A New Mathematical Programming Framework for Facility Layout Design *

Miguel F. Anjos [†]   Anthony Vannelli [‡]

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Abstract

We present a new framework for efficiently finding competitive solutions for the facility layout problem. This framework is based on the combination of two new mathematical programming models. The first model is a relaxation of the layout problem and is intended to find good starting points for the iterative algorithm used to solve the second model. The second model is an exact formulation of the facility layout problem as a non-convex mathematical program with equilibrium constraints (MPEC). Aspect ratio constraints, which are frequently used in facility layout methods to restrict the occurrence of overly long and narrow departments in the computed layouts, are easily incorporated into this new framework. Finally, we present computational results showing that both models, and hence the complete framework, can be solved efficiently using widely available optimization software. This important feature of the new framework implies that it can be used to find competitive layouts with relatively little computational effort. This is advantageous for a user who wishes to consider several competitive layouts rather than simply using the mathematically optimal layout.

Key words: Facilities planning and design, Floorplanning, VLSI Macro-Cell Layout, Combinatorial Optimization, Convex Optimization, Global Optimization.

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1 Introduction

The facility layout design (or floorplanning) problem consists in partitioning a rectangular facility of known dimensions into departments with a given (fixed) area so as to minimize the total cost associated with the (known or projected) interactions between these departments. The given pairwise costs usually reflect transportation costs and/or adjacency preferences between departments. If the height and width of the departments may vary, then finding their optimal (rectangular) shapes is also a part of the problem. This is a hard problem, in particular because any desirable layout must of course have no overlap among the areas of the different departments. In fact, even the restricted version where the shapes of the departments are fixed and the optimization is taken over a fixed finite set of possible department locations is NP-hard. (This restriction is known as the Quadratic Assignment Problem, see for example [21].) Versions of this problem occur in many environments, such as hospital layout and service center layout, as well as in other engineering applications, such as VLSI placement and design. All of these problems are known to be NP-hard. For this reason, most of the approaches in the literature are based on heuristics. Two exceptions are the exact mixed integer programming approaches of Montreuil [16] and Meller et al. [15]; however these methods can be applied only to small problems with, say, less than ten departments. For a survey on the facility layout problem, see for example [14].

The contribution of this paper is a new framework for efficiently finding competitive solutions for the facility layout problem. This framework is based on mathematical programming techniques and arises from the combination of two new models. The first model is a relaxation of the layout problem and is intended to find good starting points for the iterative algorithm used to solve the second model. The second model is an exact formulation of the facility layout problem as a mathematical program with equilibrium constraints (MPEC) [13]. Aspect ratio constraints, which are frequently used in facility layout methods to restrict the occurrence of overly long and narrow departments in the computed layouts, are easily incorporated into this new framework. Finally, we present computational results showing that both models, and hence the complete framework, can be solved efficiently using MINOS [17, 18, 19], a widely available optimization software package. This important feature of the new framework implies that it can be used to find competitive layouts with relatively little computational effort.

This paper is structured as follows. In Section 2 we briefly present some methods in the literature which are directly relevant to the ideas in our framework, and in Section 3 we introduce the model ModCoAR which is the aforementioned relaxation of the layout problem. In Section 4 we introduce the exact formulation of the facility layout problem as an MPEC, and recast it for computational purposes as a Bilinear Programming Layout (BPL) model. Finally, in Section 5, we present computational results for some facility layout problems, including the well-known Armour and Buffa problem [5], which show that our approach yields layouts that are competitive with previous results reported in the literature.

2 Background

2.1 Some Related Methods

Drezner [10] introduced the DISCON (DISpersion-CONcentration) method which assumes that the departments are labelled $1, \ldots, N$, where N is the total number of departments, and that:

1. Each department is a circle (or can be approximated by a circle) of given radius $r_i, i = 1, \ldots, N$.

2. The distance between two departments is measured as the Euclidean distance between the centres of the circles. (This measure is sometimes referred to as the CTC (centroid-to-centroid) measure [7].)

3. The non-negative costs c_{ij} per unit distance between departments i and j are given.

The DISCON method uses a formulation equivalent to

$$
\text{(DISCON)} \quad \begin{array}{c} \min\limits_{(x_i, y_i)} \sum\limits_{1 \leq i < j \leq N} c_{ij}\, d_{ij} \\ \text{subject to} \\ d_{ij} \geq r_i + r_j,\ \forall\ 1 \leq i < j \leq N, \end{array}
$$

where (x_i, y_i) denotes the centre of the ith department, and $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$.

Van Camp, Carter and Vannelli [24] introduced the NLT (Non-linear optimization Layout Technique) method where all the departments as well as the facility itself are restricted to having fixed (given) areas and rectangular shapes, but for every rectangle the height and width are optimized by the mathematical model. The NLT method uses the following model, which we denote by vCCV:

$$
\text{(vCCV)} \quad \begin{array}{l} \min\limits_{(x_i, y_i), h_i, w_i, h_F, w_F} \sum\limits_{1 \leq i < j \leq N} c_{ij}\, d_{ij} \\ \text{subject to} \\[4pt] |x_i - x_j| - \frac{1}{2}(w_i + w_j) \geq 0 \quad \text{if} \quad |y_i - y_j| - \frac{1}{2}(h_i + h_j) < 0 \\ |y_i - y_j| - \frac{1}{2}(h_i + h_j) \geq 0 \quad \text{if} \quad |x_i - x_j| - \frac{1}{2}(w_i + w_j) < 0 \\ \frac{1}{2}w_F - (x_i + \frac{1}{2}w_i) \geq 0 \quad \text{for} \quad i = 1, \ldots, N \\ \frac{1}{2}h_F - (y_i + \frac{1}{2}h_i) \geq 0 \quad \text{for} \quad i = 1, \ldots, N \\ (x_i - \frac{1}{2}w_i) + \frac{1}{2}w_F \geq 0 \quad \text{for} \quad i = 1, \ldots, N \\ (y_i - \frac{1}{2}h_i) + \frac{1}{2}h_F \geq 0 \quad \text{for} \quad i = 1, \ldots, N \\ \min(w_i, h_i) - l_i^{\min} \geq 0 \quad \text{for} \quad i = 1, \ldots, N \\ l_i^{\max} - \min(w_i, h_i) \geq 0 \quad \text{for} \quad i = 1, \ldots, N \\ \min(w_F, h_F) - l_T^{\min} \geq 0 \\ l_T^{\max} - \min(w_F, h_F) \geq 0, \end{array}
$$

where (x_i, y_i) and d_{ij} are as previously defined; w_i, h_i are the width and height of department i; $l_i^{\min}, l_i^{\max}$ are the minimum and maximum allowable lengths for the shortest side of department i; w_F, h_F are the width and height of the facility; and $l_T^{\min}, l_T^{\max}$ are the minimum

and maximum allowable lengths for the shortest side of the facility. Therefore, with the vCCV model, the user can input ranges for the shortest sides of each department and of the resulting facility, and the model optimizes all the heights and widths within the given ranges. The NLT method employs a three-stage approach:

1. Stage-1 aims to evenly distribute the centres of the departments inside the facility;

2. Stage-2 aims to reduce the overlap among departments;

3. Stage-3 determines the final solution.

Stage-3 consists of solving the complete vCCV model, whereas the problems solved at Stages 1 and 2 correspond to relaxations of the vCCV model. These relaxations approximate each department by a circle whose radius is proportional to the area of the department. We also observe that the models for all three stages are solved using a penalty-based method.

2.2 The AR and CoAR Models

The NLT method was recently improved by Anjos and Vannelli [3, 4] who introduced an attractor-repeller (AR) model. Given $\alpha > 0$, let us define for each pair i, j of circles the *target distance* t_{ij}:

$$t_{ij} := \alpha \left(r_i + r_j \right)^2, \ \forall \ 1 \leq i < j \leq N,$$

and replace stages 1 and 2 of the NLT method with the single model:

$$\min_{(x_i, y_i), h_F, w_F} \quad \sum_{1 \leq i < j \leq N} c_{ij} D_{ij} + f\left(\frac{D_{ij}}{t_{ij}} \right)$$

subject to

(AR)
$$\begin{aligned}
\tfrac{1}{2} w_F \geq x_i + r_i \quad &\text{and} \quad \tfrac{1}{2} w_F \geq r_i - x_i, \text{ for all } i = 1, \ldots, N, \\
\tfrac{1}{2} h_F \geq y_i + r_i \quad &\text{and} \quad \tfrac{1}{2} h_F \geq r_i - y_i, \text{ for all } i = 1, \ldots, N, \\
w_F^{\max} \geq \ &w_F \geq w_F^{\min} \\
h_F^{\max} \geq \ &h_F \geq h_F^{\min}
\end{aligned}$$

where $f(z) = \frac{1}{z} - 1$ for $z > 0$, and $D_{ij} = d_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$. (In our relaxations of the layout problem, we work with the squares of the distances between each pair of circles.) The motivation for this model is that the "attractor" component of our objective function, namely $\sum_{1 \leq i < j \leq N} c_{ij} D_{ij}$, makes the two circles move closer together and pulls them towards a layout where $D_{ij} = 0$, while the "repeller" component $\sum_{1 \leq i < j \leq N} f\left(\frac{D_{ij}}{t_{ij}} \right)$ prevents the circles from overlapping. We refer the reader to [3] for a detailed discussion of this paradigm. The parameters t_{ij} are key to the strategy for enforcing the separation of the circles representing the departments. The idea is that t_{ij} is the target value for D_{ij}, which is the square of the distance between the circles i and j with radii r_i and r_j respectively. The parameter $\alpha > 0$ is introduced to provide some flexibility as to how tightly the user wishes to enforce the non-overlap constraint. In theory, the AR model aims to ensure that $\frac{D_{ij}}{t_{ij}} = 1$ at optimality, so

choosing $\alpha < 1$ sets a target value t_{ij} that allows some overlap of the areas of the respective circles, which means that a "relaxed" version of the non-overlap requirement of the circles is enforced. Similarly, $\alpha = 1$ means that there should be no overlap and the circles should intersect at exactly one point on their boundaries. In practice, α is chosen empirically in such a way that we achieve a reasonable separation between all pairs of circles. By properly adjusting the value of the parameter α, we aim to find a point where $\frac{D_{ij}}{t_{ij}} \approx 1$, i.e. where the target distance is approximately attained.

We note that AR allows the user to specify bounds $w_F^{\min} \leq w_F^{\max}$ on the width of the facility, and $h_F^{\min} \leq h_F^{\max}$ on the height. In particular, if the user knows in advance that the facility should have width $\bar{w}$ and height $\bar{h}$, then these constraints can be enforced by setting $w_F^{\min} = w_F^{\max} = \bar{w}$ and $h_F^{\min} = h_F^{\max} = \bar{h}$.

The AR model has only linear constraints (on the variables x_i, y_i, h_F, and w_F). From an optimization point of view, this is a significant advantage over Stages 1 and 2 of NLT; however, both models have the disadvantage of being non-convex. Under the assumption that $c_{ij} \neq 0$ for all pairs i, j of departments, an appropriate modification of the AR objective function yields a convex model. This convex model can be thought of as a "convexification" of AR. For $c_{ij} > 0$, $t_{ij} > 0$ and $z = (x_i - x_j)^2 + (y_i - y_j)^2$, define the piecewise function

$$
f_{ij}(x_i, x_j, y_i, y_j) := \begin{cases} c_{ij}\, z + \frac{t_{ij}}{z} - 1, & z \geq \sqrt{\frac{t_{ij}}{c_{ij}}} \\[2mm] 2\,\sqrt{c_{ij}\, t_{ij}} - 1, & 0 \leq z < \sqrt{\frac{t_{ij}}{c_{ij}}}. \end{cases}
$$

It is proved in [4] that f_{ij} is convex and continuously differentiable. Using f_{ij}, we define the "convexified" AR model, denoted CoAR, as:

$$
\min_{(x_i, y_i), h_F, w_F} \quad \sum_{1 \leq i < j \leq N} f_{ij}(x_i, x_j, y_i, y_j)
$$

subject to

(CoAR)
$$
\begin{aligned}
\tfrac{1}{2} w_F \geq x_i + r_i \quad &\text{and} \quad \tfrac{1}{2} w_F \geq r_i - x_i, \text{ for all } i = 1, \ldots, N, \\
\tfrac{1}{2} h_F \geq y_i + r_i \quad &\text{and} \quad \tfrac{1}{2} h_F \geq r_i - y_i, \text{ for all } i = 1, \ldots, N, \\
w_F^{\max} \geq\ &w_F\ \geq w_F^{\min}, \\
h_F^{\max} \geq\ &h_F\ \geq h_F^{\min}.
\end{aligned}
$$

By construction, f_{ij} attains its minimum value whenever the positions of circles i and j satisfy $D_{ij} \leq \sqrt{\frac{t_{ij}}{c_{ij}}}$. This includes the case where $D_{ij} = 0$, i.e. both circles completely overlap. Of course, we do not want such a placement, therefore what we seek is an arrangement of the circles where $D_{ij} \approx \sqrt{\frac{t_{ij}}{c_{ij}}}$, that is, close to the boundary of the flat portion of f_{ij}. At these points, the minimum value of f_{ij} is still attained but the resulting overlap is minimized. This idea motivates the introduction of so-called generalized target distances in the next section.

3 Finding a Good Initial Point: The ModCoAR Model

3.1 Motivation for the New Model: Generalized Target Distances

For each pair i, j of departments, we define the *generalized target distance*

$$T_{ij} := \sqrt{\frac{t_{ij}}{c_{ij} + \epsilon}} , \qquad 1 \leq i < j \leq N, \tag{1}$$

where $\epsilon > 0$ is a sufficiently small number so that if $D_{ij} \approx T_{ij}$ then $D_{ij} \approx \sqrt{\frac{t_{ij}}{c_{ij}}}$. The addition of ϵ ensures that the assumption of non-zero costs in Theorem 3.1 of [4] is fulfilled for the "perturbed costs" $c_{ij} + \epsilon$. (Note that this definition of T_{ij} is slightly different from our original definition in [4].) The generalized target distances T_{ij} take into account both the relative size of the departments (via the value of t_{ij}) and the connection cost between them (via the value of c_{ij}). In other words, if $D_{ij} \approx T_{ij}$, then the corresponding layout of the two departments has D_{ij} proportional to both t_{ij} and $1/c_{ij}$. Indeed, it is reasonable that the distances between the circles representing the departments should be inversely proportional to c_{ij} since, from a practical point of view,

- if c_{ij} is small, then the two departments are likely to be placed far apart in the layout, and correspondingly the generalized target distance should be large; and

- if c_{ij} is large, then the opposite reasoning applies and the generalized target distance should be small.

Furthermore, when c_{ij} equals zero, the resulting generalized target distance equals $\sqrt{t_{ij}/\epsilon}$ which will be large for small ϵ. This is again sensible since if $c_{ij} = 0$ then the departments are likely to be placed far apart in an optimal layout.

3.2 A Modified CoAR Model

The concept of generalized target distances is indeed appealing, but applying it with the CoAR model is made difficult in practice by the fact that a fairly specialized algorithm is required to stop at or near the set of points on the flat portion of the objective function that are furthest from the origin.

In this section we present a new model with an objective function whose minima approximate the generalized target distances, so that $D_{ij} \approx T_{ij}$ at optimality. It can be viewed as a compromise between convexity and computational practice, in the sense that we lose the convexity of CoAR, but we gain a model which can be solved efficiently and still aims to achieve the generalized target distances. The idea leading to the ModCoAR model is to add to the objective function a term of the form

$$- \ln \left(D_{ij}/T_{ij} \right)$$

for each pair i, j of circles. This particular choice of function is inspired by the successful application of log-barrier functions in interior-point methods for convex optimization, see for

example [20, 25]. Our modified CoAR (ModCoAR) model is thus:

$$\min_{(x_i,y_i),h_F,w_F} \quad \sum_{1 \le i < j \le N} F_{ij}(x_i, x_j, y_i, y_j) - K \ln (D_{ij}/T_{ij})$$

subject to

(ModCoAR)

$$\begin{array}{ll}
\tfrac{1}{2}w_F \ge x_i + r_i & \text{and} \quad \tfrac{1}{2}w_F \ge r_i - x_i, \text{ for all } i = 1, \dots, N, \\
\tfrac{1}{2}h_F \ge y_i + r_i & \text{and} \quad \tfrac{1}{2}h_F \ge r_i - y_i, \text{ for all } i = 1, \dots, N, \\
w_F^{\max} \ge & w_F \ge w_F^{\min}, \\
h_F^{\max} \ge & h_F \ge h_F^{\min},
\end{array}$$

where

$$F_{ij}(x_i, x_j, y_i, y_j) := \begin{cases} c_{ij}\, z + \frac{t_{ij}}{z} - 1, & z \ge T_{ij} \\ 2\sqrt{c_{ij}\, t_{ij}} - 1, & 0 \le z < T_{ij} \end{cases}$$

and the constant K is some large scaling factor. (Our choice of K for computational purposes is discussed in Section 5.) The shape of the objective function of ModCoAR and the effect of K are illustrated in Figure 1. An explanation of how Figure 1 was generated is in order. For a given pair i, j of circles, the corresponding term in the objective function of ModCoAR is a function of the four variables x_i, x_j, y_i, y_j. To obtain a (partial) representation of it in two dimensions, we first fixed some particular linear paths for the two circles; these paths are depicted in Figure 1(a). We then parametrised both paths by $h = 0, 0.005, 0.010, \dots, 20$, and for each "step" h we computed the value of $F_{ij}(x_i, x_j, y_i, y_j) - K \ln (D_{ij}/T_{ij})$. Figure 1(b) shows these function values plotted versus the parameter h, and scaled by various values of K. Note how the addition of the logarithmic term changes the shape of the objective function and how the resulting minima satisfy $D_{ij} \approx T_{ij}$. (The points where $D_{ij} = T_{ij}$ are indicated by the small squares in Figure 1(b).)

4 Computing a feasible layout: The BPL Model

4.1 Reformulation of the Facility Layout Problem

In the vCCV model, the non-overlap constraints are particularly difficult to handle:

$$|x_i - x_j| - \frac{1}{2}(w_i + w_j) \ge 0 \quad \text{if} \quad |y_i - y_j| - \frac{1}{2}(h_i + h_j) < 0,$$

$$|y_i - y_j| - \frac{1}{2}(h_i + h_j) \ge 0 \quad \text{if} \quad |x_i - x_j| - \frac{1}{2}(w_i + w_j) < 0.$$

Note that this set of constraints is *disjunctive*. Indeed, for each pair of departments i, j, the above pair of constraints can be rephrased as

$$|x_i - x_j| - \frac{1}{2}(w_i + w_j) \ge 0 \quad \text{or} \quad |y_i - y_j| - \frac{1}{2}(h_i + h_j) \ge 0. \tag{2}$$

The case of linear disjunctive constraints has been well studied in the literature, primarily in the context of combinatorial optimization problems, see for example [6]. More recently,
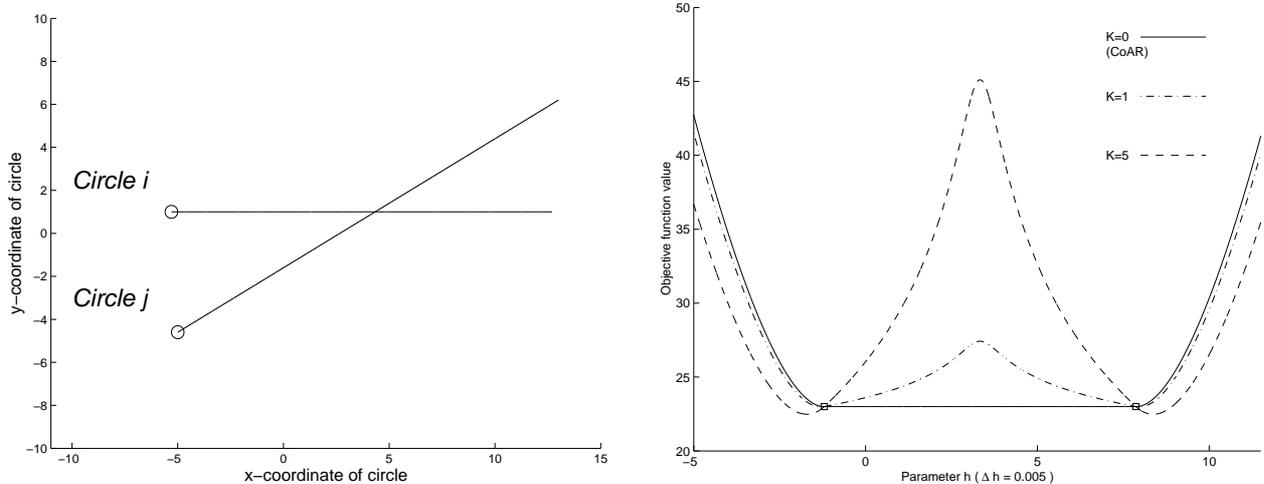
(a) Paths of circles i and j
(The circles indicate the starting points
of the parametrisation)

(b) Objective function of ModCoAR for
various K and for $t_{ij} = 9$, $c_{ij} = 16$
(The squares show where $D_{ij} = T_{ij}$)

Figure 1: Illustration of the objective function of ModCoAR

the general convex case has been studied in [9]. However, the constraints (2) are non-linear
and non-convex.

We formulate these constraints in a way that is more amenable to practical computation.
First, rewrite the constraints (2) as

$$\frac{1}{2}(w_i + w_j) - |x_i - x_j| \leq 0 \quad \text{or} \quad \frac{1}{2}(h_i + h_j) - |y_i - y_j| \leq 0,$$

which is equivalent to

$$\min \left\{ \frac{1}{2}(w_i + w_j) - |x_i - x_j|, \quad \frac{1}{2}(h_i + h_j) - |y_i - y_j| \right\} \leq 0. \tag{3}$$

For each pair of departments, introduce two new variables, X_{ij} and Y_{ij}, and let

$$X_{ij} = \max \left\{ \frac{1}{2}(w_i + w_j) - |x_i - x_j|, 0 \right\}, \qquad Y_{ij} = \max \left\{ \frac{1}{2}(h_i + h_j) - |y_i - y_j|, 0 \right\}. \tag{4}$$

Then (3) is equivalent to

$$X_{ij} Y_{ij} = 0. \tag{5}$$

In our formulation, we do not define the variables X_{ij} and Y_{ij} as in (4), but rather we require
that they satisfy the inequalities:

$$X_{ij} \geq \frac{1}{2}(w_i + w_j) - |x_i - x_j|, \quad X_{ij} \geq 0; \qquad Y_{ij} \geq \frac{1}{2}(h_i + h_j) - |y_i - y_j|, \quad Y_{ij} \geq 0.$$

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It is clear that these constraints, together with the bilinear constraint (5), are equivalent to the non-overlap constraints (2).

To formulate the facility layout problem, we need to enforce for each department the relationship $w_i h_i = a_i$, where a_i denotes the area of department i, as well as the linear constraints requiring that all departments fit inside the facility and satisfy the prescribed bounds on their dimensions. Hence we can formulate the facility layout problem as follows:

$$\min_{(x_i,y_i),h_i,w_i,h_F.w_F} \quad \sum_{1 \leq i < j \leq N} c_{ij}\, \delta(x_i,y_i,x_j,y_j)$$

subject to

$$X_{ij} \geq \tfrac{1}{2}(w_i + w_j) - |x_i - x_j| \quad \forall 1 \leq i < j \leq N$$
$$X_{ij} \geq 0 \quad \forall 1 \leq i < j \leq N$$
$$Y_{ij} \geq \tfrac{1}{2}(h_i + h_j) - |y_i - y_j| \quad \forall 1 \leq i < j \leq N$$
$$Y_{ij} \geq 0 \quad \forall 1 \leq i < j \leq N$$
$$X_{ij} Y_{ij} = 0 \quad \forall 1 \leq i < j \leq N$$
$$\tfrac{1}{2}w_F - (x_i + \tfrac{1}{2}w_i) \geq 0 \quad \text{for} \quad i = 1,\dots,N$$
$$\tfrac{1}{2}h_F - (y_i + \tfrac{1}{2}h_i) \geq 0 \quad \text{for} \quad i = 1,\dots,N$$
$$(x_i - \tfrac{1}{2}w_i) + \tfrac{1}{2}w_F \geq 0 \quad \text{for} \quad i = 1,\dots,N$$
$$(y_i - \tfrac{1}{2}h_i) + \tfrac{1}{2}h_F \geq 0 \quad \text{for} \quad i = 1,\dots,N$$
$$w_i h_i = a_i \quad \text{for} \quad i = 1,\dots,N$$
$$w_i^{\max} \geq w_i \geq w_i^{\min} \quad \text{for} \quad i = 1,\dots,N$$
$$h_i^{\max} \geq h_i \geq h_i^{\min} \quad \text{for} \quad i = 1,\dots,N$$
$$w_F^{\max} \geq w_F \geq w_F^{\min},$$
$$h_F^{\max} \geq h_F \geq h_F^{\min}$$

where $\delta(x_i,y_i,x_j,y_j)$ is the distance between departments i and j. Note that since this formulation is a non-linear programming problem, our framework can accomodate a variety of reasonable choices for the distance function, depending on the user's preference. Of course, the optimal layouts will vary depending on the choice of distance function. We report computational results with the rectilinear norm (l_1-norm) in Section 5.

The presence of the bilinear complementarity constraints

$$X_{ij} Y_{ij} = 0, \quad X_{ij} \geq 0, \quad Y_{ij} \geq 0 \tag{6}$$

means that the above model is an instance of a mathematical program with equilibrium constraints (MPEC). This class of problems has many practical applications, see for example [1, 11, 13]. MPECs are non-linear programming problems, so it is natural to attempt to solve them using standard algorithms for non-linear problems. One apparent difficulty for such algorithms is that the complementarity constraints (6) imply that, at any feasible point, either $X_{ij} = 0$ or $Y_{ij} = 0$, and hence the gradients of the active constraints are linearly dependent for all the feasible points. However, it has been observed that in general non-linear programming algorithms are often successful when applied to MPECs. A theoretical explanation of this fact is provided in [2]. We solved the above MPEC using MINOS as described below. Although other solvers could also be used, the fact remains that whichever non-linear optimization algorithm is chosen to solve the above formulation, the underlying process will be iterative and hence the final layout will heavily depend on the initial point

chosen. Our intent here is to show that the model ModCoAR provides good starting points for solving the above formulation of the layout problem, and so we now describe how we used MINOS for this purpose.

The main difficulty to overcome is that MINOS cannot be applied directly to the above formulation because the lack of a strictly feasible point causes MINOS to fail. This deficiency is a direct consequence of the constraints $X_{ij} Y_{ij} = 0$, therefore we apply a penalty-type approach to these constraints. The resulting model now has these bilinear terms "penalized", hence we call it Bilinear Penalty Layout model (BPL):

$$\min_{(x_i, y_i), h_i, w_i, h_F, w_F} \sum_{1 \leq i < j \leq N} c_{ij}\, \delta(x_i, y_i, x_j, y_j) + K\, X_{ij}\, Y_{ij}$$

subject to

(BPL)

$$X_{ij} \geq \tfrac{1}{2}(w_i + w_j) - |x_i - x_j| \quad \forall 1 \leq i < j \leq N$$
$$X_{ij} \geq 0 \quad \forall 1 \leq i < j \leq N$$
$$Y_{ij} \geq \tfrac{1}{2}(h_i + h_j) - |y_i - y_j| \quad \forall 1 \leq i < j \leq N$$
$$Y_{ij} \geq 0 \quad \forall 1 \leq i < j \leq N$$
$$\tfrac{1}{2}w_F - (x_i + \tfrac{1}{2}w_i) \geq 0 \quad \text{for} \quad i = 1, \ldots, N$$
$$\tfrac{1}{2}h_F - (y_i + \tfrac{1}{2}h_i) \geq 0 \quad \text{for} \quad i = 1, \ldots, N$$
$$(x_i - \tfrac{1}{2}w_i) + \tfrac{1}{2}w_F \geq 0 \quad \text{for} \quad i = 1, \ldots, N$$
$$(y_i - \tfrac{1}{2}h_i) + \tfrac{1}{2}h_F \geq 0 \quad \text{for} \quad i = 1, \ldots, N$$
$$w_i\, h_i = a_i \quad \text{for} \quad i = 1, \ldots, N$$
$$w_i^{\max} \geq w_i \geq w_i^{\min} \quad \text{for} \quad i = 1, \ldots, N$$
$$h_i^{\max} \geq h_i \geq h_i^{\min} \quad \text{for} \quad i = 1, \ldots, N$$
$$w_F^{\max} \geq w_F \geq w_F^{\min},$$
$$h_F^{\max} \geq h_F \geq h_F^{\min}$$

It is of course possible that the solution of BPL computed by MINOS will not satisfy all the complementarity constraints. However, it is our experience that this approach frequently yields solutions for which $X_{ij} Y_{ij} = 0$ holds for all pairs i, j, and that the corresponding layouts are competitive. The specific choice of penalty constant K we used, as well as some computational results, are reported in Section 5.

4.2 Incorporation of Aspect Ratio Constraints into BPL

In facility layout problems, it is often desirable to set bounds on the aspect ratio of the solution. The aspect ratio β_i for department i is defined as

$$\beta_i := \max\{h_i, w_i\} / \min\{h_i, w_i\}.$$

Bounding the maximum aspect ratio guarantees that no departments are excessively narrow (in either direction) in the computed layout. However, as the bounds on the aspect ratios become smaller, the layout problem becomes more constrained and the total cost of the optimal solution increases. Such aspect ratio constraints can be easily incorporated into the BPL model. In fact, this can be done in two different ways. To illustrate this, let us suppose that the aspect ratio of department i must be bounded above by a given value $\beta_i^* > 0$.

The first way to enforce this bound is to introduce a new variable β_i and three new bilinear constraints:

$$\beta_i \, w_i \geq h_i, \quad \beta_i \, h_i \geq w_i, \quad \beta_i^* \geq \beta_i.$$

Then, assuming that $w_i^{\min} > 0$ and $h_i^{\min} > 0$,

$$\beta_i^* \geq \beta_i \geq \max\{w_i/h_i, h_i/w_i\} \geq \max\{h_i, w_i\}/\min\{h_i, w_i\},$$

and so the aspect ratio constraint is enforced. This first approach has the disadvantage that it increases the number of constraints in the BPL model, and hence potentially affects the efficiency of the algorithm used to solve it.

Alternatively, bounds on the aspect ratio can be enforced via judicious choices of $w_i^{\min}$ and $h_i^{\min}$. Indeed, if we set $w_i^{\min} = h_i^{\min} = \sqrt{a_i/\beta_i^*}$, then

$$\begin{aligned} w_i \geq w_i^{\min} \quad &\Rightarrow \quad w_i^2 \geq a_i/\beta_i^* \\ &\Rightarrow \quad \beta_i^* \, w_i^2 \geq a_i = w_i \, h_i \\ &\Rightarrow \quad \beta_i^* \geq h_i/w_i, \end{aligned}$$

since $w_i^{\min} > 0$. Similarly, $h_i \geq h_i^{\min} \Rightarrow \beta_i^* \geq w_i/h_i$, and hence the aspect ratio constraint is enforced. We illustrate the application of both methods in Section 5.

5 Computational Results

We tested the models ModCoAR and BPL using MINOS 5.3 [17, 18, 19] accessed via the modelling language GAMS (release 2.25) [8] on a 300MHz SunSPARC.

To set up ModCoAR, we computed the generalized target distances (1) with $\epsilon = 0.1$; we set the radii of the approximating circles to $r_i = \sqrt{a_i/\pi}$; and we chose $K = \sum\limits_{1 \leq i < j \leq N} c_{ij}$. To solve ModCoAR, MINOS requires the user to supply an initial configuration. Since it is not clear a priori what the "best" starting configuration is, we place the centres of the N departments at regular intervals around a circle of radius $r = w_F^{\max} + h_F^{\max}$. Thus, letting $\theta_i = \frac{2\pi(i-1)}{N}$ and $r = w_F^{\max} + h_F^{\max}$, we initialize the centre (x_i, y_i) of the department to $x_i = r\cos\theta_i$, $y_i = r\sin\theta_i$, $i = 1, \ldots, N$. The model ModCoAR has $2N + 2$ variables and $4N + 4$ inequality constraints, all of which are linear, therefore only its objective function is non-linear and MINOS specifically exploits this structure by applying a reduced-gradient approach combined with a quasi-Newton algorithm. This is a significant practical advantage for the ModCoAR model because in general we can expect the algorithm to be superlinearly convergent, and thus to be capable of solving the model ModCoAR quite efficiently even for a fairly large number of departments.

To solve BPL, we take the solution of ModCoAR as the starting point. We found that setting K in the same way as for ModCoAR was generally an effective choice. (The only exception was the computation of the layout in Figure 5, where we set $K = \left(\sum\limits_{1 \leq i < j \leq N} c_{ij} \right)^2$

in BPL.) Since the solution of the BPL model depends on the choice of initial point, it is reasonable to solve the ModCoAR model for a number of different values of α and thus test a variety of starting points for the BPL model.

We now report the results obtained from applying our framework to three problems in the literature.

5.1 Bozer and Meller Small Test Problems

We first consider two examples from [7]. These small examples are interesting because we can compare the performance of our algorithm with the results reported in [7] for several other algorithms in the literature. For both examples, we required in BPL that both the height and width of each department should be at least half of the square root of the area of the department. As shown in Section 4.2, this implies that the aspect ratio is bounded above by 4, which is the condition proposed in [7] as a means to avoid unrealistic shapes within the heuristic approach based on the exact MIP (mixed integer programming) approach of Montreuil [16]. For the 9-department example, the total computation time for one run of our algorithm was 1 second of CPU time, and over several runs, we found the layout depicted in Figure 2 with rectilinear cost equal to 297.14. For the 12-department example, the total computation time for one run of our algorithm was 2 seconds of CPU time, and the best layout we found is depicted in Figure 3. Table 1 shows how our layouts compare with the layouts reported in [7].
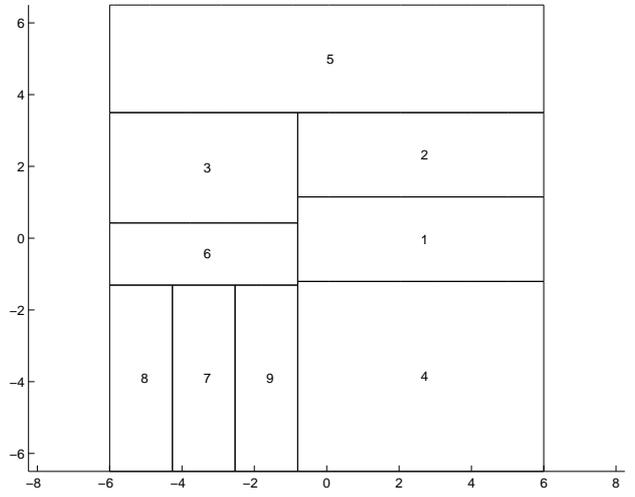
The most important feature we wish to point out is the short running time of our algorithm on these problems. This has two important consequences. Firstly, we were able to experiment with a fairly large number of values of α and find a variety of layouts for each problem within a very reasonable amount of time. More importantly, this suggests that our algorithm may be efficient for much larger problems, and this is indeed the case, as we illustrate in the next section.

Algorithm	Rectilinear cost for the 9-dept problem	Rectilinear cost for the 12-dept problem
QAP	319.00	180.00
MIP-heuristic	265.03	174.65
SFC	253.64	148.50
BPL	297.14	218.09

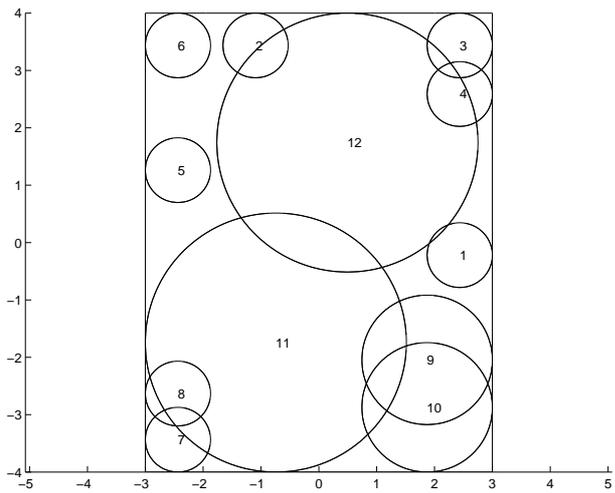Table 1: Comparison of the algorithms for the small test problems
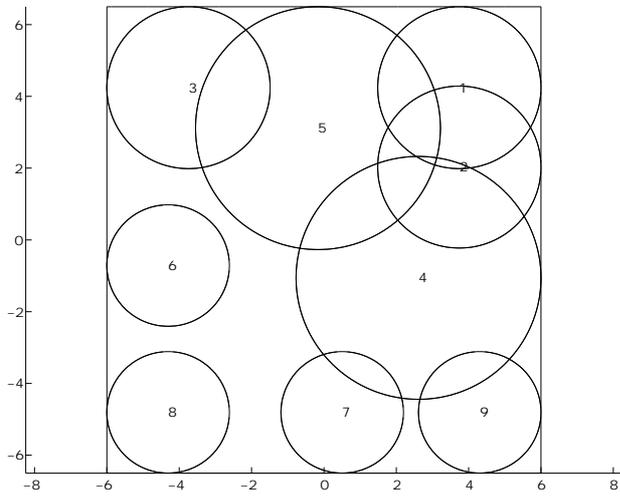
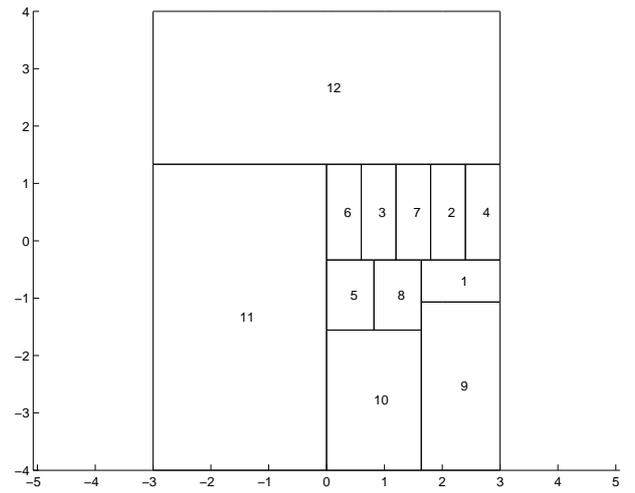Optimal ModCoAR solution for $\alpha = 2.1$ — Corresponding facility layout (Cost 297.14)

Figure 2: Best layout for the 9-department example

Optimal ModCoAR solution for $\alpha = 2.8$ — Corresponding facility layout (Cost 218.09)

Figure 3: Best layout for the 12-department example

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5.2 Armour and Buffa 20-department Problem

The two problems we considered in the previous section are fairly small, and the main advantage of our framework is the ability to efficiently find competitive layouts for large problems. Therefore we now present results for a larger problem which is well-known in the layout area, namely the Armour and Buffa 20-department problem. This problem was first described in [5] and we used the corrected cost matrix from [12, 22]. Each run of our algorithm for this problem took only about 18 seconds of CPU time.

We first set no explicit constraints on the aspect ratio of the solution, and required only that all the heights and widths of the departments be bounded below by 2. Our algorithm found the layout in Figure 4 with cost 4230.6 and largest aspect ratio of 6.67 (departments 15 and 19). The layout was found by running our algorithm approximately 20 times with different values of α between 1.0 and 3.0. In comparison, the genetic algorithm in [23] with a bound of 7 on the aspect ratio found a best layout with a cost of 5255.0 (over 10 runs of the algorithm).

We then employed our algorithm together with the first bounding method described in Section 4.2 to look for layouts with various bounds on the aspect ratio of the departments and thus obtain layouts which are directly comparable with the results in [23]. The results are summarized in Table 2, and the best layout we obtained with aspect ratio at most two is presented in Figure 5.

β_i^*	Cost of best layout in [23]	Cost of best layout found by our algorithm
5	5524.7	4591.3
4	5743.1	4786.4
3	5832.6	5140.1
2	6171.1	5224.7

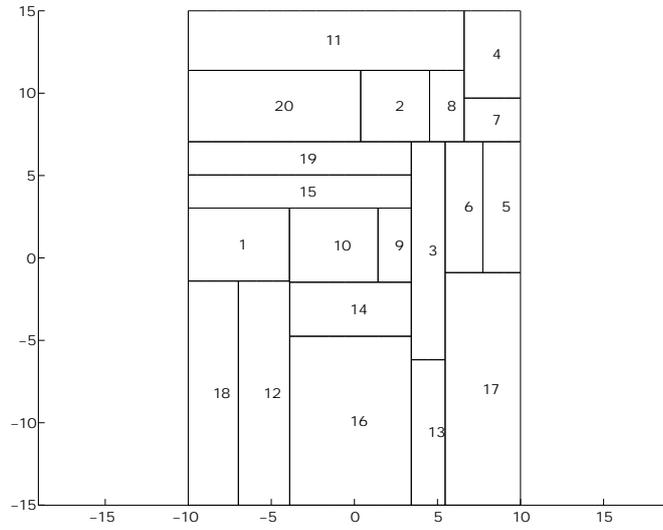Table 2: Comparison of the algorithms for the Armour and Buffa problem

Figure 4: First layout for the Armour and Buffa problem (Cost 4230.6)
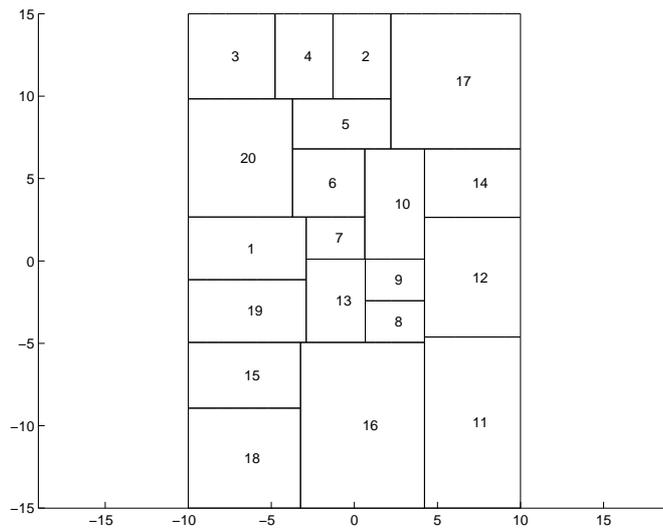

Figure 5: Best layout with $\beta_i \leq 2$ for the Armour and Buffa problem (Cost 5224.7)

6   Conclusion

We have presented a new mathematical programming framework for efficiently finding competitive solutions for the facility layout problem. This framework consists in the combination of two new models which are respectively a relaxation of the layout problem that is intended to find good starting points, and an exact formulation of the layout problem as an MPEC. Aspect ratio constraints, which are frequently used in facility layout methods to restrict the occurrence of overly long and narrow departments in the computed layouts, are easily incorporated into this new framework. We also presented computational results showing that our algorithm consistently yielded competitive layouts on several examples from the literature. Furthermore, our computational results show that this algorithm can be solved efficiently using MINOS, a widely available optimization software package. This important feature of the new algorithm implies that it can be used to find competitive layouts with relatively little computational effort, and furthermore that it is amenable to fast computation for large layout problems. This is advantageous for a user who wishes to consider several competitive layouts rather than simply using the mathematically optimal layout. Finally, our computational experience shows that the choice of α has a significant impact on the layout obtained using our algorithm. Therefore the role of α should be the subject of future research.

Acknowledgements

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