Efficient total domination in digraphs

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Abstract

We generalize the concept of efficient total domination from graphs to digraphs. An efficiently total dominating set X of a digraph D is a vertex subset such that every vertex of D has exactly one predecessor in X. Not every digraph has an efficiently total dominating set. We study graphs that permit an orientation having such a set and give complexity results and characterizations concerning this question. Furthermore, we study the computational complexity of the (weighted) efficient total domination problem for several digraph classes. In particular we deal with most of the common generalizations of tournaments, like locally semicomplete and arc-locally semicomplete digraphs.

Keywords: total domination, efficient total domination, digraphs, domination in digraphs

1. Introduction

A *digraph* is a pair $D = (V, A)$ where V is a finite set and $A \subseteq V \times V$ is an irreflexive binary relation. The elements of V are the *vertices* and the elements of A are the *arcs* of D. Since digraphs with symmetric arc set A can be considered as undirected graphs, digraphs are a natural generalization of them. There is a lot of mathematical theory on digraphs. A good introduction into the field is given by Bang-Jensen and Gutin in their book on digraphs [1].

A *dominating set* of a digraph D is a vertex subset X such that any member of $V \setminus X$ has a predecessor in X. A *total dominating set* of a digraph D is a vertex subset X such that any vertex of D has a predecessor in X. Dominating sets in digraphs are discussed in the survey book by Grinstead, Haynes and Slater [2]. A more recent paper, containing some results on domination in tournaments, is by Reid et al. [3]. However, there is not much theory on domination in digraphs yet and this field is much less studied than domination in undirected graphs. One of the possible reasons may be the following: Even for tournaments, which may be considered as one of the most famous digraph classes, it is not clear if there is an algorithm which efficiently computes the minimal size of a dominating

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set. According to our knowledge, the best exact algorithm is, essentially, brute force and runs in subexponential time [3]. So to say, a tournament has quite a lot of structure, but still not enough for the domination problem. Since the same holds for other variants of domination, there are a lot of open problems in algorithmical domination theory, and some of these problems have a high difficulty. Again for some very restricted digraph classes, like De Brujin and Kautz digraphs, some domination parameters can be explicitely computed (see for example [4, 8, 6, 7, 5]).

Efficient total domination in graphs is a somewhat studied topic in the literature (see for example [9, 10]). In this paper, we introduce efficient total domination for digraphs, a natural generalization of efficient total domination in graphs. In fact, if one restricts the attention to digraphs with symmetric arc set, one obtains the efficient total domination problem for graphs. An *efficiently total dominating set* of a digraph D is a vertex subset X such that every vertex of D has exactly one predecessor in X. So to say, the out-neighborhoods of an efficiently total dominating set of D form a partition of the vertex set of D. Another formulation is the following: Let $v_1, v_2, \dots, v_n$ be an ordering of the vertices of D and let A be the 01-adjacency matrix of D with respect to this ordering. That is, $A_{ij} = 1$ if there is an arc from v_i to v_j and $A_{ij} = 0$ otherwise. An efficiently total dominating set corresponds to a 01-vector x for which $A^t x = \mathbf{1}$, where $\mathbf{1}$ denotes the vector containing only ones. Let A^t denote the transpose matrix of A. Then an efficiently total dominating set corresponds to an exact cover of A^t and vice versa. We think that, in view of these formulations, efficient total domination in digraphs is a topic worth studying. According to our knowledge, there is not much theory on efficient domination in digraphs (besides [11]) and efficient total domination in digraphs has not been considered in the literature.

We contribute an in-depth study of the problem, offering an analysis of the relation of efficiently total dominatable digraphs and their underlying graphs. Furthermore, we study the computational complexity of the efficient total domination problem on several generalizations of tournaments. In most of the cases, we can either prove $\mathcal{NP}$-completeness or give an efficient algorithm to find even a minimum weighted efficient total dominating set.

2. Preliminaries

For standard notations we do not introduce here, the reader is always referred to the introductory chapter of [1].

If D is a digraph with no specified vertex or arc set, $V(D)$ denotes its vertices and $A(D)$ denotes its arcs. Let $D = (V, A)$ be a digraph. If U is a vertex subset, $D[U]$ denotes the induced subdigraph on U. For any vertex v of D its *out-neighborhood*, denoted by $N_D^+(v)$, is defined as the set of vertices u with $(v, u) \in A$. Such vertex u is then called an *out-neighbor* of v. The *in-neighborhood* of v, denoted by $N_D^-(v)$, is defined as the set of vertices u with $v \in N^+(u)$. Such vertex u is then called an *in-neighbor* of v. The *out-degree* d_D^+ of D is the function with $d_D^+(v) = |N_D^+(v)|$ for any $v \in V$. The *maximum out-degree* $\Delta^+(D)$

is defined as $\Delta^+(D) = \max_{v \in V} d^+(v)$. If there is a k such that $d_D^+ \equiv k$, D is said to be *k-out-regular* or just *out-regular*. The notions *in-degree* d_D^-, *maximal in-degree* $\Delta^-(D)$ and *(k-)in-regularity* are defined analogously. If D is clear from the context, we sometimes omit it from our notation, e.g. we may write $N^+(v)$ instead of $N_D^+(v)$.

An *efficiently total dominating set* (or *etd set*) of D is a set $X \subseteq V$ such that for any $v \in V$ there is exactly one vertex $x \in X$ with $(x, v) \in A$. That is, $|X \cap N^-(v)| = 1$ for any $v \in V$. If D has an etd set, D is called an *efficiently total dominatable digraph* (or an *etd digraph*). Note that not every digraph has an efficient total dominating set, e.g. acyclic digraphs. We denote the decision problem associated to the existence of efficient total dominating sets by *ETD*. If the vertices have a real-valued weight, we can consider minimum weight etd sets. The related minimization problem is denoted by *WETD*. A solution of the WETD problem is either a minimum weight etd set or the information that the input digraph is not etd.

2.1. Digraph properties and digraph classes

All of the following digraph properties and digraph classes are discussed in detail in [1]. Note that loops or parallel arcs do not play a role in this paper, and we focus on simple digraphs only.

Let $D = (V, A)$ be a digraph. Two arcs (u, v) and (v, u) are called *antiparallel*. If D does not have antiparallel arcs, it is called an *oriented graph* (*orgraph* for short). If $(u, v) \in A$, u and v are said to be *adjacent*. Thus adjacency is an irreflexive and symmetric binary relation. The *underlying graph* of D is the graph G with vertex set V defined by this adjacency relation. Hence, G is obtained from D by loosing the direction of the arcs and then identifying parallel edges. D is then called a *biorientation* of G. If furthermore D is an orgraph, D is called an *orientation* of G. D is said to be *connected* if G is connected. D is called *strongly connected* if for any two vertices u and v there is a directed path from u to v and a directed path from v to u. In particular, any strongly connected digraph is also connected. As the digraph consisting of a single arc shows, the opposite does not hold in general. For a given digraph D the *reverse digraph* D^- is obtained from D by changing the direction of each arc.

If D is the biorientation of a complete graph, it is called *semicomplete*. If D is furthermore an orgraph, it is called a *tournament*. D is called *locally out-semicomplete* (*locally in-semicomplete*) if $D[N^+(v)]$ ($D[N^-(v)]$) is semicomplete for all $v \in V$. If D is both locally out-semicomplete and locally in-semicomplete, D is simply called *locally semicomplete*. D is called *k-partite semicomplete* if it is the biorientation of a complete k-partite graph. D is called a *k-partite tournament* if it is the orientation of a complete k-partite graph. D is called *arc-locally out-semicomplete* (*arc-locally in-semicomplete*) if for every arc $(u, v) \in A$ it holds that every out-neighbor (in-neighbor) of u is identical or adjacent to every out-neighbor (in-neighbor) of v. If D is both arc-locally out-semicomplete and arc-locally in-semicomplete, D is simply called *arc-locally semicomplete*. D is called *transitive* if for all three distinct vertices u, v and w with $(u, v), (v, w) \in A$ it holds that $(u, w) \in A$. D is called *quasi-transitive* if for all three distinct vertices u, v and w with $(u, v), (v, w) \in A$ it holds that u is adjacent to w. Of course, any transitive digraph is quasi-transitive. As the directed cycle of length 3 shows, the opposite does not hold in general. A digraph is called *path-mergeable* if for any two vertices u and v the following holds: For any two directed paths P and P' from u to v that do not have common vertices (except u and v), there is a directed path P'' from u to v with $V(P'') = V(P) \cup V(P')$.

2.2. Graphs and hypergraphs

Let G be a graph. A *total dominating set* X is a vertex subset such that any vertex of G is adjacent to a member of X. Hence, $G[X]$ has minimum degree at least 1. A *pendant vertex* of G is a vertex with exactly one neighbor. The *corona* of G, denoted by $Cr(G)$, is obtained from G by simultaneously attaching a pendant vertex to any vertex of G. A graph G is $(5, 2)$-*chordal* if any cycle of length at least 5 has two chords. A *unicyclic* graph is a graph that has exactly one cycle. A graph is *planar* if it can be drawn into the plane without crossing edges. A *threshold graph* is a graph that can be constructed from the empty graph by repeatedly adding either an isolated vertex or a dominating vertex. A graph is a *split graph* if its vertices admit a partiton into a clique and a stable set. Detailed information on these graph classes are given in the survey by Brandstädt, Le and Spinrad in [12].

A *hypergraph* $H = (V, E)$ is an ordered pair where E is a nonempty finite family of nonempty finite sets and $V = \bigcup E$. The elements of V are called *vertices* and the elements of E *hyperedges*. The *bipartite incidence graph* of a hypergraph $H = (V, E)$ is the bipartite graph $(V \cup E, \{\{v, e\} : v \in e \in E\})$. A *cover* of H is a set $C \subseteq E$ such that $\bigcup C = V$. A *matching* of a hypergraph $H = (V, E)$ is a set $M \subseteq E$ such that $m \cap n = \emptyset$ for all $m \neq n \in M$. A matching M which is also a cover is a *perfect matching*. Not all hypergraphs have a perfect matching; in fact it is $\mathcal{NP}$-complete to decide if a given hypergraph has a perfect matching (see *exact cover* in Gary and Johnson [13]).

If we prove $\mathcal{NP}$-completeness of ETD for certain digraph classes in this paper, we always give a polynomial reduction from the perfect matching problem for hypergraphs. In our figures, we always choose the same hypergraph to be the instance of the perfect matching problem. It is defined by

$$\begin{aligned} V &= \{v_1, v_2, v_3\}, \\ E &= \{e_1 = \{v_1, v_2\}, e_2 = \{v_1\}, e_3 = \{v_2, v_3\}\}. \end{aligned}$$

The bipartite incidence graph of this hypergraph is displayed in Figure 1.

3. Underlying graphs of etd digraphs

3.1. Graphs having an etd biorientation

Given a graph, it is natural to ask if it can be oriented or bioriented in a way such that the resulting digraph is etd. Not every graph has an etd orientation or biorientation, e.g. the graph displayed in Figure 2.

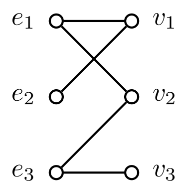

Figure 1: The bipartite incidence graph of the hypergraph used in our figures.

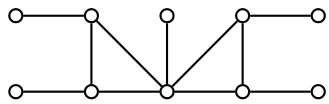

Figure 2: A graph not having an etd biorientation.

Let D be an etd digraph with etd set X. The etd condition says that every vertex of D has exactly one in-neighbor among the set X. Hence, $D[X]$ is 1-in-regular (a so-called *contrafunctional* digraph). The connected components of contrafunctional digraphs have the following structure: Any connected component has exactly one directed cycle (possibly two antiparallel arcs) and this cycle is an induced subdigraph. If a single arc of this cycle is removed, the resulting digraph is the orientation of a tree which has exactly one vertex of in-degree 0. All other vertices have in-degree 1. An example of a contrafunctional digraph is displayed in Figure 3.

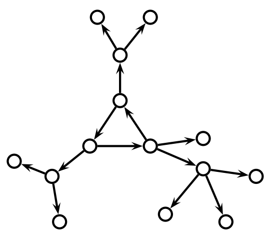

Figure 3: A connected contrafunctional digraph.

Thus, if G is the underlying graph of D, $G[X]$ is the disjoint union of graphs having at least two vertices but at most one cycle each. Furthermore, X is a total dominating set of G. In the case of D being an orientation of G, $G[X]$ is the disjoint union of unicyclic graphs only. Not every graph having an etd biorientation has a total dominating set that induces an acyclic subgraph though, e.g. the corona of a cycle.

On the other hand: If a graph G has a total dominating set X such that any connected component of $G[X]$ has at most one cycle, it also has an etd biorientation. Furthermore, if a graph G has a total dominating set X such that any connected component of $G[X]$ has exactly one cycle, it also has an etd orientation. Given such a total dominating set one efficiently constructs an etd (bi-)orientation of G as follows: The edges between the vertices of $V(G) \setminus X$ we direct in an arbitrary way. The edges between the vertices contained in the total dominating set X can easily be (bi-)oriented such that the resulting (bi-)orientation of $G[X]$ is contrafunctional. For each vertex $v \in V(G) \setminus X$ there is at least one edge joining v to a member of X, since X is a total dominating set. We direct exactly one of these edges from X to v and the other ones from v to X. Now, $|N^-(v) \cap X| = 1$ for each $v \in V(G)$.

This leads to

Lemma 1. *Let G be a graph.*

1. *G has an etd biorientation iff it has a total dominating set X such that the connected components of $G[X]$ have at most one cycle each.*
2. *G has an etd orientation iff it has a total dominating set X such that the connected components of $G[X]$ are unicyclic graphs.*

From an algorithmical point of view, the problem is intractable in general:

Theorem 1. *The following decision problems are $\mathcal{NP}$-hard: Given a graph G, does G admit an etd orientation? Does G admit an etd biorientation?*

Proof. Let $H = (V, E)$ be a hypergraph. To prove $\mathcal{NP}$-hardness, we define a graph G by

$$\begin{aligned} V(G) &= \{a, b, c, a', b', c'\} \cup V \cup E, \\ A(G) &= \{\{a, a'\}, \{b, b'\}, \{c, c'\}, \{a, b\}, \{a, c\}, \{b, c\}\} \cup \{\{a, e\} : e \in E\} \\ &\quad \cup \{\{e, f\} : e, f \in E, e \cap f \neq \emptyset\} \cup \{\{e, v\} : v \in e \in E\}, \end{aligned}$$

where $\{a, b, c, a', b', c'\}$ is assumed to be disjoint to $V \cup E$. The constructed graph G is displayed schematically in Figure 4.

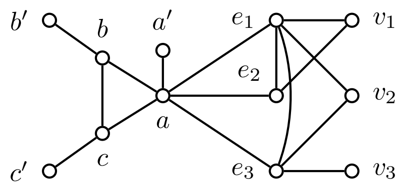

Figure 4: The constructed graph in the reduction of the proof of Theorem 1.

It is easy to see that for every total dominating set X of G, $G[X]$ is connected and $\{a, b, c\} \subseteq X$. Hence, $G[X]$ has at least one cycle. Thus by Lemma 1, G has an etd orientation iff it has an etd biorientation. We claim that there is a total dominating set Y of G such that $G[Y]$ has exactly one cycle iff H has a perfect matching. First we assume that there is a total dominating set X of G such that any connected component of $G[X]$ has exactly one cycle. Since $G[X]$ is connected, $G[X]$ has exactly one cycle. Since $\{a, b, c\} \subseteq X$ induces a cycle, $Y = X \cap E$ is a stable set. That is, Y is a matching of H. Since X is a total dominating set and V is stable, Y is also a cover of H. Hence, Y is a perfect matching of H. On the other hand, if $M \subseteq E$ is a perfect matching of H, M is a stable set in G and hence $X = \{a, b, c\} \cup M$ is a connected total dominating set such that $G[X]$ has exactly one cycle.

Lemma 1 and the fact that the perfect matching decision problem is $\mathcal{NP}$-hard complete the proof. □

As the proof of Theorem 1 shows, the problem remains intractable if one asks for etd (bi-)orientations with connected etd sets. On the other hand, the theory of the structure of total dominating subgraphs developed in [14] allows the following characterization. As Theorem 2 of [14] shows, the following holds for any graph G: Any induced subgraph of G without isolated vertices has a total dominating set X such that the connected components of $G[X]$ have at most one cycle each iff G does not contain the corona of a graph with two cycles as induced subgraph. This leads to

Theorem 2. *Let G be a graph without isolated vertices. G and any of its subgraphs without isolated vertices have an etd biorientation iff G does not contain the corona of a graph with two cycles as induced subgraph.*

3.2. Etd digraphs and their underlying graphs

A sharp non-trivial bound on the size of an etd set is given by the *stability number* of the underlying graph. This number, denoted by α, equals the size of a maximum stable set of the graph.

Theorem 3. *For each etd digraph D with underlying graph G any etd set has size at most $3\alpha(G)$. This bound is sharp for etd tournaments.*

Proof. Let D be an etd digraph with underlying graph G and let X be an etd set of D. Hence, $3\alpha(G) \geq 3\alpha(G[X]) \geq |X|$.

The bound is sharp, since for each n, each etd set of an etd tournament has size $3 = 3\alpha(K_n)$. □

The following results are obtained by an easy structural analysis leading to digraph classes on which WETD can be solved by a complete enumeration. For fixed p and q a *$\{K_{1,p}, qK_2\}$-free graph* is a graph that does not contain the complete bipartite graph $K_{1,p}$ or q disjoint copies of K_2 as induced subgraph.

Lemma 2. *For fixed p and q, the maximal size of an etd set of an etd biorientation of a $\{K_{1,p}, qK_2\}$-free graph is bounded by a constant.*

Proof. Let G be a $\{K_{1,p}, qK_2\}$-free graph and let D be an etd biorientation of G. Let X be an etd set of D. As described above, $D[X]$ is a contrafunctional digraph. Hence, $G[X]$ does not have isolated vertices and is the disjoint union of trees and unicyclic graphs. Since G is $K_{1,p}$-free, the maximum degree of $G[X]$ is p. Since G is qK_2-free, each connected component of $G[X]$ has diameter at most $3q-2$ and thus contains at most $p(p-1)^{3q-2}$ vertices. By qK_2-freeness again, $G[X]$ contains at most q connected components. All in all $|X| \leq qp(p-1)^{3q-2}$. □

This gives the following

Theorem 4. *For fixed p and q, WETD is efficiently solvable on the class of biorientations of $\{K_{1,p}, qK_2\}$-free graphs.*

We now prove $\mathcal{NP}$-completeness of ETD on (bi-)orientations of certain graph classes.

Theorem 5. *ETD is $\mathcal{NP}$-complete on the following digraph classes:*

1. *orientations of split graphs,*
2. *path-mergeable orientations of planar bipartite graphs of maximum degree 4,*
3. *strongly connected biorientations of threshold graphs,*
4. *strongly connected biorientations of complete k-partite graphs for all fixed $k \geq 2$.*

Proof. Let $H = (V, E)$ be a hypergraph on the vertices $V = \{v_1, v_2, \ldots, v_n\}$.

To see claim 1, we define an orgraph D by

$$\begin{aligned} V(D) &= V \cup E \cup \{a, b, c, d\}, \\ A(D) &= \{(a,b), (b,c), (c,a), (a,d)\} \cup \{(a,e) : e \in E\} \\ &\quad \cup \{(e,v) : v \in e \in E\} \cup \{(v_i, v_j) : 1 \leq j < i \leq n\} \\ &\quad \cup \{(v,a), (v,c), (v,d) : v \in V\}. \end{aligned}$$

The underlying graph of D is a split graph: $D[\{b,d\} \cup E]$ is arcless and $D[\{a,c\} \cup V]$ is a tournament. D is displayed schematically in Figure 5.

If $X \subseteq E$ is a perfect matching of H, then $X \cup \{a,b,c\}$ is an etd set of D. On the other hand, let X be an etd set of D. $N^-(b) = \{a\}$ gives $a \in X$. Since $N^-(d) = \{a\} \cup V$, $X \cap V = \emptyset$. Hence, $N^-(v) \cap X \subseteq E$ for all $v \in V$ and so $X \cap E$ is a perfect matching of H.

Therefore, H has a perfect matching iff D is etd and this completes the proof of claim 1.

As is shown in [15], the decision problem of the existence of a perfect matching is $\mathcal{NP}$-complete if restricted to the class of hypergraphs whose bipartite incidence graph (given by $(V \cup E, \{(e,v) : v \in e \in E\})$) is planar and has maximum degree 3. Thus we can assume that the bipartite incidence graph of H is planar and has maximum degree 3.

We define a path-mergeable orgraph D by

$$\begin{aligned} V(D) &= V \cup E \cup \{a_e, b_e, c_e, d_e : e \in E\}, \\ A(D) &= \{(a_e, b_e), (b_e, c_e), (c_e, d_e), (d_e, a_e) : e \in E\} \cup \{(a_e, e) : e \in E\} \\ &\quad \cup \{(e,v) : v \in e \in E\} \end{aligned}$$

and observe that the underlying graph of D is a planar bipartite graph of maximum degree four.

If $X \subseteq E$ is a perfect matching of H, then $X \cup \{a_e, b_e, c_e, d_e : e \in E\}$ is an etd set of D. On the other hand, let X be an etd set of D. Since

$N^-(v) = \{e : v \in e \in E\}$ for all $v \in V$, it follows that $X \cap E$ is a perfect matching of H.

Therefore, H has a perfect matching iff D is etd and this completes the proof of claim 2.

To see claim 3, we define a digraph D by

$$\begin{aligned} V(D) &= V \cup E \cup \{a, b\}, \\ A(D) &= \{(a,b),(b,a)\} \cup \{(a,e) : e \in E\} \\ &\quad \cup \{(e,v) : v \in e \in E\} \cup \{(v,e) : v \notin e \in E\} \\ &\quad \cup \{(v_i, v_j) : 1 \leq j < i \leq n\} \cup \{(v,a),(v,b) : v \in V\} \end{aligned}$$

We observe that the underlying graph of D is a threshold graph. It is constructed by iteratively adding $\{b\} \cup E$ as isolated vertices and then $\{a\} \cup V$ as dominating vertices. Since, by definition, every vertex of V is contained in at least one hyperedge and there is no empty hyperedge, D is strongly connected. D is displayed schematically in Figure 5.

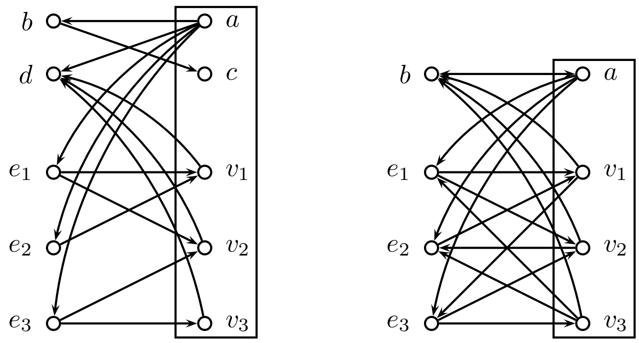

Figure 5: The constructed digraphs in the reduction of the proof of claim 1 resp. claim 3. The boxed subgraphs are acyclic tournaments in which each arc is directed upwards.

If $X \subseteq E$ is a perfect matching of H, then $X \cup \{a, b\}$ is an etd set of D. On the other hand, let X be an etd set of D. $N^-(b) = \{a\} \cup V$ gives $|X \cap (\{a\} \cup V)| = 1$. Let $x \in X \cap (\{a\} \cup V)$. Clearly any vertex of E is dominated by x. Since any vertex of H is contained in a hyperedge, $x \notin V$. Hence, $a \in X$ and $X \cap V = \emptyset$. Thus, $N^-(v) \cap X \subseteq E$ for all $v \in V$ and so $X \cap E$ is a perfect matching of H.

Therefore, H has a perfect matching iff D is etd and this completes the proof of claim 3.

To see claim 4, let $k \geq 2$ be arbitrary. We define a digraph D by

$$\begin{aligned} V(D) &= V \cup E \cup \{a, b\} \cup \{u_i : 1 \leq i \leq k-2\}, \\ A(D) &= \{(a,b),(b,a)\} \cup \{(a,e) : e \in E\} \cup \{(v,b) : v \in V\} \\ &\quad \cup \{(e,v) : v \in e \in E\} \cup \{(v,e) : v \notin e \in E\} \\ &\quad \cup \{(a,u_i) : 1 \leq i \leq k-2\} \cup \{(v,u_i) : v \in V, 1 \leq i \leq k-2\} \\ &\quad \cup \{(u_i,b) : 1 \leq i \leq k-2\} \cup \{(u_i,e) : e \in E, 1 \leq i \leq k-2\} \\ &\quad \cup \{(u_i,u_j) : 1 \leq j < i \leq n\} \end{aligned}$$

and observe that the underlying graph of D is a complete k-partite graph. Thereby, the k partitions are $\{b\} \cup E$, $\{a\} \cup V$ and $\{u_1\}$, $\{u_2\}$, $\ldots$, $\{u_{k-2}\}$. Furthermore, D is easily seen to be strongly connected.

If $X \subseteq E$ is a perfect matching of H, then $X \cup \{a, b\}$ is an etd set of D. On the other hand, let X be an etd set of D. $N^-(a) = \{b\}$ gives $b \in X$. Since $N^-(b) = \{a\} \cup V \cup \{u_i : 1 \leq i \leq k-2\}$, $X \cap (\{a\} \cup V \cup \{u_i : 1 \leq i \leq k-2\})$ contains exactly one vertex. Let x be that vertex. Since $D[X]$ is a contrafunctional digraph, it is not acyclic. If $x \neq a$, any cycle of $D[X]$ necessarily contains at least three vertices, in contradiction to $|X \cap N^-(b)| = 1$. Hence, $x = a$. Thus, $N^-(v) \cap X \subseteq E$ for all $v \in V$ and so $X \cap E$ is a perfect matching of H.

Therefore, H has a perfect matching iff D is etd and this completes the proof of claim 4. □

4. ETD and WETD in digraph classes generalizing tournaments

This section deals with the algorithmic complexity of ETD and WETD on digraph classes generalizing tournaments. All of these digraphs are rich in structure and thus some allow simple combinatorial algorithms even for WETD.

Some of our proofs make use of the following lemma:

Lemma 3.
1. *A minumum weighted etd set that induces a cycle of length 2 or 3 can be found in $\mathcal{O}(m\Delta^+ \max\{\Delta^-, \Delta^+\})$ time.*
2. *A minumum weighted etd set that induces a cycle of length 2, 3 or 4 can be found in $\mathcal{O}(m\Delta^- {\Delta^+}^2)$ time.*

Proof. All cycles of length 2 can clearly be found in $\mathcal{O}(m\Delta^+)$. Now, the etd property can be checked in $2\Delta^+$ steps for each such cycle.

The cycles of length three can be found in the following way. For each edge $a = (u, v)$, it can be checked if there is a vertex $w \in N^-(u) \cap N^+(v)$ in $\mathcal{O}(\max\{\Delta^-, \Delta^+\})$ time. Again, the etd property can be checked in $3\Delta^+$ steps for each such cycle.

The cycles of length four can be found in the following way. For each edge $a = (u, v)$, and each two $t \in N^-(u)$ and $w \in N^+(v)$, one has to check if $(w, t) \in A$. The etd property can be checked in $4\Delta^+$ steps. This completes the proof. □

4.1. Quasi-transitive digraphs and k-partite tournaments

Lemma 4. *If D is a connected etd quasi-transitive digraph, each etd set of D induces a cycle of length 2 or 3.*

Proof. We observe that the only connected contrafunctional quasi-transitive digraphs are cycles of length 2 or 3.

Let $D = (V, A)$ be a connected quasi-transitive digraph and X be an etd set of D. Thus, $D[X]$ is the disjoint union of cycles of length 2 or 3. Assume for contradiction that $D[X]$ is not a single cycle. Furthermore, assume there are two cycles in $D[X]$, say C_1 and C_2, and a vertex v with $N^-(v) \cap V(C_1) \neq \emptyset$ and

$N^+(v) \cap V(C_2) \neq \emptyset$. Since D is quasi-transitive, there is a vertex in C_1 which is adjacent to some vertex in C_2, a contradiction. Assume there are two cycles in $D[X]$, say C_1 and C_2, and two vertices, say u and v, with the following property: u is dominated by some $x \in V(C_1)$, v is dominated by some $y \in V(C_2)$, and $(u, v) \in A$. By quasi-transitivity, $(v, x) \in A$ and thus x is adjacent to y. This is a contradiction to the etd property of X. □

Furthermore, connected etd quasi-transitive digraphs are strongly connected. Lemma 3.1 and Lemma 4 give

Theorem 6. *WETD can be solved in* $\mathcal{O}(m\Delta^+ \max\{\Delta^-, \Delta^+\})$ *on quasi-transitive digraphs.*

Another easy observation is the following

Lemma 5. *If D is an etd k-partite tournament, each etd set of D induces a cycle of length 3 or 4.*

Proof. The only contrafunctional k-partite tournaments are cycles of length 3 or 4. It is clear that a k-partite tournament does not have an induced subdigraph that is the disjoint union of two cycles. □

Again, etd k-partite tournaments are strongly connected. Lemma 3.2 and Lemma 5 give

Theorem 7. *WETD can be solved in* $\mathcal{O}(m\Delta^- {\Delta^+}^2)$ *on k-partite tournaments.*

In contrast, Theorem 5 shows that ETD is $\mathcal{NP}$-complete on k-partite semicomplete digraphs for all fixed $k \geq 2$. In fact, the k-partite semicomplete digraphs constructed in the proof of Theorem 5 only have a single antiparallel arc. Hence, the existence of a single antiparallel arc leads to the intractability of the problem.

For bipartite tournaments one easily obtains the following simple characterization.

Theorem 8. *A bipartite tournament T is etd iff T^- is etd iff there is a 4-cycle $(u_1, v_1, u_2, v_2, u_1)$ in T such that $N^+(u_1) = N^-(u_2)$ and $N^+(v_1) = N^-(v_2)$.*

4.2. Locally semicomplete digraphs

Theorem 9. *WETD can be solved in* $\mathcal{O}(m\Delta^+)$ *on the class of locally out-semicomplete digraphs.*

Proof. WETD can be solved using the following procedure. Let $D = (V, A)$ be a locally out-semicomplete digraph with real-valued vertex weight c. First we determine the set B of all arcs $a = (u, v)$ of D with $N^+(u) \cap N^+(w) = \emptyset$. Then we determine the strong components of the digraph D_B defined by the arcs contained in B. Next we check the vertex set of each of these strong components for being an etd set of D. For all of those etd sets, we return as output the one with minimal total weight.

To see the correctness of the procedure, let X be an arbitrary etd set of D. Hence $N^+(u) \cap N^+(v) = \emptyset$ holds for each arc $a = (u, v)$ of $D[X]$ and thus $D[X]$ is an induced subdigraph of D_B. Since D is locally out-semicomplete, D_B is 1-out-regular and thus the strong components of D_B are exactly the contrafunctional subdigraphs of D_B (they are exactly the cycles of D_B). Therefore, $D[X]$ is a strong component of D_B and gets detected during the procedure.

Strongly connected components can be found in linear time by the algorithm of Tarjan [16]. Hence, the time of each step of the procedure is bounded by $\mathcal{O}(m\Delta^+)$. □

Furthermore, connected etd locally out-semicomplete digraphs are strongly connected.

Note that the locally out-semicomplete digraphs properly include (locally) semicomplete digraphs and (local) tournaments. We did not yet classify the complexity of ETD on locally in-semicomplete digraphs. However, Theorem 5.2 shows that for path-mergeable orgraphs ETD remains intractable. As stated in [1], this is a common superclass of locally out-semicomplete and locally in-semicomplete orgraphs.

4.3. Arc-locally semicomplete digraphs

Since any bipartite semicomplete digraph is arc-locally semicomplete, Theorem 5.4 has the following consequence:

Theorem 10. *ETD is $\mathcal{NP}$-complete on strongly connected arc-locally semicomplete digraphs.*

Another intractable problem is given by the following

Theorem 11. *ETD is $\mathcal{NP}$-complete on arc-locally in-semicomplete orgraphs.*

Proof. Let $H = (V, E)$ be a hypergraph. We define an arc-locally in-semicomplete orgraph D by

$$\begin{aligned} V(D) &= V \cup E \cup \{a, b, c\}, \\ A(D) &= \{(a, b), (b, c), (c, a)\} \cup \{(b, e) : e \in E\} \cup \{(e, v) : v \in e \in E\}. \end{aligned}$$

We observe the following: If $M \subseteq E$ is a perfect matching of H, then $\{a, b, c\} \cup M$ is an etd set of D. On the other hand, if X is an etd set of D, then $X \cap E$ is a perfect matching of H. Hence, D is etd iff H has a perfect matching. This completes the proof. □

To obtain positive results, we need further details on the structural properties of arc-locally digraphs. A recent paper by Wang and Wang [17] gives a complete description of the strongly connected arc-locally in-semicomplete digraphs. To state this characterization, we need the following notions:

An *extended cycle* of length $k \geq 2$ is a digraph $C = (V, A)$ where V admits a partition into k non-empty sets $U_1, U_2, \ldots, U_k$ such that $A = \{(u, v) : u \in U_i, v \in U_{i+1}$ for some $1 \leq i \leq k\}$ where the index is taken modulo k. It is clear that any extended cycle has an etd set. In fact, the etd sets of extended cycles are exactly the sets that contain exactly one element of U_i for each $1 \leq i \leq k$.

A *T-digraph* is a strongly connected digraph $T = (V, A)$ with the following properties. V admits a partition into the sets V_1, V_2, V_3 and V_4 such that $T[V_1]$ and $T[V_3]$ have no arcs, $|V_2| = 1$ and $T[V_4]$ is semicomplete. V_3 and V_4 may not be empty at the same time. If V_1 is empty, V_3 must be empty, too (a case the authors of [17] forgot). The arcs of T are as follows: For each $v_1 \in V_1$, $v_2 \in V_2$, $v_3 \in V_3$ and $v_4 \in V_4$ we have

- $(v_1, v_2) \in A$,
- $(v_2, v_3) \in A$, but $(v_3, v_2) \notin A$,
- $(v_3, v_1) \in A$, but $(v_1, v_3) \notin A$,
- $(v_4, v_1) \in A$, but $(v_1, v_4) \notin A$,
- $(v_4, v_3) \in A$, but $(v_3, v_4) \notin A$.

Furthermore, there may be some more arcs between V_2 and $V_1 \cup V_4$ since T is strongly connected.

Wang and Wang obtain the following characterization:

Theorem 12 (Wang and Wang [17])**.** *Let D be a strongly connected arc-locally in-semicomplete digraph. D is either semicomplete, semicomplete bipartite, an extended cycle or a T-digraph. If D has an induced cycle of length at least 5, it is an extended cycle.*

Since the reverse digraph of an arc-locally in-semicomplete digraph is an arc-locally out-semicomplete digraph and strongly connectedness is preserved by the reversing operation, we have the following

Corollary 1 (Wang and Wang [17])**.** *Let D be a strongly connected arc-locally out-semicomplete digraph. If D has an induced cycle of length at least 5, it is an extended cycle.*

Using these characterizations, we obtain the next

Theorem 13. *WETD can be solved in $\mathcal{O}(m\Delta^-{\Delta^+}^2)$ time on arc-locally out-semicomplete orgraphs.*

Proof. We observe that any connected contrafunctional arc-locally out-semicomplete orgraph is a cycle. Hence, the subdigraphs induced by etd sets in arc-locally out-semicomplete orgraphs are the disjoint union of cycles. A similar argument to the one used in the proof of Theorem 6 shows that in a connected arc-locally out-semicomplete orgraph each etd set induces a single cycle. Again, connected etd arc-locally out-semicomplete orgraphs are strongly connected.

Let D be a connected arc-locally out-semicomplete digraph. We can solve WETD using the following procedure. First we check if D is the extension of a cycle. This can be done in linear time easily. If this is the case, a minimum weight etd set is obtained by a greedy technique in linear time.

If D is not the extension of a cycle, there is no induced cycle of length at least 5 in D, by Corollary 1. Hence, we only have to search for a minimum weight etd set of D that induces a cycle of length at most four. By Lemma 3, this can be done in $\mathcal{O}(m\Delta^{-}{\Delta^{+}}^{2})$ time. If such a set does not exist, D is not etd. □

Using a similar algorithm, we obtain the same time complexity for strongly connected arc-locally in-semicomplete digraphs.

Theorem 14. *WETD can be solved in* $\mathcal{O}(m\Delta^{-}{\Delta^{+}}^{2})$ *time on strongly connected arc-locally in-semicomplete orgraphs.*

Proof. Let D be a strongly connected arc-locally in-semicomplete digraph. By Theorem 12, D is either semicomplete, semicomplete bipartite, an extended cycle or a T-digraph. In the case that D is semicomplete or semicomplete bipartite it is clear that each etd set induces a cycle of length at most 4. A few easy case distinctions show that the same holds if D is a T-digraph.

Hence, we can use the same procedure to solve WETD as for arc-locally out-semicomplete orgraphs. □

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