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by
Michael Jünger, Sebastian Leipert and Petra Mutzel

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Institut für Informatik
Universität zu Köln
Pohligstraße 1
50969 Köln

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On Computing a Maximal Planar Subgraph using PQ-Trees

Michael Jünger[a] Sebastian Leipert[b]
Petra Mutzel[c]

[a] Institut für Informatik, Universität zu Köln, mjuenger@informatik.uni-koeln.de
[b] Institut für Informatik, Universität zu Köln, leipert@informatik.uni-koeln.de
[c] Max-Planck-Institut für Informatik, Saarbrücken, mutzel@mpi-sb.mpg.de

Abstract

The problem of computing a maximal planar subgraph of a non-planar graph has been deeply investigated over the last 20 years. Several attempts have been tried to solve the problem with the help of PQ-trees. The latest attempt has been reported by Jayakumar *et al.* (1989).

In this paper we show that the algorithm presented by Jayakumar *et al.* is not correct. We show that it does not necessarily compute a maximal planar subgraph and that the same holds for a modified version of the algorithm presented by Kant (1992). Our conclusions most likely suggest not to use PQ-trees at all for this specific problem.

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1 Introduction

In Automatic Graph Drawing, a widely-used method for drawing nonplanar graphs, such as PERT-diagrams and ER-diagrams, is to transform the graph into a planar graph, and then use planar graph drawing methods. A widely used method for this transformation is to delete edges in order to get a planar subgraph, and then to reinsert the removed edges such that the number of edge crossings is small. In VLSI design the thickness problem is approximated by successively subtracting large planar subgraphs from a given nonplanar graph. However, the problem of finding the minimum number of edges that have to be removed from a given graph in order to obtain a planar subgraph, is known to be an $\mathcal{NP}$-hard problem (see Garey and Johnson, 1979).

Therefore, research has focused on computing maximal planar subgraphs. Let $G = (V, E)$ be a simple graph with n vertices and m edges then a planar subgraph G' of G is a *maximal planar* subgraph, if for all edges $e \in G - G'$ the addition of e to G' destroys planarity. Besides a trivial $O(nm)$ algorithm that can be constructed using any $O(n)$ planarity test, three different approaches are known for solving this problem.

Chiba, Nishioka, and Shirakawa (1979) presented an algorithm based on the path addition algorithm that computes a maximal planar subgraph in $O(nm)$ time. Cai, Han, and Tarjan (1993) presented later an $O(m \log n)$ algorithm that is based on the path addition algorithm as well. Di Battista and Tamassia (1989) described an algorithm that checks in

$O(\log n)$ amortized time, whether an edge can be added to G without destroying planarity, obtaining an $O(m \log n)$ time algorithm as well.

Ozawa and Takahashi (1981) have presented an $O(nm)$ algorithm using the vertex addition algorithm. Jayakumar, Thulasiraman, and Swamy (1986) showed that in general this algorithm does not determine a maximal planar subgraph. Moreover, the resulting planar subgraph may not even contain all vertices. Jayakumar, Thulasiraman, and Swamy (1989) presented an algorithm called PLANARIZE that computes a spanning planar subgraph G_p of G in $O(n^2)$ time. Furthermore, they present an algorithm called MAX-PLANARIZE that augments G_p to a subgraph G' of G by adding additional edges in $O(n^2)$ time. They claim that G' is a maximal planar subgraph of G if G_p (the result of phase 1 of the two phase algorithm) turns out to be biconnected. Kant (1992) shows that this algorithm is incorrect, and suggests a modification of the second phase of the algorithm that augments G_p to a maximal planar subgraph of G, even if G_p is not biconnected, maintaining $O(n^2)$ time requirement.

In this article, we will point out a substantial flaw in both the original and the modified two phase algorithm that was not detected previously as well as new mistakes introduced by Kant. In section 2 we give a brief introduction on PQ-trees and the planarity test using this data structure. In section 3 the principle of the planarization algorithm using the PQ-trees is described. In section 4 we show that the algorithm of Jayakumar *et al.* is incorrect giving a detailed description of the major mistake. In section 5 we discuss the attempt of Kant and make some concluding remarks in the last section.

2 Planarity test using PQ-trees

A graph is *planar*, if it can be embedded in the plane without any edge crossings. A graph is obviously planar, if and only if its biconnected components are planar. We therefore assume that G is biconnected. The planarity testing algorithm of Lempel, Even, and Cederbaum (1967) first labels the vertices of G as $1, 2 \ldots, n$ using an *st-numbering* (see Even and Tarjan, 1976). A numbering of the vertices of G by $1, 2 \ldots, n$ is an *st-numbering*, if the vertices "1" and "n" are adjacent and each other vertex j is adjacent to two vertices i and k such that $i < j < k$. The vertex 1 is denoted by s and the vertex n is denoted by t. The st-numbering induces an orientation of the graph, in which every edge is directed from the incident vertex with the higher st-number towards the incident vertex with the lower st-number. From now on we refer to the vertices of G by their st-numbers and call an edge (u, v), with $v < u$, *incoming* edge of v and *outgoing* edge of u.

For $1 \leq k \leq n$, let G_k denote the subgraph of G induced by the vertex set $V_k := \{1, 2, \ldots, k\}$. The graph G'_k arises from G_k as follows: For each edge $e = (u, v)$, where $v \in V_k$ and $u \in V \setminus V_k$, we introduce a virtual vertex u_e with label u and a virtual edge (u_e, v). Let B_k be a planar embedding of G'_k such that all virtual vertices are placed on the outer face. Then, B_k is called a *bush form*. It has been shown by Lempel *et al.* (1967) that G is planar, if and only if for every B_k, $k = 1, 2, \ldots, n-1$, there exists a bush form B'_k isomorphic to B_k, such that all virtual vertices in B'_k labeled $k+1$ appear consecutively.

The *PQ-tree* T_k corresponding to the bush form B_k is a rooted ordered tree that consists of three types of nodes:

1. Leaves in T_k represent virtual edges in B_k.
2. *P-nodes* in T_k represent cutvertices in B_k.
3. *Q-nodes* represent maximal biconnected components in B_k.

The *frontier* of a PQ-tree is the sequence of all leaves of T_k read from left to right. The frontier of a node X is the sequence of its descendant leaves read from left to right.

Let E_{k+1} denote the set of leaves in T_k that correspond to the virtual vertices labeled $k+1$. A node X is called *full*, if all leaves in its frontier are in E_{k+1}. A node X is *empty*, if its frontier does not contain any leaf of E_{k+1}. Otherwise, X is called *partial*. A node is called *pertinent*, if it is full or partial. The *pertinent subtree* is the smallest connected subtree that contains all leaves of E_{k+1} in its frontier. The root of the pertinent subtree is called *pertinent root*. Two PQ-trees are *equivalent*, if one can be obtained from the other by one or more of the following operations:

1. Permuting the children of a P-node.
2. Reversing the order of the children of a Q-node.

These operations are called *equivalence transformations* and describe *equivalence classes* on the set of all PQ-trees. An equivalence class of PQ-trees corresponds to a class of permutations called the *permissible permutations*.

It has been shown by Booth and Lueker (1976) that B'_k exists if and only if T_k can be converted into an equivalent PQ-tree T'_k such that all pertinent leaves appear consecutively in the frontier of T'_k. Booth and Lueker (1976) have defined a set of patterns and replacements called *templates* that can be used to reduce the PQ-tree such that the leaves corresponding to edges of the set E_{k+1} appear consecutively in all permissible permutations. To construct T_{k+1} from T_k they first reduce T_k by use of the templates and then replace all leaves corresponding to virtual edges incident to virtual vertices labeled $k+1$ by a P-node, whose children are the leaves corresponding to the incoming edges of the vertex $k+1$ in G.

The planarity testing algorithm now starts with T_1 and constructs a sequence of PQ-trees $T_1, T_2, \ldots$. If the graph is planar, the algorithm terminates after constructing T_{n-1}. Otherwise it terminates after detecting the impossibility of reducing some T_k, $1 \leq k < n$.

3 Principle of an approach for planarization

The basic idea of a planarization algorithm using PQ-trees presented by Jayakumar *et al.* (1989) is to construct the sequence of PQ-trees $T_1, T_2, \ldots, T_{n-1}$ by deleting an appropriate number of pertinent leaves every time the reduction fails such that the resulting PQ-tree becomes reducible. In every step of the algorithm PLANARIZE, a maximal consecutive sequence of pertinent leaves is computed by using a $[w, h, a]$-*numbering* (see Jayakumar *et al.*, 1989). All pertinent leaves that are not adjacent to the maximal pertinent sequence are removed from the PQ-tree in order to make it reducible. Hence the edges corresponding to the leaves are removed from G and the resulting graph G_p is planar.

It has been shown by Jayakumar *et al.* (1989) that the graph G_p computed by PLANARIZE is not necessarily maximal planar. The authors therefore suggest to apply a second phase called MAX-PLANARIZE, also based on PQ-trees. Knowing which edges have been removed from G to construct G_p, edges from $G - G_p$ are added back to G_p in the second phase without destroying planarity.

During the reduction of a vertex v, there may exist nonpertinent leaves that are in all permissible permutations of the PQ-tree T_{v-1} between a pertinent leaf l_v and its maximal pertinent sequence. This maximal pertinent sequence has been determined with the help of the $[w, h, a]$-numbering. In order to make the tree T_{v-1} reducible, the leaf l_v is removed from the tree and the corresponding edge is removed from the graph G, guaranteeing that the subgraph G_p will be planar. However, it may occur that the nonpertinent leaves that are positioned between l_v and its maximal pertinent sequence in T_{v-1}, are removed as well from a tree T_k, $v \leq k < n$, in order to obtain reducibility. Therefore, there is no need to remove the edge corresponding to l_v from the graph G.

In order to find leaves such as l_v, Jayakumar *et al.* (1989) use the algorithm MAX-PLANARIZE. In step i, both PLANARIZE as well as MAX-PLANARIZE reduce the same vertex i. The difference between the PQ-trees in the two algorithms is, according to the authors, that all leaves that have been deleted in PLANARIZE are ignored in MAX-PLANARIZE from the moment they are introduced into the tree until they get pertinent. This causes the nonpertinent leaves between the pertinent leaf l_v and its maximal pertinent sequence to be ignored. Hence l_v is adjacent to its maximal pertinent sequence and the corresponding edge can be added back to G_p, while the leaves between l_v and the maximal pertinent sequence are removed from the PQ-tree.

4 On the incorrectness of the algorithm

While some incorrect details of the approach of Jayakumar *et al.* have been described in a technical report by Kant (1992), who attempted to correct the algorithm, a major problem has not been detected.

Jayakumar *et al.* assume that the maximal planar subgraph G_p is biconnected for the correct application of the Lempel-Even-Cederbaum algorithm. Furthermore, as they have stated correctly, this is necessary in order to have an st-numbering. Nevertheless, the PQ-trees in MAX-PLANARIZE are constructed according to the st-numbering that was computed for the graph G.

As a matter of fact, the st-numbering of G does not imply an st-numbering of any subgraph G_p even if the subgraph G_p is biconnected. This results in two problems, of which one is crucial and cannot be dealt with even by the ideas described by Kant (1992).

Both problems are based on the fact that during the application of PLANARIZE for some vertices of V all incoming edges may be deleted from the graph while the resulting graph G_p stays biconnected.

Let $v \in V$ be such a node with no incoming edges. Since G_p is biconnected, v must have at least two outgoing edges (v, u_1) and (v, u_2). Let $w \in V$ be a vertex in G such that $w < v$, hence the leaves corresponding to the outgoing edges of w are reduced before the leaves of v. Let T_{w-1} be the PQ-tree during the application of MAX-PLANARIZE, in which

the relevant leaves corresponding to the outgoing edges of w have to be reduced. Assume that the leaves of both nodes w and v are on the outer face of the same biconnected component of the bush form that corresponds to the PQ-tree T_{w-1}. Assume further that one designated leaf w_{k+1} of the vertex w is separated by the leaves v_1 and v_2 corresponding to (v, u_1) and (v, u_2) from the leaves $w_1, w_2, \ldots, w_k$, where the latter form the maximal pertinent sequence (see Figure 1 for an illustration).

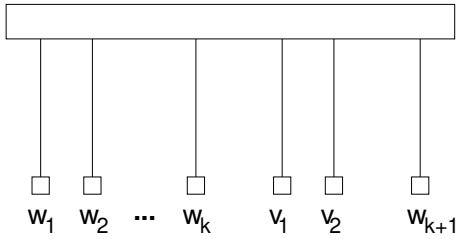

Figure 1: Leaf w_{k+1} is separated by v_1 and v_2 from its maximal pertinent sequence $w_1, w_2, \ldots, w_k$.

If (v, u_1) and (v, u_2) are the only outgoing edges of v in G_p, then the leaves v_1 and v_2 will be changed during the reduction of the PQ-tree T_{v-1} into a P-node with leaves corresponding to edges in $E \setminus E_p$. Hence, if the vertex v had been reduced before the vertex w, then MAX-PLANARIZE would have considered the leaf w_{k+1} as being adjacent to the maximal pertinent sequence $w_1, \ldots, w_k$. The edge corresponding to the leaf w_{k+1} could have been added to the graph G_p without destroying planarity. In case that none of the incoming edges of v is added to G_p in a PQ-tree T_i, $v < i < n$, the resulting graph G_p is not a maximal planar subgraph.

We now consider the second problem. The planarization algorithm of Jayakumar *et al.* (1989) does not obey an important invariant implied by the following lemma, shown by Even (1979).

Lemma 4.1 *Let $G = (V, E)$ be a planar graph with an st-numbering and let $1 \leq k \leq n$. If the edge (t, s) is drawn on the boundary of the outer face in an embedding of G, then all vertices and edges of $G - G_k$ are drawn in the outer face of the plane subgraph G_k of G.*

This result allowed Lempel, Even, and Cederbaum (1967) to transform the problem of planarity testing to the construction of a sequence of bush forms B_k, $1 \leq k \leq n$. For a planar graph G, edges and vertices that have not been introduced into the current subgraph G_k are always embedded into the outer face of G_k.

The approach of Jayakumar *et al.* (1989) does not obey this invariant in the second phase. There exist edges that have to be embedded into an inner face of some G_k, even if (t, s) is drawn on the outer face. Due to the above lemma, the correction step MAX-PLANARIZE only considers edges for reintroduction into the planar subgraph G_p that are on the outer face of the current graph G_k. Since the numbering that is used to determine the order in which the vertices are reduced does not correspond to an st-numbering of G_p in general, the algorithm of Jayakumar *et al.* (1989) ignores edges that have to be added into an inner face of the embedding of a current graph G_k. This fact is fatal, as we are about to show now.

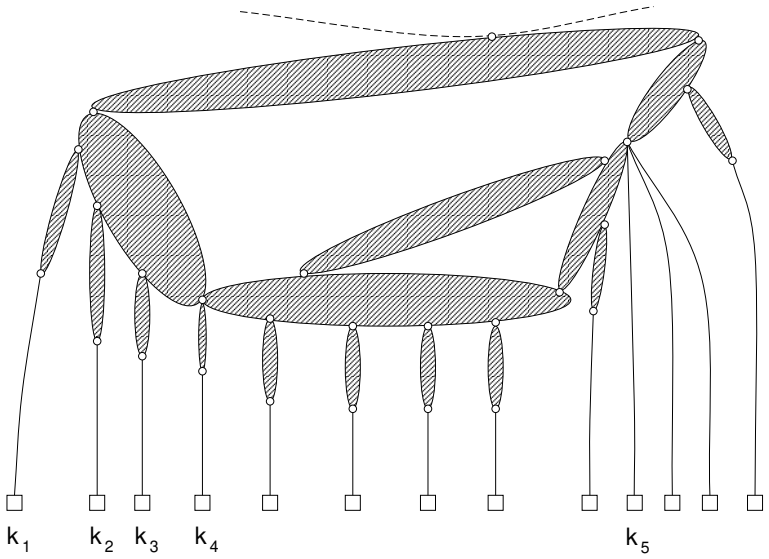

Figure 2: Part of a bush form B_{k-1}

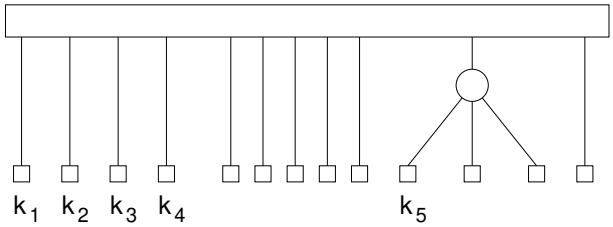

Figure 3: Part of a PQ-tree corresponding to bush form B_{k-1}

In Figure 2, a part of a bush form B_{k-1}, $1 < k \leq n$ of a graph G is shown. The virtual vertices corresponding to the vertex k are labeled $k_1, k_2, \ldots, k_5$ and all other virtual vertices are left unlabeled. The corresponding part of the PQ-tree is shown in Figure 3. Obviously, there do not exist any reversions or permutations such that the virtual vertices of k occupy consecutive positions. Hence, the graph G is not planar. Applying the $[w, h, a]$-numbering of Jayakumar *et al.* (1989) allows us to delete the virtual vertex k_5 and to reduce the other four vertices k_1, k_2, k_3, k_4. The resulting bush form B_k is planar and the relevant part is shown in Figure 4. Figure 5 shows the corresponding part of the PQ-tree. Assume now that all descendants of k have to be removed from the PQ-tree in a later step. Hence all incoming edges incident on k are removed from the tree. Now assume further that there exists a path $v_1, v_2, \ldots, v_l$ in G_p such that

- for all i, j, $1 \leq i < j \leq l$ the inequality $v_i < v_j$ holds,
- the edge (v_2, v_1) corresponds to one of the virtual edges that are between the leaf k_5 and the maximal pertinent sequence k_1, k_2, k_3, k_4 in all PQ-trees equivalent to T_{k-1},
- $v_l = t$.

This path guarantees that all outgoing edges of the vertex k cannot be embedded into the

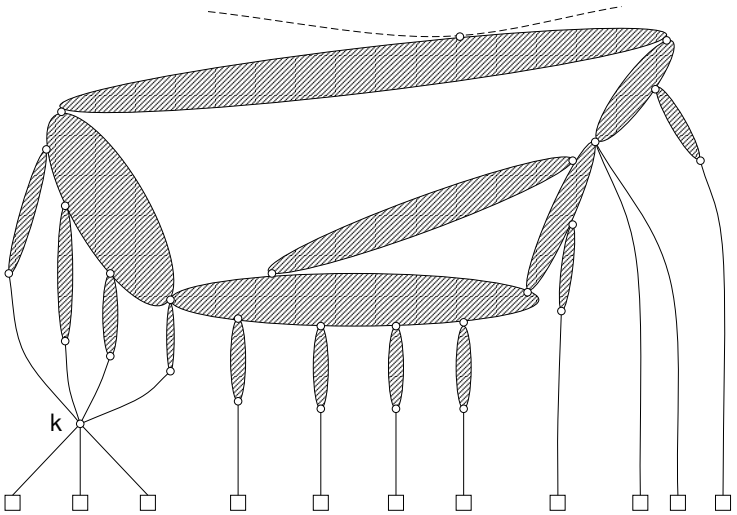

Figure 4: Part of a bush form B_k

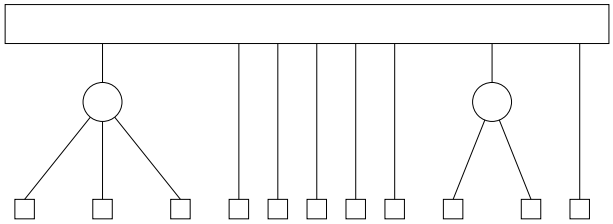

Figure 5: Part of a PQ-tree corresponding to bush form B_k

outer face of the embedding of B_{k-1} without crossing an edge on this path. Hence the edge e_{k_5} corresponding to the leaf k_5 is not considered by the algorithm MAX-PLANARIZE as being an edge that does not destroy planarity. Therefore, e_{k_5} is not added back to the planar subgraph G_p.

Nevertheless adding the edge e_{k_5} to G_p may not destroy planarity of G_p as is shown in our example in Figure 6. Since all incoming edges of the vertex k have been deleted by PLANARIZE and are not added back by MAX-PLANARIZE, it may be possible to swap the vertex k into an inner face of the embedding of B_k such that the virtual vertex k_5 can be identified with k and the edge e_{k_5} is embedded into the bush form B_k without destroying planarity.

Therefore, the strategy of using PQ-trees presented by Jayakumar *et al.* (1989) does not compute a maximal planar subgraph in general. Furthermore, we point out that the same problem holds for the modified version of this algorithm, presented by Kant (1992). This version follows a similar strategy of computing a spanning planar subgraph G_p using PLANARIZE and then adding edges that do not destroy planarity in a second phase. The order of reductions that is used to insert vertices into existing bush forms is the same as the one implied by the st-numbering on G. Hence this approach is not able to compute a maximal planar subgraph for the same reason.

Summarizing, we state the following lemma that has been shown in the discussion above.

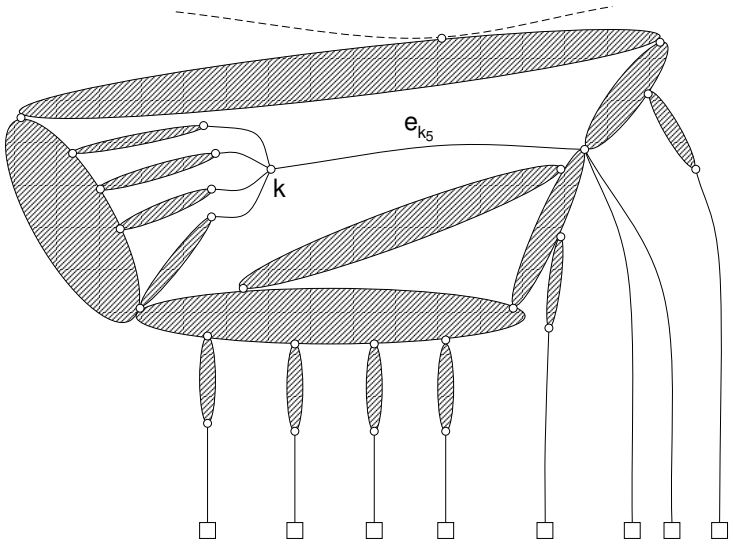

Figure 6: Part of a bush form B_k with e_{k_5} embedded

Lemma 4.2 *Let $G = (V, E)$ be a nonplanar graph. Let $G_p = (V, E_p)$, $E_p \subseteq E$, be a planar subgraph of G, such that G_p was obtained from G by*

1. *computing an st-numbering for all vertices and*
2. *applying the algorithm of Lempel, Even, and Cederbaum (1967) constructing a sequence of bush forms B_k, $1 \leq k \leq n$, by embedding a maximal number of outgoing edges of a vertex k, $1 < k \leq n$, in the outer face of B_{k-1} without crossings, deleting all other outgoing edges of k.*

Let $G'_p = (V, E'_p)$, be a planar subgraph of G such that

1. *$E_p \subseteq E'_p \subseteq E$,*
2. *the graph G'_p is computed by constructing a sequence of bush forms B'_k, $1 \leq k \leq n$, based on the st-numbering used for determining G_p, and possibly embedding outgoing edges $e \in E \setminus E_p$ of every vertex k, $1 < k \leq n$, without crossings in the outer face of B_{k-1}.*

Then the subgraph G'_p is not necessarily maximal planar.

Considering a computation of an *st*-numbering for the planar subgraph G_p in order to augment G_p to a maximal planar subgraph of G and then construct a sequence of bush forms B'_k, $1 \leq k \leq n$, is aggravated by the following two facts.

1. The graph G_p is not biconnected in general.
2. The sequence of bush forms B'_k, $1 \leq k \leq n$ is not equivalent to the bush forms B_k, constructed in the first phase PLANARIZE.

Considering a computation of an st-numbering for the planar subgraph G_p in order to augment G_p to a maximal planar subgraph of G and then construct a sequence of bush forms B'_k, $1 \leq k \leq n$, is aggravated by the fact that the graph G_p is not biconnected in general. Furthermore, the difference between the bush forms of the first phase and the second phase may result in the deletion of the edges of G_p as soon as edges of $E \setminus E_p$ are added to G_p. Adding an edge $e \in E \setminus E_p$ to G_p is able to change the corresponding bush form in such a way, that the pertinent leaves corresponding to the outgoing edges of some node v in E_p cannot form a consecutive sequence in any permissible permutations.

5 Further problems

We show now that even if the st-numbering of G is as well an st-numbering of G_p, and even if we consider the suggested modifications of Kant (1992), the algorithm presented by Jayakumar *et al.* (1989) still does not work correct.

Kant (1992) suggested a correction of the second phase by introducing sequence indicators and by delaying the decision, whether a deleted leaf can be added back to G_p, until enough information is available. In his version of MAX-PLANARIZE, a leaf l that was deleted in PLANARIZE will be a normal nonpertinent leaf, which is not ignored until it becomes pertinent. Again, the maximal pertinent sequence of a vertex i is reduced. This maximal pertinent sequence is the same as in PLANARIZE. The pertinent leaves that are not adjacent to the maximal pertinent sequence stay in the PQ-tree and their presence will be ignored in the template matching algorithm from then on. They are called *potential leaves*. If potential leaves of a vertex i remain in the tree after the reduction of i, a *sequence indicator* $< i >$ is added to the tree in order to indicate the position of the reduced pertinent sequence. The presence of the sequence indicator will be ignored as well. If a node X contains only nodes in its front, whose presence are ignored, X is called an ignored node. Applying this idea, a sequence of PQ-trees $T_1, T_2, \ldots, T_{n-1}$ is constructed during the augmentation phase, which is equivalent to the sequence of PQ-trees constructed during the first phase. Thus he makes sure that the corresponding bush forms of both phases are equivalent. In order to augment G_p to the maximal planar subgraph G'_p, edges are added to G_p, when their corresponding leaf and its sequence indicator can be reduced by deleting only other potential leaves and sequence indicators. Doing this, it is not allowed to bind empty nodes to new places, since this would change the equivalence class of the actual PQ-tree. This can be formalized as follows.

Definition 5.1 *A potential leaf l is* **near** *its sequence indicator $si(l)$, if and only if the PQ-tree T_i, $1 \leq i < n$, can be reduced, such that l and $si(l)$ are adjacent siblings, by deleting only ignored nodes and not binding empty nodes to new places. If a potential leaf l is near its sequence indicator $si(l)$, then l and $si(l)$ are called a* **near pair**.

Not every near pair, formed by a potential leaf and its sequence indicator, can be reduced. In general, we have to choose between different near pairs, since the reduction of one near pair might cause the deletion of the other. This is, in particular, typical for near pairs that intersect. Two near pairs l, $si(l)$ and l', $si(l')$ are called *intersecting* in T_i, if either l' or $si(l')$ is between l and $si(l)$ in all equivalent PQ-trees of T_i. In this case only one

edge corresponding to l or l' can be embedded into the outer face of the current bush form without causing a crossing.

Kant (1992) suggests to test for near pairs just within the maximal pertinent sequence, since by definition there will be no empty nodes inside the pertinent sequence. A near pair is found while applying the pattern matching algorithm of Booth and Lueker (1976) to the maximal pertinent sequence. Every time a pertinent node X is matched, we search for near pairs l and $si(l)$ in the frontier of X, where X is the first common ancestor of l and $si(l)$.

This search can be done effectively using two arrays PL_X and SI_X that are introduced for every internal node X in the maximal pertinent sequence. In $PL_X[i]$ children of X are stored that are ancestors of some potential leaves corresponding to edges (i, j), $j < i$, while $SI_X[i]$ contains the children that are ancestors of some sequence indicator $< i >$. It can be shown that by using the PL_X and SI_X arrays all near pairs will be found (see Kant, 1992; Leipert, 1995). After finding a near pair l, $si(l)$ we make sure that before the near pair is reduced, the first common ancestor X of l, $si(l)$ will be a Q-node. This can obviously be done without changing the equivalence class of the PQ-tree.

Kant (1992) now suggests a reduction process which is encapsulated within a procedure REDUCE and that begins with the first common ancestor X of the near pair l, $si(l)$ and goes down the PQ-tree to l and $si(l)$. A special type of node, used in the reduction, is defined first.

Definition 5.2 *An ignored P-node is said to be of* **type U**, *if all children except one child, yet unknown, must be removed with their descendants from the PQ-tree in a later step. An ignored Q-node is said to be of type U, if it has one special marked child Y, and all children of the Q-node between Y and one of the endmost children, yet unknown, must be removed with their descendants from the PQ-tree in a later step.*

We now give a brief description of the reduction process. After a near pair l, $si(l)$ is found, the following situation occurs: A Q-node X is the first common ancestor of l and $si(l)$, and the leaf l corresponds to some outgoing edge of a vertex i. There exists a sequence $Y_1, Y_2, \ldots, Y_k$, $k \geq 2$, of children of X, such that, say, Y_1 is an ancestor of l and Y_k is an ancestor of $si(l)$ and the children $Y_2, Y_3, \ldots, Y_{k-1}$ between Y_1 and Y_k are ignored.

First, all children $Y_2, Y_3, \ldots, Y_{k-1}$ and their descendants are removed from the tree by REDUCE. If a deleted leaf l' corresponds to an outgoing edge of the vertex i, then it forms a near pair with $si(l)$, so REDUCE adds it to G_p. Then REDUCE goes along the paths from Y_1 to the potential leaves of vertex i applying a top-down reduction. Observe that there might be more than one potential leaf forming a near pair with $si(l)$. The same is done with the path from Y_k to the sequence indicator $si(l)$. Since there may be other near pairs in the frontier of Y_1 and Y_k, they are reduced correspondingly. Afterwards, all ignored nodes between the outermost leaf that corresponds to an incoming edge of i, and $si(l)$ are removed from the PQ-tree and the arrays PL and SI are updated. If a deleted leaf corresponds to an outgoing edge of i, then it is added to G_p.

However, the algorithm REDUCE is not correct for three reasons:

Problem 1: Adding every outgoing edge of a vertex i that corresponds to some deleted potential leaf might result in a nonplanar subgraph. This is due to the fact that different

potential leaves may be descendants of the same type U node, or descendants of different pertinent nodes.

Problem 2: Reducing the near pairs top-down does not restrict the permissible permutations in such a way that in all permutations l and $si(l)$ form a consecutive sequence.

Problem 3: Let l, $si(l)$ be the first detected near pair in the frontier of $Y_1, Y_2, \ldots, Y_k$. Let l', $si(l')$ be some other near pair in the frontier of the same nodes. The near pairs are reduced in the order they have been detected. This implies reducing l, $si(l)$ before reducing l', $si(l')$. This is not correct since reducing l, $si(l)$ first might cause the deletion of l', $si(l')$ while the reduction of l', $si(l')$ might not cause the deletion of l, $si(l)$. Hence G' is not necessarily maximal planar.

So in order to correct REDUCE, we are confronted with solving the following three problems:

1. If there are several potential leaves that form a near pair with $si(l)$, a maximal subset of leaves has to be found, which guarantees that all leaves of the subset can be reduced together.

2. A near pair l, $si(l)$ has to be reduced, such that l and $si(l)$ form a consecutive sequence in all permissible permutations.

3. If there are several near pairs, an ordering of the near pairs has to be found in such a way that the reduction of one near pair does not hinder the reduction of the near pairs which still have to be reduced.

The first two problems have been shown to be solvable by Leipert (1995), but the last still remains unsolved.

6 Concluding Remarks

In this paper we showed that the attempt of Jayakumar *et al.* (1989) to solve the maximal planar subgraph problem with PQ-trees is not correct. The problem is due to the fact that an important invariant for planarity testing is ignored. We have further shown that even a corrected version of the algorithm applied in the best possible case, where the st-numbering of a graph G is as well an st-numbering of the planar subgraph G_p, is not correct.

Since this best case is a very rare case and since the modifications for the solved problems (see Leipert, 1995) are far beyond any reasonable implementation, we doubt that a useful algorithm based on the strategy presented by Jayakumar *et al.* (1989) can be found.

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