# On the separability of graphs

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**Abstract**

Recently, Cicalese and Milanič introduced a graph-theoretic concept called separability. A graph is said to be *k-separable* if any two non-adjacent vertices can be separated by the removal of at most $k$ vertices. The *separability* of a graph $G$ is the least $k$ for which $G$ is $k$-separable. In this paper, we investigate this concept under the following three aspects.

First, we characterize the graphs for which in any non-complete connected induced subgraph the connectivity equals the separability, so-called *separability-perfect graphs*. We list the minimal forbidden induced subgraphs of this condition and derive a complete description of the separability-perfect graphs.

We then turn our attention to graphs for which the separability is given locally by the maximum intersection of the neighborhoods of any two non-adjacent vertices. We prove that all (house,hole)-free graphs fulfill this property – a class properly including the chordal graphs and the distance-hereditary graphs. We conclude that the separability can be computed in $\mathcal{O}(m\Delta)$ time for such graphs.

In the last part we introduce the concept of edge-separability, in analogy to edge-connectivity, and prove that the class of $k$-edge-separable graphs is closed under topological minors for any $k$. We explicitly give the forbidden topological minors of the $k$-edge-separable graphs for each $0 \leq k \leq 3$.

**keywords:** separability, connectivity, HH-free graphs, forbidden topological minors.

**MSC:** 05C40, 05C75

## 1 Introduction

Recently, Cicalese and Milanič [2] introduced a graph-theoretic invariant called separability. A graph is *k-separable* if any two non-adjacent vertices can be separated by the removal of at most $k$ vertices. For example, disjoint unions of complete graphs are precisely the 0-separable graphs, acyclic graphs are 1-separable and any chordless cycle is 2-separable. The *separability number* (*separability* for short) of a graph $G$, denoted $\textsc{sep}(G)$, is defined as the least $k$ for which $G$ is $k$-separable. If $x$ and $y$ are two non-adjacent vertices of $G$, we denote by $\text{SEP}_G(x, y)$ the least $k$ such that $x$ and $y$ can be separated by the removal of $k$ vertices.

Cicalese and Milanič introduced the concept of separability to model a problem arising in the parsimony haplotyping problem from computational biology [3]. Among the many aspects of their paper [2], they investigate the computational complexity of several optimization problems (e.g. maximum clique, $k$-coloring, minimum dominating set) for graphs of bounded separability. Their main result is a decomposition theorem for the 2-separable graphs. Furthermore, they prove characterizations of the 2-separable graphs in terms of minimal forbidden induced subgraphs and minimal forbidden induced minors.

In this paper we carry on the research on the concept of separability.

In Section 2, we investigate the relationship of the connectivity of a graph and its separability. We study the graphs for which the separability equals the connectivity in any non-complete connected induced subgraph. We call these graphs *separability-perfect*. We give a full characterization in terms of forbidden induced subgraphs and derive a complete description of the separability-perfect graphs. It turns out that separability-perfectness is a rather restrictive condition. However, the class of block graphs is covered.

Cicalese and Milanič [2] asked for the relation of the separability to other graph invariants. In Section 3, we study graphs $G$ for which the separability is locally determined by the maximum intersection of the neighborhoods of any two non-adjacent vertices, i.e.

$$\text{SEP}(G) = \max\{|N(u) \cap N(v)| : u, v \in V(G), u \neq v, \{u, v\} \notin E(G)\}, \qquad (1)$$

where $N(v)$ denotes the neighborhood of a vertex $v$. We observe that the separability is computed in $\mathcal{O}(|E(G)|\Delta(G))$ time for such graphs. We show that (1) holds for HH-free graphs, i.e. graphs that do not contain as induced subgraph a house or a chordless cycle of length at least five (see Fig. 4). This class properly includes the chordal graphs and the distance-hereditary graphs. Finally, we conjecture that the theorem extends to weakly chordal graphs, a superclass of the HH-free graphs. If the conjecture holds, it would yield a new characterization of the weakly chordal graphs.

In Section 4, we introduce the concept of edge-separability. We observe that, like separability in [2], edge-separability is not monotone under edge-contraction. Hence for $k \geq 3$, the class of $k$-edge-separable graphs is not closed under taking (induced) minors. In contrast, we show that edge-separability is monotone under topological minors and give a characterization of the $k$-edge-separable graphs in terms of forbidden topological minors for each $0 \leq k \leq 3$.

## 1.1 Preliminaries

Let $P_n$ denote the induced path and $C_n$ the chordless cycle on $n$ vertices. $K_n$ is the complete graph on $n$ vertices and $K_n^-$ is obtained from $K_n$ by removing an arbitrary edge. The connectivity of a graph $G$ is denoted by $\text{CON}(G)$. The *join* of two graphs $G$ and $H$ is obtained by taking their disjoint union and then adding all possible edges from $G$ to $H$. We also say that $H$ *is joined to* $G$. A *simplicial vertex* is a vertex whose neighborhood forms a clique.

# 2 When separability equals connectivity

With the exception of complete graphs, the separability is always an upper bound for the connectivity. It seems to be natural to ask for which graphs both parameters coincide. To get a grip on the problem, we require equality for all induced subgraphs of a graph. Formally, we say that a connected graph $G$ is *separability-perfect* if and only if $\textsc{sep}(H) = \textsc{con}(H)$ for any connected induced subgraph $H$ which is not complete.

Our aim is to derive three equivalent conditions for a graph $G$ to be separability-perfect. For this, we need the following concepts.

A *cycle of cliques* is a graph obtained as follows. Start with $k \geq 3$ cliques $C^1$, $C^2$, $\ldots$, $C^k$ of size at least 2 and choose two distinct vertices from each clique. Say we choose $v_1^1$ and $v_2^1$ from $C^1$, $v_1^2$ and $v_2^2$ from $C^2$ and so on. Now identify $v_2^1$ with $v_1^2$, $v_2^2$ with $v_1^3$, and so on until finally $v_2^k$ is identified with $v_1^1$. In Fig. 1, two cycles of cliques are displayed.

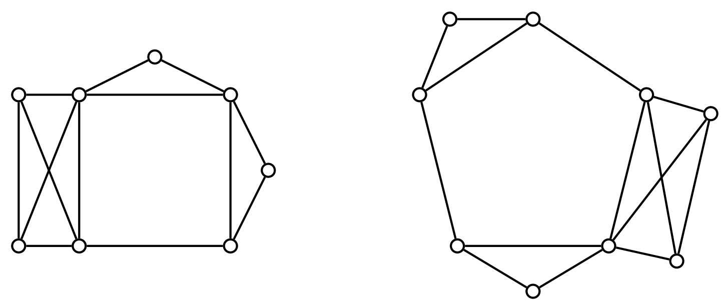

Figure 1: Two cycles of cliques.

We call a graph $F$-free if it does not contain as induced subgraphs the graphs $F^1$, $F^2$, $F^3$ (see Fig. 2) and, for any $n$, $F_n^4$ and $F_n^5$ (see Figure 3).

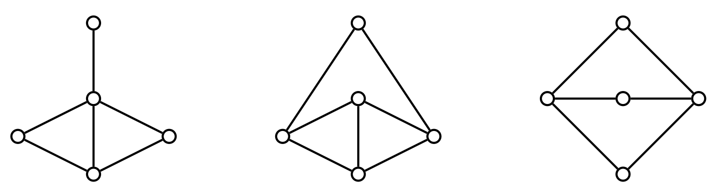

Figure 2: The graphs $F^1$, $F^2$ and $F^3$.

A graph is called a *block graph* if every 2-connected component of it is a clique. It is known that a graph is a block graph if and only if it is chordal and $K_4^-$-free (cf. [1]).

Let $\mathcal{G}$ be the class of the following graphs: $G$ is either a block graph, a cycle of cliques, a complete graph where a matching is removed, or $G$ is obtained by joining a complete graph to one of the following graphs:

(I) a disjoint union of complete graphs. In this case, the joined complete graph may not be empty.

(II) a disjoint union of two non-empty complete graphs where a non-empty matching is added between them.

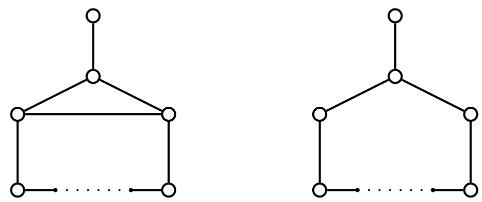

Figure 3: The configurations $F_n^4$ and $F_n^5$. In both configurations, the two vertices at the bottom are the end-vertices of a path on $n \geq 1$ vertices symbolized by the dotted line. In the case $n = 1$, the two vertices at the bottom of $F_n^4$, $F_n^5$ respectively, are identical.

(III) $C_5$.

We are now able to characterize separability-perfectness by three equivalent conditions. The first one is in terms of degrees and connectivity of the 2-connected induced subgraphs of $G$. The second lists the minimal connected non-separability-perfect graphs. The third condition gives a complete description of the separability-perfect graphs.

**Theorem 1.** *For a connected graph $G$, the following conditions are equivalent:*

*(i)* *$G$ is separability-perfect.*

*(ii)* $|N_G(x) \cap V(H)| \geq \textsc{con}(H)$ *holds for every non-complete 2-connected induced subgraph $H$ of $G$ and for every vertex $x \in V(G) \setminus V(H)$. If $N_G(x) \cap V(H)$ is not a clique in $G$, then* $|N_G(x) \cap V(H)| \geq \textsc{con}(H) + 1$.

*(iii)* *$G$ is $F$-free.*

*(iv)* $G \in \mathcal{G}$.

We give the proof of Theorem 1 after a sequence of lemmas, which settle the main step of the proof. The first one, Lemma 1, considers chordal graphs. Lemma 2 and 3 consider non-chordal graphs.

**Lemma 1.** *If $G$ is a connected $F$-free chordal graph, then $G \in \mathcal{G}$.*

*Proof.* Let $G$ be a connected $F$-free chordal graph. In the following case distinction, a vertex is called *central* if it is adjacent to all other vertices of $G$.

**Case 1.** $G$ has a central vertex.

We claim that $G$ is obtained by joining a complete graph to a disjoint union of complete graphs (type I) or is the disjoint union of two complete graphs where a non-empty matching is added (type II). To see this, let $U$ be the (non-empty) set of central vertices of $G$,

Since $F^1$ is forbidden, we know that the graph $H = G[V(G) \setminus U]$ does not contain the disjoint union of $P_3$ and $K_1$, henceforth denoted $P_3 \cup K_1$, as induced subgraph. Thus if $H$ is disconnected, any connected component of $H$ is $P_3$-free. Since connected $P_3$-free graphs are complete, $G$ is of type I.

We now assume that $H$ is connected. A classical theorem of Wolk [8] states that a connected graph without induced $P_4$ and $C_4$ has a central vertex. Since we excluded $U$, $H$ cannot have such a vertex and must, by contraposition, contain an induced $P_4$ or $C_4$. As $H$ is chordal, the latter is not possible. Let $u_1, u_2, u_3, u_4$ be the consecutive ordering of an induced $P_4$ of $H$.

Suppose that there is a vertex $x$ which does not have a neighbor in $\{u_2, u_3\}$. Then $x$ must be adjacent to $u_1$ or $u_4$, since $H$ is $P_3 \cup K_1$-free. By chordality, $x$ is adjacent to $u_1$ or $u_4$, but not both. We may assume that $x$ is adjacent to $u_1$. But then $H[\{x, u_2, u_3, u_4\}] \cong P_3 \cup K_1$, a contradiction. Hence, every vertex has a neighbor in $\{u_2, u_3\}$.

We define a partition of $V(H) \setminus \{u_2, u_3\}$ as follows: let $A$ be the set of vertices neighboring $u_2$ but not $u_3$, let $B$ be the set of vertices neighboring both $u_2$ and $u_3$ and let $C$ be the set of vertices neighboring $u_3$ but not $u_2$. We know that $u_1 \in A$ and $u_4 \in C$ and thus both sets are nonempty. Moreover, no member of $A$ is adjacent to a member of $C$, by chordality. We claim that $B$ is empty and both $A$ and $C$ are cliques.

To see this, we first suppose that $B$ is not empty. Let $b \in B$. If there is a vertex $a \in A \setminus N_H(b)$, $b$ must be adjacent to any $c \in C$. Otherwise $H[\{a, b, c, u_2\}] \cong P_3 \cup K_1$, for any $c \in C \setminus N_H(b)$. But then $H[\{a, b, c, u_2, u_3\}] \cong F_1^4$ for any $c \in C$, a contradiction.

By symmetry we know that for all $b \in B$ we have $A \cup C \subseteq N_H(b)$. Since there is no vertex in $H$ that is adjacent to all other vertices, there are non-adjacent vertices $b_1, b_2 \in B$. But then $H[\{b_1, b_2, u_1, u_3\}] \cong C_4$, a contradiction to the chordality of $H$. Therefore $B$ must be empty.

If $A$ is not a clique, there are non-adjacent vertices $a_1, a_2 \in A$. But then $H[\{a_1, a_2, u_2, u_4\}] \cong P_3 \cup K_1$. By a symmetric argumentation, $A$ and $C$ are cliques.

Recall that, since $G$ is chordal, no vertex of $A$ is adjacent to a vertex of $C$. Therefore, $H$ is the disjoint union of two complete graphs where the edge $\{u_2, u_3\}$ is added. That is, $G$ is of type II.

**Case 2.** $G$ does not have a central vertex.

If $G$ does not contain $K_4^-$ as an induced subgraph, $G$ is a block graph (cf. [1]). So we may assume that $G$ has an induced $K_4^-$ on the vertices $v_1, v_2, v_3, v_4$ such that $v_1$ is not adjacent to $v_4$. If $V(G) = \{v_1, v_2, v_3, v_4\}$, $G$ is a cycle of cliques. Hence we may assume that there is a vertex $x \in V(G) \setminus \{v_1, v_2, v_3, v_4\}$.

First, assume that $x$ has a neighbor in $\{v_1, v_2, v_3, v_4\}$. In view of $F^1$ and $F_1^4$, $x$ must have two neighbors in $\{v_1, v_2, v_3, v_4\}$. By chordality, $x$ has two neighbors in $\{v_1, v_2, v_3\}$ or two neighbors in $\{v_2, v_3, v_4\}$. By symmetry, we may assume that $x$ has two neighbors in $\{v_1, v_2, v_3\}$.

Now suppose that $x$ has a neighbor $y$ which does not have a neighbor in $\{v_1, v_2, v_3, v_4\}$. If $x$ has exactly two neighbors in $\{v_1, v_2, v_3\}$, then $G[\{x, y, v_1, v_2, v_3\}] \cong F_1^4$, a contradiction. Thus $x$ has three neighbors in $\{v_1, v_2, v_3\}$. If $x$ is adjacent to $v_4$, $G[\{x, y, v_1, v_2, v_4\}] \cong F^1$. Otherwise, $G[\{x, y, v_2, v_3, v_4\}] \cong F_1^4$. Both yields a contradiction. Since $G$ is connected, there cannot be a vertex without neighbors in $\{v_1, v_2, v_3, v_4\}$. All in all, every vertex $x \in V(G) \setminus \{v_1, v_2, v_3, v_4\}$ has at least two neighbors in $\{v_1, v_2, v_3\}$ or in $\{v_2, v_3, v_4\}$.

Suppose there are vertices $x, y \notin \{v_1, v_2, v_3, v_4\}$ such that $x$ has only one neighbor in $\{v_2, v_3, v_4\}$ and $y$ has only one neighbor in $\{v_1, v_2, v_3\}$. Since $x$ and $y$ both must have two neighbors in $\{v_1, v_2, v_3, v_4\}$, we know that $x$ is adjacent to $v_1$ and, without loss of generality, $v_2$. Conversely, $y$ is adjacent to $v_4$ and to either $v_2$ or $v_3$.

Suppose that $x$ is adjacent to $y$. If $y$ is adjacent to $v_2$, $G[\{x, y, v_1, v_3, v_4\}] \cong C_5$, and if $y$ is adjacent to $v_3$, $G[\{x, y, v_1, v_3\}] \cong C_4$. Both is contradictory to the fact that $G$ is chordal.

So $x$ is not adjacent to $y$. If $y$ is adjacent to $v_2$, $G[\{x, y, v_1, v_2, v_3\}] \cong F^1$, and if $y$ is adjacent to $v_3$, $G[\{x, y, v_1, v_2, v_3\}] \cong F_1^4$. Both yields a contradiction.

Thus we know that every vertex has two neighbors in $\{v_1, v_2, v_3\}$ or every vertex has two neighbors in $\{v_2, v_3, v_4\}$. We may assume the first case. In the remainder of the proof, $v_4$ will not be considered. So, from now on, the roles of $v_1$, $v_2$ and $v_3$ are interchangeable.

**Case 2.1.** There is a vertex $w$ adjacent to all three members of $\{v_1, v_2, v_3\}$.

Since $G$ does not have a central vertex, there must be a non-neighbor of $w$ in $G$.

First we suppose that there is a vertex $x$ with exactly two neighbors among $\{v_1, v_2, v_3\}$ such that $w$ is not adjacent to $x$. We can assume that $N(x) \cap \{v_1, v_2, v_3\} = \{v_1, v_2\}$. But then there must be a vertex, say $y$, that is not adjacent to $v_1$ or $v_2$. Since any vertex has at least two neighbors among $\{v_1, v_2, v_3\}$, we can assume that $N(y) \cap \{v_1, v_2, v_3\} = \{v_1, v_3\}$. Suppose that $x$ is adjacent to $y$. But then $G[\{x, y, v_2, v_3\}] \cong C_4$, in contradiction to the chordality of $G$. Hence, $x$ is not adjacent to $y$. So $G[\{x, y, w, v_1, v_3\}] \cong F^1$ if $y$ is not adjacent to $w$ and $G[\{x, y, w, v_2, v_3\}] \cong F_1^4$ otherwise. Both gives a contradiction.

So every vertex with exactly two neighbors in $\{v_1, v_2, v_3\}$ is adjacent to $w$ but there is a vertex $w'$ not adjacent to $w$ (since $w$ is not central). Then $w'$ is adjacent to $v_1$, $v_2$, and $v_3$. Since no two members of $\{v_1, v_2, v_3\}$ are adjacent to all other vertices, there must be $x$ and $y$, both having exactly two neighbors among $\{v_1, v_2, v_3\}$, such that $N(x) \cap \{v_1, v_2, v_3\} \neq N(y) \cap \{v_1, v_2, v_3\}$. As seen above, $x$ is not adjacent to $y$. Moreover, by the assumption of the third case, $x$ and $y$ are both adjacent to $w$ and, as seen in the second case, $x$ and $y$ must also be both adjacent to $w'$. But then $G[\{x, y, w, w'\}] \cong C_4$, a contradiction.

**Case 2.2.** Every vertex has exactly two neighbors among $\{v_1, v_2, v_3\}$.

As seen in Case 2.1, whenever two vertices $x, y \in V(G) \setminus \{v_1, v_2, v_3\}$ do not have the same neighbors in $\{v_1, v_2, v_3\}$, $x$ and $y$ cannot be adjacent. Suppose there are two vertices $x, y$ that have the same two neighbors in $\{v_1, v_2, v_3\}$, say $v_1$ and $v_2$, but are not adjacent. By assumption, there is a vertex $z$ not adjacent to both $v_1$ and $v_2$. As shown above, $z$ is not adjacent to $x$ or $y$, thus $G[\{x, y, z, v_1, v_2\}] \cong F^1$, a contradiction. Hence $G$ is a cycle of cliques. This completes the proof. □

We now come to the non-chordal graphs.

**Lemma 2.** *Let $G$ be a connected $F$-free graph. If $G$ contains a chordless cycle $C$ of length $n \geq 4$, then the following holds:*

*(i) Every vertex of $V(G) \setminus V(C)$ is either adjacent to all vertices in $C$ or to exactly two adjacent vertices in $C$.*

*(ii) If $x, y \in V(G) \setminus V(C)$, both have exactly two neighbors in $C$ and these neighbors are the same, then $x$ is adjacent to $y$.*

*(iii) If there is no vertex in $V(G) \setminus V(C)$ with exactly $n$ neighbors in the cycle and $n \geq 5$, then $G$ is a cycle of cliques.*

*Proof.* Let $C = \{v_1, v_2, \dots, v_n\}$ be an induced cycle of length $n \geq 4$ with vertices ordered consecutively.

(i) Let $x \in V(G) \setminus V(C)$. We distinguish two cases and first assume that $N(x) \cap V(C) \neq \emptyset$. Then, since $F_n^5$ is forbidden, $x$ has at least two neighbors in $V(C)$.

Suppose $x$ has exactly two neighbors in $V(C)$, w.l.o.g. $v_1$ and $v_k$, and they are not adjacent. In particular, $k \geq 3$. By symmetry, we may assume that $k \leq \lceil (n+1)/2 \rceil$. If $n = 4$, then $k = 3$ and thus $G[\{x, v_1, v_2, v_3, v_4\}] \cong F^3$, a contradiction. If $n \geq 5$, $v_k$ is not adjacent to $v_n$ and thus $G[\{x, v_1, v_2, \dots, v_k, v_n\}] \cong F_{k-2}^5$. Since $k \geq 3$, this is again a contradiction.

Now suppose that $x$ has more than two but less than $n$ neighbors in $V(C)$. W.l.o.g. we assume that $v_1$ is a neighbor of $x$ but $v_2$ is not. Let $k > 2$ be the smallest number such that $v_k$ is a neighbor of $x$. Since $x$ has more than two neighbors, $k \leq n - 1$.

Suppose $v_{k+1}$ is not a neighbor of $x$. Since $x$ has more than two neighbors in $V(C)$, $v_{k+1} \neq v_n$. Hence, $G[\{x, v_1, v_2, \dots, v_{k+1}\}] \cong F_{k-2}^5$, a contradiction.

Thus $v_{k+1}$ is a neighbor of $x$. Suppose $k + 2 \leq n$. Suppose $v_{k+2}$ is not a neighbor of $x$. Then it is not a neighbor of $v_1$ neither, since otherwise $G[\{x, v_1, v_2, v_{k+1}, v_{k+2} = v_n\}] \cong F_1^5$. Thus $G[\{x, v_1, v_2, \dots, v_{k+2}\}] \cong F_{k-1}^4$, a contradiction. Therefore, $v_{k+2}$ is a neighbor of $x$ and so $G[\{x, v_{k-1}, v_k, v_{k+1}, v_{k+2}\}] \cong F_1^4$, a contradiction.

Hence, $k + 2 > n$ and so $k + 1 = n$. It is not possile that $n \geq 5$, since we assumed $k \leq \lceil (n+1)/2 \rceil$. So $n = 4$ and thus $G[\{x, v_1, v_2, v_3, v_4\}] \cong F^2$, a contradiction. This means that if $x$ has a neighbor in $V(C)$, (i) holds.

Suppose there is a vertex $y \in V(G) \setminus V(C)$ with no neighbors in $V(C)$. Since $G$ is connected, we can choose $y$ such that it has a neighbor $x$ with $N(x) \cap V(C) \neq \emptyset$. We know that either $N(x) \cap V(C) = V(C)$ or $x$ has exactly two neighbors in $V(C)$ and they are adjacent. In the first case, $G[\{x, y, v_1, v_2, v_3\}] \cong F^1$, a contradiction. In the second, $G[\{x, y, v_1, v_2, \dots, v_n\}] \cong F_{n-2}^4$, a contradiction. Hence (i) holds.

(ii) Let $x$ and $y$ be two vertices with $N(x) \cap V(C) = N(y) \cap V(C)$, say $N(x) \cap V(C) = \{v_1, v_2\}$. Then, if $x$ is not adjacent to $y$, $G[\{x, y, v_1, v_2, v_3\}] \cong F^1$, a contradiction.

(iii) Assume there is a vertex $x \notin V(C)$ which has exactly two neighbors in $V(C)$, say $v_1$ and $v_2$. If $V(G) = V(C) \cup \{x\}$, $G$ is a cycle of cliques and we are done. Hence we can assume that there is another vertex $y$ with exactly two neighbors $v_i, v_j \in V(C)$. As before, we may assume that $2 \leq i \leq \lceil (n+1)/2 \rceil$. In particular $i \leq n - 1$ and thus $j = i + 1$.

If $y$ shares exactly one neighbor with $x$, then $i = 2$. Since $n \geq 5$, $v_4$ is not adjacent to $x$ or $y$. If $x$ and $y$ are adjacent, $G[\{x, y, v_2, v_3, v_4\}] \cong F_1^4$, a contradiction.

Now assume $x$ and $y$ do not share a neighbor in $C$. By assumption, $v_n \neq v_{i+1}$. If $x$ and $y$ are adjacent, $G[\{x, y, v_n, v_1, v_2, \dots, v_i\}] \cong F_{i-2}^4$, a contradiction. Together with (ii) this completes the proof of (iii). $\square$

**Lemma 3.** *If $G$ is a connected $F$-free graph that contains a chordless cycle of length at least 4, then $G \in \mathcal{G}$.*

*Proof.* Let $G$ be a connected $F$-free graph that contains a chordless cycle of length at least 4. We distinguish the cases that $G$ contains as induced subgraph a $C_4$, $G$ contains a $C_5$ and $G$ contains a $C_n, n \geq 6$.

**Case 1.** $G$ contains an induced $C_4$.

Let $v_1, v_2, v_3, v_4$ be a consecutive ordering of the vertices of an induced $C_4$ of $G$. By Lemma 2(i), every vertex in $V(G) \setminus \{v_1, v_2, v_3, v_4\}$ has exactly two or exactly four neighbors among the set $\{v_1, v_2, v_3, v_4\}$.

**Case 1.1.** First we assume that every vertex of $V(G) \setminus \{v_1, v_2, v_3, v_4\}$ has exactly two neighbors in $\{v_1, v_2, v_3, v_4\}$.

Let $C_{1,2}$ be the neighbors of $v_1$ and $v_2$, $C_{2,3}$ be the neighbors of $v_2$ and $v_3$, $C_{3,4}$ be the neighbors of $v_3$ and $v_4$ and finally $C_{4,1}$ be the neighbors of $v_4$ and $v_1$. By Lemma 2(ii), each of these sets is a (possibly empty) clique. In particular, if $V(G) = \{v_1, v_2, v_3, v_4\}$, $G$ is a cycle of cliques and we are done.

First we assume that both $C_{1,2} \cup C_{3,4}$ and $C_{2,3} \cup C_{4,1}$ are not empty, and let $C_{1,2}$ and $C_{2,3}$ be non-empty. Let $x \in C_{1,2}$ and $y \in C_{2,3}$. Indeed, $x$ and $y$ cannot be adjacent since otherwise $G[\{x, y, v_2, v_3, v_4\}] \cong F_1^4$. If there is a vertex $z \in C_{3,4}$, $y$ and $z$ cannot be adjacent either. Hence, $x$ and $z$ cannot be adjacent since otherwise $G[\{x, y, z, v_1, v_3, v_4\}] \cong F_2^4$, a contradiction. By symmetry, $G$ is a cycle of cliques.

Now we assume that $C_{2,3} \cup C_{4,1}$ is empty but $C_{1,2} \cup C_{3,4}$ is not. If there is no edge from $C_{1,2}$ to $C_{3,4}$, $G$ is a cycle of cliques. Hence let $x \in C_{1,2}$ and assume that $x$ has two neighbors in $C_{3,4}$, say $y$ and $z$. Then $G[\{x, y, z, v_1, v_3\}] \cong F_1^4$, a contradiction. Thus $G$ is obtained from the two disjoint complete graphs $G[C_{1,2} \cup \{v_1, v_2\}]$ and $G[C_{3,4} \cup \{v_3, v_4\}]$ by adding a matching. Hence $G$ is of type II.

**Case 1.2.** There are vertices adjacent to all four vertices $v_1, v_2, v_3, v_4$.

We denote the set of these vertices by $D$. Let $\tilde{C}_{1,2}$, $\tilde{C}_{2,3}$, $\tilde{C}_{3,4}$ and $\tilde{C}_{4,1}$ be defined as follows: $\tilde{C}_{1,2}$ contains the neighbors of $v_1$ and $v_2$ which are not in $D$, $\tilde{C}_{2,3}$ contains the neighbors of $v_2$ and $v_3$ which are not in $D$ and so on.

First we assume that $\tilde{C}_{1,2} \cup \tilde{C}_{2,3} \cup \tilde{C}_{3,4} \cup \tilde{C}_{4,1}$ is not empty. We can assume that there is a vertex $x \in \tilde{C}_{1,2}$. We observe that $x$ is adjacent to any member of $D$: if $y \in D$ is not adjacent to $x$, $G[\{x, y, v_2, v_3, v_4\}] \cong F_1^4$, a contradiction. Furthermore, $D$ is a clique: If there are two non-adjacent members of $D$, say $d_1$ and $d_2$, then $G[\{x, d_1, d_2, v_3, v_4\}] \cong F^2$, a contradiction. Now assume there is a vertex $z \in \tilde{C}_{2,3} \cup \tilde{C}_{4,1}$, say $z \in \tilde{C}_{2,3}$. Then, like $x$, $z$ is adjacent to any member of $D$. As shown in the case $D = \emptyset$, $x$ and $z$ can not be adjacent. But then, $G[\{x, y, z, v_2, v_4\}] \cong F^1$ for any $y \in D$, a contradiction. That is, if $\tilde{C}_{1,2} \cup \tilde{C}_{3,4} \neq \emptyset$, then $\tilde{C}_{2,3} \cup \tilde{C}_{4,1} = \emptyset$. By the same analysis as above we see that $G[\{v_1, v_2, v_3, v_4\} \cup \tilde{C}_{1,2} \cup \tilde{C}_{3,4}]$ is of type II. Thus $G$ is obtained from a graph of type II by joining the complete graph $G[D]$ and so again is of type II.

Now we assume that $\tilde{C}_{1,2} \cup \tilde{C}_{2,3} \cup \tilde{C}_{3,4} \cup \tilde{C}_{4,1}$ is empty. Thus $V(G) = \{v_1, v_2, v_3, v_4\} \cup D$. Now, $G[D]$ is not necessarily complete. However, in $G[D]$ any vertex is adjacent to all but at most one vertex: If there are vertices $x, y, z \in D$ such that $x$ is non-adjacent to both, $G[\{x, y, z, v_1, v_3\}] \cong F^2$ in the case $\{y, z\} \in E(G)$ and $G[\{x, y, z, v_1, v_3\}] \cong F^3$ otherwise. Both is contradictory. Hence, in $G[D]$ and thus in $G$ any vertex is adjacent to all but at most one vertex. That is, $G$ is obtained from a complete graph by removing a matching.

**Case 2.** $G$ contains $C_5$ as induced subgraph.

Let $C$ be that cycle and let $v_1, v_2, v_3, v_4, v_5$ be a consecutive ordering of the vertices of $C$. If there is no vertex in $G$ that has five neighbors in $C$, Lemma 2(iii) gives that $G$ is a cycle of cliques. Hence, we can assume that there is a vertex $x$ that has five neighbors in $C$. Assume there is another vertex $y \in V(G) \setminus V(C)$.

First suppose that $y$ does not have five neighbors among $C$. By Lemma 2(i), $y$ has exactly two neighbors in $C$ and they must be adjacent, say $y$ is adjacent to $v_1$ and $v_2$. If $x$ and $y$ are adjacent, $G[\{x, y, v_3, v_4, v_5\}] \cong F^1$. If $x$ and $y$ are not adjacent, $G[\{x, y, v_1, v_2, v_4\}] \cong F_1^4$. Both is contradictory.

Thus, $y$ has exactly five neighbors in $C$. If $x$ and $y$ are not adjacent, $G[\{x, y, v_1, v_2, v_4\}] \cong F^2$, a contradiction.

As $x$ and $y$ were arbitrary, $G$ is obtained from $C_5$ by joining a complete graph. Hence $G$ is of type III.

**Case 3.** $G$ contains $C_n$ as induced subgraph, for some $n \geq 6$.

Let $C$ be that cycle and let $v_1, v_2, \ldots, v_n$ be a consecutive ordering of the vertices of $C$. Assume for contradiction that there is a vertex $x$ that has $n$ neighbors in $C$. Then $G[\{x, v_1, v_2, v_3, v_5\}] \cong F^1$, a contradiction.

Lemma 2(iii) implies that $G$ is a cycle of cliques which completes the proof. □

We are now able to prove our main result.

*Proof of Theorem 1.* Let $G$ be a connected graph.

(i) ⇒ (ii): (i) ⇒ (ii): Assume that $G$ is separability-perfect. Let $H$ be a 2-connected induced subgraph of $G$ which is not complete and let $x \in V(G) \setminus V(H)$ be arbitrary.

First we suppose that $|N(x) \cap V(H)| = 0$, so $G[V(H) \cup \{x\}]$ is disconnected. Since $G$ is connected, there is a shortest path from $x$ to $V(H)$ in $G$, say $(x = v_0, v_1, \ldots, v_k)$ where $v_k \in V(H)$. Let $H' = G[V(H) \cup \{v_0, v_1, \ldots, v_k\}]$. Since $v_1$ is a cut-vertex of $H'$, $\text{CON}(H') = 1$. On the other hand, $\text{SEP}(H') \geq \text{SEP}(H) \geq 2$, a contradiction to the assumption that $G$ is separability-perfect.

Now let $|N(x) \cap V(H)| \geq 1$. If $|N(x) \cap V(H)| \leq \text{CON}(H) - 1$, then $\text{CON}(G[V(H) \cup \{x\}]) = |N(x) \cap V(H)| \leq \text{CON}(H) - 1$. Since the separability is monotone under vertex-deletion (cf. [2]), $\text{SEP}(G[V(H) \cup \{x\}]) \geq \text{SEP}(H)$, contradicting the separability-perfectness of $G$. Hence, $|N(x) \cap V(H)| \geq \text{CON}(H)$. If $N(x) \cap V(H)$ is not a clique in $G$ then let $y, z \in N(x) \cap V(H)$ be non-adjacent. Since $G$ is separability-perfect, $\text{SEP}_H(y, z) = \text{CON}(H)$. Thus,

$$\text{SEP}_{G[V(H) \cup \{x\}]}(y, z) = \text{SEP}_H(y, z) + 1 = \text{CON}(H) + 1.$$

If $|N(x) \cap V(H)| = \text{CON}(H)$, then $\text{CON}(G[V(H) \cup \{x\}]) = \text{CON}(H)$, again a contradiction to the fact that $G$ is separability-perfect. Therefore, $|N(x) \cap V(H)| \geq \text{CON}(H) + 1$, and (ii) follows.

(ii) ⇒ (iii): None of the graphs of the family $F$ fulfills (ii) as can be seen as follows. If we remove a vertex $x$ of minimum degree, we obtain a 2-connected graph $H$ which is not complete. In the cases of $F^1$, $F_n^4$ and $F_n^5$, for all $n$, $x$ has only one neighbor in $H$, a contradiction to (ii). In the cases of $F^2$ and $F^3$, $x$ has only two neighbors in $H$ and they are not adjacent, again contradicting (ii). Thus, the forbidden induced subgraphs of property (iii) are not separability-perfect and this proves (iii).

(iii) ⇒ (iv): This follows from the Lemmas 1 and 3.

(iv) ⇒ (i): We show that all graphs in $\mathcal{G}$ are separability-perfect.

Let $G$ be a block graph. If $G$ is a complete graph, then $G$ is separability-perfect by definition. Otherwise $\text{SEP}(G) = \text{CON}(G) = 1$. Since block graphs are closed under induced subgraphs, they are separability-perfect.

Let $G$ be a cycle of cliques. In the case of $G \cong K_3$, $G$ is separability-perfect by definition. In every other case we observe that $\textsc{sep}(G) = \textsc{con}(G) = 2$. Furthermore, if one of the simplicial vertices is removed, another cycle of cliques is obtained. If one of the non-simplicial vertices is removed, a block graph is obtained. Thus $G$ is separability-perfect.

Let $G$ be a complete graph with some matching $M$ removed. If $M = \emptyset$, $G$ is a complete graph and we are done. If $M \neq \emptyset$, $\textsc{sep}(G) = \textsc{con}(G) = |V(G)| - 2$. Since any connected induced subgraph of $G$ is again some complete graph with a matching removed, $G$ is separability-perfect.

Let $G$ be of type I, i.e. $G$ is obtained from the disjoint union of complete graphs by joining a non-empty complete graph $H$. Again, if $G$ is a complete graph, we are done. Otherwise we have $\textsc{sep}(G) = \textsc{con}(G) = |V(H)|$. Since any connected induced subgraph of $G$ is again of type I, $G$ is separability-perfect.

Let $G$ be of type II, i.e. $G$ is obtained from two disjoint complete graphs by adding a non-empty matching $M$ and joining a complete graph $H$. If $G$ is a complete graph, we are done. Otherwise we have $\textsc{sep}(G) = \textsc{con}(G) = |M| + |V(H)|$. Since any connected induced subgraph of $G$ is of type I or II, $G$ is separability-perfect.

Let $G$ be of type III, i.e. $G$ is obtained from $C_5$ by joining a complete graph $H$. Then $\textsc{sep}(G) = \textsc{con}(G) = |V(H)| + 2$. Furthermore, if a vertex of $H$ is removed, $G$ is again of type III. If a vertex from the $C_5$ is removed, $G$ is of type II. Hence $G$ is separability-perfect. ☐

# 3 Graphs with locally determined separability

Let $G = (V, E)$ be a graph. For any non-adjacent pair $x, y \in V$,

$$\textsc{sep}_G(x, y) \geq |N(x) \cap N(y)|. \tag{2}$$

As a consequence, for any non-complete graph $G$

$$\textsc{sep}(G) \geq \max\{|N(u) \cap N(v)| : u, v \in V, u \neq v, \{u, v\} \notin E\}. \tag{3}$$

The question arises, for which graphs equality holds in (3). We say that the separability number of $G$ is *locally determined* if $G$ is complete or equality holds in (3).

If the separability of a graph $G = (V, E)$ is locally determined, the separability number can easily be computed in $\mathcal{O}(|V|^2\Delta)$ time, where $\Delta$ denotes the maximal degree of $G$. The idea of this procedure is to determine, for any two non-adjacent vertices, the size of the intersection of their neighborhoods. With a little organization, we can replace the term $|V|^2$ by the number $m$ of edges:

**Lemma 4.** *For connected graphs with locally determined separability, the separability number can be computed in $\mathcal{O}(m\Delta)$ time.*

*Proof.* Let $G = (V, E)$ be a connected graph given by an adjacency list. We can decide in linear time if $G$ is a complete graph. In this case we are done.

If $G$ is not a complete graph, we use Algorithm 1 to solve the problem. For a given graph $G$, it computes the value

$$X^* = \max\{|N(u) \cap N(v)| : u, v \in V, u \neq v, \{u, v\} \notin E\}$$

and finds a non-adjacent pair of vertices $P = (x, y)$ with $|N(x) \cap N(y)| = X^*$. We now explain Algorithm 1 and thereby observe its correctness.

Initially, the value of the current optimal solution, $X^*$, is set to 0. In the loop starting in line 2, to any vertex of $G$ we assign an integer variable $CommonNeighbors$ (initially 0) and a Boolean variable $IsNeighbor$ (initially false).

In the loop starting in line 5, for any vertex $u$ the following is done: The inner loop starting in line 6 sets the value of $IsNeighbor(v)$ to true for any neighbor $v$ of $u$. In the inner loop starting in line 9, for any neighbor $w$ of any neighbor of $u$ the number of common neighbors of $u$ and $w$ is computed, provided that $IsNeighbor(w)$ is false. Thus, when the loop finishes in line 15, we have $CommonNeighbors(x) = |N(u) \cap N(x)|$ for any vertex $x$ with distance exactly two from $u$. For every other vertex, $CommonNeighbors$ is 0. Finally, the loop starting in line 16 updates $X^*$ and $P$ if necessary and afterward defaults all variables $CommonNeighbors$ and $IsNeighbor$.

**Algorithm 1** Compute local separability

**Require:** A connected non-complete graph $G = (V, E)$.
**Ensure:** $X^* = \max\{|N(u) \cap N(v)| : u, v \in V, u \neq v, \{u, v\} \notin E\}$ and a non-adjacent pair $P = (u, v)$ with $X^* = |N(u) \cap N(v)|$.

1: $X^* \leftarrow 0$
2: **for all** $u \in V$ **do**
3:   $CommonNeighbors(u) \leftarrow 0$ and $IsNeighbor(u) \leftarrow$ **false**
4: **end for**
5: **for all** $u \in V$ **do**
6:   **for all** neighbors $v$ of $u$ **do**
7:     $IsNeighbor(v) \leftarrow$ **true**
8:   **end for**
9:   **for all** neighbors $v$ of $u$ **do**
10:     **for all** neighbors $w$ of $v$ **do**
11:       **if** $IsNeighbor(w) =$ **false then**
12:         $CommonNeighbors(w) \leftarrow CommonNeighbors(w) + 1$
13:       **end if**
14:     **end for**
15:   **end for**
16:   **for all** neighbors $v$ of $u$ **do**
17:     **for all** neighbors $w$ of $v$ **do**
18:       **if** $X^* < CommonNeighbors(w)$ **then**
19:         $X^* \leftarrow CommonNeighbors(w)$ and $P \leftarrow (u, w)$
20:       **end if**
21:       $CommonNeighbors(w) \leftarrow 0$
22:     **end for**
23:     $IsNeighbor(v) \leftarrow$ **false**
24:   **end for**
25: **end for**
26: **return** $P$ and $X^*$

It remains to analyze the running time of Algorithm 1. The loop starting in line 2 is done in $\mathcal{O}(|V|)$ steps. For any vertex $u \in V$, the loop starting in line 6 is done in $\mathcal{O}(|N(u)|)$ steps. The loop starting in line 9 needs $\mathcal{O}(|N(u)|\Delta)$ steps. The same holds for the loop starting in line 16.

Since $G$ is connected, $|E| \geq |V|-1$. Thus we have $\mathcal{O}(|V|+\sum_{u\in V}|N(u)|\Delta) = \mathcal{O}(|E|\Delta)$ steps in total. $\square$

In the following, we need an easy consequence of Menger's Theorem [5].

**Lemma 5.** *Let $G$ be a graph and let $x$ and $y$ be two distinct non-adjacent vertices of $V(G)$. Then $\text{SEP}_G(x,y)$ equals the maximal number of internally vertex-disjoint paths from $x$ to $y$. Furthermore, $\text{SEP}(G)$ equals the maximal number of internally vertex-disjoint paths with the same non-adjacent end-vertices.*

If $G = (V,E)$ is a non-complete graph with $\text{SEP}(G) > 0$ for which the separability number is locally determined, then there is a non-adjacent pair $x, y \in V$ that has exactly $\text{SEP}(G)$ common neighbors. This means that any induced path from $x$ to $y$ has length exactly two. In the literature, a pair of vertices $x, y$ for which any induced path from $x$ to $y$ has length exactly two is called a *two-pair* [6, 7]. Thus we know that such a graph $G$ necessarily has a two-pair. As the following discussion shows, the notion of a two-pair is closely related to the question of locally determined separability.

A chordless cycle of length at least five is called a *hole*. The complement of a hole is called an *anti-hole*. A graph $G$ is called *weakly chordal* if it does not contain a hole or an anti-hole as induced subgraph. It is shown by Hayward, Hoàng and Maffray [6] that every non-complete weakly chordal graph has a two-pair. On the other hand, holes and anti-holes do not have a two-pair [6]. This leads to the following:

**Lemma 6.** *Let $G$ be a graph for which in any induced subgraph the separability is locally determined. Then $G$ is weakly chordal.*

We conjecture that the other direction holds as well:

**Conjecture 1.** *For any weakly chordal graph the separability is locally determined.*

Note that, since the class of weakly chordal graphs is closed under induced subgraphs, Conjecture 1 combined with Lemma 6 would give a new characterization of the class of weakly chordal graphs.

So far we can only prove a partial result of Conjecture 1, namely Theorem 2 stated below. A graph is called *HH-free* if it does not contain a hole or the house (both displayed in Fig. 4) as induced subgraph. Since any anti-hole other than $C_5$ contains a house as induced subgraph, the class of HH-free graphs is a proper subclass of the weakly chordal graphs [1]. On the other hand, the class of HH-free graphs is a proper superclass of both chordal graphs and distance-hereditary graphs [1].

Recently, a new characterization of the HH-free graphs was found by Kratsch, Spinrad and Sritharan [7]. Roughly speaking, it says that a graph is HH-free if and only if in any induced subgraph any vertex is part of a two-pair. We need the non-trivial direction of this characterization:

**Lemma 7** (Kratsch, Spinrad and Sritharan [7])**.** *Let $G$ be a connected HH-free graph. Every vertex of $G$ either is adjacent to all other vertices or is part of a two-pair.*

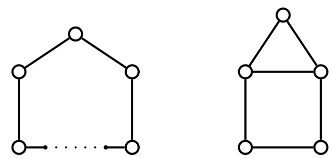

Figure 4: Hole and house. The two vertices at the bottom are the end-vertices of a path on $n \geq 2$ vertices symbolized by the dotted line.

Using Lemma 7, we obtain the following result.

**Theorem 2.** *For any HH-free graph, the separability is locally determined.*

*Proof.* Let $G$ be an HH-free graph. Since for a disjoint union of cliques the separability is locally determined, we may assume that $\text{SEP}(G) \geq 1$. Let $H = (V, E)$ be an induced subgraph of $G$ such that $\text{SEP}(H) = \text{SEP}(G)$ and the removal of any vertex from $H$ results in a graph with lower separability. Let $x, y \in V$ be two non-adjacent vertices such that $\text{SEP}_H(x, y) = \text{SEP}(H)$. We observe that $H$ is connected.

Suppose there is a proper subset $S$ of $N_H(x)$ that is a cut-set of $H$. Let $C^1, C^2, \ldots, C^k$ be the connected components of $H[V \setminus S]$. As $S$ is a proper subset of $N_H(x)$, $x$ and $y$ belong to the same component of $H[V \setminus S]$, say $C^1$. Otherwise, some vertex of $N_H(x) \setminus S$ could be removed from $H$ without a decrease of the separability. By choice of $H$, there is a path $P$ in $H$ from $x$ to $y$ that contains a vertex $u$ of a component $C^i$ for some $2 \leq i \leq k$. But then $P$ contains at least two vertices from $S$, say $v$ and $w$. Since $v$ and $w$ are neighbors of $x$, $P$ cannot be an induced path. Thus no induced path from $x$ to $y$ contains $u$. By Lemma 5, $x$ and $y$ are connected by exactly $\text{SEP}_H(x, y)$ internally vertex-disjoint paths and each of these paths can clearly be chosen to be induced. Hence for the graph $H' = H[V \setminus \{u\}]$ it holds that $\text{SEP}(H') = \text{SEP}_{H'}(x, y) = \text{SEP}_H(x, y) = \text{SEP}(H)$, a contradiction to the choice of $H$.

Therefore no proper subset of $N_H(x)$ is a cut-set of $H$. Since $H$ is connected and $x$ is not adjacent to $y$, $x$ is contained in a two-pair by Lemma 7. Let $z$ be the partner of $x$ in this two-pair. Suppose $z \neq y$. By choice of $H$, $\text{SEP}_H(x, z) < \text{SEP}(H)$, since otherwise $y$ is superfluous. Hence, the set $S = N_H(x) \cap N_H(z)$ is a proper subset of $N_H(x)$. As seen before, $S$ cannot be a cut-set of $H$. Thus there is an induced path $P$ from $x$ to $z$ in $H[V \setminus S]$. By choice of $S$, $P$ has length at least three, a contradiction to the fact that $x$ and $z$ form a two-pair. Hence, $z = y$ and so $\text{SEP}_H(x, y) = |N_H(x) \cap N_H(y)|$ by Lemma 5 and (2).

All in all,

$$\text{SEP}(G) = \text{SEP}(H) = \text{SEP}_H(x, y) = |N_H(x) \cap N_H(y)| \leq |N_G(x) \cap N_G(y)|.$$

This, together with $|N_G(x) \cap N_G(y)| \leq \text{SEP}(G)$ by (3), shows that the separability of $G$ is locally determined. $\square$

Combining Lemma 4 and Theorem 2, we obtain the following.

**Corollary 1.** *In the class of connected HH-free graphs, the separability number is computed in $\mathcal{O}(m\Delta)$ time, where $m$ denotes the number of edges and $\Delta$ denotes the maximum degree of the graph considered.*

Note that the separability can be computed in polynomial time in general [2], but probably not in $\mathcal{O}(m\Delta)$ time.

As said before, the chordal graphs form a proper subclass of the HH-free graphs. Indeed, for any chordal graph $G$ and any pair of non-adjacent vertices $x, y \in V(G)$, the set $N(x) \cap N(y)$ is a clique. Hence, Theorem 2 has the following consequence.

**Corollary 2.** *For any non-complete chordal graph $G$,*

$$\textsc{sep}(G) = \max\{n : K_{n+2}^{-} \text{ is an induced subgraph of } G\}.$$

# 4 Edge-separability

Let $G$ be a graph and let $x$ and $y$ be two distinct vertices of $V(G)$. We define the *edge-separability* of $x$ and $y$, denoted $\textsc{sep}'_G(x,y)$, as the minimal number of edges whose removal from $G$ separates $x$ and $y$. The *edge-separability* of $G$, denoted $\textsc{sep}'(G)$, is defined by

$$\textsc{sep}'(G) = \max\{\textsc{sep}'_G(x,y) : x, y \in V(G), x \neq y\}.$$

$G$ is called *$k$-edge-separable* if $\textsc{sep}'(G) \leq k$. Hence, the $k$-edge-separable graphs are those graphs where any pair of vertices can be separated by the removal of at most $k$ edges.

By Menger's Theorem [5], we immediately obtain the following equality:

**Lemma 8.** *Let $G$ be a graph and let $x$ and $y$ be two distinct vertices of $V(G)$. Then $\textsc{sep}'_G(x,y)$ equals the maximal number of edge-disjoint paths from $x$ to $y$. Furthermore, $\textsc{sep}'(G)$ equals the maximal number of edge-disjoint paths with the same end-vertices.*

Furthermore, we have the following useful observation. As usual, a *block* in a graph is a maximal two-connected component.

**Lemma 9.** *Let $G$ be a graph and let $G_1, G_2, \ldots, G_k$ be its blocks. Then*

$$\textsc{sep}'(G) = \max_{i=1,\ldots,k} \textsc{sep}'(G_i). \tag{4}$$

*Proof.* Let $G$ be a graph and let $G_1, G_2, \ldots, G_k$ be its blocks. Clearly $\textsc{sep}'(G) \geq \max_{i=1,\ldots,k} \textsc{sep}'(G_i)$.

Suppose $\textsc{sep}'(G) > \max_{i=1,\ldots,k} \textsc{sep}'(G_i)$. Let $x$ and $y$ be of minimum distance in $G$ such that $\textsc{sep}'_G(x,y) = \textsc{sep}'(G)$. By assumption, $x$ and $y$ do not belong to the same block of $G$. Hence, there is a cut-vertex $z$, distinct from $x$ and $y$, that separates $x$ from $y$. Thus, any path connecting $x$ and $y$ contains $z$. Hence, $\textsc{sep}'_G(x,z) \geq \textsc{sep}'_G(x,y) = \textsc{sep}'(G)$, in contradiction to the choice of $x$ and $y$. $\square$

Indeed, any partial subgraph of a $k$-edge-separable graph is $k$-edge-separable, too. However, the next lemma shows that the class of $k$-edge-separable graphs is closed under a stronger operation.

**Lemma 10.** *For any $k$, the class of $k$-edge-separable graphs is closed under topological minors.*

*Proof.* Let $G$ be a $k$-edge-separable graph for some $k$. We can assume that $G$ is connected. Observe that the removal of vertices and edges does not increase the edge-separability of $G$.

Now let $v \in V(G)$ be a vertex of degree two (say $N_G(v) = \{u, w\}$) and let $G'$ be the graph obtained from $G$ by contracting $v$. By definition, $G'$ contains the edge $\{u, w\}$. Suppose $\textsc{sep}'(G') > \textsc{sep}'(G)$. Thus there are two vertices $x, y \in V(G')$ such that $\textsc{sep}'_{G'}(x, y) \geq k+1$. From Lemma 8 we know that there are $k+1$ edge-disjoint paths from $x$ to $y$. By assumption, one of these paths has to contain the edge $\{u, w\}$. But then this edge can be substituted by the edges $\{u, v\}$ and $\{v, w\}$ to obtain a path in $G$ which is still edge-disjoint to the other $k$ paths. By Lemma 8, $G$ is $(k+1)$-edge-separable, a contradiction. $\square$

In contrast, $k$-edge-separable graphs are not closed under (induced) minors in general.

**Lemma 11.** *For each $k \geq 3$, the class of $k$-separable graphs is not closed under edge-contraction.*

*Proof.* Let $k \geq 3$ be arbitrary. Consider the graph $G_k$ defined by

$$\begin{aligned} V(G_k) &= \{u, v, w, x_1, x_2, \ldots, x_{k+1}\}, \\ E(G_k) &= \{\{u, x_i\} : 1 \leq i \leq k+1\} \cup \{\{v, x_i\} : 1 \leq i \leq k-1\} \\ &\quad \cup \{\{w, x_i\} : k \leq i \leq k+1\} \cup \{\{v, w\}\}. \end{aligned}$$

$G_k$ is displayed schematically in Fig. 5. We observe that

$$\begin{aligned} \textsc{sep}'_G(x_i, y) &= 2, \text{ for all } 1 \leq i \leq k+1 \text{ and } y \in V(G_k) \setminus \{x_i\}, \\ \textsc{sep}'_G(u, v) &= k, \\ \textsc{sep}'_G(u, w) &= 3, \\ \textsc{sep}'_G(v, w) &= 3. \end{aligned}$$

Now consider the graph $G'_k$ (displayed schematically in Fig. 5) obtained from $G_k$ by contracting the edge $\{v, w\}$ to a single vertex $z$. $G'_k$ is obtained from $G_k$ by a single edge-contraction, but $\textsc{sep}'_{G'_k}(u, z) = k+1$. Thus, the class of $k$-edge-separable graphs is not closed under edge-contraction. $\square$

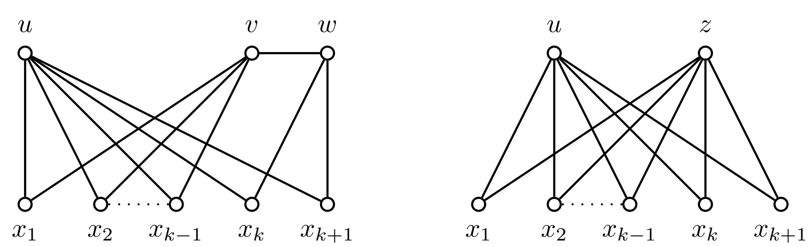


Figure 5: $G_k$ and $G'_k$ from the proof of Lemma 11.

Our next result gives the characterization in terms of forbidden topological minors for the $k$-edge-separable graphs for each $0 \leq k \leq 3$. For this, we need the following special graphs: The graph $A$ is the join of $K_2$ and the complement of $K_3$ (cf. Fig. 6).

For each $n \geq 3$, the graph $B_n$ is defined as follows:

$$\begin{aligned}
V(B_n) &= \{v_i, u_i : 1 \leq i \leq n\}, && (5)\\
E(B_n) &= \{\{v_i, v_{i+1}\}, 1 \leq i \leq n-1\} \cup \{\{v_i, u_i\}, 1 \leq i \leq n\} && (6)\\
&\quad \cup \{\{v_{i+1}, u_i\}, 1 \leq i \leq n-1\} \cup \{\{v_n, v_1\}, \{v_1, u_n\}\}. && (7)
\end{aligned}$$

As an example, $B_4$ are displayed in Fig. 6.

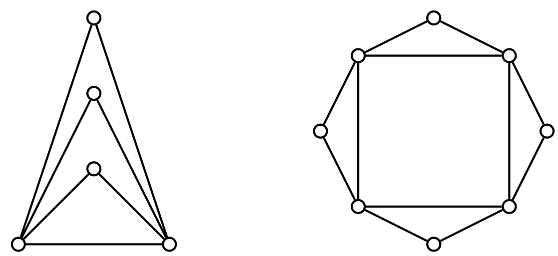

Figure 6: $A$ and $B_4$.

**Theorem 3.** *(i) A graph is 0-edge-separable if and only if it does not contain $K_2$ as topological minor. That is, $G$ consists of isolated vertices only.*

*(ii) A graph is 1-edge-separable if and only if it does not contain $K_3$ as topological minor. That is, $G$ is an acyclic graph.*

*(iii) A graph is 2-edge-separable if and only if it does not contain $K_4^-$ as topological minor. That is, any block of $G$ is an edge or a cycle.*

*(iv) A graph is 3-edge-separable if and only if it does not contain $A$ or $B_n$, for any $n \geq 3$, as topological minor.*

*Proof.* We omit the proofs of (i) and (ii) since both are straightforward.

(iii) We observe that $\text{SEP}'(K_4^-) = 3$. On the other hand, let $G$ be a graph that does not contain $K_4^-$ as topological minor. As shown by El-Mallah and Colbourn [4], any block of $G$ is an edge or a cycle. By Lemma 9, $G$ is 2-edge-separable.

(iv) We observe that $\text{SEP}'(A) = \text{SEP}'(B_n) = 4$ for any $n$. On the other hand, let $G$ be a graph with $\text{SEP}'(G) \geq 4$. We can assume that any proper topological minor of $G$ is 3-edge-separable. In this sense we call $G$ minimal. In particular, $\text{SEP}'(G) = 4$. Hence, there are two vertices $x$ and $y$ that have four edge-disjoint paths connecting them.

In the following argumentation, we need multiple indices. We use upper and lower indices which are not to be confused with arithmetical expressions (e.g. powers of numbers).

We claim that we can choose $x$ and $y$ such that there are four edge-disjoint paths connecting them, say $P^1$, $P^2$, $P^3$ and $P^4$, with the property that $P^1$ is internally vertex-disjoint to $P^2$, $P^3$ and $P^4$ and furthermore $P^2$ is internally vertex-disjoint to $P^3$ and $P^4$.

For this, choose $x$ and $y$ such that $\mathrm{SEP}'_G(x,y) = 4$ and let $P^1 = (x = p^1_1, p^1_2, \ldots, p^1_{k^1} = y)$, $P^2 = (x = p^2_1, p^2_2, \ldots, p^2_{k^2} = y)$, $P^3 = (x = p^3_1, p^3_2, \ldots, p^3_{k^3} = y)$ and $P^4 = (x = p^4_1, p^4_2, \ldots, p^4_{k^4} = y)$ be any four edge-disjoint paths connecting them. Let $i^1$ be the smallest number such that $p^1_{i^1}$ is contained in one of the other paths ($P^2$, $P^3$ or $P^4$). Let $i^2$, $i^3$ and $i^4$ be chosen accordingly.

Suppose $p^1_{i^1}$, $p^2_{i^2}$, $p^3_{i^3}$ and $p^4_{i^4}$ are mutually distinct. We can assume that $p^1_{i^1}$ is contained in $P^2$, say $p^1_{i^1} = p^2_{j^2}$.

Assume that $p^2_{i^2}$ is contained in $P^1$, say $p^2_{i^2} = p^1_{j^1}$. Then $i^1 < j^1$ and $i^2 < j^2$ by assumption. But then the paths $P^1$ and $P^2$ can be substituted by the paths

$$(x = p^1_1, p^1_2, \ldots, p^1_{i^1} = p^2_{j^2}, p^2_{j^2+1}, \ldots, p^2_{k^2} = y)$$

and

$$(x = p^2_1, p^2_2, \ldots, p^2_{i^2} = p^1_{j^1}, p^1_{j^1+1}, \ldots, p^1_{k^1} = y).$$

We observe that the non-empty partial paths $(p^1_{i^1}, p^1_{i^1+1}, \ldots, p^1_{j^1})$ and $(p^2_{i^2}, p^2_{i^2+1}, \ldots, p^2_{j^2})$ are superfluous. This contradicts the minimality of $G$.

Thus we can assume that $p^2_{i^2}$ is contained in $P^3$, say $p^2_{i^2} = p^3_{j^3}$.

By similar argumentation as above, $p^3_{i^3}$ is not contained in $P^2$. Suppose that $p^3_{i^3}$ is contained in $P^1$, say $p^3_{i^3} = p^1_{j^1}$. Then $i^1 < j^1$, $i^2 < j^2$ and $i^3 < j^3$ by assumption. But then the paths $P^1$, $P^2$ and $P^3$ can be substituted by the paths

$$(x = p^1_1, p^1_2, \ldots, p^1_{i^1} = p^2_{j^2}, p^2_{j^2+1}, \ldots, p^2_{k^2} = y),$$

$$(x = p^2_1, p^2_2, \ldots, p^2_{i^2} = p^3_{j^3}, p^3_{j^3+1}, \ldots, p^3_{k^3} = y),$$

and

$$(x = p^3_1, p^3_2, \ldots, p^3_{i^3} = p^1_{j^1}, p^1_{j^1+1}, \ldots, p^1_{k^1} = y).$$

Again this contradicts the minimality of $G$, since the non-empty partial paths $(p^1_{i^1}, p^1_{i^1+1}, \ldots, p^1_{j^1})$, $(p^2_{i^2}, p^2_{i^2+1}, \ldots, p^2_{j^2})$ and $(p^3_{i^3}, p^3_{i^3+1}, \ldots, p^3_{j^3})$ are superfluous.

Thus $p^3_{i^3}$ is contained in $P^4$, say $p^3_{i^3} = p^4_{j^4}$. By similar argumentation as above, $p^4_{i^4}$ is contained in $P^1$, say $p^4_{i^4} = p^1_{j^1}$. Then $i^1 < j^1$, $i^2 < j^2$, $i^3 < j^3$ and $i^4 < j^4$ by assumption. But then the paths $P^1$, $P^2$, $P^3$ and $P^4$ can be substituted by the paths

$$(x = p^1_1, p^1_2, \ldots, p^1_{i^1} = p^2_{j^2}, p^2_{j^2+1}, \ldots, p^2_{k^2} = y),$$

$$(x = p^2_1, p^2_2, \ldots, p^2_{i^2} = p^3_{j^3}, p^3_{j^3+1}, \ldots, p^3_{k^3} = y),$$

$$(x = p^3_1, p^3_2, \ldots, p^3_{i^3} = p^4_{j^4}, p^4_{j^4+1}, \ldots, p^4_{k^4} = y),$$

and

$$(x = p^4_1, p^4_2, \ldots, p^4_{i^4} = p^1_{j^1}, p^1_{j^1+1}, \ldots, p^1_{k^1} = y).$$

Again this contradicts the choice of $G$, since the non-empty partial paths $(p^1_{i^1}, p^1_{i^1+1}, \ldots, p^1_{j^1})$, $(p^2_{i^2}, p^2_{i^2+1}, \ldots, p^2_{j^2})$, $(p^3_{i^3}, p^3_{i^3+1}, \ldots, p^3_{j^3})$ and $(p^4_{i^4}, p^4_{i^4+1}, \ldots, p^4_{j^4})$ are superfluous.

All in all we see that $p^1_{i^1}$, $p^2_{i^2}$, $p^3_{i^3}$ and $p^4_{i^4}$ cannot be mutually distinct. We can assume $p^1_{i_1} = p^2_{i_2}$. Let $y' = p^1_{i_1} = p^2_{i_2}$. There are four edge-disjoint paths connecting $x$ and $y'$:

$$Q^1 = (x = p^1_1, p^1_2, \ldots, p^1_{i^1} = y'),$$

$$Q^2 = (x = p_1^2, p_2^2, \dots, p_{i^2}^2 = y'),$$
$$Q^3 = (x = p_1^3, p_2^3, \dots, p_{k^3}^3 = y = p_{k^1}^1, p_{k^1-1}^1, \dots, p_{i^1}^1 = y'),$$

and
$$Q^4 = (x = p_1^4, p_2^4, \dots, p_{k^4}^4 = y = p_{k^2}^2, p_{k^2-1}^2, \dots, p_{i^2}^2 = y').$$

Hence $\textsc{sep}'_G(x, y') = 4$ and furthermore $Q^1$ is vertex-disjoint to $Q^2$, $Q^3$ and $Q^4$. Moreover, $Q^2$ is vertex-disjoint to $Q^3$ and $Q^4$. This proves our claim.

Hence, we can choose $x$ and $y$ such that $P^1$ is internally vertex-disjoint to $P^2$, $P^3$ and $P^4$ and furthermore $P^2$ is internally vertex-disjoint to $P^3$ and $P^4$. If $P^3$ is internally vertex-disjoint to $P_4$, $G \cong A$ by minimality.

Assume that $P^3$ is not internally vertex-disjoint to $P_4$. Again let $P^3 = (x = p_1^3, p_2^3, \dots, p_{k^3}^3 = y)$ and $P^4 = (x = p_1^4, p_2^4, \dots, p_{k^4}^4 = y)$. Let $p_{i_1^3}^3, p_{i_2^3}^3, \dots, p_{i_j^3}^3$ with $i_1^3 < i_2^3 < \dots < i_j^3$ be the vertices of $P_3$ that are contained in $P^4$. Conversely let $p_{i_1^4}^4, p_{i_2^4}^4, \dots, p_{i_j^4}^4$ with $i_1^4 < i_2^4 < \dots < i_j^4$ be the vertices of $P_4$ that are contained in $P^3$.

Clearly $p_{i_1^3}^3 = p_{i_1^4}^4 = x$ and $p_{i_j^3}^3 = p_{i_j^4}^4 = y$. We claim that $p_{i_l^3}^3 = p_{i_l^4}^4$ for all $1 \le l \le j$. To see this, choose $m$ minimal such that $p_{i_m^3}^3 \neq p_{i_m^4}^4$. Let $p_{i_m^3}^3 = p_{i_n^4}^4$ and $p_{i_m^4}^4 = p_{i_r^3}^3$. Then $i_m^3 < i_r^3$ and $i_m^4 < i_n^4$. But then the paths $P^3$ and $P^4$ can be substituted by the paths

$$(x = p_1^3, p_2^3, \dots, p_{i_m^3}^3 = p_{i_n^4}^4, p_{i_n^4+1}^4, \dots, p_{k^4}^4 = y),$$

and
$$(x = p_1^4, p_2^4, \dots, p_{i_m^4}^4 = p_{i_r^3}^3, p_{i_r^3+1}^3, \dots, p_{k^3}^3 = y).$$

We observe that the partial paths $(p_{i_m^3}^3, p_{i_m^3+1}^3, \dots, p_{i_r^3}^3)$ and $(p_{i_m^4}^4, p_{i_m^4+1}^4, \dots, p_{i_n^4}^4)$ are superfluous. Since, $p_{i_m^3}^3 \neq p_{i_m^4}^4$, both paths are non-empty. This contradicts to the minimality of $G$.

Therefore, $p_{i_l^3}^3 = p_{i_l^4}^4$ for all $1 \le l \le j$. It is straightforward that $G \cong B_n$. $\square$

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