

On the non-unit count of interval graphs

(revised version)

A. Apke^a, R. Schrader^{a,*}

^a*Dept. of Computer Science, University of Cologne, Weyertal 80, 50931 Cologne, Germany*

Abstract

We introduce the non-unit count of an interval graph as the minimum number of intervals in an interval representation whose lengths deviate from one. We characterize a variant of the non-unit count (where all interval lengths are required to be at least one) and graphs with non-unit count 1.

Key words: interval graph, unit interval graph, comparability invariant, intersection graph

1. Introduction

Interval graphs reflect the intersection structure of intervals in the real line. For each vertex of an interval graph $G = (V, E)$ there is an interval I_v such that $(u, v) \in E$ if and only if $I_u \cap I_v \neq \emptyset$. Each such collection also defines an *interval order* as a partial order $P = (V, <)$ via $u < v$ if and only if I_u is completely left of I_v . Interval graphs and interval orders have been characterized in various ways (cf., e.g. [6, 8, 11]). For more details, the interested reader is referred to [5, 9].

A natural question, apparently first asked by R.L. Graham, is how many different interval lengths are needed to represent an interval graph. He introduced the *interval count* as the minimum number of distinct interval lengths necessary in an interval representation of an interval graph. Interval graphs with interval count 1 are the *unit interval graphs*. They were first characterized by Roberts [15] as the class of proper interval graphs or, equivalently, as the claw-free interval graphs. Shorter proofs of these characterizations and efficient recognition algorithms can be found in, e.g., [1, 4, 7, 12]. For interval graphs with interval count $k, k \geq 2$ the recognition problem seems to be open. Further results on the interval count can be found in, e.g. [2, 3].

Graham conjectured that the interval count of a graph G is at most $k + 1$ if for some vertex x the graph $G \setminus x$ has interval count k . This conjecture was proved by Leibowitz et al. [10] for $k = 1$ and disproved for $k \geq 2$. Observe that in the first case, $G \setminus x$ is a unit interval graph and x must be contained in every claw of G .

Skrien [16] and Rautenbach and Szwarcfiter [13] investigated a subclass of graphs with interval count 2. In [13] Rautenbach and Szwarcfiter give a forbidden subgraph

*Corresponding author

characterization of those graphs that have an interval representation using unit intervals and single points. They also describe a linear time recognition algorithm. In [14] the same authors characterize graphs having a representation by open and closed unit intervals.

In the following we ask a slightly different question: how many intervals in an interval representation must have a length different from one. In Section 2 we collect the basic notations and definitions. In Sections 3 and 4 we present some general results and give exact answers for two special cases where all interval lengths are required to be at least one or where all but one interval have the same length.

2. Preliminaries

Let $G = (V, E)$ be an interval graph and R a collection of intervals such that for each $v \in V$ there is an interval $I_v \in R$ and $(u, v) \in E$ if and only if I_u and I_v have a nonempty intersection. We then say that R *realizes* G . The set of all collections of intervals realizing G is denoted by $\mathcal{R}(G)$. For an interval I let $l(I)$ ($r(I)$) be its left (right) endpoint. W.l.o.g. we assume throughout that all interval endpoints are distinct. Let $|I|$ denote the length of I . The collection R of intervals also realizes the partial order $(V, <)$ given by $u < v$ if and only if $r(I_u) < l(I_v)$. $\mathcal{R}(P)$ is the set of all realizers of the interval order P .

We call two interval orders P_1, P_2 equivalent ($P_1 \sim P_2$) if they have realizers which realize the same interval graph G . The corresponding equivalence class is denoted by $\mathcal{P}(G)$. A function f operating on interval orders is a *comparability invariant* if $f(P_1) = f(P_2)$ whenever $P_1 \sim P_2$.

We call a bipartite graph $K_{1,r}$, $r \geq 3$, a *star* and a *claw*, if $r = 3$. A vertex u of a graph G is a *center* of G if u together with its neighbors $N(u)$ induces a star in G . The vertices in $N(u)$ are then called *leaves*. The set of centers of G is denoted by $Z = Z(G)$, the set of leaves by $U = U(G)$. Observe that the sets Z and U do not have to be disjoint. For an interval order P the centers $Z(P)$ of P are the centers of the corresponding interval graph.

An interval graph is a *unit interval graph* if it has a realizer in which all intervals have the same length. The corresponding partial order is then called a *semiorder*. Roberts [15] showed that unit interval graphs are characterized by the fact that they have a realization by a *proper* collection of intervals in which no interval properly contains another. Moreover, they are precisely the interval graphs having no induced *claw*.

Let P be an interval order and $R \in \mathcal{R}(P)$ a realization. The *non-unit count* $\tau(R)$ is the number of intervals in R with length different from one. Similarly, let

$$\tau(P) = \min\{\tau(R) | R \in \mathcal{R}(P)\} \tag{1}$$

and

$$\tau(G) = \min\{\tau(R) | R \in \mathcal{R}(G)\}. \tag{2}$$

Then, for an interval graph G and $P \in \mathcal{P}(G)$ we have $\tau(G) \leq \tau(P)$ and, obviously, $\tau(G) = \min\{\tau(P) | P \in \mathcal{P}(G)\}$. For unit interval graphs G we have $\tau(G) = \tau(P) = 0$ for all semiorders $P \in \mathcal{P}(G)$. This seems to suggest that the non-unit count is a comparability invariant. This, however, is not true. Figure 1 shows two interval orders with $P_1 \sim P_2$ and $\tau(P_1) = 3, \tau(P_2) = 2$.

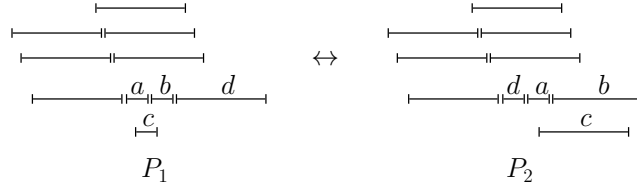


Figure 1: Two equivalent interval orders with different non-unit count

Since unit interval graphs are the claw-free interval graphs, one may conjecture that the non-unit count is the cardinality $\nu(G)$ of the smallest set $W \subseteq V$ such that $G \setminus W$ is claw-free. Again, this is not true. To see this, consider Figure 2.

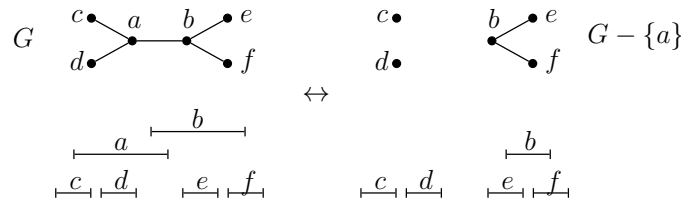


Figure 2: $G \setminus a$ is claw-free but $\tau(G) = 2$

If we remove either a or b , the resulting graph is claw-free. On the other hand, the intervals I_a and I_b have to overlap and both have to cover two non-overlapping intervals. So $\tau(G) = 2$.

Indeed, in general we have $\nu(G) \leq \tau(G)$. A related parameter is introduced in [5]. Fishburn defines $\kappa(n)$ as the maximum cardinality k such that every interval graph on n vertices contains a unit interval graph on k vertices. For interval graphs G on n vertices this obviously gives $\nu(G) \leq n - \kappa(n)$.

3. The normalized non-unit count

In this section we consider the *normalized non-unit count* where we consider only interval representations in which all intervals have length at least one. We denote the normalized non-unit count as $\tau_{>}(R), \tau_{>}(P), \tau_{>}(G)$, resp.).

Consider an interval representation $R \in \mathcal{R}(G)$ and a center $z \in Z$. Then I_z properly contains at least one interval I_x for some $x \in V \setminus Z$. Hence, if $|I_x| \geq 1$, then $|I_z| > 1$. This immediately implies $\tau_{>}(G) \geq |Z|$. Since $\tau_{>}(P) \geq \tau_{>}(G)$ for all $P \in \mathcal{P}(G)$, we also have $\tau_{>}(P) \geq |Z|$.

Let G be an interval graph G and $R \in \mathcal{R}(G)$ some realization. We call a subset $V' \subseteq V$ *R-proper* if the restriction $R[V']$ to V' is a proper collection of intervals. We adopt a technique from [1] to normalize the intervals in V' to one. We proceed in two steps. In the first step we may create intervals of length less than one. This will be corrected in the second step.

Lemma 1. *Let $P = (V, \leq)$ be an interval order and $R \in \mathcal{R}(P)$ a realization with $|I| \geq 1$ for all $I \in R$. Let $V' \subseteq V$ be R -proper. Then there exists a realization $\widehat{R} \in \mathcal{R}(P)$ with $\widehat{R} = \{\widehat{I}_x | x \in V\}$ such that $|\widehat{I}_x| = 1$ for all $x \in V'$.*

PROOF. Since V' is R -proper we may order the vertices in V' according to increasing left (resp. right) endpoints of their corresponding intervals in $R[V']$. So let $V' = \{x_1, \dots, x_{|V'|}\}$. We perform induction on $|V'|$. If $|I_{x_1}| > 1$ we do the following.

In a first step we move all endpoints p in the interval $(l(I_{x_1}), r(I_{x_1}))$ to a new point \widehat{p} in the interval $(l(I_{x_1}), l(I_{x_1}) + 1)$ via

$$p \mapsto \widehat{p} = l(I_{x_1}) + \frac{p - l(I_{x_1})}{r(I_{x_1}) - l(I_{x_1})}. \quad (3)$$

This transformation preserves the order of the endpoints so that the new intervals are still in $\mathcal{R}(P)$. Moreover, for all $p \in (l(I_{x_1}), r(I_{x_1}))$ we have $\frac{p - l(I_{x_1})}{r(I_{x_1}) - l(I_{x_1})} < 1$. Hence after the transformation there are no endpoints in the interval $[l(I_{x_1}) + 1, r(I_{x_1}))$. This allows us to move $r(I_{x_1})$ to $r(\widehat{I}_{x_1}) = l(I_{x_1}) + 1$ without changing the intersection structure of the intervals. Setting $l(\widehat{I}_{x_1}) = l(I_{x_1})$, we get $|\widehat{I}_{x_1}| = 1$.

Assume that we have already constructed a realization $R \in \mathcal{R}(P)$ with $|I_{x_1}| = \dots = |I_{x_{i-1}}| = 1$. If $|I_{x_i}| > 1$ let $m_i := \max\{l(I_{x_i}), r(\widehat{I}_{x_{i-1}})\}$. Since by induction $|\widehat{I}_{x_{i-1}}| = 1$ and $l(\widehat{I}_{x_{i-1}}) < l(I_{x_i})$ we have $r(\widehat{I}_{x_{i-1}}) < l(I_{x_i}) + 1$ and also $m_i < l(I_{x_i}) + 1$. Hence the interval $I_i := (m_i, l(I_{x_i}) + 1)$ has a positive length.

We now apply the transformation

$$p \mapsto \widehat{p} = m_i + |I_i| \frac{p - m_i}{r(I_{x_i}) - m_i}$$

to move all endpoints p in the interval $(m_i, r(I_{x_i}))$ into the interval $(m_i, l(I_{x_i}) + 1)$. As before, this transformation does not affect the order of the endpoints and there is no endpoint in the interval $[l(I_{x_i}) + 1, r(I_{x_i}))$. Hence we may move $r(I_{x_i})$ to $r(\widehat{I}_{x_i}) = l(I_{x_i}) + 1$.

This operation transforms the interval I_{x_i} to the interval \widehat{I}_{x_i} of length one. Since by the choice of m_i , the endpoints of the intervals $\widehat{I}_{x_1}, \dots, \widehat{I}_{x_{i-1}}$ are to the left of m_i and points $p < m_i$ are not moved, we still have $|I_{x_1}| = \dots = |I_{x_{i-1}}| = 1$. As the resulting set of intervals is a realization of P , the claim follows. \square

The previous construction may create intervals of length less than one. This will be corrected in the next Lemma where we link R -proper subsets V' to the set of non-centers.

Lemma 2. *Let $P = (V, \leq)$ be an interval order and $\overline{V} := V \setminus Z(P)$. Then there is an $R \in \mathcal{R}(P)$ such that \overline{V} is R -proper.*

PROOF. Consider some $R \in \mathcal{R}(P)$. If \overline{V} is not R -proper there exist $x, y \in \overline{V}$ with $I_y \subset I_x$. Then $l(I_x) < l(I_y) < r(I_y) < r(I_x)$. Let $r(I_v)$ be the right interval end next to the left of $l(I_y)$.

If $l(I_x) > r(I_v)$ or no such interval I_v exists then we may move $l(I_y)$ to the left beyond $l(I_x)$ without changing the overlap structure of the intervals. Then I_x and I_y do not properly contain each other.

If $l(I_x) \leq r(I_v)$ let $l(I_w)$ be a first left interval end to the right of $r(I_y)$. Assume such an interval I_w exists and $l(I_w) \leq r(I_x)$. Then the intervals I_w and I_x overlap and $\{x, y, v, w\}$ induces a claw with center x , in contradiction to $x \in \bar{V}$. Hence $l(I_w) > r(I_x)$ or there is no interval starting to the right of $r(I_y)$. As above we may move $r(I_y)$ to the right beyond $r(I_x)$ without changing the overlap structure of the intervals. Again, I_x and I_y do not properly contain each other.

Repeating this for all $x, y \in \bar{V}$ with $I_y \subset I_x$ we end up with a collection $R \in \mathcal{R}(P)$ such that \bar{V} is R -proper. \square

Putting the previous lemmas together, we can now show that every interval order $P \in \mathcal{P}(G)$ has an interval representation in which the centers of G precisely correspond to intervals of length greater than one.

Theorem 3. *Let $G = (V, E)$ be an interval graph. Then for all $P \in \mathcal{P}(G)$*

$$\tau_{>}(P) = |Z|$$

PROOF. According to Lemma 2 every interval order $P \in \mathcal{P}(G)$ has an interval representation $R \in \mathcal{R}(P)$ in which \bar{V} is R -proper. After scaling we may assume that $|I| \geq 1$ for all $I \in R$. Lemma 1 allows us to transform R further such that $|I_x| = 1$ for all $x \in \bar{V}$.

This transformation does not change the sequence of interval ends. So for all $z \in Z(P)$ we have an interval I_x with $x \in \bar{V}$ such that $I_x \subsetneq I_z$. Since $|I_x| = 1$ for all $x \in \bar{V}$ we must have $|I_z| > 1$.

So for all $P \in \mathcal{P}(G)$ there is an interval representation with $|I_x| = 1$ for all $x \in \bar{V}$ and $|I_z| > 1$ for all $z \in Z(P)$. Hence $\tau_{>}(P) \leq |Z(P)|$ for all $P \in \mathcal{P}(G)$. Since also $\tau_{>}(P) \geq \tau_{>}(G) \geq |Z|$ for all $P \in \mathcal{P}(G)$ and the centers of P and G are the same, we get $\tau_{>}(P) = |Z|$ for all $P \in \mathcal{P}(G)$. \square

In particular, Theorem 3 shows that $\tau_{>}$ is a comparability invariant. It also implies

Corollary 4. *Let G be an interval graph. Then $\tau_{>}(G) = |Z|$.* \square

Since centers can be found in polynomial time we can compute $\tau_{>}(G)$ efficiently.

4. Graphs with non-unit count 1

We now return to the general case and allow interval lengths smaller than one. First observe that $\tau(G) \leq \tau_{>}(G)$. Strict inequality may hold as the example in Figure 3 shows.

The centers of G are the vertices $\{e, f, g\}$. Hence $\tau_{>}(G) = 3$ while the interval representation R_2 of G needs only two non-unit intervals. The example in Figure 1 shows that τ is not a comparability invariant even if $\tau(G) = 2$.

In the following we restrict ourselves to the first nontrivial case where the length of only one interval may deviate from one. So we consider graphs G with $\tau(G) = 1$. In view of Corollary 4 we know that $|Z| = 1$ is sufficient to ensure $\tau(G) = 1$. We will analyze graphs with more than one center and characterize those with $\tau(G) = 1$. Recall that Z is the set of centers of G , U the set of leaves.

Lemma 5. *Let G be an interval graph with $\tau(G) = 1$. Then Z induces a nonempty clique in G .*

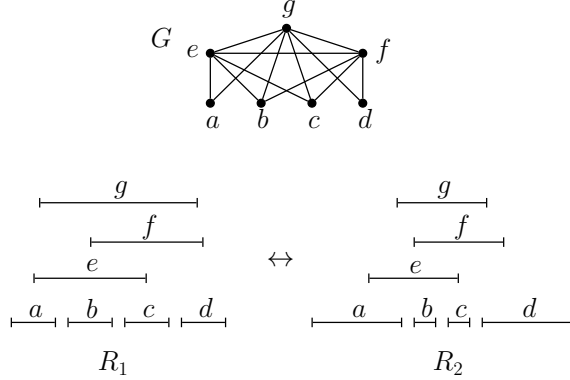


Figure 3: $\tau(G) < \tau_{>}(G)$

PROOF. If $|Z| = 0$ then G is claw-free and hence $\tau(G) = 0$. If $|Z| = 1$ the claim trivially holds. So we may assume that $|Z| > 1$. Suppose the assertion is not true. Let $z_1, z_2 \in Z$ be two nonadjacent centers.

If z_1, z_2 have two common neighbors, x, y say, then $(x, y) \in E$ since otherwise $\{z_1, x, z_2, y, z_1\}$ induces a cycle of length 4 without chord. Therefore, the two centers z_1 and z_2 have at most one leaf in common.

If z_1, z_2 have no leaf in common then, in any interval representation of G , I_{z_1} properly contains some interval I_{u_i} and I_{z_2} properly contains some interval I_{u_r} with $u_i \neq u_r$. Hence I_{z_1} and I_{u_i} have different length, and the same is true for I_{z_2} and I_{u_r} . So $\tau(G) \geq 2$.

So we may assume that z_1, z_2 have exactly one leaf in common. Let $M_1 = \{z_1, u_1, u_2, u_3\}$ and $M_2 = \{z_2, u_3, u_4, u_5\}$ be two claws with center z_1 and z_2 . We claim that $U^* = \{u_1, u_2, u_3, u_4, u_5\}$ is a stable set. Suppose not. Then, since $\{u_1, u_2, u_3\}$ and $\{u_3, u_4, u_5\}$ are stable, the set $\{u_1, u_2, u_4, u_5\}$ is not stable. W.l.o.g. let $(u_1, u_4) \in E$. Then G contains a cycle $(u_1, z_1, u_3, z_2, u_4, u_1)$ of length 5 (cf. Figure 4).

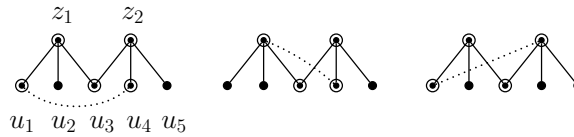


Figure 4: Two stars with nonadjacent centers and a common leaf. The dashed edge creates an induced cycle of length at least four.

We claim that this cycle has no chord, in contradiction to G being an interval graph. Obviously, $(u_1, u_3) \notin E$ and $(u_3, u_4) \notin E$. Also, by assumption, $(z_1, z_2) \notin E$. Suppose $(z_1, u_4) \in E$. Then $\{z_1, u_3, z_2, u_4\}$ induce a 4-cycle without chord, a contradiction (cf. Figure 4). Similarly, $(z_2, u_1) \notin E$. Hence U^* is a stable set.

We may then assume that in an interval representation R $r(I_{z_1}) < l(I_{z_2})$ and $r(I_{u_i}) < l(I_{u_j})$ for $i, j \in \{1, \dots, 5\}$, $i < j$ hold. Since u_3 is adjacent to both z_1 and z_2 we must have $l(I_{u_3}) < r(I_{z_1}) < l(I_{z_2}) < r(I_{u_3})$. Since I_{z_1} intersects the intervals I_{u_1} and I_{u_2}

left of I_{u_3} , it has to properly contain I_{u_2} . Hence I_{z_1} and I_{u_2} have different length. The same holds for the intervals I_{z_2} and I_{u_4} and I_{u_5} right of I_{u_3} . In both cases we have two intervals of different length, i.e. $\tau(G) > 1$. Thus z_1, z_2 are adjacent and Z is a clique. \square

We call a vertex x *short* if x is a leaf vertex contained in every claw and the following two conditions hold

$$U \text{ splits into three disconnected cliques } U_l, \{x\} \text{ and } U_r, \quad (4)$$

$$y \in N(x) \setminus Z \iff y \in \bigcap_{z \in Z} N(z) \setminus U, \quad (5)$$

We want to show that for graphs G with more than one claw the existence of a short vertex is necessary and sufficient for $\tau(G) = 1$ to hold.

Lemma 6. *Let G be an interval graph with $|Z| > 1$ and $\tau(G) = 1$. Then (4) holds.*

PROOF. Every interval representing a center properly contains at least one interval representing a leaf. Hence, if two center intervals have different lengths, these two and the shorter of the properly contained intervals have three different length, i.e. $\tau(G) > 1$. So, since $\tau(G) = 1$, all center intervals must have the same length. By $|Z| > 1$ they all have unit length. Any interval properly contained in some other then has length less than one. Hence there is only one such interval, I_x say. Moreover, I_x is properly contained in all center intervals.

Let U_l be the set of leaves whose intervals are to the left of I_x . We claim that U_l is a clique. For suppose there are two vertices $y, z \in U_l$ with $I_y \cap I_z = \emptyset$. Let I_z be the interval closer to I_x . Since I_y has a nonempty intersection with some center interval and this center interval properly contains I_x , it must also properly contain I_z . As I_x is the only properly contained leaf interval, we must have $(y, z) \in E$, i.e. U_l is a clique. Similarly, the set U_r of leaves whose intervals are to the right of I_x is a clique.

Suppose there is an interval I_y with $y \in U$ which intersects I_x . Then there must exist intervals I_{u_l}, I_{u_r} and I_z such that I_z intersects both I_{u_l} and I_{u_r} and properly contains I_y . This contradicts the fact that I_x is the only interval that is properly contained in some other interval. So U splits into three cliques $U_l, \{x\}$ and U_r . Moreover, since every claw has precisely one vertex in every clique, there is no edge between them. \square

Lemma 7. *Let G be an interval graph with $|Z| > 1$ and $\tau(G) = 1$. Then (5) holds.*

PROOF. If $y \in N(x) \setminus Z$ then $y \notin U$ and in every representation of G the interval I_y intersects I_x . Since $I_x \subseteq \bigcap_{z \in Z} I_z$, this proves one direction.

Conversely, let $y \in \bigcap_{z \in Z} N(z) \setminus U$. Suppose $y \notin N(x) \setminus Z$. Consider a claw (z, x, u_l, u_r) . Since $y \notin U$, (z, x, y, u_l) is not a claw. Hence $u_l \in N(y)$. Repeating this argument gives $u_l \subseteq N(y)$ and $u_r \subseteq N(y)$. Then I_y intersects all intervals I_{u_l} and I_{u_r} . Thus $I_y \cap I_x \neq \emptyset$, i.e. $y \in N(x)$, a contradiction. \square

The Lemmas 6 and 7 show that the existence of a short vertex is necessary for graphs with $|Z| > 1$ to have $\tau(G) = 1$. Before we prove that it is also sufficient, we state a technical lemma.

Lemma 8. *Let G be an interval graph satisfying (4). Then the following hold:*

- (i) for all $y \in N(x) \setminus Z$ either $U_l \subseteq N(y)$ or $U_r \subseteq N(y)$ but not both,
- (ii) for all $a, b \in N(x) \setminus Z$ we have $(a, b) \in E$ or $U_l \subseteq N(a)$ and $U_r \subseteq N(b)$,
- (iii) Z induces a clique,
- (iv) there are $u \in U_l, z_1, z_2 \in Z, v \in U_r$ such that $Z \subseteq N(u) \cap N(v), U_l \subseteq N(z_1)$ and $U_r \subseteq N(z_2)$

PROOF. (i) Consider some claw (z, x, u_l, u_r) with $u_l \in U_l, u_r \in U_r$. Since (z, y, u_l, u_r) is not a claw we must have $(y, u_l) \in E$ or $(y, u_r) \in E$. Assume $(y, u_l) \in E$. Then $(y, v) \notin E$ for all $v \in U_r$ since otherwise (y, u_l, x, v) induces a claw, contradicting $y \notin Z$. Hence $U_l \subseteq N(y)$ or, by symmetry, $U_r \subseteq N(y)$.

(ii) By (i) the neighborhoods of a and b contain U_l or U_r . Suppose $U_l \subseteq N(a) \cap N(b)$. Then $(a, b) \in E$ since otherwise a, x, b, u_l induce a 4-cycle.

(iii) Consider two centers $z_1, z_2 \in Z$ and claws (z_1, u_l, x, u_r) and (z_2, v_1, x, v_2) with $u_l, v_1 \in U_l$ and $u_r, v_2 \in U_r$. If $z_1, z_2 \in Z$ are not connected then x, z_1, u_l, v_1, z_2 induce a cycle of length 5 (or of length 4 if $u_l = v_1, (u_l, z_2) \in E$ or $(v_1, z_1) \in E$), a contradiction.

(iv) let (z_1, x, u_l, u_r) and (z_2, x, v_1, v_2) be two claws. Then (z_1, z_2, u_l, v_1) induces a 4-cycle unless $(z_1, v_1) \in E$ or $(z_2, u_l) \in E$. Hence for any two centers their neighbors in U_l are ordered by inclusion which proves $U_l \subseteq N(z_1)$ for some $z_1 \in Z$. The other assertions follow similarly. \square

Theorem 9. *Let G be an interval graph. Then $\tau(G) = 1$ if and only if $|Z| = 1$ or G has a short vertex.*

PROOF. If G contains a star $K_{1,r}, r \geq 4$, then U has a stable set of size $r, r \geq 4$. Now by Lemma 6 $\tau(G) = 1$ can only hold if $Z = 1$. If G contains only claws $K_{1,3}$ then the Lemmas 6 and 7 show that the vertex x is short.

Conversely, if $Z = 1$ then also $\tau(G) = 1$ by Corollary 4. So assume that G has a short vertex x . As x is contained in every claw, $G \setminus x$ is claw-free and hence a unit interval graph. Let $R \in \mathcal{R}(G \setminus x)$ be an interval representation. Since by (4) U_l, U_r are two disconnected cliques, we may assume that in R the intervals representing U_r are to the right of those representing U_l .

Z induces a clique by Lemma 8. Then Helly's Theorem implies that the intervals $I_z, z \in Z$ have a nonempty intersection. We may assume that $I := \bigcap_{z \in Z} I_z$ has length $|I| > 0$.

Let $y \in N(x) \setminus Z$. Again by Lemma 8 we know that either $U_l \subseteq N(y)$ or $U_r \subseteq N(y)$. W.l.o.g. let $U_r \subseteq N(y)$. Since $y \in N(x)$, condition (5) ensures that I_y has a nonempty intersection with I . We claim that we may assume $|I \cap I_y| > |I \cap I_u|$ for all $u \in U_r$. If this is not the case we move the interval I_y appropriately to increase $|I \cap I_y|$. Such a shift will not change the intersection structure unless we lose a neighbor a of y or create a new neighbor a of y . Since $|I_y| = |I_u|$, this can only happen if one of the cases in Figure 5 holds.

In case (a) we have $I_a \cap I = \emptyset$ and $(y, a) \in E, (u, a) \notin E$. Since y is not a center, (y, x, u, a) cannot induce a claw in G . Hence we have $(a, x) \in E$, i.e. $a \in N(x)$. Then, by (5) I_a must intersect I , a contradiction. In case (b) we have $I_a \cap I \neq \emptyset$ and $(y, a) \notin E, (u, a) \in E$. By (5) $a \in N(x)$ and so, by Lemma 8 either $U_l \subseteq N(a)$ or $U_r \subseteq N(a)$



Figure 5: A shift of I_y to increase $|I \cap I_y|$ such that (a) a neighbor a is lost or (b) a new neighbor a is created.

but not both. Since $(u, a) \in E$, we have $U_r \subseteq N(a)$. This, together with $U_r \subseteq N(y)$ and Lemma 8 (ii) implies $(y, a) \in E$, a contradiction.

We now construct the missing interval for x . Let $r = \max\{l(I_u) : u \in U_r\}$ and $l = \min\{l(I_u) : u \in U_l\}$. By the previous argument we may find an $\varepsilon > 0$ such that $I_x = [l + \varepsilon, r - \varepsilon]$ has a nonempty intersection with every $I_y, y \in N(x)$. We add the interval I_x to the interval representation R of $G \setminus x$. Let G' be the resulting graph. Obviously, $N_{G'}(u) = N_G(u)$ for all $u \neq x$. Also, by construction, $N_G(x) \subseteq N_{G'}(x)$. Conversely, if $y \in N_{G'}(x)$ then, by construction of I_x , $y \notin U$. Then, since I_v intersects $I_x \subseteq I$, (5) ensures that $y \in N_G(x)$. Hence $R \cup I_x$ is an interval representation for G using one interval of length shorter than one. Thus $\tau(G) = 1$. \square

Since centers are comparability invariant we obtain the following

Corollary 10. *Let $P_1 \sim P_2$. Then $\tau(P_1) = 1$ if and only if $\tau(P_2) = 1$.* \square

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