Characterizing Simultaneous Embedding with Fixed Edges

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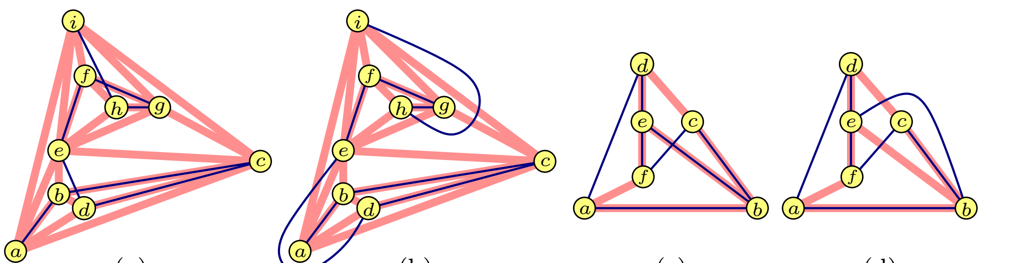
Abstract. A set of planar graphs share a simultaneous embedding if they can be drawn on the same vertex set V in the plane without crossings between edges of the same graph. Fixed edges are common edges between graphs that share the same Jordan curve in the simultaneous drawings. While any number of planar graphs have a simultaneous embedding *without* fixed edges, determining which graphs always share a simultaneous embedding *with* fixed edges (SEFE) has been open. We partially close this problem by giving a necessary condition to determine when pairs of graphs have a SEFE.

1 Introduction

In many practical applications including the visualization of large graphs and very-large-scale integration (VLSI) of circuits on the same chip, edge crossings are undesirable. A single vertex set can suffice in which multiple edge sets correspond to different edge colors or circuit layers. While the union of any pair of edge sets may be nonplanar, a planar drawing of each layer may still be possible, as crossings between edges of distinct edge sets is permitted. This corresponds to the problem of *simultaneous embedding* (SE).

Without restrictions on the types of edges used, this problem is trivial since any number of planar graphs can be drawn on the same fixed set of vertex locations [6]. However, difficulties arise once straight-line edges are required. Moving one vertex to reduce crossings in one layer can introduce additional crossings in other layers. This is the problem of *simultaneous geometric embedding* (SGE). If edge bends are allowed, then having common edges drawn the same way is important for mental map preservation. Such edges are called *fixed edges* leading to the problem of *simultaneous embedding with fixed edges* (SEFE). Since straight-line edges between a pair of vertices are also fixed, any graph that has a SGE also has a SEFE, but the converse is not true; see Fig. 1.

Deciding if two graphs have a SGE is NP-hard [2], whereas, deciding if three graphs have a SEFE is NP-complete [4]. However, deciding if two graphs have a SEFE in polynomial-time remains open. In this paper we present a necessary condition in terms of forbidden subgraphs for whether pairs of graphs always have a SEFE. While this does not yet lead to a polynomial-time decision algorithm in the general case, it does in the more restricted case of pairs of outerplanar graphs. Additionally, we characterize which pairs of planar graphs and which pairs of outerplanar graphs have a SEFE and provide simultaneous drawing algorithms when possible.



(a) (b) (c) (d)

Fig. 1. The path and planar graph in (a) do not have a SGE with straight-line edges [1], but but have a SEFE in (b). The two outerplanar graphs in (c) do not have a SEFE, but have a SE in (d) if edge (b, e) is not fixed.

* This work is supported in part by NSF grants CCF-0545743 and ACR-0222920.
** This work is supported in part by the German Science Foundation (JU204/11-1) and by the German Academic Exchange Service.

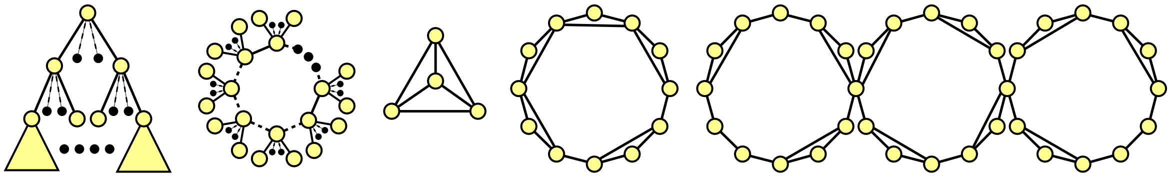


Fig. 2. Forests in (a), circular caterpillars in (b), and subgraphs of K_4 in (c) have a SEFE with any planar graph. K_3-cycles as in (d) and outerplanar graphs composed of cubic K_3-cycles as in (e) have a SEFE with any outerplanar graph.

1.1 Our Contribution

We omit the proofs of claims for the following results in this extended abstract, but details can be found in the technical report [3].

1. While most pairs of graphs whose union forms a subdivided K_5 or $K_{3,3}$ share a SEFE, we give 16 minimal forbidden pairs that do not. This gives a necessary condition for SEFE of two graphs.
2. Using this condition we show that the only graphs that always have a SEFE with any planar graph are either (i) forests, (ii) circular caterpillars (removal of all degree-1 vertices leaves a cycle), and (iii) subgraphs of K_4; see Fig. 2(a)–(c). We also show that the the only outerplanar graphs that always share a SEFE with any outerplanar graph either are (i) biconnected in which the endpoints of every chord are at a distance of two from each along the outerface (K_3-cycle) or (ii) have a cut vertex in which no two chords can be incident in the same biconnected component (each biconnected subgraph is a cubic K_3-cycle); see Fig. 2(d)–(e). In each case, we provide $O(n^2 \lg n)$ simultaneous drawing algorithms.
3. By comparing all possible subsets of vertices in a pair of outerplanar graphs against the two forbidden outerplanar pairs, we have a necessary condition for SEFE. We show that this condition also suffices, giving a $O(n^2)$ time decision algorithm for SEFE of two outerplanar graphs.

2 Forbidden Simultaneous Embeddings with Fixed Edges

We start by stating the seminal theorem by Kuratowski [5] that characterizes all planar graphs.

Theorem 1 (Kuratowski) *Every nonplanar graph has a subgraph homeomorphic to K_5 or $K_{3,3}$.*

Suppose a pair of graphs $G_1(V, E_1)$ and $G_2(V, E_2)$ does not have a SEFE as in Fig. 3(a). If deleting any edge from either graph allows a SEFE, then G_1 and G_2 are *edge minimal* as is Fig. 3(b). If a degree-2 vertex v (adjacent to u and w) in the union is not a degree-1 vertex in either G_1 or G_2, then we can *unsubdivide* the vertex by deleting v and replacing edges (u, v) and (v, w) with the edge (u, w) in G_1 and/or G_2. A pair of graphs for which this can no longer be done is *vertex minimal* as is Fig. 3(c). A *minimal forbidden pair* does not have a SEFE and is edge and vertex minimal. An *alternating edge* is a $u \rightsquigarrow v$ path in which the edges strictly alternate between being in either G_1 and G_2, but not both. An *exclusive edge* is a $u \rightsquigarrow v$ path composed of the single edge (u, v) only in G_1 or G_2, while an *inclusive edge* is composed of the single edge (u, v) in the intersection.

Claim 2 *Any pair of graphs $G_1(V, E_1)$ and $G_2(V, E_2)$ can be reduced to a pair in which every $u \rightsquigarrow v$ path is either an inclusive edge, an exclusive edge or an alternating edge.*

Any pair of graphs for which this has been done for all $u \rightsquigarrow v$ paths is called a *reduced pair.* Suppose G_1 and G_2 are a reduced pair. The *alternating edge subgraph*, denoted $G_1 \uplus G_2$, is the subgraph of $G_1 \cup G_2$ consisting only of alternating edges. The *exclusive edge subgraph* of G_1, denoted $G_1 \setminus G_2$, is the subgraph of $G_1 \cup G_2$, of exclusive edges from G_1, where $G_2 \setminus G_1$ is defined analogously. Hence, edges of $G_1 \cup G_2$ are partitioned into $G_1 \uplus G_2$, $G_1 \setminus G_2$, $G_2 \setminus G_1$, and $G_1 \cap G_2$; see Fig. 3(c)–(g). Next we see why we only need to consider crossings between nonincident edges.

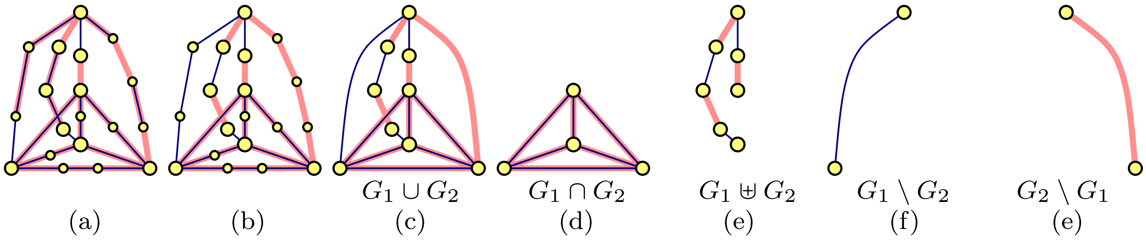


Fig. 3. Removing extra edges from (a) gives (b). Unsubdividing vertices gives (c) with the four subgraphs (d)–(g).

Observation 3 *Crossings in a nonplanar drawing between a pair of incident edges can be removed without affecting the number of crossings of nonincident edges.*

Applying this observation to K_5 and $K_{3,3}$ of Theorem 1 gives the following corollary.

Corollary 4 *(a) Every drawing of K_5 or $K_{3,3}$ has at least one crossing between nonincident edges. (b) Any K_5 and $K_{3,3}$ can be drawn with only one crossing between any pair of nonincident edges.*

We use this corollary to produce a sufficient condition for SEFE.

Lemma 5 *Suppose the union of a reduced pair G_1 and G_2 is homeomorphic to K_5 or $K_{3,3}$. Let $u \rightsquigarrow v$ and $x \rightsquigarrow y$ be nonincident paths in $G_1 \cup G_2$ but not in $G_1 \cap G_2$. G_1 and G_2 share a SEFE if either path belongs to $G_1 \uplus G_2$ or one belongs to $G_1 \setminus G_2$ and the other belongs to $G_2 \setminus G_1$.*

Applying Lemma 5, we next determine when a pair of graphs forming K_5 or $K_{3,3}$ has a SEFE.

Corollary 6 *Suppose the union of a reduced pair G_1 and G_2 is homeomorphic to K_5 or $K_{3,3}$. G_1 and G_2 do not share a SEFE if and only if (i) every nonincident edge of an alternating edge in $G_1 \uplus G_2$ is in $G_1 \cap G_2$ and (ii) every nonincident edge of an exclusive edge in $G_1 \setminus G_2$ is also in G_1.*

Using these two conditions we are able to show the following theorem.

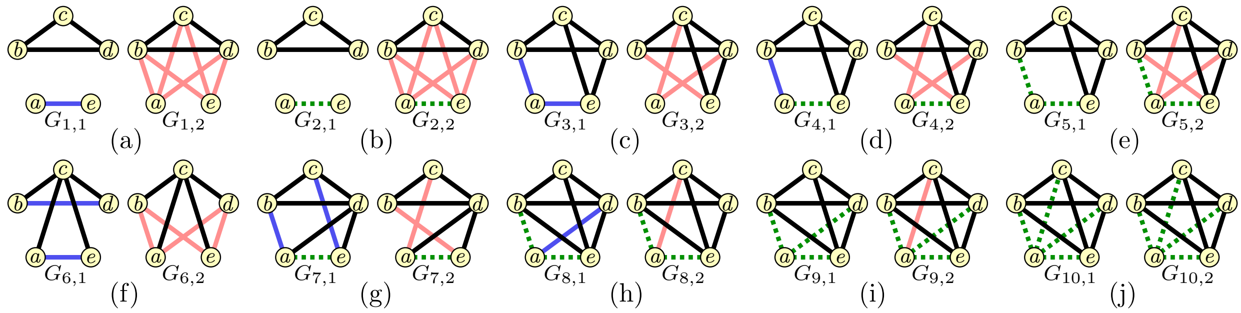


Fig. 4. Ten K_5 minimal forbidden pairs. The dark edges are in $G_1 \cap G_2$ and the dashed edges are in $G_1 \uplus G_2$.

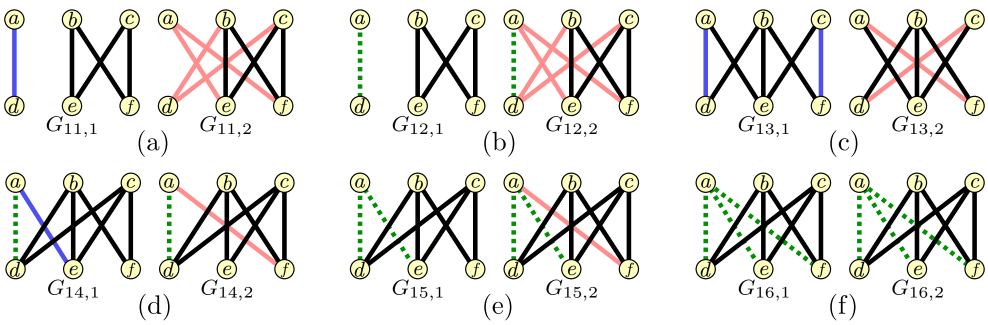


Fig. 5. Six $K_{3,3}$ minimal forbidden pairs. The dark edges are in $G_1 \cap G_2$ and the dashed edges are in $G_1 \uplus G_2$.

Theorem 7 *There are 16 minimal forbidden pairs whose union is homeomorphic to K_5 or $K_{3,3}$.*

We next consider when edge contraction does not change when a pairs of graphs has a SEFE.

Lemma 8 *Let G_1 and G_2 be a pair of graphs with a SEFE. Let G'_1 and G'_2 be the graphs after contracting edge (u, v) in $G_1 \cap G_2$ to vertex w. If there do not exist edges (u, x) in $G_1 \setminus G_2$ and (v, x) in $G_2 \setminus G_1$ (or vice versa), then G'_1 and G'_2 have a SEFE.*

We extend our definition of minimality to include edge contraction to give the next lemma.

Lemma 9 *The only minimal forbidden pairs are the 16 pairs homeomorphic to K_5 or $K_{3,3}$.*

3 Characterizing Simultaneous Embeddings with Planar Graphs

In this section we determine which graphs always have a SEFE with any planar graph and show how to produce a simultaneous drawing. Let $\mathcal{P}$ be the set of planar graphs and $\mathcal{P}_{\mathsf{SEFE}} \subset \mathcal{P}$ be the subset of forests, circular caterpillars and subgraphs of K_4.

Lemma 10 *The set $\mathcal{P}_{\mathsf{SEFE}}$ contains the graphs that can always have a SEFE with any planar graph.*

Graphs in $\mathcal{P}_{\mathsf{SEFE}}$ not only can, but always have a SEFE with any planar graph as we show next.

Theorem 11 *Pair $G_1 \in \mathcal{P}_{\mathsf{SEFE}}$, $G_2 \in \mathcal{P}$ on n vertices have a SEFE computable in $O(n^2 \lg n)$ time.*

4 Characterizing Simultaneous Embeddings with Outerplanar Graphs

We next determine which outerplanar graphs always have a SEFE with any other outerplanar graph and produce simultaneous drawings when possible. Let $\mathcal{O}$ be the set of outerplanar graphs and $\mathcal{O}_{\mathsf{SEFE}} \subset \mathcal{O}$ be the set of outerplanar graphs that consist of a single K_3-cycle or have a cubic K_3-cycle for each biconnected subgraph.

Lemma 12 *The set $\mathcal{O}_{\mathsf{SEFE}}$ contains the outerplanar graphs that can always have a SEFE with any outerplanar graph.*

We next show outerplanar graphs in $\mathcal{O}_{\mathsf{SEFE}}$ not only can, but do indeed have a SEFE with any outerplanar graph.

Theorem 13 *Pair $G_1 \in \mathcal{O}_{\mathsf{SEFE}}$, $G_2 \in \mathcal{O}$ on n vertices have a SEFE computable in $O(n^2 \lg n)$ time.*

An example layout is shown in Fig. 6 in the appendix. We conclude with our last theorem.

Theorem 14 *Deciding if a pair of outerplanar graphs have a SEFE can be done in $O(n^2)$ time.*

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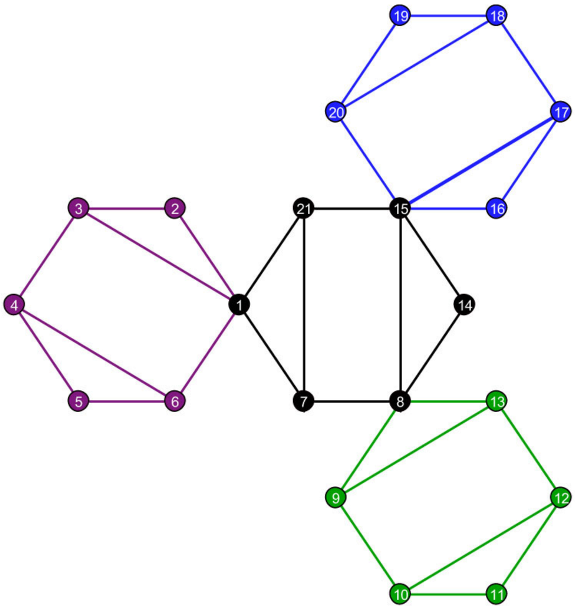
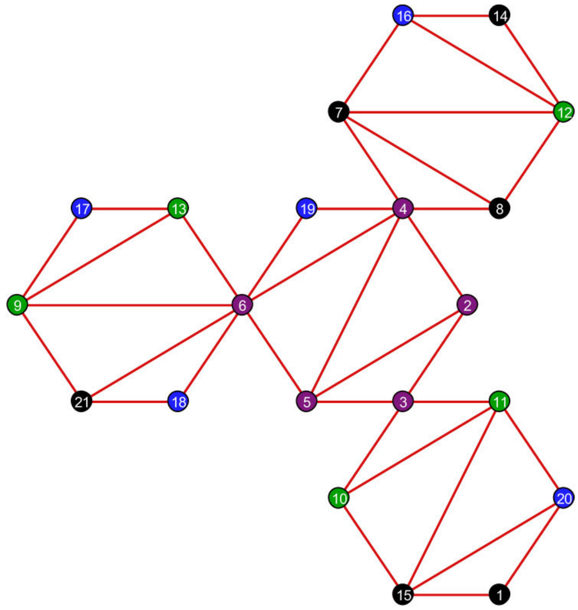
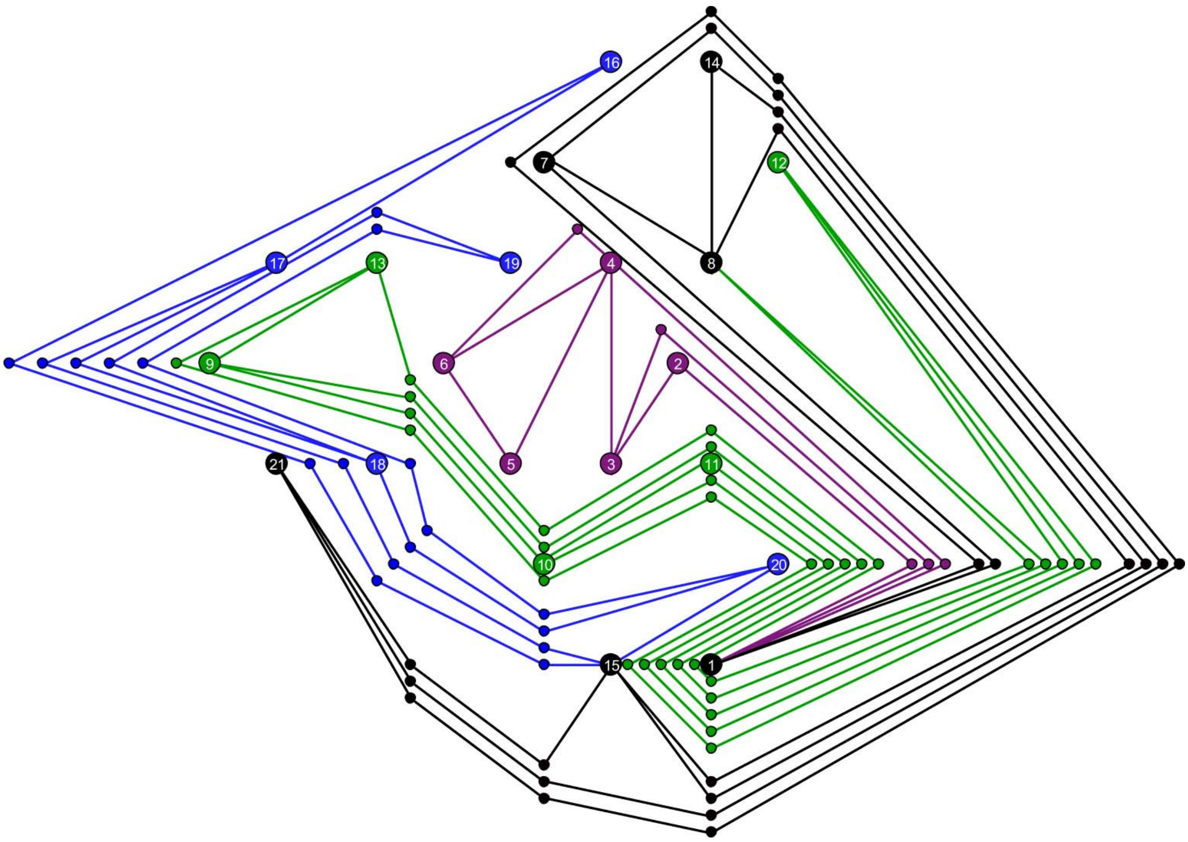
Appendix



(a) G_1



(b) G_2



(c) Planar drawing of G_1 with fixed edgess using vertex locations of G_2.

Fig. 6. SEFE of two outerplanar graphs. The edges of the four biconnected component of G_1 are colored distinctly.