have extended the definition of local Weyl modules to the setting of equivariant map algebras where $\mathfrak{g}$ is semisimple, X is affine of finite type, and the group Γ is abelian and acts freely on X. We have established a link, particularly an isomorphism, between certain categories of representations of equivariant map algebras and their untwisted analogues. We have also shown that other properties of local Weyl modules (e.g. their characterization by homological properties and a tensor product property) extend to the more general setting of equivariant map algebras. When the assumption of freeness does not hold we have investigated local Weyl modules for twisted current algebras. We have identified them with corresponding affine Demazure modules and have given an explicit construction from untwisted Weyl modules which generalize the fusion product. Therefore, we deduce from these results dimension and character formulas. On the combinatorial representation theory side, we have given an explicit realization of Kirillov-Reshetikhin crystals for the affine type A via polytopes. The advantage of this realization is mainly the fact that all formulas are explicit. This realization allows to describe explicitly the combinatorics of crystal bases of Kirillov-Reshetikhin modules.

dedicated to my parents

Contents

Introduction

Lie algebras were orginally introduced by S. Lie as an algebraic structure whose main use is in studying geometric objects such as Lie groups. The classification of simple finite-dimensional Lie algebras over $\mathbb{C}$ was provided by W. Killing and E. Cartan by the end of the 19th century and found huge applications in mathematical physics. This theory was extended in 1967 where V.G. Kac and R.V. Moody introduced independently Kac-Moody algebras. These algebras include all simple finite-dimensional Lie algebras but also many infinite-dimensional examples. Kac-Moody algebras have applications in many areas of mathematics and theoretical physics, e.g. group theory, combinatorics, differential equations, invariant theory and statistical physics. It is thus important to understand their structure. For the investigation of their structure representation theory plays a crucial role, which is one of the classical branches of mathematics. It is dedicated to the study of algebraic structures, e.g. Lie algebras, by representing their elements as linear transformations of vector spaces. In the last 50 years the research on arbitrary Lie algebras, as affine Lie algebras, quantum algebras, loop algebras or equivariant map algebras, became a highly competitive field in mathematics.
This thesis consists of three articles, which are published or will be published soon

I G. Fourier, T. Khandai, D. Kus, A. Savage
Local Weyl modules for equivariant map algebras with free abelian group actions
J. Algebra 350 (2012), 386–404

II G. Fourier, D. Kus,
Demazure modules and Weyl modules: The twisted current case,
to appear in Transactions of the AMS

III D. Kus,
Realization of affine type A Kirillov-Reshetikhin crystals via polytopes,
submitted to Journal of Combinatorial Theory, Series A

In this thesis we study the category of finite-dimensional representations of certain equivariant map algebras. Let X be a scheme and let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, both defined over an algebraically closed field k of characteristic zero. Assuming that a finite group Γ is acting on both (X and $\mathfrak{g}$) by automorphisms, the equivariant map algebra $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$ is the Lie algebra of regular maps $X \longrightarrow \mathfrak{g}$ which are equivariant with respect to the action of Γ.
A more algebraic definition of these algebras is provided via the identification with fixed point algebras of the diagonal action of Γ on $(\mathfrak{g} \otimes A)$, where the action of Γ on the coordinate ring A of X is induced by the action of Γ on X, i.e. $\mathfrak{M} = (\mathfrak{g} \otimes A)^\Gamma$.
One important class of equivariant map algebras is the class of loop algebras ($X = \mathbb{C}^*, \Gamma = \{1\}$), playing a significant role in the theory of affine Lie algebras. The classification of their irreducible finite-dimensional representations is worked out by Chari and Pressley in [5], [11], [12].
Another class of equivariant map algebras are the current algebras ($X = \mathbb{C}, \Gamma = \{1\}$), Onsager algebras and tetrahedon algebras. The irreducible finite-dimensional representations of the latter two algebras are classified in [15] and [25] respectively. Summarizing the results, it is shown that all irreducible finite-dimensional representations are evaluation representations.

A complete list of irreducible finite-dimensional representations of an arbitrary equivariant map algebra is provided in [40]. The main result is that any finite-dimensional irreducible representation is a tensor product of evaluation representations and a one-dimensional representation. In particular, not all one-dimensional representations are evaluation representations.

Since the category of finite-dimensional modules of equivariant map algebras is not semisimple (even the category of finite-dimensional modules of loop algebras is not semisimple), the set of representations which can be assembled out of the irreducible ones is far from being the whole list of finite-dimensional representations. Thus, many other classes of representations can be defined and studied.

For instance local Weyl modules, global Weyl modules, Demazure modules and Kirillov-Reshetikhin modules were investigated for (twisted) loop algebras. For details we refer to a serie of papers ([6],[7],[8],[9],[13],[14],[16],[17],[20],[21],[22]).

The local Weyl modules for loop algebras, denoted by $W(\psi)$, are parametrized by finitely supported functions ψ from $X = \mathbb{C}^*$ to P^+, the set of dominant integral weights for $\mathfrak{g}$, and have the property that any finite-dimensional highest weight module of highest weight ψ and one-dimensional highest weight space is a quotient of $W(\psi)$. Moreover, they have a nice tensor product decomposition into "smaller" local Weyl modules supported in a single point. Furthermore, it was conjectured in [14] that they play an important role in the theory of quantum affine algebras (q-deformation of the loop algebra), namely that all local Weyl modules are obtained as $q = 1$ limits of irreducible finite-dimensional representations of quantum affine algebras. The above conjecture can be reduced to computing dimensions and characters for local Weyl modules supported in a single point. Additionaly by using pull back maps, it is sufficient to compute dimensions and characters of local Weyl modules supported in zero. The conjecture is proven in the following papers:

- for the Lie algebra $\mathfrak{sl}_2$ by Chari-Pressley in [14]
- for the Lie algebra $\mathfrak{sl}_\mathfrak{n}$ by Chari-Loktev in [9]
- for simply-laced Lie algebras in [21]
- for non simply-laced Lie algebras in [38].

There are several ways to generalize the notion of local Weyl modules. By replacing $\mathbb{C}[t, t^{-1}]$ with a commutative, associative algebra ([7],[17]) one can define local and global Weyl modules as before and obtain similar properties, but character and dimension formulas are known only in certain cases.

Another way of generalizing local Weyl modules is to consider twisted current and twisted loop algebras or more general equivariant map algebras. The local Weyl modules for twisted loop algebras, also called twisted Weyl modules and denoted by $W^\Gamma(\psi)$, are studied in [8], where dimensions and characters are computed.

The more general setting of generalized Weyl modules for certain equivariant map algebras are studied in [**I**]. The initial problem here is, that for the study of representations of arbitrary equivariant map algebras, one needs new techniques, since past approaches to the study of representations for twisted loop algebras rely heavily on the representation theory of $\mathfrak{g}$. Only in a few cases one is assured of the existence of a semisimple fixed point subalgebra $\mathfrak{g}^\Gamma$ or a Cartan subalgebra of $\mathfrak{M}$ as in the classical sense. In [**I**] we develop new techniques for the study of local Weyl modules and obtain the following results:

We assume that $\mathfrak{g}$ is semisimple, X is of finite type, Γ is abelian, and Γ acts freely on X. The (twisted) loop algebras are still among these equivariant map algebras, while the (twisted) current algebras violate the freeness condition. It means, up to this point, the results of [**I**] do not apply to the twisted current case.

In the case where the action of Γ is free, all irreducible finite-dimensional representations of $M(X,\mathfrak{g})^\Gamma$ are tensor products of evaluation modules. To be more precise, the map

$$\mathcal{E}^\Gamma \longrightarrow \mathcal{S}^\Gamma, \quad \psi \mapsto \bigotimes_{i=1}^{n} V_{x_i}(\psi(x_i)),$$

is a bijection, where $\mathcal{S}^\Gamma$ denotes the set of isomorphism classes of irreducible finite-dimensional representations of $M(X,\mathfrak{g})^\Gamma$ and $\mathcal{E}^\Gamma$ is the set of Γ-invariant finitely supported functions $\psi : X_{\mathrm{rat}} \to P^+$. The $x_i's$ are representatives of each orbit in the support of ψ, i.e. $\operatorname{Supp}\psi = \{\Gamma x_1, \cdots, \Gamma x_n\}$.

Let $\mathcal{F}$ and $\mathcal{F}^\Gamma$ denote the category of finite-dimensional $(\mathfrak{g}\otimes A)$-modules and $(\mathfrak{g}\otimes A)^\Gamma$-modules respectively. Let X_* be the set of finite subsets $\mathbf{x} = \{x_1, \cdots, x_n\}$ such that $\Gamma x_i \cap \Gamma x_j = \emptyset$. For $\mathbf{x} \in X_*$, let $\mathcal{F}_\mathbf{x}$ (respectively $\mathcal{F}^\Gamma_\mathbf{x}$) denote the full subcategory of $\mathcal{F}$ (respectively $\mathcal{F}^\Gamma$) consisting of modules with support contained in $\mathbf{x}$ (respectively $\Gamma\cdot\mathbf{x}$). Then we have on the one hand an isomorphism of Lie algebras

$$(\mathfrak{g}\otimes A)^\Gamma/(\mathfrak{g}\otimes I_\eta)^\Gamma \xrightarrow{\cong} (\mathfrak{g}\otimes A)/(\mathfrak{g}\otimes I_\eta)$$

for suitable ideals $(\mathfrak{g}\otimes I_\eta)$ and $(\mathfrak{g}\otimes I_\eta)^\Gamma$ respectively and on the other hand (by using the above isomorphism), we proved that we obtain for each $\mathbf{x} \in X_*$ mutually inverse isomorphisms of categories

$$\begin{tikzcd} \mathcal{F}_\mathbf{x} \arrow[r, shift left, "\mathbf{T}_\mathbf{x}"] & \mathcal{F}^\Gamma_\mathbf{x} \arrow[l, shift left, "\mathbf{U}_\mathbf{x}"] \end{tikzcd}$$

called *twisting* and *untwisting functors*.

These functors allow us to move back and forth between the theory of finite-dimensional representations of equivariant map algebras (satisfying the aforementioned assumptions) and the corresponding theory for map algebras. In particular we define the twisted Weyl modules as follows:

Let V be a finite-dimensional irreducible module for $M(X,\mathfrak{g})^\Gamma$ and let $\mathbf{x} \in X_*$ contain one point in each Γ-orbit in the support of V. Then $\mathbf{U}_\mathbf{x}V$ is an irreducible finite-dimensional $(\mathfrak{g}\otimes A)$-module, to which is associated an untwisted local Weyl module $W(\psi)$. We then define the twisted local Weyl module associated to V to be $\mathbf{T}_\mathbf{x}W = W^\Gamma(\psi)$, and it is shown that this definition is independent of the choice of $\mathbf{x}$ (see [**I**],[Proposition 3.6]). By using the above definition, a tensor product decomposition is proven. Moreover, a justification of the definition is provided by proving a similar characterization of twisted local Weyl modules by homological properties as in [7]. More precisely, for a maximal weight $(\mathfrak{g}\otimes A)^\Gamma$-module M ([**I**] ,[Definition 4.2]) of maximal weight ψ we have $M \cong W_\Gamma(\psi)$ if and only if

$$\operatorname{Hom}_{\mathcal{F}^\Gamma}(M, V_\Gamma(\varphi)) = 0 \text{ and } \operatorname{Ext}^1_{\mathcal{F}^\Gamma}(M, V_\Gamma(\varphi)) = 0 \ \forall\ \varphi \in \mathcal{E}^\Gamma \text{ with } \operatorname{ht}(\varphi) < \operatorname{ht}(\psi).$$

The advantage of this characterization is that it can be used as a general definition of local Weyl modules for arbitrary equivariant map algebras, where the action is not necessarily free. We recall that the Weyl module conjecture for twisted current algebras is still unsolved

up to this point, since the results of [**I**] do not apply. This issue is treated particularly in [**II**], where the gap in the computation of dimension and character formulas for local Weyl modules of twisted current algebras is worked out. The techniques are the following:
Let Γ be the finite group of order 2 or 3 of non-trivial diagram automorphism of a simple Lie algebra $\mathfrak{g}$ of type A, D, E. The action by automorphisms on $X = \mathbb{C}$ is given by multiplication with ξ, a primitive 2nd or 3rd root of unity. Denote by P_0^+ the set of dominant weights of the fixed point algebra $\mathfrak{g}_0$. Then we can divide our results into two parts:
Let $\mathfrak{g}$ be not of type A_{2l}, then for $\lambda \in P_0^+$ we obtain that the graded Weyl modules $W^\Gamma(\lambda)$ can be identified with level one Demazure modules and furthermore it is isomorphic to the associated graded module of the restriction of a local Weyl module for $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. To be more precise we get the following isomorphisms,

$$W^\Gamma(\lambda) \cong D(1, \lambda) \text{ and } W^\Gamma(\lambda) \cong \mathrm{gr}(\overline{W_a(\lambda)}).$$

In the second part $\mathfrak{g}$ is assumed to be of type A_{2l}. Here the Weyl modules are described for the weights $\lambda = \lambda_1 + \lambda_2 \in P_0^+$, satisfying the property $\lambda(\alpha_l^\vee), \lambda_2(\alpha_l^\vee) \in 2\mathbb{Z}+1$.
We obtain that the Weyl module is isomorphic to a Demazure module

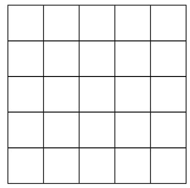
$$W^\Gamma(\lambda) \cong D(1/2, \lambda)$$

and moreover

$$W^\Gamma(\lambda) \cong \mathrm{gr}(W_a(\lambda_1) \otimes W^\Gamma(\lambda_2)).$$

In the case where $\lambda(\alpha_l^\vee)$ is even, the dimension and character of the local Weyl modules remain uncomputed and the identification with Demazure modules fails. The description of these Weyl modules and Weyl modules for arbitrary equivariant map algebras are therefore undetermined.
Another powerful tool of studying representations, as irreducible representations, Weyl modules, Demazure modules and Kirillov-Reshetikhin modules, is Kashiwara's crystal bases theory, introduced in [30]. Roughly speaking, crystal bases can be viewed as bases at $q = 0$ and they contain structures of edge-colored oriented graphs satisfying a set of axioms, called the crystal graphs. These crystal graphs have certain useful properties, for instance characters of $\mathbf{U}_q(\mathfrak{g})$-representations can be computed and the decomposition of tensor products of representations into irreducible ones can also be determined from the crystal graphs, to name just a few. It is thus an important problem to find explicit realizations of crystal graphs.
There are many such realizations of crystal graphs for irreducible representations for $\mathbf{U}_q(\mathfrak{g})$, combinatorial and geometrical, elaborated during the last decades, by way of example we refer to ([32],[33],[35],[37]).
Crystal bases for irreducible $\mathbf{U}_q'(\mathfrak{g})$ representations (quantum affine algebra) might not always exist. A certain subclass of these representations, that draw a lot of attraction during the last decades, are the so called Kirillov-Reshetikhin modules $\mathrm{KR}(m, \omega_i, a)$, where i is a node in the classical Dynkin diagram and m is a positive integer. It was first conjectured in [26], that $\mathrm{KR}(m, \omega_i, a)$ admits a crystal basis and was proven in type $A_n^{(1)}$ in [29] and in all non-exceptional cases in [41],[42]. We denote this crystal by $\mathrm{KR}^{m,i}$ and call it a Kirillov-Reshetikhin crystal.
In [**III**] we gave an explicit realization of Kirillov-Reshetikhin crystals for the affine type $A_n^{(1)}$ via polytopes. The results were the following:

The polytope introduced in [18] for all dominant integral A_n weights λ can be understood for $\lambda = m\omega_i$ as a subset of $\mathbb{R}^{i(n-i+1)}$. We denote the intersection of this polytope with $\mathbb{Z}_+^{i(n-i+1)}$ by $B^{m,i}$. We defined certain maps on $B^{m,i}$, among others the Kashiwara operators, and proved that this becomes a classical crystal of type A_n. As a set, we can identify $B^{m,i}$ with certain blocks of height $n-i+1$ and width i



where the boxes are filled, under some assumptions, with some non-negative integers (see [**III**] ,[Definition 2.1]).
Subsequently we have constructed certain local A_2 isomorphisms on our underlying polytope $B^{m,i}$ and proved that the so called Stembridge axioms are satisfied. These axioms precisely characterize the set of crystals of representations in the class of all crystals. Our first important result was therefore:
The polytope $B^{m,i}$ is as an A_n crystal isomorphic to $B(m\omega_i)$ (the one obtained from Kashiwara's crystal bases theory), i.e.

$$B^{m,i} \cong B(m\omega_i), \text{ as } \{1, \cdots, n\}\text{-crystals.}$$

As a consequence we obtain that the classical crystal structure we gave coincide with the classical KR crytal structure because in [46] it was shown that, as a $\{1, \cdots, n\}$-crystal, $\mathrm{KR}^{m,i}$ is isomorphic to $B(m\omega_i)$.
A promotion operator pr on a crystal B of type A_n is defined to be a map satisfying several conditions, namely that pr shifts the content, $pr \circ \tilde{e}_j = \tilde{e}_{j+1} \circ pr$, $pr \circ \tilde{f}_j = \tilde{f}_{j+1} \circ pr$ for all $j \in \{1, \cdots, n-1\}$ and $pr^{n+1} = id$, where $\tilde{e}_j$ and $\tilde{f}_j$ respectively are the Kashiwara operators. If the latter condition is not satisfied, but pr is still bijective, the map pr is called a weak promotion operator (see also [2]). The advantage of such (weak) promotion operators is that we can associate to a given crystal B of type A_n a (weak) affine crystal by setting $\tilde{f}_0 := pr^{-1} \circ \tilde{f}_1 \circ pr, \text{ and } \tilde{e}_0 := pr^{-1} \circ \tilde{e}_1 \circ pr.$
On the set of all semi-standard Young tableaux of rectangle shape, which is a realization of $B(m\omega_i)$, Schtzenberger defined a promotion operator, called the Schtzenberger's promotion operator [45], by using jeu-de-taquin.
In [46] it is shown that the affine crystal constructed from $B(m\omega_i)$ (realized as a set of semi-standard Young tableaux) using Schtzenberger's promotion operator is isomorphic to the Kirillov-Reshetikhin crystal $\mathrm{KR}^{m,i}$. As a result, it was of strong interest to define a bijective map on our polytope satisfying the properties of a promotion operator. In ([**III**],[Section 6]) we defined a map via an algorithm consisting of i steps, and showed that this map satisfies the conditions for a promotion operator and thus provided an explicit realization of Kirillov-Reshetikhin crystals for the affine type $A_n^{(1)}$ via polytopes. The algorithm can be implemented easily and hence provides a new method to calculate KR crystals with the computer.

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LOCAL WEYL MODULES FOR EQUIVARIANT MAP ALGEBRAS WITH FREE ABELIAN GROUP ACTIONS

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Abstract. Suppose a finite group Γ acts on a scheme X and a finite-dimensional Lie algebra $\mathfrak{g}$. The associated *equivariant map algebra* is the Lie algebra of equivariant regular maps from X to $\mathfrak{g}$. Examples include generalized current algebras and (twisted) multiloop algebras.

Local Weyl modules play an important role in the theory of finite-dimensional representations of loop algebras and quantum affine algebras. In the current paper, we extend the definition of local Weyl modules (previously defined only for generalized current algebras and twisted loop algebras) to the setting of equivariant map algebras where $\mathfrak{g}$ is semisimple, X is affine of finite type, and the group Γ is abelian and acts freely on X. We do so by defining *twisting* and *untwisting functors*, which are isomorphisms between certain categories of representations of equivariant map algebras and their untwisted analogues. We also show that other properties of local Weyl modules (e.g. their characterization by homological properties and a tensor product property) extend to the more general setting considered in the current paper.

Contents

Introduction

Partially because of their importance in the theory of quantum affine Lie algebras, loop algebras $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$, where $\mathfrak{g}$ is a semisimple Lie algebra, have been the subject of intense study over the last two decades. Their representation theory is particularly interesting because the category of finite-dimensional representations is not semisimple. In [3, 8], it was shown that the irreducible objects in these categories are highest weight in a suitable sense, and a classification was given in terms of these highest weights, which are n-tuples of polynomials, where n is the rank of $\mathfrak{g}$. In [9], it was shown that to each such n-tuple of polynomials π,

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there exists a unique largest highest weight module $W(\pi)$ of highest weight π. The modules $W(\pi)$, called *(local) Weyl modules* by analogy with the modular representation theory of (the positive characteristic version of) $\mathfrak{g}$, have the property that any finite-dimensional highest weight module of highest weight π is a quotient of $W(\pi)$.

Weyl modules for loop algebras also play an important role in the representation theory of quantum affine algebras. In particular, under a natural condition on their highest weight, the irreducible finite-dimensional representations of quantum affine algebras specialize at $q = 1$ to representations of the loop algebras. In this limit, the representations are no longer irreducible, but are quotients of the corresponding local Weyl module. It was conjectured (and proved for $\mathfrak{g} = \mathfrak{sl}_2$) in [9] that all local Weyl modules are obtained as $q = 1$ limits of irreducible finite-dimensional modules of quantum affine algebras. In particular, this conjecture implies that the local Weyl modules are the classical limits of the standard modules defined by Nakajima in [15] and further studied by Varagnolo and Vasserot in [19].

In [9], Chari and Pressley defined the *global Weyl modules* associated to dominant integral weights of $\mathfrak{g}$. These are the largest integrable highest weight modules of the given highest weight and were conjectured to be free modules for a certain commutative algebra. This motivated a series of papers [1, 6, 7, 11, 15, 16] on local Weyl modules which computed their dimension and character, identified them with tensor products of Demazure modules, and eventually lead to the proof of this conjecture as well as the aforementioned conjecture that all local Weyl modules are $q = 1$ limits of irreducible finite-dimensional modules of quantum affine algebras (for an arbitrary simple $\mathfrak{g}$).

In [10], Feigin and Loktev extended the notion of global Weyl modules to the setting of generalized current algebras $\mathfrak{g} \otimes A$, where A is a commutative associative unital algebra over the complex numbers. In the case that A is the coordinate ring of an affine variety, they also extended the definition of local Weyl modules and obtained analogues of some of the results of [9]. In particular, they proved that these modules are finite-dimensional and that every local Weyl module is the tensor product of local Weyl modules associated to a single point (a property which is also true for finite-dimensional irreducible modules).

Motivated by the methods used to study the BGG-category $\mathcal{O}$ for semisimple Lie algebras, a functorial approach to the study of the Weyl modules for generalized current algebras was adopted in [4]. There it was shown that, via homological properties, one can naturally define more general Weyl modules for the Lie algebra $\mathfrak{g} \otimes A$, where A is a commutative associative unital algebra over the complex numbers. This is done by defining the *Weyl functor* from a suitable category of modules for a commutative algebra $\mathbf{A}_\lambda$ (these modules play the role of highest weight spaces) to the category of integrable modules for $\mathfrak{g} \otimes A$ with weights bounded by a dominant integrable weight λ of $\mathfrak{g}$. Under the condition that A is finitely generated, it was shown that every local Weyl module is finite-dimensional. Furthermore, the translation of the universal property of the Weyl module into the language of homological algebra yielded a simplified proof of the tensor product property.

The algebras mentioned above all are "untwisted". There are natural twisted versions of loop algebras, related to the twisted affine Lie algebras. More precisely, the twisted loop algebras are fixed point subalgebras of untwisted loop algebras $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ under the action of certain finite-order automorphisms. Extending the ideas of [9], local Weyl modules for the twisted loop algebras were defined and studied in [5], where it was realized that they can be identified with suitably chosen local Weyl modules for untwisted loop algebras. It is thus natural to ask if twisted versions of local Weyl modules exist when one moves from loop algebras to the more general setting of generalized current algebras.

The twisted analogues of generalized current algebras are *equivariant map algebras*. Suppose $X = \operatorname{Spec} A$ is an affine scheme and $\mathfrak{g}$ is a finite-dimensional Lie algebra, both defined over an algebraically closed field of characteristic zero, and that Γ is a finite group acting on both X (equivalently, on A) and $\mathfrak{g}$ by automorphisms. Then the equivariant map algebra $(\mathfrak{g} \otimes A)^\Gamma$ is the Lie algebra of equivariant algebraic maps from X to $\mathfrak{g}$. In the current paper, we will assume that $\mathfrak{g}$ is semisimple, X is of finite type, Γ is abelian, and Γ acts freely on X. Even with these restrictions, equivariant map algebras are a large class of Lie algebras that include the above mentioned examples of (twisted) loop algebras and generalized current algebras as well as many others.

A complete classification of the irreducible finite-dimensional representations of an equivariant map algebra was given in [18]. Let X_* denote the set of finite subsets of X_{rat}, the set of rational points of X, that does not contain two points in the same Γ-orbit. For $\mathbf{x} \in X_*$, we have a surjective *evaluation map*

$$\mathrm{ev}_{\mathbf{x}}^\Gamma : (\mathfrak{g} \otimes A)^\Gamma \to \mathfrak{g}^{\mathbf{x}} = \textstyle\bigoplus_{x \in \mathbf{x}} \mathfrak{g}.$$

An *evaluation representation* is a representation of the form $\rho \circ \mathrm{ev}_{\mathbf{x}}^\Gamma$, where $\rho = \bigotimes_{x \in \mathbf{x}} \rho_x$ for representations $\rho_x : \mathfrak{g} \to \operatorname{End} V_x$, $x \in \mathbf{x}$. In the setup of the current paper, the classification of [18] says that all irreducible finite-dimensional representations are evaluation representations. We define the *support* of an irreducible finite-dimensional representation to be $\bigcup(\Gamma \cdot x)$, where the union is over the $x \in \mathbf{x}$ such that ρ_x is nontrivial. For an arbitrary finite-dimensional representation, we define its support to be the union of the supports of its irreducible constituents. This support depends only on the isomorphism class of the representation.

For an equivariant map algebra, one is not assured of the existence of a semisimple fixed point subalgebra $\mathfrak{g}^\Gamma$ or a Cartan subalgebra of $(\mathfrak{g} \otimes A)^\Gamma$ in the classical sense. Since past approaches to the study of Weyl modules for twisted loop algebras rely heavily on the representation theory of $\mathfrak{g}^\Gamma$, this is a major obstacle to generalizing such techniques to the more general setting of equivariant maps algebras. Furthermore, owing to the unavailability of the classical notion of weights for $(\mathfrak{g} \otimes A)^\Gamma$-modules, the notion of highest weight modules is not clear in this context. For these reasons, new techniques are needed.

Let $\mathcal{F}$ and $\mathcal{F}^\Gamma$ denote the category of finite-dimensional $(\mathfrak{g} \otimes A)$-modules and $(\mathfrak{g} \otimes A)^\Gamma$-modules respectively. For $\mathbf{x} \in X_*$, let $\mathcal{F}_{\mathbf{x}}$ (respectively $\mathcal{F}_{\mathbf{x}}^\Gamma$) denote the full subcategory of $\mathcal{F}$ (respectively $\mathcal{F}^\Gamma$) consisting of modules with support contained in $\mathbf{x}$ (respectively $\Gamma \cdot \mathbf{x}$). Motivated by [5, 14, 18] we define, for each $\mathbf{x} \in X_*$, mutually inverse isomorphisms of categories

$$\mathcal{F}_{\mathbf{x}} \underset{\mathbf{U}_{\mathbf{x}}}{\overset{\mathbf{T}_{\mathbf{x}}}{\rightleftarrows}} \mathcal{F}_{\mathbf{x}}^\Gamma$$

called *twisting* and *untwisting functors* (see Theorem 2.10). These functors allow us to move back and forth at will between the theory of finite-dimensional representations of equivariant map algebras (satisfying the assumptions of the current paper) and the corresponding theory for generalized current algebras. In particular, to any irreducible finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-module V, we can associate a twisted local Weyl module as follows. Let $\mathbf{x} \in X_*$ contain one point in each Γ-orbit in the support of V. Then $\mathbf{U_x}V$ is an irreducible finite-dimensional $(\mathfrak{g}\otimes A)$-module, to which is associated an (untwisted) local Weyl module W. We then define the local Weyl module associated to V to be $\mathbf{T_x}W$, and one can show that this definition is independent of the choice of $\mathbf{x}$ (see Proposition 3.6).

Apart from their role in the definition of the twisted local Weyl modules, the twisting and untwisting functors also allow us to use the characterization of local Weyl modules by homological properties given in [4] to give a similar characterization of twisted local Weyl modules. However, some subtlety is involved here. The homological characterization given in [4] involves certain categories of highest weight modules. Since the Cartan subalgebra of $\mathfrak{g}$ is not necessarily preserved by the action of the group Γ, such methods do not immediately carry over to the twisted setting. In order to circumvent this problem, we replace the usual order on weights by another partial order arising from a suitably defined *height* function on the weight lattice. Our modified homological characterization is equivalent to the one given in [4], but has the advantage that it carries over to the twisted versions.

There are several natural questions arising from our treatment of local Weyl modules for equivariant map algebras. For instance, can one define global Weyl modules (see [4, 9]) and is there an analogue of the algebra $\mathbf{A}_\lambda$ defined in [4]? Can one extend the results of the current paper to the case where the group Γ does not act freely on X? It would also be interesting to further examine the relationship between the twisting and untwisting functors defined here and connections between the representation theory of twisted and untwisted quantum affine algebras appearing in the literature (see, for example, [12]).

The paper is organized as follows. In Section 1 we recall the definition of equivariant map algebras and certain results on their finite-dimensional irreducible representations. We introduce the twisting and untwisting functors in Section 2 and prove that they are isomorphisms of categories. In Section 3 we recall the results on local Weyl modules for generalized current algebras and then introduce the notion of local Weyl modules for equivariant map algebras. We also show there that they satisfy a natural tensor product property. Finally, in Section 4 we give a characterization of the local Weyl modules by homological properties.

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1. Equivariant map algebras and their irreducible representations

In this section, we review the definition of equivariant map algebras and the classification of their irreducible finite-dimensional representations given in [18]. Let k be an algebraically closed field of characteristic zero and A be unital associative commutative finitely generated k-algebra. We let $X = \operatorname{Spec} A$, the prime spectrum of A (so X is an affine scheme of finite type). A point $x \in X$ is called a *rational point* if $A/\mathfrak{m}_x \cong k$, where $\mathfrak{m}_x$ is the ideal corresponding to x. We denote the subset of rational points of X by X_{rat}. Since A is finitely generated, we have $X_{\mathrm{rat}} = \operatorname{maxSpec} A$. Suppose Γ is a finite abelian group acting on X (equivalently, on A) and on a semisimple Lie algebra $\mathfrak{g}$ by automorphisms. Let $\mathfrak{g} \otimes A$ be the Lie k-algebra of regular maps from X to $\mathfrak{g}$. This is a Lie algebra under pointwise multiplication. The *equivariant map algebra* $(\mathfrak{g} \otimes A)^\Gamma$ consists of the Γ-fixed points of the canonical (diagonal) action of Γ on $\mathfrak{g} \otimes A$. Thus $(\mathfrak{g} \otimes A)^\Gamma$ is the subalgebra of Γ-equivariant maps. In the current paper, we are interested in the case that Γ acts freely on X, by which we mean that it acts freely on X_{rat}. We shall assume this is the case for the entirety of the paper. Following the usual abuse of notation, we will use the terms 'module' and 'representation' interchangeably.

Remark 1.1. We could consider the more general case where $\mathfrak{g}$ is finite-dimensional reductive. However, then $(\mathfrak{g} \otimes A)^\Gamma \cong ([\mathfrak{g}, \mathfrak{g}] \otimes A)^\Gamma \oplus (Z(\mathfrak{g}) \otimes A)^\Gamma$ as Lie algebras, [17, (3.4)], where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple and $Z(\mathfrak{g})$ is the centre of $\mathfrak{g}$ (and so $(Z(\mathfrak{g}) \otimes A)^\Gamma$ is an abelian Lie algebra). The representation theory of $(\mathfrak{g} \otimes A)^\Gamma$ thus essentially "splits" and so it suffices to consider the case of $\mathfrak{g}$ semisimple. See [17] for details.

We denote by X_* the set of finite subsets $\mathbf{x} \subseteq X_{\mathrm{rat}}$ for which $\Gamma \cdot x \cap \Gamma \cdot x' = \varnothing$ for distinct $x, x' \in \mathbf{x}$. For $\mathbf{x} \in X_*$, we define $\mathfrak{g}^{\mathbf{x}} = \bigoplus_{x \in \mathbf{x}} \mathfrak{g}$. The evaluation map

$$\mathrm{ev}^\Gamma_{\mathbf{x}} : (\mathfrak{g} \otimes A)^\Gamma \to \mathfrak{g}^{\mathbf{x}}, \quad \mathrm{ev}^\Gamma_{\mathbf{x}}(\alpha) = (\alpha(x))_{x \in \mathbf{x}},$$

is a Lie algebra epimorphism [18, Cor. 4.6]. To $\mathbf{x} \in X_*$ and a set $\{\rho_x : x \in \mathbf{x}\}$ of (nonzero) representations $\rho_x : \mathfrak{g} \to \mathrm{End}_k V_x$, we associate the *evaluation representation* $\mathrm{ev}^\Gamma_{\mathbf{x}}(\rho_x)_{x \in \mathbf{x}}$ of $(\mathfrak{g} \otimes A)^\Gamma$, defined as the composition

$$(\mathfrak{g} \otimes A)^\Gamma \xrightarrow{\mathrm{ev}^\Gamma_{\mathbf{x}}} \mathfrak{g}^{\mathbf{x}} \xrightarrow{\bigotimes_{x \in \mathbf{x}} \rho_x} \mathrm{End}_k \left(\bigotimes_{x \in \mathbf{x}} V_x \right).$$

If all $\rho_x, x \in \mathbf{x}$, are irreducible finite-dimensional representations, then this is also an irreducible finite-dimensional representation of $(\mathfrak{g} \otimes A)^\Gamma$, [18, Prop. 4.9]. The *support* of an evaluation representation $V = \bigotimes_{x \in \mathbf{x}} V_x$, abbreviated $\mathrm{Supp}\, V$, is the union of all $\Gamma \cdot x$, $x \in \mathbf{x}$, for which ρ_x is not the one-dimensional trivial representation of $\mathfrak{g}$.

Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and a set of simple roots for $\mathfrak{g}$. Let P and Q be the corresponding weight and root lattices respectively, and let P^+ denote the set of dominant integral weights. For $\lambda \in P^+$, let $V(\lambda)$ be the corresponding irreducible representation of $\mathfrak{g}$ of highest weight λ. In this way we identify the set of isomorphism classes of irreducible finite-dimensional $\mathfrak{g}$-modules with P^+.

It is well known that $\mathrm{Aut}\, \mathfrak{g} \cong \mathrm{Int}\, \mathfrak{g} \rtimes \mathrm{Out}\, \mathfrak{g}$, where $\mathrm{Int}\, \mathfrak{g}$ is the group of inner automorphisms of $\mathfrak{g}$ and $\mathrm{Out}\, \mathfrak{g}$ is the group of diagram automorphisms of $\mathfrak{g}$. The diagram automorphisms act naturally on P, Q, and P^+. If ρ is an irreducible representation of $\mathfrak{g}$ of highest weight $\lambda \in P$ and γ is an automorphism of $\mathfrak{g}$, then $\rho \circ \gamma^{-1}$ is the irreducible representation of $\mathfrak{g}$ of highest weight $\gamma_{\mathrm{Out}} \cdot \lambda$, where γ_{Out} is the outer part of the automorphism γ (see [2, VIII, §7.2, Rem. 1]). So the group Γ acts naturally on each P^+ via the quotient $\mathrm{Aut}\, \mathfrak{g} \twoheadrightarrow \mathrm{Out}\, \mathfrak{g}$.

Let $\mathcal{E}$ denote the set of finitely supported functions $\psi : X_{\mathrm{rat}} \to P^+$ and let $\mathcal{E}^\Gamma$ denote the subset of $\mathcal{E}$ consisting of those functions which are Γ-equivariant. Here the support of $\psi \in \mathcal{E}$ is

$$\mathrm{Supp}\, \psi = \{x \in X_{\mathrm{rat}} \mid \psi(x) \neq 0\}.$$

If $\mathbf{x} \in X_*$ and ρ_x, ρ'_x are isomorphic representations of $\mathfrak{g}$ for each $x \in \mathbf{x}$, the evaluation representations $\mathrm{ev}^\Gamma_{\mathbf{x}}(\rho_x)_{x \in \mathbf{x}}$ and $\mathrm{ev}^\Gamma_{\mathbf{x}}(\rho'_x)_{x \in \mathbf{x}}$ are isomorphic. Therefore, for $\mathbf{x} \in X_*$ and representations ρ_x of $\mathfrak{g}^x$ for $x \in \mathbf{x}$, we define $\mathrm{ev}^\Gamma_{\mathbf{x}}([\rho_x])_{x \in \mathbf{x}}$ to be the isomorphism class of $\mathrm{ev}^\Gamma_{\mathbf{x}}(\rho_x)_{x \in \mathbf{x}}$.

For $\psi \in \mathcal{E}^\Gamma$, we define $\mathrm{ev}_\psi^\Gamma = \mathrm{ev}_{\mathbf{x}}^\Gamma(\psi(x))_{x\in\mathbf{x}}$, where $\mathbf{x} \in X_*$ contains one element of each Γ-orbit in $\operatorname{Supp}\psi$. By [18, Lem. 4.13], ev_ψ^Γ is independent of the choice of $\mathbf{x}$. If ψ is the map that is identically 0 on X, we define ev_ψ^Γ to be the isomorphism class of the trivial representation of $(\mathfrak{g}\otimes A)^\Gamma$. We say that an evaluation representation is a *single orbit evaluation representation* if its isomorphism class is ev_ψ^Γ for some $\psi \in \mathcal{E}^\Gamma$ whose support is contained in a single Γ-orbit. For all of the above notation, we drop the superscript Γ when $\Gamma = \{1\}$. For instance, for a finite subset $\mathbf{x} \subseteq X_{\mathrm{rat}}$, $\mathrm{ev}_{\mathbf{x}} : \mathfrak{g}\otimes A \to \mathfrak{g}^{\mathbf{x}}$ is the corresponding evaluation map. Similarly, for $\psi \in \mathcal{E}$, ev_ψ is the corresponding isomorphism class of representations of $\mathfrak{g}\otimes A$.

Proposition 1.2 ([18, Th. 5.5])**.** *The map*

$$\mathcal{E}^\Gamma \to \mathcal{S}^\Gamma, \quad \psi \mapsto \mathrm{ev}_\psi^\Gamma, \quad \psi \in \mathcal{E}^\Gamma,$$

is a bijection, where $\mathcal{S}^\Gamma$ denotes the set of isomorphism classes of irreducible finite-dimensional representations of $(\mathfrak{g}\otimes A)^\Gamma$. In particular, all irreducible finite-dimensional representations of $(\mathfrak{g}\otimes A)^\Gamma$ are evaluation representations.

Remark 1.3. The classification of irreducible finite-dimensional representations given in [18] is much more general than Proposition 1.2. In particular, it applies in the case that $\mathfrak{g}$ is any finite-dimensional Lie algebra, Γ is any finite group (i.e. not necessarily abelian), and the action of Γ is arbitrary (i.e. Γ need not act freely on X). In this generality, all irreducible finite-dimensional representations are tensor products of evaluation representations and one-dimensional representations. However, under the more restrictive assumptions of the current paper, $(\mathfrak{g}\otimes A)^\Gamma$ is a perfect Lie algebra (i.e. $[(\mathfrak{g}\otimes A)^\Gamma, (\mathfrak{g}\otimes A)^\Gamma] = (\mathfrak{g}\otimes A)^\Gamma$) and so $(\mathfrak{g}\otimes A)^\Gamma$ has no nontrivial one-dimensional representations, [17, Lem. 6.1].

Definition 1.4 (Notation for irreducibles)**.** For $\psi \in \mathcal{E}^\Gamma$, we let $V_\Gamma(\psi)$ denote the corresponding irreducible representation of $(\mathfrak{g}\otimes A)^\Gamma$ (that is, $V_\Gamma(\psi)$ is some irreducible representation in the isomorphism class ev_ψ^Γ). For $\psi \in \mathcal{E}$, we let $V(\psi)$ denote the corresponding irreducible representation of $\mathfrak{g}\otimes A$.

Example 1.5 (Untwisted map algebras)**.** When the group Γ is trivial, $(\mathfrak{g}\otimes A)^\Gamma = \mathfrak{g}\otimes A$ is called an *untwisted map algebra*, or *generalized current algebra*. These algebras arise also for a nontrivial group Γ acting trivially on $\mathfrak{g}$ or on X. In the first case we have $(\mathfrak{g}\otimes A)^\Gamma \cong \mathfrak{g}\otimes A^\Gamma$, and in the second $(\mathfrak{g}\otimes A)^\Gamma = \mathfrak{g}^\Gamma \otimes A$.

Example 1.6 (Multiloop algebras)**.** Fix positive integers $n, m_1, \dots, m_n$. Let

$$\Gamma = \langle \gamma_1, \dots, \gamma_n : \gamma_i^{m_i} = 1,\ \gamma_i\gamma_j = \gamma_j\gamma_i,\ \forall\ 1 \le i, j \le n\rangle \cong \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_n\mathbb{Z},$$

and suppose that Γ acts on $\mathfrak{g}$. Note that this is equivalent to specifying commuting automorphisms σ_i, $i = 1, \dots, n$, of $\mathfrak{g}$ such that $\sigma_i^{m_i} = \mathrm{id}$. For $i = 1, \dots, n$, let ξ_i be a primitive m_i-th root of unity. Let $X = (k^\times)^n$ and define an action of Γ on X by

$$\gamma_i \cdot (z_1, \dots, z_n) = (z_1, \dots, z_{i-1}, \xi_i z_i, z_{i+1}, \dots, z_n).$$

Then

$$M(\mathfrak{g}, \sigma_1, \dots, \sigma_n, m_1, \dots, m_n) := (\mathfrak{g}\otimes A)^\Gamma \tag{1.1}$$

is the *multiloop algebra* of $\mathfrak{g}$ relative to $(\sigma_1, \dots, \sigma_n)$ and $(m_1, \dots, m_n)$.

Definition 1.7 ($\mathfrak{g}$-weights)**.** We can identify $\mathfrak{g}$ with the subalgebra $\mathfrak{g} \otimes k \subseteq \mathfrak{g} \otimes A$. In this way, any $(\mathfrak{g} \otimes A)$-module V can be viewed as a $\mathfrak{g}$-module. We will refer to the weights of this $\mathfrak{g}$-module as the $\mathfrak{g}$-*weights* of V (assuming V has a weight decomposition, e.g. V is finite-dimensional). For a $\mathfrak{g}$-weight λ, we let V_λ denote the corresponding weight space of V.

2. Twisting and untwisting functors

In this section, we define isomorphisms between certain categories of modules for (untwisted) map algebras $\mathfrak{g} \otimes A$ and their equivariant analogues $(\mathfrak{g} \otimes A)^\Gamma$. This isomorphism will be our key tool in defining local Weyl modules in the equivariant setting and proving their characterization via homological properties.

Recall that for a point $x \in X_{\mathrm{rat}}$, $\mathfrak{m}_x$ denotes the corresponding maximal ideal of A. For $\eta : X_{\mathrm{rat}} \to \mathbb{N} = \mathbb{Z}_{\geq 0}$ with finite support, define

$$I_\eta = \textstyle\prod_{x \in \operatorname{Supp} \eta} \mathfrak{m}_x^{\eta(x)}. \tag{2.1}$$

For a finite subset $\mathbf{x} \subseteq X$, we define $I_{\mathbf{x}} = I_\eta$, where $\eta(x) = 1$ for $x \in \mathbf{x}$ and $\eta(x) = 0$ for $x \notin \mathbf{x}$. It is straightforward to check that $\mathfrak{g} \otimes I_\eta$ is an ideal of $\mathfrak{g} \otimes A$ and so we have a *generalized evaluation map*

$$\operatorname{ev}_\eta : \mathfrak{g} \otimes A \twoheadrightarrow (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_\eta) \cong \textstyle\bigoplus_{x \in \operatorname{Supp} \eta} \mathfrak{g} \otimes (A/\mathfrak{m}_x^{\eta(x)}) \cong \bigoplus_{x \in \operatorname{Supp} \eta} (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes \mathfrak{m}_x^{\eta(x)}),$$
$$\operatorname{ev}_\eta(\alpha) = \textstyle\bigoplus_{x \in \operatorname{Supp} \eta} (\alpha + (\mathfrak{g} \otimes \mathfrak{m}_x^{\eta(x)})).$$

Let

$$\operatorname{ev}_\eta^\Gamma : (\mathfrak{g} \otimes A)^\Gamma \to \textstyle\bigoplus_{x \in \operatorname{Supp} \eta} (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes \mathfrak{m}_x^{\eta(x)})$$

denote the restriction of ev_η to $(\mathfrak{g} \otimes A)^\Gamma$. Clearly

$$\ker \operatorname{ev}_\eta^\Gamma = (\ker \operatorname{ev}_\eta) \cap (\mathfrak{g} \otimes A)^\Gamma = (\mathfrak{g} \otimes I_\eta) \cap (\mathfrak{g} \otimes A)^\Gamma = (\mathfrak{g} \otimes I_\eta)^\Gamma.$$

Recall that X_* is the set of finite subsets of X_{rat} that do not contain two points in the same Γ-orbit.

Lemma 2.1. *If* $\eta : X_{\mathrm{rat}} \to \mathbb{N}$ *satisfies* $\operatorname{Supp} \eta \in X_*$, *then*

$$(\mathfrak{g} \otimes I_\eta)^\Gamma = (\mathfrak{g} \otimes \tilde{I}_\eta)^\Gamma, \quad \textit{where} \quad \tilde{I}_\eta = \textstyle\prod_{x \in \operatorname{Supp} \eta} \prod_{\gamma \in \Gamma} \mathfrak{m}_{\gamma \cdot x}^{\eta(x)}.$$

Proof. Since $\tilde{I}_\eta \subseteq I_\eta$, we have $(\mathfrak{g} \otimes \tilde{I}_\eta)^\Gamma \subseteq (\mathfrak{g} \otimes I_\eta)^\Gamma$. Suppose $\alpha \in (\mathfrak{g} \otimes I_\eta)^\Gamma$. Then for each $x \in \operatorname{Supp} \eta$ and $\gamma \in \Gamma$, we have

$$\alpha \in (\mathfrak{g} \otimes I_\eta)^\Gamma \subseteq \mathfrak{g} \otimes I_\eta \subseteq \mathfrak{g} \otimes \mathfrak{m}_x^{\eta(x)} \implies \alpha = \gamma \cdot \alpha \in \gamma(\mathfrak{g} \otimes \mathfrak{m}_x^{\eta(x)}) = \mathfrak{g} \otimes \mathfrak{m}_{\gamma \cdot x}^{\eta(x)}.$$

Thus

$$(\mathfrak{g} \otimes I_\eta)^\Gamma \subseteq \mathfrak{g} \otimes \textstyle\bigcap_{x \in \operatorname{Supp} \eta} \bigcap_{\gamma \in \Gamma} \mathfrak{m}_{\gamma \cdot x}^{\eta(x)} = \mathfrak{g} \otimes \tilde{I}_\eta,$$

since the ideals $\mathfrak{m}_{\gamma \cdot x}$ are relatively prime. Thus $(\mathfrak{g} \otimes I_\eta)^\Gamma \subseteq (\mathfrak{g} \otimes \tilde{I}_\eta)^\Gamma$. $\square$

Proposition 2.2. *If* $\eta : X_{\mathrm{rat}} \to \mathbb{N}$ *satisfies* $\operatorname{Supp} \eta \subseteq X_*$, *then the map* $\operatorname{ev}_\eta^\Gamma$ *is surjective and hence induces an isomorphism*

$$(\mathfrak{g} \otimes A)^\Gamma / (\mathfrak{g} \otimes I_\eta)^\Gamma \xrightarrow{\cong} (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_\eta).$$

Proof. It suffices to show that for arbitrary $a \in \mathfrak{g}$, $f \in A$, $x \in \operatorname{Supp}\eta$, there exists $\alpha \in (\mathfrak{g}\otimes A)^\Gamma$ such that

$$\alpha - (a \otimes f) \in \mathfrak{g} \otimes \mathfrak{m}_x^{\eta(x)}, \quad \alpha \in \mathfrak{g} \otimes \mathfrak{m}_y^{\eta(y)} \ \forall\ y \in \operatorname{Supp}\eta \setminus \{x\}.$$

Let $n = \max_{y\in\operatorname{Supp}\eta} \eta(y)$ and let ξ be an n-th root of -1. Since the action of Γ on X is free, we can choose $f_1 \in A$ such that

$$f_1(x) = 0, \quad f_1(\gamma \cdot x) = \xi \ \forall\ \gamma \in \Gamma, \gamma \neq 1, \quad f_1(\gamma \cdot y) = \xi \ \forall\ \gamma \in \Gamma, y \in \operatorname{Supp}\eta \setminus \{x\}.$$

Then $f_1 \in \mathfrak{m}_x$. So

$$f_1^n \in \mathfrak{m}_x^n, \quad f_1^n(\gamma \cdot x) = -1 \ \forall\ \gamma \in \Gamma, \gamma \neq 1, \quad f_1^n(\gamma \cdot y) = -1 \ \forall\ \gamma \in \Gamma, y \in \operatorname{Supp}\eta \setminus \{x\}.$$

Hence

$$1 + f_1^n \in 1 + \mathfrak{m}_x^n, \quad 1 + f_1^n \in \textstyle\prod_{\gamma\in\Gamma,\,\gamma\neq 1} \mathfrak{m}_{\gamma\cdot x} \prod_{\substack{\gamma\in\Gamma \\ y\in\operatorname{Supp}\eta\setminus\{x\}}} \mathfrak{m}_{\gamma\cdot y}.$$

Recall that for any ideal I of A, the set $1 + I$ is closed under multiplication. Thus

$$(1 + f_1^n)^n \in 1 + \mathfrak{m}_x^n, \quad (1 + f_1^n)^n \in \textstyle\prod_{\gamma\in\Gamma,\,\gamma\neq 1} \mathfrak{m}_{\gamma\cdot x}^n \prod_{\substack{\gamma\in\Gamma \\ y\in\operatorname{Supp}\eta\setminus\{x\}}} \mathfrak{m}_{\gamma\cdot y}^n,$$

and so, setting $f_2 = f(1 + f_1^n)^n$, we have

$$f_2 \in f + \mathfrak{m}_x^n, \quad f_2 \in \textstyle\prod_{\gamma\in\Gamma,\,\gamma\neq 1} \mathfrak{m}_{\gamma\cdot x}^n \prod_{\substack{\gamma\in\Gamma \\ y\in\operatorname{Supp}\eta\setminus\{x\}}} \mathfrak{m}_{\gamma\cdot y}^n.$$

Define

$$\alpha = \textstyle\sum_{\gamma\in\Gamma} \gamma \cdot (a \otimes f_2) = \sum_{\gamma\in\Gamma} (\gamma \cdot a) \otimes (\gamma \cdot f_2) \in (\mathfrak{g} \otimes A)^\Gamma.$$

Since $\gamma \cdot \mathfrak{m}_y = \mathfrak{m}_{\gamma\cdot y}$ and Γ acts freely on X, we have

$$\gamma \cdot f_2 \in \mathfrak{m}_x^n \subseteq \mathfrak{m}_x^{\eta(x)} \ \forall\ \gamma \in \Gamma, \gamma \neq 1.$$

Thus

$$\alpha + \mathfrak{g} \otimes \mathfrak{m}_x^{\eta(x)} = (a \otimes f_2) + \mathfrak{g} \otimes \mathfrak{m}_x^{\eta(x)} = a \otimes f + \mathfrak{g} \otimes \mathfrak{m}_x^{\eta(x)}.$$

We also have

$$\gamma \cdot f_2 \in \mathfrak{m}_y^n \subseteq \mathfrak{m}_y^{\eta(y)} \ \forall\ \gamma \in \Gamma,\ y \in \operatorname{Supp}\eta \setminus \{x\},$$

and so

$$\alpha \in \mathfrak{g} \otimes \mathfrak{m}_y^{\eta(y)} \ \forall\ y \in \operatorname{Supp}\eta \setminus \{x\}.$$

□

Let Ξ be the character group of Γ. This is an abelian group, whose group operation we will write additively. Hence, 0 is the character of the trivial one-dimensional representation, and if an irreducible representation affords the character ξ, then $-\xi$ is the character of the dual representation.

If Γ acts on an algebra B by automorphisms, it is well-known that $B = \bigoplus_{\xi\in\Xi} B_\xi$ is a Ξ-grading, where B_ξ is the isotypic component of type ξ. It follows that $(\mathfrak{g}\otimes A)^\Gamma$ can be written as

$$(\mathfrak{g} \otimes A)^\Gamma = \textstyle\bigoplus_{\xi\in\Xi} \mathfrak{g}_\xi \otimes A_{-\xi}, \tag{2.2}$$

since $\mathfrak{g} = \bigoplus_\xi \mathfrak{g}_\xi$ and $A = \bigoplus_\xi A_\xi$ are Ξ-graded and $(\mathfrak{g}_\xi \otimes A_{\xi'})^\Gamma = 0$ if $\xi' \neq -\xi$. The decomposition (3) is an algebra Ξ-grading.

Lemma 2.3 ([17, Lem. 4.4])**.** *Suppose a finite abelian group Γ acts on a unital associative commutative k-algebra A (and hence on $X = \operatorname{Spec} A$) by automorphisms. Let $A = \bigoplus_{\xi \in \Xi} A_\xi$ be the associated grading on A, where Ξ is the character group of Γ. Then the following conditions are equivalent:*

(1) *Γ acts freely on X, and*
(2) *$\prod_{i=1}^n I_{\xi_i} = (I^n)_{\sum_{i=1}^n \xi_i}$ for all $\xi_1, \dots, \xi_n \in \Xi$ and any Γ-invariant ideal I of A. Here $I_\xi = I \cap A_\xi$ for $\xi \in \Xi$.*

For a Lie algebra L, define L^n, $n \geq 1$, by

$$L^1 = L, \quad L^n = [L, L^{n-1}], \quad n > 1.$$

The following proposition, combined with Proposition 2.2, will allow us to define and deduce properties of finite-dimensional modules for equivariant map algebras from the corresponding notions for untwisted map algebras.

Proposition 2.4. *Every finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-module is annihilated by $(\mathfrak{g} \otimes I_\eta)^\Gamma$ for some $\eta : X_{\mathrm{rat}} \to \mathbb{N}$ with $\operatorname{Supp} \eta \subseteq X_*$.*

Proof. Suppose V is a finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-module annihilated by $(\mathfrak{g} \otimes I_\eta)^\Gamma$ for some finitely supported $\eta : X_{\mathrm{rat}} \to \mathbb{N}$. By Lemma 2.1, we can find $\eta' : X_{\mathrm{rat}} \to \mathbb{N}$ with $\operatorname{Supp} \eta' \subseteq X_*$ and $(\mathfrak{g} \otimes I_{\eta'})^\Gamma \subseteq (\mathfrak{g} \otimes I_\eta)^\Gamma$. Thus it suffices to prove that every finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-module is annihilated by some $(\mathfrak{g} \otimes I_\eta)^\Gamma$.

We first prove by induction that for any Γ-invariant ideal I of A,

$$\left((\mathfrak{g} \otimes I)^\Gamma\right)^m = (\mathfrak{g} \otimes I^m)^\Gamma \quad \forall\, m \geq 1. \tag{2.3}$$

The result is trivial for $m = 1$. Assume it is true for some $m \geq 1$. Then

$$\begin{aligned}
\left((\mathfrak{g} \otimes I)^\Gamma\right)^{m+1} &= \left[(\mathfrak{g} \otimes I)^\Gamma, \left((\mathfrak{g} \otimes I)^\Gamma\right)^m\right] \\
&= \left[(\mathfrak{g} \otimes I)^\Gamma, (\mathfrak{g} \otimes I^m)^\Gamma\right] && \text{(by the induction hypothesis)} \\
&= \left[\textstyle\bigoplus_{\xi \in \Xi} \mathfrak{g}_\xi \otimes I_{-\xi}, \bigoplus_{\tau \in \Xi} \mathfrak{g}_\tau \otimes (I^m)_{-\tau}\right] \\
&= \textstyle\sum_{\xi, \tau \in \Xi} [\mathfrak{g}_\xi, \mathfrak{g}_\tau] \otimes I_{-\xi} (I^m)_{-\tau} \\
&= \textstyle\sum_{\xi, \tau \in \Xi} [\mathfrak{g}_\xi, \mathfrak{g}_\tau] \otimes (I^{m+1})_{-\xi-\tau} && \text{(by Lemma 2.3)} \\
&= \textstyle\bigoplus_{\xi \in \Xi} \mathfrak{g}_\xi \otimes (I^{m+1})_{-\xi} && \text{(since } \mathfrak{g} \text{ is semisimple)} \\
&= (\mathfrak{g} \otimes I^{m+1})^\Gamma.
\end{aligned}$$

Thus (4) holds.

Now let V be a finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-module. Then there exists a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V,$$

such that V_i / V_{i-1} is an irreducible finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-module for $1 \leq i \leq n$. By Proposition 1.2, each V_i / V_{i-1} is an evaluation module. Let $\eta_i : X_{\mathrm{rat}} \to \mathbb{N}$ be the characteristic function of the support of V_i / V_{i-1}. Then $(\mathfrak{g} \otimes I_{\eta_i})^\Gamma \cdot (V_i / V_{i-1}) = 0$. In other words, $(\mathfrak{g} \otimes I_{\eta_i})^\Gamma \cdot V_i \subseteq V_{i-1}$.

Let $\nu = \sum_{i=1}^n \eta_i$ and $\eta = n\nu$. We claim that $(\mathfrak{g} \otimes I_\eta)^\Gamma \cdot V = 0$. Since $I_\eta = I_\nu^n$, it follows from (4) that $\left((\mathfrak{g} \otimes I_\nu)^\Gamma\right)^n = (\mathfrak{g} \otimes I_\eta)^\Gamma$. Because $I_\nu \subseteq I_{\eta_i}$, we have $(\mathfrak{g} \otimes I_\nu)^\Gamma \cdot V_i \subseteq V_{i-1}$ for all

$1 \le i \le n$. Therefore

$$(\mathfrak{g} \otimes I_\eta)^\Gamma \cdot V = \left((\mathfrak{g} \otimes I_\nu)^\Gamma\right)^n \cdot V = 0. \qquad \square$$

For functions $\eta, \eta' : X_{\mathrm{rat}} \to \mathbb{N}$ with finite support, we write $\eta \le \eta'$ if $\eta(x) \le \eta'(x)$ for all $x \in X_{\mathrm{rat}}$. Clearly

$$\eta \le \eta' \implies I_\eta \supseteq I_{\eta'} \implies \mathfrak{g} \otimes I_\eta \supseteq \mathfrak{g} \otimes I_{\eta'}.$$

Thus, for $\eta \le \eta'$, we have natural projections

$$(\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_{\eta'}) \twoheadrightarrow (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_\eta), \quad (\mathfrak{g} \otimes A)^\Gamma/(\mathfrak{g} \otimes I_{\eta'})^\Gamma \twoheadrightarrow (\mathfrak{g} \otimes A)^\Gamma/(\mathfrak{g} \otimes I_\eta)^\Gamma.$$

Lemma 2.5. *If $\eta, \eta' : X_{\mathrm{rat}} \to \mathbb{N}$ are such that $\eta \le \eta'$ and $\operatorname{Supp} \eta' \subseteq X_*$, then the diagram*

$$\begin{tikzcd}
(\mathfrak{g} \otimes A)^\Gamma/(\mathfrak{g} \otimes I_{\eta'})^\Gamma \arrow[r, "\cong"] \arrow[d, two heads] & (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_{\eta'}) \arrow[d, two heads] \\
(\mathfrak{g} \otimes A)^\Gamma/(\mathfrak{g} \otimes I_\eta)^\Gamma \arrow[r, "\cong"] & (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_\eta)
\end{tikzcd} \tag{2.4}$$

is commutative, where the horizontal maps are the isomorphisms induced by evaluation as in Proposition 2.2.

Proof. This is clear from the fact that both compositions in the diagram are induced from the composition

$$(\mathfrak{g} \otimes A)^\Gamma \hookrightarrow \mathfrak{g} \otimes A \twoheadrightarrow (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_\eta).$$

$\square$

Suppose V is a finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-module. By Proposition 2.4, there exists a function $\eta : X_{\mathrm{rat}} \to \mathbb{N}$, $\operatorname{Supp} \eta \subseteq X_*$, such that $(\mathfrak{g} \otimes I_\eta)^\Gamma$ annihilates V. Therefore the action of $(\mathfrak{g} \otimes A)^\Gamma$ on V factors through $(\mathfrak{g} \otimes A)^\Gamma/(\mathfrak{g} \otimes I_\eta)^\Gamma$ and the composition

$$\mathfrak{g} \otimes A \twoheadrightarrow (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_\eta) \cong (\mathfrak{g} \otimes A)^\Gamma/(\mathfrak{g} \otimes I_\eta)^\Gamma \to \operatorname{End} V$$

defines an action of $(\mathfrak{g} \otimes A)$ on V. We denote the resulting $(\mathfrak{g} \otimes A)$-module by V^η.

Lemma 2.6. *Suppose V is a finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-module that is annihilated by $(\mathfrak{g} \otimes I_\eta)^\Gamma$ and $(\mathfrak{g} \otimes I_{\eta'})^\Gamma$ for functions $\eta, \eta' : X_{\mathrm{rat}} \to \mathbb{N}$ such that $\operatorname{Supp} \eta \cup \operatorname{Supp} \eta' \subseteq X_*$. Then $V^\eta = V^{\eta'}$ as $(\mathfrak{g} \otimes A)$-modules.*

Proof. Let $\tau = \eta + \eta'$. It is clear that $(\mathfrak{g} \otimes I_\tau)^\Gamma$ annihilates V. Since $\operatorname{Supp} \tau = \operatorname{Supp} \eta \cup \operatorname{Supp} \eta'$, it follows from Lemma 2.5 that the diagram

$$\begin{tikzcd}
 & (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_\eta) \arrow[r, "\cong"] & (\mathfrak{g} \otimes A)^\Gamma/(\mathfrak{g} \otimes I_\eta)^\Gamma \arrow[dr] & \\
\mathfrak{g} \otimes A \arrow[ur, two heads] \arrow[r, two heads] \arrow[dr, two heads] & (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_\tau) \arrow[u, two heads] \arrow[d, two heads] \arrow[r, "\cong"] & (\mathfrak{g} \otimes A)^\Gamma/(\mathfrak{g} \otimes I_\tau)^\Gamma \arrow[u, two heads] \arrow[d, two heads] \arrow[r] & \operatorname{End} V \\
 & (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_{\eta'}) \arrow[r, "\cong"] & (\mathfrak{g} \otimes A)^\Gamma/(\mathfrak{g} \otimes I_{\eta'})^\Gamma \arrow[ur] &
\end{tikzcd}$$

commutes, where the three isomorphisms in the middle are the inverses of the isomorphisms of Proposition 2.2 induced by evaluation. It follows that $V^\eta = V^\tau = V^{\eta'}$ as $(\mathfrak{g} \otimes A)$-modules. $\square$

Definition 2.7 (Categories $\mathcal{F}$, $\mathcal{F}^\Gamma$, $\mathcal{F}_{\mathbf{x}}$, and $\mathcal{F}_{\mathbf{x}}^\Gamma$)**.** Let $\mathcal{F}$ and $\mathcal{F}^\Gamma$ be the categories of finite-dimensional representations of $\mathfrak{g} \otimes A$ and $(\mathfrak{g} \otimes A)^\Gamma$ respectively. For $\mathbf{x} \in X_*$, define $\mathcal{F}_{\mathbf{x}}$ (resp. $\mathcal{F}_{\mathbf{x}}^\Gamma$) to be the full subcategory of $\mathcal{F}$ (resp. $\mathcal{F}^\Gamma$) consisting of those representations whose irreducible constituents all have support contained in $\mathbf{x}$ (resp. $\Gamma \cdot \mathbf{x}$).

Definition 2.8 (Twisting functor)**.** We have a natural *twisting functor* $\mathbf{T} : \mathcal{F} \to \mathcal{F}^\Gamma$ defined by restricting from $\mathfrak{g} \otimes A$ to $(\mathfrak{g} \otimes A)^\Gamma$. For any $\mathbf{x} \in X_*$, we have the induced functor $\mathbf{T}_{\mathbf{x}} : \mathcal{F}_{\mathbf{x}} \to \mathcal{F}_{\mathbf{x}}^\Gamma$.

Definition 2.9 (Untwisting functor)**.** Fix $\mathbf{x} \in X_*$. By Proposition 2.4, every module $V \in \mathcal{F}_{\mathbf{x}}^\Gamma$ is annihilated by some $(\mathfrak{g} \otimes I_\eta)^\Gamma$ with $\operatorname{Supp} \eta \subseteq \mathbf{x}$. By Lemma 2.6, the modules V^η are independent of the choice of η. The *untwisting functor* $\mathbf{U}_{\mathbf{x}} : \mathcal{F}_{\mathbf{x}}^\Gamma \to \mathcal{F}_{\mathbf{x}}$ is defined to be the functor that, on objects, maps V to V^η. Now suppose $V, W \in \mathcal{F}_{\mathbf{x}}^\Gamma$ and $\beta : V \to W$ is a morphism in $\mathcal{F}_{\mathbf{x}}^\Gamma$. Since $\mathcal{F}_{\mathbf{x}}^\Gamma$ is a full subcategory of $\mathcal{F}^\Gamma$, $\beta : V \to W$ is a morphism in $\mathcal{F}^\Gamma$, which means that it is a homomorphism of $(\mathfrak{g} \otimes A)^\Gamma$-modules. Choose $\eta : X_{\mathrm{rat}} \to \mathbb{N}$ with support contained in $\mathbf{x}$ such that $(\mathfrak{g} \otimes I_\eta)^\Gamma$ annihilates both V and W. Then the action of $(\mathfrak{g} \otimes A)^\Gamma$ on V and W factors through $(\mathfrak{g} \otimes A)^\Gamma / (\mathfrak{g} \otimes I_\eta)^\Gamma$. By definition, it follows that β is also a homomorphism of $(\mathfrak{g} \otimes A)$-modules from V^η to W^η. We define $\mathbf{U}_{\mathbf{x}}(\beta)$ to be this homomorphism. One easily sees that $\mathbf{U}_{\mathbf{x}}$ respects composition of morphisms and hence is a well-defined functor.

For a Γ-invariant subset Y of X_{rat}, let Y_Γ denote the set of subsets of Y containing exactly one point from each Γ-orbit in Y. For $\psi \in \mathcal{E}^\Gamma$ and $\mathbf{x} \in (\operatorname{Supp} \psi)_\Gamma$, define

$$\psi_{\mathbf{x}} : X_{\mathrm{rat}} \to P^+, \quad \psi_{\mathbf{x}}(x) = \begin{cases} \psi(x) & \text{if } x \in \mathbf{x}, \\ 0 & \text{if } x \notin \mathbf{x}. \end{cases}$$

Theorem 2.10. *For $\mathbf{x} \in X_*$, the twisting and untwisting functors have the following properties.*

(1) *The twisting functor $\mathbf{T}$ maps the isomorphism class ev_ψ for $\psi \in \mathcal{E}$, $\operatorname{Supp} \psi \subseteq X_*$, to the isomorphism class $\mathrm{ev}^\Gamma_{\psi^\Gamma}$ for $\psi^\Gamma \in \mathcal{E}^\Gamma$, where*

$$\psi^\Gamma(x) = \textstyle\sum_{\gamma \in \Gamma} \gamma \cdot \psi(\gamma^{-1} \cdot x), \quad x \in X_{\mathrm{rat}}.$$

(2) *The untwisting functor $\mathbf{U}_{\mathbf{x}}$ maps the isomorphism class ev^Γ_ψ, $\psi \in \mathcal{E}^\Gamma$. to the isomorphism class $\mathrm{ev}_{\psi_{\mathbf{x}}}$.*

(3) *The functors $\mathbf{T}_{\mathbf{x}}$ and $\mathbf{U}_{\mathbf{x}}$ are mutually inverse isomorphisms of categories.*

Proof. Part (1) follows immediately from the definition of the evaluation representations given in Section 1.

Now suppose $\mathbf{x} \in X_*$ and $V \in \mathcal{F}_{\mathbf{x}}^\Gamma$ is irreducible and corresponds to $\psi \in \mathcal{E}^\Gamma$. Let $\rho = \left(\bigotimes_{x \in \mathbf{x}} \rho_x\right) \circ \mathrm{ev}^\Gamma_{\mathbf{x}}$ be the corresponding representation. Then ρ factors through $(\mathfrak{g} \otimes A)^\Gamma / (\mathfrak{g} \otimes I_{\mathbf{x}})^\Gamma$ and so $\mathbf{U}_{\mathbf{x}}(V)$ is the $(\mathfrak{g} \otimes A)$-module given by the composition

$$\mathfrak{g} \otimes A \twoheadrightarrow (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_{\mathbf{x}}) \xrightarrow{\cong} (\mathfrak{g} \otimes A)^\Gamma / (\mathfrak{g} \otimes I_{\mathbf{x}})^\Gamma \cong \mathfrak{g}^{\mathbf{x}} \xrightarrow{\bigotimes_{x \in \mathbf{x}} \rho_x} \operatorname{End} V.$$

Since this is precisely the evaluation representation $\left(\bigotimes_{x \in \mathbf{x}} \rho_x\right) \circ \mathrm{ev}_{\mathbf{x}}$ of $\mathfrak{g} \otimes A$, which is in the isomorphism class $\mathrm{ev}_{\psi_{\mathbf{x}}}$, Part (2) follows.

Suppose $V \in \mathcal{F}_{\mathbf{x}}$. Then V is annihilated by some $\mathfrak{g} \otimes I_\eta$ and the action of $\mathfrak{g} \otimes A$ on $\mathbf{U_x T_x}(V)$ is given by

$$\mathfrak{g} \otimes A \twoheadrightarrow (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_\eta) \xrightarrow{\cong} (\mathfrak{g} \otimes A)^\Gamma/(\mathfrak{g} \otimes I_\eta)^\Gamma \xrightarrow{\cong} (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_\eta) \to \operatorname{End} V,$$

where the two isomorphisms are mutually inverse. Thus $\mathbf{U_x T_x}(V) = V$. One easily verifies that $\mathbf{U_x T_x}$ is also the identity on morphisms and is therefore the identity functor on $\mathcal{F}_{\mathbf{x}}$. Similarly, $\mathbf{T_x U_x}$ is the identity functor on $\mathcal{F}_{\mathbf{x}}^\Gamma$. This proves Part (3). □

Remark 2.11. Theorem 2.10 allows one to translate any reasonable question in the representation theory of finite-dimensional modules for equivariant maps algebras, where Γ is abelian and acts freely on X, to a corresponding question for untwisted map algebras (generalized current algebras). For instance, it can be used to reduce the computation of extensions between irreducible finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-modules to the case of extensions of $(\mathfrak{g} \otimes A)$-modules, which were considered in [13]. In this way, one can give an alternate proof of [17, Prop. 6.3].

3. Local Weyl modules

In this section, we define the local Weyl modules for equivariant map algebras. We begin by reviewing the local Weyl modules for untwisted map algebras.
Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Then we have a triangular decomposition of the untwisted map algebra

$$\mathfrak{g} \otimes A = (\mathfrak{n}^- \otimes A) \oplus (\mathfrak{h} \otimes A) \oplus (\mathfrak{n}^+ \otimes A).$$

Let $\{e_i, h_i, f_i\}_{i \in I}$ denote a set of Chevalley generators of $\mathfrak{g}$ compatible with its triangular decomposition. In particular, the f_i generate $\mathfrak{n}^-$.

Definition 3.1 (Untwisted local Weyl module)**.** Given $\psi \in \mathcal{E}$, the *(untwisted) local Weyl module* $W(\psi)$ is the $(\mathfrak{g} \otimes A)$-module generated by a nonzero vector w_ψ satisfying the relations

$$(\mathfrak{n}^+ \otimes A) \cdot w_\psi = 0, \tag{3.1}$$

$$(f_i \otimes 1)^{\lambda(h_i)+1} \cdot w_\psi = 0, \quad i \in I, \quad \text{where } \lambda = \operatorname{wt} \psi := \textstyle\sum_{x \in \mathbf{x}} \psi(x), \tag{3.2}$$

$$\alpha \cdot w_\psi = \left(\textstyle\sum_{x \in \operatorname{Supp} \psi} \psi(x)(\alpha(x)) \right) w_\psi, \quad \alpha \in \mathfrak{h} \otimes A. \tag{3.3}$$

Proposition 3.2. (1) [4, Th. 2] *For every $\psi \in \mathcal{E}$, $W(\psi)$ is a finite-dimensional $(\mathfrak{g} \otimes A)$-module.*

(2) [4, Prop. 5] *Let V be any finite-dimensional $(\mathfrak{g} \otimes A)$-module generated by a nonzero element $v \in V$ such that*

$$(\mathfrak{n}^+ \otimes A) \cdot v = 0 \quad \textit{and} \quad (\mathfrak{h} \otimes A) \cdot v = kv.$$

Then there exists $\psi \in \mathcal{E}$ such that the assignment $w_\psi \mapsto v$ extends to a surjective homomorphism $W(\psi) \twoheadrightarrow V$ of $(\mathfrak{g} \otimes A)$-modules.

For a subset $Y \subseteq X_{\mathrm{rat}}$, let

$$I_Y = \{ f \in A \mid f(x) = 0 \ \forall\, x \in Y \}.$$

For $\psi \in \mathcal{E}$, we define $I_\psi = I_{\operatorname{Supp}\psi}$. Note then that $I_\psi = I_\eta$ as in (2) for

$$\eta : X_{\mathrm{rat}} \to \mathbb{N}, \quad \eta(x) = \begin{cases} 1 & \text{if } x \in \operatorname{Supp}(\psi), \\ 0 & \text{if } x \notin \operatorname{Supp}(\psi). \end{cases}$$

Proposition 3.3. (1) [4, Prop. 9] *If $\psi \in \mathcal{E}$ with $\operatorname{wt}\psi = \lambda \in P^+$, then*

$$(\mathfrak{g} \otimes I_\psi^N) \cdot W(\psi) = 0 \quad \forall\ N \geq \lambda(h_\theta),$$

where θ is the highest root for $\mathfrak{g}$ and h_θ is the corresponding coroot.

(2) [4, Th. 3] *If $\psi, \psi' \in \mathcal{E}$ such that $\operatorname{Supp}\psi \cap \operatorname{Supp}\psi' = \varnothing$, then*

$$W(\psi + \psi') \cong W(\psi) \otimes W(\psi')$$

as $(\mathfrak{g} \otimes A)$-modules.

(3) [4, Lem. 6] *For $\psi \in \mathcal{E}$, $V(\psi)$ is the unique irreducible quotient of $W(\psi)$ (see Definition 1.4).*

Remark 3.4. In the case that A is the coordinate algebra of an affine algebraic variety, Proposition 3.2 and parts (1) and (2) of Proposition 3.3 are proven in [10] (Theorems 1, 2, and 5, and Proposition 7).

We now turn our attention to the equivariant map algebras. For a $(\mathfrak{g} \otimes A)$-module U, let $\rho_U : \mathfrak{g} \otimes A \to \operatorname{End}_k U$ be the corresponding representation.

Lemma 3.5. *Suppose $\psi \in \mathcal{E}^\Gamma$ and $\mathbf{x} \in (\operatorname{Supp}\psi)_\Gamma$. Then, for $\gamma \in \Gamma$,*

$$\rho_{W(\psi_{\mathbf{x}})} \circ \gamma^{-1} \cong \rho_{W(\psi_{\gamma \cdot \mathbf{x}})},$$

where $\gamma \cdot \mathbf{x} = \{\gamma \cdot x \mid x \in \mathbf{x}\}$.

Proof. Let $W(\psi_{\mathbf{x}})^\gamma$ be the $(\mathfrak{g} \otimes A)$-module corresponding to the representation $\rho_{W(\psi_{\mathbf{x}})} \circ \gamma^{-1}$. Recall that we identify $\mathfrak{g}$ with the subalgebra $\mathfrak{g} \otimes k$ of $\mathfrak{g} \otimes A$. Thus, via restriction, we can view $W(\psi_{\mathbf{x}})$ and $W(\psi_{\mathbf{x}})^\gamma$ as $\mathfrak{g}$-modules. Recall that $W(\psi_{\mathbf{x}})$ is a finite-dimensional $\mathfrak{g}$-module with $\operatorname{wt} W(\psi_{\mathbf{x}}) \subseteq \lambda - Q_+$, where $\lambda = \sum_{x \in \mathbf{x}} \psi(x)$. It follows that $W(\psi_{\mathbf{x}})^\gamma$ is a finite-dimensional $\mathfrak{g}$-module with $\operatorname{wt} W(\psi_{\mathbf{x}})^\gamma \subseteq \gamma \cdot \lambda - Q_+$. Furthermore, the $\gamma \cdot \lambda$ weight space of $W(\psi_{\mathbf{x}})^\gamma$ is one-dimensional since the λ weight space of $W(\psi_{\mathbf{x}})$ is one-dimensional.
We also know that $W(\psi_{\mathbf{x}})$ has unique irreducible quotient $V(\psi_{\mathbf{x}})$. By the definition of $\mathcal{E}^\Gamma$, we have that $\rho_{V(\psi_{\mathbf{x}})} \cdot \gamma^{-1} \cong \rho_{V(\psi_{\gamma \cdot \mathbf{x}})}$. Thus $W(\psi_{\mathbf{x}})^\gamma$ has unique irreducible quotient $V(\psi_{\gamma \cdot \mathbf{x}})$. Let $v \in W(\psi_{\mathbf{x}})^\gamma$ be a nonzero vector of weight $\gamma \cdot \lambda$ and let U be the smallest $(\mathfrak{g} \otimes A)$-submodule of $W(\psi_{\mathbf{x}})^\gamma$ containing v. If $U \neq W(\psi_{\mathbf{x}})^\gamma$, then U is contained in the unique maximal submodule of $W(\psi_{\mathbf{x}})^\gamma$. But this contradicts the fact that the unique irreducible quotient of $W(\psi_{\mathbf{x}})^\gamma$ has a nonzero $\gamma \cdot \lambda$ weight space. Therefore $U = W(\psi_{\mathbf{x}})^\gamma$ and so v is a cyclic vector. It then follows from Proposition 3.2(2) that $W(\psi_{\mathbf{x}})^\gamma$ is isomorphic to a quotient of $W(\psi_{\gamma \cdot \mathbf{x}})$. By symmetry, $W(\psi_{\gamma \cdot \mathbf{x}})$ is also isomorphic to a quotient of $W(\psi_{\mathbf{x}})^\gamma$. Since these modules are finite-dimensional, we conclude that $W(\psi_{\mathbf{x}})^\gamma \cong W(\psi_{\gamma \cdot x})$. ∎

Proposition 3.6. *Suppose $\psi \in \mathcal{E}^\Gamma$ and $\mathbf{x}, \mathbf{x}' \in (\operatorname{Supp}\psi)_\Gamma$. Then the restriction to $(\mathfrak{g} \otimes A)^\Gamma$-modules of the Weyl modules $W(\psi_{\mathbf{x}})$ and $W(\psi_{\mathbf{x}'})$ for $\mathfrak{g} \otimes A$ are isomorphic (as $(\mathfrak{g} \otimes A)^\Gamma$-modules).*

Proof. We first prove the result in the case that the support of ψ consists of a single Γ-orbit. Suppose $x, x' \in \operatorname{Supp}\psi$. Then there exist a unique $\gamma \in \Gamma$ such that $x' = \gamma \cdot x$. By Lemma 3.5, we have

$$\rho_{W(\psi_x)} \circ \gamma^{-1} \cong \rho_{W(\psi_{x'})}.$$

Since the restriction of the automorphism γ^{-1} to $(\mathfrak{g} \otimes A)^\Gamma$ is trivial, it follows immediately that the restrictions of $\rho_{W(\psi_x)}$ and $\rho_{W(\psi_{x'})}$ to $(\mathfrak{g} \otimes A)^\Gamma$ are isomorphic. The general result where the support of ψ is a union of Γ-orbits now follows from Proposition 3.3(2). $\square$

Definition 3.7 (Twisted local Weyl module)**.** For $\psi \in \mathcal{E}^\Gamma$, we define $W_\Gamma(\psi)$ to be the restriction to $(\mathfrak{g} \otimes A)^\Gamma$-modules of the Weyl module $W(\psi_{\mathbf{x}})$ for $\mathfrak{g} \otimes A$, for some choice of $\mathbf{x} \in (\operatorname{Supp}\psi)_\Gamma$. In other words, $W_\Gamma(\psi) := \mathbf{T}(W(\psi_{\mathbf{x}}))$. By Proposition 3.6, $W_\Gamma(\psi)$ is independent of the choice of $\mathbf{x}$ (up to isomorphism). We call $W_\Gamma(\psi)$ the *(twisted) local Weyl module* of $(\mathfrak{g} \otimes A)^\Gamma$ associated to ψ.

Lemma 3.8. *For $\psi \in \mathcal{E}^\Gamma$ and $\mathbf{x} \in (\operatorname{Supp}\psi)_\Gamma$, we have $\mathbf{U}_{\mathbf{x}}(W_\Gamma(\psi)) = W(\psi_{\mathbf{x}})$ and $\mathbf{U}_{\mathbf{x}}(V_\Gamma(\psi)) = V(\psi_{\mathbf{x}})$.*

Proof. By definition, $W_\Gamma(\psi) = \mathbf{T}_{\mathbf{x}}(W(\psi_{\mathbf{x}}))$. Thus, by Theorem 2.10, we have

$$\mathbf{U}_{\mathbf{x}}(W_\Gamma(\psi)) = \mathbf{U}_{\mathbf{x}}\mathbf{T}_{\mathbf{x}}(W(\psi_{\mathbf{x}})) = W(\psi_{\mathbf{x}}).$$

The proof of the second statement is analogous (see the proof of Theorem 2.10(2)). $\square$

Proposition 3.9 (Tensor product property)**.** *If $\psi, \psi' \in \mathcal{E}^\Gamma$ have disjoint support, then $W_\Gamma(\psi + \psi') \cong W_\Gamma(\psi) \otimes W_\Gamma(\psi')$.*

Proof. Choose $\mathbf{x} \in (\operatorname{Supp}\psi)_\Gamma$ and $\mathbf{x}' \in (\operatorname{Supp}\psi')_\Gamma$. Then $\mathbf{x} \cap \mathbf{x}' = \varnothing$ and, by Proposition 3.3(2), we have $W(\psi_{\mathbf{x}} + \psi'_{\mathbf{x}'}) \cong W(\psi_{\mathbf{x}}) \otimes W(\psi'_{\mathbf{x}'})$. Since $\mathbf{x} \cup \mathbf{x}' \in (\operatorname{Supp}(\psi + \psi'))_\Gamma$, the proposition follows after restricting to $(\mathfrak{g} \otimes A)^\Gamma$-modules. $\square$

4. Characterization of local Weyl modules by homological properties

In this section, we show that the local Weyl modules are characterized by homological properties, extending results of [4] to the equivariant setting.

For $\lambda \in P$, write $\lambda = \sum_{i \in I} k_i \alpha_i$, $k_i \in \mathbb{Q}$, as a linear combination of simple roots, and define

$$\operatorname{ht}\lambda := \textstyle\sum_{i \in I} k_i.$$

Recall the usual partial order on P given by

$$\lambda \geq \mu \iff \lambda - \mu \in Q^+.$$

It is clear that

$$\lambda > \mu \implies \operatorname{ht}\lambda > \operatorname{ht}\mu.$$

Since Γ acts on P^+ via diagram automorphisms, it preserves the set of positive roots. Therefore, for $\psi \in \mathcal{E}^\Gamma$, we have $\sum_{x \in X_{\mathrm{rat}}} \operatorname{ht}\psi_{\mathbf{x}}(x) = \sum_{x \in X_{\mathrm{rat}}} \operatorname{ht}\psi_{\mathbf{x}'}(x)$ for all $\mathbf{x}, \mathbf{x}' \in (\operatorname{Supp}\psi)_\Gamma$.

Definition 4.1 (Height function on $\mathcal{E}^\Gamma$)**.** Define the *height* of $\psi \in \mathcal{E}^\Gamma$ to be

$$\operatorname{ht}\psi = \textstyle\sum_{x \in X_{\mathrm{rat}}} \operatorname{ht}\psi_{\mathbf{x}}(x) \quad \text{for some } \mathbf{x} \in (\operatorname{Supp}\psi)_\Gamma.$$

By the above discussion, this definition is independent of the choice of $\mathbf{x}$.

For a finite-dimensional $(\mathfrak{g}\otimes A)^\Gamma$-module M and $\psi \in \mathcal{E}^\Gamma$, let $\operatorname{mult}_\psi M$ denote the multiplicity of $\operatorname{ev}_\psi^\Gamma$ in M. In other words, $\operatorname{mult}_\psi M$ is the number of (irreducible) composition factors of M in the isomorphism class $\operatorname{ev}_\psi^\Gamma$.

Definition 4.2 (Maximal weight module)**.** We call a finite-dimensional $(\mathfrak{g}\otimes A)^\Gamma$-module M a *maximal weight module* of *maximal weight* ψ if $\operatorname{mult}_\psi M = 1$ and, for all $\varphi \neq \psi$,

$$\operatorname{mult}_\varphi M \neq 0 \implies \operatorname{ht}\varphi < \operatorname{ht}\psi.$$

Lemma 4.3. *The local Weyl module $W_\Gamma(\psi)$ is a maximal weight module of maximal weight ψ.*

Proof. If $\Gamma = \{1\}$, the result follows from the fact that the $\mathfrak{g}$-weights of $W(\psi)$ lie in $\operatorname{wt}\psi - Q^+$ by Definition 3.1. Suppose now $\Gamma \neq \{1\}$ and let $\psi \in \mathcal{E}^\Gamma$. Then for any $\mathbf{x} \in (\operatorname{Supp}\psi)_\Gamma$, we have, by Lemma 3.8, $\mathbf{U}_\mathbf{x}(W_\Gamma(\psi)) = W(\psi_\mathbf{x})$. By Proposition 3.3(1), we have that all constituents of $W(\psi_\mathbf{x})$ have support contained in $\mathbf{x}$. Thus

$$\operatorname{mult}_\varphi W(\psi_\mathbf{x}) \neq 0 \implies V(\varphi) \in \mathcal{F}_\mathbf{x}.$$

By Theorem 2.10 and Lemma 3.8, we then have

$$\operatorname{mult}_\varphi W_\Gamma(\psi) = \operatorname{mult}_{\varphi_\mathbf{x}} W(\psi_\mathbf{x}).$$

Thus, for $\varphi \neq \psi$ (hence $\varphi_\mathbf{x} \neq \psi_\mathbf{x}$),

$$\begin{aligned} \operatorname{mult}_\varphi W_\Gamma(\psi) \neq 0 &\implies \operatorname{mult}_{\varphi_\mathbf{x}} W(\psi_\mathbf{x}) \neq 0 \\ &\implies \operatorname{wt}\varphi_\mathbf{x} < \operatorname{wt}\psi_\mathbf{x} \\ &\implies \operatorname{ht}\varphi = \operatorname{ht}\varphi_\mathbf{x} < \operatorname{ht}\psi_\mathbf{x} = \operatorname{ht}\psi, \end{aligned}$$

where the second implication follows again from the fact that the $\mathfrak{g}$-weights of $W(\psi_\mathbf{x})$ lie in $\operatorname{wt}\psi_\mathbf{x} - Q^+$ by Definition 3.1. $\square$

Recall that, for $\psi \in \mathcal{E}$, we have $\operatorname{wt}\psi = \sum_{x \in X_{\mathrm{rat}}} \psi(x)$. It is clear that $\operatorname{wt}\psi$ is the maximal $\mathfrak{g}$-weight occurring in $V(\psi)$. We have the following characterization of untwisted local Weyl modules in terms of homological properties.

Proposition 4.4 ([4, Prop. 8])**.** *Let M be a maximal weight $(\mathfrak{g}\otimes A)$-module of maximal weight ψ. Then $M \cong W(\psi)$ if and only if*

$$\operatorname{Hom}_\mathcal{F}(M, V(\varphi)) = 0 \quad \text{and} \quad \operatorname{Ext}^1_\mathcal{F}(M, V(\varphi)) = 0$$

for all $\varphi \in \mathcal{E}$ with $\operatorname{wt}(V(\varphi)) \subseteq (\operatorname{wt}\psi - Q^+) \setminus \{\operatorname{wt}\psi\}$.

We want to reformulate this theorem and generalize it to the case of equivariant map algebras.

Theorem 4.5. *Let M be a maximal weight $(\mathfrak{g}\otimes A)^\Gamma$-module of maximal weight ψ. Then $M \cong W_\Gamma(\psi)$ if and only if*

$$\operatorname{Hom}_{\mathcal{F}^\Gamma}(M, V_\Gamma(\varphi)) = 0 \text{ and } \operatorname{Ext}^1_{\mathcal{F}^\Gamma}(M, V_\Gamma(\varphi)) = 0 \ \forall\ \varphi \in \mathcal{E}^\Gamma \text{ with } \operatorname{ht}(\varphi) < \operatorname{ht}(\psi). \tag{4.1}$$

Proof. We first prove the theorem in the case $\Gamma = \{1\}$, where it is a slightly modified version of Proposition 4.4. In this case $(\mathfrak{g}\otimes A)^\Gamma = \mathfrak{g}\otimes A$ and $W_\Gamma(\psi) = W(\psi)$. We first want to show that $W(\psi)$ satisfies

$$\operatorname{Hom}_\mathcal{F}(W(\psi), V(\varphi)) = 0 \quad \text{and} \quad \operatorname{Ext}^1_\mathcal{F}(W(\psi), V(\varphi)) = 0$$

for all $\varphi \in \mathcal{E}$ with $\mathrm{ht}(\varphi) < \mathrm{ht}(\psi)$. Since the group Γ is trivial, all finite-dimensional $(\mathfrak{g} \otimes A)$-modules are also $\mathfrak{g}$-modules via the identification of $\mathfrak{g}$ with $\mathfrak{g} \otimes k \subseteq \mathfrak{g} \otimes A$. Thus we have weight space decompositions as $\mathfrak{g}$-modules.
Let $\lambda = \mathrm{wt}\,\psi$. Since $\mathrm{ht}\,\varphi < \mathrm{ht}\,\psi$, we have $\lambda \notin \mathrm{wt}(\varphi) - Q^+$ and so $V(\varphi)_\lambda = 0$. Since $W(\psi)$ is generated by $W(\psi)_\lambda$, this implies $\mathrm{Hom}_{\mathcal{F}}(W(\psi), V(\varphi)) = 0$.
Now suppose we have an extension of $(\mathfrak{g} \otimes A)$-modules

$$0 \to V(\varphi) \to E \to W(\psi) \to 0. \tag{4.2}$$

Let w_λ be the preimage in E of a maximal weight vector of $W(\psi)$. Since $\lambda \notin \mathrm{wt}(\varphi) - Q^+$, we have $\dim E_\lambda = 1$, and so w_λ is unique up to nonzero scalar multiple. Also, $(\mathfrak{n}^+ \otimes A) \cdot w_\lambda = 0$ and so we have an exact sequence

$$0 \longrightarrow U \longrightarrow U(\mathfrak{g} \otimes A) \cdot w_\lambda \longrightarrow W(\psi) \longrightarrow 0 \tag{4.3}$$

where U is a $\mathfrak{g} \otimes A$-module with $U_\lambda = 0$. Since $\mathrm{wt}(U(\mathfrak{g} \otimes A) \cdot w_\lambda) \subseteq \lambda - Q^+$, we have $\mathrm{wt}(U) \subseteq (\lambda - Q^+) \setminus \{\lambda\}$. Thus Proposition 4.4 implies that (11) splits, which in turn implies that (10) splits. Thus E is the trivial extension. Therefore $\mathrm{Ext}^1_{\mathcal{F}}(W(\psi), V(\varphi)) = 0$.
On the other hand, suppose M satisfies (9). We claim that M also satisfies the properties characterizing $W(\psi)$ as given in Proposition 4.4. Let $\varphi \in \mathcal{E}$ with $\mathrm{wt}(V(\varphi)) \subseteq (\lambda - Q^+) \setminus \{\lambda\}$. Then $\mathrm{wt}\,\varphi < \lambda$, hence $\mathrm{ht}(\varphi) < \mathrm{ht}(\psi)$. The claim then follows from (9). Hence the theorem is true for $\Gamma = \{1\}$.
Now consider the case of arbitrary Γ. Let $\varphi \in \mathcal{E}^\Gamma$ with $\mathrm{ht}(\varphi) < \mathrm{ht}(\psi)$. We would first like to show that

$$\mathrm{Hom}_{\mathcal{F}^\Gamma}(W_\Gamma(\psi), V_\Gamma(\varphi)) = 0 \quad \text{and} \quad \mathrm{Ext}^1_{\mathcal{F}^\Gamma}(W_\Gamma(\psi), V_\Gamma(\varphi)) = 0.$$

Let $\tau \in \mathrm{Hom}_{\mathcal{F}^\Gamma}(W_\Gamma(\psi), V_\Gamma(\varphi))$ be nonzero. Then τ is surjective since $V_\Gamma(\varphi)$ is irreducible, and so $V_\Gamma(\varphi)$ is isomorphic to a quotient of $W_\Gamma(\psi)$. By Proposition 2.4 there exists η, $\mathrm{Supp}\,\eta \subseteq X_*$, such that $W_\Gamma(\psi)$ (hence also $V_\Gamma(\varphi)$) is annihilated by $(\mathfrak{g} \otimes I_\eta)^\Gamma$. Let $\mathbf{x} = \mathrm{Supp}\,\eta$. Then

$$\mathrm{Hom}_{\mathcal{F}^\Gamma}(W_\Gamma(\psi), V_\Gamma(\varphi)) \cong \mathrm{Hom}_{\mathcal{F}^\Gamma_{\mathbf{x}}}(W_\Gamma(\psi), V_\Gamma(\varphi)).$$

Now, by Theorem 2.10 and Lemma 3.8, we have

$$\mathrm{Hom}_{\mathcal{F}^\Gamma_{\mathbf{x}}}(W_\Gamma(\psi), V_\Gamma(\varphi)) \cong \mathrm{Hom}_{\mathcal{F}_{\mathbf{x}}}(W(\psi_{\mathbf{x}}), V(\varphi_{\mathbf{x}})).$$

Since $\mathrm{ht}\,\varphi_{\mathbf{x}} = \mathrm{ht}\,\varphi < \mathrm{ht}\,\psi = \mathrm{ht}\,\psi_{\mathbf{x}}$, we conclude $\mathrm{Hom}_{\mathcal{F}_{\mathbf{x}}}(W(\psi_{\mathbf{x}}), V(\varphi_{\mathbf{x}})) = 0$ since we know the theorem is true in the untwisted case. Thus $\tau = 0$ and so $\mathrm{Hom}_{\mathcal{F}^\Gamma}(W_\Gamma(\psi), V_\Gamma(\varphi)) = 0$.
Now let

$$0 \to V_\Gamma(\varphi) \to E \to W_\Gamma(\psi) \to 0 \tag{4.4}$$

be an extension of $(\mathfrak{g} \otimes A)^\Gamma$-modules with $\mathrm{ht}\,\varphi < \mathrm{ht}\,\psi$. Since E is finite-dimensional, by Proposition 2.4 there exists η, $\mathrm{Supp}\,\eta \subseteq X_*$, such that $(\mathfrak{g} \otimes I_\eta)^\Gamma \cdot E = 0$. But this implies

$$(\mathfrak{g} \otimes I_\eta)^\Gamma \cdot W_\Gamma(\psi) = 0 \quad \text{and} \quad (\mathfrak{g} \otimes I_\eta)^\Gamma \cdot V_\Gamma(\varphi) = 0.$$

Thus (12) is an exact sequence in $\mathcal{F}^\Gamma_{\mathbf{x}}$ for $\mathbf{x} = \mathrm{Supp}\,\eta$ and hence, by Theorem 2.10 and Lemma 3.8,

$$0 \to V(\varphi_{\mathbf{x}}) \to \mathbf{U}_{\mathbf{x}} E \to W(\psi_{\mathbf{x}}) \to 0 \tag{4.5}$$

is a short exact sequence in $\mathcal{F}_{\mathbf{x}}$. Since $\operatorname{ht}\varphi_{\mathbf{x}} = \operatorname{ht}\varphi < \operatorname{ht}\psi = \operatorname{ht}\psi_{\mathbf{x}}$, (13) splits by the fact that the theorem is true in the untwisted case. Then Theorem 2.10 implies that (12) splits. So $\operatorname{Ext}^1_{\mathcal{F}^\Gamma}(W_\Gamma(\psi), V_\Gamma(\varphi)) = 0$.

On the other hand, suppose M satisfies (9). We would like to show that $M \cong W_\Gamma(\psi)$. Fix $\mathbf{x} \in (\operatorname{Supp} M)_\Gamma$. Then $M \in \mathcal{F}^\Gamma_{\mathbf{x}}$ and so $\mathbf{U}_{\mathbf{x}} M$ is a module in $\mathcal{F}_{\mathbf{x}}$. By Theorem 2.10 and Lemma 3.8, it suffices to show that $\mathbf{U}_{\mathbf{x}} M \cong W(\psi_{\mathbf{x}})$. Since M is a maximal weight module of maximal weight ψ, we have $\operatorname{Supp}\psi \subseteq \operatorname{Supp} M$, hence $\mathbf{x} \cap (\operatorname{Supp}\psi) \in (\operatorname{Supp}\psi)_\Gamma$ and $\mathbf{U}_{\mathbf{x}} M$ is a maximal weight module of maximal weight $\psi_{\mathbf{x}}$. In particular, this implies that the $\mathfrak{g}$-weight space of $\mathbf{U}_{\mathbf{x}} M$ of weight $\operatorname{wt}\psi_{\mathbf{x}}$ is one-dimensional.

Let m_ψ be a nonzero element of $(\mathbf{U}_{\mathbf{x}} M)_{\operatorname{wt}\psi_{\mathbf{x}}}$. We claim that $\mathbf{U}_{\mathbf{x}} M$ is cyclic and generated by m_ψ. Indeed, if this were not the case, then the submodule generated by v, where v is in a $\mathfrak{g}$-complement of $U(\mathfrak{g} \otimes A) \cdot m_\psi$ would have an irreducible quotient $V(\varphi)$, with $\operatorname{ht}\varphi < \operatorname{ht}\psi$ and $\operatorname{Supp}\varphi \subseteq \mathbf{x}$. Then $\mathbf{T}_{\mathbf{x}}(V(\varphi)) = V_\Gamma(\varphi^\Gamma)$ would be an irreducible object of $\mathcal{F}^\Gamma_{\mathbf{x}}$. Again by Theorem 2.10, we would have

$$\operatorname{Hom}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{U}_{\mathbf{x}} M, V(\varphi)) \neq 0 \implies \operatorname{Hom}_{\mathcal{F}^\Gamma_{\mathbf{x}}}(M, V_\Gamma(\varphi^\Gamma)) \neq 0,$$

which contradicts (9) since $\operatorname{ht}\varphi^\Gamma = \operatorname{ht}\varphi < \operatorname{ht}\psi$. By Proposition 3.2(2), $\mathbf{U}_{\mathbf{x}} M$ is a quotient of $W(\psi_{\mathbf{x}})$. It remains to show that it is not a proper quotient. We have $(\mathbf{U}_{\mathbf{x}} M)_\mu = 0$ for all $\mu > \lambda$, so $(\mathfrak{n}^+ \otimes A) \cdot m_\psi = 0$, which implies we have an exact sequence

$$0 \to U \to W(\psi_{\mathbf{x}}) \to \mathbf{U}_{\mathbf{x}} M \to 0$$

with U an object of $\mathcal{F}_{\mathbf{x}}$ satisfying $U_\lambda = 0$ and $\operatorname{wt}(U) \subseteq \lambda - Q^+$. Applying $\mathbf{T}_{\mathbf{x}}$, we have

$$0 \to \mathbf{T}_{\mathbf{x}} U \to W_\Gamma(\psi) \to M \to 0. \tag{4.6}$$

Now applying $\operatorname{Hom}_{\mathcal{F}^\Gamma_{\mathbf{x}}}(-, V_\Gamma(\varphi))$, for $\varphi \in \mathcal{E}^\Gamma$ with $\operatorname{Supp}\varphi \subseteq \mathbf{x}$, to the short exact sequence (14), we obtain the long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathcal{F}^\Gamma_{\mathbf{x}}}(M, V_\Gamma(\varphi)) \to \operatorname{Hom}_{\mathcal{F}^\Gamma_{\mathbf{x}}}(W_\Gamma(\psi), V_\Gamma(\varphi)) \to \operatorname{Hom}_{\mathcal{F}^\Gamma_{\mathbf{x}}}(\mathbf{T}_{\mathbf{x}} U, V_\Gamma(\varphi)) \\ \to \operatorname{Ext}^1_{\mathcal{F}^\Gamma_{\mathbf{x}}}(M, V_\Gamma(\varphi)) \to \cdots \end{aligned}$$

By (9), we have

$$\operatorname{Hom}_{\mathcal{F}^\Gamma_{\mathbf{x}}}(W_\Gamma(\psi), V_\Gamma(\varphi)) = \operatorname{Hom}_{\mathcal{F}^\Gamma}(W_\Gamma(\psi), V_\Gamma(\varphi)) = 0 \quad \text{and}$$
$$\operatorname{Ext}^1_{\mathcal{F}^\Gamma_{\mathbf{x}}}(M, V_\Gamma(\varphi)) = \operatorname{Ext}^1_{\mathcal{F}^\Gamma}(M, V_\Gamma(\varphi)) = 0$$

when $\operatorname{ht}\varphi < \operatorname{ht}\psi$. Thus $\operatorname{Hom}_{\mathcal{F}^\Gamma_{\mathbf{x}}}(\mathbf{T}_{\mathbf{x}} U, V_\Gamma(\varphi)) = 0$, whenever $\operatorname{ht}\varphi < \operatorname{ht}\psi$. Since all irreducible subquotients $V_\Gamma(\varphi)$ of $\mathbf{T}_{\mathbf{x}} U$ satisfy $\operatorname{Supp}\varphi \subseteq \mathbf{x}$ and $\operatorname{ht}\varphi < \psi$, we have $\mathbf{T}_{\mathbf{x}} U = 0$ and hence $U = 0$. Thus the theorem follows. ☐

The following corollary is a twisted version of Proposition 3.2(2). Condition (15) below should be thought of as a twisted analogue of the condition in Proposition 3.2(2) that M is cyclicly generated by the vector v.

Corollary 4.6. *Let M be a maximal weight $(\mathfrak{g} \otimes A)^\Gamma$-module of maximal weight $\psi \in \mathcal{E}^\Gamma$ such that*

$$\operatorname{Hom}_{\mathcal{F}^\Gamma}(M, V(\varphi)) = 0 \tag{4.7}$$

for all $\varphi \in \mathcal{E}^\Gamma$ with $\operatorname{ht}\varphi < \operatorname{ht}\psi$. Then M is a quotient of $W_\Gamma(\psi)$.

Proof. This follows from the proof of Theorem 4.5. □

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DEMAZURE MODULES AND WEYL MODULES: THE TWISTED CURRENT CASE

GHISLAIN FOURIER AND DENIZ KUS

Abstract. We study finite–dimensional respresentations of twisted current algebras and show that any graded twisted Weyl module is isomorphic to level one Demazure module for the twisted affine Kac-Moody algebra. Using the tensor product property of Demazure modules, we obtain, by analyzing the fundamental Weyl modules, dimension and character formulas. Moreover we prove that graded twisted Weyl modules can be obtained by taking the associated graded modules of Weyl modules for the loop algebra, which implies that its dimension and classical character are independent of the support and depend only on its classical highest weight. These results were known before for untwisted current algebras and are new for all twisted types.

1. Introduction

Weyl modules for loop algebras $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, where $\mathfrak{g}$ is a simple complex Lie algebra, have gained a lot of attraction during the last two decades. Starting with the analysis of finite–dimensional irreducible modules for quantum affine algebras ([9]), which are highest weight modules in a certain sense. It was natural to ask for maximal finite–dimensional modules with these highest weights since contrary to the theory of simple complex Lie algebras, the category of finite–dimensional modules is not semi–simple. In the same paper it was conjectured, that the classical limit $q = 1$ of these irreducible modules specialize to modules for the loop algebra satisfying some universal properties, the so called *local Weyl modules*. In a series of papers ([1], [6], [8], [15], [24], [25]) the character and dimension of these Weyl modules were computed. In the proofs, these modules were identified with Weyl modules for the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$. Using the tensor product property ([9]) and some pullback maps, the study was reduced to analyzing graded Weyl modules for $\mathfrak{g} \otimes \mathbb{C}[t]$, where the grading is induced by the grading of $\mathbb{C}[t]$.

One major step in the analysis of the graded Weyl modules is their identification with level one Demazure modules for simply–laced algebras ([6], [15]). With the tensor product property for Demazure modules ([14]) and the computation for fundamental Weyl and Demazure modules ([6], [14]), the character and dimension formulas were proven. In the non simply–laced case, Weyl modules admit a filtration by Demazure modules and via this filtration, the dimension and character formula were proven ([25]). One should mention that these results can also be deduced from the results in [1], [24], but there is no written proof so far in the literature.

Local Weyl modules for current and loop algebras can be parametrized by finitely supported functions from $\mathbb{C}$ (resp. $\mathbb{C}^*$) to P^+, the set of dominant integral weights for $\mathfrak{g}$. To each function one can associate a weight, which is the sum of all images, hence in P^+. To summarize

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the results above, the dimension and character of a local Weyl module are independent of the support of the parametrizing function and depend only on its weight. The graded local Weyl module of weight λ is parametrized by the function of weight λ with support in the origin only. We can also reformulate this result in terms of the global Weyl module, which is a projective module in a certain category and in general infinite–dimensional. The results on local Weyl modules are equivalent to the statement, that the global Weyl module is a free module for a certain commutative algebra $\mathbf{A}_\lambda$.

There are several ways to generalize the notion of local Weyl modules. By replacing $\mathbb{C}[t,t^{-1}]$ with a commutative, associative algebra ([4],[12]) one can define local and global Weyl modules as before, obtain similar tensor product properties, but character and dimension formulas are known only in certain cases. Even for a case as simply looking as $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathbb{C}[t_1,\dots,t_n]$ with $n \geq 4$ there is no dimension formula known.

Another way of generalizing local Weyl modules is to look at twisted current and loop algebras. Given a complex simple Lie algebra $\mathfrak{g}$ and a commutative algebra A $(= \mathbb{C}[t], \mathbb{C}[t,t^{-1}])$, both equipped with the action of a finite group Γ ($\Gamma = \mathbb{Z}/m\mathbb{Z}$) by automorphism, one can extend this action to $\mathfrak{g} \otimes A$. The fixpoint Lie algebra $(\mathfrak{g} \otimes A)^\Gamma$ is called the twisted current algebra (resp. twisted loop algebra). The twisted current algebra is a subalgebra of the twisted affine Kac-Moody algebra associated to $\mathfrak{g}$, while the twisted loop is obtained by taking the quotient by the central element of the subalgebra without derivation [2].

Local Weyl modules for the twisted loop algebra were introduced and studied in [5]. It was proven, that every Weyl module is the tensor product of Weyl modules located in a single point only. So to obtain dimension and character formulas it was sufficient to compute them for Weyl modules with support in a single point. The main theorem in [5] states that every Weyl module for the twisted loop algebra is isomorphic to the restriction of a Weyl module for the untwisted loop algebra. So all interesting information can be deduced from this isomorphism. In [16] the aforementioned global Weyl modules will be defined and studied for twisted loop algebras as well. It will be shown, that the twisted global Weyl module is a submodule of the untwisted global Weyl module, viewed as a module for the twisted loop algebra by restriction. The results about twisted local Weyl module translate again into the freeness of the twisted global Weyl module as a module for a certain commutative algebra $\mathbf{A}_\lambda^\Gamma$.

In [13] the notion of local Weyl modules was generalized to certain equivariant map algebras. Given X an affine scheme and $\mathfrak{g}$ a finite–dimensional Lie algebra, both defined over an algebraically closed field and Γ a finite group acting on X and $\mathfrak{g}$ by automorphisms, the equivariant map algebra is the Lie algebra of equivariant maps from X to $\mathfrak{g}$. In [13] several restrictions to this general case were assumed, the group action on X had to be free and abelian. But under these assumptions, again the tensor product property was proven. Furthermore it was shown, that every Weyl module for the equivariant map algebra is isomorphic to the restriction of a Weyl module for the algebra of maps from X to $\mathfrak{g}$.

In this paper we are considering the gap in the computation of dimension and character formulas for local Weyl modules of twisted current algebras. For twisted and untwisted loop and current algebras, dimension formulas for all local Weyl modules are known except for graded local Weyl modules for the twisted current algebra. Let Γ be the finite group of non–trivial diagram automorphism of a simple Lie algebra $\mathfrak{g}$, so Γ is of order 2 or 3 and $\mathfrak{g}$ of type A, D, E. In terms of equivariant map algebras, the affine scheme would be $X = \mathbb{C}$

and $\Gamma = \langle \xi \rangle$, where ξ is the multiplication by a primitive 2nd or 3rd root of unity. We see immediately that 0 is a fix point, so the group action is not free. In this setting, the results of [13] do not apply at the origin.
The goal of this paper is to compute a dimension and character formula for the local Weyl module located in 0 (the graded local Weyl module) of the twisted current algebra. The main tool are, as in [15] and [25], Demazure modules.
There are two cases to be considered, the first one is:

Theorem. Let $\mathfrak{g}$ be not of type A_{2l} and $\lambda \in P_0^+$, then the local graded $(\mathfrak{g} \otimes \mathbb{C}[t])^\Gamma$-Weyl module $W^\Gamma(\lambda)$ is isomorphic to a Demazure module of level 1.

In the proof we will use the $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$ and the $(\mathfrak{sl}_3 \otimes \mathbb{C}[t])^\Gamma$ cases (proven in [6],[9], resp. Section 7). A tensor product property for Demazure modules was proven in [14], so to obtain a character formula for Weyl modules it is sufficient to determine the *fundamental local Weyl modules*, as done in Section 5. Concluding we were able to prove an analogous result to [5], [13], that the dimension of the local Weyl module does not depend on the support but only on the highest weight.

Theorem. For $\mathfrak{g}$ not of type A_{2l} and $\lambda \in P_0^+$, the local graded $(\mathfrak{g} \otimes \mathbb{C}[t])^\Gamma$-Weyl module is isomorphic to the associated graded module of the restriction of a local Weyl module for $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

In the second case, we assume that $\mathfrak{g}$ is of type A_{2l}, then the fixpoint algebra $\mathfrak{g}_0$ is of type B_l. Here with our methods, one can only determine the local Weyl module for weights λ, where $\lambda(\alpha_l^\vee)$ is odd. In this case there is an identification with Demazure modules as before, so the graded local $(\mathfrak{g} \otimes \mathbb{C}[t])^\Gamma$-Weyl module is isomorphic to a Demazure module of level 1. Furthermore we are able to show the following:

Theorem. Let $\lambda = \lambda_1 + \lambda_2 \in P_0^+$, where $\lambda_2(\alpha_l^\vee)$ is odd, and $a \in \mathbb{C}^*$. Then

$$W^\Gamma(\lambda) \cong \mathrm{gr}(W_a(\lambda_1) \otimes W^\Gamma(\lambda_2)),$$

where $W_a(\lambda_1)$ is the local Weyl module for $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, supported in a with highest weight λ_1.

In the case where $\lambda(\alpha_l^\vee)$ is even the dimension and character of the local Weyl modules remains uncomputed, the identification with Demazure modules fails. We can state here a conjecture only

Conjecture. Let $\lambda \in P_0^+$, then the graded local Weyl module is isomorphic to the associated graded module of the restriction of a local Weyl module for $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. The dimension of a local Weyl module of highest weight λ is independent of the support of the module.

The structure of the paper is as follows, in Section 2 are basics and notations for affine Kac-Moody algebras recalled, in Section 3 for twisted current algebras. In Section 4 Demazure and Weyl modules are defined. In Section 5 we identify Demazure modules with Weyl modules and determine the "smallest" Weyl modules. In Section 6 we show that every graded Weyl module of the twisted current algebra can be obtained by taking the associated graded of the restriction of a untwisted loop module. In Section 7, the case $\mathfrak{g} = \mathfrak{sl}_3$ is treated seperately, since it is used in some of the proofs of the other cases.

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2. The affine Kac-Moody algebras

2.1. **Notation and basic results.** In this section we fix the notation and the usual technical padding. Let $\mathfrak{g} = \mathfrak{g}(A)$ be a simple complex Lie algebra of rank l associated to a Cartan matrix A of finite type, denote $I = \{1, \dots, l\}$. We fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b} \supseteq \mathfrak{h}$. Denote $\Phi \subseteq \mathfrak{h}^*$ the root system of $\mathfrak{g}$, and, corresponding to the choice of $\mathfrak{b}$, let Φ^+ be the set of positive roots and let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the corresponding basis of Φ.

For a root $\beta \in \Phi$ let $\beta^\vee \in \mathfrak{h}$ be its coroot. The basis of the dual root system (also called the coroot system) $\Phi^\vee \subset \mathfrak{h}$ is denoted $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_l^\vee\}$. The Weyl group W of Φ is generated by the simple reflections $s_i = s_{\alpha_i}$ associated to the simple roots.

Let $P = \bigoplus_{i=1}^l \mathbb{Z}\omega_i$ be the weight lattice of $\mathfrak{g}$ and let $P^+ = \bigoplus_{i=1}^l \mathbb{Z}_{\geq 0}\omega_i$ be the subset of dominant weights. The group algebra of P is denoted $\mathbb{Z}[P]$, we write $\chi = \sum a_\mu e^\mu$ (finite sum, $\mu \in P$, $a_\mu \in \mathbb{Z}$) for an element in $\mathbb{Z}[P]$, where the embedding $P \hookrightarrow \mathbb{Z}[P]$ is defined by $\mu \mapsto e^\mu$. Further we denote by $Q = \bigoplus_{i=1}^l \mathbb{Z}\alpha_i$ (respectively $Q^+ = \bigoplus_{i=1}^l \mathbb{Z}_{\geq}\alpha_i$) be the root (respectively positive root) lattice and let $\{x_i^\pm, h_i | i \in I\}$ be a set of Chevalley generators of $\mathfrak{g}$.

Let $\widehat{\mathfrak{g}}$ be the affine Kac–Moody algebra (twisted or untwisted) corresponding to the Cartan matrix $\widehat{A} = (a_{i,j})$. Note that, if $\widehat{\mathfrak{g}}$ is a untwisted affine Kac–Moody algebra associated to $\mathfrak{g}$:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Here d denotes the derivation $d = t\frac{d}{dt}$ and c is the canonical central element. Recall that the Lie bracket is defined as

$$[x \otimes t^m + \lambda c + \mu d, y \otimes t^n + \nu c + \eta d] = [x, y] \otimes t^{n+m} + \mu n y \otimes t^n + \eta m x \otimes t^m + m\delta_{m,-n}(x, y)c.$$

We assume $\widehat{\mathfrak{g}}$ is arbitrary (possibly twisted) and we fix a Cartan subalgebra $\widehat{\mathfrak{h}}$ in $\widehat{\mathfrak{g}}$ and a Borel subalgebra $\widehat{\mathfrak{b}} \supseteq \widehat{\mathfrak{h}}$, $\Pi = \{\alpha_0, \dots, \alpha_l\}$ the set of simple roots, $\Pi^\vee = \{\alpha_0^\vee, \dots, \alpha_l^\vee\}$ the set of simple coroots. Denote by $\widehat{\Phi}$ the root system of $\widehat{\mathfrak{g}}$ and let $\widehat{\Phi}^+$ be the subset of positive roots. We denote by $\widehat{P}$ the weight lattice of $\widehat{\mathfrak{g}}$ and let $\widehat{P^+}$ be the subset of dominant weights. The Weyl group $\widehat{W}$ of $\widehat{\Phi}$ is generated by the simple reflections $s_i = s_{\alpha_i}$ associated to the simple roots. Further we fix vectors $w = (a_0, \dots, a_l)^t, v = (a_0^\vee, \dots, a_l^\vee)$, such that $v\widehat{A} = \widehat{A}w = 0$. v and w are here uniquely determind up to scalars. Then it is known that the center of $\widehat{\mathfrak{g}}$ is 1-dimensional and is spanned by the canonical central element

$$c = \sum_{i=0}^l a_i^\vee \alpha_i^\vee.$$

Define further

$$\delta = \sum_{i=0}^{l} a_i\alpha_i; \quad \theta = \delta - a_0\alpha_0$$

and $d \in \widehat{\mathfrak{h}}$ which satisifes the following conditions

$$\alpha_i(d) = 0, \text{for } i = 1, \ldots, l; \quad \alpha_0(d) = 1.$$

Clearly the elements $\alpha_0^\vee, \ldots, \alpha_l^\vee, d$ form a basis of $\widehat{\mathfrak{h}}$. We have a non–degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\widehat{\mathfrak{h}}$ defined in ([19], Chapter 6)

$$(2.1) \qquad \begin{cases} \langle \alpha_i^\vee, \alpha_j^\vee \rangle = \frac{a_j}{a_j^\vee} a_{i,j} & i,j = 0, \ldots, \ell \\ \langle \alpha_i^\vee, d \rangle = 0 & i = 1, \ldots, \ell \\ \langle \alpha_0^\vee, d \rangle = \frac{a_0}{a_0^\vee} & \langle d, d \rangle = 0. \end{cases}$$

This $\widehat{W}$-invariant form induces a map

$$\nu : \widehat{\mathfrak{h}} \longrightarrow \widehat{\mathfrak{h}}^*, \quad \nu(h) : \begin{cases} \widehat{\mathfrak{h}} & \to & \mathbb{C} \\ h' & \mapsto & \langle h, h' \rangle \end{cases}$$

With the notation as above it follows for $i = 0, \ldots, l$:

$$\nu(\alpha_i^\vee) = \frac{a_i}{a_i^\vee}\alpha_i$$

Let $\Lambda_0, \ldots, \Lambda_l$ be the fundamental weights in $\widehat{P^+}$, then for $i = 1, \ldots, l$ we have

$$(2.2) \qquad \Lambda_i = \omega_i + \frac{a_i^\vee}{a_0^\vee}\Lambda_0.$$

With this we have $\widehat{P} = \sum_{i=0}^{l} \mathbb{Z}\Lambda_i + \mathbb{Z}(\delta/a_0)$ and $\widehat{P^+} = \sum_{i=0}^{l} \mathbb{Z}_{\geq 0}\Lambda_i + \mathbb{Z}(\delta/a_0)$.

2.2. **Realisation of twisted affine algebras.** In this paper we are mainly interested in twisted affine Kac-Moody algebras, which can be realised as fixed point subalgebras of so-called twisted graph automorphisms. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a graph automorphism of order m. In particular

$$m = \begin{cases} 2, & \text{if } \mathfrak{g} \text{ of type } A_{2l}, A_{2l-1}, D_{l+1} \text{ or } E_6 \\ 3, & \text{if } \mathfrak{g} \text{ is of type } D_4 \end{cases}$$

Let ξ be a primitive m^{th} root of unity, then it is well-known that there exists a decomposition of $\mathfrak{g}$ into eigenspaces. We obtain:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{m-1},$$

whereby $\mathfrak{g}_j = \{x \in \mathfrak{g} | \sigma(x) = \xi^j x\}$, $j = 0, \cdots, m-1$. The fixed point algebra $\mathfrak{g}_0$ is again a simple complex Lie algebra of type C_l, B_l, F_4 or G_2 and the eigenpaces are irreducible $\mathfrak{g}_0$-modules.

Remark 2.1. Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{g}$ such that $\sigma(\mathfrak{a}) = \mathfrak{a}$, then we get a analogue decomposition

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \cdots \oplus \mathfrak{a}_{m-1}.$$

So if $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a triangular decomposition of $\mathfrak{g}$, we obtain

$$\mathfrak{g}_j = \mathfrak{n}_j \oplus \mathfrak{h}_j \oplus (\mathfrak{n}_-)_j \text{ for all } 0 \leq j \leq m-1.$$

Now we can extend σ to a automorphism of the corresponding untwisted affine algebra given by

$$\sigma(x \otimes t^i) = \xi^{-i}\sigma(x) \otimes t^i \quad \text{for } x \in \mathfrak{g}$$

$$\sigma(c) = c; \quad \sigma(d) = d.$$

The twisted affine algebra is realized as the fixed point subalgebra

$$\widehat{\mathfrak{g}} \cong \bigoplus_{k\in\mathbb{Z}} (\mathfrak{g}_0 \otimes t^{mk}) \oplus \cdots \oplus \bigoplus_{k\in\mathbb{Z}} (\mathfrak{g}_{m-1} \otimes t^{mk+(m-1)}) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

$$= \bigoplus_{j=0}^{m-1} \bigoplus_{k\in\mathbb{Z}} (\mathfrak{g}_j \otimes t^{mk+j}) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Using the above notation we can conclude

$$\widehat{\mathfrak{h}} = \mathfrak{h}_0 \oplus (\mathbb{C}c + \mathbb{C}d) \quad \widehat{\mathfrak{h}}^* = (\mathfrak{h}_0)^* \oplus (\mathbb{C}\delta + \mathbb{C}\Lambda_0).$$

We have the following table, which describes the various possibilities for $\mathfrak{g}, \mathfrak{g}_0$, $\widehat{\mathfrak{g}}$ and the eigenspaces $\mathfrak{g}_1, \mathfrak{g}_2$.

m	$\mathfrak{g}$	$\mathfrak{g}_0$	$\widehat{\mathfrak{g}}$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	Dynkin diagram of $\widehat{\mathfrak{g}}$
2	A_2	A_1	$A_2^{(2)}$	$V(4\omega_1)$	/	$\underset{0}{\circ} \Rrightarrow \underset{1}{\circ}$
2	$A_{2l}, l \geq 2$	B_l	$A_{2l}^{(2)}$	$V(2\omega_1)$	/	$\underset{0}{\circ} \Rightarrow \underset{1}{\circ} - \cdots - \underset{l-1}{\circ} \Rightarrow \underset{l}{\circ}$
2	$A_{2l-1}, l \geq 2$	C_l	$A_{2l-1}^{(2)}$	$V(\omega_2)$	/	$\underset{1}{\circ} - \underset{2}{\overset{\circ_0 \mid}{\circ}} - \cdots - \underset{l-1}{\circ} \Leftarrow \underset{l}{\circ}$
2	$D_{l+1}, l \geq 3$	B_l	$D_{l+1}^{(2)}$	$V(\omega_1)$	/	$\underset{0}{\circ} \Leftarrow \underset{1}{\circ} - \cdots - \underset{l-1}{\circ} \Rightarrow \underset{l}{\circ}$
2	E_6	F_4	$E_6^{(2)}$	$V(\omega_1)$	/	$\underset{0}{\circ} - \underset{1}{\circ} - \underset{2}{\circ} \Leftarrow \underset{3}{\circ} - \underset{4}{\circ}$
3	D_4	G_2	$D_4^{(3)}$	$V(\omega_2)$	$V(\omega_2)$	$\underset{1}{\circ} \Rrightarrow \underset{2}{\circ} - \underset{0}{\circ}$

We put a "0" on (almost) everything related to $\mathfrak{g}_0$, e.g. denote by $\Phi_0 \subseteq (\mathfrak{h}_0)^*$ the root system of $\mathfrak{g}_0$. The recently defined element δ is the imaginary root in $\widehat{\Phi}^+$ and θ is the highest short root of the root system of $\mathfrak{g}_0$ if $\widehat{A}$ is of type $A_{2l-1}^{(2)}, D_{l+1}^{(2)}, E_6^{(2)}, D_4^{(3)}$. In the remaining twisted cases $\theta - \alpha_1$ is the highest root of the root system of $\mathfrak{g}_0$. For more details we refer to ([2],[19]).

Remark 2.2. The untwisted Kac-Moody algebras $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ can also be realised as fixed point algebras for any automorphism of order 1. We have $\mathfrak{g}_0 = \mathfrak{g}$ and the eigenspaces are the zerospaces. In this case θ is the highest root of $\mathfrak{g}$.

2.3. **The extended affine Weyl group.** Now we give a description of the Weyl group $\widehat{W}$ of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$. The Weyl group is generated by fundamental reflections $s_0, \dots, s_l$, which act on $\widehat{\mathfrak{h}}^*$ by

$$s_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i, \quad \lambda \in \widehat{\mathfrak{h}}^*.$$

Since $\delta(\alpha_i^\vee) = 0$ for all $i = 0, \dots, l$, the Weyl group $\widehat{W}$ fixes δ. Another well-known description of the affine Weyl-group is the following. Let W_0 be the subgroup of $\widehat{W}$ generated by $s_1, \dots, s_l$, i.e. W_0 can be identified with the Weyl group of the Lie algebra $\mathfrak{g}_0$, since W_0 operates trivially on $(\mathbb{C}\delta + \mathbb{C}\Lambda_0)$. Further let

$$M = \sum_{i=1}^{l} \mathbb{Z}\alpha_i \quad \text{if } \widehat{A} \text{ symmetric or } m > a_0 \tag{2.3}$$

or

$$M = \nu(\sum_{i=1}^{l} \mathbb{Z}\alpha_i^\vee) \quad \text{otherwise.} \tag{2.4}$$

For an element $\mu \in M$ let t_μ be the following endomorphism of the vector space $\widehat{\mathfrak{h}}^*$:

$$\Lambda = \lambda + b\Lambda_0 + r\delta \mapsto t_\mu(\Lambda) = \Lambda + \Lambda(c)\mu - (\langle \Lambda, \mu \rangle + \frac{1}{2}\langle \mu, \mu \rangle \Lambda(c))\delta \tag{2.5}$$

Obviously we have $t_\mu \circ t_{\mu'} = t_{\mu+\mu'}$, denote t_M the abelian group consisting of the elements t_μ, $\mu \in M$. Then $\widehat{W}$ is the semidirect product $\widehat{W} = W_0 \ltimes t_M$.
The *extended affine Weyl group* $\widehat{W}^{ext}$ is the semidirect product $\widehat{W}^{ext} = \widehat{W} \ltimes t_L$, where $L = \nu(\bigoplus_{i=1}^l \mathbb{Z}\omega_i^\vee)$ is the image of the coweight lattice. The action of an element t_μ, $\mu \in L$, is defined as above in (2.5). Let Σ be the subgroup of $\widehat{W}^{ext}$ stabilizing the dominant Weyl chamber $\widehat{C}$:

$$\Sigma = \{\sigma \in \widehat{W}^{ext} \mid \sigma(\widehat{C}) = \widehat{C}\}.$$

Then Σ provides a complete system of coset representatives of $\widehat{W}^{ext}/\widehat{W}$, so we can write in fact $\widehat{W}^{ext} = \Sigma \ltimes \widehat{W}$.
The elements $\sigma \in \Sigma$ are all of the form

$$\sigma = \tau_i t_{-\nu(\omega_i^\vee)} = \tau_i t_{-\omega_i},$$

where $\omega_i^\vee$ is a minuscule fundamental coweight. Further, set $\tau_i = w_0 w_{0,i}$, where w_0 is the longest word in W_0 and $w_{0,i}$ is the longest word in W_{ω_i}, the stabilizer of ω_i in W_0.

2.4. **Weight space decomposition and roots.** Remember that the Borel subalgebra for the twisted case is given by:

$$\widehat{\mathfrak{b}} = ((\mathfrak{h}_0 \oplus \mathfrak{n}_0) \otimes 1) \oplus \bigoplus_{k \in \mathbb{N}_{>0}} (\mathfrak{g}_0 \otimes t^{mk}) \oplus \bigoplus_{k \in \mathbb{N}} (\mathfrak{g}_1 \otimes t^{mk+1}) \oplus \dots \oplus \bigoplus_{k \in \mathbb{N}} (\mathfrak{g}_{m-1} \otimes t^{mk+(m-1)}) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Furthermore we remember that $\mathfrak{g}_j$ is a irreducible $\mathfrak{g}_0$-module for all j, so one can obtain the following weight space decomposition

$$\mathfrak{g}_j = \bigoplus_{\alpha \in (\mathfrak{h}_0)^*} (\mathfrak{g}_j)_\alpha$$

Proposition 2.3. $\mathfrak{h}_j = (\mathfrak{g}_j)_0$; $(\mathfrak{n}_-)_j = \bigoplus_{\alpha\in\Phi^-|_{\mathfrak{h}_0}}(\mathfrak{g}_j)_\alpha$; $\mathfrak{n}_j = \bigoplus_{\alpha\in\Phi^+|_{\mathfrak{h}_0}}(\mathfrak{g}_j)_\alpha$, $0 \leq j \leq m-1$.

Let $(\Phi_0)_s$ be the set of short roots and $(\Phi_0)_l$ be the set of long roots of $\mathfrak{g}_0$ and $\Phi_j = \{\alpha \in (\mathfrak{h}_0)^* | (\mathfrak{g}_j)_\alpha \neq 0\} - \{0\}$, then we get

$$\mathfrak{g}_j = \mathfrak{h}_j \oplus \bigoplus_{\alpha\in\Phi_j}(\mathfrak{g}_j)_\alpha = \mathfrak{h}_j \oplus \bigoplus_{\alpha\in\Phi_j^+}(\mathfrak{g}_j)_\alpha \oplus \bigoplus_{\alpha\in\Phi_j^-}(\mathfrak{g}_j)_\alpha,$$

whereby $\dim(\mathfrak{g}_j)_\alpha = 1$ for all $\alpha \in \Phi_j$, hence $(\mathfrak{g}_j)_{\pm\alpha} = \mathbb{C}X^{\pm}_{\alpha,j}$ for $\alpha \in \Phi_j^+$ and we have the following table [2]:

$\mathfrak{g}$	$\mathfrak{g}_0$	Φ_1	Φ_2	Dynkin diagram of $\mathfrak{g}_0$
A_2	A_1	$(\Phi_0) \cup \{2\alpha : \alpha \in \Phi_0\}$	/	$\underset{1}{\circ}$
$A_{2l}, l \geq 2$	B_l	$(\Phi_0) \cup \{2\alpha : \alpha \in (\Phi_0)_s\}$	/	$\underset{1}{\circ} - \underset{2}{\circ} - \cdots - \underset{l-1}{\circ} \Rightarrow \underset{l}{\circ}$
$A_{2l-1}, l \geq 2$	C_l	$(\Phi_0)_s$	/	$\underset{1}{\circ} - \underset{2}{\circ} - \cdots - \underset{l-1}{\circ} \Leftarrow \underset{l}{\circ}$
$D_{l+1}, l \geq 3$	B_l	$(\Phi_0)_s$	/	$\underset{1}{\circ} - \underset{2}{\circ} - \cdots - \underset{l-1}{\circ} \Rightarrow \underset{l}{\circ}$
E_6	F_4	$(\Phi_0)_s$	/	$\underset{1}{\circ} - \underset{2}{\circ} \Leftarrow \underset{3}{\circ} - \underset{4}{\circ}$
D_4	G_2	$(\Phi_0)_s$	$(\Phi_0)_s$	$\underset{1}{\circ} \Rrightarrow \underset{2}{\circ}$

3. The twisted current algebra $\mathfrak{C}(\widehat{\mathfrak{g}})$

In this section we will define the twisted current algebra $\mathfrak{C}(\widehat{\mathfrak{g}})$ and certain subalgebras, which will be needed in the following sections. The main object of this paper will be

$$\mathfrak{C}(\widehat{\mathfrak{g}}) := \bigoplus_{j=0}^{m-1}\bigoplus_{k\geq 0}(\mathfrak{g}_j \otimes t^{mk+j}).$$

The algebra $\mathfrak{C}(\widehat{\mathfrak{g}})$ can be realized by taking the fixpoints under the group of automorphisms Γ restricted to the current algebra, in detail $(\mathfrak{g}\otimes\mathbb{C}[t])^\Gamma \cong \mathfrak{C}(\widehat{\mathfrak{g}})$, hence it is called the *twisted current algebra*.

In order to give an explicit basis of $\mathfrak{C}(\widehat{\mathfrak{g}})$ we use the embedding $\mathfrak{g}_j \hookrightarrow \mathfrak{g}$ for all $0 \leq j \leq m-1$, so that we can realize the generators of the weight spaces $(\mathfrak{g}_j)_{\pm\alpha}$ as elements in $\mathfrak{g}$. This is already described in [2],[5] and [19] if α is a simple root and can be continued to arbitrary $\alpha \in \Phi_0$: Let $(\widetilde{\alpha_1}, \cdots, \widetilde{\alpha_m})$ be a m-element orbit of σ on Φ and $x^{\pm}_{\widetilde{\alpha_i}} \in \mathfrak{g}$ be root vectors such that $\sigma(x^{\pm}_{\widetilde{\alpha_i}}) = x^{\pm}_{\widetilde{\alpha_j}}$, where $j \equiv i+1 \bmod m$. Then we obtain

$$(\sum_{i=0}^{m-1}(\xi^i)^j x^{\pm}_{\sigma^i(\widetilde{\alpha_1})}) \in (\mathfrak{g}_j)_{\pm\widetilde{\alpha_1}|_{\mathfrak{h}_0}},\ 0 \leq j \leq m-1.$$

In ([2], Chapter 18.4) it is shown that the weight spaces of $\mathfrak{g}_j$ are spanned by such elements for all m-element orbits $(\widetilde{\alpha_1}, \cdots, \widetilde{\alpha_m})$. So the weight spaces $(\mathfrak{g}_j)_{\pm\alpha}$ can be described as follows: There has to be a root $\widetilde{\alpha}$, such that $\widetilde{\alpha}|_{\mathfrak{h}_0} = \alpha$ and

$$\mathbb{C}X^{\pm}_{\alpha,j} = \mathbb{C}(\sum_{i=0}^{m-1}(\xi^i)^j x^{\pm}_{\sigma^i(\widetilde{\alpha})}) \tag{3.1}$$

We set further

$$\mathbb{C}h_{\alpha,j} = \mathbb{C}(\sum_{i=0}^{m-1}(\xi^i)^j h_{\sigma^i(\widetilde{\alpha})}) \tag{3.2}$$

At this point we have adapted our notation while we denote by $h_{\alpha,0}$ the coroot of a root $\alpha \in \Phi_0$.

Lemma 3.1. *Assume $\widehat{\mathfrak{g}}$ is of type $A^{(2)}_{2l-1}, D^{(2)}_{l+1}, E^{(2)}_6$ or $D^{(3)}_4$. If α is a long root then we get an canonical isomorphism*

$$\mathfrak{sl}_2 \otimes \mathbb{C}[t] \cong \langle X^{\pm}_{\alpha,0} \otimes t^{ms}, h_{\alpha,0} \otimes t^{ms} | s \in \mathbb{N}\rangle_{\mathbb{C}} =: \mathfrak{sl}_{2,\alpha} \otimes \mathbb{C}[t^m]$$

and if α is short we have

$$\mathfrak{sl}_2 \otimes \mathbb{C}[t] \cong \langle X^{\pm}_{\alpha,j} \otimes t^{ms+j}, h_{\alpha,j} \otimes t^{ms+j} | s \in \mathbb{N}\ ,\ 0 \leq j \leq m-1\rangle_{\mathbb{C}} =: \mathfrak{sl}_{2,\alpha} \otimes \mathbb{C}[t].$$

Proof. Since the Lie algebra $\langle X^{\pm}_{\alpha,0}, h_{\alpha,0}\rangle_{\mathbb{C}}$ is canonically isomorph to $\mathfrak{sl}_2$ the first isomorphism is given by

$$x^{\pm} \otimes t^s \mapsto X^{\pm}_{\alpha,0} \otimes t^{ms}$$
$$h \otimes t^s \mapsto h_{\alpha,0} \otimes t^{ms}.$$

To verify the second isomorphism we define

$$x^{\pm} \otimes t^s \mapsto X^{\pm}_{\alpha,j} \otimes t^s,\ \text{if } s \equiv j \mod m$$
$$h \otimes t^s \mapsto h_{\alpha,j} \otimes t^s,\ \text{if } s \equiv j \mod m$$

To show that this map is an homomorphism of Lie algebras we need to check

$$[X^{+}_{\alpha,i_1}, X^{-}_{\alpha,i_2}] = h_{\alpha,i_1+i_2 \mod m},\ [h_{\alpha,i_2}, X^{\pm}_{\alpha,i_1}] = \pm 2X^{\pm}_{\alpha,i_1+i_2 \mod m} \tag{3.3}$$

Since we require α to be a short root, we know that the weight space $(\mathfrak{g}_j)_{\pm\alpha},\ 0 \leq j \leq m-1$ is non-zero and therefore we can use the description in (3.1), (3.2) with $\widetilde{\alpha}$, such that $\sigma(\widetilde{\alpha}) \neq \widetilde{\alpha}$. More than this, a case by case consideration shows $\sigma^j(\widetilde{\alpha})(\sigma^i(\widetilde{\alpha})^\vee) = 0$ and $\sigma^j(\widetilde{\alpha}) - \sigma^i(\widetilde{\alpha})$ is not a root for $i \neq j$, e.g. in type $D^{(2)}_{l+1}$ we have for an arbitrary short root $\alpha_i + \cdots + \alpha_l$ of B_l, that $\widetilde{\alpha} = \alpha_i + \cdots + \alpha_l$ and therefore $\sigma(\widetilde{\alpha}) - \widetilde{\alpha} = \alpha_{l+1} - \alpha_l$ is not a root and $\sigma(\widetilde{\alpha})(\widetilde{\alpha}^\vee) = \widetilde{\alpha}(\sigma(\widetilde{\alpha})^\vee) = 0$. The proof in the other cases is similar. We set $X^{\pm}_{\alpha,j} = (\sum_{i=0}^{m-1}(\xi^i)^j x^{\pm}_{\sigma^i(\widetilde{\alpha})}),\ h_{\alpha,j} = (\sum_{i=0}^{m-1}(\xi^i)^j h_{\sigma^i(\widetilde{\alpha})})$. The required equations in (3.3) are now immediate. □

If $\widehat{\mathfrak{g}}$ is of type $A^{(2)}_{2l}$ we obtain a similar result

Lemma 3.2. *Assume $\widehat{\mathfrak{g}}$ is of type $A^{(2)}_{2l}$ and α be a long root then we get an canonical isomorphism*

$$\mathfrak{sl}_2 \otimes \mathbb{C}[t] \cong \langle X^{\pm}_{\alpha,j} \otimes t^{ms+j}, h_{\alpha,j} \otimes t^{ms+j} | s \in \mathbb{N}\ ,\ 0 \leq j \leq m-1\rangle_{\mathbb{C}} =: \mathfrak{sl}_{2,\alpha} \otimes \mathbb{C}[t]$$

and if α is a short root, then we get an canonical isomorphism

$$\mathfrak{C}(A^{(2)}_2) \cong \langle X^{\pm}_{\alpha,j} \otimes t^{ms+j}, X^{\pm}_{2\alpha,1} \otimes t^{ms+1}, h_{\alpha,j} \otimes t^{ms+j} | s \in \mathbb{N},\ 0 \leq j \leq m-1\rangle_{\mathbb{C}}.$$

Proof. The proof of the first isomorphism is similar to Lemma 3.1 and to justify the second isomorphism we will demonstrate how to realize the elements $h_{\alpha,j}$, $X^{\pm}_{\alpha,j}$, $X^{\pm}_{2\alpha,1}$ as elements in A_{2l}. Let $\alpha = \alpha_i + \cdots + \alpha_l$ be an arbitrary short root of type B_l and $\widetilde{\alpha} = \alpha_i + \cdots + \alpha_l$ be the root considered as a root in type A_{2l}, i.e. the restriction to $\mathfrak{h}_0$ equals α. It is easy to see that $\sigma(\widetilde{\alpha}) \neq \widetilde{\alpha}$, $\sigma(\widetilde{\alpha}) - \widetilde{\alpha}$ is not a root of A_{2l} and continuing $\sigma(\widetilde{\alpha})(\widetilde{\alpha}^\vee) = \widetilde{\alpha}(\sigma(\widetilde{\alpha})^\vee) = -1$. We set

$$X^{\pm}_{\alpha,j} = (\xi)^j \sqrt{2}(x^{\pm}_{\widetilde{\alpha}} + \xi^j x^{\pm}_{\sigma(\widetilde{\alpha})}) \in (\mathfrak{g}_j)_{\pm\alpha}$$

$$X^{\pm}_{2\alpha,1} = [x^{\pm}_{\widetilde{\alpha}}, x^{\pm}_{\sigma(\widetilde{\alpha})}] \in (\mathfrak{g}_1)_{\pm 2\alpha}$$

$$h_{\alpha,j} = 2^{\delta_{0,j}}(h_{\widetilde{\alpha}} + \xi^j h_{\sigma(\widetilde{\alpha})})$$

Now, knowing the embedding in A_{2l}, it is straighforward to check the required relations. $\square$

3.1. **Filtration on $\mathfrak{C}(\widehat{\mathfrak{g}})$.** The Lie algebra $\mathfrak{C}(\widehat{\mathfrak{g}})$ has a natural grading and an associated natural filtration $F^\bullet(\mathfrak{C}(\widehat{\mathfrak{g}}))$, where $F^s(\mathfrak{C}(\widehat{\mathfrak{g}}))$ is defined to be the subspace of $\mathfrak{g}$-valued polynomials with degree smaller or equal s. One has an induced filtration also on the enveloping algebra $U(\mathfrak{C}(\widehat{\mathfrak{g}}))$ and therefore an induced filtration on arbitrary cyclic $U(\mathfrak{C}(\widehat{\mathfrak{g}}))$-modules W with cyclic vector w. Denote by W_s the subspace spanned by the vectors of the form $g.w$, where $g \in F^s(U(\mathfrak{C}(\widehat{\mathfrak{g}})))$, and denote the associated graded $\mathfrak{C}(\widehat{\mathfrak{g}})$–module by $\mathrm{gr}(W)$

$$\mathrm{gr}(W) = \bigoplus_{i \geq 0} W_i/W_{i-1}, \quad \text{where } W_{-1} = 0.$$

4. Demazure modules and Weyl modules

4.1. **Definition of Demazure modules.** For a dominant weight $\Lambda \in \widehat{P}^+$ let $V(\Lambda)$ be the irreducible highest weight module of highest weight Λ. Given an element $w \in \widehat{W}$, fix a generator $v_{w(\Lambda)}$ of the line $V(\Lambda)_{w(\Lambda)} = \mathbb{C}v_{w(\Lambda)}$ of $\widehat{\mathfrak{h}}$–eigenvectors in $V(\Lambda)$ of weight $w(\Lambda)$.

Definition 4.1. The $U(\widehat{\mathfrak{b}})$–submodule $V_w(\Lambda) = U(\widehat{\mathfrak{b}}).v_{w(\Lambda)}$ generated by $v_{w(\Lambda)}$ is called the *Demazure submodule of* $V(\Lambda)$ associated to w.

Remark 4.2.

(1) Since $\widehat{\mathfrak{h}}$ acts by multiplication with a scalar on $v_{w(\Lambda)}$, the Demazure module $V_w(\Lambda)$ is a cyclic $U(\widehat{\mathfrak{n}})$–module generated by $v_{w(\Lambda)}$.
(2) The modules $V_w(\Lambda)$ are finite–dimensional although $V(\Lambda)$ is infinite–dimensional.

To associate more generally to every element $\sigma w \in \widehat{W}^{ext} = \Sigma \ltimes \widehat{W}$ a Demazure module, recall that elements in Σ correspond to automorphisms of the Dynkin diagram of $\widehat{\mathfrak{g}}$, and thus define an associated automorphism of $\widehat{\mathfrak{g}}$, also denoted σ. For a module V of $\widehat{\mathfrak{g}}$ let $\sigma^*(V)$ be the module with the twisted action $g \circ v = \sigma^{-1}(g)v$. Then for the irreducible module of highest weight $\Lambda \in \widehat{P}^+$ we get $\sigma^*(V(\Lambda)) = V(\sigma(\Lambda))$.
So for $\sigma w \in \widehat{W}^{ext} = \Sigma \ltimes \widehat{W}$ we set

$$V_{\sigma w}(\Lambda) := V_{\sigma w \sigma^{-1}}(\sigma(\Lambda)). \tag{4.1}$$

We are mainly interested in $\mathfrak{g}_0$-stable Demazure modules. For $i \in I_0$ we have $X^-_{\alpha_i,0} v_{w(\Lambda)} = 0$ if and only if $w(\Lambda)(\alpha_i^\vee) \leq 0$. Consequently we can see that $V_w(\Lambda)$ is $\mathfrak{g}_0$-stable if and only if

$w(\Lambda)(\alpha_i^\vee) \leq 0$ for all $i \in I_0$. Assume that $\omega(\Lambda) = -\lambda + k\Lambda_0 + i\delta$, then $V_w(\Lambda)$ is stable under $\mathfrak{g}_0$ if and only if $\lambda \in P_0^+$. We define a set

$$X = \{(\lambda, k, i) \in P_0^+ \times (1/a_0^\vee)\mathbb{Z}_{>0} \times (1/a_0)\mathbb{Z} \mid \exists! \Lambda \in \widehat{P}^+ : w_0(\lambda) + k\Lambda_0 + i\delta \in \widehat{W}(\Lambda)\},$$

where w_0 is the longest word in W_0. Let $(\lambda, k, i) \in X$ and $\omega \in \widehat{W}$, such that $\omega(\Lambda) = w_0(\lambda) + k\Lambda_0 + i\delta$. Then by the above computation we get the $\mathfrak{g}_0$-stability of the Demazure module $V_w(\Lambda)$ and we denote

$$V_w(\Lambda) = D(k, \lambda)[i].$$

Remark 4.3.

(1) The $\mathfrak{g}_0$ stable Demazure modules are in fact $\mathfrak{C}(\widehat{\mathfrak{g}})$-modules.

(2) For any $\Lambda \in \widehat{P}^+$ and $i \in (1/a_0)\mathbb{Z}$, we have $V(\Lambda) \cong V(\Lambda + i\delta)$, as $\mathfrak{C}(\widehat{\mathfrak{g}})$-modules. Therefore we get

$$D(k, \lambda)[i] \cong D(k, \lambda)[i + n],$$

which justifies the notation $D(k, \lambda)$ as a $\mathfrak{C}(\widehat{\mathfrak{g}})$-module.

Remark 4.4. Whenever we speak about $D(k, \lambda)$ we will assume that $(\lambda, k) \in X$. If $\widehat{\mathfrak{g}}$ is not of type $A_{2l}^{(2)}$ $(l \geq 1)$ the set X is given by $X = P_0^+ \times \mathbb{Z}_{>0} \times \mathbb{Z}$ and else we have $P_0^+ \times \mathbb{Z}_{>0} \times \mathbb{Z} \subsetneq X$.

4.2. **Demazure character formula.** Let β be a real root of the root system $\widehat{\Phi}$. We define the *Demazure operator:*

$$D_\beta : \mathbb{Z}[\widehat{P}] \to \mathbb{Z}[\widehat{P}], \quad D_\beta(e^\lambda) = \frac{e^\lambda - e^{s_\beta(\lambda) - \beta}}{1 - e^{-\beta}}$$

Lemma 4.5.

(1) *For $\lambda, \mu \in \widehat{P}$ we have:*

$$D_\beta(e^\lambda) = \begin{cases} e^\lambda + e^{\lambda - \beta} + \cdots + e^{s_\beta(\lambda)} & \text{if } \langle \lambda, \beta^\vee \rangle \geq 0 \\ 0 & \text{if } \langle \lambda, \beta^\vee \rangle = -1 \\ -e^{\lambda + \beta} - e^{\lambda + 2\beta} - \cdots - e^{s_\beta(\lambda) - \beta} & \text{if } \langle \lambda, \beta^\vee \rangle \leq -2 \end{cases} \tag{4.2}$$

(2) *Let $\chi, \eta \in \mathbb{Z}[\widehat{P}]$. If $D_\beta(\eta) = \eta$, then*

$$D_\beta(\chi \cdot \eta) = \eta \cdot (D_\beta(\chi)). \tag{4.3}$$

Proof. For (1) see ([10], (1.5)–(1.8)) and for (2) see ([14], (2.2)). □

Since $D_{\alpha_i}(1 - e^\delta) = (1 - e^\delta)$ for all $i = 0, \ldots, n$, (4.3) shows that the ideal $I_\delta = \langle (1 - e^\delta) \rangle$ is stable under all Demazure operators D_β. Thus we obtain induced operators (we still use the same notation D_β)

$$D_\beta : \mathbb{Z}[\widehat{P}]/I_\delta \longrightarrow \mathbb{Z}[\widehat{P}]/I_\delta, \quad e^\lambda + I_\delta \mapsto D_\beta(e^\lambda) + I_\delta.$$

In the following we denote by D_i, $i = 0, \ldots, n$ the Demazure operator D_{α_i} corresponding to the simple root α_i. Recall that for any reduced decomposition $w = s_{i_1} \cdots s_{i_r}$ of $w \in \widehat{W}$ the operator $D_w = D_{i_1} \cdots D_{i_r}$ is independent of the choice of the decomposition (see [21], Corollary 8.2.10). We have the following important theorem:

Theorem 4.6 ([21] Chapter VIII)**.**

$$\operatorname{Char} V_w(\Lambda) = D_w(e^\Lambda).$$

We will need the following elementary proposition:

Proposition 4.7. Let $\lambda_1^\vee, \lambda_2^\vee$ be two dominant coweights, and set $\lambda^\vee = \lambda_1^\vee + \lambda_2^\vee$. Then

(1) $D_{t_{-\nu(\lambda_1^\vee)}} D_{t_{-\nu(\lambda_2^\vee)}} = D_{t_{-\nu(\lambda^\vee)}}$
(2) $D_{t_{-\nu(\lambda_1^\vee)}} D_{\omega_0} = D_{t_{-\nu(\lambda_1^\vee)}\omega_0}$

4.3. **Properties of Demazure modules.** Since $V_w(\Lambda) = U(\widehat{\mathfrak{b}}) \cdot v_{w(\Lambda)}$, there exists an Ideal $J \subseteq U(\widehat{\mathfrak{b}})$, such that $V_w(\Lambda) \cong U(\widehat{\mathfrak{b}})/J$. So the Demazure module can be described by generators and relations, which was done in [23]. We give here a reformulation for the twisted affine case:

Proposition 4.8 ([23])**.** Let $\Lambda \in \widehat{P}^+$ and let w be an element of the affine Weyl group of $\widehat{\mathfrak{g}}$. The Demazure module $V_w(\Lambda)$ is as a $U(\widehat{\mathfrak{b}})$-module isomorphic to the cyclic module, generated by $v \neq 0$ with respect to the following relations.
For $\beta \in \Phi_j^+$, $0 \leq j \leq m-1$ we have:

$$(X_{\beta,j}^+ \otimes t^{ms+j})^{k_\beta+1}.v = 0 \quad \text{where } s \geq 0, \quad k_\beta = max\{0, -\langle w(\Lambda), (\beta + (ms+j)\delta)^\vee \rangle\}$$
$$(X_{\beta,j}^- \otimes t^{ms+j})^{k_\beta+1}.v = 0 \quad \text{where } s > -\delta_{j,\{1,\cdots,m-1\}}, \quad k_\beta = max\{0, -\langle w(\Lambda), (-\beta + (ms+j)\delta)^\vee \rangle\}$$
$$(h \otimes t^{ms+j}).v = \delta_{j,0}\delta_{s,0} w(\Lambda)(h) v \quad \forall h \in \mathfrak{h}_j, \quad \text{where } s \geq 0, \quad d.v = w(\Lambda)(d).v, \ c.v = w(\Lambda)(c)v$$

Corollary 4.9. As a module for $\mathfrak{C}(\widehat{\mathfrak{g}})$ the Demazure module $D(k,\lambda)$ is isomorphic to the cyclic $U(\mathfrak{C}(\widehat{\mathfrak{g}}))$–module generated by a vector $v \neq 0$ subject to the following relations:
For $\beta \in \Phi_j^+$, $0 \leq j \leq m-1$ we have:

$$\mathfrak{n}_j \otimes t^j \mathbb{C}[t^m].v = 0$$
$$(X_{\beta,j}^- \otimes t^{ms+j})^{k_\beta+1}.v = 0 \quad \text{where } s \geq 0, \quad k_\beta = \max\{0, \langle \lambda, \beta^\vee \rangle - \frac{2(ms+j)}{\langle \beta, \beta \rangle} k a_0^\vee\}$$
$$(h \otimes t^{ms+j}).v = \delta_{j,0}\delta_{s,0}\lambda(h) v \quad \forall h \in \mathfrak{h}_j, \quad \text{where } s \geq 0$$

Proof. The proof is similar to the one given in ([15] Corollary 1). $\square$

Remark 4.10. Since the defining relations of $D(k,\lambda)$ respect the grading of $\mathfrak{C}(\widehat{\mathfrak{g}})$, $D(k,\lambda)$ is a graded module.

In [14] it was shown by using the Demazure operator, that $D(k,\lambda)$ decomposes as a $\mathfrak{g}$ (resp. $\mathfrak{g}_0$) module into a tensor product of "smaller" Demazure modules. We give here the result for the twisted affine case:

Theorem 4.11. [14] *Let $\lambda^\vee = \lambda_1^\vee + \lambda_2^\vee + \ldots + \lambda_r^\vee$ be a sum of dominant coweights. Then for $m \geq 0$ we have an isomorphism of $\mathfrak{g}_0$-modules between the Demazure module $V_{-\lambda^\vee}(m\Lambda_0)$ and the tensor product of Demazure modules:*

$$V_{-\lambda^\vee}(m\Lambda_0) \simeq V_{-\lambda_1^\vee}(m\Lambda_0) \otimes V_{-\lambda_2^\vee}(m\Lambda_0) \otimes \cdots \otimes V_{-\lambda_r^\vee}(m\Lambda_0).$$

Remark 4.12. This theorem holds for any special vertex k of the twisted affine diagram.

4.4. **Definition of local Weyl modules.** The representation theory of twisted current algebras is particularly interesting because the category of finite–dimensional representation is not semisimple. It makes sense to ask for the "maximal" finite–dimensional cyclic representations in this class, which leads to the definition of local Weyl modules. Please see [4] or [16] for the explanation of the term "local" in contrast to the term "global".
Let $\lambda = \sum_{i=1}^{l} m_i\omega_i \in P_0^+$ be a dominant integral weight for $\mathfrak{g}_0$. Then we define the *local graded Weyl module* $W^\Gamma(\lambda)$ in terms of generators and relations:

Definition 4.13. Let $\lambda = \sum_{i=1}^{l} m_i\omega_i$ be a dominant integral weight for $\mathfrak{g}_0$. Define $W^\Gamma(\lambda)$ to be the $U(\mathfrak{C}(\widehat{\mathfrak{g}}))$-module generated by an element w_λ with the relations:

$$\mathfrak{n}_j \otimes t^j\mathbb{C}[t^m].w_\lambda = 0,\ 0 \le j \le m-1 \tag{4.4}$$

$$(h \otimes t^{ms+j}).w_\lambda = \delta_{j,0}\delta_{s,0}\lambda(h)w_\lambda \quad \forall h \in \mathfrak{h}_j, \quad \text{where } s \ge 0 \tag{4.5}$$

$$(X^-_{\beta,0} \otimes 1)^{\lambda(\beta^\vee)+1}w_\lambda = 0,\ \text{for all positive roots } \beta \text{ of } \mathfrak{g}_0 \tag{4.6}$$

Remark 4.14. Note that the modules $W^\Gamma(\lambda)$ are graded modules since $U(\mathfrak{C}(\widehat{\mathfrak{g}}))$ is graded by the powers of t and the defining relations are graded, particulary we have

$$W^\Gamma(\lambda) \cong \bigoplus_{s\in\mathbb{Z}_+} W^\Gamma(\lambda)[s],$$

where $W^\Gamma(\lambda)[s]$ is a $\mathfrak{g}_0$-module by identifying $\mathfrak{g}_0$ with $\mathfrak{g}_0 \otimes 1 \subseteq \mathfrak{C}(\widehat{\mathfrak{g}})$.

4.5. **Properties of Weyl modules.**

Proposition 4.15.

(1) We have
$$W^\Gamma(\lambda) = \bigoplus_{\mu\in(\mathfrak{h}_0)^*} W^\Gamma(\lambda)_\mu$$
and $W^\Gamma(\lambda)_\mu \neq 0$ only if $\mu \in \lambda - Q_0^+$. Further we get $W^\Gamma(\lambda)_\mu \neq 0$ if and only if $W^\Gamma(\lambda)_{w(\mu)} \neq 0$ for all $w \in W_0$.

(2) As a $\mathfrak{g}_0$ module $W^\Gamma(\lambda)$ and $W^\Gamma(\lambda)[s]$ decompose into finite–dimensional irreducible representations of $\mathfrak{g}_0$.

(3) Let μ be a dominant integral weight, such that $\lambda - \mu$ is as well dominant integral. Then there exists a canonical homomorphism $W^\Gamma(\lambda) \to W^\Gamma(\mu) \otimes W^\Gamma(\lambda-\mu)$ mapping w_λ to $w_\mu \otimes w_{\lambda-\mu}$.

Proof. It sufficies to show that for every $v \in W^\Gamma(\lambda)_\mu$ the module $U(\mathfrak{g}_0).v$ is finite dimensional, since this proves the non-trivial statements in part (1) and (2). Part (3) is clear from the defining relations. Given $v \in W^\Gamma(\lambda)_\mu$ we obtain $\mathbf{U}(\mathfrak{g}_0).v = \mathbf{U}((\mathfrak{n}_-)_0)\mathbf{U}(\mathfrak{n}_0).v$. From part (1) we obtain that $U(\mathfrak{n}_0).v$ is finite dimensional. By the PBW-theorem $U((\mathfrak{n}_-)_0)$ is spanned by monomials, so it suffices to show that $X^-_{\beta,0} \in (\mathfrak{n}_-)_0$ acts nilpotently on v. Assume that $v \in U(\mathfrak{C}(\widehat{\mathfrak{g}}))w_\lambda$ and the action of $(\mathfrak{n}_-)_0$ on $\mathfrak{C}(\widehat{\mathfrak{g}})$, which is given by the Lie bracket is locally nilpotent. We obtain with

$$(X^-_{\beta,0} \otimes 1)^{\lambda(\beta^\vee)+1}w_\lambda = 0, \quad (X^-_{\beta,0})^N(\underbrace{u.w_\lambda}_{=v}) = \sum_{k=0}^{N} \binom{N}{k}((X^-_{\beta,0})^k u)(X^-_{\beta,0})^{N-k}w_\lambda$$

that $X^-_{\beta,0}$ acts nilpotently on v, which finally implies that $\mathbf{U}(\mathfrak{g}_0).v$ is finite dimensional. $\square$

Remark 4.16. $W^\Gamma(\lambda)$ is finite–dimensional. This will be an immediate consequence of Theorem 5.1 and Corollar 7.4.

By definition we obtain some obvious maps between Weyl modules and certain Demazure modules.

Corollary 4.17. Let λ be a dominant integral weight for $\mathfrak{g}_0$. Then for all $k \in (1/a_0^\vee)\mathbb{Z}_{>0}$, such that $(\lambda, k) \in X$, the Demazure module $D(k,\lambda)$ is a quotient of the Weyl module $W^\Gamma(\lambda)$.

Proof. This follows immediately by comparing the relations for the Weyl module in Definition 4.13 and the relations for the Demazure module in Corollary 4.9. $\square$

In this paper we want to show, that the map between Weyl and Demazure modules is in fact an isomorphism. This is already known for untwisted current algebras of simply-laced type ([6],[9],[15]). We recall the result for $\mathfrak{g} = \mathfrak{sl}_2$ here only, since this will be heavily used throughout this paper.

Theorem 4.18. *For $\mathfrak{g} = \mathfrak{sl}_2$ and $n\omega \in P^+$, we have an isomorphism of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-modules*
$$W(n\omega) \cong D(1, n\omega).$$

5. Connection between Weyl modules and Demazure modules

In this section we will show, that almost all Weyl modules are isomorphic to certain Demazure modules, e.g. the map in Corollary 4.17 is in fact an isomorphism.

Theorem 5.1. *Suppose $\widehat{\mathfrak{g}}$ is of type $A^{(2)}_{2l-1}, D^{(2)}_{l+1}, E^{(2)}_6$ or $D^{(3)}_4$, then we have an isomorphism of $\mathfrak{C}(\widehat{\mathfrak{g}})$-modules*
$$W^\Gamma(\lambda) \cong D(1/a_0^\vee, \lambda).$$
If $\widehat{\mathfrak{g}}$ is of type $A^{(2)}_{2l}$ and $\lambda = \sum\limits_{i=1}^{l} m_i\omega_i$ be a dominant weight, such that m_l is odd, we have an isomorphism of $\mathfrak{C}(\widehat{\mathfrak{g}})$-modules
$$W^\Gamma(\lambda) \cong D(1/a_0^\vee, \lambda).$$

Proof. By Corollary 4.17 we know already that the Demazure module is a quotient of the Weyl module. By comparing the defining relations in Corollary 4.9 and in Definition 4.13, we see that to prove that this map is an isomorphism, it is sufficient to show that the generator of the Weyl module is subject to the following relations:
For all $0 \le j \le m-1$, $\beta \in \Phi^+_j$:

$$(X^-_{\beta,j} \otimes t^{ms+j})^{k_\beta+1}.w_\lambda = 0, \text{ where } s \ge 0,\ k_\beta = \max\{0, \langle\lambda,\beta^\vee\rangle - \frac{2(ms+j)}{\langle\beta,\beta\rangle}\frac{1}{a_0^\vee}a_0^\vee\}. \tag{5.1}$$

Assume $\widehat{\mathfrak{g}}$ is not of type $A^{(2)}_{2l}$, then (5.1) is equivalent to :
(5.2)
$$(X^-_{\beta,j} \otimes t^{ms+j})^{k_\beta+1}.w_\lambda = 0, \text{ where } s \ge 0,\ k_\beta = \begin{cases} \max\{0, \langle\lambda,\beta^\vee\rangle - s\}, & \text{if } \beta \text{ is long} \\ \max\{0, \langle\lambda,\beta^\vee\rangle - (ms+j)\}, & \text{if } \beta \text{ is short} \end{cases}$$

Let $\beta \in \Phi^+_0$ be a long root and $V = U(\mathfrak{sl}_{2,\beta} \otimes \mathbb{C}[t^m]).w_\lambda \subseteq W^\Gamma(\lambda)$ be the $\mathfrak{sl}_{2,\beta} \otimes \mathbb{C}[t^m]$-submodule. Further let $W(\langle\lambda,\beta^\vee\rangle\omega)$ be the $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-Weyl module, which is by Theorem 4.18

isomorphic to the $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-Demazure module $D(1, \langle \lambda, \beta^\vee \rangle \omega)$. Since w_λ is a cyclic generator for V and satisfies obviously the defining relations of $W(\langle \lambda, \beta^\vee \rangle \omega)$ we obtain by Lemma 3.1 a surjective homomorphism:

$$W(\langle \lambda, \beta^\vee \rangle \omega) \cong D(1, \langle \lambda, \beta^\vee \rangle \omega) \twoheadrightarrow V \subseteq W^\Gamma(\lambda).$$

In particular, w_λ satisfies the defining relations of $D(1, \langle \lambda, \beta^\vee \rangle \omega)$, which contain the relation

$$(x^- \otimes t^s)^{\max\{0, \langle \lambda, \beta^\vee \rangle - s\}+1}.v = 0 \ \forall s \in \mathbb{N},$$

therefore again by Lemma 3.1 we obtain

$$(X^-_{\beta,0} \otimes t^{ms})^{\max\{0, \langle \lambda, \beta^\vee \rangle - s\}+1}.w_\lambda = 0$$

Now suppose β is a short root and consider the $\mathfrak{sl}_{2,\beta} \otimes \mathbb{C}[t]$-submodule $V = U(\mathfrak{sl}_{2,\beta} \otimes \mathbb{C}[t]).w_\lambda \subseteq W^\Gamma(\lambda)$. By the same reasons as above and Lemma 3.1 we get an surjective homomorphism

$$W(\langle \lambda, \beta^\vee \rangle \omega) \cong D(1, \langle \lambda, \beta^\vee \rangle \omega) \twoheadrightarrow V \subseteq W^\Gamma(\lambda),$$

and therefore w_λ satisfies again the relations of $D(1, \langle \lambda, \beta^\vee \rangle \omega)$. Using the isomorphism in Lemma 3.1 we obtain:

$$(X^-_{\beta,j} \otimes t^{ms+j})^{\max\{0, \langle \lambda, \beta^\vee \rangle - (ms+j)\}+1}.w_\lambda = 0, \ \forall s \in \mathbb{N}, 0 \le j \le m-1,$$

which proves (5.2).

To prove the theorem it remains to consider the case where $\widehat{\mathfrak{g}}$ is of type $A^{(2)}_{2l}$. We have $(\lambda, 1/2, 0) \in X$, in particulary we have $D(1/a_0^\vee, \lambda) = V_{\omega_0 t_{\lambda - \omega_l}}(\Lambda_l)$. In order to use again Corollary 4.9 we reformulate (5.1) into

(5.3)
$$(X^-_{\beta,j} \otimes t^{ms+j})^{k_\beta+1}.w_\lambda = 0, \ s \ge 0, \ k_\beta = \begin{cases} \max\{0, \langle \lambda, \beta^\vee \rangle - (ms+j)\}, & \text{if } \beta \text{ is long} \\ \max\{0, \langle \lambda, \beta^\vee \rangle - 2(ms+j)\}, & \text{if } \beta \text{ is short} \\ \max\{0, \langle \lambda, \beta^\vee \rangle - 1/2(ms+1)\}, & \text{if } \beta = 2\alpha, \ \alpha \text{ is short} \end{cases}$$

We will prove case by case that the generator of $W^\Gamma(\lambda)$ satisfies the relations in (5.3). For long roots the proof is similar to the other cases by using Lemma 3.2. So let β be a short root and $\langle X^\pm_{\beta,j} \otimes t^{ms+j}, X^\pm_{2\beta,1} \otimes t^{ms+1}, h_{\beta,j} \otimes t^{ms+j} \rangle_{\mathbb{C}}$ the Lie algebra which is isomorphic to $\mathfrak{C}(A^{(2)}_2)$ by Lemma 3.2. We consider the submodule $U(\mathfrak{C}(A^{(2)}_2)).w_\lambda \subseteq W^\Gamma(\lambda)$, which is trivially a quotient of the $A^{(2)}_2$-Weyl module $W^\Gamma(\langle \lambda, \beta^\vee \rangle \omega)$. In Section 7 Theorem 7.1 we prove (independent of Section 1-6) that $W^\Gamma(\langle \lambda, \beta^\vee \rangle \omega) \cong D(1/2, \langle \lambda, \beta^\vee \rangle \omega)$. The proof is finished with the observation, that the defining relations for $A^{(2)}_2$-Demazure module $D(1/2, \langle \lambda, \beta^\vee \rangle \omega)$ contain the relations

$$(X^-_{\beta,j} \otimes t^{ms+j})^{\max\{0, \langle \lambda, \beta^\vee \rangle - 2(ms+1)\}+1}.w = 0,$$

$$(X^-_{2\beta,1} \otimes t^{ms+1})^{\max\{0, 1/2(\langle \lambda, \beta^\vee \rangle - (ms+1))\}+1}.w = 0.$$

$\square$

5.1. **Fundamental Weyl modules.** In the previous section we have seen that Weyl modules are isomorphic to certain Demazure modules. Since most of the Demazure modules have a nice tensor product decomposition, see Theorem 4.11, we can transfer this result to most Weyl modules (only the $A_{2l}^{(2)}$ case needs more work). Using this decomposition, to compute the dimension and character of Weyl modules it is enough to describe the $\mathfrak{g}_0$ decomposition of *fundamental Weyl modules* $W^\Gamma(\omega_i)$.

Theorem 5.2. *Let $\omega_1, \cdots, \omega_l$ be the fundamental weights in P_0^+. Viewed as a $\mathfrak{g}_0$-module the fundamental Weyl modules decomposes into the direct sum of irreducible $\mathfrak{g}_0$-modules as follows:*

- *if $\widehat{\mathfrak{g}}$ is of type $A_{2l}^{(2)}$*

$$W^\Gamma(\omega_i) \cong V(\omega_i),$$
$$W^\Gamma(2\omega_l) \cong V(2\omega_l)$$

- *if $\widehat{\mathfrak{g}}$ is of type $A_{2l-1}^{(2)}$*

$$W^\Gamma(\omega_i) \cong \bigoplus_{s_{\bar{i}}+\cdots+s_i=1} V(s_{\bar{i}}\omega_{\bar{i}} + \cdots + s_{i-2}\omega_{i-2} + s_i\omega_i), \text{ where } \bar{i} \in \{0,1\} \text{ and } i = \bar{i} \mod 2$$

- *if $\widehat{\mathfrak{g}}$ is of type $D_{l+1}^{(2)}$*

$$W^\Gamma(\omega_i) \cong \bigoplus_{s_1+\cdots+s_i\le 1} V(s_1\omega_1 + \cdots + s_i\omega_i), \;\; i \neq l$$
$$W^\Gamma(\omega_l) \cong V(\omega_l)$$

- *if $\widehat{\mathfrak{g}}$ is of type $E_6^{(2)}$*

$$W^\Gamma(\omega_1) \cong \bigoplus_{s\le 1} V(s\omega_1)$$
$$W^\Gamma(\omega_2) \cong V(0) \oplus V(\omega_1)^{\oplus 2} \oplus V(\omega_2) \oplus V(\omega_4)$$
$$W^\Gamma(\omega_3) \cong V(0)^{\oplus 2} \oplus V(\omega_1)^{\oplus 4} \oplus V(\omega_2)^{\oplus 3} \oplus V(\omega_4)^{\oplus 3} \oplus V(2\omega_1) \oplus V(\omega_1+\omega_4) \oplus V(\omega_3)$$
$$W^\Gamma(\omega_4) \cong \bigoplus_{s_1+s_4\le 1} V(s_1\omega_1 + s_4\omega_4)$$

- *if $\widehat{\mathfrak{g}}$ is of type $D_4^{(3)}$*

$$W^\Gamma(\omega_1) \cong V(0) \oplus V(\omega_1) \oplus V(\omega_2)^{\oplus 2}$$
$$W^\Gamma(\omega_2) \cong \bigoplus_{s\le 1} V(s\omega_2)$$

Proof. If $\widehat{\mathfrak{g}}$ is of type $A_{2l-1}^{(2)}$ or $D_{l+1}^{(2)}$ the decomposition rule is immediate from Theorem 5.1 and Theorem 2 in [14]. By same reasons the theorem is true for $i = 2$ if $\widehat{\mathfrak{g}}$ is of type $D_4^{(3)}$ and for $i = 1, 4$ in type $E_6^{(2)}$. For $i = 1$ one can check $t_{-w_1} = w_0 s_0 s_2 s_1 s_2 s_0$ and therefore with the Demazure character formula we get

$$D_{t_{-w_1}}(e^{\Lambda_0}) = D_{w_0}(e^0 + 2e^{\omega_2} + e^{\omega_1}) \Rightarrow W^\Gamma(\omega_1) \cong V_{t_{-\omega_1}}(e^{\Lambda_0}) \cong_{G_2} V(0) \oplus V(\omega_1) \oplus V(\omega_2)^{\oplus 2},$$

which proves the claim for type $D_4^{(3)}$.
So it remains to consider the nodes $i = 2,3$ in type $E_6^{(2)}$ and the general case in type $A_{2l}^{(2)}$. In [7] Kirillov-Reshetikhin modules $KR(s\omega_i)$ respectively $KR^\sigma(s\omega_i)$ for the twisted version are defined. By inspecting the defining relations it follows that KR-modules of level 1 (e.g. $s = 1$) are precisely fundamental Weyl modules, in particular

$$W(\omega_i) \cong KR(\omega_i) \text{ and } W^\Gamma(\omega_i) \cong KR^\sigma(\omega_i).$$

Since the decomposition of KR-modules are known as $\mathfrak{g}$ respectively $\mathfrak{g}_0$-modules (see [3],[7],[18] or [20] for instance) we obtain the predicted decomposition for $i = 2,3$ in type $E_6^{(2)}$ and for the general case in type $A_{2l}^{(2)}$.
It remains to consider $W^\Gamma(2\omega_l)$, so let $\langle X^\pm_{\alpha_l,j} \otimes t^{ms+j}, X^\pm_{2\alpha_l,1} \otimes t^{ms+1}, h_{\alpha_l,j} \otimes t^{ms+j} | s \in \mathbb{N}, 0 \leq j \leq m-1\rangle_{\mathbb{C}}$ be the Lie algebra which is by Lemma 3.2 isomorphic to $\mathfrak{C}(A_2^{(2)})$. Then we obtain a surjective homomorphism

$$W^\Gamma(2\omega) \twoheadrightarrow U(\mathfrak{C}(A_2^{(2)})).w_{2\omega_l} \subseteq W^\Gamma(2\omega_l).$$

In Section 7 we will show that the $A_2^{(2)}$-Weyl module $W^\Gamma(2\omega)$ is an irreducible $\mathfrak{sl}_2$-module and hence $(X^-_{\alpha_l,0} \otimes t^2).w_{2\omega_l} = (X^-_{2\alpha_l,1} \otimes t).w_{2\omega_l} = (X^-_{\alpha_l,1} \otimes t).w_{2\omega_l} = 0$. So $W^\Gamma(2\omega_l)$ is isomorphic to the Kirillov-Reshetikhin module $KR^\sigma(2\omega_l)$, hence the decomposition is known by [7]. □

Such a similar decomposition is already known for the untwisted fundamental Weyl modules $W(\omega_i)$, see [3] or [14] for instance. This fact motivates us to compare the dimension of twisted and untwisted fundamental Weyl modules. For notational reasons, we have to extend certain linear functions $\mathfrak{h}_0 \longrightarrow \mathbb{C}$ to functions on $\mathfrak{h}$. So let $\mu \in P_0^+$ (with $\mu(\alpha_l^\vee) \in 2\mathbb{Z}_{\geq 0}$ if $\mathfrak{g}$ is of type A_{2l}). We define the extension, by abuse of notation also denoted by μ, on a basis of $\mathfrak{h}$ by:

$$\mu(h_i) = \begin{cases} \mu(\alpha_i^\vee) & \text{if } \mathfrak{g} \text{ is not of type } A_{2l} \\ 0 & \text{if } i \notin I_0 \\ (1 - \frac{\delta_{i,l}}{2})\mu(\alpha_i^\vee) & \text{if } \mathfrak{g} \text{ is of type } A_{2l} \end{cases}$$

Since there might be a confusion in notation in the A_{2l} and the l-th fundamental weight case only, we will use this identification in the remaining of the paper without further comment.

Lemma 5.3. *Let $\omega_1, \cdots, \omega_l$ be the fundamental weights in P_0^+. We set $\epsilon = (1 + \delta_{i,l})$ if $\mathfrak{g}$ is of type A_{2l} and $\epsilon = 1$ else, then we obtain*

$$\dim W^\Gamma(\epsilon\omega_i) = \dim W(\epsilon\omega_i),\ 1 \leq i \leq l.$$

Proof. Using Theorem 5.2, Theorem 2 in [14] and Lecture 24 in [17], we obtain the following straightforward calculations:

- if $\widehat{\mathfrak{g}}$ is of type $A_{2l}^{(2)}, (\mathfrak{g}, \mathfrak{g}_0) = (A_{2l}, B_l)$:

$$\dim W^\Gamma(\epsilon\omega_i) = \binom{2l+1}{i} = \dim(V_{\mathfrak{g}}(\omega_i)) = \dim W(\epsilon\omega_i)$$

- if $\widehat{\mathfrak{g}}$ is of type $A_{2l-1}^{(2)}, (\mathfrak{g}, \mathfrak{g}_0) = (A_{2l-1}, C_l)$:

$$\dim W^\Gamma(\omega_i) = \binom{2l}{\bar{i}} + \sum_{j=1}^{\frac{i-\bar{i}}{2}} \binom{2l}{\bar{i}+2j} - \binom{2l}{\bar{i}+2j-2} = \binom{2l}{i} = \dim W(\omega_i)$$

- if $\widehat{\mathfrak{g}}$ is of type $D^{(2)}_{l+1}$, $(\mathfrak{g},\mathfrak{g}_0)=(D_{l+1},B_l)$:

$$\begin{aligned}\dim W^{\Gamma}(\omega_i) &= \begin{cases} 2^i, & \text{if } i=l \\ 1+\sum_{j=1}^{i}\binom{2l+1}{j}, & i\neq l\end{cases} = \begin{cases} 2^i, & \text{if } i=l \\ \sum_{j=0}^{\frac{i-p_i}{2}}\binom{2l+2}{p_i+2j}, & i\neq l\end{cases} \\ &= \begin{cases} \dim V_{\mathfrak{g}}(\omega_l), \ \text{if } i=l \\ \dim(V_{\mathfrak{g}}(\omega_i)\oplus V_{\mathfrak{g}}(\omega_{i-2})\oplus\cdots\oplus V_{\mathfrak{g}}(\omega_{p_i})), \ i\neq l\end{cases} = \dim W(\omega_i)\end{aligned}$$

- if $\widehat{\mathfrak{g}}$ is of type $E^{(2)}_6$, $(\mathfrak{g},\mathfrak{g}_0)=(E_6,F_4)$:

$$\begin{aligned}&\dim W^{\Gamma}(\omega_1)=27=\dim V_{\mathfrak{g}}(\omega_1)=\dim W(\omega_1)\\ &\dim W^{\Gamma}(\omega_2)=378=\dim(\bigoplus_{s_2+s_6=1} V_{\mathfrak{g}}(s_2\omega_2+s_6\omega_6))=\dim W(\omega_2)\\ &\dim W^{\Gamma}(\omega_3)=3732=\dim(V_{\mathfrak{g}}(0)\oplus V_{\mathfrak{g}}(\omega_4)^{\oplus 2}\oplus V_{\mathfrak{g}}(\omega_1+\omega_6)\oplus V_{\mathfrak{g}}(\omega_3))=\dim W(\omega_3)\\ &\dim W^{\Gamma}(\omega_4)=79=\dim(\bigoplus_{s_4\le 1} V_{\mathfrak{g}}(s_4\omega_4))=\dim W(\omega_4)\end{aligned}$$

- if $\widehat{\mathfrak{g}}$ is of type $D^{(3)}_4$, $(\mathfrak{g},\mathfrak{g}_0)=(D_4,G_2)$:

$$\begin{aligned}&\dim W^{\Gamma}(\omega_1)=29=\dim(V_{\mathfrak{g}}(\omega_1)\oplus V_{\mathfrak{g}}(0))=\dim W(\omega_1)\\ &\dim W^{\Gamma}(\omega_2)=8=\dim(V_{\mathfrak{g}}(\omega_2))=\dim W(\omega_2)\end{aligned}$$

□

6. Connection between twisted and untwisted Weyl modules

In this section we will show that the Weyl modules $W^{\Gamma}(\lambda)$ can be realized as associated graded modules of certain untwisted Weyl modules for the loop algebra $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$. So consider for $a\in\mathbb{C}^*$ the Lie algebra homomorphism φ_a defined as follows:

$$\varphi_a:\mathfrak{g}\otimes\mathbb{C}[t]\longrightarrow\mathfrak{g}\otimes\mathbb{C}[t],\quad x\otimes t^m\mapsto x\otimes(t+a)^m.$$

For a $\mathfrak{g}\otimes\mathbb{C}[t]$-module W we denote by W_a be the module obtained by pulling back W through φ_a, i.e. $x\otimes t^s$ acts by $x\otimes(t+a)^s$. Further we denote by $\overline{W}$ be the module W considered as a $\mathfrak{C}(\widehat{\mathfrak{g}})$-module, obtained by the embedding

$$\mathfrak{C}(\widehat{\mathfrak{g}})\hookrightarrow\mathfrak{g}\otimes\mathbb{C}[t].$$

For the definition of the associated graded modules in the following theorem, we refer to Section 3.1:

Theorem 6.1. *Let $\lambda=\sum_{i=1}^{l}m_i\omega_i$ be a dominant $\mathfrak{g}_0$-weight. If $\widehat{\mathfrak{g}}$ is a twisted Kac-Moody algebra not of type $A^{(2)}_{2l}$ we get an isomorphism of $\mathfrak{C}(\widehat{\mathfrak{g}})$-modules:*

$$W^{\Gamma}(\lambda)\cong\operatorname{gr}(\overline{W_a(\lambda)}).$$

If $\widehat{\mathfrak{g}}$ is of type $A^{(2)}_{2l}$ and $\lambda=\lambda_1+\lambda_2\in P^+_0$, such that m_l and $\lambda_2(\alpha_l^\vee)$ are odd we get an isomorphism of $\mathfrak{C}(\widehat{\mathfrak{g}})$-modules:

$$W^{\Gamma}(\lambda)\cong\operatorname{gr}(\overline{W_a(\lambda_1)}\otimes W^{\Gamma}(\lambda_2)).$$

Proof. Let $\widehat{\mathfrak{g}}$ be not of type $A_{2l}^{(2)}$, by combining [5] and [13] it follows, that $\overline{W_a(\lambda)}$ is a cyclic $\mathfrak{C}(\widehat{\mathfrak{g}})$ module. Therefore the associated graded is again cyclic and it remains to observe, that the image of the highest weight generator $\overline{\mathbf{w}}$ satisfies for $j \in \{0, \dots, m-1\}$ and $h_j \in \mathfrak{h}_j$ the relations

$$(h_j \otimes t^{ms+j}).\overline{\mathbf{w}} = 0, \quad (s,j) \neq (0,0)$$
$$(h_0 \otimes 1)\overline{\mathbf{w}} = \lambda(h_0)\overline{\mathbf{w}}.$$

Thus we obtain a surjective homomorphism

$$W^{\Gamma}(\lambda) \twoheadrightarrow \mathrm{gr}(\overline{W_a(\lambda)}). \tag{6.1}$$

In order to compare the dimension of these modules we exploit the tensor product decomposition of $W^{\Gamma}(\lambda)$ as a $\mathfrak{g}_0$-module by combining Theorem 5.1 and Proposition 4.11. We obtain the following :

$$W^{\Gamma}(\lambda) \cong W^{\Gamma}(\omega_1)^{\otimes m_1} \otimes \cdots \otimes W^{\Gamma}(\omega_l)^{\otimes m_l} \text{ as } \mathfrak{g}_0\text{-modules.} \tag{6.2}$$

An analogue decomposition was proven in [15] for untwisted Weyl modules for the current algebra of a simply-laced simple Lie algebra and is generalized in [25] for the non simply-laced case. From this it follows immediately

$$\dim \mathrm{gr}(\overline{W_a(\lambda)}) = \dim W(\lambda) = \prod_{i=1}^{l} (\dim W(\omega_i))^{m_i}.$$

Hence by Lemma 5.3 we check that (6.1) is in fact an isomorphism.

From now on, we assume that $\widehat{\mathfrak{g}}$ is of type $A_{2l}^{(2)}$. Since $\overline{W_a(\lambda_1)}$ and $W^{\Gamma}(\lambda_2)$ are cyclic $\mathfrak{C}(\widehat{\mathfrak{g}})$-modules it follows with the usual arguments of [13] and the Chinese remainder theorem, that the tensor product is cyclic as well. Therefore we obtain similar to (6.1) a surjective homomorphism

$$W^{\Gamma}(\lambda) \twoheadrightarrow \mathrm{gr}(\overline{W_a(\lambda_1)} \otimes W^{\Gamma}(\lambda_2)). \tag{6.3}$$

With the aim to compare the dimension on both sides we notice

$$\dim \mathrm{gr}(\overline{W_a(\lambda_1)} \otimes W^{\Gamma}(\lambda_2)) = \dim W(\lambda_1) \dim W^{\Gamma}(\lambda_2) = \prod_{i=1}^{l} (\dim W(\omega_i))^{\lambda_1(\alpha_i^{\vee})} \dim W^{\Gamma}(\lambda_2).$$

Our goal now is to prove the following tensor product decomposition:

$$W^{\Gamma}(\lambda) \cong_{\mathfrak{g}_0} W^{\Gamma}(\omega_1)^{\otimes m_1} \otimes \cdots \otimes W^{\Gamma}(\omega_{l-1})^{\otimes m_{l-1}} \otimes W^{\Gamma}(2\omega_l)^{\otimes k-1} \otimes W^{\Gamma}(\omega_l), \tag{6.4}$$

where $m_l = 2k-1$ since the proposition is a immediate consequence of (6.4) and Lemma 5.3. To prove (6.4) we investigate the character of $W^{\Gamma}(\lambda)$. By Theorem 5.1 and Theorem 4.6 we obtain

$$\mathrm{Char}\, W^{\Gamma}(\lambda) = \mathrm{Char}\, V_{\omega_0 t_{\lambda-\omega_l}}(\Lambda_l) = D_{\omega_0 t_{\lambda-\omega_l}}(e^{\Lambda_l}).$$

Suppose that $V(\mu)$ is a irreducible B_l-module, such that the coefficient n_l is even, whereby $\mu = \sum_{i=1}^{l} n_i \omega_i$. The first step will be to show that $\mathrm{Char}\, V(\mu)$ is stable under the Demazure operators D_i, $i = 0, \dots, l$. The character of a finite dimensional $\mathfrak{g}_0$-module is stable under the Weyl group W and hence stable under D_i, $i = 1, \dots, l$. It remains to consider the case $i = 0$. Note that $\alpha_0 = \delta - 2\overline{\theta} = \delta - \theta$ where $\overline{\theta} = \alpha_1 + \cdots + \alpha_l$ is the highest short root of B_l. We define maps $s_{\overline{\theta}} : (\mathfrak{h}_0)^* \to (\mathfrak{h}_0)^*,\ s_{\overline{\theta}}(\lambda) = \lambda - \lambda(\overline{\theta}^{\vee})\overline{\theta}$ and $s_{\theta} : (\mathfrak{h}_0)^* \to (\mathfrak{h}_0)^*,\ s_{\theta}(\lambda) = \lambda - \lambda(\theta^{\vee})\theta$.

Since $\overline{\theta}^\vee = 2(\alpha_1^\vee + \ldots + \alpha_{l-1}^\vee) + \alpha_l^\vee$ and $\theta^\vee = \alpha_1^\vee + \ldots + \alpha_{l-1}^\vee + \frac{1}{2}\alpha_l^\vee$ we get clearly $s_{\overline{\theta}} = s_\theta$. Thus ν is a weight in $V(\mu)$ if and only if $s_\theta(\nu)$ is a weight. Assume $\nu \in (\mathfrak{h}_0)^*$ is a weight, hence $\nu = \mu - Q_0^+$ and therefore $\langle \nu, \alpha_0^\vee \rangle = \langle \nu, (\delta - \theta)^\vee \rangle = \langle \nu, -\theta^\vee \rangle \in \mathbb{Z}$. We have proved that D_0 can be defined on $\operatorname{Char} V(\mu)$ and $D_0 = D_{-\theta}$. We obtain

$$D_0(\operatorname{Char} V(\mu)) = D_{-\theta}(\operatorname{Char} V(\mu)) = \operatorname{Char} V(\mu)$$

In a second step we prove that the characters are the same by using induction on $\sum_{i=1}^{l-1} m_i + (k-1)$. So if the sum is 1 we have to show

$$D_{\omega_0 t_{\omega_i}}(e^{\Lambda_l}) = \operatorname{Char} W^\Gamma(\omega_i + \omega_l) = e^{\frac{1}{2}\Lambda_0} \operatorname{Char}\left(V_{\mathfrak{g}_0}(\omega_i) \otimes V_{\mathfrak{g}_0}(\omega_l)\right),\ i < l$$

$$D_{\omega_0 t_{2\omega_l}}(e^{\Lambda_l}) = \operatorname{Char} W^\Gamma(3\omega_l) = e^{\frac{1}{2}\Lambda_0} \operatorname{Char}\left(V_{\mathfrak{g}_0}(2\omega_l) \otimes V_{\mathfrak{g}_0}(\omega_l)\right).$$

In other words, we have to figure out the $\mathfrak{g}_0$-module decomposition of $W^\Gamma(\omega_i + \omega_l)$ respectively $W^\Gamma(3\omega_l)$. By Lemma 4.15(2) we already know that there exists such a decomposition and since the modules are finite–dimensional every $\mathfrak{g}_0$-submodule is a direct summand. So our assignment is to find all highest weight vectors, first beginning with the highest weight vectors living in $W^\Gamma(\omega_i + \omega_l)[1]$. Suppose $\alpha \in \Phi_1$, such that $(X^-_{\alpha,1} \otimes t).w$ is a highest weight vector, i.e. the element is non-zero and the upper triangular part of $\mathfrak{g}_0$ acts by zero. We want to restrict the choice of α to one possible case. Note that α is of the form $\alpha_j + \cdots + \alpha_l$ or $2(\alpha_j + \cdots + \alpha_l)$, $1 \le j \le l$ or of the form $\alpha_p + \cdots + \alpha_q$ respectively $\alpha_p + \cdots + \alpha_{q-1} + 2(\alpha_q + \cdots + \alpha_l)$, $p, q \le l-1$. If α is a short root, we obtain from Lemma 3.2

$$W^\Gamma(\langle \omega_i + \omega_l, \alpha^\vee \rangle \omega) \twoheadrightarrow U(\langle X^\pm_{\alpha,j} \otimes t^{ms+j}, X^\pm_{2\alpha,1} \otimes t^{ms+1}, h_{\alpha,j} \otimes t^{ms+j} \rangle_{\mathbb{C}} \cong \mathfrak{C}(A_2^{(2)})).w, \tag{6.5}$$

whereby $W^\Gamma(\langle \omega_i + \omega_l, \alpha^\vee \rangle \omega)$ is the Weyl module for type $A_2^{(2)}$. So if $j > i$ in the representation of α as a sum of simple roots we get $\langle \omega_i + \omega_l, \alpha^\vee \rangle = 1$. In Section 7 it is shown that $W^\Gamma(\omega)$ is irreducible and therefore $(X^-_{\alpha,1} \otimes t).w = (X^-_{2\alpha,1} \otimes t).w = 0$. Now assume $j < i$ and $(X^-_{\alpha_j + \cdots + \alpha_l, 1} \otimes t).w \neq 0$ is a highest weight vector. Hence $0 = (X^+_{\alpha_j + \cdots + \alpha_{i-1}, 0} \otimes 1)(X^-_{\alpha_j + \cdots + \alpha_l, 1} \otimes t).w = (X^-_{\alpha_i + \cdots + \alpha_l, 1} \otimes t).w$, which is a contradiction to (6.3). In almost the same manner one sees that $(X^-_{2(\alpha_j + \cdots + \alpha_l), 1} \otimes t).w$ cant't be a highest weigth vector. If α is a long root, we get with Lemma 3.2

$$W(\langle \omega_i + \omega_l, \alpha^\vee \rangle \omega) \twoheadrightarrow U(\mathfrak{sl}_{2,\alpha} \otimes \mathbb{C}[t]).w,$$

whereby $W^\Gamma(\langle \omega_i + \omega_l, \alpha^\vee \rangle \omega)$ is the Weyl module for the current algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$. So if $\alpha = \alpha_p + \cdots + \alpha_q$ we obtain again $\langle \omega_i + \omega_l, \alpha^\vee \rangle \le 1$ and therefore $(X^-_{\alpha,1} \otimes t).w = 0$. Let α be of the form $\alpha_p + \cdots + \alpha_{q-1} + 2(\alpha_q + \cdots + \alpha_l)$, such that $i \ge p$ and $(X^-_{\alpha,1} \otimes t).w$ is a non-zero highest weight vector. Therefore the upper triangular part acts by zero, especially

$$\begin{aligned} 0 &= (X^+_{\alpha_q + \cdots + \alpha_l, 0} \otimes 1)(X^+_{\alpha_p + \cdots + \alpha_{i-1}, 0} \otimes 1)(X^-_{\alpha_p + \cdots + \alpha_{q-1} + 2(\alpha_q + \cdots + \alpha_l), 1} \otimes t).w \\ &= (X^+_{\alpha_q + \cdots + \alpha_l, 0} \otimes 1)(X^-_{\alpha_i + \cdots + \alpha_{q-1} + 2(\alpha_q + \cdots + \alpha_l), 1} \otimes t).w = (X^-_{\alpha_i + \cdots + \alpha_l, 1} \otimes t).w, \end{aligned}$$

which is again a contradiction to (6.3). Hence the only possibility to get a highest weight vector of degree one is to apply $(X^-_{\alpha_i + \cdots + \alpha_l, 1} \otimes t)$ on w. Clearly we have by Section 7 $(X^-_{\alpha_i + \cdots + \alpha_l, 1} \otimes t)^2.w = (X^-_{\alpha_i + \cdots + \alpha_l, 1} \otimes t^{2s+1}).w = (X^-_{\alpha_i + \cdots + \alpha_l, 0} \otimes t^{2s}).w = 0$ for $s \ge 1$, because in (6.5) we have $\langle \omega_i + \omega_l, (\alpha_i + \ldots + \alpha_l)^\vee \rangle = 3$. Thus one can check that $(X^-_{\alpha_i + \cdots + \alpha_l, 1} \otimes t).w$ satisfies the relations (4.4), (4.5) in Definition 4.13 and has weight $\omega_{i-1} + \omega_l$ with respect to

$\mathfrak{h}_0$. Hence the calculations above show on the one hand that $(X^-_{\alpha_i+\cdots+\alpha_l,1}\otimes t).w$ is really a highest weight vector but on the other hand we get more than this, namely a surjective map

$$W^\Gamma(\omega_{i-1}+\omega_l)\twoheadrightarrow U(\mathfrak{C}(A^{(2)}_{2l}))(X^-_{\alpha_i+\cdots+\alpha_l,1}\otimes t).w$$

Since $W^\Gamma(\omega_i+\omega_l)[1]\cong_{\mathfrak{g}_0} V_{\mathfrak{g}_0}(\omega_{i-1}+\omega_l)\cong U(\mathfrak{g}_0)(X^-_{\alpha_i+\cdots+\alpha_l,1}\otimes t).w$ we obtain $W^\Gamma(\omega_i+\omega_l)=U(\mathfrak{g}_0).w\oplus U(\mathfrak{C}(A^{(2)}_{2l}))(X^-_{\alpha_i+\cdots+\alpha_l,1}\otimes t).w\cong_{\mathfrak{g}_0} V_{\mathfrak{g}_0}(\omega_i+\omega_l)\oplus W^\Gamma(\omega_{i-1}+\omega_l)/I$, for some ideal I. Using (6.3) one can check that the ideal is zero and therefore by induction we prove our claim, because for $i=1$ we get

$$\begin{aligned}\mathrm{Char}\,(V_{\omega_0 t_{\omega_1}}(\Lambda_l)\cong V_{\omega_0 s_0 s_1\ldots s_l}(\Lambda_l)) &= D_{\omega_0}D_0\ldots D_l(e^{\Lambda_l})\\ &= D_{\omega_0}(e^{\Lambda_l}+e^{\Lambda_l-\alpha_l}+\cdots+e^{\Lambda_l-\alpha_l-\cdots-\alpha_1}+e^{\Lambda_l+\omega_1})\\ &= e^{\frac{1}{2}\Lambda_0}\mathrm{Char}\,(V_{\mathfrak{g}_0}(\omega_1+\omega_l)\oplus V_{\mathfrak{g}_0}(\omega_l)) = e^{\frac{1}{2}\Lambda_0}\mathrm{Char}\,(V_{\mathfrak{g}_0}(\omega_1)\otimes V_{\mathfrak{g}_0}(\omega_l))\end{aligned}$$

Exactly the same way one can prove the existence of a surjective map

$$W^\Gamma(\omega_{l-1}+\omega_l)\twoheadrightarrow U(\mathfrak{C}(A^{(2)}_{2l}))(X^-_{\alpha_l,1}\otimes t).w$$

Furthermore a more simple calculation shows $W^\Gamma(3\omega_l)[1]\cong_{\mathfrak{g}_0} V_{\mathfrak{g}_0}(\omega_{l-1}+\omega_l)\cong U(\mathfrak{g}_0)(X^-_{\alpha_l,1}\otimes t).w$. Hence $W^\Gamma(3\omega_l)\cong_{\mathfrak{g}_0} V_{\mathfrak{g}_0}(2\omega_l)\otimes V_{\mathfrak{g}_0}(\omega_l)$, which proves finally the initial step. So let $\sum_{i=1}^{l-1} m_i+(k-1)>1$ and $m_i,\ i<l$ or $k-1$ such that one of them is bigger or equal to 1. Using Proposition 4.7 we get in the first case

$$\begin{aligned}&D_{\omega_0 t_{\lambda-\omega_l}}(e^{\Lambda_l}) = D_{t_{-\omega_i}}D_{\omega_0 t_{\lambda-\omega_l-\omega_i}}(e^{\Lambda_l})\\ &= D_{t_{-\omega_i}}(e^{\frac{1}{2}\Lambda_0}\mathrm{Char}\,(V_{\mathfrak{g}_0}(\omega_1)^{\otimes m_1}\otimes\cdots\otimes V_{\mathfrak{g}_0}(\omega_i)^{\otimes m_i-1}\otimes\cdots\otimes V_{\mathfrak{g}_0}(2\omega_l)^{k-1}\otimes V_{\mathfrak{g}_0}(\omega_l)))\\ &= \mathrm{Char}\,(V_{\mathfrak{g}_0}(\omega_1)^{\otimes m_1}\otimes\cdots\otimes V_{\mathfrak{g}_0}(\omega_i)^{\otimes m_i-1}\otimes\cdots\otimes V_{\mathfrak{g}_0}(2\omega_l)^{k-1})D_{t_{-\omega_i}}(e^{\frac{1}{2}\Lambda_0}\mathrm{Char}\,(V_{\mathfrak{g}_0}(\omega_l)))\\ &= e^{\frac{1}{2}\Lambda_0}\mathrm{Char}\,(V_{\mathfrak{g}_0}(\omega_1)^{\otimes m_1}\otimes\cdots\otimes V_{\mathfrak{g}_0}(\omega_i)^{\otimes m_i}\otimes\cdots\otimes V_{\mathfrak{g}_0}(2\omega_l)^{k-1}\otimes V_{\mathfrak{g}_0}(\omega_l)).\end{aligned}$$

In the second we obtain

$$\begin{aligned}&D_{\omega_0 t_{2(k-1)\omega_l}}(e^{\Lambda_l}) = D_{-t_{2\omega_l}}D_{\omega_0 t_{2(k-2)\omega_l}}(e^{\Lambda_l}) = D_{-t_{2\omega_l}}(e^{\frac{1}{2}\Lambda_0}\mathrm{Char}\,(V_{\mathfrak{g}_0}(2\omega_l)^{\otimes k-2}\otimes V_{\mathfrak{g}_0}(\omega_l)))\\ &= \mathrm{Char}\,(V_{\mathfrak{g}_0}(2\omega_l)^{\otimes k-2})D_{-t_{2\omega_l}}(e^{\frac{1}{2}\Lambda_0}\mathrm{Char}\,(V_{\mathfrak{g}_0}(\omega_l))) = e^{\frac{1}{2}\Lambda_0}\mathrm{Char}\,(V_{\mathfrak{g}_0}(2\omega_l)^{\otimes k-1}\otimes V_{\mathfrak{g}_0}(\omega_l)).\end{aligned}$$

□

As an immediate consequence of Theorem 6.1 and its proof we obtain explicit dimension formulas for Weyl modules. Such formulas for Weyl modules, as already mentioned, were previously known for untwisted current algebras (see [6],[15] or [25]).

Corollary 6.2. Let $\lambda=\sum_{i=1}^l m_i\omega_i$ be a decomposition of a dominant weight $\lambda\in P_0^+$.

(1) If $\widehat{\mathfrak{g}}$ is a twisted affine Kac-Moody algebra not of type $A^{(2)}_{2l}$ $(l\geq 1)$, then

$$\dim W^\Gamma(\lambda)=\prod_{i=1}^l(\dim W^\Gamma(\omega_i))^{m_i}=\prod_{i=1}^l(\dim W(\omega_i))^{m_i}.$$

(2) If $\widehat{\mathfrak{g}}$ is of type $A_{2l}^{(2)}$ and $m_l = 2k-1$, then

$$\begin{aligned}\dim W^\Gamma(\lambda) &= \prod_{i=1}^{l-1} (\dim W^\Gamma(\omega_i))^{m_i} (\dim W^\Gamma(2\omega_l))^{k-1} \dim W^\Gamma(\omega_l) \\ &= (\prod_{i=1}^{l-1} \binom{2l+1}{i}^{m_i}) \binom{2l+1}{l}^{k-1} 2^l.\end{aligned}$$

6.1. **Constructions from arbitrary local Weyl modules.** In the previous section we investigate the connection between untwisted and twisted Weyl modules. We have seen that the twisted ones can be realized as associated graded modules of certain untwisted Weyl modules located in a single point. In this section we generalize this result using untwisted Weyl modules located in a finite number of points.

Let $W^1, \cdots, W^k$ be finite–dimensional, graded and cyclic modules with cyclic vectors $w_1, \ldots, w_k$ for the current algebra and further let W be a given cyclic graded $\mathfrak{C}(\widehat{\mathfrak{g}})$-module (possibly trivial) with cyclic vector w.

Proposition 6.3. Let $a_i \in \mathbb{C}^*, 1 \leq i \leq k$ be non-zero complex numbers, such that $a_i^m \neq a_j^m$ for $i \neq j$, then $\overline{W_{a_1}^1} \otimes \cdots \otimes \overline{W_{a_k}^k} \otimes W$ is a cyclic $U(\mathfrak{C}(\widehat{\mathfrak{g}}))$-module, particulary we get

$$\overline{W_{a_1}^1} \otimes \cdots \otimes \overline{W_{a_k}^k} \otimes W = U(\mathfrak{C}(\widehat{\mathfrak{g}})).(\mathbf{w} \otimes w)$$

Proof. As W^i are finite–dimensional and graded, there exists a sufficiently large N_i such that $x \otimes t^s$ acts trivially for $s \geq N_i$. Thus the ideal $J_i := \mathfrak{g} \otimes (t-a_i)^{N_i}\mathbb{C}[t]$ acts trivially on $W_{a_i}^i$. We define $\eta : \mathbb{C}^* \to \mathbb{N}$, $a_i \mapsto N_i$, then $\operatorname{Supp} \eta$ do not contain two points in the same Γ-orbit and therefore similar to the proof of Theorem 6.1 we obtain that $\overline{W_{a_1}^1} \otimes \cdots \otimes \overline{W_{a_k}^k}$ is a cyclic $U(\mathfrak{C}(\widehat{\mathfrak{g}}))$-module. The rest is a application of the Chinese remainder theorem. □

Remark 6.4. We can consider arbitrary $\mathfrak{g}$-modules $V(\lambda_i), \lambda_i \in P^+,\ 1 \leq i \leq k$ as graded and cyclic $\mathfrak{g} \otimes \mathbb{C}[t]$-modules, where the action is given by

$$x \otimes f(t).v = f(0)x.v,\ x \in \mathfrak{g},\ f \in \mathbb{C}[t].$$

Hence if $W^i = V(\lambda_i)$, it is already shown in [22] or in a more general setting of equivariant map algebras in [26], that the tensor product in Proposition 6.3 is irreducible. Moreover it is known that all finite–dimensional irreducible modules are tensor products of evaluation modules.

In [13] local Weyl modules for equivariant map algebras were defined and a tensor product property was proven. It was shown that if W^i is an untwisted graded Weyl module, then $\overline{W_{a_i}^i}$ is an local Weyl module for $\mathfrak{C}(\widehat{\mathfrak{g}})$ supported in the point a_i. The tensor product property gives that $\overline{W_{a_1}(\lambda_1)} \otimes \cdots \otimes \overline{W_{a_r}(\lambda_r)}$ is a local Weyl module for $\mathfrak{C}(\widehat{\mathfrak{g}})$. It was shown that every local Weyl module of $\mathfrak{C}(\widehat{\mathfrak{g}})$ can be obtained in this way. The following corollary, in $A_{2l}^{(2)}$ again the odd-case is considered only, shows that the dimension and $\mathfrak{g}_0$ character is independent of the support of the local Weyl module.

Corollary 6.5. Let $\lambda = \lambda_1 + \cdots + \lambda_r$ be a decomposition of a dominant weight $\lambda \in P_0^+$ into dominant weights and let $a_1, \ldots, a_r \in \mathbb{C}^*$ s.t. $a_i^m \neq a_j^m$ for $i \neq j$.

(1) If $\widehat{\mathfrak{g}}$ is a twisted affine Kac-Moody algebra not of type $A_{2l}^{(2)}$, then we have an isomorphism of $\mathfrak{C}(\widehat{\mathfrak{g}})$-modules:
$$W^\Gamma(\lambda) \cong \mathrm{gr}(\overline{W_{a_1}(\lambda_1)} \otimes \cdots \otimes \overline{W_{a_r}(\lambda_r)})$$

(2) If $\widehat{\mathfrak{g}}$ is of type $A_{2l}^{(2)}$ and $\lambda_i(\alpha_l^\vee) \in 2\mathbb{Z}_{\geq 0}$ for $1 \leq i \leq r-1$ and $\lambda_r(\alpha_l^\vee)$ is odd, then we get an isomorphism of $\mathfrak{C}(\widehat{\mathfrak{g}})$-modules:
$$W^\Gamma(\lambda) \cong \mathrm{gr}(\overline{W_{a_1}(\lambda_1)} \otimes \cdots \otimes \overline{W_{a_{r-1}}(\lambda_{r-1})} \otimes W^\Gamma(\lambda_r))$$

Proof. By Proposition 6.3 the right hand side in (1) respectively (2) is cyclic. Hence it is easy to obtain a surjecive map of $\mathfrak{C}(\widehat{\mathfrak{g}})$-modules, which is by Theorem 6.1 clearly an isomorphism. □

Remark 6.6. As mentioned in the introduction, Weyl modules are defined in [13] in a more general way, with support in $\mathbb{C}$. And they are parametrized by finitely supported functions from $\mathbb{C}$ to P^+. With this corollary we have shown in all cases except the even case in $A_{2l}^{(2)}$, that the dimension and $\mathfrak{g}_0$ character of a local Weyl module depends only on its $\mathfrak{g}_0$ maximal weight and NOT on the support of its parametrizing function. Concluding one might be able to show that the global Weyl module is a free module for a certain algebra, which might be part of a forthcoming publication.

Remark 6.7. The same construction of an associated graded module out of finite–dimensional, graded and cyclic $\mathfrak{g} \otimes \mathbb{C}[t]$-modules is defined in [11] and is called the fusion product. In the twisted case the same construction fails, since for this, one would need a pullback map like
$$\sum_{j=0}^{m-1}(x_j \otimes t^{ms+j}) \in \mathfrak{C}(\widehat{\mathfrak{g}}) \mapsto \sum_{j=0}^{m-1}(x_j \otimes (t+a)^{ms+j}) \notin \mathfrak{C}(\widehat{\mathfrak{g}}).$$
Therefore we have constructed in our results associated graded $\mathfrak{C}(\widehat{\mathfrak{g}})$-modules out of modules coming from $\mathfrak{g} \otimes \mathbb{C}[t]$, which represent an analogue of fusion products.

6.2. **Summary of the results.** As a conclusion we summarize our results: Let $\lambda = m_1\omega_1 + \cdots + m_l\omega_l$ be a dominant weight of $\mathfrak{g}_0$ and $\epsilon = 0$ if l is odd and $\epsilon = 1$ else, then

- if $\widehat{\mathfrak{g}}$ is of type $A_2^{(2)}$ (n is odd)
$$W^\Gamma(n\omega) \cong \mathrm{gr}(\overline{W(\omega_1)}^{\otimes(k-1)} \otimes W^\Gamma(\omega)) \cong V_{s_1 t_{(n-1)\omega}}(\Lambda_1)$$
- if $\widehat{\mathfrak{g}}$ is of type $A_{2l}^{(2)}$ (m_l is odd)
$$W^\Gamma(\lambda) \cong \mathrm{gr}(\overline{W(\omega_1)}^{\otimes m_1} \otimes \cdots \otimes \overline{W(\omega_{l-1})}^{\otimes m_{l-1}} \otimes \overline{W(\omega_l)}^{\otimes(k-1)} \otimes W^\Gamma(\omega_l)) \cong V_{\omega_0 t_{\lambda-\omega_l}}(\Lambda_l)$$
- if $\widehat{\mathfrak{g}}$ is of type $A_{2l-1}^{(2)}$
$$W^\Gamma(\lambda) \cong \mathrm{gr}(\overline{W(\omega_1)}^{\otimes m_1} \otimes \cdots \otimes \overline{W(\omega_l)}^{\otimes m_l}) \cong \begin{cases} V_{\omega_0 t_\lambda}(\Lambda_0), & \text{if } m_1 + 3m_3 + \cdots + (l-\epsilon)m_{l-\epsilon} \text{ is even} \\ V_{\omega_0 t_{\lambda-\omega_1}}(\Lambda_1), & \text{else} \end{cases}$$
- if $\widehat{\mathfrak{g}}$ is of type $D_{l+1}^{(2)}$
$$W^\Gamma(\lambda) \cong \mathrm{gr}(\overline{W(\omega_1)}^{\otimes m_1} \otimes \cdots \otimes \overline{W(\omega_l)}^{\otimes m_l}) \cong \begin{cases} V_{\omega_0 t_\lambda}(\Lambda_0), & \text{if } m_l \text{ is even} \\ V_{\omega_0 t_{\lambda-\omega_l}}(\Lambda_l), & \text{else} \end{cases}$$

• if $\widehat{\mathfrak{g}}$ is of type $E_6^{(2)}$

$$W^\Gamma(\lambda) \cong \mathrm{gr}(\overline{W(\omega_1)}^{\otimes m_1} \otimes \cdots \otimes \overline{W(\omega_l)}^{\otimes m_l}) \cong V_{\omega_0 t_\lambda}(\Lambda_0)$$

• if $\widehat{\mathfrak{g}}$ is of type $D_4^{(3)}$

$$W^\Gamma(\lambda) \cong \mathrm{gr}(\overline{W(\omega_1)}^{\otimes m_1} \otimes \cdots \otimes \overline{W(\omega_l)}^{\otimes m_l}) \cong V_{\omega_0 t_\lambda}(\Lambda_0)$$

7. Proofs for the type $A_2^{(2)}$

In this section our attention is dedicated to the twisted Kac-Moody algebra $A_2^{(2)}$. In the previous sections we claim that the results hold already for $A_2^{(2)}$, so to complete our work it misses to verify the follwing main result of this section.

Theorem 7.1. *Let n be an odd integer, then the Weyl module $W^\Gamma(n\omega)$ is isomorphic to the Demazure module $D(1/2, n\omega) \cong V_{s_1 t_{(n-1)\omega}}(\Lambda_1)$.*

7.1. Properties of $W^\Gamma(n\omega)$ and minimal powers.

Lemma 7.2. *Let I^σ be the left ideal in $U(\mathfrak{C}(A_2^{(2)}))$ generated by $\mathfrak{n}_j \otimes t^j C[t^m], (h_{\alpha,0} \otimes t^{2r}), (h_{\alpha,1} \otimes t^{2r-1}), r \geq 1, 0 \leq j \leq m-1$. Then for every $k \in \mathbb{N}_+$ there exists a non-zero scalar $c_k, \tilde{c_k} \in \mathbb{C}$ such that*

(1)

$$(X^+_{\alpha,0} \otimes 1)^{2k-1}(X^-_{2\alpha,1} \otimes t)^k = \begin{cases} c_k(X^-_{\alpha,1} \otimes t^k) \mod I^\sigma, & \textit{if } k \textit{ is odd} \\ c_k(X^-_{\alpha,0} \otimes t^k) \mod I^\sigma, & \textit{if } k \textit{ is even} \end{cases} \tag{7.1}$$

(2)

$$(X^+_{2\alpha,1} \otimes t)^{k-1}(X^-_{2\alpha,1} \otimes t)^k = \tilde{c_k}(X^-_{2\alpha,1} \otimes t^{2k-1}) \mod I^\sigma \tag{7.2}$$

Proof. The first equation is a simple reformulation of Lemma 3.3 (iii) in [5]. We will prove the second equation by induction. For $k = 1$ we get trivially $\tilde{c_k} = 1$. Suppose that (2) is already true for all $p \leq k$, then

$$\begin{aligned}
&(X^+_{2\alpha,1} \otimes t)^k (X^-_{2\alpha,1} \otimes t)^{k+1} = (X^+_{2\alpha,1} \otimes t)(X^+_{2\alpha,1} \otimes t)^{k-1}(X^-_{2\alpha,1} \otimes t)^k (X^-_{2\alpha,1} \otimes t) \\
&= \tilde{c_k}(X^+_{2\alpha,1} \otimes t)(X^-_{2\alpha,1} \otimes t^{2k-1})(X^-_{2\alpha,1} \otimes t) + (X^+_{2\alpha,1} \otimes t)\mathfrak{J}(X^-_{2\alpha,1} \otimes t), \text{ for some } \mathfrak{J} \in I^\sigma \\
&\equiv -\frac{1}{2}\tilde{c_k}(h_{\alpha,0} \otimes t^{2k})(X^-_{2\alpha,1} \otimes t) \mod I^\sigma \\
&\equiv 2\tilde{c_k}(X^-_{2\alpha,1} \otimes t^{2k+1}) \mod I^\sigma
\end{aligned}$$

□

Corollary 7.3. Let $n \in \mathbb{N}$, such that $n = 2k$ if n is even and $n = 2k-1$ if n is odd. Then we have

(1) $(X^-_{2\alpha,1} \otimes t)^k w_n = 0$

(2) $\begin{cases} (X^-_{\alpha,0} \otimes t^k) w_n = (X^-_{\alpha,1} \otimes t^{k+1}) w_n = 0, & \text{if k is even} \\ (X^-_{\alpha,0} \otimes t^{k+1}) w_n = (X^-_{\alpha,1} \otimes t^k) w_n = 0, & \text{if k is odd} \end{cases}$

(3) $(X^-_{2\alpha,1} \otimes t^{2k-1}) w_n = 0$

Proof. Clearly part (2) and (3) are deductions of Lemma 7.2 and part (1). Assume now $(X^-_{2\alpha,1}\otimes t)^k w_n$ is non-zero element in $W^\Gamma(n\omega)[k]$ of weight $-2k\omega$ if n is even and $(-2k-1)\omega$ if n is odd and recall that $W^\Gamma(n\omega)[k]$ is an integrable $\mathfrak{sl}_2$-module, i.e. Proposition 4.15 is applicable. That means $W^\Gamma(n\omega)[k]_{2k\omega}\neq 0$ respectively $W^\Gamma(n\omega)[k]_{(2k+1)\omega}\neq 0$, but both are impossible, which proves part (1). $\square$

Corollary 7.4. For all $n\in\mathbb{N}$ the modules $W^\Gamma(n\omega)$ are finite–dimensional.

Proof. Proposition 4.15 implies that $W^\Gamma(n\omega)_\mu\neq 0$ only if $\mu\in n\omega-Q_0^+$ and suppose that

$$W^\Gamma(n\omega)\cong\bigoplus_{\mu\in P_0^+}V(\mu)^{n_\mu}$$

is the decomposition of $W^\Gamma(n\omega)$ into irreducible $\mathfrak{g}_0$-modules. Note that the number of elements in P_0^+ with the property $\mu\in n\omega-Q_0^+$ is finite. The corollary follows if we prove that $\dim W^\Gamma(n\omega)_\mu<\infty$, since this implies $n_\mu<\infty$. That the dimension can‘t be infinity is a direct consequence of Corollary 7.3. $\square$

As in the other cases we show that the Weyl modules are in connection with certain associated graded modules:

Proposition 7.5. Let $n\in\mathbb{N}$, such that $n=2k$ if n is even and $n=2k-1$ if n is odd. Then we get a surjective map respectively an isomorphism of $U(\mathfrak{C}(A_2^{(2)}))$-modules

$$W^\Gamma(n\omega)\left\{\begin{array}{ll}\twoheadrightarrow \mathrm{gr}(\overline{W_{a_1}(\omega_1)}\otimes\cdots\otimes\overline{W_{a_k}(\omega_1)}), & \text{if n is even}\\ \cong \mathrm{gr}(\overline{W_{a_1}(\omega_1)}\otimes\cdots\otimes\overline{W_{a_{k-1}}(\omega_1)}\otimes W^\Gamma(\omega)), & \text{if n is odd}\end{array}\right.$$

The map is given by $w_n\mapsto\underbrace{w_{\omega_1}\otimes\cdots\otimes w_{\omega_1}}_{k}$ if n is even and $w_n\mapsto\underbrace{w_{\omega_1}\otimes\cdots\otimes w_{\omega_1}}_{k-1}\otimes w_\omega$ otherwise.

Remark 7.6. We will proof the isomorphism claimed in the odd case in Section 7.2 and remind that the surjectivity of the maps in Proposition 7.5 follows by weight reasons.

Corollary 7.7. We obtain,

$$\dim W^\Gamma(n\omega)\geq\left\{\begin{array}{ll}3^{\frac{n}{2}}, & \text{if n is even}\\ 3^{\lceil\frac{n}{2}\rceil}2, & \text{if n is odd}\end{array}\right.$$

In Corollar 7.3 we proved that we can explicitly specify an integer, such that the elements with higher powers of t act by zero. In the next we will refute the question, if there exists a smaller integer with same property. To show this one can use the help of associated graded modules defined in Section 6 and Proposition 7.5.

Lemma 7.8. *Let $n\in\mathbb{N}$ like in Corollar 7.3. Then we have,*

$$\left\{\begin{array}{ll}(X^-_{\alpha,0}\otimes t^{2r})w_n\neq 0,(X^-_{\alpha,1}\otimes t^{2r+1})w_n\neq 0, & \textit{for all } r<\frac{k}{2} \textit{ if } k \textit{ is even}\\ (X^-_{\alpha,0}\otimes t^{2r})w_n\neq 0,(X^-_{\alpha,1}\otimes t^{2s+1})w_n\neq 0, & \textit{for all } r<\frac{k+1}{2},\ s<\frac{k-1}{2} \textit{ if } k \textit{ is odd}\\ (X^-_{2\alpha,1}\otimes t^{2r+1})w_n\neq 0, & \textit{if } r<k-1\end{array}\right.$$

Before we are in position to prove our main result of this section we will formulate another necessary proposition:

Proposition 7.9. Let $n \in \mathbb{N}$ as in Corollar 7.3, then we have surjective homomorphisms

$$W^\Gamma((n-2)\omega) \twoheadrightarrow \begin{cases} U(\mathfrak{C}(A_2^{(2)}))(X^-_{\alpha,1} \otimes t^{k-1})w_n, & \text{if k is even} \\ U(\mathfrak{C}(A_2^{(2)}))(X^-_{\alpha,0} \otimes t^{k-1})w_n & \text{if k is odd} \end{cases}$$

$$W^\Gamma((n-4)\omega) \twoheadrightarrow U(\mathfrak{C}(A_2^{(2)}))(X^-_{2\alpha,1} \otimes t^{2k-3})w_n$$

Proof. The proof is straightforward with Corollary 7.3. $\square$

7.2. **Proof of Theorem 7.1.**

Proof. Note that Proposition 7.5 is a direct consequence of Theorem 7.1 and the Demazure character formula (see Theorem 4.6), since this provides us

$$\dim W^\Gamma(n\omega) = \dim V_{s_1 t_{(n-1)\omega}}(\Lambda_1) = \dim V_{(s_1 s_0)^{\lceil \frac{n}{2} \rceil} s_1}(\Lambda_1) = 3^{\lceil \frac{n}{2} \rceil} 2.$$

We already know by Corollary 4.17 that the Demazure module $D(1/2, n\omega)$ is a quotient of the Weyl module $W^\Gamma(n\omega)$. So by Corollary 4.9 it remains to show that the following relations holds:

$$(X^-_{\alpha,0} \otimes t^{2r})^{max\{0,n-4r\}+1} w_n = 0 \tag{7.3}$$

$$(X^-_{\alpha,1} \otimes t^{2r+1})^{max\{0,n-2(2r+1)\}+1} w_n = 0 \tag{7.4}$$

$$(X^-_{2\alpha,1} \otimes t^{2r+1})^{max\{0,k-r-1\}+1} w_n = 0. \tag{7.5}$$

By Corollary 7.3 we can assume that the maximums are non-zero and further suppose that $(X^-_{\alpha,0} \otimes t^{2r})^{n-4r+1} w_n \neq 0$, hence $W^\Gamma(n\omega)_{(n-2(n-4r+1))\omega}[2r(n-4r+1)] \neq 0$. By Proposition 7.9 and Proposition 4.15 (1) we get that

$$W^\Gamma(n\omega)_{(n-2j)\omega}[l] = 0$$

for all l with

$$l > \begin{cases} (k-1) + \cdots + (k-j) = jk - \frac{j(j+1)}{2}, & \text{if } 0 \leq j \leq k \\ (k-1) + \cdots + (k-(n-j)) = (n-j)k - \frac{(n-j)((n-j)+1)}{2}, & \text{if } k < j \leq n \end{cases}$$

Hence,

$$2r(n-4r+1) \leq \begin{cases} jk - \frac{j(j+1)}{2}, & \text{if } 0 \leq j \leq k \\ (n-j)k - \frac{(n-j)((n-j)+1)}{2}, & \text{if } k < j \leq n \end{cases},$$

with $j = (n-4r+1)$, which contradicts $2r < k$. Exactly the same argumentation shows also (7.4) and (7.5) $\square$

Remark 7.10. An inspection of the proof of Theorem 7.1 shows, that the condition, n is odd, is not needed. Thus the relations (7.3), (7.4), (7.5) holds also in $W^\Gamma(2k\omega)$, but it is easy that they are not enough. For instance in $W^\Gamma(6\omega)$ we have already $(X^-_{\alpha,0} \otimes t^2)^2 w_6 = 0$, while relation (7.3) gives $(X^-_{\alpha,0} \otimes t^2)^3 w_6 = 0$.

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REALIZATION OF AFFINE TYPE A KIRILLOV-RESHETIKHIN CRYSTALS VIA POLYTOPES

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Abstract. On the polytope defined in [6], associated to any rectangle highest weight, we define a structure of an type A_n-crystal. We show, by using the Stembridge axioms, that this crystal is isomorphic to the one obtained from Kashiwara's crystal bases theory. Further we define on this polytope a bijective map and show that this map satisfies the properties of a weak promotion operator. This implies in particular that we provide an explicit realization of Kirillov-Reshetikhin crystals for the affine type $A_n^{(1)}$ via polytopes.

1. Introduction

Let $\mathfrak{g}$ be a affine Lie algebra and $\mathfrak{U}_q'(\mathfrak{g})$ be the corresponding quantum algebra without derivation. The irreducible representations are classified in [4],[5] in terms of Drinfeld polynomials. A certain subclass of these modules, that gained a lot of attraction during the last decades, are the so called Kirillov-Reshetikhin modules $KR(m, \omega_i, a)$, where i is a node in the classical Dynkin diagram and m is a positive integer. One of the main tools for studying such representations is Kashiwara's crystal bases theory [12]. This theory was originally defined for representations for $\mathfrak{U}_q(\mathfrak{g})$, however it can be nonetheless defined in the setting of $\mathfrak{U}_q'(\mathfrak{g})$ modules, respecting that crystal bases might not always exist. It was first conjectured in [9], that $\mathrm{KR}(m, \omega_i, a)$ admits a crystal bases and this was proven in type $A_n^{(1)}$ in [11] and in all non-exceptional cases in [19],[20]. We denote this crystal by $\mathrm{KR}^{m,i}$ and call it a Kirillov-Reshetikhin crystal.

A promotion operator pr on a crystal B of type A_n is defined to be a map satisfying several conditions, namely that pr shifts the content, $pr \circ \tilde{e}_j = \tilde{e}_{j+1} \circ pr$, $pr \circ \tilde{f}_j = \tilde{f}_{j+1} \circ pr$ for all $j \in \{1, \cdots, n-1\}$ and $pr^{n+1} = id$, where $\tilde{e}_j$ and $\tilde{f}_j$ respectively are the Kashiwara operators. If the latter condition is not satisfied, but pr is still bijective, then the map pr is called a weak promotion operator (see also [1]). The advantage of such (weak) promotion oparators are that we can associate to a given crystal B of type A_n a (weak) affine crystal by setting $\tilde{f}_0 := pr^{-1} \circ \tilde{f}_1 \circ pr$, and $\tilde{e}_0 := pr^{-1} \circ \tilde{e}_1 \circ pr$.

On the set of all semi-standard Young tableaux of rectangle shape, which is a realization of $B(m\omega_i)$ the $\mathfrak{U}_q(A_n)$-crystal associated to the irreducible module of highest weight $m\omega_i$, Schützenberger defined a promotion operator pr, called the Schützenberger's promotion operator [22], which is the analogue of the cyclic Dynkin diagram automorphism $i \mapsto i+1$ mod $(n+1)$ on the level of crystals, by using jeu-de-taquin. Given a tableaux T over the alphabet $1 \prec 2 \cdots \prec n+1$, $pr(T)$ is obtained from T by removing all letters $n+1$, adding one to each entry in the remaining tableaux, using jeu-de-taquin to slide all letters up and

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finally filling the holes with 1's (see also Section 6). One of the combinatorial descriptions of $\mathrm{KR}^{m,i}$ in the affine $A_n^{(1)}$ type was provided by Schimozono in [23]. It was shown that, as a $\{1, \cdots, n\}$-crystal, $\mathrm{KR}^{m,i}$ is isomorphic to $B(m\omega_i)$ and the affine crystal constructed from $B(m\omega_i)$ using Schützenberger's promotion operator is isomorphic to the Kirillov-Reshetikhin crystal $\mathrm{KR}^{m,i}$. The two ways of computing the affine crystal structure, one given by [11] and the other by [23], are shown to be equivalent in [21]. Another combinatorial model in this type without using a promotion operator is described in [16]. In this paper, we introduce a new realization of Kirillov-Reshetikhin crystals of type $A_n^{(1)}$.

In [6] the authors have constructed for all dominant integral A_n weights λ a polytope in $\mathbb{R}^{n\frac{n-1}{2}}$ and a basis of the irreducible A_n module of highest weight λ and have shown that the basis elements are parametrized by the integral points. For $\lambda = m\omega_i$ we can understand this polytope in $\mathbb{R}^{i(n-i+1)}$ and denote the intersection of this polytope with $\mathbb{Z}_+^{i(n-i+1)}$ by $B^{m,i}$. We define certain maps on $B^{m,i}$ and show that this becomes a crystal of type A_n. As a set, we can identify $B^{m,i}$ with certain blocks of height $n-i+1$ and width i

$$\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array},$$

where the boxes are filled, under some assumptions, with some non-negative integers (see Definition 2.1). The crystal $B^{m,i}$ has no known explicit combinatorial bijection to other combinatorial models of crystals induced by representations, such as the Young tableaux model [14] or the set of certain Nakajima monomials [18], which makes an isomorphism very complicated. Using the realization of crystal bases via Nakajima monomials, we can construct certain local A_2 isomorphisms on our underlying polytope $B^{m,i}$ and prove that the so called Stembridge axioms are satisfied. These axioms precisely characterize the set of crystals of representations in the class of all crystals. Our first important theorem is therefore the following:

Theorem A. *The polytope $B^{m,i}$ is as an A_n crystal isomorphic to $B(m\omega_i)$.*

In order to obtain an (weak) affine crystal structure we define a map *pr* on $B^{m,i}$, which is given by an algorithm consisting of i steps (see (6.1)), and show that this map satisfies the conditions for a weak promotion operator. In particular, this implies that this map *pr* is the unique promotion operator on $B^{m,i} \cong B(m\omega_i)$ and the polytope becomes an affine crystal. To be more precise, we prove the following main theorem of our paper:

Theorem B. *The associated affine crystal $B^{m,i}$ using pr is isomorphic to the Kirillov-Reshetikhin crystal $\mathrm{KR}^{m,i}$.*

Our paper is organized as follows: in Section 2 we fix some notation and present the main definitions, especially the definition of our polytope. In Section 3 we equip our main object with a crystal structure. In Section 4 Nakajima monomials are recalled and in Section 5 Theorem A is proven. Finally, in Section 6 the promotion operator is defined by an algorithm and the corresponding affine crystal is identified with the KR crystal, proving Theorem B.

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2. Notation and main definitions

Let $\mathfrak{g}$ be a complex affine Lie algebra of rank n and fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and a Borelsubalgebra $\mathfrak{b} \supseteq \mathfrak{h}$. We denote by $\Phi \subseteq \mathfrak{h}^*$ the root system of the Lie algebra, and, corresponding to the choice of $\mathfrak{b}$ let Φ^+ be the subset of positive roots. Further, we denote by $\Pi = \{\alpha_0, \cdots, \alpha_n\}$ the corresponding basis of Φ and the basis of the dual root system $\Phi^\vee \subseteq \mathfrak{h}$ is denoted by $\Pi^\vee = \{\alpha_0^\vee, \cdots, \alpha_n^\vee\}$. Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a Cartan decomposition and for a given root $\alpha \in \Phi$ let $\mathfrak{g}_\alpha$ be the corresponding root space. For a dominant integral weight λ we denote by $V(\lambda)$ the irreducible $\mathfrak{g}$-module with highest weight λ. Fix a highest weight vector $v_\lambda \in V(\lambda)$, then $V(\lambda) = \mathfrak{U}(\mathfrak{n}^-)v_\lambda$, where $\mathfrak{U}(\mathfrak{n}^-)$ denotes the universal enveloping algebra of $\mathfrak{n}^-$. For an indetermined element q we denote by $\mathfrak{U}_q'(\mathfrak{g})$ be the corresponding quantum algebra without derivation. The irreducible representations are classified in [4],[5] in terms of Drinfeld polynomials. One of the major goals in representation theory is to find nice expressions for the character of objects in the category $\mathcal{O}_{int}^q$ (see [10]). From the theory of crystal bases, introduced by Kashiwara in [12], we can compute the character of a given module M as follows:

$$chM = \sum_\mu \sharp(B_\mu)e^\mu,$$

whereby (L, B) is the crystal bases of M (see also [10]). From now on let $\mathfrak{g}$ be the affine Lie algebra

$$A_n^{(1)} = \mathfrak{sl}_{n+1} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

with index set $\hat{I} = \{0, 1, \cdots, n\} \supseteq I = \{1, \cdots, n\}$. Note that the classical positive roots are all of the form

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \text{ for } 1 \leq i \leq j \leq n.$$

Further let $P = \bigoplus_{i \in I} \mathbb{Z}\omega_i$ be the set of classical integral and $P^+ = \bigoplus_{i \in I} \mathbb{Z}_+\omega_i$ be the set of classical dominant integral weights. In order to realize the crystal graph of the so called Kirillov-Reshetikhin modules KR(m, ω_i, a), for $i \in I, m \in \mathbb{Z}_+$, we will define now the underlying combinatorial model in this paper, which we will denote by $B^{m,i}$. For more details regarding KR-modules we refer to a series of papers ([2],[3],[7]).

2.1. The polytope $B^{m,i}$.

In this subsection we will define the set $B^{m,i}$, our main object in this paper and discuss its combinatorics which is crucial for the realization of KR-crystals.

Definition. Let $B^{m,i}$ be the set of all following patterns:

$a_{1,i}$	$a_{2,i}$	$\dots$	$a_{i-1,i}$	$a_{i,i}$
$a_{1,i+1}$	$a_{2,i+1}$	$\dots$	$a_{i-1,i+1}$	$a_{i,i+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_{1,n}$	$a_{2,n}$	$\dots$	$a_{i-1,n}$	$a_{i,n}$

filled with non-negative integers, such that $\sum_{s=1}^{k} a_{\beta(s)} \leq m$ for all sequences

$$(\beta(1), \dots, \beta(k)),\ k \geq 1$$

satisfying the following: $\beta(1) = \alpha_{1,i}, \beta(k) = \alpha_{i,n}$ and if $\beta(s) = \alpha_{p,q}$ then the next element in the sequence is either of the form $\beta(s+1) = \alpha_{p,q+1}$ or $\beta(s+1) = \alpha_{p+1,q}$.

Example.

$$\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 2 & 1 \\ \hline 0 & 1 \\ \hline \end{array} \in B^{5,2}, \text{ but } \begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 0 & 1 & 3 \\ \hline 1 & 0 & 1 \\ \hline \end{array} \notin B^{5,3}.$$

Remark 2.1.1.

(1) For any element in $B^{m,i}$ the columns are numbered from 1 to i and the rows are numbered from i to n.

(2) A sequence

$$\mathbf{b} = (\beta(1), \dots, \beta(k)),\ k \geq 1$$

satisfying the rule from Definition 2.1 is called a Dyck path. The notion of a Dyck path occurs already in [6]. We will denote the set of all such paths by $\mathbf{D}$.

(3) In [16], a combinatorial model to describe the KR-crystals is developed. The parametrization of the edges is given in terms of non-negative integral matrices satisfying certain conditions based on the classical Robinson-Schensted-Knuth algorithm. These conditions can be translated into the Dyck path rule.

Remark 2.1.2. Note that the name "polytope" is justified, since $B^{m,i}$ reflects the integral points of some polytope in $\mathbb{R}^{i(n-i+1)}$:

$$B^{m,i} \cong \{(a_{r,s}) \in \mathbb{R}^{i(n-i+1)} | \sum_{s=1}^{k} a_{\beta(s)} \leq m,\ \text{for all } \mathbf{b} \in \mathbf{D}\ \} \cap \mathbb{Z}_{+}^{i(n-i+1)}.$$

What we want to show now is that the set $B^{m,i}$ carries an (affine) crystal structure. Moreover, our goal is to show that this is exactly the crystal graph of the KR-module $\mathrm{KR}(m, \omega_i, a)$, i.e. we have an isomorphism of crystals. The strongest indication that this conjecture might be true is the following modified result due to [6].

Theorem 2.1.1.

$$\dim V(m\omega_i) = \sharp B^{m,i}$$

3. Crystal Structure on $B^{m,i}$

With the purpose to show that we have a crystal structure on $B^{m,i}$, which is induced from a module we will first chop all necessary conditions of an abstract crystal. Let us start by giving the definition:

3.1. **Abstract crystals.**

Definition. Let $\hat{I}$ be a finite index set and let $A = (a_{i,j})_{i,j\in\hat{I}}$ be a generalized Cartan matrix with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. A crystal associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is a set B together with the maps $\mathrm{wt} : B \to P$, $\tilde{e}_l, \tilde{f}_l : B \to B \cup \{0\}$, and $\epsilon_l, \varphi_l : B \to \mathbb{Z} \cup \{-\infty\}$ satisfying the following properties for all $l \in \hat{I}$:

(1) $\varphi_l(b) = \epsilon_l(b) + \langle \alpha_l^\vee, \mathrm{wt}(b)\rangle$
(2) $\mathrm{wt}(\tilde{e}_l b) = \mathrm{wt}(b) + \alpha_l$ if $\tilde{e}_l b \in B$
(3) $\mathrm{wt}(\tilde{f}_l b) = \mathrm{wt}(b) - \alpha_l$ if $\tilde{f}_l b \in B$
(4) $\epsilon_l(\tilde{e}_l b) = \epsilon_l(b) - 1$, $\varphi_l(\tilde{e}_l b) = \varphi_l(b) + 1$ if $\tilde{e}_l b \in B$
(5) $\epsilon_l(\tilde{f}_l b) = \epsilon_l(b) + 1$, $\varphi_l(\tilde{f}_l b) = \varphi_l(b) - 1$ if $\tilde{f}_l b \in B$
(6) $\tilde{f}_l b = b'$ if and only if $\tilde{e}_l b' = b$ for $b, b' \in B$
(7) if $\varphi_l(b) = -\infty$ for $b \in B$, then $\tilde{f}_l b = \tilde{e}_l b$=0.

Further a crystal B is said to be semiregular if the equalities

$$\epsilon_l(b) = \max\{k \geq 0 | \tilde{e}_l^k b \neq 0\}, \quad \varphi_l(b) = \max\{k \geq 0 | \tilde{f}_l^k b \neq 0\}$$

hold.

Hence our aim is to define the Kashiwara operators $\tilde{f}_l$ and $\tilde{e}_l$, which will act on the set $B^{m,i}$ for all $l \in \{0, \cdots, n\}$, such that the properties in Definition 3.1 are fulfilled. Furthermore we will show that $B^{m,i}$ becomes a semiregular crystal. Our strategy is as follows: first we are going to define a classical crystal structure on $B^{m,i}$, which in particular means that we define the Kashiwara operators for all $l \in \{1, \cdots, n\}$. Subsequently, we show that this is precisely the crystal graph of the irreducible module $V(m\omega_i)$ and then we exploit the existence of Schützenberger's promotion operator to define the Kashiwara operators for the node 0. For more details regarding the Schützenberger promotion operator we refer to [22] or Section 6.

3.2. **Crystal Structure on $B^{m,i}$.**

As already mentioned, our aim in this section is to define the maps $\mathrm{wt}, \tilde{e}_l, \tilde{f}_l, \epsilon_l, \varphi_l$ for all $l \in I$ as in the Definition 3.1, such that the properties (1)-(7) are fulfilled. The classical crystal structure described in [16] is induced by the tensor product rule, whereby we describe the structure explicitly. So let A be an arbitrary element in $B^{m,i}$, then we define

$$\mathrm{wt}(A) = m\omega_i - \sum_{1\leq p\leq i, i\leq q\leq n} a_{p,q}\alpha_{p,q}. \tag{3.1}$$

In order to define what the Kashiwara operators are, we need much more spadework. In the following we define some useful maps and integers, such that these integers will completely determine the rule at which "place" the action is given. So we define the maps φ_l, ϵ_l :

$B^{m,i} \longrightarrow \mathbb{Z}_{\geq 0}$ for all $l \in I$ by:

$$\varphi_l(A) = \begin{cases} m - \sum_{j=1}^{i-1} a_{j,i} - \sum_{j=i}^{n} a_{i,j}, & \text{if } l = i \\ \sum_{j=1}^{p_+^l(A)} a_{j,l-1} - \sum_{j=1}^{p_+^l(A)-1} a_{j,l}, & \text{if } l > i \\ \sum_{j=p_-^l(A)}^{n} a_{l+1,j} - \sum_{j=p_-^l(A)+1}^{n} a_{l,j}, & \text{if } l < i \end{cases} \tag{3.2}$$

$$\epsilon_l(A) = \begin{cases} a_{i,i}, & \text{if } l = i \\ \sum_{j=q_+^l(A)}^{i} a_{j,l} - \sum_{j=q_+^l(A)+1}^{i} a_{j,l-1}, & \text{if } l > i \\ \sum_{j=i}^{q_-^l(A)} a_{l,j} - \sum_{j=i}^{q_-^l(A)-1} a_{l+1,j}, & \text{if } l < i, \end{cases} \tag{3.3}$$

whereby

$$p_+^l(A) = \min\{1 \leq p \leq i | \sum_{j=1}^{p} a_{j,l-1} + \sum_{j=p}^{i} a_{j,l} = \max_{1\leq q\leq i}\{\sum_{j=1}^{q} a_{j,l-1} + \sum_{j=q}^{i} a_{j,l}\}\} \tag{3.4}$$

$$q_+^l(A) = \max\{1 \leq p \leq i | \sum_{j=1}^{p} a_{j,l-1} + \sum_{j=p}^{i} a_{j,l} = \max_{1\leq q\leq i}\{\sum_{j=1}^{q} a_{j,l-1} + \sum_{j=q}^{i} a_{j,l}\}\} \tag{3.5}$$

$$p_-^l(A) = \max\{i \leq p \leq n | \sum_{j=i}^{p} a_{l,j} + \sum_{j=p}^{n} a_{l+1,j} = \max_{i\leq q\leq n}\{\sum_{j=i}^{q} a_{l,j} + \sum_{j=q}^{n} a_{l+1,j}\}\} \tag{3.6}$$

$$q_-^l(A) = \min\{i \leq p \leq n | \sum_{j=i}^{p} a_{l,j} + \sum_{j=p}^{n} a_{l+1,j} = \max_{i\leq q\leq n}\{\sum_{j=i}^{q} a_{l,j} + \sum_{j=q}^{n} a_{l+1,j}\}\}. \tag{3.7}$$

Remark 3.2.1. Note that the integers (3.2)-(3.3) for $l \neq i$ and (3.4)-(3.7) depend only on two given columns or two given rows of A. Therefore one can define these integers for any given two columns or rows $\mathbf{a}$ and $\mathbf{b}$ and denote them alternatively by $p_\pm(\mathbf{a},\mathbf{b})$,$q_\pm(\mathbf{a},\mathbf{b})$ and $\epsilon(\mathbf{a},\mathbf{b})$, $\varphi(\mathbf{a},\mathbf{b})$ respectively. For instance we will use in some places the notation $q_-(\mathbf{a_l},\mathbf{a_{l+1}})$ instead of (3.7), if $\mathbf{a_l}$ and $\mathbf{a_{l+1}}$ is the l-th column and $(l+1)$-th column respectively of A, and $\epsilon(\mathbf{a_l},\mathbf{a_{l+1}})$ instead of (3.3).

The first fact we want to note about these maps is the following lemma:

Lemma 3.2.1. *The map φ_l is uniquely determined by the map ϵ_l and conversely the map ϵ_l is uniquely determined by the map φ_l. Particularly we have*

$$\varphi_l(A) = \epsilon_l(A) + \langle \alpha_l^\vee, \mathrm{wt}(A)\rangle.$$

Proof. Assume A is an arbitrary element and let $p_+^l(A)$, $p_-^l(A)$, $q_+^l(A)$, $q_-^l(A)$ be the integers described in (3.4)-(3.7). Since the statement is obvious for $l = i$ we presume $l \neq i$. Then, because of

$$\sum_{j=1}^{p_+^l(A)} a_{j,l-1} + \sum_{j=p_+^l(A)}^{i} a_{j,l} = \sum_{j=1}^{q_+^l(A)} a_{j,l-1} + \sum_{j=q_+^l(A)}^{i} a_{j,l}$$

and

$$\sum_{j=i}^{p^l_-(A)} a_{l,j} + \sum_{j=p^l_-(A)}^{n} a_{l+1,j} = \sum_{j=i}^{q^l_-(A)} a_{l,j} + \sum_{j=q^l_-(A)}^{n} a_{l+1,j},$$

it follows that

$$\sum_{j=p^l_+(A)+1}^{q^l_+(A)} a_{j,l-1} - \sum_{j=p^l_+(A)}^{q^l_+(A)-1} a_{j,l} = \sum_{j=q^l_-(A)+1}^{p^l_-(A)} a_{l,j} - \sum_{j=q^l_-(A)}^{p^l_-(A)-1} a_{l+1,j} = 0. \tag{3.8}$$

Therefore, if $l > i$ we arrive at

$$\begin{aligned}\epsilon_l(A) &= \sum_{j=q^l_+(A)}^{i} a_{j,l} - \sum_{j=q^l_+(A)+1}^{i} a_{j,l-1} = \sum_{j=p^l_+(A)}^{i} a_{j,l} - \sum_{j=p^l_+(A)+1}^{i} a_{j,l-1} \\ &= \sum_{j=1}^{p^l_+(A)} a_{j,l-1} - \sum_{j=1}^{p^l_+(A)-1} a_{j,l} - (\sum_{j=1}^{i} a_{j,l-1} - \sum_{j=1}^{i} a_{j,l}) = \varphi_l(A) - \langle \alpha_l^\vee, \mathrm{wt}(A)\rangle\end{aligned}$$

and if $l < i$ we obtain again with (3.8)

$$\epsilon_l(A) = \sum_{j=i}^{p^l_-(A)} a_{l,j} - \sum_{j=i}^{p^l_-(A)-1} a_{l+1,j} = \varphi_l(A) - \langle \alpha_l^\vee, \mathrm{wt}(A)\rangle.$$

Thus the map ϵ_l is already determined by φ_l and conversely as well. □

For the purpose of constructing an object in the category of crystals we define the Kashiwara operators by the following rule: let A be an arbitrary element of $B^{m,i}$ filled as in Definition 2.1, then $\tilde{f}_l A$ and $\tilde{e}_l A$ respectively is defined to be 0 if $\varphi_l(A) = 0$ and $\epsilon_l(A) = 0$ respectively. Otherwise the image of A under $\tilde{f}_l$ and $\tilde{e}_l$ respectively arises from A by replacing certain boxes, namely

$$\tilde{f}_l A = \begin{cases} \text{replace } \boxed{a_{i,i}} \text{ by } \boxed{a_{i,i}+1}, & \text{if } l = i \\ \text{replace } \boxed{a_{p^l_+(A),l-1}} \text{ by } \boxed{a_{p^l_+(A),l-1}-1} \text{ and } \boxed{a_{p^l_+(A),l}} \text{ by } \boxed{a_{p^l_+(A),l}+1}, & \text{if } l > i \\ \text{replace } \boxed{a_{l,p^l_-(A)}} \text{ by } \boxed{a_{l,p^l_-(A)}+1} \text{ and } \boxed{a_{l+1,p^l_-(A)}} \text{ by } \boxed{a_{l+1,p^l_-(A)}-1}, & \text{if } l < i \end{cases} \tag{3.9}$$

$$\tilde{e}_l A = \begin{cases} \text{replace } \boxed{a_{i,i}} \text{ by } \boxed{a_{i,i}-1}, & \text{if } l = i \\ \text{replace } \boxed{a_{q^l_+(A),l-1}} \text{ by } \boxed{a_{q^l_+(A),l-1}+1} \text{ and } \boxed{a_{q^l_+(A),l}} \text{ by } \boxed{a_{q^l_+(A),l}-1}, & \text{if } l > i \\ \text{replace } \boxed{a_{l,q^l_-(A)}} \text{ by } \boxed{a_{l,q^l_-(A)}-1} \text{ and } \boxed{a_{l+1,q^l_-(A)}} \text{by } \boxed{a_{l+1,q^l_-(A)}+1}, & \text{if } l < i. \end{cases} \tag{3.10}$$

To be more accurate we should denote the Kashiwara operators by ${}_m\tilde{f}_l$ and ${}_m\tilde{e}_l$ respectively. However almost all Kashiwara operators, except ${}_m\tilde{f}_i$, are by the next lemma independent of m. Therefore the notation $\tilde{f}_l$ and $\tilde{e}_l$ respectively is justified. If there is no confusion we will also denote ${}_m\tilde{f}_i$ by $\tilde{f}_i$.

Lemma 3.2.2. *Let $m, d \in \mathbb{Z}_+$ and $A \in B^{m,i} \cap B^{d,i}$, then we have*

$$_m\tilde{f}_l A = {}_d\tilde{f}_l A \text{ and } {}_m\varphi_l(A) = {}_d\varphi_l(A) \text{ for all } l \neq i \ ,$$

$$_m\tilde{e}_l A = {}_d\tilde{e}_l A \text{ and } {}_m\epsilon_l(A) = {}_d\epsilon_l(A) \text{ for all } l \ .$$

Proof. A short investigation of (3.2),(3.3),(3.9) and (3.10) shows that they depend only on the filling of A with the exception of φ_i and $\tilde{f}_i$. □

Remark 3.2.2.
It is not clear, why these operators are well-defined. Particulary we shall show in the next lemma that the images are always contained in $B^{m,i}$.

Lemma 3.2.3. *For all $l \in I$ and $A \in B^{m,i}$ we have $\tilde{f}_l A, \tilde{e}_l A \in B^{m,i}$.*

Proof. Assume that the element $\tilde{f}_l A$ and $\tilde{e}_l A$ respectively is not contained in $B^{m,i}$, then by definition there exists a Dyck path $(\beta(1), \dots, \beta(k))$, such that

$$\sum_{s=1}^{k} a_{\beta(s)} > m. \tag{3.11}$$

This would be an impossible inequality if $l = i$; therefore we suppose that $l > i$, since the proof for $l < i$ is similar. By an inspection of the action of the Kashiwara operator $\tilde{f}_l$ we can conclude directly that (3.11) must be of the following form:

$$\sum_{s=1}^{k} a_{\beta(s)} = \sum_{s=1}^{t} a_{\beta(s)} + a_{z,l-1} + \sum_{j=z}^{p_+^l(A)} a_{j,l} + 1 + \sum_{s=t+p_+^l(A)-z+3}^{k} a_{\beta(s)}, \tag{3.12}$$

with an integer $z \in \{1, \cdots, i\}$ strictly smaller than $p_+^l(A)$ and some $t \in \{1, \cdots, k\}$. We get

$$\sum_{s=1}^{k} a_{\beta(s)} > m \geq \sum_{s=1}^{t} a_{\beta(s)} + \sum_{j=z}^{p_+^l(A)} a_{j,l-1} + a_{p_+^l(A),l} + \sum_{s=t+p_+^l(A)-z+3}^{k} a_{\beta(s)}$$

and consequently

$$\sum_{j=1}^{z} a_{j,l-1} + \sum_{j=z}^{i} a_{j,l} = \sum_{j=1}^{p_+^l(A)} a_{j,l-1} + \sum_{j=p_+^l(A)}^{i} a_{j,l},$$

which is a contradiction to the choice of $p_+^l(A)$. An inspection of the action with $\tilde{e}_l$ requires that (3.11) must be of the form:

$$\sum_{s=1}^{k} a_{\beta(s)} = \sum_{s=1}^{t} a_{\beta(s)} + \sum_{j=q_+^l(A)}^{z} a_{j,l-1} + 1 + a_{z,l} + \sum_{s=t+z-q_+^l(A)+3}^{k} a_{\beta(s)},$$

with an integer $z \in \{1, \cdots, i\}$ strictly greater than $q_+^l(A)$. Hence, together with

$$\sum_{s=1}^{k} a_{\beta(s)} > m \geq \sum_{s=1}^{t} a_{\beta(s)} + \sum_{j=q_+^l(A)}^{z} a_{j,l} + a_{q_+^l(A),l-1} + \sum_{s=t+z-q_+^l(A)+3}^{k} a_{\beta(s)},$$

we have again a contradiction to the choice of $q^l_+(A)$, namely

$$\sum_{j=1}^{z} a_{j,l-1} + \sum_{j=z}^{i} a_{j,l} = \sum_{j=1}^{q^l_+(A)} a_{j,l-1} + \sum_{j=q^l_+(A)}^{i} a_{j,l}.$$

$\square$

Consequently, we have several well-defined maps which we need so as to prove our main result of Section 3. Before we state our theorem, we proof the following helpful lemma:

Lemma 3.2.4. *Let A be an element in $B^{m,i}$, then we have*

$$\epsilon_l(A) = \max\{k \geq 0 | \tilde{e}^k_l A \neq 0\}, \quad \varphi_l(A) = \max\{k \geq 0 | \tilde{f}^k_l A \neq 0\}.$$

Proof. As usual we proof only $\varphi_l(A) = \max\{k \geq 0 | \tilde{f}^k_l A \neq 0\}$ for $l > i$ because $l = i$ is trivial and the proof in the other cases are very similar. We will proceed by induction on $p^l_+(A)$. If $p^l_+(A) = 1$ the proof is obvious so assume that $p^l_+(A) > 1$ and let $r := \min\{w \in \mathbb{Z}_+ | p^l_+(\tilde{f}^w_l A) < p^l_+(A)\}$. Then we obtain by the definition of $p^l_+(A)$ on the one hand

$$r - 1 < \sum_{j=p^l_+(\tilde{f}^r_l A)+1}^{p^l_+(A)} a_{j,l-1} - \sum_{j=p^l_+(\tilde{f}^r_l A)}^{p^l_+(A)-1} a_{j,l}$$

and on the other hand, using the definition of $p^l_+(\tilde{f}^r_l A)$, we get

$$r \geq \sum_{j=p^l_+(\tilde{f}^r_l A)+1}^{p^l_+(A)} a_{j,l-1} - \sum_{j=p^l_+(\tilde{f}^r_l A)}^{p^l_+(A)-1} a_{j,l}.$$

Hence the above inequality is actually a equality and by the induction hypothesis we can conclude

$$\max\{k \geq 0 | \tilde{f}^k_l A \neq 0\} = r + \varphi_l(\tilde{f}^r_l A) = r + \sum_{j=1}^{p^l_+(\tilde{f}^r_l A)} a_{j,l-1} - \sum_{j=1}^{p^l_+(\tilde{f}^r_l A)-1} a_{j,l} = \varphi_l(A).$$

$\square$

Now we are in position to state and to proof one of our main results in this paper, namely:

Theorem 3.2.1. The polytope $B^{m,i}$ together with the maps given by (3.1), (3.2), (3.3), (3.9) and (3.10) becomes an abstract semiregular crystal.

Proof. The idea of the proof is to check step by step the properties (1)-(7) described in Definition 3.1, whereby (2),(3) and (7) are obvious and (1),(4),(5) and the semiregularity are obvious with Lemma 3.2.1 and Lemma 3.2.4. Thus it remains to prove the correctness of condition (6), whereby we verify as usual the statement only for $l > i$:

- $\tilde{f}_l A = A'$ if and only if $\tilde{e}_l A' = A$ for $A, A' \in B^{m,i}$

Let $p_+^l(A)$ as in (3.4) and let $q_+^l(\tilde{f}_lA)$ as in (3.5). The assumption $q_+^l(\tilde{f}_lA) > p_+^l(A)$ gives

$$\sum_{j=1}^{q_+^l(\tilde{f}_lA)} a_{j,l-1} + \sum_{j=q_+^l(\tilde{f}_lA)}^{i} a_{j,l} > \sum_{j=1}^{p_+^l(A)} a_{j,l-1} + \sum_{j=p_+^l(A)}^{i} a_{j,l},$$

which is a contradiction to the maximality. The assumption $q_+^l(\tilde{f}_lA) < p_+^l(A)$ gives

$$\sum_{j=1}^{q_+^l(\tilde{f}_lA)} a_{j,l-1} + \sum_{j=q_+^l(\tilde{f}_lA)}^{i} a_{j,l} \geq \sum_{j=1}^{p_+^l(A)} a_{j,l-1} + \sum_{j=p_+^l(A)}^{i} a_{j,l},$$

which is a contradiction to the minimality of $p_+^l(A)$. Now suppose similar as above that $p_+^l(\tilde{e}_lA) > q_+^l(A)$, then

$$\sum_{j=1}^{p_+^l(\tilde{e}_lA)} a_{j,l-1} + \sum_{j=p_+^l(\tilde{e}_lA)}^{i} a_{j,l} \geq \sum_{j=1}^{q_+^l(A)} a_{j,l-1} + \sum_{j=q_+^l(A)}^{i} a_{j,l},$$

which is a contradiction to the maximality of $q_+^l(A)$ and $p_+^l(\tilde{e}_lA) < q_+^l(A)$ provides analogously

$$\sum_{j=1}^{p_+^l(\tilde{e}_lA)} a_{j,l-1} + \sum_{j=p_+^l(\tilde{e}_lA)}^{i} a_{j,l} > \sum_{j=1}^{q_+^l(A)} a_{j,l-1} + \sum_{j=q_+^l(A)}^{i} a_{j,l},$$

which is a contradiction to the maximality. Hence $p_+^l(A) = q_+^l(\tilde{f}_lA)$ and $q_+^l(A) = p_+^l(\tilde{e}_lA)$, which proves the theorem. □

Corollary 3.2.1. *The crystal $B^{m,i}$ is connected.*

Proof. It is immediate that for $A \in B^{m,i}$ with $\tilde{e}_lA = 0$ for all $l \in I$, we must have $a_{p,q} = 0$ for all $1 \leq p \leq i \leq q \leq n$. Hence, for arbitrary elements A and B there exists always a couloured path from A to B. □

4. Tensor products and Nakajima monomials

In this section, we want to recall tensor products of crystals and investigate the action of Kashiwara operators on tensor products. Furthermore, we want to introduce a crystal, the set of all Nakajima monomials, such that we can think of $B(\lambda)$, where λ is a dominant integral A_n weight, as a set of certain monomials. This theory is discovered by Nakajima [18], and generalized by Kashiwara [13] and will be important in the following sections.

4.1. **Tensor product of crystals.** Suppose that we have two abstract crystals B_1, B_2 in the sense of Definition 3.1, then we can construct a new crystal which is as a set nothing but $B_1 \times B_2$. This crystal is denoted by $B_1 \otimes B_2$ and the Kashiwara operators are given as follows:

$$\tilde{f}_l(b_1 \otimes b_2) = \begin{cases} (\tilde{f}_lb_1) \otimes b_2, \text{ if } \varphi_l(b_1) > \epsilon_l(b_2) \\ b_1 \otimes (\tilde{f}_lb_2), \text{ if } \varphi_l(b_1) \leq \epsilon_l(b_2) \end{cases}$$

$$\tilde{e}_l(b_1 \otimes b_2) = \begin{cases} (\tilde{e}_l b_1) \otimes b_2, & \text{if } \varphi_l(b_1) \geq \epsilon_l(b_2) \\ b_1 \otimes (\tilde{e}_l b_2), & \text{if } \varphi_l(b_1) < \epsilon_l(b_2). \end{cases}$$

Further, one can describe explicitely the maps wt,φ_l and ϵ_l on $B_1 \otimes B_2$, namely:

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$$

$$\varphi_l(b_1 \otimes b_2) = \max\{\varphi_l(b_2), \varphi_l(b_1) + \varphi_l(b_2) - \epsilon_l(b_2)\}$$

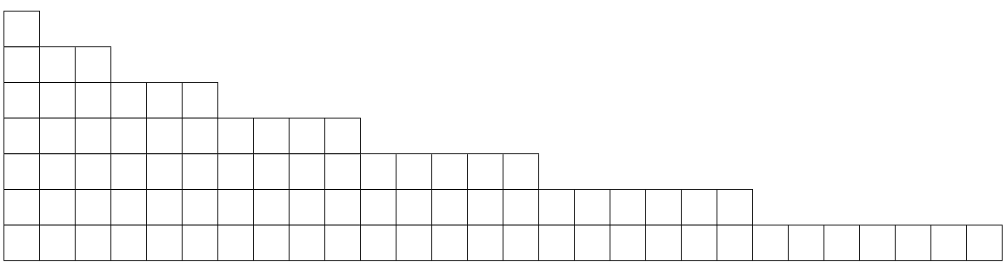
$$\epsilon_l(b_1 \otimes b_2) = \max\{\epsilon_l(b_1), \epsilon_l(b_1) + \epsilon_l(b_2) - \varphi_l(b_1)\}.$$

A very important point in representation theory is to determine crystal bases for irreducible modules over quantum algebras. This leads to many combinatorial models, discovered in a series of papers ([14],[15],[17]). Since this paper has the goal of determining the crystal graph of KR modules, we will only mention in the following remark how we could compute crystal bases for $V(\lambda)$ using tensor products of polytopes.

Remark 4.1.1. Remember that we have already a crystal structure on the sets $B^{m,i}$ for all $i = 1, \cdots, n$ and by the above considerations also on

$$B^{m_1,1} \otimes \cdots \otimes B^{m_n,n}.$$

It means we are considering patterns



with the following crystal structure: let us take such a pattern $\mathbf{A} := (A_1, \cdots, A_n)$ and fix $l \in I$, then we assign to $\mathbf{A}$ a sequence of - 's followed by a sequence of +'s

$$\mathrm{seq}(\mathbf{A}) := (\underbrace{-, \cdots, -}_{\epsilon_l(A_1)}, \underbrace{+, \cdots, +}_{\varphi_l(A_1)}, \underbrace{-, \cdots, -}_{\epsilon_l(A_2)}, \underbrace{+, \cdots, +}_{\varphi_l(A_2)}, \cdots, \underbrace{-, \cdots, -}_{\epsilon_l(A_n)}, \underbrace{+, \cdots, +}_{\varphi_l(A_n)})$$

and cancel out all $(+,-)$-pairs to obtain the so called ℓ-signature

$$\ell\text{-sgn}(\mathbf{A}) := (-, \cdots, -, +, \cdots, +). \tag{4.1}$$

Using the ℓ-signature ℓ-sgn$(\mathbf{A})$ of $\mathbf{A}$, we can describe the Kashiwara operators. Assume the left-most $+$ in the ℓ-signature corresponds to A_i and the right-most $-$ corresponds to A_j, then

$$\tilde{f}_l(A_1 A_2 \ldots A_n) = A_1 \ldots A_{i-1}(\tilde{f}_l A_i) A_{i+1} \ldots A_n$$

$$\tilde{e}_l(A_1 A_2 \ldots A_n) = A_1 \ldots A_{j-1}(\tilde{e}_l A_j) A_{j+1} \ldots A_n.$$

Hence we have defined a new crystal, and with Theorem 5.2.1 it is clear from standard arguments that the connected component of 0 (all boxes filled with 0) is isomorphic to the crystal $B(\lambda)$.

4.2. **Nakajima monomials.** For $i \in I$ and $n \in \mathbb{Z}$ we consider monomials in the variables $Y_i(n)$, i.e. we obtain the set of Nakajima monomials $\mathcal{M}$ as follows:

$$\mathcal{M} := \{ \prod_{i\in I, n\in\mathbb{Z}} Y_i(n)^{y_i(n)} | y_i(n) \in \mathbb{Z} \text{ vanish except for finitely many } (i,n)\}$$

With the goal to define the crystal structure on $\mathcal{M}$, we take some integers $c = (c_{i,j})_{i\neq j}$ such that $c_{i,j} + c_{j,i} = 1$. Let now $M = \prod_{i\in I, n\in\mathbb{Z}} Y_i(n)^{y_i(n)}$ be an arbitrary monomial in $\mathcal{M}$ and $l \in I$, then we set:

$$\mathrm{wt}(M) = \sum_i (\sum_n y_i(n))\omega_i$$

$$\varphi_l(M) = \max\{\sum_{k\leq n} y_l(k) | n \in \mathbb{Z}\}$$

$$\epsilon_l(M) = \max\{-\sum_{k>n} y_l(k) | n \in \mathbb{Z}\}$$

and

$$n_f^l = \min\{n | \varphi_l(M) = \sum_{k\leq n} y_l(k)\}$$

$$n_e^l = \max\{n | \epsilon_l(M) = -\sum_{k>n} y_l(k)\}$$

The Kashiwara operators are defined as follows:

$$\tilde{f}_l M = \begin{cases} A_l(n_f^l)^{-1}M, \text{ if } \varphi_l(M) > 0 \\ 0, \text{ if } \varphi_l(M) = 0 \end{cases}$$

$$\tilde{e}_l M = \begin{cases} A_l(n_e^l)M, \text{ if } \epsilon_l(M) > 0 \\ 0, \text{ if } \epsilon_l(M) = 0, \end{cases}$$

whereby

$$A_l(n) := Y_l(n)Y_l(n+1)\prod_{i\neq l} Y_i(n + c_{i,l})^{\langle \alpha_i^\vee, \alpha_l\rangle}.$$

The following two results due to Kashiwara [13] are essential for the process of this paper:

Proposition 4.2.1. *With the maps* wt, $\varphi_l, \epsilon_l, \tilde{f}_l, \tilde{e}_l$, $l \in I$, *the set* $\mathcal{M}$ *becomes a semiregular crystal.*

Remark 4.2.1. *A priori* the crystal structure depends on c, hence we will denote this crystal by $\mathcal{M}_c$. But it is easy to see that the isomorphism class of $\mathcal{M}_c$ does not depend on this choice. In the literature c is often chosen as

$$c_{i,j} = \begin{cases} 0, \text{ if } i > j \\ 1, \text{ else} \end{cases} \qquad \textit{or} \qquad c_{i,j} = \begin{cases} 0, \text{ if } i < j \\ 1, \text{ else.} \end{cases}$$

Proposition 4.2.2. *Let* M *be a monomial in* $\mathcal{M}$, *such that* $\tilde{e}_l M = 0$ *for all* $l \in I$. *Then the connected component of* $\mathcal{M}$ *containing* M *is isomorphic to* $B(\mathrm{wt}(M))$.

5. Stembridge axioms and isomorphism of crystals

By using the description of crystal graphs by certain monomials and maps on these, we want to show that our set $B^{m,i}$ satisfies the so called Stembridge axioms stated in [24]. These axioms give a local characterization of simply-laced crystals which are helpful if one could not find a global isomorphism. Since we could not find an isomorphism from $B^{m,i}$ to the set of all monomials describing $B(m\omega_i)$, we will identify only certain A_2-crystals. Let us first recall a slightly modified result from [24].

5.1. **Stembridge axioms.** The basic idea of the following proposition is to give a simple set of local axioms to characterize the set of crystals of representations in the class of all crystals. In particular, with these axioms one can determine whether or not a crystal is the crystal of a representation.

Proposition 5.1.1. *Let $\mathfrak{g}$ be a simply-laced Lie algebra and B be a connected crystal graph, such that the following conditions are satisfied:*

1. *If $\tilde{e}_l b$ is defined, then $\epsilon_j(\tilde{e}_l b) \geq \epsilon_j(b)$ and $\varphi_j(\tilde{e}_l b) \leq \varphi_j(b)$ for all $j \neq l$.*
2. *If $\tilde{e}_l, \tilde{e}_j b$ are defined and $\epsilon_j(\tilde{e}_l b) = \epsilon_j(b)$, then $\tilde{e}_l\tilde{e}_j b = \tilde{e}_j\tilde{e}_l b$ and $\varphi_l(b^{'}) = \varphi_l(\tilde{f}_j b^{'})$, where $b^{'} = \tilde{e}_l\tilde{e}_j b = \tilde{e}_j\tilde{e}_l b$.*
3. *If $\tilde{e}_l, \tilde{e}_j b$ are defined and $\epsilon_j(b) - \epsilon_j(\tilde{e}_l b) = \epsilon_l(b) - \epsilon_l(\tilde{e}_j b) = -1$, then $\tilde{e}_l\tilde{e}_j^2\tilde{e}_l b = \tilde{e}_j\tilde{e}_l^2\tilde{e}_j b$ and $\varphi_l(b^{'}) - \varphi_l(\tilde{f}_j b^{'}) = \varphi_j(b^{'}) - \varphi_j(\tilde{f}_l b^{'}) = -1$, where $b^{'} = \tilde{e}_l\tilde{e}_j^2\tilde{e}_l b = \tilde{e}_j\tilde{e}_l^2\tilde{e}_j b$.*
4. *If $\tilde{f}_l, \tilde{f}_j b$ are defined and $\varphi_j(\tilde{f}_l b) = \varphi_j(b)$, then $\tilde{f}_l\tilde{f}_j b = \tilde{f}_j\tilde{f}_l b$ and $\epsilon_l(b^{'}) = \epsilon_l(\tilde{e}_j b^{'})$, where $b^{'} = \tilde{f}_l\tilde{f}_j b = \tilde{f}_j\tilde{f}_l b$.*
5. *If $\tilde{f}_l, \tilde{f}_j b$ are defined and $\varphi_j(b) - \varphi_j(\tilde{f}_l b) = \varphi_l(b) - \varphi_l(\tilde{f}_j b) = -1$, then $\tilde{f}_l\tilde{f}_j^2\tilde{f}_l b = \tilde{f}_j\tilde{f}_l^2\tilde{f}_j b$ and $\epsilon_l(b^{'}) - \epsilon_l(\tilde{e}_j b^{'}) = \epsilon_j(b^{'}) - \epsilon_j(\tilde{e}_l b^{'}) = -1$, where $b^{'} = \tilde{f}_l\tilde{f}_j^2\tilde{f}_l b = \tilde{f}_j\tilde{f}_l^2\tilde{f}_j b$.*

Then B is a crystal graph induced by a representation.

5.2. **Isomorphism of A_2 crystals.** We want to make full use of the above mentioned result to prove the following main theorem of this section:

Theorem 5.2.1. We have an isomorphism of crystals
$$B^{m,i} \cong B(m\omega_i).$$

Proof. Assume $|j - r| = 1$, i.e. $r = j + 1$ and let A be an arbitrary element in $B^{m,i}$. Since $B^{m,i}$ is a connected crystal we cancel all arrows with colour $s \neq j, j+1$ and denote the remaining $(j, j+1)$-connected graph containing A by $Z_{(j,j+1)}(A)$. We define a map $\Psi : Z_{(j,j+1)}(A) \cup \{0\} \longrightarrow \mathcal{M} \cup \{0\}$ by mapping 0 to 0 and B to:

$$\begin{cases} Y_1(i)^{m-\sum_{s=i}^n b_{i,n}} \prod_{k=1}^i Y_2(k)^{b_{k,i}} Y_2(k+1)^{-b_{k,i+1}} \prod_{k=1}^i Y_1(k+1)^{-b_{k,i}}, \text{ if } j = i \\ \\ Y_2(n-1)^{m-\sum_{s=1}^i b_{s,i}} \prod_{k=i}^n Y_1(k)^{b_{i,n+i-k}} Y_1(k+1)^{-b_{i-1,n+i-k}} \prod_{k=i}^n Y_2(k)^{-b_{i,n+i-k}}, \text{ if } j = i-1 \\ \\ \prod_{k=1}^i Y_1(k)^{b_{k,j-1}} Y_1(k+1)^{-b_{k,j}} \prod_{k=1}^i Y_2(k)^{b_{k,j}} Y_2(k+1)^{-b_{k,j+1}}, \text{ if } j > i \\ \\ \prod_{k=i}^n Y_1(k)^{b_{j+1,n+i-k}} Y_1(k+1)^{-b_{j,n+i-k}} \prod_{k=i}^n Y_2(k)^{b_{j+2,n+i-k}} Y_2(k+1)^{-b_{j+1,n+i-k}}, \text{ if } j+1 < i. \end{cases}$$

We are claiming that Ψ is an $A_2 \cong \mathfrak{sl}_3(j,j+1)$ crystal isomorphism. Note that Ψ has the following properties:

• $\mathrm{wt}(\Psi(B)) = \mathrm{wt}(B)$

• $\varphi_l(\Psi(B)) = \varphi_l(B), \epsilon_l(\Psi(B)) = \epsilon_l(B)$. The proof is a case-by-case consideration, for instance if $l = j > i$, then

$$\varphi_l(\Psi(B)) = \max\{\sum_{s=1}^{r} b_{s,j-1} - \sum_{s=1}^{r-1} b_{s,j} | 1 \leq r \leq i\} = \varphi_l(B),$$

because the maximum occurs at least at $r = p^l_+(B)$.

• Ψ commutes with the Kashiwara operators: by Lemma 3.2.4 and the above computations we can conclude that Ψ commutes with all $\tilde{f}_l, \tilde{e}_l$ acting by zero on B. So assume $\tilde{f}_l B = \hat{B}$ and $\tilde{e}_l B = \hat{B}$ respectively. Our aim is to prove $\Psi(\hat{B}) = A_l(n^l_f)^{-1}\Psi(B)$, $\Psi(\hat{B}) = A_l(n^l_e)\Psi(B)$. This is again a case-by-case consideration, for instance if $l = j+1 < i$ we have $n^l_f = n+i-p^l_-(B)$ and

$$\Psi(\hat{B}) = \prod_{k=i}^{n} Y_1(k)^{\hat{b}_{j+1,n+i-k}} Y_1(k+1)^{-\hat{b}_{j,n+i-k}} \prod_{k=i}^{n} Y_2(k)^{\hat{b}_{j+2,n+i-k}} Y_2(k+1)^{-\hat{b}_{j+1,n+i-k}}$$

$$= \prod_{k=i, k\neq n^l_f}^{n} Y_1(k)^{b_{j+1,n+i-k}} \prod_{k=i}^{n} Y_1(k+1)^{-b_{j,n+i-k}} \prod_{k=i, k\neq n^l_f}^{n} Y_2(k)^{b_{j+2,n+i-k}} Y_2(k+1)^{-b_{j+1,n+i-k}}$$

$$\times Y_1(n^l_f)^{b_{j+1,p^l_-(B)}+1} Y_2(n^l_f)^{b_{j+2,p^l_-(B)}-1} Y_2(n^l_f+1)^{-b_{j+1,p^l_-(B)}-1}$$

$$= Y_1(n^l_f) Y_2(n^l_f)^{-1} Y_2(n^l_f+1)^{-1}\Psi(B) = A_l(n^l_f)^{-1}\Psi(B).$$

Hence Ψ is a strict crystal morphism.

• Ψ is bijective: since $Z_{(j,j+1)}(A)$ is connected and Ψ is a crystal morphism we get that $\mathcal{I}m(\Psi)$ is connected and contains at least, and therefore by Proposition 4.2.2 one highest weight monomial, say of weight μ. So the image is isomorphic, again by Proposition 4.2.2 to the $\mathfrak{sl}_3(j,j+1)$ crystal $B(\mu)$. Let $T \in Z_{(j,j+1)}(A)$ be a highest weight element, such that $\tilde{e}_{i_1} \cdots \tilde{e}_{i_s} A = T$, then the restriction of Ψ to $G = \{\tilde{f}_{j_1} \cdots \tilde{f}_{j_s} T | j_1, \cdots, j_s \in \{j, j+1\}\}$ is an isomorphism. However, we have $Z_{(j,j+1)}(A) = G$.

Therefore, we can conclude that A satisfies the Stembridge axioms for all $j, r \in I$ with $|j-r| = 1$, whereby the other relations are easily verified. □

Summerizing we have defined a set $B^{m,i}$ which is by Theorem 3.2.1 a crystal and by Theorem 5.2.1 actually the crystal $B(m\omega_i)$. In the following we want to collect some known facts about KR-crystals and define the Kashiwara operators $\tilde{f}_0, \tilde{e}_0$ on our underlying polytope.

6. The promotion operator

The existence of KR-crystals of type $A^{(1)}_n$ was shown in [11] and a combinatorial description was provided in [23] and [16], where the affine crystal structure in the latter work is given without using the promotion operator. The existence of KR-crystals for non-exceptional types can be found in [19],[20] and further a combinatorial description is provided in [8].

Summerizing the results for type $A_n^{(1)}$, a model for KR-crystals is given by the set of all semi-standard Young tableuax of shape $\lambda = m\omega_i$ with affine Kashiwara operators

$$\tilde{f}_0 := pr^{-1} \circ \tilde{f}_1 \circ pr, \text{ and } \tilde{e}_0 := pr^{-1} \circ \tilde{e}_1 \circ pr, \tag{6.1}$$

whereby pr is the so called Schützenbergers's promotion operator [22], which is the analogue of the cyclic Dynkin diagram automorphism on the level of crystals. The promotion operator on the set of all semi-standard Young tableaux over the alphabet $1 \prec 2 \cdots \prec n+1$ can be obtained by using jeu-de-taquin. Particulary let T be a Young tableaux, then we get $pr(T)$ by removing all letters $n+1$, adding 1 to each letter in the remaining tableaux, using jeu-de-taquin to slide all letters up and finally filling the holes with 1's. Our aim now is to define the Schützenberger promotion operator on our polytope $B^{m,i}$, to obtain a polytope realization of these crystals. Before we are in position to define such a map we will first state an important result due to [1]. For simplicity we write $\text{wt}(A) = (r_1, \cdots, r_{n+1})$, if $\text{wt}(A) = \sum_{j=1}^{n+1} r_j \epsilon_j$, whereby $\epsilon_j : \mathfrak{sl}_{n+1} \longrightarrow \mathbb{C}$ is the projection on the j-th diagonal entry.

Proposition 6.0.1. *Let $\Psi : B(m\omega_i) \longrightarrow B(m\omega_i)$ be a map, such that*

(1) Ψ shifts the content, which means if $\text{wt}(A) = (r_1, \cdots, r_{n+1})$, then $\text{wt}(\psi(A)) = (r_{n+1}, r_1, \cdots, r_n)$
(2) Ψ is bijective
(3) $\Psi \circ \tilde{f}_j = \tilde{f}_{j+1} \circ \Psi$, $\Psi \circ \tilde{e}_j = \tilde{e}_{j+1} \circ \Psi$ for all $j \in \{1, \cdots, n-1\}$,

then $\Psi = pr$.

In particular, using Theorem 5.2.1, this means that it is sufficient to define a map on $B^{m,i}$ satisfying the conditions (1)-(3). The computation of this map, which we will denote already by pr, will proceed by an algorithm consisting of i steps, where each step will give us a column of $pr(A)$. We denote by $pr(a_{r,s})$ the entries of $pr(A)$ and by $\mathbf{a}_j$ and $pr(\mathbf{a_j})$ respectively the j-th column of A and j-th column of $pr(A)$ respectively for $j = 1, \cdots, i$ and mostly we assume the notation explained in Remark 3.2.1. In order to state the algorithm we denote further by $(\mathbf{a}_j)^{\geq l}$ the column obtained from $\mathbf{a}_j$ by canceling all entries between i and $l-1$. For instance if $\mathbf{a}_j$ is the j-th column of some $A \in B^{m,4}$ we have

$$(\mathbf{a}_j)^{\geq 5} = \begin{array}{|c|}\hline 5 \\ \hline 0 \\ \hline \end{array}, \text{ if } a_j = \begin{array}{|c|}\hline 3 \\ \hline 5 \\ \hline 0 \\ \hline \end{array}.$$

If $i = 1$, then it is obvious that the map pr defined by

$$pr(a_{1,j}) = \begin{cases} m - \sum_{r=1}^{n} a_{1,r}, & \text{if } j = 1 \\ a_{1,j-1}, & \text{else} \end{cases}$$

satisfies the conditions (1)-(3). So for $n \geq i \geq 2$ we will use the following algorithm to compute pr:

6.1. **Algorithm.** For given $A \in B^{m,i}$ we implement the following steps to compute $pr(A)$:

(1) Consider the $(i-1)$-th and i-th column of A and compute inductively the integers $i \leq l_1^{i-1} < l_2^{i-1} < \cdots < l_{t_{i-1}}^{i-1} = n$, whereby

$$l_j^{i-1} = q_-((\mathbf{a}_{i-1})^{>l_{j-1}^{i-1}}, (\mathbf{a}_i)^{>l_{j-1}^{i-1}})$$

and where we undestand $l_0^{i-1} = i-1$. The i-th column of $pr(A)$ is then given by:

$$pr(a_{i,r}) = \begin{cases} \epsilon(\mathbf{a}_{i-1}, \mathbf{a}_i), & \text{if } r = i \\ \epsilon((\mathbf{a}_{i-1})^{\geq r}, (\mathbf{a}_i)^{\geq r}), & \text{if } r-1 \in \{l_1^{i-1}, \cdots, l_{t_{i-1}-1}^{i-1}\} \\ a_{i,r-1}, & \text{else.} \end{cases}$$

Further define a new column

$$\widehat{a_{i-1,r}} = \begin{cases} a_{i-1,r} + a_{i,r} - \epsilon((\mathbf{a}_{i-1})^{>r}, (\mathbf{a}_i)^{>r}), & \text{if } r \in \{l_1^{i-1}, \cdots, l_{t_{i-1}-1}^{i-1}\} \\ a_{i-1,n} + a_{i,n}, & \text{if } r = n \\ a_{i-1,r}, & \text{else.} \end{cases} \tag{6.2}$$

(2) With the aim to determine the $(i-1)$-th column of $pr(A)$ repeat step (1) with the $(i-2)$-th column of A and the new defined column (6.2). Using the integers $i \leq l_1^{i-2} < l_2^{i-2} < \cdots < l_{t_{i-2}}^{i-2} = n$ compute the $(i-2)$-th column as in step (1).

(3) Repeat step (2) as long as all columns, except the first one, from $pr(A)$ are known.

(4) The first column of $pr(A)$ is given as follows:

$$pr(a_{1,r}) = \begin{cases} m - \sum_{j=i}^{n} a_{i,j} - \sum_{j=2}^{i} pr(a_{j,i}), & \text{if } r = 1 \\ \sum_{j=1}^{i} a_{j,r-1} - \sum_{j=2}^{i} pr(a_{j,r}), & \text{if } r > 1. \end{cases}$$

Remark 6.1.1.

(1) *A priori* it is not clear, why the entries in the first column are non negative integers. This will be our first step and is proven in Proposition 6.1.1.

(2) As well it is not clear, why the image of pr lies in $B^{m,i}$. With the purpose to prove the well-definedness of pr we will show first for any $A \in B^{m,i}$ that $pr \circ \tilde{e}_j A = \tilde{e}_{j+1} \circ pr(A)$ holds for $j = 1, \cdots, n-1$, where the equation can be understood by Lemma 3.2.2 as a equation independently from the knowledge where $pr(A)$ lives.

Example. i) We pick one element A (see below) in $B^{3,3}$ and follow our algorithm. In the first step we get $l_1^2 = 3 < l_2^2 = 4 < l_3^2 = 5$ which gives us the third column:

$$A = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline \bullet & \bullet & 1 \\ \hline \bullet & \bullet & 0 \\ \hline \bullet & \bullet & 0 \\ \hline \end{array}$$

The new column is given by $\begin{array}{|c|}\hline 2 \\ \hline 0 \\ \hline 0 \\ \hline \end{array}$. So following step two we get again $l_1^1 = 3 < l_2^1 = 4 < l_3^1 = 5$ and hence

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline \bullet & \bullet & 1 \\ \hline \bullet & \bullet & 0 \\ \hline \bullet & \bullet & 0 \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline \bullet & 1 & 1 \\ \hline \bullet & 2 & 0 \\ \hline \bullet & 0 & 0 \\ \hline \end{array}$$

So our last step gives us

$$pr(A) = \begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 0 \\ \hline \end{array}$$

Another example in $B^{4,3}$ is described below:

$$\begin{array}{|c|c|c|}\hline 1&1&0\\\hline 0&1&1\\\hline 0&0&0\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline \bullet&\bullet&2\\\hline \bullet&\bullet&0\\\hline \bullet&\bullet&0\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline \bullet&1&2\\\hline \bullet&0&0\\\hline \bullet&0&0\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline 0&1&2\\\hline 2&0&0\\\hline 2&0&0\\\hline\end{array}$$

ii) Now we pick an element in $B^{7,4}$, namely

$$\begin{array}{|c|c|c|c|}\hline 1&0&1&1\\\hline 0&1&3&2\\\hline 1&0&2&0\\\hline\end{array}$$

and get $l_1^3 = 5 < l_2^3 = 6$, $l_1^2 = 4 < l_2^2 = 5 < l_3^2 = 6$ and $l_1^1 = 4 < l_2^1 = 5 < l_3^1 = 6$. The new columns after the first step and second step respectively are given by

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array} \text{ and } \begin{array}{|c|}\hline 0\\\hline 4\\\hline 2\\\hline\end{array} \text{ respectively.}$$

Thus, step by step we obtain the columns of $pr(A)$:

$$\begin{array}{|c|c|c|c|}\hline 1&0&1&1\\\hline 0&1&3&2\\\hline 1&0&2&0\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\hline \bullet&\bullet&\bullet&3\\\hline \bullet&\bullet&\bullet&1\\\hline \bullet&\bullet&\bullet&2\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\hline \bullet&\bullet&0&3\\\hline \bullet&\bullet&1&1\\\hline \bullet&\bullet&0&2\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\hline \bullet&1&0&3\\\hline \bullet&0&1&1\\\hline \bullet&1&0&2\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 0&1&0&3\\\hline 1&0&1&1\\\hline 3&1&0&2\\\hline\end{array}$$

Proposition 6.1.1. *For any $A \in B^{m,i}$ the first column of $pr(A)$ consists of non-negative integers and moreover*

$$m - \sum_{r=i}^{n} pr(a_{1,r}) = \sum_{r=1}^{i} a_{r,n}.$$

Proof. We will prove this statement by induction on i. Assume $i = 2$ and let $l_1^1 < l_2^1 < \cdots < l_{t_1}^1$ be the integers decribed in the algortihm. The first entry in the first column is exactly

$$m - \sum_{r=2}^{n} a_{2,r} - \epsilon(\mathbf{a}_1, \mathbf{a}_2) = m - \sum_{r=2}^{l_1^1} a_{1,r} - \sum_{r=l_1^1}^{n} a_{2,r} \geq 0,$$

and the other entries in the first column of $pr(A)$ are either of the form $a_{1,k}$ or of the form $a_{1,l_k^1} + a_{2,l_k^1} - \epsilon((\mathbf{a}_1)^{>l_k^1}, (\mathbf{a}_2)^{>l_k^1})$, for some k. However, by the definition of l_k^1 we know that $a_{2,l_k^1} + \cdots + a_{2,l_{k+1}^1-1} \geq a_{1,l_k^1+1} + \cdots + a_{1,l_{k+1}^1}$, which gives us $a_{2,l_k^1} \geq \epsilon((\mathbf{a}_1)^{>l_k^1}, (\mathbf{a}_2)^{>l_k^1})$. Thus all entries in the first column are non negative integers. Now we will show that for all $j \in \{0, \cdots, t_1 - 1\}$

$$m - \sum_{r=2}^{n} pr(a_{1,r}) = \sum_{r=l_j^1+1}^{l_{j+1}^1} a_{1,r} + \sum_{r=l_{j+1}^1}^{n} a_{2,r} - \sum_{r=l_j^1+2}^{n} pr(a_{1,r})$$

holds and proceed by upward induction on j. The $j = 0$ case is obvious and assuming that $j > 0$ we obtain by using the induction hypothesis

$$m - \sum_{r=2}^{n} pr(a_{1,r}) = \sum_{r=l^1_{j-1}+1}^{l^1_j} a_{1,r} + \sum_{r=l^1_j}^{n} a_{2,r} - \sum_{r=l^1_{j-1}+2}^{n} pr(a_{1,r})$$

$$= \sum_{r=l^1_{j-1}+1}^{l^1_j} a_{1,r} + \sum_{r=l^1_j}^{n} a_{2,r} - \sum_{r=l^1_{j-1}+2}^{l^1_j+1} pr(a_{1,r}) - \sum_{r=l^1_j+2}^{n} pr(a_{1,r})$$

$$= a_{1,l^1_j} + \sum_{r=l^1_j}^{n} a_{2,r} - pr(a_{1,l^1_j+1}) - \sum_{r=l^1_j+2}^{n} pr(a_{1,r})$$

$$= \sum_{r=l^1_j+1}^{l^1_{j+1}} a_{1,r} + \sum_{r=l^1_{j+1}}^{n} a_{2,r} - \sum_{r=l^1_j+2}^{n} pr(a_{1,r}),$$

which finishes the induction. According to this we complete the initial step, since

$$m - \sum_{r=2}^{n} pr(a_{1,r}) = \sum_{r=l^1_{t_1-1}+1}^{l^1_{t_1}} a_{1,r} + \sum_{r=l^1_{t_1}}^{n} a_{2,r} - \sum_{r=l^1_{t_1-1}+2}^{n} pr(a_{1,r}) = a_{1,n} + a_{2,n}.$$

Now let $i > 2$ and consider an element $B \in B^{m,i-1}$, constructed as follows. The first $(i-2)$-columns of B are the same as the ones from A and the $(i-1)$-th column is precisely the new obtained column in (6.2), which we get if we apply step (1) of the algorithm to A. In other words, we erase the i-th column of A and replace the $(i-1)$-th column of A by

$$b_{i-1,r} = \begin{cases} a_{i-1,r} + a_{i,r} - \epsilon((\mathbf{a}_{i-1})^{>r}, (\mathbf{a}_i)^{>r}), & \text{if } r \in \{l^{i-1}_1, \cdots, l^{i-1}_{t_{i-1}-1}\} \\ a_{i-1,n} + a_{i,n}, & \text{if } r = n \\ a_{i-1,r}, & \text{else} \end{cases}$$

and obtain B. One can easily check that $B \in B^{m,i-1}$, where $B^{m,i-1}$ is the polytope associated to the Lie algebra A_{n-1}. Particulary we claim the following: if we glue the i-th column of $pr(A)$ to $pr(B)$ the resulting element is again $pr(A)$, i.e.

$$pr(A) = pr(B)pr(\mathbf{a_i}).$$

By the definition of the algorithm the claim is obvious for all columns except the first one. Because of that let $\boxed{b_i}\boxed{\ldots}\boxed{b_n}$ be the transpose of the first column of $pr(B)$ and $\boxed{a_i}\boxed{\ldots}\boxed{a_n}$ be the transpose of the first column of $pr(A)$. We would like to start with the evidence of $b_i = a_i$. We have

$$b_i = m - \sum_{s=i}^{l^{i-1}_1} a_{i-1,s} - \sum_{s=l^{i-1}_1}^{n} a_{i,s} - \sum_{s=2}^{i-1} pr(a_{s,i}),$$

$$a_i = m - \sum_{s=i}^{n} a_{i,s} - \sum_{s=2}^{i} pr(a_{s,i}),$$

since the sum over all entries of the $(i-1)$-th column of B equals to $\sum_{s=i}^{l_1^{i-1}} a_{i-1,s}+\sum_{s=l_1^{i-1}}^{n} a_{i,s}$. However, this implies

$$b_i - a_i = pr(a_{i,i}) + \sum_{s=i}^{l_1^{i-1}-1} a_{i,s} - \sum_{s=i}^{l_1^{i-1}} a_{i-1,s} = pr(a_{i,i}) - \epsilon(\mathbf{a}_{i-1}, \mathbf{a}_i) = 0.$$

If $r > i$ we have

$$b_r = \sum_{s=1}^{i-2} a_{s,r-1} - \sum_{s=2}^{i-1} pr(a_{s,r}) + \begin{cases} a_{i-1,r-1} + a_{i,r-1} - \epsilon((\mathbf{a}_{i-1})^{\geq r}, (\mathbf{a}_i)^{\geq r}) \\ a_{i-1,r-1} \end{cases},$$

$$a_r = \sum_{s=1}^{i} a_{s,r-1} - \sum_{s=2}^{i} pr(a_{s,r}) = \sum_{s=1}^{i} a_{s,r-1} - \sum_{s=2}^{i-1} pr(a_{s,r}) - \begin{cases} \epsilon((\mathbf{a}_{i-1})^{\geq r}, (\mathbf{a}_i)^{\geq r}) \\ a_{i,r-1} \end{cases},$$

and thus the difference is once more zero. So by induction we can assume that the first row of $pr(B)$ consists of non-negative integers, and hence by our claim the first row of $pr(A)$ as well. Furthermore the sum over the entries in the last row of B coincides with the sum over the entries in the last row of A and thus

$$m - \sum_{r=i}^{n} pr(a_{1,r}) = \sum_{r=1}^{i} a_{r,n}.$$

□

At this point we take Lemma 3.2.2 and Remark 6.1.1 (2) up and emphasize that the action of $\tilde{f}_i$ on $pr(A)$, $A \in B^{m,i}$, is on one condition in the sense of the following lemma independent from the fact where $pr(A)$ lives. We will need this result several times in the remaining proofs.

Lemma 6.1.1. *Suppose that $A \in B^{m,i}$, $pr(A) \in B^{s,i}$ for some $s \geq m$ and $\varphi_{i-1}(A) \neq 0$, then ${}_m\tilde{f}_i$ acts on $pr(A)$ and therefore ${}_s\tilde{f}_i pr(A) = {}_m\tilde{f}_i pr(A)$.*

Proof. The sum over all entries in the i-th column of $pr(A)$ equals to the sum over all entries in the $(i-1)$-th column of A and that is why

$$\begin{aligned}\sum_{r=1}^{i-1} pr(a_{1,r}) + \sum_{r=i}^{n} pr(a_{i,r}) &= m - \langle h_{i-1}, \mathrm{wt}(A)\rangle - \epsilon_{i-1}(A) \\ &= m - \varphi_{i-1}(A) < m.\end{aligned}$$

□

6.2. **Main proofs.** This section is dedicated to the verification of the conditions (1)-(3). Initially we remark that the cases $j = i-1$ and $j = i$ in part (3) of Proposition 6.0.1 will be considered separately, which is the aim of the next proposition:

Proposition 6.2.1. *For $j = i-1, i$ and all A in $B^{m,i}$ we have*

$$pr(\tilde{e}_j A) = \tilde{e}_{j+1} pr(A).$$

Proof. We presume $j = i-1$ and $\tilde{e}_{i-1}A \neq 0$, since the condition $\tilde{e}_{i-1}A = 0$ forces $\epsilon_{i-1}(A) = pr(a_{i,i}) = \epsilon_i(pr(A)) = 0$. So let $l_1, \cdots, l_t$ be the integers from step (1) if we apply the algorithm to A and let $l'_1, \cdots, l'_{t'}$ be the integers obtained from step (1) if we apply our algorithm to $\tilde{e}_{i-1}A$. If $l'_1 = l_1$, then the algorithm gives us immediately $\tilde{e}_i pr(A) = pr(\tilde{e}_{i-1}A)$. So suppose that $l_1 > l'_1$ and let d maximal such that $l'_d < l_1$. Then, using the definition of l_1, we get on the one hand $l'_{d+1} = l_1$ and on the other hand

$$a_{i-1,l'_d+1} + \cdots + a_{i-1,l_1} - 1 = a_{i,l'_d} + \cdots + a_{i,l_1-1},$$

which means that $pr(\tilde{e}_{i-1}A)$ would not change if we skip l'_d. By repeating these arguments we can get rid of all l'_d such that $l'_d < l_1$ and consequently we can calculate $pr(\tilde{e}_{i-1}A)$ by using the sequence $l'_{d+1} = l_1 < l'_{d+2} = l_2 < \cdots < l'_{t'} = l_t$. To be more accurate we can conclude $\tilde{e}_i pr(A) = pr(\tilde{e}_{i-1}A)$.
Now let $j = i$ and in that additional separated case we will prove the required equation by induction on i. For the initial step we assume that $i = 2$ and investigate the first two rows of $pr(A)$ where these are of one of the two following forms:

$$\begin{array}{|c|c|}\hline \bullet & \epsilon_1(A) \\ \hline a_{1,2} & a_{2,2} \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline \bullet & a_{1,2} \\ \hline y & pr(a_{2,3}) \\ \hline \end{array} \quad \text{with } y = a_{1,2} + a_{2,2} - pr(a_{2,3}) > a_{1,2},$$

whereas the first case appears if and only if either $q^1_-(A) > i$ or $q^1_-(A) = i$ and $a_{2,2} = pr(a_{2,3})$. In that case, since $\epsilon_1(A) \geq a_{1,2}$, we have $q^3_+(A) = 2$ which means among other things that $\epsilon_3(pr(A)) = 0$ if $\epsilon_2(A) = a_{2,2} = 0$. Furthermore, if $\tilde{e}_2 A \neq 0$, we actually have $q^1_-(\tilde{e}_2 A) > i$ provided $q^1_-(A) > i$ and $q^1_-(\tilde{e}_2 A) = q_-((\mathbf{a}_1)^{>i}, (\mathbf{a}_2)^{>i})$ provided $q^1_-(A) = i$ and $a_{2,2} = pr(a_{2,3})$. Thus $pr(\tilde{e}_2 A)$ arises from $pr(A)$ by replacing $\epsilon_1(A)$ by $\epsilon_1(A)+1$ and $a_{2,2}$ by $a_{2,2}-1$, which proves the claim in that case. Otherwise the second case appears and there, because of $a_{2,2} > pr(a_{2,3})$, we have $\tilde{e}_2 A \neq 0$, $q^3_+(A) = 1$ and $q^1_-(\tilde{e}_2 A) = i$. So the algorthm provides us the first two rows of $pr(\tilde{e}_2 A)$:

$$\begin{array}{|c|c|}\hline \bullet + 1 & a_{1,2} \\ \hline y-1 & pr(a_{2,3}) \\ \hline \end{array},$$

where the remaining entries coincide. As a consequence we get in both cases $\tilde{e}_3 pr(A) = pr(\tilde{e}_2 A)$ so that we can devote our attention to the induction step by observing the element B from the proof of Proposition 6.1.1. Suppose first that $\tilde{e}_i A \neq 0$ and let B_{e_i} be the element obtained by same construction out of $\tilde{e}_i A$. We remember that the connection between $pr(A)$ and $pr(B)$ was $pr(A) = pr(B)pr(\mathbf{a}_i)$. By induction we can conclude among other things

$$\epsilon_i(pr(B)) = \epsilon_{i-1}(B) = \begin{cases} a_{i-1,i} + a_{i,i} - \epsilon((\mathbf{a}_{i-1})^{>i}, (\mathbf{a_i})^{>i}) & \text{if } q^{i-1}_-(A) = q^{i-1}_-(\tilde{e}_i A) = i, \\ a_{i-1,i}, & \text{else.} \end{cases}$$

The first opportunity forces $pr(a_{i,i}) = a_{i-1,i}$, $pr(a_{i,i+1}) = \epsilon((\mathbf{a}_{i-1})^{>i}, (\mathbf{a_i})^{>i}) < a_{i,i}$ where the second one forces $pr(a_{i,i}) = \epsilon_{i-1}(A) \geq a_{i-1,i}$, $pr(a_{i,i+1}) = a_{i,i}$. Moreover we have

$q_+^{i+1}(pr(A)) \in \{i, q_+^i(pr(B))\}$ and further we claim the following:

$$q_+^{i+1}(pr(A)) = q_+^i(pr(B)) \text{ if and only if } q_-^{i-1}(A) = q_-^{i-1}(\tilde{e}_i A) = i \tag{6.3}$$

Proof of (6.3): We start by supposing $q_-^{i-1}(A) = q_-^{i-1}(\tilde{e}_i A) = i$ and get

$$\epsilon_{i+1}(pr(A)) \geq \epsilon_i(pr(B)) - pr(a_{i,i}) + pr(a_{i,i+1}) = a_{i,i} > pr(a_{i,i+1}),$$

which implies $q_+^{i+1}(pr(A)) \neq i$. For the converse direction let $q_-^{i-1}(A) = q_-^{i-1}(\tilde{e}_i A) = i$ be incorrect, then another easy estimation

$$\epsilon_i(pr(B)) - pr(a_{i,i}) + pr(a_{i,i+1}) = a_{i-1,i} - \epsilon_{i-1}(A) + a_{i,i} \leq a_{i,i}$$

implies $q_+^{i+1}(pr(A)) = i$ and thus (6.3).
Assume firstly that $q_-^{i-1}(A) = q_-^{i-1}(\tilde{e}_i A) = i$ which implies $B_{e_i} = \tilde{e}_{i-1}B$ and that the i-th row of $pr(\tilde{e}_i A)$ coincides with the i-th row of $pr(A)$. According to (6.3) the gluing process commutes with the Kashiwara action. To be more precise

$$pr(\tilde{e}_i A) = pr(B_{e_i})pr(\mathbf{a}_i) = (\tilde{e}_i pr(B))pr(\mathbf{a}_i) = \tilde{e}_{i+1}(pr(B)pr(\mathbf{a}_i)) = \tilde{e}_{i+1}pr(A).$$

Lastly we assume that $q_-^{i-1}(A) = q_-^{i-1}(\tilde{e}_i A) = i$ is not fulfilled which implies immediately that $pr(\tilde{e}_i A)$ arises from $pr(A)$ by replacing $pr(a_{i,i})$ by $pr(a_{i,i}) + 1$ and $pr(a_{i,i+1})$ by $pr(a_{i,i+1}) - 1$. After that we finished our proof for all A with $\tilde{e}_i A \neq 0$, since (6.3) implies exactly $q_+^{i+1}(pr(A)) = i$. Although the ideas of the proof of the remaining case $\tilde{e}_i A = a_{i,i} = 0$ are similar we will give it nevertheless for completeness. By the reason of $0 = a_{i,i} \geq \epsilon((\mathbf{a}_{i-1})^{>i}, (\mathbf{a_i})^{>i})$ we must necessarily have $\epsilon_i(B) = a_{i-1,i}, pr(a_{i,i}) = \epsilon_{i-1}(A) \geq a_{i-1,i}$, $pr(a_{i,i+1}) = 0$ and together with $\epsilon_i(pr(B)) - pr(a_{i,i}) = a_{i-1,i} - \epsilon_{i-1}(A) \leq 0$ we find $\epsilon_{i+1}(pr(A)) = 0$. □

Hereafter we consider the remaining nodes:

Proposition 6.2.2. *The map pr described in the algorithm satisfies the condition*

$$pr \circ \tilde{e}_j = \tilde{e}_{j+1} \circ pr \textit{ for all } j \in \{1, \cdots, n-1\}.$$

Proof. By Proposition 6.2.1 it is sufficient to verify the above stated equation for all $j < i-1$ and $j > i$, where we start by assuming that $j < i-1$ and for simplicity we set $q := q_-^j(A)$. The basic idea of the proof is to compare permanently $pr(A)$ with $pr(\tilde{e}_j A)$ and reduce all assertions to the following claim:

Claim 1:

i) Let $\tilde{e}_j A \neq 0$, then there exists an integer z, such that $pr(\tilde{e}_j A)$ arises out of $pr(A)$, if we replace $pr(a_{j+1,z})$ by $pr(a_{j+1,z}) - 1$ and $pr(a_{j+2,z})$ by $pr(a_{j+2,z}) + 1$
ii) $q_-^{j+1}(pr(A)) = z$

Note that the claim will give us the proposition for all $j < i-1$, such that $\tilde{e}_j A \neq 0$. We want to emphasize here that in the proof of the claim we will also prove the statement of Proposition 6.2.2 if $\tilde{e}_j A \neq 0$ is not satisfied.

Proof of Claim 1: For simplicity we denote by $\mathbf{a}^t = \boxed{a_i \,|\, \ldots \,|\, a_n}$ the transpose of the j-th column of A and by $\mathbf{b}^t = \boxed{b_i \,|\, \ldots \,|\, b_n}$ the transpose of the $(j+1)$-th column of A. Further we will denote by $\mathbf{c}^t = \boxed{c_i \,|\, \ldots \,|\, c_n}$ the transpose of the new column (6.2) which we obtain after applying $(i-j-2)$ steps of our algorithm to A. For instance, if $j=i-2$, then $\mathbf{c}$ is precisely the i-th column of A. With the aim to obtain the $(j+2)$-th column of $pr(A)$, we apply our algorithm to the columns $\mathbf{b}$ and $\mathbf{c}$ and since there is no confusion we omit all superfluous indices and denote by $l_1, \cdots, l_t$ the integers described in the algorithm and suppose that $l_r < q \leq l_{r+1}$, where we understand again $l_0 = i-1$. Denote by $(\mathbf{c}')^t = \boxed{c'_i}\boxed{\ldots}\boxed{c'_n}$ the transpose of the $(j+2)$-th column of $pr(A)$ and by $(\mathbf{b}')^t = \boxed{b'_i}\boxed{\ldots}\boxed{b'_n}$ the transpose of the new obtained column after $(i-j-1)$ steps of our algorithm. Hence these columns, expressed in terms of the entries of $\mathbf{b}$ and $\mathbf{c}$, are of the following form:

$$(\mathbf{b}')^t = \begin{array}{|c|c|c|c|c|c|c|c|}\hline b_i & b_{i+1} & \ldots & b_{l_s-1} & b'_{l_s} & b_{l_s+1} & \ldots & b_n+c_n \\ \hline\end{array} \qquad s = 1, \cdots, t-1$$

$$(\mathbf{c}')^t = \begin{array}{|c|c|c|c|c|c|c|c|}\hline x_{l_0} & c_i & \ldots & c_{l_s-1} & x_{l_s} & c_{l_s+1} & \ldots & c_{n-1} \\ \hline\end{array} \qquad s = 1, \cdots, t-1,$$

whereby $x_{l_s} = b_{l_s+1} + \cdots + b_{l_{s+1}} - c_{l_s+1} - \cdots - c_{l_{s+1}-1}$ and $b'_{l_s} = b_{l_s} + c_{l_s} - x_{l_s}$.
With the goal to prove Claim 1 we need several minor results listed in Claim 1.1.

Claim 1.1.: Let $s := m_1 = q_-(\mathbf{a}, \mathbf{b}'), \cdots, m_p$ be the integers obtained from the algorithm if we compare $\mathbf{a}$ with $\mathbf{b}'$ and let as before $l_r < q \leq l_{r+1}$. We presume further that k is either

$$\min\{1 \leq k \leq r \,|\, \sum_{r=l_k+1}^{l_{r+1}} b_r = \sum_{r=l_k}^{l_{r+1}-1} c_r\} \tag{6.4}$$

or if the minium does not exist we set $k = r+1$. Then we have

i) $s \leq q$ and if $s < q$ then in fact $s \leq l_{k-1}$
ii) $q \in \{m_1, \cdots, m_p\}$ and $\sharp\{x | l_{k-1} < m_x < q\} = 0$
iii) $\epsilon_{j+1}(pr(A)) = \epsilon_j(A)$
iv) $q_-^{j+1}(pr(A)) \leq l_{k-1} + 1$

Proof of Claim 1.1.: Suppose $s \leq q$ in Claim 1.1.(i) is not fulfilled and let $l_p < s \leq l_{p+1}$. Then by observing the entries of $\mathbf{b}'$ we see that the sum $D := a_i + \cdots + a_s + b'_s + \cdots + b'_n$ is of the following form:

$$D = \sum_{r=i}^{s} a_r + \sum_{r=s}^{l_{p+1}} b_r + \sum_{r=l_{p+1}}^{n} c_r.$$

But

$$D \leq \sum_{r=i}^{q} a_r + \sum_{r=q}^{l_{p+1}} b_r + \sum_{r=l_{p+1}}^{n} c_r \leq \sum_{r=i}^{q} a_r + \sum_{r=q}^{l_{r+1}} b_r + \sum_{r=l_{r+1}}^{n} c_r = \sum_{r=i}^{q} a_r + \sum_{r=q}^{n} b'_r,$$

where the second last inequality is a consequence of the definition of q, particularly $\sum_{r=q}^{s-1} b_r \geq \sum_{r=q+1}^{s} a_r$ and the last inequality is by the definition of l_{r+1}, namely $\sum_{p=l_{r+1}}^{l_{p+1}-1} c_p \geq \sum_{p=l_{r+1}+1}^{l_{p+1}} b_p$. Consequently we obtain a contradiction to the definition of s and thus

$$q \geq s = q_-(\mathbf{a}, \mathbf{b}'). \tag{6.5}$$

Before we start with the proof of the second statement in (i) we would like to emphasize the following result: for all $j \in \{k, \cdots, r\}$

$$\sum_{r=l_j+1}^{l_{r+1}} b_r = \sum_{r=l_j}^{l_{r+1}-1} c_r \text{ and } b'_{l_j} = b_{l_j} \text{ hold.} \tag{6.6}$$

Let us start by proving the first part of (6.6) by induction, where the initial step is by the choice of k obvious. So assume that the first part of (6.6) holds for j. Using the definition of l_j we must have $\sum_{r=l_j+1}^{s} b_r - \sum_{r=l_j+1}^{s-1} c_r \leq c_{l_j}$ for all $s > l_j$ and since l_{j+1} is the "place" where $\sum_{r=l_j+1}^{s} b_r - \sum_{r=l_j+1}^{s-1} c_r$ is maximal we have, together with the induction hypothesis, only one opportunity, namely $\sum_{r=l_j+1}^{l_{j+1}} b_r = \sum_{r=l_j}^{l_{j+1}-1} c_r$. Hence the first part of (6.6) is proven. We proved also implicitly the second part which we can see as well as a corollary of the first part, namely we get for all $j \in \{k, \cdots, r\}$

$$\sum_{r=l_j+1}^{l_{r+1}} b_r - \sum_{r=l_j}^{l_{r+1}-1} c_r = \sum_{r=l_{j+1}+1}^{l_{r+1}} b_r - \sum_{r=l_{j+1}}^{l_{r+1}-1} c_r = 0 \Rightarrow \sum_{r=l_j+1}^{l_{j+1}} b_r = \sum_{r=l_j}^{l_{j+1}-1} c_r$$

which forces $b'_{l_j} = b_{l_j}$. Because of (6.6) we verified part (i) of Claim 1.1. since the assumption $l_{k-1} < s < q$ would end in a contradiction, namely in

$$D < D + \sum_{r=s+1}^{q} a_r - \sum_{r=s}^{q-1} b_r = D + \sum_{r=s+1}^{q} a_r - \sum_{r=s}^{q-1} b'_r = \sum_{r=i}^{q} a_r + \sum_{r=q}^{n} b'_r.$$

As a corollary of Claim 1.1. (i) we obtain that

$$q \in \{m_1, \cdots, m_p\} \text{ and } \sharp\{x | l_{k-1} < m_x < q\} = 0, \tag{6.7}$$

because if $q = s$ we are done and if not we get with the definition of m_2 and similar calculations as in the proof of Claim 1.1. (i) that $q \geq m_2 = q_-((\mathbf{a})^{>s}, (\mathbf{b}')^{>s})$ and in the case of $q > m_2$ we have $m_2 \leq l_{k-1}$. If $q = m_2$ we are done and if not we repeat these arguments until we get (6.7). For the completion of Claim 1.1. it remains to verify (iii) and (iv), whereas we start with

$$\epsilon_{j+1}(pr(A)) = \epsilon_j(A). \tag{6.8}$$

Note that the transpose of the $(j+1)$-th column of $pr(A)$ is given by

z_{m_0}	b'_i	$\dots$	b'_{m_s-1}	z_{m_s}	b'_{m_s+1}	$\dots$	b'_{n-1}

$s = 1, \cdots, p-1,$

whereby $z_{m_s} = a_{m_s+1} + \cdots + a_{m_{s+1}} - b'_{m_s+1} - \cdots - b'_{m_{s+1}-1}$. Hence any sum

$$\sum_{r=i}^{h} pr(a_{j+1,r}) + \sum_{r=h}^{n} pr(a_{j+2,r}),$$

for some $i \leq h \leq n$ such that $m_{j-1} < h \leq m_j$ and $l_p < h \leq l_{p+1}$, is of the form

$$\sum_{r=i}^{m_j} a_r - \sum_{r=h}^{m_j-1} b'_r + \sum_{r=h}^{l_{p+1}} c'_r - \sum_{r=l_{p+1}+1}^{n} b_r \tag{6.9}$$

and the expression

$$\sum_{r=i}^{h} pr(a_{j+1,r}) - \sum_{r=i}^{h-1} pr(a_{j+2,r})$$

can be written as

$$\sum_{r=i}^{m_j} a_r - \sum_{r=h}^{m_j-1} b'_r - \sum_{r=l_p+1}^{h-1} c'_r - \sum_{r=i}^{l_p} b_r. \tag{6.10}$$

Assume that h is minimal such that (6.9) is maximal, then since $b'_k \geq b_k$ for all $k = i, \cdots, n$ we have

$$\begin{aligned}
\epsilon_{j+1}(pr(A)) &= \sum_{r=i}^{m_j} a_r - \sum_{r=h}^{m_j-1} b'_r - \sum_{r=l_p+1}^{h-1} c'_r - \sum_{r=i}^{l_p} b_r \\
&\leq \sum_{r=i}^{m_j} a_r - \sum_{r=h}^{m_j-1} b_r + \begin{cases} -b_i \cdots - b_{l_p}, & \text{if } h = l_p + 1 \\ c_{l_{p+1}-1} + \cdots + c_{h-1} - b_i \cdots - b_{l_{p+1}}, & \text{else} \end{cases} \\
&\leq \sum_{r=i}^{m_j} a_r - \sum_{r=i}^{m_j-1} b_r \leq \epsilon_j(A).
\end{aligned}$$

The second last inequality is by the reason of $c_{h-1} + \cdots + c_{l_{p+1}-1} < b_h + \cdots + b_{l_{p+1}}$, which is valid by the definition of l_{p+1} and $h-1 \neq l_p$. For the converse direction we investigate (6.10) with $h = l_{k-1} + 1$, whereby we can presume with (6.7) that $q \in \{m_1, \cdots, m_p\}$, say $q = m_j$, and $l_{k-1} < h \leq l_k$, $m_{j-1} < h \leq m_j$. In addition we recall from the definition and (6.6) that $b'_k = b_k$ for $k = h, \cdots, q-1$ and obtain the reverse estimation, which will finish the proof of Claim 1.1. (iii):

$$\begin{aligned}
\epsilon_{j+1}(pr(A)) &\geq \sum_{r=i}^{q} a_r - \sum_{r=h}^{q-1} b'_r - \sum_{r=l_{k-1}+1}^{h-1} c'_r - \sum_{r=i}^{l_{k-1}} b_r \\
&= \sum_{r=i}^{q} a_r - \sum_{r=h}^{q-1} b_r - b_i \cdots - b_{h-1} \\
&= \sum_{r=i}^{q} a_r - \sum_{r=i}^{q-1} b_r = \epsilon_j(A).
\end{aligned}$$

With these calculations we get among other things also $q_-^{j+1}(pr(A)) \leq l_{k-1} + 1$, because $\epsilon_{j+1}(pr(A)) = \sum_{r=i}^{l_{k-1}+1} pr(a_{j+1,r}) - \sum_{r=i}^{l_{k-1}} pr(a_{j+2,r})$ and $q_-^{j+1}(pr(A))$ is minimal with this property.

Now we return to the goal to convince ourselves from Claim 1 and fix some notation for $\tilde{e}_j A$. Let $\mathbf{e}_j\mathbf{a}$ and $\mathbf{e}_j\mathbf{b}$ respectively be the j-th and $(j+1)$-th column respectively of $\tilde{e}_j A$. We denote by $(\mathbf{e}_j\mathbf{c}')^t = \boxed{e_jc'_i \mid \ldots \mid e_jc'_n}$ the transpose of the $(j+2)$-th column of $pr(\tilde{e}_j A)$ and by $(\mathbf{e}_j\mathbf{b}')^t = \boxed{e_jb'_i \mid \ldots \mid e_jb'_n}$ we will denote the transpose of the new obtained column after applying $(i-j-1)$ steps of the algorithm to $\tilde{e}_j A$. For the purpose of determining $\mathbf{e}_j\mathbf{c}'$ we compare $\mathbf{e}_j\mathbf{b}$ with $\mathbf{c}$ and let $n_1, \cdots, n_y$ be the integers defined in the algorithm. Suppose again that k is as in (6.4). If the minimum does not exist, then the integers do not change, i.e. $t = y$, $n_h = l_h$ for $h = 1, \cdots, t$ and otherwise k is the minimal integer such that $n_k \neq l_k$. A short calculation by using (6.6) shows that the new sequence of integers is given by $n_1 = l_1, \cdots, n_{k-1} = l_{k-1}, n_k = l_{r+1}, \cdots, n_{k+t-r-1} = l_t$ and hence

$$e_j c'_h = c'_h \text{ for } h \neq l_{k-1}+1 \text{ and } e_j c'_{l_{k-1}+1} = c'_{l_{k-1}+1} + 1$$

and

$$e_j b'_h = b'_h \text{ for } h \neq l_{k-1}, q \text{ and } e_j b'_{l_{k-1}} = b'_{l_{k-1}} - 1,\ e_j b'_q = b'_q + 1.$$

As a next step we compare the columns $\mathbf{e}_j\mathbf{a}$ with $\mathbf{e}_j\mathbf{b}'$ and determine the sequence of integers from step (1) of the algorithm, say $\overline{m}_1, \cdots, \overline{m}_x$. One can observe similar to (6.5) that $\overline{m}_1 = q_-(\mathbf{e}_j\mathbf{a}, \mathbf{e}_j\mathbf{b}') \leq q$ and as a corollary we obtain again $q \in \{\overline{m}_1, \cdots, \overline{m}_x\}$. Accordingly we have the following situation:

$$\overline{m}_1 < \overline{m}_2 < \cdots < \overline{m}_{x_0} \leq l_{k-1} < \overline{m}_{x_1} < \cdots < \overline{m}_{x_n} = q < \cdots < \overline{m}_x$$

and since by (6.7) there is no $1 \leq h \leq p$ such that $l_{k-1} < m_h < q$ we have

$$m_1 < m_2 < \cdots < m_{j-1} \leq l_{k-1} < q = m_j < \cdots < m_p.$$

The integer $\overline{m}_{x_{n-1}}$ in the aforementioned sequence $\overline{m}_1 < \cdots < \overline{m}_x$ has firstly the property $l_{k-1} < \overline{m}_{x_{n-1}} < q$ and secondly $\overline{m}_{x_{n-1}}$ is maximal with this property. Using the definition of q we obtain

$$b_{\overline{m}_{x_{n-1}}} + \cdots + b_{q-1} = a_{\overline{m}_{x_{n-1}}+1} + \cdots + a_q - 1$$

and as a consequence we get with (6.6) that the resulting new obtained column (6.2) does not change if we skip $\overline{m}_{x_{n-1}}$. Repeating these arguments we can get rid of all integers greater than l_{k-1} and less than q appearing in the sequence. Now it is obvious to see that we can replace the integer sequence $\overline{m}_1, \cdots, \overline{m}_x$ by $m_1, \cdots, m_p$ if we apply our algorithm to $\mathbf{e}_j\mathbf{a}$ and $\mathbf{e}_j\mathbf{b}'$. Thus we get two facts: the first fact is that the new obtained column at which we arrive by applying the algorithm to $\mathbf{e}_j\mathbf{a}$ and $\mathbf{e}_j\mathbf{b}'$ is the same as the one if we apply the algorithm to $\mathbf{a}$ and $\mathbf{b}'$. The second fact is that the $(j+1)$-th column of $pr(\tilde{e}_j A)$ is almost the same as the $(j+1)$-th column of $pr(A)$ except the $(l_{k-1}+1)$-th entry is one smaller. Consequently, we proved part (i) of Claim 1. Now part (ii) of Claim 1 is also proven since (6.8) forces $q_-^{j+1}(pr(A)) \geq z = l_{k-1}+1$, because otherwise we get $\epsilon_{j+1}(pr(\tilde{e}_j A)) = \epsilon_{j+1}(pr(A))$ which is a contradiction to

$$\epsilon_{j+1}(pr(\tilde{e}_j A)) = \epsilon_j(\tilde{e}_j A) = \epsilon_j(A) - 1 = \epsilon_{j+1}(pr(A)) - 1.$$

Consequently we proved our proposition for all $j < i-1$. From now on we would like to show

$$\tilde{e}_{j+1} pr(A) = pr(\tilde{e}_j A) \text{ for all } j > i. \tag{6.11}$$

If A is any element in $B^{m,i}$ such that $\varphi_1(A) = \cdots = \varphi_{i-1}(A) = 0$, then it is easy to see that the image under pr is given by

$$pr(a_{r,s}) = \begin{cases} \epsilon_{r-1}(A), & \text{if } s = i, r > 1 \\ m - \sum_{p=i}^{n} a_{1,p}, & \text{if } s = i, r = 1 \\ a_{r,s-1}, & \text{else.} \end{cases}$$

Note that $\tilde{e}_j A = 0$ implies $\tilde{e}_{j+1} pr(A) = 0$ and otherwise, using the Stembridge axioms which are fulfilled by Theorem 5.2.1 for all elements in $B^{m,i}$, we obtain $\varphi_1(\tilde{e}_j A) = \cdots = \varphi_{i-1}(\tilde{e}_j A) = 0$ and therefore by applying the algorithm to $\tilde{e}_j A$ and comparing $pr(\tilde{e}_j A)$ with $pr(A)$ we have $\tilde{e}_{j+1} pr(A) = pr(\tilde{e}_j A)$. So we proved our proposition for all A, such that $\varphi_1(A) = \cdots = \varphi_{i-1}(A) = 0$. Now let A be arbitrary and write $\mathrm{wt}(A)$ as a linear combination of simple roots $\mathrm{wt}(A) = \sum_{j \in I} k_j \alpha_j \in \sum_{j \in I} \mathbb{Q}\alpha_j$, and define

$$\mathrm{ht}(\mathrm{wt}(A)) := \textstyle\sum_{j \in I} k_j.$$

Our proof will proceed by induction on $\lceil \mathrm{ht}(\mathrm{wt}(A)) \rceil$, whereby $\lceil \cdot \rceil$ denotes the ceiling function. If the height is minimal we have the lowest weight element in $B^{m,i}$ which satisfies obiously (6.11). If $\varphi_1(A) = \cdots = \varphi_{i-1}(A) = 0$, then we are done by the above considerations and if not let $1 \leq l \leq i-1$ be any integer so that $\varphi_l(A) \neq 0$. By induction we gain $\tilde{e}_{j+1} pr(\tilde{f}_l A) = pr(\tilde{e}_j \tilde{f}_l A)$ and by earlier calculations, together with Lemma 3.2.2, Lemma 6.1.1 and Proposition 6.2.1, we can verify $pr(A) = pr(\tilde{e}_l \tilde{f}_l A) = \tilde{e}_{l+1} pr(\tilde{f}_l A) \Rightarrow \tilde{f}_{l+1} pr(A) = pr(\tilde{f}_l A)$. Thus, by using the Stembridge axioms, we can finish the proof of (6.11) since

$$\epsilon_{j+1}(pr(A)) = \epsilon_{j+1}(\tilde{f}_{l+1} pr(A)) = \epsilon_{j+1}(pr(\tilde{f}_l A)) = \epsilon_j(\tilde{f}_l A) = \epsilon_j(A)$$

and

$$\begin{aligned} \tilde{f}_{l+1} \tilde{e}_{j+1} pr(A) &= \tilde{e}_{j+1} \tilde{f}_{l+1} pr(A) = \tilde{e}_{j+1} pr(\tilde{f}_l A) = pr(\tilde{e}_j \tilde{f}_l A) \\ &= pr(\tilde{f}_l \tilde{e}_j A) = \tilde{f}_{l+1} pr(\tilde{e}_j A). \end{aligned}$$

□

At this point we are in position to state our main theorem:

Theorem 6.2.1. The map pr described in the algorithm is Schützenberger's promotion operator.

Proof. Let $A \in B^{m,i}$, then we erase all arrows with colour n and denote by $Z_{(1,\cdots,n-1)}(A)$ the connected component containing A. Let B be the $\{1, \cdots, n-1\}$ highest weight element. Then by an immediate inspection of the definiton we must have $b_{r,s} = 0$ for all (r,s) except $(r,s) = (i,n)$, which in particular means

$$pr(b_{r,s}) = \begin{cases} m - b_{i,n}, & \text{if } r = 1, s = i \\ 0, & \text{else.} \end{cases}$$

Thus we have $pr(\tilde{e}_{i_1} \cdots \tilde{e}_{i_s} A) = pr(B) \in B^{m,i}$, with some $i_1, \cdots i_s \in \{1, \cdots, n-1\}$. We claim actually that $pr(A)$ lives in $B^{m,i}$ and we will prove this statement by induction on $n := \sharp\{i_r | i_r = i-1\}$. Suppose that $pr(A) \in B^{p,i}$ for some $p \geq m$. If $n = 0$ we obtain by Lemma 3.2.2

$$pr(A) = {}_p\tilde{f}_{i_s+1} \cdots {}_p\tilde{f}_{i_1+1} pr(B) = {}_m\tilde{f}_{i_s+1} \cdots {}_m\tilde{f}_{i_1+1} pr(B) \in B^{m,i},$$

which proves the initial step. Now we assume that $l = \min\{1 \leq l \leq s | i_l = i-1\}$ and $pr(\tilde{e}_{i_{l+1}} \cdots \tilde{e}_{i_s} A) \in B^{p,i}$ for some $p \geq m$. Then we get

$$pr(\tilde{e}_{i_{l+1}} \cdots \tilde{e}_{i_s} A) = {}_p\tilde{f}_{i_l+1} \cdots {}_p\tilde{f}_{i_1+1} pr(B) = {}_p\tilde{f}_{i_l+1} \, {}_m\tilde{f}_{i_{l-1}+1} \cdots {}_m\tilde{f}_{i_1+1} pr(B).$$

So by the induction hypothesis it is sufficient to prove that

$$ {}_p\tilde{f}_{i_l+1}\ {}_m\tilde{f}_{i_{l-1}+1}\cdots{}_m\,\tilde{f}_{i_1+1}pr(B)\in B^{m,i}, \tag{6.12}$$

but since $\varphi_{i-1}(\tilde{e}_{i_l}\cdots\tilde{e}_{i_s}A)\neq 0$ we can conclude with Lemma 6.1.1 that (6.12) holds:

$$ {}_p\tilde{f}_{i_l+1}\ {}_m\tilde{f}_{i_{l-1}+1}\cdots{}_m\,\tilde{f}_{i_1+1}pr(B)=\ {}_m\tilde{f}_{i_l+1}\ {}_m\tilde{f}_{i_{l-1}+1}\cdots{}_m\,\tilde{f}_{i_1+1}pr(B)\in B^{m,i}.$$

According to that we have the well-definedness of pr, i.e. $pr: B^{m,i}\longrightarrow B^{m,i}$. The condition (1) of Proposition 6.0.1 is obviously fulfilled by construction and Proposition 6.1.1 and condition (3) is exactly Proposition 6.2.2 and the following simple calculation: by the reason of condition (1) of Proposition 6.0.1 and part (1) of Definition 3.1 we can assume without loss of generality that $\tilde{f}_jA\neq 0$ and thus

$$pr(A)=pr(\tilde{e}_j\tilde{f}_jA)=\tilde{e}_{j+1}pr(\tilde{f}_jA).$$

So the proof of part (2) of Proposition 6.0.1 will finish our main theorem. Note that for the bijectivity it is enough to prove the surjectivity. So let $A\in B^{m,i}$ be an arbitrary element and let B the highest weight element in $Z_{(2,\cdots,n)}(A)$. Then it is obvious to see that B has the property $b_{r,s}=0$ if $(r,s)\neq(1,i)$ and according to this B has a pre-image, say C. For instance one can choose C as follows:

$$c_{r,s}=\begin{cases}0, & (r,s)\neq(i,n)\\ m-b_{1,i}, & \text{if } r=i, s=n.\end{cases}$$

Therefore, since $B=\tilde{e}_{i_1}\cdots\tilde{e}_{i_s}A$ with $i_1\cdots i_s\in\{2,\cdots,n\}$, we have

$$A=\tilde{f}_{i_s}\cdots\tilde{f}_{i_1}B=\tilde{f}_{i_s}\cdots\tilde{f}_{i_1}pr(C)=pr(\tilde{f}_{i_s-1}\cdots\tilde{f}_{i_1-1}C).$$

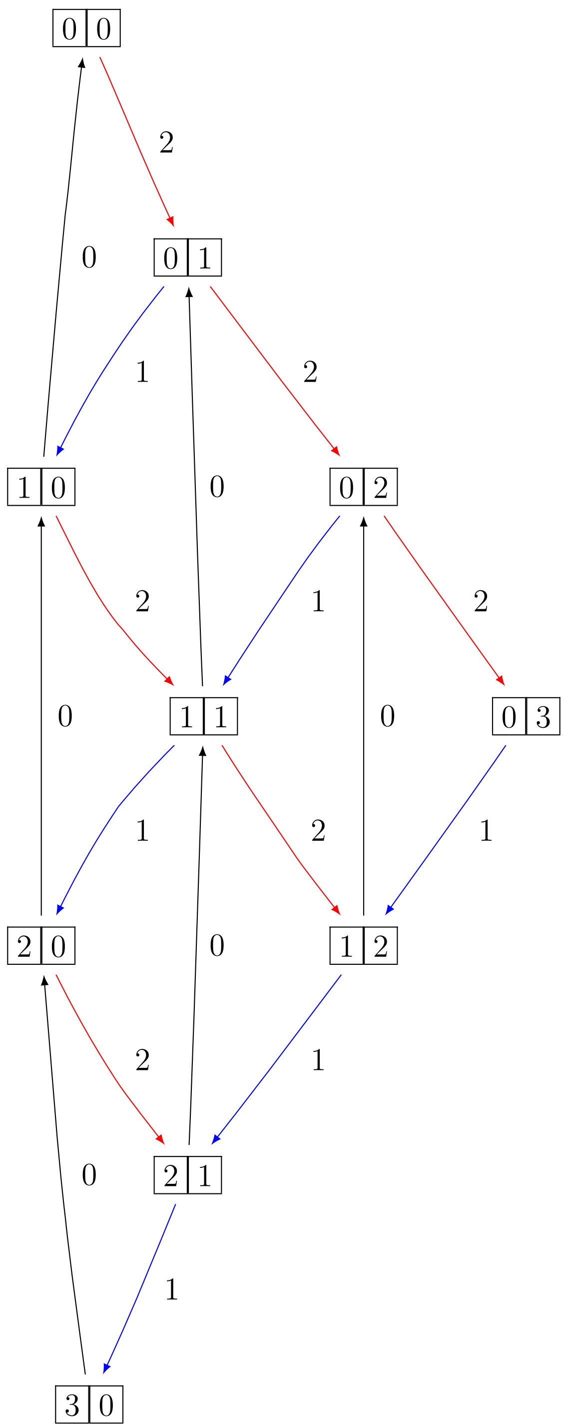
□

Remark 6.2.1. If we follow the results from [1] we can compute the inverse map of pr by composing n times pr. In particular

$$pr^{-1}=pr^n.$$

We would like to finish our paper with drawing a KR-crystal graph of type $A_2^{(1)}$.

Example. The KR-crystal $B^{3,2}$ of type $A_2^{(1)}$ looks as follows:



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Discussion

The results presented in this thesis deal in principal with the representation theory of equivariant map algebras and combinatorial representation theory, especially crystal theory. It generalizes many results known before for representations of specific equivariant map algebras as (twisted) loop and (twisted) current algebras and opens a huge area of research in representation theory.

For instance, one would like to extend the results of local Weyl modules for more general equivariant map algebras where the action of the finite group Γ is not necessarily free. In the case where $\mathfrak{g}$ is a simple complex Lie algebra of type A, D, E and Γ is the finite group of order 2 or 3 of non-trivial diagram automorphism of $\mathfrak{g}$ we have seen that we need a different approach to local Weyl modules. The techniques described in [**I**] are not applicable. The main reason therefor is that the isomorphism

$$(\mathfrak{g} \otimes A)^\Gamma / (\mathfrak{g} \otimes I_\eta)^\Gamma \xrightarrow{\cong} (\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I_\eta),$$

which we used in the case where the action of the group Γ is free, does not hold in general. In order to provide an example, we consider the Lie algebra of type A_2 and Γ the finite group of order 2, then there exists no isomorphism as in the above sense because otherwise we would obtain a 2-dimensional representation of A_2. More precisely, let $V(\omega)$ be the 2-dimensional irreducible A_1 representation. Then we can extend the action of A_1 on $V(\omega)$ to an $A_2^{(2)}$ action by letting

$$\mathfrak{sl}_2 \otimes t^2\mathbb{C}[t^2] \oplus V(4\omega) \otimes t\mathbb{C}[t^2]$$

to act by zero. This construction provides an irreducible $(\mathfrak{sl}_3 \otimes \mathbb{C}[t])^\Gamma$-module (cf. [40]) which yields a irreducible 2-dimensional representation for A_2, provided such a Lie algebra isomorphism exists.

Nevertheless local Weyl modules for the equivariant map algebra associated to these data can be computed, provided either $\mathfrak{g}$ is not of type A_{2l} or the weights have to fulfill an "odd" property (see [**II**]). They are identified with the corresponding affine Demazure modules and an explicit construction from untwisted Weyl modules which generalize the fusion product is given. Therefore dimension and character formulas were computed in [**II**].

For the remaining local Weyl modules of type $A_{2l}^{(2)}$ where the highest weight is "even" we have a conjecture, namely that the local Weyl modules are isomorphic to associated graded modules of the restrictions of local Weyl modules for loop algebras. This conjecture would follow from a dimension argument, which is for the twisted local Weyl module of highest weight $2k$ for the twisted Lie algebra $A_2^{(2)}$ exactly $\dim W^\Gamma(2k) = 3^k$. A simple case-by-case calculation proves the conjecture for $k = 1, 2, 3, 4$.

In order to prove the conjecture we can not apply the standard techniques to identify the module $W^\Gamma(2k)$ with a Demazure module and use subsequently the Demazure character formula to compute its dimension. We explain for $k = 2$ why such an identification with Demazure modules is not possible.

Let $\widehat{W}$ be the Weyl group of the twisted Lie algebra and let w be an arbitrary Weyl group element, i.e. w is of the form $w = s_1 t_{2p\omega}$ for some integer p. For an arbitrary dominant integral weight $\Lambda = a\Lambda_0 + b\Lambda_1$ we obtain

$$w(\Lambda) = (a + \frac{b}{2})\Lambda_0 - (4ap + 2bp + b)\omega.$$

That means that the complete list of Demazure modules of classical highest weight 4ω consists of $D(1,4)$ and $D(2,4)$. Together with the Demazure character formula

$$\operatorname{ch} D(1,4) = D_1 D_0(e^{\Lambda_0}) = e^{\Lambda_0} + e^{\Lambda_0-\alpha_0}(1 + e^{-\alpha_1} + e^{-2\alpha_1} + e^{-3\alpha_1} + e^{-4\alpha_1})$$

and

$$\operatorname{ch} D(2,4) = D_1(e^{4\Lambda_1}) = e^{4\Lambda_1}(1 + e^{-\alpha_1} + e^{-2\alpha_1} + e^{-3\alpha_1} + e^{-4\alpha_1}),$$

we realize that the level one Demazure module is of dimension 6 and the level two Demazure module is of dimension 5. Thus the twisted local Weyl module is in general not a Demazure module.

For the untwisted local Weyl modules for non simply-laced algebras it is shown that every local Weyl module has a Demazure flag [38]. Hence one would expect from these results that $W^\Gamma(2k)$ has a Demazure flag, however, again the $k = 2$ case yields a counterexample.

From that point of view, it is very interesting to investigate these modules for other equivariant map algebras where the action is not free. The results from [**II**] give us hope to be able to compute dimensions and characters for other algebras. For instance, another interesting algebra to look at and to study local Weyl modules is the generalized Onsager algebra. Let $X = \mathbb{C}^*$, $\mathfrak{g}$ be a simple Lie algebra, and $\Gamma = \mathbb{Z}/2\mathbb{Z}$ be a group of order 2 generated by 1 and σ. For a set of Chevalley generators $\{e_i, f_i, h_i\}$ let Γ by the standard Chevalley involution (see [4]), i.e.

$$\sigma(e_i) = -f_i,\ \sigma(f_i) = -e_i,\ \sigma(h_i) = -h_i.$$

Let σ act on $\mathbb{C}^*$ by $z \mapsto z^{-1}$. Then, the generalized Onsager algebra $\mathcal{O}(\mathfrak{g})$, considered by G. Benkart and M. Lau, is given by

$$\mathcal{O}(\mathfrak{g}) := (\mathfrak{g} \otimes \mathbb{C}[t])^\Gamma.$$

The action of Γ on the variety $\mathbb{C}^*$ is not free ($\Gamma_{-1} = \Gamma$) and thus the question what local Weyl modules (dimensions and characters) might be for these algebras is still open. The advantage of working with the Onsager algebra is mainly the fact that the fixed point subalgebra $\mathfrak{g}_0$ is in almost all cases semi-simple. We have the following table (see, for example, [27, Chapter X, 5, Tables II and III]).

$\mathfrak{g}$	$\mathfrak{g}_0$
A_l	$\mathfrak{so}_{l+1}$
$B_l, l \geq 2$	$\mathfrak{so}_{l+1} \oplus \mathfrak{so}_l$
$C_l, l \geq 2$	$\mathfrak{gl}_l$
$D_l, l \geq 4$	$\mathfrak{so}_l \oplus \mathfrak{so}_l$
E_6	C_4
E_7	A_7
E_8	D_8
F_4	$C_3 \oplus A_1$
G_2	$A_1 \oplus A_1$

The best way to avoid the question whether there exists a semisimple fixed point subalgebra or not is to use the homological chracterization of local Weyl modules proved in [**I**] as a general definition. This way of introducing yields a list of many other open questions:

- Can one describe local Weyl modules via generators and relations?
- Does a tensor product property hold?
- Is the dimension independent of the support of Ψ?
- What does it mean for the global Weyl module?

Beside studying local Weyl modules for equivariant map algebras where the action is not free, one can look at equivariant map algebras where the target space is an arbitrary finite-dimensional Lie algebra and Kac-Moody algebra respectively. Even the classification of the irreducible representations of an equivariant map algebra where $\mathfrak{g}$ is replaced by an Kac-Moody algebra is still an open question. For more details we refer to [39]. As a result one major part of forthcoming work would be to answer these questions satisfactorily.
Kirillov-Reshetikhin modules, a certain subclass of simple finite-dimensional representations of quantum affine algebras, have been studied extensively due to their application in mathematical physics (cf.[1],[6],[10],[24],[28]). One way of studying their structure is by looking at the classical limits which can be regarded as graded modules for the current algebra $\mathfrak{g}\otimes\mathbb{C}[t]$. The generalization of these algebras to the setting of equivariant map algebras seems to be quite challenging, not to mention the computation of graded character formulas. An interesting generalization for multicurrent algebras of these modules is worked out in [3] and a recursion formula for their graded character formula is provided. Here the multicurrent algebra can be regarded as a map algebra corresponding to the data $X = \mathbb{C}^n$ and $\Gamma = \{1\}$.
Likewise in combinatorial representation theory Kirillov-Reshetikhin crystals, crystal bases of KirillovReshetikhin modules have been studied extensively (cf. [23],[34],[41],[42],[43],[44]). We recall that it was first conjectured in [26] that these modules admit a crystal bases and this was proven in type $A_n^{(1)}$ in [29] and in all non-exceptional cases in [41],[42]. For the affine type $A_n^{(1)}$ a well studied realization of these crystals is given in terms of Young tableaux of rectangle shape, and the affine crystal structure is defined via a promotion operator by using jeu-de-taquin.
For nonexceptional affine types, where the KR-crystals are realized by Young tableaux as classical crystals, the affine crystal structure is given by using so-called $\pm$-diagrams ([23],[43]). For instance in type $D_n^{(1)}$ the affine crystal structure is

$$f_0 = \sigma \circ f_1 \circ \sigma,\ \ e_0 = \sigma \circ e_1 \circ \sigma,$$

where σ is the crystal analogue of the Dynkin automorphism that interchanges the 0 and 1 node. In particular, the map σ is defined on $\pm$-diagrams, which parametrize the D_{n-1} highest weight elements in the KR-crystal (see [43, Definition 4.3]).
For papers dealing with Kirillov-Reshetikhin crystals over twisted affine algebras we refer for instance to [31],[36].
In the following we collect and discuss several important open questions which will be part of the forthcoming work in the branch of combinatorial representation theory.

- Can one give a more explicit realization of Kirillov-Reshetikhin crystals for all types where the existence is proven?
- Can this realization be given in terms of polytopes?

In our results we showed that for the affine type $A_n^{(1)}$ the answer is positive. We described a polytope, such that the subset of integral points is a affine crystal isomorphic to the Kirillov-Reshetikhin crystal. The answer for all other types is still not known, however for type $C_n^{(1)}$ a natural candidate for the polytope is given in [19]. The initial problem for type C_n is the fact that the KR-crystals are on the one hand not irreducible anymore and on the other hand the components appearing in a decomposition into crystals corresponding to irreducible representations are of the form $B(\lambda)$ for more general λ's as in the A_n case (for A_n we have only $\lambda = m\omega_i$). Thus it becomes much more difficult to define Kashiwara's crystal operators on the union of C_n-polytopes. As a matter of interest we give the decomposition of the KR-crystal for all other classical types (see [6], [23]).

- For type B_n we have for $1 \leq i < n$

$$B^{m,i} = \bigcup_{\sum_j m_{i-2j}=m} B(\sum_{j=0}^{\lceil \frac{i}{2} \rceil} m_{i-2j}\omega_{i-2j})$$

  and for $i = n$

$$B^{m,n} = \bigcup_{m_n+\sum_j 2m_{n-2j}=m} B(\sum_{j=0}^{\lceil \frac{n}{2} \rceil} m_{n-2j}\omega_{n-2j})$$

- For type C_n:

$$B^{m,i} = \bigcup_{\substack{\sum_j m_j=m \\ m_j=0 \mod 2, i\neq j \\ m_i=m \mod 2}} B(\sum_{j=0}^{i} m_j\omega_j)$$

- For type D_n:

$$B^{m,i} = \bigcup_{\sum_j m_{i-2j}=m} B(\sum_{j=0}^{\lceil \frac{i}{2} \rceil} m_{i-2j}\omega_{i-2j}) \ 1 \leq i < n-1,$$

where $\omega_0 = 0$.
Because of the above decomposition of KR-crystals into classical irreducible crystals corresponding to irreducible representations, a very interesiting question arises:

- Assume that it is possible to realize the KR-crystals as a union of polytopes. Are these polytopes parametrizing a PBW-type basis of irreducible finite-dimensional representations?

Recall that for type A_n the polytope we used is given more generally in [18]. It is shown that the set of integral points is parametrizing a PBW-type basis of $V(\lambda)$ for arbitrary dominant integral highest weight λ. A similar construction is also provided for the symplectic Lie algebra in [19] where the integral points are identified with basis elements of $V(\lambda)$.
Provided that one has (partially) solved the above mentioned questions, a very popular goal in combinatorial representation theory is to find connections to other known combinatorial models. Therefore,

- What is the connection to other combinatorial models? Is it possible to give an explicit crystal isomorphism between these?

Even for type A_n the KR-crystal, realized as a polytope, has no known explicit combinatorial bijection to other combinatorial models of crystals induced by representations, such as the Young tableaux model or the set of certain Nakajima monomials. Only for type A_2 we can describe the bijection to the Young tableux model explicitly as follows.
For $i = 1$:

$$\begin{array}{|c|}\hline a\\ \hline b\\ \hline\end{array} \mapsto \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & \cdots & 1 & 2 & \cdots & 2 & 3 & \cdots & 3\\ \hline\end{array}$$

where the number of 1's is $m - a - b$, the number of 2's is a and the number of 3's is b.
For $i = 2$:

$$\begin{array}{|c|c|}\hline b & c\\ \hline\end{array} \mapsto \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & 1 & 2 & 2 & \cdots & 2 & 2\\ \hline 2 & \cdots & 2 & 3 & \cdots & 3 \\ \cline{1-6}\end{array}$$

where the number of 1's is $m-b$, the number of 2's in the first row and second row respectively is b and $m - c - b$ respectively and the number of 3's is $c + b$. We should mention that the above crystal morphism to Young tableaux can be extended to the more general polytope from [18], which is defined by distinguishing many cases. A first step to pursue these aims is to generalize the above mentioned crystal morphism to an arbitrary rank.

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Erklärung zum Eigenanteil:

In der reinen Mathematik ist die Rangfolge der Autoren sehr unüblich und schwierig zu ermitteln, ebenso die prozentuale Angabe der Eigenleistung. Die Aufzählung der Namen auf Artikeln unterliegt in der reinen Mathematik nur der alphabetischen Reihenfolge. In allen in dieser Dissertation enthaltenen Artikeln bin ich als Erstautor einzustufen. In dem Artikel *Local Weyl modules for equivariant map algebras with free abelian group actions* stammen die grundlegenden Ideen und die Durchführung dieser von Dr. G. Fourier und mir. Der Eigenanteil liegt somit bei 35%. Der Eigenanteil in dem Artikel *Demazure modules and Weyl modules: The twisted current case* liegt bei 50%. Die Ideenentwicklung als auch die Durchführung dieser enstand mit gleichem Aufwand.

Köln, im Mai 2013

(Deniz Kus)

Erklärung:

Ich versichere, dass ich die von mir vorgelegte Dissertation selbstständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit - einschließlich Tabellen, Karten und Abbildungen -, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen - noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Peter Littelmann betreut worden.

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- Realization of affine type A Kirillov-Reshetikhin crystals via polytopes (eingereicht),
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- Crystal bases as tuples of integer sequences (eingereicht)