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ADAPTIVE NONLINEAR DOMAIN DECOMPOSITION METHODS

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Abstract. In this article, different nonlinear domain decomposition methods are applied to nonlinear problems with highly-heterogeneous coefficient functions with jumps. In order to obtain a robust solver with respect to nonlinear as well as linear convergence, adaptive coarse spaces are employed. First, as an example for a nonlinearly left-preconditioned domain decomposition method, the two-level restricted nonlinear Schwarz method H1-RASPEN (Hybrid Restricted Additive Schwarz Preconditioned Exact Newton) is combined with an adaptive generalized Dryja–Smith–Widlund (GDSW) coarse space. Second, as an example for a nonlinearly right-preconditioned domain decomposition method, a nonlinear FETI-DP (Finite Element Tearing and Interconnecting - Dual Primal) method is equipped with an edge-based adaptive coarse space. Both approaches are compared with the respective nonlinear domain decomposition methods with classical coarse spaces as well as with the respective Newton-Krylov methods with adaptive coarse spaces. For some two-dimensional p-Laplace model problems with different spatial coefficient distributions, it can be observed that the best linear and nonlinear convergence can only be obtained when combining the nonlinear domain decomposition methods with adaptive coarse spaces.

Key words. Nonlinear Domain Decomposition Methods, ASPIN, RASPEN, Nonlinear Schwarz Methods, FETI-DP, Nonlinear FETI-DP, Adaptive Coarse Spaces, AGDSW

AMS subject classifications.

1. Introduction.

Nonlinear domain decomposition methods (DDMs) can be used to improve the nonlinear convergence of Newton's method applied to a discrete nonlinear problem

$$F(u) = 0. \tag{1.1}$$

Here, $u \in V$, V is some finite element space defined on a domain $\Omega \subset \mathbb{R}^d, d = 2, 3$, and $F : V \to \mathbb{R}$ is a given nonlinear function. For simplicity, we restrict ourselves to scalar problems but the results directly generalize to vector-valued problems. In general, nonlinear DDMs can be classified into nonlinearly left-preconditioned and nonlinearly right-preconditioned Newton methods. The latter usually improve the nonlinear convergence by a nonlinear elimination of certain degrees of freedom, while nonlinearly left-preconditioned methods, such as ASPIN (Additive Schwarz Preconditioned Inexact Newton) [1, 30] or RASPEN (Restricted Additive Schwarz Preconditioned Exact Newton) [2] methods, are based on a reformulation of the original nonlinear problem. Let us note that nonlinear FETI [31] (Finite Element Tearing and Interconnecting) and FETI-DP (FETI - Dual Primal) methods [19, 21] include both, first, a replacement of the original problem by an equivalent nonlinear saddle point problem and, subsequently, a nonlinear elimination process. Nevertheless, they are counted as right-preconditioned methods since only the elimination process improves the nonlinear convergence significantly. For the discussions in this article on the effect of adaptive coarse spaces on nonlinear DDMs, we consider examples from both

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classes. The nonlinear hybrid two-level Schwarz approach H1-RASPEN [13] represents the class of nonlinearly left-preconditioned approaches and Nonlinear-FETI-DP-2 [19] represents the class of nonlinearly left-preconditioned approaches.

In nonlinear Schwarz methods, the original nonlinear problem (1.1) is replaced by a modified nonlinear problem

$$G(F(u)) =: \mathcal{F}(u) = 0. \tag{1.2}$$

The modified nonlinear function $\mathcal{F}$ is constructed based on a domain decomposition approach, and the nonlinear right-preconditioner G is typically only given implicitly.

In contrast, in nonlinear FETI-DP methods, the original nonlinear problem (1.1) is first replaced by an equivalent saddle point problem A before a nonlinear right-preconditioner M is applied, which is based on a nonlinear elimination. Hence, we replace (1.1) by

$$A\left(M\left(\tilde{u}, \lambda\right)\right) = 0, \tag{1.3}$$

and the solution u^* of the original problem (1.1) can be easily computed from $M\left(\tilde{u}^*, \lambda^*\right)$, where $(\tilde{u}^*, \lambda^*)$ is the solution of (1.3).

In this paper, we focus on highly heterogeneous nonlinear problems where the nonlinear function F depends on coefficients or material parameters with large jumps. Typically, these heterogeneities badly influence both the linear and the nonlinear convergence. In order to retain robustness of linear domain decomposition methods, adaptive coarse spaces, for example, [16, 29, 8, 7, 3, 34, 5], can be employed. In these approaches, the coarse space is enriched by coarse basis functions or constraints which are constructed from certain local eigenfunctions. However, linear domain decomposition methods cannot improve the nonlinear convergence in a Newton-Krylov type iteration, which may still be deteriorated due to the heterogeneities.

Here, we will combine the aforementioned nonlinear domain decomposition methods with corresponding adaptive coarse spaces. First, we employ the two-level RASPEN approach introduced in [14, 13] to incorporate adaptive GDSW (Generalized Dryja–Smith–Widlund) coarse spaces [9, 7] into the RASPEN method in a multiplicative way. In our numerical experiments, we compare this new approach with a Newton-Krylov approach, where the linearized system is preconditioned by a linear two-level Schwarz method with the same adaptive GDSW (AGDSW) coarse space. We also consider two-level RASPEN methods with two different multiscale finite element method (MsFEM) [15] type coarse spaces, as introduced in [8, 14, 13]. Furthermore, we will discuss how the adaptive coarse space for FETI-DP and BDDC (Balancing Domain Decomposition by Constraints) methods first introduced in [29] can be implemented into the Nonlinear-FETI-DP-2 approach using a transformation of basis. In our experiments, we compare this approach with a Newton-Krylov-FETI-DP approach exploiting the same coarse space. Additionally, classical coarse spaces are considered for comparison. For different p-Laplacian model problems with highly heterogeneous coefficient functions, our numerical results indicate that only the combination of nonlinear DDMs with adaptive coarse spaces yields both robust linear and robust nonlinear convergence. This observation can be made for both classes, i.e., nonlinear left- as well as right-preconditioners.

The remainder of this paper is organized as follows: In section 2, we introduce the H1-RASPEN and Nonlinear-FETI-DP-2 nonlinear DDMs, which form the basis of the nonlinear adaptive DDMs under consideration. Whereas it is straightforward by construction to exchange the coarse space in H1-RASPEN, we will dedicate subsection 2.3

to the implementation of arbitrary coarse spaces in nonlinear FETI-DP. Next, in section 3, we introduce the AGDSW and the adaptive FETI-PD coarse spaces, which facilitate robust convergence for highly heterogeneous model problems. We describe the highly heterogeneous model problems employed in our numerical experiments and present the respective numerical results in section 4. Finally, we end with a conclusion in section 5.

2. Nonlinear Domain Decomposition Methods. In order to construct robust nonlinear domain decomposition methods, we will consider two successful nonlinear left and right domain decomposition preconditioners, that is, H1-RASPEN and Nonlinear-FETI-DP-2. In section 3, we will then introduce corresponding adaptive coarse spaces to be used within the nonlinear DDMs. Let us note that our goal is to demonstrate the general strength of the approach of combining nonlinear DDMs with adaptive coarse spaces, without attempting to directly compare the two different DDMs.

2.1. A hybrid two-level RASPEN method based on a Galerkin product. In [13], several two-level RASPEN or ASPIN approaches based on a Galerkin product have been introduced. In this paper, we will focus on one of those methods, the H1-RASPEN method, which will be briefly described in this section. To obtain the H1-RASPEN method, the RASPEN algorithm [2] is enhanced with a multiplicatively coupled nonlinear coarse correction. RASPEN itself is a restricted variant of the ASPEN method, which corresponds to the well-known ASPIN method [1] but uses exact derivatives, i.e, exact Jacobian matrices. Let us remark that there are several competing approaches to implement a nonlinear second level in RASPEN type methods. In [30], an additive coarse space is suggested for ASPIN, and a multiplicative coarse space using an FAS (full approximation scheme) approach is introduced in [2].

As already mentioned in section 1, all RASPEN type methods are based on a reformulation of (1.1) using a domain decomposition of the underlying nonlinear PDE. Specifically, for H1-RASPEN, we define the nonlinear problem

$$G(F(u)) =: \mathcal{F}_{h,1}(u) = 0, \tag{2.1}$$

where the nonlinear left-preconditioner G is given implicitly; cf. (1.2). For the definition of $\mathcal{F}_{h,1}$, we consider a decomposition of Ω into nonoverlapping subdomains Ω_i, $i = 1, ..., N$, and, by adding layers of finite elements, we obtain overlapping subdomains Ω_i', $i = 1, ..., N$. We denote the local finite element spaces associated with the overlapping subdomains by V_i, $i = 1, ..., N$. With standard restriction operators $R_i : V \to V_i$ and corresponding prolongation operators $P_i := R_i^T$, we can define nonlinear local corrections $T_i(u)$ on the overlapping subdomains Ω_i' by

$$R_i F(u - P_i T_i(u)) = 0,\ i = 1, ..., N. \tag{2.2}$$

Analogously, we can define a nonlinear coarse correction

$$\Phi^T F(u - \Phi T_0(u)) = 0, \tag{2.3}$$

where $\Phi^T : V \to V_0$ is the restriction operator to the coarse space V_0. For specific examples of choices for Φ, we refer to [13], and, for the adaptive GDSW coarse space used in this article, to subsection 3.1. Note that this approach allows for using various coarse spaces using a Galerkin product approach; in particular, coarse spaces for linear Schwarz methods can easily be employed. We will make use of this property to implement the use of an adaptive coarse spaces.

Using restricted prolongation operators $\widetilde{P}_i$, $i = 1, ..., N$, which satisfy the usual partition of unity condition

$$\sum_{i=1}^{N} \widetilde{P}_i R_i = I,$$

we can define the nonlinear reformulation

$$\mathcal{F}_{\mathrm{h},1}(u) := \sum_{i=1}^{N} \widetilde{P}_i T_i(u - \Phi T_0(u)) + \Phi T_0(u) \tag{2.4}$$

of the original nonlinear problem (1.1). Let us remark that (2.4) and (1.1) have the same solution; see [1, 2].

In the H1-RASPEN method, (2.4) is solved using Newton's method, i.e., using the iteration

$$u^{(k+1)} = u^{(k)} - \left(D\mathcal{F}_{h,1}(u^{(k)})\right)^{-1} \mathcal{F}_{h,1}\left(u^{(k)}\right), \tag{2.5}$$

with the exact Jacobian

$$D\mathcal{F}_{\mathrm{h},1}(u^{(k)}) = I - \left(I - \sum_{i=1}^{N} Q_i(v_i^{(k)})\right)\left(I - Q_0(u_0^{(k)})\right). \tag{2.6}$$

Here, for a compact notation, we have used the nonlinear Schwarz operators

$$Q_i(u) := P_i \left(R_i\, DF(u)\, P_i\right)^{-1} R_i\, DF(u),\ i = 1, ..., N,$$

and

$$Q_0(u) := \Phi \left(\Phi^T\, DF(u)\, \Phi\right)^{-1} \Phi^T\, DF(u)$$

introduced in [13]. The different linearization points are $u_0^{(k)} := u^{(k)} - \Phi T_0(u^{(k)})$ and $v_i^{(k)} := u_0^{(k)} - P_i T_i(u_0^{(k)})$ for $i = 1, ..., N$. Note that $D\mathcal{F}_{\mathrm{h},1}(u^{(k)})$ is generally nonsymmetric, such that we will solve (2.5) using the GMRES (generalized minimal residual) method [33]. For more details on the derivation of $D\mathcal{F}_{\mathrm{h},1}$ or an inexact version, we again refer to [13]. Let us note that (2.6) has the structure of the Jacobian of (1.1) preconditioned by a linear hybrid two-level restricted Schwarz preconditioner. Therefore, using an appropriate coarse space, the condition number of (2.6) is usually sufficiently small, and a Krylov method can directly be applied to the linearized system. More precisely, no additional linear preconditioner is necessary.

Let us remark that, in each Newton iteration of the H1-RASPEN method, the nonlinear coarse problem (2.3) has to be solved for the global coarse correction $T_0(u^{(k)})$ first, and afterwards, all the local nonlinear problems (2.2) have to be solved for the local corrections $T_i(u_0^{(k)})$, $i = 1, ..., N$. This can again be done using Newton's method and, for the local problems, completely in parallel. We distinguish between outer iterations, i.e., global Newton iterations as in (2.5), and inner iterations. The latter ones split up into local Newton iterations on the subdomain problems to compute the local nonlinear corrections $T_i(u_0^{(k)})$ and coarse iterations, i.e., global Newton iterations on the coarse problem to compute the correction $T_0(u^{(k)})$.

2.2. Nonlinear FETI-DP. In this section, we provide a brief overview of nonlinear FETI-DP methods and the special case of Nonlinear-FETI-DP-2, which we consider in our numerical experiments. For more details, see [19, 20, 21]. Here, we consider nonoverlapping subdomains Ω_i, $i = 1, ..., N$ and corresponding local finite element spaces W_i, $i = 1, ..., N$. To derive the aforementioned nonlinear saddle point system, we first introduce local nonlinear problems on the subdomains

$$K_i(u_i) - f_i = 0,\ i = 1, ..., N,$$

which are obtained by a finite element discretization, assuming zero Neumann type boundary conditions on the interface

$$\Gamma = \bigcup_{i \neq j} (\partial\Omega_i \cap \partial\Omega_j) \setminus \partial\Omega_D. \tag{2.7}$$

Here, we denote by $\partial\Omega_D \subseteq \partial\Omega$ the part of the boundary where Dirichlet type boundary conditions are given. Let us remark that we have the identity

$$F(u) = \overline{R}^T K(\overline{R}u) - \overline{R}^T f,$$

where $W = W_1 \times \cdots \times W_N$, $\overline{R}_i : V \to W_i$, $\overline{R} = \left(\overline{R}_1^T, ..., \overline{R}_N^T\right)^T$, $f = \left(f_1^T, ..., f_N^T\right)^T$, and $K(\overline{R}u) = \left(K_1(R_1u)^T, ..., K_N(R_Nu)^T\right)^T$. In FETI-DP, the the set of interface variables is partitioned into dual variables (index set Δ) and primal variables (index set Π). The primal variables or primal constraints can be subdomain vertices as well as (weighted) edge or face averages and can be interpreted as the coarse constraints of FETI-DP. For completeness, the degrees of freedom belonging to nodes in the interior of the subdomains are collected in the index set I. We now define the space $\widetilde{V}$ of functions, which are assembled and therefore continuous in all primal variables, but not on the remaining interface. With the restriction $\check{R} : W \to \widetilde{V}$, $\tilde{u} \in \widetilde{V}$, and the jump matrix $B : W \to \text{range}(B)$, which computes the jump across the interface of functions from W (see [26, 35] for a detailed definition), we can define

$$\widetilde{K}(\tilde{u}) := \check{R}^T K(\check{R}\tilde{u}),\ \tilde{f} := \check{R}^T f.$$

Introducing Lagrangian multipliers λ, we obtain the nonlinear saddle point system

$$A(\tilde{u}, \lambda) := \begin{pmatrix} \widetilde{K}(\tilde{u}) + \check{R}^T B^T \lambda - \tilde{f} \\ B\check{R}\tilde{u} \end{pmatrix}.$$

The familiy of nonlinear FETI-DP methods is then defined as solving

$$A(M(\tilde{u}, \lambda)) = 0$$

with Newton's method, where M is a nonlinear right-preconditioner; see [21], where this general framework has been introduced. Let us note that this constitutes the outer Newton loop of nonlinear FETI-DP. The preconditioner M is usually linear in λ, i.e., $M(\tilde{u}, \lambda) = \left(M_{\tilde{u}}\left(\tilde{u}, \lambda\right), \lambda\right)$. In general, $M_{\tilde{u}}$ is implicitly defined by a nonlinear elimination of a subset of variables of $\tilde{u}$ and several choices are discussed in [21]; see also [22] for an extension to an adaptively chosen elimination strategy. Here, we only consider the special case of Nonlinear-FETI-DP-2. In this case, $\tilde{u}$ is eliminated completely and $M_{\tilde{u}}$ is defined by

$$\widetilde{K}\left(M_{\tilde{u}}\left(\tilde{u}, \lambda\right)\right) + \check{R}^T B^T \lambda - \tilde{f} = 0. \tag{2.8}$$

Consequently, in the k-th outer Newton iteration with the iterate $(\tilde{u}^{(k)}, \lambda^{(k)})$, equation (2.8) has to be solved for $M_{\tilde{u}}\left(\tilde{u}^{(k)}, \lambda^{(k)}\right)$, using again Newton's method. We refer to this Newton iteration as the inner iteration of Nonlinear-FETI-DP-2. Let us remark that, in contrast to H1-RASPEN, a distinction between local and coarse iterations makes no sense. In particular, in each inner iteration of Nonlinear-FETI-DP-2, all subdomain problems and the FETI-DP coarse problem have to be solved, that is, a problem which is globally coupled through the coarse problem. After convergence of the outer loop against $(\tilde{u}^*, \lambda^*)$, the solution of the original system (1.1) can be obtained by computing

$$u^* = \left(\overline{R}^T \overline{R}\right)^{-1} \overline{R}^T \check{R} M_{\tilde{u}}(\tilde{u}^*, \lambda^*).$$

Let us finally remark that the Jacobian in each step of the outer Newton iteration of Nonlinear-FETI-DP-2 is identical to the Jacobian arising in a Newton-Krylov-FETI-DP approach, applying Newton's methods to (1.1) and using FETI-DP for the linear solves; only the right hand side differs. Therefore, applying a linear Dirichlet preconditioner M_D^{-1} as usual in linear FETI-DP, the linearized system in Nonlinear-FETI-DP-2 can be solved by any linear FETI-DP implementation, that is, using a Krylov subspace method to solve the preconditioned FETI-DP system iteratively. Throughout this article, we use the preconditioned conjugate gradient (pcg) method, since deriving the considered model problems always results in symmetric and positive definite Jacobian matrices.

The Dirichlet preconditioner writes

$$M_D^{-1} = \sum_{i=1}^{N} B_D^{(i)} S^{(i)} B_D^{(i)T}. \tag{2.9}$$

Here, $S^{(i)}$ is the Schur complement of $DK_i(\check{R}_i \tilde{u}_i^{(k)})$ with respect to the interface, where $DK_i(\cdot)$ is the Jacobian matrix of $K_i(\cdot)$, and $\check{R}_i$ and $\tilde{u}_i^{(k)}$ are the restrictions of $\check{R}$ and, respectively, $\tilde{u}^{(k)}$ to subdomain Ω_i. Finally, $B_D^{(i)}$ is the scaled local jump matrix $B^{(i)}$, with $B = (B^{(1)}, ..., B^{(N)})$ and $B_D = (B_D^{(1)}, ..., B_D^{(N)})$. For a description of different scalings, we refer to [27, 23]. Throughout this paper, we exclusively use ρ-scaling.

2.3. Implementation of arbitrary coarse spaces in nonlinear FETI-DP. In the linear case, there are several approaches to enforce complex coarse constraints, as, e.g., adaptive constraints, in FETI-DP. A coarse constraint always demands continuity for a (weighted) sum of interface degrees of freedom. For example, for an edge between two subdomains in two spatial dimensions, the (weighted) sum of all edge degrees should be identical for both adjacent subdomains.

Here, in the nonlinear case, we consider a transformation of basis approach to implement arbitrary coarse constraints. We denote the space containing all transformed functions by W_T and we assume to have an orthogonal matrix $T : W_T \to W$. The transformation matrix and the transformed space W_T are chosen such that each coarse constraint corresponds to a single basis function of W_T, i.e., to a single degree of freedom in W_T. Computing T is pretty simple by orthogonalizing the chosen coarse constraints in W against the standard finite element basis of W, that is, orthogonalizing the coarse constraints against the identity matrix edge by edge; see [24, 25] for details. The advantage of this approach is that one can simply enforce the coarse constraints in W_T by assembling the corresponding basis functions, or, in other words, by

choosing these degrees of freedom as primal variables. The assembly process can then again be performed using a simple restriction operator $\check{R}$. The obtained primally assembled space is referred to as $\widetilde{V}_T$ in the following. The transformation of basis approach can also be used to transform the nonlinear FETI-DP saddle point system; see also [19]. We can define a nonlinear saddle point system in the transformed space by

$$A_T(\tilde{u}_T, \lambda) := \begin{pmatrix} \widetilde{K}_T(\tilde{u}_T) + \check{R}^T T^T B^T \lambda - \tilde{f} \\ BT\check{R}\tilde{u}_T \end{pmatrix},$$

with $\tilde{u}_T \in \widetilde{V}_T$ and $\widetilde{K}_T(\tilde{u}_T) := \check{R}^T T^T K(T\check{R}\tilde{u}_T)$. With the nonlinear preconditioner $M_{\tilde{u}_T}$, implicitly defined by

$$\widetilde{K}_T\left(M_{\tilde{u_T}}\left(\tilde{u}_T, \lambda\right)\right) + \check{R}^T T^T B^T \lambda - \tilde{f} = 0, \tag{2.10}$$

Nonlinear-FETI-DP-2 is again defined by solving

$$A_T(M_{\tilde{u}_T}(\tilde{u}_T, \lambda), \lambda) = 0$$

with Newton's method. Let us remark that the iterates in all inner and outer Newton steps now fulfill the chosen coarse constraints. Therefore, the specific choice of the coarse space or the coarse constraints has a direct influence on the nonlinear convergence of Nonlinear-FETI-DP-2 and thus can be interpreted as a nonlinear coarse space.

The final solution of the original system (1.1) can finally be obtained by

$$u^* = \left(\overline{R}^T \overline{R}\right)^{-1} \overline{R}^T T \check{R} M_{\tilde{u}_T}(\tilde{u}_T^*, \lambda^*).$$

In practice, it can be quite expensive to explicitly compute the transformation matrix T for large subdomains, at least in three spatial dimensions. Additionally, the transformed Jacobian matrix $T^T\, DK(\cdot)\, T$ tends to be denser than $DK(\cdot)$, especially in three dimensions when including coarse constraints on subdomain faces. To avoid these disadvantages, one can replace the linearized systems of both the inner and outer Newton loops by equivalent linear systems operating in the original finite element space W or, respectively, $\widetilde{V}$. In this context, equivalent means that the solutions of a linearized system in W_T can be simply obtained by a multiplication of T^T with the solution of the equivalent system in W and vice versa by a multiplication with T. Therefore, neither the linear nor the nonlinear convergence is affected by the specific implementation of the linearized systems but only the computing time. Since we only consider iteration counts of our MATLAB implementations, we will not go into further details here; see [18] for a full description of the computationally more efficient reformulation used to implement the linearized systems in the inner and outer loop.

3. Adaptive coarse spaces. Classical coarse spaces, such as Lagrangian coarse spaces for Schwarz methods or vertex and edge-average constraints for FETI-DP and BDDC methods, are generally not robust for highly heterogeneous model problems, for example, with high jumps in the coefficient function; cf. subsection 4.1. When using adaptive coarse spaces, which are based on adaptively enriching the coarse space by additional functions or constraints, robust convergence can be retained. Therefore, local generalized eigenvalue problems are solved, eigenfunctions are selected based on a user chosen tolerance TOL, and coarse basis functions or constraints are constructed

based on these eigenfunctions. Finally, for adaptive GDSW and adaptive FETI-DP, respectively, we obtain a condition number bound of the form

$$\kappa \leq C \left(1 + \frac{1}{TOL}\right) \quad \text{or} \quad \kappa \leq C\,(1 + TOL)\,,$$

respectively, where the positive constant C is independent of the coefficient function of the problem. Since the tangent problems arising in the Newton iteration of the nonlinear DDMs described in section 2 are somewhat related to the preconditioned systems for the respective linear DDMs, we will use the adaptive coarse spaces also in our nonlinear DDMs.

Note that heuristic approaches, which can be constructed without the solution of local eigenvalue problems but lack a rigorous condition number bound, such as the approaches described in [6, 28] for Schwarz methods as well as the frugal coarse space [12] for FETI-DP and BDDC methods, are not considered here but could be applied as well.

3.1. Adaptive GDSW coarse space. In order to obtain a robust convergence of the H1-RASPEN method for highly heterogeneous problems, we employ the adaptive GDSW coarse space [9, 7]. As we will observe in section 4, using the AGDSW coarse space will not only significantly improve the linear convergence but also the nonlinear convergence of the two-level RASPEN method compared to using coarse spaces based on MsFEM [15] as suggested in [8, 14, 13]; the MsFEM coarse spaces do not contain adaptive coarse basis functions. Here, we will briefly recall the construction of the AGDSW coarse space in two dimensions without going into details about the application in a linear two-level Schwarz method. For more details on the AGDSW coarse space and a variant with reduced dimension, RAGDSW, we refer to [9, 7] and [10], respectively.

In order to construct the AGDSW coarse space, we first partition the interface Γ of the nonoverlapping domain decomposition into edges and vertices. The coarse basis functions Φ will then be constructed in two steps: first, the interface values Φ_Γ are defined, and then, the values in the interior of the nonoverlapping subdomains are computed using energy minimizing extensions

$$\Phi_I = -A_{II}^{-1} A_{I\Gamma} \Phi_\Gamma. \tag{3.1}$$

Here A_{II} and $A_{I\Gamma}$ are submatrices of the global matrix

$$A = \begin{bmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix}$$

with the indices corresponding to the interior degrees of freedom (DOFs) I and the indices corresponding to the interface DOFs Γ; following the definition of the interface (2.7), the Dirichlet DOFs are not part of the interface Γ and therefore as part of I. Here, we always choose $A := DF(u^{(0)})$, i.e., the linearization of the original problem in the initial value of the Newton iteration; see [13] for alternative choices for A.

The interface values Φ_Γ of the coarse basis functions are defined differently for the edges and vertices. The definition for the vertices is very simple: Let v be a vertex. Then, the corresponding basis function is chosen to be 1 at the vertex and 0 everywhere else on the interface. In order to compute the values of the edge coarse basis functions, local generalized eigenvalue problems are solve. In particular, let $\mathcal{E}$

be an edge and Ω_i and Ω_j be the two adjacent nonoverlapping subdomains. Then, we consider the two matrices

$$S_{\mathcal{E}}^{(i,j)} := A_{\mathcal{E}\,\mathcal{E}}^{(i,j)} - A_{\mathcal{E}\,\mathcal{R}}^{(i,j)} \left(A_{\mathcal{R}\,\mathcal{R}}^{(i,j)}\right)^{-1} A_{\mathcal{R}\,\mathcal{E}}^{(i,j)}$$

and $A_{\mathcal{E}\mathcal{E}}^{(i,j)}$, where $A^{(i,j)}$ is the Neumann matrix on $\Omega^{(i,j)} = \overline{\Omega_i \cup \Omega_j}$, $\mathcal{E}$ corresponds to the DOFs on the (interior) edge $\mathcal{E}$, and $\mathcal{R}$ corresponds to all remaining DOFs of $\Omega^{(i,j)}$. Using these matrices, we solve the generalized eigenvalue problem: find $(\tau_{\mathcal{E}}, \mu_{\mathcal{E}}) \in V_0^h(\mathcal{E}) \times \mathbb{R}$ such that

$$\theta^T S_{\mathcal{E}}^{(i,j)} \tau_{\mathcal{E}} = \lambda_{\mathcal{E}}^{-1}\, \theta^T A_{\mathcal{E}\,\mathcal{E}}^{(i,j)} \tau_{\mathcal{E}} \quad \forall \theta \in V_0^h(\mathcal{E}). \tag{3.2}$$

Here, $V_0^h(\mathcal{E})$ is the finite element space on the interior nodes of the edge. Now, let the eigenvalues $\lambda_{\mathcal{E}}$ of the eigenvalue problem (3.2) be sorted in nondescending order. Then, we select all eigenpairs $(\lambda_{\mathcal{E}}, \tau_{\mathcal{E}})$ with $\lambda_{\mathcal{E}}$ below a user-chosen tolerance $tol_{\mathcal{E}}$, and each of the $\tau_{\mathcal{E}}$ defines the interface values for one of the coarse edge basis functions. Note that a reduced dimension variant, can be obtained by considering other interface subsets and a slight modification of the right hand side matrix $A_{\mathcal{E}\,\mathcal{E}}^{(i,j)}$; cf. [10] for details.

After extending the interface values of the vertex and edge basis functions in an energy-minimizing way, i.e., by (3.1), we collect them as columns of the matrix Φ, which then span the AGDSW coarse space. As mentioned earlier, if this coarse space is used in a linear two-level Schwarz preconditioner to solve a two- or three-dimensional diffusion or linear elasticity problem, we obtain a condition number bound of the form

$$\kappa \leq C \left(1 + \frac{1}{tol_{\mathcal{E}}}\right)$$

for the preconditioned linear system; cf. [9, 10]. The constant C may depend on the number of overlapping subdomains each point $x \in \Omega$ can belong to. However, all constants are independent of H, h, and the contrast of the coefficient function.

In our nonlinear iteration, we reuse the coarse basis functions computed using the first linearization $A := DF(u^{(0)})$ and solve the nonlinear coarse problem in each iteration of the H1-RASPEN approach using this coarse basis.

3.2. An adaptive FETI-DP coarse space. Similar to the previous subsection, we aim for a robust nonlinear DDM by enhancing the nonlinear FETI-DP coarse space by certain adaptive constraints. In this section, we give a brief description of the adaptive coarse space introduced in [29] and first fully analyzed in [23]. First, we introduce the relevant notation and the eigenvalue problem on an edge. Second, in (3.4), we give an estimate of the condition number for two-dimensional problems where all the vertex variables are primal in the initial coarse space. This is exactly the coarse space we use in our numerical experiments. Let us remark that the adaptive constraints additionally enhance vertex constraints, i.e., the assembly in the primal vertices. As already mentioned above, there are several possibilities in the literature to implement such additional constraints for linear problems. We use a transformation of basis approach here, which is currently the only possibility to enforce additional or adaptive constraints in nonlinear FETI-DP methods.

To compute the adaptive coarse constraints, similar to AGDSW, for each edge $\mathcal{E}$ shared by the subdomains Ω_i and Ω_j, a single eigenvalue problem has to be solved. We first restrict the jump matrix B to this edge. Let $B_{\mathcal{E}} = \left(B_{\mathcal{E}}^{(i)},\, B_{\mathcal{E}}^{(j)}\right)$ be the

submatrix of $\left(B^{(i)},\ B^{(j)}\right)$ with the rows that consist of exactly one 1 and one -1 and are zero otherwise. Let $B_{D,\mathcal{E}} = \left(B^{(i)}_{D,\mathcal{E}},\ B^{(j)}_{D,\mathcal{E}}\right)$ be obtained by taking the same rows of $\left(B^{(i)}_{D},\ B^{(j)}_{D}\right)$. Furthermore, let

$$S_{ij} = \begin{pmatrix} S^{(i)} & \\ & S^{(j)} \end{pmatrix},$$

where $S^{(i)}$ and $S^{(j)}$ are the Schur complements of $DK^{(i)}(\check{R}_i\tilde{u}_i^{(k)})$ and $DK^{(j)}(\check{R}_i\tilde{u}_i^{(k)})$, respectively, with respect to the interface variables. We further define the operator $P_{D_{ij}} = B^T_{D,\mathcal{E}}B_{\mathcal{E}}$.

Then, we solve the local generalized eigenvalue problem: find $w_{ij} \in (\ker\ S_{ij})^{\perp}$

$$\langle P_{D_{ij}}v_{ij},\ S_{ij}P_{D_{ij}}w_{ij}\rangle = \mu_{ij}\langle v_{ij},\ S_{ij}w_{ij}\rangle \quad \forall v_{ij} \in (\ker\ S_{ij})^{\perp}\,. \tag{3.3}$$

For an explicit expression of the positive definite right hand side operator on the subspace $(\ker\ S_{ij})^{\perp}$, two orthogonal projections are used; see, e.g., [32]. We assume that R eigenvectors w^r_{ij}, $r = 1, ..., R$, belong to eigenvalues which are larger than a given tolerance tol. Then, we enhance the nonlinear FETI-DP coarse space with the adaptive constraints $B_{D_{ij}}S_{ij}P_{D_{ij}}w^r_{ij}$, $r = 1, ..., R$. Let us remark that, as for the AGDSW coarse space, we only compute the adaptive coarse space once, and therefore we use the Jacobian in the initial value $DK(\tilde{R}\tilde{u}^{(0)})$ for the computation of all Schur complements.

In the linear case, we obtain the condition number bound

$$\kappa \leq N_E^2 tol \tag{3.4}$$

for the preconditioned FETI-DP system if the adaptive constraints are implemented by, for example, a transformation of basis approach or using a balancing preconditioner. Here, N_E is the maximum number of edges of a subdomain and the condition number bound is thus completely independent of the coefficient function; see [23] for a full proof of this condition number estimate in two dimensions.

4. Model Problems and Numerical Results. In this section, we will first introduce the heterogeneous p-Laplace model problem, which will be considered in our numerical experiments. The different settings are then obtained by varying the coefficient distribution. Second, we will present numerical results for the nonlinear adaptive DDMs. We note that our focus is on the comparison of the nonlinear adaptive DDMs with their respective NK counterparts. We also want to analyze the positive effects of a well-chosen adaptive coarse space on the convergence of the nonlinear DDMs in comparison to their original counterparts using traditional coarse spaces. However, a fair comparison of the nonlinear Schwarz and FETI-DP approaches is difficult, and we will refrain from attempting to carry out such a comparison in this work.

4.1. Model Problems. We consider the nonlinear model problem:

$$\begin{aligned} -\alpha\Delta_p u &= 1 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

with the scaled p-Laplace operator $\alpha\Delta_p u := \operatorname{div}(\alpha|\nabla u|^{p-2}\nabla u)$. Within this article,if not stated otherwise, we use $p = 4$ and a coefficient function $\alpha : \Omega \to \mathbb{R}$ with jumps.

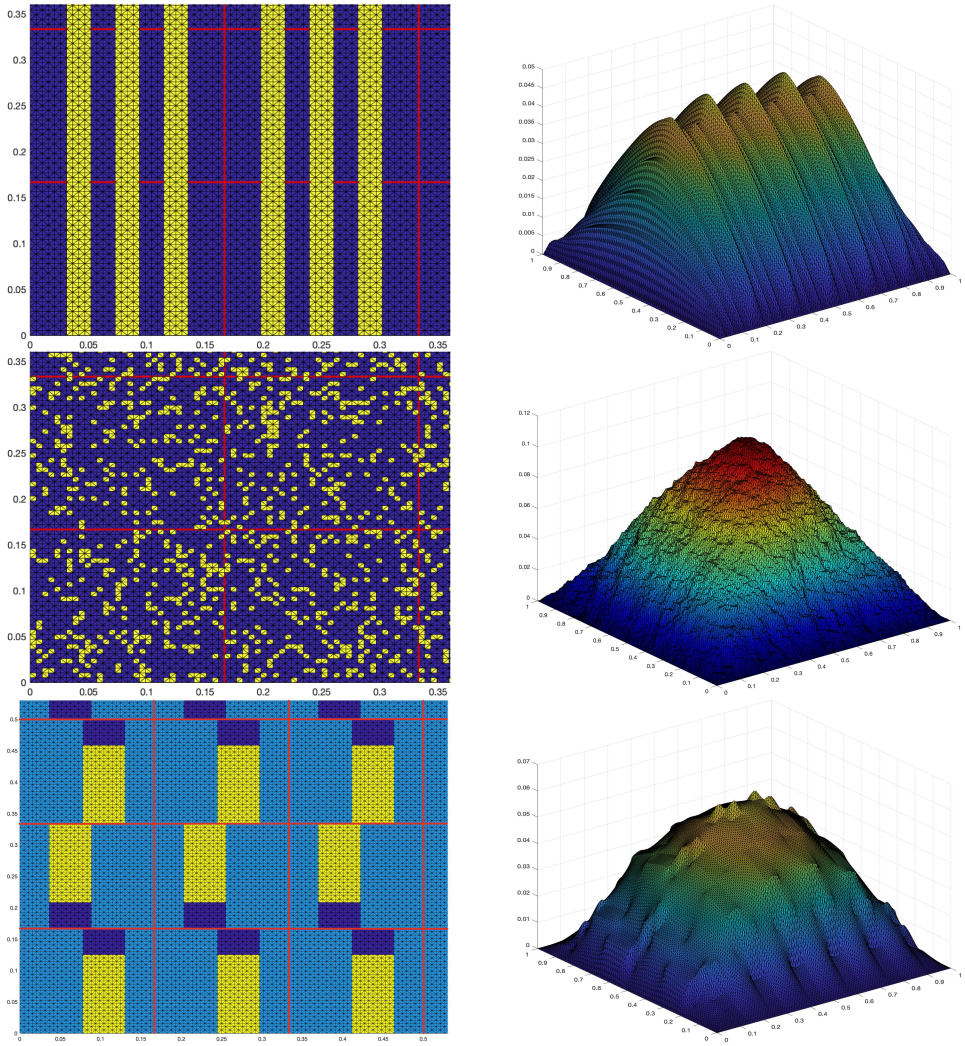

Fig. 1: Coefficient distributions and corresponding solutions of (4.1); subsections showing 4 or, respectively, 9 subdomains out of the total 36 subdomains; $H/h = 32$. **Top:** three channels cutting through each subdomain; $\alpha = 1e3$ within the yellow part and $\alpha = 1$ within the blue part. **Middle:** randomly generated coefficient distribution with 20% yellow elements ($\alpha = 1e6$); $\alpha = 1$ in the remaining blue part. **Bottom:** shifted boxes around all horizontal edges; $\alpha = 1e3$ within the yellow part, $\alpha = 1$ within the blue and light blue part, and $p = 2$ in the light blue part.

Moreover, we always use the unit square $\Omega = [0, 1] \times [0, 1]$ as the computational domain, a discretization with piecewise lineare finite elements, and a structured domain decomposition into square subdomains. However, our approaches are not restricted to this case. We consider three different heterogeneous coefficient distributions; see Figure 1 for a visualization and more details.

Table 1: Results for two of the model problems, all considered Schwarz methods, and all corresponding coarse spaces; **outer it.** gives the number of global Newton iterations; **local it.** gives the number of local Newton iterations summed up over the outer Newton iterations (average over all subdomains); **coarse it.** gives the number of nonlinear iterations on the coarse level summed up over the outer Newton iterations; **GMRES it.** gives the number of GMRES iterations summed up over the outer Newton iterations; **size cp** gives the number of coarse basis functions, i.e., the size of the coarse problem.

Channels; see Figure 1 (top) $p = 4$; $H/h = 32$; 36 subdomains; overlap $\delta = 1$;						
without globalization						
size cp	**method**	**coarse space**	**outer it.**	**local it. (avg.)**	**coarse it.**	**GMRES it. (sum)**
145	H1-RASPEN	AGDSW	5	27.0	35	77
25	H1-RASPEN	MsFEM-D	>20	-	-	-
25	H1-RASPEN	MsFEM-E	>20	-	-	-
145	NK-RAS	AGDSW	>20	-	-	-
with globalization (INB)						
145	H1-RASPEN	AGDSW	5	24.8	21	77
25	H1-RASPEN	MsFEM-D	15	75.8	62	645
25	H1-RASPEN	MsFEM-E	18	83.9	75	852
145	NK-RAS	AGDSW	13	-	-	207
Random; see Figure 1 (middle) $p = 4$; $H/h = 32$; 36 subdomains; overlap $\delta = 1$;						
without globalization						
size cp	**method**	**coarse space**	**outer it.**	**inner it. (avg.)**	**coarse it.**	**GMRES it. (sum)**
445	H1-RASPEN	AGDSW	5	29.9	38	89
25	H1-RASPEN	MsFEM-D	>20	-	-	-
25	H1-RASPEN	MsFEM-E	>20	-	-	-
445	NK-RAS	AGDSW	20	-	-	414
with globalization (INB)						
445	H1-RASPEN	AGDSW	5	28.5	31	91
25	H1-RASPEN	MsFEM-D	>20	-	-	-
25	H1-RASPEN	MsFEM-E	11	67.7	67	1097
445	NK-RAS	AGDSW	12	-	-	238

4.2. Two-level RASPEN. We compare the standard Newton-Krylov-RAS (Restricted Additive Schwarz) approach with an AGDSW coarse space to the H1-RASPEN method using three different coarse spaces, i.e., the AGDSW adaptive coarse space and two MsFEM type coarse spaces, MsFEM-D and MsFEM-E; see also [13] for a definition of the two MsFEM type coarse spaces. As for the AGDSW coarse space, we use $A := DF(u^{(0)})$ for the computation of the energy minimizing extensions in the MsFEM coarse spaces, and we keep these basis functions for the whole nonlinear iteration. We test all four approaches with or without additional globalization, that is, we either use the standard Newton method or the inexact Newton backtracking [4] (INB) method to solve all arising nonlinear problems. More precisely, in the case of

Table 2: Results for the model problem with randomized coefficient distribution using H1-RASPEN with AGDSW coarse space; effect of varying overlap $\delta = 0, 1, 2, 4$; **outer it.** gives the number of global Newton iterations; **local it.** gives the number of local Newton iterations summed up over the outer Newton iterations (average, minimum, and maximum over subdomains); **coarse it.** gives the number of nonlinear iterations on the coarse level summed up over the outer Newton iterations; **GMRES it.** gives the number of GMRES iterations averaged up over the outer Newton iterations; **size cp** gives the number of coarse basis functions, i.e., the size of the coarse problem.

Random; see Figure 1 (middle) $p = 4$; $H/h = 32$; 36 subdomains; varying overlap $\delta = 0, 1, 2, 4$					
H1-RASPEN; AGDSW; with globalization (INB)					
size overlap	**size cp**	**outer it.**	**local it. (min/max/avg)**	**coarse it.**	**GMRES it. (avg)**
$\delta = 0$	445	5	23/41/30.1	32	21.4
$\delta = 1$	445	5	23/40/28.5	31	18.2
$\delta = 2$	445	4	19/31/24.2	24	16.3
$\delta = 4$	445	7	26/50/32.9	39	13.7

Table 3: Results for the model problem with randomized coefficient distribution using H1-RASPEN with AGDSW coarse space; effect of varying tolerance $tol_{\mathcal{E}}$; **outer it.** gives the number of global Newton iterations; **local it.** gives the number of local Newton iterations summed up over the outer Newton iterations (average, minimum, and maximum over subdomains); **coarse it.** gives the number of nonlinear iterations on the coarse level summed up over the outer Newton iterations; **GMRES it.** gives the number of GMRES iterations summed up over the outer Newton iterations; **size cp** gives the number of coarse basis functions, i.e., the size of the coarse problem.

Random; see Figure 1 (middle) $p = 4$; $H/h = 32$; 36 subdomains; varying tolerance $tol_{\mathcal{E}} = 0.01, 0.1, 0.2$; $\delta = 2$					
H1-RASPEN; AGDSW; with globalization (INB)					
$tol_{\mathcal{E}}$	**size cp**	**outer it.**	**local it. (min/max/avg)**	**coarse it.**	**GMRES it. (sum)**
1e-5	364	6	25/42/30.6	33	113
1e-3	444	4	19/31/24.2	24	65
1e-1	445	4	19/31/24.2	24	65
2e-1	461	4	19/34/24.3	24	64

H1-RASPEN, we use INB to solve all local nonlinear problems, the nonlinear global problem, and the nonlinear coarse problem; see [13, subsection 7.3] for details on the INB method and the choice of parameters. Unless stated otherwise, we always use an overlap of one layer of finite elements in the first level of H1-RASPEN and choose $tol_{\mathcal{E}} = 0.1$ in the computation of the AGDSW coarse space; note that we did not optimize this parameter for the best performance. To guarantee a fair comparison, we choose a relative reduction of 10^{-6} of the residual $F(u^{(k)})$ as the stopping critirion for all tested methods. For all local or coarse solves, we terminate after a relative residual reduction of 10^{-3}.

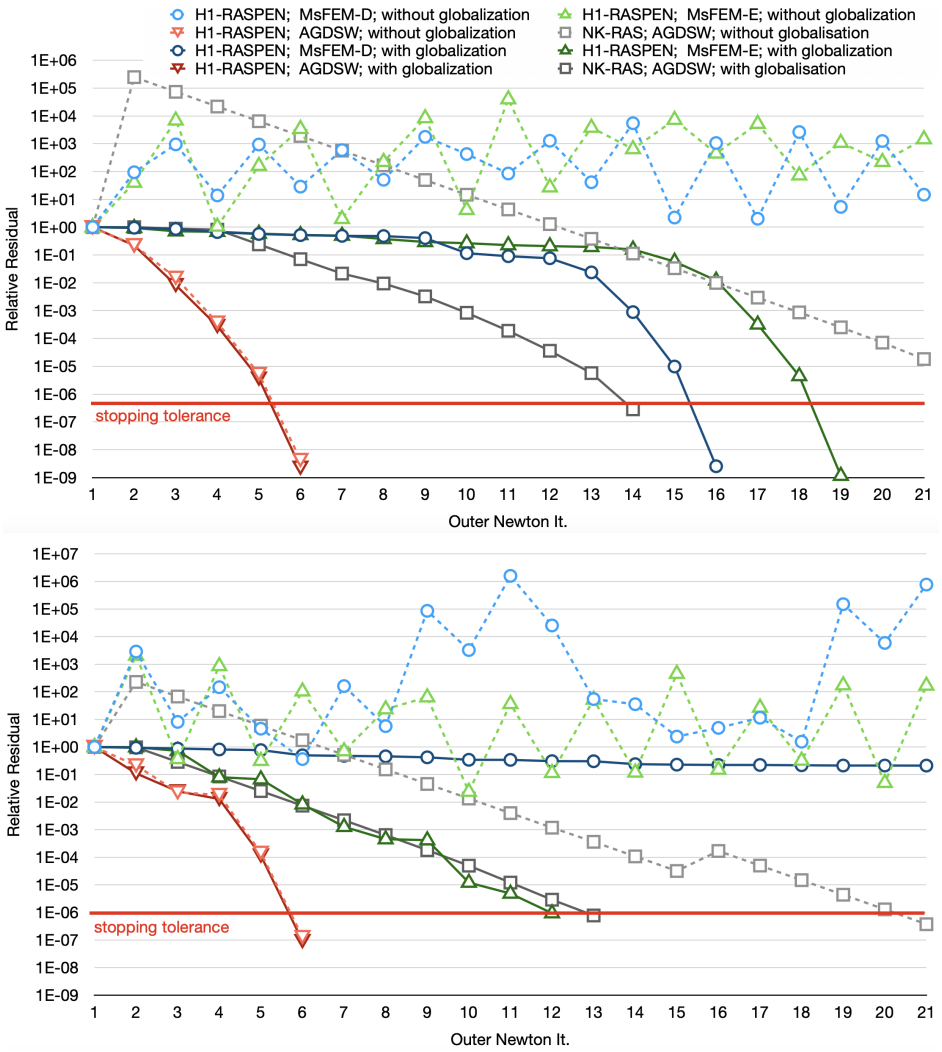

Fig. 2: Convergence behavior of the outer Newton iteration; solid lines represent methods with globalization (INB), dashed lines without globalization. **Top:** Problem with channels; see Figure 1 (top). **Bottom:** Problem with randomly generated coefficient distribution; see Figure 1 (middle). For more details, see also Table 1.

The results for the coefficient distributions in Figure 1 (top) and Figure 1 (middle) can be found in Table 1. In both cases, the H1-RASPEN method with the appropriate coarse space (AGDSW) shows the best nonlinear and linear convergence, even if the standard Newton method is used instead of INB. All other methods show a much slower nonlinear convergence; see also Figure 2. Of course, the AGDSW coarse space is, in general, larger compared to the MsFEM coarse spaces: in both cases, we have one coarse basis function for each vertex, however, the AGDSW coarse space also contains additional edge basis functions, which provide robustness with respect to coefficient jumps at the edges.

Let us remark that, for the considered model problems, it is sufficient to compute the AGDSW coarse basis functions once for the initial Jacobian matrix $DF(u^{(0)})$; thus, the comparably expensive setup has to be performed only once. Nonetheless, the condition number of all linearized systems is sufficiently small and the linear and nonlinear convergence is convincing. However, for different types of model problems, it might be necessary to recompute the coarse space in a later stage of the iteration, e.g., if new nonlinearities evolve, but for many problems this will not be necessary or, at least, only necessary in a few outer Newton iterations. A heuristic strategy how to decide when a new computation of the adaptive coarse space is necessary within a classical Newton-Krylov-FETI-DP approach has been introduced in [17] and can be simply modified for the use within adaptive nonlinear DDMs.

In Table 2, we provide additional results for a varying width of the overlap δ for the model problem with random coefficient distribution. As expected, the linear convergence accelerates with growing δ. Unexpectedly, for this specific problem, this is not the case for the nonlinear convergence, where an optimum is reached for $\delta = 2$; this effect has to be investigated in more detail in the future.

Finally, we also investigate the influence of the tolerance chosen in the local eigenvalue problems, that is, the influence of the size of the adaptive coarse problem; see Table 3. As we expected, if the adaptive coarse space gets too small, both, the linear and nonlinear convergence start to deteriorate. Moreover, as soon as all necessary basis functions corresponding to bad eigenvalues have been included in the coarse space, adding additional eigenfunctions does not lead to further significant improvements in the convergence.

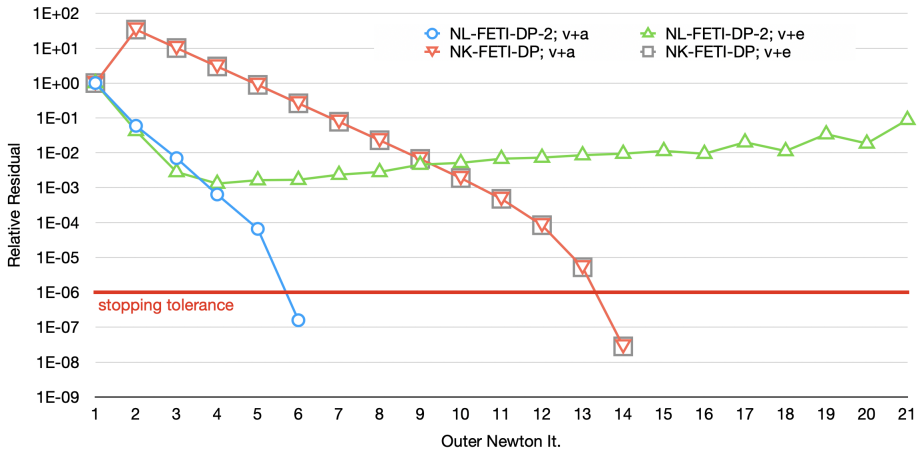

Fig. 3: Convergence behavior of the outer Newton iteration. Problem with shifted boxes; see Figure 1 (bottom). For more details, see also Table 4.

4.3. Nonlinear FETI-DP. Finally, we compare the standard Newton-Krylov-FETI-DP (NK-FETI-DP) to the Nonlinear-FETI-DP-2 (NL-FETI-DP-2) method using two different coarse spaces, that is, vertex constraints and classical edge averages as well as vertex constraints (v+e) and adaptive edge constraints (v+a). Let us remark that we do not consider an INB approach to improve the nonlinear FETI-DP methods. For the inner loop, INB could directly be implemented and has a similar positive effect on the convergence of the inner loop as previously shown for H1-RASPEN;

Table 4: Results for both model problems, all considered nonlinear FETI-DP methods, and all corresponding coarse spaces; **outer it.** gives the number of global Newton iterations; **inner it.** gives the number of inner Newton iterations summed up over the outer Newton iterations; **CG it.** gives the number of CG iterations summed up over the outer Newton iterations; **size cp** gives the number of coarse basis functions, i.e., the size of the coarse problem; **v + e** stands for vertex and edge average constraints; **v + a** stands for vertex and adaptive constraints.

Channels; see Figure 1 (top) $p = 4$; $H/h = 32$; 36 subdomains; $tol = 5$					
size cp	**method**	**coarse space**	**outer it.**	**inner it.**	**CG it. (sum)**
85	NL-FETI-DP-2	v+e	>20	-	-
99	NL-FETI-DP-2	v+a	4	29	41
85	NK-FETI-DP	v+e	>20	-	-
99	NK-FETI-DP	v+a	>20	-	-
Shifted boxes; see Figure 1 (bottom) $p = 4$; $H/h = 32$; 36 subdomains; $tol = 5$					
size cp	**method**	**coarse space**	**outer it.**	**inner it.**	**CG it. (sum)**
85	NL-FETI-DP-2	v+e	>20	-	-
269	NL-FETI-DP-2	v+a	5	28	77
85	NK-FETI-DP	v+e	13	-	477
269	NK-FETI-DP	v+a	13	-	279

cf. Table 1. In contrast, for the outer loop, it is rather complicated to find an efficient globalization approach. As for H1-RASPEN, directly applying INB and using the residual of the nonlinear saddle point problem results in many inner Newton iterations, one for each step length tested; see [13] for details. However, in contrast to H1-RASPEN, simply replacing the residual of the nonlinear saddle point system by the residual of the original problem (1.1) does not lead to satisfying results since both residuals differ drastically. Discussing more complex globalization strategies for nonlinear FETI-DP is out of the scope of this article and, as can also be observed for H1-RASPEN, adding a globalization approach does not seem necessary when an adaptive coarse space is used.

If not stated otherwise, we always choose $tol = 5$ in the computation of the adaptive FETI-DP coarse space. To guarantee a fair comparison, we again choose a relative reduction of 10^{-6} of the residual $F(u^{(k)})$ as the stopping criterion for all tested methods. Let us note that for, the Nonlinear-FETI-DP-2 approach, we use $u^{(k)} = \left(\overline{R}^T\overline{R}\right)^{-1}\overline{R}^T T\check{R}\tilde{u}_T^{(k)}$ to evaluate the global residual of the original problem. For all inner solves, we use a relative residual reduction of 10^{-3}.

The results for two of the considered model problems can be found in Table 4. In both cases, the Nonlinear-FETI-DP-2 method with the adaptive coarse space shows the best nonlinear and linear convergence. All other methods have a much slower nonlinear convergence behavior; see also Figure 3. Of course, the adaptive coarse space can be larger compared to the classical FETI-DP coarse space with vertex constraints and edge averages. As for the H1-RASPEN approach, it is sufficient to

Table 5: Results for the model problem with shifted boxes using NL-FETI-DP-2 and varying the tolerance *tol* in the computation of the adaptive coarse space; **outer it.** gives the number of global Newton iterations; **inner it.** gives the number of inner Newton iterations summed up over the outer Newton iterations; **CG it.** gives the number of CG iterations summed up over the outer Newton iterations; **size cp** gives the number of coarse basis functions, i.e., the size of the coarse problem; **tol** gives the tolerance used in the computation of the adaptive coarse space; **v + e** stands for vertex and edge average constraints; **v + a** stands for vertex and adaptive constraints.

Shifted boxes; see Figure 1 (top) $p = 4$; $H/h = 32$; 36 subdomains; varying *tol*						
tol	**size cp**	**method**	**coarse space**	**outer it.**	**inner it.**	**CG it. (sum)**
2	353	NL-FETI-DP-2	v+a	4	18	43
5	269	NL-FETI-DP-2	v+a	5	28	77
10	73	NL-FETI-DP-2	v+a	5	25	112
100	35	NL-FETI-DP-2	v+a	>20	-	-

compute the adaptive FETI-DP coarse basis functions once for the initial Jacobian matrix $DK(u^{(0)})$. Again, the condition number of all linearized systems is sufficiently small, and both the linear and the nonlinear convergence are robust. Let us again remark that it might be necessary to recompute the coarse space in a later stage of the iteration for different types of model problems.

The adaptive coarse space for the model problem with shifted boxes computed using a tolerance of $tol = 5$ is fairly large compared to the classical coarse space. Choosing a larger tolerance of $tol = 10$ drastically reduces the size of the coarse space and leads to similar results; see Table 5 for a comparison of different tolerances. Let us remark that, as in the linear case, choosing an optimal tolerance is a nontrivial task; cf. [11]. Here, we will not further investigate this topic.

5. Conclusion. In summary, our tests for the combination of the H1-RASPEN method and the AGDSW coarse space on the one hand and the Nonlinear-FETI-DP-2 method and an edge-based adaptive coarse space on the other hand indicate that the combination of adaptive coarse spaces and nonlinear domain decomposition methods has a great potential to improve the robustness and the linear as well as the nonlinear convergence. Since the setup of the adaptive coarse space may only have to be performed once, or a few times, during the nonlinear iteration, this approach greatly enhances the benefit of adaptive coarse spaces. In particular, compared to the linear case, the high setup costs of adaptive coarse spaces may be compensated more clearly in the context of nonlinear preconditioning. Of course, scaling tests with a parallel implementation are necessary for a complete conclusion. An investigation of the performance of the suggested approaches is planned for the future.

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