Polyhedral Combinatorics of QAPs with Less Objects than Locations (Extended Abstract)

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Abstract

For the classical quadratic assignment problem (QAP), where n objects have to be assigned to n locations (the $n \times n$-case), polyhedral studies have been started in the very recent years by several authors. In this paper, we investigate the variant of the QAP, where the number of locations may exceed the number of objects (the $m \times n$-case). It turns out that the polytopes that are associated with this variant are quite different from the ones associated with the $n \times n$-case. However, one can obtain structural results on the $m \times n$-polytopes by exploiting knowledge on the $n \times n$-case, since the first ones are certain projections of the latter ones. Besides answering the basic questions for the affine hulls, the dimensions, and the trivial facets of the $m \times n$-polytopes, we present a large class of facet defining inequalities. Employed into a cutting plane procedure, these polyhedral results enable us to compute optimal solutions for some hard instances from the QAPLIB for the first time without using branch-and-bound. Moreover, we can calculate for several yet unsolved instances significantly improved lower bounds.

1 Introduction

Let a set of m objects and a set of n locations be given, where $m \leq n$. We will be concerned with the following problem. Given *linear costs* $c_{(i,j)}$ for assigning object i to location j and *quadratic costs* $q_{\{(i,j),(k,l)\}}$ for assigning object i to location j and object k to location l, the task is (in the *non-symmetric case*) to find an assignment, i.e., an injective map $\varphi : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$, that minimizes

$$\sum_{i=1}^{m} \sum_{k=i+1}^{m} q_{\{(i,\varphi(i)),(k,\varphi(k))\}} + \sum_{i=1}^{m} c_{(i,\varphi(i))}.$$

In the *symmetric case* we have quadratic costs $\hat{q}_{(\{i,k\},\{j,l\})}$ for assigning the two objects i and k anyhow to the two locations j and l, and we have to find an assignment $\varphi : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$ that minimizes

$$\sum_{i=1}^{m} \sum_{k=i+1}^{m} \hat{q}_{(\{i,k\},\{\varphi(i),\varphi(k)\})} + \sum_{i=1}^{m} c_{(i,\varphi(i))}.$$

The classical *quadratic assignment problem* (introduced by Koopmans and Beckmann, 1957) is a special case of the non-symmetric formulation, where $m = n$ and $d_{\{(i,j),(k,l)\}} = f_{ik}d_{jl} + f_{ki}d_{lj}$ holds for some *flow-matrix* (f_{ik}) and some *distance-matrix* (d_{jl}). If one of the flow- or distance-matrix is symmetric then the problem is also a special case of the symmetric formulation given above.

The QAP is not only from the theoretical point of view a hard one among the classical combinatorial optimization problems (Sahni and Gonzales (1976) showed that even ϵ-approximation is $\mathcal{NP}$-hard), but it has also resisted quite well most practical attacks to solve it for larger instances.

Polyhedral approaches to the classical case with $m = n$ (the $n \times n$*-case*) have been started during the recent years by Rijal (1995), Padberg and Rijal (1996), Jünger and Kaibel (1996, 1997b), and Kaibel (1997). In Jünger and Kaibel (1997a) the first large class of facet defining inequalities for the associated polytopes is presented. These inequalities turned out to yield very effective cutting planes that allowed to solve for the first time several instances from the QAPLIB (the commonly used set of test instances compiled by Burkard, Karisch, and Rendl, 1996) to optimality without using branch-and-bound.

In this paper, we describe a polyhedral approach to the case where the number of objects m might be less than the number of locations n (the $m \times n$*-case*). We restrict our presentation to the symmetric version. In Section 2 a problem formulation in terms of certain hypergraphs is introduced and the associated polytopes are defined. The trivial facets as well as the affine hulls of these polytopes are considered in Section 3. In Section 4 rather tight relaxation polytopes are presented that are projections of certain relaxation polytopes for the $n \times n$-case. In Section 5 we describe a large class of facet defining inequalities for the polytopes that are associated with the (symmetric) $m \times n$-case. Strengthening the relaxations of Section 4 by some of these inequalities in a cutting plane procedure, we can improve for some $m \times n$-instances in the QAPLIB the lower bounds significantly. Moreover, we can solve several $m \times n$-instances by a pure cutting plane procedure. Results of these experiments are given in Section 6. We conclude with some remarks on promising further directions of polyhedral investigations for the $m \times n$-QAP in Section 7.

2 QAP-Polytopes

As indicated in the introduction, we restrict to the symmetric QAP in this paper. Since it provides convenient ways to talk about the problem, we first formulate the symmetric QAP as a problem defined on a certain hypergraph.

Throughout this paper, let $m \le n$, $\mathcal{M} := \{1, \dots, m\}$, and $\mathcal{N} := \{1, \dots, n\}$. We define a hypergraph $\hat{\mathcal{G}}_{m,n} := (\mathcal{V}_{m,n}, \hat{\mathcal{E}}_{m,n})$ on the nodes $\mathcal{V}_{m,n} := \mathcal{M} \times \mathcal{N}$ with hyperedges

$$\hat{\mathcal{E}}_{m,n} := \left\{ \{(i,j),(k,l),(i,l),(k,j)\} \in \binom{\mathcal{V}_{m,n}}{4} : i \neq k, j \neq l \right\}.$$

A hyperedge $\{(i,j),(k,l),(i,l),(k,j)\}$ is denoted by $\langle i,j,k,l \rangle$. The sets $\mathrm{row}_m ni := \{(i,j) \in \mathcal{V}_{m,n} : j \in \mathcal{N}\}$ and $\mathrm{col}_m nj := \{(i,j) \in \mathcal{V}_{m,n} : i \in \mathcal{M}\}$ are called the *i-th row* and the *j-th column* of $\mathcal{V}_{m,n}$, respectively.

We call a subset $C \subset \mathcal{V}_{m,n}$ of nodes a *clique* of the hypergraph $\hat{\mathcal{G}}_{m,n}$ if it intersects neither any row nor any column more than once. The maximal cliques of $\hat{\mathcal{G}}_{m,n}$ are the m-cliques. The set of hyperedges that is associated with an m-clique $C \subset \mathcal{V}_{m,n}$ of $\hat{\mathcal{G}}_{m,n}$ consists of all hyperedges that share two nodes with C. This set is denoted by $\hat{\mathcal{E}}_{m,n}(C)$. Solving symmetric QAPs then is equivalent to finding minimally node- and hyperedge-weighted m-cliques in $\hat{\mathcal{G}}_{m,n}$.

We denote by $x^{(\dots)} \in \mathbb{R}^{\mathcal{V}_{m,n}}$ and $z^{(\dots)} \in \mathbb{R}^{\hat{\mathcal{E}}_{m,n}}$ the characteristic vectors of subsets of $\mathcal{V}_{m,n}$ and $\hat{\mathcal{E}}_{m,n}$, respectively. Thus the following polytope encodes the structure of the symmetric QAP in an adequate fashion (where we simplify $(x^C, z^C) := (x^C, z^{\hat{\mathcal{E}}_{m,n}(C)})$):

$$\mathcal{SQAP}_{m,n} := \operatorname{conv}\left\{(x^C, z^C) : \ C \text{ is an } m\text{-clique of } \hat{\mathcal{G}}_{m,n}\right\}$$

The (mixed) integer linear programming formulation in Theorem 1 is quite basic for the polyhedral approach. Let $\Delta^{(i,j)}_{(k,l)}$ be the set of all hyperedges that contain both nodes (i,j) and (k,l). As usual, for any vector $u \in \mathbb{R}^L$ and any subset $L' \subset L$ of indices we denote $u(L') := \sum_{\lambda \in L'} u_\lambda$.

Theorem 1. *Let $1 \leq m \leq n$. A vector $(x,z) \in \mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{m,n}}$ is a vertex of $\mathcal{SQAP}_{m,n}$, i.e., the characteristic vector of an m-clique of $\hat{\mathcal{G}}_{m,n}$, if and only if it satisfies the following conditions:*

$$x(\text{row}_i) = 1 \qquad (i \in \mathcal{M}) \tag{1}$$

$$x(\text{col}_j) \leq 1 \qquad (j \in \mathcal{N}) \tag{2}$$

$$-x_{(i,j)} - x_{(k,j)} + y\left(\Delta^{(i,j)}_{(k,j)}\right) = 0 \qquad (i,k \in \mathcal{M}, i < k, j \in \mathcal{N}) \tag{3}$$

$$z_h \geq 0 \qquad (h \in \hat{\mathcal{E}}_{m,n}) \tag{4}$$

$$x_v \in \{0,1\} \qquad (v \in \mathcal{V}_{m,n}) \tag{5}$$

The proofs of this as well as of the next theorem are given in the full version of the paper. The following result shows that in order to investigate the $m \times n$-case with $m < n$ it suffices to restrict even to $m \leq n-2$. In fact, it turns out that the structures of the polytopes for $m \leq n-2$ differ a lot from those for $m = n$ or $m = n-1$.

Theorem 2. *For $n \geq 2$ the canonical orthogonal projection $\mathbb{R}^{\mathcal{V}_{n,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{n,n}} \longrightarrow \mathbb{R}^{\mathcal{V}_{n-1,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{n-1,n}}$ induces an isomorphism between the polytopes $\mathcal{SQAP}_{n,n}$ and $\mathcal{SQAP}_{n-1,n}$.*

How is the $n \times n$-case related to the $m \times n$-case with $m \leq n-2$? Obviously, $\mathcal{SQAP}_{m,n}$ arises from $\mathcal{SQAP}_{n,n}$ by the canonical orthogonal projection $\hat{\sigma}^{(m,n)} : \mathbb{R}^{\mathcal{V}_{n,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{n,n}} \longrightarrow \mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{m,n}}$. Let $W_{m,n} = \mathcal{V}_{n,n} \setminus \mathcal{V}_{m,n}$ and $F_{m,n} = \{\langle i,j,k,l\rangle \in \hat{\mathcal{E}}_{n,n} : \langle i,j,k,l\rangle \cap W_{m,n} \neq \emptyset\}$ be the sets of nodes and hyperedges that are "projected out" this way. The following connection is very useful.

Remark 3. *If an inequality $(a,b)^T(x,z) \leq \alpha$ defines a facet of $\mathcal{SQAP}_{n,n}$ and $a_v = 0$ holds for all $v \in W_{m,n}$ as well as $b_h = 0$ for all $h \in F_{m,n}$, then the "projected inequality" $(a',b')^T(x',z') \leq \alpha$ with $(a',b') = \hat{\sigma}^{(m,n)}(a,b)$ defines a facet of $\mathcal{SQAP}_{m,n}$.*

3 The Basic Facial Structures of $\mathcal{SQAP}_{m,n}$

The questions for the affine hull, the dimension, and the trivial facets are answered by the following theorem.

Theorem 4. *Let $3 \leq m \leq n-2$.*

(i) The affine hull of $\mathcal{SQAP}_{m,n}$ is precisely the solution space of the equations (1) *and* (3).

(ii) $\mathcal{SQAP}_{m,n}$ has dimension $\dim(\mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{m,n}}) - (m + mn(m-1)/2)$.

(iii) The nonnegativity constraints $(x, z) \geq 0$ define facets of $\mathcal{SQAP}_{m,n}$.

(iv) The inequalities $(x, z) \leq 1$ are implied by the equations (1) *and* (3) *together with the nonnegativity constraints $(x, z) \geq 0$.*

Proof. Part (iv) is a straightforward calculation. While (iii) follows immediately from Remark 3 and the fact that the nonnegativity constraints define facets in the $n \times n$-case (see Jünger and Kaibel, 1996), part (i) (which implies (ii)) needs some more techniques that will be explained in the full version of the paper. The key step is to project the polytope isomorphically into a lower dimensional vector space, where the vertices have a more convenient coordinate structure. □

4 Projecting a Certain Relaxation Polytope

In the $n \times n$-case, the affine hull of the polytope $\mathcal{SQAP}_{n,n}$ is described by the following equations (see Jünger and Kaibel, 1996):

$$x(\text{row}_i) = 1 \qquad (i \in \mathcal{N}) \tag{6}$$

$$x(\text{col}_j) = 1 \qquad (j \in \mathcal{N}) \tag{7}$$

$$-x_{(i,j)} - x_{(k,j)} + z\left(\Delta^{(i,j)}_{(k,j)}\right) = 0 \qquad (i, j, k \in \mathcal{N}, i \neq k) \tag{8}$$

$$-x_{(i,j)} - x_{(i,l)} + z\left(\Delta^{(i,j)}_{(i,l)}\right) = 0 \qquad (i, j, l \in \mathcal{N}, j \neq l) \tag{9}$$

The fact that the equations (7) and (9) are needed additionally in the $n \times n$-case is the most important difference to the $m \times n$-case with $m \leq n-2$.

It turned out that minimizing over the intersection $\mathcal{SEQP}_{n,n}$ of $\text{aff}(\mathcal{SQAP}_{n,n})$ and the nonnegative orthant empirically yields a very strong lower bound for the symmetric $n \times n$-QAP. In contrast to that, minimizing over the intersection $\mathcal{SEQP}_{m,n}$ of $\text{aff}(\mathcal{SQAP}_{m,n})$ and the nonnegative orthant usually gives rather poor lower bounds (for $m \leq n-2$). However, solving the corresponding linear programs is much faster in the $m \times n$ case (as long as m is much smaller than n).

In order to obtain a good lower bound also in the $m \times n$-case, one could add $n - m$ dummy objects to the instance and after that calculate the bound in the $n \times n$-model. Clearly it would be desirable to be able to compute that bound without "blowing up" the model by adding dummies. The following result provides a possibility to do so, and hence, enables us to compute good lower bounds fast in case of considerably less objects than locations. One more notational convention is needed. For two disjoint columns col_j and col_l of $\hat{\mathcal{G}}_{m,n}$ we denote by $\langle \text{col}_j : \text{col}_l \rangle$ the set of all hyperedges that share two nodes with col_j and two nodes with col_l.

Theorem 5. *Let* $3 \le m \le n-2$. *A point* $(x,z) \in \mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{m,n}}$ *is contained in* $\hat{\sigma}^{(m,n)}(\mathcal{SEQP}_{n,n})$ *if and only if it satisfies the following linear system:*

$$
\begin{aligned}
&(10) && x(\mathrm{row}_i) = 1 && (i \in \mathcal{M})\\
&(11) && x(\mathrm{col}_j) \le 1 && (j \in \mathcal{N})\\
&(12) && -x_{(i,j)} - x_{(k,j)} + z\left(\Delta^{(i,j)}_{(k,j)}\right) = 0 && (i,k \in \mathcal{M}, i<k, j \in \mathcal{N})\\
&(13) && -x_{(i,j)} - x_{(i,l)} + z\left(\Delta^{(i,j)}_{(i,l)}\right) \le 0 && (j,l \in \mathcal{N}, j<l, i \in \mathcal{M})\\
&(14) && x(\mathrm{col}_j \cup \mathrm{col}_l) - z(\langle \mathrm{col}_j : \mathrm{col}_l \rangle) \le 1 && (j,l \in \mathcal{N}, j<l)\\
&(15) && x_v \ge 0 && (v \in \mathcal{V}_{m,n})\\
&(16) && z_h \ge 0 && (h \in \hat{\mathcal{E}}_{m,n})
\end{aligned}
$$

Proof. It should be always clear from the context whether a symbol like row_i is meant to be the i-th row of $\mathcal{V}_{n,n}$ or of $\mathcal{V}_{m,n}$. The rule is that in connection with variables denoted by lower-case letters always $\mathcal{V}_{m,n}$ is the reference set, while variables denoted by upper-case letters refer to $\mathcal{V}_{n,n}$.

We leave the "only if" claim for the full version of the paper, since the "if" claim is more interesting once one has checked that the constraints (10), ... ,(16) are valid for $\mathcal{SQAP}_{m,n}$ (which can be done easily).

In order to show that the given system (10), ... ,(16) of linear constraints forces the point (x,z) to be contained in the projected polytope $\hat{\sigma}^{(m,n)}(\mathcal{SEQP}_{n,n})$, we shall exhibit a map $\phi : \mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{m,n}} \longrightarrow \mathbb{R}^{\mathcal{V}_{n,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{n,n}}$ that maps such a point (x,z) satisfying (10), ... ,(16) to a point $(X,Z) = \phi(x,z) \in \mathcal{SEQP}_{n,n}$ which coincides with (x,z) on the components belonging to $\hat{\mathcal{G}}_{m,n}$ (as a subgraph of $\hat{\mathcal{G}}_{n,n}$). Hence, the first step is to define $(X,Z) = \phi(x,z)$ as an extension of (x,z), and the second step is to prove that this (X,Z) indeed satisfies the equations (6), ... ,(9), as well as $(X,Z) \ge 0$. The following extension turns out to be a suitable choice (recall that $m \le n-2$):

$$
\begin{aligned}
X_{(i,j)} &:= \frac{1 - x(\mathrm{col}_j)}{n-m} && (i > m)\\
Z_{\langle i,j,k,l \rangle} &:= \frac{x_{(i,j)} + x_{(i,l)} - z\left(\Delta^{(i,j)}_{(i,l)}\right)}{n-m} && (i \le m, k > m)\\
Z_{\langle i,j,k,l \rangle} &:= \frac{1 - x(\mathrm{col}_j \cup \mathrm{col}_l) + z\left(\langle \mathrm{col}_j : \mathrm{col}_l \rangle\right)}{(n-m-1)(n-m)} && (i,k > m)
\end{aligned}
$$

Let $(x,z) \in \mathbb{R}^{\mathcal{V}_{m,n}} \times \mathbb{R}^{\hat{\mathcal{E}}_{m,n}}$ satisfy (10), ... ,(16), and let $(X,Z) = \phi(x,z)$ be the extension defined just above. Clearly, X is nonnegative (by (11)) and Z is nonnegative (by (13) for $i \le m$, $k > m$ and by (14) for $i,k > m$).

The validity of the equations (6), ... ,(9) for (X,Z) is shown by a series of calculations, which is left for the full version of the paper. □

Finally, we investigate the system (10), ... ,(16) with respect to the question of redundancies. From Theorem 4 we know already that the nonnegativity constraints (15) and (16) define facets of $\mathcal{SQAP}_{m,n}$ as well as that the equations (10) and (12) are needed in the linear description of the affine hull of the polytope. Thus it remains to investigate the inequalities (11), (13), and (14). And in fact, it turns out that one of these classes is redundant.

Theorem 6. *Let* $4 \leq m \leq n-2$.

(i) The inequalities (11) *are implied by the constraints* (10), (12), (13), (14), (15), *and* (16).

(ii) The inequalities(13) *and* (14) *define facets of* $\mathcal{SQAP}_{m,n}$.

Proof. The calculation for part (i) is given in the full paper, as well as as the proof of (ii), which needs the same techniques as mentioned in the proof of Theorem 4. □

5 A Large Class of Facets

In Jünger and Kaibel (1997a) a large class of facet defining inequalities for the $n \times n$-case is investigated. Many of them satisfy the requirements stated in Remark 3. We briefly introduce these inequalities here, and demonstrate in Section 6 how valuable they are for computing good lower bounds or even optimal solutions for QAPs with less objects than locations.

Let $P_1, P_2 \subseteq \mathcal{M}$ and $Q_1, Q_2 \subseteq \mathcal{N}$ be two sets of row respectively column indices with $P_1 \cap P_2 = \emptyset$ and $Q_1 \cap Q_2 = \emptyset$. Define $\mathcal{S} := (P_1 \times Q_1) \cup (P_2 \times Q_2)$ and $\mathcal{T} := (P_1 \times Q_2) \cup (P_2 \times Q_1)$. In Jünger and Kaibel (1997a) it is shown that the following inequality is valid for $\mathcal{SQAP}_{n,n}$ for every $\beta \in \mathbb{Z}$ (where $z(\mathcal{S})$ and $z(\mathcal{T})$ are the sums over all components of z that belong to hyperedges with all four endnodes in $\mathcal{S}$ respectively in $\mathcal{T}$, and $\langle \mathcal{S} : \mathcal{T} \rangle$ is the set of all hyperedges with two endnodes in $\mathcal{S}$ and the other two endnodes in $\mathcal{T}$):

$$(17) \qquad -\beta x(\mathcal{S}) + (\beta - 1)x(\mathcal{T}) - z(\mathcal{S}) - z(\mathcal{T}) + z(\langle \mathcal{S} : \mathcal{T} \rangle) \leq \frac{\beta(\beta-1)}{2}$$

Clearly the validity carries over to the $m \times n$-case. The inequality (17) is called the *4-box inequality* determined by the triple $(\mathcal{S}, \mathcal{T}, \beta)$.

In this paper, we concentrate on 4-box inequalities that are generated by a triple $(\emptyset, \mathcal{T}, \beta)$ (i.e., we have $P_1 = Q_2 = \emptyset$ or $P_2 = Q_1 = \emptyset$), which we call *1-box inequalities.* Empirically, they have turned out to be the most valuable ones within cutting plane procedures among the whole set of 4-box inequalities.

In Jünger and Kaibel (1997a), part (i) of the following theorem is proved, from which part (ii) follows immediately by Remark 3.

Theorem 7. *Let* $n \geq 7$.

(i) Let $P, Q \subseteq \mathcal{N}$ *generate* $\mathcal{T} = P \times Q \subseteq \mathcal{V}_{n,n}$, *and let* $\beta \in \mathbb{Z}$ *be an integer number such that*

- $\beta \geq 2$,
- $|P|, |Q| \geq \beta + 2$,
- $|P|, |Q| \leq n - 3$, *and*
- $|P| + |Q| \leq n + \beta - 5$

hold. Then the 1-box inequality

$$(18) \qquad (\beta - 1)x(\mathcal{T}) - z(\mathcal{T}) \leq \frac{\beta(\beta-1)}{2}$$

defined by the triple $(\emptyset, \mathcal{T}, \beta)$ *defines a facet of* $\mathcal{SQAP}_{n,n}$.

(ii) For $m \leq n$ and $P \subseteq \mathcal{M}$ the 1-box inequality (18) *defines a facet of $\mathcal{SQAP}_{m,n}$ as well.*

6 Computational Results

Using the ABACUS framework (Jünger and Thienel, 1997) we have implemented a simple cutting plane algorithm for (symmetric) $m \times n$-instances (with $m \leq n-2$) that uses (10), (12), ... ,(16) as the initial set of constraints. Thus, by Theorem 5 (and Theorem 6 (i)), the first bound that is computed is the *symmetric equation bound (SEQB)*, which is obtained by optimizing over the intersection of aff($\mathcal{SQAP}_{n,n}$) with the nonnegative orthant.

The separation algorithm that we use is a simple 2-opt based heuristic for finding violated 1-box inequalities with $\beta = 2$. We limited the experiments to this small subclass of box inequalities since on the one hand they emerged as the most valuable ones from initial tests, and on the other hand even our simple heuristic usually finds many violated inequalities among the 1-box inequalities with $\beta = 2$ (if it is called several times with different randomly chosen initial boxes $\mathcal{T}$).

The experiments were carried out on a Silicon Graphics Power Challenge computer. For solving the linear programs we used the barrier code of CPLEX 4.0, which was run in its parallel version on four processors.

We tested our code on the `esc` instances of the QAPLIB, which are the only ones in that problem library that have much less objects than locations. Note that all these instances have both a symmetric (integral) flow as well as a symmetric (integral) distance matrix, yielding that only even numbers occur as objective function values of feasible solutions. Thus, every lower bound can be rounded up to the next even integer number greater than or equal to it. Our tests are restricted to those ones among these instances that have 16 or 32 locations. All the `esc16` instances (with 16 locations) were solved for the first time to optimality by Clausen and Perregaard (1994). The `esc32` (with 32 locations) instances are still unsolved, up to three easy ones among them.

Table 1 shows the results for the `esc16` instances. The instances `esc16b`, `esc16c`, and `esc16h` are omitted since they do not satisfy $m \leq n-2$ (`esc16f` was removed from the QAPLIB since it has an all-zero flow matrix). The bounds produced by our cutting plane code match the optimal solution values for all instances but `esc16a` (see also Figure 1). Working in the $m \times n$-model speeds up the cutting plane code quite much for some instances. The running times in the $m \times n$-model are comparable with the ones of the branch-and-bound code of Clausen and Perregaard (1994).

For the `esc32` instances, our cutting plane algorithm computes always the best known lower bounds (see Table 2 and Figure 2). The three instances `esc32e`, `esc32f`, and `esc32g` have been solved to optimality for the first time by Brüngger, Marzetta, Clausen, and Perregaard (1996). Our cutting plane code is able to solve these instances to optimality within a few hundred seconds of CPU time (on four processors). These are about the same running times as Brüngger, Marzetta, Clausen, and Perregaard (1996) needed with their branch-and-bound code on a 32 processor NEC Cenju-3 machine.

The formerly best known lower bounds for the other `esc32` instances were calculated by the *triangle decomposition* bounding procedure of Karisch and Rendl (1995). The bounds obtained by the cutting plane code improve (or match, in case

name	objects	opt	SEQB	box	LPs	time (s)	speed up	CP 1994 (s)
esc16a	10	68	48	64	3	522	4.87	65
esc16d	14	16	4	16	2	269	2.74	492
esc16e	9	28	14	28	4	588	3.37	66
esc16g	8	26	14	26	3	58	14.62	7
esc16i	9	14	0	14	4	106	28.18	84
esc16j	7	8	2	8	2	25	32.96	14

Table 1: The column *objects* contains the number of objects in the respective instance, *opt* is the optimal solution value, *SEQB* is the symmetric equation bound (i.e., the bound after the first LP), *box* is the bound obtained after some cutting plane iterations, *LPs* shows the number of linear programs being solved, *time* is the CPU time in seconds, and *speed up* is the quotient of the running times for working in the $n \times n$- and in the $m \times n$-model. The last column gives the running times Clausen and Perregard needed to solve the instances on a parallel machine with 16 i860 processors.

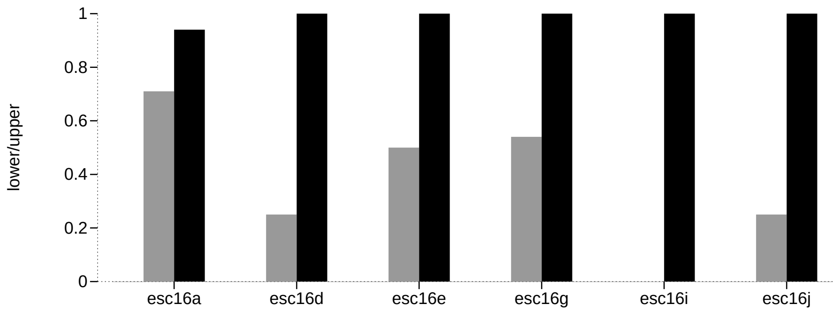

Figure 1: The bars (gray for SEQB and black for the bound obtained by the box inequalities) show the ratios of the lower and upper bounds, where the upper bounds are always the optimal solution values here.

of `esc32c`) all these bounds. The most impressive gain is the improvement of the bound quality from 0.28 to 0.68 for `esc32a`. While for the `esc16` instances switching from the $n \times n$- to the $m \times n$-model yields a significant speed up, in case of the `esc32` instances to solve the linear programs even becomes only possible in the $m \times n$-model.

Nevertheless, for the hard ones among the `esc32` instances the running times of the cutting plane code are rather large. Here, a more sophisticated cutting plane algorithm is required in order to succeed in solving these instances to optimality. This concerns the separation algorithms and strategies, the treatment of the linear programs, as well as the exploitation of sparsity of the objective functions, which will be briefly addressed in the following section.

7 Conclusion

The polyhedral studies reported in this paper have enabled us to build for the first time a cutting plane code for QAPs with less objects than locations that has a similar

name	objects	upper	prev lb	SEQB	box	LPs	time (s)
esc32a	25	130	36	40	88	3	62988
esc32b	24	168	96	96	100	4	$\star$60000
esc32c	19	642	506	382	506	8	$\star$140000
esc32d	18	200	132	112	152	8	$\star$80000
esc32e	9	2	2	0	2	2	576
esc32f	9	2	2	0	2	2	554
esc32g	7	6	6	0	6	2	277
esc32h	19	438	315	290	352	6	119974

Table 2: The column labels have the same meanings as in Table 1. Additionally, *upper* gives the objective function value of the best known feasible solution and *prev lb* denotes the best previously known lower bound. (Running times with a $\star$ are only approximately measured due to problems with the queuing system of the machine).

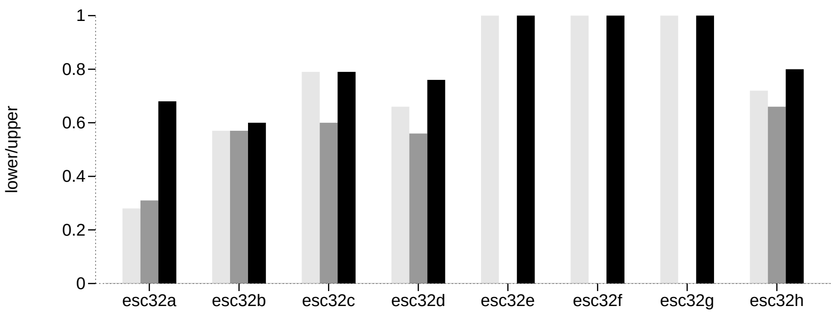

Figure 2: The dark gray and the black bars have the same meaning as in Figure 1. Additionally, the light gray bars show the qualities of the previously best known lower bounds.

performance as current parallel branch-and-bound codes for smaller instances and gives new lower bounds for the larger ones. More elaborated separation procedures (including parallelization) and a more sophisticated handling of the linear programs will surely increase the performance of the cutting plane algorithm still further.

At the moment, the limiting factor for the cutting plane approach is the size (and the hardness) of the linear programs. But if one considers the instances in the QAPLIB more closely, it turns out that the flow matrices very often are extremely sparse. If one exploits this sparsity, one can "project out" even more variables than we did by passing from the $n \times n$- to the $m \times n$-model. In our opinion, investigations of the associated projected polytopes will eventually lead to cutting plane algorithms in much smaller models, which perhaps will push the limits for exact solutions of quadratic assignment problems far beyond the current ones.

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