Tall and Monotone Complexity One Spaces of Dimension Six

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Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit sechs-dimensionalen kompakten und monotonen Hamiltonischen T-Räumen der Komplexität eins. Im Fokus steht hierbei die Frage inwiefern sich die zugrundeliegenden Mannigfaltigkeiten solcher Räume von Fano Varietäten mit komplexer Dimension drei unterscheiden.
Der berühmte Konvexitätssatz von Atiyah und Guillemin-Sternberg besagt, dass das Bild der Impulsabbildung eines kompakten Hamiltonischen T-Raums ein konvexes Polytop ist. Die Komplexität eines Hamiltonischen T-Raums ist die ganze Zahl gegeben durch die Hälfte der Dimension der zugrunde liegenden Mannigfaltigkeit minus der Dimension des Torus T. Nach einer Arbeit von Delzant sind kompakte Hamiltonsche T-Räume der Komplexität null vollständig durch die Bilder ihrer Impulsabbildungen charakterisiert. Das Analogon dieses Resultates gilt nicht für kompakte Hamiltonsche T-Räume mit einer Komplexität größer null.
Ein kompakter Hamiltonischer T-Raum der Komplexität eins wird groß genannt, wenn jeder reduzierter Raum entweder leer oder eine topologische Fläche ist. Diese großen Hamiltonischen T-Räume der Komplexität eins wurden von Karshon und Tolman intensiv untersucht.
In dieser Arbeit zeigen wir, dass das Bild der Impulsabbildung eines kompakten, großen und monotonen Hamiltonischen T-Raum der Komplexität eins besondere Eigenschaften erfüllt. Ausgehend von diesem Resultat geben wir eine komplette Klassifikation von kompakten, großen und monotonen Hamiltonischen T-Räumen der Komplexität eins. Eine Konsequenz dieser Klassifikation ist, dass jeder sechsdimensionale kompakte, große und monotone Hamiltonsche T-Raum der Komplexität eins diffeomorph zu einer Fano Varietät ist.

Abstract

In this thesis we focus on six-dimensional compact and monotone Hamiltonian T-spaces of complexity one. In particular, we are interested to which extent the underlying manifolds of such spaces differ from Fano threefolds.

The famous Convexity Theorem by Atiyah and Guillemin-Sternberg states that the image of the moment map of a compact Hamiltonian T-space is a convex polytope. The complexity of a Hamiltonian T-space is the integer given by half of the dimension of the underlying manifold minus the dimension of the torus T. By a work of Delzant compact Hamiltonian T-spaces of complexity zero are completely classified by their moment map images. The analogue of this result fails to be true for compact Hamiltonian T-spaces of complexity greater than zero.

A compact Hamiltonian T-space of complexity one is called tall if each reduced space is two-dimensional. These spaces have been intensively studied by Karshon and Tolman.

We show that the moment map image of a compact, tall and monotone Hamiltonian T-space of complexity one satisfies special properties. Using these observations we give a complete classification of such spaces in dimension six. As a consequence of this classification it turns out that any six-dimensional compact, tall and monotone Hamiltonian T-space of complexity one is diffeomorphic to a Fano threefold.

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Chapter 1

Introduction

A symplectic manifold (M, ω) is a smooth manifold M together with a closed non-degenerate two form $\omega \in \Omega^2(M)$, called symplectic form. A smooth projective variety X is a compact complex manifold that admits a holomorphic embedding $i\colon X \to \mathbb{C}P^N$ into some complex projective space. Such an embedding induces a symplectic form on X, namely the pullback of the standard symplectic form on $\mathbb{C}P^N$ via i. Not all compact symplectic manifolds can be obtained in this way. But since projective varieties carry a lot of geometric and algebraic structures it is important to understand under which condition a symplectic manifold admits the structure of a complex variety. In particular, one is interested in understanding the relation between so called positive monotone symplectic manifolds and Fano varieties. A symplectic manifold is **(positive) monotone** if its first Chern class is a (positive) multiple of the class $[\omega]$ given by the symplectic form in the second de Rham group $H^2(M; \mathbb{R})$ of M. A Fano variety is a smooth projective variety whose anticanonical line bundle is ample. These varieties have been extensively studied in algebraic geometry. For example, in any dimension there exist just finitely many Fano varieties up to diffeomorphism (see [27]) and in low dimension they are completely classified (see [17]). One can show that any Fano variety is a positive monotone symplectic manifold. But the converse fails to be true in general if the real dimension of the symplectic manifold is greater than or equal to twelve (see [7], [37]). In low (real) dimensions, namely in dimension two and four, it is proven that any compact positive monotone symplectic manifold is diffeomorphic to a Fano variety (see [14], [33], [41]). For dimension six, eight and ten this is a completely open question.

In order to understand the relations between positive monotone symplectic manifolds and Fano varieties it is natural to investigate this problem first under the assumption that the symplectic manifold and the Fano variety admit suitable symmetries. Let us introduce such symmetries. In the following we denote by T a compact torus of dimension greater than or equal to one and by $T_{\mathbb{C}}$ its universal complexification.

On the algebraic side we consider holomorphic $T_{\mathbb{C}}$-actions on smooth projective varieties. On the symplectic side we consider Hamiltonian symmetries. Namely, let T be a compact torus that acts on a symplectic manifold (M, ω) via symplectomorphisms. The action is called Hamiltonian if there exists a moment map $\phi\colon M \to \mathfrak{t}^*$, where $\mathfrak{t}^*$ denotes the dual Lie algebra of T. Moreover, if the action is also effective, the quadruple (M, ω, T, ϕ) is called a Hamiltonian T-space. By a simple symplectic argument it follows that the dimension of the torus can be at most half of the dimension of the manifold. Therefore, the non-negative integer $k := \frac{1}{2}\dim(M) - \dim(T)$ is called complexity of the action. Intuitively, this means: the smaller the complexity, the larger the symmetry. Moreover, given a compact and monotone symplectic manifold, then the existence of an effective and Hamiltonian torus action forces that the underlying symplectic manifold is positive monotone. Hence, compact and monotone Hamiltonian T-spaces are positive monotone.

The action is called toric if its complexity is zero and if so the space is called a symplectic toric manifold. From the classification result for compact symplectic toric manifolds [6] it follows that monotone compact symplectic toric manifolds are T-equivariantly symplectomorphic to toric Fano varieties. This motivates to investigate the question how different compact monotone Hamiltonian T-spaces are from Fano varieties (endowed with a holomorphic action of $T_{\mathbb{C}}$) when the complexity is positive. This question can be tackled in two ways, namely one can assume that the dimension of the Hamiltonian T-space is low or its complexity is low. In the case that the dimension is low, Fine and Panov formulated in [7] the following conjecture.

Conjecture 1.1
Let (M, ω, S^1, ϕ) be a compact and monotone Hamiltonian S^1-space of dimension six. Then M is diffeomorphic to a Fano threefold.

A recent work by Lindsay and Panov [30] supports this conjecture. For the case that the complexity is low, namely equal to one, it was shown by Sabatini and Sepe [38] that compact and monotone Hamiltonian T-spaces and Fano varieties share common topological properties. This gives rise to tackle the following weaker version of Conjecture 1.1.

Goal 1.2
Let (M, ω, T, ϕ) be a compact and monotone Hamiltonian T-space of complexity one. Then M is diffeomorphic to a Fano threefold.

Beyond the fact that dimension six is the lowest dimension in which it is not clear whether compact and monotone Hamiltonian T-spaces of complexity one are different from Fano varieties, there is a good hope that Goal 1.2 can be achieved. This relies on the following facts, which we list up as follows.

- In complex dimension three Fano $T_{\mathbb{C}}$-varieties of complexity one are completely classified in a work of Süss [40], based on the classification of Fano threefolds (see [17]).

- The works of Jupp, Wall and Zubr [21, 42, 43] provides a list of topological invariants for simply connected six-dimensional manifolds which determine their diffeomorphism types. Note that Lindsay and Panov proved that compact symplectic manifolds, which admit an effective and Hamiltonian S^1-action, are simply connected [30, Theorem1.2].

- Holm and Kessler [16] provide techniques to compute the (equivariant) cohomology ring (with respect to $\mathbb{R}$-coefficient) of a compact Hamiltonian T-space of complexity one in terms of the fixed points data.

In order to analyze Hamiltonian T-spaces, a first step is to understand the fixed points data. So let us denote the set of points which are fixed under the T-action by M^T. From a simple argument it follows that the set M^T is not empty if M is compact. The connected components of M^T are closed symplectic submanifolds of the underlying symplectic manifold (M, ω). One can show that the dimension of any of these connected components is less than or equal to $2k$, where k is the complexity of the Hamiltonian T-space. In particular, for a compact Hamiltonian T-space of complexity one any connected component of M^T is an isolated point or a compact surface. It follows that M^T either consists of only isolated points or contains at least one compact surface. In the first case one focuses on the class of so called GKM spaces and in the second case one considers the class of tall spaces.

Given a compact Hamiltonian T-space, the T-action is called **GKM**[1] if for any subtorus H of codimension one any connected component of the set of points which are fixed by H, has at most dimension two. In this case, one can associate in a canonical way a graph with certain properties, the so called GKM graph, to this space, which describes the action. From this graph some topological properties of the underlying manifold can be recovered such as its equivariant and ordinary cohomology ring with $\mathbb{R}$-coefficients. For example, Godinho and Sabatini [11] provide an algorithm which makes it possible to classify all GKM-graphs which are coming from six-dimensional compact and monotone Hamiltonian T-spaces of complexity one. Moreover, Goertsches, Konstantis and Zoller [13] proved that under certain conditions the GKM-graph of a six-dimensional compact GKM-space determines the diffeomorphism type of the underlying manifold.

A compact Hamiltonian T-space of complexity one (M, ω, T, ϕ) is called **tall** if for any $x \in \phi(M)$, the quotient space $M_x := \phi^{-1}(x)/T$ is not a point. The notion of being tall for a compact Hamiltonian T-space of complexity one was introduced by Karshon and Tolman, who studied such spaces intensively in [23, 24, 25]. For

[1]This states for the initials of Goresky, Kottwitz and MacPherson, who introduced such spaces in [15].

example, it turns out that under the tallness assumption M_x is a compact topological surface for any $x \in M$.

Roughly speaking, the category of tall type and the one of GKM type form contrary poles in the set of compact Hamiltonian T-spaces of complexity one. Both cases are very well studied in the literature. However, in this thesis we focus on the tall case. More precisely, we consider six-dimensional compact, tall and monotone Hamiltonian T-spaces of complexity one. In the next section we present the results of this thesis.

1.1 Results

In the following we will use the terminology **complexity one space** to denote Hamiltonian T-spaces of complexity one. The main result of this thesis is a complete classification of six-dimensional compact, tall and monotone complexity one spaces as follows. A work of Karshon and Tolman [23, 24, 25] provides a list of topological invariants associated to any compact and tall complexity one spaces (in any dimension) that contains enough information to determine these spaces up to isomorphisms. One of these invariants is the Duistermaat-Heckman measure (see Section 2.2.5), which is the push forward of the Liouville measure by the moment map. We will show that for compact, tall and monotone complexity one spaces of dimension six the remaining other invariants are already determined by the Duistermaat-Heckman measure. This result is the content of the following theorem.

Theorem 1.3
Let T be a two-dimensional torus. For $i = 1, 2$ let $(M_i, \omega_i, T, \phi_i)$ be a compact, tall and monotone complexity one space of dimension six with respective Duistermaat-Heckman measures $\mathfrak{m}_{DH,i}$. We have that $(M_1, \omega_1, T, \phi_1)$ and $(M_2, \omega_2, T, \phi_2)$ are isomorphic meaning that there exists a T-equivariant symplectomorphism preserving the moment maps if and only if $\mathfrak{m}_{DH,1} = \mathfrak{m}_{DH,2}$.

Using Theorem 1.3 in combination with a careful study of all possible Duistermaat-Heckman measures for this case we obtain a complete list of all isomorphy classes of six-dimensional compact, tall and monotone complexity one spaces (see Chapter 8). Analyzing each single object from this list separately we find the following.

Theorem 1.4
Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six, then the action of the two-dimensional torus T on (M, ω) extends to an effective and Hamiltonian torus action of complexity zero.

Due to the classification of compact complexity zero spaces [6], a consequence of Theorem 1.4 is the following corollary, which supports Goal 1.2.

Corollary 1.5
Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six, then M is diffeomorphic to a Fano threefold.

Hence, we have shown that the statement of Goal 1.2 is true under the additional assumption for the complexity one space to be tall.

Chapter 2
Background

2.1 Monotone Symplectic Manifolds

A symplectic manifold (M, ω) is a smooth manifold M together with a closed non-degenerate two form ω, called symplectic form. The existence of such a symplectic form forces obstructions on the topology of M. Namely, the dimension of M must be even that is $\dim(M) = 2n$ for some $n \in \mathbb{N}$ and ω^n is a volume form on M. Hence, M is orientable. Furthermore, if M is compact, then ω^i defines a non-zero element in the $2i$-th de Rham group $H^{2i}(M; \mathbb{R})$ for all $i = 1, ..., n$. Hence, the $2i$-th Betti number $b_{2i}(M)$ of M is at least one for all $i = 1, ..., n$. A symplectic manifold always admits an almost complex structure J. This is a smooth section in the endomorphism bundle of TM, $J : TM \to TM$, such that $J^2 = -\,\text{Identity}$. Such an almost complex structure allows us to see the tangent bundle TM as a complex vector bundle over M. We denote this complex vector bundle by (TM, J). Of course, such an almost complex structure J is not unique, but the following lemma holds. A proof of this lemma can be found in [35, Proposition 4.1.1].

Lemma 2.1
Let (M, ω) be a symplectic manifold. There exists an almost complex structure $J : TM \to TM$ such that J is compatible with ω, i.e. $\omega(\cdot, J\cdot)$ is a Riemannian metric on M. Moreover, the space of compatible almost complex structures is contractible.

So if J and J' are two compatible almost complex structures for (M, ω), then (TM, J) and (TM, J') are isomorphic as complex vector bundles over M. That fact allows us to define Chern classes of a symplectic manifold (M, ω).

Definition 2.2
Let (M, ω) be a symplectic manifold. The i-th Chern class $c_i(M)$ of (M, ω) is defined as the i-th Chern class of the complex vector bundle (TM, J) over M, where J is an almost complex structure compatible with (M, ω).

Definition 2.3
A symplectic manifold (M, ω) is called **monotone** if

$$c_1(M) = \lambda[\omega]$$

in $H^2(M; \mathbb{R})$ for some $\lambda \in \mathbb{R}$. Moreover, if λ is greater than zero, then (M, ω) is called **positive monotone**.

Remark 2.4
Let (M, ω) be a compact and positive monotone symplectic manifold which admits a compatible (integrable) complex structure $J : TM \to TM$, then M admits a holomorphic embedding into a projective space $\mathbb{C}P^N$, due to the Kodaira Embedding Theorem. In this case, (M, J) is Fano variety.

2.2 Hamiltonian T-spaces

Let T be a compact d-dimensional torus, i.e. $T \cong (S^1)^d$. We denote its Lie algebra by $\mathfrak{t}$ and its **lattice** by

$$\ell_T = \ker(\exp\colon \mathfrak{t} \to T).$$

Moreover, the **dual Lie algebra** of T is denoted by $\mathfrak{t}^*$ and the **dual lattice** is ℓ_T^*, where the dual lattice is also called **weight lattice**. In this work we use the following conventions.

Convention 2.5
We identify a d-dimensional torus T with the quotient space $\mathbb{R}^d/\mathbb{Z}^d$. Hence, the Lie algebra $\mathfrak{t}$ of T and its dual $\mathfrak{t}^*$ are identified with $\mathbb{R}^d$. We consider $\mathbb{R}^d/\mathbb{Z}^d$ as the subset of $\mathbb{C}^d$ given by the equations $|z_1| = ... = |z_d| = 1$, $(z_1, \dots, z_d) \in \mathbb{C}^d$. Hence, the exponential map is given by

$$\begin{aligned} \exp\colon \mathfrak{t} \cong \mathbb{R}^d &\longrightarrow T \subset \mathbb{C}^d \\ (\xi_1, ..., \xi_d) &\longmapsto (\mathrm{e}^{2\pi i \xi_1}, ..., \mathrm{e}^{2\pi i \xi_d}). \end{aligned}$$

In this setting the lattice of ℓ_T is $\mathbb{Z}^d$ and the dual lattice is

$$\ell_T^* = \{x \in \mathfrak{t}^* |\, \langle x, \xi \rangle \in \mathbb{Z} \text{ for all } \xi \in \ell_T\} \cong \mathbb{Z}^d, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is natural pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$.

Let (M, ω) be a symplectic manifold and $\psi : T \times M \to M$ be a smooth and symplectic action of T on (M, ω), i.e. $\psi(t, \cdot)\colon M \to M$ is a symplectomorphism for each $t \in T$. For simplicity we denote $\psi(t, p)$ by $t \cdot p$. Each $\xi \in \mathfrak{t}$ defines a vector field X_ξ on M given by

$$X_\xi(p) = \frac{\mathrm{d}}{\mathrm{d}\nu}|_{\nu=0} \exp(\nu\xi) \cdot p. \tag{2.2}$$

The T-action is called **Hamiltonian** if there exists a moment map $\phi\colon M \to \mathfrak{t}^*$, i.e.

ϕ is a smooth and T-invariant map such that

$$\iota_{X_\xi}\omega = -\mathrm{d}\langle \phi, \xi \rangle\,, \tag{2.3}$$

for all $\xi \in \mathfrak{t}$, where $\iota_{X_\xi}\omega$ denotes the contraction of the vector field X_ξ with ω.

Definition 2.6
A **Hamiltonian T-space** is a quadruple (M, ω, T, ϕ) consisting of the following data.

- (M, ω) is a connected symplectic manifold endowed with a smooth and symplectic T-action.
- The T-action is **effective** and **Hamiltonian**.
- $\phi \colon M \to \mathfrak{t}^*$ is a moment map for the T-action.

The integer $k = \frac{1}{2}\dim(M) - \dim(T)$ is called the **complexity** of (M, ω, T, ϕ). We call the quadruple (M, ω, T, ϕ) also a **complexity k space**.

By the Definition 2.6 it makes sense that two Hamiltonian T-spaces are considered as isomorphic if the following definition applies to these spaces.

Definition 2.7
Let (M, ω, T, ϕ) and (M', ω', T, ϕ') be two Hamiltonian T-spaces. These spaces are called isomorphic if there exists a T-equivariant symplectomorphism between (M, ω) and (M', ω') that preserves the moment maps.

Beyond that Definition 2.7 is very natural, it is in a certain way a little bit too strong. Therefore, consider the following two observations. The moment map of a Hamiltonian T-space is unique up to an irrelevant shift. Moreover, given a smooth group homeomorphism $\theta : T \to T$, then the twisted action[1] on (M, ω) is also Hamiltonian and effective. Therefore, it makes sense to soften the Definition 2.7 as follows.

Definition 2.8
Given two Hamiltonian T-spaces (M, ω, T, ϕ) and (M', ω', T, ϕ'), a **twisted isomorphism** between these spaces is a symplectomorphism $f : (M, \omega) \to (M', \omega')$ together with a smooth group homeomorphism $\theta : T \to T$ such that

$$f(t \cdot p) = \theta(t) \cdot f(p) \tag{2.4}$$

for all $t \in T$ and $p \in M$.

[1] If we denote the action of $t \in T$ by $t \cdot p$ for all $t \in T$ and $p \in M$, then the twisted action with respect to θ is given by $\theta(t) \cdot p$.

Let (M, ω, T, ϕ) be a Hamiltonian T-space. The **stabilizer** of a point p in M is $H_p\colon=\{t \in T \mid t \cdot p = p\}$, which is a subgroup of T. The induced representation of H_p on (T_pM, ω_p) is called the **isotropy representation** at p. For points which lie in the same T-orbit the following holds.

Lemma 2.9
Let (M, ω, T, ϕ) be a Hamiltonian T-space and let p and q be two points in M which lie in the same T-orbit. Then $H_p = H_q$ and the isotropy representations at p and q are naturally linearly symplectomorphic.

Proof. Since p and q lie in the same T-orbit, there exists a $t \in T$ such that $t \cdot p = q$. Therefore, $tH_pt^{-1} = H_q$, but since T is abelian, we have $H_p = H_q$. The action of t on M gives a symplectomorphism denoted by $\psi_t : M \to M$ which maps p to q. This differential $\mathrm{D}_p\, \psi_t : (T_pM, \omega_p) \to (T_qM, \omega_p)$ at p is a linear symplectomorphism with is equivariant to the respect to the representations of H_p on (T_pM, ω_p) resp. H_q on (T_qM, ω_q). Hence, the claim follows. □

This lemma gives rise for the following definition.

Definition 2.10
Let (M, ω, T, ϕ) be a Hamiltonian T-space and let $\mathcal{O}$ be a T-orbit, the **stabilizer** of $\mathcal{O}$, denoted by $H_{\mathcal{O}}$, is the stabilizer of any point $p \in \mathcal{O}$. The **isotropy representation** of $\mathcal{O}$ is the isotropy representation for any point $p \in \mathcal{O}$ up to linear symplectomorphism.

Lemma 2.11
Let (M, ω, T, ϕ) be a Hamiltonian T-space and let $\mathcal{O}$ be an T-orbit with stabilizer $H_{\mathcal{O}}$. Then $\mathcal{O}$ is an isotropic submanifold of dimension $d - h$, where d is the dimension of T and h is the dimension of $H_{\mathcal{O}}$.

Proof. The quotient $T/H_{\mathcal{O}}$ is a torus of dimension $d - h$ and its tangent space at the identity can be identify with $\mathfrak{t}/\mathfrak{h}$, where $\mathfrak{h} \subset \mathfrak{t}$ is the Lie algebra of $H_{\mathcal{O}}$. Pick a point $p \in \mathcal{O}$ and consider the orbit map

$$\Gamma_p : T \longrightarrow M,\ t \longmapsto t \cdot p. \tag{2.5}$$

The orbit map induces a smooth map $\widetilde{\Gamma_p} : T/H_{\mathcal{O}} \to M$ which is a homeomorphism onto its image $\mathcal{O}$. Now we show that $\widetilde{\Gamma_p}$ is an immersion. Fix $t \in T$ and denote its image under the natural projection $T \to T/H_{\mathcal{O}}$ by $[t]$. Then we have

$$\widetilde{\Gamma_p}([t]) = t \cdot p = \psi_t\left(\widetilde{\Gamma_p}([e])\right), \tag{2.6}$$

where $\psi_t : M \to M$ is the action of t on M and $[e]$ is the identity in $T/H_{\mathcal{O}}$. By the chain rule we obtain for the differential of $\widetilde{\Gamma_p}$ at $[t]$

$$\mathrm{D}_{[t]}\,\widetilde{\Gamma_p} \;=\; \mathrm{D}_p\,\psi_t \circ \mathrm{D}_{[e]}\,\widetilde{\Gamma_p}. \tag{2.7}$$

Since ψ_t is a diffeomorphism, $\mathrm{D}_{[t]}\,\widetilde{\Gamma_p}$ and $\mathrm{D}_{[e]}\,\widetilde{\Gamma_p}$ have the same rank. So in order to show that $\widetilde{\Gamma_p}$ is an immersion, it is left to show that $\mathrm{D}_{[e]}\,\widetilde{\Gamma_p}$ has full rank. But this follows from $\mathfrak{h} = \ker(\mathrm{D}_e\,\Gamma_p)$ and $\mathrm{D}_e\,\Gamma_p$ and $\mathrm{D}_{[e]}\,\widetilde{\Gamma_p}$ have the same image. So $\mathcal{O}$ is a submanifold of dimension $d-h$. It remains to show that $\mathcal{O}$ is isotropic. The tangent space at p is $T_p\mathcal{O} = \{X_\xi(p)\,|\,\xi \in \mathfrak{t}\}$. Since the moment map is constant on $\mathcal{O}$, we have $T_p\mathcal{O} \subset \ker(\mathrm{D}_p\psi)$. By Equation (2.3) we have $\omega_p(X_\xi(p), v) = -\langle \mathrm{D}_p\phi(v), \xi\rangle$ for all $v \in T_pM$. We conclude that

$$\ker(\mathrm{D}_p\psi) = (T_p\mathcal{O})^{\omega_p} := \{v \in T_pM\,|\,\omega_p(w,v) \text{ for all } w \in T_p\mathcal{O}\}\,.$$

Hence, we have $T_p\mathcal{O} \subset (T_p\mathcal{O})^{\omega_p}$, which completes the proof. □

Lemma 2.12
Let (M,ω,T,ϕ) be a Hamiltonian T-space and let $p \in M$ be a point, then p is a regular point of the moment if and only if the stabilizer H_p of p is finite.

Proof. Let $2n$ be the dimension of M and let d be the one of the torus T. Consider the differential of the moment map $\mathrm{D}_p\phi : T_pM \to \mathfrak{t}^*(\cong \mathbb{R}^d)$ at p. By counting the dimensions we have that p is regular point if and only if the dimension of the kernel $\ker(\mathrm{D}_p\phi)$ is equal to $2n-d$. Moreover, let $\mathcal{O}$ be the orbit trough p , its tangent space at p is

$$T_p\mathcal{O} = \{X_\xi(p)\,|\,\xi \in \mathfrak{t}\} \subset T_pM. \tag{2.8}$$

By Equation (2.3) we have $\omega_p(X_\xi(p), v) = -\langle \mathrm{D}_p\phi(v), \xi\rangle$ for all $v \in T_pM$. We conclude that $\ker(\mathrm{D}_p\psi) = (T_p\mathcal{O})^{\omega_p}$. Hence, the dimension of $\ker(\mathrm{D}_p\phi)$ is equal to $2n-d$ if and only if the dimension of $T_p\mathcal{O}$ is equal to d. Since the latter is equivalent to H_p being finite, the claim of the lemma follows. □

The next theorem that we state is the Principal Orbit Type Theorem for torus actions.

Theorem 2.13
Let M be a connected manifold endowed with a smooth and effective T-action. Then the set of points with a trivial stabilizer is open and dense in M.

A proof of this theorem can be found in [10, Corollary B.48]. A consequence of this theorem is the following corollary.

Corollary 2.14
Let (M, ω, T, ϕ) be a Hamiltonian T-space, then its complexity is a non-negative integer.

Proof. By Theorem 2.13, there exists a T-orbit $\mathcal{O}$, whose stabilizer is trivial. By Lemma 2.11 this orbit is an isotropic submanifold of (M, ω) of dimension d, where d is the dimension of the torus. Hence, d is less than or equal to half of the dimension of M, which implies that the complexity of (M, ω, T, ϕ) is non-negative. □

2.2.1 Fixed Submanifolds and the Local Normal Form

Let M be a manifold endowed with a smooth T-action, for any subgroup H of T we denote the set of H-fixed points by M^H that is $M^H := \{x \in M \mid t \cdot x = x \,\forall t \in H\}$.

Lemma 2.15
Let (M, ω) be a (compact) symplectic manifold endowed with a symplectic T-action and let H be a subgroup of T. Then any connected component of M^H is a closed (compact) symplectic submanifold of (M, ω).

A proof of this lemma can be found in [35, Lemma 5.5.7]. We call the connected components of M^T the **fixed submanifolds**.

Lemma 2.16
Let (M, ω) be a compact symplectic manifold endowed with a Hamiltonian T-action. Then the set M^T is not empty.

Proof. We prove this claim by induction over the dimension of the torus. First, let the dimension of the torus be one, i.e. it is a circle S^1. Since the action is assumed to be Hamiltonian, there exists a moment map $\phi : M \to (\mathrm{Lie}(S^1))^* \cong \mathbb{R}$, where $\mathrm{Lie}(S^1)$ denotes the Lie algebra of S^1. Since M is compact, there exists a point p at which ϕ attains its minimum. Therefore, we have

$$\omega_p(X_\xi(p), \cdot) = -\langle \mathrm{d}_p\, \phi(\cdot), \xi \rangle = 0, \tag{2.9}$$

for all $\xi \in \mathrm{Lie}(S^1)$, where

$$X_\xi(p) = \frac{\mathrm{d}}{\mathrm{d}\,\nu}|_{\nu=0} \exp(\nu\xi) \cdot p. \tag{2.10}$$

Since ω is non-degenerate, we have that $X_\xi(p)$ is equal to zero for all $\xi \in \mathrm{Lie}(S^1)$, which is equivalent to $p \in M^T$.

Now let the dimension of T be equal to d with $d \geq 2$. Suppose that the claim holds for tori of dimension smaller then d. In order to perform the induction step, fix a splitting $T \cong H \times H'$, where H is a codimensional one subtorus and H' is a subcircle of T. Since H acts also via the inclusion $H \to T$ in a Hamiltonian fashion on (M, ω), by the induction assumption the set M^H is not empty. Pick a connected component X of M^H, by Lemma 2.15 X is a compact and T-invariant symplectic submanifold of (M, ω). Hence, via the inclusion $H' \to T$ the subcircle H' acts also in a Hamiltonian fashion on (X, ω). So there exists a $p \in X \cap M^{H'}$. But since $T \cong H \times H'$, we must have $p \in M^T$. □

Corollary 2.17
Let (M, ω) a compact symplectic manifold endowed with a Hamiltonian T-action, let H be a subgroup of T and let X be a connected component of M^H. Then the intersection $X \cap M^T$ is not empty.

Proof. By Lemma 2.15 we have that X is a compact and T-invariant symplectic submanifold of (M, ω). Hence, T acts in a Hamiltonian fashion on (X, ω). Hence, by Lemma 2.16 there exists $p \in X \cap M^T$. □

Next, we recall the local normal form near fixed points for Hamiltonian T-spaces. Therefore, we fix a T-invariant almost complex structure $J \colon TM \to TM$ which is compatible with ω. Hence, for each $p \in M^T$ we have a $\mathbb{C}$-linear T-representation on $(T_pM, J_p) \cong \mathbb{C}^n$. The weights $\alpha_{p,1}, \ldots, \alpha_{p,n} \in \ell_T^*$ of this representation are called the **weights** at p. Note that the weights are unique up to permutation. The following theorem by Guillemin and Sternberg [19] and Marle [31] states that small neighborhoods of fixed points are determined by their weights up to isomorphisms.

Theorem 2.18 (Local normal forms near fixed points)
Let (M, ω, T, ϕ) be a Hamiltonian T-space of dimension $2n$. Let $p \in M^T$ be a fixed point with weights $\alpha_{p,1}, \ldots, \alpha_{p,n} \in \ell_T^*$. Then there exists a neighborhood U_p of p with complex coordinates $z_1, \ldots, z_n$ centered at p such that

- *the symplectic form is $\omega = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$,*
- *the T-action is given by*

$$\exp(\xi) \cdot (z_1, \ldots, z_n) = (e^{2\pi i \langle \alpha_{p,1}, \xi \rangle} z_1, \ldots, e^{2\pi i \langle \alpha_{p,n}, \xi \rangle} z_n) \text{ for all } \xi \in \mathfrak{t},$$

- *the moment map is $\phi(z) = \pi \sum_{j=1}^n \alpha_{p,j} |z_j|^2 + \phi(p)$.*

Remark 2.19
Let (M, ω, T, ϕ) be a Hamiltonian T-space. Since M is connected and the T-action

effective, the $\mathbb{Z}$-span of the weights at each fixed point must be equal to the dual lattice $\ell_T^* \cong \mathbb{Z}^{\dim(T)}$. Let F be a fixed submanifold. Then for any $p, q \in F$ the sets of weights at p and q are equal. So we call these weights the **weights along F** and we denote them by $\alpha_{F,1}, \ldots \alpha_{F,n}$. The dimension of F is equal to twice the number of weights which are equal to zero.

Lemma 2.20
Let (M, ω, T, ϕ) be a complexity k space. Then the dimension of each fixed submanifold is at most $2k$.

Proof. We have $\dim(M) = 2n$ so the complexity of (M, ω, T, ϕ) is $k = n - d$, where $\dim(T) = d$. Now let F be a fixed submanifold and let $\alpha_{F,1}, \ldots, \alpha_{F,n}$ be the weights along F. Since the T-action on M is effective, we have that the $\mathbb{Z}$-span of $\alpha_{F,1}, \ldots, \alpha_{F,n}$ is equal to $\ell_T^* \cong \mathbb{Z}^d$. Hence, at most k of the weights $\alpha_{F,1}, \ldots, \alpha_{F,n}$ along F are equal to zero. Since the dimension of F is equal to twice the number of weights along F that are zero, we conclude $\dim(F) \leq 2k$. $\square$

Next we like to describe the stabilizers of points near fixed submanifold. Therefore, we associate to any weight $\alpha \in \ell_T^*$ the subgroup G_α defined by

$$G_\alpha = \exp\left(\{\xi \in \mathfrak{t} \mid \langle \alpha, \xi \rangle \in \mathbb{Z}\}\right). \tag{2.11}$$

A weight $\alpha \in \ell_T^*$ is called primitive if $\alpha \neq 0$ and for any weight $\alpha' \in \ell_T^*$ and any integer $m \in \mathbb{Z}$ satisfying $\alpha = m\alpha'$ one has $m = \pm 1$. We have the following three cases.

- If $\alpha = 0$, then G_α is equal to T.
- If α is a primitive element in ℓ_T^*, then G_α is a codimensional one subtorus of T.
- If α is not zero and not a primitive element in ℓ_T^*, then there exists an $m \in \mathbb{N}_{\geq 2}$ and a primitive element $\alpha' \in \ell_T^*$ such that $\alpha = m\alpha'$. In this case, G_α is isomorphic to $H \times (\mathbb{Z}/m\mathbb{Z})$, where H is a codimensional one subtorus of T.

A simple consequence of Theorem 2.18 is the following corollary.

Corollary 2.21
Let (M, ω, T, ϕ) be a Hamiltonian T-space of dimension $2n$. Let $p \in M^T$ be a fixed point with weights $\alpha_{p,1}, \ldots, \alpha_{p,n} \in \ell_T^*$. Then there exists a neighborhood U_p of p such that for any point $q \in U_p$ there exists a subset $I \subset \{1, \ldots, n\}$ with

$$H_q = \bigcap_{i \in I} G_{\alpha_{i,p}}, \tag{2.12}$$

where H_q is the stabilizer of q.

Proof. Choose a neighborhood U_p of p with the properties as in Theorem 2.18. □

Lemma 2.22
Given a compact Hamiltonian T-space (M, ω, T, ϕ) of dimension $2n$, let $\mathcal{O}$ be an T-orbit and let $H_{\mathcal{O}}$ be its stabilizer. Then there exist $p \in M^T$ and a subset $I \subset \{1, \ldots, n\}$ such that

$$H_{\mathcal{O}} = \bigcap_{i \in I} G_{\alpha_{i,p}}\,, \tag{2.13}$$

where $\alpha_{p,1}, \ldots, \alpha_{p,n} \in \ell_T^*$ are the weights at p.

Proof. Let $X_{\mathcal{O}}$ be the connected component of $M^{H_{\mathcal{O}}}$ which contains the orbit $\mathcal{O}$. By Lemma 2.15 we have that $X_{\mathcal{O}}$ is a compact and T-invariant symplectic submanifold. In particular, the quotient torus $T/H_{\mathcal{O}}$ acts effectively and in a Hamiltonian fashion on $X_{\mathcal{O}}$. By Theorem 2.13 the subset of points in $X_{\mathcal{O}}$ for which their stabilizers are trivial with respect to the $(T/H_{\mathcal{O}})$-action is open and dense in $X_{\mathcal{O}}$. This is equivalent to the statement that the subset of points in $X_{\mathcal{O}}$ for which the stabilizers are equal to $H_{\mathcal{O}}$ with respect to the T-action is open and dense in $X_{\mathcal{O}}$. By Corollary 2.17 there exists a point $p \in M^T \cap X_{\mathcal{O}}$ and by Corollary 2.21 there exists a neighborhood U_p of p in M such that the stabilizer of any point in U_p is of the form as in the right-hand side of Equation (2.13). By combining all these arguments the claim of this lemma follows. □

We like to close this section by the following corollary of Lemma 2.22.

Corollary 2.23
Given a compact Hamiltonian T-space (M, ω, T, ϕ) of dimension $2n$ such that for any fixed point $p \in M^T$ any weight at p is primitive or equal to zero. Then the stabilizer of any T-orbit is either trivial or a subtorus of T.

Proof. Let $\mathcal{O}$ be a T-orbit and let $H_{\mathcal{O}}$ be its stabilizer. By Lemma 2.22 there exist $p \in M^T$ and a subset $I \subset \{1, \ldots, n\}$ such that Equation (2.13) holds, where $\alpha_{p,1}, \ldots, \alpha_{p,n} \in \ell_T^*$ are the weights at p. Since any of these weights is primitive or equal to zero, the right-hand side of Equation (2.13) is a subtorus or the trivial group. This completes the proof. □

2.2.2 The Convexity Theorem

In this section we recall some properties of the image of the moment map. One of the most important results for Hamiltonian T-spaces is the so called **Convexity Theorem** due to Atiyah [2], Guillemin and Sternberg [18].

Theorem 2.24 (Convexity Theorem)
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space, then the fibers of ϕ are connected and the image of the moment map $\phi(M)$ is the convex hull of the images of connected components of M^T.

Let V be a finite-dimensional real vector space and denote by V^* its dual. A subset $\mathcal{P} \subset V$ is called convex polytope if it is the convex hull of finitely many points in V. Let $\mathcal{P} \subset V$ be a convex polytope. A subset $\mathcal{F} \subset P$ is called face (of $\mathcal{P}$) if $\mathcal{F} = \mathcal{P} \cap \{v \in V \mid \langle x, v\rangle = c\}$ for some $x \in V^*$ and $c \in \mathbb{R}$ satisfying $\langle x, v\rangle \leq c$ for all $v \in \mathcal{P}$. Let $\mathcal{F}$ be a face and let W be the smallest affine subspace of V which contains $\mathcal{F}$. Then the dimension of $\mathcal{F}$ is defined to be the dimension of W and the interior of the face $\mathcal{F}$ is the interior of $\mathcal{F} \subset W$ with respect to the subset topology on W. A vertex is a face of dimension zero. An edge is a face of dimension one.

Due to the Convexity Theorem the image of the moment map of a compact Hamiltonian T-space is a convex polytope. So we call the moment map image also the **moment map polytope**, which we denote by Δ. Note that this polytope is rational in the following sense. For any edge e one can choose a primitive element $\alpha \in \ell_T^*$ so that e is contained in an one-dimensional affine subspace in the direction of α. This is a direct consequence of Theorem 2.18.

Another important property of the moment map is stated in the following theorem due to Sjamaar [39].

Theorem 2.25 (Sjamaar 1998)
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space. The moment map $\phi\colon M \to \phi(M)$ is open.

In the following we like to sum up some consequences of the Convexity Theorem. The next corollary is a direct consequence of Theorem 2.18, Theorem 2.24 and Theorem 2.25.

Corollary 2.26
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space of dimension $2n$, let F be a connected component of M^T and let $\alpha_{F,1}, \ldots \alpha_{F,n}$ be the weights along F. Consider the cone

$$\mathcal{C}_F = \mathbb{R}_{\geq 0}\text{-span}\,\{\alpha_{F,1}, \ldots, \alpha_{F,n}\} := \{s_1\alpha_{F,1} + \ldots + s_n\alpha_{F,n} \mid s_1, \ldots, s_n \in \mathbb{R}_{\geq 0}\}$$

and let $\mathcal{H}_F$ be the maximal linear subspace of $\mathfrak{t}^*$ that is contained in $\mathcal{C}_F$. Then

$$(\mathcal{H}_F + \phi(F)) \cap \phi(M)$$

is a face of the moment map polytope $\phi(M)$. The dimension of this face is equal to the dimension of $\mathcal{H}_F$. In particular, $\phi(F)$ is a vertex of $\phi(M)$ if and only if the cone $\mathcal{C}_F$ is proper.

Proof. We denote by d the dimension of the torus T and by j the dimension of the subspace $\mathcal{H}_F$. Since the $\mathbb{Z}$-span of the weights $\alpha_{F,1}, \dots \alpha_{F,n}$ is equal to $\ell_T^* \cong \mathbb{Z}^d$ (see Remark 2.19), there exists a linear isomorphism $\tau\colon \mathfrak{t}^* \to \mathbb{R}^d$ such that $\mathcal{C}_F$ resp. $\mathcal{H}_F$ is mapped one-to-one on $\mathbb{R}^j \times \mathbb{R}^{d-j}_{\geq 0}$ resp. $\mathbb{R}^j \times \{0\}$. Pick a point p in F. Due to the local normal form there exists an open neighborhood U_p of p such that $\phi(U_p)$ is an open neighborhood of $\phi(F)$ in $\mathcal{C}_F + \phi(F)$. By Theorem 2.25 the image $\phi(U_p)$ is an open subset of the moment map polytope $\phi(M)$. The affine map

$$\mathfrak{t}^* \to \mathbb{R}^d, \quad x \mapsto \tau(x - \phi(F)) \tag{2.14}$$

maps $\phi(U_p)$ to an open neighborhood of 0 in $\mathbb{R}^j \times \mathbb{R}^{d-j}_{\geq 0}$. Moreover, the intersection of $\phi(U_p)$ with the affine space $\mathcal{H}_F + \phi(F)$ is mapped to an open neighborhood of 0 in $\mathbb{R}^j \times \{0\}$. Therefore, the intersection of the affine space $\mathcal{H}_F + \phi(F)$ with the moment map polytope is indeed a j-dimensional face of the moment map polytope. □

Corollary 2.27
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space, then the preimage of any vertex of the moment map polytope is a connected component of M^T.

Proof. Let v be a vertex of the moment map polytope. By the Convexity Theorem there exists at least one connected component F of M^T which is contained in $\phi^{-1}(v)$. We need to show $\phi^{-1}(v) = F$. Also by the Convexity Theorem we have that $\phi^{-1}(v)$ is connected. Hence, we need to show that F is a connected component of $\phi^{-1}(v)$. Let $\alpha_{F,1}, ..., \alpha_{F,n}$ be the weights along F. Let $2r$ be the dimension of F, then exactly r weights along F are zero, say $\alpha_{F,n-r+1}, ..., \alpha_{F,n}$. Since $\phi(F) = v$, by Corollary 2.26 the cone

$$\mathcal{C}_F = \mathbb{R}_{\geq 0}\text{-span}\{\alpha_{F,1}, \dots, \alpha_{F,n}\} = \mathbb{R}_{\geq 0}\text{-span}\{\alpha_{F,1}, \dots, \alpha_{F,n-r}\}$$

must be proper. Pick a point $p \in F$ due to the local normal form theorem, there exists an open neighborhood U_p with complex coordinates $z_1, ..., z_n$ centered at p such that

$$\phi(z) = \pi \sum_{j=1}^{n-r} \alpha_{F,j} |z_j|^2 + v.$$

Note that in these coordinates F is given by $z_j = 0$ for $j = 1, ..., n-r$. Moreover, since the cone $\mathcal{C}_F$ is proper, in these coordinates the points which are mapped to v, belong to F. Hence, U_p is an open neighborhood of p such that the intersection of

U_p and $\phi^{-1}(v)$ is contained in F. Since $p \in F$ is chosen arbitrarily, we conclude that there exists an open neighborhood U of F such that $U \cap \phi^{-1}(v) = F$. Therefore, F is a connected component of $\phi^{-1}(v)$ and this completes the proof of this corollary. $\square$

Corollary 2.27 can be generalized in the following sense, namely the preimage of any face of the moment map polytope is a connected component of the set of points which are fixed by a subtorus. This is the content of Lemma 2.29. Before we state this lemma, we need to clarify some notation, which we do in the following definition.

Definition 2.28
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space and let $\mathcal{F}$ be a face of its moment map polytope. The **annihilator** of $\mathcal{F}$ is

$$\mathfrak{ann}(\mathcal{F}) := \{\xi \in \mathfrak{t} | \langle x - y, \xi \rangle = 0 \text{ for all } x, y \in \mathcal{F}\}, \tag{2.15}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$. This is a rational subspace of $\mathfrak{t}$ and hence $\exp(\mathfrak{ann}(\mathcal{F}))$ is a subtorus of T. We call this subtorus the **stabilizer** of $\mathcal{F}$ and it is denoted by $\mathfrak{stab}(\mathcal{F})$.

The faces of the moment map polytope correspond to symplectic submanifolds as follows.

Lemma 2.29
Let (M, ω, T, ϕ) be a Hamiltonian T-space and let $\mathcal{F}$ be a face of the moment map polytope. Then the preimage $\phi^{-1}(\mathcal{F})$ is a connected component of $M^{\mathfrak{stab}(F)}$.

Proof. Since $\mathfrak{stab}(\mathcal{F})$ is a subtorus of T, the inclusion of $\mathfrak{stab}(\mathcal{F})$ into T gives us an effective and Hamiltonian action of $\mathfrak{stab}(\mathcal{F})$ on (M, ω). The Lie algebra of $\mathfrak{stab}(\mathcal{F})$ is $\mathfrak{ann}(\mathcal{F}) \subset \mathfrak{t}$. Let $\pi \colon \mathfrak{t}^* \to (\mathfrak{ann}(\mathcal{F}))^*$ be the projection, induced by the inclusion $\mathfrak{ann}(\mathcal{F}) \to \mathfrak{t}$. The composition $\pi \circ \phi : M \to (\mathfrak{ann}(\mathcal{F}))^*$ is a moment map for the $\mathfrak{stab}(\mathcal{F})$-action on (M, ω). Note that the face $\mathcal{F}$ is mapped under π to a vertex v' of $\pi(\phi(M))$ so that $\phi(M) \cap \pi^{-1}(v') = \mathcal{F}$. Hence, $(\pi \circ \phi)^{-1}(v') = \phi^{-1}(\mathcal{F})$ and Corollary 2.27 implies that $\phi^{-1}(\mathcal{F})$ is a connected component of $M^{\mathfrak{stab}(F)}$. $\square$

Remark 2.30
Given a compact Hamiltonian T-space (M, ω, T, ϕ), let $\mathcal{F}$ be a face of its moment map polytope. Then due to Lemma 2.15 and 2.29 the preimage $\phi^{-1}(\mathcal{F})$ is a T-invariant symplectic submanifold of (M, ω) and $\mathfrak{stab}(\mathcal{F})$ acts trivially on $\phi^{-1}(\mathcal{F})$. So we obtain an action of the quotient torus $T/\mathfrak{stab}\mathcal{F}$ on $\phi^{-1}(\mathcal{F})$. This action is Hamiltonian, but it may fail to be effective.

By Lemma 2.20 the connected components of M^T of a compact complexity k space (M, ω, T, ϕ) have at most dimension $2k$. Next we show that any component of dimension equal to $2k$ is the preimage of a moment map polytope vertex.

Lemma 2.31
Let (M, ω, T, ϕ) be a compact complexity k space and let F be a $2k$-dimensional connected component of M^T, then F is the preimage of a vertex of the moment map polytope.

Proof. Let $2n$ be the dimension of (M, ω, T, ϕ). So the dimension of T is $d = n - k$. Let $\alpha_{F,1}, \ldots, \alpha_{F,n}$ be the weights along F. Since the dimension of F is equal to $2k$, exactly k weights along F are equal to zero, say $\alpha_{F,n-d+1}, \ldots, \alpha_{F,n}$. So the weights $\alpha_{F,1}, \ldots, \alpha_{F,d}$ form a $\mathbb{Z}$-basis of ℓ_T^* (see Remark 2.19). Therefore, the cone spanned by the weights along F is proper. By Corollary 2.26 the image of F is a vertex v of the moment map polytope. Hence, by Corollary 2.27 it follows $\phi^{-1}(v) = F$. $\square$

We close this section by the following corollary which generalizes Lemma 2.20 and 2.31.

Corollary 2.32
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space of dimension $2n$ and let H be a subtorus of T of dimension h. Then any connected component of M^H has at most dimension $2(n-h)$. Moreover, if X is a connected component of M^H of dimension $2(n-h)$, then $\phi(X)$ is a subset of the boundary of the moment map polytope.

Proof. The subtorus H acts (via the inclusion $H \to T$) effectively and in a Hamiltonian fashion on (M, ω) and the complexity of this action is equal to $n - h$. So by Lemma 2.20 any connected component of M^H has at most dimension $2(n-h)$. Moreover, let $\pi : \mathfrak{t}^* \to \mathfrak{h}^*$ be the projection induced be the inclusion $H \to T$. Then $\pi \circ \phi : M \to \mathfrak{h}^*$ is a moment map for the H-action. Now let X be a connected component of M^H of dimension $2(n-h)$, then due to Lemma 2.31 $\pi(\phi(X))$ is a vertex of the moment map polytope $\pi(\phi(M))$. Since the projection $\pi : \mathfrak{t}^* \to \mathfrak{h}^*$ is open, we have that $\phi(X)$ is a subset of the boundary of the moment map polytope $\phi(M)$. $\square$

2.2.3 Symplectic Toric Manifolds

In the previous section we saw that the moment map polytope of a compact Hamiltonian T-space (M, ω, T, ϕ) contains certain information on the T-action. By [6] if the complexity of a compact Hamiltonian T-space is zero, then the space can be recovered from its moment map polytope. In this case, the moment map polytope satisfies special properties, which we recall in the next definition.

Definition 2.33
Let Δ be a d-dimensional polytope in $\mathbb{R}^d$ and let v be a vertex of Δ.

(i) The vertex v is **simple** if there are exactly d edges meeting at the vertex v.

(ii) The vertex v is **rational** if the edges meeting at v are of the form $v + t\alpha_{v,i}$, where $\alpha_{v,i}$ are primitive elements of $\mathbb{Z}^d$ and $t \geq 0$.

(iii) The vertex v is **smooth** if it is simple, rational and the primitive elements $\alpha_{v,1}, \ldots, \alpha_{v,d}$ of $\mathbb{Z}^d$ as defined in (ii) form a $\mathbb{Z}$-basis of $\mathbb{Z}^d$.

Moreover, for each smooth vertex v the primitive elements $\alpha_{v,1}, \ldots, \alpha_{v,d}$ of $\mathbb{Z}^d$ defined in (ii) are called the **weights** at v. The polytope Δ is called **Delzant** if all vertices of Δ are smooth.

Example 2.34
In Figure 2.1 we list three polytopes in $\mathbb{R}^2$. The first two are Delzant. The third polytope is not Delzant since its vertex $(0,1)$ fails to be smooth.

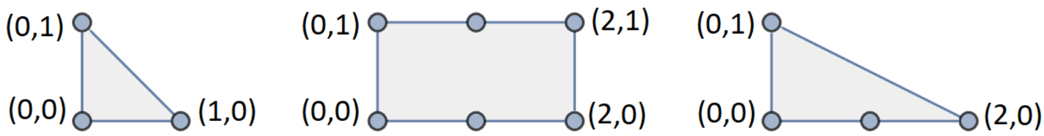


Figure 2.1: Examples of polytopes in $\mathbb{R}^2$ which are Delzant or not.
The bullets indicate integral points in $\mathbb{R}^2$. The gray colored areas indicate the polytopes.

Remark 2.35
Let T be a d-dimensional torus. Since its weight lattice $\ell_T^* \subset \mathfrak{t}^*$ is isomorphic to $\mathbb{Z}^d$, we can identify the pair $(\ell_T^*, \mathfrak{t}^*)$ with $(\mathbb{Z}^d, \mathbb{R}^d)$. So the notations in Definition 2.33 make also sense for polytopes in $\mathfrak{t}^*$.

By the work of Delzant [6] compact symplectic toric manifolds are classified as follows.

Theorem 2.36 (Delzant 1988)
Let (M, ω, T, ϕ) be a compact symplectic toric manifold, then its moment map polytope is Delzant and the manifold M admits an (integrable) complex structure $J : TM \to TM$ compatible with ω. Moreover, the following holds.

- *(Existence) Let Δ be a Delzant polytope in $\mathfrak{t}^*$, then there exists a compact symplectic toric manifold (M, ω, T, ϕ) such that the moment map polytope of (M, ω, T, ϕ) is equal to Δ, i.e. $\phi(M) = \Delta$.*

- *(Uniqueness) Let (M, ω, T, ϕ) and (M', ω', T, ϕ') be two symplectic toric manifolds of dimension $2n$ with the same moment map polytope, i.e., $\phi(M) = \phi'(M')$. Then (M, ω, T, ϕ) and (M', ω', T, ϕ') are isomorphic.*

We need the following corollary of this classification result.

Corollary 2.37
Let (M, ω, T, ϕ) be a compact symplectic toric manifold which is monotone. Then M is diffeomorphic to a Fano variety.

Proof. In a latter part (see Lemma 2.51) we show that such a space is indeed positive monotone. Moreover, due to Theorem 2.36 such a space admits an (integrable) complex structure $J : TM \to TM$ compatible with ω. Hence, M is diffeomorphic to a Fano variety (see Remark 2.4). $\square$

In the next lemma we sum up well-known properties about compact symplectic toric manifolds, which have been proven in [6].

Lemma 2.38
Let (M, ω, T, ϕ) be a compact symplectic toric manifold and let x be a point in its moment map polytope Δ, the fiber $\phi^{-1}(x)$ contains just one T-orbit $\mathcal{O}$. Moreover, the stabilizer $H_{\mathcal{O}}$ of this orbit is equal to $\mathfrak{stab}(\mathcal{F})$, where $\mathcal{F}$ is the unique face of Δ such that x lies in the interior of $\mathcal{F}$.

2.2.4 Morse-Bott-Theory for Hamiltonian T-spaces

The study of Hamiltonian T-spaces is strongly linked to Morse-Bott theory as follows. After fixing an orientation of the circle S^1 we can identify the Lie algebra $\mathrm{Lie}(S^1)$ of S^1 and its dual $(\mathrm{Lie}(S^1))^*$ with $\mathbb{R}$ and so its lattice ℓ_{S^1} resp. its dual lattice $\ell_{S^1}^*$ can be identify with $\mathbb{Z}$.

Lemma 2.39 ([35], Lemma 5.5.8)
Let (M, ω, S^1, ϕ) be a Hamiltonian S^1-space. Then the moment map $\phi \colon M \to \mathbb{R}$ is a Morse-Bott function and its critical submanifolds coincide with the fixed submanifolds of (M, ω, S^1, ϕ). Moreover, for any $p \in M^{S^1}$ the Morse-Bott index of ϕ at p is equal to twice the number of negative weights of the S^1-representation on T_pM.

This lemma is important for all Hamiltonian T-spaces, since each subcircle of T acts also in a Hamiltonian fashion on the corresponding symplectic manifold. In particular, given a Hamiltonian T-space (M, ω, T, ϕ) and $\xi \in \ell_T \setminus \{0\}$, we have that ξ generates a subcircle $S^1 = \exp(\mathbb{R}\xi)$ with a canonical orientation. Obviously, the set of S^1-fixed points M^{S^1} is a subset of the T-fixed points M^T. Let $p \in M^T$ and let $\alpha_{p,1}, ..., \alpha_{p,n}$ be the weights of the T-representation at T_pM. After a positive rescaling we can assume ξ to be primitive. So the weights of the S^1-representation on T_pM are

$$\langle \alpha_{p,1}, \xi \rangle, \ldots, \langle \alpha_{p,n}, \xi \rangle \in \ell^*_{S^1} \cong \mathbb{Z}.$$

2.2.5 Symplectic Reduction and the Duistermaat-Heckman Measure

Let (M, ω, T, ϕ) be a Hamiltonian T-space. For each $x \in \phi(M)$ the fiber $\phi^{-1}(x)$ over x is a T-invariant subset and we consider the quotient-space

$$M_x = \phi^{-1}(x)/T$$

together with the quotient topology. We call this space the **reduced space** at x. Suppose that x is a regular value for the moment map then by the Implicit Function Theorem $\phi^{-1}(x)$ is a submanifold of M and by Lemma 2.12 for each $p \in \phi^{-1}(x)$ the stabilizer H_p of p is finite. Hence, the T-action on $\phi^{-1}(x)$ is locally free. The following theorem is based on the work of Marsden and Weinstein [32] and Meyer [36].

Theorem 2.40
Given a Hamiltonian T-space (M, ω, T, ϕ) and a regular value $x \in \phi(M)$, let Γ_x denote the subgroup of T generated by all the stabilizers H_p for $p \in \phi^{-1}(x)$. Then the following holds.

- *The quotient space $M_x = \phi^{-1}(x)/T$ inherits a canonical orbifold structure from M of dimension $2k$, where k is the complexity of (M, ω, T, ϕ). Moreover, there is a unique symplectic orbifold form ω_x which satisfies $i_x^*(\omega) = \pi_x^*(\omega_x)$, where $i_x : \phi^{-1}(x) \to M$ is the inclusion and $\pi_x : \phi^{-1}(x) \to M_x$ is the projection.*

- *The projection $\phi^{-1}(x)/\Gamma_x \to M_x$ is a principal T/Γ_x-bundle.*

Note that if for any point $p \in \phi^{-1}(x)$ its stabilizer is trivial, then M_x is a manifold and ω_x is symplectic form in the standard sense. In this case, the proof of Theorem 2.40 goes back to Marsden and Weinstein [32] and Meyer [36].
This concept of symplectic reduction for Hamiltonian T-spaces is strongly related to the **Duistermaat-Heckman measure**. Namely, given a Hamiltonian T-space (M, ω, T, ϕ) with a proper moment map, its Duistermaat-Heckman measure $\mathfrak{m}_{DH}$ is

the push-forward measure of the Liouville measure on (M, ω), i.e.

$$\mathfrak{m}_{DH}(U) \colon = \int_{\phi^{-1}(U)} \frac{\omega^n}{n!}, \tag{2.16}$$

for each Borel set $U \subset \mathfrak{t}^*$. This is a well-defined measure and by the work of Duistermaat-Heckman [8] this measure is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{t}^*$, which we denote by $\mathrm{d}\,x^d$. This means that the Radon-Nikodym derivative of the Duistermaat-Heckman measure with respect to the Lebesgue measure can be represented by a Lebesgue integrable function $f_{DH} \colon \mathfrak{t}^* \to \mathbb{R}$. Hence,

$$\mathfrak{m}_{DH}(U) = \int_U f_{DH}\ \mathrm{d}\,x^d, \tag{2.17}$$

for each Borel set $U \subset \mathfrak{t}^*$. Note that such a representative is as function well-defined up to a set of Lebesgue measure equal to zero. The Duistermaat-Heckman measure is related to the concept of symplectic reduction by the Duistermaat-Heckman Theorem, which we state in the following.

Theorem 2.41 (Duistermaat-Heckman Theorem [8])
Let (M, ω, T, ϕ) be a $2n$-dimensional Hamiltonian T-space and let $x, y \in \mathfrak{t}^*$ be two regular values of the moment map ϕ which lie in the same connected component of the regular values. Then there exists a canonical diffeomorphism between M_x and M_y which allows us to identify the groups $H^2(M_x; \mathbb{R}) = H^2(M_y; \mathbb{R})$. Under this identification the cohomology classes of ω_x and ω_y are related by

$$[\omega_y] = [\omega_x] + \langle x - y\, ,\, c_1(M_x)\rangle\, , \tag{2.18}$$

where $c_1(M_x) \in H^2(M_x; \mathfrak{t}/\Lambda_x)$ is the $(\mathfrak{t}/\Lambda_x)$-valued first Chern class of the principal (T/Γ_x)-bundle $\phi^{-1}(x)/\Gamma_x \to M_x$ and Λ_x denotes the subgroup of points in $\mathfrak{t}$ which are mapped by the exponential map $\exp : \mathfrak{t} \to T$ *to Γ_x.*
Moreover, if the moment map is proper, then the Radon-Nikodym derivative of the Duistermaat-Heckman measure of (M, ω, T, ϕ) can be chosen to be equal to

$$\int_{M_x} \frac{\omega_x^k}{k!} \tag{2.19}$$

for any regular value $x \in \mathfrak{t}^*$ of the moment map, where k is the complexity of (M, ω, T, ϕ).

The Duistermaat-Heckman Theorem has important consequences for the Duistermaat-Heckman measure, which we sum up in the following remark.

Remark 2.42
Given a Hamiltonian T-space (M, ω, T, ϕ) of complexity k such that its moment map is proper, then a representative $f_{DH}\colon \mathfrak{t}^* \to \mathbb{R}$ of the Radon-Nikodym derivative of the Duistermaat-Heckman measure can be chosen to be equal to the volume of the symplectic reduction for any regular value of the moment map. This determines the representative $f_{DH}\colon \mathfrak{t}^* \to \mathbb{R}$ up to the set of singular values which has Lebesgue measure zero due to Sard's theorem. Moreover, its restriction to a connected component of the regular values is a polynomial of degree less or equal to k. This is a direct consequence of the Duistermaat-Heckman Theorem. Note that Theorem 2.41 does not give a preferred choice how to define $f_{DH}(x)$ for a singular value x of the moment map. In particular, Theorem 2.41 does not make a global statement, like f_{DH} is continuous (smooth, concave, etc.,...) on the moment map image.

Example 2.43
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space of complexity zero. Since all points which lie in the interior of the moment map polytope are regular values of the moment map ϕ, the Radon-Nikodym derivative of the Duistermaat-Heckman measure is given by the indicator function of the moment map polytope.

Example 2.44
Let (M, ω, T, ϕ) be a compact symplectic toric manifold of dimension $2n$ and let H be a codimensional one subtorus of the n-dimensional torus T. We denote the Lie algebra of H and its dual by $\mathfrak{h}$ and $\mathfrak{h}^*$. The subtorus H acts also effectively and in a Hamiltonian fashion on (M, ω) with moment map

$$\phi' = \pi \circ \phi\colon M \to \mathfrak{h}^*, \tag{2.20}$$

where $\pi\colon \mathfrak{t}^* \to \mathfrak{h}^*$ is the projection induced by the inclusion $\mathfrak{h} \to \mathfrak{t}$. Hence, (M, ω, H, ϕ') is a compact Hamiltonian H-space of complexity one. We like to compute the Duistermaat-Heckman measure of this space from the moment map polytope $\phi(M)$. For simplicity let us fix a splitting of the n-dimensional torus $T = S^1 \times .... \times S^1$ such that $H = S^1 \times \cdots \times S^1 \times \{1\}$. With respect to such a splitting we identify $\mathfrak{t}^*$ with $\mathbb{R}^n$ and $\mathfrak{h}^*$ with $\mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$. Let Δ resp. Δ' be the moment map polytope of (M, ω, T, ϕ) resp. (M, ω, H, ϕ'). Then there exist continuous and piecewise linear functions $p_{min}, p_{max}\colon \Delta' \to \mathbb{R}$ such that

$$\Delta = \left\{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, x \in \Delta' \text{ and } p_{min}(x) \le y \le p_{max}(x)\right\}. \tag{2.21}$$

Given a Borel set $U \subset \Delta'$, then

$$\int_{\phi'^{-1}(U)} \frac{\omega^n}{n!} = \int_{\pi^{-1}(U)\cap\Delta} 1 \,\mathrm{d}\,x \,\mathrm{d}\,y = \int_U (p_{max}(x) - p_{min}(x)) \,\mathrm{d}\,x, \tag{2.22}$$

where we used in the second equation that the Duistermaat-Heckman measure of (M, ω, T, ϕ) is equal to one on Δ. Hence, the Radon-Nikodym derivative of the Duistermaat-Heckman measure of (M, ω, H, ϕ') can be chosen to be equal to $(p_{max} - p_{min})$ on Δ' and equal to zero outside of Δ'.

2.3 Equivariant Cohomology

In this subsection we review some important results about the equivariant cohomology of T-spaces, which are needed in this work. (For a detailed introduction to equivariant cohomology see for instance [1] and [20]). Let M be a topological space endowed with a T-action. In the Borel-model the T-equivariant cohomology of M is defined as follows. Let ET be a contractible topological space on which T acts freely and let $BT = ET/T$ be the classifying space of T. The diagonal action of T on $M \times ET$ is free. By $M \times_T ET$ we denote the orbit space. The T-equivariant cohomology ring of M is

$$H^*_T(M; R) \colon = H^*(M \times_T ET; R), \tag{2.23}$$

where R is the coefficient ring. In particular, if the T-action on M is trivial, then $H^*_T(M; R) = H^*(M; R) \otimes H^*(BT; R)$. If $T = S^1$, then ES^1 is the unite sphere S^∞ in $\mathbb{C}^\infty$ and BS^1 is $\mathbb{C}P^\infty$. It follows that

$$H^*_{S^1}(\{\text{point}\}; R) = H^*(\mathbb{C}P^\infty; R) = R[x], \tag{2.24}$$

where x has degree 2. Moreover, if T is a d-dimensional torus, then BT is the d-times product of $\mathbb{C}P^\infty$ and so

$$H^*_T(\{\text{point}\}; R) = H^*(BT; R) = R[x_1, \ldots, x_d]. \tag{2.25}$$

If $R = \mathbb{R}$, then $\mathbb{R}[x_1, \ldots, x_d]$ can be considered as the symmetric algebra $\mathbb{S}(\mathfrak{t}^*)$ of $\mathfrak{t}^*$, where $\{x_1, \ldots, x_d\}$ is a basis of $\mathfrak{t}^*$. The projection map $M \times ET \to ET$ is T-equivariant, so we obtain a map

$$\pi\colon\ M \times_T ET \to BT. \tag{2.26}$$

This makes $M \times_T ET$ into an M-bundle over BT,

$$M \xrightarrow{r} M \times_T ET \xrightarrow{\pi} BT, \tag{2.27}$$

which induces a sequence of ring homomorphisms

$$H^*(BT;R) \xrightarrow{\pi^*} H_T^*(M;R) \xrightarrow{r^*} H^*(M;R).$$

The map π^* gives $H_T^*(M;R)$ an $H^*(BT;R)$-module structure by

$$\alpha \cdot \beta = \pi^*(\alpha) \cup \beta$$

for $\alpha \in H^*(BT;R)$ and $\beta \in H_T^*(M;R)$, where $\cup$ denotes the cup product.

Definition 2.45
The T-space M is called **equivariantly formal** (with respect to the coefficient ring R) if $H_T^*(M;R)$ is a free $H^*(BT;R)$-module such that $H_T^*(M;R)$ is isomorphic to

$$H^*(M;R) \otimes H^*(BT;R) \tag{2.28}$$

as $H^*(BT;R)$-module, as well as vector space.[2]

The next theorem is due to Kirwan [26].

Theorem 2.46
The T-space M is equivariantly formal (with respect to the coefficient ring R) if and only if the restriction map $r^*\colon H_T^*(M;R) \to H^*(M;R)$ is surjective and its kernel is the ideal generated by $\pi^*(H^2(BT;R))$.

We note that if M is equivariantly formal, then any element of $H_T^*(M;R)$ can be uniquely written as a polynomial in the variables $x_1, \dots, x_d$ with coefficient in $H^*(M;R)$, where $x_1, \dots, x_d$ is a basis of $\mathfrak{t}^*$. The restriction map $r^*\colon H_T(M;R) \to H(M;R)$ sends such a polynomial to its constant term. For example, given $\mu^T \in H_T^2(M;R)$, then

$$\mu^T = \mu \otimes 1 + \sum_{j=1}^{d} \mu_j \otimes x_j \tag{2.29}$$

and $r^*(\mu^T) = \mu$, where $\mu \in H^2(M;R)$ and $\mu_j \in H^0(M;R)$.
The following theorem is due to Ginzburg [9] and Kirwan [26].

[2] This does not mean that $H_T^*(M;R)$ and $H^*(M;R) \otimes H^*(BT)$ are isomorphic as rings.

Theorem 2.47
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space. Then M is equivariantly formal with respect to $\mathbb{R}$-coefficients. Moreover, if $H^*(M;\mathbb{Z})$ is torsion-free, then M is also equivariantly formal with respect to $\mathbb{Z}$-coefficients.

Moreover, let $i\colon M^T \to M$ be the inclusion map from the fixed point set M^T into M. This map induces a map $i^* : H_T^*(M;R) \to H_T^*(M^T;R)$. Also due to Kirwan [26] the following theorem for compact Hamiltonian T-spaces holds.

Theorem 2.48 (Kirwan's Injectivity Theorem)
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space. Then the map $i^* : H_T^*(M;R) \to H_T^*(M^T;R)$ induced by the inclusion $i\colon M^T \to M$ is injective.

2.3.1 Equivariant Chern Classes

Let M be a smooth manifold endowed with a smooth T-action. Let V be a vector bundle over M so that the T-action on M extends to a linear action on V. We obtain a vector bundle

$$V \times_T BT \to M \times_T BT. \tag{2.30}$$

The equivariant Euler class of $V \to M$, is defined to be the ordinary Euler class of the vector bundle (2.30). Moreover, if V is a complex vector bundle over M so that the T-action on M extends to a $\mathbb{C}$-linear action on V, then (2.30) is also a complex vector bundle. In this case, $c_i^T(V) \in H_T^{2i}(M;\mathbb{Z})$ are the ordinary Chern classes of (2.30). Consider the restriction map $r^* : H_T^*(M;\mathbb{Z}) \to H^*(M;\mathbb{Z})$. Then $r^*(c_i^T(V)) = c_i(V)$, where $c_i(V)$ is the i-th Chern class of V. Moreover, let $p \in M$ be a fixed point of the T-action and let $V_p \simeq \mathbb{C}^k$ be the fiber over p. So we have a T-representation on the complex vector space V_p. Let $\alpha_1, \ldots \alpha_k$ be the weights of this representation, then the restriction of the vector bundle (2.30) to $\{p\} \times_T BT \subset M \times_T BT$ is isomorphic to the vector bundle

$$\mathbb{C}^k \times_T BT \to \{p\} \times_T BT,$$

where the linear T-action on $\mathbb{C}^k$ is given by

$$\exp(\xi) \cdot (z_1, \ldots, z_k) = (\mathrm{e}^{2\pi i \langle \alpha_1, \xi \rangle}\, z_1, \ldots, \mathrm{e}^{2\pi i \langle \alpha_k, \xi \rangle}\, z_k) \tag{2.31}$$

for all ξ in the Lie algebra of T. The restriction of the total equivariant Chern class $c^T(V)$ to p is given by

$$c^T(V)|_{\{p\}} = \prod_{j=1}^{k}(1+\alpha_j) \in H_T^*(p;\mathbb{Z}). \tag{2.32}$$

In particular, $c_i^T(V)|_{\{p\}} = \sigma_i(\alpha_1, \ldots, \alpha_k)$, where σ_i denotes the i-th elementary symmetric polynomial in k variables.

2.3.2 The ABBV Localization Formula

Let M be a smooth manifold endowed with a smooth T-action and let M^T be the set of fixed points. We assume in the following that M^T is not empty. Let F be one of its connected components. The inclusion map $i_F\colon F \to M$ is T-equivariant and hence it induces a map

$$i_F^*\colon H_T^*(M;R) \to H_T^*(F;R).$$

Given μ^T in $H_T^*(M;R)$, we write $\mu^T|_F = i_F^*(\mu^T)$. Moreover, the map $\pi\colon M \times_T ET \to BT$ induces a push-forward map in equivariant cohomology

$$H_T^*(M;R) \to H^{*-\dim(M)}(BT;R), \tag{2.33}$$

which can be seen as integration along the fibers. So we denote it by $\int_M$. The following theorem due to Atiyah-Bott and Berline-Vergne (see [1], [3]) gives a formula for the map $\int_M$ in terms of the fixed point data.

Theorem 2.49 (ABBV Localization Formula)
Let M be a compact oriented manifold endowed with a smooth T-action. Given $\mu^T \in H_T^*(M;\mathbb{Q})$, we have

$$\int_M \mu^T = \sum_{F\subset M^T} \int_F \frac{\mu^T|_F}{e^T(N_F)},$$

where the sum runs over all connected components F of M^T and $e^T(N_F)$ is the equivariant Euler class of the normal bundle $N_F \to F$.

2.3.3 The Cartan Model

Let M be a smooth manifold. The de Rham cohomology ring of M can be identified with the singular cohomology ring with coefficients in $\mathbb{R}$. This is known as the de

Rham theorem. There is an equivariant version of the de Rham theorem. Namely, let M be a smooth manifold together with a smooth T-action, where T is a d-dimensional torus with Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$. Then the T-equivariant cohomology can be described by the Cartan model (see [4]) as follows. Let $(\Omega(M), \mathrm{d})$ be the de Rham complex of M and let $(\Omega^T(M), \mathrm{d})$ the subcomplex of T-invariant differential forms. The Cartan complex is $(\Omega_T(M), \mathrm{d}_T)$, where

$$\Omega_T(M) = \Omega^T(M) \otimes_{\mathbb{R}} \mathbb{S}(\mathfrak{t}^*) \tag{2.34}$$

and the differential d_T is defined as follows. Let $\{\xi_1, \dots, \xi_d\}$ be a basis of $\mathfrak{t}$ with dual basis $\{x_1, \dots, x_d\}$ and X_{ξ_j} be the vector field M generated be ξ_j for all $j = 1, \dots, d$, then[3]

$$\mathrm{d}_T(\alpha \otimes p) = \mathrm{d}\alpha \otimes p - \sum_{j=1}^{d} \iota_{X_{\xi_j}} \alpha \otimes x_j p. \tag{2.35}$$

Note that $\Omega_T(M)$ is graded by

$$\Omega_T^k(M) = \sum_{p+2q=k} \left(\Omega^p(M)\right)^T \otimes_{\mathbb{R}} \mathbb{S}^q(\mathfrak{t}^*), \tag{2.36}$$

d_T sends elements of $\Omega_T^k(M)$ to the ones of $\Omega_T^{k+1}(M)$ and $(\mathrm{d}_T)^2 = 0$. So $(\Omega_T(M), \mathrm{d}_T)$ is a cochain complex. The equivariant de Rham theorem (see [20]) states that the cohomology of the Cartan complex can be naturally identified with the equivariant cohomology with $\mathbb{R}$-coefficients described by the Borel model.

Remark 2.50
Let (M, ω, T, ϕ) be a Hamiltonian T-space of dimension $2n$ and let

$$\phi_i \colon M \to \mathbb{R}\,,\, p \mapsto \langle \phi(p), \xi_i \rangle \tag{2.37}$$

be the i-th component of the moment map with respect to the basis $\{\xi_1, \dots, \xi_d\}$ of $\mathfrak{t}$. Since ω and the components ϕ_i, $1 \leq i \leq d$, are T-invariant, we have

$$\omega_\phi \colon = \omega - \phi = \omega \otimes 1 - \sum_{i=1}^{d} \phi_i \otimes x_i \in \Omega_T^2(M), \tag{2.38}$$

where $\{x_1, \dots, x_d\} \subset \mathfrak{t}^*$ is the dual basis of $\{\xi_1, \dots, \xi_d\}$. Moreover, the form ω_ϕ is d_T-closed. Hence, $[\omega_\phi] \in H_T^2(M; \mathbb{R})$ is an equivariant extension of $[\omega] \in H^2(M; \mathbb{R})$.

[3]Note that the definition of d_T is indeed independent of the choice of the basis $\{\xi_1, \dots, \xi_d\}$.

2.3.4 Monotonicity and the Weight Sum Formula

The deep results about equivariant cohomology, which are mentioned in the sections before, have two important consequences for compact and monotone Hamiltonian T-spaces. The first consequence is that compact and monotone Hamiltonian T-spaces are positive monotone. This is the content of the following lemma. Note that this lemma is not completely new, it is proven in [12] for the case that the set of fixed points is finite. This prove can be easily generalized for the case where the set of fixed points is not discrete. For the sake of completeness we will give a proof for the generalized version.

Lemma 2.51
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space such that (M, ω) is monotone. Then (M, ω) is positive monotone.

Proof. (c.f. [12, Lemma 5.2]) Given a compact Hamiltonian T-space (M, ω, T, ϕ), then any subcircle of T acts also in a Hamiltonian fashion on (M, ω). Hence, it is enough to prove the claim for compact Hamiltonian S^1-spaces. So let (M, ω, S^1, ϕ) be a compact Hamiltonian S^1-space which is monotone, i.e. $c_1(M) = \lambda[\omega]$ in $H^2(M; \mathbb{R})$ for some $\lambda \in \mathbb{R}$. We need to show that $\lambda > 0$. After fixing an orientation for S^1, we identify the Lie algebra of S^1 with $\mathbb{R}$ and the dual lattice $\ell^*_{S^1}$ with $\mathbb{Z}$. By Lemma 2.39 $\phi\colon M \to \mathbb{R}$ is a Morse-Bott function and its set of critical points $\mathrm{Crit}(\phi)$ coincides with the set of fixed points M^{S^1}. Moreover, for any $p \in M^{S^1} = \mathrm{Crit}(\phi)$, the Morse-Bott index $\mathrm{ind}(p)$ is equal to twice the number of negative weights at p. In particular, let $p_{\min} \in M^{S^1} = \mathrm{Crit}(\phi)$ such that ϕ attains its minimum at $p_{\min}$. Then any weight at $p_{\min}$ is greater or equal to zero, but not all weights at $p_{\min}$ can be equal to zero (because the S^1-action is effective). We conclude that the sum of all weights of the S^1-action at p_{min}, denoted by $\Gamma_{p_{\min}}$, is greater than zero. Similarly, let $p_{\max} \in M^{S^1} = \mathrm{Crit}(\phi)$ such that ϕ attains its maximum at $p_{\max}$ and let $\Gamma_{p_{\max}}$ be the sum of the weights of the S^1-action at $p_{\max}$, then $\Gamma_{p_{\max}}$ is less than zero.
Now let $c_1^{S^1}(M) \in H^2_{S^1}(M, \mathbb{R})$ be the equivariant extension of the first Chern class $c_1(M) \in H^2(M; \mathbb{R})$, then

$$c_1^{S^1}(M)(p_*) = \Gamma_{p_*} x \in H^*_{S^1}(p_*; \mathbb{R}) = \mathbb{R}[x], \tag{2.39}$$

where $*$ is min or max. By Remark 2.50 $[\omega_\phi] = [\omega - \phi] \in H^2_{S^1}(M; \mathbb{R})$ is an equivariant extension of $[\omega]$, i.e. $r^*([\omega_\phi]) = [\omega]$, where $r^*\colon H^*_{S^1}(M; \mathbb{R}) \to H^*(M; \mathbb{R})$ is the restriction map. Since by Theorem 2.46 and 2.47 the kernel of r^* is the ideal generated by $\langle x\rangle \cong \pi^*(H^2_{S^1}(BS^1;\mathbb{R}))$ and

$$r^*(c_1^{S^1}(M)) = c_1(M) = \lambda[\omega] = r^*(\lambda[\omega_\phi]), \tag{2.40}$$

we have $c_1^{S^1}(M) = \lambda[\omega_\phi] + \eta x$ for some $\eta \in \mathbb{R}$. Now consider the restrictions of $c_1^{S^1}(M)$ and $[\omega_\phi]$ to the fixed points p_{min} and p_{max}, so we obtain

$$\begin{aligned} -\lambda\phi(p_{min}) - \eta = \frac{c_1^{S^1}(p_{min})}{x} &= \Gamma_{p_{min}} \\ &> \Gamma_{p_{max}} = \frac{c_1^{S^1}(p_{max})}{x} = -\lambda\phi(p_{max}) - \eta. \end{aligned}$$

Hence, λ is greater than zero. $\square$

A **symplectic Fano manifold** is a symplectic manifold, whose first Chern class is equal to the class given by the symplectic form in $H^2(M;\mathbb{R})$. Due to Lemma 2.51 the classification problem for compact and monotone Hamiltonian T-spaces reduces to the classification problem for compact Hamiltonian T-spaces whose underlyling manifolds are symplectic Fano manifolds. To see this we note the following. Given a symplectic manifold (M,ω), we rescale the symplectic form by a positive factor λ, then $\lambda\omega$ is also a symplectic form on M and the first Chern class (M,ω) and the one of $(M,\lambda\omega)$ coincides. Moreover, if a torus T acts in a Hamiltonian fashion on (M,ω) with moment map ϕ, then the T-action is also Hamiltonian with respect to $\lambda\omega$ and a moment map is given by $\lambda\phi$. Hence, by Lemma 2.51 it is not restrictive to assume that a compact and monotone Hamiltonian T-space satisfies $c_1(M) = [\omega]$. The second consequence of the results mentioned in the sections before is that for compact Hamiltonian T-spaces with $c_1(M) = [\omega]$ the moment maps satisfy (after a global shift) the **weight sum formula**, i.e. for each fixed point $p \in M^T$ we have

$$\phi(p) = -\alpha_{p,1} - \cdots - \alpha_{p,n}, \tag{2.41}$$

where $\alpha_{p,1}, ..., \alpha_{p,n}$ denote the weights at p. This is the content of the following lemma.

Lemma 2.52
Let (M,ω,T,ϕ) be a compact Hamiltonian T-space such that the first Chern class $c_1(M)$ of (M,ω) is equal to $[\omega]$ in $H^2(M;\mathbb{R})$. Then there exists a constant $\alpha_0 \in \mathfrak{t}^*$ such that for each fixed point $p \in M^T$

$$\phi(p) = -(\alpha_{p,1} + \cdots + \alpha_{p,n}) + \alpha_0, \tag{2.42}$$

where $\alpha_{p,1}, \ldots \alpha_{p,n}$ are the weights of the T-action at T_pM.

Proof. Let $c_1^T(M)$ be the equivariant extension of the first Chern class $c_1(M)$. By Remark 2.50 we have that $[\omega_\phi] = [\omega - \phi]$ is an equivariant extension of $[\omega]$ in $H_T^2(M;\mathbb{R})$. By Theorem 2.46 and Theorem 2.47 the kernel of the restriction map $r^*\colon H_T^2(M;\mathbb{R}) \to H^2(M;\mathbb{R})$ is the ideal generated by the symmetric algebra $\mathbb{S}(\mathfrak{t}^*)$ on $\mathfrak{t}^*$. So from

$$r^*(c_1^T(M)) = c_1(M) = [\omega] = r^*([\omega_\phi]) \tag{2.43}$$

we obtain that $c_1^T(M) = [\omega_\phi] + \alpha_0$ for some $\alpha_0 \in \mathfrak{t}^*$ and for each fixed point $p \in M^T$ we have

$$c_1^T(M)|_{\{p\}} = ([\omega_\phi] + \alpha_0)|_{\{p\}} = -\phi(p) + \alpha_0, \tag{2.44}$$

in $H_T^2(\{p\}\,;\mathbb{R})$. Moreover, we have

$$c_1^T(M)|_{\{p\}} = \alpha_{p,1} + \dots \alpha_{p,n}, \tag{2.45}$$

where $\alpha_{p,1}, \dots \alpha_{p,n}$ are the weights at p. Hence, the claim of this lemma follows from the Equations (2.44) and (2.45). □

Due to Lemma 2.51 and 2.52 it is not restrictive to assume that a compact and monotone Hamiltonian T-space satisfies the following definition.

Definition 2.53
Let (M, ω, T, ϕ) be a compact monotone Hamiltonian T-space. We say that (M, ω, T, ϕ) is **balanced** if the following holds.

- The first Chern class of (M, ω) is equal to the class given by the symplectic form in $H^2(M;\mathbb{R})$, i.e. $c_1(M) = [\omega]$.
- The moment map satisfies the weight sum formula.

Finally, we like to point out that the converse of Lemma 2.52 fails to be true in general. This means there exist compact Hamiltonian T-spaces which are not monotone, but their moment maps are satisfying the weight sum formula. In the next lemma we show that the converse of Lemma 2.52 holds, if the set of fixed points is isolated.

Lemma 2.54
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space with only isolated fixed points and so that the moment map satisfies the weight sum formula. Then the first Chern class $c_1(M)$ of (M, ω) is equal to $[\omega]$ in $H^2(M;\mathbb{R})$.

Proof. Let $c_1^T(M)$ be the equivariant extension of the first Chern class $c_1(M)$. By Remark 2.50 we have that $[\omega_\phi] = [\omega - \phi]$ is an equivariant extension of $[\omega]$ in $H_T^2(M;\mathbb{R})$. If the moment map satisfies the weight sum formula, we have

$$c_1^T(M)|_{\{p\}} = [\omega_\phi]|_{\{p\}}, \tag{2.46}$$

for all isolated fixed points p. Let $i^* : H_T^*(M;\mathbb{Z}) \to H_T^*(M^T;\mathbb{Z})$ be the map induced by the inclusion $i : M^T \to M$. If all fixed points are isolated, then by Kirwan's Injectivity Theorem (Theorem 2.48) the map i^* is injective. Hence, in this case, by (2.46) we have that $c_1^T(M)$ and $[\omega_\phi]$ have the same image under i^*. Therefore, $c_1^T(M) = [\omega_\phi]$, which implies $c_1(M) = [\omega]$. $\square$

Chapter 3

Basic Properties of Tall Complexity One Spaces

The main goal of this chapter is to prove basic properties about compact and tall complexity one spaces, in particular, properties of their moment map polytopes. Therefore, we first recall the definition of being tall for complexity one spaces.

Definition 3.1
A complexity one space (M, ω, T, ϕ) is called **tall**, if for any $x \in \phi(M)$ the reduced space $M_x := \phi^{-1}(x)/T$ is not a point.

We like to light out the meaning of being tall for compact complexity one spaces. Namely, let (M, ω, T, ϕ) be a compact complexity one space. By Δ_0 we denote the points in the moment map polytope $x \in \Delta$ such that the reduced space M_x is a point. Hence, (M, ω, T, ϕ) fails to be tall if and only if Δ_0 is not the empty set. Moreover, by Theorem 2.40 for any regular point $x \in \Delta$ the reduced space M_x has dimension two. Hence, Δ_0 is a subset of the set of the singular values of the moment map which has Lebesgue measure zero. Due to the next lemma Δ_0 is a subset of the boundary of the moment map polytope Δ.

Lemma 3.2 ([25, Corollary 2.4])
Let (M, ω, T, ϕ) be a compact complexity one space then Δ_0 is a subset of the boundary $\partial\Delta$ of the moment map polytope Δ. Moreover, for any $x \in \Delta \setminus \Delta_0$, the reduced space M_x is a topological surface.

Given a compact and tall complexity one space (M, ω, T, ϕ) , the moment map induces a surjective and continuous[1] map $\overline{\phi}\colon M/T \to \Delta$. The preimage of $x \in \Delta$ under $\overline{\phi}$ is the reduced space M_x, which is by Lemma 3.2 a compact surface. The following proposition states that $\overline{\phi}$ defines a surface bundle over Δ.

Proposition 3.3 ([25, Proposition 1.2])
Let (M, ω, T, ϕ) be a compact, tall complexity one space and let Δ be its moment map polytope. There exists a compact oriented surface Σ and a map $f : M/T \to \Sigma$ so that

$$(\overline{\phi}, f) : M/T \longrightarrow \Delta \times \Sigma \tag{3.1}$$

[1] Here, we consider M/T together with the quotient topology.

is a homeomorphism and the restriction $f : \phi^{-1}(x)/T \to \Sigma$ is orientation preserving for each $x \in \Delta$. Given two such maps f and f', there exists an orientation preserving homeomorphism $\theta : \Sigma' \to \Sigma$ so that f is homotopic to $\theta \circ f'$, through maps which induce homeomorphisms $M/T \to \Delta \times \Sigma$.

Due to Theorem 3.3 we can define the **genus** of a compact and tall complexity one space.

Definition 3.4
Let (M, ω, T, ϕ) be a compact, tall complexity one space. The **genus** of (M, ω, T, ϕ) is defined as the genus of M_x, where M_x is the reduced space for some point $x \in \phi(M)$.

We note that by Theorem 3.3 the genus does not depend on the choice of $x \in \phi(M)$.

Example 3.5
Let $(\widetilde{M}, \widetilde{\omega}, T, \widetilde{\phi})$ be a compact symplectic toric manifold of dimension $2n-2$, so the dimension of T is $n-1$ and let (Σ, ω_Σ) be a compact symplectic surface. We endow $\widetilde{M} \times \Sigma$ with the product T-action , i.e. T acts on first factor as above and on the second factor the action is trivial. Then $(\widetilde{M} \times \Sigma, \widetilde{\omega} \otimes \omega_\Sigma, T, \phi)$ is a compact complexity one space of dimension $2n$, where the moment map is

$$\phi : \widetilde{M} \times \Sigma \longrightarrow \mathfrak{t}^*, \quad (p, q) \longmapsto \widetilde{\phi}(p). \tag{3.2}$$

Since any non-empty fiber of the moment map of a symplectic toric manifold contains just a single orbit (Lemma 2.38), the reduced space at any point $x \in \phi(\widetilde{M} \times \Sigma)$ is homeomorphic to Σ. Hence, the space $(\widetilde{M} \times \Sigma, \widetilde{\omega} \otimes \omega_\Sigma, T, \phi)$ is tall and its genus is equal to the one of Σ.

In [23, 24, 25] Tolman and Karshon gave a list of topological invariants of compact[2] and tall complexity one spaces which contains enough information to determine these spaces up to isomorphisms. In particular, two of the these invariants are the **Duistermaat-Heckman measure** and the **genus**, the remaining other one is the **painting**. Next, we introduce the painting. Therefore, we need first to explain the **space of exceptional orbits** .

Let (M, ω, T, ϕ) be a compact complexity one space. A T-orbit is exceptional if any nearby orbit in the same fiber of the moment map has strictly smaller stabilizer. For example, if a fiber $\phi^{-1}(x)$ contains just one orbit, then this orbit is exceptional

[2] These results by Karshon and Tolman hold also for non-compact complexity one spaces, if there exists a convex subset $\mathcal{T}$ of the dual Lie algebra $\mathfrak{t}^*$ of T such that the moment map image lies in $\mathcal{T}$ and so that the moment map considered as a map from from M to $\mathcal{T}$ is proper.

(which cannot happen in the tall case by Proposition 3.3). Another simple example is that the orbit through an isolated fixed point is indeed exceptional. Given an exceptional orbit $\mathcal{O} \subset \phi^{-1}(x)$, there exists an open neighborhood U of $\mathcal{O}$ in $\phi^{-1}(x)$ so that $\mathcal{O}$ is the only exceptional orbit in U. Since the moment map fibers are compact, any fiber can contain at most finitely many exceptional orbits. The union of the exceptional orbits is called the space of exceptional orbits and is denoted by M_{exc}. It is a subspace of M/T, so we consider M_{exc} with the induced subspace topology. Let M'_{exc} be the set of exceptional orbits of an other compact complexity one space. An isomorphism between M_{exc} and M'_{exc} is a homeomorphism $i : M_{\text{exc}} \to M'_{\text{exc}}$ that respects the maps induced by moment maps and maps each orbit to an orbit with the same stabilizer and same isotropy representation. Now we define the paintings. A painting for M_{exc} is a continuous map $f : M_{\text{exc}} \to \Sigma$, where Σ is an oriented surfaces so that

$$(\overline{\phi}, f) : M_{exc} \longrightarrow \Delta \times \Sigma \tag{3.3}$$

is injective. Given two paintings $f : M_{exc} \to \Sigma$ and $f' : M'_{exc} \to \Sigma'$, these paintings are equivalent if there exist an isomorphism $i : M_{exc} \to M'_{exc}$ and an orientation preserving homeomorphism $\eta : \Sigma \to \Sigma'$ such that $\eta \circ f : M_{exc} \to \Sigma'$ and $f' \circ i : M_{exc} \to \Sigma'$ are homotopic through maps that are also paintings. Proposition 3.3 gives us a well-defined equivalent class of paintings associated to each compact and tall complexity one space.
Now we can state the main result of the work by Karshon and Tolman [23, 24, 25].

Theorem 3.6 ([25, Theorem 1.8])
Let (M, ω, T, ϕ) and (M', ω', T, ϕ') be two compact and tall complexity one spaces of dimension $2n$, then these spaces are isomorphic if and only if they have the same Duistermaat-Heckman measure, the same genus and their paintings are equivalent.

3.1 Moment Map Polytopes of Tall and Compact Complexity One Spaces

In this section we prove some properties about the moment map images of compact and tall complexity one spaces. In particular, we show that the moment map polytope of a compact and tall complexity one space is Delzant. Moreover, the essential result of this section is that the moment map polytope of a compact, tall and monotone complexity one space which is balanced, is also reflexive (see Definition 3.14). Before we concentrate on the monotone case, we verify properties of compact and tall complexity one spaces without assuming monotonicity. The following lemma gives us a necessary condition for a compact complexity one space to be tall.

Lemma 3.7
Let (M, ω, T, ϕ) be a compact and tall complexity one space and let Δ be its moment map polytope. Then for each vertex v of Δ its preimage $\phi^{-1}(v)$ is a fixed surface, whose genus is equal to the one of (M, ω, T, ϕ).

Proof. Given a vertex of the moment map polytope Δ, by Corollary 2.27 the preimage $\phi^{-1}(v)$ is a connected component of M^T. In particular, the reduced space at v is simply $\phi^{-1}(v)$. Since the complexity of (M, ω, T, ϕ) is equal to one, by Lemma 2.20 we have $\phi^{-1}(v)$ is either a point or a fixed surface. Hence, by the Definition 3.1 we conclude that the tallness of (M, ω, T, ϕ) implies that the preimage each vertex of Δ is a fixed surface. Moreover, from Proposition 3.3 it follows that for any vertex v the genus of $\phi^{-1}(v)$ is equal to the one of (M, ω, T, ϕ). $\square$

Now we like to show that the converse of Lemma 3.7 also holds. But before we prove this, we need to prove the following lemma, which we will also use in a later part of this work.

Lemma 3.8
Let (M, ω, T, ϕ) be a compact complexity one space and let v be a vertex of the moment polytope Δ such that $\phi^{-1}(v)$ is a fixed surface. Let $\mathcal{F}$ be a face of the moment map polytope which contains v. Then the preimage $\phi^{-1}(\mathcal{F})$ is a $(2f+2)$-dimensional symplectic submanifold of (M, ω), where f is the dimension of the face $\mathcal{F}$. Moreover, the set of elements in T which acts trivially on $\phi^{-1}(\mathcal{F})$ is the stabilizer $\mathfrak{stab}(\mathcal{F})$ of $\mathcal{F}$.

Proof. We denote by $2n$ the dimension of M, so the dimension of the torus T is $n-1$. Let $\mathcal{F}$ be a face of Δ of dimension f which contains the vertex v. By Lemma 2.29 the preimage of $\mathcal{F}$ is a symplectic submanifold of M. Hence, we need to show that the dimension of $\phi^{-1}(\mathcal{F})$ is equal to $2f+2$ and that the set of elements in T which acts trivially on $\phi^{-1}(\mathcal{F})$ is the stabilizer $\mathfrak{stab}(\mathcal{F})$ of $\mathcal{F}$. If $\mathcal{F}$ is the vertex v, i.e. $f=0$, there is nothing to prove. So suppose that $f \geq 1$. The preimage of v, denoted by Σ_v, is a fixed surface. Let $\alpha_{\Sigma_v,1}, \ldots, \alpha_{\Sigma_v,n}$ be the weights along Σ_v. Since Σ_v has dimension two, exactly one these weights is zero, say $\alpha_{\Sigma_v,n}$. Hence, the weights $\alpha_{\Sigma_v,1}, \ldots, \alpha_{\Sigma_v,n-1}$ form a $\mathbb{Z}$-basis of $\ell_T^*(\cong \mathbb{Z}^{n-1})$ (see Remark 2.19). Due to Corollary 2.26 there exists an open neighborhood of v in $\phi(M)$ which looks like an open neighborhood of v in the cone

$$\mathcal{C}_{\Sigma_v} = v + \mathbb{R}_{\geq 0}\text{-span}\left\{\alpha_{\Sigma_v,1}, \ldots, \alpha_{\Sigma_v,n-1}\right\}.$$

Hence, a face of dimension j that contains the vertex v must be of the form

$$\left(v + \mathbb{R}_{\geq 0}\text{-span}\left\{\alpha_{\Sigma_v,i_1}, \dots, \alpha_{\Sigma_v,i_j}\right\}\right) \cap \Delta,$$

where $1 \leq i_1 < ... < i_j \leq n-1$. Since $\mathcal{F}$ has dimension f, we can assume without loss of generality that

$$\mathcal{F} = (v + \mathbb{R}_{\geq 0}\text{-span}\,\{\alpha_{\Sigma_v,1}, \dots, \alpha_{\Sigma_v,f}\}) \cap \Delta.$$

Now pick a point $p \in \phi^{-1}(v)$. Due to Theorem 2.18 there exist complex coordinates $z_0, z_1, \dots, z_{n-1}$ centered at p such that

- the T-action is given by

  $$\exp(\xi) \cdot (z_0, \dots, z_{n-1}) = (z_0, \mathrm{e}^{2\pi i\langle \alpha_{v,1},\xi\rangle} z_1, \dots, \mathrm{e}^{2\pi i\langle \alpha_{v,n-1},\xi\rangle} z_{n-1})$$

  for all $\xi \in \mathfrak{t}$,

- the moment map is given by $\phi(z) = \pi \sum_{j=1}^{n-1} \alpha_{v,j}|z_j|^2 + v$.

Hence, in these coordinates the preimage of $\mathcal{F}$ is given by the equations $z_i = 0$ for $i = f+1, ..., n-1$. Therefore, the dimension of $\mathcal{F}$ is equal to $2(f+1)$. Note that by the above discussion for $p \in \phi^{-1}(v) \subset \phi^{-1}(\mathcal{F})$ there exists an open neighborhood U_p of p in $\phi^{-1}(\mathcal{F})$ with complex coordinates $z_0, \dots, z_f$, such that the T-action on U_p is

$$\exp(\xi) \cdot (z_0, \dots, z_f) = (z_0, \mathrm{e}^{2\pi i\langle \alpha_{v,1},\xi\rangle} z_1, \dots, \mathrm{e}^{2\pi i\left\langle \alpha_{v,f},\xi\right\rangle} z_f) \text{ for all } \xi \in \mathfrak{t}. \tag{3.4}$$

Since the weights $\alpha_{\Sigma_v,1}, \dots, \alpha_{\Sigma_v,n-1}$ form a $\mathbb{Z}$-basis of ℓ_T^*, let $\xi_1^*, \dots, \xi_{n-1}^* \subset \mathfrak{t}$ its dual basis. So the annihilator of $\mathcal{F}$ is

$$\mathfrak{ann}(\mathcal{F}) = \mathbb{R}\text{-span}\left\{\xi_{f+1}^*, \dots, \xi_{n-1}^*\right\}. \tag{3.5}$$

Let H be the subtorus of T, whose Lie algebra is spanned by $\xi_1^*, \dots, \xi_f^*$. We can identify the quotient torus $T/\mathfrak{stab}(\mathcal{F})$ with H. By using (3.4) it follows that H acts effectively on U_p. Since U_p is an open subset of $\phi^{-1}(\mathcal{F})$, the subtorus H acts effectively on $\phi^{-1}(\mathcal{F})$. □

Next we point out some remarks about the proof of Lemma 3.8.

Remark 3.9

Let (M, ω, T, ϕ) be a compact complexity one space and let v be a vertex of the moment map polytope Δ, such that $\phi^{-1}(v)$ is a fixed surface. Given a face $\mathcal{F}$ of the moment map polytope which contains v, the content of Lemma 3.8 is roughly speaking that the preimage of $\mathcal{F}$ is also a compact complexity one space. In order to see this we note the following. Let $\alpha_{\Sigma_v,1},\ldots,\alpha_{\Sigma_v,n-1}$ be the non-zero weights along $\phi^{-1}(v)$ which form a $\mathbb{Z}$-basis of ℓ_T^*, and let $\xi_1^*,\ldots,\xi_{n-1}^* \subset \mathfrak{t}$ be its dual basis. Suppose that $\mathcal{F}$ is given by

$$\mathcal{F} = (v + \mathbb{R}_{\geq 0}\text{-span}\,\{\alpha_{\Sigma_v,1},\ldots,\alpha_{\Sigma_v,f}\}) \cap \Delta.$$

Let H be the subtorus of T whose Lie algebra is spanned by $\xi_1^*,\ldots,\xi_f^*$. So H is a subtorus of dimension f. As pointed out in the proof of Lemma 3.8 we have that H acts effectively on $\phi^{-1}(\mathcal{F})$, which is a symplectic submanifold of dimension $2(f+1)$. Moreover, if we identify $\mathfrak{t}^*$ with $\mathbb{R}^{n-1}$ via

$$(t_1,\ldots,t_{n-1}) \longleftrightarrow t_1\alpha_{v,1} + \cdots + t_{n-1}\alpha_{v,n-1}, \tag{3.6}$$

a moment map for the H-action on $\phi^{-1}(\mathcal{F})$ is given by $\pi\circ\phi : \phi^{-1}(\mathcal{F}) \to \mathbb{R}^f$, where $\pi\colon \mathbb{R}^{n-1} \to \mathbb{R}^f$ is given by $\pi(t_1,\ldots,t_{n-1}) = (t_1,\ldots,t_f)$. Hence $(\phi^{-1}(\mathcal{F}),\omega,H,\pi\circ\phi)$ is a compact complexity one space of dimension $2(f+1)$.

Now we prove that the converse of Lemma 3.7 also holds.

Proposition 3.10
Let (M,ω,T,ϕ) be a compact complexity one space and let Δ be its moment map polytope. Then following two conditions are equivalent.

- *The space (M,ω,T,ϕ) is tall.*
- *For any vertex v of Δ its preimage $\phi^{-1}(v)$ is a fixed surface.*

Proof. Let (M,ω,T,ϕ) be a compact complexity one space of dimension $2n$. That the first condition implies the second one is the content of Lemma 3.7. It remains to prove the converse. Suppose that the preimage of any vertex of Δ is a fixed surface. Given a point $x\in\Delta$, we need to show that the reduced space M_x is not a point. If x is a vertex, then M_x is a fixed surface. If x lies in the interior of Δ, then M_x is not a point by Lemma 3.2. So suppose that x lies in the interior of a face $\mathcal{F}$ of Δ, whose dimension f is equal to $1,2,...,n-3$ or $n-2$. By Lemma 3.8 the preimage $\phi^{-1}(\mathcal{F})$ is a compact $(2f+2)$-dimensional symplectic submanifold and the set of elements in T which acts trivially on $\phi^{-1}(\mathcal{F})$ is the stabilizer $\mathfrak{stab}(\mathcal{F})$ of $\mathcal{F}$. Moreover, the quotient torus $T/\mathfrak{stab}(\mathcal{F})$ can be identify with a subtorus H of T, which has dimension f (see Remark 3.9). Therefore, we have

$$M_x = \phi^{-1}(x)/T = \phi^{-1}(x)/H. \tag{3.7}$$

By Remark 3.9 the H-action on $\phi^{-1}(\mathcal{F})$ is effective and Hamiltonian. Let $\phi_{\mathcal{F}}$ be a moment map for the H-action on $\phi^{-1}(\mathcal{F})$, then the quadruple $(\phi^{-1}(\mathcal{F}), \omega, H, \phi_{\mathcal{F}})$ is a compact complexity one space of dimension $2(f+1)$. Modulo a shift, we have $\phi_F = \pi \circ \phi|_{\phi^{-1}(\mathcal{F})}$, where $\pi : \mathfrak{t}^* \to \mathfrak{h}^*$ is the projection induced by the inclusion $\mathfrak{h} \to \mathfrak{t}$. Since x is a point which lies in the interior of the face $\mathcal{F}$, $\pi(x)$ is a point which lies in the interior of $\phi_{\mathcal{F}}(\phi^{-1}(\mathcal{F}))$. So by Lemma 3.2 the reduced space at $\pi(x)$ is a surface, but this reduced space is equal to (3.7). □

The next lemma describes the moment map polytopes of compact and tall complexity one spaces. Moreover, it tells us which information about the fixed point data we obtain from the moment map polytopes.

Lemma 3.11
Let (M, ω, T, ϕ) be a compact and tall complexity one space of dimension $2n$. Then the following holds.

(i) The moment map polytope Δ is a Delzant polytope of dimension $n-1$.

(ii) The assignment $v \mapsto \phi^{-1}(v)$ is a bijection between the set of vertices of Δ and the fixed surfaces contained in M^T.

(iii) Let v be a vertex, then the weights at v are equal to the non-zero weights along the fixed surface $\phi^{-1}(v)$.

Proof. *(i)* Let v be a vertex of Δ. We need to show that v is smooth. By Lemma 3.7 its preimage $\phi^{-1}(v)$ is a fixed surface, which we denote by Σ_v. Let $\alpha_{\Sigma_v,1}, \dots \alpha_{\Sigma_v,n}$ be the weights along Σ_v. Since the dimension of Σ_v is two, exactly one of these weights is zero, say $\alpha_{\Sigma,n}$. Hence, $\alpha_{\Sigma,1}, \dots, \alpha_{\Sigma,n-1}$ form a $\mathbb{Z}$-basis of $\ell_T^* \cong \mathbb{Z}^{n-1}$ (see Remark 2.19). By Corollary 2.26 an open neighborhood of v in Δ looks like an open neighborhood v in the cone

$$\mathcal{C}_{\Sigma_v} = v + \mathbb{R}_{\geq 0}\text{-span}\left\{\alpha_{\Sigma,1}, \dots, \alpha_{\Sigma,n-1}\right\}. \tag{3.8}$$

Hence, the edges meeting at v are of the form $v + t\alpha_{\Sigma_v,i}$ for $t \geq 0$ and $i = 1, ..., n-1$. So the vertex v is smooth.
(ii) This second claim is a direct consequence of Lemma 2.31 and Lemma 3.7.
(iii) This third claim follows from the proof of (i). □

Remark 3.12
The statement of Lemma 3.11 holds also for compact complexity k spaces with $k \neq 1$ in the following sense. Given a compact complexity k space (M, ω, T, ϕ) such that the preimage of any vertex of the moment map polytope has dimension $2k$, then

by using the same techniques as in Lemma 3.11, it follows that the moment map polytope is a Delzant polytope of dimension $n-k$ and the vertices of the moment map polytope are in one-to-one correspondence with the $2k$-dimensional components of M^T.

Note that for compact complexity zero spaces it is well known that their moment map polytopes are Delzant and that the assignment $v \mapsto \phi^{-1}(v)$ is a bijection between the set of vertices of Δ and the set of fixed points M^T. Moreover, the preimage of any face $\mathcal{F}$ under the moment maps have dimension $2f$, where f is the dimension of $\mathcal{F}$. In the next example we point out a necessary and sufficient condition for a compact complexity one space that comes from a compact symplectic toric manifold, which ensures that the space is tall.

Example 3.13
Let $(M, \omega, (S^1)^n, \phi)$ be a compact symplectic toric manifold of dimension $2n$, with moment map $\phi : M \to (\mathrm{Lie}((S^1)^n))^* \cong \mathbb{R}^n$. The codimensional one subtorus $(S^1)^{n-1} \subset (S^1)^n$ acts also effectively and in a Hamiltonian fashion on (M, ω). A moment map for this action is $\pi \circ \phi$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection given by $\pi(t_1, \ldots, t_n) = (t_1, \ldots, t_{n-1})$. So $(M, \omega, (S^1)^{n-1}, \pi \circ \phi)$ is a compact complexity one space. Let $\Delta \subset \mathbb{R}^n$ be the moment map polytope of $(M, \omega, (S^1)^n, \phi)$, then the moment map polytope of $(M, \omega, (S^1)^{n-1}, \pi \circ \phi)$ is $\Delta' := \pi(\Delta) \subset \mathbb{R}^{n-1}$. Given a vertex v' of Δ', its preimage under π intersected with Δ is a vertex or an edge of Δ. If $\pi^{-1}(v') \cap \Delta$ is a vertex of Δ denoted by v, then

$$(\pi \circ \phi)^{-1}(v') = \phi^{-1}(v) \tag{3.9}$$

is just a point. If $\pi^{-1}(v') \cap \Delta$ is an edge of Δ denoted by e, then

$$(\pi \circ \phi)^{-1}(v') = \phi^{-1}(e) \tag{3.10}$$

is a surface. By using Proposition 3.10 we conclude that the compact complexity one space $(M, \omega, (S^1)^{n-1}, \pi \circ \phi)$ is tall if and only if $\pi^{-1}(v') \cap \Delta$ is an edge of Δ for any vertex v' of Δ'.

3.1.1 The Monotone Case and the Proof of Proposition 3.17

Next we analyze properties of the moment map polytopes of compact tall and monotone complexity one spaces. Therefore, we first recall the definition of reflexive polytopes.

Definition 3.14
Let Δ be an d-dimensional polytope in $\mathbb{R}^d$. Then Δ is called **reflexive** if it is **integral**, i.e. the vertices of Δ belong to $\mathbb{Z}^d$ and

$$\Delta = \bigcap_{l=1}^{k} \left\{ x \in \mathbb{R}^d | \, \langle x, l_i \rangle \leq 1 \right\}, \tag{3.11}$$

where $l_1, \ldots, l_k \in \mathbb{Z}^d$ are the primitive outward normal vectors to the facets which define Δ.

Remark 3.15
Let Δ be a reflexive polytope in $\mathbb{R}^d$. From (3.11) it is easy to conclude that the origin is the only integer point that lies in the interior of Δ and all integer points which lie on the boundary of Δ must be primitive. By a result of Lagarias and Ziegler [28] there exist just finite many reflexive polytopes in any dimension d up to $\mathrm{GL}(\mathbb{Z}^d)$-transformations.

The following lemma is a slight modification of a result proven in [12, Proposition 2.10] and it describes the relation between reflexive polytopes and Delzant polytopes.

Lemma 3.16
Let Δ be a d-dimensional Delzant polytope in $\mathbb{R}^d$. Then the following conditions are equivalent.

(i) Δ is reflexive.

(ii) For each vertex v we have $v = -(\alpha_{v,1} + \cdots + \alpha_{v,d})$, where $\alpha_{v,1}, \ldots, \alpha_{v,d} \in \mathbb{Z}^d$ are the weights at v.

Based on [12, Proposition 2.10], we give an outline of the proof of Lemma 3.16.

Outline of the Proof of Lemma 3.16 . The statement of Lemma 3.16 and the one of [12, Proposition 2.10] are almost the same, beyond the fact that in [12, Proposition 2.10] it is additionally assumed, that the origin is contained in the interior of Δ to prove that the conditions (i) and (ii) are equivalent. Note if condition (i) holds, then by Remark 3.15 the origin is contained in the interior of Δ. On the other side condition (ii) also implies that the origin is contained in the interior of Δ. This holds for the following reason. For any vertex v consider the cone

$$\mathcal{C}_v = v + \mathbb{R}_{\geq 0}\text{-span} \left\{ \alpha_{v,1} \ldots \alpha_{v,d} \right\}, \tag{3.12}$$

where $\alpha_{v,1}, \ldots, \alpha_{v,d} \in \mathbb{Z}^d$ are the weights at v. The polytope Δ is equal to the intersection of these cones. The second condition of Lemma 3.16 implies that the origin lies in the interior of any $\mathcal{C}_v$, so the origin also lies in the interior of Δ. By these observations the statement of Lemma 3.16 is a simple consequence of [12, Proposition 2.10]. □

It is well-known that there exist exactly 16 reflexive polytopes in dimension two (up to $\mathrm{GL}(\mathbb{Z}^2)$-transformations). This can be proven by using fairly simple methods, see for example [5, Theorem 8.3.6]. In particular, five of these 16 polytopes are also Delzant, which we list in the following figure.

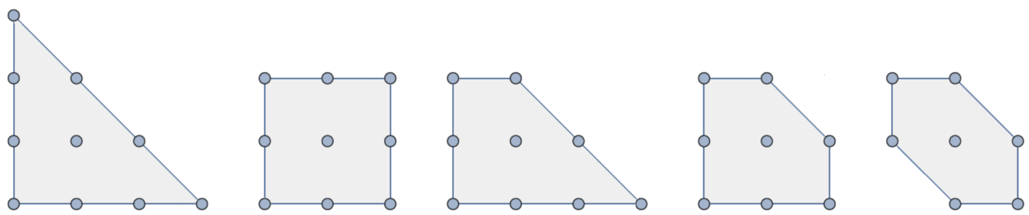


Figure 3.1: All reflexive Delzant polytopes in dimension two up to lattice transformations The bullets indicate integral points in $\mathbb{R}^2$. The gray colored areas indicate the polytopes.

Now we are able to prove Proposition 3.17, which says that the moment map polytope of a compact, tall and monotone complexity one space of dimension $2n$ which satisfies the balanced condition, is a reflexive Delzant polytope of dimension $n-1$.

Proposition 3.17
Let (M, ω, T, ϕ) be a compact tall and monotone complexity one space of dimension $2n$ which satisfies the balanced condition, then its moment map polytope is a reflexive Delzant polytope of dimension $n-1$ in $\mathfrak{t}^* \cong \mathbb{R}^{n-1}$.

Proof of Proposition 3.17 . Let (M, ω, T, ϕ) be a compact tall and monotone complexity one space of dimension $2n$ which satisfies the balanced condition. By Lemma 3.11 the moment map polytope is a Delzant polytope of dimension $n-1$ and for each vertex v we have that $\Sigma_v = \phi^{-1}(v)$ is a fixed surface and the weights at v are equal to the non-zero weights at Σ. So let $\alpha_{v,1}, \ldots, \alpha_{v,n-1}$ be non-zero weights at v. By assumption the moment map satisfies the weight sum formula, so we have

$$v = \phi(\Sigma_v) = -(\alpha_{v,1} + \cdots + \alpha_{v,n-1}) \tag{3.13}$$

for each vertex. Hence, by Lemma 3.16 the moment map polytope is also reflexive. □

Remark 3.18
Similarly, as pointed out in the Remark 3.12 of Lemma 3.11, the statement of Proposition 3.17 holds also for compact complexity k spaces with $k \neq 1$ in the following sense. Given a compact and monotone complexity k space (M, ω, T, ϕ), which is balanced such that the preimage of any vertex of the moment map polytope has dimension $2k$, it follows that the moment map polytope is a reflexive Delzant polytope of dimension $n-k$ by using the same techniques as in Lemma 3.11 and in Proposition 3.17.

Roughly speaking for compact symplectic toric manifolds the converse of Proposition 3.17 is also true. This is the content of the following proposition. Note that this proposition is not new and it has been proven in [12, Proposition 3.8] and [34, Section 3].

Proposition 3.19
Let (M, ω, T, ϕ) be a compact symplectic toric manifold of dimension $2n$, then the following conditions are equivalent.

- *The space (M, ω, T, ϕ) is monotone and satisfies the balanced condition.*
- *The moment map polytope of (M, ω, T, ϕ) is a reflexive Delzant polytope.*

3.2 Characterization of the Exceptional Orbits

In this subsection we like to have a closer look on the exceptional orbits. Recall that given a complexity one space, a T-orbit of this space is called exceptional if any nearby orbit in the same moment map fiber has a strictly smaller stabilizer. For compact and tall complexity one spaces we can characterize exceptional orbits as in the following proposition.

Proposition 3.20
Let (M, ω, T, ϕ) be a compact and tall complexity one space, let $\mathcal{O}$ be a T-orbit and let $H_{\mathcal{O}}$ be its stabilizer. Moreover, let $\mathcal{F}$ be a face of the moment map polytope, such that $\phi(\mathcal{O})$ lies in the interior of $\mathcal{F}$. Then the orbit $\mathcal{O}$ is exceptional if and only if its stabilizer is not equal to the stabilizer $\mathfrak{stab}(\mathcal{F})$ of $\mathcal{F}$.

In order to prove this proposition we use the following two lemmas. As a consequence of these lemmas, we show that the statement of this proposition is true for orbits which are mapped to the interior of the moment map polytope. At the end of this section we give a proof of Proposition 3.20.

Lemma 3.21
Given a compact complexity one space (M, ω, T, ϕ) of dimension $2n$, let $\mathcal{O}$ be a T-orbit such that $\phi(\mathcal{O})$ lies in the interior of the moment map polytope. Suppose that the stabilizer $H_{\mathcal{O}}$ of $\mathcal{O}$ is not trivial and let $X_{\mathcal{O}}$ be the connected component of $M^{H_{\mathcal{O}}}$ which contains the orbit $\mathcal{O}$. Then the dimension of $X_{\mathcal{O}}$ is equal to $2(n-1-h)$, where h is the dimension of $H_{\mathcal{O}}$.

Proof. Note that by Lemma 2.11 we have that $\mathcal{O}$ is an isotropic submanifold of dimension $n-1-h$. By Lemma 2.15 we find that $X_{\mathcal{O}}$ is a compact symplectic submanifold of (M, ω). We conclude that $\mathcal{O}$ is an isotropic submanifold of $X_{\mathcal{O}}$. Hence, the dimension of $X_{\mathcal{O}}$ is greater or equal to $2(n-1-h)$. Now we show that the dimension of $X_{\mathcal{O}}$ is indeed equal to $2(n-1-h)$. We consider therefore two cases, namely $h=0$ or $h \neq 0$. If $h=0$, we have that the dimension of $X_{\mathcal{O}}$ is equal to $2(n-1)$ or $2n$. The latter cannot occur because this would imply $X_{\mathcal{O}} = M$, which contradicts the effectiveness of the T-action on M. Now suppose that $h \geq 1$. Let $T_{\mathcal{O}}$ be the connected component of $H_{\mathcal{O}}$ which contains the identity element[3]. So $T_{\mathcal{O}}$ is a subtorus of T of dimension h. Since the following inclusions hold:

$$\mathcal{O} \subset X_{\mathcal{O}} \subset M^{H_{\mathcal{O}}} \subset M^{T_{\mathcal{O}}} \tag{3.14}$$

and $\phi(\mathcal{O})$ lies in the interior of $\phi(M)$, by Corollary 2.32 the dimension of $X_{\mathcal{O}}$ is smaller than $2(n-h)$. Hence, we conclude that the dimension of $X_{\mathcal{O}}$ is indeed equal to $2(n-1-h)$. $\square$

Lemma 3.22
Let (M, ω, T, ϕ) be a compact complexity one space and let x be a point which lies in the interior of the moment map polytope Δ. Then the fiber $\phi^{-1}(x)$ contains at most finitely many orbits with a non-trivial stabilizer.

Proof. Let $\mathcal{O}$ be an orbit in $\phi^{-1}(x)$ with a non-trivial stabilizer $H_{\mathcal{O}}$. Then the connected component $X_{\mathcal{O}}$ of $M^{H_{\mathcal{O}}}$ which contains $\mathcal{O}$ is a compact symplectic manifold of M with codimension greater than or equal to 2. Since the fiber $\phi^{-1}(x)$ is compact, there exist at most finitely many of such submanifolds. Hence, we need to show that $X_{\mathcal{O}} \cap \phi^{-1}(x) = \mathcal{O}$ is independent of the choice of the orbit $\mathcal{O}$ in $\phi^{-1}(x)$ with a non-trivial stabilizer $H_{\mathcal{O}}$. So we fix such an orbit $\mathcal{O}$. By Lemma 3.21 we have that $X_{\mathcal{O}}$ has dimension $2(n-1-h)$, where h is the dimension of $H_{\mathcal{O}}$. Now the quotient torus $T/H_{\mathcal{O}}$ acts effectively and in a Hamiltonian fashion on $X_{\mathcal{O}}$. Since the dimension of $T/H_{\mathcal{O}}$ is equal to $n-h-1$, the complexity of this action is zero. By identifying $T/H_{\mathcal{O}}$ with a subtorus in T we obtain a projection $\pi : \mathfrak{t}^* \to (\mathrm{Lie}(T/H_{\mathcal{O}}))^*$ such that

[3]This means $H_{\mathcal{O}} = T_{\mathcal{O}}$ if and only if $H_{\mathcal{O}}$ is connected.

$\pi \circ \phi : X_{\mathcal{O}} \to (\mathrm{Lie}(T/H_{\mathcal{O}}))^*$ is a moment map for the $(T/H_{\mathcal{O}})$-action. By Lemma 2.38 any non-empty fiber of $\pi \circ \phi$ contains just one orbit and $\pi(\phi(\mathcal{O})) = \pi(x)$. We conclude

$$\mathcal{O} = (\pi \circ \phi)^{-1} \left(\pi(\phi(\mathcal{O}))\right) = X_{\mathcal{O}} \cap \phi^{-1}(x), \tag{3.15}$$

which completes the proof of this lemma. □

Corollary 3.23
Let (M, ω, T, ϕ) be a compact complexity one space and let x be a point which lies in the interior of the moment map polytope Δ. An orbit which lies in the fiber $\phi^{-1}(x)$, is exceptional if and only if its stabilizer is not trivial.

Proof. The first direction of this statement is simple. Namely, if $\mathcal{O}$ is an exceptional orbit which lies in the fiber $\phi^{-1}(x)$, then any nearby orbit in the same moment map fiber has a strictly smaller stabilizer. Since by the Convexity Theorem the moment map fibers are connected and by Lemma 3.2 the fiber $\phi^{-1}(x)$ contains more then just one orbit, we must have that the stabilizer of $\mathcal{O}$ is not trivial. The second direction of this corollary follows directly from Lemma 3.22, which states that the fiber $\phi^{-1}(x)$ contains at most finitely many orbits with a non-trivial stabilizer. □

Now we are ready to prove Proposition 3.20.

Proof of Proposition 3.20. Let (M, ω, T, ϕ) be a compact and tall complexity one space of dimension $2n$. By Corollary 3.23 the statement of this proposition holds for orbits which are mapped to the interior of Δ. Let v be a vertex of Δ, then by Lemma 3.7 we have that $\phi^{-1}(v)$ is a fixed surface. Hence, all orbits (which are just points) in $\phi^{-1}(v)$ are non-exceptional. So let $\mathcal{F}$ be a face of Δ of dimension f with $1 \leq f \leq n-1$. It remains to show that an orbit which is mapped under ϕ to the interior of $\mathcal{F}$ is exceptional if and only if its stabilizer it not equal to $\mathfrak{stab}(\mathcal{F})$. Since (M, ω, T, ϕ) is tall, by Lemma 3.7 the preimage of each vertex of Δ is a fixed surface. Hence, by Lemma 3.8 the preimage $\phi^{-1}(\mathcal{F})$ is a $(2f+2)$-dimensional symplectic submanifold of (M, ω) and the set of elements in T which act trivially on $\phi^{-1}(\mathcal{F})$ is the stabilizer $\mathfrak{stab}(\mathcal{F})$ of $\mathcal{F}$. In particular, the quotient torus $T/\mathfrak{stab}(\mathcal{F})$ acts effectively and in a Hamiltonian fashion on $\phi^{-1}(\mathcal{F})$ and the complexity of this action is one. Let $\phi_{\mathcal{F}} : \phi^{-1}(\mathcal{F}) \to (\mathrm{Lie}(T/\mathfrak{stab}(\mathcal{F})))^*$ be a moment map for this action, then $(\phi^{-1}(\mathcal{F}), \omega, T/\mathfrak{stab}(\mathcal{F}), \phi_{\mathcal{F}})$ is a compact complexity one space. Furthermore, for any $p \in \phi^{-1}(\mathcal{F})$ the T-orbit and $(T/\mathfrak{stab}(\mathcal{F}))$-orbit through p are the same and

$$\phi^{-1}(\phi(p)) = \phi_{\mathcal{F}}^{-1}(\phi_{\mathcal{F}}(p)). \tag{3.16}$$

We conclude that the orbit through $p \in \phi^{-1}(\mathcal{F})$ is exceptional with respect to the T-action if and only if it is exceptional with respect to the $T/\mathfrak{stab}(\mathcal{F})$-action. Now by Corollary 3.23 it is easy to see that the orbit through a point p which is mapped under ϕ to the interior of $\mathcal{F}$ is exceptional if and only if its stabilizer is not equal to $\mathfrak{stab}(\mathcal{F})$.

$\square$

Chapter 4

Minimal Vertices, Minimal Edges and the Genus

In this section we show that the genus of a compact, tall and monotone complexity one space is equal to zero and we prove the existence of obstructions for the moment map images of isolated fixed points. That the genus of a compact, tall and monotone complexity one space is equal to zero is not a new result, since it is a direct consequence of a result made by Sabatini and Sepe [38]. The reason why we repeat this proof relies on the following. Namely, the techniques which are used by Sabatini and Sepe, enable us to find obstructions for the moment map images of isolated fixed points. Using these obstructions we can compute the isolated fixed point data (see Chapter 5). So the new results, which we present in this section, are the obstructions for the moment map images of isolated fixed points. This section is largely inspirited by the work of Sabatini and Sepe [38].

In order to prove the results of this section, we first introduce the notation of **minimal vertices** for compact and tall complexity one spaces. Recall that by Lemma 3.7 the preimage of any vertex of the moment map polytope is a fixed surface, so the following definition makes sense.

Definition 4.1
Let (M, ω, T, ϕ) be a compact and tall complexity one space. For a vertex v of the moment map polytope, we denote its preimage under ϕ by Σ_v. A vertex v of the moment map polytope is called **minimal**, if

$$\int_{\Sigma_v} \omega \leq \int_{\Sigma_{v'}} \omega,$$

for any other vertex v' of the moment map polytope.

Note that there always exists a minimal vertex. By using an exact formula for the Duistermaat-Heckman measure of compact complexity one spaces of dimension four in terms of the fixed point data, we obtain information about how the fixed surface which corresponds to a minimal vertex is embedded into M (in any dimension). Furthermore, if (M, ω) is monotone, then this surface must be a two-sphere [38, Theorem 4.5]. Next we recall this formula for the Duistermaat-Heckman measure. Let (M, ω, S^1, ϕ) be a compact Hamiltonian S^1-space of dimension four with moment map $\phi\colon M \to \mathbb{R}$. The moment map image is a compact interval, which we denote by $[t_{\min}, t_{\max}]$. The preimage $\phi^{-1}(t_{\min})$ is either an isolated fixed point or a fixed surface. If it is an isolated fixed point, we set $a_{\min} = 0$ and $c_{\min} = -\frac{1}{mn}$, where m, n are the isotropic weights at $\phi^{-1}(t_{\min})$. Otherwise, if $\phi^{-1}(t_{\min})$ is a fixed surface denoted by $\Sigma_{\min}$ we set

$$a_{\min} = \int_{\Sigma_{\min}} \omega \quad \text{and} \quad c_{\min} = \int_{\Sigma_{\min}} c_1(N_{\min}), \tag{4.1}$$

where $c_1(N_{Min})$ is the first Chern class of the normal bundle $N_{\min}$ of $\Sigma_{\min}$ in M. In the same way we define $a_{\max}$ and $c_{\max}$. Moreover, for any isolated fixed point $p \in M^{S^1}$ which is not extremal, meaning that it is not an extrema of ϕ, let m_p and n_p the weights of the S^1-action at p. We set $t_p = \phi(p)$. Moreover, let $H\colon \mathbb{R} \to \{0,1\}$ be the Heaviside step function[1] and $\theta\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto t \cdot H(t)$. In this setting a representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure is given as in the following lemma.

Lemma 4.2 ([22, Lemma 2.19])
Let (M, ω, S^1, ϕ) be a compact Hamiltonian S^1-space of dimension four. A representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure of this spaces is given by

$$\begin{aligned} f_{DH}(t) = & a_{\min} H(t - t_{\min}) - c_{\min}\theta(t - t_{\min}) \\ & + \sum_p \frac{1}{m_p n_p} \theta(t - t_{\min}) \\ & - a_{\max} H(t - t_{\max}) - c_{\max}\theta(t - t_{\max}), \end{aligned}$$

where the sum runs over all isolated fixed points $p \in M^{S^1}$ which are not extremal.

This lemma is stated in [22, Lemma 2.19] where an outline of its proof is given. Since this lemma is essential for this work, in the Appendix we give a complete proof based on this outline. Note that whenever $\phi^{-1}(t_*)$ for $* = \min$ or $* = \max$ is a fixed surface our definition of a_* is different to the one in [22] by the factor $\frac{1}{2\pi}$. This relies on the different conventions. In this work S^1 is considered as the quotient space $\mathbb{R}/\mathbb{Z}$, while in [22] S^1 is considered as the quotient space $\mathbb{R}/2\pi\mathbb{Z}$. Before we go on, we sum up simple consequences of Lemma 4.2 in the next corollary. Note that the parts (*ii*) and (*iii*) of Corollary 4.3 below are proven in [22, Lemma 2.15 and 2.19]. For the sake of completeness we will also give a proof of the parts (*ii*) and (*iii*).

Corollary 4.3
Let (M, ω, S^1, ϕ) be a compact Hamiltonian S^1-space of dimension four and let f_{DH} be the representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure as in Lemma 4.2. Then the following holds.

[1] That is $H(t) = 0$ for all $t < 0$ and $H(t) = 1$ for all $t \geq 0$.

(i) $a_{\min} - c_{\min}\theta(t_{\max} - t_{\min}) + \sum_p \frac{1}{m_p n_p}\theta(t_{\max} - t_p) = a_{\max}$,

(ii) $c_{\min} + c_{\max} = \sum_p \frac{1}{m_p n_p}$,

(iii) f_{DH} *is continuous and concave on* $[t_{\min}, t_{\max})$ *and* $\lim_{t\uparrow t_{\max}} f_{DH}(t) = a_{\max}$,

(iv) $f_{DH}(t) \geq \min\{a_{\min}, a_{\max}\}$ *for all* $t \in [t_{\min}, t_{\max})$,

where the sums run over all isolated fixed points $p \in M^{S^1}$ *which are not extremal.*

Proof. Note that for $t \geq t_{\max}$ the function f_{DH} is linear. Moreover, for any Borel set $U \subset [t_{\max}, \infty)$, we have $\phi^{-1}(U) = \emptyset$ and therefore

$$\int_U f_{DH}(t)\,\mathrm{d}\,t = \int_{\phi^{-1}(U)} \frac{\omega^2}{2!} = 0. \tag{4.2}$$

Hence, we have $f_{DH}(t) = 0$ for $t \geq t_{\max}$.
(i) This claim follows from $f_{DH}(t_{\max}) = 0$.
(ii) For $t \geq t_{\max}$ the function f_{DH} is linear with slope $-c_{\min} - c_{\max} + \sum_p \frac{1}{m_p n_p}$. Since $f_{DH}(t) = 0$ for $t \geq t_{\max}$, this slope must be equal to zero, which proves the second claim.
(iii) The function f_{DH} is continuous and piecewise linear on $[t_{\min}, t_{\max})$. Moreover, for any non-extremal isolated fixed point $p \in M^{S^1}$ one weight of the S^1-action at p is negative and one is positive, so the product $n_p m_p$ of these weights is negative. Hence, the slope of f_{DH} on $[t_{\min}, t_{\max})$ is decreasing. Therefore, f_{DH} is concave on $[t_{\min}, t_{\max})$. Moreover, $\lim_{t\uparrow t_{\max}} f_{DH}(t) = a_{\max}$ follows from (i).
(iv) By (iii) if we replace $f_{DH}(t_{\max})$ by $a_{\max}$, then f_{DH} is continuous and concave on $[t_{\min}, t_{\max}]$. Note that any continuous and concave function defined on a compact interval attains its minimum on one of the two boundary points. So we conclude $f_{DH}(t) \geq a_{\min}$ or $f_{DH}(t) \geq a_{\max}$ holds for all $t \in [t_{\min}, t_{\max})$. □

By Lemma 4.2 and its Corollary 4.3 we obtain information about how the fixed surface, which belongs to a minimal vertex of a compact and tall Hamiltonian S^1-space of dimension four, is embedded into M.

Lemma 4.4
Let (M, ω, T, ϕ) *be a compact and tall Hamiltonian* S^1*-space of dimension four and let* v *be a minimal vertex. We denote the fixed surface* $\phi^{-1}(v)$ *by* Σ_v*. Then one has*

$$\int_{\Sigma_v} c_1(N_{\Sigma_v}) \leq 0, \tag{4.3}$$

where $c_1(N_{\Sigma_v})$ *is the first Chern class of the normal bundle* N_{Σ_v} *of* Σ_v *in* M*. Moreover, if equality holds in* (4.3) *then the space has no isolated fixed points.*

Proof. The moment map polytope is a compact interval $[t_{\min}, t_{\max}]$. So we have $v = t_{\min}$ or $v = t_{\max}$. Without loss of generality we can assume that $v = t_{\min}$. Moreover, we set $v' = t_{\max}$. Since (M, ω, T, ϕ) is tall, by Lemma 3.7 the preimage $\phi^{-1}(v')$ is also a fixed surface denoted by Σ'_v. We set

$$a_{\min} = \int_{\Sigma_v} \omega \quad \text{and} \quad a_{\max} = \int_{\Sigma'_v} \omega. \tag{4.4}$$

Since v is a minimal vertex we have $a_{\min} \leq a_{\max}$. Moreover, we set $c_{\min} = \int_{\Sigma_v} c_1(N_{\Sigma_v})$. Let f_{DH} be the representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure as in Lemma 4.2. Since $a_{\min} \leq a_{\max}$, Corollary 4.3 implies $f_{DH}(t) \geq a_{\min}$ for all $t \in [t_{\min}, t_{\max})$. Moreover, we have $f_{DH}(t) = a_{\min} - (t - t_{\min}) \cdot c_{\min}$ for $t > t_{\min}$ nearby $t_{\min}$. Hence, $c_{\min} \leq 0$ holds, which proves the first statement of this lemma.
Now assume that $c_{\min} = 0$. For any non-extremal isolated fixed point $p \in M^T$ one of the weights of the S^1-action at p is negative and one is positive, so the product $n_p m_p$ of these weights is negative. So Lemma 4.2 implies that if the function f_{DH} attains its minimum at $t_{\min}$ on $[t_{\min}, t_{\max})$ and $c_{\min} = 0$, then the space (M, ω, S^1, ϕ) has no isolated fixed points. □

Next we generalize Lemma 4.4 to tall and compact complexity one spaces of dimension greater than four. Therefore, let (M, ω, T, ϕ) be a tall and compact complexity one space of dimension greater than four. By Lemma 3.8 the preimage $\Sigma_v = \phi^{-1}(v)$ resp. $M_e = \phi^{-1}(e)$ of each vertex v resp. of each edge e of the moment map polytope Δ is a symplectic submanifold of (M, ω) of dimension two resp. four. In particular, if e is an edge which contains the vertex v, then Σ_v is a symplectic submanifold of M_e. In this case, the first Chern class of the normal bundle N_e of Σ_v in M_e is denoted by $c_1(N_e)$. So Lemma 4.4 generalizes as follows.

Lemma 4.5
Let (M, ω, T, ϕ) be a tall and compact complexity one space of dimension greater than four. Let v be a minimal vertex of the moment map polytope and let e be an edge that contains v. Let $c_1(N_e)$ be the first Chern class of the normal bundle N_e of $\phi^{-1}(v) = \Sigma_v$ in $\phi^{-1}(e) = M_e$, then

$$\int_{\Sigma_v} c_1(N_e) \leq 0. \tag{4.5}$$

Moreover, if equality holds in (4.5), then there exists no isolated fixed point $p \in M^T$, such that $\phi(p) \in e$.

Note that the first statement of this lemma is a direct consequence of a result by Sabatini and Sepe [38, Lemma 3.9]. The second part of this lemma, namely whenever in (4.5) the equality holds, is new. However, we give a complete proof of this lemma.

Proof of Lemma 4.5 . The key step of this proof is to apply Lemma 4.4 to M_e. Therefore, let $\alpha_{v,1}, \ldots, \alpha_{v,n-1}$ be the non-zero weights along $\phi^{-1}(v) = \Sigma_v$. Note that these weights form a $\mathbb{Z}$-basis of $\mathfrak{t}^*$. By Lemma 3.8 and its Remark 3.9 we can assume that $e \subset v + \mathbb{R}_{\geq 0}\alpha_{v,1}$. Moreover, let $\xi_1^*, \ldots, \xi_{n-1}^* \subset \ell_T$ be the dual basis of $\alpha_{v,1}, \ldots, \alpha_{v,n-1}$, then the subcircle $\exp(\mathbb{R}\,\xi_1^*)$ of T, which we simply denote by S^1, acts effectively and in a Hamiltonian fashion on M_e. A moment map for the S^1-action on M_e is given by

$$\phi_e = \pi \circ \phi : M_e \longrightarrow \mathrm{Lie}^*(S^1) \cong \mathbb{R}, \tag{4.6}$$

where $\pi : \mathfrak{t}^* \to \mathbb{R}$ is the projection given by

$$\pi(t_1\alpha_{v,1} + \cdots + t_{n-1}\alpha_{v,n-1}) = t_1. \tag{4.7}$$

Hence, $(M_e, \omega, S^1, \phi_e)$ is a compact Hamiltonian S^1-space of dimension four. Note that this space is also tall. In particular, we can identify the moment map image of ϕ_e with e. Hence, the vertices of $\phi_e(M_e)$ are the vertices of e. So one of these vertices is v which is a minimal vertex of the moment map polytope $\phi(M)$ and we denote the other one by v'. Moreover, we have $\Sigma_v = \phi^{-1}(v) = \phi_e^{-1}(v)$ and $\Sigma_v' = \phi^{-1}(v') = \phi_e^{-1}(v')$. We conclude that v is also a minimal vertex for $(M_e, \omega, S^1, \phi_e)$. Now by applying the results of Lemma 4.4 to M_e, it follows that the inequality (4.5) holds. Moreover, if the equality in (4.5) holds, then the space $(M_e, \omega, S^1, \phi_e)$ has no isolated fixed points. The latter is equivalent to the condition that there exists no isolated fixed point $p \in M^T$, such that $\phi(p) \in e$. $\square$

Now we are able to prove the main statement of this section. Namely, we apply Lemma 4.5 to a minimal vertex of a compact, tall and monotone complexity one space.

Proposition 4.6
Given a compact tall and monotone complexity one space (M, ω, T, ϕ) of dimension $2n$ and let v be a minimal vertex of the moment map polytope Δ. Let Σ_v be the preimage of v under ϕ . Moreover, for $i = 1, ..., n-1$ we denote by e_i the edges of Δ which contain the vertex v and by $c_1(N_{e_i})$ the first Chern class of the normal bundle of Σ_v in $\phi^{-1}(e_i)$. Then the genus of Σ_v is zero and one of the following conditions is true.

(i) We have $\int_{\Sigma_v} c_1(N_{e_j}) = 0$ for all $i = 1, ..., n-1$ and there exists no isolated fixed point $p \in M^T$ such that $\phi(p)$ lies on one of the edges e_i, $i = 1, \ldots, n-1$.

(ii) There exists $j \in \{1, ..., n-1\}$ *with* $\int_{\Sigma_v} c_1(N_{e_j}) = -1$ *and* $\int_{\Sigma_v} c_1(N_{e_i}) = 0$ *for* $i \neq j$ *and there exists no isolated fixed point* $p \in M^T$ *such that* $\phi(p)$ *lies on one of the edges* e_i *with* $i \neq j$.

Here the first statement of this proposition, namely that the preimage of a minimal vertex is a surface of genus zero, is again a direct consequence of a result by Sabatini and Sepe [38, Theorem 4.5].

Proof of Proposition 4.6 . Since (M, ω) is monotone we have that $c_1(M) = \lambda[\omega]$ in $H^2(M, \mathbb{R})$ for some $\lambda > 0$ by Lemma 2.51. Hence, we have

$$\lambda \int_{\Sigma_v} \omega = \int_{\Sigma_v} c_1(M)|_{\Sigma_v}. \tag{4.8}$$

The left-hand side of (4.8) is positive because Σ_v is a symplectic submanifold of (M, ω). Hence, $\int_{\Sigma_v} c_1(M)|_{\Sigma_v}$ is positive. Note that the restriction of the first Chern class $c_1(M)$ of (M, ω) to Σ_v splits as

$$c_1(M)|_{\Sigma_v} = c_1(\Sigma_v) + c_1(N_{e_1}) + \cdots + c_1(N_{e_{n-1}}), \tag{4.9}$$

where $c_1(\Sigma_v)$ is the first Chern class of (Σ_v, ω). Therefore, we have

$$\int_{\Sigma_v} c_1(\Sigma_v) + \int_{\Sigma_v} c_1(N_{e_1}) + \cdots + \int_{\Sigma_v} c_1(N_{e_{n-1}}) > 0. \tag{4.10}$$

Since v is a minimal vertex, by Lemma 4.5 we have $\int_{\Sigma_v} c_1(N_{e_j}) \leq 0$ for all $j = 1, \ldots, n-1$. Combining this with the inequality (4.10) implies that $\int_{\Sigma_v} c_1(\Sigma_v)$ must be greater than zero. Since $\int_{\Sigma_v} c_1(\Sigma_v)$ is equal to Euler characteristic of Σ_v, we have that Σ_v is a two-sphere and $\int_{\Sigma_v} c_1(\Sigma_v) = 2$ because the two-sphere is the only compact orientable surface with a positive Euler characteristic. In particular, the genus of Σ_v is equal to zero. This proves the first statement of this proposition. Hence, it is left to show that condition (i) or (ii) holds. Since $\int_{\Sigma_v} c_1(\Sigma_v) = 2$ holds, from (4.10) it follows that

$$\int_{\Sigma_v} c_1(N_{e_1}) + \cdots + \int_{\Sigma_v} c_1(N_{e_{n-1}}) \geq -1. \tag{4.11}$$

By Lemma 4.5 we have $\int_{\Sigma_v} c_1(N_{e_j})$ is a non-positive integer for all $j = 1, \ldots, n-1$, so one of the following conditions holds.

- We have $\int_{\Sigma_v} c_1(N_{e_j}) = 0$ for all $i = 1, ..., n-1$. In this case Lemma 4.5 implies that there exists no isolated fixed point $p \in M^T$ such that $\phi(p)$ lies on one of the edges e_i, $i = 1, \ldots, n-1$.

- There exists $j \in \{1, ..., n-1\}$ with $\int_{\Sigma_v} c_1(N_{e_j}) = -1$ and $\int_{\Sigma_v} c_1(N_{e_i}) = 0$ for $i \neq j$. In this case Lemma 4.5 implies that there exists no isolated fixed point $p \in M^T$ such that $\phi(p)$ lies on one of the edges e_i with $i \neq j$.

□

We like to point out some remarks on the proof of Proposition 4.6.

Remark 4.7
Note that the assumption that (M, ω, T, ϕ) is monotone is needed in the proof of Proposition 4.6, since this assumption implies that the evaluation $\int_\Sigma c_1(M)|_\Sigma$ of the first Chern class of $c_1(M)$ to any fixed surface Σ is positive (by Lemma 2.51). Note that the results of Proposition 4.6 are still true if we soften the assumption that (M, ω, T, ϕ) is monotone in following sense. In [38] a compact complexity one space is called positive, whenever the evaluation of the first Chern class of $c_1(M)$ to any fixed surface Σ is positive. So the statement of Proposition 4.6 is still true if the monotone condition is replaced by the condition that the space is positive.

As a direct consequence of Proposition 4.6, we obtain the following corollary.

Corollary 4.8
Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space. Then its genus is equal to zero.

Proof. At least one vertex v of the moment map polytope is minimal. By Proposition 4.6 the preimage of v under ϕ is a surface of genus zero. Hence, by definition the space (M, ω, T, ϕ) has genus zero. □

Definition 4.9
Given a compact and tall complexity one space (M, ω, T, ϕ) and let e be an edge of the moment map polytope Δ. We call this edge e **minimal**, if it contains a minimal vertex v of Δ such that

$$\int_{\Sigma_v} c_1(N_e) = 0, \tag{4.12}$$

where $\phi^{-1}(v) = \Sigma_v$ and $c_1(N_e)$ is the first Chern class of the normal bundle N_e of Σ_v in $\phi^{-1}(e)$.

Corollary 4.10
Given a compact, tall and monotone complexity one space (M, ω, T, ϕ) of dimension $2n \geq 6$, let v be a minimal vertex of the moment map polytope. Then at least $n-2$ edges of the moment map polytope which contain v are minimal. Moreover, there exists no isolated fixed point $p \in M^T$ such that $\phi(p)$ lies on a minimal edge.

Proof. At least one vertex v of the moment map polytope is minimal and there are exact $n-1$ edges which contain the vertex v. By Proposition 4.6 either all of these edges are minimal or exactly $n-2$ of these edges are minimal. Moreover, by its definition and by Proposition 4.6 it follows that there exists no isolated fixed point $p \in M^T$ such that $\phi(p)$ lies on a minimal edge. $\square$

Chapter 5

The Data of The Isolated Fixed Points

Given a compact complexity one space (M, ω, T, ϕ), any connected component of M^T is an isolated point or a compact surface due to Lemma 2.15 and Lemma 2.20. In addition, if we assume that the given compact complexity one space is also tall, then due to Lemma 3.11 the connected components of M^T which are compact surfaces are in one-to-one correspondence with the vertices of the moment map polytope. The main goal of this chapter is to prove Proposition 5.14, which gives us important information about the data of the isolated fixed points (see the following definition) of a compact tall and monotone complexity one space of dimension six. Let us introduced the following definition for general Hamiltonian T-spaces.

Definition 5.1
Let (M, ω, T, ϕ) be a Hamiltonian T-space. The **data about the isolated fixed points** of this space is the unordered list of all isolated points in M^T such that each of these points is labeled with its moment map image and with its weights. Moreover, given another Hamiltonian T-space (M', ω', T, ϕ') of the same dimension, we say that (M, ω, T, ϕ) and (M', ω', T, ϕ') have the same data about the isolated fixed points if there exists a bijection between isolated points in M^T and the ones of $(M')^T$ such that each point is mapped to a point with same moment map image and with same weights.

In order to prove the main result of this chapter, namely Proposition 5.14, this chapter is divided into three sections as follows. In the first section we introduce a concept which will give us relations between the isolated fixed points of Hamiltonian T-spaces in general. In the second section we prove a list of lemmas which are needed for the proof of Proposition 5.14. In the final section of this chapter we give the proof of Proposition 5.14.

5.1 Light Weights and Light Spheres

In this section we introduce the concept of light weights and light spheres, which will give us relations between the isolated fixed points of Hamiltonian T-spaces. Before we do so we like to point out some facts of the two-dimensional components which are fixed by a codimensional one subtorus.

Given a Hamiltonian T-space (M, ω, T, ϕ) and a codimensional one subtorus H of T, the Lie algebra of H, denoted by $\mathfrak{h}$, is a rational codimensional one subspace of $\mathfrak{t}$. So we can pick an $\alpha \in \ell_T^* \setminus \{0\}$ such that $\mathfrak{h}$ is equal to

$$\ker(\alpha) = \{\xi \in \mathfrak{t} \mid \langle \alpha, \xi \rangle = 0\} . \tag{5.1}$$

Hence, H can be written as $H = \exp(\ker(\alpha))$, where $\exp : \mathfrak{t} \to T$ is the exponential map. The next lemma contains a simple fact about the connected components of M^H.

Lemma 5.2
Let (M, ω, T, ϕ) be a Hamiltonian T-space of dimension $2n$ and let $p \in M^T$ be a fixed point with weights $\alpha_{p,1}, ..., \alpha_{p,n}$. Moreover, given $\alpha \in \ell_T^* \setminus \{0\}$, let H be the codimensional one subtorus with Lie algebra $\mathfrak{h} = \ker(\alpha)$. Then the dimension of the connected component of M^H which contains p is equal to twice the number of elements in $\{\alpha_{p,1}, ..., \alpha_{p,n}\}$ which are a multiple of α.

Proof. Due to Theorem 2.18 there exists an open neighborhood U_p of p with complex coordinates $z_1, \dots, z_n$ centered at p, such that the T-action is given by

$$\exp(\xi) \cdot (z_1, \dots, z_n) = (\mathrm{e}^{2\pi i \langle \alpha_{p,1}, \xi \rangle} z_1, \dots, \mathrm{e}^{2\pi i \langle \alpha_{p,n}, \xi \rangle} z_n) \tag{5.2}$$

for all $\xi \in \mathfrak{t}$. Assume that exactly r weights at p are a multiple of α. Without loss of generality we have that for any $j = 1, \dots, r$ there exists $\lambda_j \in \mathbb{Q}$ such that $\alpha_{p,j} = \lambda_j \alpha$ holds, and for any $j = r+1, \dots, n$ and any $\lambda \in \mathbb{Q}$ we have $\alpha_{p,j} \neq \lambda \alpha$. Hence, for any $j = 1, \dots, r$ and any $\xi \in \mathfrak{h} \subset \mathfrak{t}$ we have $\langle \alpha_{p,j}, \xi \rangle = \langle \lambda_j \alpha, \xi \rangle = 0$. Moreover, for all $j = r+1, \dots, n$ there exists a $\xi \in \mathfrak{h}$ so that $\langle \alpha_j, \xi \rangle \notin \mathbb{Z}$. Hence, with respect to the complex coordinates $z_1, \dots, z_n$ for U_p, the connected component X of M^H which contains p is given by $z_{r+1}, \dots, z_n = 0$. So the dimension of X is equal to $2r$, which completes the proof. □

Next we recall a theorem about T-actions on compact oriented surfaces.

Theorem 5.3 ([20, Theorem 11.7.1])
Let X be a compact, connected oriented surface endowed with a smooth non-trivial action of an n-dimensional torus T. If the set of fixed points X^T is not empty, then this action has the following properties:

- *The set of elements in T which act trivially on X is a closed codimensional one subgroup H.*
- *X is diffeomorphic to S^2. This diffeomorphism conjugates the action of T/H on X into the standard action of S^1 on S^2 given by rotation about the z-axis.*

Note that standard action of S^1 on S^2 given by rotation about the z-axis has exactly two fixed points, the 'north pole' and the 'south pole' and S^1 acts at the south pole with weight $+1$ and at he north pol with weight -1. Hence, by Theorem 5.3 it follows that if a torus acts on a compact oriented surface in a non-trivial way such that the set of fixed points is not empty, then the action has exactly two fixed points. Moreover, if the weight at one fixed point is $\alpha \in \ell_T^* \setminus \{0\}$, then the weight at the other fixed point is $-\alpha$.

Now we introduce the notation of light weights. Therefore, let (M, ω, T, ϕ) be a Hamiltonian T-space of dimension $2n$ and let $p \in M^T$ be an **isolated fixed point** and let $\alpha_{p,1}, \ldots, \alpha_{p,n} \in \ell_T^*$ be the weights at p. Note that since p is an isolated fixed point, all these weights are different from zero.

Definition 5.4
The weight $\alpha_{p,i}$ is called **light** if $\alpha_{p,i}$ and $\alpha_{p,j}$ are linearly independent for all $j \in \left\{1, \ldots \hat{i}, \ldots, n\right\}$. Otherwise, the weight $\alpha_{p,i}$ is called **heavy**.

The next lemma is essential for the following section since it describes relations between the isolated fixed points.

Lemma 5.5
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space of dimension $2n$ and let $p \in M^T$ be an isolated fixed point with weights $\alpha_{p,1}, \ldots, \alpha_{p,n}$. Suppose that for a fixed $i \in \{1, \ldots, n\}$ the weight $\alpha_{p,i}$ is light. Let H be the codimensional one subtorus of T given by $H = \exp(\ker(\alpha_{p,i}))$ and let S be the connected component of M^H which contains p. Then the following holds. S is a T-invariant symplectic two-sphere and there exists a unique fixed point $q \in \left(S \cap M^T\right) \setminus \{p\}$. Moreover, we have the following.

- *q is an isolated fixed point and $-\alpha_{p,i}$ is a light weight at q.*
- *The image $\phi(q)$ of q lies on the open half-line $\phi(p) + \mathbb{R}_{>0}\alpha_{p,i}$.*
- *The image $\phi(S)$ is equal to the line segment through $\phi(p)$ and $\phi(q)$.*

Proof. Without loss of generality let i be equal to one. By Lemma 2.15 we have that S is a T-invariant symplectic submanifold of (M, ω). Since $\alpha_{p,1}$ and $\alpha_{p,i}$ are linearly independent for all $i = 2, \ldots, n$, by Lemma 5.2 the dimension of S is two. In particular, p is an isolated fixed point of the T-action on S. Hence, by Theorem 5.3 we find that S is a two-sphere and there exists a unique $q \in \left(S \cap M^T\right) \setminus \{p\}$. We need to show that q is isolated. Let $\alpha_{q,1}, \alpha_{q,2} \ldots, \alpha_{q,n}$ be the weights at q. Note that S is a connected component of M^H which contains q, so again by Lemma 5.2 only one weight at q can be a multiple of $\alpha_{p,1}$, say $\alpha_{q,1}$. Note that this implies that the weights $\alpha_{q,2}, \ldots, \alpha_{q,n}$ are all different from zero. Moreover, T acts on S at p with weight $\alpha_{p,1}$ and T acts on S at q with weight $\alpha_{q,1}$. Hence, we have $\alpha_{q,1} = -\alpha_{p,1}$. Therefore, also $\alpha_{q,1}$ is not equal to zero, so q is also an isolated fixed point. Since $\alpha_{q,1}$ it is the only weight at q which is a multiple of $\alpha_{p,1}$, $\alpha_{q,1}$ is a light weight at q. Moreover, by Theorem 2.24 the image $\phi(S)$ is the segment

$$\{\phi(p) + t(\phi(p) - \phi(q)) | t \in [0,1]\}\,. \tag{5.3}$$

By using Theorem 2.18 we have that points in S near p are sent to the have line $\phi(p) + \mathbb{R}_{\geq 0}\alpha_{p,1}$. We conclude that $\phi(q)$ lies on the open half line $\phi(p) + \mathbb{R}_{>0}\alpha_{p,1}$. □

Due to Lemma 5.5 the following definition make sense.

Definition 5.6
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space of dimension $2n$ and let p be an isolated fixed point with weights $\alpha_{p,1}, \ldots, \alpha_{p,n}$. Suppose that for a fixed $i \in \{1, \ldots, n\}$ the weight $\alpha_{p,i}$ is light. Let H be the codimensional one subtorus of T given by $H = \exp(\ker(\alpha_{p,i}))$. We call the connected component S of M^H which contains p the **light sphere that belongs to p and the weight $\alpha_{p,i}$.**

Remark 5.7
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space, then by Lemma 5.5 any light sphere is a two-dimensional sphere and each light sphere contains exactly two isolated fixed points of M^T. Moreover, for any isolated fixed point $p \in M^T$ we denote by ν_p the number of light weights at p. Then there exist exactly ν_p light spheres which contains p. In particular, the number of light spheres in (M, ω, T, ϕ) is equal to $\frac{1}{2}\sum_p \nu_p$, where the sum runs over all isolated fixed points.

5.2 Some Properties of the Isolated Fixed Points of Six-Dimensional Tall and Monotone Complexity One Spaces

In this section we prove a sequence of lemmas and corollaries containing information about the data of the isolated fixed points for six-dimensional compact, tall and monotone complexity one spaces which are balanced.

Lemma 5.8
Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced and let $p \in M^T$ be an isolated fixed point, then $\phi(p) \in \ell_T^*$ and $\phi(p)$ lies in the interior of an edge of the moment map polytope.

Proof. Since the space (M, ω, T, ϕ) is balanced, the moment map satisfies the weight sum formula. Hence, the image of any fixed point belongs to ℓ_T^*. Moreover, since (M, ω, T, ϕ) is tall, by Lemma 3.7 the preimage of any vertex of Δ is a fixed surface. Therefore, the image under ϕ of an isolated fixed point must lie in the interior of an edge of Δ or in the interior of Δ. We show via contradiction that the latter case can not occur. In order to perform this contradiction we use the geometric properties of Δ, the existence of a minimal edge and the relation between the isolated fixed points as described in Lemma 5.5. So in order to prove the claim of this lemma we show that the following assumption leads to a contradiction.

Assumption: There exists an isolated fixed point $p \in M^T$ such that $\phi(p)$ lies in the interior of Δ.

Since ϕ satisfies the weight sum formula, we have $\phi(p) \in \ell_T^*$. By Proposition 3.17 we have that Δ is a reflexive Delzant polytope, so the origin is the only point in ℓ_T^* which lies in the interior of Δ (see Remark 3.15). We conclude $\phi(p) = 0$. Now pick a minimal vertex v_1 and let α_0 and α_1 be the weights at v_1, i.e. the edges e_0 and e_1 of Δ which contain v_1 are given by

$$e_0 = (v_1 + \mathbb{R}_{\geq 0}\alpha_0) \cap \Delta = (-\alpha_1 + \mathbb{R}_{\geq -1}\alpha_0) \cap \Delta \text{ and} \tag{5.4}$$

$$e_1 = (v_1 + \mathbb{R}_{\geq 0}\alpha_1) \cap \Delta = (-\alpha_0 + \mathbb{R}_{\geq -1}\alpha_1) \cap \Delta, \tag{5.5}$$

where we use that $v_1 = -\alpha_0 - \alpha_1$. By Corollary 4.10 one of these edges is minimal. Without loss of generality we assume that e_0 is minimal. Note that α_0 and α_1 form a basis of $\ell_T^* \cong \mathbb{Z}^2$ and $v_1 = -\alpha_0 - \alpha_1$. Hence, we have

$$\Delta \subset \{t_0\alpha_0 + t_1\alpha_1 \,|\, t_0, t_1 \geq -1\} \tag{5.6}$$

and any point in $\ell_T^* \cap \Delta$ is of the form

$$\lambda_0\alpha_0 + \lambda_1\alpha_1 \,, \text{where } \lambda_0, \lambda_1 \in \mathbb{Z}_{\geq -1}. \tag{5.7}$$

Now let $\alpha_{p,1}, \alpha_{p,2}$ and $\alpha_{p,3}$ be the weights at p. Since $\phi(p) = 0$ lies in the interior of Δ, by Corollary 2.26 the cone $\mathbb{R}_{\geq 0}$-span $\{\alpha_{p,1}, \alpha_{p,2}, \alpha_{p,3}\}$ must be equal to $\mathfrak{t}^* \cong \mathbb{Z}^2$. We conclude that the weights $\alpha_{p,1}, \alpha_{p,2}$ and $\alpha_{p,3}$ must be pairwise linearly independent, i.e. each of these weights is light. In particular, one of these weights, say the first one, is of the form $\alpha_{p,1} = \kappa_0\alpha_0 + \kappa_1\alpha_1$ where $\kappa_1 < 0$. Due to Lemma 5.5 there exists an isolated fixed point q, such that $\phi(q)$ lies on the open half-line $\phi(p) + \mathbb{R}_{>0}\alpha_{p,1} = \mathbb{R}_{>0}(\kappa_0\alpha_0 + \kappa_1\alpha_1)$. Note that $\phi(q) \in \ell_T^* \cap \Delta$. So since $\kappa_1 < 0$, by using (5.7) we conclude that $\phi(q) = \lambda_0\alpha_0 + \lambda_1\alpha_1$, where $\lambda_1 = -1$. By using (5.4) we have that $\phi(q) \in e_0$. But this is a contradiction because e_0 is minimal that is there exists no isolated fixed point such that its image under the moment map lies on e_0. $\square$

Example 5.9
Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced and assume that its moment map polytope is a hexagon. A consequence of Lemma 5.8 is that such a space has no isolated fixed points. To see this note the following. By picking a $\mathbb{Z}$-basis of ℓ_T^* we identify this lattice with $\mathbb{Z}^2$ and so we identify $\mathfrak{t}^*$ with $\mathbb{R}^2$. By Proposition 3.17 the moment map polytope $\phi(M) = \Delta$ is a reflexive Delzant polytope in $\mathfrak{t}^* \cong \mathbb{R}^2$. There exists just one reflexive Delzant polytope in $\mathbb{R}^2$ (up to $\mathrm{GL}(\mathbb{Z}^2)$-transformation) which is a hexagon. Such a polytope has the following property, namely there exists no integer point which lies in the interior of one of its edges (see Figure 3.1). We conclude that there exists no point in ℓ_T which lies in the interior of an edge of $\phi(M)$. Hence, Lemma 5.8 implies that such a space has no isolated fixed points.

In the next lemma we describes the weights at the isolated fixed points.

Lemma 5.10
Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which satisfies the balanced condition. Let $p \in M^T$ be an isolated fixed point. The moment map image $\phi(p)$ lies in the interior of an edge e of the moment map polytope Δ. Moreover, let α_e be primitive in ℓ_T^*, such that

$$e = (\mathbb{R}\alpha_e + \phi(p)) \cap \Delta. \tag{5.8}$$

Then the weights at p are $-\alpha_e$, α_e and $-\phi(p)$. In particular, $-\phi(p)$ is a light weight at p.

In order to prove Lemma 5.10 we use that the moment map polytope of such a space is a reflexive Delzant polytope in $\mathfrak{t}^* \cong \mathbb{R}^2$. Therefore, in the following remark we point out a property of two-dimensional reflexive Delzant polytopes, which we will use in the proof of Lemma 5.10.

Remark 5.11
We list all reflexive Delzant polytopes of dimension two (up to $\mathrm{GL}(\mathbb{Z}^2)$- transformations) in Figure 3.1. So from this list it is easy to see that a reflexive Delzant polytope $\Delta \subset \mathbb{R}^2$ satisfies the following properties. Let e be an edge of Δ, then the cardinality of $e \cap \mathbb{Z}^2$ is equal to 2, 3 or 4. In particular, the number of points in $\mathbb{Z}^2$ which lie in the interior of e is at most two and if so the difference $x - y$ of two such points $x \neq y$ must be a primitive element in $\mathbb{Z}^2$.

Proof of Lemma 5.10. Due to Lemma 5.8 the moment map image $\phi(p)$ lies in the interior of an edge e of the moment map polytope Δ. So let α_e be a primitive element which satisfies Equation (5.8) and let $\alpha_{p,1}$, $\alpha_{p,2}$ and $\alpha_{p,3}$ be the weights at p. By Remark 2.19 and Corollary 2.26 the $\mathbb{Z}$-span of these weights is equal to $\ell_T^* \cong \mathbb{Z}^2$ and the maximal subspace which is contained in the cone spanned by $\alpha_{p,1}$, $\alpha_{p,2}$ and $\alpha_{p,3}$ is the line $\mathbb{R}\alpha_e$. Therefore, we can assume that α_e and $\alpha_{p,3}$ form a $\mathbb{Z}$-basis of ℓ_T^* and that there exist non-zero integers λ_1 and λ_2 of opposite signs which are coprime such that $\alpha_{p,1} = \lambda_1\alpha_e$ and $\alpha_{p,2} = \lambda_2\alpha_e$. Next we show that $\lambda_1 \neq \pm 1$ resp. $\lambda_2 \neq \pm 1$ leads to a contradiction. So without loss of generality suppose that $\lambda_1 \neq \pm 1$ and let $G_{\lambda_1\alpha_e}$ be the subgroup of T given by

$$G_{\lambda_1\alpha_e} = \exp\left(\{\xi \in \mathfrak{t} \,|\, \langle \lambda_1\alpha_e, \xi\rangle \in \mathbb{Z}\}\right). \tag{5.9}$$

Let X be the connected component of $M^{G_{\lambda_1\alpha_e}}$ which contains p. Due to Theorem 2.18 there exists a neighborhood U_p of p with complex coordinates z_1, z_2, z_3 such that the T-action is given by

$$\exp(\xi)(z_1,z_2,z_3) = (\mathrm{e}^{2\pi i\langle\lambda_1\alpha_e,\xi\rangle}\, z_1, \mathrm{e}^{2\pi i\langle\lambda_2\alpha_e,\xi\rangle}\, z_2, \mathrm{e}^{2\pi i\langle\alpha_{p,3},\xi\rangle}\, z_3) \tag{5.10}$$

for all $\xi \in \mathfrak{t}$. Since α_e and $\alpha_{p,3}$ are linearly independent and λ_1 and λ_2 are coprime with $\lambda_1 \neq \pm 1$, in these coordinates X is given by the equations $z_2 = z_3 = 0$. Hence, the dimension of X is two. Moreover, by Lemma 2.15 X is a compact and T-invariant symplectic surface. Since p is an isolated fixed point for the T-action on X and the weight at p is $\lambda_1\alpha_e$, by Theorem 5.3 we have that X is a two-sphere, there exists a unique point $q \in \left(M^T \cap X\right) \setminus \{p\}$ and the weight of the T-action for X at q is $-\lambda_1\alpha_e$. In particular, q is an isolated fixed point. Now we consider the evaluation of the first Chern class $c_1(M)$ on X. This is an integer and since $[\omega] = c_1(M)$ and X is symplectic, this integer is positive. Hence, $\int_X c_1(M) = \kappa$, where $\kappa \in \mathbb{N}_{\geq 1}$. On the other side, by applying the ABBV localization (see Theorem 2.49) we obtain

$$\int_X c_1(M) = \int_X c_1^T(M) = \frac{c_1^T(M)|_{\{p\}}}{\lambda_1\alpha_e} + \frac{c_1^T(M)|_{\{q\}}}{-\lambda_1\alpha_e} = \frac{\phi(q)-\phi(p)}{\lambda_1\alpha_e}, \tag{5.11}$$

where we use for the last equation that the restriction of the first equivariant Chern class to a fixed point is equal to the sum of the weights at this point and that ϕ satisfies the weight sum formula. We obtain

$$\phi(q) - \phi(p) = \kappa\lambda_1\alpha_e. \tag{5.12}$$

So this implies that $\phi(q)$ must also lie in the interior of e. Note that since by

Proposition 3.17 the moment map polytope is a reflexive Delzant polytope, we have that $\phi(q) - \phi(p)$ must be a primitive element in $\ell_T^* \cong \mathbb{Z}^2$ (see Remark 5.11). But this gives a contraction to Equation (5.12) since the right-hand side of this equation is not primitive. Hence, $\lambda_1 \neq \pm 1$ can not occur and for the same reason $\lambda_2 \neq \pm 1$ can not occur. Hence, (after may replacing α_e by $-\alpha_e$) we have $\alpha_{p,1} = -\alpha_e$ and $\alpha_{p,2} = \alpha_e$. Moreover, by using the weight sum formula, we obtain

$$\alpha_{p,3} = \alpha_{p,1} + \alpha_{p,2} + \alpha_{p,3} = -\phi(p). \tag{5.13}$$

□

Corollary 5.12
Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which satisfies the balanced condition and fix $\alpha \in (\ell_T^* \cap \Delta) \setminus \{0\}$. Then the number of isolated fixed points whose moment map images are equal to α is equal to the number of isolated fixed points whose moment map images are equal to $-\alpha$.

Proof. Since by Proposition 3.17 the moment map polytope Δ is a reflexive and Delzant, we have that α is a primitive element in ℓ_T^* (see Remark 3.15). Now let $p \in M^T$ be an isolated fixed point such that $\phi(p) = \alpha$. By Lemma 5.10 $-\alpha$ is a light weight at p. So let S be the light sphere that belongs to p and to the weight $-\alpha$. By Lemma 5.5 we have that S is a two-sphere and there is a unique isolated fixed point $q \in M^T$ such that $q \in (S \cap M^T) \setminus \{p\}$. Moreover, $\phi(q)$ lies on the open half-line $\alpha + \mathbb{R}_{>0}(-\alpha)$. But since ϕ satisfies the weight sum formula, we must have that $\phi(q)$ belongs also to ℓ_T^*. Note that

$$\ell_T^* \cap \Delta \cap (\alpha + \mathbb{R}_{>0}(-\alpha)) = \{0, -\alpha\} \tag{5.14}$$

holds because α is primitive and Δ does not contain a non-primitive element in ℓ_T^*. Moreover, since 0 lies in the interior of Δ by Lemma 5.8, we must have $\phi(q) = -\alpha$. Note, the same holds for fixed points whose moment map image is equal to $-\alpha$. So it is easy to conclude, that the assignment $p \mapsto q$ if q lies in the light sphere S that belongs to p and the weight $-\alpha$ defines a bijection between the isolated fixed points whose moment map images are equal to α and the isolated fixed points whose moment map images are equal to $-\alpha$. □

5.3 The Proof of Proposition 5.14

In this section we give a very precise description about the data of the isolated fixed points for compact, tall and monotone complexity one spaces of dimension six which is balanced. This is the content of Proposition 5.14 below. Roughly speaking, the 'ingredients' which we use to prove this proposition are the weight sum formula, the existence of a minimal edge and results of the former section. Therefore, let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced and let Δ be its moment map polytope, which is a reflexive Delzant polytope in $\mathfrak{t}^* \cong \mathbb{R}^2$. By Corollary 4.10 at least one edge of the moment map polytope is minimal. We pick such an edge and we use the following conventions, which we illustrate in Figure 5.1.

Convention 5.13

Let (M, ω, T, ϕ) be a compact and tall complexity one space of dimension six. Then by Lemma 3.11 the moment map polytope Δ is a two-dimensional Delzant polytope in $\ell_T^* \cong \mathbb{R}^2$. Let us pick an edge of Δ, which we denote by e_0. Let v_1 v_2 be the vertices which are contained in e_0. We denote the other edge which contains v_1 resp. v_2 by e_1 and e_2. We denote the weights of the vertex v_1 by α_0 and α_1 (see Definition 2.33) so that

$$e_0 = (v_1 + \mathbb{R}_{\geq 0}\alpha_0) \cap \Delta \text{ and } e_1 = (v_1 + \mathbb{R}_{\geq 0}\alpha_1) \cap \Delta. \tag{5.15}$$

Since v_2 is the other vertex which is contained in e_0, one weight at v_2 is $-\alpha_0$ and we denote the other by α_2. So we have

$$e_2 = (v_2 + \mathbb{R}_{\geq 0}\alpha_2) \cap \Delta. \tag{5.16}$$

Moreover, by v_3 we denote the vertex which is beyond v_1 contained in e_1 and by e_3 the second edge which contains v_3 [1]. Last, let α_3 be the weight at v_3 such that

$$e_3 = (v_3 + \mathbb{R}_{\geq 0}\alpha_3) \cap \Delta. \tag{5.17}$$

Now we can state and prove Proposition 5.14.

Proposition 5.14

Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced. Let e_0 be an edge of the moment map polytope Δ which is minimal and let v_1, v_2, e_1,e_2 and $\alpha_0, \alpha_1, \alpha_2$ as defined in Convention 5.13. Let p be an isolated fixed point, then one of the following holds.

(i) $\phi(p) = -\alpha_0$ and $\phi(p)$ lies in the interior of the edge e_1 and the weights at p are α_0, $-\alpha_1$ and $+\alpha_1$.

[1] If the moment map polytope is a triangle, then the edges e_2 and e_3 coincides.

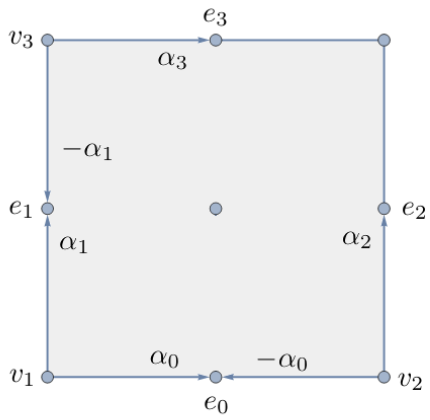


Figure 5.1: Illustration of Convention 5.13

(ii) $\phi(p) = \alpha_0$ *and* $\phi(p)$ *lies in the interior of the edge* e_2 *and the weights at* p *are* $-\alpha_0$, $-\alpha_2$ *and* $+\alpha_2$.

Moreover, the number of isolated fixed points which satisfy (i) *is equal to the number of isolated fixed points which satisfy* (ii).

Proof. Since we assume that (M, ω, T, ϕ) is balanced, by Proposition 3.17 Δ is a reflexive Delzant polytope and by Lemma 3.16 we have that

$$v_1 = -\alpha_0 - \alpha_1 \text{ and } v_2 = \alpha_0 - \alpha_2. \tag{5.18}$$

So by construction we have that the moment map polytope is contained in the set $\{t_0\alpha_0 + t_1\alpha_1 | t_0, t_1 \in \mathbb{R}_{\geq -1}\}$. Moreover, since a_0 and α_1 form a $\mathbb{Z}$-basis of ℓ_T^*, any lattice point which is contained in Δ is of the form

$$\lambda_0\alpha_0 + \lambda_1\alpha_1 \text{ , where } \lambda_0, \lambda_1 \in \mathbb{Z}_{\geq -1}. \tag{5.19}$$

Since the moment map satisfies the weight sum formula, the image of any fixed point under ϕ must be a lattice point. We conclude that if $p \in M^T$ is an isolated fixed point, then $\phi(p)$ is as in Equation (5.19). In order to prove the first claim of this proposition, we consider five cases. Namely, since the image of any isolated fixed point is of the form as in Equation (5.19), we show that $\lambda_1 \neq 0$ cannot happen (Case (a) and Case (e)). Since Δ is a reflexive polytope any non-zero lattice point that is contained in Δ is primitive. Hence, if $\lambda_1 = 0$, then we must have $\phi(p) = 0, -\alpha_0$, or α_0. We show that $\phi(p) = 0$ can not happen (Case (b)). Moreover, if $\phi(p) = -\alpha_0$ resp. $\phi(p) = \alpha_0$, then (i) resp. (ii) holds (Case (c) resp. Case (d)).

Case (a) By construction the intersection of the moment map polytope with the set of points which are of the form $t\alpha_0 - \alpha_1$ for some $t \in \mathbb{R}$ is exactly the edge e_0. Since this edge is a minimal, there exists no isolated fixed $p \in M^T$ with $\phi(p) \in e_0$. Hence, there is no isolated fixed point $p \in M^T$ with $\phi(p) = \lambda_0\alpha_0 - \alpha_1$.

Case (b) Since Δ is a reflexive Delzant polytope, the origin lies in the interior of Δ. Hence by Lemma 5.8 there exists no isolated fixed point whose moment map image is the origin.

Case (c) Suppose that $\phi(p) = -\alpha_0$. Note that since $v_1 = -\alpha_0 - \alpha_1$, we have

$$e_1 = (-\alpha_0 + \mathbb{R}_{\geq -1}\alpha_1) \cap \Delta. \tag{5.20}$$

Hence, $\phi(p) = -\alpha_0$ must lie in the interior of e_1. Moreover, by Lemma 5.10 the weights at p are α_0, $-\alpha_1$ and $+\alpha_1$.

Case (d) Suppose that $\phi(p) = \alpha_0$. Note that since $v_2 = \alpha_0 - \alpha_2$, we have

$$e_2 = (\alpha_0 + \mathbb{R}_{\geq -1}\alpha_2) \cap \Delta. \tag{5.21}$$

Hence $\phi(p) = \alpha_0$ must lie in the interior of e_2. Moreover, by Lemma 5.10 the weight at p are $-\alpha_0$, $-\alpha_2$ and $+\alpha_2$.

Case (e) We show by contradiction that there exists no isolated fixed point $p \in M^T$ with $\phi(p) = \lambda_0\alpha_0 + \lambda_1\alpha_1$ and $\lambda_1 \geq 1$. Suppose there exists such an isolated fixed point $p \in M^T$. Then by Corollary 5.12 there exists an isolated fixed point $q \in M^T$ such that $\phi(q) = -\phi(p) = -\lambda_0\alpha_0 - \lambda_1\alpha_1$. But since $\lambda_1 \geq 1$ and Δ is contained in $\{t_0\alpha_0 + t_1\alpha_1 | t_0, t_1 \in \mathbb{R}_{\geq -1}\}$, we must have $\lambda_1 = 1$. Hence, $\phi(q) = -\phi(p) = -\lambda_0\alpha_0 - \alpha_1$ which contradicts Case (a).

It follows that each isolated fixed point satisfies condition (i) or (ii). Moreover, that the number of isolated fixed points which satisfy (i) is equal to the number isolated fixed points which satisfy (ii) follows directly from Corollary 5.12. $\square$

Chapter 6

The Paintings

By using the main result of Chapter 5, namely Proposition 5.14, in this chapter we prove that the equivalent class of the painting for a compact, tall and monotone complexity one space of dimension six which is balanced is determined by its isolated fixed points data. This is the content of Proposition 6.11. In order to prove this proposition, which is the final part of this chapter, we need to prepare the proof first.

Lemma 6.1
Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced. Then the stabilizer $H_{\mathcal{O}}$ of a T-orbit $\mathcal{O}$ is trivial, equal to T or it is a subcircle of T.

Proof. By Lemma 2.20 any connected component of M^T has dimension two or it is an isolated fixed point. Given $p \in M^T$ so that p belongs to a two dimensional connected component of M^T, then exactly one weight at p is zero and the other two weights form a $\mathbb{Z}$-basis of $\ell_T^* \cong \mathbb{Z}^2$. Moreover, if p is an isolated fixed point, then due to Lemma 5.10 all weights at p are primitive. So it follows from Corollary 2.23 that the stabilizer $H_{\mathcal{O}}$ of a T-orbit $\mathcal{O}$ is trivial, equal to T or it is a subcircle of T. $\square$

Lemma 6.2
Given a compact, tall and monotone complexity one space (M, ω, T, ϕ) of dimension six which is balanced, let $\mathcal{O}$ be an exceptional T-orbit such that $\phi(\mathcal{O})$ lies in the interior of the moment map polytope Δ. Moreover, let $H_{\mathcal{O}}$ be the stabilizer of $\mathcal{O}$ and let $X_{\mathcal{O}}$ be the connected component of $M^{H_{\mathcal{O}}}$ which contains $\mathcal{O}$. Then $X_{\mathcal{O}}$ is a light sphere.

Proof. Since $\mathcal{O}$ is exceptional, its stabilizer $H_{\mathcal{O}}$ is not trivial by Corollary 3.23. Since by Lemma 5.8 there exists no fixed point which lies in the interior of Δ, Lemma 2.20 implies that $H_{\mathcal{O}}$ is a subtorus of T. Moreover, by Lemma 3.21 the dimension of $X_{\mathcal{O}}$ is equal to two. Now pick an $\alpha \in \ell_T^* \setminus \{0\}$ such that $H_{\mathcal{O}} = \exp(\ker(\alpha))$. By Corollary 2.17 there exists $p \in X_{\mathcal{O}} \cap M^T$. Let $\alpha_{p,1}, \alpha_{p,2}, \alpha_{p,3}$ be the weights at p. Since the dimension of $X_{\mathcal{O}}$ is equal to two, by Lemma 5.2 we can assume that $\alpha_{p,1}$ is a multiple of α and that $\alpha_{p,i}$ and α are linearly independent for $i = 2, 3$. Note that $\alpha_{p,1} = 0$ cannot occur because this would imply that $X_{\mathcal{O}} \subset M^T$. Hence, p is an isolated fixed point and $\alpha_{p,1}$ is a light weight. Now it follows that $X_{\mathcal{O}}$ is the light sphere that belongs to p and the weight $\alpha_{p,1}$. $\square$

Lemma 6.3
Given a compact tall and monotone complexity one space (M, ω, T, ϕ) of dimension six which is balanced and a T-orbit $\mathcal{O}$, we have that $\mathcal{O}$ is exceptional if and only if it is contained in a light sphere.

Proof. If the orbit $\mathcal{O}$ is mapped to the interior of the moment map polytope Δ then the claim follows from Corollary 3.23 and Lemma 6.2. If the orbit $\mathcal{O}$ is mapped to a vertex of the moment map polytope, then it is not exceptional and it does not lie in a light sphere. It remains to consider the case that the orbit $\mathcal{O}$ is mapped to the interior of an edge e of the moment map polytope Δ. Due to Proposition 3.20, in this case, $\mathcal{O}$ is exceptional if the stabilizer $\mathfrak{stab}(e)$ of the edge e, which is a circle, is strictly smaller than $H_{\mathcal{O}}$. Hence, if $\mathcal{O}$ is exceptional, then by Lemma 6.1 we have $H_{\mathcal{O}} = T$. So $\mathcal{O}$ is just an isolated fixed point p and by Lemma 5.10 one weight at p is light. Hence, $\mathcal{O}$ lies on a light sphere. On the other hand if $\mathcal{O}$ lies in a light sphere, it is easy to see that $\mathcal{O}$ is just an isolated fixed point p. Hence, $\mathcal{O}$ is indeed exceptional. $\square$

A simple consequence of Lemma 6.3 is the following corollary.

Corollary 6.4
Given a compact, tall and monotone complexity one space (M, ω, T, ϕ) of dimension six which is balanced and has no isolated fixed points we have that its space of exceptional orbits is empty.

Proof. If there does not exist any isolated fixed point, then there does not exist any light sphere. Hence, due to Lemma 6.3 the claim follows. $\square$

Lemma 6.5
Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced. Given two light spheres S and S' with $S \neq S'$, then $S \cap S' = \emptyset$ holds.

Proof. Suppose that $S \cap S' \neq \emptyset$. We need to show that this implies $S = S'$. Choose $p \in S \cap S'$. If p is an isolated fixed point, then by Lemma 5.10 exactly one weight at p is light. Hence, there exists exactly one light sphere that contains p. So we have $S = S'$. If p is not an isolated fixed point, then p is indeed not a fixed point. Note that by Definition 5.6 there exist subcircles H and H' of T such that S is a connected component of M^H and S' is a connected component of $M^{H'}$. Hence, $p \in M^H \cup M^{H'}$. But since p is not a fixed point, we have $H = H'$, which implies that $S = S'$. $\square$

Remark 6.6

Due to Lemma 6.3 and Lemma 6.5 we have that for any compact, tall and monotone complexity one space (M, ω, T, ϕ) of dimension six which is balanced, the connected components of its space of exceptional orbits M_{exc} are the light spheres divided by the T-action. In the next two lemmas we describe these components.

Lemma 6.7

Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced and let S be a light sphere and let p and q, $p \neq q$, be the isolated fixed points which are contained in S. Moreover, let I be the line segment through $\phi(p)$ and $\phi(q)$. Then the moment map induces a homeomorphism

$$\overline{\phi} : S/T \longrightarrow I \,. \tag{6.1}$$

Proof. Since S is a light sphere, by Lemma 5.5 the image of $\phi(S)$ is indeed equal to I. Hence, the map (6.1) is continuous and surjective. In order to prove that it $\overline{\phi}$ is a homeomorphism it remains to show that it is injective since S/T is compact and $\phi(S)$ is Hausdorff. We have that $\overline{\phi}$ is injective if and only if $\phi^{-1}(x) \cap S$ contains just one orbit. Therefore, let $\alpha_{p,1}, \alpha_{p,2}, \alpha_{p,3}$ be the weights at p. By Lemma 5.10 these weights are primitive and exactly one of them is light, say $\alpha_{p,1}$. Hence, S is the connected component of M^H which contains p, where $H = \exp(\ker(\alpha_{p,1}))$. Since $\alpha_{p,1}$ is primitive, we can pick a primitive element $\xi \in \ell_T$ such that $\langle \alpha_{p,1}, \xi \rangle = 1$. Then the subcircle $S^1 = \exp(\mathbb{R}\xi)$ acts on T_pS with weight equal to one. Hence, the S^1-action on S is effective. Moreover, a moment map for the S^1-action on S is

$$\phi_S : S \longrightarrow \mathbb{R} \cong \left(\mathrm{Lie}(S^1)\right)^*, \quad \phi_S(p) = \langle \phi(p), \xi \rangle \,. \tag{6.2}$$

So (S, ω, S^1, ϕ_S) is a compact symplectic toric manifold and by Lemma 2.38 each non-empty fiber of ϕ_S contains just one orbit. It follows that $\overline{\phi}$ is injective. ☐

Lemma 6.8

Given a compact, tall and monotone complexity one space (M, ω, T, ϕ) of dimension six which is balanced, let p be an isolated fixed point and let $\alpha_{p,1}, \alpha_{p,2}$ and $\alpha_{p,3}$ be the weights at p. Suppose that $\alpha_{p,1}$ is a light weight and let S be the light sphere that belongs to p and the weight $\alpha_{p,1}$. Moreover, put $H = \exp(\ker(\alpha_{p,1}))$. Then for any $q \in S \setminus M^T$ the following holds.

- *The stabilizer H_q of q with respect to the T-action is equal to H.*
- *The isotropy representation of H at p and q are linearly symplectomorphic.*

Proof. By Definition 5.6 we have that S is the connected component of M^H which contains p. Hence, $H \subset H_q$ for each $q \in S$. Moreover, by Lemma 6.1 the stabilizer of any point in M is trivial, equal to T or it is a subcircle of T. We conclude that $H_q = H$ for all $q \in S \setminus M^T$. Since S is connected and the H-action on S is trivial, for any two points in S the weights of the H-action at these points are the same. Hence, it follows from the local normal theorem (see Theorem 2.18) that the second statement of this lemma also holds. □

Remark 6.9

Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space (M, ω, T, ϕ) of dimension six which is balanced. If the space has isolated fixed points, then we can recover its space of exceptional orbits M_{exc} from its isolated fixed points data as follows. Due to Proposition 5.14 there exists a primitive element in $\alpha \in \ell_T^*$ such that α resp. $-\alpha$ lies in the interior of an edge e resp. e' of the moment map polytope Δ such that for any isolated fixed point p one of the following holds

(i) $\phi(p) = -\alpha$ and the weights at p are α, $-\alpha_e$ and α_e,

(ii) $\phi(p) = \alpha$ and the weights at p are $-\alpha$, $-\alpha_{e'}$ and $\alpha_{e'}$,

where α_e and $\alpha_{e'}$ are the primitive elements in ℓ_T^* defining the directions of the edges e and e'. Moreover, the numbers of isolated fixed points which satisfy (i) or (ii) are equal. Let this number be K. From the data about the isolated fixed points we can list the isolated fixed points, $p_1, \dots p_K, q_1, \dots q_K$ such that for all $j = 1, ..., K$ p_j satisfies (i) and q_j satisfies (ii) so that $p_j, q_j \in S_j$, where S_j is the light sphere that belongs to p_j and the weight α. So by Lemma 6.3 and 6.5 the connected components of M_{exc} are $S_1/T, ..., S_K/T$. Moreover, by Lemma 6.7 the moment map induces a homeomorphism from each of these connected component to the line segment through $-\alpha$ and α. Hence, as topological space M_{exc} is homeomorphic to the disjoint union of K compact intervals. Moreover, by Lemma 6.8 the stabilizer and the isotropy representation of any orbit can be recovered from (i) and (ii).

Lemma 6.10

Let (M, ω, T, ϕ) and (M', ω', T, ϕ') be two compact, tall and monotone complexity one spaces of dimension six which are balanced such that their data about the isolated fixed points are the same (see Definition 5.1).Then their spaces of exceptional orbits are isomorphic.

Proof. The proof of this lemma follows from Proposition 5.14, Lemma 6.3 and Lemma 6.8 as pointed out in Remark 6.9. □

Now we are able to prove the main proposition of this chapter as follows.

Proposition 6.11
Let (M, ω, T, ϕ) and (M', ω', T, ϕ') be two compact, tall and monotone complexity one spaces of dimension six which are balanced such that their data about the isolated fixed points are the same. Then their paintings are equivalent.

Proof. If both of these spaces do not have isolated fixed points, then the space of exceptional orbits M_{exc} of (M, ω, T, ϕ) and the space of exceptional orbits M'_{exc} of (M', ω', T, ϕ') are both empty so there is nothing to prove. So assume that these spaces have isolated points. Then by Lemma 6.10 there exists an isomorphism $i : M_{exc} \to M'_{exc}$. Pick two maps $f : M_{exc} \to \Sigma$ and $f' : M'_{exc} \to \Sigma'$ which are representing the equivalence classes of paintings of these spaces. By Corollary 4.8 Σ and Σ' are compact and oriented surfaces which are both diffeomorphic to S^2. Hence there exists an orientation preserving homeomorphism $\eta : \Sigma \to \Sigma'$. Note since M_{exc} is homeomorphic to the disjoint union of finitely many compact intervals (see Remark 6.9) the maps $\eta \circ f : M_{exc} \to \Sigma'$ and $f' \circ i : M_{exc} \to \Sigma'$ are homotopic through maps that are also paintings. This completes the proof of this proposition. $\square$

Chapter 7

Duistermaat-Heckman Measures for Tall Complexity One Spaces

In this chapter we consider the Duistermaat-Heckman measure for compact, tall and monotone complexity one spaces of dimension six. We split the task of analyzing the Duistermaat-Heckman measure into two cases, namely we consider the case that the space has no isolated fixed point (see Section 7.1) or that it has isolated fixed points (see Section 7.2). In the third section of this chapter (see Section 7.3) we prove Theorem 1.3 stated in Section 1.1.

7.1 Case: $M_{\mathrm{exc}} = \emptyset$

In this section we analyze the Duistermaat-Heckman measure for compact, tall and monotone complexity one spaces of dimension six which have **no isolated fixed points**. The main result of this section is Proposition 7.1. This section contains three parts. First we introduce the assumption which are needed to state the content of Proposition 7.1. The second part is the proof of this proposition and the third part is a remark which sums up simple consequences of this proposition.

Let us start with describing the assumptions we need to state Proposition 7.1. Throughout this section we assume that (M, ω, T, ϕ) is a **compact, tall and monotone complexity one space of dimension six** which is **balanced** such that this space has **no isolated fixed points**. Note that by Corollary 6.4 its space of exceptional orbits M_{exc} is empty. Therefore, by Proposition 3.20 the stabilizer of any point in M which is mapped to the interior of the moment map polytope Δ is trivial. Hence, by Lemma 2.12 the connected components of the regular values of the moment map ϕ are the interior of Δ and its complement $\mathfrak{t}^* \setminus \Delta$. Due to the Duistermaat-Heckman Theorem (see Theorem 2.41 and Remark 2.42) there exists a polynomial $f : \mathfrak{t}^* \to \mathbb{R}$ of degree zero or one such that a representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure is equal to f on Δ and equal to zero on $\mathfrak{t}^* \setminus \Delta$. We compute this polynomial as follows. Let us pick an edge e_0 of the moment map polytope Δ which is **minimal**. Let v_1 and v_2 be the vertices which are contained in e_0 and let α_0 and α_1 be the weights[1] at the vertex v_1

[1]Recall that by Proposition 3.17 one has that Δ is a reflexive Delzant polytope, so the weights at v_1 are defined as in Definition 2.33

such that $e_0 = (v_1 + \mathbb{R}_{\geq 0}\alpha_0) \cap \Delta$. Here we use Convention 5.13 which we illustrate in Figure 5.1. Note that α_0 and α_1 form a $\mathbb{Z}$-basis of $\ell_T^* \cong \mathbb{Z}^2$. So by using coordinates for $\mathfrak{t}^* \cong \mathbb{R}^2$ with respect to α_0 and α_1, i.e.

$$\mathbb{R}^2 \ni (t_0, t_1) \quad \longleftrightarrow \quad t_0\alpha_0 + t_1\alpha_1 \in \mathfrak{t}^*, \tag{7.1}$$

we are now able to state Proposition 7.1.

Proposition 7.1

By using the assumptions and conventions above, the polynomial $f : \mathfrak{t}^* \to \mathbb{R}$ satisfies either $f = 2$ or $f(t_0\alpha_0 + t_1\alpha_1) = 2 + t_1$ for all $t_0, t_1 \in \mathbb{R}$.

Proof. Since e_0 is a minimal edge, one of the two vertices v_1 and v_2 of Δ which are contained e_0 is minimal (see Definition 4.1 and Definition 4.9). It is not restrictive to assume that v_1 is a minimal vertex. Let e_1 be the other edge which contains the vertex v_1 (see Figure 5.1). By Lemma 3.7 and Lemma 3.8 we have that $M_{e_0} = \phi^{-1}(e_0)$ and $M_{e_1} = \phi^{-1}(e_1)$ are symplectic submanifolds of M of dimension four and $\Sigma_{v_1} = \phi^{-1}(v_1)$ is a symplectic surface in M. In particular, Σ_{v_1} is a symplectic submanifold of M_{e_0} and M_{e_1}. Therefore let $c_1(N_{e_0})$ resp. $c_1(N_{e_1})$ be the first Chern class of the normal bundle of Σ_{v_1} in M_{e_0} resp. M_{e_1}. Since the vertex v_1 is assumed to be minimal and the edge e_0 is assumed to be minimal, by Proposition 4.6 we have that $\int_{\Sigma_{v_1}} c_1(N_{e_0}) = 0$ and $\int_{\Sigma_{v_1}} c_1(N_{e_1}) \in \{0, -1\}$. Moreover, since (M, ω, T, ϕ) is assumed to be balanced, we have that $c_1(M) = [\omega]$. Therefore, we have

$$[\omega]|_{\Sigma_{v_1}} = c_1(M)|_{\Sigma_{v_1}} = c_1(\Sigma_{v_1}) + c_1(N_{e_0}) + c_1(N_{e_1}),$$

where $c_1(\Sigma_{v_1})$ is the first Chern class of (Σ_{v_1}, ω). Since Σ_{v_1} is a two sphere (see Proposition 4.6), we have that $\int_{\Sigma_{v_1}} c_1(\Sigma_{v_1}) = 2$ and hence

$$\int_{\Sigma_{v_1}} \omega = \int_{\Sigma_{v_1}} c_1(M) = \begin{cases} 2, & \text{if } \int_{\Sigma_{v_1}} c_1(N_{e_1}) = 0, \\ 1, & \text{if } \int_{\Sigma_{v_1}} c_1(N_{e_1}) = -1. \end{cases}$$

Since Δ is a reflexive Delzant polytope (see Proposition 3.17), we have $v_1 = -\alpha_0 - \alpha_1$. Using the formula given by Lemma A.11 we find

$$\begin{aligned} f(t_0\alpha_0 + t_1\alpha_1) &= f(v_1 + (t_0+1)\alpha_0 + (t_1+1)\alpha_1) \\ &= -(t_0+1)\left(\int_{\Sigma_{v_1}} c_1(N_{e_0})\right) - (t_1+1)\left(\int_{\Sigma_{v_1}} c_1(N_{e_1})\right) + \left(\int_{\Sigma_{v_1}} \omega\right) \end{aligned}$$

for all $t_0, t_1 \in \mathbb{R}$. It follows that

$$f(t_0\alpha_0 + t_1\alpha_1) = \begin{cases} 2, & \text{if } \int_{\Sigma_{v_1}} c_1(N_{e_1}) = 0, \\ 2 + t_1, & \text{if } \int_{\Sigma_{v_1}} c_1(N_{e_1}) = -1, \end{cases}$$

for all $t_0, t_1 \in \mathbb{R}$ which completes the proof of this proposition. □

Remark 7.2
Given a compact, tall and monotone complexity one space of dimension six (M, ω, T, ϕ) which is balanced such that this space has no isolated fixed points, there exists a unique polynomial $f : \mathfrak{t}^* \to \mathbb{R}$ of degree zero or one, such that a representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure is equal to f on Δ and equal to zero on $\mathfrak{t}^* \setminus \Delta$. Let E be the number of edges of Δ. Then due to Proposition 7.1 we have at most $1 + E$ possibilities for f. Namely, f is equal to 2 or by fixing an edge e_0 of Δ and assuming this edge to be minimal, f is a polynomial of degree one, such that $f^{-1}(1) \cap \Delta = e_0$.

7.2 Case: $M_{\text{exc}} \neq \emptyset$

In this section we analyze the Duistermaat-Heckman measure for compact, tall and monotone complexity one spaces of dimension six which have **isolated fixed points**. The main results of this section are Proposition 7.3 and Proposition 7.4. In this section we mainly introduce the assumption which are needed to state and prove these propositions.

So let us start with describing the assumptions we need in order to state Proposition 7.3 and Proposition 7.4. Throughout this section we assume that (M, ω, T, ϕ) is a **compact, tall and monotone complexity one space of dimension six** which is **balanced** such that this space has **isolated fixed points**. Due to Proposition 3.17 the moment map polytope Δ is reflexive and Delzant. In this section we fix an edge e_0 of Δ which is assumed to be **minimal**. Let v_1 and v_2 be the vertices which are contained in e_0 and let α_0 and α_1 be the weights of the vertex v_1. Here, we use again Convention 5.13. Let e_1 be the other edge which contains v_1 and let e_2 be the other edge which contains v_2. Moreover, let $\alpha_2 \in \ell_T^*$ so that the weights at v_2 are $-\alpha_0$ and α_2. This situation is illustrated in Figure 5.1. Due to Proposition 5.14 for any isolated fixed point p one of the following conditions holds.

(i) $\phi(p) = -\alpha_0$ and $\phi(p)$ lies on e_1. The weights at p are α_0, $-\alpha_1$ and $+\alpha_1$.

(ii) $\phi(p) = \alpha_0$ and $\phi(p)$ lies on e_2. The weights at p are $-\alpha_0$, $-\alpha_2$ and $+\alpha_2$.

Moreover, the numbers of isolated fixed points which satisfies (i) resp. (ii) are equal. We denote this number by K. Note that since we are assuming that the space has isolated fixed points, we must have $K \geq 1$. Again we are using coordinates for $\mathfrak{t}^* \cong \mathbb{R}^2$ with respect to α_0 and α_1, i.e.

$$\mathbb{R}^2 \ni (t_0, t_1) \quad \longleftrightarrow \quad t_0\alpha_0 + t_1\alpha_1 \in \mathfrak{t}^*. \tag{7.2}$$

Moreover, in this case we have three connected components of the regular values of ϕ. Namely, Δ_1 which are the interior points of Δ of the form $t_0\alpha_0 + t_1\alpha_1$ with $t_1 < 0$, Δ_2 which are the interior points of Δ of the form $t_0\alpha_0 + t_1\alpha_1$ with $t_1 > 0$ and the complement $\mathfrak{t}^* \setminus \Delta$. Due to the Duistermaat-Heckman Theorem (see Theorem 2.41 and Remark 2.42) there exist unique polynomials $f_1 : \mathfrak{t}^* \to \mathbb{R}$ and $f_2 : \mathfrak{t}^* \to \mathbb{R}$ of degree zero or one, such that a representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure is equal to f_1 on Δ_1 , equal to f_2 on Δ_2 and equal to zero on $\mathfrak{t}^* \setminus \Delta$. Now we are able to state the main propositions of this section.

Proposition 7.3
With the assumptions and conventions above the polynomial $f_1 : \mathfrak{t}^* \to \mathbb{R}$ satisfies $f_1(t_0\alpha_0 + t_1\alpha_1) = 2 + t_1$ for all $t_0, t_1 \in \mathbb{R}$.

Proposition 7.4
With the assumptions and conventions above one of the following conditions holds.

- *$f_2 = 2$ and there exist exactly two isolated fixed points, where one satisfies the condition (i) and the other one satisfies the condition (ii).*
- *$f_2(t_0\alpha_0 + t_1\alpha_1) = 2 - t_1$ for all $t_0, t_1 \in \mathbb{R}$ and there exist exactly four isolated fixed points, where two satisfy the condition (i) and the other two satisfy the condition (ii).*

where the conditions for the isolated fixed points a are

(i) $\phi(p) = -\alpha_0$ and $\phi(p)$ lies on e_1. The weights at p are α_0, $-\alpha_1$ and $+\alpha_1$.

(ii) $\phi(p) = \alpha_0$ and $\phi(p)$ lies on e_2. The weights at p are $-\alpha_0$, $-\alpha_2$ and $+\alpha_2$.

Proof of Proposition 7.3. Since e_0 is a minimal edge, one of the two vertices v_1 and v_2 of Δ which are contained e_0 is minimal (see Definition 4.1 and Definition 4.9). It is not restrictive to assume that v_1 is a minimal vertex. Let e_1 be the other edge which contains the vertex v_1 (see Figure 5.1). By Lemma 3.7 and Lemma 3.8 one has that $M_{e_0} = \phi^{-1}(e_0)$ and $M_{e_1} = \phi^{-1}(e_1)$ are symplectic submanifolds of M of dimension four and $\Sigma_{v_1} = \phi^{-1}(v_1)$ is a symplectic surface of M. In particular, Σ_{v_1} is a symplectic submanifold of M_{e_0} and M_{e_1}. Therefore, let $c_1(N_{e_0})$ resp. $c_1(N_{e_1})$ be the first Chern class of the normal bundle of Σ_{v_1} in M_{e_0} resp. M_{e_1}. Since the vertex v_1 is assumed to be minimal and the edge e_0 is assumed to be minimal, by Proposition 4.6 we have that $\int_{\Sigma_{v_1}} c_1(N_{e_0}) = 0$ and $\int_{\Sigma_{v_1}} c_1(N_{e_1}) \in \{0, -1\}$. The case $\int_{\Sigma_{v_1}} c_1(N_{e_1}) = 0$ cannot occur because this would imply that e_1 is also a minimal edge, but then by Proposition 5.14 the space has no isolated fixed points. Hence, $\int_{\Sigma_{v_1}} c_1(N_{e_1}) = -1$ and so, as shown in Proposition 7.1, we have $f_1(t_0\alpha_0 + t_1\alpha_1) = 2 + t_1$. □

Proof of Proposition 7.4. By Proposition 5.14 any isolated fixed point satisfies condition (i) or (ii) and the numbers of those which satisfy (i) is equal to the number of those which satisfy (ii). Therefore, let K be the number of isolated fixed points which satisfy condition (i). Since we assume that the space has at least one isolated fixed point, we must have $K \geq 1$. We will show that $K = 1$ or $K = 2$. Then the claim about the isolated fixed points holds. Furthermore, we will show that if $K = 1$ resp. $K = 2$ then $f_2 = 2$ resp. $f_2(t_0\alpha_0 + t_1\alpha_1) = 2 - t_1$ for all $t_0, t_1 \in \mathbb{R}$.

We denote by v_3 the second vertex that is contained in e_1. Note that $v_3 = t_{\max}\alpha_1 - \alpha_0$, where $t_{\max}$ is a positive integer. The preimage $M_{e_1} = \phi^{-1}(e_1)$ is a symplectic submanifold of dimension four. Let $\xi_0, \xi_1 \subset \ell_T$ be the dual basis of α_0, α_1, then the subcircle $S^1 = \exp(\mathbb{R}\xi_1)$ acts effectively and in a Hamiltonian fashion on M_{e_1}. A moment map for this action is $\phi_{e_1} = \pi \circ \phi : M_{e_1} \to \mathbb{R}$, where

$$\pi\colon \mathfrak{t}^* \longrightarrow \mathbb{R}, \quad t_0\alpha_0 + t_1\alpha_1 \longmapsto t_1. \tag{7.3}$$

The minimum resp. the maximum of this map is $\pi(v_1) = -1$ resp. $\pi(v_3) = t_{\max}$. By Lemma 4.2 a representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure of the space $(M_{e_1}, \omega_{e_1}, S^1, \phi_{e_1})$ is

$$f_{S^1}(t) = \begin{cases} 2 + t, & \text{if } t \in [-1, 0), \\ 2 - (K-1)t, & \text{if } t \in [0, t_{max}), \\ 0, & \text{otherwise.} \end{cases} \tag{7.4}$$

Due to Corollary 4.3, $\lim_{t \uparrow t_{\max}} f_{S^1}(t) = 2 - (K-1)t_{\max}$ is equal to the symplectic volume of the surface $\phi^{-1}(v_3)$. Hence, $2-(K-1)t_{\max}$ must be positive. Since K and $t_{\max}$ are positive integers we have $K = 1$ or $K = 2$ and if $K = 2$ holds we must have $t_{\max} = 1$. Before we analyze these cases separately we set some notation. Note that one weight at the vertex v_3 is $-\alpha_0$ and the other one we denote by α_3. Hence, the second edge which contains v_3 is $e_3 = (v_3 + \mathbb{R}_{\geq 0}\alpha_2)$ (see Figure 5.1). By Lemma 3.7 and Lemma 3.8 we find that $M_{e_1} = \phi^{-1}(e_1)$ and $M_{e_3} = \phi^{-1}(e_3)$ are symplectic submanifolds of M of dimension four and $\Sigma_{v_3} = \phi^{-1}(v_3)$ is a symplectic surface of

M. In particular, Σ_{v_3} is a symplectic submanifold of M_{e_1} and M_{e_3}. Therefore, let $c_1(N'_{e_1})$ resp. $c_1(N'_{e_3})$ be the first Chern class of the normal bundle of Σ_{v_3} in M_{e_1} resp. M_{e_3}. Note that by Lemma A.11 we have that

$$f_2(v_3 - t'_1\alpha_1 + t_3\alpha_3) = -At'_1 - Bt_3 + C \tag{7.5}$$

for all $t'_1, t_3 \in \mathbb{R}$, where $A = \int_{\Sigma_{v_3}} c_1(N'_{e_1}) =$, $B = \int_{\Sigma_{v_3}} c_1(N'_{e_3})$ and $C = \int_{\Sigma_{v_3}} \omega$. Since $[\omega] = c_1(M)$ and since Σ_{v_3} is two sphere, we have

$$C = 2 + A + B = 2 - (K-1)t_{\max}. \tag{7.6}$$

Moreover, by using Corollary 4.3 we have

$$A = 1 - K. \tag{7.7}$$

If $K = 1$, then by (7.7) we find $A = 0$ and so by (7.6) it follows $B = 0$ and $C = 2$. Hence, by (7.5) we have that $f_2 = 2$. If $K = 2$, then we have $t_{\max} = 1$. By (7.7) we find $A = -1$ and so by (7.6) it follows $B = 0$ and $C = 1$. Moreover, by using that Δ is a reflexive Delzant polytope we have that

$$v_3 = \alpha_1 - \alpha_3 = \alpha_1 - \alpha_0. \tag{7.8}$$

From (7.5) we conclude that $f_2(t_0\alpha_0 + t_1\alpha_1) = 2 - t_1$ for all $t_0, t_1 \in \mathbb{R}$. □

7.3 The Proof of Theorem 1.3

In this section we prove Theorem 1.3. Before we start with the proof of Theorem 1.3, we need the following lemma.

Lemma 7.5
Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced. Then the equivalence class of its painting is determined by its Duistermaat-Heckman measure.

Proof. Due to the Proposition 7.1, Proposition 7.3 and Proposition 7.4 we find a representative f_{DH} of the Radon-Nikodym derivative of the Duistermaat-Heckman measure of the space (M, ω, T, ϕ) which satisfies one of the following cases.

- There exists a unique polynomial $f : \mathfrak{t}^* \to \mathbb{R}$ such that $f_{DH} = f$ on the moment map polytope Δ. In this case, the space has no isolated fixed points and hence the space of exceptional orbits M_{exc} is empty (see Corollary 6.4).

- There exist two different polynomials $f_1 : \mathfrak{t}^* \to \mathbb{R}$ and $f_2 : \mathfrak{t}^* \to \mathbb{R}$ and two disjoint open non-empty subsets $\Delta_1, \Delta_2 \subset \Delta$ such that f_{DH} is equal is f_1 on Δ_1 and equal to f_2 on Δ_2. In this case, by using Proposition 7.3 and Proposition 7.4 the data about the isolated fixed points can be completely recovered from f_{DH}.

Since the equivalence class of the painting of (M, ω, T, ϕ) is determined by the data about the isolated fixed points (see Proposition 6.11), the claim of this lemma is true. $\square$

Now we are able to prove Theorem 1.3.

Proof of Theorem 1.3 . Let (M, ω, T, ϕ) and (M', ω', T, ϕ') be two compact, tall and monotone complexity one space of dimension six. We need to show that these spaces are isomorphic if and only if they have the same Duistermaat-Heckman measure. The first direction of this is trivial since Duistermaat-Heckman measure is an invariant for such spaces. In order to prove the second direction it is not restrictive to assume that both of the spaces are balanced. So suppose that these spaces are balanced and that they have the same Duistermaat-Heckman measure. Then due to Lemma 7.5 their paintings are equivalent and by Corollary 4.8 the genus of both of the spaces is zero. Due to the work of Karshon and Tolman (see Theorem 3.6) both spaces are isomorphic. $\square$

Chapter 8

Classification of Compact, Tall and Monotone Complexity One Spaces

In this section we give a complete classification of compact, tall and monotone complexity one spaces of dimension six (up to twisted isomorphisms). Note that in order to perform such a classification it is not restrictive to consider the balanced case (see discussion in Section 2.3.4). Given a compact, tall and monotone complexity one space (M, ω, T, ϕ) of dimension six which is balanced, its moment map polytope is a reflexive Delzant polytope in $\mathfrak{t}^* \cong \mathbb{R}^2$ (see Proposition 3.17). Note that there exist (up to $GL(\mathbb{Z}, 2)$-transformations) exactly five reflexive Delzant polytopes of dimension two, which we list in Figure 3.1. So by identifying ℓ_T^* with $\mathbb{Z}^2$ we can assume that the moment map polytope Δ is equal to one of the following polytopes (up to $GL(\mathbb{Z}, 2)$-transformations).

$$P_I := \Big\{(x,y) \in \mathbb{R}^2 \,|\, x \in [-1,2],\, y \in [-1, -x+1]\Big\} \tag{8.1}$$
$$P_{II} := \Big\{(x,y) \in \mathbb{R}^2 \,|\, x, y \in [-1,1]\Big\} \tag{8.2}$$
$$P_{III} := \Big\{(x,y) \in \mathbb{R}^2 \,|\, x \in [-1,2],\; -1 \leq y \leq -x+1, 1\Big\} \tag{8.3}$$
$$P_{IV} := \Big\{(x,y) \in \mathbb{R}^2 \,|\, x, y \in [-1,1] \,\text{and}\, y \leq -x+1\Big\} \tag{8.4}$$
$$P_V: := \Big\{(x,y) \in \mathbb{R}^2 \,|\, x, y \in [-1,1] \,\text{and}\, -x-1 \leq y \leq -x+1\Big\}. \tag{8.5}$$

These are all reflexive Delzant polytopes[1] in $\mathbb{R}^2$ up to $GL(\mathbb{Z}, 2)$-transformations. In this chapter we analyze each of these cases and we show that for each $i = I, II, III, IV, V$ there exist just finitely many compact, tall and monotone complexity one spaces of dimension six (up to twisted isomorphisms) which are balanced such that the moment map polytope is equal to P_i. Moreover, we show that each of these spaces is coming from a symplectic toric manifold. In order to perform this classification we go on as follows. For each $i = I, II, III, IV, V$, we list all possible Duistermaat-Heckman measures. Note that due to Proposition 7.1, Proposition 7.3 and Proposition 7.4 a representative of the Radon-Nikodym derivative f_{DH} of the Duistermaat-Heckman measure can only have one of the following forms.

- f_{DH} is equal to a polynomial on Δ.
- The polytope is divided into two parts and on each of this parts f_{DH} is equal

[1] From left to right the polytopes $P_I, P_{II}, P_{III}, P_{IV}$ and P_V are illustrated in Figure 3.1

to a polynomial.

By using this we list all possible Duistermaat-Heckman measures by going through all this possible cases. Namely, we give a picture of each polytope and we write down to each oc its parts the polynomials as above. Done this (for each case $i = I, II, III, IV, V$ separately) we show the following. Given a compact, tall and monotone complexity one space (M, ω, T, ϕ) of dimension six which is balanced such that its moment map polytope Δ is equal to P_i (for one $i \in \{I, II, III, IV, V\}$ up to $GL(\mathbb{Z}, 2)$-transformations), there exists a reflexive Delzant polytope $\tilde{\Delta}$ of dimension three, which satisfies the following.

- $\tilde{\Delta}$ is of the form

$$\tilde{\Delta} = \Big\{(x, y, z) \in \mathbb{R}^3 \,|\, (x, y) \in P_i \,\text{and}\; p_{\min}(x, y) \leq z \leq p_{\max}(x, y)\Big\}, \qquad (8.6)$$

  where $p_{\min}$ and $p_{\max}$ are function defined on P_i.

- On P_i a representative of the Radon-Nikodym derivative[2] is equal to $p_{max} - p_{min}$.

- Let $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2$ be the projection given by $\pi(x, y, z) = (x, y)$. Then for any vertex v' of P_i the set $\pi^{-1}(v') \cap \tilde{\Delta}$ is an edge of $\tilde{\Delta}$.

By showing the existence of such a reflexive Delzant $\tilde{\Delta}$ of dimension three we have by Theorem 2.36 that there exists a unique (up to isomorphisms) compact symplectic toric manifold $(M_{\tilde{\Delta}}, \omega_{\tilde{\Delta}}, T^3, \phi_{\tilde{\Delta}})$ of dimension six, whose moment map image is equal to $\tilde{\Delta}$. Moreover, by Proposition 3.19 this space is monotone and balanced. As pointed out in Example 2.44, by forgetting the S^1-factor of T^3 which belongs to the z-direction we obtain from $(M_{\tilde{\Delta}}, \omega_{\tilde{\Delta}}, T^3, \phi_{\tilde{\Delta}})$ a compact, tall and monotone complexity one space, whose Duistermaat-Heckman measure is equal to the one of (M, ω, T, ϕ). Hence, by Theorem 1.3 these spaces are isomorphic. As a consequence of the construction above we obtain Theorem 1.4 stated in Section 1.1 (see Section 8.6).

In the following we will study P_i for $i \in \{I, II, III, IV, V\}$.

Remark 8.1
We will use pictures to illustrate the classification of compact, tall and monotone complexity one spaces of dimension six which are balanced via their Duistermaat-Heckman measures. In the following pictures the bullets will denote integral points in $\mathbb{R}^2$. The gray colored area indicates the respective polytope. The expressions

[2] Or equivalent any representative of the Radon-Nikodym derivative is equal to $p_{\max} - p_{\min}$ on P_i up to a set of Lebesgue measure equal to zero

inside the polytopes describe a representative of the Radon-Nikodym derivative for a Duistermaat-Heckman measure (see Chapter 7).

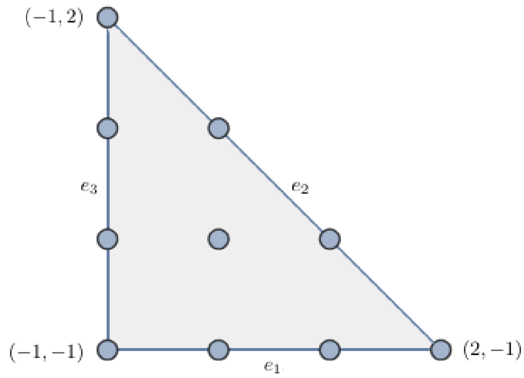


Figure 8.1: Polytope P_I

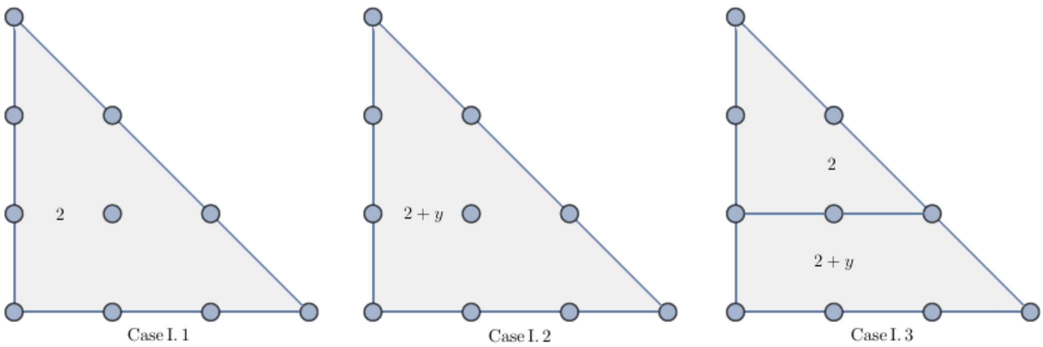


Figure 8.2: The Duistermaat-Heckman measures belonging to P_I

8.1 Case I: The Polytope P_I

Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced such that its moment map polytope Δ is equal to P_I (up to $GL(\mathbb{Z}, 2)$-transformations). Consider the polytope P_I that is

$$P_I \colon = \Big\{ (x, y) \in \mathbb{R}^2 \,|\, x \in [-1, 2],\, y \in [-1, -x+1] \Big\} .$$

This polytope has three vertices and three edges (see Figure 8.1). For reasons of symmetry we can assume that e_1 is a minimal edge. Hence, we obtain three possible Duistermaat-Heckman measures. The respective representatives of their Radon-Nikodym derivative are of the form shown in Figure 8.2. For each measure (see

Case I.1, I.2, I.3) we obtain a reflexive Delzant polytope in $\mathbb{R}^3$ listed below and illustrated in Figure 8.15 which defines a toric extension.

$$\begin{aligned}
\Delta_{I.1}\colon &= \left\{(x,y,z)\in\mathbb{R}^3 \,|\, (x,y)\in P_I,\ -1\leq z\leq 1\right\},\\
\Delta_{I.2}\colon &= \left\{(x,y,z)\in\mathbb{R}^3 \,|\, (x,y)\in P_I,\ -1\leq z\leq y+1\right\},\\
\Delta_{I.3}\colon &= \left\{(x,y,z)\in\mathbb{R}^3 \,|\, (x,y)\in P_I,\ -1\leq z\leq \min\{y+1,1\}\right\}.
\end{aligned}$$

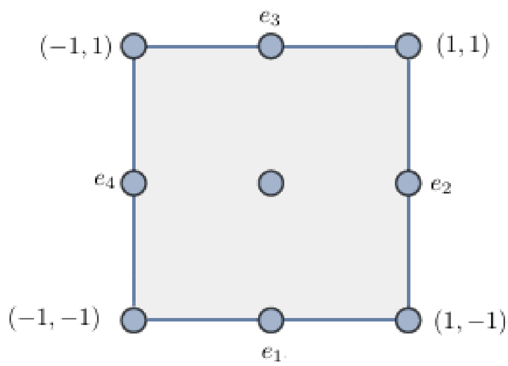


Figure 8.3: Polytope P_{II}

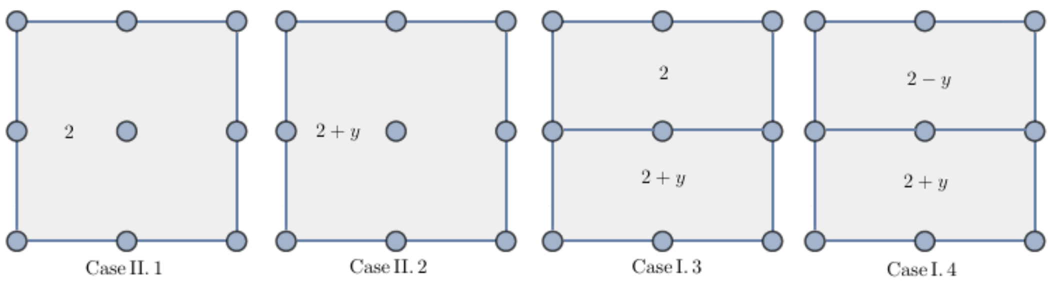


Figure 8.4: All Duistermaat-Heckman measures belonging to P_{II}

8.2 Case II: The Polytope P_{II}

Let (M,ω,T,ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced such that its moment map polytope Δ is equal to P_{II} (up to $GL(\mathbb{Z},2)$-transformations). Consider the polytope P_{II} that is

$$P_{II} = \left\{(x,y)\in\mathbb{R}^2 \,|\, x,y\in[-1,1]\right\}.$$

This polytope has four vertices and four edges (see Figure 8.3). For the reasons of symmetry we can assume that e_1 is a minimal edge. Hence, we obtain four possible

Duistermaat-Heckman measures. The respective representatives of their Radon-Nikodym derivative are of the form shown in Figure 8.4. For each measure (Case II.1, II.2, II.3, II.4) we obtain a reflexive Delzant polytope in $\mathbb{R}^3$ listed below and illustrated in Figure 8.16 which defines a toric extension.

$$\begin{aligned}
\Delta_{II.1}\colon &= \Big\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{II},\ -1 \le z \le 1\Big\},\\
\Delta_{II.2}\colon &= \Big\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{II},\ -1 \le z \le y+1\Big\},\\
\Delta_{II.3}\colon &= \Big\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{II},\ -1 \le z \le \min\{y+1,1\}\Big\},\\
\Delta_{II.4}\colon &= \Big\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{II},\ \max\{-1,y-1\} \le z \le \min\{y+1,1\}\Big\}.
\end{aligned}$$

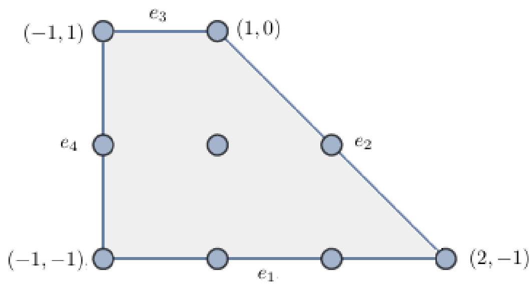


Figure 8.5: Polytope P_{III}

8.3 Case III: The Polytope P_{III}

Let (M,ω,T,ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced such that its moment map polytope Δ is equal to P_{III} (up to $GL(\mathbb{Z},2)$-transformations). Consider the polytope P_{III} that is

$$P_{III} = \Big\{(x,y) \in \mathbb{R}^2 \,|\, x \in [-1,2],\ -1 \le y \le -x+1,1\Big\}$$

This polytope has four vertices and four edges (see Figure 8.5). For reasons of symmetry we can assume that e_1 , e_3 or e_4 is a minimal edge. If e_1 is a minimal edge, then a representative of the Radon-Nikodym derivative for the Duistermaat-Heckman measure is of the form as in Figure 8.6. If e_3 is a minimal edge, then a representative of the Radon-Nikodym derivative for the Duistermaat-Heckman measure is of the form as in Case III.1 or III.4 (see Figure 8.6). Otherwise it is as in Figure 8.7. If e_4 is a minimal edge, then a representative of the Radon-Nikodym derivative for the Duistermaat-Heckman measure is of the form as in Case III.1 (see

Figure 8.6) or it is as in Figure 8.8. For each measure (Case III.1 - Case III.7) we obtain a reflexive Delzant polytope in $\mathbb{R}^3$ listed below and illustrated in Figure 8.17 which defines a toric extension.

$$
\begin{aligned}
\Delta_{III.1}\colon &= \left\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{III},\ -1 \le z \le 1\right\},\\
\Delta_{III.2}\colon &= \left\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{III},\ -1 \le z \le y+1\right\},\\
\Delta_{III.3}\colon &= \left\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{III},\ -1 \le z \le \min\{y+1,1\}\right\},\\
\Delta_{III.4}\colon &= \left\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{III},\ \max\{-1,y-1\} \le z \le \min\{y+1,1\}\right\},\\
\Delta_{III.5}\colon &= \left\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{III},\ y-1 \le z \le 1\right\},\\
\Delta_{III.6}\colon &= \left\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{III},\ \max\{-1,y-1\} \le z \le 1\right\},\\
\Delta_{III.7}\colon &= \left\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{III},\ -1 \le z \le x+1\right\}.
\end{aligned}
$$

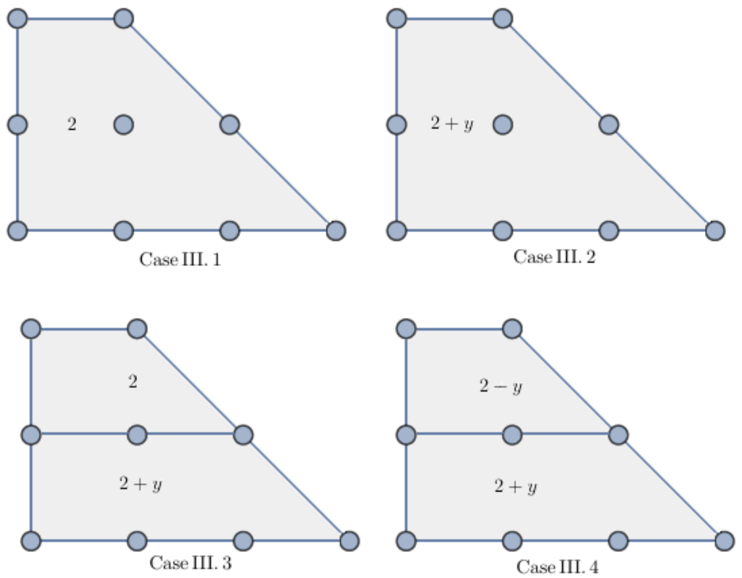


Figure 8.6: The Duistermaat-Heckman measures belonging to P_{III}, when e_1 is minimal.

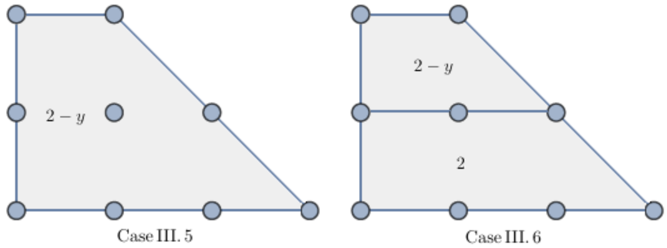


Figure 8.7: The Duistermaat-Heckman measures belonging to P_{III}, when e_3 is minimal.

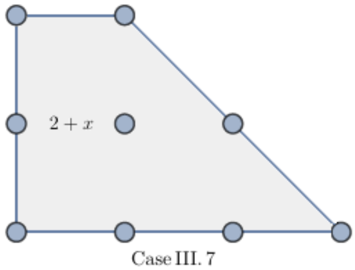


Figure 8.8: The Duistermaat-Heckman measures belonging to P_{III}, when e_4 is minimal.

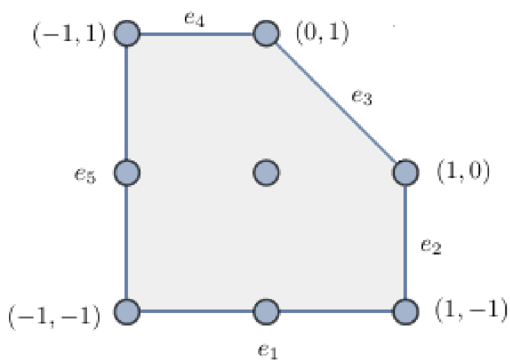


Figure 8.9: Polytope P_{IV}

8.4 Case IV: The Polytope P_{IV}

Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced such that its moment map polytope Δ is equal to P_{IV} (up to $GL(\mathbb{Z}, 2)$-transformations). Consider the polytope P_{IV} that is

$$P_{IV} = \left\{ (x, y) \in \mathbb{R}^2 \mid x, y \in [-1, 1] \,\text{and}\, y \leq -x + 1 \right\}.$$

This polytope has five vertices and five edges (see Figure 8.9). For reasons of symmetry we can assume that e_1 , e_3 or e_4 is a minimal edge. If e_1 is a minimal edge, then a representative of the Radon-Nikodym derivative for the Duistermaat-Heckman measure is of the form as in Figure 8.10. If e_4 is a minimal edge, then a representative of the Radon-Nikodym derivative for the Duistermaat-Heckman measure is of the form as in Case IV.1 (see Figure 8.10) otherwise it is as in Figure 8.12. If e_3 is a minimal edge, then a representative of the Radon-Nikodym derivative for the Duistermaat-Heckman measure is of the form as in Case IV.1 (see Figure 8.10) or it is as in Figure 8.11. For each measure (Case IV.1 - Case IV.4) we obtain a reflexive Delzant polytope in $\mathbb{R}^3$ listed below and illustrated in Figure 8.18 which defines a toric extension.

$$
\begin{aligned}
\Delta_{IV.1}\colon &= \left\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{IV},\ -1 \le z \le 1\right\},\\
\Delta_{IV.2}\colon &= \left\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{IV},\ -1 \le z \le y+1\right\},\\
\Delta_{IV.3}\colon &= \left\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{IV},\ y-1 \le z \le 1\right\},\\
\Delta_{IV.4}\colon &= \left\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_{IV},\ -1 \le z \le -x-y+1\right\}.
\end{aligned}
$$

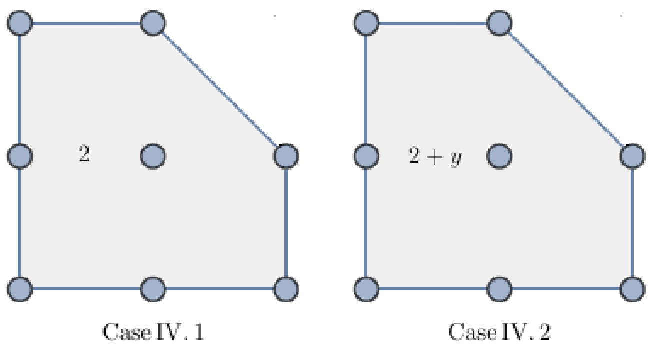


Figure 8.10: The Duistermaat-Heckman measures belonging to P_{IV}, when e_1 is minimal.

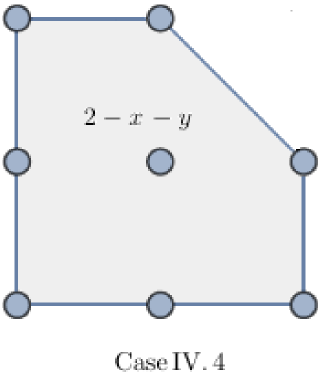


Figure 8.11: The Duistermaat-Heckman measures belonging to P_{IV}, when e_3 is minimal.

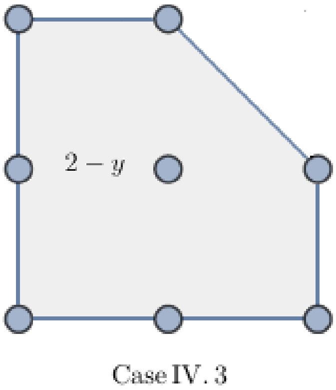


Figure 8.12: The Duistermaat-Heckman measures belonging to P_{IV}, when e_4 is minimal.

8.5 Case V: The Polytope P_V

Let (M, ω, T, ϕ) be a compact, tall and monotone complexity one space of dimension six which is balanced such that its moment map polytope Δ is equal to P_V (up to $GL(\mathbb{Z}, 2)$-transformations). Consider the polytope P_V that is

$$P_V = \left\{ (x, x) \in \mathbb{R}^2 \,|\, x, y \in [-1, 1] \text{ and } -x-1 \leq y \leq -x+1 \right\}.$$

This polytope has six vertices and six edges (see Figure 8.13). For reasons of symmetry we can assume that e_1 is a minimal edge. Hence, a representative of the Radon-Nikodym derivative for the Duistermaat-Heckman measure is of the form as in Figure 8.14. For each measure (Case V.1, Case V.2) we obtain a reflexive Delzant polytope in $\mathbb{R}^3$ listed below and illustrated in Figure 8.19 which defines a

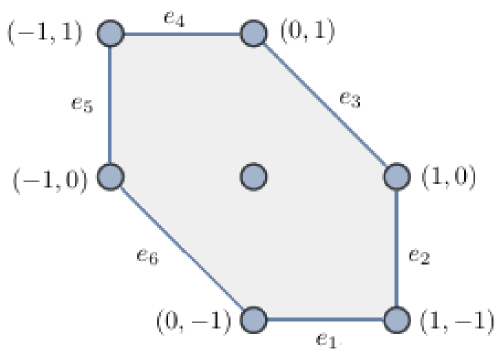


Figure 8.13: Polytope P_V

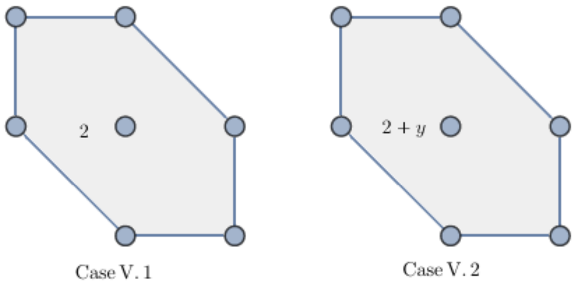


Figure 8.14: The Duistermaat-Heckman measures belonging to P_V

toric extension.

$$\Delta_{V.1}\colon = \Big\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_V,\ -1 \le z \le 1\Big\},$$
$$\Delta_{V.2}\colon = \Big\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in P_V,\ -1 \le z \le y+1\Big\}.$$

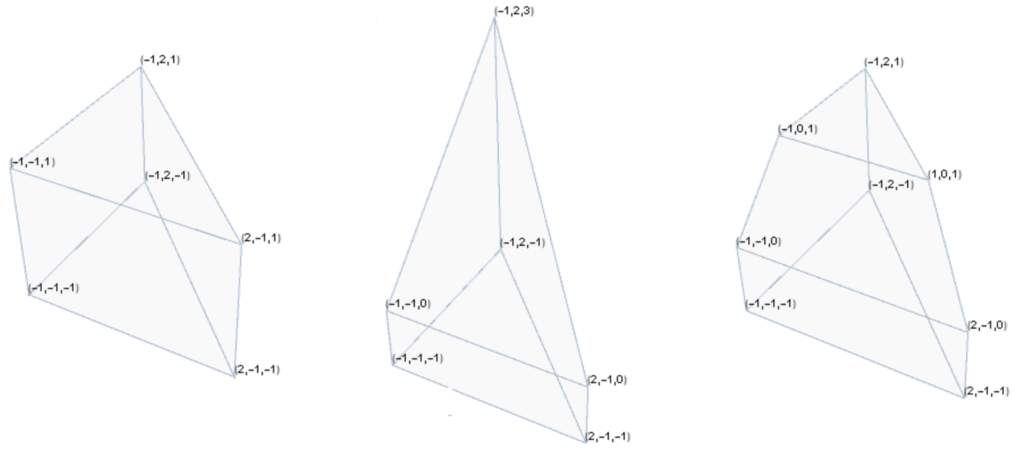
8.6 The Proofs of Theorem 1.4 and Corollary 1.5

In this section we prove Theorem 1.4 and Corollary 1.5.

Proof of Theorem 1.4. In the former sections of this chapter we classified all compact, tall and monotone complexity one spaces of dimension six. So from this it is clear that the torus action of any such a space extends to an effective and Hamiltonian action of complexity zero (see the discussion in the beginning of Chapter 8). □

Proof of Corollary 1.5 . Given a compact, tall and monotone complexity one space (M, ω, T, ϕ) of dimension six. By Theorem 1.4 the T-action on (M, ω) extends to an effective and Hamiltonian action of complexity zero. Hence, due to the classification result for compact symplectic toric manifolds (see Theorem 2.36 and Corollary 2.37) it follows that M is diffeomorphic to a Fano threefold. □

8.7 Pictures of Toric Extensions



Case III.1 Case III.2 Case III.3

Figure 8.15: Toric extensions of P_I

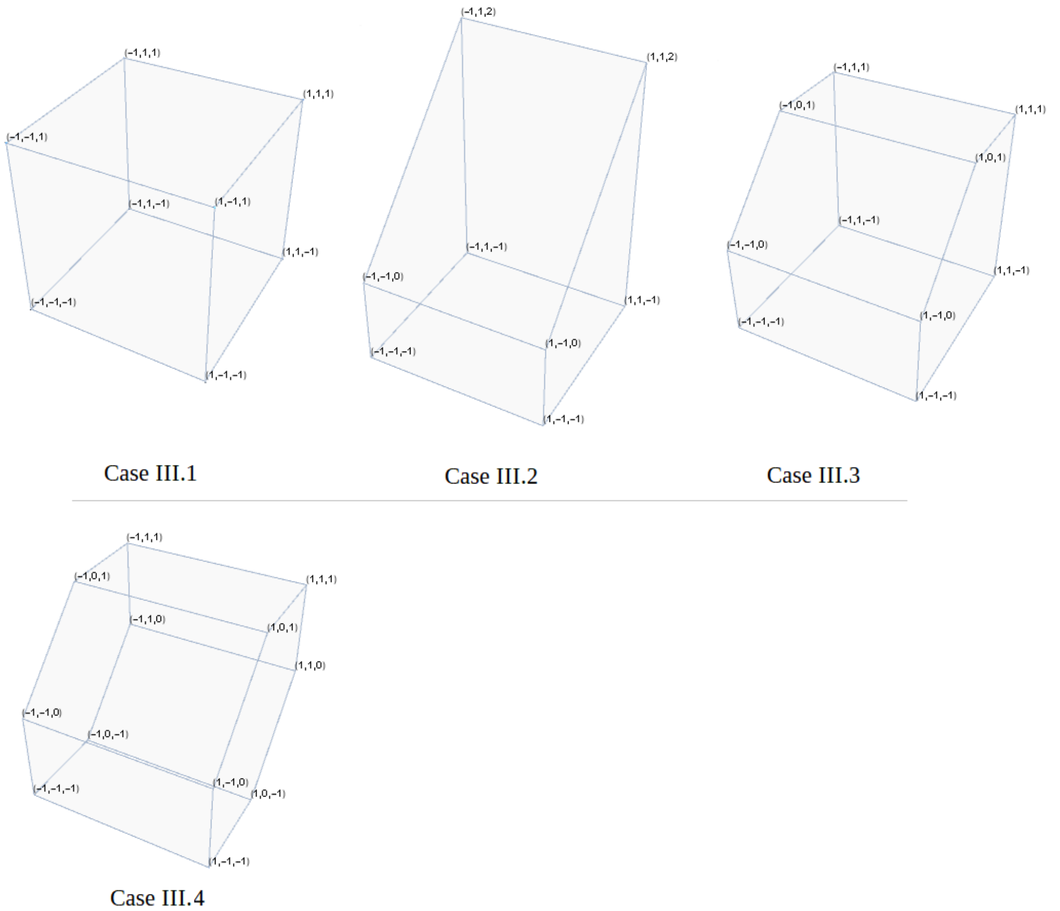


Figure 8.16: Toric extensions of P_{II}

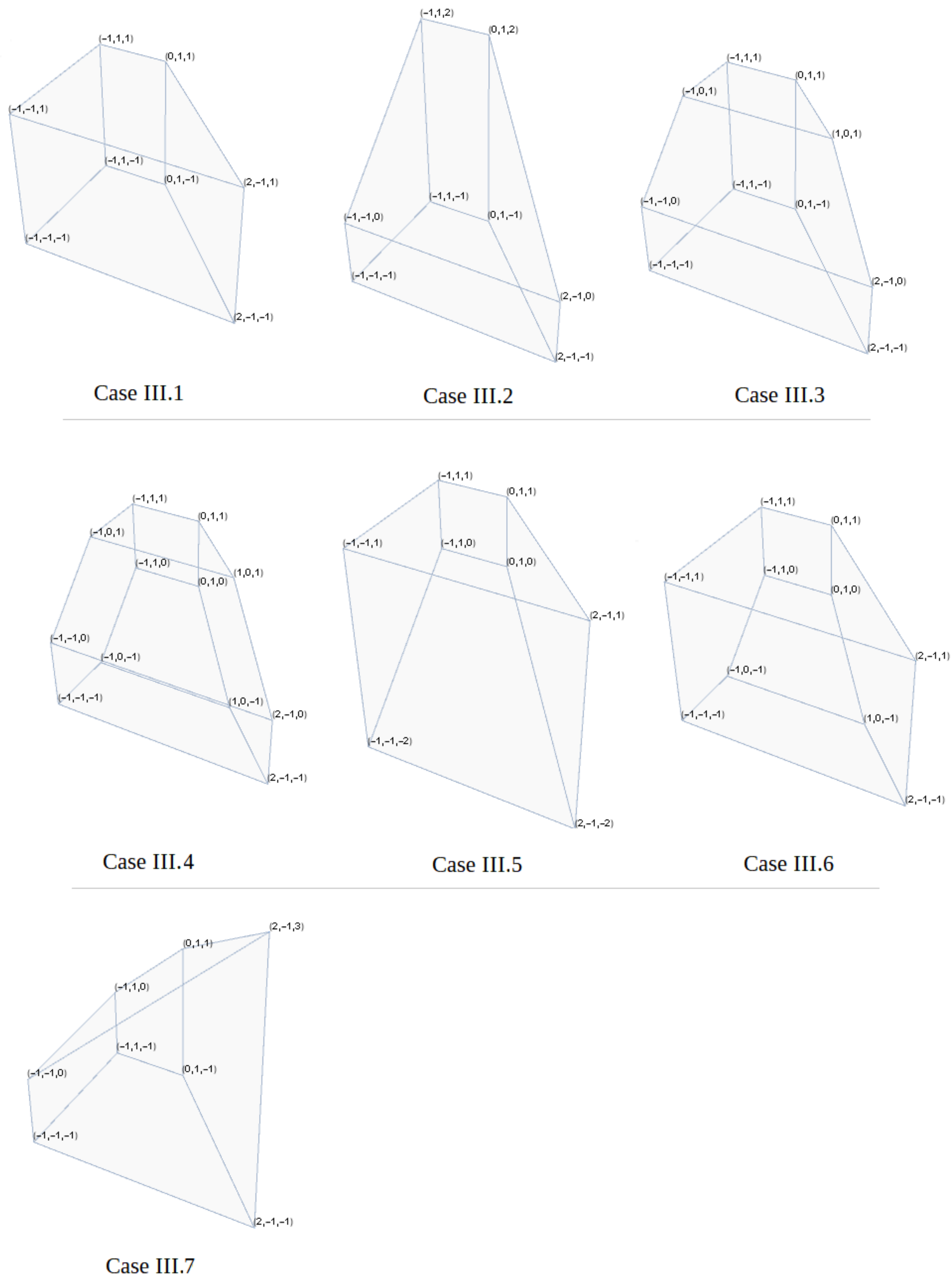


Figure 8.17: Toric extensions of P_{III}

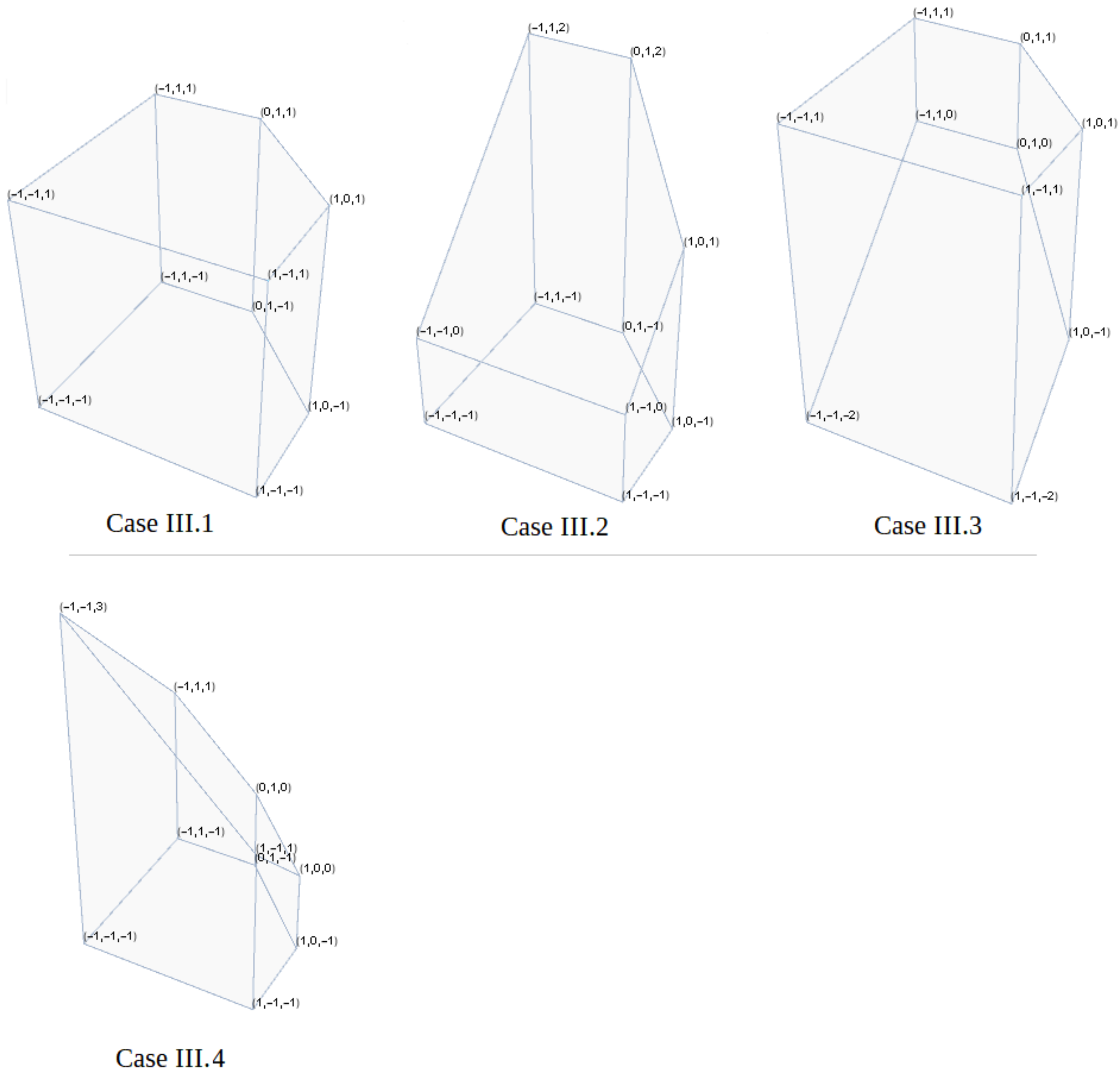


Figure 8.18: Toric extensions of P_{IV}

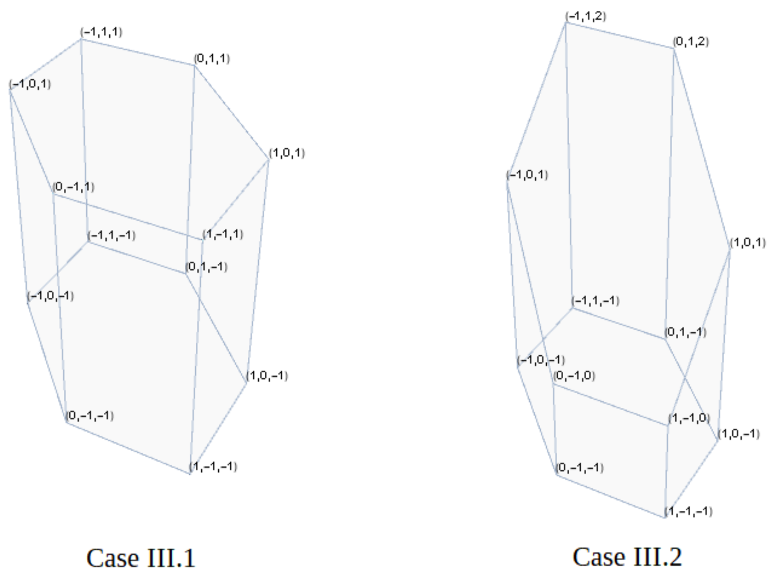


Figure 8.19: Toric extensions of P_V

Chapter A

The Fourier Transform of the Duistermaat-Heckman Measure

The main goal of this appendix is to prove Lemma 4.2 and to analyze the Duistermaat-Heckman measures of compact complexity one spaces near the vertices of their moment map polytopes. Therefore we introduce the **Fourier transform of the Duistermaat-Heckman measure** of compact Hamiltonian T-spaces. Namely, let (M, ω, T, ϕ) be a compact Hamiltonian T-space of dimension $2n$. Due to the Duistermaat-Heckman Theorem (Theorem 2.41) the Radon-Nikodym derivative of the Duistermaat-Heckman measure can be represented by an integrable function

$$f_{DH} : \mathfrak{t}^* \to \mathbb{R} \tag{A.1}$$

with respect to the Lebesgue measure $\mathrm{d}\,x^d$ on $\mathfrak{t}^*$. Recall that this means

$$\int_U f_{DH}(x)\mathrm{d}x^d = \int_{\phi^{-1}(U)} \frac{\omega^n}{n!} \tag{A.2}$$

for any Borel set $U \subset \mathfrak{t}^*$. Such a representative is well-defined up to a set of Lebesgue measure zero. In particular, such a representative can be chosen to be equal to zero on $\mathfrak{t}^* \setminus \phi(M)$. Therefore, for any continuous function $g : \mathfrak{t}^* \to \mathbb{C}$ the integral

$$\int_{\mathfrak{t}^*} g(x) \cdot f_{DH}(x)\,\mathrm{d}\,x^d \tag{A.3}$$

is well-defined and does not depend on choice of a representative of the Radon-Nikodym derivative. By these observation the following definition make sense.

Definition A.1
Let (M, ω, T, ϕ) be a compact Hamiltonian T-space of dimension $2n$ and let $f_{DH} : \mathfrak{t}^* \to \mathbb{R}$ be a representative the Radon-Nikodym derivative of the Duistermaat-Heckman measure. The **Fourier transform of the Duistermaat-Heckman measure** is the function

$$\widehat{f_{DH}} : \mathfrak{t} \to \mathbb{C}$$

given by

$$\widehat{f_{DH}}(\xi) = \int_{\mathfrak{t}^*} \mathrm{e}^{i\langle x,\xi\rangle} \cdot f_{DH}(x)\,\mathrm{d}x^d, \tag{A.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the natural pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$.

Note that even though a representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure is as function just well-defined up to a set Lebesgue measure zero, the Fourier transform of the Duistermaat-Heckman measure is a well-defined function. By using the following lemma we obtain a second expression for the Fourier transform of the Duistermaat-Heckman measure.

Lemma A.2
Let (M,ω,T,ϕ) be a compact Hamiltonian T-space of dimension $2n$ and let $f_{DH}: \mathfrak{t}^* \to \mathbb{R}$ be a representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure. Given a continuous function $g:\mathfrak{t}^* \to \mathbb{C}$, then the following equality holds

$$\int_{\mathfrak{t}^*} g(x)\cdot f_{DH}(x)\mathrm{d}x^d = \int_M (g\circ\phi)\frac{\omega^n}{n!} \tag{A.5}$$

Proof. Note that if g is a simple function, i.e. $g=\sum_{j=1}^N c_j\chi_{U_i}$, where $c_1,\ldots,c_N\in\mathbb{C}$, $U_1,\ldots,U_N$ are Borel subsets of $\mathfrak{t}^*$ and χ_{U_i} is the indicator function of U_i, then the Equation (A.5) is obviously true. Since g is continuous and M and $\phi(M)$ are compact, there exists a sequence of simple functions $\{g_i:\mathfrak{t}^*\to\mathbb{C}\}_{i\in\mathbb{N}}$ which is uniformly convergent on $\phi(M)$ with limit g and so the sequence $\{g_i\circ\phi:\mathfrak{t}^*\to\mathbb{C}\}_{i\in\mathbb{N}}$ is uniformly convergent with limit $g\circ\phi$. So we have

$$\int_{\mathfrak{t}^*} g_i(x)\cdot f_{DH}(x)\mathrm{d}x^d = \int_M (g_i\circ\phi)\frac{\omega^n}{n!} \tag{A.6}$$

and the left-hand resp. right-hand side of Equation (A.6) converges to the left-hand resp. right-hand side of Equation (A.5). Hence, the claim of this lemma follows. $\square$

A simple consequence of this lemma is that we obtain the following expression for the Fourier transform of Duistermaat-Heckman measure.

Corollary A.3
Let (M,ω,T,ϕ) be a compact Hamiltonian T-space of dimension $2n$, then the Fourier transform of the Duistermaat-Heckman measure is given by

$$\mathfrak{t}\ni\xi\longmapsto\widehat{f_{DH}}(\xi)=\int_M e^{i\langle\phi(p),\xi\rangle}\cdot\frac{\omega^n}{n!}. \tag{A.7}$$

Proof. Apply Lemma A.2 to the continuous function $\mathfrak{t}^*\ni x\mapsto \mathrm{e}^{i\langle x,\xi\rangle}$. $\square$

Note that we have two expressions for the Fourier transform of the Duistermaat-Heckman measure, namely Definition A.1 and Corollary A.3. In the following subsection we compare these expressions for compact Hamiltonian S^1-spaces.

A.1 The Duistermaat-Heckman Measure for Compact Hamiltonian S^1-spaces

Let (M, ω, S^1, ϕ) be a compact Hamiltonian S^1-space of dimension $2n$. We identify S^1 with $\mathbb{R}/\mathbb{Z}$ so that the Lie algebra is $\mathrm{Lie}(S^1) = \mathbb{R}$ and the lattice is $\ell_{S^1} = \mathbb{Z}$. We can also identify the dual Lie algebra with $\mathrm{Lie}(S^1)^*$ with $\mathbb{R}$ and the dual lattice $\ell^*_{S^1}$ with $\mathbb{Z}$. The vector field X_{S^1} on M generated by the S^1-action is given by

$$X_{S^1}(x) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\, t \cdot x, \tag{A.8}$$

where we use the identification of S^1 with $\mathbb{R}/\mathbb{Z}$. In particular, the moment map $\phi\colon M \to \mathbb{R}$ is an S^1-invariant and smooth map which satisfies

$$\iota_{X_{S^1}}\omega = -\mathrm{d}\phi. \tag{A.9}$$

By the work of Duistermaat and Heckman there exists a function $f_{DH} \in \mathcal{L}^1(\mathbb{R})$ such that for any Borel set $U \subset \mathbb{R}$

$$\mathrm{m}_{DH}(U) = \int_U f_{DH}(x)\mathrm{d}x, \tag{A.10}$$

where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}$. Moreover, this function can be chosen to be of the following form. Let

$$\phi_{\min} = C_1 < C_2 < ... < C_{k-1} < C_k = \phi_{\max} \tag{A.11}$$

be the critical values of ϕ, then

$$f_{DH}(x) = \begin{cases} f_j(x), & \text{if } x \in [C_j, C_{j+1}), \\ 0, & \text{otherwise}, \end{cases} \tag{A.12}$$

where $f_1, ..., f_{k-1}$ are polynomials of degree $\leq n-1$. We call these polynomials the **Duistermaat-Heckman polynomials**. In order to compute these polynomials we consider the Fourier transform of the Duistermaat-Heckman measure

$$\mathbb{R} \ni t \mapsto \widehat{f_{DH}}(t) = \int_{\mathbb{R}} \mathrm{e}^{itx} \cdot f_{DH}(x)\mathrm{d}x. \tag{A.13}$$

$$\tag{A.14}$$

On the one hand we can compute $\widehat{f_{DH}}$ directly by using Equation (A.12), on the other hand by Corollary A.3 we have

$$\widehat{f_{DH}}(t) = \int_{\mathbb{R}} \mathrm{e}^{itx} \cdot f_{DH}(x)\mathrm{d}x = \int_M \mathrm{e}^{it\phi(p)} \cdot \frac{\omega^n}{n!} \tag{A.15}$$

We will apply the ABBV localization formula to $\int_M \mathrm{e}^{it\phi(p)} \cdot \omega^n$. Hence, we obtain two expressions for $\widehat{f_{DH}}$. By comparing these expressions we obtain formulas for the Duistermaat-Heckman polynomials. In the next lemma we compute $\widehat{f_{DH}}$ from Equation (A.12).

Lemma A.4
Let (M, ω, S^1, ϕ) be a compact Hamiltonian S^1-space of dimension $2n$, let

$$\phi_{min} = C_1 < C_2 < ... < C_{k-1} < C_k = \phi_{max}$$

be the critical values of ϕ and let $f_1, ..., f_{k-1}$ be the corresponding Duistermaat-Heckman polynomials. Then for $t \neq 0$ the Fourier transform of the Duistermaat-Heckman measure is given by

$$\widehat{f_{DH}}(t) = \sum_{j=1}^{k} e^{iC_j t} \cdot \lambda_j(t),$$

where

$$\lambda_j(t) = \sum_{s=0}^{n-1} \left(\frac{(-1)^s}{(it)^{s+1}} \left(f_{j-1}^{(s)}(C_j) - f_j^{(s)}(C_j) \right) \right)$$

and $f_j^{(s)}$ denotes the s-th derivative of f_j for $i = 1, ..., k$.[1]

Proof. By using Equation (A.12) we obtain for $t \neq 0$ that

$$\begin{aligned} \widehat{f_{DH}}(t) &= \int_{\mathbb{R}} \mathrm{e}^{itx} \cdot f_{DH}(x)\mathrm{d}x \\ &= \sum_{j=1}^{k-1} \int_{C_j}^{C_{j+1}} \mathrm{e}^{itx} \cdot f_j(x)\mathrm{d}x \\ &= \sum_{j=1}^{k-1} \left[\mathrm{e}^{itx} \sum_{s=0}^{n-1} \left(\frac{(-1)^s}{(it)^{s+1}} f_j^{(s)}(x) \right) \right]_{x=C_j}^{x=C_{j+1}} \end{aligned}$$

For the third equation we use that the $f_1, ..., f_{n-2}$ and f_{n-1} are polynomials of degree less than or equal to $n-1$. Hence, the claim follows. □

Since we like to apply the ABBV localization formula to $\int_M \mathrm{e}^{-it\phi(p)} \cdot \omega^n$, we need to find an equivariant extension for the cohomology class in $H^{2n}(M; \mathbb{C})$ given by

[1] Here, f_0 and f_k are simply the zero function.

$\mathrm{e}^{it\phi(p)} \cdot \omega^n$. Therefore, we use the Cartan model for the equivariant cohomology (see Section 2.3). Namely, the Cartan complex $(\Omega_{S^1}(M;\mathbb{C}), \mathrm{d}_{S^1})$ of a compact Hamiltonian S^1-space is the polynomial ring in the variable X with coefficients in the ring of S^1-invariant differential forms $\Omega^{S^1}(M;\mathbb{C})$, so

$$\Omega_{S^1}(M;\mathbb{C}) = \Omega^{S^1}(M;\mathbb{C})\,[X]\,.$$

Moreover, given $\alpha \in \Omega^{S^1}(M;\mathbb{C})$ the differential d_{S^1} is defined by

$$\mathrm{d}_{S^1}(\alpha \otimes X^s) = \mathrm{d}\,\alpha \otimes X^s - \iota_{X_{S^1}}\alpha \otimes X^{s+1},$$

where $\mathrm{d}\,\alpha$ is the standard differential of α and X_{S^1} is the vector field as in (A.8).

Lemma A.5
Let (M, ω, S^1, ϕ) be a compact Hamiltonian S^1 space of dimension $2n$, then

$$\tau(t) = \sum_{s=0}^{n} \frac{(-1)^s}{(it)^s} \frac{\mathrm{e}^{it\phi}\,\omega^{n-s}}{(n-s)!} \otimes X^s \tag{A.16}$$

is for $t \neq 0$ a well-defined element in $\Omega_{S^1}(M;\mathbb{C})$ which is d_{S^1}-closed. Moreover, the cohomology class in $H^{2n}_{S^1}(M;\mathbb{C})$ given by $\tau(t)$ is an equivariant extension of the class in $H^{2n}(M;\mathbb{C})$ given by $\frac{\mathrm{e}^{it\phi(p)} \cdot \omega^n}{n!}$.

Proof. Since the symplectic form ω and the moment map ϕ are S^1-invariant, $\tau(t)$ is well-defined element in $\Omega_{S^1}(M;\mathbb{C})$ for $t \neq 0$. Next we show that $\tau(t)$ is d_{S^1}-closed. We have

$$\mathrm{d}_{S^1}(\tau(t)) = \left(\sum_{s=0}^{n} \frac{(-1)^s}{(it)^s} \frac{\mathrm{d}(\mathrm{e}^{it\phi}\,\omega^{n-s})}{(n-s)!} \otimes X^s\right) - \left(\sum_{s=0}^{n} \frac{(-1)^s}{(it)^s} \frac{\mathrm{e}^{it\phi}\,\iota_{X_{S^1}}\omega^{n-s}}{(n-s)!} \otimes X^{s+1}\right). \tag{A.17}$$

Let us consider the first term of the right-hand side of Equation (A.17). Note that $\mathrm{d}(\mathrm{e}^{it\phi}\,\omega^n) = 0$, since $\mathrm{e}^{it\phi}\,\omega^n$ is a differential form of degree $2n$. Moreover, since ω is d-closed we have

$$\mathrm{d}(\mathrm{e}^{it\phi}\,\omega^k) = it\,\mathrm{e}^{it\phi}\,\mathrm{d}\,\phi \wedge \omega^k$$

for $k = 0, ..., n-1$. Therefore, we have

$$\sum_{s=0}^{n} \frac{(-1)^s}{(it)^s} \frac{\mathrm{d}(\mathrm{e}^{it\phi}\,\omega^{n-s})}{(n-s)!} \otimes X^s = \mathrm{e}^{it\phi}\left(\sum_{s=1}^{n} \frac{(-1)^s}{(it)^{s-1}} \frac{(\mathrm{d}\,\phi \wedge \omega^{n-s})}{(n-s)!} \otimes X^s\right). \tag{A.18}$$

Now we consider the second term of the right-hand side of Equation (A.17). Note that by Equation (A.9) we have

$$\iota_{X_{S^1}}\omega^k = -k \cdot \mathrm{d}\,\phi \wedge \omega^{k-1}$$

for $k = 1, ..., n$. So we have,

$$\sum_{s=0}^{n} \frac{(-1)^s}{(it)^s} \frac{\mathrm{e}^{it\phi}\,\iota_{X_{S^1}}\omega^{n-s}}{(n-s)!} \otimes X^{s+1} = -\,\mathrm{e}^{it\phi}\left(\sum_{s=0}^{n-1} \frac{(-1)^s}{(it)^s} \frac{(\mathrm{d}\,\phi \wedge \omega^{n-s-1})}{(n-s-1)!} \otimes X^{s+1}\right) \quad \text{(A.19)}$$

$$= \mathrm{e}^{it\phi}\left(\sum_{s=1}^{n} \frac{(-1)^s}{(it)^{s-1}} \frac{(\mathrm{d}\,\phi \wedge \omega^{n-s})}{(n-s)!} \otimes X^{s}\right). \quad \text{(A.20)}$$

Form the equations (A.17), (A.18), (A.19) and (A.20) we obtain that $\mathrm{d}_{S^1}(\tau(t)) = 0$. Moreover, for $t \neq 0$ the cohomology class of $\tau(t)$ in $H^{2n}_{S^1}(M;\mathbb{Z})$ is mapped under the restriction map $r^*\colon H^{2n}_{S^1}(M;\mathbb{C}) \to H^{2n}(M;\mathbb{C})$ to the cohomology class of $\frac{\mathrm{e}^{it\phi(p)} \cdot \omega^n}{n!}$.
□

In the next lemma we apply the ABBV localization formula to the results of Lemma A.5 to obtain a second expression for the Fourier transform of the Duistermaat-Heckman measure.

Lemma A.6

Let (M, ω, S^1, ϕ) be a compact Hamiltonian S^1-space of dimension $2n$, let

$$\phi_{min} = C_1 < C_2 < ... < C_{k-1} < C_k = \phi_{max}$$

be the critical values of ϕ. For $j = 1, ..., k$ and $t \neq 0$, let

$$\gamma_j(t) = \sum_{F \subset M^{S^1} \cap \phi^{-1}(C_j)} \left(\int_F \left(\sum_{s=0}^{n} \frac{(-1)^s}{(n-s)!(it)^s} \frac{\omega^{n-s} \otimes X^s}{e^{S^1}(N_F)}\right)\right) \in H^*(BS^1, \mathbb{C}) \cong \mathbb{C}[X] \quad \text{(A.21)}$$

where the sum runs over all connected components F of M^{S^1} which are contained in $\phi^{-1}(C_j)$ and $e^{S^1}(N_F)$ is the equivariant Euler class of the normal bundle N_F to F. Then for $t \neq 0$ the Fourier transform of the Duistermaat-Heckman measure is given by

$$\widehat{f_{DH}}(t) = \sum_{j=1}^{k} e^{iC_j t} \cdot \gamma_j(t). \quad \text{(A.22)}$$

Proof. By Corollary A.3 we have

$$\widehat{f_{DH}}(t) = \int_M \frac{\mathrm{e}^{it\phi}\,\omega^n}{n!}. \tag{A.23}$$

Moreover, by Lemma A.5 the equivariant cohomology class is given by $\tau(t)$ as in (A.16) which is an equivariant extension of the cohomology class given $\frac{\mathrm{e}^{it\phi}\,\omega^n}{n!}$. So by the ABBV locatization formula (see Theorem 2.49) we have that

$$\int_M \frac{\mathrm{e}^{it\phi}\,\omega^n}{n!} = \sum_{F\subset M^{S^1}} \int_F \frac{\tau(t)}{e^{S^1}(N_F)}, \tag{A.24}$$

where the sum runs over all connected components F of M^{S^1} and $\mathrm{e}^{S^1}(N_F)$ is the equivariant Euler class of the normal bundle N_F to F. Pick a connected component F of M^{S^1} and consider the restriction of $\tau(t)$ to F. Namely, the moment map ϕ is constant on F and so $\phi(F)$ is a critiacl value of $\phi(F)$, so $\phi(F) = C_j$ for one $j = 1, \dots, k$. Hence, the restriction to F is

$$\mathrm{e}^{itC_j} \sum_{s=0}^{n} \frac{(-1)^s}{(n-s)!(it)^s} \omega^{n-s} \otimes X^s \tag{A.25}$$

and

$$\int_F \frac{\tau(t)}{e^{S^1}(N_F)} = \mathrm{e}^{itC_j} \int_F \left(\sum_{s=0}^{n} \frac{(-1)^s}{(n-s)!(it)^s} \frac{\omega^{n-s}\otimes X^s}{e^{S^1}(N_F)} \right). \tag{A.26}$$

Hence, the claim of this proposition follows from the equations (A.23), (A.24) and (A.26). □

Next we like to compare the expressions for the Fourier transform of the Duistermaat-Heckman measure as given in Lemma A.4 and Lemma A.6. Before we do so, we prove the following lemma, which helps us to compare these expressions.

Lemma A.7
Given $\alpha_1, ..., \alpha_k \in \mathbb{C}$ pairwise different and let $p_1, \dots, p_k \in \mathbb{C}[t, t^{-1}]$ be Laurent polynomials, such that

$$p_1(t)\,\mathrm{e}^{\alpha_1 t} + \cdots + p_k(t)\,\mathrm{e}^{\alpha_k t} = 0, \quad \textit{for all } t \in \mathbb{R}\setminus\{0\}. \tag{A.27}$$

Then we have $p_l = 0$ for $l = 1, \dots, k$.

Proof. First note that it is enough to prove the claim for the case that $p_1, \dots, p_k$ are polynomials because we can multiply Equation (A.27) with t^N for some $N > 1$ so that $t^N p_l$ is a polynomial for all $l = 1, ..., k$. We prove the claim for polynomials by induction over k. The claim is obviously true for $k = 1$ since $\mathrm{e}^{\alpha_1 t} \neq 0$. Suppose the claim holds for one $k \in \mathbb{N}$. We show that it also holds for $k+1$. Therefore, let $\alpha_1, ..., \alpha_{k+1} \in \mathbb{C}$ be pairwise different and let $p_1, ..., p_{k+1}$ be polynomials such that

$$p_1(t)\, \mathrm{e}^{\alpha_1 t} + ... + p_{k+1}(t)\, \mathrm{e}^{\alpha_{k+1} t} = 0 \tag{A.28}$$

holds for all $t \in \mathbb{R}$. The Equation (A.28) is equivalent to

$$p_1(t)\, \mathrm{e}^{\widetilde{\alpha_1} t} + ... + p_k(t)\, \mathrm{e}^{\widetilde{\alpha_{k+1}} t} = -p_{k+1}(t), \tag{A.29}$$

for all $t \in \mathbb{R}$, where $\widetilde{\alpha_l} = \alpha_l - \alpha_{k+1} \neq 0$. We set $m := \deg(p_{k+1}) + 1$. Since the m-th derivative of the right hand side of Equation (A.29) is equal to zero, we obtain

$$\left(\frac{\partial}{\partial t}\right)^m \left(p_1(t)\, \mathrm{e}^{\widetilde{\alpha_1} t} + ... + p_k(t)\, \mathrm{e}^{\widetilde{\alpha_{k+1}} t}\right) = 0.$$

Hence, we obtain

$$\widetilde{p_1}(t)\, \mathrm{e}^{\widetilde{\alpha_1} t} + ... + \widetilde{p_k}(t)\, \mathrm{e}^{\widetilde{\alpha_{k+1}} t} = 0, \tag{A.30}$$

where $\widetilde{p_l}(t) = \sum_{j=0}^{m} \binom{m}{j} \widetilde{\alpha_t}^j\, p_l^{(m-j)}(t)$ for all $l = 1, ..., k$. Since $\widetilde{\alpha_l} = \alpha_l - \alpha_{k+1} \neq 0$, $\widetilde{p_l}$ is polynomial with $\deg(\widetilde{p_l}) = \deg(p_l)$. By applying the induction assumption to (A.30) we have $\widetilde{p_l} = 0$ and so $\widetilde{\alpha_l}^m p_l = 0$. Therefore, we find $p_l = 0$ for all $l = 1, ..., k$ and with Equation (A.29) we have also $p_{k+1} = 0$. □

Now we are able to prove the main result of this subsection.

Proposition A.8
Let (M, ω, S^1, ϕ) be a compact Hamiltonian S^1-space of dimension $2n$, let

$$\phi_{\min} = C_1 < C_2 < ... < C_{k-1} < C_k = \phi_{\max}$$

be the critical values of ϕ and let $f_1, ..., f_{k-1}$ be the corresponding Duistermaat-Heckman polynomials. Then for all $j = 1, \ldots, k$ and all $s = 0, \ldots, n-1$, the following holds

$$f_j^{(s)}(C_j) - f_{j-1}^{(s)}(C_j) = \sum_{F \subset M^{S^1} \cap \phi^{-1}(C_j)} \int_F \frac{\omega^{n-s-1} \otimes X^{s+1}}{(n-s-1)! e^{S^1}(N_F)}, \tag{A.31}$$

where f_0 and f_k are equal to the the zero function.

Proof. Due to Lemma A.4 the Fourier transform of the Duistermaat-Heckman measure for $t \neq 0$ is $\widehat{f_{DH}}(t) = \sum_{j=1}^{k} \mathrm{e}^{iC_j t} \cdot \lambda_j(t)$, where

$$\lambda_j(t) = \sum_{s=0}^{n-1} \left(\frac{(-1)^s}{(it)^{s+1}} \left(f_{j-1}^{(s)}(C_j) - f_j^{(s)}(C_j) \right) \right).$$

By Lemma A.6 we also have $\widehat{f_{DH}}(t) = \sum_{j=1}^{k} \mathrm{e}^{iC_j t} \cdot \gamma_j(t)$, where

$$\gamma_j(t) = \sum_{F \subset M^{S^1} \cap \phi^{-1}(C_j)} \left(\int_F \left(\sum_{s=0}^{n} \frac{(-1)^s}{(n-s)!(it)^s} \frac{\omega^{n-s} \otimes X^s}{e^{S^1}(N_F)} \right) \right).$$

Since λ_j and γ_j are Laurent polynomials, by Lemma A.7 we have $\lambda_j(t) = \gamma_j(t)$ for all $j = 1, \ldots, k$ and $t \neq 0$. But the latter implies that Equation (A.31) holds for all $j = 1, \ldots, k-1$ and $s = 0, \ldots, n-1$. □

A.2 The Proof of Lemma 4.2

In this subsection we prove Lemma 4.2. Therefore, in this subsection (M, ω, S^1, ϕ) is always a compact Hamiltonian S^1-space of dimension four, and

$$t_{\min} = C_1 < C_2 < \ldots < C_{k-1} < C_k = t_{\max}$$

are the critical values of ϕ and $f_1, \ldots, f_{k-1}$ are the corresponding Duistermaat-Heckman polynomials. First we compute f_1 and f_{k-1}. We define $a_{\min}, c_{\min}$ and $a_{\max}, c_{\min}$ as in Chapter 4.

Lemma A.9
The Duistermaat-Heckman polynomials f_1 and f_{k-1} are given as follows.

$$f_1(t) = a_{\min} - c_{\min}(t - t_{\min}) \quad \textit{and} \tag{A.32}$$

$$f_{k-1}(t) = a_{\max} + c_{\max}(t - t_{\max}). \tag{A.33}$$

Proof. We set $B_{\min} = \phi^{-1}(t_{\min})$, which is a connected component of M^{S^1}. By Proposition A.8 we have

$$f_1(t_{\min}) = \int_{B_{\min}} \frac{\omega \otimes X}{e^{S^1}(N_{B_{\min}})} \quad \text{and} \tag{A.34}$$

$$f_1'(t_{\min}) = \int_{B_{\min}} \frac{1 \otimes X^2}{e^{S^1}(N_{B_{\min}})}, \tag{A.35}$$

where $e^{S^1}(N_{B_{\min}})$ is the equivariant Euler class of the normal bundle of $B_{\min}$ in M. Note that $B_{\min}$ is an isolated fixed point or a fixed surface. In the following we consider these two cases.

- If $B_{\min}$ is an isolated fixed point, then $\omega|_{B_{\min}}$ is equal to zero for dimensional reasons. Hence, by Equation (A.34) we have $f_1(t_{\min}) = 0 = a_{\min}$. Moreover, the equivariant Euler class is $e^{S^1}(N_{B_{\min}}) = mnX^2$, where m, n are the isotropic weights at $B_{\min}$. Therefore, we have

$$f_1'(t_{\min}) = \int_{B_{\min}} \frac{1 \otimes X^2}{e^{S^1}(N_{B_{\min}})} = \frac{1}{mn} = -c_{\min}. \tag{A.36}$$

- If $B_{\min}$ is a fixed surface, then the equivariant Euler class is $e^{S^1}(N_{B_{\min}}) = c_1(N_{Min}) + X$, so its formal inverse is $\frac{1}{X^2}(X - c_1(N_{Min}))$. Therefore,

$$f_1(t_{\min}) = \int_{B_{\min}} \frac{\omega \otimes X}{e^{S^1}(N_{B_{\min}})} = \int_{B_{\min}} \omega = a_{\min}\,, \quad \text{and} \tag{A.37}$$

$$f_1'(t_{\min}) = \int_{B_{\min}} \frac{1 \otimes X^2}{e^{S^1}(N_{B_{\min}})} \int_{B_{\min}} c_1(N_{Min}) = -c_{\min}\,. \tag{A.38}$$

In both cases we have $f_1(t_{\min}) = a_{\min}$ and $f_1'(t_{\min}) = -c_{\min}$. Since f_1 is a polynomial of degree zero or one, the claim of this lemma for f_1 follows. With the same argumentation the claim of this proposition is true for f_{k-1}. □

Lemma A.10
Suppose that ϕ has at least three critical values, i.e. $k \geq 3$. Then for all $j = 1, \ldots, k-2$ the following holds

$$f_{j+1}(t) = f_j(t) + \sum_{p \in M^{S^1} \cap \phi^{-1}(C_j)} \frac{1}{n_p m_p}(t - C_{j+1}), \tag{A.39}$$

where the sum runs over all isolated fixed points which are contained in $\phi^{-1}(C_{j+1})$ and n_p and m_p are the weights at $p \in \phi^{-1}(C_{j+1})$.

Proof. By Proposition A.8 we have

$$f_{j+1}(C_{j+1}) - f_j(C_{j+1}) = \sum_{F \subset M^{S^1} \cap \phi^{-1}(C_{j+1})} \int_F \frac{\omega \otimes X}{e^{S^1}(N_F)} \quad \text{and} \tag{A.40}$$

$$f_{j+1}'(C_{j+1}) - f_j'(C_{j+1}) = \sum_{F \subset M^{S^1} \cap \phi^{-1}(C_{j+1})} \int_F \frac{1 \otimes X^2}{e^{S^1}(N_F)}\,, \tag{A.41}$$

where the sum runs over all connected components of $M^{S^1} \cap \phi^{-1}(C_{j+1})$. Since C_{j+1} lies in the interior of the moment map image, all connected components of

$M^{S^1} \cap \phi^{-1}(C_{j+1})$ are isolated fixed points. So from Equation (A.40) it follows that $f_{j+1}(C_{j+1}) = f_j(C_{j+1})$. And by Equation (A.41), we have

$$f'_{j+1}(C_{j+1}) = f'_j(C_{j+1}) + \sum_{p \in M^{S^1} \cap \phi^{-1}(C_j)} \frac{1}{n_p m_p}. \tag{A.42}$$

Since f_{j+1} and f_j are polynomials of degree less than or equal to one, the claim of this lemma follows. □

Now we are able to prove Lemma 4.2.

Proof of Lemma 4.2. Due to the Duistermaat-Heckman Theorem (see Theorem 2.41) a representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure is given by

$$f_{DH}(x) = \begin{cases} f_j(x), & \text{if } x \in [C_j, C_{j+1}), \\ 0, & \text{otherwise.} \end{cases} \tag{A.43}$$

By Lemma A.9 and Lemma A.10 it follows that this function is equal to the one as in the statement of Lemma 4.2 beyond the single point $t_{\max}$. This completes the proof of this lemma.

□

A.3 The Duistermaat-Heckman Measure Near Vertices

In this section we describe the Duistermaat-Heckman measures for compact and tall complexity one spaces near the vertices of their moment map polytopes. This is the content of Lemma A.11.

Lemma A.11
Let (M, ω, T, ϕ) be a compact complexity one space of dimension $2n$ and let Σ be a fixed surface and let $\alpha_{\Sigma,1}, \ldots \alpha_{\Sigma,n-1}$ be the non-zero weights along Σ. Moreover, consider the edges $e_i = \{v + \mathbb{R}_{\geq 0}\alpha_{\Sigma,i}\} \cap \phi(M)$ of $\phi(M)$ meeting at $v = \phi(\Sigma)$ and let $c_1(N_i)$ be the first Chern class of the normal bundle of Σ inside $\phi^{-1}(e_i)$. Then there exists a representative of the Radon-Nikodym derivative of the Duistermaat-Heckman measure which satisfies

$$f_{DH}(v + t_1\alpha_{\Sigma,1} + \cdots + t_{n-1}\alpha_{\Sigma,n-1}) = -\sum_{i=1}^{n-1} t_i c_1(N_i)\,[\Sigma] \;+\; \int_\Sigma \omega \tag{A.44}$$

for $t_1, \ldots t_{n-1} \geq 0$ sufficiently small.

Proof. Note that the claim of this proposition holds for $n = 2$ by Lemma 4.2. For $n > 2$ the claim of this lemma follows from using Theorem 2.18 and that the statement is true for $n = 2$. $\square$

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