# Stochastic Optimisation of Drawdowns via Dynamic Reinsurance Controls

**I n a u g u r a l - D i s s e r t a t i o n**

zur
Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Universität zu Köln

vorgelegt von
Leonie Violetta Brinker, M. Sc.
aus Köln

2021

**Berichterstatter:**

Prof. Dr. Dr. Hanspeter Schmidli, Universität zu Köln
Prof. Dr. Alexander Drewitz, Universität zu Köln
Prof. Dr. Hansjörg Albrecher, Universität Lausanne

**Tag der mündlichen Prüfung:** 24.1.2022

# Abstract

In this work, we analyse optimisation problems related to the minimisation of severity and duration of relative losses, so-called drawdowns. We model the accumulated surplus (total income minus expenses) of an insurance company by a stochastic process $X = (X_t)_{t\geq 0}$, which is either a Cramér–Lundberg process or a diffusion. The running maximum $M = (M_t)_{t\geq 0}$ and the drawdown $\Delta = (\Delta_t)_{t\geq 0}$ of $X$ are given by $M_t = \max\{m_0, \sup_{s\in[0,t]} X_s\}$ and $\Delta_t = M_t - X_t$ at time $t \geq 0$. With this definition, we allow for an initial maximum $m_0 \in \mathbb{R}$ which has been reached before the observation starts. The drawdown process has the natural interpretation of the current decline from the last historical peak of the surplus and is, therefore, a time- and performance-adjusted measure of risk. We consider value functions based on the minimisation of the 'expected time with critical drawdown' $\mathbb{E}\big[\int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta_t > d\}} \,\mathrm{d}t \,\big|\, \Delta_0 = x\big]$, $x \geq 0$, by dynamic, proportional reinsurance controls. Here, the parameter $d > 0$ is a proxy for the size of drawdowns that is perceived as unfavourable. The 'discounting' rate $\delta > 0$ reflects the preference of postponing critical drawdowns for as long as possible.

The first chapter contains a detailed explanation of the motivation of drawdown minimisation for insurance companies and in stochastic control theory. We prove that the problem can be split into the subproblems of

i) maximising the time with uncritical drawdown, $\Delta \in [0, d]$, with a penalty for the overshoot at the exit time and

ii) minimising the time of recovery if the drawdown is currently critical, $\Delta > d$.

In the second chapter, we consider the Cramér–Lundberg model. We show that the minimal expected time in critical drawdown is the unique solution to a Hamilton–Jacobi–Bellman equation by considering a set of generalised discounted penalty functions of Gerber–Shiu type. In the third chapter, we prove that the minimal expected time in critical drawdown and optimal strategy for the diffusion model have explicit representations in terms of the Lambert W function. From these two chapters, we conclude that optimal reinsurance minimising drawdowns stabilises the surplus close to its running maximum. Especially for insurance companies, this enhanced predictability is favourable. By analysing optimally controlled processes, we discover, however, that growth of the running maximum is impeded. This can be a drawback from an economic perspective. In the fourth chapter, we therefore introduce a modified value function, including dividends as an 'incentive to grow', and solve the resulting problem for a diffusion surplus model. In our numerical examples, we consider in detail the optimal strategies (which are of feedback form in all cases). By putting the focus on a different aspect in each chapter, we highlight model-specific results: in the second chapter, we address the influence of the claim distribution, in the third chapter, the effect of costs of reinsurance and in the fourth chapter, the impact of preference (paying dividends versus avoiding drawdowns) of the insurer. In the fifth and last chapter, we give an outlook on the various possibilities for further research related.

# Zusammenfassung


In dieser Arbeit analysieren wir Optimierungsprobleme zur Minimierung von Größe und Anhaltedauer relativer Verluste, so genannter Drawdowns. Wir modellieren den akkumulierten Überschuss (gesamte Erträge abzüglich Aufwendungen) eines Versicherungsunternehmens durch einen stochastischen Prozess $X = (X_t)_{t\geq 0}$. $X$ ist ein Cramér–Lundberg- oder ein Diffusionsmodell. Das laufende Maximum $M = (M_t)_{t\geq 0}$ und der Drawdownprozess $\Delta = (\Delta_t)_{t\geq 0}$ sind zur Zeit $t \geq 0$ durch $M_t = \max\{m_0, \sup_{s\in[0,t]} X_s\}$ und $\Delta_t = M_t - X_t$ definiert. Mit dieser Definition nehmen wir an, dass es einen „initialen Rekord“ $m_0 \in \mathbb{R}$ gibt, der schon vor der Betrachtungsperiode erreicht wurde. Eine natürliche Interpretation des Drawdownprozesses ist die aktuelle, negative Abweichung vom letzten historischen Überschusshoch. „Drawdown“ ist daher ein an die Zeit und Erfolge des Unternehmens angepasster Risikoindikator. Wir betrachten Wertefunktionen, die auf der Minimierung der „erwarteten Zeit mit kritischem Drawdown“ $\mathbb{E}[\int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta_t > d\}} \,\mathrm{d}t \,|\, \Delta_0 = x]$, $x \geq 0$, durch dynamische, proportionale Rückversicherung basieren. Der Parameter $d > 0$ repräsentiert die Höhe, ab der relative Verluste das Unternehmen schädigen können. Die „Diskontierungsrate“ $\delta > 0$ drückt aus, dass kritische Drawdowns so spät wie möglich auftreten sollen.

Im ersten Kapitel motivieren wir die Minimierung von Drawdowns aus der Perspektive von Versicherungsunternehmen und in der stochastischen Kontrolltheorie. Wir beweisen, dass das Problem in die beiden Teilprobleme

i) Maximierung der Zeit mit unkritischem Drawdown, $\Delta \in [0, d]$, mit einer Strafzahlung für das Defizit zur Zeit des Austritts aus $[0, d]$ und
ii) Minimierung der Zeit bis zur Wiedererreichung des unkritischen Bereichs, wenn das aktuelle Drawdown groß ist, $\Delta > d$,

zerlegt werden kann. Im zweiten Kapitel betrachten wir das Cramér–Lundberg Modell. Wir zeigen mithilfe einer verallgemeinerten, diskontierten Straffunktion vom Gerber–Shiu Typ, dass die minimale Zeit mit kritischem Drawdown die eindeutige Lösung der assoziierten Hamilton–Jacobi–Bellman Gleichung ist. Im dritten Kapitel beweisen wir, dass die entsprechende Funktion für das Diffusionsmodell eine explizite Darstellung besitzt, die auf der Lambert'schen W Funktion basiert. Das Hauptergebnis dieser beiden Kapitel ist, dass die gefundenen optimalen Rückversicherungsstrategien den Überschussprozess bei seinem Maximum stabilisieren. Der schwankungsärmere Prozess ist leichter vorhersehbar, was besonders für Versicherungsunternehmen vorteilhaft ist. Unsere Analyse der optimal kontrollierten Prozesse ergibt allerdings auch, dass das Wachstum des laufenden Maximums gehemmt wird. Dies kann, in ökonomischer Hinsicht, ein Nachteil sein. Im vierten Kapitel führen wir daher eine modifizierte Wertefunktion ein, bei der wir Dividendenzahlungen als „Wachstumsanreiz“ einbinden. Wir lösen dieses Problem für das Diffusionsmodell. Alle Lösungen und optimalen Strategien (gegeben durch „feedback“-Funktionen) werden durch numerische Beispiele illustriert. Dabei betrachten wir jeweils unterschiedliche Aspekte: Im zweiten Kapitel steht der Einfluss der Schadenverteilung im Fokus, im dritten Kapitel die Kosten der Rückversicherungspolice und im vierten Kapitel die Präferenz des Unternehmens (Dividendenzahlungen versus Drawdownvermeidung). Im fünften und letzten Kapitel betrachten wir einige Beispiele der zahlreichen Möglichkeiten zu weiterer Forschung in diesem Bereich.

*Ich bedanke mich bei Prof. i.R. Dr. Josef G. Steinebach, dessen Vorlesungen „Wahrscheinlichkeitstheorie I & II“ mich bereits während des Bachelorstudiums geprägt haben. Danken möchte ich außerdem meinen Mentoren und persönlichen Vorbildern Privatdoz. Dr. Julia Eisenberg und Prof. Dr. Alexander Drewitz.*

*Mein besonderer Dank gilt Prof. Dr. Dr. Hanspeter Schmidli für seine exzellente fachliche und individuelle Betreuung sowie seine jederzeit gewährte Unterstützung.*

*Zudem danke ich meinen Kollegen, meinen Freunden und meiner Familie, die mich auf meinem Weg begleitet haben.*

# Contents

# Principal Notation and Abbreviations

*Basic Notation*

| | | |
|---|---|---|
| $\mathbb{N}$ | Natural numbers $\{1, 2, 3, \ldots\}$ | |
| $\mathbb{R}$ | Real numbers | |
| $\mathbb{C}$ | Complex numbers | |
| $a \wedge b$ | $\min\{a, b\}$ for $a, b \in \mathbb{R}$ | |
| $a \vee b$ | $\max\{a, b\}$ for $a, b \in \mathbb{R}$ | |
| $\mathbb{1}_A$ | Indicator function | |
| $f(x-)$ | Left limit $\lim_{a \nearrow x} f(a)$ | |
| $f(x+)$ | Right limit $\lim_{a \searrow x} f(a)$ | |
| $\mathbf{M}$, $\mathbf{M}_{k,j}$ | A matrix $\mathbf{M}$ and an entry thereof | |
| $\mathbf{t}$, $\mathbf{t}_k$ | A vector $\mathbf{t}$ and an entry thereof | |
| $\ell_f$ | Laplace transform of the function $f$ | p. 18 |
| $\mathcal{W}$ | Lambert W function | p. 55 |
| | | |
| $(\Omega, \mathcal{F}, \mathbb{P})$ | Underlying probability space | p. 8 |
| $(\mathcal{F}_t)_{t \geq 0}$ | Filtration | p. 8 |
| $\mathbb{E}$ | Expected value with respect to $\mathbb{P}$ | |
| $\mathbb{E}^x$ | Expected value given initial value $x$ | p. 9 |

*General Model*

| | | |
|---|---|---|
| $X$, $X^B$ | Surplus (under control $B$) | pp. 7,9 |
| $M$, $M^B$ | Running maximum of the surplus (under control $B$) | pp. 3,9 |
| $\Delta$, $\Delta^B$ | Drawdown of the surplus (under control $B$) | pp. 3,9 |
| $B$ | Reinsurance strategy/control $B$ | p. 8 |
| $\mathcal{B}$ | Set of admissible strategies | pp. 8,17,55 |
| $\vartheta(B)$, $\vartheta_y(B)$, $\vartheta^y(B)$ | Exit times of $\Delta^B$ | p. 12 |
| $v^B$, $v$ | Return of a strategy $B$ and value function | pp. 9,80 |
| $\delta$ | Preference rate | p. 5 |
| $d$ | Critical drawdown size | p. 5 |
| $\eta$, $\theta$ | Safety coefficients of the insurer and reinsurer | p. 8 |
| $\mathcal{A}^b$ | Abbreviation related to the generator, defined separately for each surplus model | pp. 20,56 |
| $V^B$, $V$ | (Optimised) Laplace transform, defined separately for each surplus model | pp. 17,55,55 |

*Classical Risk Model, Chapter 2*

| | | |
|---|---|---|
| $N$, $(T_k)_{k\in\mathbb{N}}$ | Poisson process and sequence of arrival times | p. 7 |
| $\lambda$ | Intensity of $N$ | p. 7 |
| $(Y_k)_{k\in\mathbb{N}}$ | Claim sizes | p. 7 |
| $G$, $\mu$ | Claim size distribution function and expected value | p. 7 |
| $\ell_Y$ | Laplace transform of $Y_1$ | p. 18 |
| $c(b)$ | Premium cash flow | p. 17 |
| $\gamma$, $\gamma(b)$, $\Psi_b$ | Exponent associated with exit times and related function | p. 22 |
| $v_C^B$, $v_C$, $\mathcal{A}_C^b$ | $v^B$, $v$ and $\mathcal{A}^b$ for penalised auxiliary problems | p. 37 |

*Diffusion Model, Chapters 3 & 4*

| | | |
|---|---|---|
| $W$ | Standard Brownian motion | p. 8 |
| $\sigma$ | Volatility of diffusion surplus model | p. 8 |
| $\mu(b)$ | Profit rate | p. 55 |
| $\kappa(b)$, $\xi(b)$, $\kappa$, $\xi$ | Exponents associated with exit times | p. 57 |
| $\beta_1$, $\beta_2$, $\chi$ | Preference weights and ratio | pp. 80,82 |

*Abbreviations*

| | |
|---|---|
| HJB equation | Hamilton–Jacobi–Bellman equation |
| PDMP | Piecewise deterministic Markov process |
| | |
| cf. | conferre |
| e.g. | exempli gratia |
| i.e. | id est |
| p., pp. | page, pages |
| Ch. | Chapter |
| Sec. | Section |
| Thm. | Theorem |
| Eq. | Equation |
| Ex. | Example |

CHAPTER 1

# Introduction

Uncertainty and reliability form the core of insurance business. An insurance contract, from the perspective of the insurance company, is an agreement to pay for future claims of the insured party in return for a predefined premium payment. While insurance premia are charged in advance of the occurrence of the insured event, at least one of the following is typically uncertain: occurrence times, size and number of claim payments. If an insurance company is not reliable, this causes a reputational damage (which can manifest itself as a competitive disadvantage) and could even lead to legal consequences and regulatory penalties. One possibility to increase the predictability of future payments and, therefore, to facilitate being reliable, is purchasing reinsurance. Reinsurance (i.e. insurance for the insurer) can be viewed as trading or exchanging risks. The insurer passes on a part of the unknown claim payments to the reinsurer and in turn pays a reinsurance premium. This results in a reduction of the 'random' liabilities but also reduces the deterministic income. Thus, an essential question for insurers (and, because of the inspirational uncertainty aspects of the problem, also for probabilists) is how to 'optimally' reinsure a contract portfolio.
The theory of stochastic control equips us with a mathematical framework for deriving substantiated solutions to optimal reinsurance problems. An introduction to optimal control theory is found in [Fleming and Soner, 1993]; an overview of different techniques to solve optimal control problems especially in the context of actuarial mathematics provides Schmidli [2008]. The key idea is to show that the target functional, or 'value function', is in a certain sense the unique solution to a Hamilton–Jacobi–Bellman (integro-differential) equation. To this purpose, one derives an associated equation, proves existence of a solution and then applies a martingale argument to verify that this solution indeed belongs to an optimal strategy and is therefore the value function.
However, before finding an optimal strategy, we have to define the notion of 'optimality'. By examining a target functional which depends on the distance to the 'high water mark' of the surplus, the so-called drawdown, this thesis aims to find both, a new question regarding optimal reinsurance and an answer.

## 1.1 Insurance, Reinsurance and Drawdowns

In actuarial sciences and mathematics, two of the most popular models for the development of the accumulated surplus of an insurer are the Cramér–Lundberg model (introduced by Lundberg [1903] and further examined by Cramér [1930 and 1955]) and its diffusion approximation. In the Cramér–Lundberg ('classical risk') model, accumulated income from insurance premia is calculated as a linear function in time and expenses for claim payments, subtracted from the income, are represented by a sum of positive random variables, governed by a Poisson process. This simple model therefore

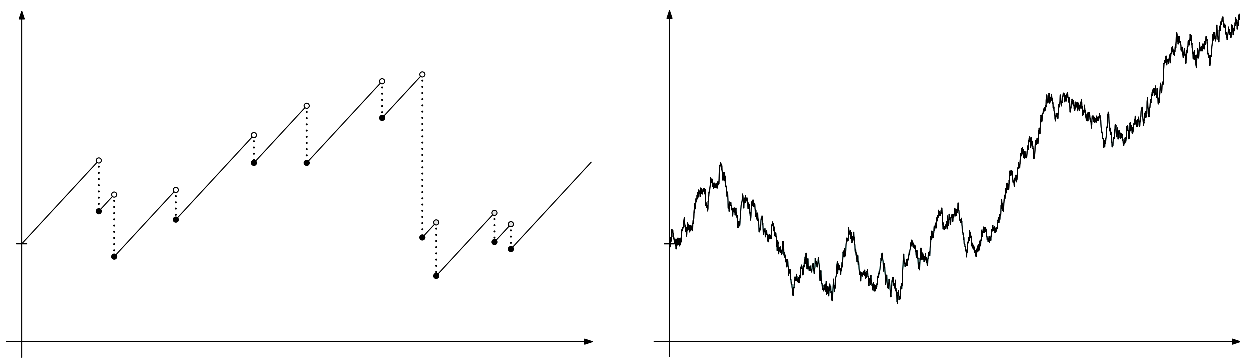


FIGURE 1.1 Exemplary sample paths of basic surplus processes: Cramér–Lundberg model (*left*) and diffusion model (*right*).

captures the key characteristics of insurance business. A typical path of this model is illustrated in Figure 1.1, on the left. The underlying idea of the 'diffusion approximation' is that for a certain re-scaling, the classical risk model converges weakly to a diffusion (compare, for example, [Iglehart, 1969] and [Grandell, 1977]). Intuitively, this is the case if the number of claims becomes infinitely large while the size of the claims goes to zero. A sketch of a path of the resulting process is shown in Figure 1.1, on the right.

A classical and well-studied approach to the optimal reinsurance problem is the minimisation of the ruin probability (i.e. the probability that the controlled surplus of the insurer becomes negative in finite time) for different surplus models. The classical risk model and its diffusion approximation allow, in many cases, insightful results. For both models, the maximisation of the survival probability (the equivalent counterpart of the minimisation of the ruin probability) is considered in [Schmidli, 2001] in the case of proportional reinsurance. 'Proportional' reinsurance can be expressed via the retention level $b \in [0, 1]$ of the insurer: if a claim of size $Y$ occurs, the insurer pays for $b \cdot Y$ and passes on the remaining $(1 - b) \cdot Y$ to the reinsurer. For this service, the reinsurer charges a premium which increases if the retention level is chosen smaller. Hence there is a trade-off between proportionally reducing the payment in the event of a claim and incurring fees. For the diffusion model, a constant retention level (which can be explicitly calculated) turns out to be optimal. For the classical risk model, optimal strategies are of 'feedback form' (that is, they are given by a function evaluated at the current surplus level) and can be calculated numerically, depending on the claim size distribution. Hipp and Vogt [2003] and Hipp and Taksar [2010] extend these methods to include different types of investment and reinsurance controls to minimise ruin probabilities for both models. A drawback of using the probability of ruin as a risk measure is that this approach ignores the deficit at the time of default: for processes with downward jumps, ruin can occur at the time of a claim payment, so that the surplus at this time is strictly negative. In reality, a large amount of debt puts the company in a worse situation than a small deficit. Gerber and Shiu [1998] introduced expected discounted penalty functions which help overcome this problem and can be interpreted as additionally measuring the severity of ruin (if it occurs). Recently, Preischl and Thonhauser [2019] analysed the problem of optimal reinsurance for value functions of the Gerber–Shiu type, depending on the capital prior to and the deficit at ruin for a classical risk model. Another possibility, proposed by Eisenberg and Schmidli [2009 and 2011], is to consider the minimisation of expected capital injections (i.e. payments made

by the insurer to 'stay in business' whenever the surplus is in technical ruin), which also serve as a measure of risk, replacing the ruin probability.
However, technical ruin is an exceptional and extreme event. In reality, one not only aims to improve the 'worst case scenario' but also the general performance and standing of a company during the time it operates. Performance-oriented approaches to optimising reinsurance include the maximisation of utility functions and dividends (see, for example, [Irgens and Paulsen, 2004] and [Azcue and Muler, 2005]). A new and somewhat different advance is based on the analysis of drawdowns of the surplus. If the *surplus* of an insurance company starting at an initial capital $\nu_0 \in \mathbb{R}$ is modelled by a stochastic process $X = (X_t)_{t\geq 0}$, its *running maximum* $M = (M_t)_{t\geq 0}$, given by

$$M_t = \max\left\{m_0,\ \sup_{s\in[0,t]} X_s\right\}, \qquad t \geq 0\,, \tag{1.1}$$

for $m_0 \geq \nu_0$, can be viewed as the history of records at which the company has outperformed itself in the past. The *(absolute) drawdown* $\Delta = (\Delta_t)_{t\geq 0}$ of $X$ is defined as

$$\Delta_t = M_t - X_t\,, \qquad t \geq 0\,, \tag{1.2}$$

and starts at the initial value $x = m_0 - \nu_0 \geq 0$. Thus, $\Delta$ can be interpreted as the relative loss since the last peak. This interpretation is one of the reasons why a large drawdown can be threatening for a company, as the following toy example illustrates.
Let us assume that we are observing two companies, insurer A and insurer B, whose surpluses develop as in Figure 1.2. The blue graph of Figure 1.2 represents the surplus of company A and the black graph represents the surplus of company B. At the time at which we start our observation, both companies have the same initial capital of 100 €. After time $T$, both have been equally profitable by increasing the accumulated surplus by 5 €. However, the drop of the surplus of company A by 25 € overshadows

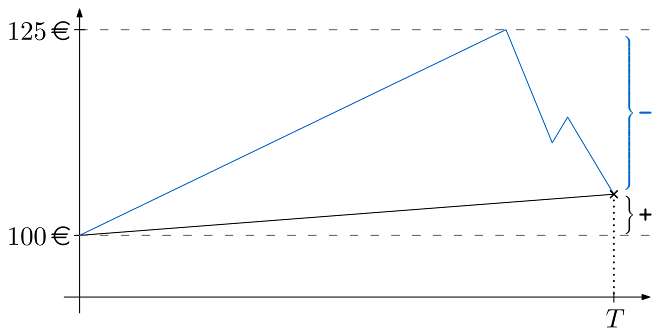



FIGURE 1.2 Surpluses of insurer A (blue) and insurer B (black) in our toy example.

its overall positive profit. This is the psychological effect of a drawdown, which, in a similar form, has also been observed in connection with decreasing dividend payments (see, for example [Albrecher et al., 2018]): while issuing dividends (or a positive profit) could generally be seen as a good sign to the market, a strongly decreasing dividend (or a large relative loss) can disappoint shareholders and is a sign of lowered performance or even managerial insufficiency in the eyes of the public. Reputational risks arising from the occurrence of a large drawdown materialise as financial losses if customers and

shareholders avoid the company in the future. On the other hand, the surplus of company B conveys a steadiness which could be read by stakeholders as managerial strength. Especially for insurers, signalling stability to policyholders, potential future customers and regulators is favourable.
Already in this simplistic example it can be seen that drawdown has two properties which are favourable for a measure of risk: firstly, it is time- and performance-adjusted (as it accounts for the historical 'high') and, secondly, it only measures the negative deviation from a record. Moreover, a large drawdown is an event which can be threatening for a company, but is not as ultimate or unusual as technical bankruptcy. In financial mathematics and economics, drawdowns are therefore used as dynamic risk indicators and form a basis for constructing new risk measures (see for example [Chekhlov et al., 2005] and [Maier-Paape and Zhu, 2018]) as well as for performance measurement (e.g. [Hamelink and Hoesli, 2004], [Schuhmacher and Eling, 2011]). Only mildly related to this monograph, but important to mention nonetheless, is that there is also a link of drawdowns to market crashes. This has been examined by Sornette [2003] and Zhang and Hadjiliadis [2012] (amongst others).
Mathematically, 'drawdown' is the result of applying a functional to the paths of a stochastic process $X$. The resulting, non-negative process $\Delta$ behaves similar to $-X$ as long as it is positive and 'glitches' along the $x$-axis whenever it arrives at zero (see Figure 1.3 on p. 6). The distributional properties of these processes have been extensively studied. A survey covering a broad scope is due to Mijatović and Pistorius [2012] who consider joint Laplace transforms of the first passage times of drawdown processes and five related quantities, such as the last supremum prior to the passage time, for spectrally negative Lévy processes (which includes the models mentioned above). Additionally, they use these results to derive explicitly the laws of certain functionals of drawdown processes. In the same framework, Landriault et al. [2017b] derive the law of the so-called 'time to recover' (i.e. the time until the historic maximum is re-reached). Moreover, they consider asymptotics of Laplace transforms of passage times of drawdown processes as the threshold tends to zero. Wang et al. [2020] consider penalised expected values similar to Gerber–Shiu functions, for which the ruin event is replaced by the first exit of the drawdown process from an interval. The drawdown process in this case is a generalisation of the above, earlier considered by Pistorius [2007], which is defined as the distance $\phi(M_t) - X_t$ at time $t$, for a measurable function $\phi : \mathbb{R} \to \mathbb{R}$. Further results, which apply specifically to the classical risk model, the diffusion approximation or special cases thereof, include [Taylor, 1975] (earliest calculation of the joint Laplace transform of the passage time and the maximum for a Brownian motion), [Lehoczky, 1977] (a generalisation of the former to Itô diffusions), [Harrison and Reiman, 1981] (on the connection to a reflected Brownian motion), [Landriault et al., 2015] (on the frequency of drawdowns for arithmetic Brownian motions), [Zhang, 2015] (on probabilities of drawdowns preceding 'drawups' for diffusion processes) and [Landriault et al., 2017a] (on drawdowns of renewal risk processes).
In the cosmos of drawdowns, the closest relative to minimising ruin probabilities is the minimisation of the probability that a 'large' drawdown occurs. That is, the probability that the drawdown process hits or exceeds a critical level $d > 0$, or, so to speak, the probability that the passage time of $d$ is finite. In a series of articles, Chen et al. [2015] and Angoshtari et al. [2015, 2016a and 2016b] consider the minimisation of the probabilities of 'proportional' drawdowns under investments and consumption. In particular, they find strategies minimising the probability of events of the form $\{\exists\, t < T : X_t < (1-\alpha)M_t\}$, where $\alpha \in [0, 1)$ and $T$ represents an infinitely long or (almost surely)

finite life time. For this definition of a drawdown, there are parallels to the risk model with tax considered by Albrecher and Hipp [2007]. In a similar setting, Han et al. [2018 and 2019] solve proportional reinsurance problems. But, similar to the ruin probability, the approach of minimising the probability of a large drawdown has blind angles. Firstly, if the surplus process contains jumps, the time at which the drawdown crosses the critical threshold can be a claim time. As in the case of ruin, the total size of the deficit, the 'severity' of the drawdown, is ignored. Secondly, as we have established above, a drawdown is usually not a once-in-a-lifetime event. Instead, the insurer will likely stay in business even after the drawdown time and, thus, needs to prepare a strategy for a quick recovery as well.

### 1.2 Scope and Overview of this Thesis

This monograph is based on [Brinker and Schmidli, 2021a, 2021b and 2022] and contributes, along with [Brinker, 2021], to the existing literature by introducing a value function which measures not only the size but also the duration period of drawdowns. That means, for preference parameters $\delta$ and $d$ (to be explained below), we consider the 'expected time with critical drawdown':

$$u(x) = \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta_t > d\}} \,\mathrm{d}t \,\Big|\, \Delta_0 = x\Big]\,, \qquad x \geq 0\,, \tag{1.3}$$

where $\Delta$ denotes the absolute drawdown. Using the expected time with critical drawdown and an extension thereof as drawdown-based risk measures, we solve the stochastic control problem of optimal (proportional) reinsurance for the Cramér–Lundberg model and its diffusion approximation.

In the definition of $u$, we choose an infinite time horizon because premature stopping (at the occurrence of large drawdown or, for example, technical ruin) could lead to intentionally exiting business in order to prevent future drawdowns. The *size* of drawdowns is reflected by the parameter $d$. We assume that there is a critical threshold $d > 0$, predefined by the insurer, such that drawdowns larger than $d$ are a threat for the company. The reason for defining such a critical size is that, due to expenses such as claim payments, the surplus of the insurance company, in fact, frequently has positive drawdowns. The insurer expects (and saves up) to pay for claims and, thus, not every single claim payment leading to a drawdown is a threat to the company. However, if there is an unexpectedly high number of claims or if outstanding payments are extremely large, this can lead to drawdowns of significant, 'critical', size. Figure 1.3 illustrates this condition with graphs of the surplus models in the top row and their corresponding drawdown processes in the bottom row. In particular, the state '$\Delta_t \leq d$' corresponds to the case in which the surplus is closer than $d$ to its running maximum at time $t$ (and the surplus and drawdown are in the respective white area). A critical drawdown means that the surplus is currently bounded away from its running maximum by at least $d$. This corresponds to the processes being in the grey areas of Figure 1.3. The choice of $d$ should reflect the preferences of the insurer. If $d$ is large, only extreme drawdowns are 'seen' by the function $u$. In this case, a large drawdown might be correlated with the unfavourable state of being undercapitalised. If $d$ is small, $u$ penalises almost all times at which the surplus is not increasing. To simultaneously measure the *duration* of drawdown phases, we 'add together' the times during which the drawdown is unfavourably large, that is $\Delta_t > d$. Additionally, we include an exponential preference at rate $\delta > 0$, to express that a large drawdown in

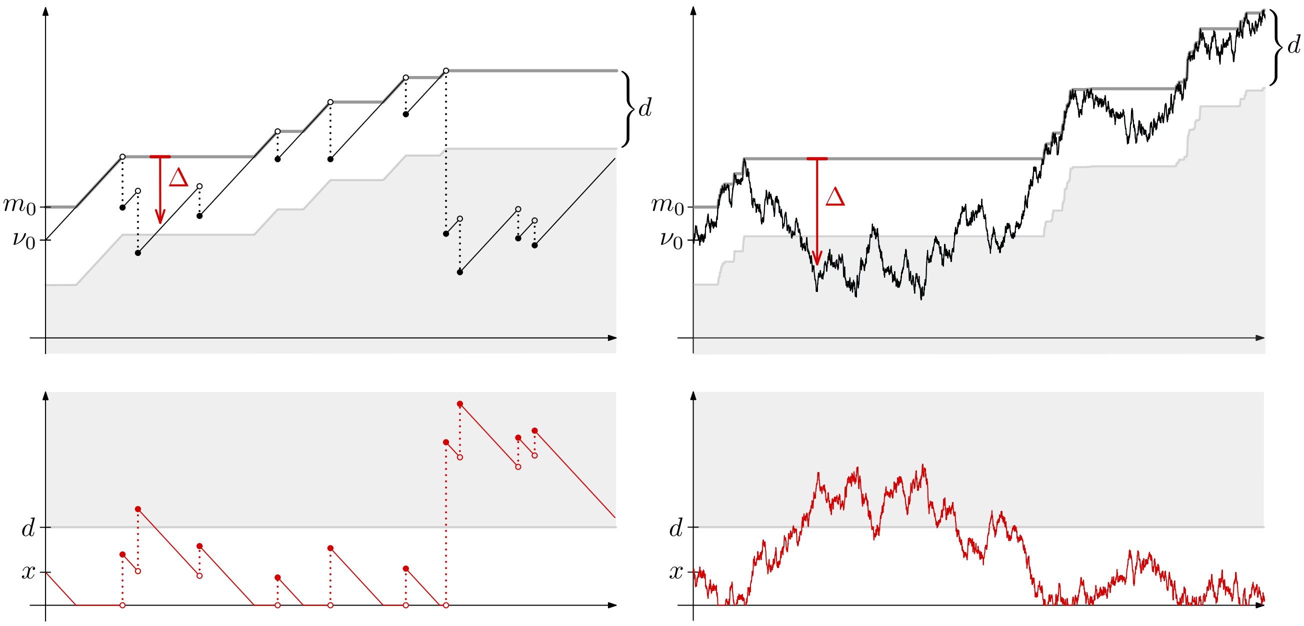



FIGURE 1.3 Surplus models (*top*, black graphs) with their running maximum (*top*, grey graphs) and respective drawdown processes (*bottom*, red graphs).

the far future is less threatening for the company than an immediate large drawdown. Similar to the choice of $d$, the definition of $\delta$ is a proxy for different corporate strategies. If $\delta$ is large, this means that early drawdowns have a much higher weight than drawdowns in the far future. In contrast, $\delta$ chosen close to zero rather fits a long-term orientation.

The motivation to consider a constant critical drawdown size (as opposed to, for example, a proportion of the running maximum) is the following. Firstly, in the models we consider, we assume that all monetary units are referenced to time zero. Otherwise, premia and claim payments should increase over time. In particular, there is no effect of inflation on the critical drawdown size $d$. Secondly, in reality, an insurance company would not be allowed (by regulators, impatient shareholders and clients) to hold an infinite surplus. This means, the surplus of the company is, in a way, 'naturally bounded'. Therefore, we can assume that the critical drawdown size cannot grow infinitely large. Thirdly, the main motivation to optimise drawdowns is to enhance stability and predictability. Thus, we expect (and prove) that the resulting controlled surplus does not fluctuate in a way which requires principal changes of $d$.

In the following two sections, we introduce the basic notation and build the mathematical framework for our analysis. We specify our surplus models and admissible control processes and review results on the relation of drawdowns to the Skorohod problem. We use these properties to derive a dynamic programming principle applicable to both models. The dynamic programming theorem is the first step towards the mathematical solution in all three settings considered in the following chapters. It allows us to split the optimisation problem into two subproblems: firstly, minimising the time with critically large drawdown and, secondly, maximising the time with uncritical drawdown (with a penalty for the overshoot). In Chapter 2, we focus on the Cramér–Lundberg model. This chapter is based on [Brinker and Schmidli, 2021b]. We start by calculating explicitly the return (i.e. the expected time in

drawdown) under simple, non-optimal controls. We prove that the minimal expected time in critical drawdown is the unique solution to a Hamilton–Jacobi–Bellman (HJB) integro-differential equation and that the optimal strategy is of feedback form. In particular, we show that the process under the feedback control exists. We then develop an algorithm to solve a discrete version of the problem. This algorithm enables us to conduct a detailed numerical study, examining the influence of input variables (such as the costs of reinsurance), different claim distributions and reinsurance strategies. In Chapter 3, based on [Brinker and Schmidli, 2022], we analyse the problem in a diffusion approximation. Again, we start by considering simple controls and derive a Hamilton–Jacobi–Bellman differential equation for the optimisation problem. However, the proof techniques used in this chapter differ widely from those in the first chapter. In particular, we prove that solutions to the equation attain an explicit representation in terms of Lambert's W function and show that this also applies to the value function. We derive optimal feedback strategies and give numerical examples. The explicit representations allow many additional conclusions regarding the value function and optimisers. From the analytic and numeric results of these two chapters, we conclude that 'pure' drawdown optimisation can lead to very strict reinsurance policies: the minimisation of drawdowns overrules all other economically relevant aspects. In Chapter 4, based on [Brinker and Schmidli, 2021a], we therefore consider an extension to the model. We propose a new target functional which measures the potential growth of the surplus while penalising the time in critical drawdown. This model has the alternative interpretation of maximising dividends (with a barrier payout strategy) while minimising the time the surplus spends critically 'far away' from the favourable position of being able to pay out dividends. We extend the proof techniques of Chapter 3 to calculate explicitly optimal feedback strategies for the diffusion approximation and analyse the influence of the model parameters. In numerical examples, we focus on the impact of the newly included incentive to grow. Chapter 4 foreshadows that our study allows for various extensions and generates opportunities for further research. In Chapter 5, we give an outlook on examples thereof. Appendix A contains some technical results and details.
This work is written in TeXstudio (Version 2.12.16) with LaTeX (MiKTeX 2.9.7400). All graphs, figures and simulations were created by the author of this work using Maple (2020.2), Inkscape (0.92 – 'Draw Freely'), Ipe (7.2.20) and RStudio (1.2.5033 – 'Orange Blossom') with R (3.6.2 – 'Dark and Stormy Night').

## 1.3 'Minimal Expected Time in Drawdown' as a New Objective Function

We start with the following conventions. By the term *surplus*, we mean the present value of the accumulated income of the insurance company minus accumulated expenses, referenced to time zero. We assume that the surplus of the insurer (without reinsurance) is either modelled by a classical risk model or an arithmetic Brownian motion. In the case of a *classical risk model* $X = (X_t)_{t\geq 0}$, we have

$$X_t = \nu_0 + pt - \sum_{k=1}^{N_t} Y_k \,, \qquad t \geq 0 \,, \tag{1.4}$$

where $p$ is the premium rate, $N = (N_t)_{t\geq 0}$ denotes a homogeneous Poisson process with intensity $\lambda > 0$ and $(Y_k)_{k\in\mathbb{N}}$ is a sequence of positive and independent identically distributed random variables which is independent of $N$. By $G$ we denote the distribution function of $Y_1$ and refer to this function as the *claim (size) distribution*. Correspondingly, the arrival times $(T_k)_{k\in\mathbb{N}}$ of $N$ are the *claim times*. We assume that $\mu = \mathbb{E}(Y_1)$ exists and that it holds $p = (1+\eta)\lambda\mu$, where $\eta > 0$ is a safety loading. This corresponds to an expected value principle for the premium calculation. If $Y_1^2$ is integrable with $\mu_2 = \mathbb{E}(Y_1^2)$, the *diffusion approximation* to this model (with coinciding first two moments) is the arithmetic Brownian motion

$$X_t = \nu_0 + \eta\lambda\mu\, t + \sqrt{\lambda\mu_2}\, W_t\,, \qquad t \geq 0 \tag{1.5}$$

(compare, for example, [Schmidli, 2008, Ch. 2]). Here, $W = (W_t)_{t\geq 0}$ denotes a standard Brownian motion. As we work with these processes separately, we assume in the diffusion case that $\lambda = \mu^{-1}$ is fulfilled (which corresponds to a change of the time unit) and write $\sigma = \sqrt{\mu_2\mu^{-1}} > 0$ for the volatility parameter to shorten notations.

In either case, we work on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is large enough to carry the respective surplus process. To simplify notation, we omit the specifications 'almost surely' or 'with probability one' if there is no risk of ambiguity. We assume that the probability space is equipped with the usual augmentation $(\mathcal{F}_t)_{t\geq 0}$ of the filtration generated by $X$. That means, for $t \geq 0$, $\mathcal{F}_t$ is the $\sigma$-algebra generated by $\mathcal{G}_t = \sigma(X_s : s \leq t)$ and the $\mathbb{P}$-negligible sets of $\mathcal{F}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous. We note that the paths of $X$ are càdlàg in both cases, such that the surplus process is progressively measurable. Additionally, we note that both of our basic surplus models have the strong Markov property with respect to this filtration. For an extensive introduction to stochastic processes and the probabilistic concepts mentioned up to this point we refer to [Feller, 1971] and [Revuz and Yor, 1991]. In the following, we use techniques from martingale theory and stochastic analysis. If not stated otherwise and explicitly referenced, the corresponding fundamental definitions and results are taken from [Protter, 2005].

### 1.3.1 Surplus Models with Proportional Reinsurance

Now we introduce reinsurance to our models. We say that a stochastic process $B = (B_t)_{t\geq 0}$ with values in $[b_0, 1] \subseteq [0, 1]$ is an *(admissible) reinsurance* or *retention level strategy*, if it is adapted to the underlying filtration and càdlàg (for the classical risk model) or progressively measurable (for the diffusion model). $b_0 \in [0, 1)$ denotes a lower bound for the retention level to be defined in each chapter, separately. We write $\mathcal{B}$ for the *set of all admissible retention level strategies*. We assume that the reinsurer uses the same premium calculation principle as the first insurer. That is, there is a parameter $\theta > 0$ representing the safety loading, such that the premium rate of the reinsurer is given by $(1-b)(1+\theta)\lambda\mu$. In order for the problem not to be trivial, we assume $\theta > \eta$ which means that reinsurance is more expensive than first insurance. This assumption ensures that there is a trade-off between a fast recovery from a large drawdown by keeping insurance premia and controlling future claim payments. If reinsurance was cheaper than first insurance, the insurer could sell the full risk to the reinsurance company as a quick, optimal and trivial solution. The *controlled surplus*

$X^B = (X_t^B)_{t\geq 0}$ under an admissible strategy $B$ takes the form

$$X_t^B = \nu_0 + \int_0^t \big[(1+\eta) - (1-B_s)(1+\theta)\big]\lambda\mu \,\mathrm{d}s - \sum_{k=1}^{N_t} B_{T_k-} Y_k\,, \qquad t \geq 0\,, \tag{1.6}$$

for the classical risk model and

$$X_t^B = \nu_0 + \int_0^t [\eta - (1-B_s)\theta]\,\mathrm{d}s + \int_0^t B_s\sigma\,\mathrm{d}W_s\,, \qquad t\geq 0\,, \tag{1.7}$$

for the diffusion model (see, for example, [Grandell, 1991] or [Schmidli, 2017]). For the constant strategy $B$ with $B_t = 1$ for all $t \geq 0$, these definitions coincide with (1.4) and (1.5) (with our notational conventions). We note that, by our definition of the set $\mathcal{B}$, all integrals in this representation are well-defined as either a (pathwise) Stieltjes- or a stochastic Itô integral. In each case, we define the *(controlled) running maximum* $M^B = (M_t^B)_{t\geq 0}$ and the *(controlled) drawdown* $\Delta^B = (\Delta_t^B)_{t\geq 0}$ by

$$M_t^B = \max\Big\{m_0,\, \sup_{s\in[0,t]} X_s^B\Big\}\,, \qquad \Delta_t^B = M_t^B - X_t^B\,, \qquad t\geq 0\,.$$

For a strategy $B$, we define the *expected time in (critical) drawdown* by

$$v^B(x) = \mathbb{E}^x\Big[\int_0^\infty \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta_t^B > d\}}\,\mathrm{d}t\Big]\,, \qquad x\geq 0\,. \tag{1.8}$$

Here and in the following we use the short notation $\mathbb{E}^x[\,\cdot\,]$ for the conditional expected value $\mathbb{E}[\,\cdot \mid Y_0 = x]$, where $Y$ is the stochastic process in question – in this case, $\Delta^B$. We refer to the function defined in Equation (1.8) as the *return function* of the strategy $B$. In Chapters 2 and 3, the *value function* of our optimisation problem is the *minimal expected time in (critical) drawdown*

$$v(x) = \inf_{B\in\mathcal{B}} v^B(x) = \inf_{B\in\mathcal{B}}\ \mathbb{E}^x\Big[\int_0^\infty \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta_t^B > d\}}\,\mathrm{d}t\Big]\,, \qquad x\geq 0\,. \tag{1.9}$$

We call an admissible strategy $B^*$ for which the infimum is attained, i.e. $v = v^{B^*}$, an *optimal strategy.* In Chapter 4, we define a different (but related) value function. All other conventions also apply to Chapter 4.

### 1.3.2 Reflection and the Skorohod Problem

The paths of the surplus process, its running maximum and its drawdown are connected by the Skorohod problem. The following definition is taken from Chaleyat-Maurel et al. [1980] and was originally formulated for general càdlàg functions. We adapt this definition for our models, which are *spectrally negative* and càdlàg, that is, all jumps of the surplus processes are directed downwards (for arbitrary strategies $B$).

Definition 1.1. [Chaleyat-Maurel et al., 1980]

*Let $y = (y_t)_{t\geq 0}$ denote a càdlàg path such that $-y$ is spectrally negative. A pair $(c, \varepsilon)$ of càdlàg paths $c = (c_t)_{t\geq 0}$ and $\varepsilon = (\varepsilon_t)_{t\geq 0}$ is a solution to the Skorohod problem for $y$ if $\varepsilon$ is non-negative with*

$\varepsilon_t = y_t + c_t$ *for all* $t \geq 0$ *and* $c$ *is continuous and non-decreasing with* $c_0 = 0$ *and*

$$\int_0^\infty \varepsilon_s \, \mathrm{d}c_s = 0 \,. \tag{1.10}$$

Intuitively, solving the Skorohod problem means finding an increasing compensator $c$ for a predefined path $y$ such that the compensated path $y+c$ stays non-negative. This compensator should be increasing or constant at any time. Moreover it should be minimal in the following sense: the condition in (1.10) implies that $c$ can only increase when $\varepsilon$ is equal to zero. This ensures that the compensated path has the same characteristics as the original path whenever it is not on the boundary. Since $c$ 'pushes' the path back into the positive half plane whenever it would normally cross the boundary, the resulting path $\varepsilon$ is also referred to as *reflected*. The condition that $-y$ is spectrally negative implies that $y$ is spectrally positive, i.e. only has upward jumps. This ensures that a (minimal) compensator $c$ is continuous. Existence and uniqueness results for the Skorohod problem and multiple variations thereof are well known. Theorem 1.2 below is a special case of Theorem 5 by Chaleyat-Maurel et al. [1980], which follows from part (b) of their proof.

Theorem 1.2. [Chaleyat-Maurel et al., 1980]
*If* $y_0 \geq 0$*, there exists a unique solution* $(c, \varepsilon)$ *to the Skorohod problem for* $y$*.* $c$ *is given by* $c_t = \max\{0, \sup_{s\in[0,t]}(-y_s)\}$ *and, accordingly,* $\varepsilon$ *is defined by* $\varepsilon_t = c_t + y_t$ *for all* $t \geq 0$*.* □

Now we associate the Skorohod problem with the running maximum and the drawdown process. We denote by $X = (X_t)_{t\geq 0}$ a spectrally negative stochastic process with càdlàg paths, starting at $\nu_0 \leq m_0$, and by $M = (X_t)_{t\geq 0}$ and $\Delta = (\Delta_t)_{t\geq 0}$ the processes defined as in (1.1) and (1.2). We denote by $y$ a specific path $t \mapsto y_t = m_0 - X_t(\omega)$ for an $\omega \in \Omega$. By $m_0 - \nu_0 \geq 0$, we conclude $y_0 \geq 0$. Then, $c_t = \max\{0, -m_0 + \sup_{s\in[0,t]} X_t(\omega)\}$ corresponds to $M_t(\omega) - m_0$ for all $t \geq 0$. This makes $\varepsilon$ the drawdown of the path $t \mapsto X_t(\omega) - m_0$. As the drawdown, per definition, is invariant to simultaneous shifts of $M$ and $X$ (for example, by $m_0$) along the vertical axis, $\varepsilon$ is equal to the drawdown $\Delta(\omega)$ of $X(\omega)$. From Theorem 1.2, we therefore derive the following corollary, which characterises the running maximum as a compensator and the drawdown as a reflected process.

Corollary 1.3. *Let* $X = (X_t)_{t\geq 0}$ *denote a spectrally negative stochastic process with càdlàg paths, starting at* $\nu_0 \leq m_0$*, and let* $M = (X_t)_{t\geq 0}$ *and* $\Delta = (\Delta_t)_{t\geq 0}$ *be defined as in* (1.1) *and* (1.2)*. For every* $\omega \in \Omega$*,* $(\tilde{c}, \varepsilon) = (M(\omega), \Delta(\omega))$ *is the unique pair of càdlàg paths such that* $\varepsilon$ *is non-negative with* $\varepsilon_t = \tilde{c}_t - X_t(\omega)$ *for all* $t \geq 0$ *and* $\tilde{c}$ *is continuous and non-decreasing with* $\tilde{c}_0 = m_0$ *and* (1.10) *with* $\tilde{c}$ *in place of* $c$*. Moreover, we have* $\varepsilon_0 = m_0 - \nu_0$*.* □

In particular, this applies to the case in which $X$ is one of our surplus models under an admissible control, given by (1.6) or (1.7). However, this more general formulation is necessary because we are going to use the corollary to prove that certain strategies are indeed admissible. By (1.10), the running maximum process can only increase at times at which the drawdown is equal to zero. This, we will use in our derivation of boundary conditions for the Hamilton–Jacobi–Bellman equations.

### 1.3.3 Divide et Impera: A Dynamic Programming Equation

The target of this section and final step of our introduction is a dynamic programming theorem which is applicable to both surplus processes. The derivation thereof is also the first step of the analysis in [Brinker and Schmidli, 2021b and 2022]. As a convention, we cite results taken from [Brinker and Schmidli, 2021a, 2021b and 2022] by providing, solely, the year key of the publication on the top right (as in Lemma 1.4, below). Results by other authors are cited with the full bibliography key (as seen in Theorem 1.2, above). We note that, in general, the ideas of the corresponding proofs in this monograph coincide with those in the publications. However, in some cases, the proofs (or substeps of proofs) were omitted in the original articles and are added here.
We start with a few preliminary observations regarding the functions defined in Equations (1.8) and (1.9).

Lemma 1.4. [2021b,2022]
*For arbitrary $B \in \mathcal{B}$, the functions $v^B$ and $v$ are bounded with values in $[0, \delta^{-1}]$, increasing and fulfil $\lim_{x\to\infty} v(x) = \lim_{x\to\infty} v^B(x) = \delta^{-1}$.*

*Proof.* For an arbitrary $B$, the indicator function attains values between zero and one. Hence, we have

$$0 \le \int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta_t^B > d\}} \, \mathrm{d}t \le \int_0^\infty \mathrm{e}^{-\delta t} \, \mathrm{d}t = \frac{1}{\delta}$$

and taking expectations yields the lower and upper bounds for $v^B$. Now consider $x > y$. For a surplus process $X^B$ starting at $\nu_0 = -y$, the drawdown process $\Delta^B$ with $\Delta_t^B = \max\{0, \sup_{s\in[0,t]} X_s^B\} - X_t^B$, $t \ge 0$, starts at $y$. For $x > y$, the drawdown $\tilde{\Delta}^B$ with $\tilde{\Delta}_t^B = \max\{0, \sup_{s\in[0,t]} \tilde{X}_s^B\} - \tilde{X}_t^B$, $t \ge 0$, of $\tilde{X}^B$ defined by $\tilde{X}_t^B = X_t^B - x + y$, $t \ge 0$, starts at $x$. We write

$$\begin{aligned}
\vartheta_0(B) &= \inf\{t \ge 0 : \Delta_t^B = 0\} = \inf\{t \ge 0 : X_t^B \ge 0\}\,, \\
\tilde{\vartheta}_0(B) &= \inf\{t \ge 0 : \tilde{\Delta}_t^B = 0\} = \inf\{t \ge 0 : X_t^B \ge x - y\}
\end{aligned}$$

for the respective first arrival at zero and observe $\tilde{\vartheta}_0(B) \ge \vartheta_0(B)$. For $t < \tilde{\vartheta}_0(B)$, we have by $\sup_{s\in[0,t]} X_s^B \le x - y$ and $x - y > 0$:

$$\tilde{\Delta}_t^B = -X_t^B + x - y \ge \left\{ \begin{matrix} \sup\limits_{s\in[0,t]} X_s^B - X_t^B\,, & t \ge \vartheta_0(B)\,, \\ -X_t^B\,, & t < \vartheta_0(B)\,, \end{matrix} \right\} = \Delta_t^B\,.$$

For $t \ge \tilde{\vartheta}_0(B)$, we have:

$$\tilde{\Delta}_t^B = \sup_{s\in[0,t]} (X_s^B - x + y) - (X_t^B - x + y) = \Delta_t^B\,.$$

This is illustrated in Figure 1.4. Thus, $\mathbb{1}_{\{\Delta_t^B > d\}} \le \mathbb{1}_{\{\tilde{\Delta}_t^B > d\}}$ for all $t \ge 0$ and we get

$$v^B(y) = \mathbb{E}^y\Big[\int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta_t^B > d\}} \, \mathrm{d}t\Big] \le \mathbb{E}^x\Big[\int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{\tilde{\Delta}_t^B > d\}} \, \mathrm{d}t\Big] = v^B(x)\,.$$

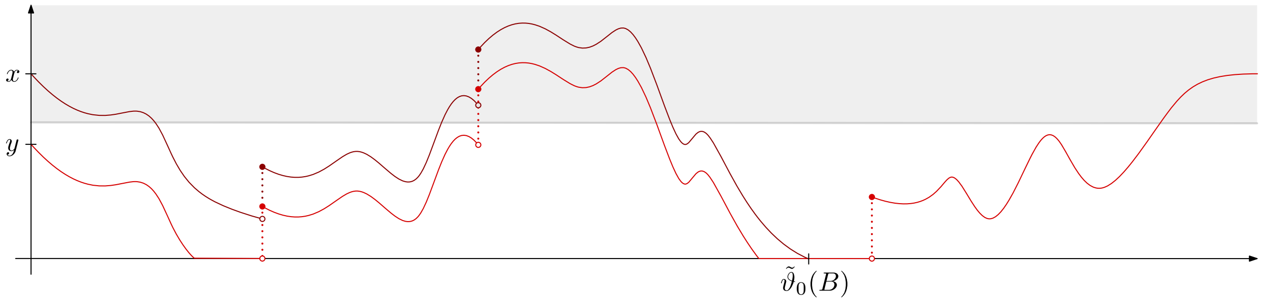



FIGURE 1.4 Sketch of paths of $\tilde{\Delta}^B$ (dark red) and $\Delta^B$ (light red).

In particular, $v^B$ is increasing. For a drawdown process starting at $x > d$, the indicator function is equal to one (at least) up to time $\vartheta_d(B) = \inf\{t \geq 0 : \Delta_t^B \leq d\}$. This means, we obtain the lower bound

$$v^B(x) \geq \mathbb{E}^x\Big[\int_0^{\vartheta_d(B)} \mathrm{e}^{-\delta t}\,\mathrm{d}t\Big] = \delta^{-1} - \delta^{-1}\,\mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta_d(B)}\big] \geq \delta^{-1} - \delta^{-1}\sup_{B\in\mathcal{B}}\mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta_d(B)}\big]\,. \tag{1.11}$$

for all $x > d$. As we will see in Chapters 2 and 3, the expected value on the far right is an exponential function that converges to zero as $x \to \infty$. This means, the right hand side converges to $\delta^{-1}$ as $x \to \infty$. By $v^B(x) \in [0, \delta^{-1}]$, we obtain $\lim_{x\to\infty} v^B(x) = \delta^{-1}$. Because $B$ was arbitrary in all scenarios, we immediately obtain that $v$ is bounded and increasing as well. We observe that the lower bound on the right hand side of (1.11) is independent of $B$. Thus, it is a lower bound for $v$, too, and the convergence statement can be deduced in the same way. □

There are two possibilities for the initial scenario: the unfavourable state of starting with a critical drawdown, $x > d$, and the state of starting with a small drawdown, $x \leq d$. Intuitively, in order to minimise the time spent in the dangerous area, the optimal strategy must force the drawdown process to re-enter the uncritical area as fast as possible. On the other hand, the process should spend as much time as possible between 0 and $d$. For $y \geq 0$, we define the times

$$\vartheta_y(B) = \inf\{t \geq 0 : \Delta_t^B \leq y\}\,, \qquad \vartheta^y(B) = \inf\{t \geq 0 : \Delta_t^B > y\}\,.$$

As hitting times of Borel sets, these are stopping times, compare for example Theorem I.4.15 of [Revuz and Yor, 1991] or, for a less technical presentation, Section 1.1.7 of [Karatzas and Shreve, 1998]. Thus, the time

$$\vartheta(B) = \max\{\vartheta_d(B), \vartheta^d(B)\} \tag{1.12}$$

of the first 'switch' into the other area under strategy $B$ is also a stopping time.

**Theorem 1.5 (Dynamic Programming).** [2021b,2022]
*The function $v$ fulfils*

$$v(x) = \begin{cases} \inf_{B \in \mathcal{B}} \mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta(B)} v(\Delta^B_{\vartheta(B)})\big]\,, & x \le d\,, \\ \delta^{-1} - (\delta^{-1} - v(d)) \cdot \sup_{B\in\mathcal{B}} \mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta(B)}\big]\,, & x > d\,. \end{cases} \tag{1.13}$$

In the proof, we use so-called '$\varepsilon$-optimal' strategies. From the definition of $v$ it follows that for every $x \ge 0$ and $\varepsilon > 0$, there exists a strategy $B \in \mathcal{B}$ (depending on $x$ and $\varepsilon$) with $v^B(x) < v(x) + \varepsilon$. We refer to this strategy as *$\varepsilon$-optimal* for $x$. We follow the approach of [Schmidli, 2008, p. 31] to prove Theorem 1.5.

*Proof of Theorem* 1.5. Let $B \in \mathcal{B}$ denote an arbitrary strategy and let $B'$ be the strategy $B$ shifted by $\vartheta(B)$ on the set $\{\vartheta(B) < \infty\}$ (conditional on $\mathcal{F}_{\vartheta(B)}$). We have

$$\begin{aligned} v^B(x) &= \mathbb{E}^x\Big[\mathbb{1}_{\{\vartheta(B)<\infty\}}\Big(\int_0^{\vartheta(B)} \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta^B_t > d\}}\,\mathrm{d}t + \mathrm{e}^{-\delta\vartheta(B)}\,\mathbb{E}^x\Big[\int_0^\infty \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta^{B'}_{\vartheta(B)+t} > d\}}\,\mathrm{d}t\,\Big|\,\mathcal{F}_{\vartheta(B)}\Big]\Big)\Big] \\ &\quad + \mathbb{E}^x\Big[\mathbb{1}_{\{\vartheta(B)=\infty\}}\int_0^\infty \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta^B_t > d\}}\,\mathrm{d}t\Big] \\ &= \mathbb{E}^x\Big[\int_0^{\vartheta(B)} \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta^B_t > d\}}\,\mathrm{d}t + \mathrm{e}^{-\delta\vartheta(B)} v^{B'}(\Delta^B_{\vartheta(B)})\Big] \\ &\ge \mathbb{E}^x\Big[\int_0^{\vartheta(B)} \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta^B_t > d\}}\,\mathrm{d}t + \mathrm{e}^{-\delta\vartheta(B)} v(\Delta^B_{\vartheta(B)})\Big]\,, \end{aligned} \tag{1.14}$$

where we could recombine the expected values on the sets $\{\vartheta(B) < \infty\}$ and $\{\vartheta(B) = \infty\}$ in the second equation because $v^{\hat{B}}$ is bounded for every strategy $\hat{B}$. Thus, taking the infimum on both sides of the inequality, we get

$$v(x) \ge \inf_{B\in\mathcal{B}} \mathbb{E}^x\Big[\int_0^{\vartheta(B)} \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta^B_t > d\}}\,\mathrm{d}t + \mathrm{e}^{-\delta\vartheta(B)} v(\Delta^B_{\vartheta(B)})\Big]\,. \tag{1.15}$$

To prove a converse inequality, we consider again $B \in \mathcal{B}$. For the drawdown of the diffusion process, $\Delta^B_{\vartheta(B)} = d$ on the set $\{\vartheta(B) < \infty\}$ follows from the continuity of paths. Since all jumps of the drawdown of the classical risk model are directed upwards, we obtain the same result for starting

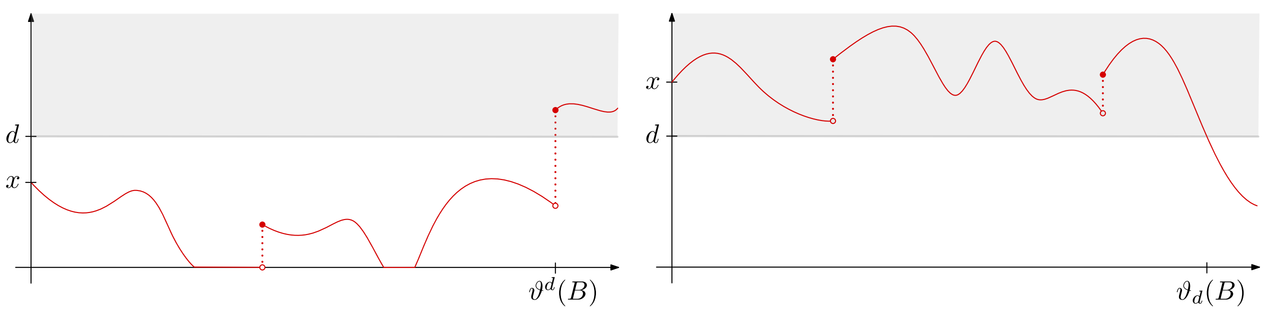



Figure 1.5 We distinguish the cases of starting below and starting above $d$.

points $x > d$ in this case. For an $\varepsilon$-optimal strategy $B^\varepsilon$ with $v^{B^\varepsilon}(d) < v(d) + \varepsilon$, we can consider the compound strategy $\tilde{B} \in \mathcal{B}$ which corresponds to $B$ for $t < \vartheta(B)$ and to $B^\varepsilon$ for $t \geq \vartheta(B)$. By $\Delta^B_{\vartheta(B)} = d$ in the aforementioned cases, we now have

$$v^{\tilde{B}'}(\Delta^B_{\vartheta(B)}) < v(\Delta^B_{\vartheta(B)}) + \varepsilon \tag{1.16}$$

on the set $\{\vartheta(B) < \infty\}$. Here, $\tilde{B}'$ again denotes the shifted strategy. As a slight extension, we (for now) assume that there is a 'universally' $\varepsilon$-optimal strategy $\tilde{B}'$, such that (1.16) is also fulfilled for the drawdown of a classical risk model starting at $x \leq d$. This case is special because, due to the upward jumps of the drawdown, we do not necessarily have $\Delta^B_{\vartheta(B)} = d$. As we prove in Chapter 2, the existence of such a strategy fulfilling (1.16) follows from the continuity properties of $v$. Similarly as above, we get:

$$\begin{aligned} v(x) \leq v^{\tilde{B}}(x) &= \mathbb{E}^x\Big[\int_0^{\vartheta(B)} \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta^B_t > d\}} \,\mathrm{d}t + \mathrm{e}^{-\delta\vartheta(B)} v^{\tilde{B}'}(\Delta^B_{\vartheta(B)})\Big] \\ &\leq \mathbb{E}^x\Big[\int_0^{\vartheta(B)} \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta^B_t > d\}} \,\mathrm{d}t + \mathrm{e}^{-\delta\vartheta(B)} v(\Delta^B_{\vartheta(B)})\Big] + \varepsilon\ . \end{aligned}$$

Now, letting $\varepsilon \to 0$, we get

$$v(x) \leq \mathbb{E}^x\Big[\int_0^{\vartheta(B)} \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta^B_t > d\}} \,\mathrm{d}t + \mathrm{e}^{-\delta\vartheta(B)} v(\Delta^B_{\vartheta(B)})\Big]\,.$$

Taking the infimum over all $B \in \mathcal{B}$ shows, in combination with (1.15), that it holds

$$v(x) = \inf_{B\in\mathcal{B}} \mathbb{E}^x\Big[\int_0^{\vartheta(B)} \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta^B_t > d\}} \,\mathrm{d}t + \mathrm{e}^{-\delta\vartheta(B)} v(\Delta^B_{\vartheta(B)})\Big]\,. \tag{1.17}$$

From this equation, we can conclude the assertion. We note that we have $\Delta^B_t \leq d$ for all $t < \vartheta(B)$ if $\Delta^B_0 = x \leq d$ and $\Delta^B_t > d$ for all $t < \vartheta(B)$ if $\Delta^B_0 = x > d$. This is illustrated in Figure 1.5. Thus, we get

$$\int_0^{\vartheta(B)} \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta^B_t > d\}} \,\mathrm{d}t = \begin{cases} 0\,, & x \leq d\,, \\ \delta^{-1}\big(1 - \mathrm{e}^{-\delta\vartheta(B)}\big)\,, & x > d\,, \end{cases}$$

which, combined with (1.17), yields (1.13). □

Equation (1.17) has the interpretation that an optimal strategy should minimise the time until $d$ is crossed and position the drawdown in such a way, that the remaining time with critical drawdown is as small as possible. Additionally, the reappearance of the function $v$ on the right hand side indicates that an optimal strategy should be composed of an optimal strategy until the first passage through $d$ and an optimal strategy for the new starting point.

Remark. We note that, up to and including Equation (1.17), the proof did not require the definition of $\vartheta(B)$. That means, for every stopping time for which an admissible strategy fulfilling (1.16) is available, Equation (1.17) can be derived. In particular, this is the case for all $\tau(B) = \max\{\vartheta_y(B), \vartheta^y(B)\}$, $y \geq 0$. This means, an optimal strategy should have an 'optimal substructure'. #

For the split at $d$, we additionally get the representation given in (1.13). In the case $x > d$, we see that (if $v(d)$ is known) the problem corresponds to maximising the Laplace transform of $\vartheta(B)$. $y \mapsto \mathrm{e}^{-\delta y}$ is a decreasing function. Thus, this essentially means that we minimise the weighted time until the drawdown is uncritical. This could be interpreted as a quick recovery. Accordingly, we refer to this problem as the *minimisation of the recovery time*. For $x \leq d$, one could similarly interpret the representation as maximising the time until a critical drawdown occurs with a penalty for the severity. We call this the *maximisation of the time to critical drawdown (with a penalty)*. The dynamic programming principle implies that these subsolutions can be stringed together. Heuristically this means, we start by forcing the process to enter the uncritical area as fast as possible, then we maximise the time in this area, after the exit we again minimise the time until re-entering and so on. With this intuition in mind, we now move on to model-specific results in the following chapters.

CHAPTER 2

# Minimal Expected Time in Drawdown for the Classical Risk Model

This chapter is based on [Brinker and Schmidli, 2021b] but we also present some results going beyond the scope of the paper. We consider the setting in which the surplus process without reinsurance is modelled by a Cramér–Lundberg process. That is, if we define

$$c(b) = [(1+\eta) - (1+\theta)(1-b)] \cdot \lambda\mu \tag{2.1}$$

using the conditions and notation of Section 1.3, the controlled surplus process $X^B$ takes the form

$$X_t^B = \nu_0 + \int_0^t c(B_s)\,\mathrm{d}s - \sum_{k=1}^{N_t} B_{T_k-} Y_k\,, \qquad t \geq 0\,.$$

$c$ is a linear function of the retention level and it holds $c(1) > \lambda\mu$. That means, if the insurer chooses not to buy reinsurance, the so-called *net profit condition* is fulfilled: expected income exceeds expected claim payments. $c(b)$ is increasing in $b$. In theory, we could allow all retention levels in $[0,1]$. However for $b_0 = \frac{\theta-\eta}{1+\theta} > 0$, retention levels $b \in [0, b_0)$ yield a strictly negative income rate $c(b) < 0$. We exclude these values from our consideration to ensure that the expenses for reinsurance do not exceed the insurer's premium income for the optimal strategy. However it should be noted that, with retention level strategies with values in $[b_0, 1]$, the net profit condition is not obligatory in our setting.

The target of this chapter is the minimisation of the expected time in drawdown, i.e. the problem posed in Equation (1.9), for the classical risk model. We proceed as follows. In Section 2.1, we consider a set of predefined strategies which we refer to as 'simple switching' strategies. For this type of strategy, we obtain explicit representations of the expected time in drawdown in terms of scale functions. In particular, we derive an integro-differential equation and prove a verification theorem connecting solutions to the equation with return functions. We consider the example of phase-type distributed claims, for which the expected time in drawdown is an exponential polynomial. In Section 2.2, we analyse in detail the optimisation problem. We start by proving a verification theorem for the original problem. Then, we consider separately the subproblems of large and small initial drawdown (in that order). In particular, for the case of critical initial drawdown, $x > d$, we show that the function $V$ defined by

$$V(x) = \sup_{B\in\mathcal{B}} \mathbb{E}^x\left[\mathrm{e}^{-\delta\vartheta(B)}\right]\,, \qquad x > d\,,$$

is an exponential function. We use this representation to construct a set of Gerber–Shiu type optimisation problems

$$v_C(x) = \inf_{B\in\mathcal{B}} \mathbb{E}^x\Big[\mathrm{e}^{-\delta\vartheta(B)} \cdot \Big(\delta^{-1} - (\delta^{-1} - C)\cdot V\big(\Delta^B_{\vartheta(B)}\big)\Big)\Big]\,, \qquad x \le d\,, \qquad C \in [0, \delta^{-1}]\,,$$

which contains our original problem for $x \le d$. We work with a Hamilton–Jacobi–Bellman equation to characterise the solutions and reconnect the strategies to find a solution to the original problem. In Section 2.3, we define a discrete version of the optimisation problem which can be solved numerically. We present numerical examples for the cases of small (exponential) claims, large (Pareto distributed) claims and deterministic claims. In Section 2.4, we discuss our findings.

For technical and notational simplicity, we assume from now on that the distribution function $G$ of $Y_1$ is continuous. The only exception of this assumption is the discrete version of the problem considered in Section 2.3. Further, we write $\ell_f(r) = \int_0^\infty f(t)\mathrm{e}^{-rt}\,\mathrm{d}t$, $r \in \mathbb{C}$, for the Laplace transform of a function $f : [0,\infty) \to \mathbb{R}$, provided that this integral exists. For the Laplace transform of $Y_1$ we write $\ell_Y(r) = \mathbb{E}(\mathrm{e}^{-rY_1})$.

## 2.1 Simple Switching Feedback Strategies

In this section, we assume that the insurer defines a fixed retention level for each of the areas, the critical and the uncritical, and switches the strategy whenever the drawdown exits the current one. That is, we define a *simple switching strategy* $B$ by

$$B_t = b(\Delta^B_t)\,, \quad t \ge 0\,, \qquad\qquad b(x) = \begin{cases} \check{b}\,, & x \le d\,,\\ \hat{b}\,, & x > d\,, \end{cases} \tag{2.2}$$

where $\check{b} \in [b_0, 1]$ is the constant retention level chosen if the current drawdown is uncritical and $\hat{b} \in [b_0, 1]$ is the constant retention level if the current drawdown is large. Figure 2.1 shows an example. Throughout this section, we consider such a strategy $B$ for a fixed pair $(\check{b}, \hat{b}) \in [b_0, 1]^2$. Because we only work with a single strategy and in order to not overload the notation, we drop the notational reference to $B$ whenever it is not crucial. That is, we simply write $X$ for the surplus $X^B$, $\Delta$ for the drawdown $\Delta^B$, $\vartheta_d$ instead of $\vartheta_d(B)$ and so on. Additionally, we write $u$ for the return $v^B$ under this strategy.

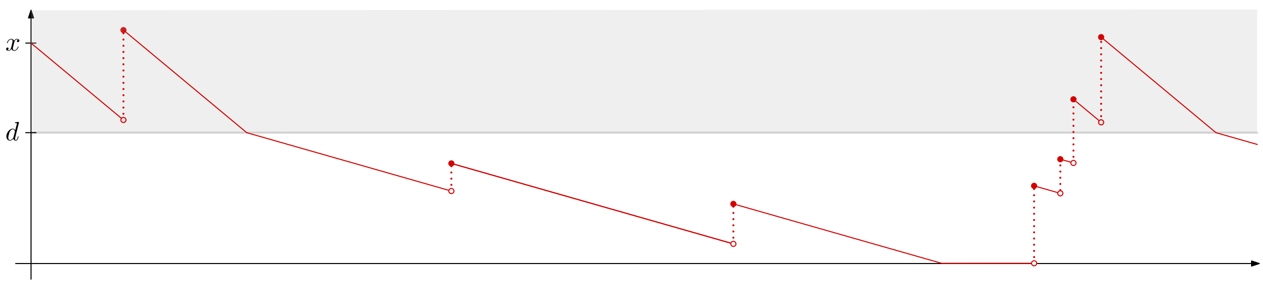



FIGURE 2.1 Path of a drawdown process under a simple switching control with $\check{b} < \hat{b}$.

A simple switching strategy is a primitive form of a feedback strategy, which depends in a measurable way on the current drawdown. The controlled drawdown process is a piecewise deterministic Markov process and, thus, the strategy $B$ is admissible. The class of piecewise deterministic Markov processes (PDMPs) was introduced by Davis [1984 and 1993]. For the sake of concision we will not give an in-depth introduction to the technical details here. Instead, we explain the construction and properties of PDMPs for the concrete case of the drawdown process under simple switching controls and, later on, extend these for optimal controls.

### 2.1.1 Controlled Drawdown as a PDMP

We assume that $c(\hat{b}), c(\check{b}) > 0$ holds (the cases $\check{b} = b_0$ and $\hat{b} = b_0$ can be treated analogously). In order to directly apply the results of the literature to our setting, we present the following construction in the notation of [Davis, 1984] which is also used by Rolski et al. [1999, Ch. 11], in a similar form. Between the jumps, the drawdown process has three different types of deterministic behaviour $\pi_c$, $\pi_u$ and $\pi_0$. By $\pi_c$ we denote the case in which the drawdown is critical, that is, larger than $d$. In this case, the drawdown evolves as $\Delta_t = \Delta_{T_{N_t}} - c(\hat{b})t$ until the next jump occurs or it arrives at $d$. This behaviour can be described by the *integral curve* $\phi_{\pi_c}(t,z) = z - c(\hat{b})t$ for $z \in M_{\pi_c} = (d,\infty)$. If there is no jump preempting, the drawdown arrives on the boundary $d$ at time $t^*(\pi_c, z) = (z-d)/c(\hat{b})$, where we write

$$t^*(\pi, z) = \sup\{t > 0 : \phi_\pi(t,z) \text{ exists and } \phi_\pi(t,z) \in M_\pi\}$$

for all $z \in M_{\pi_c}$ and $\pi = \pi_c$. Further, $\phi_{\pi_c}$ is the unique solution to the equation

$$\phi_\pi(t,z) = z + \int_0^t \mathcal{H}_\pi(\phi_\pi(s,z))\,\mathrm{d}s\,, \qquad t \le t^*(\pi,z)\,, \tag{2.3}$$

for $\pi = \pi_c$, where the constant mapping $\mathcal{H}_{\pi_c} = -c(\hat{b})$ is the associated *vector field.* Similarly, the state $\pi_u$ in which the drawdown takes values in $M_{\pi_u} = (0,d)$ (and is therefore uncritical), can be described by $\phi_{\pi_u}(t,z) = z - c(\check{b})t$. $M_{\pi_u}$ has two points on the boundary, 0 and $d$, but only 0 can be reached by an integral curve for starting points $z \in M_{\pi_u}$. In this case, we have $t^*(\pi_u, z) = z/c(\check{b})$. The analogue equation to (2.3) with $\mathcal{H}_{\pi_u} = -c(\check{b})$ is also fulfilled in this case. Lastly, we write $M_{\pi_0} = \{0\}$ and let $\phi_{\pi_0}(t,z) = z$. That means, the path stays in this point until a jump occurs. In particular, $t^*(\pi_0, 0) = \infty$ and (2.3) holds for $\mathcal{H}_{\pi_0} = 0$. Now we write $E = \{(\pi,z) : \pi \in \{\pi_0,\pi_u,\pi_c\}, z \in M_\pi\}$ and denote by $\mathcal{B}(E)$ the Borel $\sigma$-algebra of $E$. $(E, \mathcal{B}(E))$ is the *state space* of the pair $(J_t, \Delta_t)_{t\ge 0}$, where $J_t \in \{\pi_0, \pi_u, \pi_c\}$ is the type of deterministic behaviour at time $t \ge 0$. The set $\Gamma = \{(\pi_c, d), (\pi_u, 0)\}$ of points on the regimes' boundaries which can be reached by integral curves in finite time is called the *active boundary* of $E$. Intuitively, the active boundary contains the points at which the drawdown process transitions from one state to another without a jump. As the jumps are induced by a Poisson process, we have a constant *jump intensity* $\tilde{\lambda} : E \to (0,\infty)$ given by $\lambda$. Based on the jump size distribution, we can define an according *transition measure* $Q : (E \cup \Gamma) \times \mathcal{B}(E) \to [0,1]$. This means, $z \mapsto Q(z,A)$ is a measurable function of $z \in E \cup \Gamma$ for each fixed $A \in \mathcal{B}(E)$ and $A \mapsto Q(z,A)$ is a probability measure on $(E, \mathcal{B}(E))$ for each $z \in E$ (cf. [Davis, 1984]). Writing $z + A = \{z + a : a \in A\}$

for the translated set and $G_b(A) = \mathbb{P}(bY_1 \in A)$, we define

$$Q((\pi_c, z), (\pi_c, z + A)) = G_{\hat{b}}(A)\,, \qquad Q((\pi_u, z), (\pi_c, z + A)) = G_{\check{b}}(A \cap (d - z, \infty))\,,$$
$$Q((\pi_u, z), (\pi_u, z + A)) = G_{\check{b}}(A \cap (0, d - z))\,, \qquad Q((\pi_0, 0), (\pi_c, A)) = G_{\check{b}}(A \cap (d, \infty))\,,$$
$$Q((\pi_0, 0), (\pi_u, A)) = G_{\check{b}}(A \cap (0, d))$$

and, for the elements of $\Gamma$,

$$Q((\pi_c, d), \cdot) = \delta_{(\pi_u, d)}\,, \qquad Q((\pi_u, 0), \cdot) = \delta_{(\pi_0, 0)}\,,$$

where $\delta_r$ denotes the Dirac measure at $r$. With the state space, integral curves, jump intensity and transition measure, we have characterised the drawdown process as a piecewise deterministic Markov process. On the other hand, a piecewise deterministic Markov process can be constructed from these 'ingredients'. The idea is to define the process in between the jump times, independently and according to the integral curves. The 'feedback' structure (or, equivalently, the *semigroup property* $\phi_\pi(t+s, z) = \phi_\pi(t, \phi_\pi(s, z))$, $s, t \in [0, t^*(\pi, z))$ of the integral curves, generated by the equations of the form (2.3), ensures that the deterministic movements depend only on the current state. In particular, we note that even though the original definition by Davis [1984] requires all $M_\pi$ to be open sets, the definition $M_{\pi_0} = \{0\}$ does not cause any problems because the corresponding integral curve is constant. Having constructed $\Delta$ as the position component of a PDMP, we can rebuild the controlled surplus and its running maximum by defining $\mathrm{d}M_t = c(\check{b})\,\mathrm{d}t$ on the set $\{\Delta_t = 0\}$, by uniqueness of solutions to the Skorohod problem. Then, $X_t = -\Delta_t + M_t$ for all $t \geq 0$.
A special case of simple switching is a fully constant strategy with $\check{b} = \hat{b}$. In this case, the drawdown process corresponds to the M/G/1 Queue described by Davis [1984, Sec. 2.3]. As a generalisation, one could define other deterministic trajectories, associated to vector fields, such that Equation (2.3) is fulfilled for all $\pi$. Additionally, the original definition by Davis [1984] allows for countably many states. For the jump mechanism explained above, the result is again a piecewise deterministic Markov process. In fact, as we prove at the end of Section 2.2, the optimally controlled process is also of this form.
The following result is a modified version of Theorem 11.1.3 combined with Corollary 11.2.1 from [Rolski et al., 1999], tailored to our needs. For an absolutely continuous function $f : [0, \infty) \to \mathbb{R}$ with density $f'$ and $b \in [b_0, 1]$, we write

$$\mathcal{A}^b f(x) = -\delta f(x) - c(b) f'(x) + \lambda \int_0^\infty [f(x + by) - f(x)]\,\mathrm{d}G(y)\,. \tag{2.4}$$

In particular, because $f$ is absolutely continuous, it is Lebesgue almost everywhere differentiable. This means that writing $f'$ for the density is reasonable.

Lemma 2.1. [Rolski et al., 1999]
*Let $b : [0, \infty) \to [b_0, 1]$ denote a measurable function, such that $\Delta^B$ under the feedback control $B$ with $B_t = b(\Delta^B_t)$, $t \geq 0$, is a piecewise deterministic Markov process. Let $f : [0, \infty) \to \mathbb{R}$ denote a bounded,*

*absolutely continuous function with density $f'$. The process*

$$\Big(\mathrm{e}^{-\delta t} f(\Delta_t) - f(\Delta_0) - \int_0^t \mathrm{e}^{-\delta s} f'(0)\,\mathrm{d}M_s - \int_0^t \mathrm{e}^{-\delta s} \mathcal{A}^{b(\Delta_s)} f(\Delta_s)\,\mathrm{d}s\Big)_{t\geq 0} \tag{2.5}$$

*is a martingale.* □

Remark. To see that (2.5) is indeed the martingale mentioned in Corollary 11.2.1 of [Rolski et al., 1999], we observe that

$$\int_0^t \mathrm{e}^{-\delta s} f'(0)\,\mathrm{d}M_s + \int_0^t \mathrm{e}^{-\delta s} \mathcal{A}^{b(\Delta_s)} f(\Delta_s)\,\mathrm{d}s = \int_0^t \mathrm{e}^{-\delta s} \tilde{\mathcal{A}} f(\Delta_s)\,\mathrm{d}s$$

with the definition

$$\tilde{\mathcal{A}} f(x) = -\delta f(x) - c(b(x))\mathbb{1}_{\{x>0\}} f'(x) + \lambda \int_0^\infty \left[f(x+b(x)y) - f(x)\right]\,\mathrm{d}G(y)\,.$$

This is because the drift of the piecewise deterministic (drawdown) process vanishes whenever it is at zero, that is, if the maximum is increasing. Additionally, we have $\mathrm{d}M_t = c(b(0))\,\mathrm{d}t$. #

The generator of the process $(t,\Delta_t)_{t\geq 0}$ takes the form $\tilde{\mathcal{A}}$ when applied to the function $h(t,x) = \mathrm{e}^{-\delta t} f(x)$. Intuitively, the first part of $\tilde{\mathcal{A}}$ stems from discounting time, the term with the first derivative is induced by the deterministic movements and the last part is generated by the jumps. Lemma 2.1 states that $h$ belongs to the extended domain of the generator of $(t,\Delta_t)_{t\geq 0}$ – which is sufficient in our applications. However, it should be noted that the extended domain in fact contains a broader class of functions, specified in [Rolski et al., 1999, Ch. 11].
Lemma 2.1 plays an important role in our following characterisation of the expected time in drawdown under simple switching controls.

### 2.1.2 General Results

For simple switching strategies, we can split the function $u$ at $x = d$ and obtain a similar representation as in Theorem 1.5:

Lemma 2.2. [2021b]
*The function $u$ fulfils*

$$u(x) = \begin{cases} \mathbb{E}^x\left[\mathrm{e}^{-\delta\vartheta^d} u(\Delta_{\vartheta^d})\right], & x \leq d\,, \\ \delta^{-1} - \left(\delta^{-1} - u(d)\right) \cdot \mathbb{E}^x\left[\mathrm{e}^{-\delta\vartheta_d}\right], & x > d\,. \end{cases} \tag{2.6}$$

□

This is a direct consequence of Theorem 1.5 with the set $\mathcal{B}$ restricted to the single strategy currently considered. Therefore, we can treat the two cases separately to find the return function. In the case in which the drawdown already starts in the critical area, the drawdown is bounded away from zero. That means, $\vartheta_d$ is the passage time of the process $Y = (Y_t)_{t\geq 0}$ with $Y_t = x - c(\hat{b})t + \sum_{k=1}^{N_t} \hat{b} Y_k$, $t \geq 0$. There is an explicit expression for this passage time.

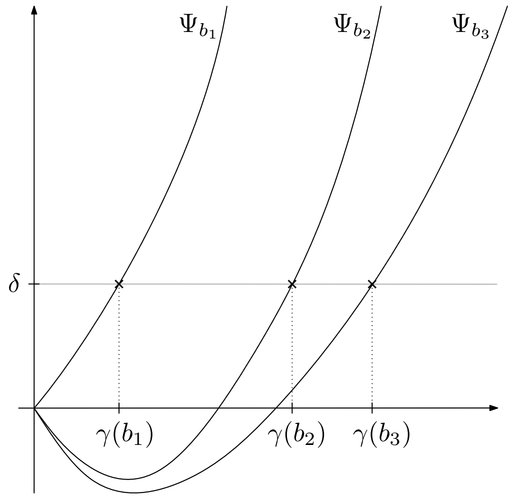



FIGURE 2.2 $r \mapsto \Psi_b(r)$ for three different values $b_i \in (b_0, 1]$, $i = 1, 2, 3$, with $b_1 > b_2 > b_3$.

LEMMA 2.3. [2021b]

*For $b \geq b_0$, we define the function $\Psi_b$ by $\Psi_b(r) = c(b)r - \lambda(1 - \ell_Y(br))$ for $r \geq 0$ and all $r < 0$ for which the right hand side exists.*

i) *For $b > b_0$, there exists a unique non-negative solution $r = \gamma(b) > 0$ to $\Psi_b(r) = \delta$.*

ii) *For all $r > 0$, the function $b \mapsto \Psi_b(r)$ defined on $[b_0, 1]$ is increasing. The function $b \mapsto \gamma(b)$ defined on $(b_0, 1]$ is decreasing.*

iii) *We have $\mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta_d}\big] = \mathrm{e}^{-\gamma(\hat{b})(x-d)}$ for $x > d$ and $\hat{b} > b_0$. For $\hat{b} = b_0$, it holds $\mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta_d}\big] = 0$ for $x > d$.*

*Proof.* i) and ii) follow from properties of the Laplace transform and are illustrated in Figure 2.2. We provide a detailed proof in the appendix, p. 111. Regarding iii), we consider the case $\hat{b} > b_0$. A direct calculation shows that the process $(\exp(-\gamma(\hat{b})Y_t - \delta t))_{t \geq 0}$ is a martingale of expectation $\mathrm{e}^{-\gamma(\hat{b})x}$. Applying the optional stopping theorem, we arrive at

$$\mathbb{E}^x\big[\mathrm{e}^{-\gamma(\hat{b})Y_{t\wedge\vartheta_d} - \delta(t\wedge\vartheta_d)}\big] = \mathrm{e}^{-\gamma(\hat{b})x}\,.$$

We have $Y_{\vartheta_d} = d$ on the set $\{\vartheta_d < \infty\}$ and $Y_t > d$ for all $t$ on its complement. Thus, letting $t \to \infty$, we obtain the assertion by dominated convergence. The case $\hat{b} = b_0$ follows directly from the fact that $c(b_0) = 0$, so that $Y$ is constant with upward jumps. This process can never reach the level $d$ which lies below its starting point. □

COROLLARY 2.4. [2021b]

*For $\hat{b} > b_0$, the function $u$ takes the form*

$$u(x) = \frac{1}{\delta} - \Big(\frac{1}{\delta} - u(d)\Big) \cdot \mathrm{e}^{-\gamma(\hat{b})(x-d)}\,, \qquad x > d\,. \tag{2.7}$$

*For $\hat{b} = b_0$, we have $u(x) = \delta^{-1}$ for $x > d$.* □

The case of uncritical initial drawdown is more complicated: $\vartheta^d$ corresponds to a jump time. Additionally, the drawdown process changes its behaviour at the arrival at zero. Therefore, if we were to use the same approach as above, this would lead to two-sided exit times. We tackle these obstacles one-by-one, starting with the case $\check{b} = b_0$ in which we only have to account for the overshoot.

Lemma 2.5. [2021b]
*For $\hat{b} > b_0$ and $\check{b} = b_0$, $u$ is the unique solution to the integral equation*

$$f(x) = \frac{\lambda}{\lambda+\delta}\int_0^{(d-x)/b_0} f(x+b_0y)\,\mathrm{d}G(y) + \frac{\lambda}{\lambda+\delta}\int_{(d-x)/b_0}^{\infty}\Big[\frac{1}{\delta} - \Big(\frac{1}{\delta} - f(d)\Big)\mathrm{e}^{-\gamma(\hat{b})(x+b_0y-d)}\Big]\,\mathrm{d}G(y)\,, \qquad x \le d\,, \tag{2.8}$$

*and it holds*

$$u(d) = \frac{\lambda}{\delta}\frac{1-\ell_Y(b_0\gamma(\hat{b}))}{\lambda[1-\ell_Y(b_0\gamma(\hat{b}))]+\delta}\,. \tag{2.9}$$

*For $\hat{b} = b_0$, the assertion holds with the interpretation '$\gamma(b_0) = \infty$' and $u(d) = \lambda(\lambda+\delta)^{-1}\delta^{-1}$.*

*Proof.* By conditioning on the time $T_1$ of the first claim, we obtain for $x \le d$:

$$u(x) = \mathbb{E}^x\big[\mathrm{e}^{-\delta T_1}u(x+b_0Y_1)\big] = \mathbb{E}^x\big[\mathrm{e}^{-\delta T_1}\big]\,\mathbb{E}^x\big[u(x+b_0Y_1)\big] = \frac{\lambda}{\lambda+\delta}\int_0^{\infty} u(x+b_0y)\,\mathrm{d}G(y)\,,$$

by independence of $Y_1$ and $T_1$. We assume that it holds $\hat{b} > b_0$. Plugging in (2.7) shows that $u|_{[0,d]}$ is a solution to Equation (2.8). For any continuous function $f : [0,d] \to \mathbb{R}$ fulfilling (2.8), a direct calculation shows

$$f(d) = \frac{\lambda}{\lambda+\delta}\cdot\Big[\frac{1}{\delta} - \Big(\frac{1}{\delta} - f(d)\Big)\ell_Y(b_0\gamma(\hat{b}))\Big]\,,$$

so $f(d)$ has to be equal to the expression on the right hand side of (2.9). Now $\mathcal{T}$ defined by

$$\mathcal{T}f(x) = \frac{\lambda}{\lambda+\delta}\int_0^{(d-x)/b_0} f(x+b_0y)\,\mathrm{d}G(y) + \frac{\lambda}{\lambda+\delta}\int_{(d-x)/b_0}^{\infty}\Big[\frac{1}{\delta} - \Big(\frac{1}{\delta} - u(d)\Big)\mathrm{e}^{-\gamma(\hat{b})(x+b_0y-d)}\Big]\,\mathrm{d}G(y)$$

is a contraction, mapping the space of continuous functions on $[0,d]$ (equipped with the supremum norm) onto itself. Thus, (2.8) has a unique solution. The assertion for the case of $\hat{b} = \check{b} = b_0$ can be proved in the same way. $\square$

For the case $\check{b} = b_0$, we therefore have fully characterised the expected time in drawdown. In particular, if we can solve the integral equation, we know the function $u$. From now on, we assume $\check{b} > b_0$. For this case, we prove a similar result in two steps. The first step is to show that $u$ solves an integro-differential equation (Lemma 2.6, below). The second step is to use a martingale argument based on Lemma 2.1, to conclude uniqueness of solutions (Lemma 2.7).

Lemma 2.6. [2021b]
*For $\check{b} > b_0$, $u$ is Lipschitz continuous and continuously differentiable for $x \in (0,d)$. Moreover, $u$ is a*

*solution to the integro-differential equation*

$$-\delta u(x) - c(\check{b})u'(x) + \lambda \int_0^\infty [u(x + \check{b}y) - u(x)] \, \mathrm{d}G(y) = 0 \,, \qquad x \in (0, d) \,. \tag{2.10}$$

*At $x = 0$, the equation holds for the derivative from the right and we have $u'(0) = 0$. At $x = d$, the equation holds for the derivative from the left. In particular,*

$$u(0) = \frac{\lambda}{\lambda + \delta} \int_0^\infty u(\check{b}y) \, \mathrm{d}G(y) \,. \tag{2.11}$$

The techniques used in the proof belong to the standard 'tool kit' of the Cramér–Lundberg model (compare for example [Schmidli, 2017]) and can be applied to its drawdown as well.

*Proof of Lemma* 2.6. Equation (2.11) follows in the same way as in the proof of Lemma 2.5. For $x \in (0, d]$ we can define $h \leq x/c(\check{b})$, such that $\Delta$ starting at $x$ cannot reach zero before time $h$. Heuristically speaking, the strong Markov property allows us to 'restart' the process at $x - c(\check{b})h$ if no claim occurs before time $h$ and at $x - c(\check{b})T_1 + \check{b}Y_1$ if there is a claim of size $Y_1$ at time $T_1 < h$. By conditioning on these events, we obtain

$$u(x) = \mathrm{e}^{-(\delta+\lambda)h} u(x - c(\check{b})h) + \int_0^h \lambda \mathrm{e}^{-(\lambda+\delta)t} \, \mathbb{E}[u(x - c(\check{b})T_1 + \check{b}Y_1)] \, \mathrm{d}t \,.$$

Because $u$ is increasing and bounded with values in $[0, \delta^{-1}]$, compare Lemma 1.4, we obtain Lipschitz continuity with Lipschitz constant $\lambda/(\delta c(\check{b}))$:

$$0 \leq u(x) - u(x - c(\check{b})h) \leq \int_0^h \frac{\lambda \mathrm{e}^{-(\lambda+\delta)t}}{\delta} \, \mathrm{d}t \leq \frac{\lambda}{\delta} h \,.$$

Dividing by $h$ and rearranging the terms, we arrive at

$$\begin{aligned} c(\check{b}) & \frac{u(x) - u(x - c(\check{b})h)}{c(\check{b})h} \\ &= \frac{\mathrm{e}^{-(\delta+\lambda)h} - 1}{h} u(x - c(\check{b})h) + \frac{1}{h} \int_0^h \lambda \mathrm{e}^{-(\lambda+\delta)t} \, \mathbb{E}[u(x - c(\check{b})t + \check{b}Y_1)] \, \mathrm{d}t \,. \end{aligned} \tag{2.12}$$

The limit for $h \to 0$ of the right hand side exists. Thus, we obtain differentiability from the left and that (2.10) is fulfilled for the left derivative $u'$. Replacing $x \in [0, d)$ by $x + c(\check{b})h$, we analogously obtain the equation for the derivative from the right. $u'(0) = 0$ follows from plugging (2.11) into (2.10) for the right hand side derivative. □

In the same way (or by direct calculation, using Corollary 2.4), we can show that $u$ solves

$$-\delta u(x) - c(\hat{b})u'(x) + \lambda \int_0^\infty [u(x + \hat{b}y) - u(x)] \, \mathrm{d}G(y) = -1 \,, \qquad x > d \,, \tag{2.13}$$

which is also fulfilled at $x = d$ for the right derivative. This means that, in general, the derivative of $u$ from the right differs from the derivative from the left at this point (for example for the choice $\check{b} = \hat{b}$).

Lemma 2.7 (Verification for Simple Switching Strategies). [2021b]
*Let $f:[0,\infty)\to\mathbb{R}$ be an absolutely continuous, bounded function with density $f'$. If $f$ solves*

$$-\delta f(x) - c(b(x))f'(x) + \lambda \int_0^\infty \big[f(x+b(x)y) - f(x)\big]\, \mathrm{d}G(y) = -\mathbb{1}_{\{x>d\}}\,, \qquad x \geq 0\,, \tag{2.14}$$

*for $b(x) = \check{b}\mathbb{1}_{\{x\in[0,d]\}} + \hat{b}\mathbb{1}_{\{x>d\}}$ and fulfils the initial condition given in Equation* (2.11)*, then it holds $f(x) = u(x)$ for all $x \geq 0$.*

*Proof.* In the notation of Lemma 2.1, Equation (2.14) is equivalent to $\mathcal{A}^{b(x)}f(x) = -\mathbb{1}_{\{x>d\}}$, $x \geq 0$. It follows from the equation that $f'$ is continuous in an environment $[0,\varepsilon)$, $\varepsilon > 0$, of zero. Thus, the initial condition implies that either $\check{b} = b_0$ (i.e. the running maximum is constant) or $f'(0) = 0$ hold. In either case, the integral with respect to $M$ in (2.5) vanishes and, thus, the process

$$\Big(\mathrm{e}^{-\delta t}f(\Delta_t) - f(\Delta_0) + \int_0^t \mathrm{e}^{-\delta s}\mathbb{1}_{\{\Delta_s>d\}}\,\mathrm{d}s\Big)_{t\geq 0}$$

is a martingale with mean zero. Building expectations, we find

$$f(x) = \mathbb{E}^x\big[\mathrm{e}^{-\delta t}f(\Delta_t)\big] + \mathbb{E}^x\Big[\int_0^t \mathrm{e}^{-\delta s}\mathbb{1}_{\{\Delta_s>d\}}\,\mathrm{d}s\Big]\,.$$

So, letting $t\to\infty$, we obtain the desired result by bounded and monotone convergence. □

With Lemmata 2.6 and 2.7, we have now characterised $u$ as the unique bounded solution to (2.14) and (2.11). Next, we are going to derive techniques to solve this equation, depending on the claim distribution.

### 2.1.3 Explicit Return Functions

In this section, we consider two ways of calculating explicit solutions for claim distributions which allow them.

#### A Naive Approach and Algorithmic Solutions

We assume that the claim distribution is a *phase-type distribution*. This means, the distribution function $G$ has a density $g$ of the form $g(y) = \mathbf{a}\mathrm{e}^{\mathbf{Q}y}\mathbf{q}\mathbb{1}_{\{y>0\}}$ fulfilling the following conditions. There is a continuous time Markov chain with transient states $\{1,\dots,n\}$ and one absorbing state $\mathfrak{t}$, such that $\mathbf{Q}$ is the $n\times n$-submatrix of the intensity matrix which belongs to the transient states. $\mathbf{a}$ is a non-negative, $n$-dimensional row vector with $\sum_{k=1}^n \mathbf{a}_k = 1$. $\mathbf{q}$ is the product of $-\mathbf{Q}$ and the $n$-dimensional, all-ones column vector $\mathbf{1}$. A survey of phase-type distributions and their applications in risk theory is found in [Bladt, 2005]. Easy examples of phase-type distributions are (mixed) exponential distributions and convolutions thereof.
For simplicity, we consider the constant strategy $(\check{b},\hat{b}) = (1,1)$ and write $c = c(1)$ and $\gamma = \gamma(1)$ for the positive solution to $\Psi_1(\gamma) = c\gamma - \lambda(1-\ell_Y(\gamma)) = \delta$. The general case could be treated in the same way, as we will see in the next section.

*Example 2.8 (Exponential Distribution).* If claims are exponentially distributed with parameter $\alpha > 0$, we have $\ell_Y(r) = \alpha(\alpha + r)^{-1}$ for $r > -\alpha$ and therefore

$$\gamma = \frac{\delta + \lambda - \alpha c}{2c} + \sqrt{\left(\frac{\delta + \lambda - \alpha c}{2c}\right)^2 + \frac{\delta\alpha}{c}}\,.$$

For $x \in (0, d)$, (2.14) becomes

$$0 = cu'(x) + (\delta + \lambda)u(x) - \lambda\alpha e^{\alpha x} \int_x^\infty e^{-\alpha y} u(y)\, dy\,. \tag{2.15}$$

$u$ and the integral term are continuously differentiable on $(0, d)$, so $u'$ is differentiable on this interval, as well. We differentiate with respect to $x$ and derive a second order differential equation:

$$\begin{aligned} cu''(x) + (\delta + \lambda)u'(x) + \lambda\alpha u(x) &= \alpha\Big(\lambda\alpha e^{\alpha x} \int_x^\infty e^{-\alpha y} u(y)\, dy\Big) \\ &= c\alpha u'(x) + \alpha(\delta + \lambda)u(x)\,. \end{aligned}$$

Thus, it should hold $cu''(x) + (\delta + \lambda - \alpha c)u'(x) - \alpha\delta u(x) = 0$ and our candidate solution is of the form

$$f(x) = C_1 e^{r_1 x} + C_2 e^{-\gamma x}\,, \qquad r_1 = \gamma - \frac{\delta + \lambda - \alpha c}{c}\,.$$

The initial condition $u'(0) = 0 = f'(0)$ implies $C_2 = r_1 C_1/\gamma$. To calculate $C_1$ in this case, we plug the function $f$ into the integro-differential equation, which yields

$$\begin{aligned} &e^{rx} C_1\Big(cr_1 + (\delta + \lambda) + \frac{\lambda\alpha}{r_1 - \alpha}\Big) + e^{-\gamma x} C_1 \frac{r_1}{\gamma}\Big(-c\gamma + (\delta + \lambda) - \frac{\lambda\alpha}{\gamma + \alpha}\Big) \\ &\quad + e^{\alpha x}\Big(-C_1\lambda\alpha e^{(r_1 - \alpha)d}\Big(\frac{1}{r_1 - \alpha} + \frac{1}{\gamma + \alpha}\Big) - \frac{\lambda\gamma}{\delta(\gamma + \alpha)} e^{-\alpha d}\Big) = 0 \end{aligned}$$

for $x \in (0, d]$. The first and second pair of brackets are zero by definition of $r_1$ and $\gamma$. Since the third term must be equal to zero as well, we conclude

$$C_1 = \frac{\gamma}{r_1 + \gamma} \frac{\alpha - r_1}{\alpha} \frac{e^{-r_1 d}}{\delta}\,, \qquad C_2 = \frac{r_1}{r_1 + \gamma} \frac{\alpha - r_1}{\alpha} \frac{e^{-r_1 d}}{\delta}\,.$$

We see here that the integro-differential equation holds for all $x \in [0, d]$, although the requirement that it is fulfilled at a single point (for example on the boundary $x = d$) already determines $C_1$. Lemma 2.7 verifies that $f$, extended by $f(x) = \delta^{-1} - (\delta^{-1} - f(d))e^{-\gamma(x-d)}$ for $x > d$, is the expected time with critical drawdown for exponential claims. ⁜

The idea of differentiating to derive an ordinary differential equation can be extended to calculate the expected time in drawdown for phase-type distributed claims. For $x \in (0, d)$, we can rewrite (2.14) by linearity of the integral:

$$cu'(x) + (\delta + \lambda)u(x) - \lambda \sum_{k=1}^{n} \mathbf{a}_k\, \mathbf{j}_k = 0\,, \tag{2.16}$$

where the vector $\mathbf{j}$ is defined by $\mathbf{j}_k = \int_x^\infty u(y)(\mathrm{e}^{\mathbf{Q}(y-x)}\mathbf{q})_k \,\mathrm{d}y$, $k = 1, \ldots, n$. Our goal is again, as it was done in the example, to replace the integral terms with linear combinations of derivatives of $u$. With the definition

$$\mathbf{b}_0 = \frac{c}{\lambda}u'(x) + \frac{\delta+\lambda}{\lambda}u(x)\,,$$

(2.16) reads $\mathbf{aj} = \mathbf{b}_0$. Assuming that $u$ is $p+1$ times continuously differentiable with $l$th derivative $u^{(l)}$, differentiation of (2.16) yields $p$ additional equations of the form

$$cu^{(l+1)}(x) + (\delta+\lambda)u^{(l)}(x) + \lambda\sum_{k=0}^{l-1} u^{(k)}(x)\,\mathbf{a}(-\mathbf{Q})^{l-1-k}\mathbf{q} - \lambda\sum_{k=1}^{n}(\mathbf{a}(-\mathbf{Q})^l)_k\,\mathbf{j}_k = 0\,,$$

$l = 1, \ldots, p$. These can be rewritten as $(\mathbf{a}(-\mathbf{Q})^l)\mathbf{j} = \mathbf{b}_l$, where

$$\mathbf{b}_l = \frac{c}{\lambda}u^{(l+1)}(x) + \frac{\delta+\lambda}{\lambda}u^{(l)}(x) + \sum_{k=0}^{l-1} u^{(k)}(x)\,\mathbf{a}(-\mathbf{Q})^{l-1-k}\mathbf{q}\,, \qquad l = 1, \ldots, p\,.$$

Differentiating $n$ times gives $n$ additional equations which form the linear equation system $(\mathbf{a}(-\mathbf{Q})^l)\mathbf{j} = \mathbf{b}_l$, $l = 1, \ldots, n$. Neither $\mathbf{a}$ nor $\mathbf{Q}$ depend on $x$, so formally solving for $\mathbf{j}$ results in a representation of $\mathbf{j}_k$ as linear combination of $u$ and its derivatives for all $k = 1, \ldots, n$. Plugging this solution into the original equation $\mathbf{aj} = \mathbf{b}_0$ yields coefficients $\mathbf{d}_l$, $l = 0, \ldots, n+1$, and an ordinary differential equation of the form $\sum_{l=0}^{n+1} \mathbf{d}_l u^{(l)}(x) = 0$. The general solution to this type of equation is a linear combination of exponential functions multiplied with powers of $x$. Plugging in the initial conditions, we thus obtain

$$u(x) = \sum_{i=0}^{k}\sum_{m=1}^{m_i} C_{m,i}\, x^{m-1}\mathrm{e}^{r_i x}\,, \tag{2.17}$$

where $r_0 = 0$ and every $r_i \in \mathbb{C}$, $i > 0$, is a solution to $\sum_{l=0}^{n+1} \mathbf{d}_l r^l = 0$ of respective algebraic multiplicity $m_i$ and for suitable constants $C_{m,i} \in \mathbb{R}$, $m = 1, \ldots, m_i$, $i = 0, \ldots, k$. In numerical examples, this 'algorithmic' method works especially well for sparse matrices. These are typical for mixtures of exponential or Erlang distributions. However, for more complicated intensity matrices, which appear, for example, in the context of phase-type approximations to other distributions, the calculation of high powers of $\mathbf{Q}$ is very inefficient. Moreover, the method strongly relies on the properties of the exponential function, so we cannot hope to extend it to a broader class of claim distributions. We therefore deal with a more versatile approach in the next section.

### Solutions via Laplace Transforms

We consider an arbitrary claim distribution, fulfilling the conditions formulated at the beginning of this chapter, and strategies with $(\check{b}, \hat{b}) \in [b_0, 1] \times (b_0, 1]$. The case $\hat{b} = b_0$ can be treated analogously, so we just state the corresponding results at the end. We define the function $w$ on $[0, d]$ by $w(x) = u(d-x)$. We know that $u$ takes the form given in (2.7) above $d$. By $w(0) = u(d)$, we obtain an integro-differential equation for $w$:

$$0 = c(\check{b})w'(x) - (\delta+\lambda)w(x) + \lambda \int_0^{x/\check{b}} w(x-\check{b}y)\,\mathrm{d}G(y)$$
$$+ \lambda \int_{x/\check{b}}^{\infty} \Big[\frac{1}{\delta} + \Big(\frac{1}{\delta} - w(0)\Big) \cdot \mathrm{e}^{-\gamma(\hat{b})(\check{b}y-x)}\Big]\,\mathrm{d}G(y)\,.$$

We extend the function $w$ from $[0,d]$ to $[0,\infty)$ through the equation. We note that this does not interfere with the expected time in drawdown because $u(x)$ is only considered for positive $x$. As a continuous and bounded function, $w$ possesses a unique Laplace transform $\ell_w$. Multiplying all terms with $\mathrm{e}^{-tx}$ for $t \in (0,\infty) \setminus \{\gamma(\hat{b})\}$ and integrating over $(0,\infty)$, we obtain an equation for $\ell_w$:

$$0 = c(\check{b})(t\ell_w(t) - w(0)) - (\delta+\lambda)\ell_w(t) + \lambda\ell_w(t)\ell_Y(t\check{b})$$
$$+ \lambda\Big(\frac{1-\ell_Y(t\check{b})}{\delta t} - \Big(\frac{1}{\delta} - w(0)\Big)\frac{\ell_Y(t\check{b}) - \ell_Y(\gamma(\hat{b})\check{b})}{\gamma(\hat{b}) - t}\Big)\,. \tag{2.18}$$

The appendix, p. 111, can be consulted for details on the derivation of this equation. Solving (2.18) for $\ell_w$ and using the definition of $\Psi_{\check{b}}$, we find

$$\ell_w(t) = \frac{1}{t}\Big(\frac{1}{\delta} + \frac{1}{\Psi_{\check{b}}(t)-\delta}\Big) - \frac{(\delta^{-1} - w(0))}{t - \gamma(\hat{b})}\Big(1 - \frac{\Psi_{\check{b}}(\gamma(\hat{b})) - \delta}{\Psi_{\check{b}}(t) - \delta}\Big) \tag{2.19}$$

for all $t \notin \{\gamma(\hat{b}), \gamma(\check{b})\}$ if $\check{b} > b_0$ and, else, for all $t \notin \{\gamma(\hat{b})\}$. This result enables us to write down $w$ in terms of the the inverse Laplace transform of $t \mapsto \frac{1}{\Psi_{\check{b}}(t)-\delta}$, i.e. the so-called *scale function* $W_{\check{b}}^{(\delta)}$, defined by

$$\int_0^\infty \mathrm{e}^{-tx} W_{\check{b}}^{(\delta)}(x)\,\mathrm{d}x = \frac{1}{\Psi_{\check{b}}(t) - \delta}\,, \qquad t > \sup\{r \geq 0 : \Psi_{\check{b}}(r) = \delta\}\,,$$

(see, for example, [Hubalek and Kyprianou, 2010] or [Kuznetsov et al., 2012]). In particular, the condition on $t$ is interpreted as '$t > 0$' for $\check{b} = b_0$ and equivalent to '$t > \gamma(\check{b})$' for $\check{b} > b_0$. We arrive at

$$w(x) = \frac{1}{\delta} + \int_0^x W_{\check{b}}^{(\delta)}(y)\,\mathrm{d}y$$
$$- \mathrm{e}^{\gamma(\hat{b})x}\Big(\frac{1}{\delta} - w(0)\Big)\Big(1 - (\Psi_{\check{b}}(\gamma(\hat{b})) - \delta)\int_0^x \mathrm{e}^{-\gamma(\hat{b})y}\, W_{\check{b}}^{(\delta)}(y)\,\mathrm{d}y\Big).$$

Now we can use the definition $u(x) = w(d-x)$ and the initial condition $u'(0) = 0$ to find an expression for $u$. That means, we have just executed the proof of the following result.

Theorem 2.9. [2021b]

*For $x \in [0,d]$, the expected time in drawdown under the strategy $(\check{b}, \hat{b})$ with $\hat{b} > b_0$ is given by*

$$u(x) = \frac{1}{\delta} + \int_0^{d-x} W_{\check{b}}^{(\delta)}(y)\,\mathrm{d}y$$
$$- \Big(\frac{1}{\delta} - u(d)\Big)\Big(1 - (\Psi_{\check{b}}(\gamma(\hat{b})) - \delta)\int_0^{d-x} \mathrm{e}^{\gamma(\hat{b})(d-x-y)}\, W_{\check{b}}^{(\delta)}(y)\,\mathrm{d}y\Big)\,, \tag{2.20}$$

*with*

$$u(d) = \frac{1}{\delta} - \frac{W_{\check{b}}^{(\delta)}(d)}{\gamma(\hat{b})\mathrm{e}^{\gamma(\hat{b})d}(1 - (\Psi_{\check{b}}(\gamma(\hat{b})) - \delta)\int_0^d \mathrm{e}^{-\gamma(\hat{b})y} W_{\check{b}}^{(\delta)}(y)\,\mathrm{d}y) - (\Psi_{\check{b}}(\gamma(\hat{b})) - \delta) W_{\check{b}}^{(\delta)}(d)}\,.$$

*For* $\hat{b} = b_0$*, it holds*

$$u(x) = \frac{1}{\delta} + \int_0^{d-x} W_{\check{b}}^{(\delta)}(y)\,\mathrm{d}y\,.$$

□

In particular, for constant strategies with $\check{b} = \hat{b}$ (and, therefore, $\Psi_{\check{b}}(\gamma(\hat{b})) = \delta$), the formulae take a simpler form because the last bracket on the right hand side of (2.20) is equal to one and the numerator of the second fraction in the representation of $u(d)$ boils down to $\gamma(\hat{b})\mathrm{e}^{\gamma(\hat{b})d}$.
Even though we have derived an appealing and (at least for constant strategies) condensed expression for the function $u$, the Laplace approach relocates the problem rather than solving it: now we have to know the scale function. However, let us consider one more time the (large) class of phase-type distributions. For this type of claim distribution, $t \mapsto \Psi_{\check{b}}(t) - \delta$ is a rational function.

*Example 2.10 (Revisiting Phase-type Distributions).* The Laplace transform of a phase-type distribution with density $g(y) = \mathbf{a}\mathrm{e}^{\mathbf{Q}y}\mathbf{q}\mathbb{1}_{\{y>0\}}$ takes the form

$$\ell_Y(r) = -\mathbf{a}(\mathbf{Q} - r\mathbf{I}^{(n)})^{-1}\mathbf{q} = \frac{\mathbf{a}\mathbf{A}^{(r)}\mathbf{q}}{(-1)^n \chi_{\mathbf{Q}}(r)}\,,$$

where $\mathbf{I}^{(n)}$ denotes the $n$-dimensional identity matrix. We write, on the far right, the inverse of $\mathbf{Q} - r\mathbf{I}^{(n)}$ as the adjugate $\mathbf{A}^{(r)}$ of this matrix divided by its determinant $(-1)^n\chi_{\mathbf{Q}}(r) = \det(r\mathbf{I}^n - \mathbf{Q})$. Here, $\chi_{\mathbf{Q}}(r) = \det(r\mathbf{I}^n - \mathbf{Q})$ is the characteristic polynomial of $\mathbf{Q}$, which is of order $n$. $\mathbf{A}_{ij}^{(r)}$ is defined as the $(i,j)$-minor of $\mathbf{Q} - r\mathbf{I}^{(n)}$. Thus, we obtain again a polynomial of the variable $r$ (maximally of order $n-1$). So, the function in question

$$\frac{1}{\Psi_{\check{b}}(t) - \delta} = \frac{\chi_{\mathbf{Q}}(\check{b}t)}{c(\check{b})t\chi_{\mathbf{Q}}(\check{b}t) - (\lambda + \delta)\chi_{\mathbf{Q}}(\check{b}t) + (-1)^n\lambda(\mathbf{a}\mathbf{A}^{(\check{b}t)}\mathbf{q})} = \sum_{i=1}^{k}\sum_{m=1}^{m_i} \frac{\tilde{C}_{i,m}}{(r_i + t)^m}$$

is a proper rational function over the real numbers. That means, we can use partial fraction decomposition to find constants $\tilde{C}_{i,k}$ such that the second equation is fulfilled. Here, every $-r_i \in \mathbb{C}$ is a root of multiplicity $m_i$ of the denominator. In [Egami and Yamazaki, 2014] and [Kuznetsov et al., 2012], one finds explicit expressions for the constants $\tilde{C}_{i,k}$. The corresponding scale function is

$$W_{\check{b}}^{(\delta)}(x) = \sum_{i=1}^{k}\sum_{m=1}^{m_i} \frac{\tilde{C}_{i,m}}{(m-1)!} x^{m-1}\mathrm{e}^{-r_i x}\,, \qquad x \geq 0\,.$$

Hence, by Theorem 2.9, we conclude that the expected time in drawdown for phase-type distributed claims under general simple switching strategies is also of the form given in (2.17) for $x \in [0, d]$ and suitable constants $C_{i,m}$. ⁕

As we will see next, one possibility to obtain a solution to the integro-differential equation for a general (positive and absolutely continuous) claim distribution is to use an approximation. A pointwise convergent approximating sequence for a given claim distribution can be chosen from the class of phase-type distributions (compare, for example, [Asmussen et al., 1996] or [Asmussen and Albrecher, 2010, Thm. A.5.14]). The obvious advantage of this method is that the approximate expected time in drawdown takes the simple form of an exponential polynomial. The following lemma applies to general approximating sequences and simple switching strategies with $\check{b}, \hat{b} \in [b_0, 1]$.

LEMMA 2.11. [2021b]
*Let $(G^{(n)})_{n\in\mathbb{N}}$ be a sequence of absolutely continuous distribution functions, concentrated on $(0, \infty)$ with $\lim_{n\to\infty} G^{(n)}(x) = G(x)$ for every $x \in (0, \infty)$. Denote by $u^{(n)}$ the expected time in drawdown under the strategy $(\check{b}, \hat{b})$ with $G^{(n)}$-distributed claims. If*

i) $\hat{b} = b_0$, *or*
ii) $\check{b} = \hat{b} \in (b_0, 1]$, *or*
iii) $(W^{\delta^{(n)}_{\check{b}}})_{n\in\mathbb{N}}$ *converges with respect to the uniform norm on* $[0, d]$,

*we have* $\lim_{n\to\infty} u^{(n)}(x) = u(x)$ *for all* $x \geq 0$.

The proof is based on our representation of $u$ by scale functions and the extended continuity theorem for Laplace transforms found in [Feller, 1971]. The distinguishment of i)–iii) is motivated by the different levels of complexity of the Laplace transform and $u(d)$. The interested reader finds a detailed proof in the appendix, p. 112. We acknowledge that the conditions stated are not exhaustive: depending on the approximating sequence there could still be convergence of the expected time with critical drawdown if neither i), ii) nor iii) are fulfilled. Moreover, it should be noted that in a numerical procedure, the calculation of an approximating sequence can be time-consuming with significant memory requirement. In addition, the calculation of the $u^{(n)}$ via Laplace transforms involves root-finding, which can cause numerical inaccuracies. We present an alternative method in Section 2.3, in which we consider numerical examples for simple switching strategies. In particular, we compare the performance of these strategies to the performance of optimal strategies. We examine the latter in the following section.

## 2.2 Solution to the Optimisation Problem

We now turn to the optimisation problem given in Equation (1.17) and start with the following general observation.

LEMMA 2.12. [2021b]
*The minimal expected time with critical drawdown $v$ is Lipschitz continuous with*

$$|v(x) - v(y)| \leq \frac{\lambda + \delta}{\delta c(1)} |x - y| \,, \qquad x, y \geq 0 \,.$$

*In particular, $v$ is absolutely continuous and differentiable almost everywhere.*

*Proof.* We consider $0 \leq y < x$ and define $h = (x - y)/c(1)$. Because $-c(1)$ is the maximal possible downward drift, this means that the drawdown process starting at $x$ cannot reach $y$ before time $h$. We choose $\varepsilon > 0$ and denote by $B^\varepsilon$ the $\varepsilon$-optimal strategy with $v^{B^\varepsilon}(y) < v(y) + \varepsilon$. We define the strategy $B$ by $B_t = \mathbb{1}_{\{T_1 \wedge t < h\}} + B^\varepsilon_{t-h}\mathbb{1}_{\{T_1 \wedge t \geq h\}}$. This corresponds to the strategy of 'no reinsurance', which we denote by $\mathit{1}$, until time $h \wedge T_1$. If a claim occurs before $h$, that is $T_1 < h$, this strategy is kept also after time $h$. If no claim occurs, the drawdown process arrives at $y$ at time $h$ and then the strategy switches to the $\varepsilon$-optimal strategy for the new initial capital. The return of this strategy fulfils:

$$\begin{aligned}
v^B(x) &= \mathbb{E}^x\Big[\Big(\int_0^{T_1} \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta^{\mathit{1}}_t > d\}}\,\mathrm{d}t\Big)\cdot\mathbb{1}_{\{T_1<h\}} + \mathrm{e}^{-\delta T_1}v^{\mathit{1}}(\Delta^{\mathit{1}}_{T_1})\cdot\mathbb{1}_{\{T_1<h\}}\Big] \\
&\quad + \mathbb{E}^x\Big[\Big(\int_0^{h} \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta^{\mathit{1}}_t > d\}}\,\mathrm{d}t\Big)\cdot\mathbb{1}_{\{T_1\geq h\}} + \mathrm{e}^{-\delta h}v^{B^\varepsilon}(y)\cdot\mathbb{1}_{\{T_1\geq h\}}\Big] \\
&\leq \mathbb{E}^x\Big[\frac{1-\mathrm{e}^{-\delta T_1}}{\delta}\cdot\mathbb{1}_{\{T_1<h\}} + \mathrm{e}^{-\delta T_1}v^{\mathit{1}}(\Delta^{\mathit{1}}_{T_1})\cdot\mathbb{1}_{\{T_1<h\}}\Big] \\
&\quad + \mathbb{E}^x\Big[\frac{1-\mathrm{e}^{-\delta h}}{\delta}\cdot\mathbb{1}_{\{T_1\geq h\}} + \mathrm{e}^{-\delta h}v^{B^\varepsilon}(y)\cdot\mathbb{1}_{\{T_1\geq h\}}\Big]\,.
\end{aligned}$$

Evaluating the terms on the right hand side and using that $v^{\mathit{1}}$ is bounded from above by $\delta^{-1}$, we find

$$\begin{aligned}
v^B(x) &\leq \delta^{-1}[1-\mathrm{e}^{-\lambda h} + (1-\mathrm{e}^{-\delta h})\mathrm{e}^{-\lambda h}] + \mathrm{e}^{-\lambda h}\mathrm{e}^{-\delta h}v^{B^\varepsilon}(y) \\
&= \delta^{-1}\big(1-\mathrm{e}^{-(\delta+\lambda)h}\big) + \mathrm{e}^{-(\delta+\lambda)h}v^{B^\varepsilon}(y)\,.
\end{aligned}$$

This means that we have:

$$\begin{aligned}
v(x) - v(y) &\leq v^B(x) - v^{B^\varepsilon}(y) + \varepsilon \leq \big(\delta^{-1} - v^{B^\varepsilon}(y)\big)\big(1-\mathrm{e}^{-(\delta+\lambda)h}\big) + \varepsilon \\
&\leq \delta^{-1}\big(1-\mathrm{e}^{-(\delta+\lambda)h}\big) + \varepsilon \leq \frac{(\lambda+\delta)h}{\delta} + \varepsilon = \frac{\lambda+\delta}{\delta c(1)}\,(x-y) + \varepsilon\,.
\end{aligned}$$

Because the left hand side is independent of $\varepsilon$, we can let $\varepsilon \to 0$. Because $v$ is increasing by Lemma 1.4, Lipschitz continuity follows. □

We recall that in the proof of the dynamic programming principle, Theorem 1.5, we assumed that there is a 'universally' $\varepsilon$-optimal strategy. With Lemma 2.12, we can now rigorously prove existence of such strategies.

Lemma 2.13. [2021b]
*For every $\varepsilon > 0$, a set of strategies $B^\varepsilon(x) \in \mathcal{B}$ can be chosen in a measurable way such that $v^{B^\varepsilon(x)}(x) < v(x) + \varepsilon$ for every $x \geq 0$.*

*Proof.* As we have just seen, $v$ is Lipschitz continuous and therefore uniformly continuous. That is, there exists $n \in \mathbb{N}$, such that $|v(x) - v(y)| < \varepsilon/2$ holds for all $x, y \geq 0$ with $|x-y| < n^{-1}$. We define a sequence $(x_k)_{k\in\mathbb{N}_0}$ of grid points given by $x_k = kn^{-1}$. For each $k$, there exists a strategy $B^{\varepsilon,k} \in \mathcal{B}$ with $v^{B^{\varepsilon,k}}(x_k) < v(x_k) + \varepsilon/2$. For every $k \geq 1$, we define $B^\varepsilon(x) = B^{\varepsilon,k}$ for all $x \in (x_{k-1}, x_k]$. Because $v$ is increasing, we have

$$v^{B^\varepsilon(x)}(x) \leq v^{B^{\varepsilon,k}}(x_k) < v(x) + v(x_k) - v(x) + \frac{\varepsilon}{2} < v(x) + \varepsilon$$

and the assertion follows. □

The dynamic programming principle implies that the optimisation problem can be split into two parts. Before we deal with the solutions to the subproblems, we show that they can be reconnected by the Hamilton–Jacobi–Bellman equation

$$\inf_{b\in[b_0,1]}\Big\{-\delta f(x)-c(b)f'(x)+\lambda\int_0^\infty[f(x+by)-f(x)]\,\mathrm{d}G(y)\Big\}=-\mathbb{1}_{\{x>d\}}\,. \tag{2.21}$$

In particular, we start by showing that if a solution to the equation can be obtained, then this solution corresponds to $v$ (provided that it is bounded, increasing and fulfils an according initial condition at zero). This could be interpreted as uniqueness of solutions to the equation. After that, we prove existence by finding the subsolutions and proving that their combination solves the Hamilton–Jacobi–Bellman equation.

### 2.2.1 Uniqueness via Associated Martingales

For an absolutely continuous and bounded function $f:[0,\infty)\to\mathbb{R}$ with density $f'$ and $b\in[b_0,1]$, we define $\mathcal{A}^b f(x)$ as in Equation (2.4). Then, the Hamilton–Jacobi–Bellman equation takes the form $\inf_{b\in[b_0,1]}\mathcal{A}^b f(x)=-\mathbb{1}_{\{x>d\}}$. We observe that $b\mapsto\mathcal{A}^b f(x)$ is continuous. This follows, for example, by Theorem IV.5.6 of [Elstrodt, 2011]. So, as $[b_0,1]$ is a compact interval, there exists a function $b^*(x):[0,\infty)\to[b_0,1]$ with $\mathcal{A}^{b^*(x)}f(x)=\inf_{b\in[b_0,1]}\mathcal{A}^b f(x)$ for a given bounded function $f$ with density $f'$. An important step of showing uniqueness is to connect this *pointwise minimiser* (i.e. the 'arg inf') to a feedback strategy.

Proposition 2.14. [2021b]

*Let $f:[0,\infty)\to\mathbb{R}$ be a bounded, absolutely continuous and increasing solution to* (2.21) *with density $f'$. We denote by $b^*:[0,\infty)\to[b_0,1]$ the pointwise minimiser. $f$ is strictly increasing and there exists a measurable version of $b^*$, such that the drawdown $\Delta^*$ under the corresponding feedback strategy $B^*$ with $B_t^*=b^*(\Delta_t^*)$, $t\ge 0$, is a piecewise deterministic Markov process and, with $b=b^*$, the process defined in* (2.5) *is a martingale.*

*Proof.* That the pointwise minimiser can be chosen in a measurable way is a direct consequence of Theorem 7.4 by Wagner [1977]. Because $f$ is increasing, we can assume that its density is non-negative. Let us assume that $f$ is not strictly increasing. Then there are at least two points $a_1<\bar{a}_1$ such that $f(a_1)=f(\bar{a}_1)$. This means, we have $f'(x)=0$ for all $x\in(a_1,\bar{a}_1)$. By (2.21), the unique minimiser is $b^*(x)=b_0$ for all $x\in(a_1,\bar{a}_1)$. In particular, $d\notin(a_1,\bar{a}_1)$ because $f'$ must have a jump at $d$. Then, it follows that

$$\int_0^\infty f(x+b_0y)\,\mathrm{d}G(y)=\int_0^\infty f(\bar{a}_1+b_0y)\,\mathrm{d}G(y)$$

for all $x\in(a_1,\bar{a}_1)$, such that $f(x+b_0y)=f(\bar{a}_1+b_0y)$ at all points of increase of $G$. This implies $\bar{a}_1=\infty$, which is a contradiction to $f$ solving Equation (2.21). By the same argument, there can only be isolated points $x$ with $b^*(x)=b_0$. Next, we prove existence of the controlled process as a piecewise deterministic Markov process. To this purpose, we first identify the set on which the deterministic paths are constant (i.e. $\{x:b^*(x)=b_0\}$). We start with initial values $x\in[0,d]$. We note that it

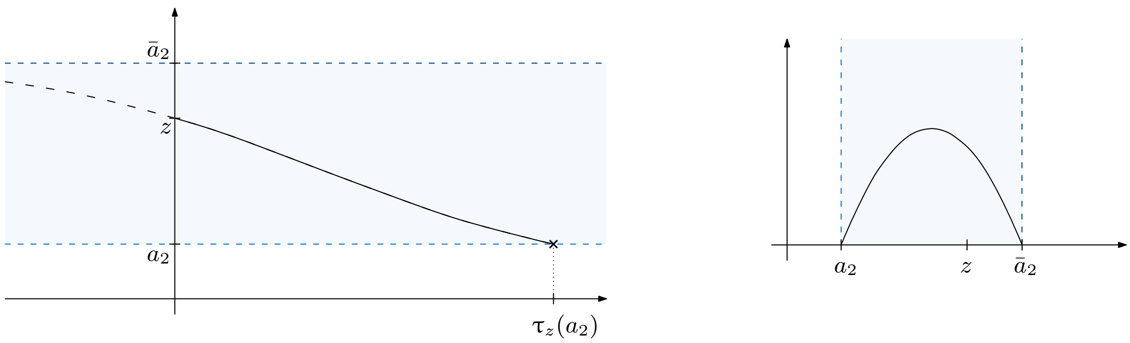



FIGURE 2.3 *Left:* Sketch of the function $t \mapsto \phi_{\pi_1}(t, z)$ (solid line). *Right:* Sketch of the corresponding function $x \mapsto c(b^*(x))$ (solid line).

follows from the Hamilton–Jacobi–Bellman equation that $H_0(x)$ with

$$H_0(x) = -(\delta + \lambda) f(x) + \lambda \int_0^\infty f(x + b_0 y) \, \mathrm{d}G(y)$$

is non-negative for all $x \in [0, d]$. A necessary condition for $b^*(x) = b_0$ is $H_0(x) = 0$. By continuity of $f$, $H_0$ is continuous. Therefore, the set $\{x : H_0(x) = 0\}$ is measurable. This means that it is no loss of generality to choose $b^*(x) = b_0$ for all $x \in \{x : H_0(x) = 0\}$, so that $b^*(x) = b_0$ is equivalent to $H_0(x) = 0$. Next, we observe that for all $x_1 \in [0, d)$ with $H_0(x_1) > 0$, there exists an open interval $I_{x_1} = (a_2, \bar{a}_2)$ with $H_0(x) > 0$ for all $x \in I_{x_1}$. In particular, $b^*(x) > b_0$ for all $x \in I_{x_1}$ (that means, there are no isolated points with $b^*(x) > b_0$). For each $x_1$, $I_{x_1}$ can be chosen maximally such that $H_0(a_2) = 0$ or $a_2 = 0$ is fulfilled. In case $H_0(d) > 0$, we analogously find a half-open interval $(a_2, d]$. Now, similarly as in the beginning of this chapter, the controlled process exists for all starting points $z$ with $b^*(z) = b_0$ up to the next jump time. In the notation of Subsection 2.1.1, we could define this as the state $\pi_0$ with $M_{\pi_0} = \{x \in [0, d] : H_0(x) = 0\}$. For $z \in M_{\pi_1} = \{x \in [0, d] : H_0(x) > 0\}$, we have $b^*(z) > b_0$ and there is a maximal interval $I_z = (a_2, \bar{a}_2)$ as defined above such that the function

$$\tau_z(u) = \int_z^u \frac{1}{-c(b^*(y))} \, \mathrm{d}y \,, \qquad u \in I_z \,,$$

fulfils $\tau_z(z) = 0$ and is absolutely continuous with a strictly negative density. Therefore, this function possesses a unique, absolutely continuous and decreasing inverse $\psi_z : (\tau_z(\bar{a}_2), \tau_z(a_2)) \to I_z$. In particular, $\psi_z(0) = z$, $\lim_{t \to \tau_z(a_2)} \psi_z(t) = a_2$ and $\psi_z(t)$ is defined for $t \in [0, \tau_z(a_2))$, i.e. up to the first time that it reaches the next point $a_2$ with $b^*(a_2) = b_0$. Figure 2.3 shows, on the left hand side, an example of a function constructed in this way for a given $x \mapsto b^*(x)$ (small graph on the right hand side). The solid line, extended by the black dashed line, corresponds to $\psi_z(t)$ with $t \in (\tau_z(\bar{a}_2), \tau_z(a_2))$ in this figure. By differentiating $\psi_z$, we see that its density is given by $\psi_z'(t) = -c(b^*(\psi_z(t)))$. Writing $\phi_{\pi_1}(t, z) = \psi_z(t)$, we find that the path of the controlled drawdown process starting from $z$ is therefore uniquely defined by

$$\phi_{\pi_1}(t, z) = z - \int_0^t c(b^*(\phi_{\pi_1}(s, z))) \, \mathrm{d}s$$

up until the first jump time or until the boundary $a_2$ is reached, that is, for $t \in [0, t^*(\pi_1, z)]$ with $t^*(\pi_1, z) = \tau_z(a_2)$. This coincides with (2.3). The cases of $z > d$ with $H_0(z) = -1$ and $H_0(z) > -1$ can be treated in the same way. In particular, the process under the feedback control exists. Then, because $b^*$ is measurable and the associated deterministic paths are decreasing, $B^*$ is an element of $\mathcal{B}$. The assertion follows by Lemma 2.1. $\square$

Remark. If one has obtained a solution $f$ to the Hamilton–Jacobi–Bellman equation, it is possible to derive additional results. Firstly, on every interval $(\underline{a}, \overline{a})$ with $b^*(x) > b_0 + \varepsilon$ for all $x \in (\underline{a}, \overline{a})$ for some $\varepsilon > 0$, we obtain the representation

$$f'(x) = \inf_{\beta \in [b_0+\varepsilon, 1]} \frac{\mathbb{1}_{\{x>d\}} + \lambda \int_0^\infty f(x + \beta y) \, \mathrm{d}G(y) - (\delta + \lambda) f(x)}{c(\beta)} .$$

Because the expression to be minimised is continuous in $(\beta, x) \in [b_0 + \varepsilon, 1] \times [0, d)$ and $(\beta, x) \in [b_0 + \varepsilon, 1] \times (d, \infty)$ and $[b_0 + \varepsilon, 1]$ is compact, $f'(x)$ is continuous for $x \in (\underline{a}, \overline{a})$ if $(\underline{a}, \overline{a}) \subset [0, d]$ or $(\underline{a}, \overline{a}) \subset (d, \infty)$. This holds for all $\varepsilon > 0$, so that we obtain the same statement for all intervals $(\underline{a}, \overline{a}) \subset [0, d]$ or $(\underline{a}, \overline{a}) \subset (d, \infty)$ with $b^*(x) > b_0$ for all $x \in (\underline{a}, \overline{a})$. If, in an application, additionally the pointwise minimiser is unique on $(\underline{a}, \overline{a})$, one can conclude from the continuity of $f'$ that $b^*$ is also continuous on this interval. However, we note that Proposition 2.14 does not require a unique minimiser. #

Our next step is to compare these feedback strategies to general admissible strategies. With Lemma 2.1 and 2.7, we have already seen a method to connect return functions to certain strategies by using martingales. However, the corresponding result relied on the properties of piecewise deterministic Markov processes and, in particular, the feedback structure of the strategy. For a general control process $B \in \mathcal{B}$, this is not possible. Therefore, we have to use a different result. The following Lemma applies to arbitrary admissible controls.

Lemma 2.15. *Let $f : [0, \infty) \to \mathbb{R}$ be a bounded, absolutely continuous and increasing solution to* (2.21) *with density $f'$. For every strategy $B \in \mathcal{B}$, the process*

$$\Big( \mathrm{e}^{-\delta t} f(\Delta_t^B) - f(\Delta_0^B) - \int_0^t \mathrm{e}^{-\delta s} f'(0) \, \mathrm{d}M_s^B - \int_0^t \mathrm{e}^{-\delta s} \mathcal{A}^{B_s} f(\Delta_s^B) \, \mathrm{d}s \Big)_{t \geq 0}$$

*is a martingale.*

This Lemma is a consequence of the martingale representation theorem in [Jacobsen, 2006, Thm. 4.6.1]. Details are found in the appendix, p. 112. In particular, Lemma 2.15 enables us to prove the following result.

Theorem 2.16 (Verification for the Minimal Expected Time in Drawdown). [2021b]
*Let $f : [0, \infty) \to \mathbb{R}$ be a bounded, absolutely continuous and increasing solution to* (2.21) *with density $f'$ and denote by $b^*$ the pointwise minimiser, measurably chosen as in Proposition* 2.14. *If either $b^*(0) = b_0$ or $f'(0) = 0$, we have $f(x) = v^{B^*}(x) = v(x)$ for all $x \geq 0$. That is, $f$ is the minimal expected time with critical drawdown and the associated feedback strategy $B^*$ is an optimal strategy.*

*Proof.* Because $f$ solves (2.21), we have $\mathcal{A}^b f(x) \geq -\mathbb{1}_{\{x>d\}}$ for all $x \geq 0$ and $b \in [b_0, 1]$. Let $B \in \mathcal{B}$ be an arbitrary admissible strategy. By $f'(0) \geq 0$, we have

$$\int_0^t \mathrm{e}^{-\delta s} f'(0)\,\mathrm{d}M_s^B + \int_0^t \mathrm{e}^{-\delta s} \mathcal{A}^{B_s} f(\Delta_s^B)\,\mathrm{d}s \geq -\int_0^t \mathrm{e}^{-\delta s} \mathbb{1}_{\{\Delta_s^B > d\}}\,\mathrm{d}s \tag{2.22}$$

for all $t \geq 0$. Hence, Lemma 2.15 implies

$$f(x) \leq \mathbb{E}^x\big[\mathrm{e}^{-\delta t} f(\Delta_t^B)\big] + \mathbb{E}^x\Big[\int_0^t \mathrm{e}^{-\delta s} \mathbb{1}_{\{\Delta_s^B > d\}}\,\mathrm{d}s\Big]\,. \tag{2.23}$$

The first term goes to zero, by boundedness of $f$, as $t \to \infty$. The second term converges to $v^B(x)$ by monotone convergence. Therefore, $f(x) \leq v(x)$. By Proposition 2.14, the drawdown $\Delta^*$ under control $B^*$ is a piecewise deterministic Markov process. Similarly as in the proof of Theorem 4.6, the initial conditions imply that either, the running maximum of the surplus following this strategy is kept constant, or, that $f'(0) = 0$. Thus, the integral with respect to the running maximum is equal to zero in both cases. Since additionally $\mathcal{A}^{b^*(x)} f(x) = -\mathbb{1}_{\{x>d\}}$ is fulfilled, repeating the above argument for the strategy $B^*$ yields equality in (2.22) and (2.23). Taking the limit $t \to \infty$ proves $f(x) = v^{B^*}(x)$. Because $B^*$ is an admissible strategy with $v^{B^*}(x) \leq v(x)$, we conclude optimality of this strategy, that is, $v^{B^*}(x) = v(x)$. $\square$

Our goal for the rest of this section is to prove that there is, indeed, a function which solves the equation.

### 2.2.2 Direct Approach to Minimising the Recovery Time

We denote by $V : (d, \infty) \to [0, 1]$ the maximal Laplace transform of the time until the drawdown is uncritical, that is,

$$V(x) = \sup_{B \in \mathcal{B}} \mathbb{E}^x\big[\mathrm{e}^{-\delta \vartheta(B)}\big]\,, \qquad x > d\,. \tag{2.24}$$

By the dynamic programming principle, the 'upper part' of our original problem is equivalent to solving this subproblem if $v(d)$ is known. We note that the set $\mathcal{B}$ in (2.24) is the same set of strategies we consider for the expected time with critical drawdown; strategies which coincide until the first passage of $d$ can be identified with one another. We derive a candidate solution by the following heuristics. With our verification theorem in mind, we expect that the optimal strategy is of feedback form. For a starting point $x$, this strategy influences drift and claim sizes such that the process travels as fast as possible from $x$ to $d$. Intuitively, it can therefore only depend on the length $x - d$ of the interval. In order to reach $d$, the process has to pass all levels $y \in (d, x)$. For a fixed $y$, the optimal strategy for initial drawdown $x$ must be a composition of the strategy with the earliest possible arrival at $y$ (quickly decreasing by $x - y$) and the respective optimal strategy for the new starting point $y$ (quickly decreasing by $y - d$). This indicates that the function $V$ should have an exponential structure: $V(x) = V(d + x - y)V(y)$ for $y \in (d, x)$. Additionally, if we split the interval into $2^n$ parts of equal length, we observe that the problem corresponds to $V(d + (x - d)2^{-n})$ in each of the subintervals. This holds for arbitrary $n \in \mathbb{N}$, so that the optimal strategy has to be constant. Thus we can use the results obtained for simple switching strategies, in particular, Lemma 2.3. In the following, we write again $\gamma = \gamma(1)$ for the positive solution to $\Psi_1(\gamma) = \delta$.

Proposition 2.17. [2021b]
*We have $V(x) = \mathrm{e}^{-\gamma(x-d)}$ for $x > d$ and a strategy which is constant and equal to 1 up to the first passage through d is optimal. The expected time in critical drawdown fulfils*

$$v(x) = \frac{1}{\delta} - \Big(\frac{1}{\delta} - v(d)\Big) \cdot \mathrm{e}^{-\gamma(x-d)}, \qquad x > d,$$

*and $\inf_{b\in[0,1]} \mathcal{A}^b v(x) = -1$ with the pointwise optimiser $b^*(x) = 1$ for all $x > d$.*

*Proof.* By the above heuristics, we are looking for a strategy $B$ which is constant for $t < \vartheta_d(B)$. We can exclude the strategy equal to $b_0$, because the drawdown will never enter the uncritical area under this strategy. We know, by Lemma 2.3, that a strategy $B$ with $B_t = b > b_0$ for $t < \vartheta_d(B)$ fulfils $\mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta_d(B)}\big] = \mathrm{e}^{-\gamma(b)(x-d)}$ for $x > d$. By Lemma 2.3 ii), $\gamma(b)$ is minimal for $b = 1$, so that $\mathrm{e}^{-\gamma(b)(x-d)}$ is maximised for $b = 1$. Thus, we expect that $f(x) = \mathrm{e}^{-\gamma(x-d)}$ corresponds to $V(x)$ for $x > d$. This function fulfils, for all $b \in [b_0, 1]$,

$$\mathcal{A}^b f(x) = -(\delta - \Psi_b(\gamma))\mathrm{e}^{-\gamma(x-d)} \leq 0, \qquad x > d, \tag{2.25}$$

because $b \mapsto \Psi_b(\gamma)$ is increasing in $b$ and $\Psi_1(\gamma) = \delta$. For $b = 1$, equality holds. By Lemma 2.15 and the optional stopping theorem, we get that

$$\Big(\mathrm{e}^{-\delta(t\wedge\vartheta_d(B))} f(\Delta^B_{t\wedge\vartheta_d(B)}) - f(\Delta^B_0) - \int_0^{t\wedge\vartheta_d(B)} \mathrm{e}^{-\delta s}\mathcal{A}^{B_s} f(\Delta^B_s)\,\mathrm{d}s\Big)_{t\geq 0}$$

is a martingale for all $B \in \mathcal{B}$. By taking expectations, we obtain

$$f(x) \geq \mathbb{E}^x\big[\mathrm{e}^{-\delta(t\wedge\vartheta_d(B))} f(\Delta^B_{t\wedge\vartheta_d(B)})\big]$$

for an arbitrary strategy $B$, with equality for all strategies with $B_t = 1$ for all $t < \vartheta_d(B)$. By $\Delta^B_{\vartheta_d(B)} = d$ on the set $\{\vartheta_d(B) < \infty\}$ and bounded convergence as $t \to \infty$, we find $f(x) \geq V^B(x)$ for the return of arbitrary strategies $B \in \mathcal{B}$. From plugging in a strategy with $B_t = 1$ for all $t < \vartheta_d(B)$, we obtain $f(x) = V(x)$. The assertion for $v$ now follows from the dynamic programming theorem and (2.25). $\square$

This means that we have now found the first part of a candidate solution. As with simple switching strategies, the case of small initial drawdown bears more difficulties.

### 2.2.3 Maximising the Time to Critical Drawdown with a Penalty

Considering the dynamic programming equation and the results of the previous section, we expect:

$$v(x) = \inf_{B\in\mathcal{B}} \mathbb{E}^x\Big[\mathrm{e}^{-\delta\vartheta(B)}\Big(\frac{1}{\delta} - \Big(\frac{1}{\delta} - v(d)\Big)\mathrm{e}^{-\gamma(\Delta^B_{\vartheta(B)}-d)}\Big)\Big], \qquad x \leq d.$$

Again, the process under consideration is reflected on the $x$-axis and enters the critical area with a jump. This means that we cannot use the same method as for critical initial drawdown. In Section 2.1, $u(d)$ could be calculated because we knew the strategy for uncritical initial drawdown. Now, $v(d)$ is unknown because it depends on the (also unknown) strategy below $d$. To overcome this additional difficulty, our approach in this section is to define a general optimisation problem with a discounted penalty function: for $C \in [0, \delta^{-1}]$, we let

$$v_C(x) = \inf_{B \in \mathcal{B}} \mathbb{E}^x \Big[ \mathrm{e}^{-\delta \vartheta(B)} p_C(\Delta^B_{\vartheta(B)}) \Big] , \qquad x \le d , \tag{2.26}$$

where $p_C : [d, \infty) \to [0, \delta^{-1}]$ is given by

$$p_C(x) = \frac{1}{\delta} - \Big(\frac{1}{\delta} - C\Big) \mathrm{e}^{-\gamma(x-d)} . \tag{2.27}$$

The function $p_C$ can be interpreted as a penalty for the overshoot at the exit time. We first notice that Lemma 2.12 (with a modified Lipschitz constant) and Lemma 2.13 remain true for the function $v_C$ on the interval $[0, d]$. That is, $v_C$ is increasing, Lipschitz continuous, differentiable almost everywhere and there exist universally $\varepsilon$-optimal strategies. We note that $p_C$ fulfils $\inf_{b \in [b_0,1]} \mathcal{A}^b p_C(x) = -1$ with $p_C(d) = C$ for $x \ge d$.

Remark. The discounted penalty function $x \mapsto \mathbb{E}^x[\mathrm{e}^{-\delta\vartheta} p_C(\Delta_\vartheta)]$ is related to a Gerber–Shiu function. To see this parallel, we consider the original process $X$ (starting at $y = d - x$) under a dividend barrier strategy with barrier $d$. Then, if $\tau$ denotes the ruin time of the ex-dividend process $U$, our problem is equivalent to minimising $\mathbb{E}^y[\mathrm{e}^{-\delta\tau}(\delta^{-1} - (\delta^{-1} - C)\mathrm{e}^{\gamma U_\tau})]$ for all initial capitals $y \in [0, d]$. We further examine this relation in Chapter 4. #

The goal of this section is to show that $v_C$ is the unique solution to a modified version of Equation (2.21) and that there exists a 'correct' constant $C_d \in [0, \delta^{-1}]$ with $v(x) = v_{C_d}(x)$ for all $x \le d$. In the modified equation, $\mathcal{A}^b f(x)$ defined in (2.4) is replaced by

$$\begin{aligned} \mathcal{A}^b_C f(x) = -(\delta + \lambda) f(x) - c(b) f'(x) + \lambda \int_0^{(d-x)/b} f(x + by) \,\mathrm{d}G(y) \\ + \lambda \int_{(d-x)/b}^{\infty} p_C(x + by) \,\mathrm{d}G(y) . \end{aligned} \tag{2.28}$$

In particular, for $f = v_C$, this expression coincides with $\mathcal{A}^b w_C(x)$, where

$$w_C(x) = v_C(x) \mathbb{1}_{\{x \le d\}} + p_C(x) \mathbb{1}_{\{x > d\}} , \qquad x \ge 0 ,$$

denotes the composition of the functions $v_C$ and $p_C$. By repeating the steps of the proofs of Proposition 2.14 and Theorem 2.16, it is possible to derive the analogue result for $v_C$.

Theorem 2.18 (Verification for $v_C$). [2021b]

*Let $f : [0, d] \to \mathbb{R}$ be a bounded, absolutely continuous solution to $\inf_{b \in [b_0,1]} \mathcal{A}^b_C f(x) = 0$ with density $f'$ and $f'(0) \ge 0$. We denote by $b^*$ the pointwise minimiser, measurably chosen as in the proof* *of Proposition 2.14. Let $B^* \in \mathcal{B}$ denote a corresponding feedback strategy with $B^*_t = b^*(\Delta^*_t)$ for $t \in [0, \vartheta^d(B^*)]$. If either $b^*(0) = b_0$ or $f'(0) = 0$, we have $f(x) = v^{B^*}_C(x) = v_C(x)$ for $x \in [0, d]$. That is, $f$ coincides with the function $v_C$ and $B^*$ is an optimal strategy.* □

The only thing that has to be modified in the proof is that the process is stopped at the first exit time for Theorem 2.18.

For general $C \in [0, \delta^{-1}]$, the composition $w_C$ defined on $[0, \infty)$ is not necessarily continuous from the right at $x = d$ because the first exit from the uncritical area (even when starting in $d$) happens at a jump time. However, for the unknown $v(d)$, we do expect continuity by $p_{v(d)} = v(x)$ for $x > d$ and $v_{v(d)}(x) = v(x)$ for $x \le d$. The following lemma implies that there exists at least one constant $C_d$ such that the composition of the functions $v_C$ and $p_C$ is continuous in $d$.

Lemma 2.19. [2021b]

*There exists $C_d \in (0, \delta^{-1})$ with $v_{C_d}(d) = C_d$, $v_C(d) \ge C$ for all $C \le C_d$ and $v_C(d) \le C$ for all $C \ge C_d$. For all $C \in [C_d, \delta^{-1}]$, the function $w_C(x) = v_C(x)\mathbb{1}_{\{x \le d\}} + p_C(x)\mathbb{1}_{\{x > d\}}$ is increasing in $x$.*

*Proof.* We prove the stronger statement that for every $x \in [0, d]$ there exists $C \in (0, \delta^{-1})$ such that $v_C(x) - C = 0$. We start by showing that $C \mapsto v_C(x) - C$ is a continuous and decreasing function of $C$. We consider two constants $C^{(1)}, C^{(2)} \in [0, \delta^{-1}]$ with $C^{(1)} < C^{(2)}$. For all strategies $B \in \mathcal{B}$, it holds

$$v^B_{C^{(1)}}(x) - C^{(1)} - (v^B_{C^{(2)}}(x) - C^{(2)}) = \Big(1 - \mathbb{E}^x\Big[\mathrm{e}^{-\delta\vartheta^d(B)}\mathrm{e}^{-\gamma(\Delta^B_{\vartheta^d(B)} - d)}\Big]\Big) \cdot (C^{(2)} - C^{(1)}) \ge 0\,.$$

This means that $C \mapsto v^B_C(x) - C$, $B \in \mathcal{B}$, is decreasing. This implies that $C \mapsto v_C(x) - C$ is also decreasing. Moreover, it follows from this equation that the functions $C \mapsto v^B_C(x) - C$ for $B \in \mathcal{B}$ are Lipschitz continuous with a common Lipschitz constant $L \le 1$. Hence, $C \mapsto v_C(x) - C$ is also Lipschitz continuous: for $C^{(1)} < C^{(2)}$, we consider an $\varepsilon$-optimal strategy $B^{\varepsilon,2}$ for $v_{C^{(2)}}$ with $v^{B^{\varepsilon,2}}_{C^{(2)}}(x) < v_{C^{(2)}}(x) + \varepsilon$. Then:

$$\begin{aligned} 0 \le v_{C^{(1)}}(x) - C^{(1)} - (v_{C^{(2)}}(x) - C^{(2)}) &\le v^{B^{\varepsilon,2}}_{C^{(1)}}(x) - C^{(1)} - (v^{B^{\varepsilon,2}}_{C^{(2)}}(x) - C^{(2)}) + \varepsilon \\ &\le L(C^{(2)} - C^{(1)}) + \varepsilon\,, \end{aligned}$$

so, letting $\varepsilon \to 0$, we derive Lipschitz continuity of $C \mapsto v_C(x) - C$. Now we plug in the boundary values $C = 0$ and $C = \delta^{-1}$. For $C = 0$, we have

$$v_0(x) - 0 = \frac{1}{\delta} \inf_{B \in \mathcal{B}} \mathbb{E}^x\Big[\mathrm{e}^{-\delta\vartheta^d(B)}\Big(1 - \mathrm{e}^{-\gamma(\Delta^B_{\vartheta^d(B)} - d)}\Big)\Big] > 0\,.$$

The strict inequality holds because we have $\Delta^B_{\vartheta^d(B)} \in (d, \infty)$ and $\vartheta^d(B) < \infty$ almost surely. The latter follows because even for the smallest possible claim sizes $b_0 Y_k$ and the largest possible premium $c(1)$, the 'exit event' $A_1 = \{\sum_{k=1}^{N_1} b_0 Y_k > d + c(1)\}$ has a strictly positive probability. By the Borel–Cantelli Lemma, therefore, $A_n = \{\sum_{k=N_{n-1}}^{N_n} b_0 Y_k > d + c(1)\}$ happens infinitely often, meaning that the interval will be exited in finite time, almost surely. For $C = \delta^{-1}$, the dependence on the overshoot disappears, such that we obtain

$$v_{1/\delta}(x) - \frac{1}{\delta} = \frac{1}{\delta}\left(\inf_{B\in\mathcal{B}} \mathbb{E}^x\left[\mathrm{e}^{-\delta\vartheta^d(B)}\right] - 1\right) < 0\,.$$

The inequality holds because the earliest exit is at the first jump: $\vartheta^d(B) \geq T_1$. By the intermediate value theorem, there exists a constant $C_x \in (0, \delta^{-1})$, such that $v_{C_x}(x) - C_x = 0$. Because $C \mapsto v_C(x) - C$ is decreasing, $v_C(x) - C \geq 0$ for $C \leq C_x$ and $v_C(x) - C \leq 0$ for $C \geq C_x$. In particular, this is true for $x = d$. The function $w_C$ is increasing on $[0, d]$ and on $(d, \infty)$, separately, for all $C \in [0, \delta^{-1}]$. By $C \geq C_d$, we additionally obtain $p_C(d) \geq v_C(d)$, from which the assertion follows. □

Remark. Lemma 2.19 also indicates that if a set of functions $v_C$ is known, one can determine the candidate solution to our original problem and the corresponding value $v(d)$ via trial and error: if $v_C(d) > C$, $C$ has been chosen too small. On the other hand, $v_C(d) < C$ implies that $C$ is too large. #

Next we show that it holds $\inf_{b\in[b_0,1]} \mathcal{A}_C^b v_C(x) = 0$ for a density $v_C'$ of $v_C$. We note that it can be assumed (without loss of generality) that every considered version of the density $v_C'$ is bounded and non-negative on $[0, d]$ because $v_C$ is Lipschitz continuous and increasing. Furthermore, $v_C$ is differentiable (i.e. upper and lower, left and right derivatives coincide) outside of a Lebesgue null set, so that we can assume that $v_C'$ corresponds to the derivative at all points outside of this set. Our three main steps (represented by Propositions 2.20, 2.21 and Theorem 2.22, below) are: firstly, deriving an initial condition at $x = 0$, secondly, showing $\inf_{b\in[b_0,1]} \mathcal{A}_C^b v_C(x) \geq 0$ and, lastly, proving the converse inequality.

Proposition 2.20. [2021b]

*The function $v_C$ fulfils the initial condition*

$$v_C(0) = \inf_{b\in[b_0,1]} \frac{\lambda}{\lambda+\delta} \int_0^\infty w_C(by)\, \mathrm{d}G(y)\,. \tag{2.29}$$

*If $C \geq C_d$, the infimum is attained at $b = b_0$.*

We shift the proof to the appendix, p. 113, and give an intuitive explanation here. Because the drift of the drawdown vanishes at zero, the process stays at this point until the first claim occurs. That means, the function $v_C$ corresponds to the value after the first jump, discounted by $\mathrm{e}^{-\delta T_1}$, where $T_1$ denotes the first claim time. The prefactor $\lambda/(\delta+\lambda)$ is generated by the Laplace transform of the $\exp(\lambda)$-distributed time $T_1$ which is independent of the claim size. In the case $C \geq C_d$, the right hand side is increasing in $b$, meaning that the minimiser is the value leading to the smallest possible claim payment, $b = b_0$. In the case $C < C_d$, the penalty for exiting is small as long as the overshoot is not too large. In particular, an early large jump from $x = 0$ into the area above (but close by) $d$ could be favoured over a postponed exit with a worse pre-exit position than $x = 0$.

Proposition 2.21. [2021b]

i) *At all $x \in [0, d]$ for which an one-sided derivative $v_C'$ of $v_C$ exists or the infimum is attained at $b = b_0$, we have $\inf_{b\in[b_0,1]} \mathcal{A}_C^b v_C(x) \geq 0$. In particular, the set $\mathcal{N}_1 = \{x \in [0, d] : \inf_{b\in[b_0,1]} \mathcal{A}^b v_C(x) < 0\}$ has Lebesgue measure zero.*

ii) *If the right derivative $v_C'(0)$ exists at $x = 0$ or if the infimum is attained at $b = b_0$, we have $\inf_{b\in[b_0,1]} \mathcal{A}_C^b v_C(0) = 0$. If the right derivative exists and the infimum is not attained at $b = b_0$, it fulfils $v_C'(0) = 0$.*

We recall that, in the proof of Lemma 2.6, we conditioned on the time of the first claim to connect the return function of a simple switching strategy with an integro-differential equation. Now, for the value function, the intuition of the following proof of Proposition 2.21 is to use a similar approach in combination with $\varepsilon$-optimal strategies.

*Proof of Proposition* 2.21. We start with i). We consider $x \in (0, d]$ and $h \le x/c(1)$, so that zero is not reached by the drawdown process before time $T_1 \wedge h$. We choose $b \in [b_0, 1]$ and consider the strategy $B$ defined by $B_t = b\mathbb{1}_{\{t<T_1\wedge h\}} + \tilde{B}_t^\varepsilon \mathbb{1}_{\{t\ge T_1\wedge h\}}$ for $t \ge 0$, provided that $\Delta^B_{T_1\wedge h} \le d$. Here, $\tilde{B}^\varepsilon$ denotes a universally $\varepsilon$-optimal strategy (as in Lemma 2.13) shifted by the time $T_1 \wedge h$. By distinguishing the cases of what happens at time $T_1 \wedge h$ (that is, *a*) no claim occurs, *b*) a small claim occurs and the drawdown stays uncritical and *c*) a large claim occurs, the drawdown exits and the overshoot is penalised), we find:

$$
\begin{aligned}
v_C(x) \le v_C^B(x) &= \mathbb{E}^x\big[\mathrm{e}^{-\delta h} v_C^{B^\varepsilon}(x - c(b)h)\mathbb{1}_{\{T_1>h\}}\big] + \mathbb{E}^x\big[\mathrm{e}^{-\delta T_1} v_C^{B^\varepsilon}(\Delta^B_{T_1})\mathbb{1}_{\{T_1\le h\}}\mathbb{1}_{\{\Delta^B_{T_1}\le d\}}\big] \\
&\quad + \mathbb{E}^x\big[\mathrm{e}^{-\delta T_1} p_C(\Delta^B_{T_1})\mathbb{1}_{\{T_1\le h\}}\mathbb{1}_{\{\Delta^B_{T_1}>d\}}\big] \\
&\le \mathrm{e}^{-(\delta+\lambda)h} v_C(x - c(b)h) + \mathbb{E}^x\big[\mathrm{e}^{-\delta T_1} w_C(\Delta^B_{T_1})\mathbb{1}_{\{T_1\le h\}}\big] + \varepsilon\,.
\end{aligned}
$$

Letting $\varepsilon \to 0$ yields

$$
v_C(x) \le \mathrm{e}^{-(\delta+\lambda)h} v_C(x - c(b)h) + \mathbb{E}^x\big[\mathrm{e}^{-\delta T_1} w_C(\Delta^B_{T_1})\mathbb{1}_{\{T_1\le h\}}\big]\,. \tag{2.30}
$$

We first assume $b = b_0$. In this case, we have $c(b) = 0$ and

$$
0 \le (\mathrm{e}^{-(\delta+\lambda)h} - 1)v_C(x) + \mathbb{E}\big[\mathrm{e}^{-\delta T_1} w_C(x + b_0Y_1)\mathbb{1}_{\{T_1\le h\}}\big]\,,
$$

so by dividing by $h$ and letting $h \to 0$, we obtain

$$
0 \le -(\delta + \lambda)v_C(x) + \lambda\,\mathbb{E}\big[w_C(x + b_0Y_1)\big]\,.
$$

By $c(b_0) = 0$, we conclude $\mathcal{A}^{b_0} v_C(x) \ge 0$. Next, we assume $b > b_0$, such that it holds $c(b) > 0$. Equation (2.30) is equivalent to

$$
\begin{aligned}
0 \le &- c(b) \cdot \frac{v_C(x) - v_C(x - c(b)h)}{c(b)h} + \frac{\mathrm{e}^{-(\delta+\lambda)h} - 1}{h} \cdot v_C(x - c(b)h) \\
&+ h^{-1} \cdot \mathbb{E}\big[\mathrm{e}^{-\delta T_1} w_C(x - c(b)T_1 + bY_1)\mathbb{1}_{\{T_1\le h\}}\big]\,.
\end{aligned}
$$

The second and third terms on the right hand side converge and the inequality implies that the difference quotient is bounded on one side. Thus, by choosing an appropriate subsequence $(h_n)_{n\in\mathbb{N}} \subset$

$(0,h)$ with $h_n \to 0$ as $n \to \infty$ such that the limit below exists, we find

$$0 \leq -\, c(b) \cdot \lim_{n\to\infty} \frac{v_C(x) - v_C(x - c(b)h_n)}{c(b)h_n} - (\delta+\lambda)v_C(x) + \lambda\, \mathbb{E}\big[w_C(x + bY_1)\big]\,.$$

This shows the inequality for the derivative from the left. Repeating the above argument for $\bar{x} = x + c(b)h$ with $x \in [0,d)$, we arrive at

$$0 \leq -\, c(b) \cdot \lim_{n\to\infty} \frac{v_C(x + c(b)h_n) - v_C(x)}{c(b)h_n} - (\delta+\lambda)v_C(x) + \lambda\, \mathbb{E}\big[w_C(x + bY_1)\big]\,, \tag{2.31}$$

again, for an appropriate subsequence $(h_n)_{n\in\mathbb{N}}$. This proves the assertion for the derivative from the right. Thus, for all $x$ at which $v_C$ is differentiable, we showed $\inf_{b\in[b_0,1]} \mathcal{A}_C^b v_C(x) \geq 0$. $\mathcal{N}_1$ is a subset of the Lebesgue null set on which $v_C$ is not differentiable. Because we have $-c(b) \leq 0$, (2.31) in combination with (2.29) implies ii). $\square$

Theorem 2.22. [2021b]
*The set* $\mathcal{N}_2 = \{x \in [0,d] : \inf_{b\in[b_0,1]} \mathcal{A}_C^b v_C(x) > 0\}$ *has Lebesgue measure zero. The function* $v_C$ *solves the Hamilton–Jacobi–Bellman equation* $\inf_{b\in[b_0,1]} \mathcal{A}_C^b v_C(x) = 0$ *for Lebesgue almost all* $x \in [0,d]$ *and fulfils the initial condition in* (2.29). *In particular, a version of the density fulfils* $\inf_{b\in[b_0,1]} \mathcal{A}_C^b v_C(x) = 0$ *for all* $x \in [0,d]$.

*Proof.* By Propositions 2.20 and 2.21, it suffices to prove the assertion for $x > 0$. For $x \in (0,d]$ we consider $h \leq x/c(1)$. We choose a strategy $B^h$ such that $v_C^{B^h}(x) < v_C(x) + h^2$. We recall that $B^h$, as an admissible strategy, is adapted to the minimal right-continuous filtration such that $X$ is adapted. This means that $t \mapsto B_t^h$ must be deterministic as long as no claim occurs. We write $R_t^{B^h} = x - \int_0^t c(B_s^h)\,\mathrm{d}s \leq x$ for the (as a consequence, also deterministic) path of $\Delta^{B^h}$ on the set $\{T_1 > h\}$. Applying the arguments of the proof of Proposition 2.21 to the strategy $B^h$, stopping at $T_1 \wedge h$ yields:

$$\begin{aligned} v_C(x) > v_C^{B^h}(x) - h^2 &= \mathrm{e}^{-(\delta+\lambda)h} v_C^{B^h}(R_h^{B^h}) + \mathbb{E}^x\big[\mathrm{e}^{-\delta T_1} w_C^{\tilde{B}^h}(\Delta_{T_1}^{B^h})\mathbb{1}_{\{T_1\leq h\}}\big] - h^2 \\ &\geq \mathrm{e}^{-(\delta+\lambda)h} v_C(R_h^{B^h}) + \mathbb{E}^x\big[\mathrm{e}^{-\delta T_1} w_C(\Delta_{T_1}^{B^h})\mathbb{1}_{\{T_1\leq h\}}\big] - h^2\,, \end{aligned} \tag{2.32}$$

where $\tilde{B}^h$ denotes the shifted strategy $B^h$. We first assume that there exists a subsequence $(h_n)_{n\in\mathbb{N}} \subset (0,h)$, $h_n \to 0$, $n \to \infty$, for which it holds $R_{h_n}^{B^{h_n}} = x$ for all sufficiently large $n$. In particular, this is equivalent to $B_t^{h_n} = b_0$ almost everywhere in $(0,h_n)$. Because $t \mapsto B_t^{h_n}$, is right continuous, we thus conclude $B_t^{h_n} = b_0$ for all $t \in (0,h_n)$. In this case, we have by (2.32):

$$0 \geq (\mathrm{e}^{-(\delta+\lambda)h_n} - 1)v_C(x) + \mathbb{E}\big[\mathrm{e}^{-\delta T_1} w_C(x + b_0Y_1)\mathbb{1}_{\{T_1\leq h_n\}}\big] - h_n^2\,.$$

Thus, letting $n \to \infty$ proves

$$0 \geq -(\delta+\lambda)v_C(x) + \lambda\, \mathbb{E}\big[w_C(x + b_0Y_1)\big]\,, \tag{2.33}$$

which corresponds to $\mathcal{A}_C^{b_0} v_C(x) \leq 0$, by $c(b_0) = 0$. If, on the other hand, there exists no such sequence, then, for all $(h_n)_{n\in\mathbb{N}} \subset (0, h)$, there exists for all $n_0 \in \mathbb{N}$ a $n \geq n_0$ such that $R_{h_n}^{B^{h_n}} < x$. In this case, we can choose a subsequence $(\tilde{h}_n)_{n\in\mathbb{N}} \subset (h_n)_{n\in\mathbb{N}}$ with $R_{\tilde{h}_n}^{B^{\tilde{h}_n}} < x$ for all $n$ and $\tilde{h}_n \to 0$ as $n \to \infty$. Now we can rewrite (2.32):

$$\begin{aligned} 0 \geq &-\frac{x - R_{\tilde{h}_n}^{B^{\tilde{h}_n}}}{\tilde{h}_n} \cdot \frac{v_C(x) - v_C(R_{\tilde{h}_n}^{B^{\tilde{h}_n}})}{x - R_{\tilde{h}_n}^{B^{\tilde{h}_n}}} \\ &+ \frac{\mathrm{e}^{-(\delta+\lambda)\tilde{h}_n} - 1}{\tilde{h}_n} v_C(R_{\tilde{h}_n}^{B^{\tilde{h}_n}}) + \tilde{h}_n^{-1}\,\mathbb{E}^x\big[\mathrm{e}^{-\delta T_1} w_C(\Delta_{T_1}^B) \mathbb{1}_{\{T_1 \leq \tilde{h}_n\}}\big] - \tilde{h}_n\,. \end{aligned} \tag{2.34}$$

Since $c(b)$ is bounded for $b \in [b_0, 1]$, we can assume (without loss of generality) that

$$\tilde{h}_n^{-1}(x - R_{\tilde{h}_n}^{B^{\tilde{h}_n}}) = \tilde{h}_n^{-1} \int_0^{\tilde{h}_n} c(B_s^{\tilde{h}_n})\,\mathrm{d}s$$

converges. Otherwise, we can choose an appropriate subsequence. In particular, the limit can be written as $c(\tilde{b})$ for some $\tilde{b} \in [b_0, 1]$. Then, the expressions $\tilde{h}_n^{-1} \int_0^{\tilde{h}_n} B_s^{\tilde{h}_n}\,\mathrm{d}s$ and $B_t^{\tilde{h}_n} \mathbb{1}_{\{t \leq \tilde{h}_n\}}$ converge to $\tilde{b}$ as $n \to \infty$. This means that the term with the expected value converges as well, by Theorem IV.5.6 of [Elstrodt, 2011]. Therefore, we obtain:

$$0 \geq -c(\tilde{b}) \cdot \liminf_{n\to\infty} \frac{v_C(x) - v_C(R_{\tilde{h}_n}^{B^{\tilde{h}_n}})}{x - R_{\tilde{h}_n}^{B^{\tilde{h}_n}}} - (\delta + \lambda) v_C(x) + \lambda\,\mathbb{E}\big[w_C(x + \tilde{b} Y_1)\big]\,.$$

for $\tilde{b} > b_0$ and (2.33) for $\tilde{b} = b_0$. In particular, for $\tilde{b} > b_0$, this limit corresponds to the derivative from the left (if it exists at $x$). Thus, we have $\mathcal{A}_C^{\tilde{b}} v_C(x) \leq 0$. That means, $\mathcal{N}_2$ is a Lebesgue null set. By Proposition 2.21, we obtain the assertion for all $x$ in $[0, d] \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$, that is, almost everywhere. Now, on the Lebesgue (measurable) null set $\mathcal{N}_1 \cup \mathcal{N}_2$, we can modify the density such that the equation is fulfilled. We note that it holds $-(\delta + \lambda) v_C(x) + \lambda \int_0^\infty v_C(x + b_0 y)\,\mathrm{d}G(y) > 0$ on this set (by $\mathcal{A}^{b_0} v_C(x) > 0$). Additionally, for every $x \in \mathcal{N}_1 \cup \mathcal{N}_2$, there exists a sequence $(x^{(n)})_{n\in\mathbb{N}}$ which converges to $x$ and such that $v_C$ is differentiable and the equation is fulfilled at $x^{(n)}$. Let $\tilde{b}^{(n)}$ denote the respective minimiser for $x^{(n)}$. By choosing an appropriate subsequence, we can assume that $(\tilde{b}^{(n)})_{n\in\mathbb{N}}$ converges to a limit $\tilde{b} \in (b_0, 1]$ and that it holds $\tilde{b}^{(n)} \in (b_0, 1]$ for all $n$. Now we can define the density $v_C'(x)$ by the equation $\mathcal{A}_C^{\tilde{b}} v_C(x) = 0$. In particular, we also have $\lim_{n\to\infty} v_C'(x^{(n)}) = v_C'(x)$, so that $\mathcal{A}_C^b v_C(x) \geq 0$ by $\mathcal{A}_C^b v_C(x^{(n)}) \geq 0$ for all $b \in [b_0, 1]$. □

Theorems 2.18 and 2.22 together imply that $v_C$ is the unique solution to the Hamilton–Jacobi–Bellman equation $\inf_{b\in[b_0,1]} \mathcal{A}_C^b v_C(x) = 0$ (with certain properties) and that the optimal strategy can be obtained from the equation. We have thus solved the optimisation problem posed in Equation (2.26) for arbitrary $C \in [0, \delta^{-1}]$.

### 2.2.4 Existence by Reconnecting the Subproblems

We are now ready to combine our findings and derive existence of a solution to the Hamilton–Jacobi–Bellman equation given in (2.21). We conclude from Lemma 2.19, Proposition 2.20 and Theorem 2.22

that there exists a constant $C_d \in (0, \delta^{-1})$ such that the composition of the functions $v_{C_d}$ and $p_{C_d}$ is absolutely continuous and solves Equation (2.21) with the required initial condition. Therefore, it follows by Theorem 2.16 that the constant $C_d$ must be unique and that the composition is the expected time with critical drawdown.

Corollary 2.23. [2021b]

*There exists a unique $C_d \in (0, \delta^{-1})$ with $v_{C_d}(d) = p_{C_d}(d)$. The expected time in critical drawdown fulfils*

$$v(x) = \begin{cases} \inf\limits_{B \in \mathfrak{B}} \mathbb{E}^x\left[\mathrm{e}^{-\delta\vartheta(B)} p_{C_d}(\Delta^B_{\vartheta(B)})\right], & x \in [0, d], \\ \delta^{-1} - (\delta^{-1} - C_d)\mathrm{e}^{-\gamma(x-d)}, & x > d. \end{cases}$$

*Restricted to the set $[0, d]$, $v$ is the unique bounded solution to $\inf_b \mathcal{A}^b_{C_d} v(x) = 0$, $x \in [0, d]$, with $v'(0) \geq 0$ and the initial condition given in (2.29) for $C = C_d$. We denote by $b^* : [0, d] \to [b_0, 1]$ the pointwise minimiser of this equation (as in Theorem 2.18). It holds $b^*(0) = b_0$ and an optimal strategy $B^*$ is of feedback form with $B^*_t = b^*(\Delta^*_t)\mathbb{1}_{\{\Delta^*_t \leq d\}} + \mathbb{1}_{\{\Delta^*_t > d\}}$, $t \geq 0$.* □

With this, we have completely (and uniquely) characterised the minimal expected time in drawdown as the solution to Equation (2.21) with the properties stated in Theorem 2.16.

Remark. The discerning reader could comment at this point that, as the minimiser inducing an optimal strategy turns out to fulfil $b^*(0) = b_0$, the maximum of the controlled process will not increase above the initial threshold. In this case, the surplus process (and the economic success of the company it models) is bounded from above. We address this aspect and its implications in Section 2.4. #

## 2.3 A Discretisation of the Problem and its Numerical Solutions

Before we come to a concluding assessment of the theoretical results, we illustrate our findings with examples for certain claim distributions. We start by formulating a discrete version of our optimisation problem. Firstly, we discretise the drawdown levels by replacing the interval $[0, d]$ with a sequence of *grid points* $(x_k)_{k=0,\dots,n}$ with $x_0 = 0$ and $x_n = d$. For simplicity, we assume that all points $x_k$ have the same distance $q = x_k - x_{k-1}$ to their predecessor. We set

$$I_0 = [0, (1-\xi)q], \qquad I_k = (x_k - \xi q, x_k + (1-\xi)q], \quad k = 1, \dots, n-1, \qquad I_n = (d - \xi q, d],$$

for a fixed $\xi \in (0, 1]$. Now we assume that the grid is sufficiently narrow and that $\xi$ is chosen appropriately such that all points within the interval $I_k$ can be identified with $x_k$. That is, all drawdowns of size $x \in I_k$ are perceived as approximately equally threatening by the insurer. In particular, $\xi = 1$ corresponds to the conservative approach of rounding up to the next grid point and $\xi = \frac{1}{2}$ represents a 'fair' rounding function.

Remark. By the intuition and in all numerical calculations considered below, the rounding mechanism does not have a significant impact on the result if the grid points are sufficiently dense. However, in some cases, the stability of the numerical procedure can be increased by choosing $\xi \neq 1$ if the matrices defined below would otherwise contain a large number of zeros. #

Next, we define a discrete version of the set of admissible strategies. As the theory shows, it suffices to consider feedback strategies which are equal to one whenever the current drawdown exceeds $d$. Thus, by a *discrete admissible strategy*, we mean a function $\underline{b} : [0, d] \to \{\varrho_i : i = 0, \ldots, a\} \subset [b_0, 1]$ which is piecewise constant on the intervals $I_k$, $k = 0, \ldots, n$. Here, $\{\varrho_i : i = 0, \ldots, a\}$ denotes a finite set of 'permitted' retention levels with $\varrho_0 = b_0$ and $\varrho_a = 1$. Because we assume that all points within $I_k$ are identified with one another, the *discrete value function* $\underline{v}$ (i.e. minimal time in drawdown casted to the grid with respect to the set of discrete admissible strategies) and all strategies $\underline{b}$ are given in terms of the values at the grid points $\mathbf{v} = (\mathbf{v}_k)_{k=0,\ldots,n}$, $\mathbf{b} = (\mathbf{b}_k)_{k=0,\ldots,n}$:

$$\underline{v}(x) = \sum_{k=0}^{n} \mathbf{v}_k \cdot \mathbb{1}_{\{x \in I_k\}}\,, \qquad \underline{b}(x) = \sum_{k=0}^{n} \mathbf{b}_k \cdot \mathbb{1}_{\{x \in I_k\}}\,, \qquad x \in [0, d]\,.$$

Similarly to the dynamic programming equation, it follows for all $k$ that $\underline{v}$ fulfils

$$\underline{v}(x) = \inf_{b \in \{\varrho_i : i=0,\ldots,m\}} \mathbb{E}^{x_k}\left[\mathrm{e}^{-\delta\tau}\underline{v}(\Delta^b_\tau) \cdot \mathbb{1}_{\{\Delta^b_\tau \le d\}} + \mathrm{e}^{-\delta\tau} p_{\mathbf{v}_n}(\Delta^b_\tau) \cdot \mathbb{1}_{\{\Delta^b_\tau > d\}}\right], \qquad x \in I_k\,, \tag{2.35}$$

for $p_C$, $C \in [0, \delta^{-1}]$, as defined in (2.27) and where $\tau$ denotes the minimum of the first exit from the interval $I_k$ and the first claim time. Either the interval is exited by passing through the lower boundary or by jumping across the upper boundary. Figure 2.4 shows a drawdown process under a discrete admissible strategy. The dashed lines represent the grid points $(x_k)_{k=0,\ldots,n}$ and the grey solid lines the boundaries of the intervals $(I_k)_{k=0,\ldots,n}$. The arrows visualise that whenever the process exits the current interval or a jump occurs, it is casted to the grid and 'restarted'. The time $\tau$ mentioned above is therefore represented by the sequence $\tau_1, \tau_2 - \tau_1, \ldots$ in the graph. By conditioning on the time $T_1$ of the first claim $Y_1$ and writing $\Delta^b_{T_1} = \Delta_0 - c(b)T_1 + bY_1$, we therefore obtain

$$\begin{aligned}\mathbf{v}_k = {} & \inf_{b \in \{\varrho_i : i=0,\ldots,m\}} \mathbf{v}_{k-1} \cdot \mathbb{E}^{x_k}\left[\mathrm{e}^{-\delta\tau}\mathbb{1}_{\{\tau < T_1\}}\right] + \sum_{j=k}^{n} \mathbf{v}_j \cdot \mathbb{E}^{x_k}\left[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{\Delta^b_{T_1} \in I_j,\, \tau = T_1\}}\right] \\ & + \mathbf{v}_n \cdot \mathbb{E}^{x_k}\left[\mathrm{e}^{-\delta T_1}\mathrm{e}^{-\gamma(\Delta^b_{T_1} - d)}\mathbb{1}_{\{\Delta^b_{T_1} > d,\, \tau = T_1\}}\right] - \delta^{-1}\,\mathbb{E}^{x_k}\left[\mathrm{e}^{-\delta T_1}\left(\mathrm{e}^{-\gamma(\Delta^b_{T_1} - d)} - 1\right)\mathbb{1}_{\{\Delta^b_{T_1} > d,\, \tau = T_1\}}\right]\end{aligned} \tag{2.36}$$

for all $k \neq 0$. For $k = 0$, the first term (starting with $\mathbf{v}_{k-1}$) disappears. For $k \neq 0$, we additionally note that $\tau$ corresponds to the deterministic times $\tau = \infty$ for $b = b_0$ and $\tau = \xi q / c(b) < \infty$ on the set $\{\tau < T_1\}$ for $b > b_0$. We observe that the coefficients of the $\mathbf{v}_k$ only depend on the distributions of $T_1$ and $Y_1$ and can be calculated for any given claim distribution. That means, $\mathbf{v}$ is determined by $n+1$ (minimised) linear equations. Specifically, the optimal strategy and the value function combined should fulfil (2.36) at every grid point. In a similar way, we can define the discrete return function of a predefined strategy and derive a characterising linear equation system. This inspires the iterative approach explained in the following section.

### 2.3.1 Construction of the Algorithm

In a preprocessing phase, we calculate the coefficients of $\mathbf{v}_j$ on the right hand side of Equation (2.36) for all $j, k \in \{0, \ldots, n\}$ and $b \in \{\varrho_i : i = 0, \ldots, a\}$ and collect these values in a three-dimensional array $\mathbf{T} \in \mathbb{R}^{(n+1)\times(n+1)\times(a+1)}$. That is, the entry $\mathbf{T}^m_{k,j}$ represents the weight of $\mathbf{v}_j$ when starting at $x_k$ with

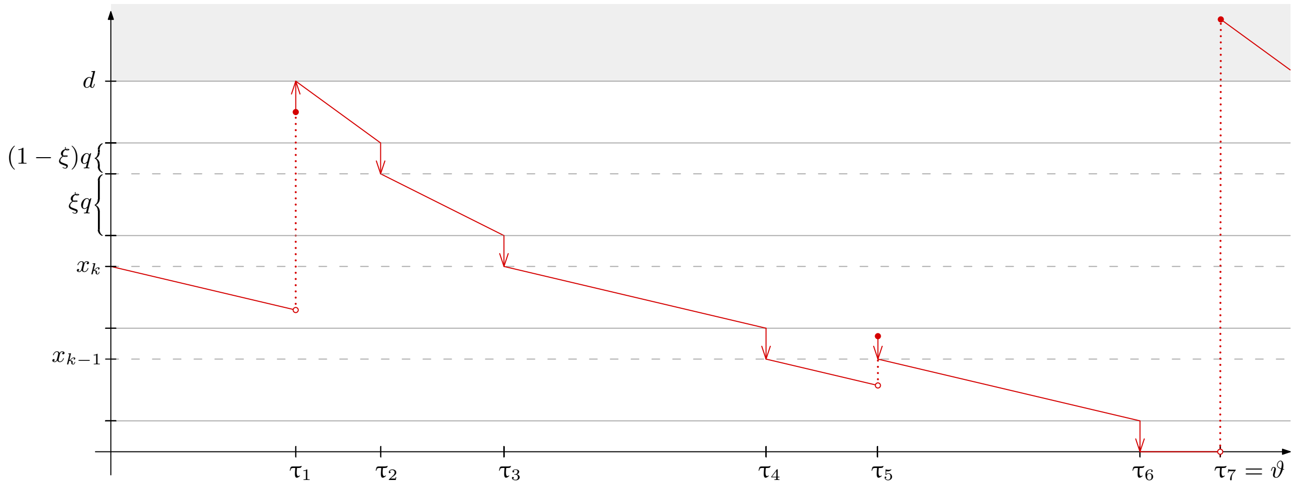



FIGURE 2.4 Drawdown process under a piecewise constant feedback strategy with rounding.

the strategy $\underline{b}(x) = \mathbf{b}_k = \varrho_m$ for $x \in I_k$. Additionally, we define the matrix $\mathbf{t} \in \mathbb{R}^{(n+1)\times(a+1)}$ by

$$\mathbf{t}_k^m = \delta^{-1}\,\mathbb{E}^{x_k}\Big[\mathrm{e}^{-\delta T_1}\big(\mathrm{e}^{-\gamma(\Delta_{T_1}^{\varrho_m}-d)} - 1\big)\mathbb{1}_{\{\Delta_{T_1}^{\varrho_m}>d,\,\tau=T_1\}}\Big]$$

such that $-\mathbf{t}_k^m$ corresponds to the last term on the right hand side of (2.36).
We start with an arbitrary admissible strategy $\mathbf{b}$. We write $\mathbf{m} \in \mathbb{R}^{n+1}$ for the vector containing the indices of the retention levels, that is $\mathbf{b}_k = \varrho_{\mathbf{m}_k}$ for $k = 0, \ldots, n$. In *Step 1* of the algorithm, we calculate the return $\mathbf{v}$ of the predefined strategy $\mathbf{b}$. We define the matrix $\mathbf{U} \in \mathbb{R}^{(n+1)\times(n+1)}$ and the vector $\mathbf{u} \in \mathbb{R}^{n+1}$ by $\mathbf{U}_{k,j} = \mathbf{T}_{k,j}^{\mathbf{m}_k}$ and $\mathbf{u}_k = \mathbf{t}_k^{\mathbf{m}_k}$ for $k, j = 0, \ldots, n$. Then, we define $\mathbf{v}$ as the return of this strategy, that is, the solution to the linear equation system $(\mathbf{U} - \mathbf{I}^{(n+1)})\mathbf{v} = \mathbf{u}$. We note that this results in $n+1$ equations to find $n+1$ variables. To be completely thorough, one would have to check if the rows of this equation system are linearly independent. However, the upper Hessenberg structure of the matrix and complexity of its entries suggest that this will in general be the case. In *Step 2*, we improve $\mathbf{b}$ for a given $\mathbf{v}$. That means, we choose $\mathbf{b}_k$ so that the right hand side of (2.36) is minimal, or, equivalently, we let

$$\mathbf{m}_k = \operatorname*{arg\,inf}_{m\in\{0,\ldots,a\}} (\mathbf{T}^m\mathbf{v} - \mathbf{t}^m)_k\ , \qquad k = 0, \ldots, n\,,$$

and define $\mathbf{b}_k = \varrho_{\mathbf{m}_k}$. Next, we calculate the return of the new strategy. We iterate Step 1 and Step 2 until no further improvement of $\mathbf{b}$ in the second step is possible. Then, $\mathbf{v}$ is the return of the strategy $\mathbf{b}$ and both functions combined solve (2.36). In particular, $\mathbf{v}$ is the value function and $\mathbf{b}$ is optimal.
In practice, the part of the algorithm causing the largest proportion of the total computation time is the preprocessing phase in which the entries of $\mathbf{T}$ are defined. This is also the only part of the procedure which depends on the claim size distribution (which also has an effect on the runtime by the complexity of the weights). The matrix $\mathbf{U}$ defined in Step 1 is 'almost triangular' which allows an efficient calculation of a solution. The interested reader finds additional information on the exact definition of the involved matrices in the appendix, p. 114.

We observe that, even though our analysis of the previous sections specifically dealt with continuous distribution functions $G$, this discrete version of the problem can be transferred to discontinuous claim distributions, such as $G(y) = \mathbb{1}_{\{y \geq \zeta\}}$ for $\zeta > 0$. It is easy to see that, also in this case, the optimal strategy for minimising the time spent in the critical area is the strategy of maximal drift (with $B_t = 1$ for $t < \vartheta(B)$) and that Lemma 2.3 remains true.

### 2.3.2 Description of the Numerical Study

For all our examples, we use the parameter set given in Table 2.1. That means, the safety loading $\eta = 0.2$ yields a premium income rate of $1 + \eta$ if no reinsurance is bought. In a unit interval, we expect one claim, $\lambda = 1$. The relatively small preference factor $\delta = 0.3$ could be interpreted as a long term orientation. The critical size of drawdowns $d = 0.8$ has to be interpreted in combination with the claim size distribution. In our examples below, this reflects a rather strong aversion towards drawdowns: if $\mu \approx 0.5$, an example of an event leading to a large drawdown is the occurrence of two average size claims in a row. $\xi = 0.99$ also corresponds to a cautious approach. We consider three scenarios for the external factor of costs of reinsurance (represented by the reinsurance safety loading $\theta$). That means, we let $\theta \in \{0.33, 0.8, 1.1\}$ and compare the resulting strategies. For each of the three considered claim distributions we compare the performance of

i) the strategy of 'no reinsurance', i.e. the feedback strategy induced by $b(x) = 1$, $x \geq 0$,
ii) a feedback strategy which linearly increases with the drawdown, that is, $b_\ell(x) = (b_0 + (1 - b_0)xd^{-1})\mathbb{1}_{\{x \leq d\}} + \mathbb{1}_{\{x > d\}}$, $x \geq 0$,
iii) simple switching with $\check{b} = b_0$ and $\hat{b} = 1$, associated with $b_s(x) = b_0\mathbb{1}_{\{x \leq d\}} + \mathbb{1}_{\{x > d\}}$, $x \geq 0$, and
iv) the optimal strategy connected to $b^*(x)$, $x \geq 0$.

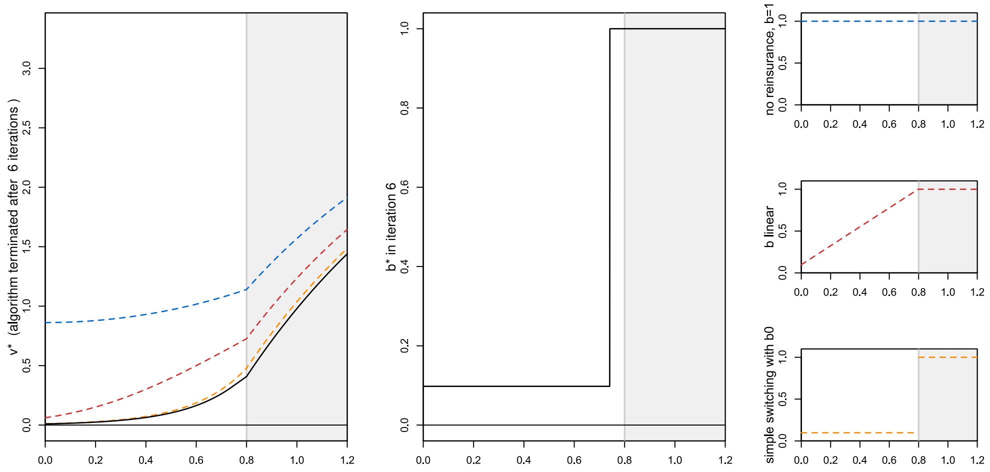
We note that the paths of the drawdown process under the linear strategy are determined by exponential functions between the claims (this follows by the same construction as in the proof of Proposition 2.14). The return functions of discrete versions of these strategies can be calculated with the algorithm described above. We compare these return functions and, additionally, consider simulations of the paths of the surplus process under the respective stochastic control for the case $\theta = 0.8$. To ensure comparability across different claim distributions, we use the same initial values and claim time sequence for all examples. Moreover, we recall that the absolute position of the surplus and its running maximum is not relevant for the optimisation problem as it only depends on their relative distance, the drawdown. In particular, the simulated paths could be shifted from $X_0 = -x$ and $M_0 = 0$ to arbitrary starting points for the surplus and historic maximum as long as their distance remains unchanged.

| $\eta$ | $\lambda$ | $\delta$ | $d$ | $\theta$ | | $n$ | $a$ | $\xi$ |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 1 | 0.3 | 0.8 | (A) | 0.33 | 1000 | 200 | 0.99 |
| | | | | (B) | 0.8 | | | |
| | | | | (C) | 1.1 | | | |

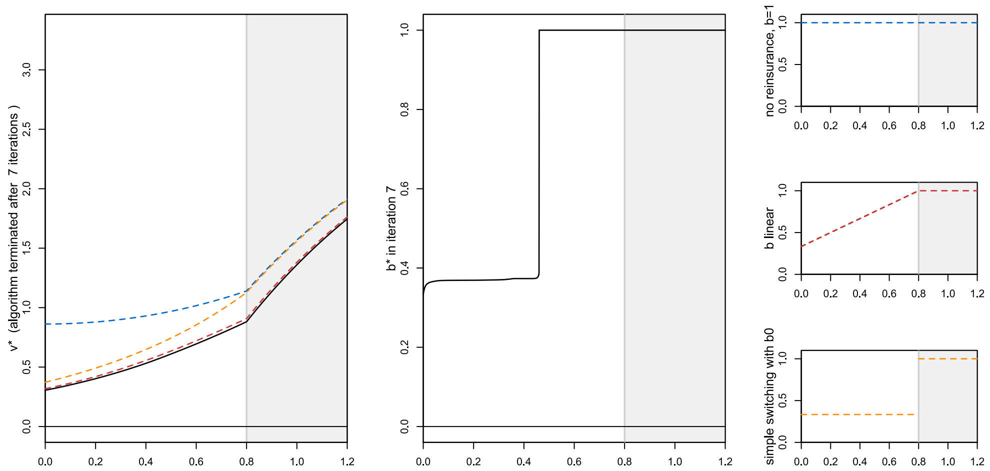
TABLE 2.1 Parameters of insurance surplus, preference and costs of reinsurance.

On the following double pages, we present the return functions and $b, b_\ell, b_s, b^*$ on the left page and the path simulations on the right page. In Figures 2.5, 2.7 and 2.9 on the respective left page, each 'row' corresponds to one choice of $\theta$, that is, the graphs of the first row (A) belong to the cheapest scenario, $\theta = 0.33$, the graphs of the middle row (B) correspond to $\theta = 0.8$ and the last row (C) belongs to $\theta = 1.1$. In each row, the middle graph belongs to the function $b^*$ inducing the optimal feedback strategy and the small graphs on the right show the functions $b$ (dashed, blue line), $b_\ell$ (dashed, red line), $b_s$ (dashed, yellow line). The leftmost graph illustrates the value function (black solid line) and the return functions of the alternative strategies (in the same line type and colour as the corresponding feedback functions). Figures 2.6, 2.8 and 2.10 display path simulations under the feedback strategies $b$, $b_\ell$, $b_s$ and $b^*$. If the graph of the surplus process (black and blue line) is in the grey area, this corresponds to a critical drawdown. In particular, the graphs are colour coded with respect to the retention level: a black point in the graph corresponds to the retention level 1 chosen at that time, whereas a blue tinted point visualises that the retention level is strictly smaller than 1. For a clear presentation of the current retention level at the jump times, the jumps of the path simulations are displayed as solid lines (instead of dotted lines as in Figure 2.4, for example).

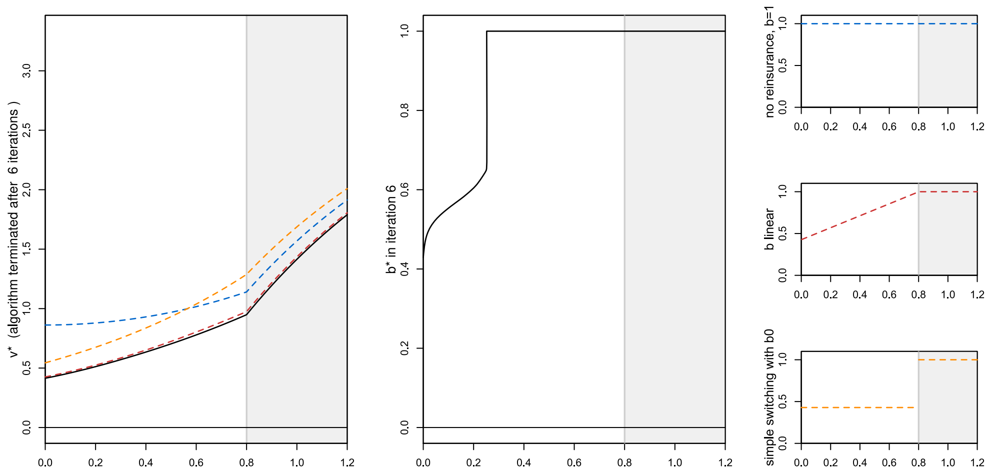
### 2.3.3 Exponential Claims




(A) $\theta = 0.33$




(B) $\theta = 0.8$




(C) $\theta = 1.1$

FIGURE 2.5 Return functions, pointwise optimisers and alternative feedback functions for exponential claims (from *left* to *right* in every row).

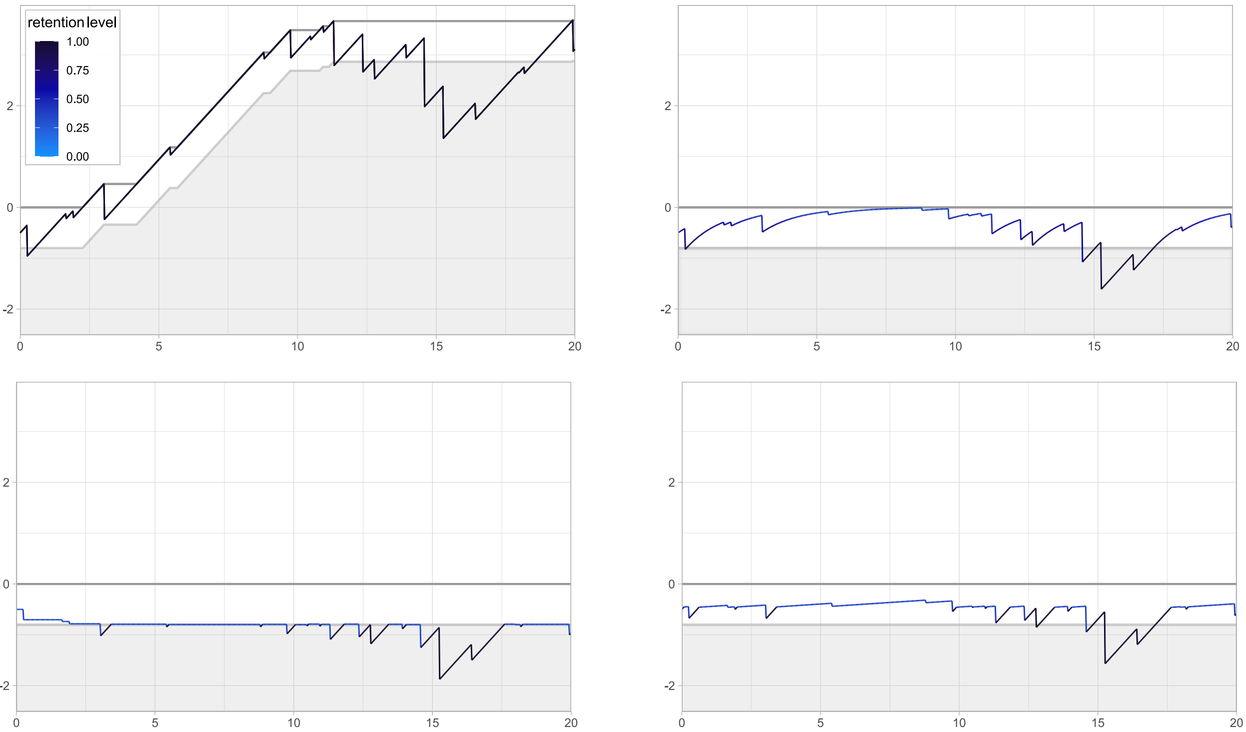
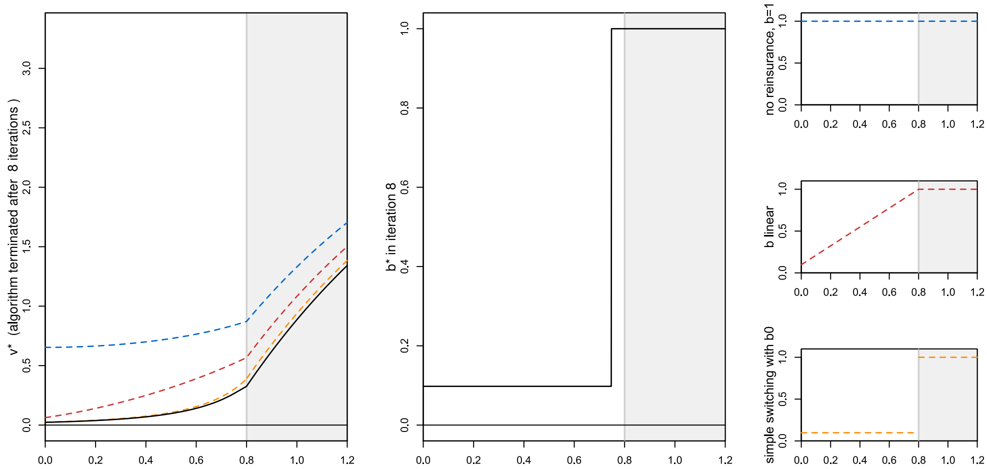


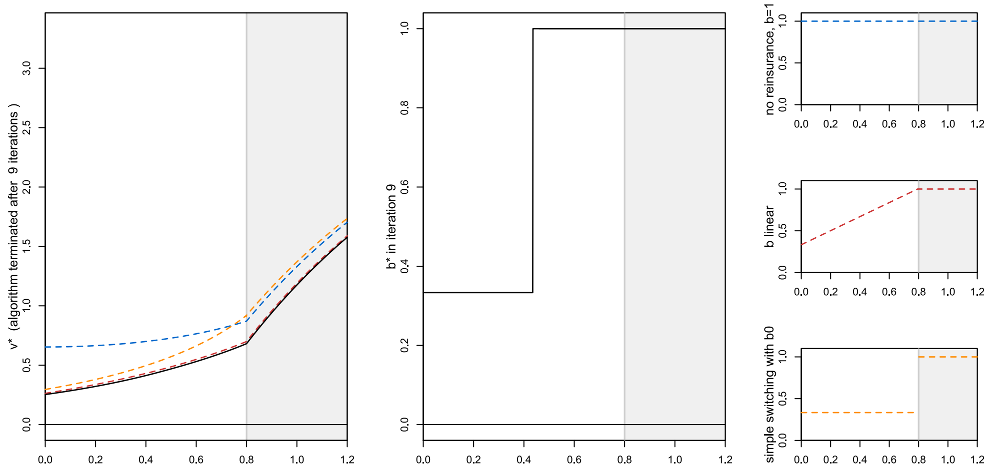

FIGURE 2.6 Path simulations for exponential claims and $\theta = 0.8$. *Top, left:* no reinsurance, *top, right:* linear feedback strategy, *bottom, left:* simple switching with $b_0$, *bottom, right:* optimal feedback strategy.

We consider the claim distribution function $G(y) = 1 - \mathrm{e}^{-\alpha y}$, $y > 0$, for $\alpha = 2$. That means, $\mu = \alpha^{-1} = 0.5$ is the expected claim size. The parameter set given in Table 2.1 yields $\gamma \approx 1.0868$ as derived in Example 2.8. In the case $\theta = 0.33$, in the top row of Figure 2.5, we see that the optimal strategy corresponds to a very low retention level for smaller uncritical drawdowns and shoots up towards 1 as the process approaches the critical boundary. This corresponds to 'playing it safe' in an area close to zero and, if this area is exited by a jump, pushing the process back (at the risk of even larger jumps). Additionally, the value function is convex and increases quickly between $x = 0$ and $x = d$. This allows the interpretation that for an 'almost critical' drawdown, the strategy of maximal drift is chosen because the area close to the boundary is too dangerous. The simple switching strategy, despite seeming to be a radical choice, leads to a smaller time in drawdown than the other alternative strategies and resembles this behaviour the most. In the cases $\theta = 0.8$ and $\theta = 1.1$, we see that the constant strategy $b$ and its return function (dashed, blue) remain the same in all graphs as they are independent of $\theta$. The optimal strategy, $b_\ell$ and $b_s$ change with $\theta$. This is because of the restriction to retention levels from the set $[b_0, 1]$ with $b_0 = \frac{\theta - \eta}{1+\theta}$ (which is increasing in $\theta$). Additionally, the claim reduction by proportional reinsurance is more costly (in terms of drift reduction) for larger $\theta$. As a consequence, more expensive reinsurance leads to a generally longer time in critical drawdown. For the optimal strategy, the area in which a maximal drift is chosen grows larger with $\theta$. Below that level, the graph attains an 'S'-shape. From the set of alternative strategies, the strategy induced by $b_\ell$ is now the best choice. This is also visible in the path simulations of Figure 2.6 which are based on the strategies for $\theta = 0.8$. The critical drawdown phase starting approximately at time 14.5 for the 'extreme' strategies associated with $b$ and $b_s$ is significantly shorter for the strategies induced by $b_\ell$ and $b^*$.

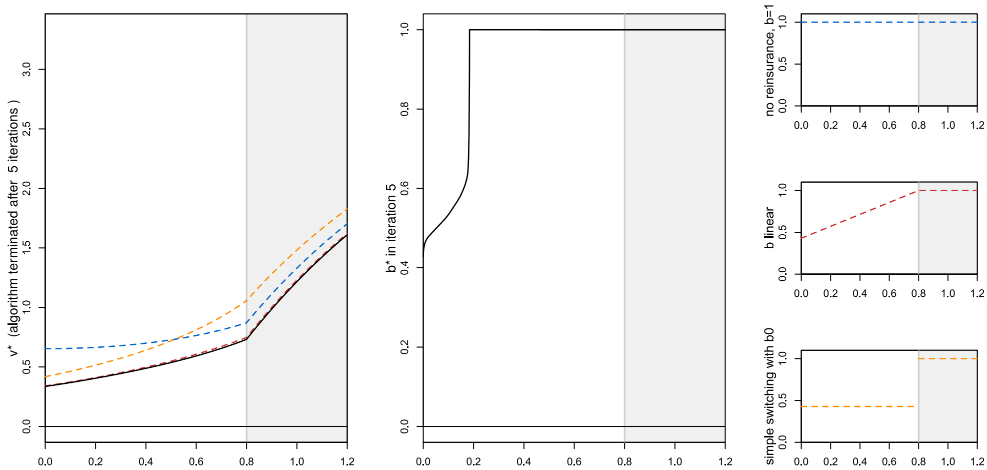
### 2.3.4 Pareto Distributed Claims




(A) $\theta = 0.33$




(B) $\theta = 0.8$




(C) $\theta = 1.1$

FIGURE 2.7 Return functions, pointwise optimisers and alternative feedback functions for Pareto claims (from *left* to *right* in every row).

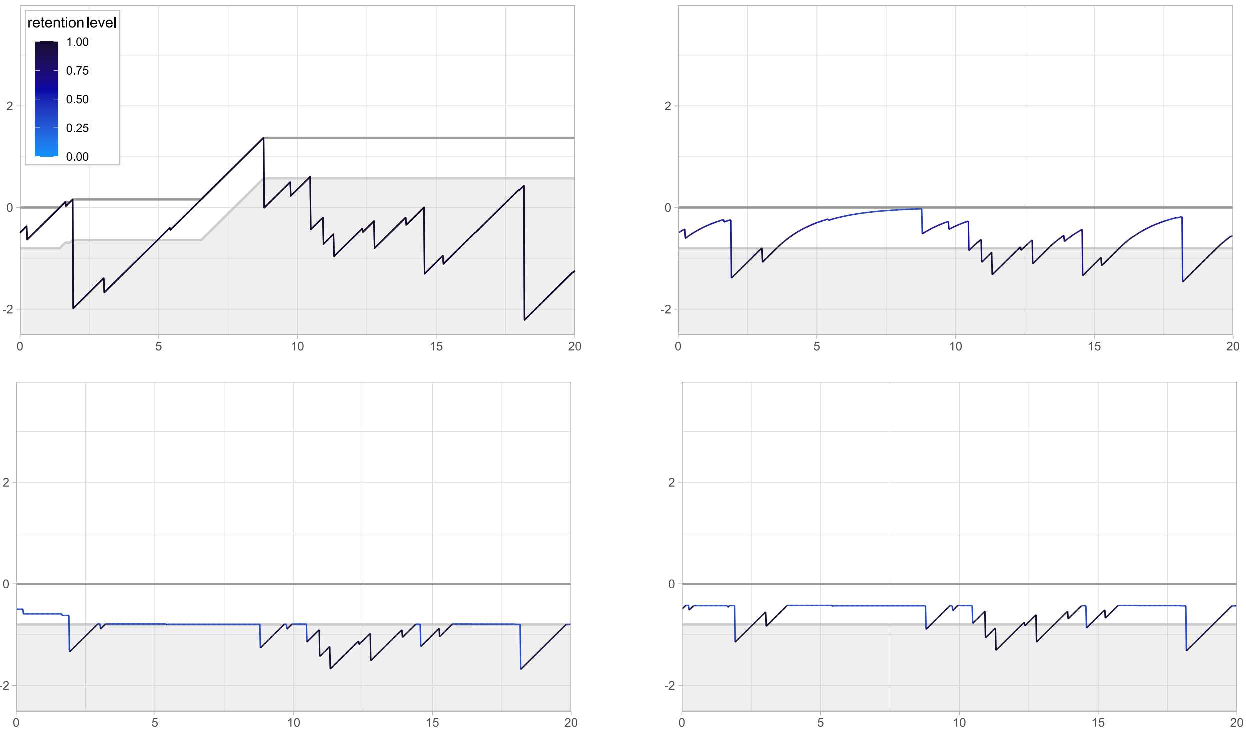


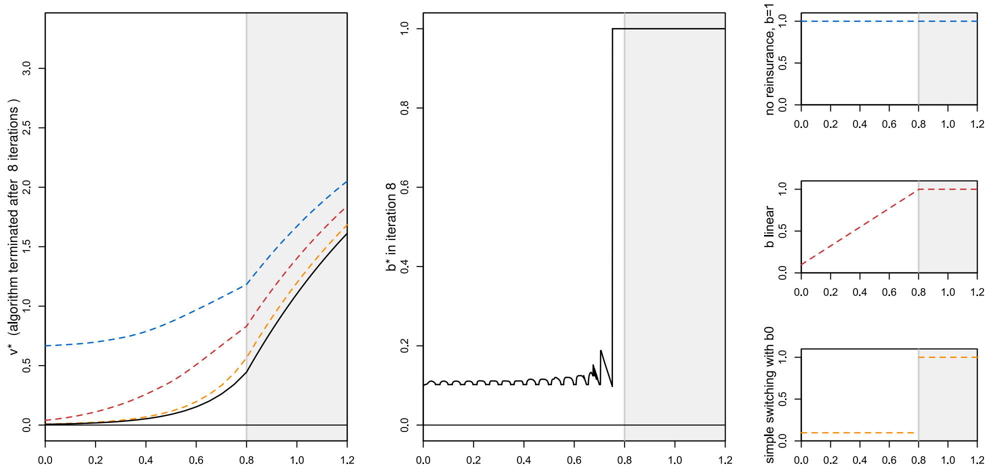

FIGURE 2.8 Path simulations for Pareto claims and $\theta = 0.8$. *Top, left:* no reinsurance, *top, right:* linear feedback strategy, *bottom, left:* simple switching with $b_0$, *bottom, right:* optimal feedback strategy.

Next, we assume that we have $G(y) = 1 - \beta^\alpha(\beta + y)^{-\alpha}$, $y > 0$, for $\alpha = 2$ and $\beta = 0.45$. With this definition, the expected value of the claims $\mu = \beta/(\alpha - 1) = \beta$ is slightly smaller than in the previous example. However, by choice of $\alpha$, the variance is infinite. According to Lemma 2.3, the parameter $\gamma \approx 1.0304$ (which can be calculated numerically) is determined by the equation

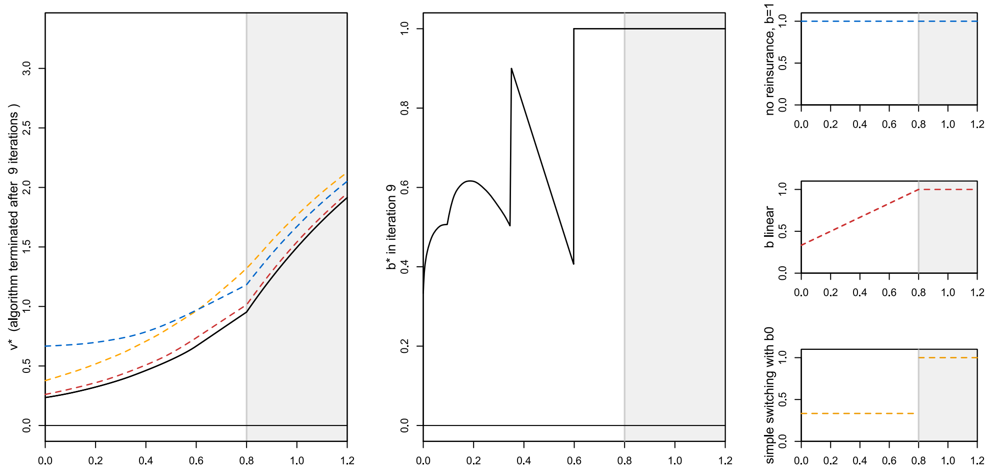
$$\delta = c(1)\gamma - \lambda\big(1 - \alpha(\beta\gamma)^\alpha \mathrm{e}^{\beta\gamma}\Gamma(-\alpha, \beta\gamma)\big)\,,$$

where $\Gamma(-\alpha, x) = \int_x^\infty \mathrm{e}^{-t} t^{-(\alpha+1)}\,\mathrm{d}t$, $x > 0$, denotes the incomplete Gamma function. The return and feedback functions of Figure 2.7 look similar to the case of exponential claims. For $\theta = 0.8$ and $\theta = 1.1$, however, the shape of the optimiser $b^*$ differs. In particular, for $\theta = 1.1$, we see a convex 'ramp' instead of the 'S'-shape of the function $b^*$ for small $x$. This means that the retention level rapidly increases as the drawdown grows larger. Another significant difference can be observed in the path simulations of Figure 2.8. In comparison to the case of exponential claims, we see that the surplus without reinsurance is subject to much more severe and long-lasting drawdowns. As the claim occurrence times are unchanged, this is due to the different claim distribution. In comparison to the surplus without reinsurance (top, left), the surplus with optimal control (bottom, right) even is at a higher level at the end of the time interval. The main reason for this is that the impact of the exceptionally large claims (at the approximate times 2, 8.75, 14.5 and 18) is weakened by reinsurance. Comparing the optimally controlled path to the simple switching path (bottom, left), we see that the optimal strategy has a similar mechanism. The retention level 'switches' (shoots up) already when the drawdown is still in the uncritical area. This fits to the graph in the centre of Figure 2.7 and results in a reduction of the time in critical drawdown.

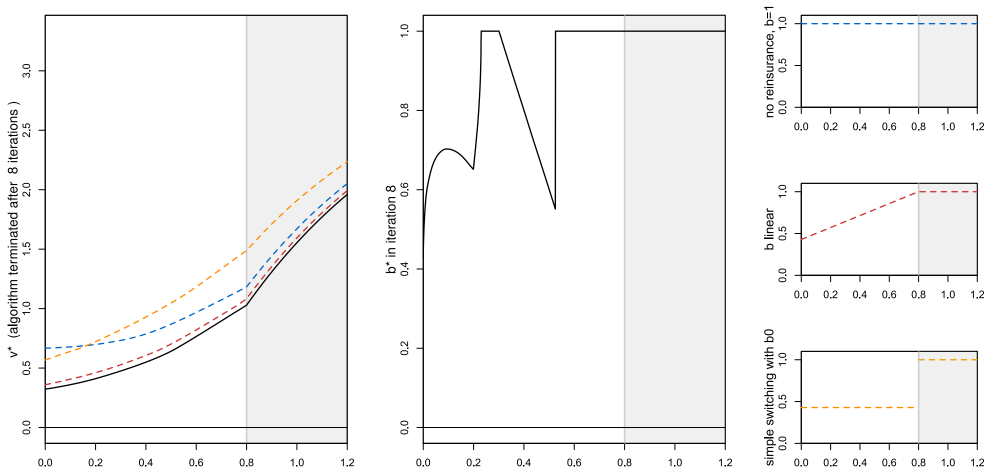
### 2.3.5 Deterministic Claims




(A) $\theta = 0.33$




(B) $\theta = 0.8$




(C) $\theta = 1.1$

FIGURE 2.9 Return functions, pointwise optimisers and alternative feedback functions for deterministic claims (from *left* to *right* in every row).

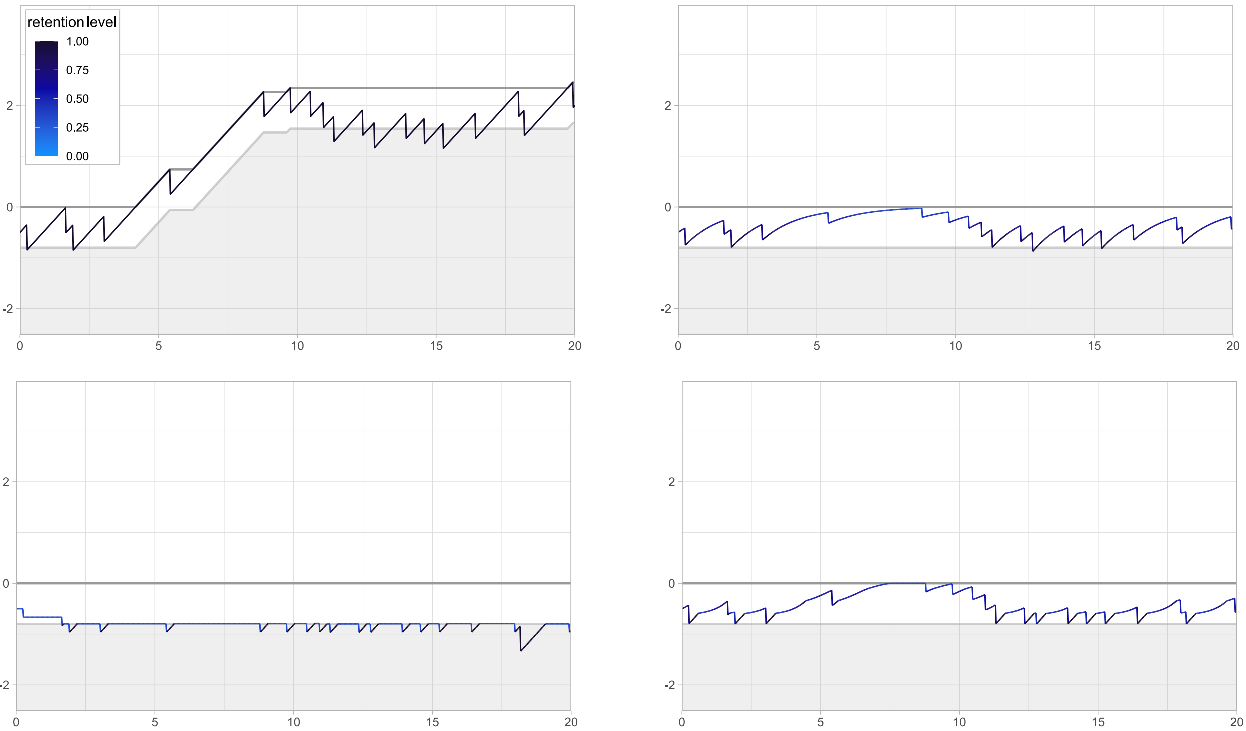



FIGURE 2.10 Path simulations for deterministic claims and $\theta = 0.8$. *Top, left:* no reinsurance, *top, right:* linear feedback strategy, *bottom, left:* simple switching with $b_0$, *bottom, right:* optimal feedback strategy.

Up to this point, we have dealt with unbounded claim distributions with support on $(0, \infty)$. This means that from the current position of the drawdown, any higher level can be reached with the next jump. With the example of deterministic claims, we now consider a bounded distribution. We describe the claim size distribution by defining $\zeta = 0.5$ and $\mathbb{P}[Y_1 = \zeta] = 1$. Analogously to Subsection 2.2.2, we obtain $\gamma \approx 1.2940$ as the positive solution to the equation

$$\delta = c(1)\gamma - \lambda(1 - \mathrm{e}^{-\gamma\zeta})\,. \tag{2.37}$$

Details of this calculation are found in the appendix, p. 116. In this case, the drawdown process is more predictable and easier to control for the insurer. For example, if $\zeta$ is sufficiently small, it is possible to choose a retention level which prevents the drawdown from exiting the uncritical area at the next jump time. This leads to a partition of the uncritical area into different 'bands'. That means, retention levels of strong drift and retention levels leading to a significant claim reduction are alternated. As seen in the graphs of the optimal strategies in Figure 2.9, this effect intensifies if costs of reinsurance increase. A possible explanation is that in the bands with low retention levels, one aims to stabilise the process and chooses a retention level which ensures that the uncritical area is not exited. In the bands with higher retention levels, one chooses a strategy which quickly pushes the process into the next 'low retention level'-band, that is, the next 'safety zone'. The kinks between claims of the path of the optimally controlled process (bottom, right graph of Figure 2.10) correspond to the local minima and maxima of $b^*$ for $\theta = 0.8$ (graph in the centre of Figure 2.9). We observe that the surplus without control (top, left in Figure 2.10) has a few critical drawdowns after time 10, whereas critical drawdowns of the optimally controlled surplus are completely prevented.

## 2.4 Key Findings and Concluding Remarks

In this chapter, we analysed the optimisation problem of minimising the expected time with critical drawdown defined in Equation (1.9) for a classical risk model. Our first step was to prove a verification theorem, Theorem 2.16, which implies that a bounded, increasing solution to the connected inhomogeneous Hamilton–Jacobi–Bellman integro-differential equation corresponds to the value function if it fulfils the initial condition of $b^*(0) = b_0$ (i.e. the 'drift' component of the equation vanishes if $x = 0$). In a way, this result yields uniqueness of solutions to the equation. To prove existence, we started by splitting the problem at the critical level $x = d$, according to the dynamic programming principle proved in the introduction. For the separate problem of minimising the recovery time, we showed that the optimal strategy is constant by analysing (explicit) Laplace transforms of the first passage through $d$. For the complementary subproblem of maximising the time in the uncritical area, reflection on the $x$-axis and an unknown penalty for the overshoot complicated the procedure. Here we used the explicit representation obtained in the first part to construct a set of Gerber–Shiu optimisation problems containing our original target function. We showed that the corresponding value functions are uniquely characterised as solutions to homogeneous Hamilton–Jacobi–Bellman equations. We found out that the optimal strategies are of feedback form. Then, we reconnected the subsolutions to derive the same result for the expected time in critical drawdown.

In our numerical examples, we further examined the influence of costs of reinsurance and the claim distribution. We saw that, if the parameter $\theta$ increases, less reinsurance is bought, so that the minimal expected time in drawdown increases. Further, we observed that the heavier the claims, the more the insurer is willing to 'give up' drift in order to reduce possible claim payments. In the case of bounded claims, the optimal strategy sensitively controls the size of the next claim and, therefore, lower and higher retention levels are alternated. In the considered examples, it is optimal to leave the critical level as quickly as possible, especially if reinsurance is expensive.

In our probabilistic analysis of the problem, we found out that all optimal strategies have two key characteristics in common. Firstly, if the drawdown is already critically large, it is optimal not to purchase reinsurance until the uncritical area is re-entered. That means, the goal of a quick recovery is reached by choosing the strategy of maximal downward drift of the drawdown process (at the risk of full-sized jumps). Secondly, when the drawdown is currently at zero, that is, if the surplus is at its maximum, the lowest retention level is optimal. This fits the intuition because the reflection barrier absorbs the drift of the process, so that controlling the next claim size becomes the only objective. This policy has the effect that the surplus is balanced in the uncritical area close to its running maximum. However, as remarked in Section 2.2 and also obvious in the path simulations, this strategy also prevents the running maximum from increasing. This means that the controlled surplus never outgrows its initial maximum. Stability (valued over growth) is an intrinsic aspect of our optimisation criterion and could also be perceived as a drawback thereof. Therefore, we expect this approach to work best if there are external bounds on the surplus, for example due to taxation or regulatory laws, or if it is subject to extraordinarily large claims. In a way, the latter is visualised by the exemplary simulated paths with Pareto distributed claims in the previous section. In Chapter 4, we additionally consider an extension which accounts for the potential of future records.

CHAPTER 3

# Minimal Expected Time in Drawdown for the Diffusion Approximation

This chapter is based on [Brinker and Schmidli, 2022] and contains also some complementary results. We consider a scenario in which the surplus is modelled by a diffusion approximation and retention level strategies with values in $[0,1]$. We define

$$\mu(b) = \eta - (1-b)\theta\,, \qquad b \in [0,1]\,. \tag{3.1}$$

That means, in the notation of Section 1.3, the surplus with reinsurance $X^B$, $B \in \mathcal{B}$, is the diffusion process given by

$$X_t^B = \nu_0 + \int_0^t \mu(B_s)\,\mathrm{d}s + \int_0^t B_s\sigma\,\mathrm{d}W_s\,, \qquad t \geq 0\,, \tag{3.2}$$

in this chapter.

Our goal is the minimisation of the expected time with critical drawdown $v$, defined in Equation (1.9), for this model. As a short introduction we start off by calculating the expected discounted time in drawdown of an arithmetic Brownian motion without optimisation in Section 3.1. This corresponds to the case in which the insurer decides on a certain retention level at time zero. We present a different technique in this section which does not rely on splitting at the critical line. Additionally, we consider simple switching controls. In Section 3.2, we turn to the optimal control problem. Using the dynamic programming approach, we derive a 'natural candidate' for the value function, i.e. a solution to the associated Hamilton–Jacobi–Bellman equation. For critical initial drawdown, we find out that the maximal Laplace transform of the passage time of $d$

$$V(x) = \sup_{B\in\mathcal{B}} \mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta(B)}\big]\,, \qquad x > d\,,$$

is an exponential function (as in the previous chapter). For uncritical initial drawdown, we calculate explicitly the minimal Laplace transform of the first exit from $[0,d]$

$$V(x) = \inf_{B\in\mathcal{B}} \mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta(B)}\big]\,, \qquad x \leq d\,,$$

in terms of the upper branch of the Lambert W function $\mathcal{W}$. By this we mean the unique inverse of $x \mapsto x\mathrm{e}^x$ with values in $[-1,\infty)$ which is defined for $x \in [-\mathrm{e}^{-1},\infty)$ (compare, for example, [Corless et al., 1996]). For the pointwise minimiser of this function in the Hamilton–Jacobi–Bellman equation, we show that the surplus under the connected feedback control exists as a solution to a (reflected) stochastic differential equation. This enables us to prove that the return of the composition of the optimal strategies in the subproblems is indeed the value function $v$. We use the explicit results to derive further properties of the value function and its optimiser, in particular, the impact of the parameter $\theta$. In Section 3.3, we present numerical examples of value functions, strategies and path simulations. We finish with concluding remarks in Section 3.4.

## 3.1 Warm-Up: Constant Controls and Existence

We start with the consideration of simple reinsurance strategies. As in Chapter 2, we connect these controls to their return functions by martingale methods.

### 3.1.1 Motivation of the Differential Equation and an Approximation

In this section, we consider a strategy $B$ with $B_t = b \in (0, 1]$ for all $t \geq 0$ and write $X$ for the surplus, $M$ for the running maximum and $\Delta$ for the drawdown of this strategy. Additionally, we denote by $u$ the return $v^B$. We exclude the case $b = 0$, in which the drawdown corresponds to the deterministic function $t \mapsto x + (\theta - \eta)t$. In this case, we clearly have $u(x) = \delta^{-1}$ for $x > d$ and $u(x) = \delta^{-1}\mathrm{e}^{-\delta(d-x)/(\theta-\eta)}$ for $x \leq d$. Moreover, writing $b_1 = (\theta - \eta)/\theta > 0$ for the retention level with $\mu(b_1) = 0$, we have $\liminf_{t\to\infty} X_t = -\infty$, by the law of the iterated logarithm, for $b \leq b_1$. This means that the drawdown process (eventually) spends an infinite time in the critical area. However, in contrast to the classical risk model and the case $b = 0$, we do not simply get $u(x) = \delta^{-1}$ for $x > d$ because the diffusion process can re-enter the uncritical area with positive probability. This case is included in the considerations below.

We motivate our approach in the following way. For a twice continuously differentiable function $f : [0, \infty) \to [0, \delta^{-1}]$ with bounded first derivative, we obtain by Itô's formula applied to $\tilde{f}(t, x) = \mathrm{e}^{-\delta t} f(x)$ and the process $((t, \Delta_t))_{t\geq 0}$ that it holds

$$\begin{aligned}\mathrm{e}^{-\delta t} f(\Delta_t) - f(\Delta_0) = &\int_0^t \mathrm{e}^{-\delta s} f'(0)\,\mathrm{d}M_s - \int_0^t \mathrm{e}^{-\delta s}\sigma b f'(\Delta_s)\,\mathrm{d}W_s \\ &+ \int_0^t \mathrm{e}^{-\delta s}\Big[-\delta f(\Delta_s) - \mu(b) f'(\Delta_s) + \frac{\sigma^2 b^2}{2} f''(\Delta_s)\Big]\,\mathrm{d}s\,.\end{aligned}$$

Here we used that $\Delta$ is equal to zero in all points of increase of $M$, by Corollary 1.3. Thus, by boundedness of $f'$, the process

$$\Big(\mathrm{e}^{-\delta t} f(\Delta_t) - f(\Delta_0) - \int_0^t \mathrm{e}^{-\delta s} f'(0)\,\mathrm{d}M_s - \int_0^t \mathrm{e}^{-\delta s}\mathcal{A}^b f(\Delta_s)\,\mathrm{d}s\Big)_{t\geq 0}$$

is a martingale, where we write

$$\mathcal{A}^b f(x) = -\delta f(x) - \mu(b) f'(x) + \frac{\sigma^2 b^2}{2} f''(x) \tag{3.3}$$

for $x \geq 0$. Thus, if $f$ was a solution to

$$-\delta f(x) - \mu(b) f'(x) + \frac{\sigma^2 b^2}{2} f''(x) = -\mathbb{1}_{\{x>d\}}\,, \qquad x \geq 0\,, \tag{3.4}$$

with $f'(0) = 0$, we could verify that $f = u$ holds (by the same arguments as in the proof of Lemma 2.7). However, we observe that the right hand side of (3.4) is not continuous. This means that a classical – twice continuously differentiable – solution cannot exist. In this section, our idea is therefore to approximate the shifted unit step on right hand side of (3.4) by a sequence $(H_k)_{k\in\mathbb{N}}$ of continuous functions so that we can calculate a sequence of classical solutions $(f_k)_{k\in\mathbb{N}}$. We show that this sequence converges to a limit $f$ as $k \to \infty$. Then, we use the martingale approach above with $f_k$ in place of $f$ and employ an additional convergence argument to verify that the limit function is indeed equal to $u$. We introduce the following notation which will be frequently used in the present and the subsequent chapter.

Notation 3.1. *For $b \in (0,1]$, we define the strictly positive constants*

$$\kappa(b) = \frac{-\mu(b) + \sqrt{[\mu(b)]^2 + 2\delta b^2\sigma^2}}{b^2\sigma^2}\,, \qquad \xi(b) = \frac{\mu(b) + \sqrt{[\mu(b)]^2 + 2\delta b^2\sigma^2}}{b^2\sigma^2}\,,$$

*so that $x \in \{-\kappa(b), \xi(b)\}$ solves the quadratic equation $-\mu(b)x + (\sigma b x)^2/2 = \delta$. We note that $\xi(b_1) = \kappa(b_1)$, $\xi(b) > \kappa(b)$ for $b > b_1$ and $\xi(b) < \kappa(b)$ for $b < b_1$. We use the abbreviations $\kappa = \kappa(1)$ and $\xi = \xi(1)$.*

With this notation, we define the functions $H_k : [0,\infty) \to [0,1]$ and $f_k : [0,\infty) \to (-\infty, \delta^{-1})$ by

$$\begin{aligned} H_k(x) &= \frac{1}{e^{2k(d-x)} + 1}\,,\\ f_k(x) &= \frac{\kappa(b)e^{\xi(b)x} + \xi(b)e^{-\kappa(b)x}}{\delta[\kappa(b)+\xi(b)]e^{\xi(b)d}} - \frac{\kappa(b)\xi(b)}{\delta[\kappa(b)+\xi(b)]}\int_0^x \big(e^{\xi(b)(x-z)} - e^{-\kappa(b)(x-z)}\big)H_k(z)\,\mathrm{d}z\,, \qquad x \geq 0\,, \end{aligned}$$

for every $k \in \mathbb{N}$. The constant preceding the integral has the alternative representation of $1/\sqrt{[\mu(b)]^2 + 2\delta\sigma^2 b^2}$. We collect all preliminary results on these functions in the following lemma.

Lemma 3.2. *For every $k \in \mathbb{N}$, $H_k$ is continuous and $f_k$ is twice continuously differentiable. We have $\mathcal{A}^b f_k(x) = -H_k(x)$ for all $x \geq 0$ and $f_k'(0) = 0$. For every $x \geq 0$, $\lim_{k\to\infty} H_k(x) = \frac{1}{2}\mathbb{1}_{\{x=d\}} + \mathbb{1}_{\{x>d\}}$ and $\lim_{k\to\infty} f_k(x) = f(x)$, where*

$$f(x) = \begin{cases} C_1 e^{\xi(b)x} + C_2 e^{-\kappa(b)x}\,, & x \leq d\,,\\ \delta^{-1} - [\delta^{-1} - (C_1 e^{\xi(b)d} + C_2 e^{-\kappa(b)d})]e^{-\kappa(b)(x-d)}\,, & x > d\,, \end{cases}$$

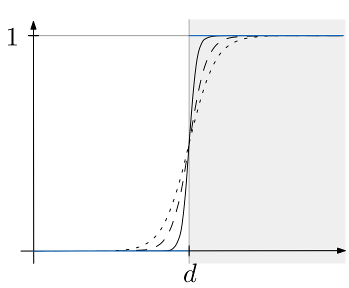
*for*

$$C_1 = \frac{\kappa(b)}{\delta[\kappa(b)+\xi(b)]e^{\xi(b)d}}\,, \qquad C_2 = \frac{\xi(b)C_1}{\kappa(b)}\,.$$

*$f : [0,\infty) \to (0, \delta^{-1})$ is strictly increasing and continuously differentiable on $(0,\infty)$ with $f'(0) = 0$. Additionally, $f|_{[0,d]}$ and $f|_{(d,\infty)}$ are classical solutions to $\mathcal{A}^b f(x) = -\mathbb{1}_{\{x>d\}}$ on the respective domain*

*if $f''(d)$ is interpreted as a one-sided derivative.*

The proof follows by straightforward calculation and can be found in the appendix, p. 117. Figure 3.1 shows a sketch of $H_k$ and $f_k$ for different choices of $k$ (black graphs) and their limits (blue graphs). Plotting $f_k$ requires numerical evaluation of the integral. Two possibilities are to use an approximation (for example the trapezoidal rule) or to rewrite the integral in terms of functions for which value tables are available (as described in the appendix, p. 119).



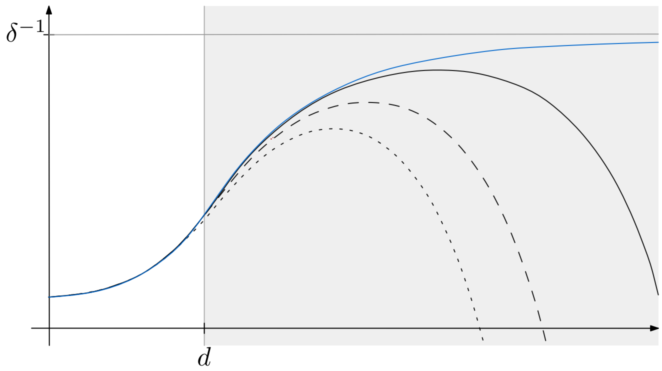




FIGURE 3.1 Approximation $H_k$ to the indicator function on right hand side of (3.4) (*left*) and corresponding solution $f_k$ (*right*) for different $k$.

LEMMA 3.3. *For $f$ as defined in Lemma 3.2, it holds $f(x) = u(x)$ for all $x \geq 0$.*

*Proof.* As in the introduction, we write $\vartheta^y = \vartheta^y(B) = \inf\{t \geq 0 : \Delta_t > y\}$ for the first entry of the drawdown into $(y, \infty)$ and define a sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ by $T_n = \vartheta^n \wedge n$ for $n \in \mathbb{N}$. For $x < n$, we have $\Delta_s \in [0, n]$ for all $s \leq \vartheta^n$. By Itô's formula it follows that

$$\Big(\mathrm{e}^{-\delta(t\wedge\vartheta^n)} f_k(\Delta_{t\wedge\vartheta^n}) - f_k(\Delta_0) + \int_0^{t\wedge\vartheta^n} \mathrm{e}^{-\delta s} H_k(\Delta_s)\,\mathrm{d}s\Big)_{t\geq 0}$$

is a martingale, where we have already used that $f'(0) = 0$ and $\mathcal{A}^b f_k(x) = -H_k(x)$, $x \geq 0$. We note that stopping at $\vartheta^n$ is important because the derivative $f_k'$ is continuous (and thus, bounded on bounded intervals) but unbounded from below on $[0, \infty)$. This is also visible in Figure 3.1. Taking expectations, we find

$$f_k(x) = \mathbb{E}^x\big[\mathrm{e}^{-\delta(t\wedge\vartheta^n)} f_k(\Delta_{t\wedge\vartheta^n})\big] + \mathbb{E}^x\Big[\int_0^{t\wedge\vartheta^n} \mathrm{e}^{-\delta s} H_k(\Delta_s)\,\mathrm{d}s\Big].$$

We note that, for all $k \in \mathbb{N}$ and $x \in [0, n]$, it holds by the mean value theorem:

$$|f_k(x)| \leq \frac{\kappa(b)\mathrm{e}^{\xi(b)n} + \xi(b)\mathrm{e}^{-\kappa(b)n}}{\delta(\kappa(b) + \xi(b))\mathrm{e}^{\xi(b)d}} + \frac{n(1 + \mathrm{e}^{\xi(b)n})\kappa(b)\xi(b)}{\delta(\kappa(b) + \xi(b))}.$$

Thus, $(f_k)_{k\in\mathbb{N}}$ is uniformly bounded on $[0, n]$. Moreover, $H_k$ takes values in $[0, 1]$ and the integral in the second expected value takes values in $[0, \delta^{-1}]$. Therefore, by bounded convergence, we find

$$f(x) = \mathbb{E}^x\big[\mathrm{e}^{-\delta(t\wedge\vartheta^n)} f(\Delta_{t\wedge\vartheta^n})\big] + \mathbb{E}^x\Big[\int_0^{t\wedge\vartheta^n} \mathrm{e}^{-\delta s} \mathbb{1}_{\{\Delta_s > d\}}\,\mathrm{d}s\Big]$$

where Fubini's theorem shows that the limit $H(\Delta_s)$ can be replaced by the indicator function. Next, we take the limit $n \to \infty$, such that (pathwise) $\vartheta^n \to \infty$ and consequently $\Delta_{t\wedge\vartheta^n} \to \Delta_t$ for any fixed $t \geq 0$. By bounded convergence for both terms, we get

$$f(x) = \mathbb{E}^x\big[\mathrm{e}^{-\delta t} f(\Delta_t)\big] + \mathbb{E}^x\Big[\int_0^t \mathrm{e}^{-\delta s} \mathbb{1}_{\{\Delta_s > d\}}\, \mathrm{d}s\Big]\,.$$

Now, the assertion follows by taking the limit $t \to \infty$, boundedness of $f$ and monotone convergence. $\square$

We observe that the exponential function in the representation of $u$ in Lemma 3.2 for $x > d$ corresponds to the Laplace transform of the passage time of an arithmetic Brownian motion of drift $-\mu(b)$ and volatility $\sigma b$. This can be proved in the same way as Lemma 2.3 iii) via corresponding martingales (see also Proposition 3.10, below). For $x \in [0, d]$, the function has a similar structure as the Laplace transform of a two-sided exit time of this process (compare [Borodin and Salminen, 2002, p. 309, Eq. (3.0.5)]). This does not come as a surprise because, similar to the proof of the dynamic programming principle with the set $\mathcal{B}' = \{B\}$, one can show:

Lemma 3.4. *The function $u$ fulfils*

$$u(x) = \begin{cases} u(0)\cdot \mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta_0}\mathbb{1}_{\{\vartheta_0<\vartheta^d\}}\big] + u(d)\cdot \mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta^d}\mathbb{1}_{\{\vartheta_0>\vartheta^d\}}\big]\,, & x \leq d\,,\\ \delta^{-1} - (\delta^{-1} - u(d))\cdot \mathbb{E}^x\big[\mathrm{e}^{-\delta\vartheta_d}\big]\,, & x > d\,. \end{cases}$$

$\square$

Figure 3.2 shows, on the left, examples of the return of different constant strategies for the parameter set of Table 3.1 (given and further explained at the end of this chapter, p. 75) with $\theta = 0.8$. The considered retention levels are $b = 1$ (black), $b = 0.8$ (dark blue), $b = 0.5$ (medium blue) and $b = 0.2$ (light blue). We observe that none of the return functions lies below the others for all $x$. Thus, we can already conclude that none of the considered strategies is optimal.

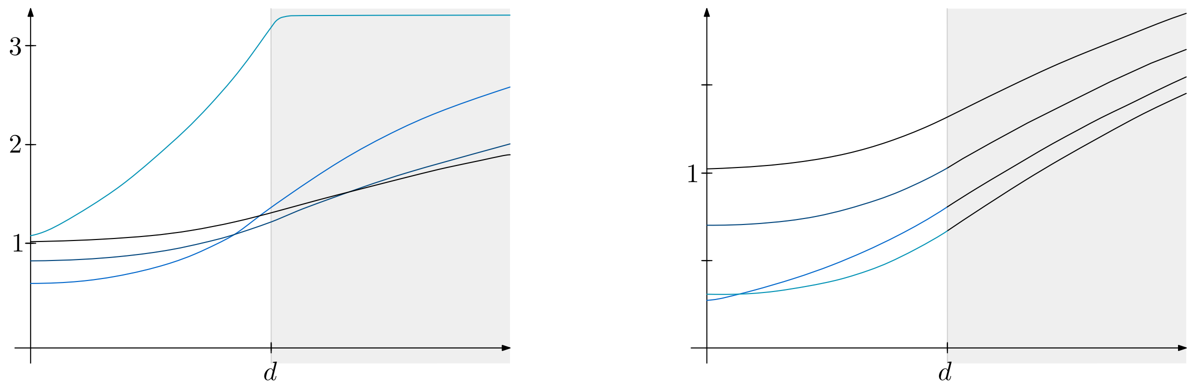



Figure 3.2 Return functions of constant controls with $b \in \{0.1, 0.5, 0.8, 1\}$ (*left*) and simple switching controls with $\check{b} \in \{0.1, 0.5, 0.8, 1\}$ and $\hat{b} = 1$ (*right*).

### 3.1.2 A Note on Simple Switching and General Feedback Controls

In view of the non-optimality of the considered constant controls, a canonical question is if this can be improved by considering 'regime dependent' strategies. We recall that a simple switching strategy $B$ is induced by a function $b(x) = \check{b}\mathbb{1}_{\{x\leq d\}} + \hat{b}\mathbb{1}_{\{x>d\}}$, so that $B_t = b(\Delta_t^B)$, $t \geq 0$. For the classical risk model, we argued at this point that the controlled process exists as a piecewise deterministic Markov process. For the diffusion case, it is also not trivial that the controlled process exists. Indeed, the drawdown under the simple switching control is defined by the relation

$$\Delta_t^B = M_t^B - \int_0^t \mu(b(\Delta_s^B))\,\mathrm{d}s - \int_0^t b(\Delta_s^B)\sigma\,\mathrm{d}W_s\,, \qquad t \geq 0\,,$$

which is a *reflected stochastic differential equation* with discontinuous coefficients. The following definition is taken from [Pilipenko, 2014, Ch. 1] and lightly modified to fit our notation.

Definition 3.5. [Pilipenko, 2014]

*Let $\tilde{\mu} : [0,\infty) \to \mathbb{R}$ and $\tilde{\sigma} : [0,\infty) \to \mathbb{R}$ be measurable functions. A pair $(\mathcal{C},\mathcal{E})$ of continuous and adapted processes $\mathcal{C} = (\mathcal{C}_t)_{t\geq 0}$ and $\mathcal{E} = (\mathcal{E}_t)_{t\geq 0}$ is a solution to the stochastic differential equation*

$$\mathcal{E}_t = x + \mathcal{C}_t - \int_0^t \tilde{\mu}(\mathcal{E}_s)\,\mathrm{d}s - \int_0^t \tilde{\sigma}(\mathcal{E}_s)\,\mathrm{d}W_s\,, \qquad t \geq 0\,, \tag{3.5}$$

*with reflection at zero if all integrals are well-defined, $\mathcal{E}$ is non-negative and $\mathcal{C}$ is non-decreasing with $\mathcal{C}_0 = 0$ and (1.10) for $(\mathcal{C},\mathcal{E})$ in place of $(c,\varepsilon)$.*

Remark. Usage of term 'reflected' in this context is motivated by the connection of maximum and drawdown to the Skorohod problem: the definition implies that $\mathcal{C}$ is the compensator for $\mathcal{E}$. That is, if a solution exists, $(\mathcal{C},\mathcal{E})$ is (pathwise) the unique solution to the Skorohod problem for the process $(-Y)_{t\geq 0}$ with $Y_t = -\mathcal{E}_t + \mathcal{C}_t$ for $t \geq 0$. In particular, by Corollary 1.3, this corresponds to the controlled surplus process up to distance-preserving shifts of the initial running maximum and surplus. #

We note that the process in Definition 3.5 is a solution in the 'strong' sense, i.e. a solution for a predefined Brownian motion on a given probability space. That means, we can say that Equation (3.5) has the property of *pathwise uniqueness* if any two solutions $\mathcal{E}^1$ and $\mathcal{E}^2$ fulfil $\mathcal{E}_t^1 = \mathcal{E}_t^2$ for all $t \geq 0$, almost surely. However, as our return and value functions are based on expected values, a 'weak' solution will generally be sufficient. That means, there exists a filtered probability space with the properties stated in the introduction with a Brownian motion and a process fulfilling the equation. *Weak uniqueness* therefore refers to uniqueness in law. For an exact definition and distinction of weak and strong solutions (which merely serve as technical tools here) we refer to [Protter, 2005], specifically pp. 204,246.

For reflected diffusion equations, such as in Definition 3.5, existence and uniqueness results are well studied. For example, similar to the basic result for stochastic differential equations without reflection, there exists a unique strong solution to (3.5) for continuous coefficients $\tilde{\mu}$ and $\tilde{\sigma}$ fulfilling a Lipschitz- and a linear growth condition, cf. [Pilipenko, 2014, Thm. 1.2.1].

For simple switching controls, the coefficients of (3.5) are discontinuous but still relatively 'uncomplicated'. If we assume that it holds $\check{b} > 0$ (such that the volatility component is bounded away from zero in the uncritical area), we obtain weak existence by Theorem 4.1 of [Rozkosz and Słomiński, 1997] and pathwise uniqueness by Corollary 4.3 of [Semrau, 2009]. This means that we can apply the Yamada–Watanabe type Theorem 333 of [Situ, 2005] which states that these two properties combined imply strong existence and uniqueness. In the following section, we prove a general verification theorem which states that a candidate function with certain properties is equal to the return function of a feedback strategy if the process under the feedback control exists. As it will turn out, the function defined below fulfils these conditions. As a consequence, we obtain:

THEOREM 3.6. *Writing* $u : [0,\infty) \to [0,\delta^{-1}]$ *for the return function of a simple switching control with* $(\check{b},\hat{b}) \in (0,1] \times [0,1]$*, we have*

$$u(x) = \begin{cases} C_1 e^{\xi(\check{b})x} + C_2 e^{-\kappa(\check{b})x}\,, & x \le d\,,\\ \delta^{-1} - [\delta^{-1} - (C_1 e^{\xi(\check{b})d} + C_2 e^{-\kappa(\check{b})d})]e^{-\kappa(\hat{b})(x-d)}\,, & x > d\,, \end{cases}$$

*where*

$$C_1 = \frac{\kappa(\check{b})\kappa(\hat{b})}{\delta[\kappa(\check{b})(\kappa(\hat{b})+\xi(\check{b}))e^{\xi(\check{b})d} - \xi(\check{b})(\kappa(\check{b})-\kappa(\hat{b}))e^{-\kappa(\check{b})d}]}\,, \qquad C_2 = \frac{\xi(\check{b})C_1}{\kappa(\check{b})}\,,$$

*with the interpretation of '*$\kappa(\hat{b}) = \infty$*' if* $\hat{b} = 0$*.* $u$ *is increasing and continuously differentiable on* $(0,\infty)$ *with* $u'(0) = 0$*.* $u|_{[0,d]}$ *and* $u|_{(d,\infty)}$ *are classical solutions to* $\mathcal{A}^{b(x)}u(x) = -\mathbb{1}_{\{x>d\}}$ *on the respective domain if* $u''(d)$ *is interpreted as a one-sided derivative.* □

Some examples of these return functions are displayed on the right hand side of Figure 3.2.
In the degenerate case $\check{b} = 0$, the controlled drawdown corresponds to the deterministic function $t \mapsto x + (\theta - \eta)t$ for $x \le d$ up to the first passage through $d$ at time $(d-x)(\theta-\eta)^{-1}$. By $\theta > \eta$, the drift is strictly positive. Thus, starting at $x > d$ with $\hat{b} > 0$, this could be viewed as a process with reflection at $d$. For the sake of conciseness, we exclude this 'outlier' from our consideration and move on to the optimisation problem.

## 3.2 Solution to the Optimisation Problem

In principle, we now follow the same 'roadmap' as in Chapter 2. That means, we derive a Hamilton–Jacobi–Bellman equation and prove that certain solutions thereof can be verified to be the function $v$, thus (in a way) obtaining uniqueness. Then we consider separately the two subproblems. The main part of this section is devoted to finding a strategy which maximises the time in the uncritical area. In the end, we reconnect the solutions. However, as our 'warm-up' already indicates, the proof techniques and results differ strongly from those of the second chapter (with the small exception of Subsection 3.2.2).

### 3.2.1 The HJB Equation Connected and a General Verification Theorem

The results of this section are based on [Brinker and Schmidli, 2021a] and formulated in such a way that they fit to the case considered in [Brinker and Schmidli, 2022]. We now prove that a solution (in a sense to be specified) $f : [0,\infty) \to \mathbb{R}$ to the Hamilton–Jacobi–Bellman equation

$$\inf_{b\in[0,1]} \Big\{ -\delta f(x) - \mu(b)f'(x) + \frac{\sigma^2 b^2}{2} f''(x) \Big\} = -\mathbb{1}_{\{x>d\}} \tag{3.6}$$

can be verified to be the value function in Equation (1.9). With the abbreviation introduced in (3.3), we could alternatively write $\inf_{b\in[0,1]} \mathcal{A}^b f(x) = -\mathbb{1}_{\{x>d\}}$. Similarly as in the constant control case, we note that there cannot exist a classical solution to this equation as the right hand side is discontinuous at $x = d$. Hence, if we refer to a function $f$ as a *solution* to (3.6), we mean the composition of two classical solutions to the homogeneous and inhomogeneous equation, connected at $x = d$ by a smooth fit. By $f''(d)$ we denote the derivative of $f'$ from the left at $d$, if not stated otherwise. A simple, yet important, observation is the following. For a strictly increasing function $f$ plugged into (3.6), the term to be maximised with respect to $b$ is given by

$$\mathfrak{I}_f(b) = -\mu(b)f'(x) + \frac{\sigma^2 b^2}{2} f''(x)$$

for a fixed $x$. If $f'(x) \geq 0$ and $f''(x) > 0$ are fulfilled at this point then, in view of

$$\mathfrak{I}_f'(b) = -\theta f'(x) + \sigma^2 b f''(x)\,, \qquad \mathfrak{I}_f''(b) = \sigma^2 f''(x)\,,$$

the optimum is attained at

$$b_f(x) = \frac{\theta f'(x)}{\sigma^2 f''(x)} \tag{3.7}$$

if this value is in $[0, 1]$. If, on the other hand, $f'(x) \geq 0$ with $f''(x) \leq 0$, the optimum is attained at $b = 1$. The next two lemmata are interesting on their own and help characterise the solutions to (3.6) we are looking for. We first show that $b = 0$ is usually not optimal.

LEMMA 3.7. [2021a]

*If $f : [0,\infty) \to \mathbb{R}$ is a strictly increasing solution to* (3.6)*, then there is no non-empty interval $(\underline{x}, \overline{x}) \subset [0,\infty)$ such that the pointwise optimiser $b^*$ is given by $b^*(x) = 0$ for all $x \in (\underline{x}, \overline{x})$.*

*Proof.* Suppose $b^*(x) = 0$ is the minimiser on $(\underline{x}, \overline{x}) \subset [0, d]$. Then the solution is of the form $f(x) = C\mathrm{e}^{x\delta/(\theta-\eta)}$ for $x \in (\underline{x}, \overline{x})$. Because $f$ is increasing, $C > 0$, and therefore $f''(x) > 0$ for $x \in (\underline{x}, \overline{x})$. This means that $\mathfrak{I}_f(b)$ is minimised at

$$b^*(x) = \Big(\frac{\theta f'(x)}{\sigma^2 f''(x)} \vee 0\Big) \wedge 1 = \Big(\frac{\theta(\theta-\eta)}{\sigma^2\delta} \vee 0\Big) \wedge 1 = \frac{\theta(\theta-\eta)}{\sigma^2\delta} \wedge 1 > 0\,.$$

This is a contradiction. In the same way we can argue for the case $(\underline{x}, \overline{x}) \subset (d,\infty)$, in which $b^*(x) = 0$ yields $f(x) = \delta^{-1} + C\mathrm{e}^{x\delta/(\theta-\eta)}$. □

We next show that a strictly increasing solution to (3.6) cannot change from convex to concave except at $d$.

LEMMA 3.8. [2021a]

*Let $f : [0,\infty) \to \mathbb{R}$ be a strictly increasing solution to* (3.6)*. At a point $\bar{x} \neq d$ with $f''(\bar{x}) = 0$, the function changes from concave to convex. Every bounded, strictly increasing solution $f$ to* (3.6) *is concave for $x > d$.*

*Proof.* We first notice that $f''(x) = 0$ cannot hold on an interval of positive length. Assume that there is an interval $(y_0, y_1)$ such that $y_1 < d$ or $y_0 > d$ and $f''(x) = 0$ for $x \in (y_0, y_1)$. Then the infimum is attained at $b^*(x) = 1$ for all $x \in (y_0, y_1)$. By solving the resulting equation, we obtain $f''(x) = C\delta[\mu(1)]^{-1}f'(x)$ for some constant $C \in \mathbb{R}$. But this implies that $C = 0$ and $f'(x) = 0$, which is a contradiction. Let $\bar{x}$ be a point with $f''(\bar{x}) = 0$. Suppose that $f'(\bar{x}) = 0$. Then $f(\bar{x}) = \delta^{-1}\mathbb{1}_{\{\bar{x}>d\}}$. Since $f$ is strictly increasing, there can at most be one such point on $(0, d)$ and $(d, \infty)$, respectively. If $f'(\bar{x}) > 0$, then the infimum of $\mathcal{J}_f(b)$ is taken in $b = 1$ in an environment of $\bar{x}$ by the continuity of $|\theta f'(x)/(\sigma^2 f''(x))|$, which tends to $\infty$. Note that this implies that $\mathcal{J}_f(b)$ is increasing in $b$ for any sign of $f''(x)$ in this environment. This implies that $f''$ is differentiable in an environment of $\bar{x}$. In particular, this shows that points with $f''(\bar{x}) = 0$ must be isolated points as we show next. Assume that there is such a point. Due to the continuity of the second derivative (except at $x = d$, which is excluded), either $f''(x) \geq 0$ or $f''(x) < 0$ on $(\bar{x} - \varepsilon, \bar{x})$ with a sufficiently small $\varepsilon > 0$. In any case, we can assume that $b = 1$ is optimal on $(\bar{x} - \varepsilon, \bar{x})$. If $f''(x) \geq 0$ on this interval, $f'(x)$ and $f(x)$ are both increasing, implying that $f''(x)$ is increasing, too, by

$$\frac{\sigma^2}{2} f''(x) = -\mathbb{1}_{\{x>d\}} + \delta f(x) + \mu(1)f'(x) \,,$$

where the indicator function is constant because we have excluded $d$. This contradicts $f''(\bar{x}) = 0$. We conclude that $f''(x) < 0$ on $(\bar{x} - \varepsilon, \bar{x})$. Let us suppose that, additionally, $f''(x) < 0$ on $(\bar{x}, \bar{x} + \tilde{\varepsilon})$ for some $\tilde{\varepsilon} > 0$. Then, the supremum is attained at $b = 1$ for $x \in (\bar{x} - \varepsilon, \bar{x} + \tilde{\varepsilon})$. By

$$\frac{\sigma^2}{2} f'''(x) = \delta f'(x) + \mu(1)f''(x) \,,$$

the third derivative of $f$ exists and is continuous on this interval. We get $f'''(\bar{x}) = 0$, so that it follows from the differential equation that $\delta f'(\bar{x}) = 0$ (and $\delta f(\bar{x}) = 0$). This is, again, a contradiction because $f'$ is strictly increasing on $(\bar{x} - \varepsilon, \bar{x} + \tilde{\varepsilon})$. Thus, the only possible case is that $f''$ changes its sign from negative to positive in $\bar{x}$. In particular, there can be at most one change of the sign on each interval, $(0, d)$ and $(d, \infty)$.
Now we prove the second statement. Assume that there exists an inflection point $\bar{x} > d$. Then, by the arguments above, $f''(x) > 0$ for all $x > \bar{x}$. The convexity implies that for fixed $x_1$ and $x_2$ with $x_2 > x_1 > \bar{x}$ and any $x > x_2$, we have:

$$f(x) \geq f(x_1) + (f(x_2) - f(x_1))\frac{x - x_1}{x_2 - x_1} \,,$$

where the right hand side diverges to $\infty$ as $x \to \infty$ because $f$ is strictly increasing. Then $f$ would not be bounded from above, which is a contradiction. $\square$

We formulate the following theorem in such a way that it can also be utilised to connect a predefined (feedback) strategy to its return. For example, this applies to the function $u$ defined in Theorem 3.6. For this purpose, we introduce the following notation. For $x \geq 0$, denote by $\mathcal{I}(x) \subseteq [0, 1]$ a compact set of possible retention levels to choose from when the drawdown is currently $x$. By $\mathcal{B}_\mathcal{I}$ we denote the set of strategies $B \in \mathcal{B}$ with $B_t \in \mathcal{I}(\Delta_t^B)$, $t \geq 0$. Our optimisation problem then corresponds to defining $\mathcal{I}(x) = [0,1]$ for all $x \geq 0$ and $\mathcal{B}_\mathcal{I} = \mathcal{B}$.

Theorem 3.9 (General Verification Theorem). [2021a]
*We assume $f : [0,\infty) \to \mathbb{R}$ is a bounded solution to*

$$\inf_{b \in \mathcal{I}(x)} \Big\{ -\delta f(x) - \mu(b) f'(x) + \frac{\sigma^2 b^2}{2} f''(x) \Big\} = -\mathbb{1}_{\{x>d\}} \tag{3.8}$$

*with $f'(0) \geq 0$. We have $f(x) \leq v^B(x)$ for every strategy $B \in \mathcal{B}_\mathcal{I}$ and all $x \geq 0$. If, in addition, for each $x$ there is a minimiser $b^*(x) \in \mathcal{I}(x)$ in (3.8) such that the surplus process $X^*$ and the corresponding drawdown process $\Delta^*$ under the feedback strategy $B^*$ with $B_t^* = b^*(\Delta_t^*)$, $t \geq 0$, exist and either $f'(0) = 0$ holds or the controlled running maximum $M^*$ is constant, then we have $f(x) = v^{B^*}(x) = \inf_{B \in \mathcal{B}_\mathcal{I}} v^B(x)$ for all $x \geq 0$.*

*Proof.* From the differentiability properties of $f$ we conclude that we can apply a generalised version of Itô's formula (compare, for example, [Elworthy et al., 2007, Thm. 2.1 and Eq. (2.23)]) to find:

$$\begin{aligned} e^{-\delta t} f(\Delta_t^B) - f(\Delta_0^B) &= \int_0^t e^{-\delta s} f'(\Delta_s^B)\, \mathrm{d}M_s^B - \int_0^t e^{-\delta s} \sigma B_s f'(\Delta_s^B)\, \mathrm{d}W_s + \int_0^t e^{-\delta s} \mathcal{A}^{B_s} f(\Delta_s^B)\, \mathrm{d}s \\ &\geq \int_0^t e^{-\delta s} f'(0)\, \mathrm{d}M_s^B - \int_0^t e^{-\delta s} \sigma B_s f'(\Delta_s^B)\, \mathrm{d}W_s - \int_0^t e^{-\delta s} \mathbb{1}_{\{\Delta_s^B > d\}}\, \mathrm{d}s \end{aligned} \tag{3.9}$$

for $B \in \mathcal{B}_\mathcal{I}$. Here we have used that the drawdown is equal to zero whenever the running maximum increases. Now, by its continuity on $[0,\infty)$, $f'$ is bounded on every compact interval $[0,n]$, $n \in \mathbb{N}$. We define a sequence of finite stopping times $(T_n)_{n\in\mathbb{N}}$ by $T_n = \vartheta^n(B) \wedge n$ for $n \in \mathbb{N}$ and we have that $T_n \to \infty$ as $n \to \infty$ (pathwise). For every $n$, the stochastic integral on the right hand side of (3.9) stopped at $T_n$ is a martingale of expectation 0. Taking expectations and using $f'(0) \geq 0$, we find:

$$f(x) \leq \mathbb{E}^x \big[ e^{-\delta t \wedge T_n} f(\Delta_{t\wedge T_n}^B) \big] + \mathbb{E}^x \Big[ \int_0^{t \wedge T_n} e^{-\delta s} \mathbb{1}_{\{\Delta_s^B > d\}}\, \mathrm{d}s \Big].$$

By bounded (first term) and monotone (second term) convergence, we firstly let $n \to \infty$ and then $t \to \infty$, obtaining $f(x) \leq v^B(x)$. Provided that $\Delta^*$ exists, we have $B^* \in \mathcal{B}_\mathcal{I}$ and we can repeat the argument with the strategy $B^*$ to obtain $f(x) = v^{B^*}(x)$ and therefore $f(x) = \inf_{B\in\mathcal{B}_\mathcal{I}} v^B(x)$. □

We can connect a feedback strategy induced by a function $b(x)$ to its return in the following way. For the feedback strategy $B$ associated with $b$, the retention level $B_t = b(\Delta_t^B)$ at time $t$ is determined by the function $b$ and the current drawdown. Suppose the surplus process $X^B$ under this feedback control $B$ exists (as a solution to a stochastic differential equation) and $f$ is a bounded solution to

$$-\delta f(x) - \mu(b(x)) f'(x) + \frac{\sigma^2 [b(x)]^2}{2} f''(x) = -\mathbb{1}_{\{x>d\}}$$

with $f'(0) = 0$ or such that $M^B$ is constant, then it follows from Theorem 4.6 with $\mathcal{I}(x) = \{b(x)\}$ for all $x \geq 0$ that $f$ is the return of this feedback control.

### 3.2.2 Minimising the Recovery Time

We consider in this section the function $V : (d, \infty) \to [0, 1]$ defined by

$$V(x) = \sup_{B \in \mathcal{B}} \mathbb{E}^x \big[\mathrm{e}^{-\delta \vartheta(B)}\big] \,. \tag{3.10}$$

By the same heuristics as in Chapter 2, we expect that the optimal strategy is constant and induces a maximal drift. This is indeed the case. We recall the definition $\kappa = \kappa(1)$ from Notation 3.1.

PROPOSITION 3.10. [2022]

*We have $V(x) = \mathrm{e}^{-\kappa(x-d)}$ for $x \geq d$ and a strategy which is constant and equal to $1$ up to the first passage through $d$ is optimal. The function $v$ fulfils*

$$v(x) = \frac{1}{\delta} - \Big(\frac{1}{\delta} - v(d)\Big)\mathrm{e}^{-\kappa(x-d)} \,, \qquad x > d \,,$$

*and* $\inf_{b \in [0,1]} \mathcal{A}^b v(x) = -1$ *with the pointwise optimiser* $b^*(x) = 1$ *for all* $x > d$.

*Proof.* Following the constant strategy $B$ with $B_t = 1$ up to time $\vartheta_d(B)$, we have $\Delta_t^B = -\mu(1)t - \sigma W_t$ for $t \leq \vartheta_d(B)$ and the process

$$\Big(\mathrm{e}^{-\kappa \Delta^B_{t \wedge \vartheta_d(B)} - \delta(t \wedge \vartheta_d(B))}\Big)_{t \geq 0}$$

is a martingale by the optional stopping theorem. Thus, taking expectations and letting $t \to \infty$, we obtain $\mathbb{E}^x[\mathrm{e}^{-\kappa d - \delta \vartheta_d(B)}] = \mathrm{e}^{-\kappa x}$ by bounded convergence. Therefore, $f(x) = \mathrm{e}^{-\kappa(x-d)}$ is the return $V^B(x)$ of this strategy for all $x > d$. On the other hand, let $B$ be an arbitrary admissible strategy. It follows that $\mathcal{A}^b f(x) \leq 0$ for all $x \geq d$ and $b \in [0, 1]$ by

$$-\delta - (\theta - \eta - \theta b)\kappa + \frac{\sigma^2 b^2}{2}\kappa^2 \leq -\delta + \eta\kappa + \frac{\sigma^2}{2}\kappa^2 = 0 \,.$$

Additionally, we get by the classical version of Itô's formula that the process

$$\Big(\mathrm{e}^{-\delta(t \wedge \vartheta_d(B))} f(\Delta^B_{t \wedge \vartheta_d(B)}) - f(\Delta^B_0) - \int_0^{t \wedge \vartheta_d(B)} \mathrm{e}^{-\delta s} \mathcal{A}^{B_s} f(\Delta^B_s) \,\mathrm{d}s\Big)_{t \geq 0}$$

is a martingale. From here, the assertion is deduced by following closely the proof of Proposition 2.17 and using continuity of paths of the drawdown process. □

### 3.2.3 Maximising the Time to Critical Drawdown

In this section, we consider the function $V : [0, d] \to [0, 1]$ defined by

$$V(x) = \inf_{B \in \mathcal{B}} \mathbb{E}^x \big[\mathrm{e}^{-\delta \vartheta(B)}\big] \,. \tag{3.11}$$

In the same way as Theorem 3.9, we obtain the following result.

Theorem 3.11 (Verification Theorem for $V$). [2022]
*We suppose $f : [0,d] \to [0,1]$ is a bounded solution to $\inf_{b\in[0,1]} \mathcal{A}^b f(x) = 0$ with $f'(0) \geq 0$ and $f(d) = 1$. We have $f(x) \leq V^B(x)$ for every strategy $B \in \mathcal{B}$ and all $x \in [0,d]$. If, in addition, for each $x$ there is a minimiser $b^*(x) \in [0,1]$ such that the surplus process $X^*$ and the corresponding drawdown process $\Delta^*$ under the feedback strategy $B^*$ with $B^*_t = b^*(\Delta^*_t)$, $t \geq 0$, exist and either $f'(0) = 0$ holds or $M^*$ is constant, then $f(x) = V^{B^*}(x) = V(x)$ for all $x \in [0,d]$.* □

We already know, by Lemma 3.8, that the second derivative of a candidate solution can only change its sign once and if it does, the function goes from concave to convex. The following technical result follows from the fact that if $f$ is concave on an interval, $b = 1$ is optimal, so that $f$ solves an ordinary differential equation. A detailed proof is found in the appendix, p. 119.

Lemma 3.12. [2022]
*Let $f : [0,d] \to \mathbb{R}$ be a non-negative and strictly increasing solution to $\inf_{b\in[0,1]} \mathcal{A}^b f(x) = 0$. There is no non-empty interval $(\underline{x}, \overline{x}) \subset [0,d]$ such that $f''(x) \leq 0$ for all $x \in (\underline{x}, \overline{x})$.*

This lemma implies that we can expect our value function $V$ to be convex. Intuitively, this means that the time up to the first critical drawdown is longer and decreases slower if the initial value is small. Partly, this can be explained with the effect that two paths with different small (as opposed to 'almost critical') initial values are more likely to meet before they reach the critical line for the first time. Another explanation is that the exponential preference discounting puts more weight on early time intervals. That means, close to the critical line, small differences of the initial drawdown have a greater impact than close to zero because the exit times are typically earlier.

Remark. We recall that also the functions in the numerical examples of Chapter 2 were convex on $[0,d]$. However, we did not consider convexity in our proofs for the general penalised overshoot problem $v_C$ in Chapter 2 because the above explanation only applies to sufficiently large $C \geq C_d$. #

Assuming convexity, we now construct a candidate function by a (partly) heuristic approach. Once we have finished the construction, we will use the explicit representation to fill in the missing information.

### Solving the Homogeneous HJB Equation

For the sake of clarity of presentation, we assign numbers to the different steps of the construction.
*Step 1:* As noted in Subsection 3.2.1, for an increasing and convex function $f : [0,\infty) \to (0,\infty)$, the pointwise minimiser $b^*$ takes the form $b^*(x) = b_f(x) = \theta f'(x)/(\sigma^2 f''(x))$ if this value lies in $[0,1]$. Plugging this expression into the Hamilton–Jacobi–Bellman equation yields the non-linear, modified equation

$$-\delta f(x) + (\theta - \eta) f'(x) - \frac{\theta^2}{2\sigma^2} \frac{[f'(x)]^2}{f''(x)} = 0\,. \tag{3.12}$$

Now we follow the approach of Højgaard and Taksar [1998] by using a substitution to solve this equation. To this purpose, we firstly notice that $x \mapsto -\ln(f'(x))$ is strictly decreasing and therefore

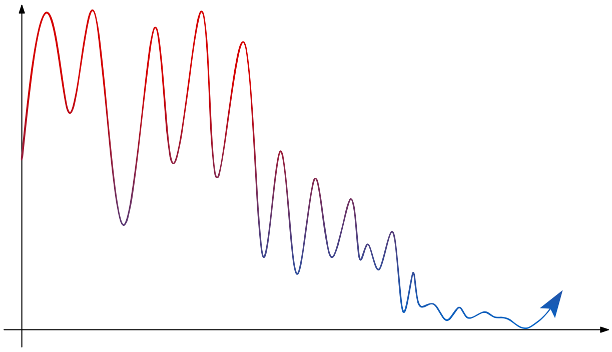


FIGURE 3.3 As the controlled drawdown approaches zero, it is forced away from the reflection barrier.

has an inverse function $Y$ with the same properties. With this definition, we have $f'(Y(z)) = \mathrm{e}^{-z}$ and $f''(Y(z)) = -\mathrm{e}^{-z}/Y'(z)$. Equation (3.12) takes the form

$$-\delta f(Y(z)) + (\theta-\eta)\mathrm{e}^{-z} + \frac{\theta^2}{2\sigma^2}\mathrm{e}^{-z}Y'(z) = 0\,. \tag{3.13}$$

Differentiating once again, we obtain an ordinary differential equation for $Y$, independent of $f$:

$$-\delta \mathrm{e}^{-z}Y'(z) - (\theta-\eta)\mathrm{e}^{-z} - \frac{\theta^2}{2\sigma^2}\mathrm{e}^{-z}Y'(z) + \frac{\theta^2}{2\sigma^2}\mathrm{e}^{-z}Y''(z) = 0\,,$$

or, equivalently:

$$-(\theta^2 + 2\delta\sigma^2)Y'(z) + \theta^2 Y''(z) = 2\sigma^2(\theta-\eta)\,.$$

The general solution to this equation is

$$Y(z) = C_1\mathrm{e}^{\zeta z} - \rho z - C_2\,, \qquad \zeta = \frac{2\delta\sigma^2+\theta^2}{\theta^2} > 1\,, \quad \rho = \frac{2\sigma^2(\theta-\eta)}{2\delta\sigma^2+\theta^2} > 0\,. \tag{3.14}$$

*Step 2:* This means, our next task is to derive appropriate constants $C_1$ and $C_2$. The initial conditions of Theorem 3.11 imply that it should hold $f'(0) = 0$ unless $X^*$ never exceeds the initial maximum $M_0^* = m_0$. If we assume that it holds $f'(Y(z)) = \mathrm{e}^{-z} = 0$, this means $z = \infty$ and $Y(\infty) = 0$. This is not possible because both, $\zeta$ and $\rho$, are strictly positive. As a consequence, we expect that $X^*$ never grows beyond the initial maximum. This can only happen if the drift becomes negative and, simultaneously, the volatility of the process goes to zero whenever $X^*$ approaches its maximum. That means, $b_f(x)$ has to converge to zero (in a certain way to be specified below), as $x$ approaches zero. This is illustrated in Figure 3.3 in which the graph represents the controlled drawdown process with high (red) and low (blue) retention levels. The combination $b_f(0) = 0$ and $f'(0) = 0$ would imply $f(0) = 0$, by (3.12), which is not possible. Therefore, we assume $\lim_{x\to 0} f''(x) = \infty$. That means, by (3.12) and $b_f(x) \to 0$ as $x \to 0$, we have the initial condition $f'(0) = \delta f(0)/(\theta-\eta)$ (assuming that $f(0)$ is known). From this, we derive initial conditions for $Y$: we let $z_0 = \ln[(\theta-\eta)/(\delta f(0))]$ such that

$$f'(Y(z_0)) = \mathrm{e}^{-z_0} = \frac{\delta f(0)}{\theta-\eta}$$

implies $Y(z_0) = 0$ and, thus, $Y'(z_0) = \mathrm{e}^{-z_0}/f''(0) = 0$. With this, we arrive at

$$Y(z) = \frac{\rho}{\zeta}\big[\mathrm{e}^{\zeta(z-z_0)} - \zeta(z - z_0) - 1\big]\,,$$

which is convex for $z \in \mathbb{R}$.

*Step 3:* Our next step is to find the inverse function $Z$ (and its domain and image) for $Y$ such that we can define the corresponding function $f$ by $f'(x) = f'(Y(Z(x))) = \mathrm{e}^{-Z(x)}$. We observe that $Y$ is not bijective as we have $\lim_{z\to\infty} Y(z) = \lim_{z\to-\infty} Y(z) = \infty$. Therefore, we have to choose which 'branch' of $Y$ to invert. Because $f$ is supposed to be convex, $f'$ should be increasing. That means, $Z$ (and also $Y$) should be decreasing. Explicit calculations show that $Y'(z) < 0$ for $z \in (-\infty, z_0)$. In particular, we define $Z : [0,\infty) \to (-\infty, z_0]$ as the inverse of $Y : (-\infty, z_0] \to [0,\infty)$. We prove in the appendix, p. 120, that $Z$ can be written in terms of the upper branch $\mathcal{W}$ of the Lambert W function:

$$Z(x) = -\frac{x}{\rho} - \frac{1}{\zeta} - \ln\Big(\frac{\delta f(0)}{\theta - \eta}\Big) - \frac{\mathcal{W}\big(-\mathrm{e}^{-(1+\zeta/\rho x)}\big)}{\zeta}\,. \tag{3.15}$$

The argument of the Lambert W function is an element of $[-\mathrm{e}^{-1}, 0]$ for all $x \geq 0$. This means, $Z$ is determined by the part of $\mathcal{W}$ in the grey rectangle on left hand side of Figure 3.4. Moreover, it holds $Z(0) = z_0$. With this definition, we choose the ansatz

$$f(x) = f(0) + \int_0^x \mathrm{e}^{-Z(y)}\,\mathrm{d}y\,. \tag{3.16}$$

Using a substitution in the integral, one can show that it holds

$$f'(x) = \frac{\delta f(0)}{\theta - \eta} Q(x)\,, \qquad f(x) = \frac{\delta f(0)}{\theta - \eta} P(x) \tag{3.17}$$

with the definitions

$$Q(x) = \Big[-\mathcal{W}\Big(-\mathrm{e}^{-(1+\zeta/\rho x)}\Big)\Big]^{-1/\zeta}\,, \qquad P(x) = \Big[1 + \frac{\mathcal{W}(-\mathrm{e}^{-(1+\zeta/\rho x)})}{1-\zeta}\Big]\rho\, Q(x)\,, \qquad x \geq 0\,. \tag{3.18}$$

We prove this identity in the appendix, p. 120.

*Step 4:* Lastly, we have a closer look at the function $b_f$, to see whether or not $f$ also solves the Hamilton–Jacobi–Bellman equation, i.e. to determine for which $x$ we have $b_f(x) \in [0,1]$. By noticing that it holds

$$b_f(x) = \frac{\theta f'(x)}{\sigma^2 f''(x)} = -\frac{\theta Y'(Z(x))}{\sigma^2} = \frac{\theta\rho}{\sigma^2}\Big[1 + \mathcal{W}\Big(-\mathrm{e}^{-(1+\zeta/\rho x)}\Big)\Big]\,, \tag{3.19}$$

we find that $b_f$ is a non-negative, strictly increasing function with $b_f(0) = 0$ and $\lim_{x\to\infty} b_f(x) = \theta\rho\sigma^{-2}$. That means, if $\theta\rho\sigma^{-2} \leq 1$, we have $b_f(x) \in [0,1]$ for all $x$. Otherwise, there exists a unique $x_0 > 0$, given by

$$x_0 = \frac{\rho}{\zeta}\ln\Big(\frac{\theta\rho}{\theta\rho - \sigma^2}\Big) - \frac{\sigma^2}{\theta\zeta}\,, \tag{3.20}$$

such that $b_f(x) \in [0,1]$ for all $x \le x_0$, $b_f(x_0) = 1$ and $b_f(x) > 1$ for all $x > x_0$. With this, our construction is completed.
As seen in the last step of the construction, we have to distinguish two cases, depending on whether $\theta\rho\sigma^{-2} \le 1$ is fulfilled. Plugging in the definition of $\rho$, we obtain that this can be written as a condition for the parameter $\theta$, which is 'externally' determined by the reinsurer. This inspires the following definition.

NOTATION 3.13. *We distinguish the cases of cheap reinsurance with $\theta \le \sigma^2\xi = \eta + \sqrt{\eta^2 + 2\delta\sigma^2}$ (or, equivalently, $\theta\rho \le \sigma^2$) and expensive reinsurance with $\theta > \sigma^2\xi$ (that is, $\theta\rho > \sigma^2$).*

Since $\theta$ represents the safety loading of the reinsurance premium and $\eta$ is the safety loading charged by the insurer, these inequalities can be interpreted as the relation of the respective prices of re- and first insurance. It should be acknowledged that, in this context, 'expensive' means that the reinsurance safety loading is more than twice as large as the first insurance safety loading. The case of 'cheap' reinsurance thus covers most realistic scenarios. However, including the case of expensive reinsurance in our considerations allows further insights on the influence of pricing (in Section 3.3). Moreover, it will be useful to assign a name to the term on the far right of Equation (3.19) as a function of $x$:

NOTATION 3.14. *We define $r : [0,\infty) \to [0, \theta\rho\sigma^{-2}]$ by*

$$r(x) = \frac{\theta\rho}{\sigma^2}\Big[1 + \mathcal{W}\Big(-\mathrm{e}^{-(1+\zeta/\rho x)}\Big)\Big], \qquad x \ge 0\,. \tag{3.21}$$

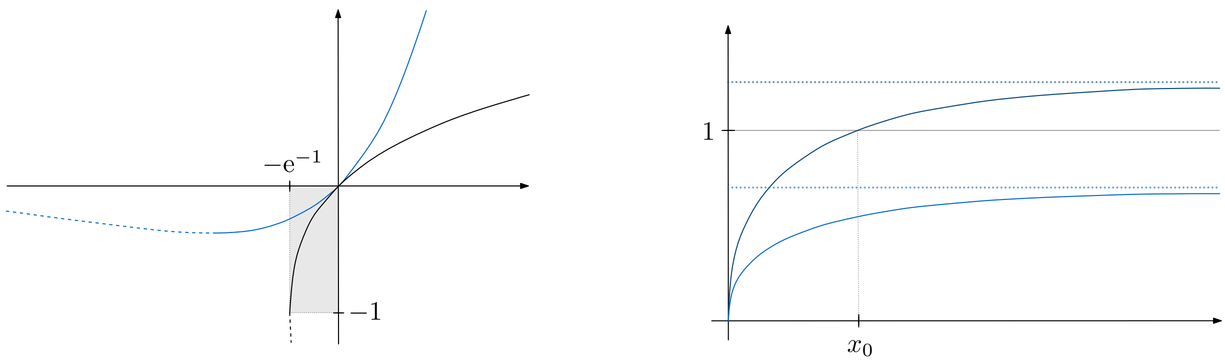



FIGURE 3.4 *Left:* The functions $x \mapsto x\mathrm{e}^x$ (blue graph) and $\mathcal{W}$ (black solid graph). *Right:* $r$ (solid) and $\theta\rho\sigma^{-2}$ (dotted) for $\theta$ categorised as cheap (light blue) and expensive (dark blue).

From our construction, we deduce:

PROPOSITION 3.15. [2022]
*We assume that reinsurance is cheap or that reinsurance is expensive with $x_0 \ge d$. The function $f : [0,d] \to [0,1]$ given by $f(x) = P(x)/P(d)$ fulfils $f(d) = 1$, $f'(0) > 0$ and is a solution to $\inf_{b\in[0,1]} \mathcal{A}^b f(x) = 0$ for all $x \in [0,d]$. The pointwise minimiser $b^*$ is given by $b^*(x) = r(x)$ for $x \in [0,d]$.* □

With a few extra arguments (found in the appendix, p. 121), we can extend the solution to $[0,d]$ in the case of expensive reinsurance with $x_0 < d$. In particular, because $b_f$ is increasing, the ansatz is to choose the minimiser $b = 1$.

Proposition 3.16. [2022]

*Assume that reinsurance is expensive with $x_0 < d$. We define $f : [0,d] \to [0,1]$ by*

$$f(x) = \begin{cases} C_0 P(x)\,, & x \le x_0\,, \\ C_1 \mathrm{e}^{\xi x} + C_2 \mathrm{e}^{-\kappa x}\,, & x > x_0\,, \end{cases}$$

*with*

$$\begin{aligned} C_0 &= \frac{\kappa\xi\sigma^2(\xi+\kappa)}{Q(x_0)[(\kappa\sigma^2+\theta)\kappa\mathrm{e}^{\xi(d-x_0)} - (\sigma^2\xi-\theta)\xi\mathrm{e}^{-\kappa(d-x_0)}]}\,, \\ C_1 &= \frac{C_0 Q(x_0)\mathrm{e}^{-\kappa(d-x_0)}+\kappa}{\xi\mathrm{e}^{\xi x_0}\mathrm{e}^{-\kappa(d-x_0)}+\kappa\mathrm{e}^{\xi d}}\,, \qquad C_2 = \frac{1-C_1\mathrm{e}^{\xi d}}{\mathrm{e}^{-\kappa d}}\,. \end{aligned} \tag{3.22}$$

*This function fulfils $f(d) = 1$, $f'(0) > 0$ and is a solution to $\inf_{b\in[0,1]} \mathcal{A}^b f(x) = 0$ for all $x \in [0,d]$. The pointwise minimiser $b^*$ is given by $b^*(x) = r(x)\mathbb{1}_{\{x\in[0,x_0]\}} + \mathbb{1}_{\{x\in(x_0,d]\}}$.*

In view of our verification theorem, this means that $f$ can be shown to be the value function $V$ if we are able to prove existence of the controlled process. To this purpose, we derive next some important properties of the function $r$ determining the optimiser.

### Properties of the Minimiser and Existence

We note that $r$, defined in Notation 3.14, is independent of $f$ and $d$. The following lemma summarises our findings 'hidden' in the construction and further technical properties of this function.

Lemma 3.17. [2022]

i) *$r$ is strictly increasing and concave with $r(0) = 0$ and $\lim_{x\to\infty} r(x) = \theta\rho\sigma^{-2}$.*
ii) *In the case of expensive reinsurance, $\theta > \sigma^2\xi$, there exists a unique $x_0 > 0$, defined as in (3.20), with $r(x_0) = 1$. Interpreted as a function of the parameter $\theta$, $x_0 : (\sigma^2\xi, \infty) \to (0,\infty)$ is strictly decreasing with $\lim_{\theta\to\sigma^2\xi} x_0(\theta) = \infty$ and $\lim_{\theta\to\infty} x_0(\theta) = 0$. In particular, there exists a unique $\theta_d \in (\sigma^2\xi,\infty)$ with $x_0(\theta_d) = d$.*
iii) *There exists a finite constant $C > 0$ such that $r(x) \le C\sqrt{x}$ holds for all $x \ge 0$. For every $\varepsilon > 0$ there is a finite constant $c > 0$ such that $r(x) \ge c\sqrt{x}$ for all $x \in [0,\varepsilon]$. In particular, $r$ is ½-Hölder continuous.*

Statement ii) implies that the condition $x_0 < d$ could be viewed as another condition on $\theta$, i.e. $\theta > \theta^d$. This means, Proposition 3.15 covers the cases of cheap and 'moderately' expensive reinsurance and Proposition 3.16 corresponds to the case of 'extremely' expensive reinsurance. iii) implies that $\sqrt{x}$ is an asymptotically sharp bound for $r(x)$ as $x \to 0$, which we will use below. The proof of Lemma 3.17 is found in the appendix, p. 121.

Now we show existence of the controlled processes. In consideration of our construction, we expect that the running maximum never increases (compare Figure 3.3). Therefore, we start with a regular stochastic differential equation without reflection.

Lemma 3.18. [2022]

*We define $b^* : \mathbb{R} \to [0,1]$ by $b^*(x) = r(x)\mathbb{1}_{\{x\in[0,x_0]\}} + \mathbb{1}_{\{x>x_0\}}$ with the interpretation $[0,x_0] = [0,\infty)$*

*if $x_0$ does not exist. On the set $[0, d]$, this function coincides with $b^*$ defined in Propositions* 3.15 *or* 3.16*, respectively. There exists a unique strong solution to the stochastic differential equation*

$$\mathcal{E}_t = x + \int_0^t [(\theta - \eta) - \theta b^*(\mathcal{E}_s)] \, \mathrm{d}s - \int_0^t \sigma b^*(\mathcal{E}_s) \, \mathrm{d}W_s \,, \qquad t \geq 0 \,. \tag{3.23}$$

Many existence and uniqueness results impose conditions on the linear growth of the drift component. However, Lemma 3.17 implies that there is no linear bound for our drift close to zero. One possibility would be to approximate the differential equation and construct a series of solutions which converge to a limiting process. This is done for a comparable stochastic differential equation in Chapter 6.2 of [Ikeda and Watanabe, 1989]. However, the specific information on $b^*$ allows us to use a combination of different shortcuts for diffusion equations.

*Proof of Lemma* 3.18. We define $\alpha(x) = (\theta - \eta) - \theta b^*(x)$ and $\beta(x) = -\sigma b^*(x)$, $x \in \mathbb{R}$, and observe that both of these functions are bounded and continuous. Theorem 2.2 together with Remark 2.1 in [Ikeda and Watanabe, 1989] ensures existence of a weak solution. Since $\beta$ is Hölder continuous and $\alpha$ is non-increasing it follows from [Yamada, 1973, Ex. 1.1], that Equation (3.23) has the property of pathwise uniqueness. Hence, there is a unique strong solution by Theorem 1.1 of Ikeda and Watanabe [1989]. □

Heuristically, it is clear that the solution $\mathcal{E}$ from the previous lemma is non-negative due to the definition of $b^*$ (recall, again, Figure 3.3). The next lemma specifies the behaviour of $\mathcal{E}$ at its reflecting boundary.

Lemma 3.19. *Denote by $\mathcal{E}$ the solution to* (3.23) *in Lemma* 3.18*.*

i) *$\mathcal{E}$ is non-negative with $\mathcal{E}_t \geq 0$ for all $t \geq 0$, almost surely, for all initial values $x \geq 0$.*
ii) *We denote by $\tau_0$ the time of the first arrival $\tau_0 = \inf\{t \geq 0 : \mathcal{E}_t = 0\}$ at zero. If reinsurance is cheap and $\theta^2 < 2\delta\sigma^2$, we have $b^*(x) = r(x)$, $x \geq 0$, and*

$$\mathbb{P}^x[\tau_0 < \infty] = \Big[1 - \frac{\sigma^2}{\theta\rho} r(x)\Big]^{(2\delta\sigma^2 - \theta^2)/(2\delta\sigma^2 + \theta^2)} \,, \qquad x > 0 \,. \tag{3.24}$$

*Otherwise, $\tau_0$ is almost surely finite.*

*Proof.* We start with i). For $x \geq 0$, it follows from Theorem 4.53 in [Engelbert and Schmidt, 1991] that the stochastic differential equation

$$Z_t = x + \int_0^t [-\theta b^*(Z_s)] \, \mathrm{d}s + \int_0^t [-\sigma b^*(Z_s)] \, \mathrm{d}W_s \tag{3.25}$$

has a unique strong solution $Z$ which is non-negative in the above sense. Intuitively, the reason (and difference to (3.23)) is that $Z$, given by Equation (3.25), is 'trapped' in zero: once $Z$ reaches the x-axis, drift and volatility vanish. We note that it holds $(\theta - \eta) - \theta b^*(x) \geq -\theta b^*(x)$, that is, the drift component in (3.23) is larger than the one in (3.25). Thus, the comparison theorem in [Ikeda and Watanabe, 1989, Thm. 1.1] implies $\mathcal{E}_t \geq Z_t \geq 0$ for all $t \geq 0$, almost surely. We prove assertion ii) in the appendix, p. 122, because it is rather complementary than necessary for our existence result (but interesting nonetheless). □

Remark. As it will turn out, the re-appearing constant $\theta\rho\sigma^{-2}$ is a critical retention level. Another critical value is the retention level $b_1 = (\theta-\eta)\theta^{-1} \in (0,1)$ of 'zero drift'. The inequality $\theta\rho\sigma^{-2} \geq b_1$ is equivalent to $\theta^2 \geq 2\delta\sigma^2$ and can only be harmed if reinsurance is cheap and $\eta^2 < 2\delta\sigma^2$. For $\theta^2 \geq 2\delta\sigma^2$, the retention level $\theta\rho\sigma^{-2}$ leads to a negative drift of the process $\mathcal{E}$ with non-zero volatility $\theta\rho\sigma^{-1}$. Otherwise, the drift of the process is positive for this retention level. This observation yields an intuitive explanation for the bound for $\theta$ in Lemma 3.19 ii). #

Corollary 3.20. [2022]

*For $b^*$ as defined in Propositions 3.15 or 3.16, respectively, the surplus process $X^*$ (with initial capital $X_0^* = \nu_0$), its drawdown $\Delta^*$ and running maximum $M^*$ (starting at $M^* = m_0$) under the feedback control $B^*$ with $B_t^* = b^*(\Delta_t^*)$, $t \geq 0$, exist. In particular, $B^*$ is an admissible strategy. The respective function $f$ defined in Propositions 3.15, 3.16 is the value function, i.e. $f(x) = V^{B^*}(x) = V(x)$ for all $x \in [0,d]$, and $B^*$ is optimal.*

*Proof.* Let $x = m_0 - \nu_0$. By Lemmata 3.18 and 3.19 i), it follows that $\mathcal{E}$ and $\mathcal{C}$ with $\mathcal{C}_t = 0$, $t \geq 0$, form a solution to the corresponding reflected equation of Definition 3.5. It follows from the uniqueness of solutions to the Skorohod problem, Corollary 1.3, that $\mathcal{E}$ is the drawdown and $\mathcal{C}$ is the running maximum of the process $\mathcal{X} = -\mathcal{E}$, that is,

$$\mathcal{X}_t = -(m_0-\nu_0) + \int_0^t [\eta - (1-\theta b^*(\mathcal{E}_s)\theta]\,\mathrm{d}s + \int_0^t \sigma b^*(\mathcal{E}_s)\,\mathrm{d}W_s\,, \qquad t \geq 0\,.$$

In particular, $X^*$ is defined by $X_t^* = \mathcal{X}_t + m_0$, $t \geq 0$, with the constant running maximum $M^*$ with $M_t^* = m_0$, $t \geq 0$. Because $b^*$ is continuous, $B^*$ is continuous and adapted and, thus, progressively measurable. This means that the strategy is admissible. The assertion follows by Theorem 3.11. ☐

With this, we have explicitly calculated a strategy to postpone the first critical drawdown for as long as possible. We discuss consequences and interpretations of Corollary 3.20 in Section 3.3 at the end of this chapter. Next, we return to our original problem of minimising the expected time in drawdown. We note that, by Theorem 1.5 and because the paths of the drawdown of a diffusion process are continuous, we have $v(x) = v(d)V(x)$, $x \in [0,d]$. The following result is therefore a direct consequence of Corollary 3.20.

Corollary 3.21. [2022]

*The function $v$ fulfils $v(x) = v(d)V(x)$ and $\inf_{b\in[0,1]} \mathcal{A}^b v(x) = 0$ for $x \in [0,d]$ with the pointwise optimiser $b^*$ defined as in Propositions 3.15 or 3.16.* ☐

### 3.2.4 Minimal Expected Time in Critical Drawdown

By Proposition 3.10 and Corollary 3.21, we know that – up to the unknown value $v(d)$ – $v$ is composed of a classical solution to $\inf_{b\in[0,1]} \mathcal{A}^b v(x) = -1$ for $x > d$ and a classical solution to $\inf_{b\in[0,1]} \mathcal{A}^b v(x) = 0$ for $x \leq d$. The corresponding pointwise minimiser in the Hamilton–Jacobi–Bellman equation is

$$b^*(x) = \begin{cases} \theta\rho\sigma^{-2}\big[1 + \mathcal{W}(-\mathrm{e}^{-(1+\zeta/\rho x)})\big]\,, & x \in [0, x_0 \wedge d]\,, \\ 1\,, & x > x_0 \wedge d\,, \end{cases} \tag{3.26}$$

with the interpretation '$x_0 = \infty$' if reinsurance is cheap.

Lemma 3.22. [2022]

*Let $b^* : \mathbb{R} \to [0,1]$ be defined as in Equation (3.26) for $x \geq 0$ and, otherwise, equal to zero. The surplus process $X^*$ (with initial capital $X_0^* = \nu_0$), its drawdown $\Delta^*$ and running maximum $M^*$ (starting at $M^* = m_0$) under the feedback control $B^*$ with $B_t^* = b^*(\Delta_t^*)$, $t \geq 0$, exist. $M^*$ is constant and $B^*$ is an admissible strategy.*

In the case of expensive reinsurance with $x_0 \leq d$, $b^*$ coincides with the 'old' optimiser from the previous section and the assertion follows from Corollary 3.20. Otherwise, $b^*$ has a jump at $x = d$ such that the classical results by Ikeda and Watanabe [1989] used in the proof of Lemma 3.18 cannot be applied. One possibility is to adapt the techniques used by Halidias and Kloeden [2006] in the proof of their Theorem 3.1 to the case of a decreasing drift and a ½-Hölder continuous volatility (for diffusion equations). Another possibility is to apply Theorem 1 of [Kyprianou and Loeffen, 2010] by connecting $\Delta^*$ to a so-called 'refracted' Lévy process. We provide details on both methods in the appendix, p. 124. Additionally, it should be noted that this special case is the only scenario in which we 'just' get a progressively measurable $B^*$ instead of a continuous strategy. In particular, $b^*(d) \neq b^*(d+)$ causes a jump of $t \mapsto B_t^*(\omega)$ if $t \mapsto \Delta_t^*(\omega)$ approaches $d$.
In view of our verification theorem, the natural candidate for $v(d)$ is the constant $C_d = \kappa/[\delta(V'(d-) + \kappa)] \in (0, \delta^{-1})$ leading to a smooth fit of $f(x) = C_d V(x)$, $x \leq d$, and $g(x) = \delta^{-1} - (\delta^{-1} - C_d)\mathrm{e}^{-\kappa(x-d)}$, $x > d$. Here, $V$ is the maximised Laplace transform of the first exit from $[0,d]$ defined in (3.11). Indeed, the desired explicit representation of the minimal expected time in drawdown and an optimal strategy follow directly by Theorem 3.9, Proposition 3.10, Corollary 3.21 and Lemma 3.22. We recall that $Q$ and $P$ are defined in terms of the Lambert W function, by (3.18).

Theorem 3.23. [2022]

*The function $v$ is the unique solution to (3.6) fulfilling the conditions stated in Theorem 3.9. An optimal strategy is the feedback strategy induced by the pointwise minimiser $b^*$ given in Equation (3.26), above. In particular, the controlled running maximum is constant.*
*Moreover, $v$ has the following explicit representations:*

i) *If reinsurance is cheap or expensive with $x_0 \geq d$, we have*

$$v(x) = \begin{cases} C_d P(x)/P(d)\,, & x \in [0,d]\,, \\ \delta^{-1} - (\delta^{-1} - C_d)\mathrm{e}^{-\kappa(x-d)}\,, & x > d\,, \end{cases}$$

*with*

$$C_d = \frac{\kappa P(d)}{\delta(Q(d) + P(d)\kappa)} = v(d)\,.$$

ii) *If reinsurance is expensive with $x_0 > d$, we have, for the constants $C_0$, $C_1$ and $C_2$ given in*

*Equation* (3.22),

$$v(x) = \begin{cases} C_d C_0 P(x)\,, & x \in [0, x_0]\,, \\ C_d(C_1 \mathrm{e}^{\xi x} + C_2 \mathrm{e}^{-\kappa x})\,, & x \in (x_0, d]\,, \\ \delta^{-1} - (\delta^{-1} - C_d)\mathrm{e}^{-\kappa(x-d)}\,, & x > d\,, \end{cases}$$

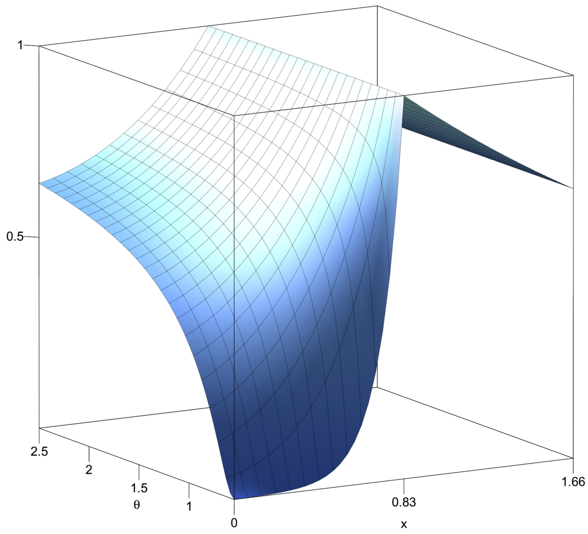
*with*

$$C_d = \frac{\kappa}{\delta(C_1 \xi \mathrm{e}^{\xi d} - C_2 \kappa \mathrm{e}^{-\kappa d} + \kappa)} = v(d)\,.$$

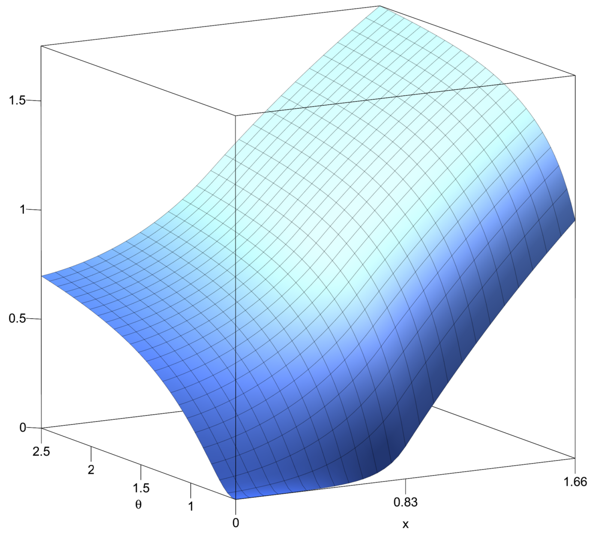
□

## 3.3 Numerical Examples

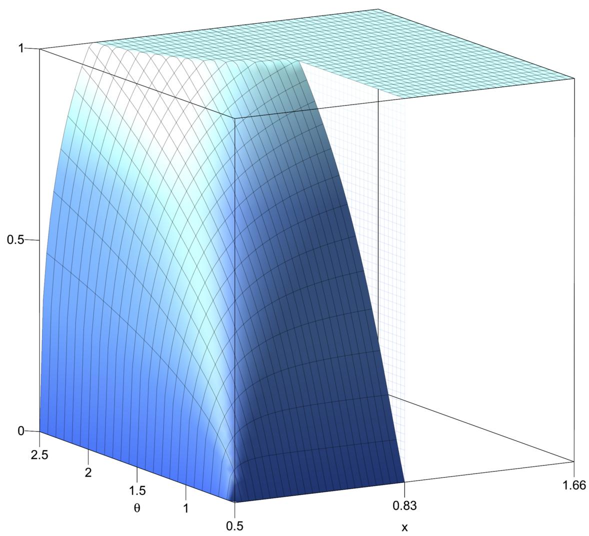
For our numerical study, we consider the parameter set given in Table 3.1 on p. 75. We assume in all cases that the drift of the insurance surplus is $\eta = 0.5$ with volatility $\sigma = 1$. The parameters $\delta = 0.3$ and $d = 0.83$ could be interpreted as a long term orientation with a moderately high tolerance for drawdowns. The price of reinsurance is an external variable which cannot be changed by the insurer and, as we have seen, influences if and to what extent reinsurance is bought. Thus, a naturally arising question is, at which maximal price reinsurance is still cost-effective. The explicit representations allow us to consider the value function and the pointwise minimiser as functions of $x$ and $\theta$ and find an answer to this question for the diffusion model. We consider $(x, \theta) \in [0, 2d] \times [\eta + 10^{-2}, 2.5]$. The graph in Figure 3.5(A) belongs to the value functions of our subproblems: the respective, optimised Laplace transform $V$ of the passage time of $d$ which takes values in $[0, 1]$. For $x > d$, we define $V$ in (3.10) as the minimal Laplace transform of the recovery time and, for $x \leq d$, we define $V$ in (3.11) as the maximal Laplace transform of the time to critical drawdown. In particular, small values of $V$ in Figure 3.5(A) suggest that the controlled process stays in the initial regime for a long time whereas values close to 1 indicate that the drawdown will soon transition to the adjoining area. Because for $x > d$, the optimal strategy is independent of $\theta$, $V$ (given by $(x, \theta) \mapsto \mathrm{e}^{-\kappa(x-d)}$) is independent of $\theta$ as well. For $x \leq d$, the optimal strategy is given in terms of the Lambert W function and $V$ is increasing and convex. We recall that this reflects that the time until the favourable area is left is shorter and decreases faster for initial drawdowns close to $d$. Additionally, we can see in Figure 3.5(A) that this effect is stronger if reinsurance is cheap. Figure 3.5(B) shows the value function $v$ of our original optimisation problem of minimising the expected time in drawdown, defined in (1.9), as a function of $x$ and $\theta$. This function is convex for $x \leq d$ and concave for $x > d$. In particular, for $x > d$, the function depends on $\theta$ only by the value on the boundary, $v(d)$. As $x \to \infty$, it converges to $\delta^{-1} \approx 3.3333$ in our example. As for the classical risk model, we observe that the time in critical drawdown is generally longer for larger 'price' parameters $\theta$. The pointwise optimiser, defined as in Equation (3.26), for all three problems is displayed in Figure 3.5(C). For critical $x > d$, the optimal retention level is always equal to one, that means, no reinsurance should be bought. For smaller $x$, the optimiser takes the shape induced by the Lambert W function. In particular, for all $\theta$ for which $x_0(\theta)$ is not defined or




(A) Optimised Laplace transform $V$ of $\vartheta(B)$.




(B) Minimal expected time in critical drawdown $v$.



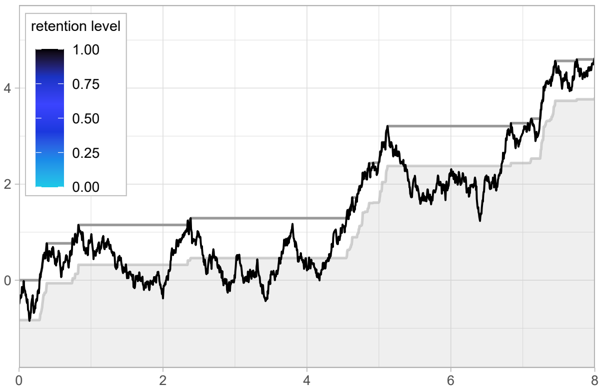

(C) Pointwise minimiser $b^*$.

FIGURE 3.5 Value functions and pointwise minimiser in dependence of $x \in [0, 2d]$ and $\theta \in [\eta + 10^{-2}, 2.5]$ for the parameter set of Table 3.1.

| $\eta$ | $\sigma$ | $\delta$ | $d$ |
|---|---|---|---|
| 0.5 | 1 | 0.3 | 0.83 |

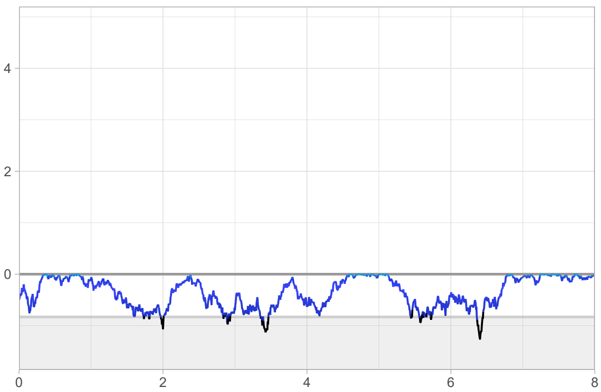
| $\sqrt{2\delta\sigma^2}$ | $\eta + \sqrt{\eta^2 + 2\delta\sigma^2}$ | $\theta_d$ |
|---|---|---|
| 0.7746 | 1.4220 | 1.5790 |

TABLE 3.1 Parameters of insurance surplus and preference (*left*) and critical reinsurance prices (*right*).

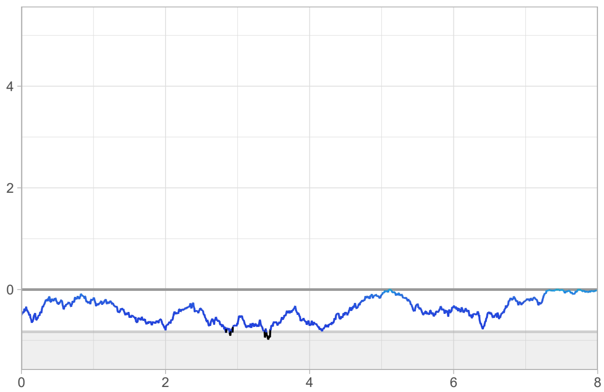



(A) No reinsurance

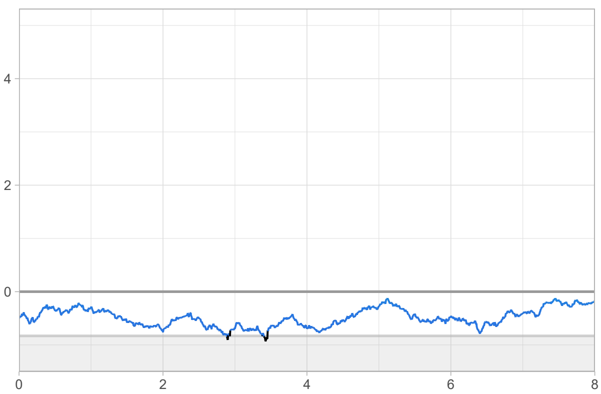
(B) $\theta = 1.6$



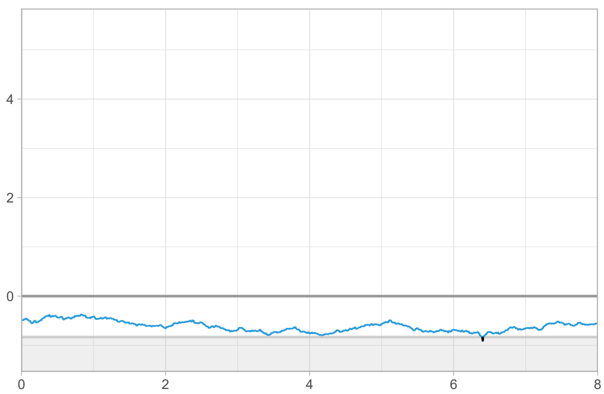
(C) $\theta = 1.2$



(D) $\theta = 0.8$



(E) $\theta = 0.7$



(F) $\theta = 0.6$

FIGURE 3.6 Path simulations without reinsurance and with optimal feedback strategies for different $\theta$.

which fulfil $x_0(\theta) > d$, this is the case whenever the drawdown is uncritical, $x \in [0, d]$. This causes a discontinuity of $x \mapsto b^*(x, \theta)$ at $x = d$. In our example, the largest safety loading qualifying as 'cheap' reinsurance is 1.4220, as seen on the right in Table 3.1. That means, for $\theta \leq 1.4220$, $x_0(\theta)$ is not defined and it is always optimal to buy reinsurance in the uncritical area (independently of the critical drawdown size $d$). For our explicit $d$, we can determine another critical safety loading: $\theta_d \approx 1.5790$. This is the smallest $\theta$ fulfilling $x_0(\theta) \leq d$. For $\theta \geq \theta_d$, the optimal retention level is equal to one for $x \in [x_0(\theta), d]$. In particular, the boundary of the flattened area on the top of the graph of $b^*$ is the curve $\theta \mapsto (x_0(\theta), \theta)$.

The properties of $v$ and the influence of $\theta$ are also visible in the path simulations of Figure 3.6. These are colour coded in the same way as in the previous chapter, i.e. small retention levels are represented by a blue tint of the graph. We use the same realisations of the driving Brownian motion's increments for all graphs. We compare a simulation of a path of the surplus without reinsurance, (A), to paths under optimal reinsurance for different $\theta \in \{1.6, 1.2, 0.8, 0.7, 0.6\}$, (B)–(F). The largest value, $\theta = 1.6$, is considered 'expensive', all other values are 'cheap'. A special choice is $\theta = 0.6 < \sqrt{2\delta\sigma^2} \approx 0.7746$. Here, the maximal retention level $\theta\rho\sigma^{-2} = 0.1250$ for uncritical drawdowns leads to a negative drift $\mu(0.1250) = -0.0250$ of the controlled surplus process. As we would expect (in view of Lemma 3.19 ii)), this path does not arrive at the initial maximum within the observed time span. In this example, reinsurance is so cheap that one accepts the resulting small positive drift of the drawdown in order to eliminate (almost all) volatility. Intuitively, we therefore have a smooth transition to the degenerate case with $\theta = \eta$, in which the insurer would sell all risk to never have a critical drawdown.

In general, in the cases with $\theta^2 > 2\delta\sigma^2$, the drift of the controlled surplus is negative in the upper part of the favourable area and positive in the lower part (as for $\theta \in \{1.6, 1.2, 0.8\}$). In the cases with $\theta^2 < 2\delta\sigma^2$, the drift is always negative (as for $\theta \in \{0.6, 0.7\}$). Roughly speaking, this means that in the first case, the process is stabilised within the favourable area, whereas in the second case, it tends towards the critical line until it exits. Then, it is pushed back (with maximal drift) into the uncritical area.

We note that the influence of $\theta$ is just one aspect which can be analysed via the explicit representation. We have seen that the optimal strategies only depend on $d$ by the position of the 'cut'. In particular, the strategy for small initial drawdowns postpones reaching any larger level for as long as possible. If $\sigma$ is large, generally more reinsurance should be bought in the uncritical area. This is comparable to our findings of the previous chapter for large claims. Similarly, a short-time oriented, optimal insurer has a higher tolerance for reinsurance costs because the main objective is to avoid immediate drawdowns.

## 3.4 Key Findings and Concluding Remarks

In this chapter, we found an explicit solution to the optimisation problem of minimising the expected time with critical drawdown defined in Equation (1.9). We considered firstly the return functions of constant and simple switching controls. We discovered that feedback controlled drawdown processes are determined by reflected stochastic differential equations. For the diffusion setting, this concept replaced the piecewise deterministic Markov processes of Chapter 2. After proving a general verification theorem, Theorem 3.9, we considered the subproblems derived from the dynamic programming principle, Theorem 1.5. In particular, we saw that these correspond to optimising the Laplace transform of the passage of $d$ for the diffusion approximation. For the problem of minimising the time until entering the uncritical area, we could build on our knowledge obtained in Chapter 2 to quickly obtain that the optimal strategy is to choose a maximal drift. For the problem of maximising the time to the first critical drawdown, we constructed an explicit solution to the homogeneous Hamilton–Jacobi–Bellman equation by, firstly, modifying the equation using the 'technical' optimiser and a substitution, secondly, solving the modified equation, thirdly, calculating a candidate function by re-substitution and, lastly, analysing the admissibility of the optimiser. We reconnected the subsolutions with a smooth fit to obtain an explicit representation of $v$ and the function $b^*$ inducing an optimal feedback control. In our numerical examples, we examined further the dependence on the parameter $\theta$ as a proxy for costs of reinsurance.

Although we considered a different model (requiring different mathematical methods) than in Chapter 2, our key observations coincide on the content level. That is, if the drawdown is critical, no reinsurance is bought in order to leave the unfavourable area as fast as possible. Below the critical line, there is a trade-off between controlling the volatility and lowering the drawdown by drift. This leads to a retention level strategy which increases with the drawdown. In particular, the running maximum of the surplus is kept constant under the optimal strategy, which is not reasonable from an economic perspective. Still, preventing large drawdowns is preferable. In the following Chapter, we therefore extend our value function to find a new conception of 'optimality'.

CHAPTER 4

# Maximal Growth with a Drawdown Penalty

In Chapters 2 and 3 it was shown that optimal drawdown-focused decision making has the drawback that stability overrules all other economically important values. What adds to this general impression is that a (in a way) comparable result was obtained in [Brinker, 2021] for the case of optimal investments. In this article, it is shown that the amount invested in an independent Black–Scholes asset of positive drift should increase with the drawdown. Figure 4.1 shows sketches of the surplus under the optimal reinsurance (left) and the optimal investment control (right), side by side. The black colour corresponds to no control, blue to a decreased retention level and, similarly, green to an increased invested amount. In particular, when the surplus is at its maximum, nothing should be invested. This contradicts the economic intuition of investing when the process is at a high point to

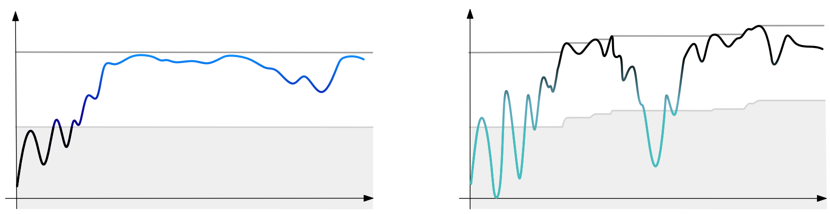


FIGURE 4.1 Sketch of optimally controlled surplus processes under reinsurance (*left*, blue) and investments (*right*, green).

maximise the profit. Though the proof techniques in [Brinker, 2021] differ, the results for optimal investment have therefore similar implications as our analysis of optimal reinsurance. In all cases mentioned, optimal strategies lead to rigorous policies preventing large relative losses at all costs. In particular, the minimisation of drawdowns supplants the potential for current and future surpluses. Companies favouring stability over profits and dividends might be preferable from the perspective of the regulator but, from an economic point of view, this behaviour is not realistic. Therefore, in this chapter (which is based on [Brinker and Schmidli, 2021a]), we introduce a drawdown-targeted optimisation problem that additionally accounts for the potential of growth. Combining the two components, potential growth and penalised large drawdowns, we aim at finding economically attractive reinsurance strategies leading to a sustainable increase of the running maximum and the company's surplus. The optimisation problem that we present has the alternative interpretation of maximising dividend payments while simultaneously minimising the time during which the ex-dividend process is 'far away' from the dividend barrier.

We consider again the diffusion approximation to the surplus of an insurance company under proportional reinsurance, as defined in (3.2) at the beginning of Chapter 3, with the set $\mathcal{B}$ of progressively measurable strategies with values in $[0,1]$. For $B \in \mathcal{B}$ and $x \geq 0$ we consider the reward-penalty-function

$$v^B(x) = \mathbb{E}^x\Big[\beta_1 \int_0^\infty \mathrm{e}^{-\delta t}\, \mathrm{d}M_t^B - \beta_2 \int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta_t^B > d\}}\, \mathrm{d}t\Big].$$

This corresponds to the time the process spends in critical drawdown with a reward for an increasing maximum of the surplus. The weights $\beta_1 \geq 0$ and $\beta_2 \geq 0$ express the relation between the benefit of an upturn of the surpluses maximum (that is, the company outperforms itself) and the possible damage due to large drawdowns. Our aim is to maximise the 'return' by finding an optimal retention level strategy:

$$v(x) = \sup_{B \in \mathcal{B}} \mathbb{E}^x\Big[\beta_1 \int_0^\infty \mathrm{e}^{-\delta t}\, \mathrm{d}M_t^B - \beta_2 \int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta_t^B > d\}}\, \mathrm{d}t\Big]. \tag{4.1}$$

Figure 4.2 illustrates the two opposing aspects of this value function. On one hand, we 'add together'

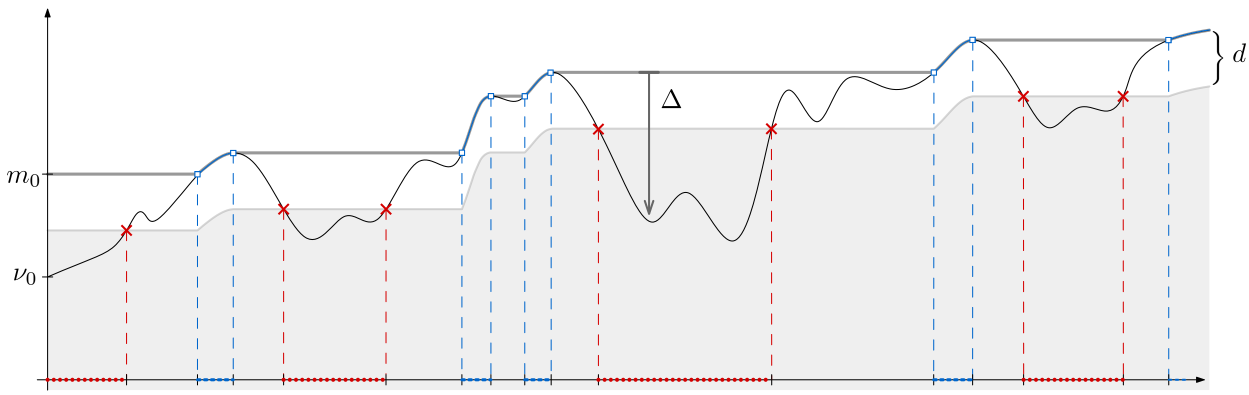



FIGURE 4.2 $v^B$ rewards growth of the running maximum (blue) and penalises large drawdowns (red).

the phases with a critical drawdown (marked in red) at a preference rate, as before. On the other hand, we integrate with respect to the growing running maximum during the phases marked in blue. A different interpretation of the abstract concept 'growth of the running maximum' is the following. As mentioned in the introduction of this monograph, it is not realistic to assume that a company can gather infinite capital: for example, shareholders demand dividends if a company generates large profits. Hence, we assume that the insurer in our optimisation problem pays dividends. Inspired by the results of optimal dividend control problems, we assume that dividends are issued according to a barrier strategy (see, for example, [Schmidli, 2008, Sec. 2.5] and references therein). That is, whenever the surplus exceeds a predefined level $y > 0$, all additional earnings are paid. The accumulated dividend process $D^B = (D_t^B)_{t \geq 0}$ and ex-dividend surplus $U^B = (U_t^B)_{t \geq 0}$ are given by

$$D_t^B = \sup_{s \in [0,t]} \left(X_s^B - y\right)^+, \qquad U_t^B = X_t^B - D_t^B, \qquad t \geq 0.$$

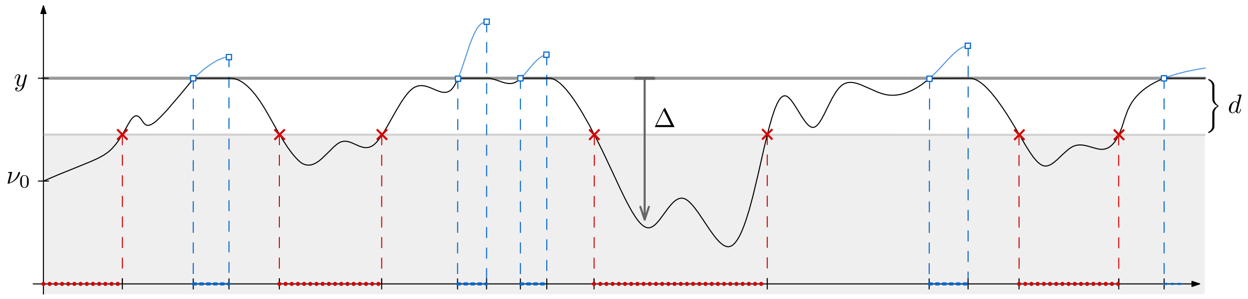



FIGURE 4.3 $w^B$ rewards dividend payments and penalises large, negative deviations from the dividend barrier.

Issuing dividends is generally a good signal to the public and can increase the market value of the company. Hence, we assume that the insurer aims to maximise dividends. On the other hand, if the surplus is bounded away from the dividend barrier, this means that no dividends will be paid for a long period. If the company fails to recover from low surplus levels and has to pass over a payment, this could lead to a depreciation of the company's market value. For this reason, an interesting target functional is the expected value of accumulated discounted dividends minus a penalty for the time during which the ex-dividend surplus is further away than $d$ from the dividend barrier:

$$
\begin{aligned}
w^B(\nu_0) &= \mathbb{E}\Big[\beta_1 \int_0^\infty \mathrm{e}^{-\delta t}\, \mathrm{d}D_t^B - \beta_2 \int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{U_t^B < y-d\}}\, \mathrm{d}t \,\Big|\, U_0^B = \nu_0\Big]\,, \qquad \nu_0 \le y\,,\\
w^B(\nu_0) &= \beta_1(\nu_0 - y) + w^B(y)\,, \qquad \nu_0 > y. \qquad (4.2)
\end{aligned}
$$

We note that choosing $d = y$ in this scenario leads to the optimisation problem of maximising dividends and penalising the time in 'technical ruin' (i.e. with $U^B \le 0$). This optimisation target could, for example, occur if a certain capital requirement has to be met. Now, defining the drawdown process $\Delta^B$ as the distance of the ex-dividend surplus to the dividend barrier

$$
\Delta_t^B = y - U_t^B\,, \qquad t \ge 0\,, \tag{4.3}
$$

we can rewrite the first line of (4.2) for $x = y - \nu_0$ as

$$
w^B(x) = \mathbb{E}^x\Big[\beta_1 \int_0^\infty \mathrm{e}^{-\delta t}\, \mathrm{d}D_t^B - \beta_2 \int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta_t^B > d\}}\, \mathrm{d}t\Big]\,, \qquad x \ge 0\,.
$$

Since $w^B(\nu_0)$ is fixed by (4.2) for $x = y - \nu_0 < 0$, it suffices to consider

$$
w(x) = \sup_{B\in\mathcal{B}} \mathbb{E}^x\Big[\beta_1 \int_0^\infty \mathrm{e}^{-\delta t}\, \mathrm{d}D_t^B - \beta_2 \int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{\Delta_t^B > d\}}\, \mathrm{d}t\Big] \tag{4.4}
$$

for $x \ge 0$ with the appropriate boundary condition. In particular, $w(x) = v(y - x)$ for $x \le y$. Figure 4.3 illustrates this value function. In this sketch, the solid black and grey lines represent the ex-dividend surplus and the dividend barrier. When the ex-dividend surplus is in the grey area, this corresponds to a large drawdown from the dividend barrier. Our intuitive explanation shows that both problems are included in our considerations. We focus on the value function defined in Equation

(4.1) as the proofs can analogously be carried out for the function given in (4.4). In particular, the optimisation problems are equivalent in the sense that the functions inducing optimal feedback strategies coincide.

This chapter is organised as follows. In Section 4.1, we deduce preliminary results from the degenerate cases $\beta_1 = 0$ and $\beta_2 = 0$ such as boundedness of the value function. We conclude that for large initial drawdown, $x \geq d$, the optimal strategy is the strategy of maximal drift (i.e. $B_t = 1$ up to the first time with uncritical drawdown). We derive the Hamilton–Jacobi–Bellman equation connected to (4.1) and prove a general verification theorem. In Section 4.2, we calculate value functions and optimal strategies by solving the Hamilton–Jacobi–Bellman equation explicitly. We prove that optimal strategies are of feedback form, determined by the pointwise maximiser. We identify different types of maximisers depending on the *preference ratio* $\chi = \beta_1/\beta_2$ of the weights applied to dividends (growth) and drawdowns. We prove existence of the processes under the respective feedback controls to conclude that the corresponding strategies are optimal. Additionally, we analyse the effect of the price of reinsurance on these types of strategies. In Section 4.3, we provide numerical examples and path simulations. Here we focus on the impact of the preference ratio $\chi = \beta_1/\beta_2$. We finish with some concluding remarks in Section 4.4.

## 4.1 Preliminary Results

The optimal strategy and its return depend heavily on the weights $\beta_1$ and $\beta_2$ attached to the growth reward and the drawdown penalty. We use the 'degenerate' cases $\beta_1 = 0$ and $\beta_2 = 0$ to give a first intuition of the impacts of the weights and to derive first results on the value function for arbitrary weights. We recall the following definition of $\kappa$ and $\xi$ from Chapter 3, Notation 3.1:

NOTATION 4.1. *We write*

$$\kappa = \frac{-\eta + \sqrt{\eta^2 + 2\delta\sigma^2}}{\sigma^2}\,, \qquad \xi = \frac{\eta + \sqrt{\eta^2 + 2\delta\sigma^2}}{\sigma^2}\,. \tag{4.5}$$

### 4.1.1 Implications of the Cases $\beta_1 = 0$ and $\beta_2 = 0$

The case $\beta_1 > 0$ with $\beta_2 = 0$ has the interpretation of maximising growth of the process (and, respectively, dividend payments) without a drawdown penalty. In many optimisation problems, the time of ruin marks the end of the period of collecting dividends. This leads to a trade-off between the payment of dividends and the risk of early ruin. In our case, since the time horizon is infinite, the insurer has no reason to be cautious. Noting that

$$\int_0^\infty \mathrm{e}^{-\delta t}\,\mathrm{d}M_t^B = \delta \int_0^\infty \mathrm{e}^{-\delta t} M_t^B\,\mathrm{d}t\,,$$

we conclude that the optimal strategy maximises $\mathbb{E}^x[M_t^B]$. This is attained for the constant maximal drift strategy $B$ with $B_t = 1$, $t \geq 0$. The return of this strategy is the value function $v$ with

$$v(x) = \frac{\beta_1 \mathrm{e}^{-\kappa x}}{\kappa}\,, \qquad x \geq 0\,.$$

A rigorous proof can be executed by following the steps of the proofs of Theorems 4.6 and 4.12, below. The function $v$ is decreasing in this case. This is intuitively clear because the larger the initial drawdown, the further away the surplus is from the 'dividend' barrier, i.e. the running maximum. By the same argument, the return $v^B$ of any other retention level strategy $B$ is decreasing. The convexity of $v$ indicates that this effect is stronger for small initial drawdowns (under the optimal strategy).
The case $\beta_1 = 0$ with $\beta_2 > 0$ (that is, solely the drawdown penalty is taken into account) leads to the optimisation problem considered in Chapter 3. Hence, the functions $v$ and $v^B$ are decreasing and attain values in $(-\beta_2\delta^{-1}, 0)$ for $\beta_1 = 0$.
From the consideration of these extreme cases we conclude:

Lemma 4.2. [2021a]
*For arbitrary $\beta_1, \beta_2 \geq 0$ and any admissible strategy $B$, the functions $v^B$ and $v$ are bounded and decreasing. In particular,*

$$v^B(x),\, v(x) \in \Big[-\frac{\beta_2}{\delta}, \frac{\beta_1}{\kappa}\Big]\,, \qquad x \geq 0\,.$$

□

If the drawdown process starts above the critical level $d$, the running maximum cannot increase until the process enters the uncritical area for the first time. This means that, for $x > d$ and $t$ smaller than $\vartheta_d(B)$, the only objective is to minimise the time to $\vartheta_d(B)$. In the previous chapter, we have seen that the constant strategy with $B_t = 1$, $t < \vartheta_d(B)$, is optimal for large initial drawdowns. This yields:

Lemma 4.3. [2021a]
*For arbitrary $\beta_1, \beta_2 \geq 0$, we have*

$$v(x) = -\frac{\beta_2}{\delta} + \Big(v(d) + \frac{\beta_2}{\delta}\Big)\mathrm{e}^{-\kappa(x-d)}\,, \qquad x > d\,.$$

*This corresponds to the return of a strategy $B$ with $B_t = 1$ for all $t < \vartheta_d(B)$.* □

A rigorous proof of the statement of Lemma 4.3 follows in Section 4.2.

### 4.1.2 General Verification and the Case of Critical Initial Drawdown

The Hamilton–Jacobi–Bellman equation connected to the problem posed in Equation (4.1) takes the form

$$\sup_{b\in[0,1]}\Big\{-\delta f(x) - \mu(b)f'(x) + \frac{\sigma^2 b^2}{2}f''(x)\Big\} = \beta_2 \mathbb{1}_{\{x>d\}} \tag{4.6}$$

for $f : [0,\infty) \to \mathbb{R}$. Again, as there exists no twice continuously differentiable function solving the equation, with a *solution* to (4.6) we mean the composition of two classical solutions to the homogeneous and inhomogeneous equation, connected at $x = d$ by a smooth fit. By $f''(d)$ we denote the derivative of $f'$ from the left at $d$, if not stated otherwise. Similarly as in Section 3.2, the term to be maximised with respect to $b$ is

$$\mathfrak{I}_f(b) = -\mu(b)f'(x) + \frac{\sigma^2 b^2}{2}f''(x)$$

for a strictly decreasing function $f$ inserted into (4.6). We observe that, by $f'(x) < 0$, the optimum is attained at $b = 1$ for a fixed $x$ if either $f''(x) \geq 0$ or if $f''(x) < 0$ with $\theta f'(x) \leq \sigma^2 f''(x)$. Also, these are the only cases in that the optimum is $b = 1$. Analogously as in the previous chapter (Lemmata 3.7 and 3.8), we obtain the following two results.

Lemma 4.4. [2021a]

*If $f : [0, \infty) \to \mathbb{R}$ is a strictly decreasing solution to (4.6), then there is no non-empty interval $(\underline{x}, \overline{x}) \subset [0, \infty)$ such that the optimiser is given by $b^*(x) = 0$ for all $x \in (\underline{x}, \overline{x})$.* □

Lemma 4.5. [2021a]

*Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing solution to (4.6). At a point $\bar{x} \neq d$ with $f''(\bar{x}) = 0$ the function changes from convex to concave. Every bounded, strictly decreasing solution $f$ to (4.6) is convex for $x > d$.* □

Moreover, in the notation of Theorem 3.9, we also have a generalised verification theorem (with new, generalised boundary conditions).

Theorem 4.6 (General Verification Theorem). [2021a]

*We assume that $f : [0, \infty) \to \mathbb{R}$ is a bounded solution to*

$$\sup_{b \in \mathcal{I}(x)} \Big\{ -\delta f(x) - \mu(b) f'(x) + \frac{\sigma^2 b^2}{2} f''(x) \Big\} = \beta_2 \mathbb{1}_{\{x > d\}} \tag{4.7}$$

*with $f'(0) \leq -\beta_1$. We have $f(x) \geq v^B(x)$ for every strategy $B \in \mathcal{B}_\mathcal{I}$ and all $x \geq 0$. If, in addition, for each $x$ there is a minimiser $b^*(x) \in \mathcal{I}(x)$ in (4.7) such that the process $X^*$ and the corresponding drawdown process $\Delta^*$ under the feedback strategy $B^*$ with $B^*_t = b^*(\Delta^*_t)$, $t \geq 0$, exist and either $f'(0) = -\beta_1$ holds or $M^*$ is constant, then we have $f(x) = v^{B^*}(x) = \sup_{B \in \mathcal{B}_\mathcal{I}} v^B(x)$ for all $x \geq 0$.*

*Proof.* By the generalised Itô formula of [Elworthy et al., 2007, Thm. 2.1 and Eq. (2.23)], it follows:

$$\begin{aligned} \mathrm{e}^{-\delta t} f(\Delta^B_t) - f(\Delta^B_0) &= \int_0^t \mathrm{e}^{-\delta s} f'(\Delta^B_s)\, \mathrm{d}M^B_s - \int_0^t \mathrm{e}^{-\delta s} \sigma B_s f'(\Delta^B_s)\, \mathrm{d}W_s \\ &\quad + \int_0^t \mathrm{e}^{-\delta s} \Big( -\delta f(\Delta^B_t) - \mu(B_s) f'(\Delta^B_s) + \frac{\sigma^2 B_s^2}{2} f''(\Delta^B_s) \Big)\, \mathrm{d}s \\ &\leq \int_0^t \mathrm{e}^{-\delta s} f'(0)\, \mathrm{d}M^B_s - \int_0^t \mathrm{e}^{-\delta s} \sigma B_s f'(\Delta^B_s)\, \mathrm{d}W_s + \int_0^t \mathrm{e}^{-\delta s} \beta_2 \mathbb{1}_{\{\Delta^B_s > d\}}\, \mathrm{d}s\,. \end{aligned} \tag{4.8}$$

Stopping at $T_n = \vartheta^n(B) \wedge n$, $n \in \mathbb{N}$, we find (by boundedness of $f'$ on bounded intervals):

$$f(x) \geq \mathbb{E}^x \big[ \mathrm{e}^{-\delta(t \wedge T_n)} f(\Delta^B_{t \wedge T_n}) \big] + \mathbb{E}^x \Big[ \beta_1 \int_0^{t \wedge T_n} \mathrm{e}^{-\delta s}\, \mathrm{d}M^B_s - \beta_2 \int_0^{t \wedge T_n} \mathrm{e}^{-\delta s} \mathbb{1}_{\{\Delta^B_s > d\}}\, \mathrm{d}s \Big],$$

for all $x \geq 0$. Here we used that it holds $f'(0) \leq -\beta_1$. By the same arguments as in the proof of Theorem 3.9, we can let $n \to \infty$ and then $t \to \infty$ to find $f(x) \geq v^B(x)$. Applying the same steps to the strategy $B^*$, we obtain equality and, thus, $f(x) = \sup_{B \in \mathcal{B}_\mathcal{I}} v^B(x)$. □

The following two results can be verified directly and will be useful in the remainder of this chapter.

Lemma 4.7. [2021a]
*Any function $g : (d, \infty) \to \mathbb{R}$ of the form*

$$g(x) = -\frac{\beta_2}{\delta} + \Big(C + \frac{\beta_2}{\delta}\Big)\mathrm{e}^{-\kappa(x-d)}\,, \qquad x \geq d\,,$$

*with $C \geq -\beta_2\delta^{-1}$ is a bounded, decreasing and convex solution to*

$$\sup_{b\in[0,1]} \Big\{ -\delta g(x) - \mu(b)g'(x) + \frac{\sigma^2 b^2}{2} g''(x)\Big\} = \beta_2$$

*for all $x > d$ with bounded first and second derivatives. In particular, $v|_{(d,\infty)}$ has these properties.* □

Corollary 4.8. [2021a]
*A decreasing function $f : [0, \infty) \to \mathbb{R}$ that solves*

$$\sup_{b\in[0,1]} \Big\{ -\delta f(x) - \mu(b)f'(x) + \frac{\sigma^2 b^2}{2} f''(x)\Big\} = 0$$

*for all $x \leq d$ and fulfils $f'(d) = -\kappa(f(d) + \beta_2\delta^{-1})$, can be extended to a bounded solution to (4.6) by a decreasing and convex function $g : (d, \infty) \to \mathbb{R}$ of the form*

$$g(x) = -\frac{\beta_2}{\delta} + \Big(f(d) + \frac{\beta_2}{\delta}\Big)\mathrm{e}^{-\kappa(x-d)}\,, \qquad x > d\,.$$

□

## 4.2 Optimal Proportional Reinsurance

As in the previous chapter, we use the following definitions to shorten notation and facilitate interpretation.

Notation 4.9. *We write*

$$\zeta = \frac{2\delta\sigma^2 + \theta^2}{\theta^2} > 1\,, \qquad \rho = \frac{2\sigma^2(\theta - \eta)}{2\delta\sigma^2 + \theta^2} > 0 \tag{4.9}$$

*and distinguish the cases of cheap reinsurance with $\theta \leq \sigma^2\xi = \eta + \sqrt{\eta^2 + 2\delta\sigma^2}$ (which is equivalent to $\theta\rho \leq \sigma^2$) and expensive reinsurance with $\theta > \sigma^2\xi$ (i.e. $\theta\rho > \sigma^2$).*

### 4.2.1 Optimality of Operating without Reinsurance

As we have seen in our preliminary considerations, there is no incentive to be cautious if only dividends are taken into account (i.e. $\beta_2 = 0$). The corresponding optimal strategy is withholding the full premium income in the company. Intuitively, this should also be the case if the condition is weakened in the sense that 'dividends are just much more important than the threat of a large drawdown'. We now calculate conditions on the model parameters for this statement to be true. We start by evaluating the reward-penalty-function of the constant strategy $B$ with $B_t = 1$, $t \geq 0$.

PROPOSITION 4.10. [2021a]

*The function $u : [0,\infty) \to \mathbb{R}$ defined as*

$$u(x) = \begin{cases} C_1 \mathrm{e}^{\xi x} + C_2 \mathrm{e}^{-\kappa x}\,, & x \le d\,, \\ -\frac{\beta_2}{\delta} + \Big(C_1 \mathrm{e}^{\xi d} + C_2 \mathrm{e}^{-\kappa d} + \frac{\beta_2}{\delta}\Big)\mathrm{e}^{-\kappa(x-d)}\,, & x > d\,, \end{cases}$$

*with*

$$C_1 = -\frac{\beta_2 \kappa}{\delta \mathrm{e}^{\xi d}(\xi + \kappa)}\,, \qquad C_2 = \frac{C_1 \xi + \beta_1}{\kappa} \tag{4.10}$$

*is the return of the constant strategy $B$ with $B_t = 1$, $t \ge 0$.*

*Proof.* As an arithmetic Brownian motion, the surplus process under constant controls exists. The composite function $u$ solves (4.7) with $\mathcal{I}(x) = \{1\}$ for all $x \ge 0$ and $u'(0) = -\beta_1$. Thus, the assertion follows by Theorem 4.6. ☐

From now on, we express the weight of growth in comparison to drawdowns in terms of the ratio $\chi = \beta_1/\beta_2$. The above Lemma and following analysis also apply to the degenerate case $\beta_2 = 0$ mentioned in the introduction (with the interpretation $\chi = \infty$).

NOTATION 4.11. *We define the critical preference ratios*

$$\chi_{c,1} = \frac{\kappa\xi\big[(\theta + \sigma^2\kappa)\mathrm{e}^{-\xi d} - (\theta - \sigma^2\xi)\mathrm{e}^{\kappa d}\big]}{\delta(\theta + \sigma^2\kappa)(\kappa + \xi)}\,, \qquad \chi_{e,1} = \frac{\kappa\xi\sigma^2 \mathrm{e}^{-\xi d}}{\delta(\theta + \sigma^2\kappa)}\,.$$

REMARK. We note that we have $\chi_{c,1} = \chi_{e,1}$ for $\theta = \sigma^2\xi$, that is, the price of reinsurance lies on the boundary between the cheap and expensive area. In general, we have $\chi_{c,1} > 0$ (at least for cheap reinsurance) and $\chi_{e,1} > 0$. #

THEOREM 4.12. [2021a]

*$u : [0,\infty) \to \mathbb{R}$ as defined in Proposition 4.10 solves Equation (4.6) with the constant pointwise optimiser $b^*(x) = 1$ for all $x \ge 0$ if and only if the parameter set fulfils one of the following conditions:*

i) *Reinsurance is cheap and $\chi \ge \chi_{c,1}$.*

ii) *Reinsurance is expensive and $\chi \ge \chi_{e,1}$.*

*In these cases, $u$ is equal to the value function and an optimal strategy $B^*$ is constant with $B^*_t = 1$ for all $t \ge 0$.*

*Proof.* In view of Lemma 4.7, Proposition 4.10 and the verification theorem, the only thing left to prove is that $u$ solves the homogeneous part of Equation (4.7) for $\mathcal{I}(x) = [0,1]$ for all $x \in [0,d]$ (if and only if one of the conditions is fulfilled). This is the case if the optimiser of the equation is indeed $b = 1$. The term to be maximised with respect to $b$ is given by $\mathcal{J}_u(b)$, defined as above. The inequality $\theta u'(x) \le \sigma^2 u''(x)$ is equivalent to

$$\Big(1 - \frac{\sigma^2\xi}{\theta}\Big)\xi C_1 \le \Big(1 + \frac{\sigma^2\kappa}{\theta}\Big)\kappa \mathrm{e}^{-(\kappa+\xi)x} C_2\,. \tag{4.11}$$

The cases of cheap and expensive reinsurance lead to different signs of the bracket term on the left hand side. One can show, by distinguishing these cases, that the proposed bounds on $\chi$ are necessary and sufficient for one of the conditions, $u''(x) \geq 0$ or $u''(x) < 0$ with (4.11), to be fulfilled for all $x \in [0, d]$. The details of this calculation are found in the appendix, p. 126. □

Intuitively, the lower bounds on $\chi = \beta_1/\beta_2$ indicate that, if future growth is (or dividends are) much more important than the threat of a large drawdown, then the insurer will choose the same strategy as in the example without drawdowns. Further, the price of reinsurance influences the critical weights. From the explicit expressions, we conclude that the more expensive the reinsurance premium, the sooner (in terms of the importance of dividends in comparison to drawdowns) the insurer will refuse a reinsurance contract.

If the conditions stated in Proposition 4.12 are harmed, the return of the constant strategy of never buying reinsurance does not solve Equation (4.6). In view of the verification theorem, it is reasonable to assume that this strategy is therefore not optimal for $\chi$ smaller than $\chi_{c,1}$ and $\chi_{e,1}$, respectively. This means that the drawdown penalty forces the insurer to buy reinsurance. As we show in the following, the strategies for different ratios $\chi$ are non-constant and of feedback form. By Lemma 4.5, the 'natural candidates' for the value function are decreasing and concave solutions to the homogeneous Hamilton–Jacobi–Bellman equation for $x \in [0, d]$ that are smoothly extended to $[0, \infty)$. In the following, we derive a family of solutions related to the Lambert W function $\mathcal{W}$ (which made its first appearance in the introduction of Chapter 3). Value functions, optimal strategies and optimality criteria will be expressed in terms of these solutions.

### 4.2.2 Solutions to the Homogeneous HJB Equation for General Initial Conditions

We characterise in detail explicit solutions to the Hamilton–Jacobi–Bellman equation

$$\sup_{b\in[0,1]} \Big\{ -\delta f(x) - \mu(b) f'(x) + \frac{\sigma^2 b^2}{2} f''(x) \Big\} = 0\,, \qquad x \geq 0\,. \tag{4.12}$$

We write $b_f(x) = \theta f'(x)/(\sigma^2 f''(x))$ if this expression is defined. Moreover, we introduce the following notation.

Notation 4.13. *For $a \in [0, \infty)$ and $\gamma \in [0, 1]$, we write*

$$E_{\gamma,a}(x) = \frac{\gamma\sigma^2 - \theta\rho}{\theta\rho} \cdot \exp\Big[\frac{\gamma\sigma^2 - \theta\rho}{\theta\rho} - \frac{\zeta}{\rho}(x - a)\Big]\,, \qquad x \geq a\,.$$

*For $\theta\rho \neq \sigma^2\gamma$, we additionally define*

$$Q_{\gamma,a}(x) = \Big[\frac{\gamma\sigma^2 - \theta\rho}{\theta\rho\, \mathcal{W}(E_{\gamma,a}(x))}\Big]^{1/\zeta}\,, \qquad P_{\gamma,a}(x) = \Big[1 + \frac{\mathcal{W}(E_{\gamma,a}(x))}{1-\zeta}\Big]\rho\, Q_{\gamma,a}(x)\,, \qquad x \geq a\,.$$

*We write $E_\gamma(x) = E_{\gamma,0}(x)$, $Q_\gamma(x) = Q_{\gamma,0}(x)$, $P_\gamma(x) = P_{\gamma,0}(x)$ for $x \geq 0$.*

We note that $Q_0$ and $P_0$ coincide with the functions $Q$ and $P$ defined in Equation (3.18) of the preceding chapter for the degenerate case $\beta_1 = 0$. The next lemma states conditions under which a solution to (4.12) with the initial values $f'(a) = \alpha < 0$ and $b_f(a) = \gamma \in [0, 1]$ (at some $a \geq 0$) exists.

Lemma 4.14. [2021a]

*For given $a \in [0,\infty)$, $\gamma \in [0,1]$ and $\alpha < 0$, we let*

$$f(a) = \alpha\frac{2(\theta-\eta)-\theta\gamma}{2\delta}\,, \qquad f(x) = f(a) - \int_a^x \mathrm{e}^{-Z(y)}\,\mathrm{d}y\,, \qquad x > a\,, \tag{4.13}$$

*and*

$$Z(x) = -\frac{x-a}{\rho} - \frac{\theta\rho-\sigma^2\gamma}{\theta\zeta\rho} - \ln(-\alpha) - \frac{\mathcal{W}(E_{\gamma,a}(x))}{\zeta}\,. \tag{4.14}$$

i) *If $\theta\rho < \sigma^2\gamma$, the function $f$ is a strictly concave and decreasing solution to* (4.12) *for $x \in [a,\infty)$ fulfilling the initial conditions $f'(a) = \alpha$ and $b(a) = \gamma$. The pointwise maximiser is given by*

$$b_f(x) = \frac{\theta f'(x)}{\sigma^2 f''(x)} = \frac{\theta\rho}{\sigma^2}\Big[1 + \mathcal{W}(E_{\gamma,a}(x))\Big]\,, \tag{4.15}$$

*strictly decreasing in $x$ and converges to its lower bound $\theta\rho\sigma^{-2}$ as $x \to \infty$.*

ii) *If $\theta\rho = \sigma^2\gamma$, the function $f$ is a strictly concave and decreasing solution to* (4.12) *for $x \in [a,\infty)$ fulfilling the initial conditions $f'(a) = \alpha$ and $b(a) = \gamma$. The pointwise maximiser is equal to $b_f$ and constant with $b_f(x) = \theta\rho\sigma^{-2}$.*

iii) a) *If $\theta\rho > \sigma^2\gamma$ and $\theta\rho \le \sigma^2$, the function $f$ is a strictly concave and decreasing solution to* (4.12) *for $x \in [a,\infty)$ that fulfils the initial conditions $f'(a) = \alpha$ and $b(a) = \gamma$. The pointwise maximiser is given by* (4.15)*, strictly increasing in $x$ and converges to its upper bound $\theta\rho\sigma^{-2}$ as $x \to \infty$.*
b) *If $\theta\rho > \sigma^2\gamma$ and $\theta\rho > \sigma^2$, $f$ is a strictly concave and decreasing solution to* (4.12) *for $x \in [a, x_\gamma]$ fulfilling the initial conditions $f'(a) = \alpha$ and $b(a) = \gamma$ with*

$$x_\gamma = a + \frac{\rho}{\zeta}\ln\Big(\frac{\theta\rho-\sigma^2\gamma}{\theta\rho-\sigma^2}\Big) - \frac{\sigma^2(1-\gamma)}{\theta\zeta}\,. \tag{4.16}$$

*The pointwise maximiser is given by* (4.15)*, strictly increasing in $x$ and equal to* 1 *at $x_\gamma$.*

In a simpler form, we have already seen parts of the proof in our construction in Section 3.2 for $\gamma = 0$. For the general case we need a more sophisticated approach. However, as the underlying idea of the proof remains the same, we postpone the technical details to the appendix, p. 127.
Lemma 4.14 is an important tool as it yields explicit solutions to the Hamilton–Jacobi–Bellman equation for arbitrary starting points. In particular, the initial condition for the second derivative is presented as an initial condition for the pointwise maximiser. Thus, the lemma enables us to construct a corresponding solution to the Hamilton–Jacobi–Bellman equation based on the intuition of what the optimal strategy 'should' look like. The following result will be useful to show existence of processes under the corresponding feedback controls.

Lemma 4.15. [2021a]

*Under the conditions of Lemma* 4.14*, $b_f : [a,\infty) \to \mathbb{R}$ given in* (4.15) *is continuously differentiable and convex when it is decreasing, case* i)*, and concave when it is increasing, case* iii)*. In case* i)*, its derivative is absolutely bounded. In case* iii)*, its derivative is absolutely bounded if (and only if) $\gamma > 0$.*

The proof of Lemma 4.15 is based on properties of $\mathcal{W}$ and is found in the appendix, p. 128. We note that the special case 'iii) with $\gamma = 0$' (in which the derivative is unbounded) also follows by

Lemma 3.17, which states that $b_f(x)$ is of order $\Theta(\sqrt{x})$ as $x \to a$.
In the case $\theta\rho = \sigma^2\gamma$, the function $f$ defined in (4.13) has the simple representation $f(x) = \alpha\rho e^{(x-a)/\rho}$.
In the remaining two cases, we can write $f$ in terms of the functions $Q_{\gamma,a}$ and $P_{\gamma,a}$, as we show in the appendix, p. 129:

LEMMA 4.16. [2021a]
*Under the conditions of Lemma* 4.14 *and* $\theta\rho \neq \sigma^2\gamma$, *we have* $f(x) = \alpha P_{\gamma,a}(x)$, $f'(x) = \alpha Q_{\gamma,a}(x)$, *and* $f''(x)\sigma^2 b_f(x) = \alpha\theta Q_{\gamma,a}(x)$ *for* $x \geq a$.

In the next section, we use these results to find solutions to the inhomogeneous Hamilton–Jacobi–Bellman equation.

### 4.2.3 Optimal Feedback Controls

We construct solutions to the Hamilton–Jacobi–Bellman equation for the different cases of cheap and expensive reinsurance. At the end of this section we prove existence of the processes under the feedback controls and conclude that the candidates are indeed the value functions.

#### Optimal Strategies for Cheap Reinsurance

Throughout this section, we assume that reinsurance is cheap; $\theta\rho \leq \sigma^2$. We introduce the following critical ratios.

NOTATION 4.17. *For* $\theta\rho < \sigma^2$, *we write*

$$\chi_{c,2} = \frac{\kappa}{\delta(P_1(d)\kappa + Q_1(d))}$$

*and for* $\theta\rho \leq \sigma^2$, *we additionally define*

$$\chi_{c,3} = \frac{\kappa}{\delta e^{d/\rho}(\kappa\rho + 1)}, \qquad \chi_{c,4} = \frac{\kappa}{\delta(P_0(d)\kappa + Q_0(d))}.$$

REMARK. For $\theta\rho = \sigma^2$, we have $\rho^{-1} = \xi$ and therefore $\chi_{c,1} = \chi_{c,3}$ (where $\chi_{c,2}$ is undefined). However, similarly to the proof of Lemma A.3 of the appendix, p. 131, one can show that $\lim_{\theta\rho \nearrow \sigma^2} \chi_{c,2} = \chi_{c,1}$. For this reason, we interpret intervals such as $[\chi_{c,2}, \chi_{c,1})$ or $(\chi_{c,3}, \chi_{c,2}]$ as empty sets if $\theta\rho = \sigma^2$, in the following. #

LEMMA 4.18. [2021a]
*We have* $\chi_{c,1} > \chi_{c,2} > \chi_{c,3} > \chi_{c,4} > 0$ *for* $\theta\rho < \sigma^2$. *For* $\theta\rho = \sigma^2$, *it holds* $\chi_{c,1} = \chi_{c,3} > \chi_{c,4} > 0$. □

This lemma is a direct consequence of two rather technical results found in the appendix, p. 130 (Lemmata A.2 and A.3).
To present our findings as clearly as possible, we start with an outline of the cases to distinguish for cheap reinsurance. The case $\chi \in [\chi_{c,2}, \chi_{c,1})$ is closest to the case $\chi \geq \chi_{c,1}$. One might presume that the strategy with maximal drift is kept when the drawdown is close to zero and that reinsurance is bought if the drawdown grows towards the critical value $d$. We prove below that this is indeed the case and

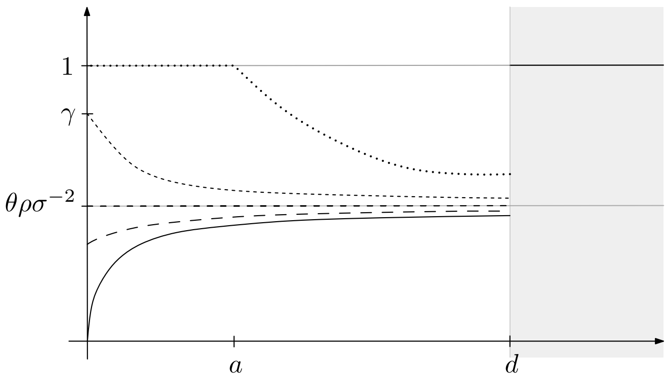



FIGURE 4.4 Optimal reinsurance strategies for cheap reinsurance.

that the pointwise maximiser $b^*$ of the Hamilton–Jacobi–Bellman equation (that induces the optimal feedback strategy) takes the form represented by the dotted line in Figure 4.4. If the ratio of dividend to drawdown penalty is further reduced, the strategy with maximal drift will never be chosen, even if the drawdown process is currently in zero. We distinguish three cases that are represented by the dashed lines in Figure 4.4. For $\chi \in (\chi_{c,3}, \chi_{c,2})$, the optimal retention level for uncritical drawdowns is given as a decreasing function of the current drawdown (short dashes in Figure 4.4), bounded from below by $\theta\rho\sigma^{-2}$. For $\chi = \chi_{c,3}$, the optimal retention level is constant and equal to $\theta\rho\sigma^{-2}$ (medium length dashes in Figure 4.4). This corresponds to a simple switching strategy. For $\chi \in (\chi_{c,4}, \chi_{c,3})$, the influence of the weight of the dividends is still large enough for the insurer to choose $b^*(0) > 0$ (long dashed line in Figure 4.4). In this case, the retention level increases with the drawdown and is bounded from above by $\theta\rho\sigma^{-2}$. For $\chi \leq \chi_{c,4}$, the weight of the drawdown penalty dominates the value function: we obtain that $b^*$ corresponds to the pointwise optimiser of the case without dividends considered in Chapter 3 (solid line in Figure 4.4).

LEMMA 4.19. [2021a]

*We assume $\chi \in (\chi_{c,2}, \chi_{c,1})$. There exists a unique $a \in (0, d)$ such that the function $f : [0, \infty) \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} C_1 e^{\xi x} + C_2 e^{-\kappa x}\,, & x \in [0, a), \\ (\xi C_1 e^{\xi a} - \kappa C_2 e^{-\kappa a}) P_{1,a}(x)\,, & x \in [a, d], \\ -\beta_2 \delta^{-1} + ([\xi C_1 e^{\xi a} - \kappa C_2 e^{-\kappa a}] P_{1,a}(d) + \beta_2 \delta^{-1}) e^{-\kappa(x-d)}\,, & x > d, \end{cases}$$

*with*

$$C_1 = -\frac{\beta_1(\sigma^2\kappa + \theta)}{((\sigma^2\kappa + \theta) + (\sigma^2\xi - \theta)e^{(\xi+\kappa)a})\xi} \quad \text{and} \quad C_2 = \frac{\xi C_1 + \beta_1}{\kappa}$$

*is a bounded solution to (4.6) with $f'(0) = -\beta_1$. The pointwise maximiser is given by*

$$b^*(x) = \begin{cases} 1\,, & x \in [0, a)\,, \\ \theta\rho\sigma^{-2}[1 + \mathcal{W}(E_{1,a}(x))]\,, & x \in [a, d]\,, \\ 1\,, & x > d\,, \end{cases}$$

*and continuous in every point except in $x = d$. $b^*$ is strictly decreasing and convex for $x \in (a, d)$. If $\chi = \chi_{c,2}$, the assertion holds for $a = 0$.*

*Proof.* We first construct a solution on the interval $[0, a]$, then extend it to $[0, d]$ and afterwards to $[0, \infty)$. Suppose that there exists $a \in [0, d)$ such that the optimiser is equal to one for all $x \in [0, a]$. Similarly to Proposition 4.10, this implies that the function for $x \in [0, a]$ with derivative $-\beta_1$ in zero should be of the form

$$h_0(x) = C\mathrm{e}^{\xi x} + \frac{C\xi + \beta_1}{\kappa}\mathrm{e}^{-\kappa x}$$

for some constant $C$. The condition $\sigma^2 h_0''(x) \geq \theta h_0'(x)$ is equivalent to

$$C \geq -\frac{\beta_1(\sigma^2\kappa + \theta)}{\xi((\sigma^2\kappa + \theta) + (\sigma^2\xi - \theta)\mathrm{e}^{(\xi+\kappa)x})}\,. \tag{4.17}$$

The right hand side is increasing in $x$. In particular, for $C = C_1$, the condition is fulfilled for all $x \in [0, a]$ and $a$ is the rightmost point with this property. Thus, we assume $C = C_1$. By $C_1 < \beta_1/((\mathrm{e}^{(\xi+\kappa)a} - 1)\xi)$ and $C_1 < -\beta_1\kappa/((\xi - \kappa)\xi)$, $h_0$ is strictly decreasing and concave for $x \in [0, a]$. This means, $h_0$ solves the Hamilton–Jacobi–Bellman equation for all $x \in [0, a]$ with optimiser $b^*(x) = 1$. According to Lemma 4.14 i), the function $h_1(x) = h_0'(a)P_{1,a}(x)$ is an extension of $h_0$ that solves the equation on $[a, d]$ with the optimiser defined above. Because the first derivatives of $h_0$ and $h_1$ coincide at $x = a$ and we have

$$1 = \frac{\theta h_0'(a)}{\sigma^2 h_0''(a)} = \frac{\theta h_1'(a)}{\sigma^2 h_1''(a)}\,,$$

the second derivatives coincide as well. Since both functions solve the Hamilton–Jacobi–Bellman equation at $x = a$ with the same optimiser, we also obtain $h_0(a) = h_1(a)$. In view of Lemma 4.7 and Corollary 4.8, we choose the ansatz $g(x) = -\beta_2\delta^{-1} + (h_1(d) + \beta_2\delta^{-1})\mathrm{e}^{-\kappa(x-d)}$ and look for a smooth fit. Continuity of the first derivative at $x = d$ implies $h_1'(d) = -\kappa(h_1(d) + \beta_2\delta^{-1})$ which is equivalent to

$$h_0'(a)Q_{1,a}(d) = -\kappa(h_0'(a)P_{1,a}(d) + \beta_2\delta^{-1})\,. \tag{4.18}$$

By plugging in the definitions of $h_0'(a)$ and $C_1$, we obtain

$$\chi = \frac{\kappa\big[(\sigma^2\kappa + \theta) + (\sigma^2\xi - \theta)\mathrm{e}^{(\kappa+\xi)a}\big]}{\delta\sigma^2(\kappa + \xi)\mathrm{e}^{\xi a}\big[\kappa P_{1,a}(d) + Q_{1,a}(d)\big]}\,. \tag{4.19}$$

The right hand side is strictly increasing in $a$ (compare Lemma A.2), equal to $\chi_{c,2}$ for $a = 0$ and equal to $\chi_{c,1}$ for $a = d$. Thus, for any $\chi \in [\chi_{c,2}, \chi_{c,1})$, there exists a unique $a \in [0, d)$, implicitly given by (4.19), so that $f$ as defined above is a solution to (4.6). $\square$

A by-product of the proof is the following corollary, which states that the 'plateau' $[0, a]$ with $b^*(x) = 1$ is growing with $\chi$. This means, the point $a$ marking the sharp bend of the dotted graph of Figure 4.4 slides to the left if $\chi \in [\chi_{c,2}, \chi_{c,1}]$ is chosen smaller and is equal to zero for $\chi = \chi_{c,2}$.

Corollary 4.20. [2021a]

*The function $a : [\chi_{c,2}, \chi_{c,1}] \to [0, d]$ implicitly given by*

$$\chi = \frac{\kappa\big[(\sigma^2\kappa + \theta)\mathrm{e}^{-\xi a(\chi)} + (\sigma^2\xi - \theta)\mathrm{e}^{\kappa a(\chi)}\big]}{\delta\sigma^2(\kappa + \xi)\big[\kappa P_{1,a(\chi)}(d) + Q_{1,a(\chi)}(d)\big]} \tag{4.20}$$

*is increasing with $a(\chi_{c,2}) = 0$ and $a(\chi_{c,1}) = d$.* □

We next deal with the cases represented by the three dashed graphs of Figure 4.4.

Lemma 4.21. [2021a]

i) *For $\chi \in (\chi_{c,3}, \chi_{c,2})$, there exists a unique $\gamma \in (\theta\rho\sigma^{-2}, 1)$ such that the function $f : [0, \infty) \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} -\beta_1 P_\gamma(x)\,, & x \in [0, d]\,, \\ -\beta_2\delta^{-1} + \Big(-\beta_1 P_\gamma(d) + \beta_2\delta^{-1}\Big)\mathrm{e}^{-\kappa(x-d)}\,, & x > d\,, \end{cases} \tag{4.21}$$

*is a bounded solution to (4.6) with $f'(0) = -\beta_1$. The pointwise maximiser is given by*

$$b^*(x) = \begin{cases} \theta\rho\sigma^{-2}[1 + \mathcal{W}(E_\gamma(x))]\,, & x \in [0, d]\,, \\ 1\,, & x > d\,, \end{cases} \tag{4.22}$$

*and continuous in every point except in $x = d$. $b^*$ fulfils $b^*(0) = \gamma$ and is strictly decreasing and convex on $[0, d)$.*

ii) *For $\chi = \chi_{c,3}$ the statement holds for the function*

$$f(x) = \begin{cases} -\beta_1\rho\mathrm{e}^{x/\rho}\,, & x \in [0, d]\,, \\ -\beta_2\delta^{-1} + \Big(-\beta_1\rho\mathrm{e}^{d/\rho} + \beta_2\delta^{-1}\Big)\mathrm{e}^{-\kappa(x-d)}\,, & x > d\,, \end{cases}$$

*and the maximiser given by (4.22), which is piecewise constant with $b^*(x) = \theta\rho\sigma^{-2}$ for $x \in [0, d]$.*

iii) *For $\chi \in [\chi_{c,4}, \chi_{c,3})$, there exists a unique $\gamma \in [0, \theta\rho\sigma^{-2})$ such that the statement holds for the function $f : [0, \infty) \to \mathbb{R}$ defined in (4.21) and the maximiser given by (4.22) which fulfils $b(0) = \gamma$ and is strictly increasing and concave on $[0, d)$. In particular, if $\chi = \chi_{c,4}$, the assertion is fulfilled for $\gamma = 0$.*

*Proof.* In the cases i) and iii) it suffices to prove that there exists for every $\chi$ in the respective interval a unique $\gamma(\chi)$ such that $f$ defined above is continuously differentiable at $x = d$. Then, the desired properties of $f|_{[0,d]}$ and $f|_{(d,\infty)}$ follow from Lemma 4.7, Corollary 4.8 and Lemma 4.14 i), iii). The condition of smooth fit is equivalent to

$$\chi = \frac{\kappa}{\delta(P_\gamma(d)\kappa + Q_\gamma(d))}\,. \tag{4.23}$$

We prove in the appendix, Lemma A.3 i), that the right hand side is defined for all $\gamma \in [0, 1] \setminus \{\theta\rho\sigma^{-2}\}$, positive, continuous with a removable discontinuity at $\gamma = \theta\rho\sigma^{-2}$ and strictly increasing. For $\gamma = 1$, it is equal to $\chi_{c,2}$. As $\gamma \searrow \theta\rho\sigma^{-2}$ and as $\gamma \nearrow \theta\rho\sigma^{-2}$, it converges to $\chi_{c,3}$. It is equal to $\chi_{c,4}$ for $\gamma = 0$. Thus, for any $\chi \in (\chi_{c,3}, \chi_{c,2})$ there exists a unique $\gamma \in (\theta\rho\sigma^{-2}, 1)$, defined by the relation (4.23), such that the function $f$ is a solution to the Hamilton–Jacobi–Bellman equation. The same holds for $\chi \in (\chi_{c,4}, \chi_{c,3})$ and $\gamma \in (0, \theta\rho\sigma^{-2})$. In case ii), $\chi = \chi_{c,3}$, an explicit calculation shows that the smooth fit condition is fulfilled as well. The result follows again from Lemma 4.7, Corollary 4.8 and Lemma 4.14 ii). $\square$

Similarly as above, a result hidden in the proof is the following.

Corollary 4.22. [2021a]
*The function $\gamma : [\chi_{c,4}, \chi_{c,2}] \setminus \{\chi_{c,3}\} \to [0,1] \setminus \{\theta\rho\sigma^{-2}\}$ implicitly given by*

$$\chi = \frac{\kappa}{\delta(P_{\gamma(\chi)}(d)\kappa + Q_{\gamma(\chi)}(d))} \tag{4.24}$$

*is strictly increasing with $\gamma(\chi_{c,4}) = 0$, $\lim_{\chi \nearrow \chi_{c,3}} \gamma(\chi) = \theta\rho\sigma^{-2}$, $\lim_{\chi \searrow \chi_{c,3}} \gamma(\chi) = \theta\rho\sigma^{-2}$ and $\gamma(\chi_{c,2}) = 1$. The function is continuous except for a removable discontinuity at $\chi_{c,3}$.* $\square$

This means, if $\chi \in [\chi_{c,4}, \chi_{c,2}]$ is reduced, the starting point $b^*(0) = \gamma$ 'slides down' the $y$-axis (compare Figure 4.4) and we have $b^*(0) = 0$ for $\chi = \chi_{c,4}$. Lastly, we consider the case $\chi < \chi_{c,4}$ which is represented by the solid line in Figure 4.4.

Lemma 4.23. [2021a]
*For $\chi \in [0, \chi_{c,4})$, the function $f : [0, \infty) \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} -\beta_2\chi_{c,4}P_0(x)\,, & x \in [0,d]\,, \\ -\beta_2\delta^{-1} + \Big(-\beta_2\chi_{c,4}P_0(d) + \beta_2\delta^{-1}\Big)\mathrm{e}^{-\kappa(x-d)}\,, & x > d\,, \end{cases}$$

*is a bounded solution to (4.6) with $f'(0) = -\beta_2\chi_{c,4} < -\beta_1$. The pointwise maximiser is given by (4.22) with $\gamma = 0$ and continuous in every point except in $x = d$. The pointwise maximiser $b^*$ fulfils $b^*(0) = 0$ and is strictly increasing and concave on $[0, d)$.*

*Proof.* A direct calculation proves that the smooth fit condition is fulfilled. The result therefore follows by Lemma 4.7, Corollary 4.8 and Lemma 4.14 iii). $\square$

We note that the function $f$ defined in Lemma 4.23 is a multiple of the value function from Chapter 3, so that we could alternatively prove this result with Corollary 3.23.

### Optimal Strategies for Expensive Reinsurance

In this section, we calculate optimisers and value functions under the assumption that $\theta\rho > \sigma^2$ is fulfilled (expensive reinsurance).

Notation 4.24. *For $\theta\rho > \sigma^2$ and $\gamma \in [0,1]$, we define $x_\gamma$ as in (4.16). In particular, $x_1 = 0$. For $\gamma = 0$, $x_\gamma$ coincides with the definition of $x_0$ in Equation (3.20).*

We recall that, by Lemma 3.17 ii), $x_0$ can be interpreted as a strictly decreasing function of $\theta$ and that there is a unique $\theta_d = x_0^{-1}(d) \in (\sigma^2\xi, \infty)$. As it turns out, the structure of optimal strategies for the cases $\theta \in (\sigma^2\xi, \theta_d]$ and $\theta > \theta_d$ differs slightly. The following Lemma can be verified by an explicit calculation of derivatives.

Lemma 4.25. [2021a]
*$x_\gamma$ is continuous and strictly decreasing in $\gamma$. For $\theta \in (\sigma^2\xi, \theta_d]$, there exists a unique $\bar\gamma \in [0,1)$ with $x_{\bar\gamma} = d$. We have $\bar\gamma = \theta\rho\sigma^{-2}[1 + \mathcal{W}(E_{1,d}(0))]$. For $\theta \in (\theta_d, \infty)$, we have $x_\gamma < d$ for all $\gamma \in [0,1]$.* □

We define the following critical preference ratios for the case of expensive reinsurance.

Notation 4.26. *We assume $\theta\rho > \sigma^2$. If $\theta \le \theta_d$, we define*

$$\chi_{e,2} = \frac{\kappa\xi\sigma^2}{\delta(\sigma^2\kappa + \theta)Q_{\bar\gamma}(d)} \quad \text{and} \quad \chi_{e,3} = \frac{\kappa}{\delta(\kappa P_0(d) + Q_0(d))}.$$

*If $\theta > \theta_d$, we write*

$$\chi_{e,2}^d = \frac{\kappa\xi\sigma^2 \mathrm{e}^{\xi x_0}}{\delta(\sigma^2\kappa + \theta)\mathrm{e}^{\xi d} Q_0(x_0)}.$$

Remark. We note that, for $\theta = \theta_d$, we have $\bar\gamma = 0$, $x_0 = d$ and therefore $Q_{\bar\gamma}(d) = Q_0(x_0)$ and $\chi_{e,2} = \chi_{e,3} = \chi_{e,2}^d$. Moreover, it holds $\lim_{\theta\rho \searrow \sigma^2} \chi_{e,2} = \lim_{\theta\rho \nearrow \sigma^2} \chi_{c,1}$ and the definition of $\chi_{e,3}$ coincides with $\chi_{c,4}$. Hence, the 'transitions' between the cases of cheap reinsurance, expensive reinsurance with $\theta \le \theta_d$ and expensive reinsurance with $\theta > \theta_d$ are continuous. #

Lemma 4.27. [2021a]

i) *For $\theta < \theta_d$, we have $\chi_{e,1} > \chi_{e,2} > \chi_{e,3} > 0$ and for $\theta = \theta_d$, $\chi_{e,1} > \chi_{e,2} = \chi_{e,3} > 0$.*
ii) *For $\theta > \theta_d$, we have $\chi_{e,1} > \chi_{e,2}^d > 0$.* □

This follows from Lemma A.3 and Lemma A.4 of the appendix, pp. 131, 132.
As in the case of cheap reinsurance, we start with an overview of the core results of this section. If reinsurance is 'moderately' expensive, that means $\theta \le \theta_d$, we obtain in the following that there are three different types of increasing optimisers depending on the size of $\chi < \chi_{e,1}$. These are displayed in the left graph of in Figure 4.5. If $\chi \in [\chi_{e,2}, \chi_{e,1})$, the optimiser is positive in zero ($b^*(0) > 0$) and there exists a point $x_\gamma < d$ from which on it is equal to one (dotted line). $\chi = \chi_{e,2}$ is the critical value such that the corresponding pair $(\bar\gamma, x_{\bar\gamma})$ fulfils $x_{\bar\gamma} = d$ (short dashed, black line). For $\chi \in (\chi_{e,3}, \chi_{e,2})$ we have $b^*(0) > 0$ and $b^*(d-) < 1$ (long dashed line). For $\chi \in [0, \chi_{e,3}]$, we additionally have $b^*(0) = 0$

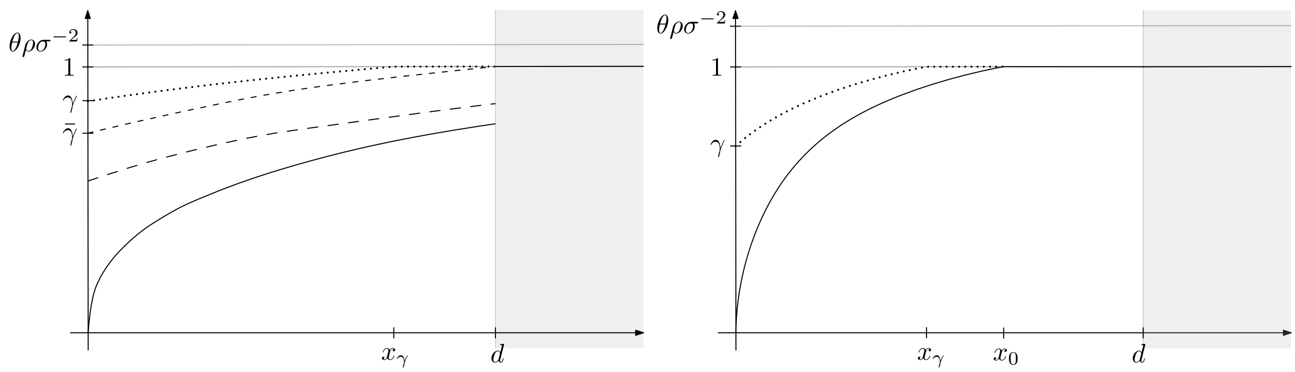



Figure 4.5 Optimal reinsurance strategies for expensive reinsurance with $\theta \le \theta_d$ (*left*) and $\theta > \theta_d$ (*right*).

(solid line). This optimiser corresponds to the one obtained in Chapter 3 for pure drawdown control. If reinsurance is even more expensive with $\theta > \theta_d$, there are two possibilities for $\chi < \chi_{e,1}$. These are displayed in the right graph of in Figure 4.5. For $\chi \in (\chi^d_{e,2}, \chi_{e,1})$, we get $b^*(0) > 0$ and $x_\gamma < d$ with $b^*(x_\gamma) = 1$ (dotted line). For $\chi \in [0, \chi^d_{e,2}]$, $b^*(0) = 0$ and $b^*(x) = 1$ is fulfilled for all $x \in [x_0, d]$ (solid line). Again, $b^*$ coincides with the function obtained in Chapter 3.

**The case $\theta \leq \theta_d$**

LEMMA 4.28. [2021a]

*For $\chi \in (\chi_{e,2}, \chi_{e,1})$, there exists a unique $\gamma \in (\bar\gamma, 1)$ with $x_\gamma \in (0, d)$ such that the function $f : [0,\infty) \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} -\beta_1 P_\gamma(x)\,, & x \in [0, x_\gamma]\,, \\ C_1 e^{\xi x} + C_2 e^{-\kappa x}\,, & x \in (x_\gamma, d]\,, \\ -\beta_2\delta^{-1} + (C_1 e^{\xi d} + C_2 e^{-\kappa d} + \beta_2\delta^{-1}) e^{-\kappa(x-d)}\,, & x > d\,, \end{cases}$$

*with*

$$C_1 = -\frac{\beta_1 Q(x_\gamma)(\theta + \kappa\sigma^2)}{\sigma^2 \xi(\kappa+\xi) e^{\xi x_\gamma}} \quad \text{and} \quad C_2 = -\frac{\beta_1 Q(x_\gamma)(\theta - \xi\sigma^2)}{\sigma^2 \kappa(\kappa+\xi) e^{-\kappa x_\gamma}}$$

*is a bounded solution to (4.6) with $f'(0) = -\beta_1$. The pointwise maximiser is given by*

$$b^*(x) = \begin{cases} \theta\rho\sigma^{-2}(1 + \mathcal{W}(E_\gamma(x)))\,, & x \in [0, x_\gamma]\,, \\ 1\,, & x > x_\gamma\,, \end{cases} \tag{4.25}$$

*and continuous. $b^*$ is strictly increasing and concave on $(0, x_\gamma)$. If $\chi = \chi_{e,2}$, the assertion holds for the pair $\bar\gamma$ and $x_{\bar\gamma} = d$.*

*Proof.* The third case of Lemma 4.14 implies that, for all $\gamma \in [0,1]$, the function $h_0(x) = -\beta_1 P_\gamma(x)$ solves the homogeneous Hamilton–Jacobi–Bellman equation for $x \in [0, x_\gamma]$ with the proposed optimiser which fulfils $b^*(0) = \gamma$ and $b^*(x_\gamma) = 1$. For $\gamma \geq \bar\gamma$, we have $x_\gamma \leq d$. Similarly to Lemma 4.19, we extend $h_0$ on $(x_\gamma, d]$ by $h_1(x) = C_1 e^{\xi x} + C_2 e^{-\kappa x}$. Here, $C_1$ and $C_2$ are defined as above and chosen such that the first and second derivatives coincide at $x = x_\gamma$. Now, the condition of smooth fit at $x = d$ of $h_1$ extended by $g(x) = -\beta_2/\delta + (h_1(d) + \beta_2/\delta) e^{-\kappa(x-d)}$ becomes

$$\chi = \frac{\kappa\xi\sigma^2}{\delta(\sigma^2\kappa + \theta) e^{\xi d}} \frac{e^{\xi x_\gamma}}{Q_\gamma(x_\gamma)}\,. \tag{4.26}$$

By Lemma A.4, the right hand side is continuous and strictly increasing in $\gamma$, equal to $\chi_{e,2}$ for $\gamma = \bar\gamma$ and equal to $\chi_{e,1}$ for $\gamma = 1$. In particular, for any $\chi \in [\chi_{e,2}, \chi_{e,1}]$, there is a unique $\gamma \in [\bar\gamma, 1]$ such that equality holds.

This means, the only thing left to check is whether $h_1$ solves the Hamilton–Jacobi–Bellman equation with maximiser 1 on $(x_\gamma, d]$ in the case $x_\gamma < d$. This can be shown by observing that $h_1$ is strictly decreasing and concave on that interval and that the 'technical' maximum of $\mathcal{J}_{h_1}(b)$ fulfils $b_{h_1}(x) \geq b_{h_1}(x_\gamma) = 1$ for all $x \in [x_\gamma, d]$. We provide details in the appendix, p. 130. Because $h_1$ solves the homogeneous differential equation for $\mathcal{I}(x) = \{1\}$ for all $x \geq 0$, this implies that $h_1$ is in fact a solution to the Hamilton–Jacobi–Bellman equation for $x \in [x_\gamma, d]$. Therefore, $f$ solves (4.6) by Lemma 4.7. $\square$

These are the cases represented by the dotted line and the short-dashed line on the left hand side of Figure 4.5. Next, we consider functions associated with the long-dashed line of the graph.

Lemma 4.29. [2021a]
*For $\chi \in (\chi_{e,3}, \chi_{e,2})$, there exists a unique $\gamma \in (0, \bar{\gamma})$ such that the function $f : [0, \infty) \to \mathbb{R}$ defined by* (4.21) *is a bounded solution to* (4.6) *with $f'(0) = -\beta_1$. The pointwise maximiser is given by* (4.22) *and continuous in every point except in $x = d$. $b^*$ is strictly increasing and concave on $(0, d)$. If $\chi = \chi_{e,3}$, the assertion holds for $\gamma = 0$.*

*Proof.* Similarly to the proof of Lemma 4.21, we obtain that the condition of smooth fit at $x = d$ is

$$\chi = \frac{\kappa}{\delta(\kappa P_\gamma(d) + Q_\gamma(d))} \,.$$

By Lemma A.3, the right hand side is positive and strictly increasing in $\gamma$. The rest of the assertion is a direct consequence of Lemma 4.7, Corollary 4.8 and Lemma 4.14 iii). $\square$

The analogue to Corollary 4.22 is the following result, which is related to the proofs of Lemmata 4.28 and 4.29.

Corollary 4.30. [2021a]
*The function $\gamma : [\chi_{e,3}, \chi_{e,1}] \to [0, 1]$ implicitly given by*

$$\chi = \frac{\kappa}{\delta(\kappa P_{\gamma(\chi)}(d) + Q_{\gamma(\chi)}(d))} \cdot \mathbb{1}_{[\chi_{e,3}, \chi_{e,2})}(\chi) + \frac{\kappa\xi\sigma^2}{\delta(\sigma^2\kappa + \theta)\mathrm{e}^{\xi d}} \frac{\mathrm{e}^{\xi x_{\gamma(\chi)}}}{Q_{\gamma(\chi)}(x_{\gamma(\chi)})} \cdot \mathbb{1}_{[\chi_{e,2}, \chi_{e,1}]}(\chi)$$

*is continuous and strictly increasing with $\gamma(\chi_{c,3}) = 0$, $\gamma(\chi_{e,2}) = \bar{\gamma}$ and $\gamma(\chi_{e,1}) = 1$. The function $x_\gamma : [\chi_{e,2}, \chi_{e,1}] \to [0, d]$ given by $x_\gamma(\chi) = x_{\gamma(\chi)}$ is continuous and strictly decreasing.* $\square$

This means, if $\chi \in [\chi_{e,3}, \chi_{e,1}]$ is reduced, the starting point $b^*(0) = \gamma$ 'slides down' the $y$-axis and, simultaneously, $x_\gamma$ is shifted to the right on the $x$-axis in the left graph of Figure 4.5. Lastly, we consider the maximiser corresponding to the solid line in the left graph of Figure 4.5.

Lemma 4.31. [2021a]
*For $\chi \in [0, \chi_{e,3})$, the function $f : [0, \infty) \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} -\beta_2 \chi_{e,3} P_0(x) \,, & x \in [0, d] \,, \\ -\beta_2\delta^{-1} + \Big(-\beta_2\chi_{e,3}P_0(d) + \beta_2\delta^{-1}\Big)\mathrm{e}^{-\kappa(x-d)} \,, & x > d \,, \end{cases}$$

*is a bounded solution to* (4.6) *with $f'(0) = -\beta_2\chi_{e,3} < -\beta_1$. The pointwise maximiser $b^*$ is given by* (4.22) *with $\gamma = 0$ and continuous in every point except in $x = d$. $b^*$ is strictly increasing and concave on $(0, d)$.*

*Proof.* An explicit calculation proves the continuity of the derivative of $f$ at $x = d$. Hence, the assertion follows by Lemma 4.7, Corollary 4.8 and Lemma 4.14 iii). $\square$

**The case $\theta > \theta_d$**

The following statements can be shown by repeating the arguments of the proof of Lemma 4.28.

LEMMA 4.32. [2021a]

*For $\chi \in (\chi^d_{e,2}, \chi_{e,1})$, there exists a unique $\gamma \in (0,1)$ with $x_\gamma \in (0,d)$ such that the function $f : [0,\infty) \to \mathbb{R}$ defined as in Lemma 4.28 is a bounded solution to (4.6) with $f'(0) = -\beta_1$. The pointwise maximiser $b^*$ is given by (4.25) and is continuous. $b^*$ is strictly increasing and concave on $(0, x_\gamma)$. If $\chi = \chi^d_{e,2}$, the assertion holds for the pair $\gamma = 0$ and $x_0$.* □

This case corresponds to the dotted line in the right graph of Figure 4.5. The next result follows from Lemma A.4 of the appendix, p. 132.

COROLLARY 4.33. [2021a]

*The function $\gamma : [\chi^d_{e,2}, \chi_{e,1}] \to [0,1]$ implicitly given by*

$$\chi = \frac{\kappa\xi\sigma^2}{\delta(\sigma^2\kappa + \theta)\mathrm{e}^{\xi d}} \frac{\mathrm{e}^{\xi x_{\gamma(\chi)}}}{Q_{\gamma(\chi)}(x_{\gamma(\chi)})}$$

*is continuous and increasing with $\gamma(\chi^d_{e,2}) = 0$ and $\gamma(\chi_{e,1}) = 1$. The function $x_\gamma : [\chi^d_{e,2}, \chi_{e,1}] \to [0, x_0]$ given by $x_\gamma(\chi) = x_{\gamma(\chi)}$ is continuous and strictly decreasing.* □

Hence, we get the same behaviour as in the case $\theta \le \theta_d$. Lastly, we consider the case in which the drawdown penalty dominates.

LEMMA 4.34. [2021a]

*For $\chi \in [0, \chi^d_{e,2})$ the function $f : [0,\infty) \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} -\beta_2\chi^d_{e,2}P_0(x)\,, & x \in [0, x_0]\,,\\ C_1\mathrm{e}^{\xi x} + C_2\mathrm{e}^{-\kappa x}\,, & x \in (x_0, d]\,,\\ -\beta_2\delta^{-1} + \Big(C_1\mathrm{e}^{\xi d} + C_2\mathrm{e}^{-\kappa d} + \beta_2\delta^{-1}\Big)\mathrm{e}^{-\kappa(x-d)}\,, & x > d\,, \end{cases}$$

*where*

$$C_1 = \frac{-\beta_2\kappa}{\delta\mathrm{e}^{\xi d}(\kappa+\xi)} \quad \text{and} \quad C_2 = \frac{\xi\mathrm{e}^{(\kappa+\xi)x_0}(\theta - \sigma^2\xi)}{\kappa(\theta+\sigma^2\kappa)}C_1\,,$$

*is a bounded solution to (4.6) with $f'(0) = -\beta_2\chi^d_{e,2} < -\beta_1$. The pointwise maximiser $b^*$ is given by (4.25) with $\gamma = 0$ and is continuous. $b^*$ is strictly increasing and concave on $(0, x_0)$.* □

Now that we have identified solutions to (4.6) for cheap and expensive reinsurance, we move on to verifying that these solutions correspond to optimal strategies.

**Verification**

In view of Theorem 4.6, we have to prove existence of the processes under the respective feedback controls to conclude that our candidate solutions are return functions of admissible strategies. We combine strategies of the same 'type'. The first result applies to the functions defined in Lemmata 4.29 and 4.32. These are the cases in which the optimiser $b^*$ is bounded away from zero and continuous and which are, with regard to the related stochastic differential equations, the easiest to handle.

PROPOSITION 4.35. [2021a]

*The processes under the feedback controls induced by the functions defined for expensive reinsurance with $\theta \leq \theta_d$ and $\chi > \chi_{e,2}$ or $\theta > \theta_d$ and $\chi > \chi^d_{e,2}$ exist.*

*Proof.* For the respective optimiser $b^*$, we consider the reflected stochastic differential equation of Definition 3.5 with $\tilde{\mu}(x) = \mu(b^*(x))$ and $\tilde{\sigma}(x) = \sigma b^*(x)$, $x \geq 0$. By Lemma 4.15, $b^*$ is everywhere continuously differentiable except at $x = x_\gamma$, where it is continuous. Additionally, the derivative of $b^*$ is bounded. Hence, $b^*$ is Lipschitz continuous and fulfils a linear growth condition. Thus, Equation (3.5) has a unique strong solution $(\mathcal{E}, \mathcal{C})$ by [Pilipenko, 2014, Thm. 2.1.1] where $\mathcal{E}$ corresponds to the drawdown under the feedback control and $\mathcal{C}$ to the controlled running maximum with initial value zero. From this, the controlled surplus can be obtained for arbitrary initial capitals. □

Next, we consider optimisers that are bounded away from zero and fulfil a Lipschitz and linear growth condition except at $x = d$ where they are discontinuous. This result covers the cases of Lemmata 4.19, 4.21 and 4.28 and the argument coincides with the one for simple switching controls in Chapter 3.

PROPOSITION 4.36. [2021a]

*The processes under the feedback controls induced by the functions defined for cheap reinsurance with $\chi > \chi_{c,4}$ and for expensive reinsurance with $\theta \leq \theta_d$ and $\chi \in (\chi_{e,3}, \chi_{e,2}]$ exist.*

*Proof.* By Lemma 4.15 and the same arguments as above, $b^*$ is Lipschitz continuous on $[0, d)$ and $(d, \infty)$ with a bounded derivative. If $b^*$ jumps at $x = d$, weak existence of the controlled drawdown as a solution to (3.5) in Definition 3.5 follows by Theorem 4.1 of [Rozkosz and Słomiński, 1997]. As $b^*$ is bounded away from zero, Corollary 4.3 of [Semrau, 2009] ensures pathwise uniqueness. The Yamada–Watanabe type Theorem 333 of [Situ, 2005] therefore implies strong existence and uniqueness. □

Lastly, we consider the cases of Lemmata 4.23, 4.31, 4.34. The optimisers coincide with those of pure drawdown control and are associated with solutions $f$ to the Hamilton–Jacobi–Bellman equation with $f'(0) < \beta_1$. Therefore, the result follows from Lemma 3.22 from the previous chapter.

PROPOSITION 4.37. [2021a]

*The processes under the feedback controls induced by the functions defined for cheap reinsurance with $\chi \leq \chi_{c,4}$ and expensive reinsurance with $\chi \leq \chi_{e,2}$ and $\chi \leq \chi^d_{e,2}$ exist. The running maximum of the respective controlled process is constant.* □

We obtain the following theorem as a consequence of the existence results and the verification theorem.

THEOREM 4.38. [2021a]

*For cheap and expensive reinsurance, the functions we have obtained in this section are indeed the value functions for the respective choices of $\chi$. For all cases, an optimal strategy is given by the feedback strategy induced by the associated pointwise maximiser.* □

## 4.3 Numerical Examples

For our numerical study, we consider the parameter set given in Table 4.1. We assume in all cases that the drift of the insurance surplus is $\eta = 0.5$ with volatility $\sigma = 1$. Further, we choose the preference parameters $\delta = 0.3$ and $d = 0.83$, so these key parameters coincide with those of Chapter 3. We fix

| $\eta$ | $\sigma$ | $\delta$ | $d$ | $\theta$ | $\beta_1$ | | | | $\beta_2$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 1 | 0.3 | 0.83 | 0.6 | (A) | 1.58 | (D) | 0.28 | 1 |
| | | | | | (B) | 0.8 | (E) | 0.2 | |
| | | | | | (C) | 0.4 | (F) | 0.016 | |

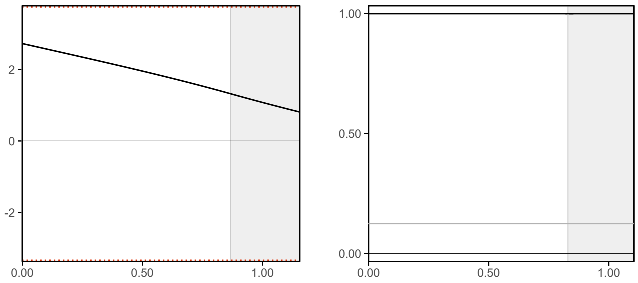
TABLE 4.1 Parameters of insurance surplus, preference and costs of reinsurance and weights of the value function.

the (cheap) reinsurance safety loading $\theta = 0.6$ for which we are going to compare different preference ratios $\chi$. In particular, the optimal strategies are either constant and equal to one or take one of the shapes displayed in Figure 4.4. The critical values $\chi_{c,i}$ for this parameter set are given in Table 4.2. This means that for $\chi > \chi_{c,1} \approx 1.5715$, the strategy of never purchasing reinsurance is optimal, for $\chi \geq \chi_{c,2} \approx 0.3153$, the optimiser is equal to one at least at those times when the drawdown is zero and so on. For the sake of clarity of presentation we choose $\beta_2 = 1$ in all cases so that we have $\beta_1 = \chi$. In the following, we consider value functions, optimisers and path simulations for different values of $\chi$: $\chi \in \{1.58, 0.8, 0.4, 0.28, 0.2, 0.016\}$. With this choice, we focus on parameter sets for which the insurer has some interest in increasing the maximum surplus ($\chi \geq \chi_{c,3}$ in five out of six cases) and include the extreme cases $\chi \geq \chi_{c,1}$ and $\chi \leq \chi_{c,4}$.

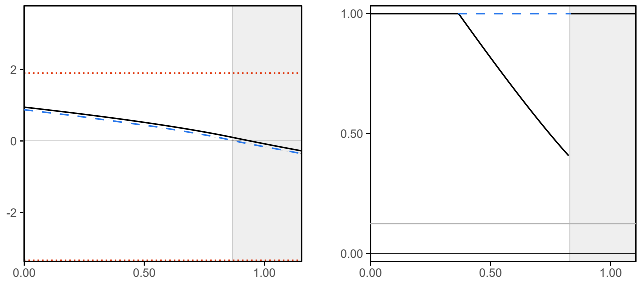
Figure 4.6 shows the value functions and optimisers for the optimal control problem defined in Equation (4.1). In particular, for $\chi = 1.58$, the left image of Figure 4.6(A) displays the return $v$ of the optimal strategy that is constant and equal to one. The grey area corresponds to $x > d$, that is, a large drawdown. The red, dotted lines on the very bottom and very top of this graph represent the a priori bounds of Lemma 4.2. The right image of Figure 4.6(A) displays the maximiser $b^*$ of the Hamilton–Jacobi–Bellman equation that induces the optimal feedback strategy. Again, the area with $x > d$ is shaded in grey. The grey, horizontal line corresponds to the value $\theta\rho\sigma^{-2} = 0.1250$. Figures 4.6(B) and 4.6(C) show graphs of these functions (solid lines) for $\chi = 0.8, 0.4 \in (\chi_{c,2}, \chi_{c,1})$. These strategies are of the type considered in Lemma 4.19, that is, they are equal to one in an environment of zero. Figures 4.6(D) and 4.6(E) display the graphs for $\chi = 0.28, 0.2 \in (\chi_{c,3}, \chi_{c,2})$, which are strictly decreasing and of the type defined in Lemma 4.21 i). For $\chi = 0.016 < \chi_{c,4}$, the graph of the optimal strategy in Figure 4.6(F) corresponds to the optimal strategy for pure drawdown optimisation considered in Chapter 3. In the cases (B)–(F), the return of the optimal strategies is higher than the return of the constant strategy of never buying reinsurance which is represented by the blue, dashed line in the (respective) left graph. The optimal retention level tends towards the critical value $\theta\rho\sigma^{-2}$

| $\chi_{c,1}$ | $\chi_{c,2}$ | $\chi_{c,3}$ | $\chi_{c,4}$ |
|---|---|---|---|
| 1.5715 | 0.3153 | 0.0241 | 0.0165 |

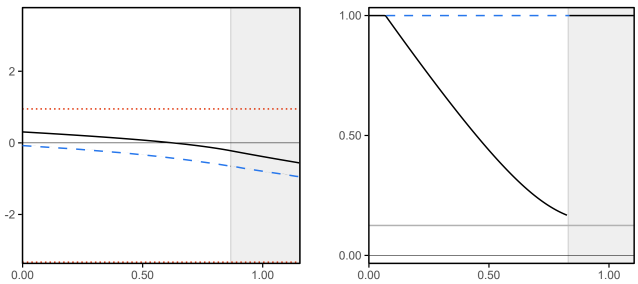
TABLE 4.2 Critical preference ratios for the parameter set of Table 4.1.



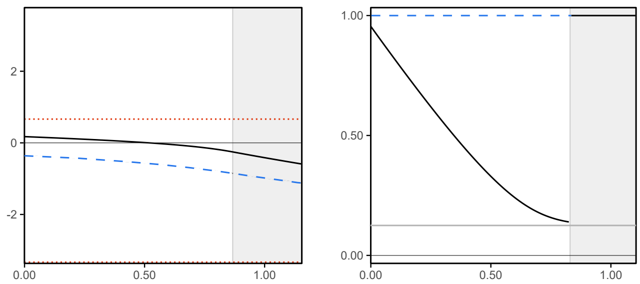

(A) $\chi = 1.58$



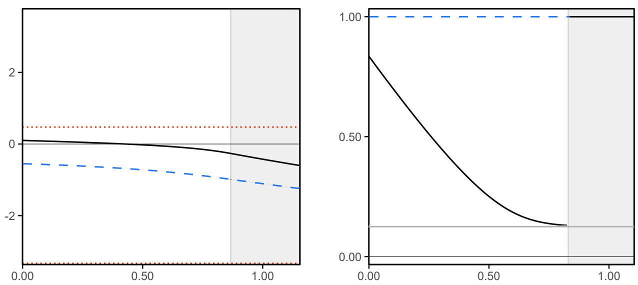

(B) $\chi = 0.8$



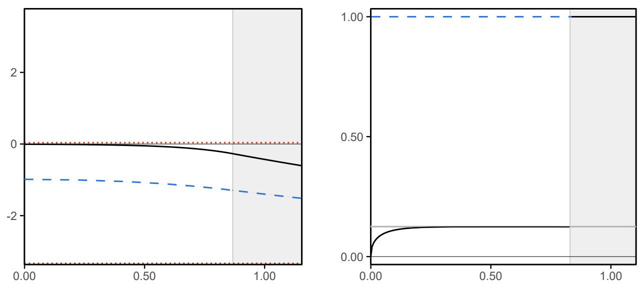

(C) $\chi = 0.4$




(D) $\chi = 0.28$




(E) $\chi = 0.2$




(F) $\chi = 0.016$

FIGURE 4.6 Value functions (*left*) and optimal strategies (*right*) for different $\chi$.

represented by the grey line in the respective right graph.

Comparing the different cases included in Figure 4.6, we observe that the value functions and the upper a priori bound generally attain lower values as $\chi$ is chosen smaller. This is just due the fact that $\beta_2$ is kept constant while $\beta_1$ decreases. Additionally, the distance to the return of the constant strategy increases as $\chi$ decreases. Intuitively, this is because the optimal strategy differs more from the constant strategy in the cases of smaller $\chi$.

For our path simulation, we assume that the initial drawdown is $x = 0.5$. The graphs of Figure 4.7, (A)–(F), correspond to possible paths of the surplus following the optimal strategies induced by the functions displayed in Figure 4.6. We use the same data set of increments of the Brownian motion as in Chapter 3, that means, the external scenario remains unchanged. Again, the colour of the graph of the surplus path indicates the current retention level of the insurer. Therefore, the black graph representing the surplus without reinsurance for $\chi = 1.58$, Figure 4.7(A), coincides with the graph in Figure 3.6(A) of the previous chapter. For the extreme case on the other side of the spectrum, $\chi = 0.016$, the optimal strategy coincides with the optimal 'pure' drawdown control. Thus, the light blue graph of Figure 4.7(F) is the same as the one pictured in Figure 3.6(F). For the 'intermediate' weights and strategies, Figures 4.7(B)–(E) show that reinsurance is bought whenever the drawdown


(A) $\chi = 1.58$

(B) $\chi = 0.8$

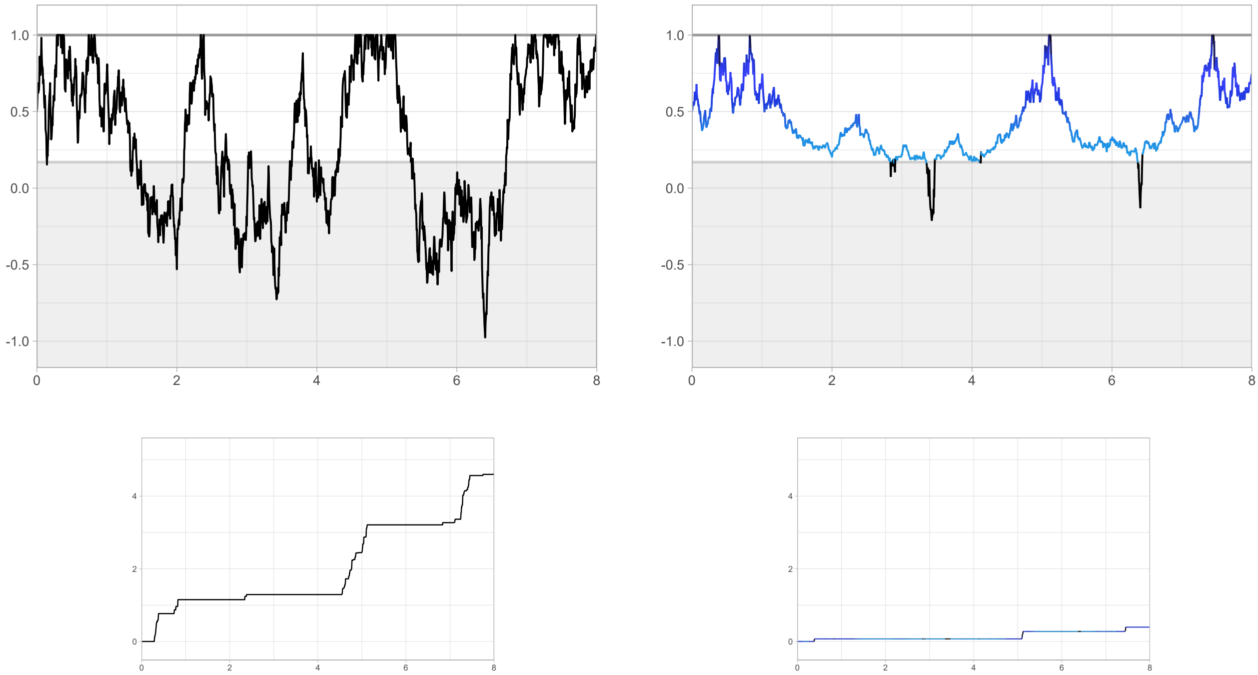
(C) $\chi = 0.4$

(D) $\chi = 0.28$

(E) $\chi = 0.2$

(F) $\chi = 0.016$

FIGURE 4.7 Path simulations for the optimal feedback strategies.




(A) $\chi = 1.58$ (D) $\chi = 0.28$

FIGURE 4.8 Path simulations of ex-dividend processes (top) and accumulated dividends (bottom) for different values of $\chi$.

approaches the unfavourable area and that it forces the process to stay in the uncritical area for a longer time. In Figure 4.7(B), this even leads to a higher surplus level (and higher historical record) at the end of the time interval than in the case without reinsurance. In Figures 4.7(C)–(F), we see that stricter controls corresponding to smaller $\chi$ lead to even less drawdowns at the cost of a lower record level.

As stated in the introduction of this chapter, our optimisation has the additional interpretation of 'maximal dividends with enhanced stability'. This is the function defined in Equation (4.2). Choosing the dividend barrier $y = 1$ and keeping the remaining parameters of Table 4.1, the value function $w$ penalises the state of a surplus lower than $y - d = 0.17$. In particular, $w(x)$ corresponds to $v(y - x)$ (displayed in Figure 4.6) for $x \leq y$ and is extended by $\beta_1(x - y) + w(y)$ for $x > y$. In path simulations, it can be observed that the accumulated dividend processes correspond to the running maxima of the processes without dividend payments (as per definition) and that the retention level strategies, indeed, coincide at every point in time (see Figure 4.8).

## 4.4 Key Findings and Concluding Remarks

In this chapter, we considered an extension to our central optimal control problem (1.9). In particular, we defined in (4.1) a new target function, which measures the discounted growth (or, equivalently, accumulated dividends) with a penalty for the time in critical drawdown. This choice was motivated by the fact that 'pure drawdown' controls stabilise the surplus process at the cost of never outperforming (and, in some cases, never reaching) the historical high water mark again. We started by proving a general verification theorem for the related Hamilton–Jacobi–Bellman equation. For the homogeneous part of this equation, we found (by refining the methods of Chapter 3) a set of solutions with non-constant optimisers related to the Lambert W function. For the inhomogeneous part, we derived that the optimal strategy for large initial drawdown is again the strategy of 'maximal drift'. This allowed us to approach the optimal control problem from a behaviouristic perspective. That is, instead of solving the Hamilton–Jacobi–Bellman equation itself, we started with a 'reasonable' pointwise optimiser $b^*$ with respect to the preference ratio $\chi = \beta_1/\beta_2$. Then, we calculated a solution to the modified differential equation and derived conditions under which this solution also solves the maximised equation. We got a first intuition of what a 'reasonable' optimiser looks like by considering the return function of the constant strategy $B$ with $B_t = 1$, $t \geq 0$. Here we found out that this strategy is optimal if the preference ratio is large enough, i.e. $\chi \geq \chi_{c,1}$ for cheap and $\chi \geq \chi_{e,1}$ for expensive reinsurance. Correspondingly, we derived for both cases critical upper bounds for $\chi$ such that the maximiser corresponded to the 'pure drawdown' control of Chapter 3. Distinguishing cheap and expensive reinsurance, we found a spectrum of optimisers and value functions connecting these extreme cases. Moreover, we calculated explicitly the critical preference ratios for certain strategy 'types' and showed that the transitions between all cases are continuous.

Our analysis and numerical examples show that, with the considered extension, we overcome the problem of non-increasing surplus brought up in Chapters 2 and 3. With the preference weights $\beta_1$ and $\beta_2$, it is possible to sensitively manage the dynamics of the surplus under the resulting feedback control. In particular, the optimal strategies combine the positive aspects of drawdown control, such as enhanced stability, with the opportunity of increasing profits and dividends.

Apart from the possibilities discussed in the following chapter, there are two specific aspects of this model raising interesting questions for further research. Firstly, because the definition of our problem was inspired by the case considered in Chapter 3, we transferred the assumption $\theta > \eta$, i.e. reinsurance is more expensive than first insurance. In Chapter 3, $\theta \leq \eta$ leads to trivial strategies. With the possibility of paying dividends included, however, this is not the case. From an economic point of view, the case $\theta = \eta$ of free risk trade could therefore be a scenario worth analysing. Though the case $\theta < \eta$ is not as realistic, the related stochastic differential equations are still intriguing from a mathematical perspective. Secondly, we assumed without further comments that the preference discounting for dividends is the same as for the drawdown penalty. If $\delta$ is interpreted as reflecting the general time value of money, this is a reasonable assumption. Another possibility would be to allow different discounting factors $\delta_1$ (for growth or dividends) and $\delta_2$ (for the drawdown penalty). This setting allows for two different 'time lines' for the perception of the benefit from dividends and the threat caused by drawdowns. In particular, this results in a two-dimensional problem because the Hamilton–Jacobi–Bellman equation and the optimal strategy are time dependent. Such problems are usually hard to solve. However, in view of the smooth transitions between the different categories of $\chi$, an approach could be to compose a two-dimensional strategy by 'glueing' one-dimensional strategies together. We discuss further extensions of the preference model in the following chapter.

CHAPTER 5

# Opportunities for Future Research

Drawdown-targeted optimal reinsurance enforces stability and, therefore, increases predictability of stochastic surplus models, as our analysis shows. The optimal strategies connected are anti-cyclic: reinsuring to minimise the time with critical drawdown (without any additional motives) means to 'play it safe' whenever the surplus is close to its historical peak and to take a risk when the drawdown is already critical. This equalises the surplus in the favourable area and, should a critical drawdown occur, induces a quick recovery. In combination with an incentive to grow (such as dividends), the proposed drawdown performance measure can be utilised to find balanced reinsurance policies leading to a 'sustainably' increasing profit. With this chapter, we conclude this monograph by addressing the various possibilities of further research related.

We start with extensions inspired by (some obstacles of) transferring our model results to a realistic setting. The classical risk process and its diffusion approximation with proportional reinsurance are very simple and versatile models. However, in an application, a more complex model could be favoured and more 'tools' for drawdown control might be available. Therefore, canonical extensions are to allow a broader class of surplus models (e.g. spectrally negative Lévy processes), to consider different types of reinsurance contracts (e.g. excess of loss) or to combine different control tools (e.g. reinsurance and investments or dividends).
Throughout this work, we assumed that the parameters $d$ (critical drawdown size) and $\delta$ (exponential time preference rate) are predefined. With regard to applications, a significant challenge is to examine how to 'correctly' choose these parameters. Moreover, in reality, the risk preference of a company

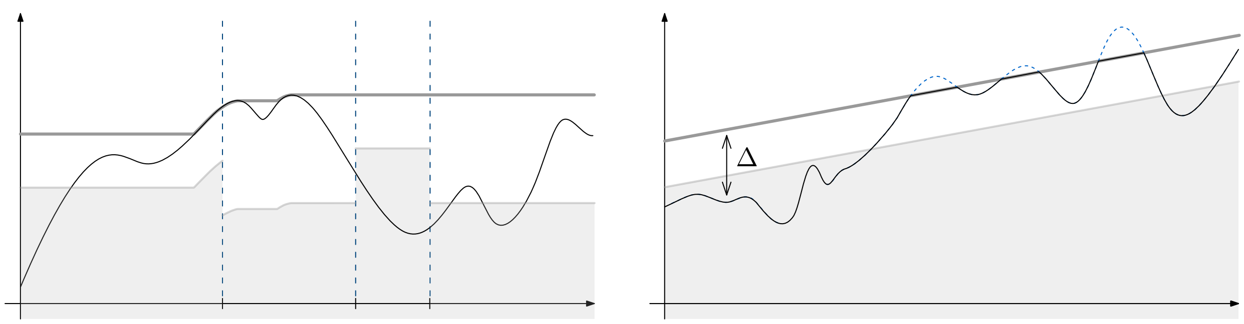



Figure 5.1 *Left:* The tolerance for drawdowns changes over time. *Right:* A time-dependent target function $f$ replaces the running maximum in an alternative problem.

could vary over time (for example because of a new manager or changed market situation). Therefore, a preference rate or critical drawdown size changing over time would be an interesting variant of the problem (compare Figure 5.1). One possibility (preserving the Markovian structure of the problem) is to consider a regime switching model, such as in [Jiang and Pistorius, 2012] or [Brinker and Eisenberg, 2021]. That means, the preference parameters switch between different 'states' according to a time-homogeneous Markov chain. This method leads to a system of Hamilton–Jacobi–Bellman equations with one equation for each state.
Another aspect which would have to be adjusted in an application is the structure of optimal policies. As we have seen, the optimal strategies are of feedback form in our models. However, in reality, an insurer cannot change the retention level continuously in time without added costs. That means, reinsurance strategies would have to be discretised to match target dates or periods. Administrative costs related to changes of the reinsurance coverage are also a suitable addition to the problem.

In Chapter 4, we have already seen one possibility to combine drawdown minimisation with other incentives in a value function. In the following, we consider further ideas for drawdown-oriented optimisation problems based on our analysis and applicable to different models or control tools.

**Drawdown from an Economic Target**

In the spirit of the alternative interpretation of the problem considered in Chapter 4, one may analyse the time the surplus spends 'far away' from a time-dependent target value $f(t)$ (given as a deterministic function or adapted stochastic process) representing an economic goal. This economic goal could, for example, be a managerial (previsible) decision or be influenced by factors which cannot be foreseen such as development of the market index or performance of a competitor. That means, if $X = (X_t)_{t\geq 0}$ denotes the surplus process, the value function is the expected (discounted) time during which the process $\Delta = (\Delta_t)_{t\geq 0}$ given by

$$\Delta_t = \big(f(t) - X_t\big)^+ , \qquad t \geq 0 ,$$

exceeds a critical level $d$, minimised with respect to reinsurance or investments. The right graph of Figure 5.1 shows an example. If $f(t)$ increases over time, this target function already accounts for the potential of growth. This effect can be reinforced by 'rewarding' the time in which $\Delta$ is equal to zero. For the constant function $f(t) = y \in \mathbb{R}$, this corresponds to the problem considered in Chapter 4. However, in general, the problem will be time-dependent.

**Dividend Optimisation without Regrets**

Another variant of the model in Chapter 4 would be to consider the problem as an optimisation of the dividend policy (with or without reinsurance). That means, for an adapted increasing process $D$ representing the accumulated dividends, we define the ex-dividend maximum $M = (M_t)_{t\geq 0}$ and drawdown $\Delta = (\Delta_t)_{t\geq 0}$ by

$$M_t = \max\left\{m_0, \sup_{s\leq t}(X_s - D_s)\right\}, \qquad \Delta_t = M_t - (X_t - D_t), \qquad t \geq 0,$$

and consider

$$\beta_1 \int_0^\infty \mathrm{e}^{-\delta t}\, \mathrm{d}D_t - \beta_2 \int_0^\infty \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta_t > d\}}\, \mathrm{d}t .$$

This is the value of accumulated dividends minus a penalty for the time in which the company's reputation is at risk because of large relative losses. In general, $M$ changes over time depending on $D$, so that $\Delta$ and $M$ have to be tracked. Still, we note that the return of barrier strategies can be calculated analogously to Chapter 4. In this way, one could find an optimal strategy (at least) from the set of barrier strategies.

### Ruin and Drawdown Penalties

In all considered models, the strategy of not purchasing reinsurance is optimal in the critical area. As we found out, this is the best strategy for quickly re-entering the uncritical area. But for a company whose surplus is already at very low level, such a 'high risk' strategy might not be favourable. For example, the surplus path in the left graph of Figure 5.1 starts closer to the bankruptcy line than to the uncritical area. One possibility to overcome this problem is to include ruin in the value function.

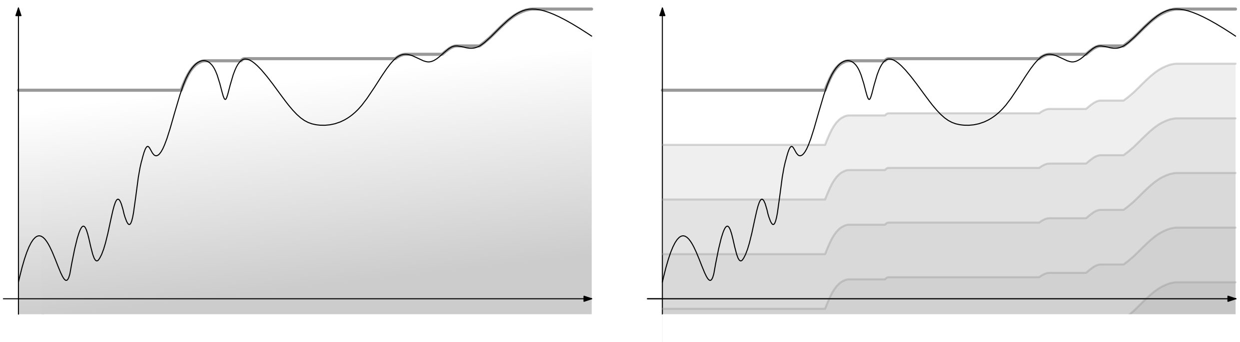


FIGURE 5.2 Two penalty functions increasing with the drawdown size.

In this case, one has to ensure that an early, deliberate ruin to prevent future drawdowns is not possible. That means, ruin has to be penalised, for example with the value $\delta^{-1}$ of staying in critical drawdown forever. More generally, one could minimise expressions such as

$$\int_0^\tau \mathrm{e}^{-\delta t}\mathbb{1}_{\{\Delta_t > d\}}\, \mathrm{d}t + \mathrm{e}^{-\delta\tau}\phi(X_\tau), \qquad \text{for } \tau = \{t \geq 0 : X_t < 0\},$$

or, similarly as in [Schmidli and Vierkötter, 2017],

$$\int_0^\infty \mathrm{e}^{-\delta t}\left[\mathbb{1}_{\{\Delta_t > d\}} + \phi(X_t)\right] \mathrm{d}t ,$$

where $\phi$ is a decreasing function. In these cases, an increasing maximum postpones the time of ruin or lowers the penalising term in the future. A disadvantage of including the surplus in the value function is that these two-dimensional problems (depending on $X$ and $\Delta$) are mathematically less tractable. A different and promising approach is the following. Low surplus levels are caused by particularly large drawdowns. However, our original value function penalises all 'critical' drawdowns above $d$ in the same way. Instead, one could define an increasing and convex penalty function $\varphi$ expressing the negative impact of extreme drawdowns and minimise

$$\int_0^\infty \mathrm{e}^{-\delta t}\varphi(\Delta_t)\,\mathrm{d}t\,.$$

Of course, here one has to prove that the integral is finite for the chosen $\varphi$. A simple example is $\varphi(x) = \mathbb{1}_{\{x>d\}}(x-d)^2$. This function resembles our original problem in the way that there is a tolerance for uncritical drawdowns. Relative losses exceeding the critical value are counted with a quadratic penalty. The left graph of Figure 5.2 illustrates this. A different possibility is shown in the right image of Figure 5.2: there are different critical drawdown sizes with constant penalties. As before, the white area of both images corresponds to a small drawdown and the darker areas symbolise the less favourable states.

### Drawdown Control for Extreme Events

In our considerations of the classical risk model, we have seen that drawdown minimising strategies are especially effective in the case of an extraordinarily high number or large size of claims. This is a useful property in the context of time-inhomogeneous claim intensities. For example, a Poisson process counting claims can be replaced by a Cox process with a shot-noise intensity to model 'catastrophic' events (such as floods or hurricanes) triggering a large number of claims (compare [Dassios and Jang, 2003] and [Albrecher and Asmussen, 2006]). That is, we assume that the intensity of the process $N=(N_t)_{t\geq 0}$ in the definition of the classical risk model is itself a stochastic process $\lambda=(\lambda_t)_{t\geq 0}$, given by

$$\lambda_t = \lambda_0 \mathrm{e}^{-\delta t} + \sum_{i=1}^{\tilde{N}_t} Z_i \mathrm{e}^{-\delta(t-T_i)}\,, \qquad t\geq 0\,,$$

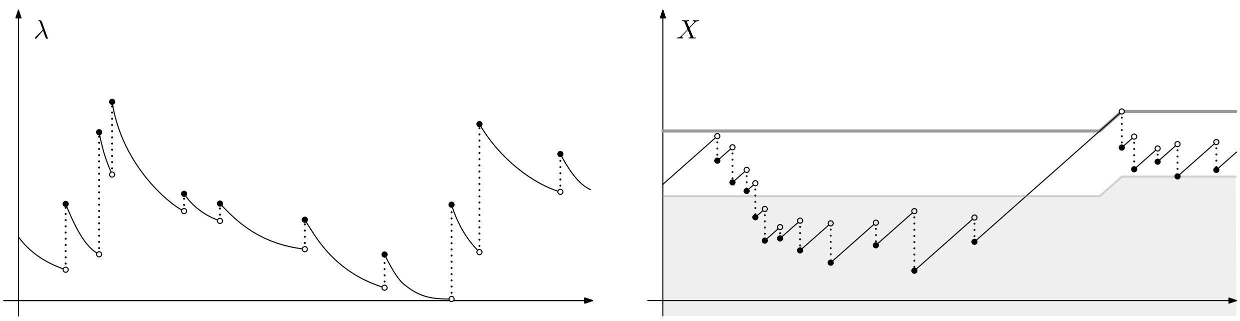



FIGURE 5.3 The claim arrival intensity changes over time and an 'extreme event' causes a critical drawdown.

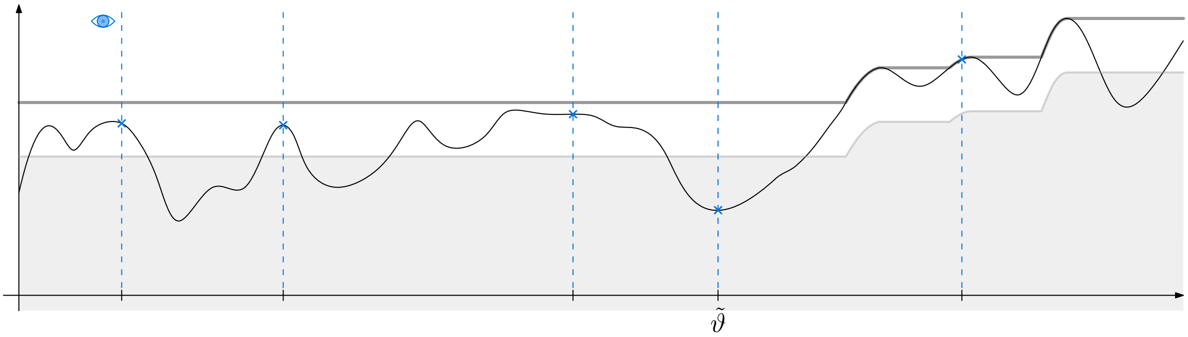



FIGURE 5.4 The process is observed at discrete points in time.

where $\tilde{N} = (\tilde{N}_t)_{t\geq 0}$ is a homogeneous Poisson process with arrival times $(T_i)_{i\in\mathbb{N}}$ and $(Z_i)_{i\in\mathbb{N}}$ is an independent sequence of independently identically distributed random variables determining the size of the 'shots'. The resulting surplus process is still a piecewise deterministic Markov process. Figure 5.3 is a sketch of possible paths of the intensity and surplus. To consider optimisation problems in this 'doubly stochastic' setting is an intriguing and significant challenge. Moreover, how to reinsure (based on the given information) to lower the resulting relative losses for extreme events is also an economically relevant question. As a first approach, one could assume that the surplus process and the intensity are observed. This leads to a two-dimensional problem. However, in reality, the insurer can only monitor the intensity indirectly by counting claim arrivals of the surplus process. That means, one would additionally have to use a filtering argument to estimate the unknown intensity.

### Revisiting Reputational Risks

Completing the circle, we consider an idea inspired by our motivation in Chapter 1. Here we stated that large drawdowns could damage a company's reputation because stakeholders would loose trust in the management. However, it is reasonable to assume that (potential) customers and shareholders do not monitor the surplus of the company continuously in time. Therefore, another extension to the model worth analysing is to allow 'inattentive' stakeholders who only check on the company at discrete observation times (as considered by Albrecher et al. [2011]). In this setting, critical drawdowns in between observation times remain unnoticed. If we denote the sequence of observation times by $(S_k)_{k\in\mathbb{N}}$, the expression to minimise (in expectation) is

$$\sum_{k=1}^{\infty} \mathrm{e}^{-\delta S_k} \mathbb{1}_{\{\Delta_{S_k} > d\}} .$$

This is illustrated in Figure 5.4: the vertical lines represent the observation times, such that the first critical drawdown which 'counts' is the one at time $\tilde{\vartheta}$. In addition, if the times between observations are memoryless (i.e. exponentially distributed), $\tilde{\vartheta}$ corresponds to the time to critical drawdown with exponential Parisian delay (compare [Dassios and Wu, 2008]), which is interesting on its own.

# Appendix: Details, Proofs and Technical Results

## A.1 Addendum to Chapter 2

*Proof of Lemma* 2.3 i), ii). We start by showing i). We observe that $\Psi_b(r)$ is defined (at least) for all $r \geq 0$. We have $\Psi_b(0) = 0$, $\Psi_b''(r) = \lambda b^2 \,\mathbb{E}(Y_1^2 \mathrm{e}^{-rbY}) > 0$ and $\lim_{r\to\infty} \Psi_b(r) = \infty$. If $\Psi_b'(0) \geq 0$ then $\Psi_b$ is strictly increasing on $(0, \infty)$ and $\gamma(b) > 0$ with $\Psi_b(\gamma(b)) = \delta$ exists and is unique. Otherwise, if $\Psi_b'(0) < 0$, there exists a unique $r_0 > 0$ with $\Psi_b(r_0) = 0$ because of the strict convexity of $\Psi_b$. Then, $\Psi_b'(r_0) > 0$ and $\Psi_b(r) < 0$ for all $r \in (0, r_0)$, so that there exists a unique $\gamma(b) > r_0$ with the desired properties. Now we prove ii). We notice that $b \mapsto \Psi_b(r)$ is increasing in $b \in [b_0, 1]$ for all $r > 0$: it holds

$$\frac{\mathrm{d}}{\mathrm{d}b} \Psi_b(x) = (1+\theta)\lambda\mu r + \lambda \ell_Y'(br) r \,,$$

where the right hand side is increasing in $b$ and positive at $b = b_0$ by $\ell_Y'(rb_0) = -\,\mathbb{E}(Y \mathrm{e}^{-rb_0 Y}) > -\mu$, so $(1+\theta)\lambda\mu r + \lambda \ell_Y'(b_0 r) r > \theta\lambda\mu r > 0$. Hence, by $\Psi_b(0) = 0$, $b \mapsto \gamma(b)$ decreases for $b \in (b_0, 1]$. $\square$

*Proof of Equation* (2.18). We note that

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-tx} \int_0^{x/\check{b}} w(x - \check{b}y)\, \mathrm{d}G(y)\, \mathrm{d}x &= \int_0^\infty \int_{\check{b}y}^\infty \mathrm{e}^{-tx} w(x - \check{b}y)\, \mathrm{d}x\, \mathrm{d}G(y) \\
&= \int_0^\infty \int_0^\infty \mathrm{e}^{-t(z+\check{b}y)} w(z)\, \mathrm{d}z\, \mathrm{d}G(y) = \Big(\int_0^\infty \mathrm{e}^{-t\check{b}y}\, \mathrm{d}G(y)\Big)\Big(\int_0^\infty \mathrm{e}^{-tz} w(z)\, \mathrm{d}z\Big) \\
&= \ell_w(t) \ell_Y(t\check{b})
\end{aligned}$$

and

$$\int_0^\infty \mathrm{e}^{-tx} \int_{x/\check{b}}^\infty G(y)\, \mathrm{d}x = \int_0^\infty \int_0^{\check{b}y} \mathrm{e}^{-tx}\, \mathrm{d}x\, \mathrm{d}G(y) = \frac{1}{t} \int_0^\infty (1 - \mathrm{e}^{-t\check{b}y})\, \mathrm{d}G(y) = \frac{1 - \ell_Y(t\check{b})}{t}$$

and

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-tx} \int_{x/\check{b}}^\infty \mathrm{e}^{-\gamma(\hat{b})(\check{b}y - x)}\, \mathrm{d}G(y)\, \mathrm{d}x &= \int_0^\infty \mathrm{e}^{-\gamma(\hat{b})\check{b}y} \int_0^{\check{b}y} \mathrm{e}^{-(t-\gamma(\hat{b}))x}\, \mathrm{d}x\, \mathrm{d}G(y) \\
&= -\frac{1}{\gamma(\hat{b}) - t} \int_0^\infty (\mathrm{e}^{-\gamma(\hat{b})\check{b}y} - \mathrm{e}^{-t\check{b}y})\, \mathrm{d}G(y) = \frac{\ell_Y(t\check{b}) - \ell_Y(\gamma(\hat{b})\check{b})}{\gamma(\hat{b}) - t} \,.
\end{aligned}$$

The equation follows from plugging in these expressions. □

*Proof of Lemma* 2.11. By adding a superscript '$(n)$' to the already existing notation we indicate the quantities related to the claim distribution $G^{(n)}$. By the extended continuity theorem in [Feller, 1971], the Laplace transforms of the approximating sequence converge pointwise to the Laplace transform of $Y_1$. Therefore, $\Psi_b^{(n)}(t) = c(b)t - \lambda(1 - \ell_{Y^{(n)}}(b)) \to \Psi_b(t)$ as $n \to \infty$ for all $t \geq 0$. We first show that $\gamma^{(n)}(b)$ converges to $\gamma(b)$ if it exists, that is, if $b \in (b_0, 1]$. To this purpose, we recall that $\Psi_b^{(n)}$ and $\Psi_b$ are continuous and convex. In particular, these functions are increasing on the subinterval of $(0, \infty)$ on which they are positive. Now we use a similar technique as in [Asmussen and Albrecher, 2010, Corollary A5.17] to prove the convergence of the $\gamma^{(n)}(b)$. For all $\varepsilon > 0$, it holds $\Psi_b^{(n)}(\gamma(b) + \varepsilon) \to \Psi_b(\gamma(b) + \varepsilon) > \delta$, such that there exists $n_\varepsilon$ with $\Psi_b^{(n)}(\gamma(b) + \varepsilon) > \delta$ for all $n \geq n_\varepsilon$. By $\delta = \Psi_b^{(n)}(\gamma^{(n)}(b))$, we obtain $\gamma(b) + \varepsilon > \gamma^{(n)}(b)$, which holds for all $n \geq n_\varepsilon$. This implies that we have $\sup_{n \geq n_\varepsilon} \gamma^{(n)}(b) \leq \gamma(b) + \varepsilon$ and, as a consequence: $\limsup_{n \to \infty} \gamma^{(n)}(b) \leq \gamma(b)$. Analogously, one can show the opposite inequality: $\liminf_{n \to \infty} \gamma^{(n)}(b) \geq \gamma(b)$. Therefore, in case i) with $\hat{b} = b_0$, the Laplace transforms

$$\ell_{w^{(n)}}(t) = \frac{1}{t}\Big(\frac{1}{\delta} + \frac{1}{\Psi_{\check{b}}^{(n)}(t) - \delta}\Big),$$

converge to $\ell_w$ (which takes the same form) on an interval $(a, \infty)$, $a \geq 0$. Again by the extended continuity theorem, we can conclude $u^{(n)}(x) = w^{(n)}(d - x) \to w(d - x) = u(x)$ as $n \to \infty$ for all $x \in [0, d]$. In case ii), (2.19) takes the form

$$\ell_{w^{(n)}}(t) = \frac{1}{t}\Big(\frac{1}{\delta} + \frac{1}{\Psi_{\check{b}}^{(n)}(t) - \delta}\Big) - \frac{(\delta^{-1} - w^{(n)}(0))}{t - \gamma^{(n)}(\hat{b})}.$$

Thus, in this case, we additionally have to show that $w^{(n)}(0)$, that is $u^{(n)}(d)$, converges. As in case i), we conclude convergence of $W^{\delta}{}_{\check{b}}^{(n)}(x)$ to $W_{\check{b}}^{\delta}(x)$ for every $x \in [0, d]$ from the convergence of $\Psi_{\check{b}}^{(n)}$ to $\Psi_{\check{b}}$. In particular, this is fulfilled at $x = d$. Thus, it follows from the representation of $u^{(n)}(d)$, $u(d)$ given in Theorem 2.9, that $\ell_{w^{(n)}}(t) \to \ell_w(t)$, as above, and thus that $u^{(n)}(x) \to u(x)$ for all $x \in [0, d]$. Case iii) follows analogously: we can assume that $(\gamma^{(n)})_{n \in \mathbb{N}}$ is monotone by choosing an adequate subsequence. In combination with the uniform convergence of $W^{\delta}{}_{\check{b}}^{(n)}$, the integral in the denominator of $u^{(n)}(d)$ in (2.20) converges. Again we obtain convergence of the Laplace transforms and of $u^{(n)}(d)$, so that the assertion can be concluded. In all cases, we have $\lim_{n \to \infty} u^{(n)}(x) = u(x)$ for all $x > d$ by Equation (2.7) and convergence at $x = d$. □

*Proof of Lemma* 2.15. We first note that $\Delta^B$ and $X^B$ are processes of locally bounded variation, since the size and number of jumps in a finite time interval are almost surely finite. $M^B$ additionally is increasing and continuous. We have, by change of variables:

$$
\begin{aligned}
\mathrm{e}^{-\delta t} f(\Delta_t^B) - f(\Delta_0^B) = \int_0^t -\delta \mathrm{e}^{-\delta s} f(\Delta_{s-}^B)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\delta s} f'(\Delta_{s-}^B)\,\mathrm{d}M_s^B \\
- \int_0^t \mathrm{e}^{-\delta s} f'(\Delta_{s-}^B)\,\mathrm{d}\,(X_s^B)^c + \sum_{s\le t} \mathrm{e}^{-\delta s}\big[f(\Delta_s^B) - f(\Delta_{s-}^B)\big]\,,
\end{aligned}
$$

where $(X^B)^c$ denotes the continuous part of the respective path of $X^B$. Since the running maximum can only increase if the drawdown is equal to zero, we can replace the integrand of the integral with respect to $M^B$ with $\mathrm{e}^{-\delta s} f'(0)$. Additionally plugging in the continuous part of $X^B$, we arrive at

$$
\begin{aligned}
\mathrm{e}^{-\delta t} f(\Delta_t^B) - f(\Delta_0^B) = \int_0^t \mathrm{e}^{-\delta s} f'(0)\,\mathrm{d}M_s^B + \int_0^t \mathrm{e}^{-\delta s}\big(-\delta f(\Delta_{s-}^B) - c(B_{s-}) f'(\Delta_{s-}^B)\big)\,\mathrm{d}s \\
+ \sum_{k=1}^{N_t} \mathrm{e}^{-\delta T_k}\big(f(\Delta_{T_k-}^B + B_{T_k-}Y_k) - f(\Delta_{T_k-}^B)\big)\,. \qquad \text{(A.1)}
\end{aligned}
$$

Now the last term on the right hand side is a pure jump process. This process can be expressed as a stochastic integral with respect to the random counting measure $\mu^\circ$ associated to the compound Poisson process determined by $N$ and $(Y_k)_{k\in\mathbb{N}}$. The processes $S^y = (S_s^y)_{s\ge 0}$, $y \ge 0$, with

$$
S_s^y = \mathrm{e}^{-\delta s}\big(f(\Delta_{s-}^B + B_{s-}y) - f(\Delta_{s-}^B)\big)\,, \qquad s \ge 0\,,
$$

are predictable. Since $f$ is bounded, the absolute value of $S_s^y$ is bounded by $\mathrm{e}^{-\delta s}$ multiplied with a constant. Thus, by the martingale representation theorem (compare [Jacobsen, 2006, Thm. 4.6.1]), the process $O = (O_t)_{t\ge 0}$ with

$$
O_t = \sum_{k=1}^{N_t} \mathrm{e}^{-\delta T_k}\big(f(\Delta_{T_k-}^B + B_{T_k-}Y_k) - f(\Delta_{T_k-}^B)\big) - \int_0^t \lambda \int_0^\infty S_s^y\,\mathrm{d}G(y)\,\mathrm{d}s\,, \qquad t \ge 0\,,
$$

is a martingale of expectation zero. By (A.1), we have

$$
\begin{aligned}
O_t = \mathrm{e}^{-\delta t} f(\Delta_t^B) - f(\Delta_0^B) - \int_0^t \mathrm{e}^{-\delta s} f'(0)\,\mathrm{d}M_s^B \\
- \int_0^t \mathrm{e}^{-\delta s}\Big(-\delta f(\Delta_s^B) - c(B_s) f'(\Delta_s^B) + \lambda \int_0^\infty f(\Delta_s^B + B_s y) - f(\Delta_s^B)\,\mathrm{d}G(y)\Big)\,\mathrm{d}s\,.
\end{aligned}
$$

Since we integrate with respect to the Lebesgue measure and every path $s \mapsto \Delta_s^B(\omega)$ is càdlàg and has at most countably many jumps, it is possible to replace the left limits '$s-$' with '$s$' in the integrand. □

*Proof of Proposition* 2.20. The drawdown process stays in zero up to the first claim time $T_1$ for all $B \in \mathfrak{B}$ because we have $-c(B_t) \le -c(b_0) = 0$, $t \ge 0$. For a given strategy $B$, we write $\tilde{B}$ for the strategy shifted by the time of the first claim, that is $\tilde{B}_t = B_{T_1+t}$, $t \ge 0$. Distinguishing the cases of staying in $[0,d]$ and exiting at the first claim time, we obtain:

$$\begin{aligned}
v_C^B(0) &= \mathbb{E}^0\big[\mathrm{e}^{-\delta T_1} v_C^{\tilde B}(B_{T_1}{-}Y_1)\mathbb{1}_{\{\Delta^B_{T_1}\le d\}}\big] + \mathbb{E}^0\big[\mathrm{e}^{-\delta T_1} p_C(B_{T_1}{-}Y_1)\mathbb{1}_{\{\Delta^B_{T_1}> d\}}\big] \\
&\ge \mathbb{E}^0\big[\mathrm{e}^{-\delta T_1} w_C(B_{T_1}{-}Y_1)\big] \ge \inf_{b\in[b_0,1]} \mathbb{E}^0\big[\mathrm{e}^{-\delta T_1} w_C(bY_1)\big]\,.
\end{aligned}$$

Because this is fulfilled for arbitrary strategies $B\in\mathcal{B}$, we obtain (using independence of $Y_1$ and $T_1$):

$$v_C(0) \ge \inf_{b\in[b_0,1]} \mathbb{E}^0\big[\mathrm{e}^{-\delta T_1} w_C(bY_1)\big] = \frac{\lambda}{\lambda+\delta}\inf_{b\in[b_0,1]}\int_0^\infty w_C(by)\,\mathrm{d}G(y)\,.$$

By [Elstrodt, 2011, Thm. IV.5.6], $b\mapsto \mathbb{E}^0[w_C(bY_1)]$ is continuous and $[b_0,1]$ is compact. Thus, the infimum on the far right is attained at some $b'\in[b_0,1]$. To show the opposite inequality we define the strategy $B$ by $B_t=b'$, $t<T_1$, and $B_t=B^\varepsilon_{t-T_1}$, $t\ge T_1$, where $B^\varepsilon$ fulfils $v_C(x) > v_C^{B^\varepsilon}(x)-\varepsilon$ for all $x\in[0,d]$. We recall that such a universally $\varepsilon$-optimal strategy exists. Now we have

$$\begin{aligned}
v_C(0) \le v_C^B(0) &= \mathbb{E}^0\big[\mathrm{e}^{-\delta T_1} v_C^{B^\varepsilon}(b'Y_1)\mathbb{1}_{\{\Delta^B_{T_1}\le d\}}\big] + \mathbb{E}^0\big[\mathrm{e}^{-\delta T_1} p_C(b'Y_1)\mathbb{1}_{\{\Delta^B_{T_1}> d\}}\big] \\
&\le \mathbb{E}^0\big[\mathrm{e}^{-\delta T_1} w_C(b'Y_1)\big]+\varepsilon
\end{aligned}$$

and letting $\varepsilon\to 0$ shows (2.29). Because, by Lemma 2.19, $w_C$ is an increasing function for $C\ge C_d$, this representation implies $b'=b_0$. $\square$

*Calculation of* $\mathbf{T}$ *and* $\mathbf{t}$ *in Subsection* 2.3.1. We distinguish the cases of zero drift (i.e. $k=0$ or $m=0$) and the cases of non-zero drift ($k>0$ and $m>0$). In order to avoid repetition, we identify frequently reappearing expressions which only have to be calculated once. Then we show how $\mathbf{T}$ and $\mathbf{t}$ can be defined based on these terms. If either $k=0$ or $m=0$ with $k<n$, we obtain, similarly to Equation (2.36):

$$\begin{aligned}
\mathbf{v}_k &= \mathbf{v}_k\cdot\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{\varrho_m Y_1\le(1-\xi)q\}}\Big] \\
&\quad+\sum_{j=k+1}^{n-1}\mathbf{v}_j\cdot\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{(j-k)q-\xi q<\varrho_m Y_1\le(j-k)q+(1-\xi)q\}}\Big] \\
&\quad+\mathbf{v}_n\cdot\Big(\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{(n-k)q-\xi q<\varrho_m Y_1\le(n-k)q\}}\Big]+\mathrm{e}^{(n-k)q}\,\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathrm{e}^{-\gamma\varrho_m Y_1}\mathbb{1}_{\{\varrho_m Y_1>(n-k)q\}}\Big]\Big) \\
&\quad+\delta^{-1}\Big(\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{\varrho_m Y_1>(n-k)q\}}\Big]-\mathrm{e}^{(n-k)q}\,\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathrm{e}^{-\gamma\varrho_m Y_1}\mathbb{1}_{\{\varrho_m Y_1>(n-k)q\}}\Big]\Big)\,.
\end{aligned}$$

For $k=n$ and $m=0$, this equation holds with the first and second terms set to zero. We observe that in all cases, the expressions only depend on the distance of the grid points. Thus, writing for $m=0,\dots,n$

$$\begin{aligned}
\mathcal{D}[p,m] &= \mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{pq-\xi q<\varrho_m Y_1\le pq+(1-\xi)q\}}\Big]\,, \\
&= \frac{\lambda}{\lambda+\delta}\,\mathbb{P}\Big[\frac{pq-\xi q}{\varrho_m}<Y_1\le\frac{pq+(1-\xi)q}{\varrho_m}\Big]\,,\qquad p=0,\dots,n-1\,,
\end{aligned}$$

and for $m=0$ (with $k=0,\dots,n$) and $k=0$ (with $m=0,\dots,a$)

$$\begin{aligned}
\mathcal{R}[k,m] &= \mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{(n-k)q-\xi q<\varrho_m Y_1\le (n-k)q\}}\Big] = \frac{\lambda}{\lambda+\delta}\,\mathbb{P}\Big[\frac{(n-k)q-\xi q}{\varrho_m}<Y_1\le\frac{(n-k)q}{\varrho_m}\Big]\,,\\
\mathcal{J}_1[k,m] &= \mathrm{e}^{(n-k)q}\,\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathrm{e}^{-\gamma\varrho_m Y_1}\mathbb{1}_{\{\varrho_m Y_1>(n-k)q\}}\Big] = \frac{\lambda}{\lambda+\delta}\mathrm{e}^{(n-k)q}\,\mathbb{E}\Big[\mathrm{e}^{-\gamma\varrho_m Y_1}\mathbb{1}_{\{\varrho_m Y_1>(n-k)q\}}\Big]\,,\\
\mathcal{J}_2[k,m] &= \mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{\varrho_m Y_1>(n-k)q\}}\Big] = \frac{\lambda}{\lambda+\delta}\,\mathbb{P}\Big[Y_1>\frac{(n-k)q}{\varrho_m}\Big]\,,
\end{aligned}$$

we get, in the zero drift case:

$$\begin{aligned}
\mathbf{v}_k &= \sum_{j=k}^{n-1}\mathbf{v}_j\cdot\mathcal{D}[j-k,m]+\mathbf{v}_n\big(\mathcal{R}[k,m]+\mathcal{J}_1[k,m]\big)+\delta^{-1}\Big(\mathcal{J}_2[k,m]-\mathcal{J}_1[k,m]\Big)\,, \qquad k<n\,,\\
\mathbf{v}_n &= \mathbf{v}_n\big(\mathcal{R}[n,m]+\mathcal{J}_1[n,m]\big)+\delta^{-1}\Big(\mathcal{J}_2[n,m]-\mathcal{J}_1[n,m]\Big)\,.
\end{aligned}$$

For $k\notin\{0,n\}$, $m\neq 0$, we have:

$$\begin{aligned}
\mathbf{v}_k = {} & \mathbf{v}_{k-1}\cdot\mathrm{e}^{-(\delta+\lambda)\xi q/c(\varrho_m)}+\mathbf{v}_k\cdot\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{-c(\varrho_m)T_1+\varrho_m Y_1\le(1-\xi)q\}}\mathbb{1}_{\{T_1<\xi q/c(\varrho_m)\}}\Big]\\
&+\sum_{j=k+1}^{n-1}\mathbf{v}_j\cdot\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{(j-k)q-\xi q<-c(\varrho_m)T_1+\varrho_m Y_1\le(j-k)q+(1-\xi)q\}}\mathbb{1}_{\{T_1<\xi q/c(\varrho_m)\}}\Big]\\
&+\mathbf{v}_n\cdot\Big(\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{(n-k)q-\xi q<-c(\varrho_m)T_1+\varrho_m Y_1\le(n-k)q\}}\mathbb{1}_{\{T_1<\xi q/c(\varrho_m)\}}\Big]\\
&\qquad+\mathrm{e}^{(n-k)q}\,\mathbb{E}\Big[\mathrm{e}^{-(\delta-\gamma c(\varrho_m))T_1}\mathrm{e}^{-\gamma\varrho_m Y_1}\mathbb{1}_{\{-c(\varrho_m)T_1+\varrho_m Y_1>(n-k)q\}}\mathbb{1}_{\{T_1<\xi q/c(\varrho_m)\}}\Big]\Big)\\
&+\delta^{-1}\Big(\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{-c(\varrho_m)T_1+\varrho_m Y_1>(n-k)q\}}\mathbb{1}_{\{T_1<\xi q/c(\varrho_m)\}}\Big]\\
&\qquad-\mathrm{e}^{(n-k)q}\,\mathbb{E}\Big[\mathrm{e}^{-(\delta-\gamma c(\varrho_m))T_1}\mathrm{e}^{-\gamma\varrho_m Y_1}\mathbb{1}_{\{-c(\varrho_m)T_1+\varrho_m Y_1>(n-k)q\}}\mathbb{1}_{\{T_1<\xi q/c(\varrho_m)\}}\Big]\Big)\,,
\end{aligned}$$

with the interpretation of $\sum_{j=n}^{n-1}=0$, where we used that $x_j-x_k=(j-k)q$ and $d=nq$. For $k=n$ and $m\neq 0$, this equation holds with the second and third terms set to zero. We write, for $m=1,\dots,a$,

$$\mathcal{E}[p,m]=\mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{pq-\xi q<-c(\varrho_m)T_1+\varrho_m Y_1\le pq+(1-\xi)q\}}\mathbb{1}_{\{T_1<\xi q/c(\varrho_m)\}}\Big]\,, \qquad p=0,\dots,n-1\,,$$

and for $m=1,\dots,a$, $k=1,\dots,n$,

$$\begin{aligned}
\mathcal{R}[k,m] &= \mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{(n-k)q-\xi q<-c(\varrho_m)T_1+\varrho_m Y_1\le(n-k)q\}}\mathbb{1}_{\{T_1<\xi q/c(\varrho_m)\}}\Big]\,,\\
\mathcal{J}_1[k,m] &= \mathrm{e}^{(n-k)q}\,\mathbb{E}\Big[\mathrm{e}^{-(\delta-\gamma c(\varrho_m))T_1}\mathrm{e}^{-\gamma\varrho_m Y_1}\mathbb{1}_{\{-c(\varrho_m)T_1+\varrho_m Y_1>(n-k)q\}}\mathbb{1}_{\{T_1<\xi q/c(\varrho_m)\}}\Big]\,,\\
\mathcal{J}_2[k,m] &= \mathbb{E}\Big[\mathrm{e}^{-\delta T_1}\mathbb{1}_{\{-c(\varrho_m)T_1+\varrho_m Y_1>(n-k)q\}}\mathbb{1}_{\{T_1<\xi q/c(\varrho_m)\}}\Big]\,,
\end{aligned}$$

so that, in the negative drift case:

$$\begin{aligned}\mathbf{v}_k &= \mathbf{v}_{k-1}\cdot \mathrm{e}^{-(\delta+\lambda)\xi q/c(\varrho_m)} + \sum_{j=k}^{n-1}\mathbf{v}_j\cdot \mathcal{E}[j-k,m] + \mathbf{v}_n\big(\mathcal{R}[k,m]+\mathcal{J}_1[k,m]\big)\\ &\qquad + \delta^{-1}\Big(\mathcal{J}_2[k,m]-\mathcal{J}_1[k,m]\Big)\,,\qquad k<n\,,\\ \mathbf{v}_n &= \mathbf{v}_{n-1}\cdot \mathrm{e}^{-(\delta+\lambda)\xi q/c(\varrho_m)} + \mathbf{v}_n\big(\mathcal{R}[k,m]+\mathcal{J}_1[k,m]\big) + \delta^{-1}\Big(\mathcal{J}_2[k,m]-\mathcal{J}_1[k,m]\Big)\,.\end{aligned}$$

So, if $\mathcal{D}$, $\mathcal{E}$, $\mathcal{R}$, $\mathcal{J}_1$ and $\mathcal{J}_2$ are known for the corresponding claim distribution, $\mathbf{T}$ and $\mathbf{t}$ are given by

$$\mathbf{T}^m_{k,j} = \begin{cases} \mathrm{e}^{-(\delta+\lambda)\xi q/c(\varrho_m)}\,, & j=k-1 \text{ and } m=1,\dots,a\,,\ \text{and } k=1,\dots,n\,,\\ \mathcal{E}[j-k,m]\,, & j=k,\dots,n-1 \text{ and } m=1,\dots,a\,,\ \text{and } k=1,\dots,n-1\,,\\ \mathcal{D}[j-k,m]\,, & j=k,\dots,n-1 \text{ and } m=0 \text{ or } k=0,\\ \mathcal{R}[j-k,m]+\mathcal{J}_1[k,m]\,, & j=n\,,\\ 0\,, & \text{else}\,,\end{cases}$$

and

$$\mathbf{t}^m_k = \delta^{-1}\big(\mathcal{J}_1[k,m]-\mathcal{J}_2[k,m]\big)\,,\qquad k=0,\dots,n \text{ and } m=0,\dots,a\,.$$

For the cases of deterministic and exponential claims, all expected values can be explicitly calculated. For the Pareto distribution, the calculation is not trivial. Here we used the incomplete Gamma function $\Gamma$, given by

$$\Gamma(s,\underline{x},\overline{x}) = \int_{\underline{x}}^{\overline{x}} \mathrm{e}^{-z} z^{s-1}\,\mathrm{d}z\,,\qquad s>0\,,$$

which is available as a pre-implemented function in many programming languages. In particular, by substituting $z=Q(B+Ct)/C$, we can transform integrals of the form

$$\begin{aligned}\int_{\underline{A}}^{\overline{A}} \mathrm{e}^{-Qt}\frac{1}{(B+Ct)^\alpha}\,\mathrm{d}t &= \mathrm{e}^{QB/C}\frac{Q^{\alpha-1}}{C^\alpha}\cdot\Gamma\Big[1-\alpha,\tfrac{QB}{C}+Q\underline{A},\tfrac{QB}{C}+Q\overline{A}\Big]\,,\\ \int_{\underline{A}}^{\overline{A}} \mathrm{e}^{-Qt}\frac{1}{(B+Ct)^{\alpha+1}}\,\mathrm{d}t &= \mathrm{e}^{QB/C}\frac{Q^{\alpha}}{C^{\alpha+1}}\cdot\Gamma\Big[-\alpha,\tfrac{QB}{C}+Q\underline{A},\tfrac{QB}{C}+Q\overline{A}\Big]\,,\end{aligned}$$

which appear in the calculation of $\mathcal{R}$, $\mathcal{J}_1$ and $\mathcal{J}_2$. □

*Calculation of $\gamma$ for Deterministic Claims in Subsection* 2.3.5. Similar to Lemma 2.3, we have that $\gamma$ is determined by Equation (2.37). Rearranging the terms, we find

$$\frac{(\delta+\lambda-c(1)\gamma)\zeta}{\lambda} = \zeta\mathrm{e}^{-\gamma\zeta}\,,$$

so substituting $z = (\delta + \lambda - c(1)\gamma)\zeta/\lambda$, we obtain

$$-\frac{z\lambda}{c(1)}\mathrm{e}^{-z\lambda/c(1)} = -\frac{\lambda\zeta}{c(1)}\mathrm{e}^{-\zeta(\delta+\lambda)/c(1)}\,.$$

We recall that the Lambert W function $\mathcal{W} : [-\mathrm{e}^{-1}, \infty) \to [-1, \infty)$ is the unique inverse of $x \mapsto x\mathrm{e}^x$ with values in $[-1, \infty)$. Applying this function to both sides of the equation, we arrive at

$$-\frac{z\lambda}{c(1)} = \mathcal{W}\Big[-\frac{\lambda\zeta}{c(1)}\mathrm{e}^{-\zeta(\delta+\lambda)/c(1)}\Big]\,.$$

We plug in the explicit expression of $z$ and rearrange the terms to find:

$$\gamma = \frac{\delta+\lambda}{c(1)} + \frac{1}{\zeta}\mathcal{W}\Big(-\frac{\lambda\zeta}{c(1)}\mathrm{e}^{-\zeta(\delta+\lambda)/c(1)}\Big) = \frac{1}{\zeta}\Big[\frac{\delta+\lambda}{(1+\eta)\lambda} + \mathcal{W}\Big(-\frac{1}{1+\eta}\mathrm{e}^{-(\delta+\lambda)/((1+\eta)\lambda)}\Big)\Big]\,.$$

To obtain the last equality, we additionally used the definition of $c(1)$. Now, with the explicit parameters given in Subsection 2.3.5, one can use an implementation of the Lambert W function to obtain an approximation to $\gamma$. □

## A.2 Addendum to Chapter 3

*Proof of Lemma* 3.2. To see that every $f_k$ solves the corresponding ordinary differential equation, we first observe that $l : [0, \infty) \to \mathbb{R}$ defined by

$$l(x) = \frac{\kappa(b)\mathrm{e}^{\xi(b)x} + \xi(b)\mathrm{e}^{-\kappa(b)x}}{\delta(\kappa(b)+\xi(b))\mathrm{e}^{\xi(b)d}} = C_1\mathrm{e}^{\xi(b)x} + C_2\mathrm{e}^{-\kappa(b)x}$$

solves the homogeneous equation $\mathcal{A}^b l(x) = 0$ by definition of $\xi(b)$ and $\kappa(b)$. Moreover, we have

$$f_k(x) = l(x) - \frac{\kappa(b)\xi(b)}{\delta(\kappa(b)+\xi(b))}(g_k - j_k)(x)\,,$$

where we write

$$g_k(x) = \int_0^x \mathrm{e}^{\xi(b)(x-z)}H_k(z)\,\mathrm{d}z\,, \qquad j_k(x) = \int_0^x \mathrm{e}^{-\kappa(b)(x-z)}H_k(z)\,\mathrm{d}z\,.$$

These functions fulfil

$$\begin{aligned} g_k'(x) &= \xi(b)g_k(x) + H_k(x)\,, \qquad & g_k''(x) &= [\xi(b)]^2 g_k(x) + \xi(b)H_k(x) + H_k'(x)\,, \\ j_k'(x) &= -\kappa(b)j_k(x) + H_k(x)\,, \qquad & j_k''(x) &= [\kappa(b)]^2 j_k(x) - \kappa(b)H_k(x) + H_k'(x)\,, \end{aligned}$$

so that we have

$$\begin{aligned} (g_k - j_k)'(x) &= \xi(b)g_k(x) + \kappa(b)j_k(x)\,, \\ (g_k - j_k)''(x) &= [\xi(b)]^2 g_k(x) - [\kappa(b)]^2 j_k(x) + (\kappa(b)+\xi(b))H_k(x)\,. \end{aligned}$$

From the definition of $\xi(b)$ it follows that $(\sigma b\xi(b))^2 = 2(\delta + \mu(b)\xi(b))$ and an analogous equation holds for $-\kappa(b)$. Thus, multiplying both sides of the second order equation with $\sigma^2 b^2/2 = \delta/(\kappa(b)\xi(b))$, we find

$$\begin{aligned}\frac{\sigma^2 b^2}{2}(g_k - j_k)''(x) &= (\delta + \mu(b)\xi(b))g_k(x) - (\delta - \mu(b)\kappa(b))j_k(x) + \frac{\sigma^2 b^2(\kappa(b)+\xi(b))}{2}H_k(x)\\ &= \delta(g_k - j_k)(x) + \mu(b)(g_k - j_k)'(x) + \frac{\delta(\kappa(b)+\xi(b))}{\kappa(b)\xi(b)}H_k(x)\,.\end{aligned}$$

So, it follows:

$$\mathcal{A}^b f_k(x) = \frac{\kappa(b)\xi(b)}{\delta(\kappa(b)+\xi(b))}\Big(\delta(g_k - j_k)(x) + \mu(b)(g_k - j_k)'(x) - \frac{\sigma^2 b^2}{2}(g_k - j_k)''(x)\Big) = -H_k(x)\,.$$

Except for the convergence of $(f_k)_{k\in\mathbb{N}}$, all other properties follow directly from the explicit representations of the functions. To see $\lim_{k\to\infty} f_k(x) = f(x)$ for all $x \geq 0$, we firstly note that for $x \leq d$, we have $d - z \geq x - z > 0$ for all $z \in [0, x)$. Hence, $k \mapsto H_k(z)$ is decreasing for all $z \in [0, x)$. Thus, it follows by monotone convergence that

$$\lim_{k\to\infty}\int_0^x \big(\mathrm{e}^{\xi(b)(x-z)} - \mathrm{e}^{-\kappa(b)(x-z)}\big)H_k(z)\,\mathrm{d}z = 0\,.$$

Hence, we have convergence for $x \leq d$ by

$$\begin{aligned}\lim_{k\to\infty} f_k(x) &= \frac{\kappa(b)\mathrm{e}^{\xi(b)x} + \xi(b)\mathrm{e}^{-\kappa(b)x}}{\delta(\kappa(b)+\xi(b))\mathrm{e}^{\xi(b)d}} - \frac{\kappa(b)\xi(b)}{\delta(\kappa(b)+\xi(b))}\lim_{k\to\infty}\int_0^x \big(\mathrm{e}^{\xi(b)(x-z)} - \mathrm{e}^{-\kappa(b)(x-z)}\big)H_k(z)\,\mathrm{d}z\\ &= \frac{\kappa(b)\mathrm{e}^{\xi(b)x} + \xi(b)\mathrm{e}^{-\kappa(b)x}}{\delta(\kappa(b)+\xi(b))\mathrm{e}^{\xi(b)d}} = C_1\mathrm{e}^{\xi(b)x} + C_2\mathrm{e}^{-\kappa(b)x}\,.\end{aligned}$$

For $x > d$, we have $d - z > 0$ for $z \in [0, d)$ and $d - z < 0$ for $z \in (d, x)$ and $k \mapsto H_k(z)$ is increasing for all $z \in [d, x)$. Splitting the integral at $x = d$, we get (again by monotone convergence):

$$\begin{aligned}\lim_{k\to\infty}\int_0^x \big(\mathrm{e}^{\xi(b)(x-z)} - \mathrm{e}^{-\kappa(b)(x-z)}\big)H_k(z)\,\mathrm{d}z &= \int_d^x \big(\mathrm{e}^{\xi(b)(x-z)} - \mathrm{e}^{-\kappa(b)(x-z)}\big)\,\mathrm{d}z\\ &= \frac{\mathrm{e}^{\xi(b)(x-d)} - 1}{\xi(b)} + \frac{\mathrm{e}^{-\kappa(b)(x-d)} - 1}{\kappa(b)}\,,\end{aligned}$$

from which we obtain convergence for $x > d$ by

$$\begin{aligned}&\lim_{k\to\infty} f_k(x) = \frac{\kappa(b)\mathrm{e}^{\xi(b)x} + \xi(b)\mathrm{e}^{-\kappa(b)x}}{\delta(\kappa(b)+\xi(b))\mathrm{e}^{\xi(b)d}} - \frac{\kappa(b)\xi(b)}{\delta(\kappa(b)+\xi(b))}\Big[\frac{\mathrm{e}^{\xi(b)(x-d)} - 1}{\xi(b)} + \frac{\mathrm{e}^{-\kappa(b)(x-d)} - 1}{\kappa(b)}\Big]\\ &= \frac{1}{\delta} + \frac{\kappa(b)\mathrm{e}^{\xi(b)(x-d)}}{\delta(\kappa(b)+\xi(b))} + \frac{\xi(b)\mathrm{e}^{-\kappa(b)x}}{\delta(\kappa(b)+\xi(b))\mathrm{e}^{\xi(b)d}} - \frac{\xi(b)\mathrm{e}^{-\kappa(b)(x-d)}}{\delta(\kappa(b)+\xi(b))} - \frac{\kappa(b)\mathrm{e}^{\xi(b)(x-d)}}{\delta(\kappa(b)+\xi(b))}\\ &= \frac{1}{\delta} + \frac{\xi(b)\mathrm{e}^{-\kappa(b)(x-d)}\mathrm{e}^{-\kappa(b)d}}{\delta(\kappa(b)+\xi(b))\mathrm{e}^{\xi(b)d}} - \frac{\xi(b)\mathrm{e}^{-\kappa(b)(x-d)}}{\delta(\kappa(b)+\xi(b))} = \frac{1}{\delta} - \Big[\frac{1}{\delta} - \Big(\frac{\kappa(b)\mathrm{e}^{\xi(b)d} + \xi(b)\mathrm{e}^{-\kappa(b)d}}{\delta(\kappa(b)+\xi(b))\mathrm{e}^{\xi(b)d}}\Big)\Big]\cdot\mathrm{e}^{-\kappa(b)(x-d)}\,,\end{aligned}$$

which completes the proof. □

*Alternative representation of $f_k$ in Lemma* 3.2. We denote by ${}_2F_1$ the Gaussian hypergeometric function, defined by

$$ {}_2F_1(c_1, c_2, c_1+1, h) = c_1 \int_0^1 t^{c_1-1}(1-th)^{-c_2}\,\mathrm{d}t = \frac{c_1}{h^{c_1}} \int_0^h t^{c_1-1}(1-t)^{-c_2}\,\mathrm{d}t \tag{A.2} $$

for $|h| < 1$, $c_1 > 0$ and $c_2 \in \mathbb{R}$, cf. [Paris, 2010, Eq. (8.17.7)]. We firstly consider the function $g_k$ (in the notation of the preceding proof of Lemma 3.2). We note that $H_k$ fulfils $H_k'(x) = 2k\mathrm{e}^{2k(d-z)}[H(z)]^2$. Substituting $h = H_k(z)$ in the integral, we therefore arrive at

$$ \begin{aligned} g_k(x) &= \int_0^x \mathrm{e}^{\xi(b)(x-z)} H_k(z)\,\mathrm{d}z = \frac{\mathrm{e}^{\xi(b)(x-d)}}{2k} \int_{H_k(0)}^{H_k(x)} h^{(1-\xi(b)/2k)-1}(1-h)^{\xi(b)/2k-1}\,\mathrm{d}h \\ &= \frac{\mathrm{e}^{\xi(b)(x-d)}}{2k} \Big[ \frac{[H_k(x)]^{1-\xi(b)/2k}}{1-\xi(b)/2k}\, {}_2F_1\Big(1-\frac{\xi(b)}{2k}, 1-\frac{\xi(b)}{2k}, 2-\frac{\xi(b)}{2k}, H_k(x)\Big) \\ &\qquad - \frac{[H_k(0)]^{1-\xi(b)/2k}}{1-\xi(b)/2k}\, {}_2F_1\Big(1-\frac{\xi(b)}{2k}, 1-\frac{\xi(b)}{2k}, 2-\frac{\xi(b)}{2k}, H_k(0)\Big) \Big] \end{aligned} $$

for all $k > \xi(b)/2$. Similarly, we get

$$ \begin{aligned} j_k(x) &= \frac{\mathrm{e}^{-\kappa(b)(x-d)}}{2k} \int_{H_k(0)}^{H_k(x)} h^{(1+\kappa(b)/2k)-1}(1-h)^{-\kappa(b)/2k-1}\,\mathrm{d}h \\ &= \frac{\mathrm{e}^{-\kappa(b)(x-d)}}{2k} \Big[ \frac{[H_k(x)]^{\kappa(b)/2k+1}}{\kappa(b)/2k+1}\, {}_2F_1\Big(\frac{\kappa(b)}{2k}+1, \frac{\kappa(b)}{2k}+1, \frac{\kappa(b)}{2k}+2, H_k(x)\Big) \\ &\qquad - \frac{[H_k(0)]^{\kappa(b)/2k+1}}{\kappa(b)/2k+1}\, {}_2F_1\Big(\frac{\kappa(b)}{2k}+1, \frac{\kappa(b)}{2k}+1, \frac{\kappa(b)}{2k}+2, H_k(0)\Big) \Big]\,. \end{aligned} $$

By (A.2), the integrals can alternatively be approximated with an implementation of the incomplete Beta function. □

*Proof of Lemma* 3.12. If $f''(x) \le 0$ holds for all $x \in (\underline{x}, \overline{x})$, then $\mathcal{J}_f(b)$ is minimised by $b = 1$ on this interval because $f'(x) \ge 0$ and $-\mu(b)$ is decreasing in $b$. Thus, we have

$$ -\delta f(x) - \eta f'(x) + \frac{\sigma^2}{2} f''(x) = 0\,. $$

All solutions to this equation are of the form $f(x) = C_1\mathrm{e}^{\xi x} - C_2\mathrm{e}^{-\kappa x}$. Since the function is non-negative and the second derivative is non-positive, this implies

$$ C_1 \frac{\xi^2}{\kappa^2} \mathrm{e}^{(\kappa+\xi)x} \le C_2 \le C_1 \mathrm{e}^{(\kappa+\xi)x}\,. $$

By $\xi > \kappa$, this can only be true if $C_1 \le 0$. Then we have $C_2 \le 0$ as well. Therefore, $f'(x) = C_1\xi\mathrm{e}^{\xi x} + C_2\kappa\mathrm{e}^{-\kappa x} \le 0$. This is a contradiction as the function is supposed to be strictly increasing. □

*Proof of Equation* (3.15). $x = Y(z)$ is equivalent to

$$\zeta(z - z_0) + 1 + \frac{\zeta}{\rho} x = \mathrm{e}^{\zeta(z-z_0)} \,.$$

Thus, substituting $w = \zeta(z - z_0) + 1 + \zeta/\rho x$, we obtain

$$-w\mathrm{e}^{-w} = -\mathrm{e}^{-(1+\zeta/\rho x)} \,.$$

Noting that the right hand side takes values in $[-\mathrm{e}^{-1}, 0]$ for all $x \geq 0$, we find

$$-w = \mathcal{W}\big(-\mathrm{e}^{-(1+\zeta/\rho x)}\big) \,.$$

So, it follows from plugging in the explicit representation of $w$:

$$z = z_0 - \frac{1}{\zeta} - \frac{x}{\rho} - \frac{\mathcal{W}\big(-\mathrm{e}^{-(1+\zeta/\rho x)}\big)}{\zeta} \,,$$

which corresponds to the desired expression. $\square$

*Proof of Equation* (3.17). By plugging in the explicit representation of $Z$, we obtain:

$$f(x) = f(0) + \frac{\delta f(0)}{\theta - \eta} \int_0^x \Big[\mathrm{e}^{1+\zeta/\rho y} \mathrm{e}^{\mathcal{W}\left(-\mathrm{e}^{-(1+\zeta/\rho y)}\right)}\Big]^{1/\zeta} \,\mathrm{d}y \,. \tag{A.3}$$

By definition of the Lambert W function, we have $\exp(\mathcal{W}(x)) = x/\mathcal{W}(x)$, so that we can rewrite the term in the integral:

$$f(x) = f(0) + \frac{\delta f(0)}{\theta - \eta} \int_0^x \Big[-\mathcal{W}\big(-\mathrm{e}^{-(1+\zeta/\rho y)}\big)\Big]^{-1/\zeta} \,\mathrm{d}y \,.$$

From this equation, we conclude the representation of $f'$ in terms of $Q$. We note that $\mathcal{W}$ fulfils $\mathcal{W}'(x) = \mathcal{W}(x)/(x(1 + \mathcal{W}(x)))$ for $x \neq 0$, as found in [Corless et al., 1996]. Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathcal{W}\big(-\mathrm{e}^{-(1+\zeta/\rho y)}\big) = -\frac{\zeta}{\rho} \frac{\mathcal{W}\big(-\mathrm{e}^{-(1+\zeta/\rho y)}\big)}{(1 + \mathcal{W}\big(-\mathrm{e}^{-(1+\zeta/\rho y)}\big))}$$

and substituting $z = \mathcal{W}(-\exp(-(1 + \zeta/\rho y)))$ in the integral, we find

$$\begin{aligned} f(x) &= f(0) + \frac{\delta f(0)}{\theta - \eta} \frac{\rho}{\zeta} \int_{-1}^{\mathcal{W}(-\exp(-(1+\zeta/\rho x)))} (-z)^{-(1/\zeta+1)} (1 + z) \,\mathrm{d}z \\ &= f(0) + \frac{\delta f(0)}{\theta - \eta} \rho \Big(\Big[1 + \frac{\mathcal{W}\big(-\mathrm{e}^{-(1+\zeta/\rho x)}\big)}{1 - \zeta}\Big] Q(x) - \frac{\zeta}{\zeta - 1}\Big) \,. \end{aligned}$$

Now we note that we have $((\zeta-1)(\theta-\eta))^{-1}\delta\rho\zeta = 1$, to conclude

$$f(x) = \frac{\delta f(0)}{\theta-\eta}\rho\Big[1+\frac{\mathcal{W}(-\mathrm{e}^{-(1+\zeta/\rho x)})}{1-\zeta}\Big]Q(x) = \frac{\delta f(0)}{\theta-\eta}P(x)\,,$$

which corresponds to Equation (3.17). □

*Proof of Proposition* 3.16. We give a slightly different argument than in [Brinker and Schmidli, 2022], which is based on the differential equation. We observe that $Q_0(x) > 0$ is fulfilled for all $x \geq 0$ and, by $\theta > \xi\sigma^2$,

$$\begin{aligned}(\kappa^2\sigma^2+\theta\kappa)\mathrm{e}^{(\xi+\kappa)(d-x_0)} - (\sigma^2\xi^2-\theta\xi) > (\kappa^2\sigma^2+\theta\kappa) - \sigma^2\xi^2 - \theta\xi = [\sigma^2(\kappa-\xi)+\theta](\xi+\kappa)\\ > \sigma^2\kappa(\xi+\kappa) > 0\,,\end{aligned}$$

so that $C_0 > 0$. That means, the equation is fulfilled for all $x \in [0, x_0]$. Similarly, one can show that $C_1, C_2 > 0$ holds, implying $f''(x) > 0$ for $x \in [x_0, d]$. By $f'(x_0) > 0$, it therefore holds $f'(x) > 0$ on $[x_0, d]$. Thus, the optimiser is given by $b_f(x) = \theta f'(x)/\sigma^2 f''(x)$. The constants are chosen such that it holds: $f(d) = 1$, $f'(x_0+) = C_0P_0'(x)$ and $b_f(x_0+) = 1$, i.e. $\theta f'(x_0+) = \sigma^2 f''(x_0+)$. Therefore, $b_f(x_0+) = b_f(x_0-) = 1$. We have

$$\frac{\sigma^2}{2}f''(x) = \eta f'(x) + \delta f(x)\,,$$

so that we can calculate the third derivative of $f$. In particular, $f'''$ is positive and increasing. We note that it holds

$$[f''(x)]^2 - f'''(x)f'(x) = C_1C_2\mathrm{e}^{(\xi-\kappa)x}\kappa\xi(\kappa+\xi)^2 > 0$$

and, therefore,

$$\frac{\theta}{\sigma^2}(b_f(x))' = \Big(\frac{f'(x)}{f''(x)}\Big)' = \frac{[f''(x)]^2 - f'''(x)f'(x)}{[f''(x)]^2} > 0\,.$$

Hence, $b_f$ is increasing on $[x_0, d]$. In particular, we have $b_f(x) \geq b_f(x_0) = 1$, so that $f$ solves the Hamilton–Jacobi–Bellman equation with optimiser $b(x) = 1$ for $x \in [x_0, d]$. □

*Proof of Lemma* 3.17. Regarding i), we obtain that $r$ is increasing and concave by an explicit calculation based on the first and second derivative. The limit follows from the definition of the Lambert W function. Then, ii) follows from the implicit function theorem. We prove iii) in the following way. For the upper bound, we note that $r(x) \leq C\sqrt{x}$ for some $C \geq 0$ may in general be rewritten as

$$\mathcal{W}\Big(-\mathrm{e}^{-(1+\zeta/\rho x)}\Big) \leq \frac{C\sigma^2}{\theta\rho}\sqrt{x} - 1\,.$$

Since both sides are larger or equal to $-1$, applying the function $x \mapsto x\mathrm{e}^x$ to both sides yields the equivalent inequality:

$$1 \geq \Big(1 - \frac{\sigma^2 C}{\theta\rho}\sqrt{x}\Big)\mathrm{e}^{\sigma^2 C/(\theta\rho)\sqrt{x}+\zeta/\rho x}\,. \tag{A.4}$$

For $x = 0$, we have equality of both sides for all $C \geq 0$. The derivative of the right hand side is equal to

$$\frac{\sigma^4 \mathrm{e}^{\zeta/\rho x+(\sigma^2 C)/(\theta\rho)\sqrt{x}}}{2\theta^2\rho^2} \cdot \Big(-\frac{2\theta\zeta C}{\sigma^2}\sqrt{x} + \frac{2\theta^2\zeta\rho}{\sigma^4} - C^2\Big)\,.$$

The first term is positive. The bracket term is non-positive for all $x \geq 0$ if $C \geq \theta\sqrt{2\zeta\rho}\sigma^{-2}$. In this case the right hand side of (A.4) is decreasing. This means that (A.4) holds for all $x > 0$ as well. We also conclude from this analysis that $C$ cannot be chosen any smaller: the right hand side of (A.4) would be strictly increasing in a small environment around zero. By equality at zero, this would mean that the inequality is harmed for small $x$.
Since $r$ is bounded by $\theta\rho\sigma^{-2}$ and $\sqrt{x} \to \infty$ as $x \to \infty$, there can be no global lower bound of square root order. By the same argument as above, one may choose $c \leq \theta\sigma^{-2}(-\zeta\sqrt{\varepsilon} + \sqrt{\zeta^2\varepsilon + 2\zeta\rho})$ such that the right hand side of (A.4) is increasing for all $x \in [0, \varepsilon]$. Then, (A.4) holds in the opposite direction, implying $r(x) \geq c\sqrt{x}$ on this interval. However, we note that this bound is not necessarily sharp.
We know that $r$ is increasing and concave. Thus, for every $h \geq 0$ it holds that $r'(x+h) \leq r'(x)$ for all $x \geq 0$. By $r(0) = 0$ we conclude:

$$r(x+h) - r(x) \leq r(h) \leq C\sqrt{h}\,.$$

This means that $r$ is Hölder continuous with exponent ½. □

*Proof of Lemma* 3.19 ii). We use a classic technique which is, for example, repeatedly applied in [Lamberton and Lapeyre, 1996]. We define the function

$$s(x) = \int_1^x \exp\Big(-\int_1^y \frac{2[\theta-\eta-\theta b^*(z)]}{\sigma^2(b^*(z))^2}\,\mathrm{d}z\Big)\,\mathrm{d}y\,,$$

which is twice continuously differentiable for $x \in (0,\infty)$ and solves the differential equation

$$\frac{\sigma^2[b^*(x)]^2}{2}s''(x) + [\theta-\eta-\theta b^*(x)]s'(x) = 0\,.$$

Now we let $0 < a < x < b < \infty$ and denote by $\tau_a^b = \inf\{t \geq 0 : \mathcal{E}_t \notin (a,b)\}$ the exit time from the interval $(a,b)$. By Itô's formula, the process $(s(\mathcal{E}_{\tau_a^b\wedge t}) - s(x))_{t\geq 0}$ is a $\mathbb{P}^x$-martingale because $|s(y)-s(x)|$ is bounded for $y \in [a,b]$. By boundedness of $s(y)$ and since $s'(y)b^*(y)$ is bounded away from zero on

$[a,b]$, we have

$$C > \mathbb{E}^x\big[(s(\mathcal{E}_{\tau_a^b\wedge t}) - s(x))^2\big] = \mathbb{E}^x\Big[\int_0^{t\wedge\tau_a^b} \sigma^2\big(s'(\mathcal{E}_s)b^*(\mathcal{E}_s)\big)^2\,\mathrm{d}s\Big]$$
$$\geq \mathbb{E}^x\Big[\int_0^{t\wedge\tau_a^b} \sigma^2\Big(\min_{y\in[a,b]} s'(y)b^*(y)\Big)^2\,\mathrm{d}s\Big] > 0$$

for some constant $C>0$, for all $t\geq 0$. Letting $t\to\infty$ on both sides and using monotone convergence to pull the limit into the expectation, we find that $\tau_a^b$ must be finite almost surely. Otherwise the inequality would be harmed. By the martingale property and bounded convergence we have

$$\mathbb{E}^x[s(\mathcal{E}_{\tau_a^b})] = s(a)\,\mathbb{P}^x[\mathcal{E}_{\tau_a^b}=a] + s(b)(1-\mathbb{P}^x[\mathcal{E}_{\tau_a^b}=a]) = s(x)\,,$$

and, hence,

$$\mathbb{P}^x[\mathcal{E}_{\tau_a^b}=a] = \frac{s(b)-s(x)}{s(b)-s(a)}\,.$$

We start with the case of cheap reinsurance. In particular, $b^*$ coincides with $r$. Using the substitution

$$\frac{\mathrm{d}r(z)}{\mathrm{d}z} = \frac{\theta\zeta}{\sigma^2}\frac{(\theta\rho\sigma^{-2}-r(z))}{r(z)}$$

in the exponent of the integrand of $s$, we find

$$-\int_1^x \frac{2[(\theta-\eta)-\theta r(z)]}{\sigma^2(r(z))^2}\,\mathrm{d}z = -\frac{2}{\theta\zeta}\int_{r(1)}^{r(x)} \frac{(\theta-\eta)-\theta r}{r(\theta\rho\sigma^{-2}-r)}\,\mathrm{d}r$$
$$= -\ln(r(x)) + \frac{2\delta\sigma^2-\theta^2}{2\delta\sigma^2+\theta^2}\ln\Big(\frac{\theta\rho}{\sigma^2}-r(x)\Big) + \ln(r(1)) - \frac{2\delta\sigma^2-\theta^2}{2\delta\sigma^2+\theta^2}\ln\Big(\frac{\theta\rho}{\sigma^2}-r(1)\Big)\,.$$

Both logarithms exist for all $x>0$ by $r(x)\nearrow\theta\rho\sigma^{-2}$ as $x\to\infty$. Let us assume that $\theta^2\neq 2\delta\sigma^2$. It follows that

$$s(x) = \underbrace{\frac{r(1)}{\big[\theta\rho\sigma^{-2}-r(1)\big]^{(2\delta\sigma^2-\theta^2)/(2\delta\sigma^2+\theta^2)}}}_{=:C_1}\cdot\int_1^x \frac{1}{r(y)}\Big[\frac{\theta\rho}{\sigma^2}-r(y)\Big]^{(2\delta\sigma^2-\theta^2)/(2\delta\sigma^2+\theta^2)}\,\mathrm{d}y\,.$$

Substituting $r(x)$ in the same way as above, we arrive at:

$$s(x) = C_1\int_{r(1)}^{r(x)} \frac{\sigma^2}{\theta\zeta}\Big[\frac{\theta\rho}{\sigma^2}-r\Big]^{(-2\theta^2)/(2\delta\sigma^2+\theta^2)}\,\mathrm{d}r$$
$$= -C_1\frac{\sigma^2\theta}{2\delta\sigma^2-\theta^2}\Big[\frac{\theta\rho}{\sigma^2}-r(x)\Big]^{(2\delta\sigma^2-\theta^2)/(2\delta\sigma^2+\theta^2)} + C_2$$

for

$$C_2 = \frac{\sigma^2 r(1)}{\theta\zeta}\frac{2\delta\sigma^2+\theta^2}{2\delta\sigma^2-\theta^2}\,.$$

If $\theta^2 = 2\delta\sigma^2$, we analogously find

$$s(x) = -C_1 \frac{\sigma^2}{\theta\zeta} \ln\Big(\frac{\theta\rho}{\sigma^2} - r(x)\Big) + C_2$$

for different constants $C_1 > 0$ and $C_2$. By $r(0) = 0$, there exists a limit $\lim_{x\to 0} s(x) = s(0) \in (-\infty, \infty)$. For $x \to \infty$, $r(x) \to \theta\rho\sigma^{-2}$, so if $\theta^2 < 2\delta\sigma^2$, we have $\lim_{x\to\infty} s(x) = C_2$. If $\theta^2 \geq 2\delta\sigma^2$, we have $\lim_{x\to\infty} s(x) = \infty$. This means that, by continuity from below, it follows for every $b > 0$:

$$\mathbb{P}^x[\mathcal{E}_{\tau_0^b} = 0] = \lim_{a\downarrow 0} \frac{s(b) - s(x)}{s(b) - s(a)} = \frac{s(b) - s(x)}{s(b) - s(0)} \,. \tag{A.5}$$

Similarly, continuity from above yields, for $b \to \infty$,

$$\begin{aligned}
\mathbb{P}^x[\tau_0 < \infty] &= \mathbb{P}^x\Big[ \bigcup_{b\in\mathbb{Q}^+} \{\mathcal{E}_{\tau_0^b} = 0\} \Big] = \lim_{b\to\infty} \frac{s(b) - s(x)}{s(b) - s(0)} \\
&= \begin{cases} \Big[1 - \frac{\sigma^2}{\theta\rho} r(x)\Big]^{(2\delta\sigma^2-\theta^2)/(2\delta\sigma^2+\theta^2)} & \text{if } \theta^2 < 2\delta\sigma^2 \,, \\ 1 \,, & \text{if } \theta^2 \geq 2\delta\sigma^2 \,. \end{cases}
\end{aligned}$$

For expensive reinsurance, we have $b^*(x) = r(x)\mathbb{1}_{\{x\leq x_0\}} + \mathbb{1}_{\{x>x_0\}}$. We can split the integral at $x_0$ to obtain, in the same way as above, a finite limit $s(0)$ as $x \to 0$. By $b^*(x) = 1$ for $x > x_0$, we have $\lim_{x\to\infty} s(x) = \lim_{x\to\infty} C_1 \mathrm{e}^{2\eta\sigma^{-2}x} - C_2 = \infty$ for some constants $C_1 > 0$ and $C_2 \in \mathbb{R}$. $\mathbb{P}^x[\tau_0 < \infty] = 1$ follows in the same way as above. Noticing that $\mathcal{E}$ is the drawdown under the feedback control, see Corollary 3.20, we could also calculate probabilities of the form $\mathbb{P}^x[\vartheta_0 < \vartheta^d]$ by plugging $b = d$ into Equation (A.5). $\square$

*Proof of Lemma* 3.22. We note that, since the behaviour of the controlled drawdown (if it exists) on $[0, d)$ is the same as in the case of continuous drift (considered in Lemma 3.18), the running maximum is constant.

For existence in the case case $b^*(d) < b^*(d+)$, we give two different arguments – both of which are of interest on their own. The first argument is based on the approach of Halidias and Kloeden [2006] and specifically applies to diffusion equations with discontinuous (and, in our case, decreasing) drift component. The second argument, taken from [Kyprianou and Loeffen, 2010], applies to Lévy processes from which we subtract a deterministic drift whenever the process is above a certain level.

For our first construction, we start with an alteration of the problem. To overcome the discontinuity of the volatility component, we define two auxiliary functions. Firstly, let

$$r_d(x) = \begin{cases} 0 \,, & x < 0 \,, \\ r(x) \,, & x \in [0, d] \,, \\ r(d) \,, & x > d \,, \end{cases} \tag{A.6}$$

which corresponds to the function $r$ extended with constant values outside of $[0, d]$. In particular, $r_d$

is equal to $b^*$ on $[0,d]$. We note that we have $r_d(d) \in (0,1)$. Secondly. we define

$$h(x) = \begin{cases} \theta - \eta - \theta r_d(x)\,, & x \le d\,, \\ -\eta r_d(d)\,, & x > d\,. \end{cases}$$

With this construction, if we assume for a moment that there exists a (strong) solution $Y = (Y_t)_{t\ge 0}$ to the stochastic differential equation

$$Y_t = x + \int_0^t h(Y_s)\,\mathrm{d}s - \int_0^t \sigma r_d(Y_s)\,\mathrm{d}W_s\,, \qquad t \ge 0\,, \tag{A.7}$$

the process $\Delta = (\Delta_t)_{t\ge 0}$ defined by

$$\Delta_t = Y_t \mathbb{1}_{\{Y_t \le d\}} + \Big(d + \frac{Y_t - d}{r_d(d)}\Big)\mathbb{1}_{\{Y_t > d\}}\,, \qquad t \ge 0\,, \tag{A.8}$$

has the desired properties. This corresponds to 'stretching' the process $Y$ vertically whenever it exceeds $d$. That means, we have to show that $Y$ solving (A.7) exists. In particular, the volatility coefficient of the altered Equation (A.7) is now continuous whereas the drift still has a jump at $x = d$. To this purpose, we follow the proof of Theorem 3.1 of [Halidias and Kloeden, 2006] with some modifications. We omit the technical details because they go beyond the scope of this monograph and, more importantly, are not 'sufficiently different' from the original proof by Halidias and Kloeden [2006]. However, we give a general outline of the proof and list all necessary changes such that it can be executed by following closely the steps of their proof. We define the processes $\check{Y} = (\check{Y}_t)_{t\ge 0}$ and $\hat{Y} = (\hat{Y}_t)_{t\ge 0}$ by

$$\begin{aligned} \hat{Y}_t &= x + \int_0^t (\theta - \eta)\,\mathrm{d}s - \int_0^t \sigma r_d(\hat{Y}_s)\,\mathrm{d}W_s\,, \qquad t \ge 0\,, \\ \check{Y}_t &= x + \int_0^t (-\eta r_d(d))\,\mathrm{d}s - \int_0^t \sigma r_d(\check{Y}_s)\,\mathrm{d}W_s\,, \qquad t \ge 0\,. \end{aligned}$$

We note that $\check{Y}$ and $\hat{Y}$ exist and that it holds $\hat{Y}_t \ge \check{Y}_t$ for all $t \ge 0$, almost surely, by Theorems IV 1.1, 2.2, 3.2 and VI 1.1 of [Ikeda and Watanabe, 1989]. This means that we can define the order interval $\mathcal{K}$ of processes $U$ with trajectories between those of $\check{Y}$ and $\hat{Y}$ and certain continuity and integrability properties, specified in [Halidias and Kloeden, 2006]. For every $U \in \mathcal{K}$, $\hat{Y}$ is a so called *upper* and $\check{Y}$ is a so called *lower solution* to

$$Y_t = x + \int_0^t h(U_s)\,\mathrm{d}s - \int_0^t \sigma r_d(Y_s)\,\mathrm{d}W_s\,, \qquad t \ge 0\,. \tag{A.9}$$

This means, in effect, that $\check{Y}$ and $\hat{Y}$ fulfil the equation with '$\ge$' and '$\le$' in place of '$=$'. Now, Halidias and Kloeden [2006] show in their Theorem 2.2 that for every pair $Y^1, Y^2 \in \mathcal{K}$ of an upper and a lower solution, Equation (A.9) has a unique solution $Y \in \mathcal{K}$ with trajectories between those of $Y^1$ and $Y^2$. This theorem, though formulated for coefficients of linear growth, remains true in our case. The only point that has to be changed in the proof of Theorem 2.2 is the reference for existence and uniqueness. For example, Theorems 1.1 and 1.4 of [Lan and Wu, 2014] are an appropriate choice for our absolutely

bounded drift and ½-Hölder continuous volatility coefficient. Next, Halidias and Kloeden [2006] prove that the map $\mathcal{S} : \mathcal{K} \to \mathcal{K}$ assigning the solution $Y = \mathcal{S}(U)$ to the process $U$ is increasing and must, by Corollary 3.2 in [Heikkilä, 1993], have a fixed point in $\mathcal{K}$. Similarly, because $h$ is a decreasing function, one can show that $\mathcal{S}$ is a decreasing map in our setting. Thus, also by the result of Heikkilä [1993], there is a fixed point $Y^* \in \mathcal{K}$ fulfilling $Y^* = \mathcal{S}(Y^*)$. For this fixed point $Y^*$, Equation (A.7) is valid. We note that the strong solution obtained in this way is not necessarily unique.
A different proof is possible by applying Theorem 1 by Kyprianou and Loeffen [2010]. Similarly as above, we assume that $r_d$ is given by (A.6). We consider the process $\tilde{Y} = (\tilde{Y}_t)_{t\geq 0}$ defined as the unique strong solution to

$$\tilde{Y}_t = x + \int_0^t [-\eta + \theta(1 - r_d(\tilde{Y}_s))]\,\mathrm{d}s - \int_0^t \sigma r_d(\tilde{Y}_s)\,\mathrm{d}W_s\,, \qquad t \geq 0\,, \tag{A.10}$$

which exists by the same arguments as the process considered in Lemma 3.18. Then, we consider the equation

$$Y_t = \tilde{Y}_t - \theta(1 - r_d(d)) \int_0^t \mathbb{1}_{\{Y_s > d\}}\,\mathrm{d}s\,, \qquad t \geq 0\,. \tag{A.11}$$

This corresponds to a *refracted* Lévy process as considered in [Kyprianou and Loeffen, 2010]. It follows by their Theorem 1 that there exists a unique strong solution $Y = (Y_t)_{t\geq 0}$. We note that we could not directly apply Theorem 305 of [Situ, 2005] (as mentioned in Remark 2 of Kyprianou and Loeffen [2010]) because our volatility component is not bounded away from zero. The process $\Delta = (\Delta_t)_{t\geq 0}$ defined by (A.8) has the properties we expect from the drawdown under the feedback control. □

### A.3 Addendum to Chapter 4

*Proof of Theorem 4.12, Technical Details.* We write $f(x) = C_1 \mathrm{e}^{\xi x} + C_2 \mathrm{e}^{-\kappa x}$ for the extension of $u|_{[0,d]}$ to $\mathbb{R}$. Since we have $C_1 < 0$, $f''(x) \geq 0$ for some $x$ is only possible if $C_2 > 0$. We distinguish the cases of cheap and expensive reinsurance.
We start with i). We assume $\chi \geq \chi_{c,1}$ and $\theta\rho < \sigma^2$. Then, in particular,

$$\chi > \frac{\kappa\xi \mathrm{e}^{-\xi d}}{\delta(\kappa + \xi)} \tag{A.12}$$

and, thus, $C_2 > 0$. In this case, $f''(x)$ is strictly decreasing and there exists a unique $\bar{x} \in \mathbb{R}$ with $f''(\bar{x}) = 0$. For all $x \leq \bar{x}$, $f''(x) \geq 0$ and, thus, $\mathcal{J}_u(b)$ is maximised at $b = 1$. If $\bar{x} < d$, we have $f''(x) < 0$ for $x \in (\bar{x}, d]$. Now $\chi \geq \chi_{c,1}$ is equivalent to (4.11) at $x = d$. The inequality holds, in this case, for all $x \in (\bar{x}, d]$, as the right hand side of (4.11) is positive and decreasing in $x$ and the left hand side is positive and independent of $x$. Therefore, $b = 1$ is optimal for all $x \in [0, d]$. Assume now that $b = 1$ is optimal for all $x \in [0, d]$ and that reinsurance is cheap. We show first that (A.12) must hold. If (A.12) is harmed, then $C_2 \leq 0$, and thus $f''(x) < 0$ for all $x$. Then, (4.11) is harmed as well (for all $x$, as the right hand side is negative) which is a contradiction to $b = 1$ being optimal. Thus, (A.12) must at least be fulfilled. By the same arguments as above, optimality of $b = 1$ for all $x \in [0, d]$ is now equivalent to $\chi \geq \chi_{c,1}$.

Now we show ii). We assume $\chi \geq \chi_{e,1}$ and that $\theta\rho \geq \sigma^2$. If the stronger condition (A.12) holds with '$\geq$' in place of $\chi_{e,1}$, then $C_2 \geq 0$. Hence, the left hand side of (4.11) is negative and the right hand side is non-negative for all $x$. This means that for every $x$ we either have $f''(x) > 0$ or $f''(x) \leq 0$ and (4.11), so $b = 1$ is optimal. If $\chi \geq \chi_{e,1}$ and (A.12) holds with '$<$' in place of '$>$', we have $C_2 < 0$, $f''(x) < 0$ and the right hand side of (4.11) is negative and increasing. This means, (4.11) if fulfilled for all $x \in [0, d]$, precisely if it is fulfilled at $x = 0$. This condition is equivalent to $\chi \geq \chi_{e,1}$. The converse statement follows analogously. ☐

*Proof of Lemma* 4.14. For a strictly concave and decreasing solution $f$ to (4.12), $\mathfrak{J}_f(b)$ is maximised at $b_f(x) = \theta f'(x)/(\sigma^2 f''(x))$ if this value lies within $[0, 1]$. Plugging this optimiser into the equation we arrive at the non-linear differential equation

$$-\delta f(x) + (\theta - \eta) f'(x) - \frac{\theta^2}{2\sigma^2} \frac{[f'(x)]^2}{f''(x)} = 0\,. \tag{A.13}$$

The function $-\log(-f'(y))$ is strictly decreasing and therefore invertible. Assuming continuity, there is a function $Y : \mathbb{R} \to [0, \infty)$ such that $f'(Y(z)) = -\mathrm{e}^{-z}$ holds. Plugging this into (A.13) and differentiating again, we receive an equation that is solved by functions of the form $Y(z) = C_1 \mathrm{e}^{\zeta z} - \rho z - C_2$. A detailed calculation of this step can be executed analogously to Section 3.2 of Chapter 3. The first initial condition implies that for $z_0$ with $Y(z_0) = a$ we have $-\mathrm{e}^{-z_0} = \alpha$ and, hence, $z_0 = -\ln(-\alpha)$. The second initial condition in combination with $f''(Y(z)) = \mathrm{e}^{-z}/Y'(z)$ and the assumption that the maximum is attained at $b_f(Y(z_0))$ yields $b_f(Y(z_0)) = -\sigma^{-2}\theta Y'(-\ln(-\alpha)) = \gamma$. Therefore, the constants $C_1$ and $C_2$ are given by

$$C_1 = \frac{\theta\rho - \sigma^2\gamma}{\zeta\theta}(-\alpha)^{\zeta}\,, \qquad C_2 = \frac{\theta\rho - \sigma^2\gamma}{\theta\zeta} + \rho\ln(-\alpha) - a\,. \tag{A.14}$$

We have $\alpha < 0$ by assumption, so the sign of $C_1$ is determined by the term $\theta\rho - \sigma^2\gamma$. Technically, $Y(z)$ is defined for all $z \in \mathbb{R}$. Since our goal is to calculate $f'(x)$, we have to find an inverse function $Z$ of $Y$. If $C_1 > 0$, the function $Y$ is not bijective, so here we have to decide which branch of $Y$ to invert. The function $z \mapsto f'(Y(z)) = -\mathrm{e}^{-z}$ is increasing. Since $f'(x)$ is supposed to be decreasing, $Y$ should be decreasing as well. We distinguish three cases:

i) $\theta\rho < \gamma\sigma^2$, and thus $C_1 < 0$. $Y$ as defined above is decreasing on $(-\infty, \infty)$. Inverting the function yields

$$Z(x) = -\frac{x + C_2}{\rho} - \frac{1}{\zeta}\mathcal{W}\Big(-\frac{\zeta C_1}{\rho}\mathrm{e}^{-\zeta\rho^{-1}(x+C_2)}\Big)\,. \tag{A.15}$$

Because $C_1 < 0$, the argument of the function $\mathcal{W}$ is positive. Plugging in the constants $C_1$ and $C_2$ proves that this is the function $Z$ defined in Equation (4.14). We note that the case $\gamma\sigma^2 > \theta\rho$ can only occur if $\theta\rho < \sigma^2$.

ii) $\gamma\sigma^2 = \theta\rho$, and thus $C_1 = 0$. $Y(z)$ is decreasing on $(-\infty, \infty)$ with inverse $Z(x) = -(x + C_2)/\rho$. By $\mathcal{W}(0) = 0$, this coincides with the representation (4.14). We note that this case can only occur if $\theta\rho \le \sigma^2$.

iii) $\gamma\sigma^2 < \theta\rho$, and thus $C_1 > 0$. $Y(z)$ is decreasing for $z \in (-\infty, z_1]$ with $z_1 = -\ln(-\alpha) + \zeta^{-1}\ln(\theta\rho/(\theta\rho - \gamma\sigma^2))$. We note that $z_0 \in (-\infty, z_1]$. If we invert the function $Y$ on this interval, we obtain again $Z(x)$ of the form (A.15). This time, the argument of $\mathcal{W}$ is negative with values in $[-\mathrm{e}^{-1}, 0)$.

In all three cases $f'(x) = -\mathrm{e}^{-Z(x)}$ solves the equation that results from differentiating (A.13). For the anti-derivative to solve (A.13), we have to calculate $f(a)$ such that $f$ fulfils the equation at $x = a$. This implies

$$-\delta f(a) + (\theta - \eta)\alpha - \frac{\theta\alpha\gamma}{2} = 0\,,$$

from which $f(a)$ can be obtained. This corresponds to (4.13). From the construction it follows that $f$ is decreasing and concave. We noted above that any concave and decreasing solution $g$ to (A.13) solves Equation (4.12) for all $x \ge a$ with $\theta g'(x) \ge \sigma^2 g''(x)$. Hence, the last part of this proof is to consider $b_f(x) = b_f(Y(Z(x))) = -\theta\sigma^{-2}Y'(Z(x))$ for $f$ as constructed above in each of the three cases:

i) $\gamma\sigma^2 > \theta\rho$. We have

$$b_f(x) = \frac{\theta\rho}{\sigma^2}\Big[1 + \mathcal{W}\Big(-\frac{\zeta C_1}{\rho}\mathrm{e}^{-\frac{\zeta}{\rho}(x+C_2)}\Big)\Big] = \frac{\theta\rho}{\sigma^2}\Big[1 + \mathcal{W}(E_{\gamma,a}(x))\Big]\,.$$

Since $\mathcal{W}$ is (strictly) increasing, $b_f$ is (strictly) decreasing. This implies $b_f(x) < b_f(a) = \gamma \le 1$ for $x > a$. Moreover, we observe that $b_f(x) \to \theta\rho\sigma^{-2}$ as $x \to \infty$ and that $b_f(x) > \theta\rho\sigma^{-2}$ for all $x \ge a$ by $\mathcal{W}(y) > 0$ for $y > 0$.

ii) $\gamma\sigma^2 = \theta\rho$. We have $b_f(x) = \theta\rho\sigma^{-2}$. We have $\gamma \in [0, 1]$, so it holds $b_f(x) \in [0, 1]$ for all $x \in [a, d]$. By $\mathcal{W}(0) = 0$, this also coincides with the representation (4.15).

iii) $\gamma\sigma^2 < \theta\rho$. $b_f(x)$ takes the same form as in case i) and is strictly increasing with $b_f(a) \in [0, 1]$. Again, we have $b_f(x) \to \theta\rho\sigma^{-2}$ as $x \to \infty$. Thus, if $\theta\rho \le \sigma^2$, $b_f(x) \in [0, 1]$ for all $x \ge a$. On the other hand, if $\theta\rho > \sigma^2$, the limit of $b_f(x)$ is larger than one and there exists a unique $x_\gamma > a$ with $b_f(x_\gamma) = 1$ which is given by

$$x_\gamma = a + \frac{\rho}{\zeta}\ln\Big(\frac{\theta\rho - \gamma\sigma^2}{\theta\rho - \sigma^2}\Big) - \frac{\sigma^2(1-\gamma)}{\theta\zeta}\,.$$

With this, the proof is complete. □

*Proof of Lemma* 4.15. We start with i), that is $\gamma\sigma^2 > \theta\rho$. We already know, from the proof of Lemma 4.14, that $\mathcal{W}$ is (strictly) increasing and that $b_f$ is (strictly) decreasing. As $E_{\gamma,a}(x) > 0$, $\mathcal{W}(E_{\gamma,a}(x)) > 0$ for $x \ge a$. We have

$$b_f'(x) = \frac{\theta\rho}{\sigma^2}\frac{\mathcal{W}(E_{\gamma,a}(x))}{1 + \mathcal{W}(E_{\gamma,a}(x))}\frac{E_{\gamma,a}'(x)}{E_{\gamma,a}(x)} = -\frac{\theta\zeta}{\sigma^2}\frac{\mathcal{W}(E_{\gamma,a}(x))}{1 + \mathcal{W}(E_{\gamma,a}(x))}\,. \tag{A.16}$$

As the numerator is larger than zero, we can rewrite this as

$$b_f'(x) = -\frac{\theta\zeta}{\sigma^2}\frac{1}{1+1/\mathcal{W}(E_{\gamma,a}(x))}\,.$$

Since $\mathcal{W}(E_{\gamma,a}(x))$ is decreasing, we conclude that $b_f'$ is increasing, such that $b_f$ is convex. In particular, as $\gamma > \theta\rho\sigma^{-2} > 0$, we have

$$|b_f'(x)| \leq |b_f'(a)| = \frac{\theta\zeta}{\sigma^2}\Big(1 - \frac{\theta\rho}{\gamma\sigma^2}\Big) < \frac{\theta\zeta}{\sigma^2}$$

as an upper bound for the absolute value of the derivative.
Now we prove the assertion for iii), that is $\gamma\sigma^2 < \theta\rho$. We already know that $b_f$ is optimal and strictly increasing for $x \geq a$ if $\theta\rho \leq \sigma^2$ and for $x \in [a, x_\gamma]$ if $\theta\rho > \sigma^2$. The derivative of $b_f$ takes the form (A.16). $\mathcal{W}(E_{\gamma,a}(x))$ is negative and increasing. The denominator is positive and increasing. Hence, $b_f$ is concave. If $\gamma > 0$, we obtain in the same way as above:

$$|b_f'(x)| \leq |b_f'(a)| = \frac{\theta\zeta}{\sigma^2}\Big(\frac{\theta\rho}{\gamma\sigma^2} - 1\Big)\,.$$

In the case $\gamma = 0$, we get for $x \in (a, x_\gamma)$:

$$|b_f'(x)| = \frac{\theta\zeta}{\sigma^2}\frac{1}{1+1/\mathcal{W}\big(-\mathrm{e}^{-1}\mathrm{e}^{-(x-a)\zeta/\rho}\big)}\,. \tag{A.17}$$

As $x \searrow a$, $\mathcal{W}\big(-\mathrm{e}^{-1}\mathrm{e}^{-(x-a)\zeta/\rho}\big) \searrow -1$, meaning that the denominator of the right hand side of (A.17) goes to zero (from above). In particular, the right hand side is unbounded as $x \searrow a$. ☐

*Proof of Lemma* 4.16. From the definition of the Lambert W function it follows that

$$-\mathrm{e}^{-Z(x)} = \alpha\Big[\frac{\theta\rho}{\gamma\sigma^2-\theta\rho}E_{\gamma,a}(x)\Big]^{-1/\zeta}\mathrm{e}^{\mathcal{W}(E_{\gamma,a}(x))/\zeta} = \alpha\Big[\frac{\gamma\sigma^2-\theta\rho}{\theta\rho}\big[\mathcal{W}(E_{\gamma,a}(x))\big]^{-1}\Big]^{1/\zeta},$$

such that $Q_{\gamma,a}(x) = -\mathrm{e}^{-Z(x)}\alpha^{-1} = f'(x)\alpha^{-1}$. In particular, $\mathcal{W}(E_{\gamma,a}(a)) = (\sigma^2\gamma - \theta\rho)\theta^{-1}\rho^{-1}$ and $Q_{\gamma,a}(a) = 1$. $P_{\gamma,a}$ is the anti-derivative of $Q_{\gamma,a}$ with $P_{\gamma,a}(a) = f(a)\alpha^{-1}$. This can be obtained by substituting $z = \mathcal{W}(E_{\gamma,a}(x))$ with

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{z}{1+z}\frac{E_{\gamma,a}'(x)}{E_{\gamma,a}(x)} = -\frac{z}{1+z}\frac{\zeta}{\rho}$$

in the integral over $Q_{\gamma,a}(x)$. This step is similar to the proof of Equation (3.17), p. 120. The statement involving $b_f$ now follows from the construction of $f$ in the proof of Lemma 4.14. ☐

Lemma A.1. [2021a]
*Under the conditions of Lemma* 4.14 *and* $\theta\rho \neq \sigma^2\gamma$, *we have* $\kappa P_{\gamma,a}(x) + Q_{\gamma,a}(x) > 0$ *for* $x \geq a$.

*Proof.* We know that $P_{\gamma,a}$ is increasing and convex (because $f$ is decreasing and concave and $\alpha$ is negative) and therefore $\kappa P_{\gamma,a}(x) + Q_{\gamma,a}(x)$ is increasing in $x$. We have $Q_{\gamma,a}(a) = 1$ and thus

$$\kappa P_{\gamma,a}(a) + Q_{\gamma,a}(a) = \kappa\rho\Big[1 + \frac{\gamma\sigma^2 - \theta\rho}{\theta\rho(1-\zeta)}\Big] + 1 = \kappa\rho + \frac{\kappa(\sigma^2 - \theta\rho)}{\theta(1-\zeta)} + 1\,.$$

Plugging in the explicit representations of $\kappa, \zeta$ and $\rho$, we find:

$$\begin{aligned}\kappa\rho + \frac{\kappa(\sigma^2 - \theta\rho)}{\theta(1-\zeta)} + 1 \quad &= \quad \frac{(-\eta + \sqrt{2\delta\sigma^2 + \eta^2})(\theta - 2\eta) + 2\delta\sigma^2}{2\delta\sigma^2} \\ &\geq \quad \frac{(\eta - \sqrt{\eta^2 + 2\delta\sigma^2})^2}{2\delta\sigma^2} > 0\end{aligned}$$

where we have used $\theta \geq 0$ and $\delta, \sigma > 0$. Hence, we get $\kappa P_{\gamma,a}(x) + Q_{\gamma,a}(x) \geq \kappa P_{\gamma,a}(a) + Q_{\gamma,a}(a) > 0$. $\square$

*Proof of Lemma* 4.28*, Technical Details.* This follows in a similar way as in the proof of Proposition 3.16, p. 121. We consider the function $h_1$ on the interval $(x_\gamma, d]$ for $x_\gamma < d$. We observe that $C_1$ and $C_2$ are negative by $Q(x_\gamma) > 0$ and $\theta > \sigma^2\xi$. This implies that $h_1$ is strictly concave on $[x_\gamma, d]$. Moreover, by $1 = \theta h_1'(x_\gamma)/(\sigma^2 h_1''(x_\gamma))$, we have $h_1'(x_\gamma) < 0$ and thus $h_1'(x) < 0$ for $x \in [x_\gamma, d]$. This means, the maximiser of the Hamilton–Jacobi–Bellman equation is given by $b_{h_1}(x) \wedge 1$. Differentiating $b_{h_1}$, we get

$$[h_1''(x)]^2 - h_1'''(x)h_1'(x) = C_1 C_2 \mathrm{e}^{(\xi-\kappa)x}\kappa\xi(\kappa+\xi)^2 \geq 0\,,$$

so

$$b_{h_1}(x) = \frac{\theta h_1'(x)}{\sigma^2 h_1''(x)} \geq \frac{\theta h_1'(x_\gamma)}{\sigma^2 h_1''(x_\gamma)} = 1$$

for $x \in [x_\gamma, d]$. $\square$

Lemma A.2. [2021a]

*Let $\theta\rho < \sigma^2$. The function $R : [0, d] \to [\chi_{c,2}, \chi_{c,1}]$ defined by*

$$R(a) = \frac{(\sigma^2\kappa + \theta) + (\sigma^2\xi - \theta)\mathrm{e}^{(\kappa+\xi)a}}{\delta\sigma^2(\kappa+\xi)\mathrm{e}^{\xi a}\big[\kappa P_{1,a}(d) + Q_{1,a}(d)\big]}$$

*is strictly increasing. We have $R(0) = \chi_{c,2}$ and $R(d) = \chi_{c,1}$. In particular, $\chi_{c,1} > \chi_{c,2}$.*

*Proof.* The numerator is positive and strictly increasing in $a$ and the function in the denominator, $\tilde{R}(a) = \mathrm{e}^{\xi a}\big[\kappa P_{1,a}(d) + Q_{1,a}(d)\big]$, is positive and decreasing in $a$. The latter follows by taking the derivative: we have

$$\begin{aligned}
\mathrm{e}^{-\xi a}\frac{\mathrm{d}}{\mathrm{d}a}\big[\mathrm{e}^{\xi a}[\kappa P_{1,a}(d)+Q_{1,a}(d)]\big] &= \xi\kappa P_{1,a}(d)+(\xi-\kappa)P'_{1,a}(d)-P''_{1,a}(d)\\
&= \frac{2}{\sigma^2}\Big(\delta P_{1,a}(d)+\eta P'_{1,a}(d)-\frac{\sigma^2}{2}P''_{1,a}(d)\Big)\\
&< \frac{2}{\sigma^2}\sup_{b\in[0,1]}\Big\{\delta P_{1,a}(d)+\mu(b)P'_{1,a}(d)-\frac{\sigma^2 b^2}{2}P''_{1,a}(d)\Big\}=0
\end{aligned}$$

for $a<d$ (and equality for $a=d$). For $a=d$, we note that

$$\chi_{1,c}=\frac{(\sigma^2\kappa+\theta)\mathrm{e}^{-\xi d}+(\sigma^2\xi-\theta)\mathrm{e}^{\kappa d}}{\delta\sigma^2(\kappa+\xi)\big[P_{1,d}(d)\kappa+Q_{1,d}(d)\big]}=R(d)\,.$$

Hence, plugging in the remaining boundary value $a=0$ proves the assertion. □

Lemma A.3. [2021a]

i) *Let $\theta\rho<\sigma^2$. The function $S:[0,\theta\rho\sigma^{-2})\cup(\theta\rho\sigma^{-2},1]\to[\chi_{c,4},\chi_{c,3})\cup(\chi_{c,3},\chi_{c,2}]$ defined by*

$$S(\gamma)=\frac{\kappa}{\delta(P_\gamma(d)\kappa+Q_\gamma(d))}$$

*is positive and strictly increasing. It holds $S(0)=\chi_{c,4}$, $\lim_{\gamma\searrow\theta\rho\sigma^{-2}}S(\gamma)=\chi_{c,3}=\lim_{\gamma\nearrow\theta\rho\sigma^{-2}}S(\gamma)$ and $S(1)=\chi_{c,2}$. In particular, $\chi_{c,2}>\chi_{c,3}>\chi_{c,4}>0$. For $\theta\rho=\sigma^2$, the corresponding result holds for $S:[0,\theta\rho\sigma^{-2})\to[\chi_{c,4},\chi_{c,3})$ and we have $\chi_{c,3}>\chi_{c,4}>0$.*

ii) *Let $\theta\rho>\sigma^2$. The function $S:[0,\bar\gamma]\to[\chi_{e,3},\chi_{e,2}]$ defined as above is positive and strictly increasing. We have $S(0)=\chi_{e,3}$ and $S(\bar\gamma)=\chi_{e,2}$. In particular, $\chi_{e,2}>\chi_{e,3}>0$.*

*Proof.* We assume $\theta\rho<\sigma^2$ and $\gamma\in[0,\theta\rho\sigma^{-2})\cup(\theta\rho\sigma^{-2},1]$. Taking derivatives, we see that $y\mapsto\mathcal{W}(y\mathrm{e}^yC)-y$ is strictly decreasing for $y\in\mathbb{R}$ for any $C\in[0,1)$. We note that

$$Q_\gamma(d)=\mathrm{e}^{d/\rho}\Big[\Big(\exp\Big(\mathcal{W}\Big(\frac{\gamma\sigma^2-\theta\rho}{\theta\rho}\mathrm{e}^{\frac{\gamma\sigma^2-\theta\rho}{\theta\rho}}\mathrm{e}^{-\zeta/\rho d}\Big)-\frac{\gamma\sigma^2-\theta\rho}{\theta\rho}\Big)\Big]^{1/\zeta}\tag{A.18}$$

and therefore conclude that $\gamma\mapsto Q_\gamma(d)$ is decreasing. Now, we have

$$P_\gamma(d)\kappa+Q_\gamma(d)=\Big(\kappa\rho+\kappa\rho\frac{\mathcal{W}(E_\gamma(x))}{1-\zeta}+1\Big)\,Q_\gamma(d)\,.$$

Since the left hand side is positive by Lemma 4.16 and $Q_\gamma(d)$ is positive, the term in brackets must also be positive. By $1-\zeta<0$ the bracket term is also decreasing in $\gamma$. Hence, $P_\gamma(d)\kappa+Q_\gamma(d)$ is decreasing in $\gamma$. The monotonicity is strict in all cases. Hence, $S(\gamma)$ is strictly increasing in $\gamma$. The statements for $\gamma=0$ and $\gamma=1$ follow from explicitly evaluating the function. From the representation (A.19) it follows that $\lim_{\gamma\to\theta\rho\sigma^{-2}}Q_\gamma(d)=\mathrm{e}^{d/\rho}$ and, hence, $\lim_{\gamma\to\theta\rho\sigma^{-2}}P_\gamma(d)=\rho\mathrm{e}^{d/\rho}$ which proves the statements for $\gamma$ let towards $\theta\rho\sigma^{-2}$. The cases $\theta\rho=\sigma^2$ and $\theta\rho>\sigma^2$ can be treated analogously. □

LEMMA A.4. [2021a]

i) *Let $\theta\rho > \sigma^2$ and $\theta \leq \theta_d$. The function $T : [\bar{\gamma}, 1] \to [\chi_{e,2}, \chi_{e,1}]$ defined by*

$$T(\gamma) = \frac{\kappa\xi\sigma^2}{\delta(\sigma^2\kappa + \theta)\mathrm{e}^{\xi d}} \frac{\mathrm{e}^{\xi x_\gamma}}{Q_\gamma(x_\gamma)}$$

*is strictly increasing. We have $T(\bar{\gamma}) = \chi_{e,2}$ and $T(1) = \chi_{e,1}$. In particular, $\chi_{e,1} > \chi_{e,2}$.*

ii) *Let $\theta\rho > \sigma^2$ and $\theta > \theta_d$. The function $T : [0, 1] \to [\chi^d_{e,2}, \chi_{e,1}]$ as defined above is strictly increasing. It holds $T(0) = \chi^d_{e,2}$ and $T(1) = \chi_{e,1}$. In particular, $\chi_{e,1} > \chi^d_{e,2}$.*

*Proof.* We have

$$T(\gamma) = \frac{\kappa\xi\sigma^2}{\delta(\sigma^2\kappa + \theta)\mathrm{e}^{\xi d}} \Big[\frac{\theta\rho - \sigma^2\gamma}{\theta\rho - \sigma^2}\Big]^{(\xi\rho-1)/\zeta} \mathrm{e}^{\xi\sigma^2(\gamma-1)/(\zeta\theta)}\,, \tag{A.19}$$

by $Q_\gamma(x_\gamma) = [(\theta\rho - \gamma\sigma^2)/(\theta\rho - \sigma^2)]^{1/\zeta}$. We conclude that, by $\xi\rho - 1 < 0$ for $\theta\rho > \sigma^2$, the right hand side is strictly increasing in $\gamma$, equal to $\chi_{e,2}$ for $\gamma = \bar{\gamma}$ and equal to $\chi_{e,1}$ for $\gamma = 1$. In particular, for any $\chi \in [\chi_{e,2}, \chi_{e,1}]$, there is a unique $\gamma \in [\bar{\gamma}, 1]$ with $T(\gamma) = \chi$. The second case is proved analogously. $\square$

# Bibliography


Albrecher, H. and S. Asmussen, 2006. 'Ruin probabilities and aggregate claims distributions for shot noise Cox processes'. In: *Scandinavian Actuarial Journal* 2, 86–110.

Albrecher, H., N. Bäuerle and M. Bladt, 2018. 'Dividends: from refracting to ratcheting'. In: *Insurance: Mathematics and Economics* 83, 47–58.

Albrecher, H., E.C.K. Cheung and S. Thonhauser, 2011. 'Randomized observation periods for the compound Poisson risk model: dividends'. In: *ASTIN Bulletin* 41, 645–672.

Albrecher, H. and C. Hipp, 2007. 'Lundberg's risk process with tax'. In: *Blätter der DGVFM* 28, 13–28.

Angoshtari, B., E. Bayraktar and V.R. Young, 2015. 'Minimizing the expected lifetime spent in drawdown under proportional consumption'. In: *Finance Research Letters* 15, 106–114.

Angoshtari, B., E. Bayraktar and V.R. Young, 2016a. 'Minimizing the probability of lifetime drawdown under constant consumption'. In: *Insurance: Mathematics and Economics* 69, 210–223.

Angoshtari, B., E. Bayraktar and V.R. Young, 2016b. 'Optimal investment to minimize the probability of drawdown'. In: *Stochastics* 88, 946–958.

Asmussen, S. and H. Albrecher, 2010. *Ruin Probabilities.* Second edition. Singapore: World Scientific.

Asmussen, S., O. Nerman and M. Olsson, 1996. 'Fitting phase-type distributions via the EM algorithm'. In: *Scandinavian Journal of Statistics* 23, 419–441.

Azcue, P. and N. Muler, 2005. 'Optimal reinsurance and dividend distribution policies in the Cramér–Lundberg model'. In: *Mathematical Finance* 15, 261–308.

Bladt, M., 2005. 'A review on phase-type distributions and their use in risk theory'. In: *ASTIN Bulletin* 35, 145–161.

Borodin, A.N. and P. Salminen, 2002. *Handbook of Brownian Motion – Facts and Formulae.* Second edition. Basel: Birkhäuser.

Brinker, L.V., 2021. 'Minimal expected time in drawdown through investment for an insurance diffusion model'. In: *Risks* 9, 17.

Brinker, L.V. and J. Eisenberg, 2021. 'Dividend optimisation: a behaviouristic approach'. In: *Insurance: Mathematics and Economics,* forthcoming.

Brinker, L.V. and H. Schmidli, 2021a. 'Maximal growth with a penalty for the time in drawdown in a diffusion model under proportional reinsurance'. Preprint.

Brinker, L.V. and H. Schmidli, 2021b. 'Optimal drawdowns in the classical risk model under proportional reinsurance'. Manuscript submitted for publication.

Brinker, L.V. and H. Schmidli, 2022. 'Optimal discounted drawdowns in a diffusion approximation under proportional reinsurance'. In: *Journal of Applied Probability* 59, forthcoming.

Chaleyat-Maurel, M., N. El Karoui and B. Marchal, 1980. 'Réflexion discontinue et systèmes stochastiques'. In: *Annals of Probability* 8, 1049–1067.

Chekhlov, A., S. Uryasev and M. Zabarankin, 2005. 'Drawdown measure in portfolio optimization'. In: *International Journal of Theoretical and Applied Finance* 8, 13–58.

Chen, X., D. Landriault, B. Li and D. Li, 2015. 'On minimizing drawdown risks of lifetime investments'. In: *Insurance: Mathematics and Economics* 65, 46–54.

Corless, R.M., G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth, 1996. 'On the Lambert W function'. In: *Advances in Computational Mathematics* 5, 329–359.

Cramér, H., 1930. *On the Mathematical Theory of Risk*. Stockholm: Scandia Jubilee Volume.

Cramér, H., 1955. *Collective Risk Theory*. Stockholm: Scandia Jubilee Volume.

Dassios, A. and J.-W. Jang, 2003. 'Pricing of catastrophe reinsurance and derivatives using the Cox process with shot noise intensity'. In: *Finance and Stochastics* 7, 73–95.

Dassios, A. and S. Wu, 2008. *Parisian ruin with exponential claims*. Technical report. London School of Economics and Political Science, LSE Library. URL: `http://eprints.lse.ac.uk/32033/1/Parisian_ruin_with_exponential_claims_%28LSERO%29.pdf` (visited on 20/09/2021).

Davis, M.H.A., 1984. 'Piecewise-deterministic Markov processes: a general class of non-diffusion stochastic models'. In: *Journal of the Royal Statistical Society* 46, 353–388.

Davis, M.H.A., 1993. *Markov Models and Optimization*. London: Chapman & Hall.

Egami, M. and K. Yamazaki, 2014. 'Phase-type fitting of scale functions for spectrally negative Lévy processes'. In: *Journal of Computational and Applied Mathematics* 264, 1–22.

Eisenberg, J. and H. Schmidli, 2009. 'Optimal control of capital injections by reinsurance in a diffusion approximation'. In: *Blätter der DGVFM* 30, 1–13.

Eisenberg, J. and H. Schmidli, 2011. 'Minimising expected discounted capital injections by reinsurance in a classical risk model'. In: *Scandinavian Actuarial Journal* 2011, 155–176.

Elstrodt, J., 2011. *Maß- und Integrationstheorie*. Seventh edition. Heidelberg: Springer.

Elworthy, K.D., A. Truman and H. Zhao, 2007. 'Generalized Itô formulae and space-time Lebesgue–Stieltjes integrals of local times'. In: *Séminaire de Probabilités XL*. Ed. by C. Donati-Martin, M. Émery, A. Rouault and C. Stricker. Vol. 1899. Lecture Notes in Mathematics. Berlin: Springer, 117–136.

Engelbert, H.J. and W. Schmidt, 1991. 'Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations (Part III)'. In: *Mathematische Nachrichten* 151, 149–197.

Feller, W., 1971. *An Introduction to Probability Theory and Its Applications, Volume II*. New York: Wiley.

Fleming, W.H. and H.M. Soner, 1993. *Controlled Markov Processes and Viscosity Solutions*. New York: Springer.

Gerber, H.U. and E.S. Shiu, 1998. 'On the time value of ruin'. In: *North American Actuarial Journal* 2, 48–72.

Grandell, J., 1977. 'A class of approximations of ruin probabilities'. In: *Scandinavian Actuarial Journal* 1977, 37–52.

Grandell, J., 1991. *Aspects of Risk Theory.* Springer Series in Statistics: Probability and its Applications. New York: Springer.

Halidias, N. and P.E. Kloeden, 2006. 'A note on strong solutions of stochastic differential equations with a discontinuous drift coefficient'. In: *Journal of Applied Mathematics and Stochastic Analysis*, 1–6.

Hamelink, F. and M. Hoesli, 2004. 'Maximum drawdown and the allocation to real estate'. In: *Journal of Property Research* 21, 5–29.

Han, X., Z. Liang and K. Yuen, 2018. 'Optimal proportional reinsurance to minimize the probability of drawdown under thinning-dependence structure'. In: *Scandinavian Actuarial Journal* 10, 863–889.

Han, X., Z. Liang and C. Zhang, 2019. 'Optimal proportional reinsurance with common shock dependence to minimise the probability of drawdown'. In: *Annals of Actuarial Science* 13, 268–294.

Harrison, J.M. and M.I. Reiman, 1981. 'Reflected Brownian motion on an orthant'. In: *Annals of Probability* 9, 302–308.

Heikkilä, S., 1993. 'On fixed points of multifunctions in ordered spaces'. In: *Applicable Analysis* 51, 115–127.

Hipp, C. and M. Taksar, 2010. 'Optimal non-proportional reinsurance control'. In: *Insurance: Mathematics and Economics* 47, 246–254.

Hipp, C. and M. Vogt, 2003. 'Optimal dynamic XL reinsurance'. In: *ASTIN Bulletin* 33, 193–207.

Højgaard, B. and M. Taksar, 1998. 'Optimal proportional reinsurance policies for diffusion models'. In: *Scandinavian Actuarial Journal*, 166–180.

Hubalek, F. and E. Kyprianou, 2010. 'Old and new examples of scale functions for spectrally negative Lévy processes'. In: *Sixth Seminar on Stochastic Analysis, Random Fields and Applications.* Ed. by R. Dalang, M. Dozzi and F. Russo. Basel: Birkhäuser, 119–146.

Iglehart, D.L., 1969. 'Diffusion approximations in collective risk theory'. In: *Journal of Applied Probability* 6, 285–292.

Ikeda, N. and S. Watanabe, 1989. *Stochastic Differential Equations and Diffusion Processes.* Second edition. Amsterdam: North Holland.

Irgens, C. and J. Paulsen, 2004. 'Optimal control of risk exposure, reinsurance and investments for insurance portfolios'. In: *Insurance: Mathematics and Economics* 35, 21–51.

Jacobsen, M., 2006. *Point Process Theory and Applications.* Boston: Birkhäuser.

Jiang, Z. and M.R. Pistorius, 2012. 'Optimal dividend distribution under Markov regime switching'. In: *Finance and Stochastics* 16, 449–476.

Karatzas, I. and S. Shreve, 1998. *Brownian Motion and Stochastic Calculus.* Second edition. New York: Springer.

Kuznetsov, A., A.E. Kyprianou and V. Rivero, 2012. 'The theory of scale functions for spectrally negative Lévy processes'. In: *Lévy Matters II.* Ed. by S. Cohen, A. Kuznetsov, A.E. Kyprianou and V. Rivero. Berlin: Springer, 97–186.

Kyprianou, A.E. and R.L. Loeffen, 2010. 'Refracted Lévy processes'. In: *Annales de l'Institut Henri Poincaré* 46, 24–44.

Lamberton, D. and B. Lapeyre, 1996. *Introduction to Stochastic Calculus Applied to Finance.* London: Chapman & Hall.

Lan, G. and J. Wu, 2014. 'New sufficient conditions of existence, moment estimations and non confluence for SDEs with non-Lipschitzian coefficients'. In: *Stochastic Processes and their Applications* 124, 4030–4049.

Landriault, D., B. Li and S. Li, 2017a. 'Drawdown analysis for the renewal insurance risk process'. In: *Scandinavian Actuarial Journal*, 267–285.

Landriault, D., B. Li and H. Zhang, 2015. 'On the frequency of drawdowns for Brownian motion processes'. In: *Journal of Applied Probability* 52, 191–208.

Landriault, D., B. Li and H. Zhang, 2017b. 'On magnitude, asymptotics and duration of drawdowns for Lévy models'. In: *Bernoulli* 23, 432–458.

Lehoczky, J.P., 1977. 'Formulas for stopped diffusion processes with stopping times based on the maximum'. In: *Annals of Probability* 5, 601–607.

Lundberg, F., 1903. *I. Approximerad Framställning av Sannolikhetsfunktionen. II. Återförsäkring av Kollektivrisker*. Uppsala: Almqvist & Wiksell.

Maier-Paape, S. and Q.J. Zhu, 2018. 'A general framework for portfolio theory. Part II: drawdown risk measures'. In: *Risks* 6, 76.

Mijatović, A. and M.R. Pistorius, 2012. 'On the drawdown of completely asymmetric Lévy processes'. In: *Stochastic Processes and Their Applications* 122, 3812–3836.

Paris, R.B., 2010. 'Incomplete Gamma and related functions'. In: *NIST Handbook of Mathematical Functions*. Ed. by F.W.J. Olver, D.M. Lozier, R.F. Boisvert and C.W. Clark. Cambridge: Cambridge University Press, 173–192.

Pilipenko, A., 2014. *An Introduction to Stochastic Differential Equations with Reflection*. Potsdam: Potsdam University Press.

Pistorius, M.R., 2007. 'An excursion-theoretical approach to some boundary crossing problems and the Skorokhod embedding for reflected Lévy processes'. In: *Séminaire de Probabilités XL*. Ed. by C. Donati-Martin, M. Émery, A. Rouault and C. Stricker. Vol. 1899. Lecture Notes in Mathematics. Berlin: Springer, 287–307.

Preischl, M. and S. Thonhauser, 2019. 'Optimal reinsurance for Gerber–Shiu functions in the Cramér–Lundberg model'. In: *Insurance: Mathematics and Economics* 87, 82–91.

Protter, P.E., 2005. *Stochastic Integration and Differential Equations*. Second edition. Berlin: Springer.

Revuz, D. and M. Yor, 1991. *Continuous Martingales and Brownian Motion*. Berlin: Springer.

Rolski, T., H. Schmidli, V. Schmidt and J.L. Teugels, 1999. *Stochastic Processes for Insurance and Finance*. Chichester: Wiley.

Rozkosz, A. and L. Słomiński, 1997. 'On stability and existence of solutions of SDEs with reflection at the boundary'. In: *Stochastic Processes and Their Applications* 68, 285–302.

Schmidli, H., 2001. 'Optimal proportional reinsurance policies in a dynamic setting'. In: *Scandinavian Actuarial Journal*, 55–68.

Schmidli, H., 2008. *Stochastic Control in Insurance*. London: Springer.

Schmidli, H., 2017. *Risk Theory*. Cham: Springer.

Schmidli, H. and M. Vierkötter, 2017. 'On optimal dividends with exponential and linear penalty payments'. In: *Insurance: Mathematics and Economics* 72, 265–270.

Schuhmacher, F. and M. Eling, 2011. ‘Sufficient conditions for expected utility to imply drawdown-based performance rankings’. In: *Journal of Banking & Finance* 35, 2311–2318.

Semrau, A., 2009. ‘Discrete approximations of strong solutions of reflecting SDEs with discontinuous coefficients’. In: *Bulletin of the Polish Academy of Sciences* 57, 169–180.

Situ, R., 2005. *Theory of Stochastic Differential Equations with Jumps and Applications*. New York: Springer.

Sornette, D., 2003. *Why Stock Markets Crash: Critical Events in Complex Financial Systems*. Princeton: Princeton University Press.

Taylor, H.M., 1975. ‘A stopped Brownian motion formula’. In: *Annals of Probability* 3, 234–246.

Wagner, D.H., 1977. ‘Survey of measurable selection theorems’. In: *SIAM Journal on Control and Optimization* 15, 859–903.

Wang, W., P. Chen and S. Li, 2020. ‘Generalized expected discounted penalty function at general drawdown for Lévy risk processes’. In: *Insurance: Mathematics and Economics* 91, 12–25.

Yamada, T., 1973. ‘On a comparison theorem for solutions of stochastic differential equations and its applications’. In: *Journal of Mathematics of Kyoto University* 13, 497–512.

Zhang, H., 2015. ‘Occupation times, drawdowns, and drawups for one-dimensional regular diffusions’. In: *Advances in Applied Probability* 47, 210–230.

Zhang, H. and O. Hadjiliadis, 2012. ‘Drawdowns and the speed of market crash’. In: *Methodology and Computing in Applied Probability* 14, 739–752.

## Eidesstattliche Erklärung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne die Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Werken dem Wortlaut oder dem Sinn nach entnommen wurden, sind als solche kenntlich gemacht. Ich versichere an Eides statt, dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen und eingebundenen Artikeln und Manuskripten - noch nicht veröffentlicht worden ist sowie, dass ich eine Veröffentlichung der Dissertation vor Abschluss der Promotion nicht ohne Genehmigung des Promotionsausschusses vornehmen werde. Die Bestimmungen dieser Ordnung sind mir bekannt. Darüber hinaus erkläre ich hiermit, dass ich die Ordnung zur Sicherung guter wissenschaftlicher Praxis und zum Umgang mit wissenschaftlichem Fehlverhalten der Universität zu Köln gelesen und sie bei der Durchführung der Dissertation zugrundeliegenden Arbeiten und der schriftlich verfassten Dissertation beachtet habe und verpflichte mich hiermit, die dort genannten Vorgaben bei allen wissenschaftlichen Tätigkeiten zu beachten und umzusetzen. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.

Köln, 10. November 2021

Leonie Violetta Brinker

*Teilpublikationen:*

1. Optimal discounted drawdowns in a diffusion approximation under proportional reinsurance. *Journal of Applied Probability*, in Kürze erscheinend, 2022. (mit H. Schmidli)
2. Optimal drawdowns in the classical risk model under proportional reinsurance, 2021. Eingereicht. (mit H. Schmidli)
3. Maximal growth with a penalty for the time in drawdown in a diffusion model under proportional reinsurance, 2021. In Vorbereitung. (mit H. Schmidli)

*Verwendete Hilfsmittel und Programme:*

Diese Arbeit wurde in TeXstudio 2.12.16 verfasst. Die Grafiken auf Seite 75 wurden mit Maple, Version 2020.2, erstellt. Alle anderen Simulationen und Plots wurden in R, Version 3.6.2 – „Dark and Stormy Night", programmiert und in RStudio, Version 1.2.5033 – „Orange Blossom", verarbeitet. Dabei verwendet wurden die Pakete expint, ggplot2, ggpubr, gridExtra, lamW und NLRoot. Zur Nachbearbeitung der Bilder und Erstellung von Skizzen wurden Ipe, Version 7.2.20, und Inkscape, Version 0.92 – „Draw Freely", genutzt.