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Integer Multicommodity Flows with Reduced Demands

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Abstract

Given a supply graph $G = (V, E)$, a demand graph $H = (T, D)$, edge capacities $c : E \mapsto \mathbb{N}$ and requests $r : D \mapsto \mathbb{N}$, we consider the problem of finding integer multiflows subject to c, r. Korach and Penn constructed approximate integer multiflows for planar graphs, but no results were known for the general case. Via derandomization we present a polynomial-time approximation algorithm. There are two cases:

a) The main result is: For fractional solvable instances (G, H, c, r) and each $0 < \epsilon \leq \frac{9}{10}$ our algorithm finds in polynomial-time an integer multiflow subject to c, such that for all $d \in D$ the $d-$th flow value satisfies $f(d) \geq (1 - \epsilon)r(d)$, provided that capacities and requests are not too small, i.e $c, r = \Omega(\frac{1}{\epsilon^2}\log(|E| + |D|))$. In particular, if $c, r \geq 36\lceil\log 2(|E| + |D| + 1)\rceil$ we have a strongly polynomial-time algorithm and the first $\frac{1}{2}-$factor approximation.

b) If the problem is not fractionally solvable we can reduce it to the case mentioned above decreasing the requests in an optimal way. This can be done by linear programming and the results of a) apply.

The design and analysis of the algorithm require new techniques for randomized rounding as well as for derandomization. One key tool is an *algorithmic* version of the classical Angluin-Valiant inequality (a variant of the well known Chernoff-Hoeffding bound) estimating the tail of *weighted* sums of Bernoulli trials, which was not previously known and might be of independent interest in computational probability theory.
The significance of our rounding algorithm is emphasized by the fact that there is a rich combinatorial theory exhibiting many examples of fractionally solvable problems, but finding approximate integer solutions even for fractionally solvable problems is NP-hard as it is shown in this paper.

Keywords: randomized algorithms, derandomization, integer programming, multicommodity flows.

Classification: 60C05, 60E15, 68Q25, 90C35, 05C85, 68R10, 90C35.

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1 Introduction

(a) The Problem:
According to [4] the feasibility multicommodity flow problem is stated as follows: Let $G = (V, E)$ be the supply graph and $H = (T, D)$ be the demand graph with $T \subseteq V$. The vertices of H are the terminals and the edges $(q_1, s_1), \ldots, (q_k, s_k)$ of H are called commodities or demand edges. For each demand edge $d = (q_d, s_d) \in D$ let σ_d be an orientation of G forming the directed graph (V, A_d) and let $F(d)$ be an integer $(q_d, s_d)-$flow in (V, A_d). Then the $|D|$-tuple of flows $F = (F(d))_{d \in D}$ is called a *multicommodity flow.*

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Given a capacity function $c : E \mapsto \mathbb{N}$ and a demand (or request) function $r : D \mapsto \mathbb{N}$ the multicommodity flow is said to be subject to c, if for each edge $e \in E$ the sum of the flows through e (in both directions) does not exeed $c(e)$ and is subject to r if for each demand edge $d \in D$ the $d-$th flow value $f(d)$ is at least $r(d)$. Let henceforth denote (G, H, c, r) an instance of the multicommodity flow problem.

Finding integer multicommodity flows subject to c and r is well-known to be NP-hard [4]. But Korach and Penn [6] found an interesting integer approximate solution of the multicommdity flow problem for planar graphs: If G and $G \cup H$ are planar and if the cut condition holds, then the multicommodity flow problem $(G, H, c, r-1)$, where each request has been reduced by one unit, can be solved in polynomial-time. In fact they proved a stronger result but stated in this form the Korach/Penn result motivates the following interesting approximation problem.

(1.1) Reduced Demand Multiflow Problem

Let (G, H, c, r) be a multicommodity flow problem. Find a rational non-negative function $K : D \mapsto \mathbb{Q}$ with minimum $\sum_{d \in D} K(d)$ such that the reduced problem $(G, H, c, r-K)$ can be integrally solved.

Let K_I denote the minimal (in the sense above) such K and let denote K_R the minimal function, if we allow fractional solvability. In other words, we ask for the best possible approximate integer multiflow.

(b) The Results:

Since (1.1) can be formulated as an integer linear program, its linear programming relaxation can be solved in polynomial-time and gives an efficiently computable fractional "lower" bound K_R on K_I. But we show in section 4 that the decision version of (1.1) is NP-complete, even if (G, H, c, r) is fractionally solvable. In contrast to this we exhibit a large class of non-planar instances of the problem, where surprisingly good approximate integer flows can be constructed in polynomial-time:

Theorem *Let (G, H, c, r) be a multicommodity flow problem. Let $0 < \epsilon < \frac{9}{10}$ and suppose that $c(e), r(d) - K_R(d) \geq \frac{6(2-\epsilon)}{\epsilon^2} \lceil \log(2(|E| + |D| + 1)) \rceil$ for all $e \in E$ and $d \in D$. Then an integer multiflow F can be found in polynomial-time such that $f(d) \geq (1-\epsilon)(r(d) - K_R(d))$ for all demand edges $d \in D$.* □

For *fractionally solvable* problems this gives for instances with $c, r \geq \frac{6(2-\epsilon)}{\epsilon^2} \lceil \log(2(|E|+|D|+1)) \rceil$ the estimate $f(d) \geq (1-\epsilon) r(d)$ for all demands d and in particular a $\frac{1}{2}$-factor approximation:

Corollary *Let (G, H, c, r) be a fractionally solvable multicommodity flow problem. Suppose that $c(e), r(d) \geq 36 \lceil \log(2(|E| + |D| + 1)) \rceil$ for all $e \in E$ and $d \in D$. Then an integer flow F can be found in strongly polynomial-time such that $f(d) \geq \frac{r(d)}{2}$ for all demand edges $d \in D$.* □

The dominating part of running time is the time to solve the corresponding linear programming relaxation.

In general, this seems to be the best possible approximation factor. We can construct for each integer $K \in \mathbb{N}$ a non-planar instance for which the fractional problem is solvable, but the integer problem is not solvable even if we decrease each request value by K. In particular, this example shows $K_I(d) \geq \frac{1}{2}(r(d) - 1)$ for all demand edges $d \in D$.

Note that in the *RAM*-model of computation the theorem above cannot not be proved by the cosh-algorithm of Spencer [1] or the basic pessimistic estimator method of Raghavan [12].

(c) The Methods:

In the design and analysis of our algorithms we wish to use randomized rounding and afterwards derandomization. But here neither randomized rounding nor derandomization are directly applicable: Randomized rounding as proposed by Raghavan and Thompson [11] operates on instances where fractional solutions from the *unit interval* are given. Here the fractional flows can take arbitrary rational values. Representing the fractional flows in terms of directed paths P with associated path values $\lambda(P)$, the direct approach would be to cut off the integer part $\lfloor \lambda(P) \rfloor$ and round the remaining fractional value to 0 or 1. Consequently we must now consider the decreased edge capacities $\tilde{c}(e) = c(e) - \sum_{e \in P} \lfloor \lambda(P) \rfloor$. Unfortunatly, due to the Angluin-Valiant inequality randomized rounding can be analysed only for packing problems, where enough packing space is available. In our problem we must assume $\tilde{c}(e) = \Omega(\log(|E| + |D|)$. So even if $c(e) = \Omega(\log(|E| + |D|)$, it may happen that $\tilde{c}(e)$ is too small and the method fails. An other "solution" would be to split of each P into $2\lfloor\lambda\rfloor + 1$ parallel paths with values from $(0, 1)$, which also reduces the problem to the 0-1 case, but for the prize that we would increase the number of variables in the randomized rounding procedure by the maximal path value. In consequence the complexity of the rounding procedure would depend on the magnitude of numbers appearing in the input and this would not be a polynomial-time procedure anymore. We show that we have to introduce for every $e \in E$ and every $d \in D$ at most $O(\epsilon^{-2}\log(|E| + |D|))$ $0 - 1$ random variables.

Furthermore the derandomiztion method of pessimistic estimators as introduced by Raghavan [12] for approximating packing integer programs does not give for problems with *rational* weights a polynomial-time algorithm on usual models of computation, for example the RAM model. Unfortunately, the calculation of pessimistic estimators in our algorithm requires exponentiation of rational numbers to the power of rational numbers. There is no obvious way to avoid such numerical problems. We solve this problem extending the derandomization method of conditional probabilities in an interesting way. We introduce the concept of so called *approximate* pessimistic estimators, which are low degree polynomials in polynomial-time computable rational numbers and prove an algorithmic version of the Angluin-Valiant inequality. This enables us to find events concentrated around the mean of *weighted* linear sums of Bernoulli trials in polynomial-time.

(d) Related Work

Raghavan [12] investigated the problem of finding maximal 0-1 multiflows. If $c = \Omega(\log |E|)$ he constructed an integral flow with total sum M satisfying $M \geq \gamma M_{opt}(1-D)$, where M_{opt} is the integer maximum, $\gamma \geq \frac{1}{2}$ a constant and D a function depending on $|E|$, γ and M_{opt}. If $c = \Omega(\log |E|)$ Raghavan showed that D is constant, hence proved an *implicit* constant factor. We can prove by the methods developed in this paper, especially the algorithmic Angluin-Valiant inequality, a $\frac{1}{2}$-factor approximation of M_{opt}, provided that $c = \Omega(\log |E|)$. This removes the $\gamma D M_{opt}$ term and shows an *explicit* constant factor. In fact, for $0 < \epsilon \leq \frac{9}{10}$ and $c = \Omega(\frac{1}{\epsilon^2}\log |E|)$ we have $M \geq (1-\epsilon)M_{opt}$ ([15]).

Furthermore the algorithmic version of the *weighted* Angluin-Valiant inequality opens a way to solve *weighted* packing integer programs without the previous restriction to 0-1 cases ([15]).

2 Randomized Flow Generation

For each commodity $d \in D$ and each edge $\{u, v\} \in E$ let us introduce integer variables x^d_{uv} and x^d_{vu}, where x^d_{uv} represents the flow value of the commodity d on edge $\{uv\}$ in direction from u to v and vice versa. The reduced demand multiflow problem (1.1) is equivalent to the following integer linear program:

(2.1) Multicommodity Flow with Reduced Demands as an Integer Linear Program

$$\min \textstyle\sum_{d \in D} K(d)$$

such that:

$$\begin{array}{rcll} \sum\limits_{\{v \in V : \{q_d, v\} \in E\}} x^d_{q_d v} - x^d_{v q_d} & \geq & r(q_d, s_d) - K(q_d, s_d) & (\forall d \in D) \\ \sum_{d \in D} x^d_{uv} + x^d_{vu} & \leq & c(\{u, v\}) & (\forall \{u, v\} \in E) \\ \sum\limits_{\{v \in V - \{q_d, s_d\} : \{u,v\} \in E\}} x^d_{uv} & = & \sum\limits_{\{v \in V - \{q_d, s_d\} : \{u,v\} \in E\}} x^d_{vu} & (\forall d \in D, u \in V - \{q_d, s_d\}) \end{array}$$

Let us denote by (MLP) the fractional relaxation, where the flows x^d_{uv} are rational numbers. The fractional solution K_R together with the coresponding flows can be constructed in polynomial-time with standard LP-algorithms and of course $\sum K_R(d) \leq \sum K_I(d)$ (see [4]).

In the following we use the reformulation of the multicommodity flow problem in terms of *directed paths*. Let $\Gamma = \{P_1, \ldots, P_s\}$ be the set of paths defined as follows: Each path $P \in \Gamma$ is a $(q_d, s_d)-$ path for a commodity $d = (q_s, s_d) \in D$ and can be extended to a circle in $G \cup H$ adding the demand edge (q_d, s_d). Each path $P \in \Gamma$ is associated to a nonnegative integer (in case of fractional flows to a rational number) $\lambda(P)$. The value of the flow for a commodity d is equal to the sum of those $\lambda(P)$ for which P is a (q_d, s_d)-path.

Given edge capacities c and a request function r the multicommodity flow is subject to c, if for each edge $e \in E$ the sum of the $\lambda(P)$ for which P contains e is at most $c(e)$ and it is subject to r, if for each demand edge $d = (q_d, s_d)$ the sum of those $\lambda(P)$, for which P is a $(q_d, s_d)-$path is at least $r(d)$.

Having solved the LP-relaxation of (2.1), we represent the fractional multicommodity flow by directed paths following Malhotra et al. [7]. Raghavan and Thompson [11] used the same idea for randomly *maximizing* multicommodity flow. The idea of the algorithm is very simple: For every commodity d we assign a direction to every edge. Then we try to find a directed path starting at q_d, ending in s_d, such that every edge on the path has a strictly positive weight. We calculate the minimum edge weight on this path. This minimum value is substracted from every edge weight in this path and will be assigned to the path. Edges with zero weight will be deleted and we try to find the next path. After finding at most $|E|$ paths for every commodity the procedure terminates.

In the following let Γ_d be the set of paths representing the commodity d, and let $\Gamma = \bigcup_{d \in D} \Gamma_d$.

(2.2) Path Generation Algorithm (GENPATH)

INPUT: A fractional optimal multicommodity flow solving (2.1) and the function $K_R : D \mapsto \mathbb{Q}$.
OUTPUT: For each demand $d \in D$ a set Γ_d, path values $\lambda(P)$ for each path $P \in \Gamma_d$, the set $\Gamma = \bigcup_d \Gamma_d$.

```
begin
for each d in D do
    {* Form a directed graph G_d where E_d is a set of directed edges
       from E as follows:                                           *}
  Let d = (q_d, s_d), set Γ_d := ∅, E_d := ∅.
  while there is v ∈ V with x_{q_d,v} > 0 do
    for each e in E do
      {* assign a direction to e:                                   *}
      let e = {u, v}:
      if x^d_{uv} = x^d_{vu} then next e ∈ E.
      if x^d_{uv} > x^d_{vu} then
        direction(e) = (u, v), x^d(e) = x^d_{uv} − x^d_{vu}.
      if x^d_{uv} < x^d_{vu} then
        direction(e) = (v, u), x^d(e) = x^d_{vu} − x^d_{uv}.
      E_d := E_d ∪ {e}.
    end for
    Discover a directed path P = {q_d, ..., s_d} in G_d
    using depthfirst search discarding loops.
    Set λ(P) = min{x^d(e(j)), 1 ≤ j ≤ p}.
    for 1 ≤ j ≤ p,
      x^d(e_j) := x^d(e_j) − λ(P), Γ_d := Γ_d ∪ {P}.
    for each e in E_d do
      if x^d(e) = 0 then E_d := E_d\{e}
  end while
end for
end
```

It is clear, that the **while** loop is executed at most $|E|$ times for every demand, as there is always at least one edge which is excluded from E_d. Thus, the algorithm will find a representation of the fractional multicommodity flow with at most $|D||E|$ paths. If we reduce every path value λ by its fractional part $(\lambda - \lfloor\lambda\rfloor)$ we obtain an integer solution where every path value has been reduced by at most 1. So if (G, H, c, r) is fractionally solvable, $(G, H, c, r - |E|)$ trivially has an integer solution.

A simple randomized procedure to turn the fractional path values into integer ones is to flip for each path a biased coin independently deciding whether $\lambda(P)$ should be truncated to $\lfloor\lambda(P)\rfloor$ or rounded up to $\lceil\lambda(P)\rceil$. As mentioned in the introduction such roundings cannot be analysed by the probabilistic methods given so far. An intuitive better idea is to perform a more flexible rounding procedure in which by chance some rounded path values could become much bigger or smaller than $\lceil\lambda(P)\rceil$. One extreme way to do so is to split each path value $\lambda(P)$ into $2\lfloor\lambda(P)\rfloor$ "path segments" of value 0.5 and one segment of value $\lambda(P)-\lfloor\lambda(P)\rfloor$. Then rounding the value of the segments to 0 or 1 randomly with probabilities equal to the segment values the expected total path value is $\lambda(P)$. But this is not a polynomial-time rounding algorithm as the number of trials depends on $r(d)$. Our strategy is to split off each path value into a fixed integer part and a sufficiently big roundable part of size $\Omega(\frac{1}{\epsilon^2}\log(|E|+|D|))$. The following algorithm shows that for each $e\in E$ and $d\in D$ we must introduce at most $O(\frac{1}{\epsilon^2}\log(|E|+|D|))$ 0-1 random variables.

(2.3) Path Splitting Algorithm SPLITPATH(ϵ)
INPUT: The set of directed paths Γ, associated path values $\lambda(P)$, $P\in\Gamma$, and a rational number $0<\epsilon\leq 1$.
OUTPUT: New path values
$\lambda(P),\lambda_0(P),\ldots,\lambda_{N(P)}(P)$ for each $P\in\Gamma$.

begin
For each $P\in\Gamma$ set
  $\lambda_0(P):=\lambda(P)-\lfloor\lambda(P)\rfloor$,
  $\lambda(P)=\lambda(P)-\lambda_0(P)$ and $N(P):=0$.
(a) Set $r_\epsilon=\frac{6(2-\epsilon)}{\epsilon^2}\lceil\log(2(|E|+|D|+1))\rceil$.
for each d **in** D **do**
  while $r(d)-K_R(d)-\sum_{P\in\Gamma_d}\lambda(P)<r_\epsilon$ **do**
    choose $P\in\Gamma_d$ with $\lfloor\lambda(P)\rfloor\geq 1$.
    set $\lambda(P)=\lambda(P)-1$,
    $\lambda_{N(P)+1}(P)=\lambda_{N(P)+2}(P)=0.5$,
    $N(P)=N(P)+2$.
  end while
end for.
(b) Set $c_\epsilon=\frac{6(2-\epsilon)}{\epsilon^2}\lceil\log(2(|E|+|D|+1))\rceil$.
for each e **in** E **do**
  while $c(e)-\sum_{e\in P\in\Gamma}\lambda(P)<c_\epsilon$ **do**
    choose $d\in D$ and $P\in\Gamma_d$
      with $e\in P$ and $\lceil\lambda(P)\rceil\geq 1$.
    set $\lambda(P)=\lambda(P)-1$,
    $\lambda_{N(P)+1}(P)=\lambda_{N(P)+2}(P)=0.5$
    $N(P)=N(P)+2$.
  end while
end for.
end

As a result of the algorithm SPLITPATH(ϵ) we have a representation of the multicommodity flow with at most $O(|E||D|\epsilon^{-2})$ path values and the original $\lambda(P)$ has been decreased such that for every $e\in E$ and $d\in D$

$$c(e)-\sum_{e\in P\in\Gamma}\lambda(P)\geq c_\epsilon$$

and

$$r(d)-K_R(d)-\sum_{P\in\Gamma_d}\lambda(P)\geq r_\epsilon. \tag{1}$$

We are ready to perform randomized rounding.
(2.4) Randomized Integer Flow Generation R-FLOW(ϵ)

Let $0 < \epsilon \le 1$. Let $P \in \Gamma$ and $\lambda(P)$ and $\lambda_i(P)$ $(i = 0, 1, \ldots, N(P))$ generated by SPLITPATH(ϵ). For every path $P \in \Gamma$ and every $i = 0, 1, \ldots, N(P)$ do independently

1. Set
   - $\lambda_i^I(P) = 1$ with probability $(1 - \frac{\epsilon}{2})\lambda_i(P)$.
   - $\lambda_i^I(P) = 0$ with probability $1 - (1 - \frac{\epsilon}{2})\lambda_i(P)$.
2. For each $P \in \Gamma$ set $\lambda^I(P) = \lambda(P) + \sum_{i=0}^{N(P)} \lambda_i^I(P)$.

Algorithm R-FLOW(ϵ) outputs for each path an integer path value. We proceed to the analysis of such an integer multiflow. In Lemma (2.2) we show that the flow is feasible with respect to c and in Lemma (2.3) we prove that it conveys enough commodities. We invoke the Angluin-Valiant inequality in order to estimate deviation of sums of weighted Bernoulli trials from their mean. McDiarmid's proof of the Angluin-Valiant inequality ([8] , proof of corollary 5.1 and 5.2) gives also the following "conditional probability" formulation:

Lemma 2.1 *Let $a_1, \ldots, a_n$ be real numbers with $0 \le a_j \le 1$ for each j and let $\psi_1, \ldots, \psi_n$ be independent $0-1$ valued random variables. Let $\tilde{p}_j = E(\psi_j)$,*
$\tilde{q}_j = 1 - \tilde{p}_j$, $\psi = \sum_{j=1}^n a_j \psi_j, p = \frac{1}{n}E(\psi), q = 1 - p$ and $0 < \beta < 1$.
Define $s^+ = \frac{q(1+\beta)}{q-p\beta}$, $s^- = \frac{q+p\beta}{q(1-\beta)}$ and for $1 \le l \le n$ let $x_1, \ldots, x_l \in \{0, 1\}$.
Then we have

$$\begin{aligned}
&(i)Prob(\psi > (1+\beta)np | \psi_1 = x_1, \ldots, \psi_l = x_l) \\
&\le \quad e^{-(1+\beta)pn \ln s^+} e^{\sum_{j=1}^l a_j x_j \ln s^+} \prod_{j=l+1}^{n} [\tilde{p}_j e^{a_j \ln s^+} + 1 - \tilde{p}_j] \\
&\le \quad e^{-\frac{\beta^2 np}{3}}.
\end{aligned}$$

$$\begin{aligned}
&(ii)Prob(\psi < (1-\beta)np | \psi_1 = x_1, \ldots, \psi_l = x_l) \\
&\le \quad e^{-(1-\beta)pn \ln s^-} e^{-\sum_{j=1}^l a_j x_j \ln s^-} \prod_{j=l+1}^{n} [\tilde{p}_j e^{-a_j \ln s^-} + 1 - \tilde{p}_j] \\
&\le \quad e^{-\frac{\beta^2 np}{2}}.
\end{aligned}$$

□

Lemma 2.2 *Let $0 < \epsilon \le 1$. Suppose that $c(e) \ge \frac{6(2-\epsilon)}{\epsilon^2}\lceil \log(2(|E| + |D| + 1)) \rceil$ for all $e \in E$. Then with probability at least $1 - \frac{|E|}{2(|E|+|D|+1)}$ R-FLOW(ϵ) finds for each $P \in \Gamma$ an integer path value $\lambda^I(P)$ such that for all $e \in E$*

$$\sum_{e \in P \in \Gamma} \lambda^I(P) \le c(e).$$

Proof. Since we are rounding only a part of the path values, we have to consider only decreased edge capacities $\tilde{c}(e)$ defined by

$$\tilde{c}(e) := c(e) - \sum_{e \in P \in \Gamma} \lambda(P). \tag{2}$$

Define for every edge $e \in E$ the random variable

$$\chi(e) = \sum_{e \in P \in \Gamma} \sum_{i=1}^{N(P)} \lambda_i^I(P),$$

Then a straight forward calculation shows

$$\mathbb{E}(\chi(e)) \leq (1 - \frac{\epsilon}{2})\tilde{c}(e). \tag{3}$$

Taking $\beta := \frac{\epsilon}{2-\epsilon}$ we have $0 < \beta \leq 1$ and with (1)

$$\frac{6(2-\epsilon)}{\epsilon^2} \ln 2(|E| + |D| + 1) \leq \tilde{c}(e). \tag{4}$$

The Angluin-Valiant inequality (Lemma 2.1 (i)), (3) and (4) imply

$$\begin{aligned} \mathbb{P}(\chi(e) > \tilde{c}(e)) &= \mathbb{P}(\chi(e) > (1+\beta)(1-\frac{\epsilon}{2})\tilde{c}(e)) \\ &\leq \frac{1}{2(|E| + |D| + 1)}. \end{aligned} \tag{5}$$

This together with (2) implies for all $e \in E$ with probability at least $1 - \frac{|E|}{2(|E|+|D|+1)}$

$$\begin{aligned} \sum_{e \in P \in \Gamma} \lambda^I(P) &= \sum_{e \in P \in \Gamma} \lambda(P) + \chi(e) \\ &\leq \sum_{e \in P \in \Gamma} \lambda(P) + \tilde{c}(e) \;=\; c(e). \end{aligned} \tag{6}$$

□

In the next lemma we estimate the rounded flows.

Lemma 2.3 *Let $0 < \epsilon \leq 1$. Suppose that $r(d) - K_R(d) \geq \frac{6(2-\epsilon)}{\epsilon^2}\lceil \log(2(|E| + |D| + 1)) \rceil$ for all $d \in D$. Then with probability at least $1 - \frac{|D|}{2(|E|+|D|+1)}$ R-FLOW(ϵ) finds for each $P \in \Gamma$ an integer path value $\lambda^I(P)$ such that for all $d \in D$ we have*

$$\sum_{P \in \Gamma_d} \lambda^I(P) \geq (1-\epsilon)(r(d) - K_R(d))$$

Proof. Define the reduced request $\tilde{r}(d)$ by

$$\tilde{r}(d) := r(d) - K_R - \sum_{P \in \Gamma_d} \lambda(P). \tag{7}$$

Define for every edge $e \in D$ the random variable

$$\chi(d) = \sum_{P \in \Gamma_d} \sum_{i=1}^{N(P)} \lambda_i^I(P),$$

Then

$$\mathbb{E}(\chi(d)) = (1 - \frac{\epsilon}{2}) \sum_{P \in \Gamma_d} \sum_{i=1}^{N(P)} \lambda_i(P) = (1 - \frac{\epsilon}{2})\tilde{r}(d). \tag{8}$$

Put $\gamma := \sqrt{\frac{6 \ln(|E|+|D|+1)}{(2-\epsilon)\tilde{r}(d)}}$. Then by (1)

$$6\lceil \ln(2(|E| + |D| + 1)) \rceil \leq \tilde{r}(d). \tag{9}$$

which implies $0 < \gamma \leq 1$. With the Angluin-Valiant inequality (Lemma 2.1 (ii)), (8), (9) it is not hard to prove

$$\begin{aligned} \mathbb{P}(\chi(d) &< (1-\epsilon)\tilde{r}(d)) \\ &\leq \mathbb{P}(\chi(d) < (1-\gamma)(1-\frac{\epsilon}{2})\tilde{r}(d)) \leq \frac{1}{2(|E| + |D| + 1)}. \end{aligned} \tag{10}$$

(8) and (10) imply for all $d \in D$ with probability at least $1 - \frac{|D|}{2(|E|+|D|+1)}$

$$\sum_{P \in \Gamma_d} \lambda^I(P) = \sum_{P \in \Gamma_d} \lambda(P) + \chi(d) \geq (1-\epsilon)(\tilde{r}(d) - K_R(d)).$$

□

Theorem 2.4 *Let (G, H, c, r) be a multicommodity flow problem and let $0 < \epsilon \leq 1$ with $c(e) \geq \frac{6(2-\epsilon)}{\epsilon^2}\lceil \log(2(|E|+|D|+1)) \rceil$ for all $e \in E$ and $r(d) - K_R(d) \geq \frac{6(2-\epsilon)}{\epsilon^2}\lceil \log(2(|E|+|D|+1)) \rceil$ for all $d \in D$. Then with probability at least $1 - \frac{|E|+|D|}{2(|E|+|D|+1)}$ R-FLOW(ϵ) finds an integer multicommodity flow F subject to c such that for all $d \in D$*

$$f(d) \geq (1-\epsilon)(r(d) - K_I(d)).$$

□

For fractionallly solvable problems we have $f(d) \geq (1-\epsilon)r(d)$ and for $\epsilon = \frac{1}{2}$:

Corollary 2.5 *Let (G, H, c, r) be a fractionally solvable multicommodity flow problem with $c(e) \geq 36\lceil \log 2(|E|+|D|+1) \rceil$ for all $e \in E$ and $r(d) \geq 36\lceil \log 2(|E|+|D|+1) \rceil$ for all $d \in D$. Then with probability at least $1 - \frac{|E|+|D|}{2(|E|+|D|+1)}$ R-FLOW(ϵ) finds an integer multicommodity flow F subject to c such that for all $d \in D$*

$$f(d) \geq \frac{1}{2}r(d).$$

□

3 Algorithmic Angluin-Valiant Inequality and Derandomization

In this section we give a derandomized version of R-FLOW(ϵ). The fundamental inequalities of Hoeffding [5] and Angluin-Valiant[2] gives remarkable tight bounds on the tail of the distribution of the weighted sum of Bernoulli trials. These inequalities prove the existence of certain structures, but does not supply an efficient way of finding such structures, which is the main problem in the theory of derandomization. In his work on approximate packing integer programs Raghavan [12] was able to derive an alogrithmic version of the Angluin-Valiant inequality for *unweighted* sums of Bernoulli trials. The problem in the weighted case remained open, because there the computation of the conditional probabilities under consideration requires the computation of the exponential function (see [12], pg. 138).

In the following we show that this is not necessary and establish an algorithmic version also in the weighted case. This constitutes the essential tool to derandomize R-FLOW(ϵ). We omit the technically difficult proof and refer to [15]

Let $X_1, \ldots, X_n$ be $0-1$ random variables defined through $Prob(X_i = 1) = \tilde{x_i}$ and $Prob(X_i = 0) = 1 - \tilde{x_i}$ for some rational $0 \leq \tilde{x_i} \leq 1$. Let w_{ij} be rational non-negative weights, $1 \leq i \leq m$, $1 \leq j \leq n$, $0 \leq w_{ij} \leq 1$ and denote by ψ_i the random variables

$$\psi_i = \sum_{j=1}^{n} w_{ij} X_j$$

Let $p_i = \frac{\mathbb{E}(\psi_i)}{n}$ and $q_i = 1 - p_i$ and let $0 \leq \beta_i \leq 1$ be a rational number for each $1 \leq i \leq m$. Denote by $E_i^{(+)}$ the event

$$\text{“}\psi_i \leq (1+\beta_i)\mathbb{E}(\psi_i)\text{''}$$

and by $E_i^{(-)}$ the event

$$\text{“}\psi_i \geq (1-\beta_i)\mathbb{E}(\psi_i)\text{''}$$

Furthermore let

$$E = E_1 \wedge \ldots \wedge E_m,$$

where E_i is either $E_i^{(+)}$ or $E_i^{(-)}$. For each i, $(i = 1, \ldots, m)$ let δ_i be a rational number $0 < \delta_i \leq 1$ with the property: If E_i is the event "$\psi_i \geq (1+\beta_i)\mathbb{E}(\psi_i)$" then

$$\exp\left(-\frac{\beta_i^2 \mathbb{E}(\psi_i)}{3}\right) \leq \delta_i$$

and if E_i is the event "$\psi_i \leq (1-\beta_i)\mathbb{E}(\psi_i)$" then

$$\exp\left(-\frac{\beta_i^2 \mathbb{E}(\psi_i)}{2}\right) \leq \delta_i$$

Hence by the Angluin-Valiant inequality (Lemma 2.1) the event E hold with probability at least $1 - \sum \delta_i$. The basic problem we analyse is to find a $0-1$ vector $x \in \{0,1\}^n$ in *deterministic polynomial-time*, for which the event E holds. This problem can be sovled in the *RAM*-model of computation by the following theorem. The essential idea of the proof is the use of low degree Taylor-polynomials of elementary functions, like exp, log, $\sqrt{\ }$ for the construction of a new class of pessimistic estimators. We have

Theorem 3.1 *Let $E = E_1 \wedge \ldots \wedge E_m$ be an event, where E_i denotes either $E_i^{(+)}$ or $E_i^{(-)}$ as defined above. Let $0 < \delta < 1$ be a rational number with $\delta + \sum_{i=1}^n \delta_i < 1$ and suppose that $\beta_i \leq \frac{n-1}{n}$ for all $i = 1, \ldots, m$. Then a vector $x \in \{0,1\}^n$ for which the event E holds can be constructed in $O\left(mn^2 \log \frac{mn}{\delta}\right)$-time.* □

Let $m = |E| + |D|$ and let $L(\epsilon) = \max(L, \frac{1}{\epsilon^4} \log^2 m \log \frac{m}{\epsilon})$, where L is the encoding length of the integer programming formulation of the reduced demand multiflow problem. The deterministic counterpart to Theorem 2.4 then is

Theorem 3.2 *Let (G, H, c, r) be a multicommodity flow problem and let $0 < \epsilon \leq \frac{9}{10}$ with $c(e) \geq \frac{6(2-\epsilon)}{\epsilon^2}\lceil \log(2(|E| + |D| + 1)) \rceil$ for all $e \in E$ and $r(d) - K_R(d) \geq \frac{6(2-\epsilon)}{\epsilon^2}\lceil \log(2(|E| + |D| + 1)) \rceil$ for all $d \in D$. Then we can find in time $O\left(L(\epsilon) m^3\right)$ an integer multicommodity flow F subject to c such that for all $d \in D$*

$$f(d) \geq (1-\epsilon)(r(d) - K_I(d)).$$

Proof. Let n denote the number of Bernoulli trials performed in the randomized rounding procedure. After having executed GENPATH and SPLITPATH(ϵ) n is fixed. Since we introduced for each $e \in E$, $d \in D$ at most $O\left(\frac{\log(|E|+|D|)}{\epsilon^2}\right)$ random variables, we have $n = O(\epsilon^{-2}|E||D| \log(|E| + |D|))$. Let $E = \{e_1, \ldots, e_{|E|}\}$ and $D = \{d_{|E|+1}, \ldots, d_{|E|+|D|}\}$. For $1 \leq i \leq |E|$ let denote E_i the event "$\chi(e_i) > \tilde{c}(e_i)$". For $|E| + 1 \leq i \leq |E| + |D|$ let β_i be a rational number with

$$\sqrt{\frac{3\lceil \log 2(|E| + |D| + 1) \rceil}{(2-\epsilon)\tilde{r}(d_i)}} \leq \beta_i \leq \sqrt{\frac{6\lceil \log 2(|E| + |D| + 1) \rceil}{(2-\epsilon)\tilde{r}(d_i)}}. \tag{11}$$

We will later see how to determine such a β_i efficiently. The event E_i, $|E| \leq i \leq |E| + |D|$, then is

$$\text{“}\chi(d_i) \leq (1-\beta_i)(1-\frac{\epsilon}{2})\tilde{r}(d_i)''.$$

Let $E := E_1 \wedge \ldots \wedge E_{|E|+|D|}$. By Theorem 2.4 $Prob(E^c) \leq \frac{1}{2}\frac{|E|+|D|}{|E|+|D|+1}$, so in order to apply Theorem 3.1 we choose $\delta = \frac{1}{2}\frac{|E|+|D|}{|E|+|D|+1}$, m and n as above. Assuming that $m \geq 10$ and using $\epsilon \leq \frac{9}{10}$, (1) and (11) it is not difficult to prove that $\beta_i \leq \frac{n-1}{n}$ for all $i = 1, \ldots, m$. According to Theorem 3.1 we can perform the rounding in time $O(mn^2 \log \frac{mn}{\delta}) = O\left(\frac{m^3}{\epsilon^4} \log^2 m \log \frac{m}{\epsilon}\right)$ and obtain for all $e_i \in E$ and $d_i \in D$

$$\chi(e_i) \leq \tilde{c}(e_i)$$

and

$$\chi(d_i) \geq (1 - \frac{\epsilon}{2})(1 - \beta_i)\tilde{r}(d_i).$$

We add to each fixed integer part of a path value its rounded part. As in the proof of Lemma 2.2 and 2.3 the integer multiflow F is subject to c and $f(d) \geq (1-\epsilon)(r(d) - K_R(d))$ for all d.

The computation of β_i :
Choose $\gamma_i = \frac{1}{4}\sqrt{\frac{3\lceil \log 2(|E|+|D|+1)\rceil}{(2-\epsilon)\tilde{r}(d_i)}}$. Then iteratively halving the interval $\left[0, \frac{3\lceil \log 2(|E|+|D|+1)\rceil}{(2-\epsilon)\tilde{r}(d_i)}\right]$ we can find in $O(\log(\gamma_i^{-1}))-$time a rational β_i such that

$$0 \leq \beta_i - \sqrt{\frac{3\lceil \log 2(|E|+|D|+1)\rceil}{(2-\epsilon)\tilde{r}(d_i)}} \leq \gamma_i$$

which implies (11). Since $O(\log(\gamma_i^{-1})) = O(\log \frac{m}{\epsilon})$ we are done. □

Corollary 3.3 *Let $0 < \epsilon \leq \frac{9}{10}$ and let (G, H, c, r) be a multicommodity flow problem as above but with $K_R = 0$. Then we can find in polynomial-time an integer multiflow F subject to c such that for all $d \in D$*

$$f(d) \geq (1-\epsilon)r(d).$$

□

Since the multiflow problem can be fractionally solved in strongly polynomial time by Tardos' algorithm we have:

Corollary 3.4 *Let (G, H, c, r) be a fractionally solvable multicommodity flow problem with $c(e), r(d) \geq 36\lceil \log 2(|E|+|D|+1)\rceil$ for all $e \in E$ and $d \in D$. Then in strongly polynomial time we can find an integer multicommodity flow F subject to c such that for all $d \in D$*

$$f(d) \geq \frac{1}{2}r(d).$$

□

4 NP-completeness

Lemma 4.1 *[10] There is no fixed integer $K \in I\!N$ such that every fractional-solvable multicommodity flow problem posess an integer solution, when the requests are reduced by K.*

Proof. Assume that there is a fixed K, for which every fractionally solvable multicommodity flow problem has an integer solution, when each request is reduced by K. The idea of the proof is visualized by Figure 1:

The figure shows a grid-graph, where each grid-node is blown-up in the described way. Here we have 2 commodities each requesting $r(d) = 2K + 1$. Obviously such many commodities can be delivered using fractional values. But note that for integer values only a request with total sum of $2K+1$ can be conveyed. So the integer multicommodity flow can only be solved, if the requests is reduced by at least $K+1$. The method can be extended for arbitrary K and arbitrary many pairs of source-sink-pairs by copying the graph in Figure 1. □

Note that in this construction the supply-graph G is planar. This shows that the Korach/Penn result is only valid when the union of the supply graph and the demand graph is planar.

This construction can be used to prove the $NP-$completness of (1.1), even if the problem is fractionally solvable: We show by a reduction to the original integer multi-flow-problem:

Theorem 4.2 *The demand reduced multiflow problem with fractionally solvable inputs is NP-complete.*

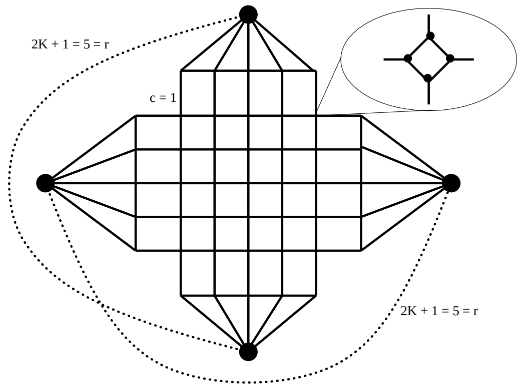

Figure 1: An example with $K = 2$

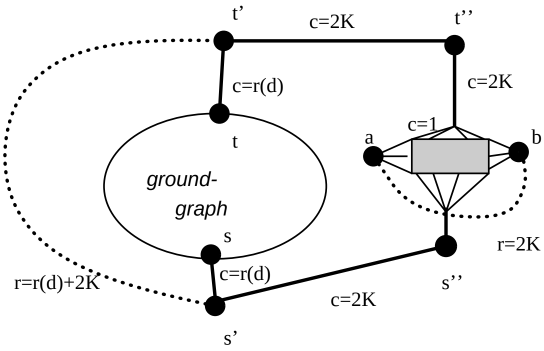

Figure 2: NP-completeness

Proof. Let $K \in \mathbb{N}$ be an integer and suppose we are given a fractionally solvable multicommodity-flow problem. For each source-sink-pair (s, t) construct an auxillary graph as shown in Figure 2. The new demand edge now is (s', t') requesting a value of $r(s, t) + 2K$. Connect s'' and t'' to the "grid" with $2K$ edges, all having capacity 1. Introduce a new demand edge (a, b) requesting a value of $2K$. It is clear, that the resulting graph remains fractional solvable. It is easy to see that the new graph has an integer solution, where the request is reduced by K if and only if the original problem has a integer solution: To saturate both reduced integer flows form a to b and from s'' to t'' we have to push one flow, say (a, b) through the grid and the other "around" the grid. So there is a flow of $\frac{K}{2}$ from s'' trough a to t'' and the other $\frac{K}{2}$ through b to t''. There is no way to convey any other flow than the original one through the ground graph. $\square$

5 Conclusion and Open Problems

1. We have presented a deterministic approximation algorithm, in particular a $\frac{1}{2}$-factor algorithm finding feasible *integer* multicommodity flows. The running time of our algorithm is dominated by the time needed to solve a LP. The analysis of the algorithm shows that we have to require $c, r = \Omega(\log(|E| + |D|))$. As the example in section 4 shows, there are fractional solvable problems, where the approximation factor is less than $\frac{1}{2}$. The interesting open problem is to give a $\frac{1}{2}$-factor approximation if $c, r = O(\log(|E| + |D|)$ or even $c, r = O(1)$.

2. Note that with similiar methods these results hold also for the integer maximum $0 - 1$

multiflow problem.

3. Better approximation results might be possible in special cases, for example for planar graphs. Here several question arises, for example, whether better Korach/Penn type results can be proved, for planar graphs with stronger cut-conditions.

4. The algorithmic version of the Angluin-Valiant inequality might be usefull to attack other packing-type problems, especially those with weights.

5. The non-intractability of approximation problems from the class MAX-SNP motivates the question, whether or not there is a *polynomial-time* approximation scheme for the integer multiflow problem.

References

[1] N. Alon, J. Spencer, P. Erdös; *The prababilistic method.* John Wiley & Sons, Inc. 1992.

[2] D. Angluin, L.G. Valiant: *Fast probabilistic algorithms for Hamiltonion circuits and matchings.* J. Computer and System Sciences, Vol. 18, (1979), 155–193.

[3] A. Frank: *Packing paths, circuits and cuts - a survey.* In Paths, Flows and VLSI-Layout, B. Korte, L. Lovasz, H.-J. Prömel, A. Schrijver (eds.), Springer Verlag, (1990), pp.47 - 97.

[4] M. Grötschel, L. Lovász, A. Schrijver; *Geometric algorithms and combinatorial optimixation.* Springer-Verlag (1988)

[5] W. Hoeffding; *On the distribution of the number of success in independent trials.* Annals of Math. Stat. 27, (1956), 713-721.

[6] E. Korach, M. Penn; *Tight integral duality gap in the chinese postman problem* Technical report, Computer Science Department, Israel Institute of Technology, Haifa, Revised Version, December 1989.

[7] V. M. Malhotra, M. P. Kumar, S. N. Maheshwari; *An $O(|V|^2)$ Algorithm for Finding Maximum Flows in Networks.* Information Processing Letters 7 (1978), 277–278.

[8] C. McDiarmid; *On the Method of Bounded Differences.* Surveys in Combinatorics, 1989. J. Siemons, Ed.: London Math. Soc. Lectures Notes, Series 141, Cambridge University Press, Cambridge, England 1989.

[9] K. Mehlhorn; *Data structures and algorithms 1: Sorting and Searching.* Sringer-Verlag (1984)

[10] F. Pfeiffer; *Personal communication*, Bonn 1991.

[11] P. Raghavan, C. D. Thompson; *Randomized Rounding: A technique for provably good algorithms and algorithmic proofs.* Combinatorica 7 (4), (1987), 365-373.

[12] P. Raghavan; *Probabilistic construction of deterministic algorithms: Approximating packing integer programs.* Jour. of Computer and System Sciences 37, (1988), 130-143.

[13] A. Srivastav, P. Stangier; *The relationship between fractional and integral graph partitioning.* Working Paper; Research Institute of Discrete Mathematics, University of Bonn, (1992).

[14] A. Srivastav, P. Stangier; *Weighted fractional and integral $k-$matching in hypergraphs.* Working Paper; Research Institute of Discrete Mathematics, University of Bonn (1992).

[15] A. Srivastav; *An Algorithmic Version of the Chernoff-Hoeffding Inequality and New Apllications to Packing Integer Programs.* Working Paper; Research Institute of Discrete Mathematics, University of Bonn (1993).