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A Basic Study of the QAP-Polytope *

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Abstract

We investigate a polytope (the *QAP-Polytope*) beyond a "natural" integer programming formulation of the Quadratic Assignment Problem (QAP) that has been used successfully in order to compute good lower bounds for the QAP in the very recent years. We present basic structural properties of the QAP-Polytope, partially independently also obtained by Rijal (1995). The main original contribution of this work is the representation of the QAP-Polytope in a space different from the one in which it is defined naturally. This representation provides us with a much simpler way to derive the dimension of the QAP-Polytope, as well as to investigate the facial structure of it. Furthermore, it leads to some interesting observations concerning the combinatorial structure of the QAP-Polytope.

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1 Introduction

The *Quadratic Assignment Problem* (*QAP*) is one of the classical $\mathcal{NP}$-hard combinatorial optimization problems, like, e.g., the Traveling Salesman Problem (TSP) or the Max Cut Problem. The task is, to find an assignment π of n objects having certain amounts of flow between each pair of them (stored in a matrix $A = (a_{ik})$) to n locations having certain distances between each pair of them (stored in a matrix $B = (b_{jl})$) such that

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{\pi(i)\pi(k)}$$

is minimized. In this form, the problem was introduced by Koopmans and Beckmann (1957). Actually, they put also a linear term $\sum_{i=1}^{n} C_{i\pi(i)}$ into the objective function, modelling costs that arise when placing a certain object in a certain location (independently from the placement of the other objects).

For the following slightly more general formulation, Lawler (1963) proposed a linearization underlying the approach by polyhedral combinatorics. In this formulation, an

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instance of the QAP is given by the size $n \in \mathbb{N}$ and objective function coefficients $g_{ijkl} \in \mathbb{R}$ $(i, j, k, l = 1, \ldots, n)$ and $h_{ij} \in \mathbb{R}$ $(i, j = 1, \ldots, n)$. Let Π_n be the set of all $n \times n$ permutation matrices. Then, the task is to solve

$$\begin{aligned} \min \quad & \sum_{i,j,k,l=1}^{n} g_{ijkl} x_{ij} x_{kl} + \sum_{i,j=1}^{n} h_{ij} x_{ij} \\ \text{s.t.} \quad & X = (x_{ij})_{i,j=1,\ldots,n} \in \Pi_n . \end{aligned}$$

Sahni and Gonzales (1976) showed that even ϵ-approximation of the QAP is $\mathcal{NP}$-hard. As a result, most approaches for exact algorithms for the QAP are branch&bound type methods requiring in particular good lower bounds on the optimal objective function value. Consequently, there has been much effort in deriving procedures that compute (tight) lower bounds.

However, unlike e.g. for the TSP or the Max Cut Problem, until a few years ago, there have been no approaches to the QAP from the direction of polyhedral combinatorics, although there were some attempts of polyhedral kind in the sense that the QAP was formulated as a linear mixed integer program in several different ways (Burkard (1990) gives an overview). During the last three or four years, a few people have started to work with an approach that admits a treatment by means of polyhedral combinatorics. First computational results (in particular by Johnson (1992) and Resende, Ramakrishnan, and Drezner (1994)) showed that this approach yields very good lower bounds.

We (and independently from us Rijal (1995)) started to investigate the polytope beyond that approach. We describe the approach in a detailed way in Section 2, where we give the definition of the QAP-Polytope $\mathcal{QAP}_n$ as well as some first structural properties of it. In Section 3, we present a polytope in a different space that is isomorphic to $\mathcal{QAP}_n$. This different representation of the QAP-Polytope has some features that appear to be nicer than the corresponding ones in the original representation. Clearly, since both polytopes are isomorphic, these features are not of a combinatorial kind, but connected to the structure of the coordinate vectors of the representations. In Section 4, we will derive the dimension of the QAP-Polytope exploiting our second representation. Finally, Section 5 samples the results obtained concerning the affine hulls of both representations.

We will use the following notations. For any set M, the set of all subsets of M of cardinality 2 is denoted by $\binom{M}{2}$. For a graph $G = (V, E)$ with node set V and edge set E, the set $E(W)$ for a subset $W \subseteq V$ of nodes contains all edges having both endpoints in W and $\delta(W)$ contains all edges having precisely one endpoint in W. For two disjoint subsets $W_1, W_2 \subseteq V$ of nodes, $(W_1 : W_2)$ is the set of all edges having one node in W_1 and one in W_2. For singletons $\{v\}$ we often omit the brackets and write, e.g., $\delta(v)$. If $x \in \mathbb{R}^V$ is a vector whose components are associated with the nodes in V, and, again, $W \subseteq V$ is a subset of the nodes, then $x(W)$ is the sum of all components of x belonging to elements in W. An analogous definition holds for a vector $y \in \mathbb{R}^E$ and a subset $F \subseteq E$. For any two vectors $a, b \in \mathbb{R}^m$, the notation $a \leq b$ means that $a_i \leq b_i$ for all components $i = 1, \ldots, n$. If R is a subset of the index set of a matrix A then $A_{R,\bullet}$ means the submatrix of A consisting just of the rows of A corresponding to R. The analogue holds for a subset of the column indices. All vectors are meant to be columns, and for a subset R of the indices of a vector v, V_R clearly means the vector consisting of the components of v corresponding to R. Finally, $\mathrm{conv}(\Omega)$, $\mathrm{aff}(\Omega)$, and $\mathrm{lin}(\Omega)$ denote the convex, affine, and linear hull, respectively, of Ω.

2 Definition of the QAP-Polytope $\mathcal{QAP}_n$

Consider the graph $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$ having nodes

$$\mathcal{V}_n := \{(i,j) \mid i,j \in \{1,\ldots,n\}\}$$

and edges

$$\mathcal{E}_n := \{\{(i,j),(k,l)\} \in \binom{\mathcal{V}_n}{2} \mid i \neq k, j \neq l\}.$$

For ease of notation, we define

$$[i,j,k,l] := \{(i,j),(k,l)\}$$

for all edges $\{(i,j),(k,l)\} \in \mathcal{E}_n$. Figure 2 shows the canonical drawing of $\mathcal{G}_n$ (for $n = 4$) indicated by the matrix-structure of its node set.

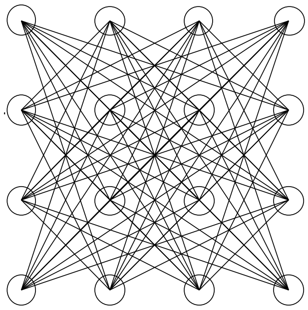


Figure 1: The graph $\mathcal{G}_n$ has all possible edges but the "horizontal" and the "vertical" ones.

Obviously, the graph $\mathcal{G}_n$ has the following two properties:

(i) The size of a maximum clique of $\mathcal{G}_n$ is n, i.e., the clique-number of $\mathcal{G}_n$ is $\omega(\mathcal{G}_n) = n$.

(ii) The maximum cliques (i.e., the n-cliques) of $\mathcal{G}_n$ correspond to the $n \times n$ permutation matrices.

Let an instance

$$\text{(QAP)} \qquad \begin{array}{ll} \min & \displaystyle\sum_{i,j,k,l=1}^{n} g_{ijkl} x_{ij} x_{kl} + \sum_{i,j=1}^{n} h_{ij} x_{ij} \\ \text{s.t.} & X = (x_{ij})_{i,j=1,\ldots,n} \in \Pi_n \end{array}$$

of the QAP be given ($h \in (\mathbb{R}^n)^2$, $g \in (\mathbb{R}^n)^4$). Define $(h', g') \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ by setting $h'_{(i,j)} := h_{ij} + g_{ijij}$ ($(i,j) \in \mathcal{V}_n$) and $g'_{[i,j,k,l]} := g_{ijkl} + g_{klij}$ ($[i,j,k,l] \in \mathcal{E}_n$). The set of (node sets of) n-cliques of $\mathcal{G}_n$ is denoted by

$$\mathcal{CLQ}_n^n := \{C \subseteq \mathcal{V}_n \mid C \text{ } n\text{-clique of } \mathcal{G}_n\}.$$

Solving (QAP) now is equivalent to finding among the n-cliques of $\mathcal{G}_n$ one having minimal node- and edge-weight (with respect to the weights (h', g')), i.e., solving (QAP) is equivalent to solving

$$\text{(QAP')} \qquad \begin{array}{ll} \min & h'(C) + g'(\mathcal{E}_n(C)) \\ \text{s.t.} & C \in \mathcal{CLQ}_n^n. \end{array}$$

We will use the following notations in the sequel: For any subset $W \subseteq \mathcal{V}_n$ of nodes of $\mathcal{G}_n$, we denote by $x^W \in \mathbb{R}^{\mathcal{V}_n}$ the characteristic vector of W, i.e., for $v \in \mathcal{V}_n$, we set

$$x_v^W := \begin{cases} 1, & \text{if } v \in W \\ 0, & \text{if } v \notin W. \end{cases}$$

Analogously, for any subset $F \subseteq \mathcal{E}_n$ of edges of $\mathcal{G}_n$, we denote the characteristic vector of F by $y^F \in \mathbb{R}^{\mathcal{E}_n}$, i.e., for $e \in \mathcal{E}_n$, we define

$$y_e^F := \begin{cases} 1, & \text{if } e \in F \\ 0, & \text{if } e \notin F. \end{cases}$$

We write x^v instead of $x^{\{v\}}$ and y^e instead of $y^{\{e\}}$.

The QAP-Polytope is

$$\mathcal{QAP}_n := \operatorname{conv}\Big(\{(x^C, y^{\mathcal{E}_n(C)}) \mid C \in \mathcal{CLQ}_n^n\}\Big).$$

For any $\alpha \in \{1, \dots, n\}$, we define

$$\operatorname{row}_\alpha^{(n)} := \{(\alpha, j) \mid j = 1, \dots, n\}$$

to be the *α-th row* of $\mathcal{V}_n$, and

$$\operatorname{col}_\alpha^{(n)} := \{(i, \alpha) \mid i = 1, \dots, n\}$$

to be the *α-th column* of $\mathcal{V}_n$. For $v = (i, j) \in \mathcal{V}_n$, we set $r(v) := i$ and $c(v) := j$.

Observation 1. *The graph $\mathcal{G}_n$ is invariant under permutations of the rows, permutations of the columns, and transposition of the node set, where transposition means mapping (i, j) to (j, i) for all $(i, j) \in \mathcal{V}_n$. Clearly, the automorphisms of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ induced by the corresponding coordinate permutations are symmetries of $\mathcal{QAP}_n$, i.e., they induce distance preserving affine transformations that map $\mathcal{QAP}_n$ into itself.*

From Observation 1 one immediately derives a first property of the QAP-Polytope.

Theorem 1. *All cones induced at the vertices of $\mathcal{QAP}_n$ are isomorphic.*

We will now show, that $\mathcal{QAP}_n$ has famous relatives, which are certain *Boolean Quadric Polytopes / Cut Polytopes.* Let $K_m = (V_m, E_m)$ be the complete graph on m nodes. Similarly to the definitions for the QAP-Polytope, for any subset $W \subseteq V_m$ of nodes we denote the characteristic vector of W by $X^W \in \mathbb{R}^{V_m}$ and for any subset $F \subseteq E_m$ we denote the characteristic vector of F by $Y^F \in \mathbb{R}^{E_m}$(we use capital letters for vectors in the space belonging to K_m).

The *Boolean Quadric Polytope* of the complete graph with m nodes is

$$\mathcal{BQ}_m := \operatorname{conv}\Big(\{(X^C, Y^{E_m(C)}) \mid C \subseteq V_m\}\Big).$$

This polytope was introduced by Padberg (1989). It was shown to be isomorphic to the *Cut Polytope* of the complete graph with $m+1$ nodes (i.e., the convex hull of all characteristic vectors – with respect just to edges – of cuts in that graph) by De Simone (1989).

Let $\overline{\mathcal{QAP}_n}$ be the canonical embedding of $\mathcal{QAP}_n$ into the space $\mathbb{R}^{V_{n^2}} \times \mathbb{R}^{E_{n^2}}$. The following theorem was independently also discovered by Rijal (1995).

Theorem 2. *$\overline{\mathcal{QAP}_n}$ is a face of $\mathcal{BQ}_{n^2}$. Hence, $\mathcal{QAP}_n$ is isomorphic to a face of $\mathcal{BQ}_{n^2}$ and to a face of the Cut Polytope of the complete graph with n^2+1 nodes.*

Proof. Obviously, $\overline{\mathcal{QAP}_n}$ is contained in the face

$$\mathcal{F} := \{(X,Y) \in \mathcal{BQ}_{n^2} \mid Y_{\{(i,j),(k,l)\}} = 0 \text{ for all } \{(i,j),(k,l)\} \in E_{n^2} \setminus \mathcal{E}_n\}$$

of $\mathcal{BQ}_{n^2}$. But $X(V_{n^2}) \leq n$ is a valid inequality for $\mathcal{F}$, and the face of $\mathcal{F}$ defined by the corresponding equation is precisely $\overline{\mathcal{QAP}_n}$. Hence $\overline{\mathcal{QAP}_n}$ is a face of a face of $\mathcal{BQ}_{n^2}$. $\square$

It is a well known result (found by Barahona and Mahjoub (1986)) that the Cut Polytope of the complete graph has diameter one. Clearly, this fact and Theorem 2 yield immediately:

Theorem 3. *The diameter of $\mathcal{QAP}_n$ equals one (for $n \geq 2$).*

Next, we will derive a linear description of a polytope $\mathcal{P}_n \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ whose integer points are precisely the vertices of $\mathcal{QAP}_n$. We observe that for any row or column $S \subseteq \mathcal{V}_n$ the equation

$$x(S) = 1 \tag{1}$$

holds for all points $(x,y) \in \mathcal{QAP}_n$. If furthermore $v \in \mathcal{V}_n \setminus S$ then also

$$y(v : S) - x_v = 0 \tag{2}$$

is valid for $\mathcal{QAP}_n$. We denote the system of all equations of types (1) and (2) by $A^{(n)}(x,y) = b^{(n)}$. It consists of $2n + 2n^2(n-1)$ equations. We define

$$\mathcal{P}_n := \{(x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid A^{(n)}(x,y) = b^{(n)}, y \geq 0\}$$

Remark 1. *For all $(x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ we have*

$$(x,y) \in \mathcal{P}_n \quad \Longrightarrow \quad y \leq 1,\ x \geq 0,\ x \leq 1.$$

The following theorem was discovered first by Johnson (1992), and later, independently, by Drezner (1994), Rijal (1995), and the authors.

Theorem 4. $\mathcal{QAP}_n = \operatorname{conv}\Big(\mathcal{P}_n \cap (\mathbb{Z}^{\mathcal{V}_n} \times \mathbb{Z}^{\mathcal{E}_n})\Big)$

Proof. Clearly, $\mathcal{QAP}_n \subseteq \operatorname{conv}\big(\mathcal{P}_n \cap (\mathbb{Z}^{\mathcal{V}_n} \times \mathbb{Z}^{\mathcal{E}_n})\big)$ does hold. To show $\mathcal{QAP}_n \supseteq \operatorname{conv}\big(\mathcal{P}_n \cap (\mathbb{Z}^{\mathcal{V}_n} \times \mathbb{Z}^{\mathcal{E}_n})\big)$, let $(x,y) \in \mathcal{P}_n \cap (\mathbb{Z}^{\mathcal{V}_n} \times \mathbb{Z}^{\mathcal{E}_n})$. Since the equations of type (1) hold for the $\{0,1\}$-vector (x,y) (cf. Remark 1), x is the characteristic vector of an n-clique $C \subseteq \mathcal{V}_n$ of $\mathcal{G}_n$ and y is the characteristic vector of some subset $F \subseteq \mathcal{E}_n$ of the edges of $\mathcal{G}_n$. Hence, it remains to prove that $F = \mathcal{E}_n(C)$. Let $e = [i,j,k,l] \in F$ be an arbitrary edge. Because the equations $y((i,j) : \text{row}_k^{(n)}) - x_{(i,j)} = 0$ and $y((k,l) : \text{row}_i^{(n)}) - x_{(k,l)} = 0$ hold, we conclude $(i,j),(k,l) \in C$, and hence $F \subseteq \mathcal{E}_n(C)$. On the other hand, adding up $(n-1)$ times all equations of type (1) and once all equations of type (2), and dividing the resulting equation by 4, one obtains $y(\mathcal{E}_n) = \frac{1}{2}n(n-1)$, which implies $|F| = |\mathcal{E}_n(C)|$, and the proof is complete. $\square$

Remark 2. *Theorem 8 will show that all inequalities in the linear description of $\mathcal{P}_n$ are facet defining for $\mathcal{QAP}_n$. Theorem 11 will exhibit a maximal redundant subset of the equations in the above linear description.*

It follows from (the easy part of) Theorem 4 that

$$\min\{(g,h)^T(x,y) \mid (x,y) \in \mathcal{P}_n\} \leq \min\{(g,h)^T(x,y) \mid (x,y) \in \mathcal{QAP}_n\}$$

for all $(g,h) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$. We call the lower bound for the QAP obtained by solving the linear program $\min\{(g,h)^T(x,y) \mid (x,y) \in \mathcal{P}_n\}$ the *Equation Bound.*

It was proved by Johnson (1992) and later, independently, by Drezner (1994) that the Equation Bound is always at least as good as the *Gilmore/Lawler Bound* proposed independently by Gilmore (1962) and Lawler (1963). Furthermore, computational experiments – most extensively done by Resende, Ramakrishnan, and Drezner (1994) – showed that the Equation Bound is the best known lower bound for most of the instances in the QAPLIB (Burkard, Karisch, and Rendl, 1994) of size at most 30. The Equation Bound was not computed for larger instances up to now due to the huge and very difficult linear programs that arise. (There are instances of size 16 in the QAPLIB that are still not solved to optimality!) For approximating this bound, Johnson (1992) developed a nice Langrangean relaxation procedure.

The theoretical and practical results concerning the Equation Bound motivated our further study of $\mathcal{QAP}_n$.

3 Another Representation: $\mathcal{QAP}_n^\star$

Starting the investigation of $\mathcal{QAP}_n$, it turns out rather quickly that the vertices of $\mathcal{QAP}_n$ are very difficult to handle with respect to such goals like showing the affine independency of a certain subset of them. This is mainly due to the fact that among these vectors there are no pairs that have just "slightly differing" supports.

For this reason, we searched for another representation of the QAP-Polytope, i.e., we tried to map $\mathcal{QAP}_n$ isomorphically into another space, in which the vertices "look nicer". For this purpose, recall that the system $A^{(n)}(x,y) = b^{(n)}$ of $2n + 2n^2(n-1)$ equations is valid for $\mathcal{QAP}_n$, indicating a lot of redundant variables. Note also that $\mathcal{G}_{n-1}$ is contained in $\mathcal{G}_n$ in the sense that $\mathcal{V}_{n-1} \subseteq \mathcal{V}_n$ and $\mathcal{E}_{n-1} \subseteq \mathcal{E}_n$.

Proposition 1. *Let M be a subset of the row indices of $A^{(n)}(x,y) = b^{(n)}$ corresponding to a set of equations consisting of*

(a) all but one of the equations $x(S) = 1$ $(S \in \{\mathrm{row}_1^{(n)}, \dots, \mathrm{row}_n^{(n)}, \mathrm{col}_1^{(n)}, \dots, \mathrm{col}_n^{(n)}\})$

(b) all equations $y(w : \mathrm{row}_i^{(n)}) - x_w = 0$ $(w \in \mathcal{V}_{n-1}, i \in \{1, \dots, n-1\} \setminus \{r(w)\})$

(c) all equations $y(w : \mathrm{col}_j^{(n)}) - x_w = 0$ $(w \in \mathcal{V}_{n-1}, j \in \{1, \dots, n-1\} \setminus \{c(w)\})$

(d) for every node $w \in \mathcal{V}_{n-1}$ either $y(w : \mathrm{row}_n^{(n)}) - x_w = 0$ or $y(w : \mathrm{col}_n^{(n)}) - x_w = 0$, but not both

(e) for every pair $v \in \mathrm{row}_n^{(n)} \setminus \{(n,n)\}$, $v' \in \mathrm{col}_n^{(n)} \setminus \{(n,n)\}$ *either* $y(v : \mathrm{row}_{r(v')}^{(n)}) - x_v = 0$ *or* $y(v' : \mathrm{col}_{c(v)}^{(n)}) - x_{v'} = 0$, *but not both.*

Then $A_{M,\bullet}^{(n)}(x,y) = b_M^{(n)}$ *is equivalent to the following system.*

$$
\begin{array}{lllr}
(1) & x_v & = 1 - x(\mathrm{col}_{c(v)}^{(n-1)}) & (v \in \mathrm{row}_n^{(n)} \setminus \{(n,n)\}) \\
(2) & x_v & = 1 - x(\mathrm{row}_{r(v)}^{(n-1)}) & (v \in \mathrm{col}_n^{(n)} \setminus \{(n,n)\}) \\
(3) & x_{(n,n)} & = x(\mathcal{V}_{n-1}) - (n-2) & \\
(4) & y_{\{v,w\}} & = x_w - y(w : \mathrm{col}_{c(v)}^{(n-1)}) & (v \in \mathrm{row}_n^{(n)} \setminus \{(n,n)\}, \\
 & & & w \in \mathcal{V}_{n-1} \setminus \mathrm{col}_{c(v)}^{(n)}) \\
(5) & y_{\{v,w\}} & = x_w - y(w : \mathrm{row}_{r(v)}^{(n-1)}) & (v \in \mathrm{col}_n^{(n)} \setminus \{(n,n)\}, \\
 & & & w \in \mathcal{V}_{n-1} \setminus \mathrm{row}_{r(v)}^{(n)}) \\
(6) & y_{\{(n,n),w\}} & = y(w : \mathcal{V}_{n-1}) - (n-3)x_w & (w \in \mathcal{V}_{n-1}) \\
(7) & y_{\{v,v'\}} & = 1 - \left(x(\mathrm{col}_{c(v)}^{(n-1)} \cup \mathrm{row}_{r(v')}^{(n-1)}) - y(\mathrm{col}_{c(v)}^{(n-1)} : \mathrm{row}_{r(v')}^{(n-1)})\right) & \\
 & & & (v \in \mathrm{row}_n^{(n)} \setminus \{(n,n)\}, \\
 & & & v' \in \mathrm{col}_n^{(n)} \setminus \{(n,n)\})
\end{array}
$$

Proof. We show that any equation in (1), ... , (7) can be obtained from a linear combination of equations in (a), ... , (e). Noting that (a), ... , (e) and (1), ... , (7) have the same number of equations (and that the system (1), ... , (7) has full row rank), this will prove the proposition.

Since we know that in the node-edge-incidence matrix of the complete bipartite graph $K_{n,n}$ any single row can be linearly combined by the other rows, we also can linearly combine all equations in (1), (2), and (3) from the equations in (a). Equations (4) and (5) are precisely the equations in (b) and (c). We can combine any equation in (6) from an appropriate one in (d) and some suitable ones either in (b) or in (c). Similarly, we can combine any equation in (7) from one equation in (e), some equations either in (b) or in (c), and one equation in (a). □

Let $W := \mathrm{row}_n^{(n)} \cup \mathrm{col}_n^{(n)}$, $F := \mathcal{E}_n(W) \cup \delta(W)$, and $\overline{W} := \mathcal{V}_n \setminus W$, $\overline{F} := \mathcal{E}_n \setminus F$. Let the system of equations (1), ... , (7) in Proposition 1 be

$$(x_W, y_F) = A'(x_{\overline{W}}, y_{\overline{F}}) + b'.$$

Define the affine map $\Phi : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ via $\Phi(x,y) := (x', y')$ with

$$
\begin{array}{lcl}
(x'_W, y'_F) & := & (x_W, y_F) - (A'(x_{\overline{W}}, y_{\overline{F}}) + b'), \\
(x'_{\overline{W}}, y'_{\overline{F}}) & := & (x_{\overline{W}}, y_{\overline{F}})
\end{array}
$$

for all $(x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$. Obviously, Φ is an affine transformation of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$, i.e., a one-to-one affine map. Proposition 1 implies the following.

Lemma 1. *On the affine space*

$$\mathcal{A} := \{(x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid A^{(n)}(x,y) = b^{(n)}\}$$

(containing $\mathcal{QAP}_n$), Φ coincides with the canonical projection

$$\pi^{\mathrm{row}_n^{(n)} \cup \mathrm{col}_n^{(n)}} : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow U := \{(x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid x_W = 0, y_F = 0\}.$$

Clearly, U can be identified with $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$. So we have shown that the canonical projection $\pi : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$ transforms the polytope $\mathcal{QAP}_n \subseteq \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ into an isomorphic polytope

$$\mathcal{QAP}^\star_{n-1} \subseteq \mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}.$$

Hence, investigating the combinatorial structures (face lattices) of the polytopes $\mathcal{QAP}_n$ ($n \geq 2$) is equivalent to investigating the combinatorial structures of the polytopes $\mathcal{QAP}^\star_n$ ($n \geq 1$).

Remark 3. *The following describes the relationship between $\mathcal{QAP}_n$ and $\mathcal{QAP}^\star_{n-1}$ more precisely.*

(i) π induces a one-to-one mapping κ between the set $\mathcal{CLQ}^n_n$ of n-cliques of $\mathcal{G}_n$ and the set

$$\mathcal{C}^\star_{n-1} := \{C \subseteq \mathcal{V}_{n-1} \mid C \text{ is } (n-1)\text{- or } (n-2)\text{-clique of } \mathcal{G}_{n-1}\}$$

of $(n-1)$- and $(n-2)$-cliques of $\mathcal{G}_{n-1}$ (cf. Figure 2).

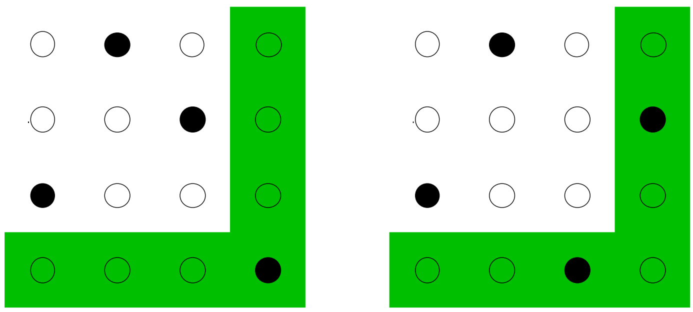


Figure 2: The effect of the projection π.

(ii) Let $\mathcal{F} \subseteq \mathcal{QAP}_n$ be a face of $\mathcal{QAP}_n$ with vertices corresponding to a set $F \subseteq \mathcal{CLQ}^n_n$, and let $\mathcal{F}^* \subseteq \mathcal{QAP}^\star_{n-1}$ be the corresponding (via π) face of $\mathcal{QAP}^\star_{n-1}$. Then the vertices of $\mathcal{F}^*$ correspond to the projections $\kappa(F)$ of the n-cliques in F. In particular: The vertices of $\mathcal{QAP}^\star_{n-1}$ are the characteristic vectors (with respect to nodes and edges) of the $n-1$- and $(n-2)$-cliques of $\mathcal{G}_{n-1}$.

(iii) Let $(u,v)^T(x,y) \leq \omega$ be a valid (resp. facet defining) inequality for $\mathcal{QAP}_n$, and let $(\hat{u},\hat{v}) \in \mathbb{R}^{\mathcal{V}_{n+1}} \times \mathbb{R}^{\mathcal{E}_{n+1}}$ be the vector arising from (u,v) by "zero-lifting". From part (ii) of this remark it follows that $(\hat{u},\hat{v})^T(\hat{x},\hat{y}) \leq \omega$ is valid (resp. facet defining) for $\mathcal{QAP}_{n+1}$, if, and only if, $(u,v)^T(x,y) \leq \omega$ is valid (resp. facet defining) for $\mathcal{QAP}^\star_n$.

Observation 2. *As for $\mathcal{QAP}_n$, permutations of the rows, permutations of the columns, and transposition of the node set induce symmetries of $\mathcal{QAP}_n^\star$.*

Next, as we did for $\mathcal{QAP}_n$, we will derive a linearly described polytope $\mathcal{P}_n^\star \subseteq \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$, whose integer points are the vertices of $\mathcal{QAP}_n^\star$. Let $\alpha_1, \alpha_2 \in \{1, \dots, n\}$ be two distinct numbers. Then, for all points $(x, y) \in \mathcal{QAP}_n^\star$

$$x(\mathrm{row}_{\alpha_1}^{(n)}) + x(\mathrm{row}_{\alpha_2}^{(n)}) - y(\mathrm{row}_{\alpha_1}^{(n)} : \mathrm{row}_{\alpha_2}^{(n)}) = 1$$

and

$$x(\mathrm{col}_{\alpha_1}^{(n)}) + x(\mathrm{col}_{\alpha_2}^{(n)}) - y(\mathrm{col}_{\alpha_1}^{(n)} : \mathrm{col}_{\alpha_2}^{(n)}) = 1$$

hold. Let $A^{\star(n)}(x, y) = b^{\star(n)}$ be the system of all these equations. Also, for all $v \in \mathcal{V}_n$, $i \in \{1, \dots, n\} \setminus r(v)\}$, $j \in \{1, \dots, n\} \setminus c(v)\}$ and $(x, y) \in \mathcal{QAP}_n^\star$

$$y(v : \mathrm{row}_i^{(n)}) - x_v \leq 0 \qquad \text{and} \qquad y(v : \mathrm{col}_j^{(n)}) - x_v \leq 0$$

are valid. Hence, $\mathcal{QAP}_n^\star$ is contained in the polytope

$$\mathcal{P}_n^\star := \left\{ (x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \;\middle|\; \begin{array}{rcll} A^{\star(n)}(x,y) & = & b^{\star(n)} & \\ y(v : \mathrm{row}_i^{(n)}) - x_v & \leq & 0 & (v \in \mathcal{V}_n,\ i \neq r(v)) \\ y(v : \mathrm{col}_j^{(n)}) - x_v & \leq & 0 & (v \in \mathcal{V}_n,\ j \neq c(v)) \\ y & \geq & 0 & \end{array} \right\}.$$

Remark 4. *For all $(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$, we have*

$$(x, y) \in \mathcal{P}_n^\star \quad \Longrightarrow \quad x \leq 1,\ x \geq 0,\ y \leq 1.$$

Lemma 2. *Let $C \subseteq \mathcal{V}_n$ be an arbitrary clique of $\mathcal{G}_n$. If $A^{\star(n)}(x^C, y^{\mathcal{E}_n(C)}) = b^{\star(n)}$ then C is of size n or $n-1$.*

Proof. Clearly, $|C| \leq n$. Suppose, $|C| < n-1$, then there must be two rows, without loss of generality $\mathrm{row}_1^{(n)}$ and $\mathrm{row}_2^{(n)}$, such that $C \cap \mathrm{row}_1^{(n)} = \emptyset$ and $C \cap \mathrm{row}_2^{(n)} = \emptyset$, contradicting $x^C(\mathrm{row}_1^{(n)}) + x^C(\mathrm{row}_2^{(n)}) - y^{\mathcal{E}_n(C)}(\mathrm{row}_1^{(n)} : \mathrm{row}_2^{(n)}) = 1$. □

Theorem 5. $\mathcal{QAP}_n^\star = \mathrm{conv}\Big(\mathcal{P}_n^\star \cap (\mathbb{Z}^{\mathcal{V}_n} \times \mathbb{Z}^{\mathcal{E}_n})\Big)$

Proof. Clearly, $\mathcal{QAP}_n^\star \subseteq \mathrm{conv}\big(\mathcal{P}_n^\star \cap (\mathbb{Z}^{\mathcal{V}_n} \times \mathbb{Z}^{\mathcal{E}_n})\big)$. To show $\mathrm{conv}\big(\mathcal{P}_n^\star \cap (\mathbb{Z}^{\mathcal{V}_n} \times \mathbb{Z}^{\mathcal{E}_n})\big) \subseteq \mathcal{QAP}_n^\star$, let $(x, y) \in \mathcal{P}_n^\star \cap (\mathbb{Z}^{\mathcal{V}_n} \times \mathbb{Z}^{\mathcal{E}_n})$. Since (x, y) is a $\{0,1\}$-vector (cf. Remark 4), x is the characteristic vector of some $C \subseteq \mathbb{R}^{\mathcal{V}_n}$, and y is the characteristic vector of some $F \subseteq \mathbb{R}^{\mathcal{E}_n}$. From $x(\mathrm{row}_1^{(n)}) + x(\mathrm{row}_2^{(n)}) - y(\mathrm{row}_1^{(n)} : \mathrm{row}_2^{(n)}) = 1$ and $y(v : \mathrm{row}_1^{(n)}) - x_v \leq 0$ for all $v \in \mathrm{row}_2^{(n)}$, we can conclude $x(\mathrm{row}_1^{(n)}) \leq 1$. An analogous argument holds for all other rows as well as for all columns, and hence, C must be a clique of $\mathcal{G}_n$. Therefore, by Lemma 2, it suffices to show $F = \mathcal{E}_n(C)$.

If $\{v, w\} \in F$ then $y(v : \mathrm{row}_{r(w)}^{(n)}) - x_v \leq 0$ implies $v \in C$, and $y(w : \mathrm{row}_{r(v)}^{(n)}) - x_w \leq 0$ implies $w \in C$, which proves $F \subseteq \mathcal{E}_n(C)$. On the other hand, for any $\{v, w\} \in \mathcal{E}_n(C)$, due to $x(\mathrm{row}_{r(v)}^{(n)}) + x(\mathrm{row}_{r(w)}^{(n)}) - y(\mathrm{row}_{r(v)}^{(n)} : \mathrm{row}_{r(w)}^{(n)}) = 1$, there must be an edge in $F \cap (\mathrm{row}_{r(v)}^{(n)} : \mathrm{row}_{r(w)}^{(n)})$. This shows $|\mathcal{E}_n(C)| \leq |F|$, and the proof of the theorem is complete. □

Remark 5. *From Theorem 8 (coming up in Section 4) one can conclude that all inequalities in the linear description of $\mathcal{P}_n^\star$ are facet defining for $\mathcal{QAP}_n^\star$. Theorem 10 (in Section 5) will show that precisely one (arbitrary) equation in that linear description is redundant.*

In Theorem 2 we showed that the canonical embedding $\overline{\mathcal{QAP}_n}$ of $\mathcal{QAP}_n$ into the space $\mathbb{R}^{V_{n^2}} \times \mathbb{R}^{E_{n^2}}$ is a face of the Boolean Quadric Polytope $\mathcal{BQ}_{n^2}$. Analogously, let $\overline{\mathcal{QAP}_n^\star}$ be the canonical embedding of $\mathcal{QAP}_n^\star$ into $\mathbb{R}^{V_{n^2}} \times \mathbb{R}^{E_{n^2}}$.

Theorem 6. *$\overline{\mathcal{QAP}_n^\star}$ is a face of $\mathcal{BQ}_{n^2}$. Hence, $\mathcal{QAP}_n^\star$ is also isomorphic to a face of $\mathcal{BQ}_{n^2}$ and to a face of the Cut Polytope of the complete graph with n^2+1 nodes.*

Proof. Analogously to the proof of Theorem 2, $\overline{\mathcal{QAP}_n^\star}$ is contained in the face

$$\mathcal{F} := \{(X,Y) \in \mathcal{BQ}_{n^2} \mid Y_{\{(i,j),(k,l)\}} = 0 \text{ for all } \{(i,j),(k,l)\} \in E_{n^2} \setminus \mathcal{E}_n\}$$

of $\mathcal{BQ}_{n^2}$. Let $\overline{A^{\star(n)}}$ be the matrix obtained by adding a zero-column to $A^{\star(n)}$ for each edge in $E_{n^2} \setminus \mathcal{E}_n$. Then, $\overline{A^{\star(n)}}(X,Y) \leq b^{\star(n)}$ is valid for $\mathcal{F}$, since the vertices of $\mathcal{F}$ correspond to the cliques of $\mathcal{G}_n$. The face of $\mathcal{F}$ defined by $\overline{A^{\star(n)}}(X,Y) = b^{\star(n)}$ contains $\overline{\mathcal{QAP}_n^\star}$. But then, using Lemma 2, one deduces that this face is precisely $\overline{\mathcal{QAP}_n^\star}$, which proves the theorem. $\square$

Remark 6. *Due to Theorems 2 and 6, $\mathcal{QAP}_n$ is isomorphic both to a face of $\mathcal{BQ}_{n^2}$ as well as to a face of $\mathcal{BQ}_{(n-1)^2}$ (for $n \geq 2$) and to faces of the corresponding Cut Polytopes, too.*

By construction, $\mathcal{QAP}_n^\star$ is isomorphic to $\mathcal{QAP}_{n+1}$, but "living" in the same space as $\mathcal{QAP}_n$. So, what is the relationship between $\mathcal{QAP}_n$ and $\mathcal{QAP}_n^\star$? By observing that

$$x(\mathcal{V}_n) \leq n$$

is valid for $\mathcal{QAP}_n^\star$, one deduces immediately:

Observation 3. *$\mathcal{QAP}_n$ is a face of $\mathcal{QAP}_n^\star$.*

However, the relationship is indeed much stronger.

Proposition 2. *Let $n \geq 2$. There are $n+1$ affine maps $\phi_\alpha : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ ($\alpha = 0, \ldots, n$) such that for the $n+1$ images $\mathcal{Q}_\alpha := \phi_\alpha(\mathcal{QAP}_n)$ ($\alpha = 0, \ldots, n$) of $\mathcal{QAP}_n$ the following hold:*

(i) *Every $\mathcal{Q}_\alpha$ is isomorphic to $\mathcal{QAP}_n$.*

(ii) *Each $\mathcal{Q}_\alpha$ is a face of $\mathcal{QAP}_n^\star$.*

(iii) *The $\mathcal{Q}_\alpha$ have pairwise empty intersection.*

(iv) $\mathcal{QAP}_n^\star = \operatorname{conv}\left(\bigcup_{\alpha=0}^{n} \mathcal{Q}_\alpha\right)$

Proof. For any row or column $S \in \{\mathrm{row}_1^{(n)}, \dots, \mathrm{row}_n^{(n)}, \mathrm{col}_1^{(n)}, \dots, \mathrm{col}_n^{(n)}\}$, let $\pi^S : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \{(x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid x_S = 0,\ y_{\delta(S)} = 0\}$ be the canonical projection. Then, the map $\pi^{\mathrm{row}_n^{(n)} \cup \mathrm{col}_n^{(n)}}$ in Lemma 1 decomposes into

$$\pi^{\mathrm{row}_n^{(n)} \cup \mathrm{col}_n^{(n)}} = \pi^{\mathrm{col}_n^{(n)}} \circ \pi^{\mathrm{row}_n^{(n)}}.$$

Since Lemma 1 showed that $\pi^{\mathrm{row}_n^{(n)} \cup \mathrm{col}_n^{(n)}}$ performs an isomorphic transformation of $\mathcal{QAP}_n$, so does $\pi^{\mathrm{row}_n^{(n)}}$, too. Clearly, there is nothing special about $\mathrm{row}_n^{(n)}$, and therefore, the same holds for all

$$\phi_\alpha := \pi^{\mathrm{row}_\alpha^{(n)}} \quad (\alpha = 1, \dots, n).$$

Finally, define ϕ_0 to be the identical map on $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$. Hence, all the

$$\mathcal{Q}_\alpha := \phi_\alpha(\mathcal{QAP}_n) \quad (\alpha = 0, \dots, n)$$

are isomorphic to $\mathcal{QAP}_n$, what proves part *(i)*. Parts *(ii)*, *(iii)*, and *(iv)* follow from the observation, that for any $\alpha \in \{1, \dots, n\}$, the vertices of $\mathcal{Q}_\alpha$ correspond to the $(n-1)$-cliques of $\mathcal{G}_n$ having no node in common with the α-th row of $\mathcal{V}_n$. □

From Proposition 2 and the isomorphism between $\mathcal{QAP}_{n+1}$ and $\mathcal{QAP}_n^\star$, the following "inductive construction" of $\mathcal{QAP}_{n+1}$ follows.

Theorem 7. *For $n \geq 1$ there are $n+1$ affine maps $\iota_\alpha : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_{n+1}} \times \mathbb{R}^{\mathcal{E}_{n+1}}$ $(\alpha = 0, \dots, n)$ such that for the $n+1$ images $\mathcal{Q}_\alpha := \iota_\alpha(\mathcal{QAP}_n)$ $(\alpha = 0, \dots, n)$ of $\mathcal{QAP}_n$ the following hold:*

(i) *Every $\mathcal{Q}_\alpha$ is isomorphic to $\mathcal{QAP}_n$.*

(iI) *Each $\mathcal{Q}_\alpha$ is a face of $\mathcal{QAP}_{n+1}$.*

(iiI) *The $\mathcal{Q}_\alpha$ have pairwise empty intersection.*

(iv) $\mathcal{QAP}_{n+1} = \mathrm{conv}\left(\bigcup_{\alpha=0}^{n} \mathcal{Q}_\alpha\right)$

In the remaining part of this section, we will investigate the system $A^{\star(n)}(x,y) = b^{\star(n)}$ describing an affine subspace

$$\mathcal{A}^* := \{(x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid A^{\star(n)}(x,y) = b^{\star(n)}\}$$

that contains $\mathcal{QAP}_n^\star$. It will turn out (cf. Section 5) that, indeed, $\mathcal{A}^*$ is the affine hull of $\mathcal{QAP}_n^\star$. $A^{\star(n)}(x,y) = b^{\star(n)}$ consists of the $\frac{1}{2}n(n-1)$ equations

$$x(\mathrm{row}_i^{(n)}) + x(\mathrm{row}_k^{(n)}) - y(\mathrm{row}_i^{(n)} : \mathrm{row}_k^{(n)}) = 1 \quad (1 \leq i < k \leq n) \tag{3}$$

and of the $\frac{1}{2}n(n-1)$ equations

$$x(\mathrm{col}_j^{(n)}) + x(\mathrm{col}_l^{(n)}) - y(\mathrm{col}_j^{(n)} : \mathrm{col}_l^{(n)}) = 1 \quad (1 \leq j < l \leq n). \tag{4}$$

Now, let's consider just the "y-part" $M := A^{\star(n)}_{\bullet,\mathcal{E}_n}$ of the matrix A^*. We define a (total) ordering of the edges $\mathcal{E}_n$ by requiring that each edge $[i,j,k,l] \in \mathcal{E}_n$ with $i < k$ and $j < l$ has as its successor the edge $[i,l,k,j]$, and by ordering the edges $\{[i,j,k,l] \in \mathcal{E}_n \mid i < k, j < l\}$

lexicographically according to the quadrupels (i,k,j,l). After permuting the columns of M with respect to this ordering of $\mathcal{E}_n$, M is the following $n(n-1)\times|\mathcal{E}_n|$ matrix (for $n=3$):

$$\begin{pmatrix}
1&1&1&1&1&1&&&&&&&&&&&& \\
&&&&&&1&1&1&1&1&1&&&&&& \\
&&&&&&&&&&&&1&1&1&1&1&1 \\
1&1&&&&&1&1&&&&&1&1&&&& \\
&&1&1&&&&&1&1&&&&&1&1&& \\
&&&&1&1&&&&&1&1&&&&&1&1
\end{pmatrix}$$

We are interested in the bases of the matrix M, i.e., in maximal subsets of linearly independent columns of M. Since columns corresponding to edges $[i,j,k,l]$ and $[i,l,k,j]$ are identical, we can identify them for our considerations. But then, the resulting $n(n-1)\times\frac{1}{2}|\mathcal{E}_n|$ matrix M' is the node-edge incidence matrix of the complete bipartite graph $K_{\frac{n(n-1)}{2},\frac{n(n-1)}{2}} = (\mathcal{R}_n\cup\mathcal{C}_n,(\mathcal{R}_n:\mathcal{C}_n))$, $(\mathcal{R}_n\cap\mathcal{C}_n=\emptyset)$,

A pair $\{[i,j,k,l],[i,l,k,j]\}$ of edges $[i,j,k,l],[i,l,k,j]\in\mathcal{E}_n$ is called a *crossing pair.* We interpret the "left shore" $\mathcal{R}_n$ of $K_{\frac{n(n-1)}{2},\frac{n(n-1)}{2}}$ with the set of all possible pairs of rows of $\mathcal{V}_n$, and the "right shore" $\mathcal{C}_n$ with the set of all possible pairs of columns of $\mathcal{V}_n$. The edges $(\mathcal{R}_n:\mathcal{C}_n)$ are then in one-to-one correspondence with the set of crossing pairs of $\mathcal{E}_n$.

It is a well-known fact (discovered by Balinski and Russakoff, 1974) that the set of bases of the node-edge-incidence matrix of the complete bipartite graph $K_{m,m}$ is precisely the set of node-edge-incidence matrices of spanning trees of $K_{m,m}$. This yields the following characterization of all bases of $A^{\star(n)}$ that do not contain columns corresponding to nodes of $\mathcal{G}_n$.

Proposition 3.

(i) Precisely one (arbitrary) equation in $A^{\star(n)}(x,y)=b^{\star(n)}$ *is redundant, in particular* $\operatorname{rank}(A^{*})=n(n-1)-1$.

(ii) A subset $B\subseteq\mathcal{E}_n$ *of edges of* $\mathcal{G}_n$ *corresponds to a basis of* $A^{\star(n)}$ *if and only if*

- *(a)* $|B|=n(n-1)-1$
- *(b) There is no crossing pair contained in* B.
- *(c) There is no sequence* $(e_0,f_0,e_1,f_1,\ldots,e_{k-1},f_{k-1})$ $(k\geq 2)$ *of edges in* B *such that* e_α *and* f_α *connect the same rows of* $\mathcal{V}_n$ *and* f_α *and* $e_{(\alpha+1)\bmod k}$ *connect the same columns of* $\mathcal{V}_n$ *for all* $\alpha=0,\ldots,k-1$.

4 Dimension and trivial facets of $\mathcal{QAP}_n$ / $\mathcal{QAP}_n^\star$

Theorem 8. *For* $n\geq 3$ *the following three statements hold.*

(i) The dimension of the QAP-Polytope is

$$\dim(\mathcal{QAP}_n)=(n-1)^2+\frac{(n-1)^2(n-2)^2}{2}-((n-1)(n-2)-1).$$

(ii) The inequalities

$$y_e \geq 0 \qquad (e \in \mathcal{E}_n)$$

are facet defining for $\mathcal{QAP}_n$.

(iii) The inequalities

$$\begin{aligned} y_e &\leq 1 &\qquad (e \in \mathcal{E}_n) \\ x_v &\geq 0 &\qquad (v \in \mathcal{V}_n) \\ x_v &\leq 1 &\qquad (v \in \mathcal{V}_n) \end{aligned}$$

are implied by the equation system $A^{(n)}(x, y) = b^{(n)}$ and the nonnegativity of the y-variables.

The results described in this theorem were found independently also by Rijal (1995). His proof is a "direct" one (i.e., using affinely independent vectors and a dimension argument to show that a certain space is generated), while ours is an "indirect" proof (i.e., using affine/linear combinations to show that this space is generated). Our use of the representation $\mathcal{QAP}_n^\star$ of the QAP-Polytope simplifies the proof enormously, since the presence of characteristic vectors of cliques of two different sizes allows us to construct "much nicer" linear combinations.

Statement *(iii)* of Theorem 8 is just a repetition of Remark 1. Parts *(i)* and *(ii)* follow from the following Theorem describing the dimension and the trivial facets of $\mathcal{QAP}_n^\star$.

Theorem 9. *For $n \geq 2$ the following two statements hold.*

(i)

$$\dim(\mathcal{QAP}_n^\star) = n^2 + \frac{n^2(n-1)^2}{2} - (n(n-1) - 1)$$

(ii) The inequalities

$$y_e \geq 0 \qquad (e \in \mathcal{E}_n)$$

are facet defining for $\mathcal{QAP}_n^\star$.

Proof of Theorem 9. We will give a single proof covering both parts of the theorem. Due to the symmetries of $\mathcal{QAP}_n^\star$ described in Observation 2, we can restrict the proof of part *(ii)* to the case $e = [n, n-1, n-1, n]$. Consider the face

$$F := \mathcal{QAP}_n^\star \cap \{(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid y_{[n,n-1,n-1,n]} = 0\}$$

of $\mathcal{QAP}_n^\star$. Define

$$\mathcal{F} := \{(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid A^{\star(n)}(x, y) = b^{\star(n)}, y_{[n,n-1,n-1,n]} = 0\}.$$

Claim 1: If we prove $\mathcal{F} = \mathrm{aff}(F)$ then the whole theorem follows.
Proof of Claim 1. From Proposition 3 we know that $\dim(\mathcal{QAP}_n^\star) \leq n^2 + \frac{n^2(n-1)^2}{2} - (n(n-1)-1)$. Since F is a proper face of $\mathcal{QAP}_n^\star$, we know that $A^{\star(n)}(x, y) = b^{\star(n)}$ does not imply

$y_{[n,n-1,n-1,n]} = 0$, and hence, again, by Proposition 3, $\dim(\mathcal{F}) = n^2 + \frac{n^2(n-1)^2}{2} - n(n-1)$. Now, $\mathcal{F} = \operatorname{aff}(F)$ implies that

$$\begin{aligned} n^2 + \frac{n^2(n-1)^2}{2} - n(n-1) &= \dim(F) \\ &< \dim(\mathcal{QAP}_n^\star) \\ &\leq n^2 + \frac{n^2(n-1)^2}{2} - (n(n-1)-1), \end{aligned}$$

which proves both parts of the theorem. □

Claim 2: For $n \geq 2$ we have $\mathcal{F} = \operatorname{aff}(F)$.

Proof of Claim 2. The proposition does hold for all $n \geq 2$. However, in order to simplify the proof, we will restrict ourselves to the case $n \geq 5$, what means that in particular the four lemmata stated below apply. (The cases $n = 2, 3, 4$ can be checked by computer.) Clearly, $\mathcal{F} \supseteq \operatorname{aff}(F)$ holds. Hence, it suffices to show $\dim(\mathcal{F}) \leq \dim(F)$. As already deducted in the proof of Claim 1, $\dim(\mathcal{F}) = \dim(\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}) - n(n-1)$. For $\mathcal{L} := \{(x,y) - (x',y') \mid (x,y),(x',y') \in F\}$ we have $\dim(\operatorname{lin}(\mathcal{L})) = \dim(F)$, and therefore, it suffices to exhibit a set $\mathcal{B} \subseteq \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ of $n(n-1)$ vectors that satisfies

$$\operatorname{lin}(\mathcal{L} \cup \mathcal{B}) = \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}. \tag{5}$$

The next four lemmata will provide us with a suitable collection of vectors in $\operatorname{lin}(\mathcal{L})$. Figure 3 illustrates these vectors, as well as some notations used in the respective proofs.

Lemma 3. *If $n \geq 2$, and if $v_1, \ldots, v_n \in \mathcal{V}_n$ form an n-clique of $\mathcal{G}_n$, and $[n, n-1, n-1, n] \notin \mathcal{E}_n(\{v_1, \ldots, v_n\}$ then*

$$(x^{v_1}, 0) + \sum_{\alpha=2}^{n} (0, y^{\{v_1, v_\alpha\}}) \in \mathcal{L}.$$

Proof of Lemma 3. Let $C_1 := \{v_1, \ldots, v_n\}$ and $C_2 := \{v_2, \ldots, v_n\}$. Since $(x^{C_1}, y^{\mathcal{E}_n(C_1)})$, $(x^{C_2}, y^{\mathcal{E}_n(C_2)}) \in F$, we conclude that $(x^{v_1}, 0) + \sum_{\alpha=2}^{n} (0, y^{\{v_1, v_\alpha\}}) = ((x^{C_1}, y^{\mathcal{E}_n(C_1)}) - (x^{C_2}, y^{\mathcal{E}_n(C_2)}) \in \operatorname{lin}(\mathcal{L})$. □

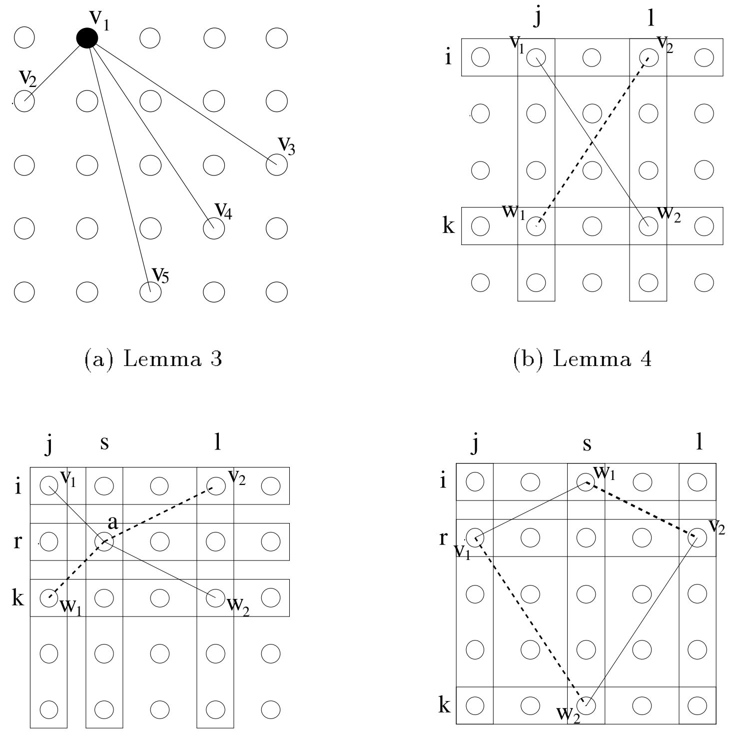
Lemma 4. *If $n \geq 4$, $[i,j,k,l] \in \mathcal{E}_n$ and $[n, n-1, n-1, n] \notin \{[i,j,k,l], [i,l,k,j]\}$ then*

$$(0, y^{[i,j,k,l]}) - (0, y^{[i,l,k,j]}) \in \operatorname{lin}(\mathcal{L}).$$

Proof of Lemma 4. Let $v_1 := (i,j)$, $v_2 := (i,l)$, $w_1 := (k,j)$, and $w_2 := (k,l)$. Since $n \geq 4$, we can find $u_1, \ldots, u_{n-2} \in \mathcal{V}_n$, such that $C_1 := \{v_1, w_2, u_1, \ldots, u_{n-2}\}$ as well as $C_2 := \{v_2, w_1, u_1, \ldots, u_{n-2}\}$ form n-cliques of $\mathcal{G}_n$ and $[n, n-1, n-1, n] \notin \mathcal{E}_n(C_\alpha)$ for $\alpha = 1, 2$. Define $C_3 := C_1 \setminus \{v_1\}$, $C_4 := C_1 \setminus \{w_2\}$, $C_5 := C_2 \setminus \{v_2\}$, and $C_6 := C_2 \setminus \{w_1\}$. Then $(x^{C_\alpha}, y^{\mathcal{E}_n(C_\alpha)}) \in F$ for $\alpha = 1, \ldots, 6$, and hence, $(0, y^{[i,j,k,l]}) - (0, y^{[i,l,k,j]}) = (x^{C_1}, y^{\mathcal{E}_n(C_1)}) - (x^{C_2}, y^{\mathcal{E}_n(C_2)}) - (x^{C_3}, y^{\mathcal{E}_n(C_3)}) - (x^{C_4}, y^{\mathcal{E}_n(C_4)}) + (x^{C_5}, y^{\mathcal{E}_n(C_5)}) + (x^{C_6}, y^{\mathcal{E}_n(C_6)}) \in \operatorname{lin}(\mathcal{L})$. □

Lemma 5. *If $n \geq 5$ and $r, i, k \in \{1, \ldots, n\}$ as well as $s, j, l \in \{1, \ldots, n\}$ are pairwise distinct and $[n, n-1, n-1, n] \notin \{[r,s,i,j], [r,s,k,j], [r,s,k,l], [r,s,i,l]\}$ then*

$$(0, y^{[r,s,i,j]}) - (0, y^{[r,s,k,j]}) + (0, y^{[r,s,k,l]}) - (0, y^{[r,s,i,l]}) \in \operatorname{lin}(\mathcal{L}).$$



(a) Lemma 3

(b) Lemma 4

(c) Lemma 5

(d) Lemma 6

Figure 3: The vectors of Lemmata 3 – 6

Proof of Lemma 5. Let $a := (r, s)$, $v_1 := (i, j)$, $v_2 := (i, l)$, $w_1 := (k, j)$, and $w_2 := (k, l)$. Since $n \geq 5$, we can find $u_1, \ldots, u_{n-3} \in \mathcal{V}_n$ such that $C_1 := \{a, v_1, w_2, u_1, \ldots, u_{n-3}\}$ as well as $C_2 := \{a, v_2, w_1, u_1, \ldots, u_{n-3}\}$ form n-cliques of $\mathcal{G}_n$ and $[n, n-1, n-1, n] \notin \mathcal{E}_n(C_\alpha)$ for $\alpha = 1, 2$. Define $C_3 := C_1 \setminus \{a\}$ and $C_4 := C_2 \setminus \{a\}$. Then $(x^{C_\alpha}, y^{\mathcal{E}_n(C_\alpha)}) \in F$ for $\alpha = 1, \ldots, 4$, and hence, $(0, y^{[r,s,i,j]}) - (0, y^{[r,s,k,j]}) + (0, y^{[r,s,k,l]}) - (0, y^{[r,s,i,l]}) = (x^{C_1}, y^{\mathcal{E}_n(C_1)}) - (x^{C_3}, y^{\mathcal{E}_n(C_3)}) - (x^{C_2}, y^{\mathcal{E}_n(C_2)}) + (x^{C_4}, y^{\mathcal{E}_n(C_4)}) \in \operatorname{lin}(\mathcal{L})$. □

Lemma 6. *If $n \geq 5$ and $r, i, k \in \{1, \ldots, n\}$ as well as $s, j, l \in \{1, \ldots, n\}$ are pairwise distinct and $[n, n-1, n-1, n] \notin \{[i, s, r, j], [r, j, k, s], [k, s, r, l], [r, l, i, s]\}$ then*

$$(0, y^{[i,s,r,j]}) - (0, y^{[r,j,k,s]}) + (0, y^{[k,s,r,l]}) - (0, y^{[r,l,i,s]}) \in \operatorname{lin}(\mathcal{L}).$$

Proof of Lemma 6. Let $v_1 := (r, j)$, $v_2 := (r, l)$, $w_1 := (i, s)$, and $w_2 := (k, s)$. Since $n \geq 5$, we can find $u_1, \ldots, u_{n-3} \in \mathcal{V}_n$ such that $C_1 := \{w_1, v_1, u_1, \ldots, u_{n-3}\}$, $C_2 := \{v_1, w_2, u_1, \ldots, u_{n-3}\}$, $C_3 := \{w_2, v_2, u_1, \ldots, u_{n-3}\}$, and $C_4 := \{v_2, w_1, u_1, \ldots, u_{n-3}\}$ form $(n-1)$-cliques of $\mathcal{G}_n$ and $[n, n-1, n-1, n] \notin \mathcal{E}_n(C_\alpha)$ for $\alpha = 1, \ldots, 4$. Hence, $(0, y^{[i,s,r,j]}) - (0, y^{[r,j,k,s]}) + (0, y^{[k,s,r,l]}) - (0, y^{[r,l,i,s]}) = (x^{C_1}, y^{\mathcal{E}_n(C_1)}) + (x^{C_3}, y^{\mathcal{E}_n(C_3)}) - (x^{C_2}, y^{\mathcal{E}_n(C_2)}) - (x^{C_4}, y^{\mathcal{E}_n(C_4)}) \in \operatorname{lin}(\mathcal{L})$. □

Now, back on the level of the proof of Claim 2, we choose $B := \{[1j2l] \mid 1 \leq j < l \leq n\} \cup \{[i1k2] \mid 1 \leq i < k \leq n\}$ and $\mathcal{B} := \{(0, y^e) \mid e \in B \cup \{[n, n-1, n-1, n]\}$. We will prove equation (5) by showing $(0, y^e) \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for all $e \in \mathcal{E}_n$ and $(x^v, 0) \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$

for all $v \in \mathcal{V}_n$. This will be done in five steps whose "effects" are shown in Figure 4. Figure 5 illustrates the notations in the proofs of Steps 2, 3, and 4 In order to simplify the notations, we define $R_\alpha := \mathrm{row}_\alpha^{(n)}$ and $C_\alpha := \mathrm{col}_\alpha^{(n)}$ for all $\alpha = 1, \ldots, n$.

Step 1: $(0, y^e) \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for all $e \in (R_1 : R_2) \cup (C_1 : C_2)$
Proof. This follows by Lemma 4. □

Step 2: $(0, y^e) \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for all $e \in ((R_1 \cup R_2) : (C_1 \cup C_2))$
Proof. Let $e = [i, j, k, l] \in (R_i : C_l)$ $(i, l \in \{1, 2\})$. If $j \in \{1, 2\}$ or $k \in \{1, 2\}$ then we are already done by Step 1. So assume $j, k \notin \{1, 2\}$. Let $\{\bar{i}\} := \{1, 2\} \setminus \{i\}$ and $\{\bar{l}\} := \{1, 2\} \setminus \{l\}$. By Lemma 6, $(0, y^{[i,j,k,l]}) - (0, y^{k,l,i,\bar{l}]}) + (0, y^{[i,\bar{l},\bar{i},l]}) - (0, y^{[\bar{i},l,i,j]}) \in \mathrm{lin}(\mathcal{L})$, and hence, using Step 1, $(0, y^{[i,j,k,l]}) \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$. □

Step 3: $(0, y^e) \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for all $e \in \delta(R_1 \cup R_2 \cup C_1 \cup C_2)$
Proof. Let $e = [i, j, k, l] \in \delta(R_1 \cup R_2 \cup C_1 \cup C_2)$ with $i \in \{1, 2\}$ and $k, l \notin \{1, 2\}$. (The case $j \in \{1, 2\}$ and $k, l \notin \{1, 2\}$ can be shown analogously.) Let $\{\bar{i}\} := \{1, 2\} \setminus \{i\}$ and choose $\bar{j} \in \{1, 2\} \setminus \{j\}$. By Lemma 5, $y^{[i,j,k,l]} - y^{[i,j,\bar{i},l]} + y^{[i,j,\bar{i},\bar{j}]} - y^{[i,j,k,\bar{j}]} \in \mathrm{lin}(\mathcal{L})$, and hence, by Steps 1 and 2, $y^{[i,j,k,l]} \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$. □

Step 4: $(0, y^e) \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for all $e \in \mathcal{E}_n(\mathcal{V}_n \setminus (R_1 \cup R_2 \cup C_1 \cup C_2))$
Proof. Let $e = [i, j, k, l] \in \mathcal{E}_n(\mathcal{V}_n \setminus (R_1 \cup R_2 \cup C_1 \cup C_2))$. If $e = [n, n-1, n-1, n]$ the claim is clear, so assume $e \neq [n, n-1, n-1, n]$. By Lemma 5, $y^{[i,j,k,l]} - y^{[i,j,1,l]} + y^{[i,j,1,1]} - y^{[i,j,k,1]} \in \mathrm{lin}(\mathcal{L})$, and hence, because of Step 3, $y^{[i,j,k,l]} \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$. □

Step 5: $(x^v, 0) \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for all $v \in \mathcal{V}_n$
Proof. This follows using Lemma 3 and the fact that by Steps 1–4, $(0, y^e) \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for all $e \in \mathcal{E}_n$. □

This completes the proof of Claim 2, and hence, the proof of Theorem 9 is done. □

5 Affine hull of $\mathcal{QAP}_n$ / $\mathcal{QAP}_n^\star$

Proposition 3 and Theorem 9 imply the following.

Theorem 10.

(i) $\mathrm{aff}(\mathcal{QAP}_n^\star) = \{(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid A^{\star(n)}(x, y) = b^{\star(n)}\}$

(ii) Precisely one arbitrary equation in $A^{\star(n)}(x, y) = b^{\star(n)}$ is redundant.

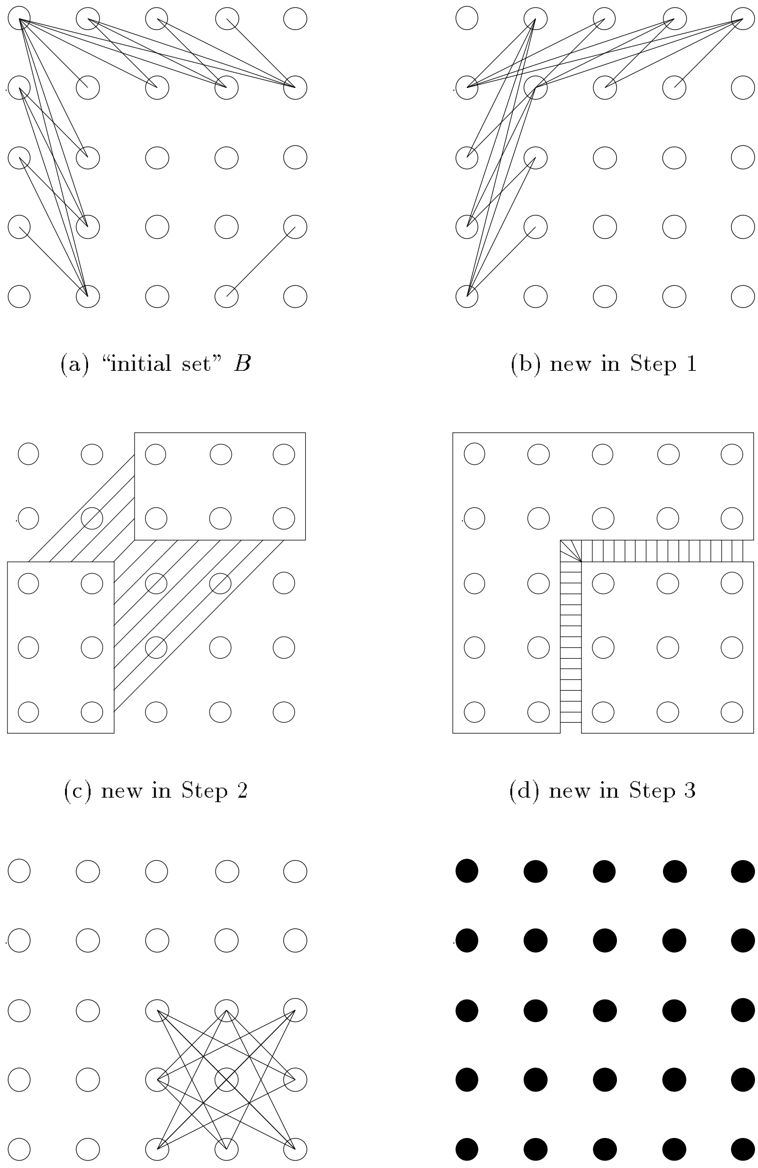
Finally, we want to show that also $A^{(n)}(x, y) = b^{(n)}$ describes $\mathrm{aff}(\mathcal{QAP}_n)$ completely, and to exhibit a minimal subset of these equations describing $\mathrm{aff}(\mathcal{QAP}_n)$. An analogous theorem can be also found in Rijal (1995).

Theorem 11.

(i) $\mathrm{aff}(\mathcal{QAP}_n) = \{(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid A^{(n)}(x, y) = b^{(n)}\}$

(ii) Let $r, c \in \{1, \ldots, n\}$ and let T be a subset of the equations $A^{(n)}(x, y) = b^{(n)}$ consisting of

(a) one of the equations $x(S) = 1$ $(S \in \{\mathrm{row}_1^{(n)}, \ldots, \mathrm{row}_n^{(n)}, \mathrm{col}_n^{(n)}, \ldots, \mathrm{col}_n^{(n)}\})$



(a) "initial set" B

(b) new in Step 1

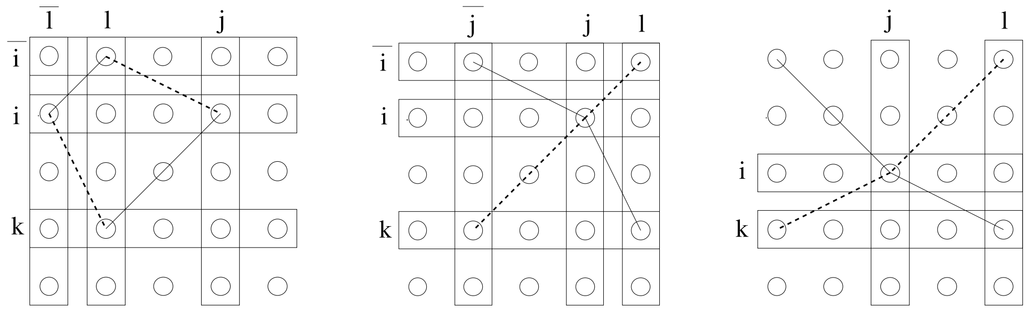
(c) new in Step 2

(d) new in Step 3

(e) new in Step 4

(f) new in Step 5

Figure 4: The vectors y^e in $\mathcal{B}$ and the ones newly shown to be in $\mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for each of the steps in the proof.



(a) proof of Step 2 (b) proof of Step 3 (c) proof of Step 4

Figure 5: The notations used in the proofs od Steps 2, 3, and 4.

(b) for every $v \in \mathcal{V}_n \setminus (\mathrm{row}_r^{(n)} \cup \mathrm{col}_c^{(n)})$ either $y(v : \mathrm{row}_r^{(n)}) - x_v = 0$ or $y(v : \mathrm{col}_c^{(n)}) - x_v = 0$, but not both

(c) for every pair $j \in \{1, \ldots, n\} \setminus \{c\}$, $k \in \{1, \ldots, n\} \setminus \{r\}$ either $y((r, j) : \mathrm{row}_k^{(n)}) - x_{(r,j)} = 0$ or $y((k, c) : \mathrm{col}_j^{(n)}) - x_{(k,c)} = 0$, but not both

(d) for every pair $j, l \in \{1, \ldots, n\} \setminus \{c\}$ $(j \neq l)$ either $y((r, j) : \mathrm{col}_l^{(n)}) - x_{(r,j)} = 0$ or $y((r, l) : \mathrm{col}_j^{(n)}) - x_{(r,l)} = 0$, but not both

(e) for every pair $i, k \in \{1, \ldots, n\} \setminus \{r\}$ $(i \neq k)$ either $y((i, c) : \mathrm{row}_k^{(n)}) - x_{(i,c)} = 0$ or $y((k, c) : \mathrm{row}_i^{(n)}) - x_{(k,c)} = 0$, but not both

(f) either for one pair j, l as in d) the equation not yet chosen in (d) or either for one pair i, k as in e) the equation not yet chosen in (e), but not both

(g) all equations $y((r, c) : \mathrm{row}_i^{(n)}) - x_{(r,c)} = 0$ $(i \in \{1, \ldots, n\} \setminus \{r\})$

(h) all equations $y((r, c) : \mathrm{col}_j^{(n)}) - x_{(r,c)} = 0$ $(j \in \{1, \ldots, n\} \setminus \{c\})$

Then, removing the set T from the system $A^{(n)}(x, y) = b^{(n)}$ yields a minimal system of equations having the same solution space as $A^{(n)}(x, y) = b^{(n)}$, and hence, the resulting system is a minimal system of equations for $\mathcal{QAP}_n$.

Proof. Clearly, in order to proof part *(ii)*, we can restrict to $r = n$, $c = n$. Let $\overline{T}$ be the set consisting of all equations in $A^{(n)}(x, y) = b^{(n)}$ but the ones in T. Then $\overline{T}$ partitions into $\overline{T} = M \cup L$ $(M \cap L = \emptyset)$, where M meets the requirements of Proposition 1 and L consists of all equations considered in *(d)* and *(e)* but not chosen for T in *(d)*, *(e)*, or *(f)*. Now, let $H := \mathrm{row}_n^{(n)} \cup \mathrm{col}_n^{(n)}$ be the set of variables eliminated by the transition from $\mathcal{QAP}_n$ to $\mathcal{QAP}_{n-1}^{\star}$. Then, the matrix $A_{M,\bullet}$ can be transformed by row operations (i.e., adding a multiple of a row to another row as well as permuting rows) and permutations of the columns to the following matrix

$$A' := \left(\begin{array}{c|c} I_{|H|} & * \\ \hline 0 & \tilde{A} \end{array} \right),$$

where $I_{|H|}$ is the $|H| \times |H|$ identity matrix and $\tilde{A}$ is a matrix obtained from the matrix $A^{\star(n-1)}$ of the system $A^{\star(n-1)}(x,y) = b^{\star(n-1)}$ by removing one arbitrary row. Hence, A' has full row rank (due to Proposition 3), and this rank is precisely $\dim(\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}) - \dim(\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}) - (n(n-1)-1)$. From this, both parts of the theorem follow by using Theorem 8. □

6 Conclusion

The study presented in this paper covers only investigations of basic properties of the polytope we (and also (Rijal, 1995)) have defined to be "the" QAP-Polytope, like considerations of the affine hull, the trivial inequalities, or relationships between this polytope and the Boolean Quadric / Cut Polytope. In order to use insight into the structure of the QAP-Polytope in a computational way, it will be necessary to investigate the facial structure more closely. We hope, that the results and the methods presented in this paper (in particular the relationship between $\mathcal{QAP}_n$ and $\mathcal{QAP}^\star_{n-1}$) will be useful for further research in that direction.

References

Balinski, M. and Russakoff, A. (1974). On the assignment polytope. *SIAM Review*, **16**, 516–525.

Barahona, F. and Mahjoub, A. (1986). On the cut polytope. *Mathematical Programming*, **36**, 157–173.

Burkard, R. (1990). *Discrete Location Theory*, chapter 9: Locations with Spatial Interactions: The Quadratic Assignment Problem, pages 387–437.

Burkard, R., Karisch, S., and Rendl, F. (1994). QAPLIB - A quadratic assignment problem library. Technical Report 287, Technische Universität Graz, Institut für Mathematik.

De Simone, C. (1989). The cut polytope and the boolean quadric poplytope. *Discrete Mathematics*, **79**, 71–75.

Drezner, Z. (1994). Lower bounds based on linear programming for the quadratic assignment problem. Technical report, Dept. of Management Science, School of Business Administration and Economics, California State University.

Gilmore, P. (1962). Optimal and suboptimal algorithms for the quadratic assignment problem. *Journal of the Society for Industrial and Applied Mathematics*, **10**(2), 305–313.

Johnson, T. (1992). *New Linear-Programming Based Solution Procedures for the Qauadratic Assignment Problem*. Ph.D. Dissertation, Clemson University, Clemson, SC.

Koopmans, T. and Beckmann, M. (1957). Assignment problems and the location of economic activities. *Econometrica*, **25**, 53–76.

Lawler, E. (1963). The quadratic assignment problem. *Management Science*, **9**, 586–599.

Padberg, M. (1989). The boolean quadric poplytope: Some characteristics and facets. *Mathematical Programming*, **45-1**, 139–172.

Resende, M., Ramakrishnan, K., and Drezner, Z. (1994). Computing lower bounds for the quadratic assignment problem with an interior point algorithm for linear programming. Technical report, Mathematical Sciences Research Center, AT&T Bell Laboratories, Murray Hill.

Rijal, M. (1995). *Scheduling, Design and Assignment Problems with Quadratic Costs*. Ph.D. Dissertation, New York University.

Sahni, S. and Gonzales, T. (1976). P-complete approximation problems. *Journal of the Association for Computing Machinery*, **23**, 555–565.