A note on connected dominating sets of distance-hereditary graphs

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Abstract

A vertex subset of a graph is a dominating set if every vertex of the graph belongs to the set or has a neighbor in it. A connected dominating set is a dominating set such that the induced subgraph of the set is a connected graph. A graph is called distance-hereditary if every induced path is a shortest path.

In this note, we give a complete description of the (inclusionwise) minimal connected dominating sets of connected distance-hereditary graphs in the following sense: If G is a connected distance-hereditary graph that has a dominating vertex, any minimal connected dominating set is a single vertex or a pair of two adjacent vertices. If G does not have a dominating vertex, the subgraphs induced by any two minimal connected dominating sets are isomorphic. In particular, any inclusionwise minimal connected dominating set of a connected distance-hereditary graph without dominating vertex has minimal size. In other words, connected distance-hereditary graphs without dominating vertex are connected well-dominated. Furthermore, we show that if G is a distance-hereditary graph that has a minimal connected dominating set X of size at least 2, then for any connected induced subgraph H it holds that the subgraph induced by any minimal connected dominating set of H is isomorphic to an induced subgraph of $G[X]$.

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1 Introduction

A vertex subset of a graph G is a *dominating set* if every vertex of the graph belongs to the set or has a neighbor in it. A *connected dominating set* is a dominating set X such that the induced subgraph of the set, henceforth denoted $G[X]$, is a connected graph. If no proper subset of X is a connected dominating set, X is called a *minimal connected dominating set.* Let U be a vertex subset of G and let $u \in U$. A vertex $v \in N(u)$ that does not belong to U or have a neigbor among U is called a *private neighbor of* u (with respect to U). For any minimal connected dominating set X of G the following holds: Any vertex $x \in X$ either is cut-vertex of $G[X]$ or has a private neighbor. Among the applications of connected dominating sets is the routing of messages in mobile ad hoc networks (see [1] for a recent survey). The *distance* of two vertices of a connected graph is the minimal number of edges of a path connecting the two vertices. A graph G is called *distance-hereditary* if for every connected induced subgraph the distance of any two vertices is the same as in G. That is, every induced path of G is a shortest path. In particular, distance-hereditary graphs are *hole-free*, i.e. every chordless cycle has length at most 4. Distance-hereditary graphs were introduced and first studied by Bandelt and Mulder [2] in 1986. There are a lot of alternative characterizations known for distance-hereditary graphs, some of which were discovered by Bandelt and Mulder [2] and D'Atri and Moscarini [3]. One of these characterizations is in terms of minimal forbidden induced subgraphs. Bandelt and Mulder [2] showed that a graph is distance-hereditary iff it does not contain one of the graphs displayed in Figure 1.

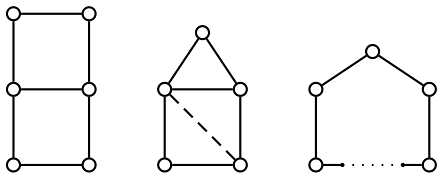

Figure 1: The forbidden induced subgraphs of the distance-hereditary graphs. The dashed line is an optional edge. The dotted line is a path of arbitrary length at least one.

The problem of computing minimum connected dominating sets is known to be $\mathcal{NP}$-complete in general, but for distance-hereditary graphs it can be solved efficiently as was shown by D'Atri and Moscarini [3] and Brandstädt and Dragan [5]. Further, connected dominating sets that form a clique, so-called dominating cliques, can be computed efficiently for distance-hereditary graphs as was shown by Dragan [4] and Chang and Yeh [6].

2 The results

Our first result gives a complete description of the connected dominating sets of distance-hereditary graphs:

Theorem 1. *Let G be a connected distance-hereditary graph. If G has a dominating vertex, any minimal connected dominating set is a single vertex or a pair of two adjacent vertices. If G does not have a dominating vertex, the subgraphs induced by any two minimal connected dominating sets are isomorphic.*

Proof. Let G be a connected distance-hereditary graph.

Assume that there is a minimal connected dominating set Z of G with $|Z| \geq 3$. For each non-cutting vertex x of $G[Z]$ choose a private neighbor n_x and let P be the collection of these private neighbors. Assume for contradiction that P is not a stable set. Since G is hole-free, there is an adjacent pair $x, y \in Z$ of vertices that are not cut-vertices of $G[Z]$ such that n_x is adjacent to n_y. Since $|Z| \geq 3$, x and y belong to a chordless cycle C of $G[Z]$. But then $G[V(C) \cup \{n_x, n_y\}]$ is not distance-hereditary, this is straightforward in view of Figure 1. Hence, G can not be distance-hereditary, a contradiction. Thus P is a stable set and so $G[Z \cup P]$ contains P_4 as subgraph. Since G is distance-hereditary, it has diameter at least 3 and can therefore not have a dominating vertex.

All in all, if a distance-hereditary graph has a dominating vertex, any minimal connected dominating set is a single vertex or a pair of two adjacent vertices.

We can now assume that G does not have a dominating vertex. Assume for contradiction that there are two minimal connected dominating sets of G that do not have isomorphic induced subgraphs. Among the pairs of minimal connected dominating sets that do not have isomorphic subgraphs, choose X and Y such that $|X \setminus Y|$ is minimal. Since Y is a minimal connected dominating set, there is a vertex $x \in X \setminus Y$. Let $P(x)$ be the (possibly empty) set of private neighbors of x with respect to X and let $X' = X \cup P(x)$. Note, that x is a cut-vertex of $G[X']$, since $|X| \geq 2$ by assumption. Let $X_1, X_2, \ldots, X_k$ be the connected components of $G[X'] - x$. Since Y is a connected dominating set and $x \notin Y$, x is not a cut-vertex of G. Thus we can choose a set $S \subseteq Y \setminus X'$ inclusionwise minimal with respect to the property that the graph $G[X' \cup S] - x$ is connected. Let $s \in S$ be arbitrary. Assume that there is a component of $G[X'] - x$, say X_i, that s does not have a neighbor among. By choice of S, $G[X' \cup S] - \{x, s\}$ is disconnected. Since S is minimal, there is a component of $G[X'] - x$, say X_j, that is not contained in the component of $G[X' \cup S] - \{x, s\}$ that X_i belongs to. Let $x_i \in N(x) \cap X_i$ and let $x_j \in N(x) \cap X_j$. Then any path from x_i to x_j in $G[X' \cup S] - x$ contains s and thus has length at least 3, since s is not adjacent to x_i. However, the distance of x_i and x_j in G is 2, as $x_i, x_j \in N(x)$. Since this contradicts the choice of G, s has a neighbor among X_i for any $1 \leq i \leq k$ and so $|S| = 1$.

Further, $N(s) \cap (X' \setminus \{x\}) \subseteq N(x)$: Otherwise, we can choose $u \in (N(s) \cap X') \setminus N(x)$ and $v \in N(s)$ from different components of $G[X'] - x$. The distance of u and v in G is 2, but at least 3 in $G[X']$, a contradiction. On the other hand, $N(x) \cap X' \subseteq N(s)$, as can be seen by a symmetric argumentation.

Hence, $N(s) \cap (X' \setminus \{x\}) = N(x) \cap X'$. In particular, $Z = (X \setminus \{x\}) \cup \{s\}$ fulfills $G[Z] \cong G[X]$. Further, $P(x) \subseteq N(s)$ and thus Z is a connected dominating set of G. Therefore $|X \setminus Y|$ is not minimal, a contradiction to the choice of X and Y. □

A distance-hereditary graph that has a dominating vertex and a minimal connected dominating set of size 2 is the 4-wheel, obtained from a 4-cycle by adding a dominating vertex. This graph has a dominating vertex, but any two adjacent vertices from the 4-cycle also form a minimal connected dominating set.

Graphs with the property that all minimal dominating sets have the same size are usually called *well-dominated* [9, 10, 11]. In this sense, we say that a graph is *connected well-dominated* if all minimal connected dominating sets have the same size. A direct consequence of Theorem 1 is that all minimal connected dominating sets of a distance-hereditary graph without dominating vertex have the same size.

Corollary 1. *Any connected distance-hereditary graph that does not have a dominating vertex is connected well-dominated.*

Using a similar argumentation as in the proof of Theorem 1, we obtain the following.

Theorem 2. *Let G be a connected distance-hereditary graph and let H be any connected induced subgraph of G. If X is a minimal connected dominating set* *of G which is not a single vertex, then the subgraph induced by any minimal connected dominating set of H is isomorphic to an induced subgraph of $G[X]$.*

Proof. Let G, X and H be as in the theorem. Assume for contradiction that there is a minimal connected dominating set Y of H such that $H[Y]$ is not isomorphic to a subgraph of $G[X]$. Since $Y \subseteq V(H) \subseteq V(G)$, we can choose a set Y' with the property $G[Y'] \cong H[Y]$ such that $|Y' \setminus X|$ is minimal. Since $G[Y']$ is not isomorphic to a subgraph of $G[X]$, there is a vertex $y \in Y' \setminus X$. Let $P(y)$ be the (possibly empty) set of private neighbors of y with respect to Y' and let $Y'' = Y' \cup P(y)$. Note, that y is a cut-vertex of $G[Y'']$. Using the argumentation from the proof of Theorem 1 we obtain a contradiction to the choice of Y'. This completes the proof. □

3 Extending the results

A graph is called $(5,2)$-*chordal* if any cycle of length at least 5 has two chords. If any cycle of length at least 5 has two crossing chords, the graph is called $(5,2)$-*crossing-chordal*. It is discovered by Howorka [8], that a graph is distance-hereditary iff it is $(5,2)$-*crossing-chordal*. Hence, the class of distance-hereditary graphs is a proper subclass of the $(5,2)$-chordal graphs and the two classes have many properties in common. However, Theorem 1 does not extend to the class of $(5,2)$-chordal graphs, as the graph displayed in Figure 2 shows. The graph does not have a dominating vertex, but the subgraphs induced by the minimal connected dominating sets $\{b,c\}$ and $\{b,e,f\}$ are not isomorphic. In particular, this graph is not connected well-dominated.

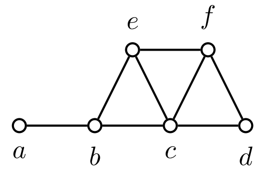

Figure 2: A $(5,2)$-chordal graph with minimal connected dominating sets $\{b,c\}$ and $\{b,e,f\}$.

It remains an open problem to find a forbidden induced subgraph characterization of the connected graphs, any connected induced subgraph of which has the property mentioned in Theorem 1. For an example of a graph that has this property but is not distance-hereditary consider any chordless cycle of length at least 5.

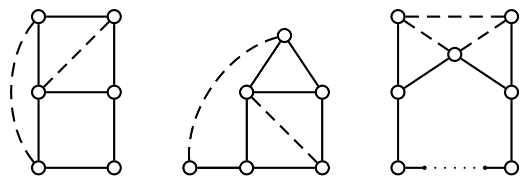

Figure 3: Dashed lines are optional edges. The dotted line is a path of arbitrary length at least one.

However, all of the graphs displayed in Figure 3 do not have this property, since they do not have a dominating vertex and are not connected well-dominated. Compared to the forbidden induced subgraphs of the distance-hereditary graphs (displayed in Figure 1), we think that it seems to be unlikely that Theorem 1 can be extended to a reasonable superclass of the distance-hereditary graphs.

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