On the existence of total dominating subgraphs with a prescribed additive hereditary property

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Abstract

Recently, Bacsó and Tuza gave a full characterization of the graphs for which every connected induced subgraph has a connected dominating subgraph satisfying an arbitrary prescribed hereditary property. Using their result, we derive a similar characterization of the graphs for which any isolate-free induced subgraph has a total dominating subgraph that satisfies a prescribed additive hereditary property. In particular, we give a characterization for the case where the total dominating subgraphs are disjoint union of complete graphs. This yields a characterization of the graphs for which every isolate-free induced subgraph has a vertex-dominating induced matching, a so-called induced paired-dominating set.

Keywords: total domination, dominating subgraphs, hereditary properties, dominating cliques

1. Introduction

For any graph G, $V(G)$ denotes its set of vertices and $E(G)$ denotes is set of edges. A *dominating set* of a graph G is a subset $X \subseteq V(G)$ such that each vertex in $V(G) \setminus X$ has a neighbor in X. There is a lot of literature dealing with the concept of domination problems. An introduction into the field of domination in graphs is the book by Haynes, Hedetniemi and Slater [1]. Among the many variants of domination is the concept of total domination. A *total dominating set* X of a graph G is a vertex subset that each vertex of G has a neighbor in X. In the following, we use the term subgraph for subgraphs induced by vertex subsets only. Denoting by $G[X]$ the subgraph induced by X, X is total dominating if X is dominating and $G[X]$ does not have isolated vertices. We say that $G[X]$ is *isolate-free.* If X is a total dominating set, we call $G[X]$ a *total dominating subgraph* of G. Note that any isolate-free graph has a total dominating set. According to our knowledge, total dominating sets were introduced and first studied by Cockayne, Dawes and Hedetniemi [2]. There

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is now a huge amount of papers dealing with this topic, see Henning [3] for a recent survey.

Another variant of domination is connected domination. A *connected dominating set* X is a dominating set such that $G[X]$ is connected. $G[X]$ is called a *connected dominating subgraph* of G. Recently, Bacsó [5] and Tuza [6] independently gave a full characterization of the graphs for which every connected induced subgraph has a connected dominating subgraph satisfying an arbitrary prescribed hereditary property. Their Theorems settle a problem that was implicitely stated 20 years ago (for a history of the problem, see Bacsó [5] or Tuza [6]). Let D be a class of connected graphs. Tuza [6] defines $Dom(D)$ as the class of connected graphs for which every connected subgraph H has a connected dominating subgraph that is isomorphic to a member of D. For example, $Dom(\{K_n : n \in \mathbb{N}\})$ is the set of connected graphs such that any connected subgraph has a dominating clique. The *leaf graph* $F(G)$ of a graph G is the graph obtained from G by attaching a pendant vertex to any vertex which is not a cut vertex. For an example of a leaf graph, see Figure 1. Leaf graphs play a central role in the characterization of $Dom(\mathcal{G})$, since if G is a connected graph, any connected dominating subgraph of $F(G)$ induces G as subgraph. Denoting by P_n (C_n) the path (cycle) on n vertices, Tuza [6] (and independently Bacsó [5]) showed the following.

Theorem 1 (Tuza [6]). *Let D be a nonempty class of connected graphs closed under taking connected subgraphs. The minimal forbidden subgraphs of $Dom(D)$ are the cycle C_{t+2} if $P_t \notin D$ but $P_{t-1} \in D$ and the leaf graphs of the minimal forbidden subgraphs of D.*

For example, if $\mathcal{T}$ is the class of trees, i.e. the set of connected graphs not containing a cycle, then $Dom(\mathcal{T})$ is the set of connected graphs which are $\{F(C_k) : k \geq 3\}$-free. This result was previously discovered by Rautenbach [7].

In this paper, we aim for a characterization similar to Theorem 1 considering total domination. If $\mathcal{G}$ is a graph class we denote by $Total(\mathcal{G})$ the set of isolate-free graphs for which every isolate-free subgraph H has a total dominating subgraph T that is isomorphic to some member of $\mathcal{G}$. We say T is contained in $\mathcal{G}$, for short. Note that $Total(\mathcal{G})$ is, in some sense, the total domination equivalent to $Dom(\mathcal{G})$. Like in Theorem 1, we restrict our attention to graph classes that are hereditary, i.e. that are closed under taking induced subgraphs. Since total dominating subgraphs need not to be connected, we further restrict the graph classes to be additive, i.e. closed under disjoint union of graphs. Among the most prominent additive hereditary graph classes are acyclic graphs, chordal graphs, perfect graphs and disjoint unions of complete graphs. Note that the minimal forbidden subgraphs of an additive hereditary graph class are connected, a fact which should be kept in mind throughout the paper. Furthermore, we say that an additive hereditary graph class is *non-trivial* if it is non-empty and contains K_2. This is not a real restriction, since the only non-empty additive hereditary graph class that is not non-trivial is the class of graphs without any edges. Up to the fact that a graph with isolated vertices does not have a total dominating set, we can treat the set $Total(\mathcal{G})$ as an additive hereditary graph class itself. In this sense, we say that an isolate-free graph is a forbidden subgraph of $Total(\mathcal{G})$ if it is not contained in $Total(\mathcal{G})$, but each of its isolate-free proper subgraphs is. We want to characterize $Total(\mathcal{G})$ in terms of minimal forbidden subgraphs, for arbitrary non-trivial additive hereditary properties $\mathcal{G}$. A first application of our results can be found in [8].

2. Auxiliary results

For our first observation, we need the following concept. Let G be a graph. The *corona* of G, denoted by $Cr(G)$, is obtained from G by attaching a pendant vertex to any vertex of G. For an example of a corona graph, see Figure 1. We observe that corona graphs play an important role in the characterization of $Total(\mathcal{G})$: If G is an isolate-free graph, any total dominating subgraph of $Cr(G)$ contains G as subgraph. This observation leads us to the following Lemmas:

Lemma 1. *Let $\mathcal{G}$ be a non-trivial additive hereditary graph class. If F is a minimal forbidden subgraph of $\mathcal{G}$, the corona of F is a minimal forbidden subgraph of $Total(\mathcal{G})$.*

Proof. Let $\mathcal{G}$ be a non-trivial additive hereditary graph class and let F be a minimal forbidden subgraph of $\mathcal{G}$. Since any total dominating subgraph T contains F as subgraph, we have $Cr(F) \notin Total(\mathcal{G})$. On the other hand, any isolate-free proper subgraph of $Cr(F)$ has a total dominating subgraph which is the disjoint union of proper subgraphs of F and thus is contained in $\mathcal{G}$, by choice of F. Hence, any isolate-free proper subgraph of $Cr(F)$ is contained in $Total(\mathcal{G})$ and so $Cr(F)$ is a minimal forbidden subgraph of $Total(\mathcal{G})$. □

Lemma 2. *Let $\mathcal{G}$ be a non-trivial additive hereditary graph class. If G is a minimal forbidden subgraph of $Total(\mathcal{G})$, then any total dominating subgraph of G is connected.*

Proof. Let $\mathcal{G}$ be a non-trivial additive hereditary graph class and let G be a minimal forbidden subgraph of $Total(\mathcal{G})$. Let T be a total dominating subgraph of G such that the number c of connected components of T is maximal. Under this condition, let T have a minimal number f of connected components that are not contained in $\mathcal{G}$. Let T' be a connected component of T for which $T' \notin \mathcal{G}$. Let X be the set of those vertices of $V(G)$ which are dominated by vertices of T' only. Let T'' be any total dominating subgraph of $G[X]$. T'' is connected, since otherwise we can substitute the component T' of T by T'' and obtain a total dominating subgraph of G with more than c connected connected components. Furthermore, $T'' \notin \mathcal{G}$, since otherwise we can substitute the component T' of T by T'' and obtain a total dominating subgraph of G with c connected components, less than f of which are not contained in $\mathcal{G}$. Hence, any total dominating subgraph of $G[X]$ is connected and furthermore $G[X] \notin Total(\mathcal{G})$. By minimality, $G[X] = G$. □

For the next Lemma we need some more notation. Let G be an isolate-free graph. The *support vertex* of a pendant vertex is its unique neighbor. The *support graph* $Supp(G)$ of G is obtained from G by attaching a pendant vertex to any of the cut-vertices of the connected components of G, except for the support vertices. For an example of a support graph, see Figure 1.

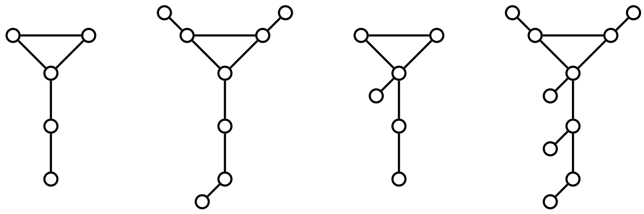

Figure 1: G, $F(G)$, $Supp(G)$ and $Cr(G)$

The following relationship is immediate from the definition:

Observation 1. *For any isolate-free graph G, $Supp(F(G)) \cong Cr(G)$.*

In the following, a *private neighbor* of a vertex x with respect to some vertex set S is a vertex $y \notin S$ such that the only neighbor of y in S is x. We now come to our next Lemma.

Lemma 3. *Let G be an isolate-free graph such that every total dominating subgraph is connected. Then for every total dominating subgraph T of G there is a superset S of $V(T)$ such that $G[S] \cong Supp(T)$.*

Proof. Let G be as in the Lemma and let T be any total dominating subgraph of G. Assume for contradiction that there is no superset S of $V(T)$ with $G[S] \cong Supp(T)$. Let S be a maximal total dominating superset of $V(T)$ such that $G[S]$ is a proper subgraph of $Supp(T)$. Hence, $G[S]$ has a cut-vertex x that is not a support vertex. Let $S' = S \setminus \{x\}$. $G[S']$ is not connected and does not have an isolated vertex, but G does not have a total dominating subgraph that is not connected. Therefore, S' cannot be a dominating set of G. Thus there is a private neighbor y of x with respect to S. Furthermore, $S'' = S \cup \{y\}$ is a total dominating set of G and a proper superset of S. $G[S'']$ is still a subgraph of $Supp(T)$, which is a contradiction to the choice of S. □

The main step of the proof of Theorem 1 is the following Lemma. A slightly weaker version is stated there, but in fact, the following is proved:

Lemma 4 (Tuza [6])**.** *Let G be any connected graph that does not have a dominating induced path. There is a connected dominating subgraph C and a superset S of $V(C)$ such that $G[S] \cong F(C)$.*

This Lemma will be very useful for our proofs.

3. Graph classes containing all paths

In this section, we deal with graph classes containing all paths. Combining Lemma 4 and the Lemmas 1, 2 and 3, we derive our first main result. It shows that the corona graphs are the only minimal forbidden subgraphs of $Total(\mathcal{G})$.

Theorem 2. *Let $\mathcal{G}$ be a non-trivial additive hereditary graph class containing all paths. Then the minimal forbidden subgraphs of $Total(\mathcal{G})$ are the corona graphs of the minimal forbidden subgraphs of $\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be a non-trivial additive hereditary graph class that contains all paths. Let G be a minimal forbidden subgraph of $Total(\mathcal{G})$. By Lemma 2, any total dominating subgraph of G is connected. Further, there is a connected dominating subgraph C of G such that there is a superset S of $V(C)$ with $G[S] \cong F(C)$. Otherwise, by Lemma 4, G has a dominating induced path and thus $G \in Total(\mathcal{G})$. Thus there is a connected dominating subgraph C of G and a superset S of $V(C)$ with $G[S] \cong F(C)$. As G is a minimal forbidden subgraph of $Total(\mathcal{G})$ and $\mathcal{G}$ is non-trivial, C properly contains K_2 as subgraph. Hence, C is a total dominating subgraph and thus $C \notin \mathcal{G}$. By Lemma 3, $Supp(G[S]) \cong Supp(F(C))$ is a subgraph of G, and by Observation 1, $Supp(F(C)) \cong Cr(C)$. Since $Cr(C) \notin Total(\mathcal{G})$, $G \cong Cr(C)$ and furthermore C is a minimal forbidden subgraph of $\mathcal{G}$. Hence, G is the corona graph of a minimal forbidden subgraph of $\mathcal{G}$.

Lemma 1 completes the proof. □

Note that if G is a 2-connected graph, then $F(G) \cong Cr(G)$. We denote by $\mathcal{C}$ the class of connected graphs. Together with Theorem 1, the above observation leads to the following.

Corollary 1. *Let $\mathcal{G}$ be a non-trivial additive hereditary graph class such that any minimal forbidden subgraph of $\mathcal{G}$ is 2-connected. Then*

$$Total(\mathcal{G}) \cap \mathcal{C} = Dom(\mathcal{G} \cap \mathcal{C}) \setminus \{K_1\}. \tag{1}$$

In words, for any $G \in Total(\mathcal{G})$ it holds that any connected subgraph of G has a connected dominating subgraph contained in $\mathcal{G}$.

Proof. Let $\mathcal{G}$ be as in the Lemma. In particular, $\mathcal{G}$ contains all paths. Let $G \in Total(\mathcal{G})$ and let H be a connected subgraph of G. By Theorem 2, G (and thus H) does not contain the corona of a minimal forbidden subgraph of $\mathcal{G}$. Since any minimal forbidden subgraph of $\mathcal{G}$ is 2-connected, H does not contain the leaf graph of a minimal forbidden subgraph of $\mathcal{G}$. By Theorem 1, $H \in Dom(\mathcal{G} \cap \mathcal{C})$. This completes the proof. □

For an example, let $\mathcal{A}$ be the class of acyclic graphs. Clearly the minimal forbidden subgraphs of $\mathcal{A}$ are 2-connected. By (1), if G is an isolate-free graph such that any isolate-free subgraph has an acyclic total dominating subgraph, any connected subgraph of G has a dominating subgraph that is a tree.

Another consequence of Theorem 2 is the following relation:

Corollary 2. *Let $\mathcal{F}$ be a non-empty family of non-trivial additive hereditary graph classes such that for every $\mathcal{G} \in \mathcal{F}$, $\mathcal{G}$ contain all paths. Then*

$$Total\left(\bigcap_{\mathcal{G}\in\mathcal{F}} \mathcal{G}\right) = \bigcap_{\mathcal{G}\in\mathcal{F}} Total(\mathcal{G}). \tag{2}$$

Proof. Let $\mathcal{F}$ be as in the Corollary. Clearly "$\subseteq$" holds in (2). By Theorem 2, the minimal forbidden subgraphs of $Total(\mathcal{G})$ are the coronas of the minimal forbidden subgraphs of $\mathcal{G}$ for any $\mathcal{G} \in \mathcal{F}$. Since every $\mathcal{G} \in \mathcal{F}$ contain all paths, $\bigcap_{\mathcal{G}\in\mathcal{F}} \mathcal{G}$ contains all paths, too. Thus the minimal forbidden subgraphs of $Total(\bigcap_{\mathcal{G}\in\mathcal{F}} \mathcal{G})$ are the coronas of the minimal forbidden subgraphs of $\bigcap_{\mathcal{G}\in\mathcal{F}} \mathcal{G}$, by Theorem 2. But any minimal forbidden subgraph of $\bigcap_{\mathcal{G}\in\mathcal{F}} \mathcal{G}$ is a minimal forbidden subgraph of some $\mathcal{G} \in \mathcal{F}$. Hence, "$\supseteq$" holds in (2). □

As an example, let $\mathcal{A}$ be the class of acyclic graphs and let $\mathcal{B}$ be the class of claw-free graphs. Corollary 2 gives $Total(\mathcal{A}) \cap Total(\mathcal{B}) = Total(\mathcal{A} \cap \mathcal{B})$. Hence, if G is an isolate-free graph such that any isolate-free subgraph has an acyclic total dominating subgraph and a claw-free total dominating subgraph, then any isolate-free subgraph of G has a total dominating subgraph which is a claw-free acyclic graph, i.e. the disjoint union of paths.

Note that a formular similar to (2) holds in the case of connected domination, i.e.

$$Dom\left(\bigcap_{\mathcal{G}\in\mathcal{F}} \mathcal{G}\right) = \bigcap_{\mathcal{G}\in\mathcal{F}} Dom(\mathcal{G})$$

is true for any non-empty family $\mathcal{F}$ of classes of connected graphs closed under taking connected induced subgraphs. Here, the restriction of the classes $\mathcal{G} \in \mathcal{F}$ to contain all paths is not necessary. In contrast to this, the results of the next section suggest that for total domination the general case is more difficult.

4. Graph classes not containing all paths

As the discussion in Section 4.1 shows, the case of additive hereditary graph classes $\mathcal{G}$ which do not contain all paths is not that easy. In fact, (2) does not necessarily hold if at least one of the classes does not contain all paths. We therefore think that there might not be a closed formula like Theorem 2 for the minimal forbidden subgraphs of $Total(\mathcal{G})$ in the general case. However, this question might be seen as a challenging open problem.

In some cases we are able to give partial characterizations for $Total(\mathcal{G})$ or sufficient conditions for a graph to be contained in this set. These results are presented in section 4.2.

4.1. Further forbidden subgraphs

Our first example for the violation of (2) is the graph G displayed in Figure 2.

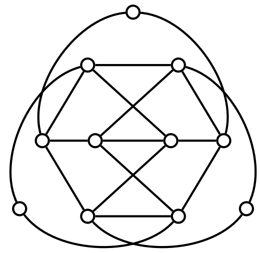

Figure 2: The graph G.

It is not hard to check that $G \notin Total(\{C_4, P_4, K_{1,3}\}$-free graphs). On the other hand, G does not contain $Cr(C_4)$, $Cr(P_4)$ or $Cr(K_{1,3})$ as subgraph. By Theorem 2, $G \in Total(C_4$-free graphs) and $G \in Total(K_{1,3}$-free graphs). By Theorem 3, $G \in Total(P_4$-free graphs). Hence, (2) is violated by the family

$$\mathcal{F} = \{C_4\text{-free graphs}, P_4\text{-free graphs}, K_{1,3}\text{-free graphs}\}.$$

As the following observation shows, this is not the only exception: There are infinitely many families of graph classes violating (2). For any $k \geq 3$ and $2 \leq i \leq k-1$ let T_k^i be the graph obtained from the path P_k by attaching a pendant vertex to the i-th vertex of P_k. Note that T_k^i is P_{k+1}-free for any $2 \leq i \leq k-1$. Let $\mathcal{T}_k = \{T_k^i : 2 \leq i \leq k-1\}$ be the collection of these graphs.

In the following, for $v \in V(G)$, $N_G(v)$ ($N_G[v]$) denotes the open (closed) neighborhood of v in G.

Observation 2. *For any $k \geq 5$, $Total(P_k$-free graphs $\cap\, \mathcal{T}_{k-1}$-free graphs) is a proper subset of $Total(P_k$-free graphs) $\cap\, Total(\mathcal{T}_{k-1}$-free graphs).*

Proof. Let $k \geq 5$ and G be the graph constructed as follows: Let C be a cycle on $k+2$ vertices. For any two vertices u and v in C with $N_C[u] \cap N_C[v] = \emptyset$ we add a vertex $x_{u,v}$ and connect it to u and v. That is,

$$\begin{aligned} V(G) &= V(C) \cup \{x_{u,v} : N_C[u] \cap N_C[v] = \emptyset\}, \\ E(G) &= E(C) \cup \{\{u, x_{u,v}\}, \{v, x_{u,v}\} : N_C[u] \cap N_C[v] = \emptyset\}. \end{aligned}$$

To see that $G \notin Total(P_k$-free graphs $\cap\, \mathcal{T}_{k-1}$-free graphs), let T be any total dominating subgraph of G. For contradiction, we assume that T is P_k-free and $\mathcal{T}_{k-1}$-free. We observe that for any vertex $x_{u,v}$, T contains u or v. Hence, $G[V(C) \cap V(T)]$ contains P_{k-1} as subgraph. By assumption, $G[V(C) \cap V(T)]$ does not contain P_k as subgraph. Hence, there are three vertices in C, say u, v and w, such that $N_G(u) \cap N_G(w) = \{v\}$ and $u, w \notin V(T)$. Since v must be dominated, there is a vertex $t \in V(C) \cap V(T)$ such that $N_C[v] \cap N_C[t] = \emptyset$ and $x_{v,t} \in V(T)$. Therefore, $G[((V(T) \cap V(C)) \setminus \{v\}) \cup \{x_{v,t}\}] \in \mathcal{T}_{k-1}$, in contradiction to the assumption. G and its dominating subgraph T_{k-1}^i are displayed schematically in Figure 3.

We observe that G is $Cr(P_k)$-free and $Cr(T_{k-1}^i)$-free for all $2 \leq i \leq k-2$. By Theorem 3, $G \in Total(P_k$-free graphs), and by Theorem 2, $G \in Total(\mathcal{T}_{k-1}$-free graphs). □

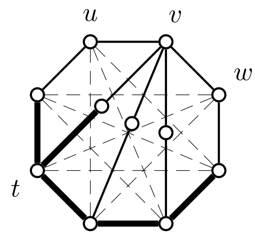

Figure 3: G in the case $k = 6$. The bold edges mark its dominating subgraph T_5^2. The dashed lines stand for subdivided edges.

4.2. Partial characterizations and sufficient conditions

For the main result of this section we need the following:

Lemma 5. *Let $\mathcal{G}$ be a non-trivial additive hereditary graph class that does not contain all paths and let $k \geq 3$ be minimal such that $P_k \notin \mathcal{G}$. Any minimal forbidden subgraph G of $Total(\mathcal{G})$ which is not the corona of a minimal forbidden subgraph of $\mathcal{G}$ has the following properties:*

1. *G contains $Cr(P_{k-1})$ and the cycle C_i as subgraph for all $5 \leq i \leq k+2$.*
2. *If $k \geq 4$, G has a total dominating subgraph which is isomorphic to T_{k-1}^i for any $2 \leq i \leq k-2$.*

Proof. Let $k \geq 3$ and $\mathcal{G}$ be a graph class with the properties of the Lemma. Let G be a minimal forbidden subgraph of $Total(\mathcal{G})$ which is not the corona of a minimal forbidden subgraph of $\mathcal{G}$ In particular, G is $Cr(P_k)$-free. By Lemma 2, any total dominating subgraph of G is connected.

As the proof of Theorem 2 shows, we only have to deal with the case that there is no connected dominating subgraph C with superset S of $V(C)$ such that $G[S] \cong F(C)$. (Otherwise, $G \cong Cr(C)$.) By Lemma 4, G has a dominating induced path P. We choose P to be minimal. By choice of G, P contains at least k vertices. We denote the vertices of P by $v_1, v_2, \ldots, v_r$ consecutively, i.e. v_1 and v_r are the endvertices of P. As P is minimal, v_1 and v_r have at least one private neighbor each. By assumption, if S_{v_1} (resp. S_{v_r}) is the set of private neighbors of v_1 (resp. v_r), then any vertex of S_{v_1} is adjacent to any vertex of S_{v_r}.

Let $x \in S_{v_1}$ and $y \in S_{v_r}$ be arbitrary. By choice of G again, $r \geq k$. We observe that $G[V(P) \cup \{x\}] \cong P_{r+1}$ and $V(P) \cup \{x\}$ is a total dominating set of G. By Lemma 3, there is a superset S of $V(P) \cup \{x\}$, such that $G[S] \cong Supp(P_{r+1}) \cong Cr(P_{r-1})$. Hence, $k = r$, since G is $Cr(P_k)$-free. In particular, G contains $Cr(P_{k-1})$ as subgraph, which proves the first part of claim 1.

If $P \cong P_3$, then $k = 3$. Furthermore, $G[\{x, v_1, v_2, v_3, y\}] \cong C_5$. Hence, the proof is finished in this case and so we can assume $P \cong P_r$ with $r = k \geq 4$.

We denote the vertices of $S \setminus (V(P) \cup \{x\})$ by $w_2, w_3, \ldots, w_{k-2}$, according to the index of their support vertices in $G[S]$. Assume there is a vertex $w_i \in S \setminus (V(P) \cup \{x\})$ that is not adjacent to y. By definition, $i \geq 2$ and v_i is the support vertex of w_i in $G[S]$. We observe that $(V(P) \setminus \{v_r\}) \cup \{x\}$ is a total dominating set of G. By induction, any k consecutive vertices of the cycle $G[V(P) \cup \{x, y\}]$ dominate G. Hence, $T = (V(P) \setminus \{v_{i-1}\}) \cup \{w_i, x, y\}$ is a total dominating set of G with $G[T] \cong P_{k+2}$. By Lemma 3, $Supp(P_{k+2}) \cong Cr(P_k)$ is a subgraph of G, a contradiction.

Therefore, any vertex of $S \setminus (V(P) \cup \{x\})$ is adjacent to y. For any $2 \leq i \leq k-2$ let $V_i = \{y, x, v_1, v_2, \dots, v_i, w_i\}$. We observe that $G[V_i] \cong C_{i+3}$ for any $2 \leq i \leq k-2$. Furthermore, $G[V(P) \cup \{x, y\}] \cong C_{k+2}$. This proves 1.

To see claim 2, recall $k \geq 4$. Let $U = \{x, v_1, v_2, \dots, v_k\}$ and let $2 \leq i \leq k-2$ be arbitrary. Since $U \setminus \{v_i\}$ is disconnected and $G[U \setminus \{v_i\}]$ does not have an isolated vertex, it is not a dominating set, by Lemma 2. Hence, there is a private neighbor u_i of v_i with respect to U. Let $W = (V(P) \setminus \{v_k\}) \cup \{u_i\}$ and observe that $G[W] \cong T^i_{k-1}$.

Assume for contradiction that W is not a dominating set of G. Hence, there is a common neighbor of x and v_k, say z, that is not dominated by W.

Assume v_{i-1} does not have a private neighbor with respect to $U \cup \{u_i, z\}$. Then $G[U \cup \{u_i, z\} \setminus \{v_{i-1}\}] \cong P_{k+2}$ is a dominating induced path of G. By Lemma 3, G contains $Cr(P_k)$ as induced subgraph, a contradiction. Hence, v_{i-1} has a private neighbor, say u_{i-1}, with respect to $U \cup \{u_i, z\}$. Inductively, we obtain a stable set $\{u_1, u_2, \dots, u_i\}$ such that, for any $1 \leq j \leq i$, u_j is a private neighbor of v_j with respect to $U \cup \{z\}$. This situation is displayed in Figure 4.

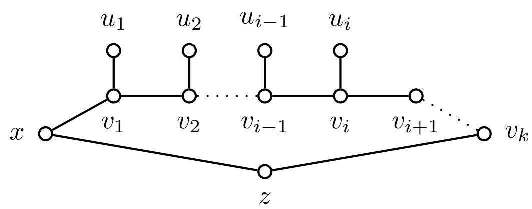

Figure 4: The situation of the proof of Lemma 5.2.

Hence, $T = V(P) \cup \{u_1, z\}$ is a total dominating set of G with $G[T] \cong P_{k+2}$. By Lemma 3 again, G contains $Cr(P_k)$ as induced subgraph, a contradiction. This implies claim 2. □

Theorem 3. *Let $\mathcal{G}$ be a non-trivial additive hereditary graph class that does not contain all paths and let k be minimal such that $P_k \notin \mathcal{G}$.*

1. *If $k = 3$, then the minimal forbidden subgraphs of $Total(\mathcal{G})$ are C_5 and the coronas of the minimal forbidden subgraphs of $\mathcal{G}$.*
2. *If $k \geq 4$ and $\mathcal{G} \cap \mathcal{T}_{k-1} \neq \emptyset$, then the minimal forbidden subgraphs of $Total(\mathcal{G})$ are the coronas of the minimal forbidden subgraphs of $\mathcal{G}$.*
3. *If $k \geq 4$, then $Total(\mathcal{G})$ contains all graphs that do not contain a corona of the minimal forbidden subgraphs of $\mathcal{G}$ as subgraph and do not contain any graph of $\{C_i : 5 \leq i \leq k+2\} \cup \{Cr(P_{k-1})\}$ as subgraph.*

Proof. Let $\mathcal{G}$ be a non-trivial additive hereditary graph class that does not contain all paths and let k be minimal such that $P_k \notin \mathcal{G}$.

To see part 1, let $P_3 \notin \mathcal{G}$. By Lemma 1, the coronas of the minimal forbidden subgraphs of $Fb(\mathcal{G})$ is a subset of the minimal forbidden subgraphs of $Total(\mathcal{G})$. On the other hand, any minimal forbidden subgraph H of $Total(\mathcal{G})$ which is not the corona of a minimal forbidden subgraph of $\mathcal{G}$ contains C_5 as subgraph, by Lemma 5.1. Finally, C_5 is easily checked to be a minimal forbidden subgraph of $Total(\mathcal{G})$ and this completes the proof.

To see Part 2, let $k \geq 4$, $\mathcal{G} \cap \mathcal{T}_{k-1} \neq \emptyset$ and let G be a minimal forbidden subgraph of $Total(\mathcal{G})$. Assume G is not the corona of a minimal forbidden subgraph of $\mathcal{G}$. By Lemma 5.2, G has a total dominating subgraph T that is isomorphic to T^i_{k-1} for any $2 \leq i \leq k-2$. By assumption, $\mathcal{G} \cap \mathcal{T}_{k-1} \neq \emptyset$ and hence G cannot be a minimal forbidden subgraph of $Total(\mathcal{G})$, a contradiction.

For part 3, let G be a graph that does not contain a corona of the minimal forbidden subgraphs of $\mathcal{G}$ as subgraph and does not contain any graph of $\{C_i : 5 \leq i \leq k+2\} \cup \{Cr(P_{k-1})\}$ as subgraph. Then G cannot contain a minimal forbidden subgraph of $Total(\mathcal{G})$, since Lemma 5.1 says that any forbidden subgraph that is not a corona contains any member of $\{C_i : 5 \leq i \leq k+2\} \cup \{Cr(P_{k-1})\}$ as subgraph. Thus $G \in Total(\mathcal{G})$. □

In particular, we obtain the following special case.

Corollary 3. *Let $\mathcal{G}$ be a non-trivial additive hereditary graph class. An isolate-free C_5-free graph G is contained in $Total(\mathcal{G})$ iff G does not contain the corona of a minimal forbidden subgraph of $\mathcal{G}$ as subgraph.*

As another consequence of Theorem 3, we obtain the characterization of the case where only paths are forbidden:

Corollary 4.
1. *The minimal forbidden subgraphs of $Total(P_3$-free graphs$)$ are C_5 and $Cr(P_3)$.*
2. *If $k \geq 4$, then the minimal forbidden subgraph of $Total(P_k$-free graphs$)$ is $Cr(P_k)$.*

Note that Corollary 4.1 is the total domination equivalent to a Theorem of Bacsó and Tuza [9] (and independently Cozzens and Kelleher [10]) about dominating cliques: There it is shown that a connected graph G and any of its connected subgraphs have a dominating clique iff G is P_5-free and C_5-free.

By definition, any connected component of a total dominating subgraph contains K_2 as subgraph. Hence, it is a natural question whether a given graph has a total dominating set X such that the connected components of $G[X]$ are isomorphic to K_2. Then X is called an *induced paired-dominating set*. Apparently, this concept was introduced and first studied by Haynes, Lawson and Studer [11], later by Zelinka [12] and by Telle [4] as *dominating induced matchings*. Telle [4] shows that the decision problem associated to the existence of induced-paired dominating sets is NP-complete. On the other hand, Theorem 3 gives the following characterization:

Corollary 5. *Any isolate-free subgraph of an isolate-free graph G has an induced paired-dominating set iff G is $\{C_5, Cr(K_3), Cr(P_3)\}$-free.*

Note that this is a forbidden subgraph characterization with a finite set of forbidden graphs. Hence, the property of Corollary 5 has a decision problem which is efficiently solvable.

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