The complexity of connected domination and total domination by restricted induced graphs

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Abstract

Given a graph class $\mathcal{G}$, it is natural to ask whether a given graph has a connected or a total dominating set inducing a graph in $\mathcal{G}$ and, if so, what is the minimal size of such a set. We give a sufficient condition on $\mathcal{G}$ for the intractability of this problem. This condition is fulfilled by a wide range of graph classes.

Keywords: connected domination, total domination, dominating subgraphs, computational complexity, combinatorial problems

1. Introduction

Domination problems are among the most studied topics in graph theory and combinatorial optimization today, partly due to their importance in location problems and network design. A very good introduction into the topic is given by Haynes, Hedetniemi and Slater [1]. One of the questions that have been posed in this area is if for a given graph a certain type of domination can be realized by a vertex set with additional properties. One example is acyclic domination, according to our knowledge first studied by Hedetniemi, Hedetniemi and Rall [2]. Here one asks for the minimal size of a dominating set which induces an acyclic subgraph. Clearly, each graph has a dominating set inducing an acyclic subgraph, e.g. an inclusionwise maximal independent set. The related concept of tree domination, as discussed for example by Chen, McRae, Sun [3] and Rautenbach [4], asks for the existence of a connected acyclic dominating set. An example of a graph having no connected acyclic dominating set is the net, i.e. the 1-corona of K_3. In fact, Rautenbach [4] showed that the corresponding decision problem is $\mathcal{NP}$-complete. Another example is the problem of dominating cliques where one is interested in the existence (and the minimal size) of a dominating set that induces a complete graph. The concept was introduced by Cozzens and Kelleher [5] in 1990 and it is now a well-studied problem (see [6, 7, 8] for recent developments and applications). Recently, Bacsó [9] and

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Tuza [10] independently gave a full characterization of the graphs such that every connected induced subgraph has a connected dominating set satisfying an arbitrary prescribed hereditary property. Their theorems settle a problem that was implicitely stated 20 years ago (for a history of the problem, see Tuza [10]).

Also for total domination, the structural properties of the subgraphs induced by the total dominating sets have been investigated in the literature. Among the papers concerned with this question are the following examples. Henning [11] studies total dominating sets containing given sets of vertices. Telle [12] investigates generalized dominating sets with additional degree conditions imposed on the dominating vertices. Goddard, Haynes and Knisley [13] study some classical graph parameters where the parameters are restricted to measure the size of sets whose induced subgraphs have certain hereditary properties. Finally, one of the authors [14] studies graphs that hereditarily have a total dominating set the induced subgraph of which satisfies certain prescribed additive hereditary properties.

In this paper, we deal with the algorithmic complexity of the decision problem and the minimal size problem of connected dominating and total dominating sets such that the induced subgraphs belong to a given graph class. We show that for a wide range of graph classes both problems are $\mathcal{NP}$-hard. In some cases $\mathcal{NP}$-hardness of the decision problem holds even if the instances are restricted to be $K_{1,5}$-free graphs. Similar, the minimal size problem often remains $\mathcal{NP}$-hard even if the instances are restricted to be bipartite graphs of maximum degree 4.

2. Preliminaries

We consider undirected, simple graphs G with vertex set $V(G)$ and edge set $E(G)$. For $X \subseteq V(G)$ the subgraph induced by X is denoted by $G[X]$. For a given graph G and $k \in \mathbb{N}$ the *k-corona* of G is the graph obtained from G by attaching a path of length k to each vertex of G. For example, the 0-corona of G is just G itself. A *pendant vertex* is a vertex with exactly one neighbor.

Let $G = (V, E)$ be a graph. A *dominating set* is a set $X \subseteq V$ such that each vertex of $V \setminus X$ is adjacent to at least one vertex of X. A *connected dominating set* is a dominating set X such that $G[X]$ is connected. A *total dominating set* is a dominating set X such that $G[X]$ has no isolated vertices. A graph without isolated vertices is called *isolate-free*.

A *hypergraph* $H = (V, E)$ is an ordered pair where E is a nonempty finite family of nonempty finite sets and $V = \bigcup E$. The elements of V are called *vertices* and the elements of E *hyperedges*. A *cover* of H is a set $C \subseteq E$ such that $\bigcup C = V$. It is a well-known $\mathcal{NP}$-complete problem to find for a given hypergraph H the smallest k such that H has a cover of cardinality k (see *set covering* in [16]). A *matching* of a hypergraph $H = (V, E)$ is a set $M \subseteq E$ such that $m \cap n = \emptyset$ for all $m \neq n \in M$. A matching M which is also a cover is a *perfect matching*. The *three dimensional matching problem* (see [16]) is as follows. Given a hypergraph $H = (V, E)$ with a partition $V = R \cup S \cup T$ with $|R| = |S| = |T| = p(H)$ for some $p(H) \in \mathbb{N}$ such that each hyperedge contains exactly one vertex of each block of the partitioning. The task is to decide if H has a matching consisting of p elements, i.e. a perfect matching. It is mentioned by Dyer and Frieze [17] that three dimensional matching remains $\mathcal{NP}$-complete even when restricted to those hypergraphs for which each vertex is contained in at most three hyperedges.

3. Existence of dominating sets with restricted induced graphs

All the graph classes and their inclusions we consider in this paper can be found in the book of Brandstädt, Le and Spinrad [15]. A simple observation establishes a first connection between recognition of graph classes and total domination:

Lemma 1. *Let $\mathcal{G}$ be a class of isolate-free graphs closed under deleting pendant vertices. If $\mathcal{G}$ is $\mathcal{NP}$-hard to recognize then the existence of a total dominating set inducing a graph of $\mathcal{G}$ is also $\mathcal{NP}$-hard.*

Proof. Obviously, an isolate-free graph G belongs to $\mathcal{G}$ iff its 1-corona has a total dominating set inducing a graph of $\mathcal{G}$. □

Since the majority of graph classes is recognizable in polynomial time, Lemma 1 is not really satisfying. To state our main results, we need the following concept. We call a graph class $\mathcal{G}$ *suitable* if there is a sequence $G_1, G_2, \dots$ of graphs in $\mathcal{G}$ with the following property. For all $k = 1, 2, \dots$ there is an independent set S_k in G_k of size $|S_k| = k$ such that for all for $S \subseteq S_k$ $G_k[V \setminus S]$ is connected and contained in $\mathcal{G}$. Furthermore, for fixed k, G_k and S_k can be computed in polynomial time.

Lemma 2. *Let $\mathcal{G}$ be suitable graph class. If for each $k \in \mathbb{N}$ and $x, y \in S_k$ no graph of $\mathcal{G}$ contains the graph obtained from $G_k[(V(G_k) \setminus S_k) \cup \{u, v\}]$ by adding the edge $\{u, v\}$ as induced subgraph, then the existence of a connected dominating or total dominating set inducing a graph of $\mathcal{G}$ is $\mathcal{NP}$-hard.*

Furthermore, if for each $k \in \mathbb{N}$ G_k is $K_{1,4}$-free and the neighborhood of each member of S_k forms a clique, then the two problems remain $\mathcal{NP}$-hard even if the instances are restricted to be $K_{1,5}$-free graphs.

Proof. Let $\mathcal{G}$, $G_1, G_2, \dots$ and $S_1, S_2, \dots$ fulfill the condition of the Lemma and let $H = (V, E)$ be a hypergraph with $|E| = k$ hyperedges. We construct in polynomial time a graph G which has a connected dominating set inducing a graph of $\mathcal{G}$ iff it has a total dominating set inducing a graph of $\mathcal{G}$ iff H has a perfect matching.

Compute G_k and S_k. Let $G_k = (V_k, E_k)$. Let $U_k = V_k \backslash S_k$ and let $\phi : E \to S_k$ be any bijective function. Let $P = \{p_u : u \in U_k\}$ be a disjoint copy of U_k. We define a graph G by

$$\begin{aligned} V(G) &= P \cup V_k \cup V, \\ E(G) &= \{\{p_u, u\} : u \in U_k\} \cup E_k \cup \{\{\phi(e), \phi(f)\} : e \neq f \in E, e \cap f \neq \emptyset\} \\ &\quad \cup \{\{\phi(e), v\} : v \in e \in E\}. \end{aligned}$$

G is obtained from G_k and H as follows: Each vertex $u \in U_k$ has a pendant vertex $p_u \in P$ attached. Further, S_k is identified with E (via ϕ) and then connected to V by the bipartite incidence graph of the hypergraph H.

Let $M \subseteq E$ be a perfect matching of H and let $X = U_k \cup \phi(M)$. We observe that X is a connected total dominating set of G, since the members of $P \cup S_k$ are dominated by U_k and the members of V are dominated by $\phi(M)$ (as M is in particular a cover of H). Further, $\phi(M)$ is an independent set since the elements of M are pairwise non-intersecting. Thus $G[X] = G_k[X]$. By definition, $G_i[V \setminus S] \in \mathcal{G}$ for any subset $S \subseteq S_k$. Hence $G_k[X] \in \mathcal{G}$ and so $G[X] \in \mathcal{G}$.

Now let G have a connected or total dominating set X such that $G[X] \in \mathcal{G}$. Since P is a set of pendant vertices, $U_k \subseteq X$. Assume there are vertices $x, y \in X \cap S_k$ adjacent in G. Then $G[U_k \cup \{x, y\}]$ is an induced subgraph of $G[X]$ obtained from $G_k[U_k \cup \{x, y\}]$ by adding the edge $\{x, y\}$. Therefore $G[X]$ does not belong to $\mathcal{G}$ by the assumption of the lemma, a contradiction. Let $Y = X \cap V_k$ and observe that $G[Y] = G_k[Y] \in \mathcal{G}$. Furthermore $G[Y]$ is connected, since $U_k \subseteq Y$ and $S_k \setminus Y$ does not separate G_k and thus not $G[Y]$. $G[Y]$ has no isolated vertices since it is connected and has at least one vertex in U_k and one in S_k. Thus Y is a connected total dominating set of G and $G[Y] \in \mathcal{G}$. Since $G[Y \cap S_k]$ is an independent set, the set of hyperedges $M = \phi^{-1}(Y \cap S_k)$ is pairwise non-intersecting by definition of $E(G)$. As $Y \cap S_k$ dominates V, M is a perfect matching in H.

Since the existence of perfect matchings in hypergraphs is $\mathcal{NP}$-complete, the existence of a connected or total dominating set inducing a graph of $\mathcal{G}$ is $\mathcal{NP}$-hard.

Now we assume that for each $k \in \mathbb{N}$ G_k is $K_{1,4}$-free and the neighborhood of each member of S_k forms a clique. We can choose H to be an instance of the three dimensional matching problem such that each vertex is contained in at most three hyperedges. Then the graph G from the above construction is seen to be $K_{1,5}$-free as follows. The only vertices for which this is not trivial are the members of S_k. Let $v \in S_k$. The neighbors of v can be covered by at most four cliques: The first clique are the neighbors in U_k. By choice of H, we can assume that v has three neighbors, say u_1, u_2, u_3, among V. The second clique is $N(u_1 \cup \{u_1\})$, the third is $N(u_2) \cup \{u_2\}$ and the fourth is $N(u_3) \cup \{u_3\}$. □

Note that if $\mathcal{G}$ admits a polynomial recognition algorithm, then $\mathcal{NP}$-completeness holds. We now give an (incomplete) list of examples for Lemma 2.

Theorem 1. *The existence of a connected or total dominating set inducing a graph of the following graph classes is $\mathcal{NP}$-hard, even if the instances are restricted to be $K_{1,5}$-free graphs:*

(a) *perfect graphs; Meyniel and Gallai graphs; (p, q)-chordal graphs, fixed $p \geq 4, q \geq 1$; weakly chordal graphs; doubly chordal graphs; strongly chordal graphs.*

(b) *parity graphs; distance-hereditary graphs; ptolemaic graphs.*

(c) *bipartite graphs; planar bipartite graphs; chordal bipartite graphs; bipartite permutation graphs; acyclic graphs.*

(d) *asteroidal-triple-free graphs; co-comparability graphs; trapezoid graphs; permutation graphs.*

(e) *(proper) interval graphs; (proper) unit interval graphs.*

(f) *(proper) circular arc graphs; (proper) unit circular arc graphs.*

(g) *unicyclic graphs; cacti.*

The existence of a connected or total dominating set inducing a graph of the following graph classes is $\mathcal{NP}$-hard:

(h) $K_{1,r}$*-free graphs, fixed* $r \geq 3$.

(i) r*-colorable graphs,* $r \geq 2$.

(j) *triangle-free graphs; complete bipartite graphs; stars* ($\{K_{1,n} : n \in \mathbb{N}\}$).

Proof. For each graph class we give polynomially computable sequences $G_1, G_2, \ldots$ and $S_1, S_2, \ldots$ which obviously fulfill the condition of Lemma 2.

(a) & (b) Note that all classes are C_{2r+1}-free for some $r \in \mathbb{N}$. For all k let G_k be the graph obtained as follows. Start with a K_k and subdivide each edge exactly once. Add all possible edges between the vertices that subdivide the former edges of K_k. Now, this graph has k vertices of degree $k-1$ (the vertices of the former K_k). Attach a path of length $r-1$ to each of these vertices. Let S_k be the set of end-vertices of these paths. Now, any two vertices $x, y \in S_k$ are connected by a path of length $2r$.

(c) Note that all classes do not allow odd cycles as subgraphs. For all $k \in \mathbb{N}$ let G_k be the complete binary tree with $2^{\lceil \log_2(k) \rceil}$ leaves. Let S_k be any k leaves of G_k.

(d) Note that all classes are asteroidal-triple-free and thus in particular $(7,1)$-chordal. For all $k \in \mathbb{N}$ let G_k be the graph obtained from P_{4k-3} by attaching a pendant vertex to the $4i+1$th vertex for all $0 \leq i \leq k-1$ and S_k the pendant vertices of G_k.

(e) For all $k \geq 2$ let G_k be the graph obtained from the path $P_{2k} = v_0 v_1 \ldots v_{2k-1}$ by adding vertices $s_0, \ldots, s_{k-1}$ such that $S_k = \{s_0, \ldots, s_{k-1}\}$ is an independent set and s_i is adjacent to v_j iff $j-1 \leq 2i \leq j$. Let $G_1 = G_2$ and $S_1 = \{s_0\}$.

(f) For all $k \geq 2$ let G_k be the graph obtained from the cycle $C_{2k} = v_0 v_1 \ldots v_{2k-1} v_0$ by adding vertices $s_0, \ldots, s_{k-1}$ such that $S_k = \{s_0, \ldots, s_{k-1}\}$ is an independent set and s_i is adjacent to v_j iff $j-1 \leq 2i \leq j$. Let $G_1 = G_2$ and $S_1 = \{s_0\}$.

(g) For all $k \geq 3$ let G_k be the 1-corona of C_k and S_k the set of pendant vertices of G_k. Let $G_1 = G_2 = G_3$, S_1 contain a single pendant vertex and S_2 contain exactly two pendant vertices.

(h) For all $r \geq 3$ and $k \in \mathbb{N}$ let G_k be the graph obtained from k disjoint copies of $K_{1,r-2}$ and the complete graph $K_{k(r-1)}$ by adding a perfect matching from the vertices of the disjoint copies of $K_{1,r-2}$ to the vertices of $K_{k(r-1)}$. Let S_k be the roots of the disjoint copies of $K_{1,r-1}$.

(i) Note that for all $r \geq 2$, the r-colorable graphs do not include K_{r+1}. For all $k \in \mathbb{N}$ let G_k be the graph obtained from K_r by adding k non-adjacent twins of a fixed vertex and S_k the set of added twins.

(j) Note that all classes are triangle-free. For all $k \in \mathbb{N}$ let $G_k \cong K_{1,k}$ and S_k be the leaves of G_k. $\square$

4. Minimal dominating sets with restricted induced graphs

Theorem 2. *Computing the minimal size of a connected dominating or total dominating set inducing a graph of a given suitable graph class $\mathcal{G}$ is $\mathcal{NP}$-hard if restricted to the class of graphs for which such set exists.*

Proof. Let $\mathcal{G}$, $G_1, G_2, \ldots$ and $S_1, S_2, \ldots$ fulfill the condition of the theorem and let $H = (V, E)$ be a hypergraph with $|E| = k$ hyperedges. We construct in polynomial time a graph G which has a connected total dominating set X such that $G[X] \in \mathcal{G}$ and furthermore the size of a minimal connected or total dominating set equals the size of a minimal cover of H (up to a constant).

Compute $G_k = (V_k, E_k)$ and $S_k \subseteq V_k$. Let $U_k = V_k \setminus S_k$ and let $\phi : E \to S_k$ be any bijective function. Let $P = \{p_u : u \in U_k\}$ be a disjoint copy of U_k. We define a graph G by

$$
\begin{aligned}
V(G) &= P \cup V_k \cup V, \\
E(G) &= \{\{p_u, u\} : u \in U_k\} \cup E_k \cup \{\{\phi(e), v\} : v \in e \in E\}.
\end{aligned}
$$

Let X be any connected or total dominating set of G. It is easy to see that X necessarily contains U_k to dominate P. Since G_k is connected and no $S \subseteq S_k$ separates G_k the same holds for $G[V_k] = G_k$ and thus S_k is dominated by U_k. Since V is an independent set in G, V is dominated by $X \cap S_k$ and thus $Y = X \cap V_k$ is a connected total dominating set of G and furthermore $\phi^{-1}(Y \setminus U_k)$ is a cover of H. Therefore a minimal cover of H contains at most $|Y| - |U_k|$ hyperedges.

If $C \subseteq E$ is a cover of H, then, since $G(V_k)$ is connected and $\phi(C) \setminus S_k$ does not separate $G[X] = G_k[U_k \cup \phi(C)]$, $X = U_k \cup \phi(C)$ is a connected dominating set of G. As $X \cap U_k \neq \emptyset$ and $X \cap S_k \neq \emptyset$, $G[X]$ consists of at least two vertices and thus X is also a total dominating set. Furthermore $G[X] \in \mathcal{G}$, since $G[X] = G[U_k \cup \phi(C)] = G_k[U_k \cup \phi(C)]$ and $G_k[U_k \cup \phi(C)] \in \mathcal{G}$ by definition. Thus a minimal connected or total dominating set of G inducing a graph of $\mathcal{G}$ contains at most $|C| + |U_k|$ vertices.

Hence, the minimal size of a connected or total dominating set of G inducing a graph of $\mathcal{G}$ equals the minimal size of a cover of H up to the constant $|U_k|$. Since determining the minimal size of a cover in a hypergraph is $\mathcal{NP}$-complete, determining the minimal size of a connected or total dominating set inducing a graph of $\mathcal{G}$ is $\mathcal{NP}$-hard. $\square$

Instead of giving a list of suitable graph classes, we give a corollary which has a less general condition but a stronger consequence.

Corollary 1. *Let $\mathcal{G}$ be a graph class which contains the 1-corona of P_k for all $k \in \mathbb{N}$. Then computing the minimal size of a connected dominating or total dominating set inducing a graph of $\mathcal{G}$ is $\mathcal{NP}$-hard if restricted to the class of bipartite graphs of maximum degree four for which such set exists.*

Proof. For all $k \in \mathbb{N}$ let G_k be the 1-corona of $P_{2k} = v_0 v_1 \ldots v_{2k-1}$ and S_k the set of pendant vertices of G_k whose neighbor v_i has even index. As described above, the hypergraph H in the proof of Theorem 2 can be chosen as an instance of three dimensional matching. Clearly the graph G constructed in the proof has a total or connected dominating set of size $k + p(H)$ iff H has a three dimensional matching. Furthermore the constructed graph is bipartite and has maximum degree 4. □

Note that since all classes mentioned in Theorem 1 satisfy the condition of Lemma 2 they satisfy the condition of Theorem 2, too. Furthermore the class of planar and outerplanar graphs are also covered by Corollary 1.

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