The Thickness of Graphs without K_5 - Minors *

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Abstract

The thickness problem on graphs is $\mathcal{NP}$-hard and only few results concerning this graph invariant are known. Using decomposition theorems of Wagner and Truemper, we show that the thickness of graphs without K_5-minors is less than or equal to two. Therefore, the thickness of this class of graphs can be determined with a planarity testing algorithm in linear time.

Key words: Thickness, crossing number, skewness, graph-minor, 2-sum, Δ-sum

1 Introduction

The **thickness** $\theta(G)$ of a graph $G = (V, E)$ is the minimum number k such that G is the union of k planar subgraphs (here, by "union of k planar subgraphs" we mean that the edge-set E can be partitioned into k sets so that the graph induced by each set is planar). Therefore, the thickness is one measure of the degree of nonplanarity of a graph.

Clearly, $\theta(G) = 1$ if and only if G is planar. The thickness problem, asking for the thickness of a given graph G, is $\mathcal{NP}$-hard ([Man83]), so there is little hope to find a polynomial time algorithm for the thickness problem on general graphs. However, for some graph classes, the thickness can be determined in polynomial time. For example, the thickness is known for complete and complete bipartite

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graphs, see, e.g. [BW78]. In some cases, there are (often relatively poor) bounds on the thickness of a graph ([DHS91] and [Hal91]).

The thickness problem has applications in VLSI-design. In electronic circuits, components are joined by means of conducting strips. These may not cross, since this would lead to undesirable signals. In this case, an insulated wire must be used. For that reason, circuits with a large number of crossings are decomposed into several layers without crossings, which are then pasted together. The goal is to use as few layers as possible. In this application it would be desirable to know the thickness of a hypergraph whose nodes are cells to be placed and whose hyperedges correspond to the nets connecting the cells. If the thickness problem could be solved for graphs, it would be a useful engineering tool in the layout of electronic circuits.

We have restricted our attention to the class of graphs without K_5-minors. Our method to determine the thickness of this class of graphs is based on a decomposition theorem of Wagner [Wag37] and Truemper [Tru92]. The paper is organized as follows. The concept of graph decomposition is introduced in section 2. In section 3 we prove the main theorem of this paper. Finally, in section 4 we give negative results on using our approach for the two other graph invariants crossing number and skewness.

2 Decomposition of Graphs

In this section, we present the 2- and Δ-sums of graphs. Furthermore, we describe a recursive construction process for graphs without K_5-minors based on Wagner's decomposition theorem, which is essential for the proof of the main theorem.

For that purpose, let $G = (V, E)$ be a connected graph. G is called a **2-sum** of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted $G = G_1 \bigoplus_2 G_2$, if the identification of an arbitrary edge e_1 of G_1 with an arbitrary edge e_2 of G_2 and subsequent deletion of this edge produces G. Analogously, G is called a **Δ-sum** of G_1 and G_2, denoted $G = G_1 \bigoplus_\Delta G_2$, if identification of a triangle of G_1 with a triangle of G_2 and subsequent deletion of this triangle produces G (see Figure 1). Conversely, if $G = G_1 \bigoplus_2 G_2$ or $G = G_1 \bigoplus_\Delta G_2$, we say that G_1 and G_2 are a 2- resp. Δ-sum decomposition of G. Let $\bigoplus \in \{\bigoplus_2, \bigoplus_\Delta\}$. If, for $k \geq 2$, $G = (((G_1 \bigoplus G_2) \bigoplus G_3) \bigoplus \cdots) \bigoplus G_k$, we call the graphs G_i $(1 \leq i \leq k)$ **building blocks** of G.

A "modern" version of a theorem by Wagner [Wag37], given by Truemper [Tru92], allows us to restrict our attention to certain building blocks for all 2-connected graphs without K_5-minors.

Theorem 2.1 (Truemper, 1992)
Any 2-connected graph without K_5-minors is planar, isomorphic to $K_{3,3}$ or V_8 or may be constructed recursively by 2-sums and Δ-sums. The building blocks of

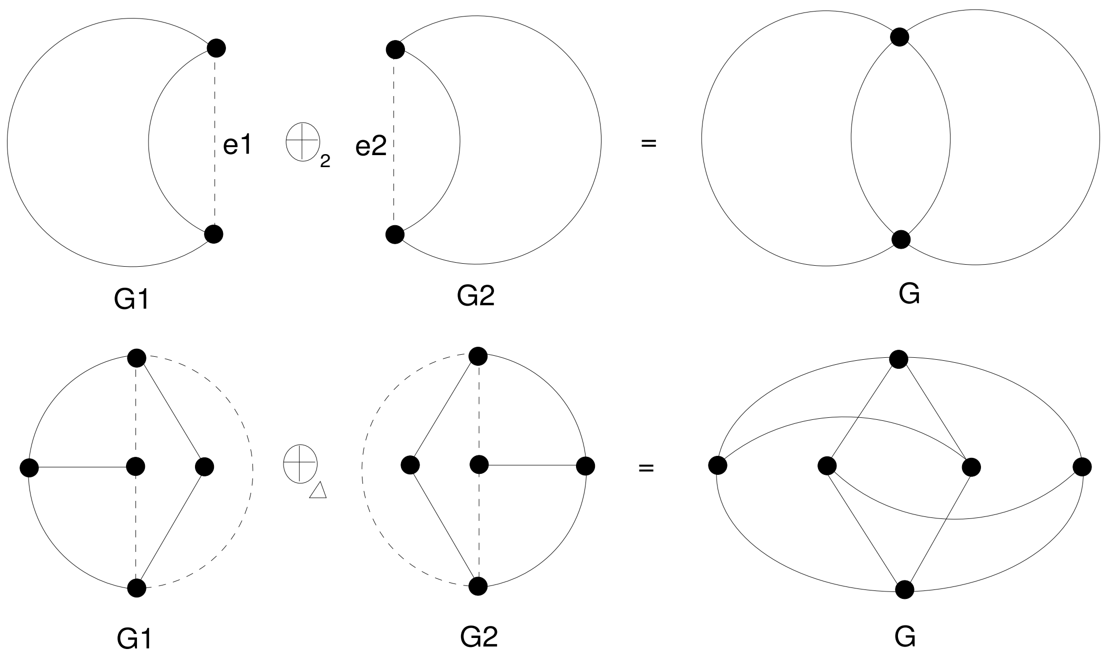

Figure 1: 2- and Δ-sum

such a construction are planar graphs and graphs isomorphic to $K_{3,3}$ or V_8 in the case of a 2-sum and planar graphs only in the case of a Δ-sum.

As a preparation for the proof of the main theorem, we have to deal with the structure of the two graphs $K_{3,3}$ and V_8 (see Figure 2). Both graphs are not planar and have crossing number one, which can be seen from the embeddings of Figure 3. Obviously their thickness equals two. We call the edges of V_8 between two succeeding vertices circle-edges, and the others diagonal-edges.

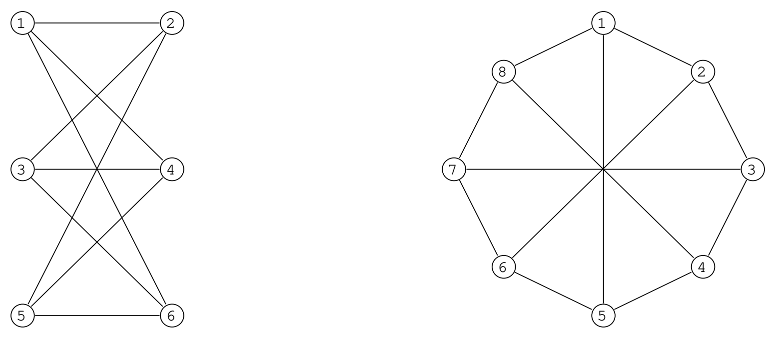

Figure 2: $K_{3,3}$ and V_8

We make the convention that a graph G_1 always represents the graph already obtained by means of 2- resp. Δ-sums, and a graph G_2 is chosen according to the specification of the building blocks given in Truempers theorem. Moreover, we identify a graph with a drawing in the plane. According to the application in VLSI-design the planar graphs, whose union is the original graph, are embedded on different layers.

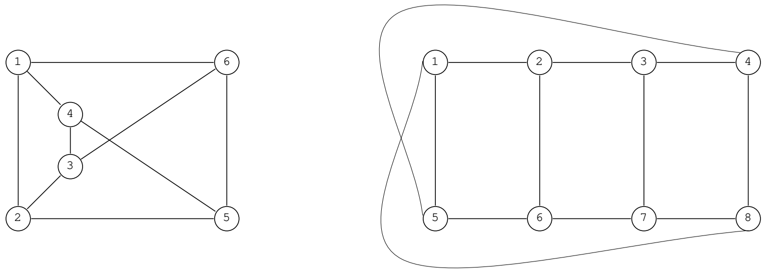

Figure 3: Embedding of $K_{3,3}$ and V_8 with one crossing

3 Thickness Theorem

We are prepared to prove the main theorem of this paper.

Theorem 3.1 *If G is a graph without K_5-minors, then $\theta(G) \leq 2$.*

In the proof we will make use of two lemmas.

Lemma 3.2 *If G_1 is a graph and G_2 is a planar graph or isomorphic to $K_{3,3}$ or V_8, then the following holds for any 2-sum $G = G_1 \bigoplus_2 G_2$:*

$$\theta(G) = \max\{\theta(G_1), \theta(G_2)\}.$$

Proof. Let G_1 be divided into $\theta(G_1)$ planar subgraphs, each of them embedded on a different layer. Denote by e_1 resp. e_2 the edges of G_1 resp. G_2, which are identified in a 2-sum. We can assume without loss of generality that e_1 is embedded on the first layer. We have to deal with two cases.

If G_2 is planar, then, by means of stereographic projection, we can obtain an embedding of G_2 in the plane, in which e_2 bounds the outer face. Thus, the whole graph G_2 can be embedded in one of the two faces, bounded by edge e_1 on layer 1 of graph G_1. Then the edge e_1 is identified with the edge e_2 and subsequently deleted (see Figure 4). The thickness of the resulting graph has not increased, i.e., we have $\theta(G) = \theta(G_1)$.

If G_2 is isomorphic to $K_{3,3}$ or V_8, an analogous approach is possible. The deletion of any edge of $K_{3,3}$ results in a planar graph, and the deletion of any circle-edge of V_8 yields a planar graph. Let e_3 be such an edge, which, by the above, can be chosen to be nonadjacent to e_2.

As in the first case we can embed the graph $G_2 - e_3$ on the first layer. Since e_2 and e_3 are nonadjacent, none of the terminal-vertices of e_3 is, after identification

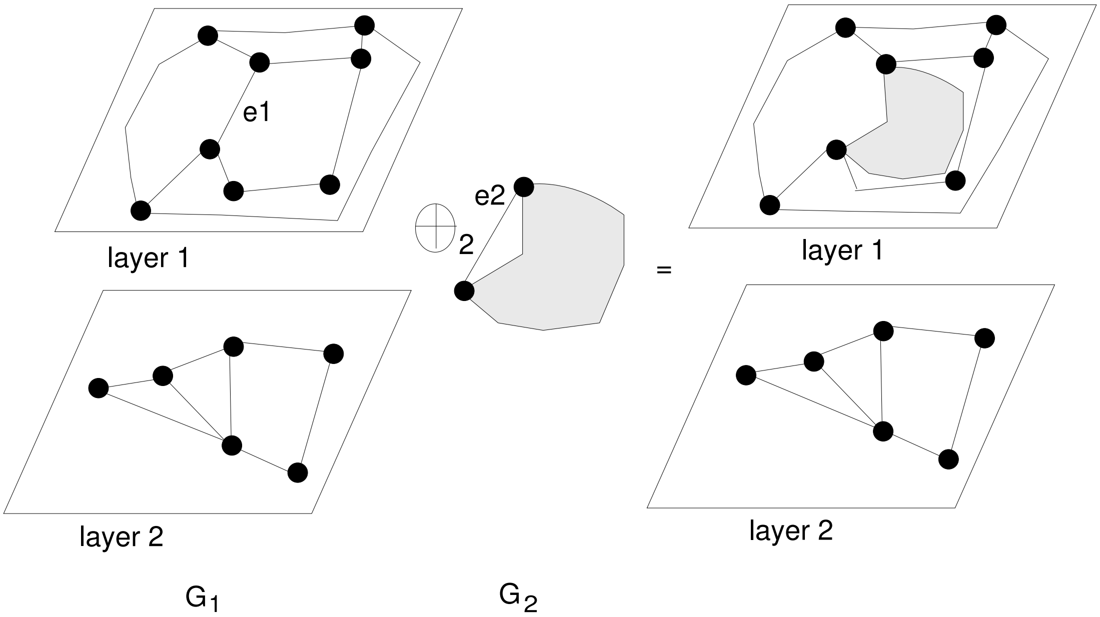

Figure 4: 2-sum with a planar graph

of e_1 with e_2, a vertex of G_1, therefore the deleted edge e_3 can be embedded on layer 2 without destroying planarity (see Figure 5).

If G_1 is planar and G_2 non-planar, we have $\theta(G_1 \bigoplus_2 G_2) = \theta(G_2)$. If G_1 is nonplanar, the thickness does not increase for any 2-sum. Consequently, $\theta(G_1 \bigoplus_2 G_2) = \max\{\theta(G_1), \theta(G_2)\}$. $\Box$

Lemma 3.3 *If G_1 is a graph and G_2 is a planar graph, then the following holds for any Δ-sum $G = G_1 \bigoplus_\Delta G_2$:*

(i) If G_1 is nonplanar, then $\theta(G) = \theta(G_1)$,
(ii) If G_1 is planar, then $\theta(G) \leq 2$.

Proof. If we start with a planar embedding of the graph G_2, then, after deletion of the triangle T defining the Δ-sum, G_2 can be partitioned into the planar subgraphs G_{out}, embedded in the outer side of T, and G_{in}, embedded in the inner side of T.

Consider the graph $G_2' := G_2 - G_{in}$. In G_2', the triangle T bounds a face, which can be made the outer face via stereographic projection. Now the graph G_{in} can be reinserted into G_2' in such a way that all vertices of G_{in} lie in one face of G_2', which is bounded by an edge of the triangle T.

This representation is not necessarily free of crossings, but the triangle T now bounds the outer face (see Figure 6). In the following, we refer to such a representation of a planar graph as a "Δ-representation" of the graph.

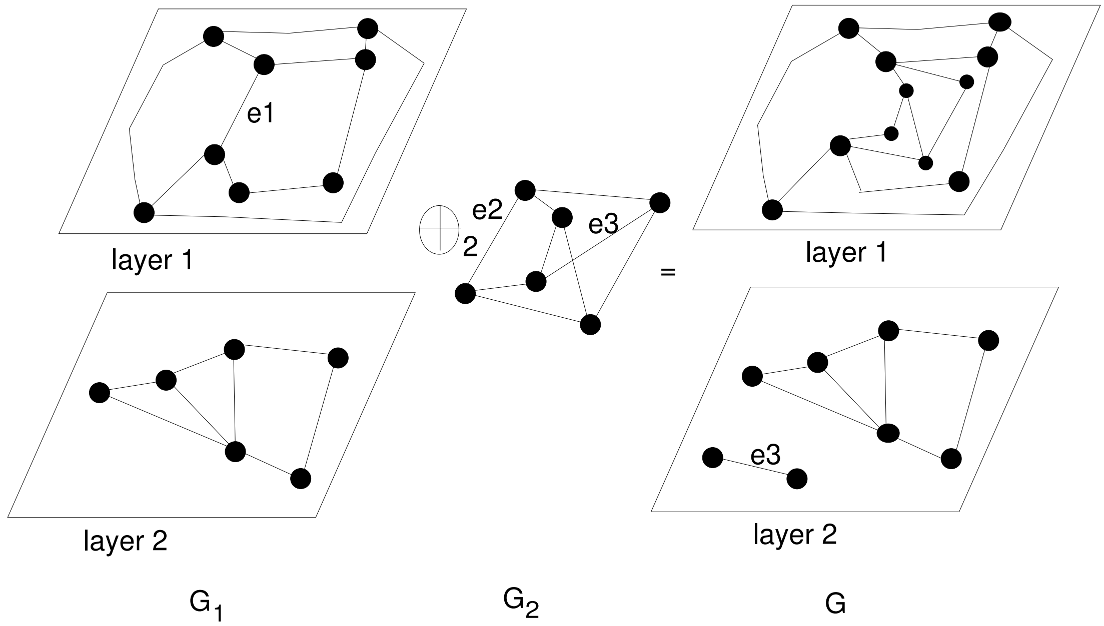

Figure 5: 2-sum with a $K_{3,3}$

In order to prove (i), we assume that we have a decomposition of G_1 into $t = \theta(G_1)$ planar layers $l_1, l_2, \ldots, l_t$. Let v_1, v_2, and v_3 be the vertices of G_1, which are identified in a Δ-sum with G_2, and $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, and $e_3 = (v_3, v_1)$ be the edges of the Δ-clique. Suppose edge e_1 is embedded on layer l_k, $k \in \{1, 2, \ldots, t\}$, then let F_1 be one of the faces adjacent to e_1 in a planar embedding of l_k. The insertion of the graph G_2, given in Δ-representation, is done by placing the vertices of G_{in} in face F_1. Moreover, all those G_{in}-edges, that are not adjacent to v_3 can be embedded on this layer. The remaining edges (v_3, u), $u \in G_{in}$ can be embedded without crossings into any face adjacent to node v_3 in layer $l_h \neq l_k$, $h \in \{1, 2, \ldots, t\}$. The insertion of G_{out} together with its edges not adjacent to v_1 is done analogously into the face F_2 adjacent to e_2 in a planar embedding of $l_{k'}$, if e_2 is embedded on layer $l_{k'}$. The remaining edges (v_1, u), $u \in G_{out}$ can be embedded into any face adjacent to node v_1 in layer $l_{h'} \neq l_{k'}$, $h' \in \{1, 2, \ldots, t\}$. Figure 7 (a) illustrates the situation in the case that all three edges are embedded on layer l_k.

Therefore, the thickness does not increase in a Δ-sum of a nonplanar graph G_1 with a planar graph G_2.

In (ii), both G_1 and G_2 are planar. If the triangle defining the Δ-sum is the boundary of a face of G_1, and if G_2, given in Δ-representation, has no crossings, then the embedding of G_2 can be inserted into the embedding of G_1 without violating planarity, and we have $\theta(G_1 \bigoplus_\Delta G_2) = 1$.

Otherwise, the graph G_2 is embedded as in part (i), which results in a thickness of at most 2 for G. □

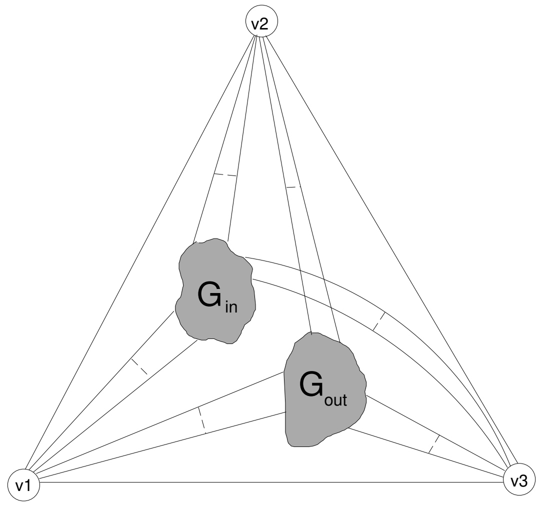

Figure 6: Δ-representation of a planar graph

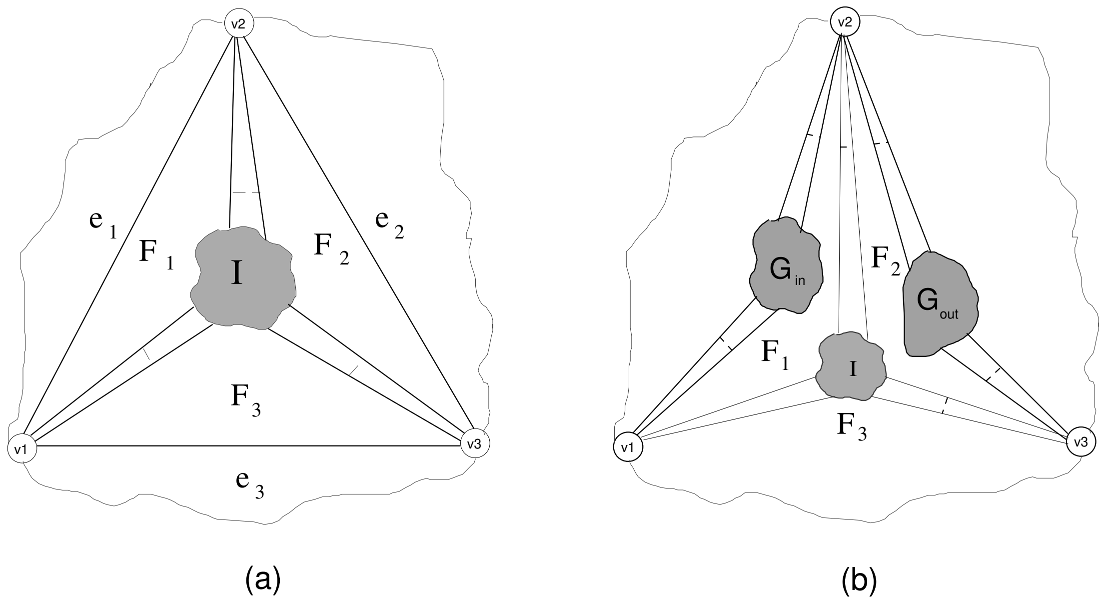

Figure 7: (a) Shape of layer l_k if all three edges e_1, e_2, e_3 are embedded on this layer. (b) Layer l_k after the insertion of G_{in} and G_{out}

Proof of Theorem 3.1 Using induction, we show that if G_k is a graph, which is produced by k 2- and Δ-sums with the building blocks described in section 2, then $\theta(G_k) \leq 2$.

The claim is trivial for $k = 1$, because planar graphs as well as $K_{3,3}$ and V_8 have a thickness less than or equal to two. Consider the graph $G_k := G_{k-1} \bigoplus H$, obtained from G_{k-1} by a 2- resp. Δ-sum with a graph H chosen according to Theorem 2.1. In the case of a 2-sum we obtain from Lemma 3.2 that

$\theta(G_k) = \max\{\theta(G_{k-1}), \theta(H)\}$, and thus $\theta(G_k) \leq 2$. In the case of a Δ-sum with a nonplanar graph G_{k-1} we obtain from Lemma 3.3 (i) that $\theta(G_k) = \theta(G_{k-1})$. If G_{k-1} is planar, Lemma 3.3 (ii) applies directly. Truemper's decomposition theorem says that every 2-connected graph without K_5-minors can be obtained by a sequence of 2- resp. Δ-sums, i.e., $G = G_k$ for a $k \in \mathbb{N}$. Therefore, the theorem is proved for 2-connected graphs. If G is not 2-connected, the theorem applies for every 2-connected block of the graph and hence for the whole graph (note that the blocks are disjoint up to the cut vertices). □

As a corollary, we obtain that the thickness problem in the class of graphs without K_5-minors is solvable in linear time.

Corollary 3.4 *The thickness of a graph G without K_5-minors can be determined in linear time in the number of nodes of G.*

Proof. Apply a linear time planarity testing algorithm [HT74] to G. If G is planar, then $\theta(G) = 1$, otherwise $\theta(G) = 2$. □

4 Other Invariants

One may think that applying certain sum operations might also be applicable to control other topological invariants of graphs, such as the **crossing number** $\nu(G)$ or the **skewness** $\mu(G)$ of a graph G. The crossing-number $\nu(G)$ of a given graph G is the minimum number of pairwise intersections of edges when G is drawn in the plane. The skewness is the minimum number of edges which have to be deleted from the graph G to make it planar.

Unfortunately, such a transfer is not possible, since by a 2-sum there is neither additivity of the crossing number resp. skewness of the building blocks nor a fixed value as for the thickness. We prove this by giving counterexamples.

Theorem 4.1 *For each $n \in \mathbb{N}$ there exist graphs G_1 and G_2 such that, for any graph $G = G_1 \bigoplus_2 G_2$, the following holds:*

$$\nu(G) > \nu(G_1) + \nu(G_2) + n.$$

Proof. For $n \in \mathbb{N}$, denote by M_{n+4} the planar graph shown in Figure 8 with $n+4$ vertices and $2n+5$ edges. Start with the graph $K_{3,3}$ (embedding of Figure 3) and take sucessively 2-sums with seven edges of the $K_{3,3}$ and M_{n+4} as shown in Figure 9. The resulting graph H has crossing number one. Take a further 2-sum of H and M_{n+4} by identifying the edges e and f_1.

In every drawing of the graph, the edge f_2 crosses a complete subgraph $M_{n+4} - e$ and therefore at least $n + 2$ edges. Therefore, we have $\nu(H \bigoplus_2 M_{n+4}) = n + 2 > \nu(H) + \nu(M_{n+4}) + n$. □

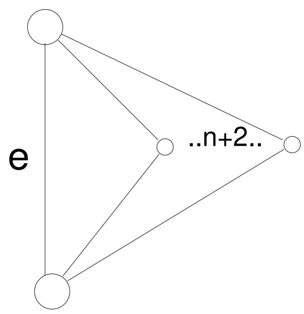

Figure 8: Graph M_{n+4}

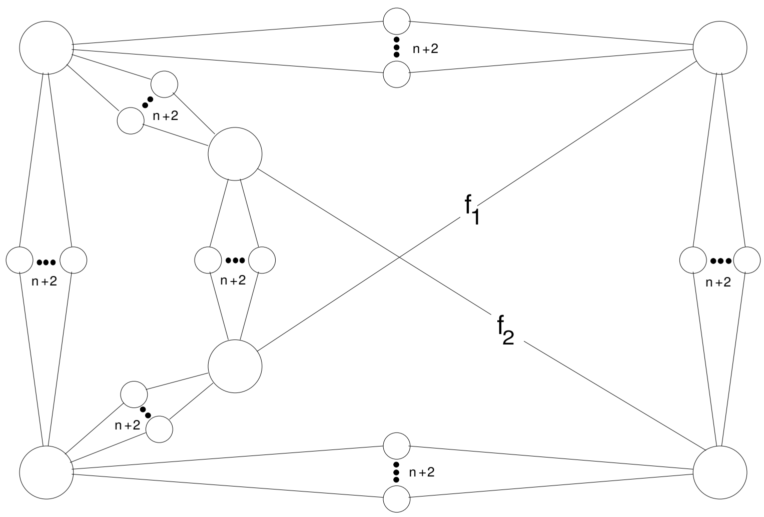

Figure 9: Graph H

An example of the nonadditivity of the skewness can be obtained by a slight modification of the proof of Theorem 4.1.

Theorem 4.2 *For each $n \in \mathbb{N}$ there exist graphs G_1 and G_2 such that the following holds for the graph $G = G_1 \bigoplus_2 G_2$:*

$$\mu(G) > \mu(G_1) + \mu(G_2) + n.$$

Proof. Take 2-sums of eight edges of $K_{3,3}$ with M_{n+4}. The skewness of the resulting graph equals one. A further 2-sum of the remaining edge of $K_{3,3}$ with M_{n+4} gives the graph F of Figure 10. In order to achieve planarity, a graph $M_{n+4} - e$ must be removed, i.e., the skewness is $n + 2$. $\square$

Since we only used building blocks according to Theorem 2.1, the above theorems are valid even if we restrict ourselves to graphs without K_5-minors.

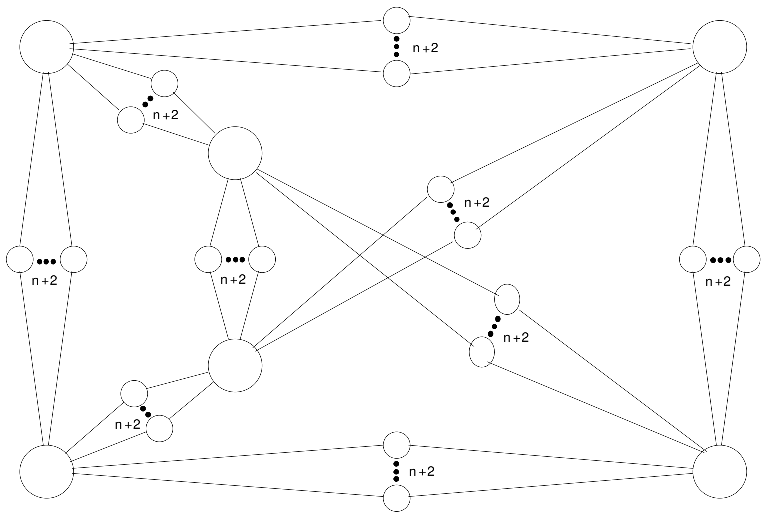

Figure 10: Graph F

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