

# The Transitive Minimum Manhattan Subnetwork Problem in 3 Dimensions

Birgit Engels

*Zentrum für angewandte Informatik,  
University of Cologne, Germany*

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## Abstract

We consider the Minimum Manhattan Subnetwork (MMSN) Problem which generalizes the already known Minimum Manhattan Network (MMN) Problem: Given a set  $P$  of  $n$  points in the plane, find shortest rectilinear paths between all pairs of points. These paths form a network, the total length of which has to be minimized. From a graph theoretical point of view, a MMN is a 1-spanner with respect to the  $L_1$  metric. In contrast to the MMN problem, a solution to the MMSN problem does not demand  $L_1$ -shortest paths for all point pairs, but only for a given set  $R \subseteq P \times P$  of pairs. The complexity status of the MMN problem is still unsolved in  $\geq 2$  dimensions, whereas the MMSN was shown to be  $NP$ -complete considering general relations  $R$  in the plane. We restrict the MMSN problem to transitive relations  $R_T$  (*Transitive* Minimum Manhattan Subnetwork (TMMSN) Problem) and show that the TMMSN problem in 3 dimensions is  $NP$ -complete.

*Key words:* Manhattan network, 1-spanner, grid graph, 3 dimensions

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## 1 Introduction

The Minimum Manhattan Network Problem was first introduced in 2001 by Gudmundson et al. [5] and can be briefly described as follows: Given a set  $P$  of points in the plane and two orthogonal directions ( $X$ - and  $Y$ -axis), connect all pairs  $(p, q)$  of points of  $P$ . For *every* pair  $(p, q)$  a path from  $p$  to  $q$  must be a shortest path with respect to the Manhattan or  $L_1$  metric and only consist of axis parallel line segments. The set of line segments, containing all shortest paths is called *Manhattan Network (MN)*. We measure the length of such a network by summing up the lengths of all line segments and call a solution to

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*Email address:* engels@zpr.uni-koeln.de (Birgit Engels).

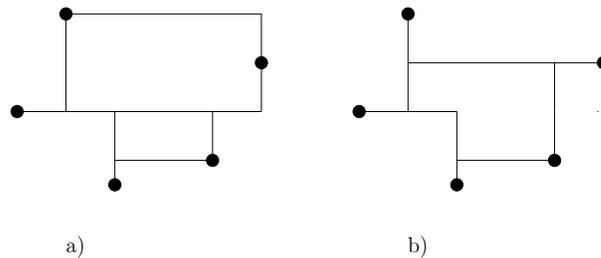


Fig. 1. a) A Manhattan Network; b) A Minimum Manhattan Network.

the MMN problem, i.e. a MN with minimum length, a *Minimum Manhattan Network (MMN)* of  $P$  (Fig. 1).

The MMN problem, as well as the closely related rectilinear Steiner Tree and Steiner Arborescence problem, has application in VLSI design, where connecting the chip components with minimum total wire length is desirable. Especially a MMN solution is useful, as it minimizes the total wire length while the connections between pairs of components are as short (and thus as fast) as possible. Further, Lam et. al. [8] use an MMN approach as a preprocessing step to accelerate the Viterbi algorithm for an alignment problem.

Thus, efficient algorithms for these problems would be highly appreciable, but both of the above Steiner problems already turned out to be *NPO*-complete (see [4], [12]). Up to now, the complexity status of the MMN problem is still unknown, but mostly suspected to be *NP*-hard as well. Hence, the previous work on the MMN problem solely features approximation algorithms: Gudmundson et al. [5] presented an 8-approximation in time  $O(n \log n)$  and a 4-approximation in time  $O(n^3)$  (which is used in [8]). Benkert et al. [1] introduced a 3-approximation in time  $O(n \log n)$ . Kato et al. [6] proposed a 2-approximation in time  $O(n^3)$ . However, the proof of the correctness seems to be incomplete [1]. Chepoi et al. [2] gave a 2-approximation based on an LP with  $O(n^3)$  variables and constraints dominating the running time. The LP-formulation was given by K.Nouioua [9], who also developed a 3-approximation which runs in  $O(n \log n)$  time [10]. The best approximation factor so far was achieved by Seibert and Unger [13] who proposed a 1.5-approximation in time  $O(n^3)$ , although the correctness of their algorithm and the completeness of its analysis is discussed critically in [3].

The problem considered in this paper differs from the original MMN problem in that the point pairs to be connected by shortest  $L_1$ -paths are explicitly given due to a relation  $R \subseteq P \times P$ . This problem is mentioned in [7] as the *F*-restricted Minimum Manhattan Network problem. A special case of the FMMN problem arises for  $R_{onetoall} = \{\{p, q_i\}\}$  and the point set  $P = \{p, q_1, \dots, q_{n-1}\}$ , e.g. every point  $q_i$  has to be connected to the point  $p$  by a shortest  $L_1$ -path. The resulting problem is known as the Rectilinear Steiner Minimum Arborescence (RSMA) problem mentioned above. It was shown to

be  $NP$ -complete in  $\mathbb{R}^2$  by W. Shi and C. Su [12].

But this result features a rather special relation, whereas we consider all transitive relations  $R_T$ , i.e.  $R$  such that  $\forall\{p, q\}, \{p, r\} \in R : \{q, r\} \in R$ . The transitivity of the considered relations result in a kind of double layer MMN problem: We solve the MMN problem for subsets of points of  $P$  under the restriction that not only the sum of the shortest path lengths has to be minimized as in the original MMN problem, but also the sum of the lengths of the subset-MMNs is to be minimized (see Section 2 and Figure 2). The set of all transitive relations  $R_T$  includes  $R_{alltoall} = \{\{p, q\} | p \neq q \in P\}$ . This means that the original MMN problem is a special case of TMMSN. On the other hand the set  $R_T$  obviously does not contain  $R_{onetoall}$  and RSMA is no special case of TMMSN. Thus the question of the complexity of the TMMSN problem is open.

The remaining part of the article is organized as follows. In Section 2, we give some formal definitions regarding the different problem formulations. Then we proceed to an  $NP$ -completeness proof for the decision version of TMMSN in Section 3. We conclude with a short summary in Section 4.

## 2 Definitions

In this section, we briefly introduce some definitions and notations used throughout the paper. Let  $P$  be a set of points  $p = (x_p, y_p, z_p)$  in  $\mathbb{R}^3$  and consider three pairwise orthogonal directions ( $x$ -,  $y$ - and  $z$ -axis).

**Definition 1** *Let  $s$  be a line segment and  $S$  a set of such segments. Then  $|s|$  denotes the length of  $s$ , and we define the total length of  $S$  as  $|S| = \sum_{s \in S} |s|$ .*

**Definition 2**  $\pi_{pq}$  *denotes a set of axisparallel line segments which form an arbitrary shortest path between two points  $p, q \in P$  with respect to the  $L_1$ -metric.*

**Definition 3** *Let  $B(P)$  be the bounding box of  $P$ , i.e. the smallest axis aligned cuboid that contains all points of  $P$ , and let  $\partial B(P)$  be its boundary.*

In the following section we refer to  $\pi_{pq}$  as  $L_1$ -shortest path or simply (shortest) path. The length of a  $L_1$ -shortest path is always given by  $|\pi_{pq}| = |p_x - q_x| + |p_y - q_y| + |p_z - q_z|$ . Points lying on  $\partial B(P)$  are considered as “contained in  $B(P)$ ”.  $B(P)$  can degenerate to a rectangle or even a line segment. Further, all  $\pi_{pq}, p, q \in P$ , are contained in  $B(P)$ . Recall the Minimum Manhattan Network (MMN) Problem in  $\mathbb{R}^2$  which transfers most easily into three dimensions as follows :

Given a set  $P$  of points in  $\mathbb{R}^3$  and three pairwise orthogonal directions ( $x$ -,  $y$ - and  $z$ -axis), a *Minimum Manhattan Network* of  $P$ ,  $MMN(P)$ , is a set of axis parallel line segments with the following properties:

- $\forall p, q \in P : \exists \pi_{pq} \subseteq MMN(P)$ .
- $|MMN(P)|$  is minimal amongst the lengths of all sets of axis parallel line segments containing at least one shortest  $L_1$  path between each pair of points  $p, q \in P$ .

We call a given set  $P$  an *instance* of the MMN problem.

Analogously define the Minimum Manhattan Subnetwork (MMSN) Problem:

Given a set  $P$  of points in  $\mathbb{R}^3$ , 3 pairwise orthogonal directions ( $x$ -,  $y$ - and  $z$ -axis) and a relation  $R \subseteq P \times P$ , a *Minimum Manhattan Subnetwork* of  $P$  with regard to  $R$ ,  $MMSN([P, R])$ , is a set of axis parallel line segments with the following properties:

- $\forall \{p, q\} \in R : \exists \pi_{pq} \subseteq MMSN([P, R])$ .
- $|MMSN([P, R])|$  is minimal amongst the lengths of all of all sets of axis parallel line segments containing at least one shortest  $L_1$  path between each pair of points  $\{p, q\} \in R$ .

We call a given pair  $[P, R]$  an *instance* of the MMSN problem.

The Transitive MMSN is then defined as the Minimum Manhattan Subnetwork Problem for instances  $[P, R_T]$ . Due to the transitivity of  $R$ , the set  $P$  can be partitioned uniquely into subsets  $P_1, \dots, P_k$  of points such that:

- $\forall p, q \in P_i : \{p, q\} \in R$
- $\forall p \in P_i, q \in P_j, i \neq j : \{p, q\} \notin R$

We call  $\mathcal{P} = P_1, \dots, P_k$  the  $R$ -induced partition of  $P$ . Points forming a set  $P_i, 1 \leq i \leq k$  have to be *fully interconnected*.

To solve the TMMSN problem it may seem to be sufficient to solve independent MMN problems for all  $P_i$ , but this impression is misleading: Consider the points  $P = \{p, q, r, s\}$  and the (transitive) relation  $R = \{\{p, q\}, \{r, s\}\}$  which induces the partition  $\mathcal{P} = P_1 \cup P_2, P_1 = \{p, q\}, P_2 = \{r, s\}$ . Solving the two MMN problems on  $P_1$  and  $P_2$  may yield a “solution” for the TMMSN( $[P, R]$ ) like the one in Figure 2 a), whereas the solution in Figure 2 b) is minimal.

This problem only arises for instances  $[P, R]$ , where shortest paths between point pairs from different sets  $P_i$  can share line segments (like the paths  $\pi_{pq}$  and  $\pi_{rs}$  in Figure 2). Thus combining solutions for the MMN problem on the point sets  $P_i$  yields a valid solution TMMSN( $[P, R]$ ) if paths  $\pi_{pq}, p, q \in P_i$  and  $\pi_{rs}, r, s \in P_{j \neq i}$  cannot share any line segments. This holds if:

$$\forall P_i, P_{j \neq i} : B(P_i) \cap B(P_j) = \emptyset.$$

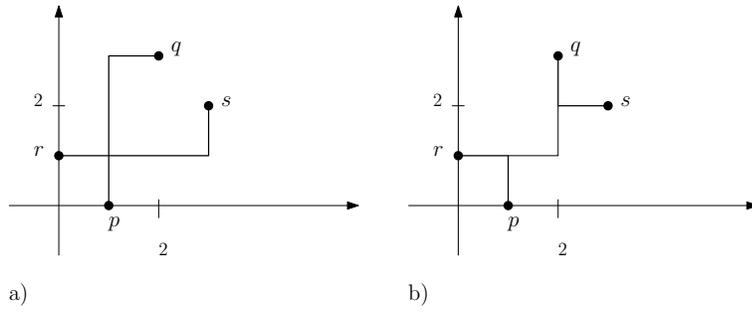


Fig. 2. a) Combining MMN-solutions for  $P_i$ ; b) TMMSN solution for  $[P, R]$ .

We call point sets  $P_i$  and  $P_j$  with the above property *geometrically independent*.

Finally, we formulate the decision version of the TMMSN problem:

Given an instance  $[P, R]$  and a constant  $c$ , is  $|TMMSN([P, R])| \leq c$ ?

We call a triple  $[P, R, c]$  an *instance* of the  $TMMSN([P, R])$  decision problem.

### 3 TMMSN is NP-complete

We show that the TMMSN decision problem in three dimensions is NP-hard. For this a polynomial time reduction from E3SAT is presented. Since the decision version of the TMMSN is obviously in  $NP$ ,<sup>1</sup> we obtain the  $NP$ -completeness result. The reduction works as follows. At first we construct an instance  $[P_\alpha, R_\alpha]$  of the TMMSN problem from a given E3SAT instance  $\alpha$  in polynomial time. We then determine a lower bound  $c_\alpha$  to the length of the minimal transitive Manhattan subnetwork of this instance. Finally we prove that  $\alpha$  is satisfiable if and only if the length of the solution attends the lower bound, i.e.  $|TMMSN([P_\alpha, R_\alpha])| = c_\alpha$ .

**Construction of  $[P_\alpha, R_\alpha]$ .** Consider the structure of an instance of the E3SAT problem: Given a Boolean formula  $\alpha$  with  $n$  clauses,  $c_1, \dots, c_n$ , over  $m$  variables,  $x_1, \dots, x_m$ , where each clause consists of three literals; that is:

$$\alpha = c_1 \wedge \dots \wedge c_n$$

$$\text{with } c_i = (l_i^1 \vee l_i^2 \vee l_i^3) \text{ and } l_i^1, l_i^2, l_i^3 \in \{x_j, \neg x_j; 1 \leq j \leq m\}.$$

<sup>1</sup> Ask an oracle for the  $TMMSN([P, R])$ , which is a set of  $O(n^2)$  line segments, sum up their length to  $l$  and check whether  $l < c$ , which only needs polynomial time due to the size of  $TMMSN([P, R])$ .

Let  $\text{occ}(x_j)$  denote the number of occurrences of variable  $x_j$  (literal  $x_j$  or  $\neg x_j$ ) in  $\alpha$ . Then  $P_\alpha$  consists of the following point sets:

- For each variable  $x_j$ ,  $1 \leq j \leq m$ , we add a set  $P_j = E_j \cup A_j$  of points with  $|E_j| = 4$  and  $|A_j| = 2(m+1)\text{occ}(x_j)$  to  $P_\alpha$ .
- For each clause  $c_i$ ,  $1 \leq i \leq n$ , we add  $m+1$  point sets  $P_i^k = \{p_i^k, q_i^k, r_i^k, s_i^k\}$ ,  $1 \leq k \leq m+1$  to  $P_\alpha$ .

Further, we define  $R_\alpha$  such that:

- For each variable  $x_j$ ,  $1 \leq j \leq m$ :  $P_j \times P_j \in R_\alpha$ .
- For each clause  $c_i$ ,  $1 \leq i \leq n$  and  $1 \leq k \leq m+1$ :  $\{\{p_i^k, q_i^k\}, \{r_i^k, s_i^k\}\} \in R_\alpha$ .

Next we define the placement of the points  $P_j$  and  $P_i^k$  in  $\mathbb{R}^3$ . We start with  $E_j = \{e_j^1, e_j^2, e_j^3, e_j^4\}$ , where:

$$\begin{aligned} e_j^1 &= (0, 0, j), \quad e_j^2 = (5(m+1)n, 0, j), \\ e_j^3 &= (5(m+1)n, 2, j), \quad e_j^4 = (0, 2, j). \end{aligned}$$

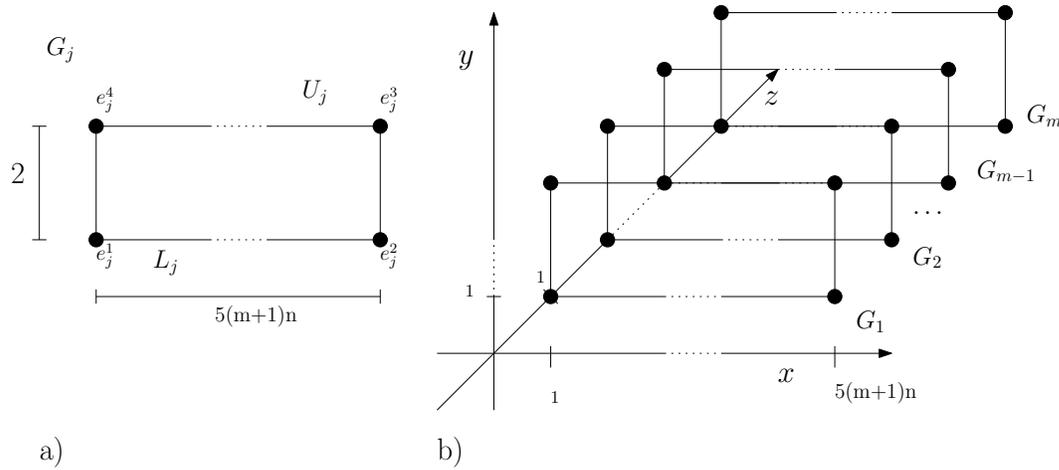


Fig. 3. a) Single basic variable gadget  $G_j$ . b) Arrangement of basic gadgets in  $\mathbb{R}^3$ .

Thus  $E_j$  determines the corner points of a rectangle of height 2 and length depending on the number of variables and clauses of  $\alpha$  and which lies in the plane  $z = j$  of  $\mathbb{R}^3$  (Fig. 3). The shortest  $L_1$ -paths between pairs of neighboring corner points (in counter-clockwise order) are reduced to the straight line segments between the points, e.g.  $\pi_{e_j^1 e_j^2} = \{e_j^1, e_j^2\}$ . Thus the rectangle  $e_j^1, e_j^2, e_j^3, e_j^4$  identifies with  $\partial B(E_j)$  when we consider the two-dimensional boundary. We call the points  $E_j$  together with  $\partial B(E_j)$  the *basic variable gadget*. The *basic variable gadget* together with the points  $A_j$  forms the *variable gadget*  $G_j$ . We abbreviate:  $L_j := \pi_{e_j^1 e_j^2}$  and  $U_j := \pi_{e_j^3 e_j^4}$ .

Before we proceed with the coordinates of the points from  $A_j$  (which will all lie on  $L_j$  or  $U_j$ ), we place the four points of each  $P_i^k, 1 \leq i \leq n, 1 \leq k \leq m+1$  as follows:

$$\begin{aligned} p_i^k &= (5(k-1)n + 5(i-1) + 2, -1, 0), \\ q_i^k &= (5(k-1)n + 5(i-1) + 4, 3, m+1), \\ r_i^k &= (5(k-1)n + 5(i-1) + 4, 0.5, m+1), \\ s_i^k &= (5(k-1)n + 5(i-1) + 4, 1.5, m+1). \end{aligned}$$

$B(P_i^k)$  is a cube of width 2 height 4 and depth  $m+1$  with  $p_i^k$  as front lower left and  $q_i^k$  as back upper right point. Together with the points  $P_i^k$ ,  $B(P_i^k)$  forms the *clause gadget*  $G_i^k$ . All  $L_j$  and  $U_j$  pass through each  $G_i^k$  in  $x$ -direction (see Fig. 4), i.e.  $B(P_i^k) \cap L_j = l$ , where  $l$  is a line segment parallel to the  $x$ -axis with  $|l| = 2$  and  $B(P_i^k) \cap U_j = u$  for a line segment  $u$  with the same properties as  $l$ .

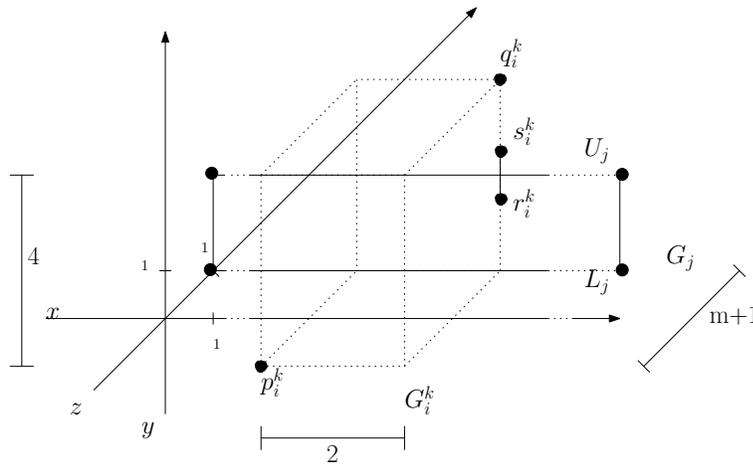


Fig. 4. A clause gadget  $G_i^k$  (cube defined by  $p_i^k$  and  $q_i^k$ ) with  $L_j, U_j$  passing through. (Note that  $\pi_{s_i^k r_i^k}$  is not crossed by any  $L_j$  or  $U_j$  because with  $z$ -coordinate  $m+1$  the points  $s_i^k$  and  $r_i^k$  are located behind the variable gadgets  $G_j$ .)

From the placement of the points it is evident that all  $m+1$  clause gadgets  $G_i^k$  are copies of  $G_i^1$  shifted in  $x$ -direction which we need for technical reasons. The  $G_i^k, 1 \leq i \leq n, 1 \leq k \leq m+1$ , follow each other with a horizontal spacing of 3 along the  $x$ -axis in the following order:

$$[\text{Clause Gadget Order}] G_1^1, \dots, G_n^1, G_1^2, \dots, G_n^2, \dots, G_1^{m+1}, \dots, G_n^{m+1}.$$

Now we determine the placement of the  $2(m+1)occ(x_j)$  remaining points from  $A_j$  on each variable gadget. These points are essential as they represent the occurrence of a variable  $x_j$  as literal  $x_j$  or  $\neg x_j$  respectively in a clause  $c_i$ . Their

positions influence the structure of the  $\text{TMMSN}[P_\alpha, R_\alpha]$  and the lower bound on its length. The lower bound to  $|\text{TMMSN}[P_\alpha, R_\alpha]|$  is then attained by the solution if we can save a fixed length for every clause. In this case, every clause is satisfiable by a consistent truth assignment. This again is true if and only if  $\alpha$  is satisfiable. We achieve this by a placement of the points of  $A_j, 1 \leq j \leq m$ , with respect to the following rule.

**Placement Rule:**

- If literal  $x_j$  occurs in  $c_i$ , we place  
 $\overline{p_{i,j}^k} = (5n(k-1) + 5(i-1) + 3, 0, j)$  (on  $L_j$ ) and  
 $p_{i,j}^k = (5n(k-1) + 5i, 2, j)$  (on  $U_j$ ) for  $1 \leq k \leq m+1$ .
- If literal  $\neg x_j$  occurs in  $c_i$ , we place  
 $\overline{p_{i,j}^k} = (5n(k-1) + 5(i-1) + 1, 0, j)$  (on  $L_j$ ) and  
 $p_{i,j}^k = (5n(k-1) + 5(i-1) + 3, 2, j)$  (on  $U_j$ ) for  $1 \leq k \leq m+1$ .

This means that for each variable  $x_j$  that occurs in clause  $c_i$  there is one point of  $A_j$  (on  $L_j$  or  $U_j$ ) *inside* each  $G_i^k$  and one point to the left *or* to the right of  $G_i^k$ . Whether the point inside  $G_i^k$  is located on  $L_j$  or  $U_j$  depends on which literal of  $x_j$  occurs in  $c_i$ . As an example consider formula  $\alpha$  with only one variable and two clauses  $\alpha = (a) \wedge (\neg a)$  for the sake of simplicity, although this is no exact E3SAT formula (see Fig. 5).

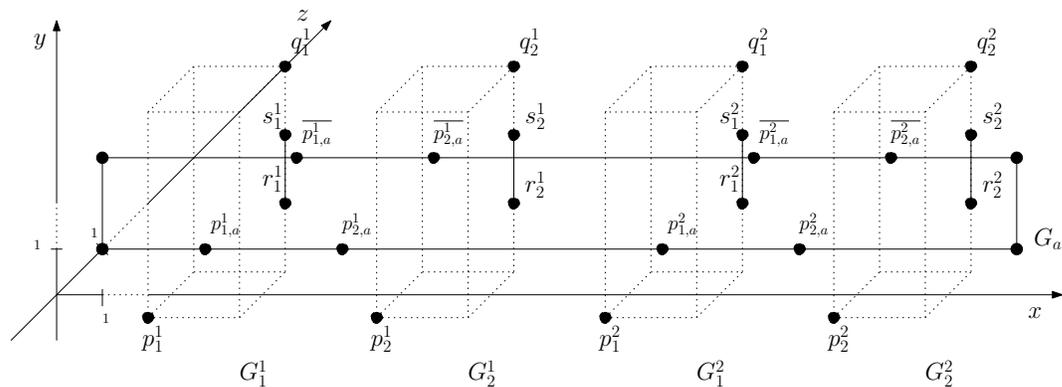


Fig. 5. Example:  $P_\alpha$  for  $\alpha = (a) \wedge (\neg a)$ .

We have determined the coordinates for all points of  $P_\alpha$  which together with the relation  $R_\alpha$  completes the construction of the instance  $[P_\alpha, R_\alpha]$ . The size of our construction is in  $O(m^2n)$  and thus with  $m \leq 3n$  polynomial in  $O(n^3)$ .

**Lower bound on  $|\text{TMMSN}([P_\alpha, R_\alpha])|$  and structural properties.** Proceeding with the reduction, we give a lower bound  $c_\alpha$  to the length of the solution to the TMMSN problem on  $[P_\alpha, R_\alpha]$ . Let  $M$  be any Manhattan network on  $[P_\alpha, R_\alpha]$ , i.e. a set of axis parallel line segments containing at least

one shortest path for each point pair from  $R_\alpha$  (which need not be of minimal length amongst such sets). Further, denote with  $\pi_{uv}^*$  a shortest path between a point pair  $\{u, v\} \in R_\alpha$ , that is contained in the solution  $M$ .<sup>2</sup>

Each network  $M$  can be partitioned into the following three sets of segments  $U_M$ ,  $V_M$  and  $C_M$ :

The set  $U_M \subseteq M$  contains all fixed line segments. These are all segments that have to be contained in the solution, because they constitute unique  $L_1$ -shortest paths i.e. the paths between the corner points of the variable gadgets and the paths between each two points  $r_i^k$  and  $s_i^k$ :

$$U_M = \bigcup_{1 \leq j \leq m} \partial B(E_j) \cup \bigcup_{1 \leq i \leq n, 1 \leq k \leq m+1} \pi_{r_i^k s_i^k}. \quad (1)$$

The set  $V_M \subseteq M$  contains all segments from  $M$  that contribute to  $L_1$ -shortest paths between point pairs  $\{u, v\} \in \bigcup_{1 \leq j \leq m} A_j \times A_j$  without segments that already occur in  $U_M$  :

$$V_M = \bigcup_{u \neq v \in A_j, 1 \leq j \leq m} \pi_{uv}^* \setminus U_M. \quad (2)$$

The set  $C_M \subseteq M$  contains all segments from  $M$  that contribute to  $L_1$ -shortest paths between the points pairs  $(p_i^k, q_i^k)$  without segments that already occur in  $U_M$  or  $V_M$ :

$$C_M = \bigcup_{1 \leq i \leq n, 1 \leq k \leq m+1} \pi_{p_i^k q_i^k}^* \setminus U_M \cup V_M. \quad (3)$$

Let  $\min_X$ ,  $X \in \{U, V, C\}$  be defined as:  $\min_X = \min_M |X_M|$ . Then  $|M| \geq c_\alpha$  with  $c_\alpha := \min_U + \min_V + \min_C$  is true for all  $M$  and especially:

**Property 1**  $|TMMSN([P_\alpha, R_\alpha])| \geq c_\alpha$

Thus we can determine the desired lower bound  $c_\alpha$  as the sum of  $\min_U$ ,  $\min_V$ ,  $\min_C$  by minimizing the lengths of  $U_M$ ,  $V_M$  and  $C_M$  independently over all  $M$ . The set  $U_M$  contains the fixed segments of  $\partial B(E(x_j))$  with length  $10(m+1)n+4$  for each of the  $m$  variable gadgets and the line segments  $\overline{r_i^k s_i^k}$  of length 1 for the  $(m+1)n$  clause gadgets. Thus the length of  $U_M$  is always  $|U_M| = \min_U$  and  $\min_U$  only depends on  $m$  and  $n$ :

$$\min_U = (10(m+1)n+4)m + (m+1)n. \quad (4)$$

<sup>2</sup> In general there are different  $\pi_{uv}^*$ , but in cases where we discuss the geometric structure of one such  $\pi_{uv}^*$  in detail, it will be unique in  $M$ .

As the structure of  $U_M$  is also the same for all  $M$ , we set  $U = U_M$ . To determine  $\min_V$ , we further partition  $V_M$ : Let  $V_j$  be the subset of  $V_M$  containing all segments from the paths  $\pi_{uv}^* \setminus U$ ,  $\{u, v\} \in A_j \times A_j$ . Then  $V_M = \bigcup_{1 \leq j \leq m} V_j$  and  $|V_M| = \sum_{1 \leq j \leq m} |V_j|$ . The latter holds, because  $A_j$  and  $A_{l \neq j}$  are geometrically independent by the placement of the parallel planes on the z-Axis. Therefore  $\pi_{uv}^*$ ,  $u, v \in A_j$ , and  $\pi_{wt}^*$ ,  $w, t \in A_{l \neq j}$  can not share any segments.

Thus it is sufficient to determine the minimum length of  $V_j$  for all variables. For this, we consider a basic variable gadget which is the (2-D) rectangle  $\partial B(E_j)$  including the corner points. All points  $A_j$  also lie on  $\partial B(E_j)$ , especially on  $L_j$  and  $U_j$ . Moreover all pairs  $\{p, q\}$  of (*x*-)neighbouring points, i.e.  $p = (x_p, y_p, z_p)$  and  $q = (x_q, y_q, z_q)$  such that  $p, q \in A_j$  and  $\exists r = (x_r, y_r, z_r) \in A_j : x_p < x_r < x_q$ , are located on different of the two segments  $L_j$  and  $U_j$ . We call such a  $\{p, q\}$  an *alternating point pair* or state that points (of a set) *alternate*, if all neighboring point pairs are alternating point pairs.

**Property 2** *The points of  $A_j$  alternate on  $L_j$  and  $U_j$ .*

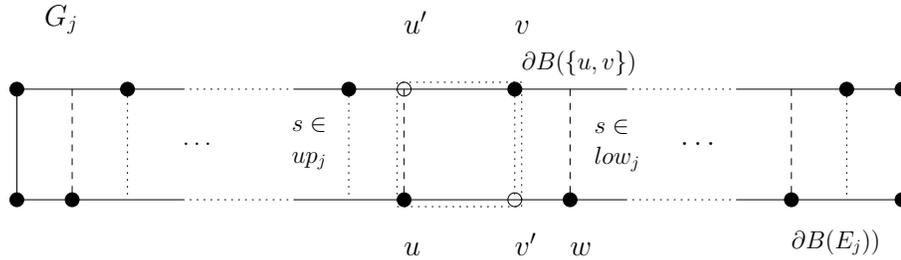


Fig. 6. Alternating points of  $A_j$ : All shortest paths  $\pi_{uv}^*$  between (*x*-)neighbouring points contain induced segments.

This can be seen by the following inductive argument: Let  $c_i$  be the first clause of  $\alpha$  containing  $x_j$  as a literal, which results in a point  $p_{i,j}^k$  located on  $L_j$  and  $\overline{p_{i,j}^k}$  on  $U_j$  to the right of  $p_{i,j}^k$  as determined by the placement rule. Let  $c_l$  be the next clause of  $\alpha$  where variable  $x_j$  occurs. If literal  $x_j$  occurs in  $c_l$ , a point  $p_{l,j}^k$  is again located on  $L_j$  (and to the right of  $\overline{p_{i,j}^k}$ ) followed by  $\overline{p_{l,j}^k}$  on  $U_j$ : The alternation is kept. The same is true, if  $\neg x_j$  occurs in  $c_l$ , because  $\overline{p_{l,j}^k}$  is located on  $U_j$  inside  $G_l^k$ , but  $p_{l,j}^k$  is located to the left of  $G_l^k$ , between  $\overline{p_{i,j}^k}$  and  $\overline{p_{l,j}^k}$ . The same is true for all other possible sequences of occurrence of a variable  $x_j$ .

With Property 2, it suffices to consider the interconnection of alternating points on the lower and upper horizontal edge of a given 2D rectangle to determine the minimal length of each  $V_j$ . This situation corresponds to the *original* MMN problem on the point set  $P_j$ . All points on  $L_j$  ( $U_j$  resp.) are already interconnected by  $L_1$ -shortest paths via  $L_j$  ( $U_j$  resp.).  $L_j$  and  $U_j$  are contained in  $U$  and thus do not contribute to  $|V_j|$ . Further, we only need to connect a point  $u$  on  $L_j$  to both its neighbours on  $U_j$  directly. All other points

on  $U_j$  are then connected to  $u$  via its neighbours automatically. Thus  $V_j$  only contains vertical segments  $s$  of length  $|s| = 2$  which start on  $L_j$  and end on  $U_j$  (or vice versa). The minimal length of  $V_j$  is then determined by the minimal number  $\min_s(V_j)$  of such segments used.

**Lemma 1**  $\min_s(V_j) = (m + 1)\text{occ}(x_j)$

**PROOF.** Consider the neighbors  $u$  on  $L_j$  and  $v$  on  $U_j$  to the right of  $u$  and their vertical projections  $u'$  on  $U_j$  and  $v'$  on  $L_j$  (Fig. 6). Let  $s$  be a vertical segment which starts on  $L_j$  and ends on  $U_j$  (or vice versa). Any such segment contributing to  $\pi_{uv}$  has to be situated in  $B(\{u, v\})$  which identifies with the rectangle  $(u, v', v, u')$  in this case. Thus  $s$  can contribute to at least one  $\pi_{uv}$ . Let  $w$  be the right neighbour of  $v$  on  $L_j$ . Then a segment  $s$  contributing to  $\pi_{vw}$  has again to lie within  $B(\{v, w\})$ . Further  $B(\{u, v\}) \cap B(\{v, w\}) = \overline{vv'}$  and  $\overline{vv'}$  can contribute to both  $\pi_{uv}$  and  $\pi_{vw}$ . We call such a segment, starting from a point  $q$  on either  $L_j$  or  $U_j$  respectively and ending in its vertical projection  $q'$  on  $U_j$  resp.  $L_j$ , an *induced* segment (by  $q$ ). Thus any induced segment can contribute to two shortest paths  $\pi_{pq}$  and  $\pi_{qr}$ ,  $p$  and  $r$  being  $q$ 's left and right neighbor. For any three rectangles  $B(\{u, v\})$ ,  $B(\{w, t\})$ ,  $B(\{x, y\})$ ,  $u, v, w, t, x, y \in A_j$  the intersection is empty (even if  $v = w$  and  $t = x$ ). Thus no segment  $s$  can contribute to more than two shortest paths between neighboring points in  $A_j$ . As  $|A_j| = 2(m+1)\text{occ}(x_j)$ , we have to connect  $2(m+1)\text{occ}(x_j) - 1$  neighbouring point pairs. Using induced segments leads to the following minimal number of segments needed:

$$\left\lceil \frac{2(m+1)\text{occ}(x_j) - 1}{2} \right\rceil = \left\lceil \frac{2(m+1)\text{occ}(x_j)}{2} \right\rceil - \left\lfloor \frac{1}{2} \right\rfloor = (m+1)\text{occ}_x$$

□

We can now determine the minimal length of  $V_M$  as the sum of all  $\min_s(V_j)$  multiplied by the length of the segments  $|s| = 2$ . With Lemma 1,  $\min_s(V_j)$  depends on the number of occurrences of variable  $x_j$  in  $\alpha$  which is not known for the general  $\alpha$ . On the other hand the total number of variable occurrences in  $\alpha$  is  $3n$  as the definition of E3SAT claims three literals in each of  $n$  clauses. Thus we obtain  $|V_M| = \min_V$ , which again only depends on  $n$  and  $m$ :

$$\min_V = 2 \sum_{1 \leq j \leq m} \min_s(V_j) = 2 \sum_{1 \leq j \leq m} (m+1)\text{occ}(x_j) = 2(m+1)3n. \quad (5)$$

Before we proceed, we make some remarks on the structure of the sets  $V_j \subset V_M$  of minimal length. The proof of Lemma 1 suggests only to use induced segments, but does not specify which segments should be chosen. Let  $S =$

$s_1, \dots, s_{2occ(x_j)}$  be the  $x$ -ordered sequence of segments induced by the points of  $A_j$ . Of course,  $V_j$  can not only contain the left half of  $S$ , as they would only complete half the shortest paths  $\pi_{uv}^*$ ,  $u, v \in A_j$ . But we obtain all necessary shortest paths, if we take every second next induced segment. Starting from the left, these are all segments induced by a point on  $L_j$  and starting from the right we obtain all segments induced by a point on  $U_j$ . We will refer to the set of segments induced by all points of  $A_j$  on  $L_j$  as the set  $low_j$  (dashed in Fig. 6) and to the set of segments induced by all points of  $A_j$  on  $U_j$  as the set  $up_j$  (dotted in Fig. 6) respectively. We refer to both sets as the *parities* of the variable gadget  $G_j$ . The parities of  $G_j$  identify with the two possible *nice minimum vertical covers* of  $A_j$  as defined in [1] and [6] and obviously together with the rectangle edges of  $G_j$  each of the sets contains all desired  $\pi_{pq}, \{pq\} \in A_j$ .

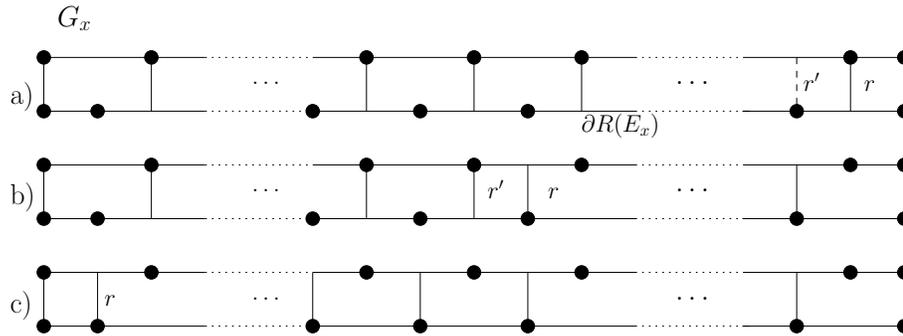


Fig. 7. a)  $V_j = up_j$ ; b) Parity change between  $r$  and  $r'$ ; c)  $V_j = low_j$ .

Later on, the parities of  $V_j$  are interpreted as the possible truth values of  $x_j$ . Unfortunately,  $low_j$  and  $up_j$  are not the only possible sets  $V_j$  with minimum length. As there are  $2(m+1)occ(x_j) - 1$  point pairs  $\{u, v\} \in A_j \times A_j$  to be fully interconnected, only  $(m+1)occ(x_j) - 1$  induced segments are necessary and the last  $\pi_{uv}^*$  can be completed by a non-induced segment. Further, not all induced segments in  $V_j$  have to belong to the same parity of  $G_j$  (see Fig. 7). This would lead to an inconsistency of a derived truth assignment. We define:

**Definition 4** Assume the segments of  $V_j$  are sorted from left to right (by  $x$ -order of the inducing points). Then any two subsequent segments are considered as a parity change if they are of different parities, i.e. one segment belongs to  $low_j$  and the other to  $up_j$  or vice versa (see  $r$  and  $r'$  in Fig. 7b).

**Lemma 2** Any set  $V_j$  of minimal length contains at most one parity change or one non-induced segment.

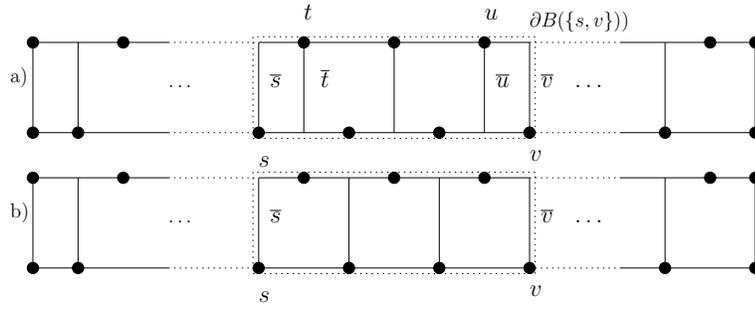


Fig. 8. a) Two parity changes in  $V_j$ ; b) No parity change in  $V_j$  with  $|V_j| = |V'_j| - 2$ :  $V'_j$  was not minimal.

**PROOF.** First, we see that  $V_j$  cannot contain two non-induced segments. If we assume otherwise, we know that each of them contributes to only one of the  $2(m+1)\text{occ}(x_j) - 1$  shortest paths. Even if we use only induced segments for the remaining  $2(m+1)\text{occ}(x_j) - 3$  paths we need the following total number of segments:

$$2 + \left\lceil \frac{2(m+1)\text{occ}(x_j) - 3}{2} \right\rceil = 2 + \left\lceil \frac{2(m+1)\text{occ}(x_j)}{2} \right\rceil - \left\lfloor \frac{3}{2} \right\rfloor =$$

$$2 + (m+1)\text{occ}_x - 1 = (m+1)\text{occ}_x + 1 = m_s(V_j) + 1$$

This contradicts the minimality of  $V_j$  by Lemma 1 and we just have to consider two cases left: First, assume  $V_j$  contains two parity changes. Let  $\bar{s}, \bar{t}$  and  $\bar{u}, \bar{v}$  be the two pairs of subsequent segments forming the parity changes and  $s, t, u, v$  the points from which the segments extend (Fig. 8 a). W.l.o.g. the segment  $\bar{s}$  belongs to  $low_j$ ,  $\bar{t}$  and  $\bar{u}$  belong to  $up_j$  and  $\bar{v}$  again to  $low_j$ : We can divide  $V_j$  into three parts. Segments from  $low_j$  situated at the left and right end of  $G_j$  and segments from  $up_j$  in between. We look at the part of  $G_j$  containing the segments of  $up_j$  which is specified by  $B(\{s, v\})$  (Fig. 8 a): The left and right sides of  $B(\{s, v\})$ ,  $\bar{s}$  and  $\bar{v}$  are both from  $low_j$ , which means  $s$  and  $v$  are both on  $L_j$ . As the points on  $L_j$  and  $U_j$  are alternating (Property 2),  $B(\{s, v\})$  contains one more point on  $U_j$  than on  $L_j$  inside. Moreover  $V_j$  contains *all* segments extending from those points on  $U_j$ , as there are no segments from  $low_j$  inside  $B(\{s, v\})$  and otherwise there would be a pair of neighboring points  $(x, y)$  inside  $B(\{s, v\})$  such that  $\pi_{xy} \not\subseteq U \cup V_j$ . Thus, we can exchange all segments from  $up_j$  inside  $B(\{s, v\})$  for all segments from  $low_j$  inside  $B(\{s, v\})$ , which is one segment less (Fig. 8 b). This contradicts the minimality of  $V_j$ .

Second, assume  $V_j$  contains a parity change and a non-induced segment. Then the non-induced segment induces a parity change in the following sense: Let  $s$  be the non-induced segment,  $u$  the next point of  $A_j$  to the left and  $v$  the next point of  $A_j$  to the right of  $s$  in  $x$ -order (see Fig. 9). W.l.o.g.  $u$  lies on  $L_j$  and  $v$  on  $U_j$ . Then  $s$  only contributes to  $\pi_{uv}^* \in U \cup V_j$ . Thus neither  $\bar{u} \in V_j$  nor  $\bar{v} \in V_j$ . Otherwise  $s$  would be superfluous and contradict the minimality of  $V_j$ .

Now let  $t \in A_j$  be the next point to the left of  $u$  and  $w \in A_j$  the next point to the right of  $v$ . As we have seen above, there cannot be any more non-induced segment in  $V_j$  ( $\bar{s}'$  or  $\bar{s}''$  in Fig.9),  $\bar{t} \in V_j$  and  $\bar{w} \in V_j$ . Otherwise there would be no  $\pi_{tu}$  and  $\pi_{vw}$  in  $U \cup V_j$  (see Fig. 9). Now neither the structure nor the number of locally needed segments changes, if  $s$  is shifted to the left or to the right until  $s = \bar{u}$  or  $s = \bar{v}$ . In both cases, we either have a second parity change and thus again case 1 or  $u$  or  $v$  coincide with corner points from  $E_j$  and thus  $s$  has been superfluous from the beginning. This again contradicts the minimality of  $V_j$ .

□

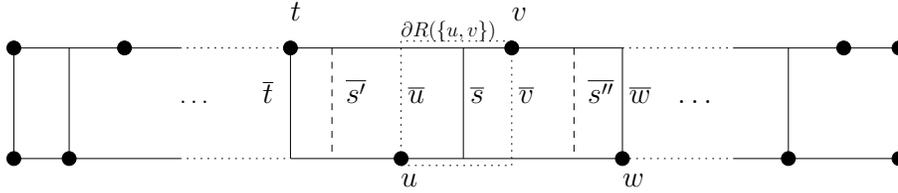


Fig. 9. Non-induced segment  $\bar{s}$  causes a parity change.

Each parity change causes the later derived truth assignment to be inconsistent. Thus we must avoid any parity change, but with Lemma 2 the derived truth value from the whole construction for each variable can still be inconsistent. Yet we show that there is always at least one part of our construction that has the desired property and neither contains a non-induced segment nor a parity change.

**Definition 5** First we define the eight corner points  $b_1^k, \dots, b_8^k$  of a cuboid  $cub^k, 0 \leq k \leq m + 1$  as follows:

$$\begin{aligned} cub^k = \{ & b_1^k = (5kn, -1, 0), b_2^k = (5(k+1)n, -1, 0) \\ & b_3^k = (5(k+1)n, 3, 0), b_4^k = (5kn, 3, 0) \\ & b_5^k = (5kn, -1, m+1), b_6^k = (5(k+1)n, -1, m+1) \\ & b_7^k = (5(k+1)n, 3, m+1), b_8^k = (5kn, 3, m+1) \} \end{aligned}$$

Then let a block  $b^k, 1 \leq k \leq m + 1$  be defined as:  $b^k = B(P_\alpha) \cap B(cub^k)$ .

**Lemma 3** There exists a block  $b^* \in \{b^1, \dots, b^k\}$  such that no parity change and no non-induced segments occurs on the parts of all variable gadgets within  $b^*$ .

**PROOF.** Remember that  $[P_\alpha, R_\alpha]$  contains  $m+1$  copies of each clause gadget in the clause gadget order. With the definition of the blocks  $b^k$ ,  $1 \leq k \leq m+1$  we now “virtually” divide the whole instance  $[P_\alpha, R_\alpha]$  into parts of length  $5n$  such that every block  $b^k$  contains the copy  $G_i^k$  of any clause gadget. As we know from Lemma 2, we can only have one parity change *or* one non-induced segment on each of the  $m$  variable gadgets, but we constructed  $m+1$  blocks  $b^k$ . Thus it is impossible to scatter the  $m$  possible parity changes/non-induced segments over all blocks and we obtain at least one block  $b^*$  containing the clause gadgets  $G_i^*$  without parity changes and non-induced segments.  $\square$

Any truth assignment derived only from parities of the sets  $V_j$  in  $b^*$  must consequently be consistent by Lemma 3.

Now we determine the minimal length of the last set  $C_M$ , which consists of parts of the paths  $\pi_{p_i^k q_i^k}^*$ . As they are mutually geometrically independent, we consider each  $\pi_{p_i^k q_i^k}^*$  separately. The length of each  $\pi_{p_i^k q_i^k}^*$  comprises of the distances between  $p_i^k$  and  $q_i^k$  in x- y- and z-direction. By definition of a  $L_1$ -shortest path:  $|\pi_{p_i^k q_i^k}^*| = 2 + 4 + (m+1)$  and by the definition of  $C_M$  we only count line segments  $s \in \pi_{p_i^k q_i^k}^*$  for  $\min_C$  if  $s \notin U \cup V_M$ . Thus, apart from its pure length, we also have to consider the structure, i.e. geometric location, of such a path as we did for  $\pi_{uv}^*$ ,  $\{u, v\} \in A_j \times A_j$ .

Generally  $\pi_{p_i^k q_i^k}^*$  can only contain segments located inside  $G_i^k$ , which is spatially bounded by  $\partial B(P_i^k)$ . As  $U$  already contains the  $2m$  line segments  $L_j, U_j, 1 \leq j \leq m$  which cross each clause gadget completely in x-direction,  $\pi_{p_i^k q_i^k}^*$  can run along one of these segments to bridge the  $x$ -distance between  $p_i^k$  and  $q_i^k$ . This will be the case in an optimal solution, as on the one hand the total length of the solution gets smaller. On the other hand, the paths contributing to  $C_M$  cannot influence  $U$  in turn, because the line segments of  $U$  are fixed by unique shortest paths.

In  $y$ -direction  $\pi_{p_i^k q_i^k}^*$  has two possible courses: There may be a vertical segment  $s \in V_M$  located inside  $G_i^k$  or not (see Fig. 10). In the first case,  $\pi_{p_i^k q_i^k}^*$  can run along  $s$ , such that only half of the  $y$ -distance between  $p_i^k$  and  $q_i^k$  has to be bridged by line segments in  $C_M$ . Thus the length of  $s$  does not contribute to  $\min_C$ . In the second case,  $\pi_{p_i^k q_i^k}^*$  can still run along  $\pi_{r_i^k, s_i^k}^* \in U$ , which is also not counted for  $\min_C$ . But this only saves  $C_M$  a line segment of length 1. Therefore, in any optimal solution,  $\pi_{p_i^k q_i^k}^*$  will always run along  $s \in V_M$  located in  $G_i^k$ , if such a segment exists (Fig. 10a) and along  $\pi_{r_i^k, s_i^k}^* \in U$  if not (Fig.

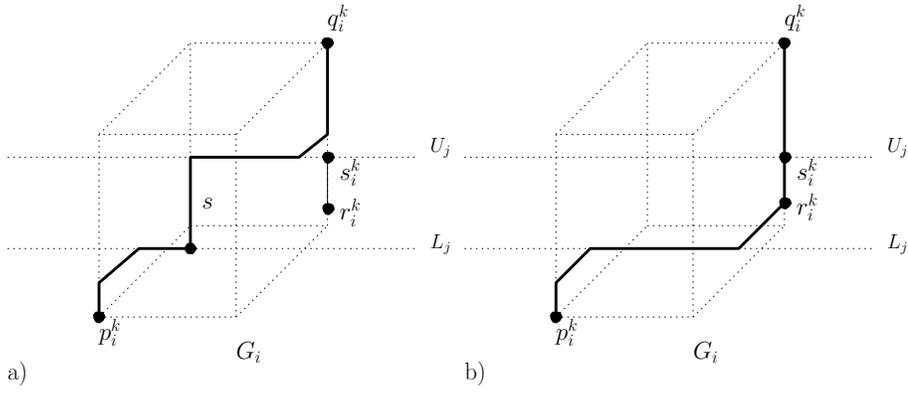


Fig. 10. A path  $\pi_{p_i^k, q_i^k}$ .

10b). In turn  $C_M$  may influence the position of segments in  $V_M$ , but not their number, as an additional segment in  $V_M$  costs a length of 2, whereas it only saves a length of 1 in  $C_M$ , taking the use of  $\pi_{r_i^k, s_i^k}^*$  into account. This leads to the following property which will be essential in the final step of the reduction.

**Property 3**  $|C_M| = \min_C \Leftrightarrow \forall G_i^k \exists s \in V : s \text{ is located in } G_i^k$ .

Since neither  $U$  nor  $V_M$  contains any line segment in  $z$ -direction, because the respective point pairs all share the same  $z$ -coordinate,  $C_M$  has to contain such segments of length  $m+1$  for each  $\pi_{p_i^k, q_i^k}^*$  in any solution and the choice of those line segments is independent from  $U$  and  $V_M$ .

Finally each  $C_M$  must afford no line segments in  $x$ -direction, at least line segments of total length 2 in  $y$ -direction and  $m+1$  in  $z$ -direction for each  $\pi_{p_i^k, q_i^k}^*$ , which determines  $\min_C$  to:

$$\min_C = 2(m+1)n + (m+1)^2n. \quad (6)$$

This completes the computation of the lower bound  $c_\alpha$  to the length of an optimal solution  $\text{TMMSN}([P_\alpha, R_\alpha])$  and by equations 4, 5 and 6 we obtain:

$$c_\alpha = \min_U + \min_V + \min_C \quad (7)$$

$$= 10(m+1)mn + 4m + (m+1)n + 6(m+1)n \quad (8)$$

$$+ 2(m+1)n + (m+1)^2n \quad (9)$$

$$= (m+1)^2n + 10(m+1)mn + 9(m+1)n + 4m. \quad (10)$$

**|TMMSN([P<sub>α</sub>, R<sub>α</sub>])| attending the lower bound.** Let  $M$  be an optimal solution  $\text{TMMSN}([P_\alpha, R_\alpha])$  and  $V = V_M$ ,  $C = C_M$ . In the above paragraph we have determined the lower bound  $c_\alpha$  on  $|M|$ , which is attended, if the length of each of the sets  $U$ ,  $V$  and  $C$  equals its minimum length  $\min_U, \min_V, \min_C$  over

all  $M$ . For  $U$  and  $V$ , this is possible for every instance  $[P_\alpha, R_\alpha]$ , as  $\min_U$  and  $\min_V$  only depend on  $n$  and  $m$ . But with Property 6  $|C|$  also depends on the geometric structure of  $V \subset \text{TMMSN}([P_\alpha, R_\alpha])$ . The latter again depends on the actual position of the segment inducing points  $\bigcup_{1 \leq j \leq m} A_j$ , which is chosen due to the placement rule, i.e. due to the occurrence of literals in clauses of  $\alpha$ . We show:

**Theorem 4**  $|\text{TMMSN}([P_\alpha, R_\alpha])| = c_\alpha \Leftrightarrow \alpha$  is satisfiable.

**PROOF.**

$|\text{TMMSN}([P_\alpha, R_\alpha])| = c_\alpha \Rightarrow \alpha$  is satisfiable. From  $|\text{TMMSN}([P_\alpha, R_\alpha])| = c_\alpha$  and Property 1 follows that  $|U| = \min_U$ ,  $|V| = \min_V$ ,  $|C| = \min_C$ . With Property 3 there must be a segment  $\bar{s} \in V$  located in every clause gadget  $G_i^k$ , especially in every clause gadget  $G_i^*$ . Considering  $b^*$  from Lemma 3,  $\bar{s}$  must be an induced segment, as there are no non-induced segments in  $b^*$ . Let  $V_j^*$  be the subset of segments of  $V_j$  located inside  $b^*$ , then the following truth assignment  $\nu$  is well defined and consistent, as no parity changes or induced segments occur inside  $b^*$ .

- $V_j^* = \text{low}_j \Rightarrow \nu(x_j) = 1$ .
- $V_j^* = \text{up}_j \Rightarrow \nu(x_j) = 0$ .

Let  $\bar{s}$  located in  $G_i^*$  be induced by a point  $s \in A_j$  then  $s$  is on  $L_j$ , if variable  $x_j$  occurs in clause  $c_i$  of  $\alpha$  and on  $U_j$ , if  $\neg x_j$  occurs in  $c_i$  due to the placement rule. Thus  $\bar{s} \in \text{low}_j$  if  $x_j$  occurs in  $c_i$  and  $\bar{s} \in \text{up}_j$  if  $\neg x_j$  occurs in  $c_i$ . With the above truth assignment  $\nu(x_j) = 1$  if  $x_j$  occurs in  $c_i$  and  $\nu(x_j) = 0 \Leftrightarrow \nu(\neg x_j) = 1$  if  $\neg x_j$  occurs in  $c_i$ . In both cases  $c_i$  is satisfied by  $\nu$ . With Property 3 this holds for every clause of  $\alpha$  and thus  $\nu$  satisfies  $\alpha$ .

$|\text{TMMSN}([P_\alpha, R_\alpha])| = c_\alpha \Leftarrow \alpha$  is satisfiable. If  $\alpha$  is satisfiable, there exists a truth assignment  $\nu$  such that every clause of  $\alpha$  is satisfied, i.e. for each clause  $c_i$  there exists one variable  $x_j$  occurring in  $c_i$  such that  $\nu(x_j)$  satisfies  $c_i$ . We now construct  $\text{TMMSN}([P_\alpha, R_\alpha])$  by choosing the line segments of the sets  $V$  and  $C$  partly depending on  $\nu$ . ( $U$  is fixed by  $[P_\alpha, R_\alpha]$ .) Choose  $V = \bigcup_{1 \leq j \leq m} V_j$  as follows:

- $\nu(x_j) = 1 \Rightarrow V_j = \text{low}_j$ .
- $\nu(x_j) = 0 \Rightarrow V_j = \text{up}_j$ .

Thus  $|V| = \min_V$  is minimal due to Lemma 1. Moreover,  $C$  can also be chosen with minimum length: If  $x_j$  ( $\neg x_j$ ) occurs in  $c_i$ , a point  $s$  is placed inside  $G_i^k$  on  $L_j$  ( $U_j$ ). Let  $x_j$  be the variable satisfying  $c_i$  under  $\nu$ , i.e.  $\nu(x_j) = 1$  ( $\nu(\neg x_j) = 1$ ), if  $x_j$  ( $\neg x_j$ ) occurs in  $c_i$ . Then by the placement rule and the definition of  $V$  above the induced segment  $\bar{s} \in V_j \subset V$  is located inside  $G_i^k$ . This holds for every  $G_i^k$ ,  $1 \leq i \leq n$ ,  $1 \leq k \leq m + 1$ , and a line segment  $\bar{s} \in V$

and by Property 3  $|C| = \min_C$ . Thus the sets  $U, V, C$  can be chosen such that they are of minimum length and  $|\text{TMMSN}([P_\alpha, R_\alpha])|$  attains the lower bound  $c_\alpha = \min_U + \min_V + \min_C$ .

□

Since the construction of the instance  $[P_\alpha, R_\alpha]$  contains  $4 + 2(m + 1)\text{occ}(x_j)$  points for each variable  $x_j$  and  $4(m + 1)n$  points for each clause  $c_i$  of  $\alpha$  and  $m \leq 3n$ , we need  $4m + 2(m + 1)3n + 4(m + 1)n^2 \leq 12n + 18(n + 1)n + 12(n + 1)n^2 \in O(n^3)$  points. All points having integer coordinates within the range of their number, the construction is polynomial in the input size. Thus together with Theorem 4 we have proved the *NP*-hardness of the TMMSN decision problem. As the TMMSN decision problem also is in *NP* we state:

**Proposition 5** *The TMMSN decision problem is NP-complete.*

## 4 Conclusion

We have introduced the TMMSN problem, a generalization of the MMN, but not the RSMA problem. We have proved the TMMSN problem to be *NPO*-complete in three dimensions, which is the complexity first result regarding  $> 2$  dimensions.

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