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On the Two-Connected Planar Spanning Subgraph Polytope*

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Abstract

The problem of finding in a complete edge-weighted graph a two-connected planar spanning subgraph of maximum weight is important in automatic graph drawing. We investigate the problem from a polyhedral point of view.

Keywords: Planar graphs, two-connected graphs, polyhedra, facets.

1 Introduction

We assume familiarity with basic notions of graph theory (see, for instance, [1]) and with elementary notions of polyhedral combinatorics (see, for instance, [6]). Our graphs will be undirected and simple (no loops and no multiple edges). As usual, K_n denotes the complete graph with n vertices; $K_{n,m}$ denotes the complete bipartite graph with $n+m$ vertices and $n \times m$ edges. Let G be a graph; G is *connected* if for every pair of distinct vertices there exists a path in G joining them; G is *two-connected* if for every vertex v of G, the graph $G - v$ is connected; G is *planar* if it can be embedded in the plane. A subgraph H of a G is *spanning* if the vertex sets of H and G are the same. *Subdivision* of an edge uv of G consists of removing edge uv, and adding a new vertex w and the two edges uw and vw; w is called *subdivision vertex*. If G and H are two graphs, we say that G contains a subdivision of H, if H arises by subdivision of the edges of some subgraph of G. As usual, $\delta(u)$ denotes the set of all edges that are incident in the vertex u.

In automatic graph drawing the following problem arises: find in a complete graph with weights on its edges a two-connected planar spanning subgraph with weight as

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large as possible. This problem is NP-hard and it was introduced in [5]. In this paper we shall study this problem from a polyhedral point of view.

For this purpose, let n be an integer greater than or equal to four. Let $S(K_n)$ denote the set of the incidence vectors of all spanning subgraphs of K_n that are both planar and two-connected and let $P(K_n)$ denote the convex hull of $S(K_n)$; $P(K_n)$ is known as the *two-connected planar subgraph polytope*. In [5], a first version of a branch and cut algorithm based on the partial knowledge of the facet-defining structure of $P(K_n)$, found in [4], [5], and [7], was designed and tested. The partial knowledge of $P(K_n)$ comes from the investigations of two other related polytopes, namely the convex hull $Q_1(K_n)$ of the incidence vectors of all subgraphs of K_n that are planar [4], and the convex hull $Q_2(K_n)$ of the incidence vectors of all spanning subgraphs of K_n that are two-connected [7]. Indeed, $P(K_n) \subseteq Q_1(K_n) \cap Q_2(K_n)$, and so every inequality valid for $Q_i(K_n)$ $(i = 1, 2)$ is also valid for $P(K_n)$. In [5] it was shown that every facet-defining inequality of $Q_1(K_n)$ is also facet-defining for $P(K_n)$ and that some facet-defining inequalities of $Q_2(K_n)$ are also facet-defining for $P(K_n)$. It is not known whether every facet-defining inequality for $Q_2(K_n)$ is also facet-defining for $P(K_n)$.

The purpose of this paper is to investigate the structure of $P(K_n)$ that does not arise from the structures of $Q_1(K_n)$ and $Q_2(K_n)$. Clearly, not every facet-defining inequality for $P(K_n)$ is necessarily facet-defining for $Q_1(K_n)$ or for $Q_2(K_n)$. In fact, we shall show that there exist facet-defining inequalities for $P(K_n)$ that are valid for neither $Q_1(K_n)$ nor for $Q_2(K_n)$.

2 Facets arising from subdivisions of K_5 and $K_{3,3}$

The two graphs K_5 and $K_{3,3}$ play a central role in planarity: Kuratowski [3] showed that a graph is planar if and only if it contains no subdivisions of K_5 or $K_{3,3}$. Subdivisions of K_5 and of $K_{3,3}$ will play a central role also in this paper.

Consider the complete graph K_5 with vertices $1, 2, 3, 4, 5$. Subdivide each edge ij of K_5 $(1 \leq i < j \leq 5)$ N_{ij} times, with $N_{ij} \geq 1$; let $ij^1, ij^2, \cdots, ij^{N_{ij}}$ denote the corresponding subdivision vertices. Denote by $G = (V, E)$ the resulting graph. Note that for every $1 \leq i < j \leq 5$, the graph G contains the edges $(i, ij^1), (ij^1, ij^2), \cdots, (ij^{N_{ij}-1}, ij^{N_{ij}})$, and $(ij^{N_{ij}}, j)$, and it does not contain the edge (i, j). We shall refer to each of the five vertices $1, 2, 3, 4, 5$ of G as a *white* vertex and to each of all others (vertex ij^k) as a *black* vertex. Let N denote the total number of black vertices of G. Note that, by assumption, $N \geq 10$. Figure 1 shows a graph G with 20 black vertices.

For every $1 \leq i < j \leq 5$, let $K_{(i,j)}$ denote the complete graph with vertex set $\{i, ij^1, \cdots, ij^{N_{ij}}, j\}$, and let $G^+ = (V, E^+)$ be the graph obtained from G by adding every edge of each $K_{(i,j)}$. Write $n = N + 5$ and let K_n denote the complete graph with vertex set V. Let F denote the set of all edges of K_n that are not edges of G^+.

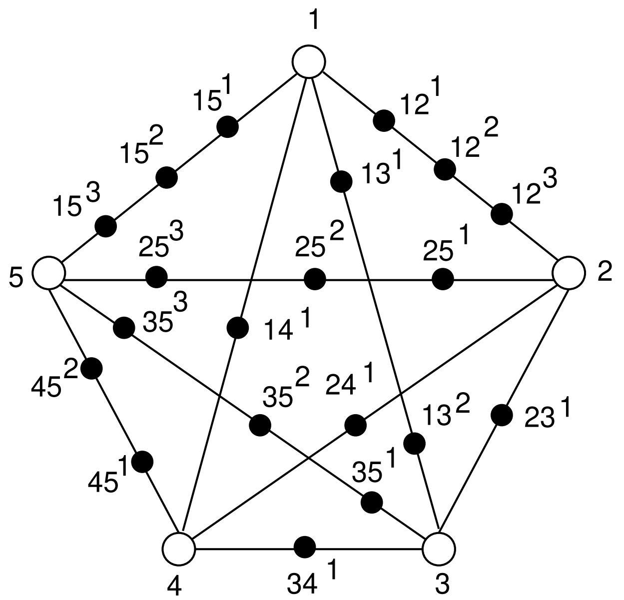

Figure 1: A subdivision of K_5

The following theorem shows that the inequality $x(F) \geq 1$ defines a facet of $P(K_n)$. Note that such an inequality is not valid for $Q_1(K_n)$ (because for every edge e of G, the graph $G - e$ is planar and its incidence vector y is such that $y(F) = 0$), and is not valid for $Q_2(K_n)$ (because the graph G is two-connected and its incidence vector y is such that $y(F) = 0$).

Theorem 1 *For every complete graph K_n, with $n \geq 15$, the inequality $x(F) \geq 1$ defines a facet of $P(K_n)$.*

Proof. To prove the validity, let y be the an arbitrary point in $S(K_n)$ and let H be the subgraph of K_n corresponding to y. We only need show that $y(F) \geq 1$. For this purpose, assume that $y(F) < 1$, and so $y(F) = 0$. But then, every edge of H must be an edge of the graph G^+, which is impossible: G^+ is not planar and no spanning planar subgraph of G^+ is two-connected. Since y was an arbitrary point in $S(K_n)$, it follows that the inequality is valid over $P(K_n)$.

Now let $c \in \{0,1\}^{\binom{n}{2}}$ such that $c_e = 0$ for every edge e of G^+ and $c_e = 1$ for every other edge; in other words, $c_e = 1$ if and only if $e \in F$, and so the inequality $x(F) \geq 1$ reads $c^T x \geq 1$. Let $x^1, x^2, \cdots, x^t$ be points in $S(K_n)$ such that $c^T x^i = 1$, for all $i = 1, \cdots, t$; and let $\overline{c}$ be a vector such that $\overline{c}x^i = \overline{c}x^j$ for all choices of i and j. Clearly, to show that $c^T x \geq 1$ defines a facet of $P(K_n)$ we only need show that $\overline{c}$ is a multiple of c. For this purpose, set $T = \{x^1, x^2, \cdots, x^t\}$.

First, we shall show that $\overline{c}_e = 0$ for every edge e of G^+. Let e be an arbitrary such an edge. Without loss of generality, we can assume that e is an edge of the complete graph $K_{(1,2)}$. Let u and v denote the vertices $34^{N_{34}}$ and 12^1, respectively. Consider the subgraph H of K_n obtained from G by deleting edge $(u, 4)$ and by adding edge (u, v) (see Figure 2); and let y denote its incidence vector. Clearly, $y \in T$.

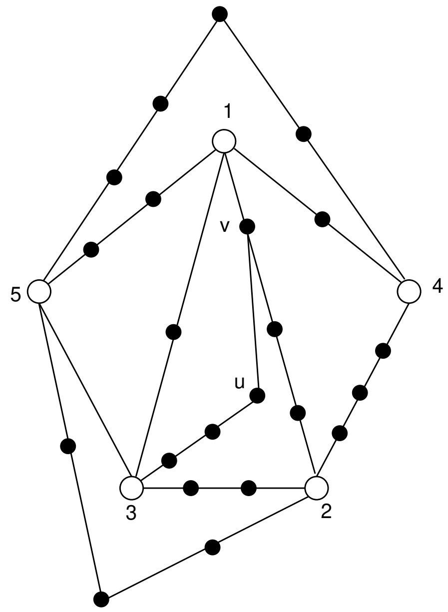

Figure 2: The graph H

Now, if $e \notin E$ (i.e., e is an edge of $K_{(1,2)}$ and is not an edge of G), then we let H^1 denote the subgraph of K_n obtained from H by adding the edge e. Since the incidence vector y^1 of H^1 belongs also to T, by assumption, $\overline{c}y = \overline{c}y^1$, and so $\overline{c}_e = 0$, and we are done. Hence, assume that $e \in E$. If one endpoint of e is a white vertex then, without loss of generality, we can assume that $e = (1, v)$ (in case $e = (2, 12^{N_{12}})$ it is sufficient to set $v = 12^{N_{12}}$). In this case, we let H^2 denote the subgraph of K_n obtained from H by deleting edge e. Since the incidence vector y^2 of H^2 belongs also to T, by assumption, $\overline{c}y = \overline{c}y^2$, and so $\overline{c}_e = 0$, and again we are done. Otherwise, both endpoints of the edge e in E are black, and so $e = (12^k, 12^{k+1})$, with $1 \leq k \leq N_{12} - 1$. Let H' denote the subgraph of K_n obtained from H by deleting edge (u, v) and adding edge $(u, 12^k)$, and let y' denote its incidence vector. (Note that, if $k = 1$ then $y' = y$, and so H and H' are in fact the same graph.) Clearly $y' \in T$. Now, let H^3 be the subgraph of K_n obtained from H' by deleting edge e and by adding edge $f = (12^{k+1}, 1)$. Since the incidence vector y^3 of H^3 belongs also to T, by assumption, $\overline{c}y' = \overline{c}y^3$, and so $\overline{c}_e = \overline{c}_f$. But $\overline{c}_f = 0$ (since f is an edge of $K_{(1,2)}$ and is not an edge of G), and so $\overline{c}_e = 0$, and again we are done. Hence we have shown that $\overline{c}_e = 0$ for every edge e of G^+.

Now to finish the proof, we only need show that $\overline{c}_e$ has the same value for every

$e \in F$. For this purpose, let $u = ij^k$ be an arbitrary black vertex of K_n; without loss of generality, we can assume that $u = 12^k$, with $1 \le k \le 12^{N_{12}}$. We propose to show that $\overline{c}_e = \overline{c}_f$ for every pair of arbitrary edges e, f in $F \cap \delta(u)$. Note that as soon as this is accomplished, we are done, since every edge in F has a black endpoint and since u was chosen arbitrary among all black vertices.

Consider the graph H^L in Figure 3 and the graph H^R in Figure 4, where all black vertices 12^i, with $i = 1, 2, \cdots, 12^{N_{12}}$ are present.

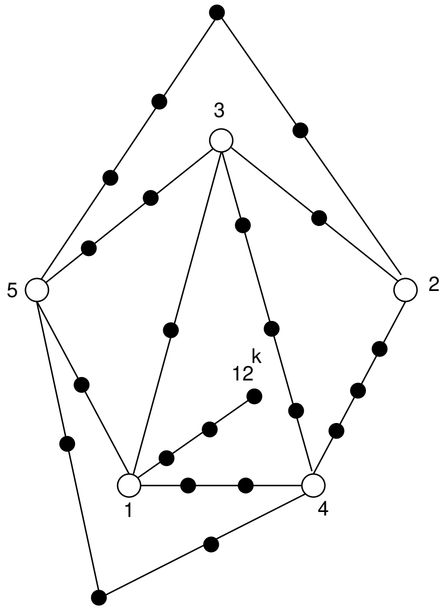

Figure 3: The graph H^L

Let L denote the subset of V of all vertices that do not belong to the complete graphs $K_{(1,2)}, K_{(2,3)}, K_{(2,4)}$, and $K_{(2,5)}$; and let R denote the subset of V of all vertices that do not belong to the complete graphs $K_{(1,2)}, K_{(1,3)}, K_{(1,4)}$, and $K_{(1,5)}$. Write $e = (u, v)$ and $f = (u, w)$; clearly, both v and w are in $L \cup R$. If both vertices v and w are in L (or in R), consider the graphs H^4 and H^5 obtained from H^L (or H^R) by adding edge e and edge f, respectively; let y^4 and y^5 denote the corresponding incidence vectors. Since both y^4 and y^5 are in T, by assumption, $\overline{c}y^4 = \overline{c}y^5$, and so $\overline{c}_e = \overline{c}_f$. But then, since $R \cap L \neq \emptyset$, and since $\overline{c}_g = 0$ for every $g \notin F$, it follows that $\overline{c}_e = \overline{c}_f$ for every choice of e and f in $F \cap \delta(u)$. The theorem follows. ∎

A different class of facet-defining inequalities for the polytope $P(K_n)$ can be obtained in a similar way from the complete bipartite graph $K_{3,3}$. Let $1, 2, 3, 4, 5, 6$ denote the vertices of $K_{3,3}$ and let Q denote its edge-set, i.e. $Q = \{14, 15, 16, 24, 25, 26, 34, 35, 36\}$. Subdivide each edge ij of $K_{3,3}$, N_{ij} times, with $N_{ij} \ge 1$; and let ij^1, ij^2, $\cdots$, $ij^{N_{ij}}$ denote the corresponding subdivision vertices. Denote by $G = (V, E)$ the

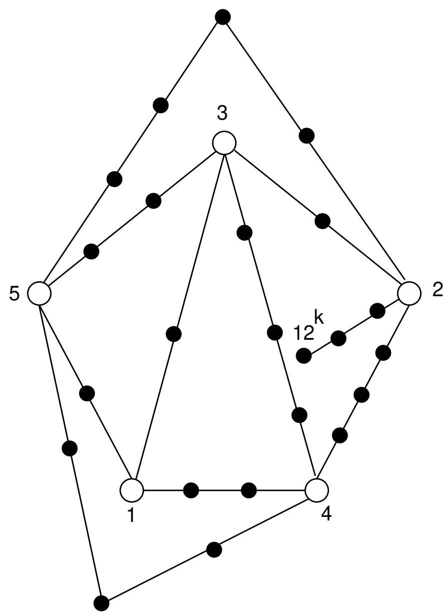

Figure 4: The graph H_R

resulting graph. Note that G contains all the edges (i, ij^1), (ij^1, ij^2), $\cdots$, $(ij^{N_{ij}-1}, ij^{N_{ij}})$, and $(ij^{N_{ij}}, j)$, and it does not contain the edge (i,j), for every $ij \in Q$. Figure 5 shows a graph G having 16 subdivision vertices. Note that, by assumption, the total number of subdivision vertices is at least nine. Set $M = \{12, 13, 23, 45, 46, 56\}$, and let $G^+ = (V, E^+)$ be the graph obtained from G by adding all edges in M and by adding every edge of each $K_{(i,j)}$, for all $ij \in Q$, where $K_{(i,j)}$ denote the complete graph with vertex set $\{i, ij^1, \cdots, ij^{N_{ij}}, j\}$. Let K_n denote the complete graph with vertex set V, and let F' denote the set of all edges of K_n that are not edges of G^+. The following theorem shows that the inequality $x(F') \geq 1$ defines a facet of $P(K_n)$. Note that also this inequality is not valid for $Q_1(K_n)$ (because for every edge e of G, the graph $G - e$ is planar and its incidence vector y is such that $y(F') = 0$), and is not valid for $Q_2(K_n)$ (because the graph G is two-connected and its incidence vector y is such that $y(F') = 0$).

Theorem 2 *For every complete graph K_n with $n \geq 15$, the inequality $x(F') \geq 1$ defines a facet of $P(K_n)$.*

Proof. The proof is similar to the proof of Theorem 1. The only difference is that we have to show that $\overline{c}_e = 0$ also for every edge $e \in M$, and this is easy. ∎

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