Compact and Extended Formulations for Range Assignment Problems*

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Abstract. We devise two new integer programming models for range assignment problems arising in wireless network design. Building on an arbitrary set of feasible network topologies, e.g., all spanning trees, we explicitly model the power consumption at a given node as a weighted maximum over edge variables. We show that the standard ILP model is an extended formulation of the new models. For all models, we derive complete polyhedral descriptions in the unconstrained case where all topologies are allowed. These results give rise to tight relaxations even in the constrained case. We can show experimentally that the compact formulations compare favorably to the standard approach.

Key words: wireless networks, range assignment, submodular functions

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1 Introduction

Range assignment problems play a central role in the design of ad hoc wireless networks. When assigning transmission ranges to the nodes of the network, it is a natural objective to minimize the overall energy that is needed to ensure certain connectivity properties of the network. In traditional wired networks, the transmission costs are roughly proportional to the length of all connections installed, so that the aim is to minimize the total length of all connections. On the other hand, in wireless networks, the transmission costs depend on the range assigned to the node. The main difference lies in the so-called *multi-cast advantage*: if a node v reaches another node w, then it also reaches each node u that is closer to v than w, at no additional cost.

Recently, several approaches to range assignment problems have been presented, comprising both exact algorithms [8, 9] and heuristics [1, 2, 4]. Exact approaches are often implemented by a branch-and-cut framework. The main task here is to model the problem as an integer linear program. It is natural to model the underlying networks by binary edge variables, where the restrictions on the network topology are expressed by linear constraints. However, other than in the wired case, the power consumption is not a linear function in the edge variables any more: the power consumption in a node v is now determined by the furthest node w that can be reached from v, i.e., by the most expensive edge vw that is adjacent to v in the network. In other words, the objective function can now be expressed as a sum of weighted maxima, each one taken over the variables corresponding to edges adjacent to a given node v.

In any integer programming approach to range assignment problems, the objective function thus has to be linearized. This is usually done by discretizing the power consumption [1, 2, 4]: obviously, in any optimal solution the range assigned to v equals the distance of v to some other node w. If a binary variable is introduced for each potential range, together with the constraint that at most one of these variables is one, the objective function becomes linear.

In this paper, we consider two more natural ways to address the nonlinear objective function: we either introduce a continuous variable y_v for each node v, modeling directly the range assigned to v, or we introduce a single continuous variable y representing the whole objective function. We show that the standard model can be considered an extended formulation of both these models.

We are mainly interested in a comparison of all three models, under arbitrary constraints on the network topologies. We first show that the two new models have positive polyhedral properties: even if they have an exponential number of facets in general, the separation problems can be solved in $O(m \log m)$ time, when m is the number of potential edges in the network. At the same time, the smaller number of variables compared to the standard model is clearly an advantage. We present results of an experimental evaluation of all three models for the min-power symmetric connectivity problem and the min-power multicast problem. These results show that the more compact models lead to faster running times in general on both problem classes.

In the next section, we describe all three models in more detail. In Section 3, we investigate the relation between these models. Their polyhedral properties in the unconstrained case are discussed in Section 4. Finally, in Section 5 we present the results of our experimental evaluation.

2 Models for Range Assignment Problems

We consider the following general formulation for range assignment problems:

$$\begin{array}{rl} \min & \sum_{u \in V} \max \{c_{uv} x_{uv} \mid uv \in E\} \\ \text{s.t.} & x \in X \ . \end{array} \tag{1}$$

Here we allow an arbitrary set $X \subseteq \{0,1\}^E$ of incidence vectors of edge-induced subgraphs of $G = (V, E)$. We assume however that an integer programming formulation for X is known. In wireless network problems, several sets X may be of interest, for example the set of all spanning trees in G or, more generally, the set of all spanning k-edge-connected subgraphs of G, for $k \geq 1$. In these cases G is an undirected graph. An example for range assignment problems on directed graphs is the multicast problem, where uni-directional connections between a source node and one or more terminal nodes have to be established.

The cost c_{vw} of an edge can be any nonnegative value. A typical assumption in wireless networking is that the cost for transmitting a signal directly from node v to node w is the square of the Euclidean distance between the two nodes, i.e., $c_{vw} = ||v - w||^2$. However, in practice the cost is not necessarily determined by the distance, so that it is useful to allow arbitrary values of c_{vw}.

In the following, we discuss different polyhedral approaches to Problem (1). Since the objective function in (1) is non-linear, the main modeling task is to choose a suitable linearization of the problem. Different choices may give rise to different polyhedral models, three of which we present and compare in the remainder of this section. Note that in the following we do not make any use of the underlying graph G. All results remain true if E is an arbitrary finite set, and if X corresponds to an arbitrary set of subsets of E.

2.1 The Standard Model

The standard approach to linearizing Problem (1) found in the wireless networking literature is to introduce new binary variables into the model [1, 2]. These variables model the possible values of the non-linear terms in optimal solutions. The resulting problem then reads

$$\begin{array}{lll} \min & \sum_{uv \in E} c_{uv} y_{uv} & \\ \text{s.t.} & \sum_{uv \in E} y_{uv} \leq 1 & \text{for all } u \in V \\ & \sum_{c_{uw} \geq c_{uv}} y_{uw} \geq x_{uv} & \text{for all } uv \in E \\ & x \in X & \\ & y \in \{0,1\}^E \ . & \end{array} \tag{2}$$

In this model, the binary variable y_{uv} is thus set to one if and only if the transmission power of node u is just enough to reach node v. Note that in many applications the first set of constraints consists of equations. This is valid if all feasible edge-induced subgraphs are connected. In this case, every node has to reach at least one other node. In general, this is not true, so that for some u all variables y_{uv} can be zero. The number of variables in this model is $2|E|$.

A closely related model appearing in the literature [5] uses binary variables in an incremental way: again, a variable $y'_{uv} \in \{0,1\}$ is used for every pair of nodes u and v, now set to one if and only if node u can reach node v. It is easy to see that the two models are isomorphic by the linear transformation

$$y'_{uv} = \sum_{c_{uw} \geq c_{uv}} y_{uw} \ .$$

Because of this, the two models are equivalent from a polyhedral point of view. Hence it suffices to consider the first model in the following.

2.2 A Mixed-Integer Model

A more direct way to derive a linear reformulation of Problem (1) is to introduce a single new variable y_v modeling the power consumption of every node v. In other words, y_v replaces the non-linear term $\max\{c_{vw}x_{vw} \mid vw \in E\}$. This leads to the following mixed-integer formulation:

$$\begin{array}{ll} \min & \sum_{v \in V} y_v \\ \text{s.t.} & y_v \geq \max\{c_{vw}x_{vw} \mid vw \in E\} \;\text{ for all } v \in V \\ & x \in X \\ & y \in \mathbb{R}^V \ . \end{array} \tag{3}$$

Compared to model (2), this formulation is more compact, it contains $|E| + |V|$ variables. The constraints in (3) are non-linear, but can be replaced by

$$y_v \geq c_{vw}x_{vw} \text{ for all } vw \in E \ .$$

Later, we will discuss a stronger reformulation of the non-linear constraints.

2.3 A Compact Mixed-Integer Model

An even more compact way to deal with the non-linear objective function in (1) is to replace the whole objective function by a single variable $y \in \mathbb{R}$:

$$\begin{array}{ll} \min & y \\ \text{s.t.} & y \geq \sum_{v \in V} \max\{c_{vw}x_{vw} \mid vw \in E\} \\ & x \in X \\ & y \in \mathbb{R} \ . \end{array} \tag{4}$$

This model contains only $|E| + 1$ variables. Again, we will show later that the non-linear constraints in (4) can be modeled by a well-behaved class of linear inequalities.

3 Polyhedral Relations

In the following, we investigate the polyhedral properties of the different models presented in Section 2. First, we show how the corresponding polyhedra are related to each other. For this, let $P_1(X)$, $P_2(X)$, and $P_3(X)$ denote the polyhedra given as the convex hulls of feasible solutions in the models (2), (3), and (4), respectively. In particular, we have

$$\begin{aligned} &P_1(X) \subseteq [0,1]^E \times [0,1]^E, \\ &P_2(X) \subseteq [0,1]^E \times \mathbb{R}^V, \text{ and} \\ &P_3(X) \subseteq [0,1]^E \times \mathbb{R}\,. \end{aligned}$$

Note that $P_1(X)$ is a convex hull of binary vectors, so in particular it is a polytope and all its integral points are vertices. On the other hand, the polyhedra $P_2(X)$ and $P_3(X)$ are unbounded by definition. It is easy to see that $P_2(X)$ arises from the convex hull of

$$\left\{(x,y) \in X \times \mathbb{R}^V \mid y_v = \max\left\{c_{vw}x_{vw} \mid vw \in E\right\}\right\}$$

by adding arbitrary non-negative multiples of unit vectors for the variables y_v. Similarly, $P_2(X)$ arises from the convex hull of

$$\left\{(x,y) \in X \times \mathbb{R} \mid y_v = \sum_{v \in V} \max\left\{c_{vw}x_{vw} \mid vw \in E\right\}\right\}$$

by adding arbitrary non-negative multiples of the unit vector for y.

Theorem 1. *The convex hull of all vertices of $P_2(X)$ is a projection of an integer subpolytope of $P_1(X)$.*

Proof. Consider the projection π_1 given by

$$y_v = \sum_{vw \in E} c_{vw} y_{vw}\,.$$

Let $(x,y) \in X \times \mathbb{R}^V$ be a vertex of $P_2(X)$. Then $y_v = \max\left\{c_{vw}x_{vw} \mid vw \in E\right\}$ for all $v \in V$. Thus setting $y_{vw} = 1$ for exactly one w with $y_v = c_{vw}$ yields a vertex of $P_1(X)$ that is mapped to (x,y) under π_1. □

Note that in general $P_1(X)$ contains vertices that are not mapped to the convex hull of vertices of $P_2(X)$. These vertices cannot be optimal for any of the considered objective functions.

Theorem 2. *The polyhedron $P_3(X)$ is a projection of the polyhedron $P_2(X)$.*

Proof. It is easily checked that the projection π_2 given by $y = \sum_{v \in V} y_v$ yields the desired result. □

These results show that for every reasonable objective function the optimal faces of all three polyhedra are projections of each other. The first model can thus be considered an extended formulation of the second and third one, and the second model can be considered an extended formulation of the third one.

4 Polyhedral Properties in the Unconstrained Case

For general $X \subseteq \{0,1\}^E$, Problem (1) is known to be NP-hard. This is true, e.g., when X models all connected subgraphs of G [3]. In this case, Problem (1) specializes to the so-called min-power symmetric connectivity problem.

In particular, optimizing over any polyhedron $P_1(X)$, $P_2(X)$, or $P_3(X)$ is NP-hard for general X. However, in the unconstrained case $X = \{0,1\}^E$ we can give complete polyhedral descriptions of all three polytopes. Combining these descriptions with an arbitrary linear description of X, one obtains three different LP-relaxations of Problem (1). All three relaxations are much tighter in general than a standard LP-relaxation.

Let $P_i = P_i(\{0,1\}^E)$ for $i = 1,2,3$. First note that the LP-relaxation of the extended formulation (2) is integer in the unconstrained case:

Theorem 3. *All vertices of the polytope*

$$\begin{aligned} P' = \Big\{(x,y) \mid & \sum_{uv \in E} y_{uv} \leq 1 \text{ for all } u \in V, \\ & \sum_{c_{uw} \geq c_{uv}} y_{uw} \geq x_{uv} \text{ for all } uv \in E, \\ & 0 \leq x \leq 1,\ 0 \leq y \leq 1\Big\} \end{aligned}$$

are integer. In particular, the polytope P' coincides with P_1.

Proof. Consider the isomorphism ϕ defined by $y'_{uv} = \sum_{c_{uw} \geq c_{uv}} y_{uw}$. Under ϕ, the polytope P' is mapped onto

$$\begin{aligned} \phi(P') = \Big\{(x,y') \mid & \; y'_{uv} \geq y'_{uw} \text{ for all } uv, uw \in E \text{ with } c_{uv} \leq c_{uw}, \\ & \; y'_{uv} \geq x_{uv} \text{ for all } uv \in E, \\ & \; 0 \leq x \leq 1,\ 0 \leq y' \leq 1\Big\}, \end{aligned}$$

which is the so-called incremental model. Since $\phi(P')$ is defined by a totally unimodular matrix, all its vertices are integer. As ϕ^{-1} preserves integrality, the result follows. $\square$

Theorem 3 implies that optimizing an objective function with non-negative costs over any of the polytopes P_1, P_2, and P_3 is possible in polynomial time. In the remainder of this section, we will show that we can actually separate from P_2 and P_3 in $O(|E| \log |E|)$ time. We need the following general result:

Theorem 4. *Let $E = \{1, \ldots, n\}$ and let $f \colon 2^E \to \mathbb{R}$ be a submodular function with $f(\emptyset) \geq 0$. Then the separation problem for the polyhedron*

$$P_f = \operatorname{conv}\{(x,y) \in \{0,1\}^E \times \mathbb{R} \mid y \geq f(x)\}$$

can be solved in $O(n \log n)$ time. The facets of P_f are either induced by trivial inequalities $0 \leq x_i \leq 1$, $i \in E$, or by an inequality $y \geq a^\top x$ with

$$a_i = f(S_i) - f(S_{i-1}) \text{ for all } i = 1, \dots, n$$

where $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E$.

Theorem 5. *For every $v \in V$ and for arbitrary $c \in \mathbb{R}^E$, the function*

$$f_v(x) = \max \{c_{vw} x_{vw} \mid vw \in E\}$$

is submodular. In particular, the function $f(x) = \sum_{v \in V} f_v(x)$ is submodular.

Proof. By definition, f_v is submodular if

$$f_v(A \cup B) + f_v(A \cap B) \leq f_v(A) + f_v(B)$$

for arbitrary sets $A, B \subseteq E$. We distinguish two cases:

(a) if $f_v(A) \geq f_v(B)$, then $f_v(A \cup B) = f_v(A)$ and $f_v(A \cap B) \leq f_v(B)$
(b) if $f_v(A) \leq f_v(B)$, then $f_v(A \cup B) = f_v(B)$ and $f_v(A \cap B) \leq f_v(A)$

In both cases, it follows that $f_v(A \cup B) + f_v(A \cap B) \leq f_v(A) + f_v(B)$. Finally, the function $f(x) = \sum_{v \in V} f_v(x)$ is submodular, because it is a positive sum of submodular functions [6]. □

Corollary 1. *The separation problem for P_3 can be solved in $O(|E| \log |E|)$ time.*

The separation algorithm for P_3 obtained from these results proceeds as follows: Given a fractional point $(x^\star, y^\star) \in [0,1]^E \times \mathbb{R}$, sort the elements of E in non-increasing order according to their value in $x^\star$. Starting with the empty set, iteratively construct a chain of subsets $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E$ by adding the elements in this order. The potentially violated inequality $y \geq a^\top x$ is then constructed by setting $a_i = f(S_i) - f(S_{i-1})$.

Our next aim is to show that Corollary 1 also holds for P_2. First note that Theorem 5 yields a complete description of the polytope P_{f_v}, for each $v \in V$. For the following, define

$$P = \bigcap_{v \in V} P_{f_v} \ ,$$

where each P_{f_v} is trivially extended from $\{0,1\}^E \times \mathbb{R}$ to $\{0,1\}^E \times \mathbb{R}^V$. Then

$$P_2 = \operatorname{conv}\left(\left(\{0,1\}^E \times \mathbb{R}^V \right) \cap P \right) .$$

It thus remains to show that the vertices of P have integer x-entries. In other words, the facets of P_2 are precisely the facets of the single polyhedra P_{f_v}.

Lemma 1. *For each $vw \in E$, the intersection $P_{f_v} \cap \{x_{vw} = 0\}$ is again of the form P_g with $g(x) = \max \{d_{vu} x_{vu} \mid vu \in E\}$, for suitable d.*

Proof. Set $d_{vw} = 0$ and $d_{vu} = c_{vu}$ for $vu \in E$ with $u \neq w$. $\square$

Lemma 2. *For each $vw \in E$, the intersection $P_{f_v} \cap \{x_{vw} = 1\}$ is a translation of a polyhedron P_g with $g(x) = \max\{d_{vu}x_{vu} \mid vu \in E\}$, for suitable d.*

Proof. Set $d_{vu} = \max\{c_{vu} - c_{vw}, 0\}$ for all $vu \in E$. Then $P_{f_v} \cap \{x_{vw} = 1\}$ is the translation of P_g by the vector $(0, c_{vw})$. $\square$

Lemma 3. *If $(x, y) \in P$ with $x \in (0,1)^E$, then (x, y) is not a vertex of P.*

Proof. We first observe that $(x, y) \in P$ implies $(x, y) \in P_{f_v}$ for all $v \in V$. Choose $\varepsilon > 0$ such that $0 \leq x_{vw} \pm \varepsilon \leq 1$ for all $vw \in E$. Denote by $\mathbf{1}$ the vector $(1, \ldots, 1) \in \mathbb{R}^E$. Define

$$\bar{c}_v = f_v(E) = \max\{c_{vw} \mid vw \in E\} .$$

We claim that the points $z_1 = (x - \varepsilon\mathbf{1}, y - \varepsilon\bar{c})$ and $z_2 = (x + \varepsilon\mathbf{1}, y + \varepsilon\bar{c})$ belong to P. It suffices to show that they belong to P_{f_v} for all $v \in V$.

Fix $v \in V$. The choice of ε makes sure that z_1 and z_2 satisfy the trivial inequalities $0 \leq x_{vw} \leq 1$. It is easy to see that the remaining facet-inducing inequalities of Theorem 4 also hold for z_1 and z_2: for a given inequality $a^\top x \leq y_v$, we have

$$a^\top(x \pm \varepsilon\mathbf{1}) = a^\top x \pm \varepsilon a^\top \mathbf{1} \leq y_v \pm \varepsilon a^\top \mathbf{1} = y_v \pm \varepsilon f_v(E) \mp \varepsilon f_v(\emptyset) = y_v \pm \varepsilon \bar{c}_v .$$

So $(x, y) = \frac{1}{2}(z_1 + z_2)$ and $z_1, z_2 \in P$, thus (x, y) cannot be a vertex of P. $\square$

Corollary 2. *The vertices of P are exactly the points*

$$\left\{(x, y) \in \{0,1\}^E \times \mathbb{R}^V \mid y_v = \max\{c_{vw}x_{vw} \mid vw \in E\}\right\} .$$

In particular, we have $P = P_2$.

Proof. It is clear that every such point is a vertex of P. We show that every vertex (x', y') of P is of this form. Since y_v is not bounded from above, every vertex must satisfy $y_v = \max\{c_{vw}x_{vw} \mid vw \in E\}$. Now assume that at least one component of x' is fractional. Consider

$$\begin{aligned} S &= \{(x, y) \in P \mid x_{vw} = 1 \text{ if } x'_{vw} = 1, \text{ and } x_{vw} = 0 \text{ if } x'_{vw} = 0\} \\ &= \bigcap_{v \in V} \{(x, y) \in P_{f_v} \mid x_{vw} = 1 \text{ if } x'_{vw} = 1, \text{ and } x_{vw} = 0 \text{ if } x'_{vw} = 0\} . \end{aligned}$$

By Lemma 1 and Lemma 2, the polyhedron S is isomorphic to an intersection of polytopes P_g with $g(x) = t^g + \max\{d^g_{vu}x_{vu} \mid vu \in E\}$ for suitable d^g, t^g. Applying Lemma 3 to this intersection, it follows that (x', y') is not a vertex of S. Consequently, (x', y') is not a vertex of P. $\square$

Corollary 3. *The separation problem for P_2 can be solved in $O(|E| \log |E|)$ time.*

The separation algorithm for P_2 that is obtained from these results proceeds as follows: A sequence of subsets is constructed, as in the separation algorithm for P_3. Then, for every node $v \in V$, a potentially violated inequality $y_v \geq a^\top x$ is constructed independently.

5 Experimental Results

After having compared the general range assignment models (2), (3), and (4) from a polyhedral point of view, we next present the results of an experimental comparison. In these experiments, we consider two different classes of network topologies, arising in the symmetric connectivity and in the multicast scenario, respectively; see Section 5.1 and Section 5.2. In Section 5.3, we describe the framework of our experimental evaluation. Finally, the results of the evaluation are presented and discussed in Section 5.4.

5.1 Min-Power Symmetric Connectivity

So far all results presented were valid for arbitrary network topologies X. If G is an undirected graph and X is the set of all connected subgraphs of G, the range assignment problem (1) specializes to the so-called *min-power symmetric connectivity problem*, which has been studied extensively [2, 4, 9].

In this case, all three IP-formulations (2), (3), and (4) can be significantly strengthened. First of all, the set X can be restricted to the set of spanning trees in G without loss of generality. This is equivalent to introducing an additional constraint $\sum_{e \in E} x_e = |V| - 1$. The model remains correct: even if there may be several edges adjacent to any given node, only the adjacent edge with the largest weight contributes to the objective function. Thus, for each cycle C in a connected subgraph, at least one edge of C does not have any influence on the objective function value. When this edge is removed, the objective value does not change and the graph remains connected. Removing edges iteratively, we finally end up with a spanning tree without having changed the value of the objective function. In particular, this stronger formulation does not change the optimum of our problem, but improves the quality of the bounds obtained from the LP-relaxations and thus reduces running time.

Another way to strengthen the model is related to the fact that in a connected subgraph (on at least two nodes) each node has at least one adjacent edge. For the standard model, this means that the constraints

$$\sum_{uv \in E} y_{uv} \leq 1 \text{ for all } u \in V$$

can be strengthened to equations

$$\sum_{uv \in E} y_{uv} = 1 \text{ for all } u \in V\,.$$

In the mixed-integer models (3) and (4), we can effectively eliminate one variable from every maximum term. As the transmission power for each node v has to be at least the smallest weight $c_v^{\min}$ of the adjacent edges, this constant can be extracted from the corresponding maximum term. The constraints

$$y_v \geq \max\{c_{vw} x_{vw} \mid vw \in E\} \text{ for all } v \in V$$

of model (3) become

$$y_v \geq c_v^{\min} + \max\{(c_{vw} - c_v^{\min})x_{vw} \mid vw \in E\} \text{ for all } v \in V\,,$$

so that at least one entry in the maximum can be removed. Analogously, in the compact model (4), the constraint that bounds the overall transmission power from below,

$$y \geq \sum_{v \in V} \max\{c_{vw}x_{vw} \mid vw \in E\}\,,$$

can be strengthened to

$$y \geq \sum_{v \in V} \left(c_v^{\min} + \max\{(c_{vw} - c_v^{\min})x_{vw} \mid vw \in E\}\right).$$

Both replacements lead to stronger LP-relaxations if the separation algorithms derived in Section 4 are now applied to the remaining maximum terms.

5.2 Min-Power Multicast

In the min-power symmetric connectivity problem, a link between two nodes u and v is established if the transmission power of node v is large enough to reach u and vice versa. If signals must be transmitted from a source to one or more receiving nodes and it is not necessary that the recipients can send signals back to the source, this can be modeled with a directed graph G as follows: one of the nodes is a designated source node s and every other node of G is either a terminal or a relay node. All nodes can receive and transmit signals. The task is to establish uni-directional connections from s to every terminal. These connections correspond to directed paths from s to the terminals that may use one or more relay nodes. The single paths do not have to be disjoint.

The resulting problem is called the *min-power multicast problem* [8, 9]. As special cases, it includes the *min-power unicast problem*, where only one terminal node exists, and the *min-power broadcast problem*, where all nodes except for the source are terminals. All these problems can again be modeled in the general framework (1). In this case, all edges of G are directed and X is the set of all Steiner-arborescences in G.

The given connectivity constraints can again be used to strengthen the three different formulations, however to a lesser extent than in the symmetric case. Relay nodes do not need to have any incoming or outgoing edges. Terminals need to have at least one incoming edge, but this does not have a direct influence on any maximum term. Only the fact that at least one edge has to leave the source node provides a way to strengthen the models: in the standard model, the constraint

$$\sum_{sv \in E} y_{sv} \leq 1$$

can be strengthened to

$$\sum_{sv \in E} y_{sv} = 1\,,$$

while the maximum

$$\max\{c_{sw}x_{sw} \mid sw \in E\}$$

can be replaced by

$$c_s^{min} + \max\{(c_{sw} - c_v^{min})x_{sw} \mid sw \in E\}$$

in both compact models.

5.3 The Experimental Framework

To compare the performance of the three models for range assignment problems discussed in this paper, we focus on the two problems introduced in the previous sections, the *min-power symmetric connectivity problem* and the *min-power multicast problem*. We implemented straightforward branch-and-cut algorithms for these problems that are based on the polyhedral results of Section 4. For the implementation, we used the optimization tool SCIL [7]. For both problem classes, we used the stronger formulations described in Section 5.1 and Section 5.2.

For the min-power symmetric connectivity problem, we modeled the spanning tree constraint by adding the equation $\sum_{e \in E} x_e = |V| - 1$ statically and by separating subtour elimination constraints dynamically. We enforced the Steiner arborescence constraint in the min-power multicast problem using

$$\begin{aligned} x(V, v) &= 1 \quad \text{for all } v \in T && (5)\\ x(V, v) &\geq 1 \quad \text{for all } v \in V \setminus (T \cup \{s\}) && (6)\\ x(V \setminus S, S) &\geq 1 \quad \text{for all } S \subseteq V \setminus \{s\} \text{ with } S \cap T \neq \emptyset \text{ and } |S| \geq 2\,, && (7)\end{aligned}$$

where s is the source node, $T \subseteq V \setminus \{s\}$ is the set of terminal nodes, and $x(A, B)$ denotes the sum of those variables corresponding to edges with source node in $A \subseteq V$ and target node in $B \subseteq V$. Constraints (5) and (6) were added at the beginning of the optimization process, constraints of type (7) were separated.

In the standard model, the maximum constraints are explicitly given by the inequalities containing the y variables. For the mixed-integer models, we used the inequalities and separation routines described in Section 4.

To ensure the comparability of all three models, no other enhancements, such as preprocessing or primal heuristics, were implemented. For the same reason, we did not separate any model or problem specific constraints as, e.g., the crossing inequalities introduced by Althaus et al. in [1].

We tested all algorithms on instances that were generated as proposed in [1]. For a given number n, the n nodes were placed on a 10000×10000 grid randomly. In this way, 50 instances of each size n were generated. The edge weights, i.e., the costs for establishing a direct link between two given nodes v and w, were chosen as $c_{vw} = ||v - w||^2$, where $||v - w||$ denotes the Euclidean distance between v and w. To cover all possible scenarios for the min-power multicast problem, we considered the cases of transmitting from a source to a single terminal (unicast), to $k = \lfloor \frac{n}{2} \rfloor$ terminals, or to all other nodes (broadcast).

All tests were done on 2.33GHz Intel Xeon PCs. We set a cpu time limit of 10 hours per instance. Instances that could not be solved to optimality within this time limit were not considered in the computation of averages in the following statistics.

5.4 Results

The results of our tests are shown in Table 1. For each model, the number of instances that could be solved to optimality is shown in the first column, followed by the average number of subproblems in the branch-and-bound tree, the average number of linear programs solved during the optimization, the average running time, and the percentage of time spent on separating inequalities. The upper part of the table reports the results for the min-power symmetric connectivity problem, the lower part those for the min-power multicast problem. Here the very first column not only contains the total number of nodes, but also the number of terminals for each class of instances.

Table 1 shows that, when solving very small instances of the min-power symmetric connectivity problem, the mixed-integer models do not perform considerably better than the standard model. The average computation time needed by the compact mixed-integer model is even significantly higher than that of the other two models. This also holds true for larger instances and is due to the very large number of linear programs that need to be solved in this model. Each separation step for the maximum constraint in (4) yields at most one violated inequality, while in model (3) up to $|V|$ cutting planes are found per iteration, aside from inequalities coming from network topology constraints.

For larger instances, the mixed-integer model gains a clear advantage over the standard approach. More instances can be solved in time and considerably less subproblems are needed on average. The time spent on separation is only slightly higher than in the standard model, showing that the separation procedure described in Section 4 is indeed fast and effective.

When solving instances of the min-power multicast problem, the dominance of the mixed-integer models is even more apparent. Despite its drawbacks described in the previous paragraph, the compact model works better than the standard approach for the unicast and multicast instances with up to 15 nodes. For larger instances, running times rise sharply, but in terms of the number of solved instances the compact mixed-integer model still compares favorably to the standard model.

The second model is obviously the model of choice for range assignment problems on directed graphs. We could solve all but one instance within the time limit, whereas with the standard approach only 42 multicast and 43 broadcast instances could be solved to optimality for $n = 20$. Also running times were generally shorter, in some cases significantly, which is also reflected in the lower average number of subproblems needed.

	compact mixed-integer model (4)					mixed-integer model (3)					standard model (2)				
$n/\lvert T\rvert$	solved	#sub	#LP	avg t/s	%sep	solved	#sub	#LP	avg t/s	%sep	solved	#sub	#LP	avg t/s	%sep
10	50	42.56	1259.70	0.77	40.87	50	37.52	73.58	0.11	14.83	50	76.00	65.18	0.13	9.13
15	50	2551.92	275403.24	594.41	45.97	50	1776.28	3827.56	55.87	11.64	49	1354.10	1261.33	10.00	10.09
20	39	1690.38	565548.82	2661.91	44.40	44	3645.09	10085.43	253.23	8.23	41	8433.15	8045.54	240.40	6.65
10/1	50	29.16	316.36	0.15	21.25	50	24.64	60.32	0.05	5.38	50	201.16	268.30	0.33	1.57
10/4	50	68.08	745.14	0.47	35.15	50	48.64	105.26	0.12	12.04	50	364.28	414.48	0.82	7.17
10/9	50	81.56	919.54	0.79	45.09	50	68.52	141.96	0.23	14.98	50	203.72	203.34	0.44	11.73
15/1	50	67.20	3956.82	4.43	25.49	50	40.80	160.58	0.31	10.54	50	976.92	2010.72	9.56	6.39
15/7	50	266.80	18713.22	34.54	39.01	50	230.72	768.44	2.45	16.44	50	2322.04	2655.68	48.01	9.71
15/14	50	413.16	28133.62	69.33	46.91	50	362.52	1246.94	5.36	22.62	50	3598.36	3560.64	46.01	11.51
20/1	50	235.64	87074.96	649.63	26.46	50	136.84	687.96	2.60	7.55	50	2757.64	6193.02	61.11	3.59
20/9	48	2469.46	563046.83	2272.49	41.08	50	3756.44	16030.14	535.49	16.63	42	12746.24	16391.24	536.40	6.73
20/19	48	3008.88	679026.58	3913.96	49.78	49	2619.90	13361.43	135.99	25.01	43	9184.77	10114.53	316.92	10.86

Table 1. Experimental results for the min-power symmetric connectivity problem (top) and the min-power multicast problem (bottom)

6 Conclusion

We introduced two natural models for general range assignment problems and compared them with the standard model. From the theoretical point of view, the new models have two advantages: the number of variables is smaller and the corresponding polyhedra have less vertices. On the other hand, a separation algorithm has to be applied in order to arrive at the same LP-relaxation that is given by a single LP in the standard model. Experimentally, the new models lead to faster algorithms both in the symmetric connectivity case and in the directed multicast case.

We would like to point out that our approach is in fact much more general: we can not only deal with general weighted maxima in the objective function that do not necessarily arise from network design problems, but we can even allow arbitrary submodular objective functions. In other applications, the number of possible values of each y_v-variable may be much larger than $|E|$, so that the discretization approach may become infeasible from a practical point of view, while the efficiency of the new models does not depend on the number of useful choices for y_v.

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