Fourier Analysis on Non-Compact Symmetric Superspaces of Rank One

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„Mögen hätt' ich schon wollen,
aber dürfen hab ich mich nicht getraut!“

Karl Valentin, [Val90]

Kurzzusammenfassung

Die vorliegende Arbeit gliedert sich in zwei Themenbereiche. Der erste Teil befasst sich mit der Bestimmung der Asymptotik sphärischer Superfunktionen auf nicht-kompakten symmetrischen Superräumen niedrigen Ranges. Diese spielt eine wichtige Rolle in der Harmonischen Analysis solcher Superräume. Sie wird durch die c-Funktion nach Harish-Chandra beschrieben. Die Kenntnis der c-Funktion ermöglicht es, eine explizite Reihenentwicklung sphärischer Superfunktionen zu bestimmen, mittels derer sich das Wachstumsverhalten dieser Superfunktionen abschätzen lässt. Die wesentliche Problematik bei der Bestimmung der c-Funktion besteht darin, dass gewisse notwendige Integrationsformeln sich nur für kompakt getragene Integranden verallgemeinern lassen.

Der zweite Teil befasst sich mit der Herleitung einer Rücktransformationsformel für die Fouriertransformation auf dem Raum $\mathrm{SOSp}^+_{cs}(1, 1 + p|2q)/\,\mathrm{SOSp}_{cs}(1 + p|2q)$. Der Unterschied zum klassischen Fall besteht hierbei darin, dass die c-Funktion Nullstellen mit positiven Realteil aufweisen kann, weswegen die Residuen von $\frac{1}{c}$ zu zusätzlichen Termen in der Rücktransformationsformel führen. Im Beweis dieser Formel werden Polarkoordinaten benötigt, von denen im Allgemeinen zu erwarten ist, dass sie zu sogenannten Randtermen führen. Zu diesem Zweck wird dieser symmetrische Raum mit der Superpoincarékugel identifiziert, da sich auf dieser eine geeignete Polarintegrationsformel herleiten lässt.

Abstract

This thesis covers two topics. The first part studies the asymptotic behaviour of spherical super functions on non-compact symmetric superspaces of low rank. This plays an important role in the harmonic analysis of such spaces. It is described by Harish-Chandra's c-function. The c-function is the first step in order to obtain an explicit series expansion of spherical super functions. This expansion allows to estimate the growth behaviour of these functions. The main difficulty in determining the c-function is that the necessary integration formulas generalise to the super case only for compactly supported integrands.

The second part focuses on the Fourier transform on the symmetric superspace $\mathrm{SOSp}^+_{cs}(1, 1 + p|2q)/\,\mathrm{SOSp}_{cs}(1 + p|2q)$ to obtain a Fourier inversion formula. In distinction to the classical setting, the c-function might have zeros with positive real part in the super case. Therefore, the residues of $\frac{1}{c}$ lead to additional terms in the inversion formula. The proof of this formula makes it necessary to work with polar coordinates which in general produce boundary terms. For this purpose, this space will be identified with the Poincaré super ball. On this space, a necessary polar integration formula can be derived easily.

Contents

1. Introduction

In the study of harmonic analysis non-compact symmetric spaces G/K, the algebra $\Gamma(\mathcal{D}_{G/K})^K$ of G-invariant differential operators on G/K is of special interest. Functions on G/K, which happen to be K-invariant joint eigenfunctions of all elements of $\Gamma(\mathcal{D}_{G/K})^K$, are called spherical functions. As it turns out, the class of spherical functions is exhausted by functions ϕ_λ, given by

$$\phi_\lambda(g) \coloneqq \int_{K/M} D\dot{k} e^{(\lambda-\varrho)(H(gk))}$$

for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Here, $\mathfrak{a}^*_{\mathbb{C}}$ is the set of complex linear functionals on a Cartan subspace $\mathfrak{a}$ of $\mathfrak{g}$ and ϱ is the Weyl vector of a positive root system. Moreover, the function $H\colon G \to \mathfrak{a}$ determines the $\mathfrak{a}$ component of an element $g \in G$ in the corresponding Iwasawa decomposition.

The asymptotic behaviour of spherical functions is of special interest. It is described by Harish-Chandra's c-function, which is given by

$$c(\lambda) \coloneqq \lim_{t\to\infty} e^{(\lambda-\varrho)(th)} \phi_\lambda(e^{th}),$$

for $h \in \mathfrak{a}^+$. The c-function has various different applications, as for example in the Cartan–Helgason Theorem. It is also necessary for the formulation of the Fourier inversion formula in that it determines the Plancherel measure. Such a formula is, besides its independent mathematical interest, useful in physics. For example, it can be used to calculate the mean conductance of a quasi-one-dimensional disordered conductor (*cf.* [Zir91a, Zir91b]).

One of the objectives of this thesis is to derive the c-function in the super setting on spaces where there is only one simple positive restricted root α. Namely, this happens for the spaces $\mathrm{U}_{cs}(1, 1+p|q)/\mathrm{U}(1) \times \mathrm{U}_{cs}(1+p|q)$, $\mathrm{SOSp}^+_{cs}(1, 1+p|2q)/\mathrm{SOSp}_{cs}(1+p|2q)$ and $\big(\mathrm{GL}_{cs}(1|1) \times \mathrm{GL}_{cs}(1|1)\big)/\mathrm{GL}_{cs}(1|1)$. For the former two spaces, the c-function turns out to be

$$c(\lambda) = \frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{\lambda+\frac{m_\alpha}{2}+1}{2}\right)\Gamma\left(\frac{\lambda+\frac{m_\alpha}{2}+m_{2\alpha}}{2}\right)}.$$

Here, λ is identified with the number $\lambda(h_0)$, where $\alpha(h_0) = 1$. On the third space, the c-function takes on a different form. The restricted root α is odd and isotropic, which leads to

$$c(\lambda) = \lambda(h_0)$$

with $h_0 \in \ker\alpha$. These findings are the first step towards a formula for the c-function on general symmetric superspaces. As it was recently shown in [AS13], the general problem can be reduced to these cases of low rank by a procedure called rank reduction.

After the c-function is determined, an explicit series expansion for ϕ_λ on $A^+ = \exp(\mathfrak{a}^+)$ will be given in the ortho-symplectic case. Analogously to the classical setting, this will be done by determining the eigenfunctions of the radial part of the Laplace operator. Here, some interesting observations can be made. If α has even non-positive multiplicity, the expansion of ϕ_λ terminates (*cf.* Corollary 3.2.3). Moreover, if the restricted root α is purely odd (this is the case $p = 0$), the spherical functions are not symmetric as it is the case for even α (Proposition 3.1.14). The same happens for $\big(\mathrm{GL}_{cs}(1|1) \times \mathrm{GL}_{cs}(1|1)\big)/\,\mathrm{GL}_{cs}(1|1)$, which is the only other considered case of a purely odd restricted root (Proposition 3.1.17).

Once this is done, the Fourier transform can be defined:

$$\mathcal{F}f(\lambda, k) := \int_{G/K} D\dot{g}\, f(g) e^{(\lambda-\varrho)(H(g^{-1}k))}$$

This will be performed only for $G = \mathrm{SOSp}^+_{cs}(1, 1 + p|2q)$ and $K = \mathrm{SOSp}_{cs}(1 + p|2q)$. With the inverse Fourier transform

$$\mathcal{J}\varphi(g) := \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \int_{K/M} D\dot{k}\, \varphi(\lambda, k) e^{(-\lambda-\varrho)(H(g^{-1}k))},$$

one would suspect that $\mathcal{J}\mathcal{F}f = f$ like in the classical case. However, this is only true if $m_\alpha \geq 0$.

Already in the case $m_\alpha = -1$, a correction term needs to be added as it has been shown in [Zir91a]. This is due to the fact that $c(\lambda)$ has zeros for $\mathrm{Re}\,\lambda > 0$. Since the reciprocal of $c(\lambda)$ occurs in the inverse Fourier transform, this leads to collecting some residues in the proof of the inversion formula. This is similar to the case of non-Riemannian symmetric spaces (*cf.* [HS95, Part II, Lecture 8]). These residues add up to the inverse Fourier transform of the constant function 1 (which indeed exists). Generally, the inversion formula is of the form

$$\mathcal{J}\mathcal{F}f = f + (f * \mathcal{J}1),$$

where $*$ denotes the convolution. A necessary tool for the proof of this formula is a polar integration formula. For this purpose, G/K will be identified with the unit ball $\mathbb{B}^{1+p|2q}$ in $\mathbb{A}^{1+p|2q}$. This is in analogy to the Poincaré ball model. This will be done in order to apply a polar integration formula which can be derived on $\mathbb{A}^{1+p|2q}$. Since this needs a coordinate change, boundary terms are obtained by a formula from [AHP12]. These integration formulas take very different forms for even and odd negative multiplicities. This is due to the fact that the volume of K/M vanishes if and only if m_α is negative and odd.

Unfortunately, it was only possible to prove the inversion formula for positive and odd negative multiplicities. At the time of submission of this thesis, the original proof for even negative multiplicities turned out to be faulty because a certain estimate was not strong enough to allow for interchanging some derivations and integrals.

In the unitary setting, m_α is always even. The principal behaviour in this setting is like in the one for odd m_α in the ortho-symplectic model, since the symmetric superspace $\mathrm{U}_{cs}(1, 1 + p|q)/\mathrm{U}(1) \times \mathrm{U}_{cs}(1 + p|q)$ may be identified with $\mathbb{B}^{1+2p|2q}$. This is the reason for focusing on the ortho-symplectic case, since only there the volume of K/M can be non-zero for negative multiplicities. It seems to be straight-forward to adapt the proof from the ortho-symplectic case to this setting.

The content of this thesis is structured as follows. In Chapter 2, basic concepts on superspaces are explained. In particular, in Section 2.3, an explicit formula for polar integration on $\mathbb{A}^{p|2q}$ is derived. Chapter 3 covers the calculation of the c-function on the mentioned spaces as well as the determination of the series expansion of ϕ_λ in the ortho-symplectic setting and some deduced estimates of ϕ_λ. Finally, Chapter 4 aims to prove the Fourier inversion formula for the ortho-symplectic setting. The ideas in Chapters 3 and 4 follow mainly the concepts of [Hel62, Hel84, Hel94].

2. Preliminaries

2.1. Basics on *cs* Spaces

This section gives a brief synopsis of the theory of *cs* spaces. It is based on the concepts of sheaves and categories. For the former [Bre97, Ive84] can be recommended to the reader, whereas the later is briefly summarised in Appendix A.

The concept of *cs* manifolds was introduced by Joseph Bernstein (*cf.* [DM99]). The category of real super manifolds is equivalent to a subcategory of the category of *cs* manifolds. Each real super manifold becomes a *cs* manifold by complexifying structure sheaves and morphisms. However, easy examples show that the so obtained subcategory is not a full subcategory. The advantage of considering the bigger category of *cs* manifolds lies in the easier application to physical problems (*cf.* [Zir91a]). Also, from the mathematical point of view, *cs* manifolds are to be preferred. In certain situations, *e.g.* when it comes to integration theory, is it reasonable to consider a complex super manifold as a *cs* manifold rather than as a real super manifold (*cf.* [DM99]).

Although the majority of the standard literature (as [Ber87, Lei80, Man97, DM99]) is written in the context of super manifolds, almost all results can be carried over to *cs* manifolds. The first part of this section follows widely (but in a much smaller setting) [AHW14a, AHW14b].

2.1.1 Definition. A $\mathbb{C}$-*superspace* is a pair $X = (X_0, \mathcal{O}_X)$, where X_0 is a topological space and $\mathcal{O}_X$ is a sheaf of unital super-commutative super algebras over $\mathbb{C}$. Furthermore, the stalks $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at any point $x \in X_0$ are assumed to be local super-commutative rings, whose maximal ideals are denoted as $\mathfrak{m}_{X,x}$. Elements of $\mathcal{O}_X(U)$ for any open set $U \subseteq X_0$ are referred to as *super functions*.

A *morphism* $\varphi = (\varphi_0, \varphi^\sharp)\colon X \to Y$ of $\mathbb{C}$-superspaces consists of a pair of a continuous function φ_0 and a morphism of sheaves $\varphi^\sharp\colon \mathcal{O}_Y \to \varphi_{0,*}\mathcal{O}_X$, which induces local ring homomorphisms on the stalks, *i.e.* $\varphi^\sharp_x(\mathfrak{m}_{Y,\varphi_0(x)}) \subseteq \mathfrak{m}_{X,x}$. In the following, no difference will be made between the equivalent morphisms $\mathcal{O}_Y \to \varphi_{0,*}\mathcal{O}_X$ and $\varphi_0^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. The category of $\mathbb{C}$-superspaces will be denoted by $\mathbf{SSp}_{\mathbb{C}}$.

2.1.2 Definition. A real super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is called a *cs* vector space ("*c*" for "complex", "*s*" for "super") if there is a complex structure on $V_{\bar{1}}$ which is compatible with the real structure. Morphisms of *cs* vector spaces are even linear maps which are additionally complex linear on the odd part. The category of *cs* vector spaces will be denoted **csVec**. The complexification of a *cs* vector space V is the complex super vector space $V_{\mathbb{C}} \coloneqq (V_{\bar{0}} \otimes_{\mathbb{R}} \mathbb{C}) \oplus V_{\bar{1}}$. Considering $V_{\mathbb{C}}$ as *cs* vector space, V can be understood as sub *cs* vector space *via* identifying it with $(V_{\bar{0}} \otimes 1) \oplus V_{\bar{1}}$.

For any two *cs* vector spaces V and W, the *cs* vector space of *inner homs* is defined to be:

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathbf{csVec}}(V,W) &\coloneqq \mathrm{Hom}_{\mathbf{csVec}}(V,W) \oplus \big(\mathrm{Hom}_{\mathbb{R}}(V_{\bar{0}}, W_{\bar{1}}) \oplus \mathrm{Hom}_{\mathbb{R}}(V_{\bar{1}}, W_{\bar{0}})\big)\\ &\cong \mathrm{Hom}_{\mathbf{csVec}}(V,W) \oplus \big(\mathrm{Hom}_{\mathbb{C}}(V_{\bar{0}} \otimes \mathbb{C}, W_{\bar{1}}) \oplus \mathrm{Hom}_{\mathbb{C}}(V_{\bar{1}}, W_{\bar{0}} \otimes \mathbb{C})\big)\\ &\cong \{\lambda \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}}) \mid \lambda(V_{\bar{0}}) \subseteq W_{\bar{0}}\}\end{aligned}$$

The dual *cs* vector space of V is $V^* \coloneqq \underline{\mathrm{Hom}}(V,\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(V_{\bar{0}},\mathbb{R}) \oplus \mathrm{Hom}_{\mathbb{C}}(V_{\bar{1}},\mathbb{C})$. Similarly, a *multi-linear morphism* of *cs* vector spaces is an even complex multi-linear morphism $\lambda\colon V_{1,\mathbb{C}} \times \cdots \times V_{k,\mathbb{C}} \to W_{\mathbb{C}}$, such that $\lambda(V_{1,\bar{0}} \times \cdots \times V_{k,\bar{0}}) \subseteq W_0$. The set of all such morphisms will be denoted $\mathrm{Mult}_{\mathbf{csVec}}(V_1, \dots, V_k; W)$.

The tensor product $V \otimes W$ is defined *via*

$$(V \otimes W)_{\bar{0}} \coloneqq (V_{\bar{0}} \otimes_{\mathbb{R}} W_{\bar{0}}) \oplus_{\mathbb{R}} (V_{\bar{1}} \otimes_{\mathbb{C}} W_{\bar{1}}), \quad (V \otimes W)_{\bar{1}} \coloneqq (V_{\bar{1}} \otimes_{\mathbb{R}} W_{\bar{0}}) \oplus_{\mathbb{C}} (V_{\bar{0}} \otimes_{\mathbb{R}} W_{\bar{1}}).$$

The complex structures on the odd part of $V \otimes W$ is given by the complex structures on $V_{\bar{1}}$ and $W_{\bar{1}}$. The space $\underline{\mathrm{Hom}}(V,W)$ is indeed the inner hom space in the category of *cs* vector spaces, since $\mathrm{Hom}(U \otimes V, W) = \mathrm{Hom}\,(U, \underline{\mathrm{Hom}}(V,W))$.

For a given a *cs* vector space V, let $\mathbb{A}(V) \coloneqq (V_{\bar{0}}, \mathcal{O}_V)$, with $\mathcal{O}_V \coloneqq \mathcal{C}^{\infty}_{V_{\bar{0}}} \otimes_{\mathbb{C}} \bigwedge V_{\bar{1}}^*$. $\mathbb{A}(V)$ is called the corresponding *affine* $\mathbb{C}$-superspace. Here, $\mathcal{C}^{\infty}_{V_{\bar{0}}}$ denotes the sheaf of complex valued smooth functions on $V_{\bar{0}}$. $\mathbb{A}$ becomes a functor $\mathbb{A}\colon \mathbf{csVec} \to \mathbf{SSp}_{\mathbb{C}}$ by setting $\mathbb{A}(\lambda) \coloneqq \big(\lambda_{\bar{0}}, \mathbb{A}(\lambda)^{\sharp}\big)$ for $\lambda \in \mathrm{Hom}(V,W)$ with $\mathbb{A}(\lambda)^{\sharp}(f) \coloneqq (f \circ \lambda)$ for $f \in \mathcal{C}^{\infty}_{W_{\bar{0}}}$ and $\mathbb{A}(\lambda)^{\sharp}(v_i^*) \coloneqq v_i^* \circ \lambda$ for $v_i^* \in V_{\bar{1}}^*$.

The $\mathbb{C}$-superspace associated to $V = \mathbb{R}^p \oplus \mathbb{C}^q$ will be denoted by the symbol $\mathbb{A}^{p|q}$. It is called the *affine space* of dimension $p|q$.

2.1.3 Definition. Let X be a $\mathbb{C}$-superspace whose underlying space X_0 is Hausdorff. Then X is called a *cs* manifold of dimension $\dim X = p|q$ if it is locally isomorphic to $\mathbb{A}^{p|q}$. The super dimension of X is defined to be $\operatorname{sdim} X \coloneqq p - q$. The full subcategory of *cs* manifolds of $\mathbf{SSp}_{\mathbb{C}}$ will be denoted **csMan**. Clearly, $\mathbb{A}^{p|q}$ is a *cs* manifold by definition. It is easy to see that the category **csMan** admits finite products and that the terminal object is given by $* = \mathbb{A}^{0|0}$.

From now on, all superspaces and morphisms will be considered in **csMan** unless something else is stated.

2.1.4 Definition. A morphism $\varphi\colon X \to Y$ is called an *embedding* if φ_0 is an embedding and $\varphi_0(X_0)$ is closed in some open $U \subseteq Y_0$, such that $\varphi^{\sharp}\colon \mathcal{O}_Y|_U \to \varphi_{0,*}\mathcal{O}_X$ is surjective. If φ_0 is an open (closed) map, φ is said to be an *open* (*closed*) *embedding*.

2.1.5 Definition. For any open subset $U \subseteq X_0$, the *open subspace* $X|_U := (U, \mathcal{O}_X|_U)$ is called the *restriction* of X to U. It comes with an embedding $j_{X|_U}\colon X|_U \to X$, given by $j_{X|_U,0}\colon U \hookrightarrow X_0$ and $j^\sharp_{X|_U}\colon \mathcal{O}_X|_U = j^{-1}_{X|_U,0}\mathcal{O}_Y \to \mathcal{O}_X$. Accordingly, the restriction of a morphism φ is $\varphi|_U := \varphi \circ j_{X|_U}$.

2.1.6 Definition. Let $f \in \mathcal{O}_X(U)$ be a super function and $x \in U$. The *value* of f at x is the unique $f(x) \in \mathbb{C}$, such that $f_x - f(x) \in \mathfrak{m}_{X,x}$. This allows to define real super functions: A super function $f \in \mathcal{O}_X(U)$ is called *real* if $f(x) \in \mathbb{R}$ for all $x \in U$. The subsheaf of $\mathcal{O}_X$ formed by real super functions is denoted $\mathcal{O}_{X,\mathbb{R}}$.

By defining a subsheaf $\mathcal{N}_X$ of $\mathcal{O}_X$ *via* $\mathcal{N}_X(U) := \{f \in \mathcal{O}_X(U) \,|\, \forall x \in U : f(x) = 0\}$, the so-called *reduced space* $(X_0, \mathcal{O}_X/\mathcal{N}_X)$ of X is obtained. The reduced space has the structure of a real manifold and is denoted by the same symbol as the topological space: X_0. The open embedding $j_{X_0}\colon M_0 \to M$, inherited from the projection $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{N}_X$, is called the *canonical embedding*.

2.1.7 Definition. The *standard coordinate system* on $\mathbb{A}^{p|q}$ is the tuple $t = (t_a)_{a=1}^{p+q}$ of super functions on $\mathbb{A}^{p|q}$, given by $t_a := (\mathrm{pr}_a\colon \mathbb{R}^p \to \mathbb{R}) \in \mathcal{O}_{\mathbb{A}^{p|q},\mathbb{R},\bar{0}}$ for $a = 1, \dots, p$ and $t_a := (\mathrm{pr}_{a-p}\colon \mathbb{C}^q \to \mathbb{C}) \in \mathcal{O}_{\mathbb{A}^{p|q},\bar{1}}$ for $a > p$. To each morphism $\varphi\colon X \to Y$, where Y is an open subspace of $\mathbb{A}^{p|q}$, a tuple $(\varphi_a)_{a=1}^{p+q}$ is associated via $\varphi_a := \varphi^\sharp(t_a)$. The super functions φ_i are called *component super functions* of φ. This construction is compatible with the canonical embedding in the sense that the component functions $\varphi_{a,0}$ of φ_0 are indeed the reduced super functions of the φ_i. Thanks to Proposition 2.1.9, it will be reasonable to identify morphisms and such tuples.

2.1.8 Definition. An open embedding $x\colon X|_{U_x} \to \mathbb{A}^{\dim X}$ for open $U_x \subseteq X_0$ is called a *chart* or *coordinate system*. *Coordinates* are the component functions of a coordinate system. Sometimes it is convenient to distinguish between even and odd coordinate functions of x and write $x = (u, \xi)$, where $u = (u_i)_{i=1}^p = (x_a)_{a=1}^p$ and $\xi = (\xi_j)_{j=1}^q = (x_a)_{a=p+1}^{p+q}$. The set U_x is said to be a *coordinate neighbourhood*. If U_x can be chosen to be equal to X_0, one calls x a global coordinate system.

2.1.9 Proposition. *Let X, Y be* cs *manifolds with $\dim X = p|q$, let $y = (v, \eta)$ be a coordinate system on Y, and let $f_1, \dots, f_p, g_1, \dots, g_q$ be real super functions, where the f_i are even and the g_j are odd. Then there exists a morphism $\varphi\colon X \to Y$ such that $\varphi^\sharp(v_i) = f_i$ and $\varphi^\sharp(\eta_j) = g_i$ if and only if $f_0(X_0) \subseteq y(U_y)$.*

2.1.10 Proposition. *Let S be a* cs *manifold and V be a super vector space. Then there exists a bijection*

$$\mathbb{A}(V)(S) = \mathrm{Hom}_{\boldsymbol{csMan}}\left(S, \mathbb{A}(V)\right) \cong \mathrm{Hom}_{\boldsymbol{cs\,Vec}}\left(V^*, \Gamma(\mathcal{O}_{S,\mathbb{R}})\right) \cong \left(V \otimes \Gamma(\mathcal{O}_{S,\mathbb{R}})\right)_{\bar{0}}$$

which is natural in S and V. It is given by $\varphi \mapsto \sum_a v_a \otimes \varphi^\sharp(v_a^*)$ for a graded basis (v_a) of V with dual basis (v_a^*). For open subspaces $U \subseteq \mathbb{A}(V)$, the bijection restricts to

$$\begin{aligned} U(S) &\cong \{f \in \operatorname{Hom}(V^*, \Gamma(\mathcal{O}_{S,\mathbb{R}})) \,|\, \forall (s \in S_0)\, \exists (u \in U_0)\, \forall (v' \in V^*) : f(v')(s) = v'(u)\} \\ &\cong \left\{ f = \sum\nolimits_i v_i \otimes f_i \in (\Gamma(V \otimes \mathcal{O}_{S,\mathbb{R}}))_{\bar{0}} \,\middle|\, \forall (s \in S_0)\, \exists (u \in U_0) : u = \sum\nolimits_i v_i f_i(s) \right\}. \end{aligned}$$

2.1.11 Remark. For morphisms $\lambda \in \operatorname{Hom}_{\mathbf{csVec}}(V, W)$, the proposition above shows that the induced morphism $\mathbb{A}(\lambda)$ is $\Gamma(\mathcal{O}_{S,\mathbb{R}})_{\bar{0}}$-linear on S-points: $\mathbb{A}(\lambda)(v \otimes f) = \lambda(v) \otimes f$ for $v \otimes f \in (V \otimes \Gamma(\mathcal{O}_S))_{\bar{0}}$. This suggests to define for $\lambda \in \operatorname{Mult}_{\mathbf{csVec}}(V_1, \ldots, V_k; W)$ a morphism $\mathbb{A}(\lambda) \in \operatorname{Hom}_{\mathbf{csMan}}(\mathbb{A}(V_1) \times \cdots \times \mathbb{A}(V_k), \mathbb{A}(W))$ *via*

$$(v_1 \otimes f_1, \ldots, v_k \otimes f_k) \longmapsto \lambda(v_1, \ldots, v_k) \otimes f_1 \cdots f_k.$$

Here, the right hand side of this map should be read as $w \otimes f = w_r \otimes_{\mathbb{R}} f + w_i \otimes_{\mathbb{R}} if$ if $w = (w_r \otimes 1) + (w_i \otimes i) \in W_{\mathbb{C},\bar{0}}$. Since $w_i \neq 0$ only if f is nilpotent, this definition makes sense.

2.1.12 Definition. In view of Proposition 2.1.10, it is reasonable to define a functor

$$\mathbb{A}^{\mathbb{C}} \colon \mathbf{csVec} \to \mathbf{csMan}^{\vee}$$

via

$$\mathbb{A}(V)(S) := \operatorname{Hom}_{\mathbb{C}}\left(V_{\mathbb{C}}^*, \Gamma(\mathcal{O}_S)\right) \cong \left(V_{\mathbb{C}} \otimes_{\mathbb{C}} \Gamma(\mathcal{O}_S)\right)_{\bar{0}},$$

for any *cs* vector space V and any *cs* manifold S. Furthermore, one sets

$$\mathbb{A}(\lambda)_S(f) := f \circ \lambda^*$$

for *cs* linear maps $\lambda \colon V \to W$ and $f \in \mathbb{A}(V)(S)$.

As for the functor $\mathbb{A}$, it is reasonable to set $\mathbb{A}^{\mathbb{C},p|q} := \mathbb{A}^{\mathbb{C}}(\mathbb{R}^p \oplus \mathbb{C}^q)$. In particular, one has $\mathbb{A}^{\mathbb{C},1|1}(S) = \Gamma(\mathcal{O}_S)$.

2.1.13 Remark. Note that, unlike $\mathbb{A}(V)(-)$, the functor $\mathbb{A}^{\mathbb{C}}(V)$ can in general not be represented by an object in **csMan**. In particular, $\mathbb{A}(V_{\mathbb{C}}) \not\cong \mathbb{A}^{\mathbb{C}}(V)$ as objects in $\mathbf{csMan}^{\vee}$. The reason for this lies in the fact that $\Gamma(\mathcal{O}_X)$ is not the complexification of the *cs* vector space $\Gamma(\mathcal{O}_{S,\mathbb{R}})$.

However, $\mathbb{A}^{\mathbb{C}}(V)$ can be represented by the $\mathbb{C}$-superspace $(V_{\mathbb{C}}, \mathcal{O})$. Here, the sheaf $\mathcal{O}$ is given by $\mathcal{O} := \mathcal{C}^{\omega}_{V_{\mathbb{C},\bar{0}}} \otimes_{\mathbb{C}} \bigwedge V_{\bar{1}}^*$, where $\mathcal{C}^{\omega}_{V_{\mathbb{C},\bar{0}}}$ denotes the sheaf of holomorphic functions on $V_{\mathbb{C},\bar{0}}$.

Similarly to the setting of *cs* manifolds, a $\mathbb{C}$-superspace X is called a *complex super manifold* if X is locally isomorphic to some $\mathbb{A}^{\mathbb{C},p|q}$ and X_0 is Hausdorff.

2.1.14 Corollary.

$$\mathbb{A}(V)(S) = \left\{ s \in \mathbb{A}^{\mathbb{C}}(V)(S) \,\middle|\, s_0 \in \mathbb{A}(V_0)(S_0) \right\}$$

2.1.15 Remark. In applications it proves to be useful to consider morphisms only in the setting of super points. The spaces $\mathbb{A}^{0|q}$ are called *super points*. The full subcategory of super points of **csMan** will be denoted **SPt**. Similar to the construction of the Yoneda embedding (*cf.* Definition A.6), each *cs* manifold X determines a functor $X(-)\colon \mathbf{SPt} \to \mathbf{Man}$. Here, **Man** denotes the category of ordinary real smooth manifolds.

The obtained functor $h_{\mathbf{SPt}}\colon \mathbf{csMan} \to \mathbf{Man}^{\vee}$ is, like the Yoneda embedding, faithful. However, it is not full. This means that morphisms of *cs* manifolds are completely determined by their values on super points, but not every natural transformation on super points defines a morphism. This can be seen as an analogy to the fact that smooth maps on manifolds are completely determined by their value on ordinary points.

For a more detailed view of the super points approach to superspaces, one may also consult [SW11].

2.1.16 Definition. If a morphism $\gamma\colon X \to X_0$ is a retraction of the canonical embedding j_{X_0}, it will be called *retraction* without referring to j_{X_0}.

On an affine space $\mathbb{A}(V)$, the projection $\mathrm{pr}_1\colon V = V_{\bar{0}} \oplus V_{\bar{1}} \to V_{\bar{0}}$ induces a retraction. This retraction will be referred to as the *standard retraction* on $\mathbb{A}(V)$.

2.1.17 Remark. Assume that X admits a global coordinate system (u, ξ). In this case, a retraction γ is easily defined by requiring $u_0 \circ \gamma = \gamma^{\sharp}(u_0) = u$. One calls γ the retraction *associated* with (u, ξ). Conversely, given a global coordinate system v_0 on the reduced space X_0, a global coordinate system (v, ξ) is obtained by setting $v \coloneqq v_0 \circ \gamma$ and choosing some odd coordinates. Such a coordinate system is called *adapted* to γ.

Although global coordinate systems do not always exist on *cs* manifolds, retractions do [RS83, Lemma 3.2]. This is achieved by taking retractions on coordinate neighbourhoods and gluing them together by using a partition of unity. In general there is no unique retraction on a supermanifold.

Given a global coordinate system (u, ξ) with associated retraction γ, each super function f on X possesses a unique decomposition

$$f = \sum_{\nu \in \mathbb{Z}_2^q} \gamma^{\sharp}(f_\nu)\xi^\nu, \quad f_\nu \in \Gamma(\mathcal{O}_{X_0}).$$

Here, q is the odd dimension of X and $\xi^\nu \coloneqq \xi_1^{\nu_1} \cdots \xi_q^{\nu_q}$.

2.1.18 Definition. The tangent space $T_x X = \underline{\mathrm{Der}}(\mathcal{O}_{X,\mathbb{R},x}, \mathbb{R})$ of a *cs* manifold X at $x \in X_0$ is the subspace of *derivations* in $\underline{\mathrm{Hom}}_{\mathbf{csVec}}(\mathcal{O}_{X,\mathbb{R},x}, \mathbb{R})$. These are the elements $\delta \in \underline{\mathrm{Hom}}(\mathcal{O}_{X,\mathbb{R},x}, \mathbb{R})$ satisfying

$$\delta(fg) = \delta(f)g(x) + (-1)^{|\delta||g|} f(x)\delta(g).$$

Such elements are called *tangent vectors* at x. T_xX is a *cs* vector space whose even part is given by even derivations: $\mathrm{Der}(\mathcal{O}_{X,\mathbb{R},x}) \cong \mathrm{Der}(\mathcal{O}_{X_0,\mathbb{R},x}) = T_xX_0$. The complexification of T_xX is isomorphic the complex super vector subspace of derivations in $\underline{\mathrm{Hom}}_{\mathbb{C}}(\mathcal{O}_{X,x}, \mathbb{C})$.

Given a morphism $\varphi\colon X \to Y$ and $x \in X_0$, the *differential* at x is $T_x\varphi\colon T_xX \to T_{\varphi(x)}Y$, with $T_x\varphi(\delta)(f) = \delta(\varphi^\sharp(f))$. The morphism φ is called *immersion at x* if $T_x\varphi$ is injective and *immersion* if this is the case for all $x \in X_0$. If $T_x\varphi$ is surjective at x, then φ is called a *submersion at x*. It is a *submersion* if this is true for all $x \in X_0$.

In order to understand expressions like $\int_X \omega(x)f(s,x)$, the concept of relative *cs* manifolds needs to be introduced.

2.1.19 Definition. Assuming that there is a fixed *cs* manifold Y, a cs *manifold over Y* or *relative* cs *manifold* is a *cs* manifold X, together with a submersion $p_X\colon X \to Y$. One simply writes X/Y for this statement.

The *relative dimension* is defined to be $\dim_Y X \coloneqq \dim X - \dim Y$. Similarly, the *relative superdimension* is given *via* $\mathrm{sdim}_Y X \coloneqq \mathrm{sdim}\, X - \mathrm{sdim}\, Y$.

A morphism $\varphi\colon X/Y \to Z/Y$ is a morphism of *cs* manifolds, such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "p_X"'] & & Z \arrow[dl, "p_Z"] \\ & Y & \end{tikzcd}$$

The category of *cs* manifolds over Y is denoted $\mathbf{csMan}_Y$. Clearly, a *cs* manifold is always a *cs* manifold over $*$ and each morphism of *cs* manifolds is a morphism over $*$.

The simplest relative *cs* manifolds are of the form $X_Y \coloneqq Y \times X$ with $p_{X_Y} = \mathrm{pr}_Y$. The spaces $\mathbb{A}_Y^{p|q}$ are of special interest, since they are the model spaces for relative *cs* manifolds.

2.1.20 Proposition. *Let $\varphi\colon X \to Y$ be a morphism. If φ is an immersion at x, it is a section at x (*i.e. *there are neighbourhoods U of x and V of $f_0(x)$, with $f_0(U) \subseteq V$ and a morphism $\psi\colon Y|_U \to X|_U$ such that $\psi \circ \varphi|_U = \mathrm{id}_{X|_U}$). This local condition shows that (global) immersions are monomorphisms.*

*If φ is an submersion at x, it is a retraction at x (*i.e. *there are neighbourhoods U of x and V of $f_0(x)$, with $f_0(U) \subseteq V$ and a morphism $\psi\colon Y|_U \to X|_U$ such that $\varphi \circ \psi = \mathrm{id}_{Y|_U}$). In particular, surjective submersions are epimorphisms. Moreover, considering X as relative Y-space, there exists a neighbourhood W of x and an open embedding $\Phi\colon X|_W/Y \to \mathbb{A}_Y^{\dim_Y X}/Y$.*

If φ is an immersion as well as an submersion at x, then φ is an open embedding in a neighbourhood of x, i.e. *it is a local isomorphism at x. In the case where φ_0 is in addition injective, an immersive and submersive φ becomes an isomorphism onto its image $Y|_{\varphi_0(X_0)}$.*

2.1.21 Definition. An open embedding $x : X|_{U_x}/Y \to \mathbb{A}_Y^{\dim_Y X}$, with $U_x \subseteq X_0$ open, is called a *relative chart*, *relative coordinate system* or *fibre coordinate system.* If U_x can be chosen to be the whole space, the cart x is called a *global chart* or *global coordinate system. Relative coordinates* are the component functions of $\mathrm{pr}_2 \circ x$, where $\mathrm{pr}_2 \colon \mathbb{A}_Y^{p|q} = Y \times \mathbb{A}^{\dim_Y X} \to \mathbb{A}^{\dim_Y X}$ is the projection onto the second component. Again, if convenient, even and odd relative coordinates will be distinguished: $x = (u, \xi)$. Note that $x_0 = u_0$ by considering x and u as morphisms.

2.1.22 Definition. Let $\varphi\colon X \to Y$ be a morphism of *cs* manifolds and γ_X, γ_Y be retractions on X and Y. Then γ_X and γ_Y are called compatible under φ if the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\gamma_X"'] & Y \arrow[d, "\gamma_Y"] \\
X_0 \arrow[r, "\varphi_0"'] & Y_0
\end{tikzcd}$$

2.1.23 Lemma. *Suppose that φ in above definition is an immersion and that γ_Y is fixed. Then there is at most one retraction γ_X, compatible with γ_Y under φ.*

Similarly, if φ is a surjective submersion and γ_X is fixed, then there is at most one retraction γ_Y, compatible with γ_X under φ.

PROOF. In both cases, the assertion follows from $\varphi_0 \circ \gamma_X = \gamma_Y \circ \varphi$ since φ_0 is a monomorphism in the first case and φ is an epimorphism in the second case. □

2.1.24 Remark. Although uniqueness of compatible retractions in the preceding lemma can be proven, it cannot be assumed that such retractions generally exist. For example, the retraction γ on $\mathbb{A}^{2|2}$, given by $\gamma(s) := (s_1 + s_2 s_3 s_4, s_2)$ for an S-point $s \in_S \mathbb{A}^{2|2}$ has a counterpart neither under the immersion $\mathbb{A}^{1|2} \to \mathbb{A}^{2|2}, (s_1, s_2, s_3) \mapsto (s_1, s_1, s_2, s_3)$, nor under the submersion $\mathbb{A}^{2|2} \to \mathbb{A}^{1|2}, (s_1, s_2, s_3, s_4) \mapsto (s_1, s_3, s_4)$.

2.1.25 Definition. A retraction of a *cs* manifold X, which is also a relative *cs* manifold X/Y, is called a *relative retraction*, *retraction over* Y or simply *retraction of* X/Y if there exists a retraction γ_Y on $p_X(X) := Y|_{p_{X,0}(X_0)}$, compatible with γ over p_X. Note that $p_{X,0}$ is an open map, hence $p_X(X)$ is well-defined.

2.1.26 Remark. It should be noted that in general neither can the retraction γ_Y be assumed to be extendible onto Y for non-surjective $p_{X,0}$, nor can such a continuation be assumed to be unique.

2.1.27 Definition. For a relative *cs* manifold X/Y, the *relative tangential sheaf* consists of the derivations on X which are $\mathcal{O}_Y$-invariant: $\mathcal{T}_{X/Y} := \underline{\mathrm{Der}}_{p_{X,0}^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$. The dual sheaf $\Omega^1_{X/Y} := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}_{X/Y}, \mathcal{O}_X)$ is called the *relative cotangential sheaf*. In the case $Y = *$, let $\mathcal{T}_X := \mathcal{T}_{X/*}$ and $\Omega^1_X := \Omega^1_{X/*}$.

2.1.28 Definition. On $\underline{\mathrm{Hom}}_{p_{X,0}^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ the *super commutator* is given by

$$[D, D'] := D \circ D' - (-1)^{|D||D'|} D' \circ D$$

on local sections. Consider the subsheaf $\mathcal{D}^0_{X/Y} \cong \mathcal{O}_X$ of $\underline{\mathrm{Hom}}_{p_{X,0}^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ obtained by left multiplication with super functions. The subsheaf $\mathcal{D}^n_{X/Y}$ of *differential operators of order at most n* of $\underline{\mathrm{Hom}}_{p_{X,0}^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ is defined recursively as follows: The local sections of $\mathcal{D}^n_{X/Y}$ are the D such that $[D, f]$ is a local section of $\mathcal{D}^{n-1}_{X/Y}$ for any local section f of $\mathcal{O}_X$. This gives rise to the sheaf of *differential operators* $\mathcal{D}_{X/Y} := \bigcup_{n=0}^{\infty} \mathcal{D}^n_{X/Y}$. Here, the union has to be understood in the sense of sheaves, *i.e.* one needs to sheafificate after taking the union of the sets of local sections.

2.1.29 Proposition. *Given a global relative coordinate system x on X/Y, there exists a unique $\mathcal{O}_X$-basis $(\partial_{x_i})_i = \left(\frac{\partial}{\partial x_i}\right)_i$ of $\Gamma(\mathcal{T}_{X/Y})$, such that $\partial_{x_i}(x_j) = \delta_{ij}$. In particular, if γ denotes the retraction associated with $x = (u, \xi)$ and $f = \sum_\nu \gamma^\sharp(f_\nu)\xi^\nu$, then*

$$\frac{\partial f}{\partial u_i} = \sum_\nu \gamma^\sharp\left(\frac{\partial f_\nu}{\partial u_{i,0}}\right)\xi^\nu, \qquad \frac{\partial f}{\partial \xi_j} = \sum_{\nu_j \neq 0} \gamma^\sharp(f_\nu)\xi^{\nu - e_j}(-1)^{\nu_1 + \ldots + \nu_{j-1}}. \tag{2.1}$$

Furthermore, these coordinate derivations give rise to a basis of $\mathcal{D}_{X/Y}$: For any $n \in \mathbb{N}_0$, the set of all

$$\partial_x^i := \partial_{x_{p+q}}^{i_{p+q}} \cdots \partial_{x_1}^{i_1}$$

with $i \in \mathbb{N}_0^p \times \mathbb{Z}_2^q$ and $|i| := i_1 + \ldots + i_{p+q} \leq n$ is a basis of $\mathcal{D}^n_{X/Y}$.

2.1.30 Proposition (Taylor's formula)**.** *Let x be a global fibre coordinate system on a relative* cs *manifold X/Y of dimension $p|q$ and $o \in \mathrm{pr}_{\mathbb{A}^p} \circ x_0(X_0)$. Then for any $f \in \Gamma(\mathcal{O}_X)$ and any $k \in \mathbb{N}_0$*

$$f \equiv \sum_{|i| \leq k} (x - o)^i \partial_{x=o}^i f \mod \left\langle (x - o)^j \,\middle|\, |j| = k + 1 \right\rangle_{\Gamma(\mathcal{O}_X)}.$$

with $i, j \in \mathbb{N}_0^p \times \mathbb{Z}_2^q$. Furthermore, $\partial_{x=o}^i := (p_X \times o)^\sharp \circ (x^\sharp)^{-1} \circ \partial_{x_i}$, where o is identified with the morphism $* \to \mathbb{A}^{p|q}$, $o^\sharp(f) = f(o)$ and

$$(x - o)^i := x_{p+q}^{i_{p+q}} \cdots x_{p+1}^{i_{p+1}} (x_p - o_p)^{i_p} \cdots (x_1 - o_1)^{i_1}.$$

The expression $\langle (x - o)^j \,|\, |j| = k + 1 \rangle_{\Gamma(\mathcal{O}_X)}$ denotes the ideal in $\Gamma(\mathcal{O}_X)$, generated by $(x - o)^j$ for $|j| = k + 1$.

2.1.31 Definition. Let $\varphi\colon X \to Y$ be a morphism of *cs* manifolds and $\mathcal{E}$ be an $\mathcal{O}_Y$-module. The *pullback* is defined to be the sheaf

$$\varphi^* \mathcal{E} := \mathcal{O}_X \otimes_{\varphi_0^{-1}} \varphi_0^{-1} \mathcal{E}.$$

The $\otimes$ in this notation denotes the tensor product of sheaves, *i.e.* the sheafification of the presheaf inherited from the tensor product of graded super algebras.

2.1.32 Proposition (Base change)**.** *Let X/Y be a relative* cs *manifold and $\varphi\colon Z \to Y$ be a morphism. The fibre product $Z \times_Y X$ (*cf. *Definition A.11) exists and is a* cs *manifold over Z* via *the canonical morphism $p_Z := \mathrm{pr}_Z\colon Z \times_Y X \to Z$.*

Fibre coordinate systems x on X/Y induce fibre coordinate systems $\mathrm{id}_Z \times_{\mathrm{id}_Y} x$ on $(Z \times_Y X)/Z$. Their component super functions are given by $\mathrm{pr}_X^\sharp(x_i)$, with the canonical morphism $\mathrm{pr}_X\colon Z \times_Y X \to X$. Therefore, it makes sense to denote the induced coordinate system by $\mathrm{pr}_X^\sharp(x)$.

2.1.33 Corollary. *The base change induces an isomorphism of $\mathcal{O}_{Z\times_Y X}$-sheaves*

$$\mathrm{pr}_X^\sharp\colon p_Z^*\mathcal{T}_{X/Y} \longrightarrow \mathcal{T}_{Z\times_Y X/Z},$$

satisfying $\mathrm{pr}_X^\sharp(D)\,\mathrm{pr}_X^\sharp(f) = \mathrm{pr}_X^\sharp\,(D(f))$. Locally, it takes the form

$$f\partial_{x_i} \longmapsto \mathrm{pr}_X^\sharp(f)\partial_{\mathrm{pr}_X^\sharp(x_i)}.$$

This isomorphism can be extended to an $\mathcal{O}_{Z\times_Y X}$-isomorphism $\mathrm{pr}_X^*\,\mathcal{D}_{X/Y} \to \mathcal{D}_{Z\times_Y X/Z}$.

2.1.34 Definition. Suppose X/Y is a relative manifold and $\rho = (\rho_1, \dots, \rho_n)$ a family of smooth functions on X_0. Then ρ is said to be a family of *boundary functions* over Y_0 if for every $x \in X_0$ and every subfamily $\tilde\rho = (\rho_{i_1}, \dots, \rho_{i_k})$ with $\tilde\rho(x) = 0$, the function $\tilde\rho$ is submersive at x and $T_x p_{X,0}|_{\operatorname{Ker} T_x\tilde\rho}$ is surjective.

The set $X_{0,\rho} := \bigcap_{i=1}^n \{\rho_i > 0\}$ is called a *manifold with corners*. For any subfamily $\tilde\rho$ of ρ, the set $H = H_{\tilde\rho} = \bigcap_{\rho_i\in\tilde\rho}\{\rho_i = 0\} \cap \bigcap_{\rho_i\notin\tilde\rho}\{\rho_i > 0\}$ is called a boundary manifold if it is not empty. In this case, let $\rho_{H_0} := \tilde\rho$. The collection of all boundary manifolds is denoted by $B(X_0, \rho) = B(\rho)$. The disjoint union of all boundary manifolds equals the topological boundary of $X_{0,\rho}$ in X_0.

A family $\tau = (\tau_1, \dots, \tau_n)$ of super functions is called a family of *boundary super functions* if the underlying functions τ_0 are a family of boundary functions. The open subspace $X_\tau := X|_{X_{0,\tau_0}}$ is called a *cs* manifold with corners in X.

2.1.35 Proposition. *Let a family of boundary super functions τ on X/Y be given and $H_0 \in B(X_0, \rho)$. There exist a unique* cs *manifold H/Y with underlying space H_0 and a unique immersion $\iota_H\colon H \to X$ over Y, such that $\tau_H \circ \iota_H = 0$.*

If there is a relative retraction γ on X/Y, such that τ satisfies $\gamma^\sharp(\tau_{i,0}) = \tau_i$ for all $\tau_i \in \tau$, the space H only depends on γ in the following sense: Any other family of boundary super functions τ', which are compatible with γ and satisfy $B(\tau_0') = B(\tau_0)$, gives the same cs *manifold over H_0. In this case, there exists a unique relative retraction γ_H on H/Y, compatible with γ under ι_H.*

Proof. The condition of $T_x p_{X,0}|_{\operatorname{Ker} T_x\tilde\rho}$ being surjective induces a submersive morphism $(p_Y, \tau_{0,H_0})\colon X|_U \to Y \times \mathbb{A}^{k,0}$ on a sufficiently small neighbourhood U of H_0. Here, $k = \operatorname{codim} H_0 = \dim X_0 - \dim H_0$. Defining $Y \to Y \times \mathbb{A}^{k,0}$ on generalized points by $y \mapsto (y, 0)$ yields $Y \times_{Y\times\mathbb{A}^{k,0}} X$. The restriction of this space to H_0 gives H. $\square$

2.1.36 Definition. The spaces obtained in the above proposition are called *boundary* cs *manifolds.* The collection of all boundary *cs* manifolds is denoted $B(X,\tau)=B(\tau)$. Let $\tau_H := (\tau_i)_{\tau_{i,0}\in\tau_{H_0}}$.

2.1.37 Corollary. *Corollary 2.1.33 and Proposition 2.1.35 induce the following morphism of sheaves:*

$$\iota_H^*\mathcal{T}_{X/Y} \longrightarrow \iota_H^*\mathcal{T}_{X/Y\times A^{k,0}} \xrightarrow{\sim} \mathcal{T}_{H/Y},\ D \longmapsto D|_{H,\tau}\,.$$

If $x=(\tau_H,\tilde{x})$ is a fibre coordinate system, then $(f\partial_{x_i})|_{H,\tau}=\iota_H^\sharp(f)\partial_{\iota^\sharp(x_i)}$ for $x_i\notin\tau_H$ and $(f\partial_{x_i})|_{H,\tau}=0$ otherwise.

2.2. Integration on *cs* Manifolds

This section summarises the concept of integration on (relative) *cs* manifolds. Integration of Berezin densities with non-compact support will be explained, as well as boundary terms that occur under changes of retractions. This follows [AHP12]. However, it is necessary to extend these boundary term formulas to relative *cs* manifolds.

Unless otherwise stated, X/Y will be assumed to be a relative *cs* manifold.

2.2.1 Definition. The *sheaf of relative Berezin densities* $|\mathrm{Ber}|_{X/Y}$ is the $\mathcal{O}_X$-module sheaf of absolute Berezinians of $\Omega_{X/Y}$. This sheaf is is locally described by

$$|\mathrm{Ber}|_{X/Y}(U_x)=\left\{|Dx|f \mid f\in\mathcal{O}_X(U_x)\right\},$$

whenever $x=(x,\xi)$ is a fibre coordinate system on X/Y. The symbol $|Dx|$ is supposed to be of parity q mod 2, where q denotes the odd relative dimension of X/Y. It behaves under coordinate changes as $|Dx|=|Dy|\,|\mathrm{Ber}|\left(\frac{\partial x}{\partial y}\right)$, where $y=(v,\eta)$ denotes another fibre coordinate system. Here,

$$|\mathrm{Ber}|\begin{pmatrix} R & S\\ T & V\end{pmatrix} := \operatorname{sgn} j_{X_0}^\sharp(\det R)\cdot\det(R-SV^{-1}T)\det V^{-1},$$

and

$$\frac{\partial x}{\partial y} := \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_{p+q}}{\partial y_1}\\ \vdots & \ddots & \vdots\\ \frac{\partial x_1}{\partial y_{p+q}} & \cdots & \frac{\partial x_{p+q}}{\partial y_{p+q}}\end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial v} & \frac{\partial \xi}{\partial v}\\ \frac{\partial u}{\partial \eta} & \frac{\partial \xi}{\partial \eta}\end{pmatrix}.$$

If $Y=*$, the sheaf $|\mathrm{Ber}|_X := |\mathrm{Ber}|_{X/*}$ is called the sheaf of *Berezin densities.*

Sections $\omega\in\Gamma(|\mathrm{Ber}|_{X/Y})$, such that $p_{X,0}\colon \operatorname{supp}\omega\to Y_0$ is a proper map, are called *compactly supported along the fibres.* The set of all these sections will be denoted $\Gamma_{cf}(|\mathrm{Ber}|_{X/Y})$.

2.2.2 Example. Let $\lambda\colon V \to \mathbb{R}^p \oplus \mathbb{C}^q$ be an isomorphism of super vector spaces. Then $\mathbb{A}(\lambda)$ is a global coordinate system on $\mathbb{A}(V)$ and $D\lambda := |D\lambda|$ is a Berezin density. Berezin densities obtained in this manner will be called *Lebesgue Berezin densities*. Lebesgue Berezin densities are unique up to constant multiples.

2.2.3 Corollary. *The isomorphism* $\mathrm{pr}_X^* \mathcal{T}_{X/Y} \to \mathcal{T}_{Z\times_Y X/Z}$ *from Corollary 2.1.33 gives rise to an isomorphism* $\mathrm{pr}_X^\sharp\colon \mathrm{pr}_X^* |\mathrm{Ber}|_{X/Y} \to |\mathrm{Ber}|_{Z\times_Y X/Z}$. *Locally, it is given by*

$$|Dx|f \longmapsto |D\,\mathrm{pr}_X^\sharp(x)|\,\mathrm{pr}_X^\sharp(f).$$

In the case of $Y = *$, *this simply means*

$$|\mathrm{Ber}|_{X_Z/Z} \cong \mathrm{pr}_X^* |\mathrm{Ber}|_X\,.$$

2.2.4 Corollary. *Let* τ *be a family of boundary super functions on* X/Y *and* $H \in B(\tau)$. *The morphism from Corollary 2.1.37 induces a morphism of sheaves*

$$|_{H,\tau}\colon \iota_{H,0}^\sharp |\mathrm{Ber}|_{X/Y} \to |\mathrm{Ber}|_{H/Y}\,.$$

Locally it takes the form $(|Dx|f)|_{H,\tau} = |D\iota^\sharp(\tilde{x})|\iota_H^\sharp(f)$, *where* $x = (\tau_H, \tilde{x})$ *is a fibre coordinate system.*

2.2.5 Proposition. *There is a unique right connection*

$$\nabla\colon\ |\mathrm{Ber}|_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{D}^1_{X/Y} \to |\mathrm{Ber}|_{X/Y},$$

of the sheaf $\mathcal{D}^1_{X/Y} = \mathcal{O}_X \otimes \mathcal{T}_{X/Y}$ *on* $|\mathrm{Ber}|_{X/Y}$. *It is determined by the requirement that* $\nabla(|Dx| \otimes \partial_{x_i}) = 0$ *for any local fibre coordinate system* x *and all* i. *Right connection in this setting means that it is a right-*$\mathcal{O}_X$*-linear morphism. This right connection can be continued to an* $\mathcal{O}_X$*-linear right action of* $\mathcal{D}_{X/Y}$ *on* $|\mathrm{Ber}|_{X/Y}$, i.e.

$$\nabla(\nabla(\omega \otimes D) \otimes D') = \nabla(\omega \otimes DD').$$

The first part of this proposition is the content of [Lei80, Lemma 2.4.6], whereas the second part is due to [Che94].

2.2.6 Definition. Let γ be a relative retraction on X and suppose that there is a global coordinate system (v, η) on Y adapted to γ_Y. Recall that γ_Y denotes the retraction on Y compatible with γ under p_X. Furthermore, assume that there is a fibre coordinate system (u, ξ) on X adapted to γ, such that (v, u, η, ξ) is a global coordinate system on X. The *fibre integral with respect to* γ of a relative Berezin density $\omega = |D(u, \xi)|f$ with $f = \sum_{\mu,\nu} \gamma^\sharp(f_{\nu,\mu})\xi^\mu\eta^\nu$ is set to be

$$\sideset{_Y}{_X^\gamma}{\fint} \omega := \sum_\nu \gamma_Y^\sharp \left(\sideset{_{Y_0}}{_{X_0}}{\fint} |du_0| f_{\nu,(1,\dots,1)} \right) \eta^\nu,$$

whenever all $du_0 f_{\nu,(1,\dots,1)}$ are absolutely fibre integrable. In this case, ω is said to be *(fibre) integrable with respect to* γ. If γ is understood, it will be omitted in notation: $\sideset{_Y}{_X}{\fint} := \sideset{_Y}{_X^\gamma}{\fint}$.

2.2.7 Proposition. *The definition of the fibre integral is independent of the choice of adapted coordinates and extends to relative* cs *manifolds in general. It gives rise to a morphism of* $p_{X,0}^{-1}\mathcal{O}_Y$ *module presheaves* ${}_Y\!\fint_X^\gamma\colon \mathrm{Ber}^{\mathrm{int}}_{X/Y} \to p_{X,0}^{-1}\mathcal{O}_Y$*. Here,* $\mathrm{Ber}^{\mathrm{int}}_{X/Y}$ *denotes the presheaf of integrable relative Berezin densities.*

Proof. The first thing to notice is the $\mathcal{O}_Y$-linearity of the integral, *i.e.*

$$\forall f \in \Gamma(\mathcal{O}_Y): \quad {}_Y\!\fint_X^\gamma p_X^\sharp(f)\omega = f\, {}_Y\!\fint_X^\gamma \omega.$$

This is due to the compatibility of the retractions γ and γ_Y. It already proves the invariance of the integral under coordinate changes on Y.

Now assume that (u', ξ') to be a second global coordinate system adapted to γ and write $\omega = |D(u', \xi')|g$ with $g = \sum_{\mu,\nu} \gamma^\sharp(g_{\nu,\mu})\xi^\mu\eta^\nu$. Equation (2.1) on page 12 shows $\frac{\partial u_i}{\partial \xi'_j} = 0$ as well as $\frac{\partial u_i}{\partial u'_k} = \gamma^\sharp(\frac{\partial u_{i,0}}{\partial u'_{k,0}})$, hence

$$|\mathrm{Ber}|\left(\frac{\partial x'}{\partial x}\right) = |\det|\left(\frac{\partial u'}{\partial u}\right)\det\left(\frac{\partial \xi'}{\partial \xi}\right)^{-1} = \gamma^\sharp\left(|\det|\left(\frac{\partial u'_0}{\partial u_0}\right)\right)\det\left(\frac{\partial \xi'}{\partial \xi}\right)^{-1}.$$

Furthermore, each odd coordinate can be written as $\xi'_j = \sum_k \xi_k \frac{\partial \xi'_j}{\xi_k} \mod \mathcal{N}^2$. Accordingly, $\xi'_1 \cdots \xi'_q = \det\left(\frac{\partial \xi'}{\partial \xi}\right)\xi_1 \cdots \xi_q$ and therefore

$$\begin{aligned}
{}_Y\!\fint_X^\gamma |D(u', \xi')|\gamma^\sharp(g_{\nu,(1,\ldots,1)})\xi'_1 \cdots \xi'_q &= {}_Y\!\fint_X^\gamma |D(u', \xi')|\gamma^\sharp\left(g_{\nu,(1,\ldots,1)}\,|\det|\left(\frac{\partial u'_0}{\partial u_0}\right)\right)\xi_1 \cdots \xi_q \\
&= \gamma_Y^\sharp\left({}_{Y_0}\!\fint_{X_0} |du_0|g_{\nu,(1,\ldots,1)}\,|\det|\left(\frac{\partial u'_0}{\partial u_0}\right)\right) \\
&= \gamma_Y^\sharp\left({}_{Y_0}\!\fint_{X_0} |du'_0|g_{\nu,(1,\ldots,1)}\right).
\end{aligned}$$

It remains to show that the integral vanishes for summands with $\mu \neq (1, \ldots, 1)$. Each such summand can be obtained by derivating along an odd fibre coordinate, which means $|D(u', \xi')|\gamma^\sharp(f_{\nu,\mu})\xi^\mu = \nabla(\omega' \otimes \partial_{\xi'_j})$ for some j and $\omega \in |\mathrm{Ber}|_{X/Y}$. Using again $\frac{\partial u_i}{\partial \xi'_j} = 0$ and the chain rule,

$$\nabla\left(\omega' \otimes \partial_{\xi'_j}\right) = \sum\nolimits_k \nabla\left(\omega'\frac{\partial \xi'_j}{\partial \xi_k} \otimes \partial_{\xi_k}\right)$$

is obtained. The fibre integral of this Berezin density vanishes by definition, showing that Definition 2.2.6 does not depend on the choice of coordinates.

Now, extending the integral to *cs* manifolds in general can be accomplished by using the standard argumentation with partitions of unity. □

2.2.8 Remark. The second part of the proof of Proposition 2.2.7 is a slight modification of the proof of [Lei80, Theorem 2.4.5].

2.2.9 Remark. Let U be an open dense subset of X_0. Then it is clear by definition, that ${}_Y\!\fint_X^\gamma \omega = {}_Y\!\fint_{X|_U}^\gamma \omega|_U$. This observation becomes very useful if one has non-global coordinate systems, adapted to a global retraction, with dense domains.

2.2.10 Corollary. *If $\omega \in \Gamma_{cf}(|\mathrm{Ber}|_{X/Y})$, the fibre integral of ω does not depend on the choice of γ.*

2.2.11 Definition. Given an $\mathcal{O}_Y$-module presheaf $\mathcal{E}$ over Y_0, it is reasonable to define the presheaf

$$p_{X,0}^{-1}\mathcal{E} \otimes_{p_{X,0}^{-1}\mathcal{O}_Y} |\mathrm{Ber}|_{X/Y}^{\mathrm{int}}$$

over X_0 by taking tensor products of local sections. In particular, if Y is considered to be a relative *cs* manifold over Z, this yields $|\mathrm{Ber}|_{X/Z}^{\mathrm{int}_Y} := p_{X,0}^{-1} |\mathrm{Ber}|_{Y/Z} \otimes_{p_{X,0}^{-1}\mathcal{O}_Y} |\mathrm{Ber}|_{X/Y}$. This presheaf can be considered as sub-presheaf of $p_{X,0}^{-1} |\mathrm{Ber}|_{X/Z}$.

2.2.12 Corollary. *The definition of the integral extends, thanks to its $\mathcal{O}_Y$-linearity, for any $\mathcal{O}_Y$-module presheaf $\mathcal{E}$ over Y_0 to a morphism of presheaves*

$${}_Y\!\fint_X^\gamma : p_{X,0}^{-1}\mathcal{E} \otimes_{p_{X,0}^{-1}\mathcal{O}_Y} |\mathrm{Ber}|_{X/Y}^{\mathrm{int}} \longrightarrow p_{X,0}^{-1}\mathcal{E}.$$

In particular, this takes the form

$${}_Y\!\fint_X^\gamma : \ |\mathrm{Ber}|_{X/Z}^{\mathrm{int}_Y} \longrightarrow p_{X,0}^{-1} |\mathrm{Ber}|_{Y/Z} .$$

The following Fubini formula is easily obtained by checking it on coordinate neighbourhoods.

2.2.13 Corollary. *Suppose X/Y and Y/Z to be relative* cs *manifolds and let γ be a relative retraction on X/Y, such that γ_Y is a relative retraction on $p_X(X)/Z$. Furthermore, let $\omega \in \Gamma(\,|\mathrm{Ber}|_{X/Z}^{\mathrm{int}_Y})$, such that ${}_Y\!\fint_X^\gamma \omega \in \Gamma(\,|\mathrm{Ber}|_{p_X(X)/Z}^{\mathrm{int}})$. Then*

$${}_Z\!\fint_X^\gamma \omega = {}_Z\!\fint_{p_X(X)}^{\gamma_Y} {}_Y\!\fint_X^\gamma \omega.$$

Suppose γ to be a relative retraction on X/Y and let $\varphi\colon Z \to Y$ be a morphism. If γ_Z is a retraction on $p_Z(Z \times_Y X)$, compatible with γ_Y *via* φ, then the fibre product $\gamma_Z \times_{\gamma_Y} \gamma_X$ (*cf.* Definition A.11) is a retraction on $Z \times_Y X$. This leads to the following proposition.

2.2.14 Proposition. *The fibre integral is invariant under base change* via φ *in the following sense*

$${}_Z\!\fint_{Z\times_Y X}^{\gamma_Z \times_{\gamma_Y} \gamma} \mathrm{pr}_X^\sharp(\omega) = \varphi^\sharp \left({}_Y\!\fint_X^\gamma \omega \right)$$

for $\omega \in \Gamma(|\mathrm{Ber}|_{X/Y}^{\mathrm{int}})$.

Proof. Assume that X/Y admits a global fibre coordinate system $x = (u, \xi)$, adapted to γ. Due to the $\mathcal{O}_Y$ linearity of the integral, it suffices to assume $\omega = |Dx|\gamma^\sharp(f)\xi^\nu$. Keeping in mind that $\gamma \circ \mathrm{pr}_X = \mathrm{pr}_{X_0} \circ(\gamma_Z \times_{\gamma_Y} \gamma)$, that $\mathrm{pr}_X^\sharp(x)$ is adapted to $\gamma_Z \times_{\gamma_Y} \gamma$, and that $\varphi_0 \circ \gamma_Z = \gamma_Y \circ \varphi$, the proof reduces to the purely even case, which is trivial:

$$
\begin{aligned}
{}_Z\!\!\fint_{Z\times_Y X}^{\gamma_Z\times_{\gamma_Y}\gamma} \mathrm{pr}_X^\sharp\left(|Dx|\gamma^\sharp(f)\xi^\nu\right) &= {}_Z\!\!\fint_{Z\times_Y X}^{\gamma_Z\times_{\gamma_Y}\gamma} |D\,\mathrm{pr}_X^\sharp(x)|(\gamma_Z\times_{\gamma_Y}\gamma)^\sharp(\mathrm{pr}_{X_0}^\sharp(f))\,\mathrm{pr}_X^\sharp(\xi)^\nu \\
&= \delta_{\nu,(1,\ldots,1)}\gamma_Z^*\left({}_{Z_0}\!\!\fint_{Z_0\times_{Y_0}X_0} |d\,\mathrm{pr}_{X_0}^\sharp(x_0)|\,\mathrm{pr}_{X_0}^\sharp(f)\right) \\
&= \delta_{\nu,(1,\ldots,1)}\gamma_Z^*\circ\varphi_0^*\left({}_{Y_0}\!\!\fint_{X_0}|dx_0|f\right) \\
&= \varphi^\sharp\left({}_Y\!\!\fint_X^\gamma \omega\right). \qquad\square
\end{aligned}
$$

2.2.15 Remark. The case of the relative *cs* manifold $(Y \times X)/Y$ is of special interest. Let γ_X and γ_Y be retractions on X and Y. Then ${}_Y\!\fint_{Y\times X}^{\gamma_Y\times\gamma_X}$ does not depend on the choice of γ_Y. This can be shown as follows. Consider $Y \times X$ as *cs* manifold over $Y \times X_0$ via the morphism $\mathrm{id}_Y \times\gamma_X$. Then

$$
{}_Y\!\!\fint_{Y\times X}^{\gamma_Y\times\gamma_X}\omega = {}_Y\!\!\fint_{Y\times X_0}^{\gamma_Y\times \mathrm{id}_X}\ {}_{Y\times X_0}\!\!\fint_{Y\times X}^{\gamma_Y\times\gamma_X}\omega
$$

by Corollary 2.2.13

Assume that X admits a global fibre coordinate system $x = (u, \xi) = \mathrm{id}_Y \times x'$, adapted to $\gamma_Y \times \gamma_X$. In this case, the property of being adapted to $\gamma_Y \times \gamma_X$ depends only on γ_X. Therefore, $\partial_{\xi_j} \circ (\gamma'_Y \times \gamma_X)^\sharp = 0$ for any retraction γ'_Y on Y. This shows the independence of ${}_{Y\times X_0}\!\fint_{Y\times X}^{\gamma_Y\times\gamma_X}$ from the choice of γ_Y.

Therefore, one may assume $X = X_0$. Elements of $\Gamma(\mathrm{Ber}_{(Y\times X_0)/Y})$ can be considered as $\Gamma(\mathcal{O}_X)$-valued densities on X_0 in this case. Here, ${}_Y\!\fint_{X_0}^{\gamma_Y\times\mathrm{id}_{X_0}}$ is the Blochner integral of such densities (*cf.* [AS14, Appendix B1]), which means in particular that it is independent from the choice of γ_Y.

A similar argument shows that in this special case the assumption of φ being compatible with the retractions on Y and Z can be dropped in Proposition 2.2.14. Therefore, the following definition is justified.

2.2.16 Definition. Let γ be a retraction on X, let $\omega \in \Gamma(|\mathrm{Ber}|_X^{\mathrm{int}})$ and $f \in \mathcal{O}_{S\times X}$. Considering f as a morphism $S \times X \to \mathbb{A}_S^{p|q}$, and setting $f(s, x) := f \circ (s \times x) = (s \times x)^\sharp(f)$ for any $s \in_T S$ and $x := \mathrm{id}_X$, let

$$
\int_X^\gamma \omega(x)f(s,x) := {}_T\!\!\fint_{T\times X}^{\gamma_T\times\gamma}\mathrm{pr}_X^\sharp(\omega)f(s,x) = s^\sharp\left({}_S\!\!\fint_{S\times X}^{\gamma_S\times\gamma}\mathrm{pr}_X^\sharp(\omega)f\right).
$$

Here, γ_T and γ_S are retractions on T and S, respectively (not necessarily compatible).

This definition is natural in T, *i.e.*

$$\int_X^\gamma \omega(x)f(s(t),x) = \Big(\int_X^\gamma \omega(x)f(s,x)\Big)(t)$$

for generalised points t of T.

2.2.17 Remark. In the language of Definition 2.2.16, Corollary 2.2.13 takes the following form:

$$\int_{Y\times X}^{\gamma_Y\times\gamma_X} (\varpi\otimes\omega)(y,x)f(s,y,x) = \int_Y^{\gamma_Y} \varpi(y)\int_X^{\gamma_X} \omega(x)f(s,y,x)$$

for *cs* manifolds X, Y and S, Berezin densities $\omega \in \Gamma(|\mathrm{Ber}|_X)$ and $\varpi \in \Gamma(|\mathrm{Ber}|_Y)$, and retractions γ_X on X and γ_Y on Y.

2.2.18 Definition. For a relative isomorphism $\varphi\colon Z/Y \to X/Y$, the *pullback* of relative Berezin densities is obtained by requiring $\varphi^\sharp(|Dx|f) := |D(x\circ\varphi)|\varphi^\sharp(f)$ for fibre coordinate systems x. It rises to a morphism of sheaves $\varphi^\sharp\colon \varphi_0^{-1}|\mathrm{Ber}|_{X/Y} \to |\mathrm{Ber}|_{Z/Y}$.

2.2.19 Proposition. *Given a relative isomorphism $\varphi\colon Z/Y \to X/Y$ and relative retractions γ_X on X and γ_Z on Z, compatible under φ, the following equality holds:*

$$\vphantom{\int}_Y\!\!\fint_Z^{\gamma_Z} \varphi^\sharp(\omega) = \vphantom{\int}_Y\!\!\fint_X^{\gamma_X} \omega,$$

whenever one of both sides exists.

In view of Definition 2.2.16 this takes in the case $Y = *$ the form

$$\int_Z^{\gamma_Z} \varphi^\sharp(\omega)(z)f(s,\varphi(z)) = \int_X^{\gamma_X} \omega(x)f(s,x),$$

for $f \in \Gamma(\mathcal{O}_{S\times X})$ and generalised points s of S.

For a relative morphism $\varphi\colon X/Y \to X/Y$, it might happen that $\varphi^\sharp$ is a differential operator. Thanks to the $p_{X,0}^{-1}\mathcal{O}_Y$-linearity of $\varphi^\sharp$ this means $\varphi^\sharp \in \mathcal{D}_{X/Y}$. The following proposition relates the right action of $\mathcal{D}_{X/Y}$ on $\mathrm{Ber}_{X/Y}$ to the pullback of Berezin densities. It is the key ingredient in the derivation of boundary terms, occurring under changes of retractions.

2.2.20 Proposition. *Suppose $\varphi\colon X/Y \to X/Y$ to be a isomorphism with $\varphi^\sharp \in \Gamma(\mathcal{D}_{X/Y})$. Then $\varphi^\sharp(\nabla(\omega\otimes\varphi^\sharp)) = \omega$ for any $\omega \in \Gamma(|\mathrm{Ber}|_{X/Y})$.*

The proof of this proposition is based on partial integration, Corollary 2.2.10 and Proposition 2.2.19.

Let γ, γ' be retractions on X/Y, such that $\gamma_Y = \gamma'_Y$, and suppose that X admits a global fibre coordinate system $x = (u,\xi)$, adapted to γ. Then $\varphi^\sharp := \sum_{i\in\mathbb{N}_0^q} \frac{1}{i!}(v-u)\partial_u^i$ defines an isomorphism $\varphi\colon X/Y \to X/Y$, such that γ and γ' are compatible *via* φ. Here, $v := \gamma'^\sharp(u_0)$ and $(v-u)^i := (v_1-u_1)^{i_1}\cdots(v_p-u_p)^{i_p}$ with $p = \dim_{Y_0} X_0$. In this setting the following formula is obtained by applying Propositions 2.2.19 and 2.2.20.

2.2.21 Corollary.

$$ {}_Y\!\!\fint_X^{\gamma'} \omega = {}_Y\!\!\fint_X^{\gamma} \nabla(\omega \otimes \varphi^\sharp) = {}_Y\!\!\fint_X^{\gamma} \omega + \sum_{i \in \mathbb{N}_0^q \setminus \{0\}} \frac{1}{i!} {}_Y\!\!\fint_X^{\gamma} \nabla(\omega (v-u)^i \otimes \partial_u^i), $$

if the right hand side exists.

By applying the Fundamental Theorem of Calculus, a global formula can be deduced after fixing some notation. Given a family of boundary super functions $\tau = (\tau_1, \ldots, \tau_n)$ and $H \in B(\tau)$, let $J_H := \{j \in \mathbb{N}_0^n \mid j_i = 0 \Leftrightarrow \rho_i \notin \rho_H\}$. For $j \in \mathbb{N}_0^n$, set $j{\downarrow} := (j_1{\downarrow}, \ldots, j_n{\downarrow})$, where $s{\downarrow} := \max(0, s-1)$.

Furthermore, it can be shown that there exist $D_i \in \Gamma(\mathcal{T}_{X/Y})$ for $i = 1, \ldots, n$, such that $D_i(\tau_k) = 0$ on $U_i \cap U_k$, where U_i is a neighbourhood of the boundary *cs* manifold, given by τ_i.

2.2.22 Proposition. *Let $\omega \in \Gamma_{cf}(|\mathrm{Ber}|_{X/Y})$, let ρ be a family of boundary functions on X_0/Y_0 and γ, γ' be relative retractions on X/Y with $\gamma_Y = \gamma'_Y$. Then*

$$ {}_Y\!\!\fint_{X_\rho}^{\gamma'} \omega = {}_Y\!\!\fint_{X_\rho}^{\gamma} \omega + \sum_{H \in B(\gamma^\sharp(\rho))} \sum_{j \in J_H} {}_Y\!\!\fint_H^{\gamma_H} \nabla\Big(\omega \big(\gamma'^\sharp(\rho) - \gamma^\sharp(\rho)\big)^j \otimes D^{j\downarrow}\Big)\Big|_{H, \gamma^\sharp(\rho)}. $$

Here, $D = (D_1, \ldots, D_n)$ is a family of derivations adjusted to $\gamma^\sharp(\rho)$, as described above.

2.2.23 Remark. The space $X = \mathbb{A}_+^k \times Z$, where Z is an ordinary *cs* manifold and $\mathbb{A}_+^k := \mathbb{A}^k|_{\mathbb{R}_+^k}$ is of special interest. Let γ be a retraction on $\mathbb{A}^k \times Z$, let $\omega \in \Gamma(|\mathrm{Ber}|_Z)$ and let $d\lambda$ be the Lebesgue measure on $\mathbb{A}^k$. Furthermore, let $f \in \Gamma_{cf}(\mathcal{O}_{S \times \mathbb{A}^p \times Z})$. Then, the proposition above takes the following form.

$$ \begin{aligned} &\int_{\mathbb{A}_+^k \times Z}^{\gamma} (d\lambda \otimes \omega)(t,z) f(s,t,z) - \int_{\mathbb{A}_+^k} d\lambda(t) \int_Z \omega(z) f(s,t,z) \\ &\qquad = \sum_{l=1}^{k} \sum_{\sigma \in S_k} \sum_{j \in \mathbb{N}^l} (-1)^l \frac{1}{j!} \partial^{j\downarrow}_{t_{\sigma \le l} = 0} \int_{\mathbb{A}_+^{k-l}} d\lambda(t_{\sigma > l}) \int_Z \omega(z) \big(t - \gamma(t,z)\big)^j_{\sigma \le l} f(s,t,z), \end{aligned} $$

where

$$ \begin{aligned} \partial^j_{t_{\sigma \le l} = 0} &:= \partial^{j_1}_{t_{\sigma^{-1}(1)} = 0} \cdots \partial^{j_l}_{t_{\sigma^{-1}(l)} = 0}, \\ t_{\sigma > l} &:= (t_{\sigma^{-1}(l+1)}, \ldots, t_{\sigma^{-1}(k)}), \\ \big(t - \gamma(t,z)\big)^j_{\sigma \le l} &:= \Big(t_{\sigma^{-1}(1)} - \mathrm{pr}_{\sigma^{-1}(1)} \circ \gamma(t,x)\Big)^{j_1} \cdots \Big(t_{\sigma^{-1}(l)} - \mathrm{pr}_{\sigma^{-1}(l)} \circ \gamma(t,z)\Big)^{j_l}. \end{aligned} $$

Here, it should be noted that the derivations can be pulled out of the integral due to the compact support along the fibres. For the same reason, it is not necessary to fix a retraction on Z.

2.2.24 Remark. In applications of Proposition 2.2.22 it proves beneficial that the boundary functions may be replaced by $\rho_i - c_i$ for constants $c_i \in \mathbb{R}$. This can be performed, since the c_i cancel each other out in the formula for boundary terms.

2.3. Integration Along Polar Coordinates on $\mathbb{A}^{p|2q}$

In this section, let $D\lambda \coloneqq (-2\pi)^{-q}|Dy| \in \Gamma(|\mathrm{Ber}|_{\mathbb{A}^{p|2q}})$ be the Lebesgue Berezin density. Here, $y = (v, \eta)$ denotes the standard coordinate system on $\mathbb{A}^{p|2q}$. Let γ be the retraction associated with y, *i.e.* $\gamma(x) = x_{\bar{0}} \coloneqq (x_1, \ldots, x_p)$ for $x \in \mathbb{A}^{p|2q}(T) = \mathcal{O}_{\mathbb{R},\bar{0}}(T)^p \times \mathcal{O}_{\bar{1}}(T)^q$. One should avoid to confuse the notation $x_{\bar{0}} \in \mathbb{A}^p(T)$ with the underlying morphism $x_0 \in \mathbb{A}^p(T_0)$. Furthermore, $\|\cdot\|^2 \colon \mathbb{A}^{p|2q} \to \mathbb{A}^1_+$ shall be given by

$$\|x\|^2 \coloneqq \sum_{i=1}^{p} x_i^2 + 2\sum_{j=1}^{q} x_{p+2j-1}x_{p+2j}$$

for $x \in_T \mathbb{A}^{p|2q}$. Using the positive square root, this yields

$$\|\cdot\| \coloneqq \sqrt{} \circ \|\cdot\|^2 \colon \mathbb{A}^{p|2q}_{\neq 0} \to \mathbb{A}^1_+,$$

where $\mathbb{A}^{p|2q}_{\neq 0} \coloneqq \mathbb{A}^{p|2q}|_{\mathbb{R}^p\setminus\{0\}}$.

2.3.1 Definition. A super function $f \in \Gamma(\mathcal{O}_{S\times\mathbb{A}^{p|2q}})$ for $p > 0$ will be called *rotationally invariant in the second component* if there exists an $f^\circ \in \Gamma(\mathcal{O}_{S\times\mathbb{A}^1_+})$ such that $f(s, x) = f^\circ(s, \|x\|)$, for $(s, x) \in_T S \times \mathbb{A}^{p|2q}_{\neq 0}$.

Since $\|\cdot\|$ is not well-defined on $\mathbb{A}^{0|2q}$, the definition of rotational invariance has to be modified in this case. Here, f is called rotational invariant if there exists $g \in \Gamma(\mathcal{O}_{S\times\mathbb{A}^1})$, with $f(s, x) = g(s, \|x\|^2)$ for $(s, x) \in_T \mathbb{A}^{0|2q}_{\neq 0}$, is needed. In this case, it is reasonable to define $f^\circ(s, t) \coloneqq g(s, t^2)$ analogously.

2.3.2 Remark. In the case $p > 0$, the super function f° in Definition 2.3.1 can also be extended to a super function on $S \times \mathbb{A}^1$ which is even in the second component. Such an extension is given by $f^\circ(s, t) \coloneqq f(s, te_1)$, where $t \in_T \mathbb{A}^1$ and e_1 is the first basis vector of $\mathbb{R}^p$. Due to the evenness in the second component, there exists an extension g of $f^\circ \circ (\mathrm{id}_S \times \sqrt{})$ onto $S \times \mathbb{A}^1$. This extension fulfils $f(s, x) = g(s, \|x\|^2)$ for all $(s, x) \in_T S \times \mathbb{A}^{p|2q}$.

2.3.3 Lemma. *Let $f \in \mathcal{O}_{S\times\mathbb{A}^{p|2q}}$ be rotationally invariant in the second component. Then*

$$\int_{\mathbb{A}^{p|2q}}^{\gamma} D\lambda(x) f(s, x) = \begin{cases} \frac{\pi^{\frac{p-2q}{2}}(-1)^q}{\Gamma(\frac{p}{2})} \int_{\mathbb{A}^1_+} dr\, r^{\frac{p}{2}-1} \partial_r^q f^\circ(s, \sqrt{r}), & p > 0, \\ (-\pi)^{-q} \partial_{r=0}^q f^\circ(s, \sqrt{r}), & p = 0, \end{cases}$$

Here, one side exists if and only if the other exists.

Proof. Applying Taylor-expansion to g from above yields

$$g(s, t + t') \equiv \sum_{k=1}^{q} \frac{1}{k!} t'^k \partial_{t'=0}^k g(s, t + t') \equiv \sum_{k=1}^{q} \frac{1}{k!} t'^k \partial_t^k g(s, t) \mod \langle (t')^{q+1} \rangle,$$

with $t = t' = \mathrm{id}_{\mathbb{A}^1}$. Hence,

$$f(s,y) = g(s, \|v\|^2 + \|\eta\|^2) = \sum_{k=1}^{q} \frac{1}{k!} \partial_2^k g(s, \|v\|^2) \|\eta\|^{2k},$$

where $y = \mathrm{id}_{\mathbb{A}^{p|q}}$, $v = \mathrm{id}_{\mathbb{A}^p}$, $\eta = \mathrm{id}_{\mathbb{A}^{0|q}}$ and $(\partial_2 g)(s,t) := \partial_t g(s,t)$.

The expression $\|\eta\|^{2k}$ contains $\eta_1 \cdots \eta_{2q}$ if and only if $k = q$. In this case, it equals $2^q q! \eta_1 \cdots \eta_{2q}$. For $p = 0$ this means that the integral equals $(-\pi)^{-q} \partial_{t=0}^q g(s,t)$. If $p > 0$, the integral takes the form

$$\int_{\mathbb{A}^{p|2q}}^{\gamma} D\lambda(y) f(s,y) = (-\pi)^{-q} \int_{\mathbb{A}^p} dv_0 \, \partial_2^q g(s, \|v_0\|^2).$$

Polar coordinates are easily used on this integral for $p \geq 2$:

$$\begin{aligned}
(-\pi)^{-q} \int_{\mathbb{A}^p} dv_0 \, \partial_2^q g(s, \|v_0\|^2) &= \frac{2\pi^{\frac{p-2q}{2}} (-1)^q}{\Gamma(\frac{p}{2})} \int_{\mathbb{A}^1_+} dr \, r^{p-1} \partial_2^q g(s, r^2) \\
&= \frac{\pi^{\frac{p-2q}{2}} (-1)^q}{\Gamma(\frac{p}{2})} \int_{\mathbb{A}^1_+} dr \, r^{\frac{p}{2}-1} \partial_r^q g(s, r).
\end{aligned}$$

The case $p = 1$ is just as simple:

$$\begin{aligned}
(-\pi)^{-q} \int_{\mathbb{A}^1} dv \, \partial_2^q g(s, v^2) &= 2(-\pi)^{-q} \int_{\mathbb{A}^1_+} dv \, \partial_2^q g(s, v^2) \\
&= \frac{\pi^{\frac{1-2q}{2}} (-1)^q}{\Gamma(\frac{1}{2})} \int_{\mathbb{A}^1_+} dr \, r^{-\frac{1}{2}} \partial_r^q g(s, r).
\end{aligned}$$

□

2.3.4 Corollary. *Let $k \leq \min(\frac{p}{2}, q)$ and $f \in \Gamma_{cf}(\mathcal{O}_{\mathbb{A}_S^{p|2q}})$ be rotationally invariant. Then*

$$\int_{\mathbb{A}^{p|2q}}^{\gamma} D\lambda(y) f(s,y) = \int_{\mathbb{A}^{p-2k|2q-2k}}^{\gamma} D\lambda(y) f^\circ(s, \|y\|).$$

Proof. Use Lemma 2.3.3 and partial integration for $k < \frac{p}{2}$. In addition, the fundamental theorem of calculus needs to be applied if $k = \frac{p}{2}$. □

2.3.5 Corollary. *Let $f \in \Gamma_{cf}(\mathcal{O}_{\mathbb{A}_S^{p|2q}})$ be rotationally invariant. Then*

$$\int_{\mathbb{A}^{p|2q}}^{\gamma} D\lambda(y) f(s,y) = \begin{cases}
\frac{2\pi^{\frac{p-2q}{2}}}{\Gamma(\frac{p-2q}{2})} \int_{\mathbb{A}^1_+} dr \, r^{p-2q-1} f^\circ(s,r), & p-2q > 0, \\
(-\pi)^{\frac{p-2q}{2}} \partial_{r=0}^{\frac{2q-p}{2}} f^\circ(s, \sqrt{r}) & \\
\qquad = (-\pi)^{\frac{p-2q}{2}} \frac{(\frac{2q-p}{2})!}{(2q-p)!} \partial_{r=0}^{2q-p} f^\circ(s,r), & p-2q \leq 0 \textit{ even}, \\
(-\pi)^{\frac{p-1-2q}{2}} \int_{\mathbb{A}^1_+} dr \, r^{-\frac{1}{2}} \partial_r^{\frac{2q+1-p}{2}} f^\circ(s, \sqrt{r}), & p-2q < 0 \textit{ odd}.
\end{cases}$$

Proof. The only case where anything needs to be proven is the case of $p = 0$. Here, Faà di Bruno's formula yields

$$\partial^{2q}_{r=0} f^\circ(s, r) = \partial^{2q}_{r=0} f^\circ(s, \sqrt{r^2}) = \sum_{k_1+2k_2=2q} \frac{(2q)!}{k_1!k_2!} \partial^{k_1+k_2}_{t=0} f^\circ(s, \sqrt{t}) 0^{k_1} 1^{k_2}. \tag{2.2}$$

All summands except for $k_2 = q$ vanish. □

In the following, it will be useful to define some notation for coordinate systems x on $\mathbb{A}^{p|q}$. Let $\xi^*_j := \xi_{j-1}$ for j even and $\xi^*_j := -\xi_{j+1}$ for j odd. If one considers ξ^* as row vector and ξ as column vector, then $\xi^*\xi = \|\cdot\|^2 \circ \xi$. On the right hand side of this equation, ξ has to be understood as a morphism $\mathbb{A}^{p|2q} \to \mathbb{A}^{0|2q}$.

The first thing to do, in order to obtain some kind of polar integration formula, is to define polar coordinates. With this aim, the symbol $\rho := \|\cdot\|$ will be used in the following for the radial coordinate.

Let $u_0 = (\rho_0, \tilde{u}_0) \colon \mathbb{R}_+ \times]-\pi, \pi[\times]-\frac{\pi}{2}, \frac{\pi}{2}[^{p-2} \to \mathbb{R}^p \setminus (]-\infty, 0[\times \mathbb{R}^{p-1})$ be the ordinary polar coordinate system,

$$\begin{aligned}
v_{1,0} &= \rho_0 \cos u_{2,0} \cos u_{3,0} \cdots \cos u_{p,0}, \\
v_{i,0} &= \rho_0 \sin u_{i,0} \cos u_{i+1,0} \cdots \cos u_{p,0}, \quad i = 2, \ldots, p-1, \\
v_{p,0} &= \rho_0 \sin u_{p,0}.
\end{aligned}$$

Let γ' be the retraction on $\mathbb{A}^{p|2q}_{\neq 0}$ which is given by $\gamma'(t) := \gamma(t) \frac{\|t\|}{\|\gamma(t)\|} = \frac{\|t\| t_{\bar{0}}}{\|t_{\bar{0}}\|}$ for $t \in_T \mathbb{A}^{p|2q}_{\neq 0}$ and set $u := \gamma^\sharp(u_0)$ and $\xi_j := \frac{\eta_j}{\rho}$ for $j = 1, \ldots, 2q$. From now on, the coordinate system $x = (u, \xi)$ will be referred to as *polar coordinates on* $\mathbb{A}^{p|2q}$.

It is clear by definition that $\rho^2 = \|v\|^2 + \eta^*\eta = \|v\|^2 + \rho^2 \xi^*\xi$, hence $\|v\| = \rho\sqrt{1 - \xi^*\xi}$. Furthermore, the linearity of $v_{i,0}$ shows $v_i(t) = v_{i,0}(\gamma(t)) = \frac{\|\gamma(t)\|}{\|t\|} v_{i,0}(\gamma'(t))$, and therefore $v_i = \frac{\gamma^\sharp(\rho_0)}{\rho} \gamma'^\sharp(v_{i,0})$. Keeping in mind that $\gamma^\sharp(\rho_0) = \|v\|$, this proves

$$\begin{aligned}
v_1 &= \rho\sqrt{1 - \xi^*\xi} \cos u_2 \cos u_3 \cdots \cos u_p, \\
v_i &= \rho\sqrt{1 - \xi^*\xi} \sin u_i \cos u_{i+1} \cdots \cos u_p, \quad i = 2, \ldots, p-1, \\
v_p &= \rho\sqrt{1 - \xi^*\xi} \sin u_p, \\
\eta_j &= \rho\xi_j, \quad j = 1, \ldots, 2q.
\end{aligned}$$

2.3.6 Lemma. *The Berezinian of the coordinate change to polar coordinates is*

$$|\mathrm{Ber}| \left(\frac{\partial y}{\partial x} \right) = \rho^{p-1-2q} \cos u_3 \cos^2 u_4 \cdots \cos^{p-2} u_p (1 - \xi^*\xi)^{\frac{p-2}{2}}.$$

Proof. Note that $\frac{\partial \xi^*\xi}{\partial \xi_j} = -2\xi_j^*$, hence $\frac{\partial\sqrt{1-\xi^*\xi}}{\partial \xi_i} = \frac{\xi_i^*}{\sqrt{1-\xi^*\xi}}$. Writing $|A| := |\mathrm{Ber}|(A)$, this yields

$$
\begin{aligned}
\left|\frac{\partial y}{\partial x}\right| &= \begin{vmatrix}
\frac{v_1}{\rho} & \frac{v_2}{\rho} & \cdots & \cdots & \frac{v_p}{\rho} & \xi_1 & \cdots & \xi_{2q} \\
-v_1\frac{\sin u_2}{\cos u_2} & v_2\frac{\cos u_2}{\sin u_2} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \vdots \\
-v_1\frac{\sin u_p}{\cos u_p} & \cdots & \cdots & -v_{p-1}\frac{\sin u_p}{\cos u_p} & v_p\frac{\cos u_p}{\sin u_p} & 0 & \cdots & 0 \\
\frac{\xi_1^* v_1}{1-\xi^*\xi} & \cdots & \cdots & \cdots & \frac{\xi_1^* v_p}{1-\xi^*\xi} & \rho & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\xi_{2q}^* v_1}{1-\xi^*\xi} & \cdots & \cdots & \cdots & \frac{\xi_{2q}1^* v_p}{1-\xi^*\xi} & 0 & \cdots & \rho
\end{vmatrix} \\
&= \rho^{-1-2q}\left|v_1\cdots v_p\frac{\sin u_2}{\cos u_2}\cdots\frac{\sin u_p}{\cos u_p}\right| \begin{vmatrix}
1 & 1 & \cdots & \cdots & 1 & \xi_1 & \cdots & \xi_{2q} \\
-1 & \frac{\cos^2 u_2}{\sin^2 u_2} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \vdots \\
-1 & \cdots & \cdots & -1 & \frac{\cos^2 u_p}{\sin^2 u_p} & 0 & \cdots & 0 \\
\frac{\xi_1^*}{1-\xi^*\xi} & \cdots & \cdots & \cdots & \frac{\xi_1^*}{1-\xi^*\xi} & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\xi_{2q}^*}{1-\xi^*\xi} & \cdots & \cdots & \cdots & \frac{\xi_{2q}1^*}{1-\xi^*\xi} & 0 & \cdots & 1
\end{vmatrix} \\
&= \rho^{-1-2q}\left|v_1\cdots v_p\frac{\sin u_2}{\cos u_2}\cdots\frac{\sin u_p}{\cos u_p}\right| \begin{vmatrix}
1 & 0 & \cdots & \cdots & 0 & \xi_1 & \cdots & \xi_{2q} \\
-1 & \frac{1}{\sin^2 u_2} & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \vdots \\
-1 & 0 & \cdots & 0 & \frac{1}{\sin^2 u_p} & 0 & \cdots & 0 \\
\frac{\xi_1^*}{1-\xi^*\xi} & 0 & \cdots & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\xi_{2q}^*}{1-\xi^*\xi} & 0 & \cdots & \cdots & 0 & 0 & \cdots & 1
\end{vmatrix} \\
&= \rho^{p-1-2q}\cos u_3\cdots\cos^{p-2}u_p(1-\xi^*\xi)^{\frac{p}{2}} \begin{vmatrix}
1 & \xi_1 & \cdots & \xi_{2q} \\
\frac{\xi_1^*}{1-\xi^*\xi} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\xi_{2q}^*}{1-\xi^*\xi} & 0 & \cdots & 1
\end{vmatrix} \\
&= \rho^{p-1-2q}\cos u_3\cdots\cos^{p-2}u_p(1-\xi^*\xi)^{\frac{p-2}{2}}.
\end{aligned}
$$

$\square$

2.3.7 Definition. The *super sphere* is the sub *cs* manifold of $\mathbb{A}^{p|2q}$ defined by the boundary function $\rho - 1$. This means that the T-points of $S^{p-1|2q}$ are given by

$$S^{p-1|2q}(T) = \left\{ s \in_T \mathbb{A}_1^{p|2q} \,\middle|\, \|s\| = 1 \right\}.$$

Abusing notation, $\tilde{x} = (\tilde{u}, \xi)$ will be used for the coordinate system $\iota^\sharp_{S^{p-1|2q}}(\tilde{x})$. As well, γ' will be used for the retraction on $S^{p-1|2q}$, induced by γ'. Of course, $\tilde{x}$ is no global coordinate system on $S^{p-1|2q}$. However, it is defined on a subset of S^{p-1} whose complement has measure 0 and it is adapted to the global retraction γ. Therefore, $\tilde{x}$ can be used for integration on $S^{p-1|2q}$.

2.3.8 Corollary. *The induced Berezin density on $S^{p-1|2q}$ is given by*

$$DS := D\lambda\big|_{S^{p-1|2q},\rho-1} = |D(\tilde{u},\xi)|(-2\pi)^{-q} \cos u_2 \cos^2 u_3 \cdots \cos^{p-1} u_p (1 - \xi^*\xi)^{\frac{p-2}{2}}$$

on the open dense subset, where $\tilde{x}$ is defined.

With a similar calculation as in the proof of Lemma 2.3.3, it is possible to calculate the volume of the super sphere:

$$\begin{aligned}\mathrm{Vol}(S^{p-1|2q}) &:= \int_{S^{p-1|2q}} DS = (-2\pi)^{-q} 2^q \partial^q_{r=0} (1-r)^{\frac{p-2}{2}} \mathrm{Vol}(S^{p-1}) \\ &= \pi^{-q} \frac{\Gamma(\frac{p}{2})}{\Gamma(\frac{p-2q}{2})} \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} = \frac{2\pi^{\frac{p-2q}{2}}}{\Gamma(\frac{p-2q}{2})}.\end{aligned}$$

In particular, the volume vanishes for $p \leq 2q$ even. This has recently been obtained, by different methods, in [Gro13].

From now on let $f \in \Gamma_{cf}(\mathcal{O}_{\mathbb{A}_T^{p|2q}})$ and $p > 0$. Then

$$\int^\gamma_{\mathbb{A}^{p|2q}} D\lambda(y) f(t,y) = \int^\gamma_{\mathbb{A}^1_+ \times S^{p-1|2q}} (d\lambda \otimes DS)(r,s)\, r^{p-2q-1} f(t, rs).$$

This is achieved by applying Proposition 2.2.19, where the isomorphism is given by $\mathbb{A}^1_+ \times S^{p-1|2q} \to \mathbb{A}^{p|2q}_{\neq 0}$, $(r,s) \mapsto rs$. The retraction γ is identified with the retraction $(r,s) \mapsto (r\|s_{\bar{0}}\|, \frac{s_{\bar{0}}}{\|s_0\|})$. However, this retraction cannot be extended onto $\mathbb{A}^1 \times S^{p-1|2q}$. Therefore, in order to apply the boundary term formula in the form of Remark 2.2.23, a shift away from 0 has to be inserted:

$$\begin{aligned}\int^\gamma_{\mathbb{A}^{p|2q}} D\lambda(y) f(t,y) = \lim_{\varepsilon \to 0^+} \Bigg(&\int_{\mathbb{A}^1_{>\varepsilon}} dr\, r^{p-2q-1} \int_{S^{p-1|2q}} DS(s) f(t, rs) \\ &- \sum_{k>0} \frac{1}{k!} \partial^{k-1}_{r=\varepsilon} \int_{S^{p-1|2q}} DS(s) r^{p-2q-1} (r - r\|s_{\bar{0}}\|)^k f(t, rs) \Bigg)\end{aligned} \tag{2.3}$$

Here, $\mathbb{A}^1_{>\varepsilon} := \mathbb{A}^1\big|_{]\varepsilon,\infty[}$.

The integrals $\int_{S^{p-1|2q}} DS(s)(1 - \|s_{\bar{0}}\|)^k f(t, rs)$ yield super functions which are well-defined on $\mathbb{A} \times S^{p-1|2q}$. If p is bigger than $2q$, then $\partial^{k-1}_{r=0} r^{p-2q-1+k} = 0$.

2.3.9 Proposition. *For $p > 2q$, the polar integration formula takes the form*

$$\int_{\mathbb{A}^{p|2q}}^{\gamma} D\lambda(y) f(t,y) = \int_{\mathbb{A}^1_+} dr \int_{S^{p-1|2q}} DS(s) r^{p-2q-1} f(t, rs).$$

For the cases where $p \leq 2q$ further efforts need to be made. Taylor's formula (2.1.30) applied to f at 0 yields

$$f = \sum_{l=0}^{2q-p} f_l + g, \qquad f_l = \sum_{|i|=l} \frac{1}{i!} y^i \partial^i_{y=0} f, \qquad g \in \left\langle y^j \,\middle|\, |j| = 2q - p + 1 \right\rangle_{\Gamma\left(\mathcal{O}_{T\times\mathbb{A}^{p|2q}}\right)}.$$

The first thing to notice is that $f_l(t, rs) = r^l f_l(t,s)$ and $f_l(t, -s) = (-1)^l f_l(t,s)$ for generalised points (t,r,s) of $T \times \mathbb{A}^1_+ \times S^{p-1|2q}$. Therefore,

$$\int_{S^{p-1|2q}} DS(s)(1 - \|s_{\bar{0}}\|)^k f_l(t,s) = 0,$$

if l is odd. This is due to the fact that DS is invariant under the morphism $s \mapsto -s$ on $S^{p-1|2q}$, since $D\lambda$ and ρ are invariant under the morphism $\mathbb{A}^{p|2q} \to \mathbb{A}^{p|2q}$, $x \mapsto -x$. In addition, $g(t,rs) = r^{p-2q+1} \tilde{g}(t,r,s)$ for some $\tilde{g} \in \Gamma\left(\mathcal{O}_{T\times\mathbb{A}^1\times S^{p-1|2q}}\right)$, hence

$$\lim_{\varepsilon\to 0} \partial^{k-1}_{r=\varepsilon} \int_{S^{p-1|2q}} DS(s) r^{p-2q-1}(r - r\|s_{\bar{0}}\|)^k g(t, rs) = 0.$$

Putting these considerations together, Equation (2.3) reduces to

$$\int_{\mathbb{A}^{p|2q}}^{\gamma} D\lambda(y) f(t,y) = \lim_{\varepsilon\to 0+} \left(\int_{\mathbb{A}^1_{>\varepsilon}} dr\, r^{p-2q-1} \int_{S^{p-1|2q}} DS(s) f(t,rs) \right.$$
$$\left. - \sum_{k>0} \sum_{2l\leq 2q-p} \frac{1}{k!} \partial^{k-1}_{r=\varepsilon} r^{p-2q-1+k+2l} \int_{S^{p-1|2q}} DS(s)(1 - \|s_{\bar{0}}\|)^k f_{2l}(t,s) \right).$$

Since $-\frac{1}{k!}\partial^{k-1}_{r=\varepsilon} r^{p-2q-1+k+2l} = \frac{1}{2q-p-2l}(-1)^k \binom{2q-p-2l}{k} \varepsilon^{p-2q+2l}$, the second line reduces to

$$\sum_{2l\leq 2q-p} \varepsilon^{p-2q+2l} \int_{S^{p-1|2q}} DS(s) \frac{\|s_{\bar{0}}\|^{2q-p-2l} - 1}{2q-p-2l} f_{2l}(t,s). \tag{2.4}$$

Here, $\frac{\|s_{\bar{0}}\|^0 - 1}{0}$ shall be understood as $\log(\|s_{\bar{0}}\|)$.

Now, it is time to recall that $DS = |D(\tilde{u},\xi)| h (1 - \xi^*\xi)^{\frac{p-2}{2}}$ with some super function h, satisfying $h = \gamma'^\sharp(h_0)$. Furthermore, $\|s_{\bar{0}}\| = \sqrt{1 - \xi^*\xi}(s)$. The integral of

$$DS \sqrt{1-\xi^*\xi}^{2q-p-2l} \iota^\sharp_{S^{p-1|2q}}(f_{2l}) = |D(\tilde{u},\xi)| h (1 - \xi^*\xi)^{q-l-1} \iota^\sharp_{S^{p-1|2q}}(f_{2l})$$

vanishes, since the order of $\iota^\sharp_{S^{p-1|2q}}(f_{2l})$ in ξ is at most $2l$. If p is even, the integral of $DS\iota^\sharp_{S^{p-1|2q}}(f_{2l})$ vanishes for the same reason, showing that in this case only the summand for $2l = 2q - p$ remains.

2.3.10 Proposition. *For $p \leq 2q$ with p even, the polar integration formula is given by*

$$\int_{\mathbb{A}^{p|2q}}^{\gamma} D\lambda(y) f(t,y) = \int_{\mathbb{A}^1_+} dr\, r^{p-2q-1} \int_{S^{p-1|2q}} DS(s) f(t,rs) + \frac{\partial_{r=0}^{2q-p}}{(2q-p)!} \int_{S^{p-1|2q}} DS(s) \log(\|s_{\bar{0}}\|) f(t,rs).$$

Proof. This follows directly from the fact that $f_{2q-p}(t,s) = \frac{1}{(2q-p)!}\partial_{r=0}^{2q-p} f(t,rs)$. □

2.3.11 Remark. It is important to note that the integration formula in Proposition 2.3.10 only exist in this iteratively written form. The integrand on the right hand side is not integrable over $\mathbb{A}^1_+ \times S^{p-1|2q}$, since negative powers of r would let this integral diverge.

2.3.12 Corollary. *Applying Proposition 2.3.10 to any rotationally invariant function and comparing the result with Corollary 2.3.5 shows*

$$\int_{S^{p-1|2q}} DS(s) \log(\|s_{\bar{0}}\|) = (-\pi)^{\frac{p-2q}{2}} \left(\frac{2q-p}{2}\right)!.$$

2.3.13 Proposition. *For $p \leq 2q$ with p odd, the polar integration formula takes the following form:*

$$\int_{\mathbb{A}^{p|2q}}^{\gamma} D\lambda(y) f(t,y) = \frac{\Gamma(\frac{p-2q}{2})(-1)^{\frac{p-1-2q}{2}}}{2\pi} \int_{\mathbb{A}^1_+} dr\, r^{-\frac{1}{2}} \partial_r^{\frac{2q+1-p}{2}} \int_{S^{p-1|2q}} DS(s) f(t,\sqrt{r}s)$$

Proof. It suffices to take a closer look at

$$\int_{\mathbb{A}^{p|2q}}^{\gamma} D\lambda(y) f(t,y) = \lim_{\varepsilon\to 0^+} \Bigg(\int_{\mathbb{A}^1_{>\varepsilon}} dr\, r^{p-2q-1} \int_{S^{p-1|2q}} DS(s) f(t,rs) + \sum_{2l \leq 2q-p} \frac{\varepsilon^{p-2q+2l}}{p-2q+2l} \frac{\partial_{r=0}^{2l}}{(2l)!} \int_{S^{p-1|2q}} DS(s) f(t,rs) \Bigg). \quad (2.5)$$

The super function f°, given by $f^\circ(t,r) := \frac{1}{\mathrm{Vol}(S^{p-1|2q})} \int_{S^{p-1|2q}} DS(s) f(t,rs)$, is even in r. Therefore, $(t,y,s) \mapsto f^\circ(t,\|y\|,s)$ can be extended to a super function on $T \times \mathbb{A}^{p|2q}$ which is rotationally invariant in the second component. Replacing $f(t,y)$ by $f^\circ(t,\|y\|)$ leaves the right hand side of Equation (2.5) invariant. Therefore, Corollary 2.3.5 proves the claim. □

2.4. Lie *cs* Groups, Symmetric Spaces and their Decompositions

2.4.1 Definition. A *Lie* cs *algebra* is a *cs* vector space $\mathfrak{g}$ together with a $\mathbb{C}$-bilinear map $[\cdot,\cdot] \colon \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$, such that the following conditions are satisfied

- $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \subseteq \mathfrak{g}_{\bar{0}}$,
- $[x, y] = (-1)^{|x||y|}[y, x]$,
- $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$

for homogeneous $x, y, z \in \mathfrak{g}_{\mathbb{C}}$. For this, recall that $\mathfrak{g}_{\mathbb{C}} \coloneqq (\mathfrak{g}_{\bar{0}} \otimes \mathbb{C}) \oplus \mathfrak{g}_1$.

A *morphism* of Lie *cs* algebra is a morphism of *cs* vector spaces $\lambda\colon \mathfrak{g} \to \mathfrak{h}$, such that $[\lambda(x), \lambda(y)] = \lambda([x, y])$ for all $x, y \in \mathfrak{g}$. The category of Lie *cs* algebras will be denoted **Liecsalg**.

Note that the map $\mathbb{A}([\cdot, \cdot])\colon \mathbb{A}(\mathfrak{g}) \times \mathbb{A}(\mathfrak{g}) \to \mathbb{A}(\mathfrak{g})$ induces a Lie algebra structure over $\Gamma(\mathcal{O}_{S,\mathbb{R},\bar{0}})$ on S-points of $\mathbb{A}(\mathfrak{g})$. Accordingly, $\mathbb{A}(\lambda)$ behaves as $\Gamma(\mathcal{O}_{S,\mathbb{R},\bar{0}})$-Lie algebra morphism on S-points for a Lie *cs* algebra morphism λ.

2.4.2 Definition. A *Lie* cs *group* is a group object in the category **csMan** (*cf.* Definition A.9). The category of Lie *cs* groups is denoted **LiecsGrp**. For a Lie *cs* group G, one uses the corresponding fracture letter $\mathfrak{g}$ for the tangential space T_eG. The linear map $T_e\varphi$ for a morphism φ of Lie *cs* groups is denoted $d\varphi$.

In the following, G will always denote a Lie *cs* group with multiplication m, neutral element e and inversion i. Since morphisms $* \to X$ for *cs* manifolds X are always of the form $f \mapsto f(x)$ for some $x \in X_0$, it is indeed reasonable to speak of e as the neutral element.

2.4.3 Remark. Recall Remark 2.1.15. Since $G(\mathbb{A}^{0|q})$ is an ordinary manifold, it can be considered as Lie group with the induced maps from e, m and ι.

2.4.4 Proposition. *There are unique morphisms*

$$\mathrm{Ad}_G\colon G \times \mathbb{A}(g) \to \mathbb{A}(g), \quad \mathrm{ad}_{\mathfrak{g}}\colon \mathbb{A}(\mathfrak{g}) \times \mathbb{A}(\mathfrak{g}) \to \mathbb{A}(\mathfrak{g}), \quad \exp_G\colon \mathbb{A}(\mathfrak{g}) \to G,$$

such that the induced morphisms on super points are the adjoint representation, the Lie bracket and the exponential map in the classical sense. In particular, the exponential map is a local isomorphism at $0 \in \mathfrak{g}_{\bar{0}}$. Furthermore, there is a unique Lie cs *algebra structure on $\mathfrak{g}$ such that* $\mathrm{ad} = \mathbb{A}([\cdot, \cdot])$.

To actually prove the existence of these morphisms much more effort is needed, especially for the exponential map (*cf.* [GW12, CCF11]). If convenient, subscripted G and $\mathfrak{g}$ will be omitted in notation and $\exp(x)$ will be abbreviated as e^x for $x \in_S \mathbb{A}(\mathfrak{g})$. Furthermore, to avoid an excess of notation, the symbol $\mathfrak{g}$ will also be used for the affine space $\mathbb{A}(\mathfrak{g})$ of $\mathfrak{g}$ as long as this is clear from the context. As well, for a tangential map $d\varphi$ the map $\mathbb{A}(d\varphi)$ will be denoted $d\varphi$.

2.4.5 Corollary. *The above defined morphisms are compatible with morphisms of Lie* cs *groups* $\varphi\colon G \to H$, i.e.

$$\begin{aligned} \mathrm{Ad}_H\left(\varphi(g), d\varphi(x)\right) &= d\varphi(\,\mathrm{Ad}_G(g,x)), \\ e^{d\varphi(x)} &= \varphi(e^x), \\ [d\varphi(x), d\varphi(y)] &= d\varphi([x,y]) \end{aligned}$$

for $g \in_S G$ *and* $x, y \in_S \mathfrak{g}$*. In particular,* $d\varphi$ *is a morphism of Lie* cs *algebras. Moreover, the following conditions are satisfied:*

$$\begin{aligned} \mathrm{Ad}(gh,x) &= \mathrm{Ad}\left(g, \mathrm{Ad}(h,x)\right) \\ e^{\mathrm{Ad}(g,x)} &= g e^x g^{-1}, \\ \mathrm{Ad}(e^x, y) &= \sum_{i=1}^{\infty} \frac{1}{i!} \mathrm{ad}(x)^i y \end{aligned}$$

for $g, h \in_S G$ *and* $x, y \in_S \mathfrak{g}$*.*

2.4.6 Corollary (Baker-Campbell-Hausdorff formula)**.** *Consider the morphism*

$$\cdot * \cdot \coloneqq C\colon\ \mathbb{A}(\mathfrak{g})|_U \times \mathbb{A}(\mathfrak{g})|_U \to \mathbb{A}(\mathfrak{g}),$$

given on S*-points* via

$$x * y \coloneqq x + \sum_{\substack{k,m \geq 0 \\ p_i + q_i > 0}} \frac{(-1)^k}{(k+1)(q_1 + \ldots + q_k + 1)} \frac{(\mathrm{ad}x)^{p_1} (\mathrm{ad}y)^{q_1} \cdots (\mathrm{ad}x)^{p_k} (\mathrm{ad}y)^{q_k} (\mathrm{ad}x)^m}{p_1! q_1! \cdots p_k! q_k! m!} y.$$

Here, U *is the subset of* $\mathfrak{g}_{\bar{0}}$ *on which the classical Hausdorff series converges. Then*

$$\exp(x * y) = \exp(x)\exp(y).$$

Proof of Corollaries 2.4.5 and 2.4.6. Some of the expressions to check are only well-defined if the topology on $\Gamma(\mathcal{O}_S)$ is understood. A summary of this topic can be found in [AS14, Appendix C]. Then, it suffices to check these equations on super points, where they are evident. □

2.4.7 Corollary. $G \cong G_0 \times g_{\bar{1}}$ via *the isomorphism* $G_0(S) \times \mathfrak{g}_{\bar{1}}(S), (g,x) \mapsto g e^x$ *as* cs *manifolds. Similarly,* $(g,x) \mapsto e^x g$ *defines another isomorphism.*

Proof. Let the morphism be denoted φ. The tangential map of φ at e is given by

$$\mathfrak{g}_0(S) \times \mathfrak{g}_1(S) \to \mathfrak{g}(S),\ (x_{\bar{0}}, x_{\bar{1}}) \mapsto x_{\bar{0}} + x_{\bar{1}}.$$

Since this is an isomorphism, the same is true for φ on a neighbourhood of e. Now, the left G_0-invariance of φ immediately proves the claim. □

2.4.8 Definition. A retraction γ on a *cs* Lie group G is said to be G_0-invariant if $\gamma(gh) = g\gamma(h)$ for all $g \in_S G_0$, $h \in_S G$.

2.4.9 Corollary. *Each Lie* cs *group admits a unique G_0-invariant retraction γ_G, compatible with the standard retraction $\gamma_{\mathfrak{g}}$ on $\mathfrak{g}$ under the exponential function. This retraction will be called the* standard retraction *on G. Moreover, these retractions are compatible under morphisms of Lie* cs *groups.*

In general, standard retractions are not morphisms of Lie cs *groups.*

Proof. In view of Corollary 2.4.7, it suffices to set $\gamma_G(ge^x) := g$ for $g \in_S G_0$, $x \in_S \mathfrak{g}_{\bar{1}}$. □

2.4.10 Definition. A cs *group pair* $(G_0, \mathfrak{g})$ consists of a Lie group G_0, a Lie *cs* algebra $\mathfrak{g}$ and a representation $\mathrm{Ad}\colon G_0 \to \mathrm{Hom}_{\mathbf{Liecsalg}}(\mathfrak{g}, \mathfrak{g})$, such that the following is true

- $\mathfrak{g}_{\bar{0}}$ is the Lie algebra of G_0,
- $\mathrm{Ad}|_{G_0 \times \mathfrak{g}_{\bar{0}}}$ is the adjoint action of G_0 on $\mathfrak{g}_{\bar{0}}$,
- $d\,\mathrm{Ad}(x)(y) = [x, y]$ for $x \in \mathfrak{g}_{\bar{0}}$ and $y \in \mathfrak{g}$.

A *morphism* of *cs* group pairs $(\varphi_0, d\varphi)\colon (G_0, \mathfrak{g}) \to (H_0, \mathfrak{h})$ consist of a Lie group morphism $\varphi_0\colon G_0 \to H_0$ and a Lie *cs* algebra morphism $d\varphi\colon \mathfrak{g} \to \mathfrak{h}$, such that the restriction of $d\varphi$ is the differential of φ_0 and

$$\mathrm{Ad}\left(\varphi(g), d\varphi(x)\right) = d\varphi(\,\mathrm{Ad}(g, x))$$

for $g \in G_0$ and $x \in \mathfrak{g}$. The category of *cs* group pairs will be denoted **csGrpPr**.

2.4.11 Proposition. *The categories* ***LiecsGrp*** *and* ***csGrpPr*** *are equivalent* via *the functor which sends a Lie* cs *group G to the pair $(G_0, \mathfrak{g})$ with $\mathrm{Ad}_G|_{G_0 \times \mathfrak{g}}$ as representation of G_0 on $\mathfrak{g}$. Morphisms φ are sent to $(\varphi_0, d\varphi)$ by this functor.*

A proof for this non-trivial fact can be found in [CCF11, Chapter 7].

2.4.12 Remark. Proposition 2.4.11 shows in particular, that a morphism of Lie *cs* groups is an isomoprhism if and only if φ_0 and $d\varphi$ are isomorphisms.

2.4.13 Definition. A *complex Lie super group* is a group object in the category of complex super manifolds.

The complexification of a Lie *cs* group G is a complex Lie super group $G_{\mathbb{C}}$ together with a morphism of group objects $j_G\colon G \to G_{\mathbb{C}}$ (in $\mathbf{SSp}_{\mathbb{C}}$) with the following universal property: If H is a complex Lie super group and $\varphi\colon G \to H$ is a morphism of group objects, then there is a unique morphism of complex Lie super groups $\varphi_{\mathbb{C}}$, such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "j_G"'] & H \\ G_{\mathbb{C}} \arrow[ur, "\varphi_{\mathbb{C}}"'] & \end{tikzcd}$$

The complexification of a Lie *cs* group is unique up to unique isomorphisms.

2.4.14 Proposition. *The complexification of a Lie* cs *group always exists.*

Sketch of the Proof. This can be easily verified by using the fact that there is also an equivalence of complex group pairs and complex Lie super groups. In particular, the complexification of a Lie *cs* group G is given by the complex group pair $(G_{0,\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$, where $G_{0,\mathbb{C}}$ is the complexification of the underlying group. Since the sheaves of super functions are given by

$$\mathcal{O}_G \cong \mathrm{Hom}_{U(\mathfrak{g}_{\bar{0},\mathbb{C}})}\left(U(\mathfrak{g}_{\mathbb{C}}), \mathcal{C}^\infty\right), \qquad \mathcal{O}_{G_{\mathbb{C}}} \cong \mathrm{Hom}_{U(\mathfrak{g}_{\bar{0},\mathbb{C}})}\left(U(\mathfrak{g}_{\mathbb{C}}), \mathcal{C}^\omega\right),$$

(*cf.* [CCF11, Proposition 7.4.9]), the morphism $j_G^\sharp$ can be defined to be the inclusion of the latter into the former. The underlying map $j_{G,0}\colon G_0 \to G_{0,\mathbb{C}}$ is just the ordinary inclusion. $\square$

2.4.15 Example. Let $\mathfrak{gl}_{\mathbb{C}}(p|q)$ be the complex super vector space of square matrices written as blocks $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of size $p+q$ with entries in $\mathbb{C}$. The grading on $\mathfrak{gl}_{\mathbb{C}}(p|q)$ is given by these blocks, *i.e.* $\mathfrak{gl}_{\mathbb{C},\bar{0}}(p|q) = \mathfrak{gl}_{\mathbb{C}}(p) \oplus \mathfrak{gl}_{\mathbb{C}}(q)$. The space $\mathfrak{gl}_{\mathbb{C}}(p|q)$ is the complexification of $\mathfrak{gl}_{cs}(p|q) := \left(\mathfrak{gl}_{\mathbb{R}}(p) \oplus \mathfrak{gl}_{\mathbb{R}}(q)\right) \oplus \mathfrak{gl}_{\mathbb{C}}(p|q)_{\bar{1}}$. The *cs* vector space $\mathfrak{gl}_{cs}(p|q)$ is turned into a Lie *cs* algebra by setting $[x,y] := xy + (-1)^{|x||y|}yx$ for homogeneous elements x and y of $\mathfrak{gl}_{\mathbb{C}}(p|q)$.

Furthermore, let

$$\begin{aligned} \mathrm{GL}_{\mathbb{C}}(p|q) &:= \mathbb{A}^{\mathbb{C}}(\mathfrak{gl}_{cs}(p|q))\Big|_{\mathrm{GL}_{\mathbb{C}}(p)\times \mathrm{GL}_{\mathbb{C}}(q)}, \\ \mathrm{GL}_{cs}(p|q) &:= \mathbb{A}(\mathfrak{gl}_{cs}(p|q))\big|_{\mathrm{GL}_{\mathbb{R}}(p)\times \mathrm{GL}_{\mathbb{R}}(q)}. \end{aligned}$$

Consider the bilinear map $: \mathfrak{gl}_{\mathbb{C}}(p|q) \times \mathfrak{gl}_{\mathbb{C}}(p|q) \to \mathfrak{gl}_{\mathbb{C}}(p|q)$ obtained by matrix multiplication. This map induces a multiplication map m on $\mathrm{GL}_{\mathbb{C}}(p|q)$ and $\mathrm{GL}_{cs}(p|q)$, turning these spaces into a complex Lie super group and a Lie *cs* group, respectively.

Let S be a *cs* manifold. According to Definition 2.1.12 and Corollary 2.1.14, S-points of these spaces can be seen as block matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where the entries of A and D are in $\Gamma(\mathcal{O}_{S,\bar{0}})$ and the entries of B and C are contained in $\Gamma(\mathcal{O}_{S,\bar{1}})$. In the case of $\mathfrak{gl}_{cs}(p|q)$, the entries of these matrices are additionally real valued, whereas in the case of $\mathrm{GL}_{\mathbb{C}}(p|q)$

and $\mathrm{GL}_{cs}(p|q)$ they have values in $\mathrm{GL}_{\mathbb{C}}(p) \times \mathrm{GL}_{\mathbb{C}}(q)$ and $\mathrm{GL}_{\mathbb{R}}(p) \times \mathrm{GL}_{\mathbb{R}}(q)$, respectively. It is clear, that the multiplication map is on S-points given by matrix multiplication, as well.

These spaces satisfy indeed the fact that $\mathfrak{gl}_{cs}(p|q)$ is the Lie *cs* algebra of $\mathrm{GL}_{cs}(p|q)$ and that $\mathfrak{gl}_{\mathbb{C}}(p|q)$ is the Lie *cs* algebra of $\mathrm{GL}_{\mathbb{C}}(p|q)$, since this is the case on super points. Furthermore, in both cases $\mathrm{Ad}(g, x) = gxg^{-1}$ and $\exp(x) = \sum_{i=0}^{\infty} \frac{1}{i!} x^i$ for S-points g and x.

Moreover, $\mathrm{GL}_{\mathbb{C}}(p|q)$ is the complexification of $\mathrm{GL}_{cs}(p|q)$.

The fact that S-points of $\mathrm{GL}_{cs}(p|q)$ are precisely those of $\mathrm{GL}_{\mathbb{C}}(p|q)$ with values in $\mathrm{GL}_{\mathbb{R}}(p) \times \mathrm{GL}_{\mathbb{R}}(q)$ can be generalised.

2.4.16 Proposition. *The generalised points of a Lie* cs *group G can be obtained as*

$$G(S) = \{s \in G_{\mathbb{C}}(S) \mid s_0 \in G_0(S_0)\}$$

for any cs *manifold S.*

Sketch of the Proof. The following diagram commutes:

$$\begin{tikzcd} \mathbb{A}(\mathfrak{g}) \arrow[r] \arrow[d, "\exp_G"'] & \mathbb{A}^{\mathbb{C}}(\mathfrak{g}) \arrow[d, "\exp_{G_{\mathbb{C}}}"] \\ G \arrow[r, "j_G"] & G_{\mathbb{C}} \end{tikzcd}$$

Therefore the claim can be proven locally, using Corollary 2.1.14. The general assertion is obtained by shifting with elements of G_0 and gluing. □

The following explanations follow mainly [All12, AS13].

2.4.17 Definition. A symmetric super pair is a pair $(\mathfrak{g}, \theta)$, where $\mathfrak{g}$ is a Lie *cs* algebra and θ is an involutive automorphism of $\mathfrak{g}$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{k} := \ker(\mathrm{id}_{\mathfrak{g}} - \theta)$ and $\mathfrak{p} := \ker(\mathrm{id}_{\mathfrak{g}} + \theta)$ is called the *polar decomposition* of $(\mathfrak{g}, \theta)$.

Let (G, K) be a pair of Lie *cs* group G and a sub Lie *cs* group K together with an involution θ on is Lie *cs* algebra $\mathfrak{g}$, such that $\mathfrak{k}$ is the Lie *cs* algebra of K. Then (G, K) is said to admit a global polar decomposition if the map

$$K \times \mathfrak{p} \longrightarrow G, \ (k, x) \longmapsto ke^x \tag{2.6}$$

defines an isomorphism of *cs* manifolds.

2.4.18 Proposition. *A pair (G, K) admits a global polar decomposition if and only if this is true on the underlying pair (G_0, K_0).*

Proof. The morphism in Equation (2.6) is always a local isomorphism, since its derivative is given by $\mathfrak{k} \times \mathfrak{p} \longrightarrow \mathfrak{g}$, $(y, x) \longmapsto y + x$. The claim follows from Remark 2.4.12. $\square$

2.4.19 Definition. A symmetric super pair $(\mathfrak{g}, \theta)$ is called *reductive* if the following is true: $\mathfrak{g}$ is a semisimple $\mathfrak{g}_{\bar{0}}$-module, $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}_{\bar{0}}$, and there is a super symmetric non-degenerate $\mathfrak{g}$-invariant form $b \in \mathrm{Mult}(\mathfrak{g}, \mathfrak{g}; \mathbb{R})$.

An Abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p}_{\bar{0}}$ is called an *even Cartan subspace* if $\mathfrak{a}$ consists entirely of semi-simple elements of $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{p} = [\mathfrak{k}, \mathfrak{a}]$. A symmetric super pair, which admits an even Cartan subspace is said to be of *even type*.

A reductive Lie symmetric super pair which satisfies $\mathfrak{g}_{\mathbb{C}} = \mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) \oplus [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ is called *strongly reductive*.

2.4.20 Definition. Let $(\mathfrak{g}, \theta)$ be a reductive symmetric super pair of even type with even Cartan subspace $\mathfrak{a}$. Then

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha, \tag{2.7}$$

where for $\alpha \in \mathfrak{a}^*$

$$\mathfrak{g}^\alpha \coloneqq \{x \in \mathfrak{g} \,|\, (\forall h \in \mathfrak{a}) : [h, x] = \alpha(h)x\}, \quad \Sigma = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \,|\, \mathfrak{g}^\alpha \neq 0\},$$

and $\mathfrak{m} \coloneqq \mathfrak{g}^0 \cap \mathfrak{k}$. Of course, having such a decomposition for Abelian $\mathfrak{a} \subseteq \mathfrak{p}_0$ already implies that $\mathfrak{a}$ is an even Cartan subspace.

Elements of Σ are called *restricted roots*. Σ itself is said to be a *root system*. The elements of

$$\Sigma_{\bar{j}} \coloneqq \{\alpha \in \mathfrak{a}^* \setminus \{0\} \,|\, \mathfrak{g}^\alpha_{\bar{j}} \neq 0\}$$

for $\bar{j} \in \mathbb{Z}_2$ are called *even/odd*, respectively.

A subset $\Sigma^+ \subseteq \Sigma$ is called *positive* if $\Sigma^+ \dot{\cup} (-\Sigma^+) = \Sigma$ and $\Sigma \cap (\Sigma^+ + \Sigma^+) \subseteq \Sigma^+$. If Σ^+ is fixed, restricted roots $\alpha \in \Sigma^+$ are called *positive*. In this case $\alpha \in \Sigma^+$ will be abbreviated $\alpha > 0$ and $\alpha \in -\Sigma^+$ by $\alpha < 0$. Positive restricted roots α for which $\frac{\alpha}{2}$ is no restricted root are called *simple*. The Element $\varrho \coloneqq \frac{1}{2} \sum_{\alpha > 0} m_\alpha \alpha \in \mathfrak{a}^*$, where $m_\alpha \coloneqq \mathrm{sdim}\, \mathfrak{g}^\alpha$, is called the *Weyl vector* of Σ^+.

By setting $\mathfrak{n} \coloneqq \bigoplus_{\alpha > 0} \mathfrak{g}^\alpha$ and $\bar{\mathfrak{n}} \coloneqq \bigoplus_{\alpha < 0} \mathfrak{g}^\alpha$, Equation (2.7) takes the form

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Furthermore, there is the so-called *Iwasawa decomposition* of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Let $\Sigma^+_{\bar{j}} \coloneqq \Sigma^+ \cap \Sigma_{\bar{j}}$. Then $\Sigma_{\bar{0}}$ is a root system on $\mathfrak{g}_{\bar{0}}$ and $\Sigma^+_{\bar{0}}$ is a positive root system on $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{n}_{\bar{0}} = \mathfrak{n} \cap \mathfrak{g}_{\bar{0}} = \bigoplus_{\alpha \in \Sigma^+_{\bar{0}}} \mathfrak{g}^\alpha_{\bar{0}}$, hence $\mathfrak{g}_{\bar{0}} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\bar{0}}$ is a classical Iwasawa decomposition.

It makes sense to define the *centraliser* of $\mathfrak{a}$ in K to be the sub Lie *cs* group of K, associated to the *cs* group pair $(M_0, \mathfrak{m})$, where $M_0 := Z_{K_0}(\mathfrak{a})$ is the ordinary centraliser. Note that M indeed centralises $\mathfrak{a}$, since $\mathrm{Ad}(me^x, h) = \mathrm{Ad}(m, [x, h]) = h$ for $m \in_S G_0$, $x \in_S \mathfrak{g}_{\bar{1}}$ and $h \in_S \mathfrak{a}$.

2.4.21 Proposition. *Let G be a Lie* cs *group with an involution θ on $\mathfrak{g}$, such that $(\mathfrak{g}, \theta)$ is reductive and of even type. The morphism*

$$\bar{N} \times M \times A \times N \longmapsto G, \; (\bar{n}, m, a, n) \longmapsto \bar{n}man$$

is an open embedding if this is true for the underlying spaces. Here $A := \exp(\mathfrak{a})$. Furthermore, N and $\bar{N}$ are the Lie cs *groups associated with the* cs *group pairs $\big(\exp(\mathfrak{n}_{\bar{0}}), \mathfrak{n}\big)$ and $\big(\exp(\bar{\mathfrak{n}}_{\bar{0}}), \bar{\mathfrak{n}}\big)$.*

2.4.22 Proposition (Global Iwasawa decomposition)**.** *Let G be a Lie* cs *group with an involution θ on $\mathfrak{g}$, such that $(\mathfrak{g}, \theta)$ is reductive and of even type. Furthermore, let K be a sub Lie* cs *group of G, such that its Lie* cs *algebra equals k. Then, the morphisms*

$$\begin{aligned} K \times A \times N &\longrightarrow G, \; (k, a, n) \longmapsto kan, \\ N \times A \times K &\longrightarrow G, \; (n, a, k) \longmapsto nak \end{aligned}$$

are isomorphisms if and only if this is already true for the underlying morphisms.

Proof of Propositions 2.4.21 and 2.4.22. The morphisms induce isomorphisms on tangent spaces, which proves the assertions. □

2.4.23 Definition. If G admits an Iwasawa decomposition $G = KAN$, it is common to write $g = k(g)e^{H(g)}n(g)$. This defines a morphisms of *cs* manifolds $k\colon G \to K$, as well as $H\colon G \to \mathbb{A}(\mathfrak{a})$ and $n\colon G \to N$. Then the decomposition of $g = n_1(g)e^{A(g)}u(g)$ with respect to the NAK-decomposition satisfies $u(g) = n(g^{-1})^{-1}$, $A(g) = -H(g^{-1})$ and $u(g) = k(g^{-1})^{-1}$.

2.4.24 Remark. The morphisms from the definition above satisfy the following equations:

$$\begin{aligned} H(gm) &= H(g), && (2.8) \\ k(gm) &= k(g), && (2.9) \\ H(ga) &= H(g) + H(a), && (2.10) \\ H(gh) &= H\big(gk(h)\big) + H(h), && (2.11) \\ k(gh) &= k\big(gk(h)\big), && (2.12) \end{aligned}$$

for all $g \in_S G$, $a \in_S A$, $m \in_S M$ and $n \in_S N$.

These can seen by

$$
\begin{aligned}
kanm &= (km)a(m^{-1}nm),\\
H(ga) &= H\big(k(g)e^{H(g)}n(g)a\big) = H\big(k(g)e^{H(g)}a(a^{-1}n(g)a)\big) = H(g) + H(a),\\
H(gh) &= H\big(gk(h)e^{H(h)}n(h)\big) = H\big(gk(h)\big) + H(h),\\
k(gan) &= k\big(k(g)e^{H(g)}a(a^{-1}n(g)a)n\big) = k(g) \quad \Rightarrow \quad k(gh) = k\big(gk(h)\big).
\end{aligned}
$$

In the first equation, the fact $\mathrm{Ad}(m,n) \in_S \mathfrak{n}$ for all $m \in_S M$, $n \in_S \mathfrak{n}$ was used.

2.4.25 Definition. Let G be a Lie *cs* group and X a *cs* manifold. A *left action* of G on X is a morphism $\alpha\colon G \times X \to X$, which induces a left action of $G(S)$ on $X(S)$ on S-points.

An action α is said to be *transitive* if the morphism $\alpha_x \coloneqq \alpha \circ (\mathrm{id}_G, x)\colon G \to X$ is a surjective submersion for one (and hence for all) $x \in X_0 = X(*)$. If X admits a transitive G-action, it is said to be a *symmetric superspace*. The underlying manifold of a symmetric superspace X is a symmetric space. If this symmetric space happens to be non-compact, X is also said to be *non-compact*.

2.4.26 Proposition ([CCF11, Proposition 8.4.7])**.** *Let $\alpha\colon G \times X \to X$ be an action. Then the* stabiliser *$G_x \coloneqq \alpha_x^{-1}(x) = G \times_X *$ of x in G exist and is a closed subgroup of G. The generalised points of G_x have the form $G_x(S) = \{g \in_S X \mid gx = x\}$ and the Lie* cs *algebra of G_x is given by the kernel of $T_e\alpha\colon \mathfrak{g} \to T_xX$.*

2.4.27 Definition. Let G be a Lie *cs* group. A sub Lie *cs* group H of G is said to be closed in G if this is the case for the underlying Lie groups. In this case, one defines $G/H \coloneqq (G_0/H_0, \mathcal{O}_{G/H})$ with

$$
\mathcal{O}_{G/H}(U) = \Big\{ f \in \mathcal{O}_{G/H}(p_{G,0}^{-1}(U)) \,\Big|\, (m|_{G\times H})^\sharp(f) = p_1^\sharp(f) \Big\}, \tag{2.13}
$$

where $p_{G,0}\colon G_0 \to G_0/H_0$ is the classical projection and $p_1\colon G \times H \to G$ is the projection onto the first component. A canonical morphism $p_G\colon G \to G/H$ is easily obtained *via* $p_G^\sharp(f) \coloneqq f$.

After proving that G/H is indeed a *cs* manifold, it is clear that super functions on G/H are precisely the super functions f on G which satisfy $f(gh) = f(g)$ for $g \in_S G$ and $h \in_S H$. Moreover, the map p_G turns G into a *cs* manifold over G/H.

Similarly, one can define the *cs* manifold $H\backslash G$ with underlying space $H_0\backslash G_0$ by requiring $(m|_{H\times G})^\sharp(f) = p_2^\sharp(f)$ in Equation (2.13).

2.4.28 Proposition. *The superspace G/H is a* cs *manifold. Moreover, for any right H-invariant morphism $\varphi\colon G \to X$ there exist $\tilde{\varphi}\colon G/H \to X$, such that the following* *diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "p_G"] & X \\ G/H \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

If φ is submersive, so is $\tilde{\varphi}$. Additionally, if $H = \varphi^{-1}(x)$ for $x \in X$, then $\tilde{\varphi}$ becomes a local isomorphism. In particular if G acts transitively on a cs *manifold X then $X \cong G/G_x$ for any $x \in X_0$.*

Similar statements are true for $H\backslash G$.

For the proof of this proposition, consult [CCF11, Chapter 9].

2.4.29 Remark. The multiplication map on G induces a transitive left G-action on G/H and a transitive right G-action on $H\backslash G$. The inversion morphism i_G induces an isomorphism $G/H \to H\backslash G$. This isomorphism is left G-invariant, considering the right action on $H\backslash G$ as left action (*i.e.* $g.x := x.g^{-1}$).

2.4.30 Corollary. *In case of an existing global Iwasawa decomposition on G, the morphism $G \to N \times A$, given by $g \mapsto \big(n_1(g), e^{A(g)}\big)$ for $g \in_S G$, induces an isomorphism*

$$G/K \longrightarrow N \times A,$$

since this is the case on the underlying spaces.

2.4.31 Corollary. *In the situation of Proposition 2.4.21, let MAN be the sub Lie* cs *group of G given by $(M_0AN_0, \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n})$. The right M-invariant morphism $G \to K/M$ induces an isomorphism $G/MAN \to K/M$ due to Proposition 2.4.28. Moreover, $\bar{N}$ is embedded in G/MAN. This leads to an embedding $\bar{N} \to K/M$ which will be denoted k, too.*

2.4.32 Proposition. *The morphism*

$$K \times A^+ \longrightarrow G,\ (k,a) \longmapsto g$$

induces an open embedding with dense image $K/M \times A^+ \to G/K$. Here, $A^+ := \exp(\mathfrak{a}^+)$, where

$$\mathfrak{a}^+ := \big\{h \in \mathfrak{a} \,\big|\, (\forall \alpha \in \Sigma_{\bar{0}}^+) : \alpha(h) > 0\big\}.$$

Since this fact will only used to outline an idea in Equation 5, its proof will be omitted.

2.5. Integration on Lie *cs* Groups and Symmetric Superspaces

In this section, invariant Berezin densities on symmetric spaces (as introduced in [AH10]) will be explained. Afterwards, integration formulas induced by the decompositions from the preceding section will be discussed, following [All12].

2.5.1 Definition. Let $\alpha\colon G \times X \to X$ be an action. Considering the relative *cs* manifold X_G/G, a Berezin density $\omega \in |\mathrm{Ber}|_X$ is said to be G-invariant if

$$(\alpha, \mathrm{id}_X)^\sharp \left(\mathrm{pr}_X^\sharp(\omega)\right) = \mathrm{pr}_X^\sharp(\omega).$$

2.5.2 Lemma. *A Berezin density $\omega \in \Gamma(|\mathrm{Ber}|_X)$ is G-invariant if and only if for all $f \in \Gamma_{cf}(\mathcal{O}_{X_G})$ and all $g \in G$, the following is true:*

$$\int_X \omega(x) f(gx) = \int_X \omega(x) f(x).$$

It has to be noted that the condition of f being compactly supported along the fibres cannot be omitted, since in general no retraction of the form $\gamma'_G \times \gamma_X$ is compatible with $\gamma_G \times \gamma_X$ under (α, id). However, there is a weaker version where this is possible.

2.5.3 Corollary. *Let $\omega \in \Gamma(|\mathrm{Ber}|_X)$ be G-invariant and γ be a G_0-invariant retraction on X. Then*

$$\int_X^\gamma \omega(x) f(gx) = \int_X^\gamma \omega(x) f(x)$$

for all $f \in \Gamma(\mathcal{O}_{X_G})$ and all $g \in_S G_0$.

From now on, the integral on a Lie *cs* group G will be considered to be the one coming from the standard retraction on G unless something else is stated.

2.5.4 Definition. Let G be a Lie *cs* group and H a closed subgroup. If G/H admits a G-invariant Berezin density, it is unique up to multiplication by a constant. In this case, G/H is called *unimodular*.

Similarly, G is called *unimodular* if it admits a Berezin density which is invariant with respect to the action of $G \times G$ on G, given by $(g_1, g_2).g \coloneqq g_1 g g_2^{-1}$ on generalised points. This is equivalent to the statement that the symmetric space $(G \times G)/G$ is unimodular.

The invariant Berezin density on a unimodular Lie *cs* group G will be denoted by Dg (g being the small letter for G). Accordingly, the invariant Berezin density on G/H will be denoted $D\dot{g}$.

2.5.5 Proposition ([All12, Proposition A.2])**.** *Let G be a Lie* cs *group and H a closed subgroup. Then*

1. *If G and H are unimodular, then so is G/H.*

2. *If G and H are unimodular, then so is the Lie* cs *group $G \times H$.*

3. *If $\mathfrak{g}$ is nilpotent, Abelian, or strongly reductive, and G_0 is connected, then G is unimodular.*

2.5.6 Proposition ([AH10, Corollary 5.12])**.** *Let G and H be unimodular. Then*

$$\int_G Dg\, f(s,g) = \int_{G/H} D\dot{g} \int_H Dh\, f(s,gh)$$

for $f \in \Gamma(\mathcal{O}_{G_S})$, assuming that the densities are normalised accordingly.

2.5.7 Proposition. *Let G be unimodular. Then*

$$\exp^\sharp(Dg) = D\lambda\, |\mathrm{Ber}| \left(\frac{1-e^{-\mathrm{ad}x}}{\mathrm{ad}x} \right)$$

on the open neighbourhood U of 0 where $\exp$ is an isomorphism. Here, $D\lambda$ is an adequately normalised Lebesgue Berezin density on $\mathfrak{g}$, $x = \mathrm{id}_{\mathfrak{g}}$ and $\frac{1-e^z}{z}$ denotes the formal power series $\sum_{i=0}^\infty \frac{(-z)^i}{(i+1)!}$.

In particular, this shows $\exp^\sharp(Dg) = D\lambda$ if $\mathfrak{g}$ is nilpotent (or Abelian).

Proof. For $x \in_S \mathfrak{g}|_U$ and a sufficiently small neighbourhood V of 0, the morphism $C(x,\cdot) \in_{S\times\mathfrak{g}|_V} \mathfrak{g}$ satisfies

$$d(C(x,\cdot)) = \frac{\mathrm{ad}x}{1-e^{-\mathrm{ad}x}}$$

on $S \times \mathfrak{g}$. This is true, since the right hand side converges (similarly to $e^{-\mathrm{ad}x}$) and the equation is true for super points of $S \times \mathfrak{g}$ by the classical theory.

Now, let $\exp^\sharp(Dg) = D\lambda\, \rho$ for some $\rho \in \mathcal{O}_{\mathfrak{g}}(U)$. Possibly after shrinking V, one may assume $C(-x,y) \in_{S\times T} \mathfrak{g}|_V$ for all $y \in_T \mathfrak{g}|_V$. Then for any compactly supported $f \in \mathcal{O}_{\mathfrak{g}}(V)$ the following is true:

$$\begin{aligned}\int_{\mathfrak{g}} D\lambda\, \rho f &= \int_G Dg\, f(\exp^{-1} g) = \int_G Dg\, f\big(\exp^{-1}(e^{-x}g)\big) \\ &= \int_{\mathfrak{g}} D\lambda(y)\, \rho(y) f\big(C(-x,y)\big) = \int_{\mathfrak{g}} D\lambda(y)\, |\mathrm{Ber}|\, (dC(x,\cdot))\rho\big(C(x,y)\big) f(y).\end{aligned}$$

Here, the identity $C(x, C(-x,y)) = y$ was applied. Since f was chosen arbitrarily, this implies $\rho(y) = |\mathrm{Ber}|\, (dC(x,\cdot))\rho\big(C(x,y)\big)$ for $y \in_T \mathfrak{g}|_V$. Setting $y = 0$ shows

$$\rho(x) = |\mathrm{Ber}| \left(\frac{1-e^{-\mathrm{ad}x}}{\mathrm{ad}x} \right),$$

hence the claim.

If G is nilpotent, U can be chosen to equal $\mathfrak{g}_{\bar{0}}$. Moreover, $\mathrm{ad}x$ is nilpotent for any $x \in_S \mathfrak{g}$, hence $\frac{1-e^{-\mathrm{ad}x}}{\mathrm{ad}x} = 1 + n$, where n is nilpotent. Then

$$\mathrm{Ber}(1+n) = e^{\mathrm{str}(\log(1+n))} = e^{\sum_{i>0} \frac{1}{n}(-1)^{n+1}\,\mathrm{str}(n^i)} = e^0 = 1$$

concludes the proof. □

From now on, let G be a Lie *cs* algebra such that the decompositions from Section 2.4 exist and such that the subgroups K and M are unimodular. In particular, this implies that G and G/K are unimodular.

2.5.8 Proposition. *The pullback of the invariant Berezin density $D\dot{g}$* via *the Iwasawa isomorphism is $Dk \otimes da \otimes Dn$ (up to normalisation). In particular,*

$$\int_G Dg\, f(s,g) = \int_K Dk \int_A da \int_N Dn\, f(s,kan) e^{2\varrho(\log a)}$$

for $f \in \Gamma_{cf}(\mathcal{O}_{G_S})$.

2.5.9 Lemma ([AS13, Lemma 4.2])**.** *Let $f \in \Gamma_{cf}(\mathcal{O}_{S\times K/M})$ and $g \in \Gamma_{cf}(\mathcal{O}_{T\times K/M})$. Then*

$$\int_K Dk\, f(s, k(g^{-1}k))g(t,k) = \int_K Dk\, f(s,k)g(t,k(gk))e^{-2\varrho(H(gk))}.$$

The proof of this lemma in [AS13, Lemma 4.2] is only performed in the case $S = *$, generalising it is straightforward.

2.5.10 Corollary. *Let $f \in \Gamma(\mathcal{O}_{S\times K/M})$. Then*

$$\int_{K/M} D\dot{k}\, f(s, k(g^{-1}k)) = \int_{K/M} D\dot{k}\, f(s,k) e^{-2\varrho(H(gk))}.$$

In the classical case, this equation follows directly from Lemma 2.5.9 by using Proposition 2.5.6. However, it is not completely trivial in the super setting, since the volume of K might vanish.

Proof of Corollary 2.5.10. Let $h \in \Gamma_c(\mathcal{O}_K)$, such that $\int_K Dl\, h(l) = 1$. Then

$$\begin{aligned}
\int_{K/M} D\dot{k}\, f\left(s, k(g^{-1}k)\right) &= \int_K Dl\, h(l) \int_{K/M} D\dot{k} f(s, k(g^{-1}lk)) \\
&= \int_{K/M} D\dot{k} \int_K Dl\, h(lk^{-1}) f(s, k(g^{-1}l)) \\
&= \int_{K/M} D\dot{k} \int_K Dl\, h\big(k(gl)k^{-1}\big) f(s, l) e^{-2\varrho(H(gl))} \\
&= \int_K Dl \int_{K/M} D\dot{k}\, h\big(k(glk)k^{-1}\big) f(s, lk) e^{-2\varrho(H(glk))} \\
&= \int_K Dl \int_{K/M} D\dot{k}\, h\big(k(gk)k^{-1}l\big) f(s, k) e^{-2\varrho(H(gk))} \\
&= \int_{K/M} D\dot{k}\, f(s, k) e^{-2\varrho(H(gk))} \int_K Dl\, h(l)
\end{aligned}$$

by the following arguments: In Lines 1 and 5, left K-invariance of $D\dot{k}$ is used. Lines 2 and 4 utilize right K-invariance of Dl. In Line 3, Lemma 2.5.9 is applied. In the last line, the left K-invariance of $D\dot{k}$ is used. Furthermore, it should be noted that the integrand in Lines 4 and 5 is indeed right M-invariant due to Equation (2.9). □

2.5.11 Proposition ([AS13, Proposition 4.4])**.** *The pullback of the invariant Berezin density $D\dot{k}$ on K* via *the open embedding k from Corollary 2.4.31 is $D\bar{n}\, e^{-2\varrho(H(\bar{n}))}$.*

2.5.12 Remark. Note that the standard retraction on $\bar{N}$ is in general not compatible with a global retraction on K/M *via* k. This means that in order to obtain

$$(\forall f \times \Gamma(\mathcal{O}_{S\times K/M})) : \int_{K/M} D\dot{k}\, f(s, k) = \int_{\bar{N}}^{\gamma} D\bar{n}\, f(s, k(\bar{n})) e^{-2\varrho(H(\bar{n}))},$$

γ needs to be chosen in the right way. Otherwise this equation is only true for those f that have compact support along fibres.

2.5.13 Proposition. *Recall the embedding from Proposition 2.4.32. The pullback of $D\dot{g}$ via this morphism is given by $D\dot{k}\, da\, \delta(a)$ where $\delta(a) = \prod_{\alpha>0} \sinh^{m_\alpha} \alpha(\log a)$.*

Since this fact will only used to outline an idea in Equation 5, its proof will be omitted.

2.5.14 Remark. For the morphism in Proposition 2.4.32, a similar problem as in Remarks 2.5.12 occurs. Given a global retraction γ on G/K, the pullback of γ is in general not of the form $\gamma' \times \mathrm{id}_{A^+}$. This means, that a integral formula of the form

$$\int_{G/K} D\dot{g}\, f(s, g) = \int_{K/M} D\dot{k} \int_{A^+} da\, f(s, ka)\delta(a)$$

would be wrong, unless boundary terms are added (*cf.* Remark 2.2.23, Proposition 2.3.10 and [Bun93]). Although, explicit formulas can be given for such boundary terms, there is right now no know general formula, which is also manageable in applications.

3. Spherical Super Functions

A key ingredient in the study of non-compact symmetric superspaces G/K are the so-called *spherical super functions*. These are the super functions ϕ_λ for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ which are given by

$$\phi_\lambda(g) := \int_{K/M} D\dot{k}\, e^{(\lambda-\varrho)(H(gk))}$$

for $g \in_S G$. Thanks to Equation (2.8), this is well-defined. Due to the left K-invariance of H and the left K-invariance of $D\dot{k}$, the sperical super functions are K-bi-invariant super functions on G/K and therefore uniquely determined by their values on A. Especially the analysis of the asymptotic behaviour of spherical super functions, described by Harish-Chandra's c-function, has a vast amount of applications. In Chapter 4, the c-function will emerge in the formula for the inverse Fourier transform.

In order to obtain a general formula for the c-function, one has to execute a procedure called rank reduction (*cf.* [AS13]). This reduces the problem to certain spaces of low rank. Here, the rank is the dimension of $\mathfrak{a}$ in the root space decomposition. Section 3.1 is devoted to the derivation of the c-function in these low rank cases. In Section 3.2, an expansion of ϕ_λ will be obtained in one of these cases. This expansion plays an important role in the proof of the Fourier inversion formula.

3.1. Harish-Chandra's c-function

Unless something else is stated, G/K will be a symmetric superspace, $\mathfrak{a}$ will be an even Cartan subspace and a fixed positive root system Σ^+ is understood.

3.1.1 Definition. Let $h \in \mathfrak{a}$, such that $\alpha(h) > 0$ for all $\alpha > 0$. If all roots are even, this means $h \in \mathfrak{a}^+$. The *c-function* is defined as follows:

$$c(\lambda) := \lim_{t\to\infty} e^{(\lambda-\varrho)(th)}\phi_\lambda(e^{th}), \tag{3.1}$$

for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ with $\operatorname{Re}\lambda(h) > 0$, provided the limit exists. The independence of this definition from the choice of h is not obvious per se. However, in the rank one cases, this is clear, since such choices only differ by a constant multiple.

3.1.2 Lemma.

$$\phi_\lambda(hg^{-1}) = \int_{K/M} D\dot{k}\, e^{(\lambda-\varrho)(H(hk))} e^{(-\lambda-\varrho)(H(gk))}$$

for $g, h \in_S G$.

Proof. Applying Equation (2.11) twice gives

$$H(hg^{-1}k) = H\big(hk(g^{-1}k)\big) + H(g^{-1}k) = H\big(hk(g^{-1}k)\big) - H\big(gk(g^{-1}k)\big),$$

hence

$$\phi_\lambda(hg^{-1}) = \int_{K/M} D\dot{k}\, e^{(\lambda-\varrho)(H(hk(g^{-1}k)))} e^{-(\lambda-\varrho)(H(gk(g^{-1}k)))}.$$

Now, Corollary 2.5.10 proves the claim with $f(g,h,k) = e^{(\lambda-\varrho)(H(hk))} e^{-(\lambda-\varrho)(H(gk))}$. □

3.1.3 Corollary. *Setting $h = e$ in Lemma 3.1.2 shows $\phi_\lambda(g^{-1}) = \phi_{-\lambda}(g)$.*

3.1.4 Corollary. *Let $D\dot{k}_r$ be the invariant measure on $M\backslash K$. Then*

$$\phi_\lambda(g) = \int_{M\backslash K} D\dot{k}_r\, e^{(\lambda+\varrho)(A(kg))}.$$

Proof. Note, that the isomorphism $\varphi\colon K/M \to M\backslash K$, given by $k \mapsto k^{-1}$ (*cf.* Remark 2.4.29), satisfies $\varphi^\sharp(D\dot{k}_r) = D\dot{k}$, provided the Berezin densities are normalised adequately. Therefore,

$$\begin{aligned}\int_{M\backslash K} D\dot{k}_r\, e^{(\lambda+\varrho)(A(kg))} &= \int_{K/M} D\dot{k}\, e^{(\lambda+\varrho)(A(k^{-1}g))} = \int_{K/M} D\dot{k}\, e^{(\lambda+\varrho)(-H(g^{-1}k))} \\ &= \phi_{-\lambda}(g^{-1}) = \phi_\lambda(g),\end{aligned}$$

using Definition 2.4.23. □

The Unitary Case

In the following, let $\mathfrak{g} = \mathfrak{u}_{cs}(1, 1+p|q)$ be the Lie *cs* algebra, where $\mathfrak{g}_{\bar{0}} = \mathfrak{u}(1,p) \oplus \mathfrak{u}(q)$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{gl}_{\mathbb{C}}(2+p|q)_{\bar{1}}$. Obviously, the complexification of $\mathfrak{u}_{cs}(1,1+p|q)$ is $\mathfrak{gl}_{\mathbb{C}}(2+p|q)$. Furthermore, let $G = \mathrm{U}_{cs}(1, 1+p|q)$ be the Lie *cs* group corresponding to the *cs* group pair

$$\big(\mathrm{U}(1,1+p) \times \mathrm{U}(q), \mathfrak{u}_{cs}(1,1+p|q)\big).$$

An involution $\vartheta\colon \mathfrak{g} \to \mathfrak{g}$ is given by

$$\vartheta(X) \coloneqq \sigma X \sigma,$$

where

$$\sigma := \left(\begin{array}{c:c|c} -\mathbb{1}_1 & 0 & 0 \\ \hdashline 0 & \mathbb{1}_{1+p} & 0 \\ \hline 0 & 0 & \mathbb{1}_q \end{array}\right). \tag{3.2}$$

Under this involution $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with

$$\mathfrak{k} = \left(\begin{array}{c:c|c} * & 0 & 0 \\ \hdashline 0 & * & * \\ \hline 0 & * & * \end{array}\right), \quad \mathfrak{p} = \left(\begin{array}{c:c|c} 0 & * & * \\ \hdashline * & 0 & 0 \\ \hline * & 0 & 0 \end{array}\right).$$

Here, dashed lines in the matrices indicate the action of the involution, whereas full lines signify the grading.

A super symmetric non-degenerate even bilinear form on $\mathfrak{g}$ can be defined with the super trace: $b(X, Y) := \operatorname{str}(XY)$. Here, $\operatorname{str}\left(\begin{smallmatrix} R & S \\ T & V \end{smallmatrix}\right) = \operatorname{tr} R - \operatorname{tr} V$, where the blocks indicate the grading.

Let $\mathfrak{a} \subset \mathfrak{p}_{\bar{0}}$ be the even subspace which is generated by the element

$$h_0 = \left(\begin{array}{c:cc|c} 0 & 1 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right).$$

From now on, matrices will be written in the form of columns (and rows) of size 1, 1, p and q. Since h_0 is even, $\operatorname{ad}(h_0)$ acts on $\mathfrak{gl}_{\mathbb{C}}(2+p|q)$ in the same way as $\operatorname{ad}(h_0)$ acts on $\mathfrak{gl}_{\mathbb{C}}(2+p+q)$. This means that the root decomposition is formally the same as in the classical case.

There are 4 roots: $\alpha, 2\alpha, -\alpha, -2\alpha$, where $\alpha(h_0) = 1$. The root spaces are

$$
\begin{aligned}
\mathfrak{g}_{\alpha} &= \left\{ x \in \mathfrak{g} \,\middle|\, x = \left(\begin{array}{c:cc|c} 0 & 0 & B_0 & B_1 \\ \hdashline 0 & 0 & B_0 & B_1 \\ -C_0 & C_0 & 0 & 0 \\ \hline -C_1 & C_1 & 0 & 0 \end{array}\right) \right\}, \\
\mathfrak{g}_{2\alpha} &= \left\{ x \in \mathfrak{g} \,\middle|\, x = \left(\begin{array}{c:cc|c} -A & A & 0 & 0 \\ \hdashline -A & A & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \right\}, \\
\mathfrak{g}_{-\alpha} &= \left\{ x \in \mathfrak{g} \,\middle|\, x = \left(\begin{array}{c:cc|c} 0 & 0 & -B_0 & -B_1 \\ \hdashline 0 & 0 & B_0 & B_1 \\ C_0 & C_0 & 0 & 0 \\ \hline C_1 & C_1 & 0 & 0 \end{array}\right) \right\}, \\
\mathfrak{g}_{-2\alpha} &= \left\{ x \in \mathfrak{g} \,\middle|\, x = \left(\begin{array}{c:cc|c} -A & -A & 0 & 0 \\ \hdashline A & A & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \right\}, \\
\mathfrak{m} &= \left\{ x \in \mathfrak{g} \,\middle|\, x = \left(\begin{array}{c:cc|c} D & 0 & 0 & 0 \\ \hdashline 0 & D & 0 & 0 \\ 0 & 0 & E & F \\ \hline 0 & 0 & G & H \end{array}\right) \right\}.
\end{aligned} \tag{3.3}
$$

In the following, α and 2α will be considered to be the positive restricted roots, hence $\mathfrak{n} = \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha}$ and $\bar{\mathfrak{n}} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$. Furthermore, it is clear, that $m_{\alpha} = 2(p-q)$ and $m_{2\alpha} = 1$, hence $\varrho = (1 + p - q)\alpha$.

Let $K \coloneqq \mathrm{U}(1) \times \mathrm{U}_{cs}(1+p|q)$ be the sub Lie *cs* group of G with underlying space $K_0 = \mathrm{U}(1) \times \mathrm{U}(1+p) \times \mathrm{U}(q)$ and Lie *cs* algebra $\mathfrak{k}$. G admits an Iwasawa decomposition $G = KAN$, since this is the case for $\mathrm{U}(1, 1+p)$. An essential tool in the calculation of the c-function is Proposition 2.5.11. In order to apply this proposition, it is necessary to determine $H|_{\bar{N}} \colon \bar{N} \to \mathbb{A}(\mathfrak{a})$.

Due to Proposition 2.1.10, S-points of $\mathbb{A}(\mathfrak{n})$ and $\mathbb{A}(\bar{\mathfrak{n}})$ are given by

$$
\begin{aligned}
\mathbb{A}(\mathfrak{n})(S) &= \left\{ X = M \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} M^\top \sigma \;\middle|\; \begin{array}{l} A \in_S \mathbb{A}^{\mathbb{C},1|0}, \\ B^\top \in_S \mathbb{A}^{\mathbb{C},p|q}, \\ C \in_S \mathbb{A}^{\mathbb{C},p|q}, \\ \operatorname{im} j_{S_0}^\sharp(X) \subseteq \mathfrak{n}_{\bar{0}} \end{array} \right\}, \\
\mathbb{A}(\bar{\mathfrak{n}})(S) &= \left\{ X = \sigma M \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} M^\top \;\middle|\; \begin{array}{l} A \in_S \mathbb{A}^{\mathbb{C},1|0}, \\ B^\top \in_S \mathbb{A}^{\mathbb{C},p|q}, \\ C \in_S \mathbb{A}^{\mathbb{C},p|q}, \\ \operatorname{im} j_{S_0}^\sharp(X) \subseteq \bar{\mathfrak{n}}_{\bar{0}} \end{array} \right\},
\end{aligned}
\tag{3.4}
$$

with $M^\top = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & \mathbb{1}_{p+q} \end{smallmatrix}\right)$. In this notation, B is considered as row-vector with entries in $\Gamma(\mathcal{O}_S)$, whereas C is a column vector. Furthermore $\operatorname{im} j_{S_0}^\sharp(X) \subset \mathfrak{g}_{\bar{0}}$ means that the underlying function $j_{S_0}^\sharp(A)$ of A is has values in $i\mathbb{R}$ and that $j_{S_0}^\sharp(B_i) = -\overline{j_{S_0}^\sharp(C_i)}$ for $i = 1, \ldots, p$.

The calculations to obtain $H|_{\bar{N}}$ are formally the same as in the classical case. The only difference is that one has to work with generalised points instead of ordinary points. The complexification of $\mathrm{U}_{cs}(1, 1+p|q)$ is $\mathrm{GL}_{\mathbb{C}}(2+p|q)$. Therefore, the S-points of N and $\bar{N}$ are matrices in $\mathrm{GL}_{\mathbb{C}}(2+p|q)(S)$. The S-points of N and $\bar{N}$ are of the form

$$
\begin{aligned}
n_{A,B,C} &= \exp\left(M \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} M^\top \sigma \right) = \mathbb{1}_{2+p+q} + M \begin{pmatrix} A + \frac{1}{2}BC & B \\ C & 0 \end{pmatrix} M^\top \sigma \\
&= \begin{pmatrix} 1 - A - \frac{1}{2}BC & A + \frac{1}{2}BC & B \\ -A - \frac{1}{2}BC & 1 + A + \frac{1}{2}BC & B \\ -C & C & \mathbb{1}_{p+q} \end{pmatrix}, \\
\bar{n}_{A,B,C} &= \exp\left(\sigma M \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} M^\top \right) = \mathbb{1}_{2+p+q} + \sigma M \begin{pmatrix} A + \frac{1}{2}BC & B \\ C & 0 \end{pmatrix} M^\top \\
&= \begin{pmatrix} 1 - A - \frac{1}{2}BC & -A - \frac{1}{2}BC & -B \\ A + \frac{1}{2}BC & 1 + A + \frac{1}{2}BC & B \\ C & C & \mathbb{1}_{p+q} \end{pmatrix},
\end{aligned}
$$

with A, B, C as in Equation (3.4), since $\exp_N$ and $\exp_{\bar{N}}$ are isomorphisms and

$$
\begin{aligned}
M^\top \sigma M &= \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{p+q} \end{pmatrix}, \\
\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{p+q} \end{pmatrix} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} &= \begin{pmatrix} BC & 0 \\ 0 & 0 \end{pmatrix}, \\
\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{p+q} \end{pmatrix} \begin{pmatrix} BC & 0 \\ 0 & 0 \end{pmatrix} &= 0.
\end{aligned}
$$

From this

$$\vartheta(n_{A,B,C}) = \bar{n}_{A,B,C}, \qquad \vartheta(\bar{n}_{A,B,C}) = n_{A,B,C}$$

can be deduced immediately.

The identities

$$\begin{pmatrix}1 & 1 & 0\end{pmatrix} n_{A,B,C} = \begin{pmatrix}1-2A-BC & 1+2A+BC & 2B\end{pmatrix}, \tag{3.5}$$

$$\begin{pmatrix}1 & 1 & 0\end{pmatrix} \bar{n}_{A,B,C} = \begin{pmatrix}1 & 1 & 0\end{pmatrix}, \tag{3.6}$$

follow from $(1,1,0)\sigma M = 0$ and $(1,1,0)M = (2,0)$. Similarly,

$$n_{A,B,C}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \qquad \bar{n}_{A,B,C}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1-2A-BC\\1+2A+BC\\2C\end{pmatrix} \tag{3.7}$$

are obtained.

Now, let

$$\bar{n}_{A,B,C} = k(\bar{n}_{A,B,C})\exp(h_0 \otimes h)n_{F,G,H},$$

with $h \in \Gamma(\mathcal{O}_S)$ such that $h_0 \otimes h = H(\bar{n}_{A,B,C})$ and $n_{F,G,H} \in_S N$. Applying ϑ on both sides of this equation yields

$$\vartheta(\bar{n}_{A,B,C}) = k(\bar{n}_{A,B,C})\exp(-h_0 \otimes h)\vartheta(n_{F,G,H}),$$

hence

$$\vartheta(\bar{n}_{A,B,C})^{-1}\bar{n}_{A,B,C} = \vartheta(n_{F,G,H})^{-1}\exp(h_0 \otimes 2h)n_{F,G,H},$$

or equivalently

$$n_{-A,-B,-C}\bar{n}_{A,B,C} = \bar{n}_{-F,-G,-H}\exp(h_0 \otimes 2h)n_{F,G,H}.$$

Thanks to Equations (3.6) and (3.7), multiplying $(1,1,0)$ from the left and its transpose from the right reduces the right hand side to

$$\begin{pmatrix}1 & 1 & 0\end{pmatrix}\exp(h_0 \otimes h)\begin{pmatrix}1\\1\\0\end{pmatrix} = 2e^h,$$

since $(1,1,0)h_0 = (1,1,0)$. Therefore, Equations (3.5) and (3.7) lead to

$$\begin{aligned} e^{2h} &= \frac{1}{2}\begin{pmatrix}1 & 1 & 0\end{pmatrix} n_{-A,-B,-C}\bar{n}_{A,B,C}\begin{pmatrix}1\\1\\0\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}1+2A-BC & 1-2A+BC & -2B\end{pmatrix}\begin{pmatrix}1-2A-BC\\1+2A+BC\\2C\end{pmatrix} \\ &= (1-BC)^2-4A^2. \end{aligned}$$

3.1.5 Lemma. *The restriction of H to N is given by*

$$H(\bar{n}_{A,B,C}) = h_0 \otimes \frac{1}{2}\ln\Big((1-BC)^2-4A^2\Big)$$

for $\bar{n}_{A,B,C} \in_S \bar{N}$.

The classical approach to derive c is to write

$$\phi_\lambda(g) = \int_{K/M} D\dot{k}\, e^{(\lambda-\varrho)(H(gk))} = \int_{\bar{N}} D\bar{n}\, e^{(\lambda-\varrho)(H(gk(\bar{n})))}.$$

However, if $pq \neq 0$, this cannot be performed, since the retractions on K/M and $\bar{N}$ are not compatible. There are examples where the integral on the right hand side does not even exist. The next guess might be to interchange the limit in Equation (3.1) with $\int_{K/M}$, which should be easy to perform, since K/M is compact. Yet, this is not permitted either, since the limit of $e^{(\lambda-\varrho)\circ H(e^{th_0}ke^{-th_0})}$ does not exit. Even in the classical case the limit would not be smooth outside of $k(\bar{N})$.

Consequently, the only possible way is to apply Proposition 2.5.11 with the support of the integrand forced to be compact inside $k_0(\bar{N}_0)$. In order to do this let $\chi \in \mathcal{C}_c^\infty(]0,\infty[)$ such that $\chi \equiv 1$ on a neighbourhood of e. Define $\aleph \in \Gamma(\mathcal{O}_{K/M})$ by

$$\aleph(k(\bar{n}_{A,B,C})) \coloneqq \chi\big((1-BC)^2-4A^2\big)$$

for $\bar{n}_{A,B,C} \in_S \bar{N}$. This super function has compact support in the image of $k\colon \bar{N} \to K/M$ and is therefore well-defined on K/M. Then

$$\phi_\lambda(g) = \int_{K/M} D\dot{k}\, e^{(\lambda-\varrho)(H(gk))}\aleph(k) + \int_{K/M} D\dot{k}\, e^{(\lambda-\varrho)H(gwk)}(1-\aleph)(wk) \tag{3.8}$$

for any $w \in M_0' = N_{K_0}(\mathfrak{a}) = \{m \in K_0 \mid \mathrm{Ad}(m,\mathfrak{a}) \subseteq \mathfrak{a}\}$ due to the left K-invariance of Dk. It is clear that the first of these two integrals can be pulled back to $\bar{N}$. In order to be able to do the same for the second integral, a good choice for w has to be made.

The Weyl group M_0'/M_0 consists in the considered case of two elements. Let $w \in M_0'$ be a representative of the non-trivial Weyl group element. The Bruhat decomposition (*cf.* [Hel62, Theorem 1.3]) shows that the complement of $k(\bar{N}_0)$ in K_0/M_0 is indeed wM_0. Since $1-\aleph$ vanishes in a neighbourhood of eM_0, this shows that the second integrand in Equation (3.8) has also support inside $k(\bar{N}_0)$. Now, Equation (2.11) shows $H\big(gk(\bar{n})\big) = H(g\bar{n}) - H(\bar{n})$, hence by Proposition 2.5.11

$$\begin{aligned}\phi_\lambda(g) = &\int_{\bar{N}} D\bar{n}\, e^{(\lambda-\varrho)(H(g\bar{n}))} e^{-(\lambda+\varrho)(H(\bar{n}))} \aleph\big(k(\bar{n})\big) \\ &+ \int_{\bar{N}} D\bar{n}\, e^{(\lambda-\varrho)(H(gw\bar{n}))} e^{-(\lambda+\varrho)(H(\bar{n}))} (1-\aleph)\big(wk(\bar{n})\big).\end{aligned} \tag{3.9}$$

For the following considerations, it is necessary to assume $g \in A$. For $t \in \mathbb{R}$

$$e^{th_0} = \begin{pmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & \mathbb{1}_{p+q} \end{pmatrix}.$$

It is clear that $e^{th_0}\bar{n}_{A,B,C}e^{-th_0} = \bar{n}_{e^{-2t}A,e^{-t}B,e^{-t}C}$ for $\bar{n}_{A,B,C} \in_S \bar{N}$, since

$$e^{th_0}\begin{pmatrix} -A & -A & -B \\ A & A & B \\ C & C & 0 \end{pmatrix} e^{-th_0} = \begin{pmatrix} -e^{-2t}A & -e^{-2t}A & -e^{-t}B \\ e^{-2t}A & e^{-2t}A & e^{-t}B \\ e^{-t}C & e^{-t}C & 0 \end{pmatrix}.$$

This shows

$$\begin{aligned} H(e^{th_0}\bar{n}_{A,B,C}e^{-th_0}) &= H(\bar{n}_{e^{-2t}A,e^{-t}B,e^{-t}C}) \\ &= h_0 \otimes \frac{1}{2}\ln\big((1-e^{-2t}BC)^2 - 4e^{-4t}A^2\big),\end{aligned} \tag{3.10}$$

thus

$$\begin{aligned} H(e^{th_0}w\bar{n}_{A,B,C}e^{-th_0}) &= H(e^{-th_0}\bar{n}_{A,B,C}e^{th_0}) - 2t \\ &= h_0 \otimes \frac{1}{2}\ln\big((e^{-2t}-BC)^2 - 4A^2\big)\end{aligned} \tag{3.11}$$

by Equation (2.10).

In particular, Equation (3.10) shows $\aleph(k) = \lim_{t\to\infty}\chi\left(e^{-2\alpha(H(e^{th_0}ke^{-th_0}))}\right)$ for $k \in_S K$, using $H(ak(\bar{n})a^{-1}) = H(a\bar{n}a^{-1}) - H(\bar{n})$. Accordingly,

$$(1-\aleph)\big(wk(\bar{n}_{A,B,C})\big) = (1-\chi)\left(\frac{(1-BC)^2-4A^2}{(BC)^2-4A^2}\right)$$

for $\bar{n}_{A,B,C} \in_S \bar{N}$, thanks to Equation (3.11). Since $(1-\chi) \equiv 1$ at ∞, this equation makes sense everywhere on $\bar{N}$.

Let $\varphi\colon \mathbb{A}^{1+2p|2q} = \mathbb{A}^1 \times \mathbb{A}^{p|q} \times \mathbb{A}^{p|q} \to \mathbb{A}(\bar{\mathfrak{n}})$ be the isomorphism given by

$$(a,b,c) \longmapsto \begin{pmatrix} -\frac{a}{2i} & -\frac{a}{2i} & -(b_0 - ic_0)^\top & -(b_1 - c_1)^\top \\ \frac{a}{2i} & \frac{a}{2i} & (b_0 - ic_0)^\top & (b_1 - c_1)^\top \\ b_0 + ic_0 & b_0 + ic_0 & 0 & 0 \\ b_1 + c_1 & b_1 + c_1 & 0 & 0 \end{pmatrix}.$$

Since this isomorphism is linear, the pullback a Lebesgue Berezin density on $\mathfrak{g}$ *via* φ is a Lebesgue density on $\mathbb{A}^{1+2p|2q}$, hence the same is true for the pullback of $D\dot{k}$ *via* $\exp\circ\varphi$, thanks to Proposition 2.5.7.

By pulling the integrals on $\bar{N}$ back *via* $\exp\circ\varphi$, Equation (3.9) takes the form

$$\begin{aligned}
&\phi_\lambda(e^{th_0})e^{-(\lambda-\varrho)(th_0)} \\
&\quad= \int_{\mathbb{A}^{1+2p|2q}} D\mu(s,y)\, \frac{\left((1+e^{-2t}\|y\|^2)^2 + e^{-4t}s^2\right)^{\frac{1}{2}(\lambda-\varrho)(h_0)}}{\left((1+\|y\|^2)^2+s^2\right)^{\frac{1}{2}(\lambda+\varrho)(h_0)}} \chi\big((1+\|y\|^2)^2+s^2\big) \\
&\qquad+ \int_{\mathbb{A}^{1+2p|2q}} D\mu(s,y)\, \frac{\left((e^{-2t}+\|y\|^2)^2+s^2\right)^{\frac{1}{2}(\lambda-\varrho)(h_0)}}{\left((1+\|y\|^2)^2+s^2\right)^{\frac{1}{2}(\lambda+\varrho)(h_0)}} (1-\chi)\left(\frac{(1+\|y\|^2)^2+s^2}{\|y\|^4+s^2}\right)
\end{aligned}$$

Assuming $D\dot{k}$ to be normalised adequately, $D\mu$ denotes the Lebesgue Berezin density from Section 2.3.

From now on, $\lambda(h_0)$ will be abbreviated λ as long the meaning is clear from the context. Since the Lebesgue Berezin density on $\mathbb{A}^{1+2p|2q} = \mathbb{A}^1 \times \mathbb{A}^{2p|2q}$ is the same as the tensor product of the Lebesgue Berezin densities on the factors, Corollary 2.3.4 can be applied. In particular, this means that ϕ_λ only depends on m_α and $m_{2\alpha}$. Therefore, if $m_\alpha > 0$, one can assume $q = 0$ and use the well-known formula for the c-function in the classical case (*cf.* [Hel84, Chapter IV, Theromem 6.4]).

3.1.6 Lemma. *Let* $m_\alpha > 0$ *and* $\operatorname{Re}\lambda > 0$. *Then*

$$c(\lambda) = c_0 \frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{\lambda+\frac{m_\alpha}{2}+1}{2}\right)\Gamma\left(\frac{\lambda+\frac{m_\alpha}{2}+m_{2\alpha}}{2}\right)}$$

for some constant c_0.

In the case $m_\alpha \leq 0$, Corollary 2.3.5 can be brought to use.

3.1.7 Lemma. *Let* $m_\alpha \leq 0$ *and* $\operatorname{Re}\lambda > 0$. *Then*

$$\begin{aligned}
c(\lambda) =& C\int_0^\infty ds\, \partial_{r=0}^{-\frac{m_\alpha}{2}} \left((1+r)^2+s^2\right)^{-\frac{\lambda+\varrho}{2}} \chi\big((1+r)^2+s^2\big) \\
&+ C\int_0^\infty ds\, \partial_{r=0}^{-\frac{m_\alpha}{2}} \frac{(r^2+s^2)^{\frac{\lambda-\varrho}{2}}}{\left((1+r)^2+s^2\right)^{\frac{\lambda+\varrho}{2}}} (1-\chi)\left(\frac{(1+r)^2+s^2}{r^2+s^2}\right)
\end{aligned} \tag{3.12}$$

for some constant C. *Furthermore,* c *is a meromorphic function for* $\operatorname{Re}\lambda > 0$.

PROOF. The integrands have compact support and are bounded even after deriving in λ. Thus, it is no problem to interchange $\lim_{t\to\infty}$ with the integrals and the same can be done for ∂_λ. ☐

In order to finally derive $c(\lambda)$, the function χ needs to be removed. For this, the following estimation will repeatedly be used.

3.1.8 Lemma.

$$\left|\partial_r^k\left((1+r)^2+s^2\right)^{\frac{z}{2}}\right| < c_k\left((1+r)^2+s^2\right)^{\frac{\operatorname{Re} z-k}{2}}$$

for all $r, s \geq 0$, where $k \in N_0$ and c_k is some not further specified (z-dependent) constant.

PROOF. As one can see by induction,

$$\partial_r^k\left((1+r)^2+s^2\right)^{\frac{z}{2}} = p_k\left(\frac{1+r}{s}\right)s^k\left((1+r)^2+s^2\right)^{\frac{z-2k}{2}},$$

where p_k is a polynomial of order at most k. Since $\lim_{t\to\infty}(t^2+1)^{-\frac{k}{2}}p_k(t)$ exists, there is a constant c_k such that $|(t^2+1)^{-\frac{k}{2}}p_k(t)| < c_k$ for all $t \geq 0$. This implies

$$\begin{aligned}\left|\partial_r^k\left((1+r)^2+s^2\right)^{\frac{z}{2}}\right| &= \left|p_k\left(\frac{1+r}{s}\right)\left(\left(\frac{1+r}{s}\right)^2+1\right)^{-\frac{k}{2}}\left((1+r)^2+s^2\right)^{\frac{z-k}{2}}\right| \\ &< c_k\left((1+r)^2+s^2\right)^{\frac{\operatorname{Re} z-k}{2}}.\end{aligned}$$ ☐

3.1.9 Lemma. *If $m_\alpha \leq 0$, and $\operatorname{Re}\lambda(H_0) > 0$, then*

$$c(\lambda) = C\int_0^\infty ds\,\partial_{r=0}^{1-\varrho}\left((1+r)^2+s^2\right)^{-\frac{\lambda+\varrho}{2}}. \tag{3.13}$$

PROOF. First of all, note that $\frac{m_\alpha}{2} = 1-\varrho$ and consider the second integral in Equation (3.12). The substitution $r = su$ yields

$$\int_0^\infty ds\,\partial_{r=0}^{1-\varrho}\frac{(r^2+s^2)^{\frac{\lambda-\varrho}{2}}}{\left((1+r)^2+s^2\right)^{\frac{\lambda+\varrho}{2}}}(1-\chi)\left(\frac{(1+r)^2+s^2}{r^2+s^2}\right) \tag{3.14}$$

$$= \int_0^\infty ds\,\partial_{u=0}^{1-\varrho}s^{\varrho-1}\frac{(s^2u^2+s^2)^{\frac{\lambda-\varrho}{2}}}{\left((1+su)^2+s^2\right)^{\frac{\lambda+\varrho}{2}}}(1-\chi)\left(\frac{(1+su)^2+s^2}{s^2u^2+s^2}\right) \tag{3.15}$$

$$= \int_0^\infty ds\,\partial_{u=0}^{1-\varrho}s^{\lambda-1}\frac{(u^2+1)^{\frac{\lambda-\varrho}{2}}}{\left((1+su)^2+s^2\right)^{\frac{\lambda+\varrho}{2}}}(1-\chi)\left(\frac{(s^{-1}+u)^2+1}{u^2+1}\right) \tag{3.16}$$

The next step is to interchange $\int ds$ and ∂_u^q. Since only the limit $u \to 0$ is of interest, u may assumed to be small. Let $\varepsilon < 1$ be small enough such that $(1-\chi) \equiv 0$ on the interval $[1, 1+4\varepsilon^2]$. Then the function

$$[0,\varepsilon] \times \mathbb{R}_+, (u,s) \mapsto (1-\chi)\left(\frac{(s^{-1}+u)^2+1}{u^2+1}\right)$$

is of compact support, since $\frac{(s^{-1}+u)^2+1}{u^2+1} \leq 1+4\varepsilon^2$ for $s > \frac{1}{\varepsilon}$. Therefore, $\partial_{u=0}^{1-\varrho}$ and $\int_\delta^\infty ds$ can be interchanged for arbitrarily small δ.

Let $\delta < 1$ be small enough such that $(1-\chi)(\frac{(s^{-1}+u)^2+1}{u^2+1}) = 1$ for $s < \delta$ and $u < \varepsilon$. Now, by Lemma 3.1.8

$$\left| s^{\lambda-1} \partial_u^k \big((1+su)^2+s^2\big)^{-\frac{\lambda+\varrho}{2}} \right| < c_k s^{\operatorname{Re}\lambda - 1 + k}\big((1+su)^2+s^2\big)^{-\frac{\operatorname{Re}\lambda+\varrho+k}{2}} \leq 5^{-\frac{\varrho}{2}} c_k s^{\operatorname{Re}\lambda-1}$$

for $u \leq \varepsilon$, $s \leq \delta$, and $k \leq q$. Since $s^{\operatorname{Re}\lambda-1}$ is integrable on $[0,\delta]$, one may also interchange $\int_0^\delta ds$ and $\partial_{u=0}^{1-\varrho}$. Therefore, the right hand side of Equation (3.16) equals

$$\begin{aligned}
&\partial_{u=0}^{1-\varrho} \int_0^\infty ds\, s^{\lambda-1} \frac{(u^2+1)^{\frac{\lambda-\varrho}{2}}}{\big((1+su)^2+s^2\big)^{\frac{\lambda+\varrho}{2}}} (1-\chi)\left(\frac{(s^{-1}+u)^2+1}{u^2+1}\right) \\
&\qquad = \partial_{u=0}^{1-\varrho} \int_0^\infty \frac{ds}{s} \frac{\big(s(u^2+1)\big)^{\frac{\lambda-\varrho}{2}}}{\big(s^{-1}+2u+s(u^2+1)\big)^{\frac{\lambda+\varrho}{2}}} (1-\chi)\left(\frac{s^{-1}+2u+s(u^2+1)}{s(u^2+1)}\right).
\end{aligned}$$

Substituting $t = s^{-1}(u^2+1)^{-1}$ leads to

$$\begin{aligned}
&\partial_{u=0}^{1-\varrho} \int_0^\infty \frac{dt}{t} \frac{t^{-\frac{\lambda-\varrho}{2}}}{\big(t^{-1}+2u+t(u^2+1)\big)^{\frac{\lambda+\varrho}{2}}} (1-\chi)\Big(t\big(t^{-1}+2u+t(u^2+1)\big)\Big), \\
&\qquad\qquad = \partial_{u=0}^{1-\varrho} \int_0^\infty dt\, t^{\varrho-1}\big((1+tu)^2+t^2\big)^{-\frac{\lambda+\varrho}{2}} (1-\chi)\big((1+tu)^2+t^2\big).
\end{aligned}$$

Note that $\frac{ds}{s}$ and $s^{-1}+2u+s(u^2+1)$ are invariant under this substitution.

Again, derivation and integral need to be interchanged. Possibly after shrinking ε one may assume $(1-\chi)\big((1+tu)^2+t^2\big) = 0$ for $t \leq \varepsilon$ and $u \leq \varepsilon$. Therefore, it suffices to consider the integral $\int_\varepsilon^\infty dt$. Obviously, there is no problem in interchanging $\int_\varepsilon^R dt$ and $\partial_{u=0}^{1-\varrho}$, where R is big enough such that $(1-\chi)\big((1+tu)^2+t^2\big) = 1$ for $t \geq R$. Again, by Lemma 3.1.8,

$$\left| \partial_u^k t^{\varrho-1}\big((1+tu)^2+t^2\big)^{-\frac{\lambda+\varrho}{2}} \right| < c_k t^{\varrho-1+k}\big((1+tu)^2+t^2\big)^{-\frac{\operatorname{Re}\lambda+\varrho+k}{2}} \leq 5^{-\frac{\varrho}{2}} c_k t^{-\operatorname{Re}\lambda-1}.$$

which is integrable on $[R,\infty[$ for $\operatorname{Re}\lambda > 0$. Thus, $\int_R^\infty dt$ and $\partial_{u=0}^{1-\varrho}$ can also be interchanged. Therefore,

$$\begin{aligned}
&\int_0^\infty ds\, \partial_{r=0}^{1-\varrho} \frac{(r^2+s^2)^{\frac{\lambda-\varrho}{2}}}{((1+r)^2+s^2)^{\frac{\lambda+\varrho}{2}}}(1-\chi)\left(\frac{(1+r)^2+s^2}{r^2+s^2}\right)\\
&\qquad = \int_0^\infty dt\, \partial_{u=0}^{1-\varrho} t^{\varrho-1}\big((1+tu)^2+t^2\big)^{-\frac{\lambda+\varrho}{2}}(1-\chi)\big((1+tu)^2+t^2\big)\\
&\qquad = \int_0^\infty dt\, \partial_{r=0}^{1-\varrho}\big((1+r)^2+t^2\big)^{-\frac{\lambda+\varrho}{2}}(1-\chi)\big((1+r)^2+t^2\big),
\end{aligned}$$

by substituting $u=\frac{r}{t}$. This combined with Equation (3.12) yields the assertion. In particular, the right hand side of Equation (3.13) exists, since this is true for both summands. $\square$

3.1.10 Theorem. *The c-function for* $\mathrm{U}_{cs}(1,1+p|2q)$ *is given by*

$$c(\lambda) = c_0 \frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{\lambda+\frac{m_\alpha}{2}+1}{2}\right)\Gamma\left(\frac{\lambda+\frac{m_\alpha}{2}+m_{2\alpha}}{2}\right)}$$

for some constant c_0 *and* $\operatorname{Re}\lambda > 0$.

Proof. The case $m_\alpha > 0$ is already clear by Lemma 3.1.6. Let $m_\alpha \leq 0$. The c-function is meromorphic, thanks to Lemma 3.1.7. Therefore, an argument using analytic continuation allows to assume $\operatorname{Re}\lambda > -\varrho$ for the following.

Thanks to Lemma 3.1.9

$$c(\lambda) = C\int_0^\infty ds\, \partial_{r=0}^{1-\varrho}\big((1+r)^2+s^2\big)^{-\frac{\lambda+\varrho}{2}} = C\partial_{r=0}^{1-\varrho}\int_0^\infty ds\big((1+r)^2+s^2\big)^{-\frac{\lambda+\varrho}{2}}.$$

The second equality comes from Lemma 3.1.8, since

$$\left|\partial_r^k\big((1+r)^2+s^2\big)^{-\frac{\lambda+\varrho}{2}}\right| < c_k\big((1+r)^2+s^2\big)^{-\frac{\operatorname{Re}\lambda+\varrho+k}{2}} \leq c_k(1+s^2)^{-\frac{\operatorname{Re}\lambda+\varrho}{2}}$$

for $k \leq 1-\varrho$. This is integrable by assumption.

Substitution with $s=(1+r)\sqrt{t}$ yields

$$\begin{aligned}
c(\lambda) &= \frac{C}{2}\partial_{r=0}^{1-\varrho}(1+r)^{-\lambda-\varrho+1}\int_0^\infty dt\, t^{-\frac{1}{2}}(1+t)^{-\frac{\lambda+\varrho}{2}} \\
&= \frac{C}{2}(1-\varrho)!\binom{-\lambda-\varrho+1}{1-\varrho}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{\lambda+\varrho-1}{2})}{\Gamma(\frac{\lambda+\varrho}{2})} \\
&= \frac{C\sqrt{\pi}(-1)^{1-\varrho}}{2}(1-\varrho)!\binom{\lambda-1}{1-\varrho}\frac{\Gamma(\frac{\lambda+\varrho-1}{2})}{\Gamma(\frac{\lambda+\varrho}{2})} \\
&= \frac{C\sqrt{\pi}(-1)^{1-\varrho}}{2}\frac{\Gamma(\lambda)\Gamma(\frac{\lambda+\varrho-1}{2})}{\Gamma(\lambda+\varrho-1)\Gamma(\frac{\lambda+\varrho}{2})} \\
&= C(-2)^{1-\varrho}\pi\frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma(\frac{\lambda+\varrho}{2})\Gamma(\frac{\lambda+\varrho}{2})}.
\end{aligned}$$

Here, the beta function $\int_0^\infty t^{x-1}(1+t)^{-y}dt = \frac{\Gamma(x)\Gamma(y-x)}{\Gamma(y)}$ for $\mathrm{Re}\, y > \mathrm{Re}\, x > 0$ has been applied as well as $\binom{-x}{k} = (-1)^k\binom{x+k-1}{k}$ for $k\in\mathbb{N}$. Furthermore, the duplication formula $\Gamma(z) = \frac{1}{\sqrt{\pi}}2^{z-1}\Gamma(\frac{z}{2})\Gamma(\frac{z+1}{2})$ came to use. The claim follows from $m_{2\alpha}=1$ and $m_\alpha = 2(\varrho-1)$. □

The Ortho-Symplectic Case

Let $\mathfrak{osp}_{\mathbb{C}}(1,1+p|2q)$ be the complex sub Lie super algebra of $\mathfrak{gl}_{\mathbb{C}}(2+p|2q)$, given by

$$\mathfrak{osp}_{\mathbb{C}}(1,1+p|2q) = \left\{ x\in\mathfrak{gl}_{\mathbb{C}}(2+p|2q) \,\middle|\, x^{\mathrm{st}^3}J + Jx = 0 \right\},$$

where $\left(\begin{smallmatrix} R & S \\ T & V\end{smallmatrix}\right)^{\mathrm{st}^3} = \left(\begin{smallmatrix} R^\top & T^\top \\ -S^\top & V^\top\end{smallmatrix}\right)$ and

$$J = \left(\begin{array}{cc|c} -\mathbb{1}_1 & 0 & 0 \\ 0 & \mathbb{1}_{1+p} & 0 \\ \hline 0 & 0 & J_q \end{array}\right). \tag{3.17}$$

Here J_q denotes the $2q\times 2q$ matrix with q copies of the 2×2 matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$ on the diagonal. Therefore, $\mathfrak{osp}_{\mathbb{C}}(1,1+p|2q)$ consists of matrices of the form

$$\left(\begin{array}{cc|c} 0 & X_{12} & X_{13} \\ X_{12}^\top & X_{22} & X_{23} \\ \hline J_qX_{13}^\top & -J_qX_{23}^\top & X_{33} \end{array}\right) \tag{3.18}$$

with $X_{22}\in\mathfrak{so}_{\mathbb{C}}(1+p)$ and $X_{33}\in\mathfrak{sp}_{\mathbb{C}}(2q)$.

In this subsection the *orthosymplectic* Lie *cs* algebra

$$\mathfrak{g} = \mathfrak{osp}_{cs}(1,1+p|2q) \coloneqq \mathfrak{osp}_{\mathbb{C}}(1,1+p|2q)\cap\mathfrak{u}_{cs}(1,1+p|2q)$$

will play the central role. Note that $\mathfrak{g}_{\bar{0}} = \mathfrak{so}_{\mathbb{R}}(1+p) \times \mathfrak{usp}(2q)$, where $\mathfrak{usp}(2q)$ is the compact form of $\mathfrak{sp}_{\mathbb{C}}(2q)$. Again, the involution $\vartheta\colon \mathfrak{g} \to \mathfrak{g}$ will be given by

$$\vartheta(X) \coloneqq \sigma X \sigma$$

with σ from Equation (3.2).

Let $G = \mathrm{SOSp}^{+}_{cs}(1, 1+p|2q)$ be the sub Lie *cs* group of $\mathrm{U}_{cs}(1, 1+p|2q)$ given by the underlying Lie group $\mathrm{SO}_{\mathbb{R}}(1, 1+p) \times \mathrm{USp}(2q)$ and the Lie *cs* algebra $\mathfrak{osp}_{cs}(1, 1+p|2q)$. Here, $\mathrm{USp}(2q) \coloneqq \mathrm{U}(2q) \cap \mathrm{Sp}(2q)$. Let $K = \mathrm{SOSp}_{cs}(1+p|2q)$ be the sub Lie *cs* group of G with underlying space $\{1\} \times \mathrm{SO}_{\mathbb{R}}(1+p) \times \mathrm{USp}(2q)$ and Lie *cs* algebra $\mathfrak{k}$ from $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Let $\mathfrak{a} \subseteq \mathfrak{p}_{\bar{0}}$ again be the even Cartan subalgebra generated by the element

$$h_0 = \left(\begin{array}{c:cc|c} 0 & 1 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right).$$

Since σ and J commute, the restricted root space decomposition on $\mathfrak{g}$ can be easily obtained by restricting the decomposition from Equation (3.3) to $\mathfrak{osp}_{cs}(1, 1+p|2q)$. This time there are only two restricted roots: α and $-\alpha$, where $\alpha(h_0) = 1$. The root spaces are

$$\begin{aligned}
\mathfrak{g}_\alpha &= \left\{ x \in \mathfrak{g} \;\middle|\; \left(\begin{array}{c:cc|c} 0 & 0 & B_0 & B_1 \\ \hdashline 0 & 0 & B_0 & B_1 \\ B_0^\top & -B_0^\top & 0 & 0 \\ \hline J_q B_1^\top & -J_q B_1^\top & 0 & 0 \end{array}\right) \right\}, \\
\mathfrak{g}_{-\alpha} &= \left\{ x \in \mathfrak{g} \;\middle|\; \left(\begin{array}{c:cc|c} 0 & 0 & B_0 & B_1 \\ \hdashline 0 & 0 & -B_0 & -B_1 \\ B_0^\top & B_0^\top & 0 & 0 \\ \hline J_q B_1^\top & J_q B_1^\top & 0 & 0 \end{array}\right) \right\}, \\
\mathfrak{m} &= \left\{ x \in \mathfrak{g} \;\middle|\; \left(\begin{array}{c:cc|c} 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & E & F \\ \hline 0 & 0 & -J_q G^\top & H \end{array}\right) \right\}.
\end{aligned}$$

There is no root 2α, since the intersection of $\mathfrak{g}_{2\alpha}$ from Equation (3.3) with $\mathfrak{g}$ is $\{0\}$. This can also be obtained by just counting dimensions.

The root α will be considered to be the positive one, hence $\mathfrak{n} = \mathfrak{g}_\alpha$ and $\bar{\mathfrak{n}} = \mathfrak{g}_{-\alpha}$. As one can see from Equation (3.19), $m_\alpha = p - 2q$, and $m_{2\alpha} = 0$ hence $\varrho = \frac{p-2q}{2}\alpha$.

As in Equation (3.4) the S-points of $\mathbb{A}(\mathfrak{n})$ and $\mathbb{A}(\bar{\mathfrak{n}})$ are of the form

$$\begin{aligned}
\mathbb{A}(\mathfrak{n})(S) &= \left\{ X = M \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} M^\top \sigma \;\middle|\; \begin{array}{l} C \in_S \mathbb{A}^{p|2q}, \\ B = (-C_0^\top, -C_1^\top J_q), \\ \operatorname{im} j_{S_0}^\sharp(X) \subset \mathfrak{n}_{\bar{0}} \end{array} \right\}, \\
\mathbb{A}(\bar{\mathfrak{n}})(S) &= \left\{ X = \sigma M \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} M^\top \;\middle|\; \begin{array}{l} C \in_S \mathbb{A}^{p|2q}, \\ B = (-C_0^\top, -C_1^\top J_q), \\ \operatorname{im} j_{S_0}^\sharp(X) \subset \bar{\mathfrak{n}}_{\bar{0}} \end{array} \right\},
\end{aligned} \tag{3.19}$$

This means that the S-points of N and $\bar{N}$ are the form $\mathfrak{n}_{0,B,C}$ and $\bar{\mathfrak{n}}_{0,B,C}$. Therefore, the H-function is, as in the $\mathrm{U}_{cs}(1, 1 + p|q)$-case, given by

$$H(\bar{n}_{0,B,C}) = h_0 \otimes \frac{1}{2} \ln((1 - BC)^2) h_0 = h_0 \otimes \ln(1 - BC).$$

As well,

$$\begin{aligned}
H(e^{th_0} k(\bar{n}_{0,B,C}) e^{-th_0}) &= h_0 \otimes \ln \frac{1 - e^{-2t} BC}{1 - BC}, \\
H(e^{th_0} w k(\bar{n}_{0,B,C}) e^{-th_0}) &= h_0 \otimes \ln \frac{e^{-2t} - BC}{1 - BC},
\end{aligned}$$

where $w \in M_0'$ again denotes a representative of the non-trivial element of the Weyl group. Note that this only makes sense if $p > 0$. Otherwise, the root α is not even and the Weyl group would be trivial in this case. Therefore, let $p > 0$ for the time being. The case $p = 0$ will be postponed to the end of this subsection.

Let χ be as before and define $\aleph \in \Gamma(\mathcal{O}_{K/M})$ by

$$\aleph(k(\bar{n}_{0,B,C})) \coloneqq \chi(1 - BC)$$

for $\bar{n}_{0,B,C} \in_S \bar{N}$. Both $\aleph$ and $k \mapsto (1 - \aleph)(wk)$ have compact support inside $k_0(\bar{N}_0)$ and

$$(1 - \aleph)(wk(\bar{n}_{0,B,C})) = (1 - \chi) \left(\frac{1 - BC}{-BC} \right).$$

Therefore,

$$\begin{aligned}
\phi_\lambda(g) &= \int_{K/M} D\dot{k}\, e^{(\lambda - \varrho)(H(gk))} \aleph(k) + \int_{K/M} D\dot{k}\, e^{(\lambda - \varrho) H(gwk)} (1 - \aleph)(wk) \\
&= \int_{\bar{N}} D\bar{n}\, e^{(\lambda - \varrho)(H(g\bar{n}))} e^{-(\lambda + \varrho)(H(\bar{n}))} \aleph(k(\bar{n})) \\
&\quad + \int_{\bar{N}} D\bar{n}\, e^{(\lambda - \varrho)(H(gw\bar{n}))} e^{-(\lambda + \varrho)(H(\bar{n}))} (1 - \aleph)(wk(\bar{n})).
\end{aligned}$$

Let $\varphi\colon \mathbb{A}^{p|2q} \to \mathbb{A}(\bar{\mathfrak{n}})$ be the (linear) isomorphism with

$$\varphi(a) = \begin{pmatrix} 0 & 0 & a^\top \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & J_q \end{pmatrix} \\ 0 & 0 & -a^\top \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & J_q \end{pmatrix} \\ a & a & 0 \end{pmatrix}.$$

Pulling back the integrals over $\bar{N}$ *via* $\exp\circ\varphi$ then yields

$$\begin{aligned}\phi_\lambda(e^{th_0})e^{-(\lambda-\varrho)(th_0)} &= \int_{\mathbb{A}^{p|2q}} D\mu(y)\, \frac{(1+e^{-2t}\|y\|^2)^{\lambda-\varrho}}{(1+\|y\|^2)^{\lambda+\varrho}} \chi(1+\|y\|^2) \\ &\quad + \int_{\mathbb{A}^{p|2q}} D\mu(y)\, \frac{(e^{-2t}+\|y\|^2)^{\lambda-\varrho}}{(1+\|y\|^2)^{\lambda+\varrho}} (1-\chi)\left(\frac{1+\|y\|^2}{\|y\|^2}\right),\end{aligned}$$

assuming $D\dot{k}$ to be normalised adequately. Again, $D\mu$ denotes the Lebesgue Berezin density on $\mathbb{A}^{p|2q}$.

Applying Corollary 2.3.4 shows, that ϕ_λ and therefore $c(\lambda)$ only depend on m_α and not on p and q separately in this case, too. Therefore, if $m_\alpha > 0$, the c-function can be taken from [Hel84, Chapter IV, Theorem 6.4].

3.1.11 Lemma. *Let $m_\alpha > 0$. Then*

$$c(\lambda) = c_0 \frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{\lambda+\frac{m_\alpha}{2}+1}{2}\right)\Gamma\left(\frac{\lambda+\frac{m_\alpha}{2}+m_{2\alpha}}{2}\right)}$$

for some constant c_0 and $\operatorname{Re}\lambda > 0$.

Corollary 2.3.5 shows that one needs to differ between even and odd cases for $m_\alpha \leq 0$.

3.1.12 Lemma. *Let $m_\alpha \leq 0$ be even and $p > 0$. Then*

$$c(\lambda) = c_0' \frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)}$$

for some constant c_0' and $\operatorname{Re}\lambda > 0$.

Proof. Corollary 2.3.5 shows

$$\begin{aligned}c(\lambda) =& C\partial_{r=0}^{-\varrho}\left((1+r)^{-(\lambda+\varrho)}\chi(1+r)\right) \\ &+ C\partial_{r=0}^{-\varrho}\left(r^{\lambda-\varrho}(1+r)^{-(\lambda+\varrho)}(1-\chi)\left(\frac{1+r}{r}\right)\right).\end{aligned}$$

for some constant C. The functions $\chi(1+r)$ and $(1-\chi)(\frac{1+r}{r})$ are constant for r near zero. Furthermore, $\partial_{r=0}^k r^{\lambda-\varrho} = 0$ for $k \leq -\varrho$, hence

$$c(\lambda) = C(-\varrho)!\binom{-(\lambda+\varrho)}{-\varrho} = C(-1)^\varrho(-\varrho)!\binom{\lambda-1}{-\varrho} = C(-1)^\varrho\frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)}. \qquad \square$$

3.1.13 Lemma. *Let $m_\alpha < 0$ be odd. Then*
$$c(\lambda) = c_0' \frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)}$$
for some constant c_0' and $\operatorname{Re}\lambda > 0$.

Proof. By Corollary 2.3.5,
$$\begin{aligned} c(\lambda) =& C\int_0^\infty dr\, r^{-\frac12}\partial_r^{-\varrho+\frac12}\left((1+r)^{-(\lambda+\varrho)}\chi(1+r)\right) \\ &+ C\int_0^\infty dr\, r^{-\frac12}\partial_r^{-\varrho+\frac12}\left(r^{\lambda-\varrho}(1+r)^{-(\lambda+\varrho)}(1-\chi)\left(\frac{1+r}{r}\right)\right). \end{aligned} \tag{3.20}$$

Similar to the proof of Equation 3.13, the second integral will be rewritten in order to get rid of χ. This will be done by partial integration. Note, that $(1-\chi)\left(\frac{1+r}{r}\right) = 1$ for small r. Therefore,
$$\begin{aligned} &\lim_{r\to 0}\left(\partial_r^{k-1}r^{-\frac12}\right)\partial_r^{-\varrho+\frac12-k}\left(r^{\lambda-\varrho}(1+r)^{-(\lambda+\varrho)}(1-\chi)\left(\frac{1+r}{r}\right)\right) \\ &\qquad = \lim_{r\to0} r^{\frac12-k}\sum_{l=0}^{-\varrho+\frac12-k} c_l r^{\lambda-\varrho-(-\varrho+\frac12-k-l)}(1+r)^{-(\lambda-\varrho)-l} \\ &\qquad = \lim_{r\to0}\sum_{l=0}^{-\varrho+\frac12-k} c_l r^{\lambda+l}(1+r)^{-(\lambda-\varrho)-l} = 0 \end{aligned}$$
with adequate c_l for $\operatorname{Re}\lambda > 0$ and $1 \le k \le -\varrho + \frac12$. This means that the second integral equals
$$\begin{aligned} &\left(-\varrho+\frac12\right)!\binom{-\varrho}{-\varrho+\frac12}\int_0^\infty dr\, r^{\varrho-1}r^{\lambda-\varrho}(1+r)^{-(\lambda+\varrho)}(1-\chi)\left(\frac{1+r}{r}\right) \\ &\qquad = \left(-\varrho+\frac12\right)!\binom{-\varrho}{-\varrho+\frac12}\int_0^\infty \frac{dr}{r} r^{\lambda}(1+r)^{-(\lambda+\varrho)}(1-\chi)\left(\frac{1+r}{r}\right) \\ &\qquad = \left(-\varrho+\frac12\right)!\binom{-\varrho}{-\varrho+\frac12}\int_0^\infty \frac{ds}{s} s^{-\lambda}\left(1+\frac1s\right)^{-(\lambda+\varrho)}(1-\chi)(s+1) \\ &\qquad = \left(-\varrho+\frac12\right)!\binom{-\varrho}{-\varrho+\frac12}\int_0^\infty ds\, s^{\varrho-1}(1+s)^{-(\lambda+\varrho)}(1-\chi)(1+s). \end{aligned}$$
Here, the substitution $s = r^{-1}$ was applied. Furthermore, partially integrating again yields
$$\begin{aligned} &\int_0^\infty dr\, r^{-\frac12}\partial_r^{-\varrho+\frac12}\left((1+r)^{-(\lambda+\varrho)}(1-\chi)(1+r)\right) \\ &\qquad = \left(-\varrho+\frac12\right)!\binom{-\varrho}{-\varrho+\frac12}\int_0^\infty ds\, s^{\varrho-1}(1+s)^{-(\lambda+\varrho)}(1-\chi)(1+s), \end{aligned}$$

since

$$\lim_{r\to\infty}\left(\partial_r^{k-1}r^{-\frac{1}{2}}\right)\partial_r^{\frac{1}{2}-\varrho-k}\left((1+r)^{-(\lambda+\varrho)}(1-\chi)(1+r)\right)=C'\lim_{r\to\infty}r^{\frac{1}{2}-k}(1+r)^{-\lambda-\frac{1}{2}+k}=0$$

for $1\leq k<-\varrho$. Note that $(1-\chi)(1+r)$ vanishes for small r and equals 1 for big r.

Combining these insights leads to

$$\begin{aligned}c(\lambda)&=C\int_0^\infty dr\,r^{-\frac{1}{2}}\partial_r^{-\varrho+\frac{1}{2}}(1+r)^{-(\lambda+\varrho)}\\&=C\left(-\varrho+\frac{1}{2}\right)!\binom{-(\lambda+\varrho)}{-\varrho+\frac{1}{2}}\int_0^\infty dr\,r^{-\frac{1}{2}}(1+r)^{-(\lambda+\frac{1}{2})}\\&=C(-1)^{-\varrho+\frac{1}{2}}\left(-\varrho+\frac{1}{2}\right)!\binom{\lambda-\frac{1}{2}}{-\varrho+\frac{1}{2}}\frac{\Gamma(\frac{1}{2})\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})}\\&=C(-1)^{-\varrho+\frac{1}{2}}\sqrt{\pi}\frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)}.\end{aligned}$$

□

The case where $p=0$ is still open. This situation is rather interesting, since it is the only one yet where there are only purely odd roots. Fortunately, it is pretty easy to deal with. The underlying spaces of K/M and $\bar{N}$ are both trivial, *i.e.* isomorphic to the terminal object. Therefore, both spaces admit only one retraction, hence they are compatible *via* k. This leads to the following result.

3.1.14 Proposition. *If $p=0$, the spherical function ϕ_λ is given by*

$$\phi_\lambda(e^{th_0})=c_1e^{\lambda t}\sum_{k=0}^{-\varrho}\binom{\lambda-\varrho}{k}\binom{-\lambda-\varrho}{-\varrho-k}e^{(-\varrho-2k)t}$$

for some constant c_1.

Proof.

$$\begin{aligned}\phi_\lambda(e^{th_0})e^{-(\lambda-\varrho)t}&=\int_{K/M}D\dot{k}\,e^{(\lambda-\varrho)(H(e^{th_0}ke^{-th_0}))}\\&=\int_{\bar{N}}D\bar{n}\,e^{(\lambda-\varrho)(H(e^{th_0}\bar{n}e^{-th_0}))}e^{-(\lambda+\varrho)(H(\bar{n}))}\\&=\int_{\mathbb{A}^{0|2q}}D\mu(y)(1+e^{-2t}\|y\|^2)^{\lambda-\varrho}(1+\|y\|^2)^{-(\lambda+\varrho)}\\&=C\partial_{r=0}^{-\varrho}(1+e^{-2t}r)^{\lambda-\varrho}(1+r)^{-(\lambda+\varrho)}\\&=C(-\varrho)!\sum_{k=0}^{-\varrho}\binom{\lambda-\varrho}{k}\binom{-\lambda-\varrho}{-\varrho-k}e^{-2kt}.\end{aligned}$$

□

3.1.15 Corollary. *If $p=0$, the c-function is given by*

$$c(\lambda)=c_0'\frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)}$$

for some constant c_0' and $\operatorname{Re}\lambda>0$.

Since $\Gamma(\lambda+\varrho)=\pi^{-\frac{1}{2}}2^{\lambda+\varrho-1}\Gamma(\frac{\lambda+\varrho}{2})\Gamma(\frac{\lambda+\varrho+1}{2})$, one can merge Lemmas 3.1.11 to 3.1.13 and Corollary 3.1.15.

3.1.16 Theorem. *The c-function for $\mathrm{SOSp}^+_{cs}(1,1+p|2q)$ is given by*

$$c(\lambda)=c_0\frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{\lambda+\frac{m_\alpha}{2}+1}{2}\right)\Gamma\left(\frac{\lambda+\frac{m_\alpha}{2}+m_{2\alpha}}{2}\right)}$$

for some constant c_0 and $\operatorname{Re}\lambda>0$.

The case $\mathrm{GL}_{cs}(1|1)\times\mathrm{GL}_{cs}(1|1)$

Let $\mathfrak{g}_{\mathbb{C}}=\mathfrak{gl}_{\mathbb{C}}(1|1)\times\mathfrak{gl}_{\mathbb{C}}(1|1)$ with the induced Lie super algebra structure. Note that $\mathfrak{g}_{\mathbb{C},\bar{0}}$ is Abelian. The elements of $\mathfrak{g}_{\mathbb{C}}$ can be written as double matrices of the form

$$\left(\begin{array}{cc|cc} A & B & E & F\\ C & D & G & H\end{array}\right).$$

Let $\mathfrak{g}$ be the Lie *cs* algebra given by

$$\mathfrak{g}=\left\{\left(\begin{array}{cc|cc} A & B & E & F\\ C & D & G & H\end{array}\right)\in\mathfrak{g}_{\mathbb{C}}\,\middle|\,E=-\bar{A},\ D\in\mathbb{R},\ H\in\mathbb{R}\right\}$$

and let G be the sub Lie *cs* group of $\mathrm{GL}_{\mathbb{C}}(1|1)\times\mathrm{GL}_{\mathbb{C}}(1|1)$ with underlying space

$$G_0=\left\{\left(\begin{array}{cc|cc} z & 0 & \frac{1}{\bar{z}} & 0\\ 0 & r & 0 & s\end{array}\right)\in\mathfrak{g}_{\mathbb{C}}\,\middle|\,z\in\mathbb{C}\backslash\{0\},\ r,s\in\mathbb{R}_+\right\}$$

and Lie *cs* algebra $\mathfrak{g}$.

An involution ϑ on $\mathfrak{g}$ can be defined by

$$\vartheta\left(\begin{array}{cc|cc} A & B & E & F\\ C & D & G & H\end{array}\right)=\left(\begin{array}{cc|cc} E & F & A & B\\ G & H & C & D\end{array}\right).$$

Therefore,

$$\mathfrak{k}=\left\{x\in\mathfrak{g}\,\middle|\,\left(\begin{array}{cc|cc} A & B & A & B\\ C & D & C & D\end{array}\right)\right\},\quad \mathfrak{p}=\left\{x\in\mathfrak{g}\,\middle|\,x=\left(\begin{array}{cc|cc} A & B & -A & -B\\ C & D & -C & -D\end{array}\right)\right\}.$$

Let K be the sub Lie cs group of G with Lie cs algebra $\mathfrak{k}$ and

$$K_0 = \left\{ \left(\begin{array}{cc|cc} z & 0 & z & 0 \\ 0 & r & 0 & r \end{array} \right) \in \mathfrak{g}_{\mathbb{C}} \,\middle|\, |z| = 1,\ r \in \mathbb{R}_+ \right\}.$$

There is no other choice for the even Cartan subalgebra than $\mathfrak{a} = \mathfrak{p}_{\bar{0}}$, since $\mathfrak{p}_{\bar{0}}$ is Abelian. Let

$$h_1 = \left(\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad h_2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{array} \right)$$

be a basis of $\mathfrak{a}$ and suppose $\alpha \in \mathfrak{a}_{\mathbb{C}}^*$ to be a restricted root. Then

$$\begin{aligned} \alpha(a_1h_1 + a_2h_2) \left(\begin{array}{cc|cc} A & B & E & F \\ C & D & G & H \end{array} \right) &= \alpha(a_1h_1 + a_2h_2) \left(\begin{array}{cc|cc} A & B & E & F \\ C & D & G & H \end{array} \right) \\ &= \left[\left(\begin{array}{cc|cc} a_1 & 0 & -a_1 & 0 \\ 0 & a_2 & 0 & -a_2 \end{array} \right), \left(\begin{array}{cc|cc} A & B & E & F \\ C & D & G & H \end{array} \right) \right] \\ &= (a_1 - a_2) \left(\begin{array}{cc|cc} 0 & B & 0 & -F \\ -C & 0 & G & 0 \end{array} \right). \end{aligned}$$

This means that there are only two possible roots: α and $-\alpha$ with

$$\alpha(a_1h_1 + a_2h_2) = a_1 - a_2.$$

The corresponding root spaces are

$$\begin{aligned} \mathfrak{g}_\alpha &= \left\{ \left(\begin{array}{cc|cc} 0 & B & 0 & 0 \\ 0 & 0 & G & 0 \end{array} \right) \,\middle|\, B, G \in \mathbb{C} \right\}, \\ \mathfrak{g}_{-\alpha} &= \left\{ \left(\begin{array}{cc|cc} 0 & 0 & 0 & F \\ C & 0 & 0 & 0 \end{array} \right) \,\middle|\, C, F \in \mathbb{C} \right\}, \\ \mathfrak{m} = \mathfrak{k} &= \left\{ \left(\begin{array}{cc|cc} iA & 0 & iA & 0 \\ 0 & D & 0 & D \end{array} \right) \,\middle|\, A, D \in \mathbb{R} \right\}. \end{aligned}$$

In the following, suppose that α is the positive root, hence $\mathfrak{n} = \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \mathfrak{g}_{-\alpha}$, $m_\alpha = -1$ and therefore $\varrho = -\alpha$. The Lie cs group G admits an Iwasawa decomposition, since this is the case for G_0. Each element in G_0 can be decomposed as

$$\begin{aligned} k_0 \left(\begin{array}{cc|cc} z & 0 & \frac{1}{\bar{z}} & 0 \\ 0 & r & 0 & s \end{array} \right) &= \left(\begin{array}{cc|cc} \frac{z}{|z|} & 0 & \frac{z}{|z|} & 0 \\ 0 & \sqrt{rs} & 0 & \sqrt{rs} \end{array} \right), \\ H_0 \left(\begin{array}{cc|cc} z & 0 & \frac{1}{\bar{z}} & 0 \\ 0 & r & 0 & s \end{array} \right) &= \left(\begin{array}{cc|cc} \log|z| & 0 & -\log|z| & 0 \\ 0 & \frac{1}{2}\log(\frac{r}{s}) & 0 & -\frac{1}{2}\log(\frac{r}{s}) \end{array} \right). \end{aligned}$$

Note that $N_0 = \{e\}$.

The S-points of N and $\bar{N}$ are of the form

$$n_{B,G} = \left(\begin{array}{cc|cc} 1 & B & 1 & 0 \\ 0 & 1 & G & 1 \end{array}\right), \quad \bar{n}_{C,F} = \left(\begin{array}{cc|cc} 1 & 0 & 1 & F \\ C & 1 & 0 & 1 \end{array}\right),$$

with $B, C, F, G \in \Gamma(\mathcal{O}_S)_{\bar{1}}$.

Again, $H \colon \bar{N} \longrightarrow \mathfrak{a}$ will be derived using the equation

$$\vartheta(\bar{n}_{C,F})^{-1}\bar{n}_{C,F} = \vartheta(n_{B,G})^{-1}\exp(2H(\bar{n}_{C,F}))n_{B,G}, \qquad \bar{n}_{C,F} \in_S \bar{N}.$$

The left hand side equals

$$\left(\begin{array}{cc|cc} 1 & -F & 1 & 0 \\ 0 & 1 & -C & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & 0 & 1 & F \\ C & 1 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1-FC & -F & 1 & F \\ C & 1 & -C & 1-CF \end{array}\right),$$

whereas the right hand side yields

$$\begin{aligned}
&\left(\begin{array}{cc|cc} 1 & 0 & 1 & -B \\ -G & 1 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} e^{2t_1} & 0 & e^{-2t_1} & 0 \\ 0 & e^{2t_2} & 0 & e^{-2t_2} \end{array}\right)\left(\begin{array}{cc|cc} 1 & B & 1 & 0 \\ 0 & 1 & G & 1 \end{array}\right) \\
&\qquad = \left(\begin{array}{cc|cc} 1 & 0 & 1 & -B \\ -G & 1 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} e^{2t_1} & e^{2t_1}B & e^{-2t_1} & 0 \\ 0 & e^{2t_2} & e^{-2t_2}G & e^{-2t_2} \end{array}\right) \\
&\qquad = \left(\begin{array}{cc|cc} e^{2t_1} & Be^{2t_1} & e^{-2t_1} - BGe^{-2t_2} & -Be^{-2t_2} \\ -Ge^{2t_1} & e^{2t_2} - GBe^{2t_1} & Ge^{-2t_2} & e^{-2t_2} \end{array}\right),
\end{aligned}$$

with $H(\bar{n}_{C,F}) = h_1 t_1 + h_2 t_2$ for $t_1, t_2 \in \Gamma(\mathcal{O}_S)_{\bar{0}}$. Therefore, $e^{2t_1} = 1 - FC$ and furthermore $e^{-2t_2} = 1 - CF = 1 + FC$, hence $t_1 = t_2 = -\frac{1}{2}FC$. This shows

$$H(\bar{n}_{C,F}) = -\frac{1}{2}FCh_0 = \frac{1}{2}CFh_0$$

for $\bar{n}_{C,F} \in_S \bar{N}$, with $h_0 := h_1 + h_2$.

The coordinates ξ_1, ξ_2 on $\bar{N}$, given by

$$\xi_1(\bar{n}_{C,F}) = C, \quad \xi_2(\bar{n}_{C,F}) = F,$$

satisfy $D\bar{n} = |D\xi|$, provided the normalisation of $D\bar{n}$ is suitable. The spaces K/M and $\bar{N}$ are of purely odd dimension, hence both admit only one retraction. These retractions have to be compatible under k. Therefore,

$$\phi_\lambda(e^h) = \int_{K/M} D\dot{k}\, e^{(\lambda-\varrho)(e^h k)} = e^{(\lambda-\varrho)(h)} \int_{\bar{N}} D\bar{n}\, e^{(\lambda-\rho)(H(e^h \bar{n} e^{-h}))} e^{-(\lambda+\varrho)(H(\bar{n}))} \tag{3.21}$$

for $h \in \mathfrak{a}$.

Note that $e^h \bar{n}_{C,F} e^{-h} = \bar{n}_{e^{-\alpha(h)}C, e^{-\alpha(h)}F}$, since

$$\left(\begin{array}{cc|cc} e^{t_1} & 0 & e^{-t_1} & 0 \\ 0 & e^{t_2} & 0 & e^{-t_2} \end{array}\right)\left(\begin{array}{cc|cc} 1 & 0 & 1 & F \\ C & 1 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} e^{-t_1} & 0 & e^{t_1} & 0 \\ 0 & e^{-t_2} & 0 & e^{t_2} \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & 1 & e^{t_2-t_1}F \\ e^{t_2-t_1}C & 1 & 0 & 1 \end{array}\right).$$

This leads to the following proposition.

3.1.17 Proposition.

$$\phi_\lambda(e^h) = c_0 \lambda(h_0) e^{\lambda(h)} \sinh \alpha(h)$$

for some constant c_0 and any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

PROOF. Since the root is even, $\varrho(h_0) = -\alpha(h_0) = 0$. Furthermore,

$$e^{(\lambda-\varrho)(H(\bar{n}_{C,F}))} = e^{\lambda(H(\bar{n}_{C,F}))} = e^{\frac{1}{2}\lambda(h_0)CF} = 1 + \frac{1}{2}\lambda(h_0)CF,$$

hence, by Equation (3.21),

$$\begin{aligned} \phi_\lambda(e^h) &= e^{(\lambda-\varrho)(h)} \int_{\bar{N}} D\xi \left(1 + \frac{1}{2}\lambda(h_0)e^{-2\alpha(h)}\xi_1\xi_1\right)\left(1 - \frac{1}{2}\lambda(h_0)\xi_1\xi_2\right) \\ &= Ce^{(\lambda-\varrho)(h)}\lambda(h_0)(e^{-2\alpha(h)} - 1). \end{aligned}$$ □

From this proposition, it can directly be seen that

3.1.18 Corollary. *The c-function for G is given by*

$$c(\lambda) = c_0 \lambda(h_0)$$

for some constant c_0 and any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

3.2. Harish-Chandra's Spherical Function Expansion

In the following, suppose $G = \mathrm{SOSp}^+_{cs}(1, 1+p|2q)$. An expansion for ϕ_λ in the case $p = 0$ was easily obtained in Proposition 3.1.14. This section aims to derive such an expansion for general p. The standard procedure to do so is to solve a differential equation (*cf.* [Hel84, Chapter IV, §5]).

Again, let $h_0 \in \mathfrak{a}$ be the element with $\alpha(h_0) = 1$ and identify $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\lambda(h_0)$ where it is convenient.

3.2.1 Proposition. *Spherical super functions are eigenfunctions of the differential operator $\Delta(L)$ on A:*

$$\Delta(L)\phi_\lambda = (\lambda^2 - \varrho^2)\phi_\lambda. \tag{3.22}$$

Here, $\Delta(L)$ is given by

$$\Delta(L)(f)(e^{th_0}) := (\partial_t^2 + m_\alpha \coth(t)\partial_t) f(e^{th_0}).$$

For this proposition one needs to check that the differential operator $\Delta(L)$ is the radial part of the Laplace operator L on G/K (*cf.* [GV88, §4.2]). The claim then follows by calculating the eigenvalue of ϕ_λ for L (*cf.* [All12]). Moreover, it can be seen that the ϕ_λ exhaust the set of joint eigenfunctions of all K-invariant differential operators. However, for the application in mind this is not necessary.

The function $t \mapsto e^{(\lambda-\varrho)t}$ satisfies $(\partial_t^2 + m_\alpha\partial_t)e^{(\lambda-\varrho)t} = (\lambda^2 - \varrho^2)e^{(\lambda-\varrho)t}$. Therefore, it is reasonable to to make a perturbation ansatz. Let Φ_λ be a function on A^+ solving Equation (3.22), with

$$\Phi_\lambda(e^{th_0}) = e^{(\lambda-\varrho)t}\sum_{l=0}^{\infty}\gamma_l(\lambda)e^{-2lt}. \tag{3.23}$$

3.2.2 Lemma. *The coefficients γ_l from Equation* (3.23) *are given by*

$$\gamma_l(\lambda) = \gamma_0(\lambda)\prod_{m=0}^{l-1}\frac{(m+\varrho)(m+\varrho-\lambda)}{(m+1)(m+1-\lambda)} \tag{3.24}$$

$$= \gamma_0(\lambda)c(-\lambda)(-1)^l\binom{-\varrho}{l}\frac{-\lambda}{(l-\lambda)c(l-\lambda)}. \tag{3.25}$$

for $\lambda \notin \mathbb{N}$. Moreover, the series which defines Φ_λ converges absolutely on each interval $[\varepsilon,\infty[$ for $\varepsilon > 0$ and Φ_λ is an eigenfunction of $\Delta(L)$.

Proof. With

$$\begin{aligned}\partial_t^2 + m_\alpha\coth(t)\partial_t &= (\partial_t^2 + m_\alpha\partial_t) + m_\alpha(\coth(t) - 1)\partial_t \\ &= (\partial_t^2 + m_\alpha\partial_t) + 2m_\alpha\sum_{k=1}^{\infty}e^{-2kt}\partial_t,\end{aligned}$$

applying $\Delta(L) - (\lambda^2 - \varrho^2)$ to $\frac{1}{4}\Phi_\lambda$ yields

$$\begin{aligned}0 &= \sum_{l=0}^{\infty}\gamma_l(\lambda)\left(l(l-\lambda)e^{(\lambda-\varrho-2l)t} + \sum_{k=1}^{\infty}(\lambda-\varrho-2l)\varrho e^{(\lambda-\varrho-2l-2k)t}\right) \\ &= \sum_{l=1}^{\infty}e^{(\lambda-\varrho-2l)t}\left(l(l-\lambda)\gamma_l(\lambda) + \sum_{m=1}^{l}(\lambda-\varrho-2(l-m))\varrho\gamma_{l-m}(\lambda)\right),\end{aligned}$$

hence

$$\begin{aligned}
l(l-\lambda)\gamma_l(\lambda) &= \sum_{m=0}^{l-1}(-\lambda+\varrho+2m)\varrho\gamma_m(\lambda) \\
&= (-\lambda+\varrho+2l-2)\varrho\gamma_{l-1}(\lambda)+(l-1)(l-1-\lambda)\gamma_{l-1}(\lambda) \\
&= \big((-\lambda+\varrho+l-1)\varrho+(l-1)\varrho-(l-1)\varrho+(l-1)(l-1-\lambda+\varrho)\big)\gamma_{l-1}(\lambda) \\
&= (l-1+\varrho-\lambda)(l-1+\varrho)\gamma_{l-1}(\lambda)
\end{aligned}$$

for $l>0$. This shows

$$\begin{aligned}
\gamma_l(\lambda) &= \gamma_0(\lambda)\prod_{m=0}^{l-1}\frac{(m+\varrho)(m+\varrho-\lambda)}{(m+1)(m+1-\lambda)} \\
&= \gamma_0(\lambda)(-1)^l\binom{-\varrho}{l}\frac{\Gamma(l+\varrho-\lambda)\Gamma(1-\lambda)}{\Gamma(l+1-\lambda)\Gamma(\varrho-\lambda)} \\
&= \gamma_0(\lambda)(-1)^l\binom{-\varrho}{l}c(-\lambda)\frac{-\lambda}{(l-\lambda)c(l-\lambda)}.
\end{aligned}$$

Since

$$\lim_{l\to\infty}\left|\frac{\gamma_l(\lambda)}{\gamma_{l+1}(\lambda)}\right| = \lim_{l\to\infty}\frac{(l+1)|l+1-\lambda|}{|\varrho+l||l+\varrho-\lambda|}=1,$$

the series for ϕ_λ converges absolutely on $[\varepsilon,\infty[$. Therefore, derivation and infinite sum can be interchanged, proving that ϕ_λ indeed is an eigenfunction. □

From now on, $\gamma_0(\lambda)$ will be chosen to equal $c(\lambda)$.

3.2.3 Corollary. *If $m_\alpha \le 0$ is even, the series terminates and*

$$\Phi_\lambda(e^{th_0}) = c_1 e^{\lambda t}\sum_{l=0}^{-\varrho}\binom{\lambda-\varrho}{l}\binom{-\lambda-\varrho}{-\varrho-l}e^{(-\varrho-2l)t}.$$

for some constant c_1. In particular, ϕ_λ is well-defined on A. If $p=0$, this means that $\Phi_\lambda=\phi_\lambda$.

Proof. Clearly, $\gamma_l(\lambda)=0$ for $l>-\varrho$. Recall that

$$c(\lambda)=\frac{c_0 2^{\varrho-1}}{\sqrt{\pi}}\frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)}=\frac{c_0 2^{\varrho-1}}{\sqrt{\pi}}\prod_{k=1}^{-\varrho}(\lambda-k),$$

hence

$$\begin{aligned}\gamma_k(\lambda) &= (-1)^l \binom{-\varrho}{l} c(\lambda) \prod_{m=0}^{l-1} \frac{m+\varrho-\lambda}{m+1-\lambda} \\ &= \frac{c_0 2^{\varrho-1}}{\sqrt{\pi}} (-1)^l \binom{-\varrho}{l} \prod_{k=l+1}^{-\varrho} (\lambda-k) \prod_{m=0}^{l-1} (\lambda-\varrho-m) \\ &= \frac{c_0(-2)^\varrho(-\varrho)!}{2\sqrt{\pi}} \binom{-\lambda-\varrho}{-\varrho-l} \binom{\lambda-\varrho}{l}. \end{aligned}$$

□

3.2.4 Proposition. *If $p \neq 0$ and $\lambda \notin \frac{1}{2}\mathbb{Z}$, the spherical functions are given by*

$$\phi_\lambda = \Phi_\lambda + \Phi_{-\lambda}$$

on A^+.

Proof. Both Φ_λ and $\Phi_{-\lambda}$ are solutions for the differential equation $\Delta(L)f = (\lambda^2 - \varrho^2)f$. Since they are obviously linearly independent, there are constants $b_1, b_2 \in \mathbb{C}$ such that $\phi_\lambda = b_1\Phi_\lambda + b_2\Phi_{-\lambda}$. Let $w \in M_0'$ be a representative of the non-trivial Weyl group element. Then for any $a \in A^+$

$$\phi_\lambda(a) = \phi_{-\lambda}(a^{-1}) = \phi_{-\lambda}(waw^{-1}) = \phi_{-\lambda}(a) = b_1\Phi_{-\lambda}(a) + b_2\Phi_\lambda(a)$$

due to the K-bi-invariance of ϕ_λ and Corollary 3.1.3, hence $b_1 = b_2$. Now, $b_1 = 1$ follows from

$$b_1 c(\lambda) = \lim_{t\to 0} \Phi_\lambda(e^{th_0})e^{-(\lambda-\varrho)t} = \lim_{t\to 0} \phi_\lambda(e^{th_0})e^{-(\lambda-\varrho)t} = c(\lambda).$$

□

The following lemma will be important in Chapter 4.

3.2.5 Lemma. *The residues of $\frac{\Phi_\lambda}{c(\lambda)c(-\lambda)}$ are located at $\lambda \neq 0$ with $\lambda = \varrho + k$ for $k \in \mathbb{N}_0$. They are given by*

$$\operatorname{Res}_{\lambda=\varrho+k} \frac{\Phi_\lambda(e^{th_0})}{c(\lambda)c(-\lambda)} = \frac{(-1)^k(\varrho+k)}{c_0\Gamma(1-\varrho)} \sum_{l=0}^{k} \binom{-\varrho}{l}\binom{-\varrho}{k-l} e^{(k-2l)t} \tag{3.26}$$

$$= \frac{\varrho+k}{c_0\Gamma(1-\varrho)k!} \partial_{y=0}^k \left(1 - 2y\cosh t + y^2\right)^{-\varrho}, \tag{3.27}$$

where c_0 is the constant from Theorem 3.1.16.

Proof. Since

$$\frac{\gamma_l(\lambda)}{c(\lambda)c(-\lambda)} = (-1)^l \binom{-\varrho}{l} \frac{-\lambda}{(l-\lambda)c(l-\lambda)} = (-1)^l \binom{-\varrho}{l} \frac{-\lambda\Gamma(l+\varrho-\lambda)}{c_0\Gamma(l+1-\lambda)},$$

it is clear that the residues of Φ_λ are located at $\varrho + \mathbb{N}_0$. Any residue at $\lambda = 0$ is eliminated by the single λ in the equation. The residue of $\Gamma(l + \varrho - \lambda)$ at $\lambda = \varrho + k$ is $\frac{(-1)^{l+k+1}}{(k-l)!}$ for $l \leq k$ and 0 otherwise. Therefore,

$$\mathrm{Res}_{\lambda=\varrho+k} \frac{\gamma_l(\lambda)}{c(\lambda)c(-\lambda)} = \frac{(-1)^k(\varrho+k)}{c_0\Gamma(1-\varrho)} \binom{-\varrho}{l} \binom{-\varrho}{k-l}$$

for $l \leq k$ and $\mathrm{Res}_{\lambda=\varrho+k} \frac{\gamma_l(\lambda)}{c(\lambda)c(-\lambda)} = 0$ for $l > k$. This proves Equation (3.26).

Equation (3.27) can be obtained by considering the following exponential series:

$$\begin{aligned}\sum_{k=0}^{\infty} y^k \sum_{l=0}^{k} \binom{-\varrho}{l} \binom{-\varrho}{k-l} e^{(k-2l)t} &= \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} y^{m+l} \binom{-\varrho}{l} \binom{-\varrho}{m} \left(e^t\right)^{m-l} \\ &= \left(1+ye^t\right)^{-\varrho} \left(1+ye^{-t}\right)^{-\varrho} = \left(1+2y\cosh t + y^2\right)^{-\varrho}\end{aligned}$$

for y small enough. This proves the claim. □

3.3. An Estimate for Spherical Super Functions

In this section, an estimate for the growth behaviour of spherical super functions on $\mathrm{SOSp}^+_{cs}(1, 1+p|2q)$ will be given. This estimate is necessary in the proof of the Fourier inversion formula. It will generalise the following classical result.

3.3.1 Proposition (Chapter IV, Theorem 8.1)**.** *The spherical function ϕ_λ is bounded if and only if $\lambda \in i\mathfrak{a}^* + [-\varrho, \varrho]$.*

Since spherical super functions depend on m_α, but not on the dimension, this proposition is also true in the super case, as long as $m_\alpha > 0$. However, if $m_\alpha < 0$ this does not make sense. As one can see by Corollary 3.2.3 and Proposition 3.2.4, spherical super functions are not bounded at all at least for even $m_\alpha < 0$.

The following proposition gives an analogon to Proposition 3.3.1 in the case $m_\alpha \leq 0$.

3.3.2 Proposition. *Let $p > 0$ and $\varrho \leq 0$. For $\delta > 0$ and $K \geq 0$ there exists a constant C such that*

$$|\partial_t^k \phi_\lambda(e^{th_0})| \leq C(1+|\lambda|)^{-\varrho+k} \cosh(\mathrm{Re}\,\lambda t) \cosh^{-\varrho} t \tag{3.28}$$

for $k \leq -\varrho$, whenever $\mathrm{Re}\,\lambda \in [-K, K]$ and $(\forall n \in \mathbb{Z}_{\neq 0}) : |\lambda - n| > \delta$.

3.3.3 Lemma. *Let $\delta, K \geq 0$ and $\varrho \leq 0$. Then there exists a constant $C_0 > 0$ such that*

$$|\Phi_\lambda(e^h)| \leq C_0 |\gamma_0(\lambda)| e^{(\mathrm{Re}\,\lambda - \varrho)(h)} \tag{3.29}$$

for $h \in \mathfrak{a}^+$ and all λ with $\mathrm{Re}\,\lambda \leq K$ and $(\forall n \in \mathbb{N}) : |\lambda - n| \geq \delta$.

Proof. First,

$$|\gamma_l(\lambda)| \le C|\gamma_0(\lambda)| \left|\binom{-\varrho}{l}\right|$$

for all l. For this let $l \ge K+1-\varrho$ and choose $C>0$ such that above estimate is true for all smaller l. Note that this requires the minimum distance δ of λ from all natural numbers. The estimate

$$|l+\varrho-\operatorname{Re}\lambda| = l+\varrho-\operatorname{Re}\lambda \le l+1-\operatorname{Re}\lambda$$

yields

$$|l+\varrho-\lambda| \le |l+1-\lambda|,$$

hence

$$|\gamma_l(\lambda)| = |\gamma_{l-1}(\lambda)|\frac{l-1+\varrho}{l}\frac{|l+\varrho-\lambda|}{|l+1-\lambda|} \le C|\gamma_0(\lambda)| \left|\binom{-\varrho}{l}\right|$$

by induction. Applying this to Φ_λ gives

$$\begin{aligned}
|\Phi_\lambda(e^{th_0})| &\le e^{(\operatorname{Re}\lambda-\varrho)t}\sum_{l=0}^{\infty}|\gamma_l(\lambda)|e^{-2lt} \\
&\le C|\gamma_0(\lambda)|e^{(\operatorname{Re}\lambda-\varrho)t}\sum_{l=0}^{\infty}\left|\binom{-\varrho}{l}\right|e^{-2lt} \\
&\le C|\gamma_0(\lambda)|e^{(\operatorname{Re}\lambda-\varrho)t}\left(\sum_{l=0}^{\lceil-\varrho\rceil}\left(\left|\binom{-\varrho}{l}\right| \mp \binom{-\varrho}{l}\right)e^{-2lt} \pm \sum_{l=0}^{\infty}\binom{-\varrho}{l}(-1)^l e^{-2lt}\right) \\
&\le C_0|\gamma_0(\lambda)|e^{(\operatorname{Re}\lambda-\varrho)t}. \qquad (3.30)
\end{aligned}$$

Note that the series in the third line equals $(1-e^{-2t})^{-\varrho} \le 1$. □

3.3.4 Lemma. *Under the assumptions of Lemma 3.3.3*

$$|\partial_t^k\Phi_\lambda(e^{th_0})| \le C_k|\gamma_0(\lambda)|(1+|\lambda|)^k e^{(\operatorname{Re}\lambda-\varrho)t} \qquad (3.31)$$

for $k \le -\varrho$ and some constant C_k, which depends on δ and K.

Proof. Since

$$\begin{aligned}
|\partial_t^k e^{(\lambda-\varrho-2l)t}| &= |\lambda-\varrho-2l|^k e^{(\operatorname{Re}\lambda-\varrho-2l)t} \\
&\le (1+|\lambda|+2l-\varrho-1)^k e^{(\operatorname{Re}\lambda-\varrho-2l)t} \\
&\le C_k'(1+|\lambda|)^k l^k e^{(\operatorname{Re}\lambda-\varrho-2l)t},
\end{aligned}$$

the series $\sum_{l=0}^{\infty}|\gamma_l(\lambda)|l^k e^{-2lt}$ needs to be estimated. This can be performed by adapting the estimate from (3.30), using an upper bound of $\partial_t^k(1-e^{-2t})^{-\varrho}$. □

3.3.5 Remark. Lemma 3.3.3 and Lemma 3.3.4 remain true for $\varrho > 0$ and $k > -\varrho$. However, in these cases one has to assume $t > \varepsilon > 0$ to get Equations (3.29) and (3.31) with an ε-dependent constant.

If $m_\alpha \leq 0$ is even, the constants in Equations (3.29) and (3.31) become independent of K and δ. This is due to the fact that the series Φ_λ terminates by Corollary 3.2.3.

3.3.6 Lemma. *For fixed ϱ let c be as in Theorem 3.1.16. For each $\delta, K > 0$ there exists a constant C_δ such that*

$$\frac{1}{|c(\lambda)|} \leq C_\delta (1 + |\lambda|)^\varrho$$

for all λ with $\operatorname{Re}\lambda \geq -K$ and $(\forall n \in \mathbb{N}_0) : |\lambda + \varrho + n| > \delta$. If m_α is even, the estimate holds for all λ with $|\lambda - n| \geq \delta$ for $n = 1, \dots, -\varrho$.

Similarly,

$$|c(\lambda)| \leq C_\delta (1 + |\lambda|)^{-\varrho}$$

for all λ with $\operatorname{Re}\lambda \geq -K$ and $(\forall n \in \mathbb{N}_0) : |\lambda + n| > \delta$.

Proof. Recall that $c(\lambda) = c_0' \frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)}$ for some constant c_0'. The assertion follows immediately, since the c-function has only finitely many zeros with $\operatorname{Re}\lambda \geq -K$ and

$$\lim_{|\lambda| \to \infty} \frac{\Gamma(\lambda)\lambda^\varrho}{\Gamma(\lambda + \varrho)} = 1$$

for $\arg\lambda \in \,]-\pi, \pi[$. $\square$

Proof of Proposition 3.3.2. First Lemmas 3.3.4 and 3.3.6 can be combined to get (3.28) for $t > 0$, $\operatorname{Re}\lambda > 0$ by using $\frac{1}{2}e^t \leq \cosh t \leq e^t$ and $e^{\lambda t} \leq 2\cosh(\lambda t)$. The assertion then follows by using symmetry and continuity of $\phi_\lambda(e^{th_0})$ in λ and t. $\square$

3.3.7 Corollary. *For $\delta > 0$, $\varrho \leq 0$ and $K \in \mathbb{R}$, there exists a constant C such that*

$$\left| \frac{\partial_t^k \Phi_\lambda(e^{th_0})}{c(\lambda)c(-\lambda)} \right| \leq C(1 + |\lambda|)^{\varrho + k} e^{(\operatorname{Re}\lambda - \varrho)t}$$

for $t > 0$ and $k \leq -\varrho$, whenever $\operatorname{Re}\lambda \leq K$ and $(\forall n \in \mathbb{N}_0 \backslash \{-\varrho\}) : |\lambda + \varrho + n| > \delta$.

Proof. This can be obtained from Equation (3.25) as follows.

$$\frac{1}{c(l - \lambda)} \leq C_\delta (1 + |l - \lambda|)^\varrho \leq C_\delta (1 + |\lambda|)^\varrho$$

for $l > 2K$, since $|l - \lambda|^2 = l(l - 2\operatorname{Re}\lambda) + |\lambda|^2 \geq |\lambda|^2$. An estimate like the one in Equation (3.30) concludes the proof. $\square$

3.3.8 Corollary. *For $\delta > 0$, $\varrho \leq 0$ and $K \in \mathbb{R}$, there exists a constant C such that*

$$\left| \frac{\partial_t^k \phi_\lambda(e^{th_0})}{c(\lambda)c(-\lambda)} \right| \leq C(1 + |\lambda|)^{\varrho + k} \cosh(\operatorname{Re} \lambda t) \cosh^{-\varrho} t.$$

for $k \leq -\varrho$, whenever $\operatorname{Re} \lambda \in [-K, K]$ and $(\forall n \in \mathbb{Z} \backslash \{-\varrho\}) : |\lambda + \varrho + n| > \delta$.

4. The Fourier Transform

This chapter aims to generalise the Fourier transform on symmetric spaces, as described in [Hel84], to an example in the setting of *cs* spaces. Suppose G admits an Iwasawa decomposition KAN. Define for $f \in \Gamma_{cf}(\mathcal{O}_{G/K})$ a super function $\hat{f} = \mathcal{F}f \in \Gamma(\mathcal{O}_{\mathfrak{a}^*_{\mathbb{C}} \times K/M})$ *via*

$$\mathcal{F}f(\lambda, k) := \int_{G/K} D\dot{g}\, f(g) e^{(\lambda-\varrho)(H(g^{-1}k))}.$$

The super function $\mathcal{F}f$ is said to be the *Fourier transform* of f. This super function is right M-invariant, since this is the case for H due to Equation (2.9). The *inverse Fourier transform* $\mathcal{J}\varphi \in \Gamma(\mathcal{O}_{G/K})$ of a super function $\varphi \in \Gamma(\mathcal{O}_{\mathfrak{a}^*_{\mathbb{C}} \times K/M})$ is defined by

$$\mathcal{J}\varphi(g) := \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \int_{K/M} D\dot{k}\, \varphi(\lambda, k) e^{(-\lambda-\varrho)(H(g^{-1}k))},$$

provided this iterative integral exists. Here, c denotes the c-function for G. The super function $\mathcal{J}\varphi$ is indeed right K-invariant due to the left K-invariance of H.

In the classical setting $\mathcal{F}$ and $\mathcal{J}$ are indeed inverse to each other. However, in the super setting this is not the case any more. Here, a correction needs to be added. In the following let $G = \mathrm{SOSp}^+_{cs}(1, 1+p|2q)$, as in the second part of Section 3.1.

4.1. The Spherical Transform

The Fourier transform of a left K-invariant $f \in \Gamma_{cf}(\mathcal{O}_{G/K})$ does not depend on K. This is due to the left invariance of $D\dot{g}$. Vice versa, the inverse Fourier transform of $\varphi \in \Gamma(\mathcal{O}_{\mathfrak{a}^*_{\mathbb{C}}})$ is left K-invariant if it exists. In particular,

$$\mathcal{J}\varphi(g) = \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \phi_\lambda(g)\varphi(\lambda).$$

In the following, a few insights on the inverse Fourier transform will be obtained in this setting.

4.1.1 Definition. Let

$$\|\varphi\|_{k,R} := \sup_{\lambda \in \mathfrak{a}^*_{\mathbb{C}}} (1+|\lambda|)^k |\varphi(\lambda)| e^{-R|\operatorname{Re}\lambda|}$$

for $R > 0$, $k \in \mathbb{N}_0$ and $\varphi \in \Gamma(\mathcal{O}_{\mathfrak{a}^*_{\mathbb{C}}})$ and

$$\mathrm{PW}_R \coloneqq \left\{\varphi \in \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}) \,\middle|\, (\forall w \in W)\,(\forall \lambda \in \mathfrak{a}^*_{\mathbb{C}}) : \varphi(\lambda) = \varphi(w\lambda),\ (\forall k \in \mathbb{N}_0) : \|\varphi\|_{k,R} < \infty\right\}.$$

Here, $\mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}})$ denotes the set of holomorphic functions on $\mathfrak{a}^*_{\mathbb{C}}$. Moreover, the condition $\varphi(\lambda) = \varphi(w\lambda)$ is equivalent to $\varphi(\lambda) = \varphi(-\lambda)$ for $p > 0$. If $p = 0$, this is not the case, since the Weyl group is trivial.

4.1.2 Lemma. *The function $\mathcal{J}\varphi(a)$ is invariant under $a \mapsto a^{-1}$ on A if $p > 0$. Moreover,*

$$\mathcal{J}\varphi(a) = |W| \int_{i\mathfrak{a}^*_{\mathbb{R}}} \frac{d\lambda}{|c(\lambda)|^2} \Phi_\lambda(a)\varphi(\lambda)$$

for $a \in A^+$. Here, $|W|$ denotes the order of the Weyl group.

Proof. Using the invariance of $d\lambda$ under $\lambda \mapsto -\lambda$ yields

$$\begin{aligned}\mathcal{J}\varphi(a) &= \int_{i\mathfrak{a}^*_{\mathbb{R}}} \frac{d\lambda}{|c(\lambda)|^2} \phi_\lambda(a)\varphi(\lambda) = \int_{i\mathfrak{a}^*_{\mathbb{R}}} \frac{d\lambda}{c(\lambda)c(-\lambda)} \phi_{-\lambda}(a^{-1})\varphi(\lambda) \\ &= \int_{i\mathfrak{a}^*_{\mathbb{R}}} \frac{d\lambda}{|c(\lambda)|^2} \phi_\lambda(a^{-1})\varphi(\lambda) = \mathcal{J}\varphi(a^{-1}).\end{aligned}$$

The second assertion follows the same way from $\phi_\lambda = \Phi_\lambda + \Phi_{-\lambda}$ and $\varphi(\lambda) = \varphi(-\lambda)$ by

$$\int_{i\mathfrak{a}^*_{\mathbb{R}}} \frac{d\lambda}{|c(\lambda)|^2} \Phi_{-\lambda}(a)\varphi(\lambda) = \int_{i\mathfrak{a}^*_{\mathbb{R}}} \frac{d\lambda}{|c(\lambda)|^2} \Phi_\lambda(a)\varphi(\lambda).$$ □

4.1.3 Proposition. *If $\varrho \leq 0$ and $\|\varphi\|_{n,R} < \infty$ for some n and R, then*

$$\partial_t^k \mathcal{J}\varphi(e^{th_0}) = \int_{i\mathfrak{a}^*_{\mathbb{R}}} \frac{d\lambda}{|c(\lambda)|^2} \partial_t^k \phi_\lambda(e^{th_0})\varphi(\lambda)$$

for $k < -\varrho + n - 1$. In particular, $\mathcal{J}1$ exists if $\varrho < 0$. Moreover, $\mathcal{J}\varphi$ is smooth on A if $\varphi \in PW_R$.

Proof. This follows directly from Proposition 3.3.1 and Corollary 3.3.8. □

The statement about the smoothness is of course also true if $\varrho > 0$ due to Proposition 3.3.1. The interesting part of this proposition is that it shows that the inverse Fourier transform of the constant function $\mathcal{J}1$ is continuous if $\varrho < -1$. This insight will play an important role in the proof of the Fourier inversion formula.

4.1.4 Proposition. *Let $\|\varphi\|_{n,R} < \infty$ for some $n > \varrho$, $R \geq 0$. Then*

$$\mathcal{J}\varphi(e^{th_0}) = 2\pi|W| \sum_{\substack{k \in \mathbb{N}_0 \\ k < -\varrho}} \mathrm{Res}_{\lambda = \varrho + k} \frac{\Phi_\lambda(e^{th_0})}{c(\lambda)c(-\lambda)} \varphi(\varrho + k) \tag{4.1}$$

for $t > R$.

PROOF. Let $\delta > 0$ and C as in Corollary 3.3.7. For $K > |\varrho|$ let β_K be the path given by $\beta(s) = iKe^{is}$, $s \in [0, \pi]$. Then

$$i \int_{-iK}^{iK} \frac{\Phi_\lambda(e^{th_0})}{c(\lambda)c(-\lambda)} \varphi(\lambda) d\lambda + \int_{\beta_K} \frac{\Phi_\lambda(e^{th_0})}{c(\lambda)c(-\lambda)} \varphi(\lambda) d\lambda$$
$$= 2\pi i \sum_{k<-\varrho} \varphi(\varrho + k) \operatorname{Res}_{\lambda=\varrho+k} \frac{\Phi_\lambda(e^{th_0})}{c(\lambda)c(-\lambda)}$$

by Lemma 3.2.5. Moreover, Corollary 3.3.7 yields

$$\begin{aligned} \left| \int_{\beta_K} \frac{\Phi_\lambda(e^{th_0})}{c(\lambda)c(-\lambda)} \varphi(\lambda) d\lambda \right| &\leq C \int_{\beta_K} (1 + |\lambda|)^{\varrho - n} e^{\operatorname{Re} \lambda (t-R) - \varrho t} d|\lambda| \\ &= C(1+K)^{\varrho-n} e^{-\varrho t} 2K \int_0^{\frac{\pi}{2}} e^{-(t-R)K \sin s} ds \\ &\leq C(1+K)^{\varrho-n} e^{-\varrho t} 2K \int_0^{\frac{\pi}{2}} e^{-(t-R)K \frac{s}{2}} ds \\ &= C(1+K)^{\varrho-n} e^{-\varrho t} \frac{4}{t-R} \left(1 - e^{-(t-R)K\frac{\pi}{4}}\right) \\ &\leq C(1+K)^{\varrho-n} e^{-\varrho t} \frac{4}{t-R}. \end{aligned}$$

This vanishes for $K \to \infty$, hence the integral over β_K also vanishes for $K \to \infty$. □

4.1.5 Definition. Let

$$\tilde{\mathcal{J}}\varphi(a) := \mathcal{J}\varphi(a) - 2\pi|W| \sum_{k<-\varrho} \operatorname{Res}_{\lambda=\varrho+k} \frac{\Phi_\lambda(a)}{c(\lambda)c(-\lambda)} \varphi(\varrho + k)$$

for $a \in A$. If $p > 0$, the function $\tilde{\mathcal{J}}\varphi$ is invariant under $a \mapsto a^{-1}$ by Lemma 4.1.2 and Equation (3.27).

The following corollary is the first step towards a Payley-Wiener theorem.

4.1.6 Corollary. *If $p > 0$, $\tilde{\mathcal{J}}$ maps the space PW_R to the space $\mathcal{C}_R^\infty(A)$ of smooth functions on A with support inside* $\exp([-R, R]h_0)$.

4.1.7 Lemma. *If ϱ is negative, $\mathcal{J}1$ exists on $A \setminus \{0\}$ and is given by*

$$\mathcal{J}1(e^{th_0}) = -\frac{\pi|W|}{c_0\Gamma(1-\varrho)\Gamma(-2\varrho)} \partial_{y=0}^{-2\varrho-1} \frac{(1 - 2y^2 \cosh t + y^4)^{-\varrho}}{(1-y)^2}$$

with the constant c_0 from Theorem 3.1.16. In particular, $\mathcal{J}1$ has a smooth extension in $e \in A$ which will also be denoted $\mathcal{J}1$.

If m_α is odd,

$$\mathcal{J}1(e^{th_0}) = -\frac{\sqrt{\pi}|W|2^{\frac{1}{2}-\varrho}}{c_0\Gamma(\frac{1}{2}-\varrho)}(1-\cosh t)^{-\varrho-\frac{1}{2}}.$$

Proof. Note that in the case $m_\alpha = -1$, the existence of $\mathcal{J}1$ needs to be checked. However, this rather easy, since the integrand behaves like $\frac{\sin(\operatorname{Im}\lambda)}{\operatorname{Im}\lambda}$ for big $|\operatorname{Im}\lambda|$.

By Proposition 4.1.4 and Lemma 3.2.5,

$$\begin{aligned}
\mathcal{J}1(e^{th_0}) &= \frac{2\pi|W|}{c_0\Gamma(1-\varrho)}\sum_{k<-\varrho}\frac{\varrho+k}{k!}\partial_{y=0}^k\left(1-2y\cosh t+y^2\right)^{-\varrho}\\
&= \frac{\pi|W|}{c_0\Gamma(1-\varrho)}\sum_{k<-\varrho}\frac{2(\varrho+k)}{(2k)!}\partial_{y=0}^{2k}\left(1-2y^2\cosh t+y^4\right)^{-\varrho}\\
&= -\frac{\pi|W|}{c_0\Gamma(1-\varrho)}\sum_{k=0}^{-2\varrho-1}\frac{-2\varrho-k}{k!}\partial_{y=0}^k\left(1-2y^2\cosh t+y^4\right)^{-\varrho}
\end{aligned}$$

since $\partial_{y=0}^{2k}f(y^2) = \frac{(2k)!}{k!}\partial_{y=0}^k f(y)$ (*cf.* Equation (2.2)) and $\partial_{y=0}^{2k+1}f(y^2) = 0$. Furthermore,

$$\frac{1}{(-2\varrho-k-1)!}\partial_{y=0}^{-2\varrho-k-1}(1-y)^{-2} = -2\varrho-k,$$

which proves the first claim.

Now, let m_α be odd. The operator $\partial_{y=0}^{-2\varrho-1}$ vanishes on anti-symmetric functions, since $-2\varrho-1$ is even. Therefore, decomposing $(1-y)^{-2} = \frac{1+y^2}{(1-y^2)^2} + \frac{2y}{(1-y^2)^2}$ yields

$$\begin{aligned}
\mathcal{J}1(e^{th_0}) &= -\frac{\pi|W|}{c_0\Gamma(1-\varrho)\Gamma(-2\varrho)}\partial_{y=0}^{-2\varrho-1}\frac{(1+y^2)\left(1-2y^2\cosh t+y^4\right)^{-\varrho}}{(1-y^2)^2}\\
&= -\frac{\pi|W|}{c_0\Gamma(1-\varrho)\Gamma(-\varrho+\frac{1}{2})}\partial_{y=0}^{-\varrho-\frac{1}{2}}\frac{(1+y)\left(1-2y\cosh t+y^2\right)^{-\varrho}}{(1-y)^2}
\end{aligned}$$

Note, that

$$\begin{aligned}
&\frac{\partial_{x=1}^k}{k!}\frac{\partial_{y=0}^{-\varrho-\frac{1}{2}}}{(-\varrho-\frac{1}{2})!}\frac{1-y}{(1+y)^2}(1+2yx+y^2)^{-\varrho}\\
&= 2^k\binom{-\varrho}{k}\frac{\partial_{y=0}^{-\varrho-\frac{1}{2}}}{(-\varrho-\frac{1}{2})!}y^k(1-y)(1+y)^{-2\varrho-2k-2}\\
&= 2^k\binom{-\varrho}{k}\left(\frac{\partial_{y=0}^{-\varrho-\frac{1}{2}-k}}{(-\varrho-\frac{1}{2}-k)!}-\frac{\partial_{y=0}^{-\varrho-\frac{1}{2}-k-1}}{(-\varrho-\frac{1}{2}-k-1)!}\right)(1+y)^{-2\varrho-2k-2}
\end{aligned}$$

This vanishes for all $k \neq -\varrho - \frac{1}{2}$. For $k = -\varrho - \frac{1}{2}$ it equals $2^{-\varrho-\frac{1}{2}}\binom{-\varrho}{-\varrho-\frac{1}{2}}$. Therefore,

$$\begin{aligned}\mathcal{J}1(e^{th_0}) &= -\frac{\pi|W|}{c_0\Gamma(1-\varrho)}(-1)^{-\varrho-\frac{1}{2}}2^{-\varrho-\frac{1}{2}}\binom{-\varrho}{-\varrho-\frac{1}{2}}(\cosh t - 1)^{-\varrho-\frac{1}{2}} \\ &= -\frac{\sqrt{\pi}|W|2^{\frac{1}{2}-\varrho}}{c_0\Gamma(\frac{1}{2}-\varrho)}(1-\cosh t)^{-\varrho-\frac{1}{2}}.\end{aligned}$$ □

4.1.8 Lemma. *Let m_α be odd. Then*

$$\partial^k_{r=0}\mathcal{J}1(e^{\tanh^{-1}(r)h_0}) = \partial^k_{t=0}\mathcal{J}1(e^{th_0}) = \begin{cases}(-1)^{\frac{1}{2}-\varrho}\frac{\sqrt{\pi}|W|2(-2\varrho-1)!}{c_0\Gamma(\frac{1}{2}-\varrho)} & k = -2\varrho-1, \\ 0 & k < -2\varrho-1.\end{cases}$$

Proof. Faà di Bruno's formula shows

$$\partial^n_{t=0}(\cosh t - 1)^{-\varrho-\frac{1}{2}} = \sum_{k_1+2k_2+\ldots+nk_n=n} c_k 0^{k_1}(2!)^{-k_2}0^{k_3}(4!)^{-k_4}\cdots.$$

with $c_k := \binom{-\varrho-\frac{1}{2}}{k_1+\ldots+k_n}\frac{(k_1+\ldots+k_n)!n!}{k_1!\cdots k_n!}0^{-\varrho-\frac{1}{2}-k_1-\ldots-k_n}$. The only non-vanishing summand for $n \leq -2\varrho - 1$ is the one with $k_2 = -\varrho - \frac{1}{2}$, which is only possible for $n = -2\varrho - 1$. This proves the assertion for $\mathcal{J}1(e^{th_0})$. For $\mathcal{J}1(e^{\tanh^{-1}(r)h_0})$, the claim follows again by applying Faà di Bruno's formula. □

4.2. The Inversion Formula

This section aims to prove an Fourier inversion formula for $G = \mathrm{SOSp}^+_{cs}(1, 1+p|2q)$.

4.2.1 Definition. Let $f_1 \in \Gamma(\mathcal{O}_{S\times G/K})$, $f_2 \in \Gamma(\mathcal{O}_{T\times G/K})$ such that one of both is compactly supported along fibres and f_2 is K-bi-invariant. The *convolution* of such super functions $f_1 * f_2 \in \Gamma(\mathcal{O}_{S\times T\times G/K})$ is defined via

$$(f_1 * f_2)(s,t,h) := \int_{G/K} D\dot{g}\, f_1(s,hg)f_2(t,g^{-1}) = \int_{G/K} D\dot{g}\, f_1(s,g)f_2(t,g^{-1}h)$$

for generalised points (s,t,h) of $S \times T \times G$.

In the following, let $\mathcal{J}1$ be the K-bi-invariant super function from Lemma 4.1.7 for $\varrho < 0$. In case $\varrho \geq 0$, set $\mathcal{J}1 := 0$.

4.2.2 Theorem. *Let m_α be odd or non-negative and $f \in \Gamma_{cf}(\mathcal{O}_{S\times G/K})$. Then*

$$\mathcal{J}\mathcal{F}f = C_0 f + (f * \mathcal{J}1)$$

for a constant C_0 which does not depend on f.

4.2.3 Lemma. *It suffices to prove Theorem 4.2.2 for $e \in G_0$ and $S = *$.*

PROOF. Let $F \in \Gamma_{cf}(\mathcal{O}_{S\times G\times G/K})$ be defined by $F(s,g,h) := f(s,gh)$ for generalised points $(s,g,h) \in_T S \times G \times G$. Then $(F * \mathcal{J}1)(s,g,h) = (f * \mathcal{J}1)(s,gh)$ by definition. Moreover,

$$\begin{aligned}\mathcal{J}\mathcal{F}f(s,h) &= \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \int_{K/M} D\dot{k}\, e^{(-\lambda-\varrho)(H(h^{-1}k))} \int_{G/K} D\dot{g}\, e^{(\lambda-\varrho)(H(g^{-1}k))} f(g) \\ &= \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \int_{G/K} D\dot{g}\, f(g) \int_{K/M} D\dot{k}\, e^{(\lambda-\varrho)(H(g^{-1}k))} e^{(-\lambda-\varrho)(H(h^{-1}k))}. \\ &= \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \int_{G/K} D\dot{g}\, f(g)\phi_\lambda(g^{-1}h)\end{aligned}$$

by Lemma 3.1.2. Therefore,

$$\mathcal{J}\mathcal{F}f(s,h) = \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} (f * \phi_\lambda)(s,h) = \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} (F * \phi_\lambda)(s,h,e) = \mathcal{J}\mathcal{F}F(s,h,e),$$

showing that suffices to assume $h = e$.

The claim for S follows, since integration is continuous on $\Gamma_{cf}(\mathcal{O}_{S\times G/K})$ and

$$\mathcal{J}\mathcal{F}(f_1 \otimes f_2) = f_1 \otimes (\mathcal{J}\mathcal{F}f_2)$$

for $f_1 \otimes f_2 \in \Gamma_c(\mathcal{O}_S) \otimes \Gamma_c(\mathcal{O}_{G/K})$. $\square$

In the following, it will be necessary to identify G/K with the unit ball in $\mathbb{A}^{p+1|2q}$ to use the formulas from Section 2.3. In order to do so, a few considerations need to be made.

4.2.4 Lemma. *The S-points of $G = \mathrm{SOSp}^+_{cs}(1, 1+p|2q)$ are precisely those generalised points $g \in_S \mathrm{GL}_\mathbb{C}(2+p|2q)$ which satisfy*

$$g^{\mathrm{s}\top} J g = J, \tag{4.2}$$

such that g_0 has values in $G_0 = \mathrm{SO}^+_\mathbb{R}(1,1+p) \times \mathrm{USp}(2q)$. Here, $\left(\begin{smallmatrix} R & S \\ T & V \end{smallmatrix}\right)^{\mathrm{s}\top} = \left(\begin{smallmatrix} R^\top & T^\top \\ -S^\top & V^\top \end{smallmatrix}\right)$ and J is the matrix from Equation (3.17)*.*

PROOF. Consider the non-degenerate super symmetric bilinear form b on $V = \mathbb{C}^{p+2|2q}$, given by $b(v,w) := v^{\mathrm{s}\top^3} Jw$. Such a bilinear form can always be identified with the element

$$v_b := \sum_{i,j} e_i^* \otimes b(e_i, e_j) e_j^* \in S^2(V^*)_{\bar{0}}.$$

Here, $S^2(V^*)$ denotes the vector space of symmetric 2-tensors in V^* and $(e_i)_i$ is a graded basis of V^* with dual basis $(e_i^*)_i$. This definition is independent from the choice of a certain basis. Moreover, the actions of $\mathrm{GL}_{\mathbb{C}}(2+p|2q)$ on $\mathbb{A}^{\mathbb{C}}\big(S^2(V^*)\big)$ and the space of bilinear forms, given by $g.(v^*\otimes w^*) \coloneqq (v^*\circ g^{-1})\otimes(w^*\circ g^{-1})$ and $(g.b)(v,w) \coloneqq b(gv,gw)$, are compatible in the sense that $g.v_b = v_{gb}$.

Let the complex Lie super group $H \coloneqq \mathrm{OSp}_{\mathbb{C}}(1,1+p|2q) \coloneqq G_{v_b}$ be the stabiliser of v_b. Its generalised points are precisely the ones which leave b invariant, hence those which satisfy Equation (4.2). At this point it should be noted that $\mathbb{A}^{\mathbb{C}}\big((\cdot)^{\mathrm{s}\top^3}\big) = (\cdot)^{\mathrm{s}\top}$ as linear maps. The corresponding complex Lie super algebra is the subset of all $x\in\mathfrak{gl}_{\mathbb{C}}(2+p|2q)$ such that $b(xv,w) = -(-1)^{|v||x|}b(v,xw)$, hence $\mathfrak{osp}_{\mathbb{C}}(1,1+p|2q)$.

Since $H_0 = \mathrm{O}_{\mathbb{C}}(1,1+p)\times\mathrm{Sp}_{\mathbb{C}}(2q)$, it is clear that the complexification of G is the identity component of H. Now, Proposition 2.4.16 concludes the proof. ☐

4.2.5 Remark. Similarly, one can obtain that the S-points of $K = \mathrm{SOSp}_{cs}(1+p|2q)$ are those of G which are of the form $\left(\begin{smallmatrix}1&0\\0&k\end{smallmatrix}\right)$.

4.2.6 Lemma. *Let $\mathbb{B}^{1+p|2q} \coloneqq \mathbb{A}^{p+1|2q}|_{\mathbb{B}^{1+p}}$ be the unit ball in $\mathbb{A}^{p+1|2q}$. Then*

$$\alpha\colon G\times\mathbb{B}^{1+p|2q}\longrightarrow\mathbb{B}^{1+p|2q},\quad \left(\begin{pmatrix}A&B\\C&D\end{pmatrix},x\right)\longmapsto\frac{Dx+C}{Bx+A}$$

defines a transitive action. Here, the notation of the matrices is in accordance with the involution, ignoring the grading. Moreover, the stabiliser of $o\coloneqq 0\in\mathbb{B}^{1+p}$ is K. In particular, $G/K\cong\mathbb{B}^{1+p|2q}$.

Proof. The compatibility of α with the group structure is clear. Transitivity is clear on the underlying spaces. The tangent map of α_o is surjective, since

$$T_e\alpha_o\left(\begin{pmatrix}0&B\\C&0\end{pmatrix}\right) = C$$

for $\left(\begin{smallmatrix}0&B\\C&0\end{smallmatrix}\right)\in\mathfrak{g}$. Let $g = \left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in_S G$, such that $g.o = o$. Then

$$0 = g.o = \frac{C}{A},$$

hence $C=0$. Equation (4.2) then implies $B=0$ and $A=1$, thus $g\in_S K$. The converse is clear. ☐

4.2.7 Lemma. *If $p>0$, the action*

$$\alpha\colon K\times S^{p|2q}\longrightarrow S^{p|2q},\quad \left(\begin{pmatrix}1&0\\0&k\end{pmatrix},s\right)\longmapsto ks$$

is transitive and therefore $K/M\cong S^{p|2q}$.

PROOF. First, this map is indeed an action on $S^{p|2q}$, since K leaves $\|\cdot\|$ invariant on $\mathbb{A}^{p+1|2q}$. Since $S^p = \mathrm{SO}_{\mathbb{R}}(1+p)/\mathrm{SO}_{\mathbb{R}}(p)$, the action is transitive on the underlying spaces. Considering the point $e_1 \in S^p$, the transitivity of the action is easily shown. The stabiliser of e_1 is $M = \mathrm{SOSp}_{cs}(p|2q)$, which proves the claim. □

4.2.8 Lemma. *The following diagram commutes.*

$$\begin{tikzcd} A^+ \times K \arrow[r, "(e^{th_0},k)\mapsto(\tanh(t),ke_1)"] \arrow[d, "(a,k)\mapsto ka"'] & \mathbb{A}^1|_{]0,1[} \times S^{p|2q} \arrow[d, "(r,s)\mapsto rs"] \\ G \arrow[r, "g\mapsto g.o"] & \mathbb{B}^{1+p|2q} \end{tikzcd}$$

In the next part, the Berezin densities $D\lambda$ and DS from Section 2.3 will be used.

4.2.9 Lemma. *The Berezin density $D\lambda(1-\|\cdot\|^2)^{-1-\varrho}$ on $\mathbb{B}^{1+p|2q}$ is left G-invariant. Similarly, the Berezin density DS on $S^{p|2q}$ is left K-invariant.*

PROOF. The Lebesgue Berezin density $D\lambda$ is invariant under K, since $\mathrm{Ber}\, k = 1$ for all $k \in_S K$. The same is true for $\|\cdot\|$, which defines $S^{p|2q}$. Therefore, $DS = D\lambda\,|_{S^{p|2q},\|\cdot\|}$ is also invariant under the action of K. In the same manner, $D\lambda(1-\|\cdot\|)^{-1-\varrho}$ is invariant under K. Let x denote the standard coordinate system on $\mathbb{B}^{1+p|2q}$. Then

$$\mathrm{Ber}\left(\frac{\partial e^{th_0}x}{\partial x}\right) = (x_1\sinh t + \cosh t)^{-2-p+2q} = \left(\frac{1-\|e^{th_0}x\|^2}{1-\|x\|^2}\right)^{1+\varrho}$$

shows that this Berezin density is also invariant under A. □

In the following assume that $D\dot{g}$ and $D\dot{k}$ are normalised such that they are compatible with $D\lambda$ and DS

4.2.10 Corollary. *Let $f \in \Gamma_{cf}(\mathcal{O}_{S\times G/K})$. Then for $p > 0$*

$$\int_{G/K} D\dot{g}\, f(s,g) = \begin{cases} \int_0^1 dr\, r^{2\varrho}(1-r^2)^{-1-\varrho}\int_{K/M} D\dot{k}\, f(s,ka_r) & m_\alpha \in \mathbb{N}_0, \\ C\int_0^1 dr\, r^{-\frac{1}{2}}\partial_r^{-\varrho}\left((1-r)^{-1-\varrho}\int_{K/M} D\dot{k}\, f(s,ka_{\sqrt{r}})\right) & \frac{m_\alpha}{2} \in -\mathbb{N}, \\ \int_0^1 dr\, r^{2\varrho}(1-r^2)^{-1-\varrho}\int_{K/M} D\dot{k}\, f(s,ka_r) & \\ \quad + \frac{\partial_{r=0}^{-2\varrho-1}}{(-2\varrho-1)!}(1-r^2)^{-1-\varrho}\int_{K/M} D\dot{k}\log(\|(ke_1)_{\bar{0}}\|)f(s,ka_r) & \textit{else}, \end{cases}$$

with $a_r := e^{\tanh^{-1}(r)h_0}$ and $C = \frac{\Gamma(\varrho+\frac{1}{2})(-1)^\varrho}{2\pi}$.

For $p = 0$ the formula takes the form

$$\int_{G/K} D\dot{g}\, f(s,g) = C\int_0^1 dr\, r^{-\frac{1}{2}}\partial_r^{-\varrho}\left((1-r)^{-1-\varrho}\int_{K/M} D\dot{k}\, f(s,ka_{\sqrt{r}}) + f(s,a_{-\sqrt{r}})\right).$$

Proof. The claim follows directly from Propositions 2.3.9, 2.3.10 and 2.3.13. The different form of the formula for $p = 0$ is due to the fact that $S^{0|2q}$ consists of two super points, whereas K/M consists of only one. □

Proof of Theorem 4.2.2 for $m_\alpha \geq 0$.

$$\int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \int_{G/K} D\dot{g}\, f(g)\phi_{-\lambda}(g) = \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \int_0^1 dr \left(r^{2\varrho}(1-r^2)^{-1-\varrho} f^\circ(r)\phi_{-\lambda}(a_r) \right)$$

with $f^\circ(r) \coloneqq \int_{K/M} D\dot{k}\, f(s, ka_r)$. The right hand side of the equation above only depends on m_α for any function f°. Therefore, one may assume $q = 0$ and obtain from the classical result (*cf.* [Hel84, Chapter IV, Theorem 7.5])

$$\int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \int_{G/K} D\dot{g}\, f(g)\phi_{-\lambda}(g) = C f^\circ(0) = C_0 f(e)$$

for constants C and C_0. □

4.2.11 Lemma. *Let $m_\alpha < 0$ be odd. Then $\partial^k_{t=0}\phi_\lambda(e^{th_0}) = 0$ for all $k \leq -2\varrho$. Similarly, $\partial^k_{r=0}\phi_\lambda(a_r) = 0$ for all $k \leq -2\varrho$.*

Proof. Since $\phi_\lambda(e) = \int_{K/M} D\dot{k} = \mathrm{Vol}(S^{p|2q}) = 0$, the assertions are trivial if $k = 0$ or $\lambda = \pm\varrho$. In the case of odd k, the claims follow from the invariance of $\phi_\lambda(a)$ under $a \mapsto a^{-1}$.

Write $\coth t = \frac{1}{t} + f(t)$, where f is analytic at 0. Then

$$\lim_{t\to 0} \left((\partial^l_t \coth t) g(t) \right) = \lim_{t\to 0} (-1)^l l! \frac{g(t)}{t^{l+1}} + (\partial^l_{t=0} f(t)) g(0) = \frac{(-1)^l}{l+1} \partial^{l+1}_{t=0} g(t)$$

for any function g with $\partial^m_{t=0} g(t) = 0$ for $m \leq l$.

Now, suppose the first assertion to be true for all $k \leq 2n+1$ with $n < -\varrho - 1$. Then

$$\begin{aligned}
0 &= \partial^{2n}_{t=0}(\lambda^2 - \varrho^2)\phi_\lambda(e^{th_0}) = \partial^{2n}_{t=0}(\Delta(L)(\phi_\lambda))(e^{th_0}) \\
&= \partial^{2(n+1)}_{t=0}\phi_\lambda(e^{th_0}) + 2\varrho \lim_{t\to 0} \sum_{l=0}^{2n} \binom{2n}{l} (\partial^l_t \coth t)\, \partial^{2n+1-l}_t \phi_\lambda(e^{th_0}) \\
&= \partial^{2(n+1)}_{t=0}\phi_\lambda(e^{th_0}) + 2\varrho \sum_{l=0}^{2n} \binom{2n}{l} \frac{(-1)^l}{l+1} \partial^{2n+2}_{t=0}\phi_\lambda(e^{th_0}) \\
&= \left(1 + \frac{2\varrho}{2n+1}\right) \partial^{2(n+1)}_{t=0}\phi_\lambda(e^{th_0}) = \frac{2n+1+2\varrho}{2n+1} \partial^{2(n+1)}_{t=0}\phi_\lambda(e^{th_0}),
\end{aligned}$$

hence $\partial^{2(k+1)}_{t=0}\phi_\lambda(e^{th_0}) = 0$. In the last line, the identity $\sum_{l=0}^{2n} \binom{2n}{l} \frac{(-1)^l}{l+1} x^{l+1} = \frac{1-(1-x)^{2n+1}}{2n+1}$ was used.

By applying Faà di Bruno's formula on $\phi_\lambda(a_r) = \phi_\lambda(e^{\tanh^{-1}(r)h_0})$, the second claim follows immediately. □

Proof of Theorem 4.2.2 for $m_\alpha < 0$ odd. By Corollary 4.2.10

$$\begin{aligned}
\mathcal{J}\mathcal{F}f(e) &= \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \int_{G/K} D\dot{g}\, f(g)\phi_{-\lambda}(g) \\
&= \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \int_0^1 dr \left(r^{2\varrho}(1-r^2)^{-1-\varrho}\phi_{-\lambda}(a_r) \int_{K/M} D\dot{k}\, f(ka_r) \right) \\
&+ \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \frac{\partial_{r=0}^{-2\varrho-1}}{(-2\varrho-1)!} \left((1-r^2)^{-1-\varrho}\phi_{-\lambda}(a_r) \int_{K/M} D\dot{k} \log(\|(ke_1)_{\bar{0}}\|) f(ka_r) \right).
\end{aligned}$$

Thanks to Lemma 4.2.11, the second summand vanishes after applying the product rule. Therefore,

$$\mathcal{J}\mathcal{F}f(e) = \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \int_0^1 dr \left(r^{2\varrho}(1-r^2)^{-1-\varrho}\phi_{-\lambda}(a_r) \int_{K/M} D\dot{k}\, f(ka_r) \right).$$

On the other hand,

$$\begin{aligned}
(f * \mathcal{J}1)(e) &= \int_{G/K} D\dot{g}\, f(g)\mathcal{J}1(g^{-1}) \\
&= \int_0^1 dr \left(r^{2\varrho}(1-r^2)^{-1-\varrho} \int_{i\mathfrak{a}^*} \frac{d\lambda}{|c(\lambda)|^2} \phi_\lambda(a_r) \int_{K/M} D\dot{k}\, f(ka_r) \right) \\
&\quad + \frac{\partial_{r=0}^{-2\varrho-1}}{(-2\varrho-1)!} \left((1-r^2)^{-1-\varrho}\mathcal{J}1(a_r) \int_{K/M} D\dot{k} \log(\|(ke_1)_{\bar{0}}\|) f(ka_r) \right).
\end{aligned}$$

Here, the integrals over r and λ in the second line can be interchanged, since the integrand can be estimated adequately due to Corollary 3.3.8. Moreover, $\partial_r^k \mathcal{J}1(a_r)$ vanishes for $k < -2\varrho - 1$ by Lemma 4.1.8. Therefore, Corollary 2.3.12 shows

$$\begin{aligned}
(f * \mathcal{J}1)(e) &= \mathcal{J}\mathcal{F}f(e) + \frac{\partial_{r=0}^{-2\varrho-1}\mathcal{J}1(a_r)}{(-2\varrho-1)!} \int_{K/M} D\dot{k} \log(\|(ke_1)_{\bar{0}}\|) f(ke). \\
&= \mathcal{J}\mathcal{F}f(e) - \frac{4}{c_0}\pi^{1+\varrho} f(e).
\end{aligned}$$

□

4.2.12 Remark. The idea to this proof generalises concepts of [Zir91a], where the case of $m_\alpha = -1$ was covered.

4.2.13 Remark (The first part of the proof of Theorem 4.2.2 for $m_\alpha < 0$ even)**.** Define

$$f^\circ(r) := (1-r^2)^{-1-\varrho} \int_{K/M} D\dot{k}\, f(ka_r)$$

for $r \in]-1,1[$. Let T be the distribution on $]-1,1[$ given by

$$Tg := \int_{i\mathfrak{a}^*} \frac{d\lambda}{c(\lambda)c(-\lambda)} \int_0^1 dr\, r^{-\frac{1}{2}} \partial_r^{-\varrho} \phi_\lambda(a_{\sqrt{r}}) g(\sqrt{r}) - \int_0^1 dr\, r^{-\frac{1}{2}} \partial_r^{-\varrho} g(\sqrt{r}) \mathcal{J}1(a_{\sqrt{r}}).$$

Since $Tf^\circ = C^{-1}\mathcal{J}\mathcal{F}f(e) - C^{-1}(f * \mathcal{J}1)(e)$ for $p > 0$ by Corollary 4.2.10, it suffices to check $Tg = C_0 g(0)$ for some constant C_0

The operator T satisfies $\operatorname{supp} T \subseteq \{0\}$. This is easy to check, since for any compactly supported g with $0 \notin \operatorname{supp} g$:

$$\begin{aligned}
&\int_{i\mathfrak{a}^*} \frac{d\lambda}{c(\lambda)c(-\lambda)} \int_0^1 dr\, r^{-\frac{1}{2}} \partial_r^{-\varrho} \phi_\lambda(a_{\sqrt{r}}) g(\sqrt{r}) \\
&\qquad = (-1)^\varrho (-\varrho)! \binom{-\frac{1}{2}}{-\varrho} \int_{i\mathfrak{a}^*} \frac{d\lambda}{c(\lambda)c(-\lambda)} \int_0^1 dr\, r^{-\frac{1}{2}-\varrho} \phi_\lambda(a_{\sqrt{r}}) g(\sqrt{r}) \\
&\qquad = (-1)^\varrho 2(-\varrho)! \binom{-\frac{1}{2}}{-\varrho} \int_{i\mathfrak{a}^*} \frac{d\lambda}{c(\lambda)c(-\lambda)} \int_0^1 dr\, r^{-2\varrho} \phi_\lambda(a_r) g(r) \\
&\qquad = (-1)^\varrho 2(-\varrho)! \binom{-\frac{1}{2}}{-\varrho} \int_0^1 dr\, r^{-2\varrho} \int_{i\mathfrak{a}^*} \frac{d\lambda}{c(\lambda)c(-\lambda)} \phi_\lambda(a_r) g(r) \\
&\qquad = \int_0^1 dr\, r^{-\frac{1}{2}} \partial_r^{-\varrho} g(\sqrt{r}) \mathcal{J}1(a_{\sqrt{r}})
\end{aligned}$$

Here, the integrals over r an λ can be interchanged due to Corollary 3.3.8. Therefore $Tg = 0$.

In order to finish this proof, one needs to show $|Tg| \le \sup_{r \in \mathbb{R}^+} |g(r)|$. For this, the canonical procedure would be to show

$$Tg = \int_{i\mathfrak{a}^* - M} \frac{d\lambda}{c(\lambda)c(-\lambda)} \int_0^1 dr\, r^{-\frac{1}{2}} \partial_r^{-\varrho} \phi_\lambda(a_{\sqrt{r}}) g(\sqrt{r})$$

for some $M > -\varrho$. This can be done, since the integration over r commutes with taking residues of the integrand.

The idea is to estimate $|Tg|$ by using

$$\left| \frac{\partial_r^k \phi_\lambda(a_{\sqrt{r}})}{c(\lambda)c(-\lambda)} \right| \le C_k (1 + |\lambda|)^{\varrho + 2k} \tag{4.3}$$

for $|t| \le R$ and $|\operatorname{Re} \lambda| \le M$, which follows from Corollary 3.3.8 by Faà di Bruno's formula. However, this leads nowhere. In fact, this idea would only work if ϱ in Equation (4.3) was replaced by 2ϱ. Unfortunately this is not possible, even in the case $m_\alpha = -2$.

So far, there is no idea to overcome this obstacle. The fact that ϕ_λ is given by a finite sum of exponential functions might be useful for a solution. One can show

$$\phi_\lambda(e^{th_0}) = c_1 e^{(\lambda - \varrho)t} P_{-\varrho}^{(-\lambda, 2\varrho - 1)}(1 - 2e^{-2t}),$$

where the $P_n^{(\alpha,\beta)}$ denote the Jacobi polynomials.

5. Outlook

As it was pointed out in the introduction, a general formula for the c-function has already been derived in [AS13]. Obtaining a series expansion for spherical super functions in higher rank cases is then straightforward. Therefore, the main obstacle in proving a general Fourier inversion formula is the determination of a polar integration formula for the dense open embedding $K/M \times A^+ \to G/K$ from Proposition 2.4.32. During the preparation of this thesis, an idea to solve this problem arose. This idea shall be outlined in the following.

In Section 2.3, the development of a polar integration formula was only possible due to the linearity of the standard retraction on $\mathbb{A}^{p|2q}$. Therefore, it is reasonable to try to obtain a retraction on G/K that is linear in some sense. Recall the decomposition $G \cong K \times \mathfrak{p}$ from Proposition 2.4.18. The vector space $\mathfrak{p}$ admits a transitive action of G, given by $g.x = \log_G\left(ge^x\theta(g^{-1})\right)$ with the Cartan involution θ on G. This is indeed well-defined, hence G/K may be identified with $\mathfrak{p}$. Under this identification, the morphism $K/M \times \mathfrak{a}^+ \to G/K = \mathfrak{p}$ takes the form

$$(\dot{k}, h) \longmapsto 2\,\mathrm{Ad}(k)(h).$$

Let γ be the retraction on $K/M \times A^+$ which is compatible with the canonical retraction on $\mathfrak{p}$ under this isomorphism. The canonical retraction is linear. Since Ad is linear in the second component, the same is true for γ, considered as an retraction on $K/M \times \mathfrak{a}^+$.

For brevity, assume $\mathfrak{g}$ to have only even roots and enumerate the simple roots as $\alpha_1, \ldots, \alpha_k$. Then $\rho_i(kM_0, h) \coloneqq \alpha_i(h)$ for $i = 1, \ldots, k$ defines a family of boundary functions on $K_0/M_0 \times \mathfrak{a}^+$. Let $h_1, \ldots, h_k$ be the basis of $\mathfrak{a}$ which is dual to the simple roots. Then $\rho_i\big(\gamma(k,h)\big) = \alpha_i(h)\rho_i\big(\gamma(k,h_i)\big)$ implies with Remark 2.2.23 and Proposition 2.5.13 that

$$\begin{aligned}\int_{G/K} D\dot{g}\, f(g) = \lim_{\varepsilon\to 0}\Bigg(&\int_{\mathbb{A}^k_{>\varepsilon}} d\lambda(t) \int_{K/M} D\dot{k}\, \delta(e^{th_0}) f(ke^{th_0}) \\ &+ \sum_{l=1}^{k} \sum_{\sigma\in S_k} \sum_{j\in\mathbb{N}^l} (-1)^l \frac{1}{j!} \partial^{j\downarrow}_{t_{\sigma\le l}=\varepsilon_{\sigma\le l}} \int_{\mathbb{A}^{k-l}_{>\varepsilon_{\sigma>l}}} d\lambda(t_{\sigma>l}) \int_{K/M} D\dot{k}\, n^j_{\sigma\le l} t^j_{\sigma\le l} \delta(e^{th_0}) f(ke^{th_0})\Bigg).\end{aligned} \tag{5.1}$$

Here, the restriction of $\mathbb{A}^{k-l}$ to $]\varepsilon_{\sigma^{-1}(1)}, \infty[\times\cdots\times]\varepsilon_{\sigma^{-1}(l)}, \infty[$ is denoted $\mathbb{A}^{k-l}_{>\varepsilon_{\sigma>l}}$ for $\varepsilon \in \mathbb{R}^k_+$. Furthermore, $th_0 \coloneqq t_1h_1 + \ldots + t_kh_k$ and

$$n^j_{\sigma\le l} \coloneqq \Big(1 - \rho_{\sigma^{-1}(1)} \circ \gamma\big(k, h_{\sigma^{-1}(1)}\big)\Big)^{j_1} \cdots \Big(1 - \rho_{\sigma^{-1}(l)} \circ \gamma\big(k, h_{\sigma^{-1}(l)}\big)\Big)^{j_l}.$$

If δ has no singularities at the boundary of A^+ (which is for example the case if the multiplicities of all roots are non-negative), the second line of Equation (5.1) vanishes for $\varepsilon \to 0$, and one obtains the formula

$$\int_{G/K} D\dot{g}\, f(g) = \int_{A^+} da \int_{K/M} D\dot{k}\, \delta(a) f(ka), \tag{5.2}$$

similar to Proposition 2.3.9.

For negative multiplicities, a procedure as in the proof of Propositions 2.3.10 and 2.3.13 becomes necessary. First, one needs to obtain for each $\sigma \in S_n$ an expansion of the super function $\delta(e^{th_0}) f(ke^{th_0})$ on $K/M \times \mathbb{A}_+^{k-l} \times \mathbb{A}_+^l$ in terms of $t_{\sigma \leq l}$ at 0. Let the coefficients of this expansion be denoted f_n^σ for $n \in \mathbb{N}_0^l$. This leads as in Equation (2.4) to integrals of the form

$$\int_{K/M} D\dot{k}\, b_{\sigma^{-1}(1),n_1} \cdots b_{\sigma^{-1}(l),n_l} f_l^\sigma(k, t_{\sigma > l}), \tag{5.3}$$

where

$$b_{i,k} := \begin{cases} \log\big(\rho_i \circ \gamma(k, h_i)\big) & k = -1 - m_\alpha - m_{2\alpha}, \\ \frac{1 - \left(\rho_i \circ \gamma(k, h_i)\right)^{-1-m_{\alpha_i}-m_{2\alpha_i}-k}}{1 + m_{\alpha_i} + m_{2\alpha_i} + k} & \text{else.} \end{cases}$$

Although it is easy to check by Weyl group invariance that the integrals in Equation (5.3) vanish unless $n \in 2\mathbb{N}^l$, it becomes rather difficult to obtain further insights on them. In Section 2.3, the further approach was to show that most of these integrals equal $\int_{K/M} D\dot{k}\, f_l^\sigma(k, t_{\sigma > l})$ or even vanish. This was rather easy to see by using polar coordinates. However, in the general case, the only available coordinates so far are the ones given by the map $k\colon \bar{N} \to K/M$, under which retractions on K/M are not compatible with the canonical one on $\bar{N}$. Therefore, the major task in order to obtain a general polar coordinate formula is to find suitable coordinates on K/M.

A. Appendix: Categories

This section should summarise the concepts of category theory, used in this thesis. For a broader view on the topic, [Lan98] may be recommended to the reader.

A.1 Definition. A category $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}\ \mathcal{C}$ of objects,
- a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for any pair of objects $X, Y \in \mathcal{C}$,
- a composition map $\circ\colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ for any triple of objects $X, Y, Z \in \mathrm{Ob}\ \mathcal{C}$,

satisfying the following two conditions:

- for any three $f \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$
- for any object $X \in \mathrm{Ob}\ \mathcal{C}$, there exists a morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, called the *identity morphism*, such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$ for any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

It is easy to check that the identity morphism is unique. For convenience, one writes Hom instead of $\mathrm{Hom}_{\mathcal{C}}$ if the category is understood and $f\colon X \to Y$ instead of $f \in \mathrm{Hom}(X, Y)$.

For any two morphisms $f\colon X \to Y$, $g\colon Y \to X$ with $f \circ g = \mathrm{id}_Y$, one calls f a *retraction* of g. Vice versa, g is called a *section* of f.

If g is both a section and a retraction of f, it is uniquely determined by these properties. f and g are called *isomorphisms* and one writes $f^{-1} := g$. The objects X and Y are then said to be isomorphic: $X \cong Y$.

If f is a retraction, it is also an *epimorphism*, *i.e.* $g \circ f = h \circ f$ implies $g = h$. If f is a section, it is a *monomorphism*, *i.e.* $f \circ g = f \circ h$ implies $g = h$.

A.2 Definition. A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is a category for which all objects $X \in \mathrm{Ob}\ D$ are also objects in $\mathcal{C}$. Furthermore $\mathrm{Hom}_{\mathcal{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is required for all $X, Y \in \mathrm{Ob}\ \mathcal{D}$. If this inclusion is an equality for all Objects, $\mathcal{D}$ is said to be a *full* subcategory.

A.3 Definition. For any index set I and objects $X_i \in \mathrm{Ob}\ \mathcal{C}$, an object Y is the product of $(X_i)_{i \in I}$ if it satisfies the following universal property. There exist morphisms $\mathrm{pr}_i\colon Y \to X_i$ for all $i \in I$ such that for each $Z \in \mathrm{Ob}\ C$ with morphisms $f_i\colon Z \to X_i$ there exists a unique morphism $f\colon Z \to Y$ such that for each $i \in I$, the following diagram commutes:

$$\begin{tikzcd} & Y \arrow[d, "\mathrm{pr}_i"] \\ Z \arrow[ur, "f"] \arrow[r, "f_i"'] & X_i \end{tikzcd}$$

If the product exists it is denoted $\prod_{i\in I} X_i$. For finite products one uses the symbol "$\times$", *e.g.* $X_1 \times X_2$. Note that products are preserved under functors.

A terminal object $* \in \mathrm{Ob}\,\mathcal{C}$ is an object such that for each $X \in \mathrm{Ob}\,\mathcal{C}$, there exists a unique morphism $X \to *$. Terminal objects, so they exist, are unique up to unique isomorphisms. In a category which admits finite products a terminal object always exists. It is given by the product over the index set $\emptyset$.

A.4 Definition. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories consists of two assignments:

- each $X \in \mathrm{Ob}\,\mathcal{C}$ is mapped to $F(X) \in \mathrm{Ob}\,\mathcal{D}$,
- either each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is mapped to $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ or all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ are mapped to $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(Y),F(X))$.

Functors fulfilling the first condition are called covariant, whereas those satisfying the second are said to be contravartiant. Furthermore, the following is assumed to be true for all composable morphisms f and g: $F(f \circ g) = F(f) \circ F(g)$ in the covariant case and $F(f \circ g) = F(g) \circ F(f)$ in the contravariant case.

Functors which are injective on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are called faithful, those which are surjective on the sets of morphisms are called full.

A.5 Definition. Given two functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a natural transformation from F to G is a collection of morphisms $\eta = (\eta_X)_{X\in \mathrm{Ob}\,\mathcal{C}}$, with $\eta_X\colon F(X) \to G(X)$, such that for all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

Two natural transformations $\eta\colon F \to G$, $\delta\colon G \to H$ are combined to a natural transformation $\delta \circ \eta\colon F \to H$ by composing the respective maps.

Given a category $\mathcal{C}$, let $\mathcal{C}^\vee$ denote the category whose objects are given by contravariant functors $\mathcal{C} \to \mathbf{Sets}$. Here, **Sets** is the category of sets, where the morphisms are represented by ordinary maps between these sets. The morphisms in $\mathcal{C}^\vee$ are the natural transformation between functors, with the obvious identity morphism.

A.6 Definition. Given $X \in \mathcal{C}$, a natural contravariant functor $X(-)\colon \mathcal{C} \to \textbf{Sets}$ arises by setting $X(S) \coloneqq \mathrm{Hom}_{\mathcal{C}}(S, X)$ for $S \in \mathrm{Ob}\,\mathcal{C}$ and $X(f)\colon X(Z) \to X(Y), g \mapsto g \circ f$ for $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$.

This gives rise to a functor $h\colon \mathcal{C} \to \mathcal{C}^\vee$ *via* $h(X) \coloneqq X(-)$, where natural transformations $h(f)\colon X(-) \to Y(-)$ for $f\colon X \to Y$ are given by $h(f)_S\colon X(S) \to Y(S), s \mapsto f \circ s$. The functor h is known as the Yoneda embedding.

A.7 Proposition (Yoneda Lemma)**.** *Given a functor $F\colon C \to$ **Sets** and $X \in \mathrm{Ob}\,\mathcal{C}$. Then*

$$\mathrm{Hom}_{\mathcal{C}^\vee}\left(X(-), F\right) \cong F(X).$$

By setting $F \coloneqq Y(-)$ for $Y \in \mathrm{Ob}\,\mathcal{C}$, this shows in particular

$$\mathrm{Hom}_{\mathcal{C}^\vee}\left(X(-), Y(-)\right) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

hence $h\colon \mathcal{C} \to \mathcal{C}^\vee$ is a fully faithful functor. This explains the name of the Yoneda embedding

A.8 Remark. Although the Yoneda Lemma might look unimpressive at first sight, it gives a very useful tool into one's hand. For applications it is profitable to introduce the notion of S-points. For any object $X \in \mathrm{Ob}\,\mathcal{C}$, elements s of $X(S) = \mathrm{Hom}_{\mathcal{C}}(S, X)$ are called *S-points* or *generalised points* of X. One writes $s \in_S X$ instead of $x \in X(S)$ and $f(s)$ instead of $f \circ s$.

Now, the Yoneda Lemma states that instead of dealing with rather complicated morphisms $f\colon X \to Y$ it suffices to analyse them on the level of S-points. Even more, to define such a morphism it is enough to define it on S-points: Given maps (between sets) $f_S\colon X(S) \to Y(S)$, such that for any $g\colon S \to T$ the following diagram commutes

$$\begin{tikzcd} X(S) \arrow[r, "f_S"] \arrow[d, "X(g)"'] & Y(S) \arrow[d, "Y(g)"] \\ X(T) \arrow[r, "f_T"'] & Y(T) \end{tikzcd},$$

there exists a unique corresponding morphism $f\colon X \to Y$.

In applications it it therefore reasonable to define morphisms $f\colon X \to Y$ by defining $f(x)$ for $x \in_S X$ with implicitly stating that this is natural in x. The benefit of this method becomes very clear in the context of group objects.

A.9 Definition. A group object in a category $\mathcal{C}$ which allows finite products, is an object G together with morphisms $m_G\colon G \times G \to G$, $i_G\colon G \to G$, $e_G\colon * \to G$, such that the following diagrams commute:

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{id}_G \times m_G"] \arrow[d, "m_G \times \mathrm{id}_G"'] & G \times G \arrow[d, "m_G"] \\
G \times G \arrow[r, "m_G"'] & G
\end{tikzcd}
\quad
\begin{tikzcd}
& G \times G \arrow[rr, "\mathrm{id}_G \times i_G"] & & G \times G \arrow[dr, "m_G"] & \\
G \arrow[ur, "\delta_G"] \arrow[rr] \arrow[dr, "\delta_G"'] & & * \arrow[rr, "e_G"] & & G \\
& G \times G \arrow[rr, "i_G \times \mathrm{id}_G"'] & & G \times G \arrow[ur, "m_G"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
G \times * \arrow[rr, "\mathrm{id}_G \times e_G"] & & G \times G \arrow[dl, "m_G"] \\
& G \arrow[ul] \arrow[dl] & \\
* \times G \arrow[rr, "e_G \times \mathrm{id}_G"'] & & G \times G \arrow[ul, "m_G"']
\end{tikzcd}
$$

Here, δ_G denotes the diagonal morphism, which arises from the identity morphism in both components.

A morphism of group objects $G \to H$ is a morphism which is compatible with the multiplication and neutral morphisms, *i.e.* $m_H \circ (\varphi \times \varphi) = \varphi \circ m_G$ and $\varphi \circ e_G = e_H$. This defines the subcategory of group objects in $\mathcal{C}$.

A.10 Remark. Thanks to the Yoneda Lemma there is only one thing to check in order to prove that an object is indeed a group object: If for any object S, the set $G(S)$, together with the induced maps m_S and i_S is a group with neutral element $e_S \colon S \to * \to G$, then G is a group object. Moreover, with this approach it is easily shown that the inversion morphism i_G is compatible with morphisms of group objects.

A.11 Definition. Given a commuting diagram

$$
\begin{tikzcd}
P \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}
$$

in a category $\mathcal{C}$, the object P is called the fibre product of X and Y over Z if the diagram is universal in the following way: For any other such diagram with Q in place of P, there exists a unique morphism $Q \to P$ such that

$$
\begin{tikzcd}
Q \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}
$$

commutes, as well. P is unique up to unique isomorphisms by this condition, which justifies to write $X \times_Z Y \coloneqq P$.

Given a commuting diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dd] & & X \arrow[dr, "f"] \arrow[dd] & \\
& X' \times_{Z'} Y' \arrow[rr] \arrow[dd] & & X' \arrow[dd] \\
Y \arrow[rr] \arrow[dr, "g"] & & Z \arrow[dr, "h"] & \\
& Y' \arrow[rr] & & Z'
\end{tikzcd}$$

the induced morphism $X \times_Z Y \to X' \times_{Z'} Y'$ is denoted $f \times_h g$.

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Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von PD Dr. Alexander Alldridge betreut worden.

Köln, den 4. Dezember 2013 Wolfgang Palzer

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