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For my parents

Nichts kommt mir weniger wahrscheinlich vor, als daß ein Wissenschaftler, oder Mathematiker, der mich liest, dadurch in seiner Arbeitsweise ernstlich beeinflußt werden sollte. (In sofern sind meine Warnungen wie die Plakate an den Kartenschaltern der englishen Bahnhöfe “Is your journey really necessary?” Als ob Einer, der das liest sich sagen würde “On second thoughts, *no*”.)

Ludwig Wittgenstein

Abstract

The main result of this thesis is that the two higher Stasheff–Tamari orders are equal, as was originally conjectured by Edelman and Reiner in 1996. These are two orders on the set of triangulations of a cyclic polytope—the first introduced by Kapranov and Voevodsky, and the second introduced by Edelman and Reiner.

Our first step in proving the conjecture is to give new combinatorial interpretations of the higher Stasheff–Tamari orders which make them easier to compare. As a necessary prequel to these combinatorial interpretations, we characterise triangulations of $(2d+1)$-dimensional cyclic polytopes in terms of their d-simplices. The proof itself is then by induction on the number of vertices of the cyclic polytope. As a technical tool for this proof, we develop a theory for expanding triangulations of cyclic polytopes at any vertex, which is of independent interest.

We apply our results in representation theory of algebras, building on the work of Oppermann and Thomas, who show how triangulations of even-dimensional cyclic polytopes arise in the representation theory of the higher Auslander algebras of type A. Indeed, triangulations of even-dimensional cyclic polytopes are in bijection with both tilting modules and cluster-tilting objects. We choose to work in the slightly different framework of d-silting complexes, which we show are also in bijection with triangulations of even-dimensional cyclic polytopes. We show that the higher Stasheff–Tamari orders in even dimensions correspond to natural orders on d-silting complexes, which originally arose in the work of Riedtmann and

Schofield concerning partial orders on tilting modules.

This algebraic interpretation of the even-dimensional orders allows us to show that odd-dimensional triangulations correspond to equivalence classes of d-maximal green sequences, which we introduce as the higher-dimensional versions of classical maximal green sequences. We are then able to interpret the higher Stasheff–Tamari orders on equivalence classes of d-maximal green sequences. The orders obtained are very natural, but have not been studied before. The equivalence of the higher Stasheff–Tamari orders shows that these algebraic orders are equal for the higher Auslander algebras of type A.

We prove a pair of results on mutation, one on mutating cluster-tilting objects in higher cluster categories and the other on mutating triangulations of even-dimensional cyclic polytopes. The criterion for mutating triangulations works by associating quivers to the triangulations. These quivers originate from the cluster-tilting objects which correspond to the triangulations. We further use these quivers to characterise $2d$-dimensional triangulations which do not possess any interior $(d+1)$-simplices.

Finally, another open question we resolve comes from Dimakis and Müller-Hoissen. These authors introduce orders known as 'the higher Tamari orders' in the context of studying KP solitons. We show that, as conjectured, these are indeed the same posets as the higher Stasheff–Tamari orders. Since the higher Tamari orders are explicitly defined as a quotient of the higher Bruhat orders, this provides a quotient map from the higher Bruhat orders to the higher Stasheff–Tamari orders. Indeed to make this precise, we develop some new theory concerning quotient posets.

Erklärung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne die Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Werken dem Wortlaut oder dem Sinn nach entnommen wurden, sind als solche kenntlich gemacht. Ich versichere an Eides statt, dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie—abgesehen von unten angegebenen Teilpublikationen und eingebundenen Artikeln und Manuskripten—noch nicht veröffentlicht worden ist sowie, dass ich eine Veröffentlichung der Dissertation vor Abschluss der Promotion nicht ohne Genehmigung des Promotionsausschusses vornehmen werde. Die Bestimmungen dieser Ordnung sind mir bekannt. Darüber hinaus erkläre ich hiermit, dass ich die Ordnung zur Sicherung guter wissenschaftlicher Praxis und zum Umgang mit wissenschaftlichem Fehlverhalten der Universität zu Köln gelesen und sie bei der Durchführung der Dissertation zugrundeliegenden Arbeiten und der schriftlich verfassten Dissertation beachtet habe und verpflichte mich hiermit, die dort genannten Vorgaben bei allen wissenschaftlichen Tätigkeiten zu beachten und umzusetzen. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.

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Chapter 1

Introduction

This thesis solves an open problem in combinatorics and shows how this problem may be applied in the representation theory of algebras. Namely, the principal result of this thesis is that the two higher Stasheff–Tamari orders on triangulations of cyclic polytopes are equal. We then show how the higher Stasheff–Tamari orders may be interpreted naturally in the representation theory of the higher Auslander algebras of type A. Consequently, the equality of the higher Stasheff–Tamari orders provides new results about the representation theory of these algebras. Other results we prove include criteria for mutating triangulations of cyclic polytopes and cluster-tilting objects, and that the higher Stasheff–Tamari orders are equal to the higher Tamari orders.

1.1 Cyclic polytopes and their triangulations

We begin the thesis by proving results on cyclic polytopes and their triangulations in Chapter 2. Cyclic polytopes are a family of convex polytopes. A convex polytope is a bounded region of $\mathbb{R}^{\delta}$ cut out by linear inequalities. Thus, convex polytopes arise naturally in applied mathematics, namely when studying constrained optimisation [Sch86]. The key information captured by convex polytopes consists in their facial structure: which subsets of vertices span faces. In two dimensions convex polytopes are all alike. Their vertices may be labelled by numbers such that vertices labelled by cyclically consecutive numbers span an edge. But, in dimensions higher than two, the facial structure of a polytope may be more complex.

Given the possible complexity of high-dimensional polytopes, it is natural to seek well-behaved families of polytopes. Cyclic polytopes form one such family. Indeed, cyclic polytopes satisfy the Upper Bound Theorem of McMullen: they have the largest number of k-dimensional faces possible for every value of k, given their dimension and number of vertices [McM70; BB80; Sta75; AK85]. The facial structure of a cyclic polytope can in fact be described by a simple combinatorial criterion known as 'Gale's Evenness Criterion' [Gal63; Grü03; ER96]. A curve in $\mathbb{R}^\delta$ is called a *δ-order curve* if every affine hyperplane intersects it in at most δ points. It is known that the convex hull of any finite set of points lying on a δ-order curve is a cyclic polytope [MS71; CD00] and, conversely, that for every cyclic polytope there exists an order δ curve passing through its vertices [Stu87]. In fact, every sufficiently large collection of points in general position in $\mathbb{R}^\delta$ contains the vertices of a cyclic polytope [CD00]; and cyclic polytopes are precisely the polytopes which have this property [Bjö+99].

Being natural combinatorial objects, cyclic polytopes appear in many different areas of mathematics. Their duals have been used in game theory to construct games whose Nash equilibria are difficult to compute [Ste97; SS06; SS16]. Sturmfels has shown that cyclic polytopes are equivalent to totally positive matrices [Stu88b], which are of significant interest in both pure mathematics and applications [And87; Lus98; Pos06]. In theoretical physics, cyclic polytopes arise as examples of amplituhedra—celebrated objects introduced by Arkani-Hamed and Trnka to facilitate the computation of scattering amplitudes [AT14]. Cyclic polytopes were at the forefront of the development of Stanley–Reisner theory in Stanley's generalisation of the Upper Bound Theorem [Sta75], and have subsequently been studied in this context [TH96].

In this thesis, we shall be particularly interested in triangulations of cyclic polytopes. Triangulations of cyclic polytopes are often used to define higher-dimensional analogues of structures that exist for lower-dimensional triangulations. The example of this *par excellence* is the application of triangulations of cyclic polytopes to define higher Segal spaces [DK19]—see also [Pog17; DJW19]. The definition of the original Segal spaces can be seen as involving line dissections, which are triangulations of one-dimensional cyclic polytopes [Seg74; Rez01]. Another appearance of cyclic polytopes in algebraic K-theory occurs in [HM97]. Repeatedly applying the BCFW recursion [Bri+05] to compute scattering amplitudes produces a triangulation of the cyclic polytope in the case where the amplituhedron is a cyclic polytope [BT18]. In integrable systems, regular triangulations of cyclic polytopes describe the evolution of a class of solitary waves modelled by the Kadomtsev–Petviashvili equation [DM12; Wil21c]—see also [Hua15; KK21; GPW19]. Triangulations of cyclic polytopes have also been shown to be in bijection with other combinatorial objects, such as snug partitions [Tho02] and persistent graphs [FR21]. In general, finding a formula for the number of triangulations of a cyclic polytope is an open problem [KV91, 5.2], although specific cases have been solved [AS02].

The first result we prove about triangulations of cyclic polytopes in Chapter 2 comprises a description of $(2d+1)$-dimensional triangulations in terms of their d-simplices. This gives the other half of the picture from [OT12], where triangulations of $2d$-dimensional cyclic polytopes were described in terms of their d-simplices. Namely, a triangulation of a $2d$-dimensional cyclic polytope can be described as a maximal-size set of non-intersecting d-simplices [OT12], just as a triangulation of a convex polygon is a maximal set of non-intersecting arcs. To describe odd-dimensional triangulations we define two new properties for collections of d-simplices which we call being 'supporting' (Definition 2.2.11) and being 'bridging' (Definition 2.2.13). We show that if a collection of d-simplices is supporting and bridging, then one can build a triangulation of a $(2d+1)$-dimensional cyclic polytope out of them. This gives the following theorem [Wil21a].

Theorem 1.1.1 (Theorem 2.2.3). *Triangulations of the $(2d+1)$-dimensional cyclic polytope are given by sets of d-simplices which are supporting and bridging.*

In Chapter 2 we also prove a technical result concerning expanding and contracting triangulations of cyclic polytopes. This will be a key tool in the proof of our main result in Chapter 3.

1.2 The higher Stasheff–Tamari orders

We first meet the higher Stasheff–Tamari orders in Chapter 3. These are two *a priori* different partial orders on the set of triangulations of a cyclic polytope. In two dimensions, where cyclic polytopes are simply convex polygons, both partial orders coincide with the Tamari lattice, a widely occurring partial order in mathematics. The Tamari lattice arises when considering weak associativity conditions [Tam62], which are often of mathematical interest. Here triangulations of convex polygons correspond to the different possibilities for performing a binary operation on a string. Homotopy associativity of H-spaces in algebraic topology was studied by Stasheff [Sta63] using the associahedron, a polytope whose 1-skeleton is the Tamari lattice and which was also originally considered by Tamari [Tam51; Sta12]. In mathematical physics, weak associativity conditions occur in open string field theory [Moo55; KK74; Hat+86], and in the Biedenharn–Elliott identities [Bie53; Ell53]. The Tamari lattice can be realised in algebra as a partial order on tilting modules [BK04] or torsion classes [Tho12] for the type A path algebra; this is related to the fact that triangulations of convex polygons correspond to clusters in the type A cluster algebra [FZ02a; FZ03a] or cluster-tilting objects in the type A cluster category. In this thesis, we look at a higher-dimensional version of this correspondence.

The sequence counting the number of objects of the Tamari lattice is the Catalan numbers, which is known to enumerate over two hundred different sequences of combinatorial objects [Sta15]. Many of these combinatorial objects also provide nice interpretations of the Tamari lattice. The extensive reach of the Tamari lattice into different areas of mathematics is exhibited in the Tamari memorial festschrift [MPS12].

The first higher Stasheff–Tamari order was introduced by Kapranov and Voevodsky in 1991 [KV91] as a natural example of a strictly ordered n-category produced by a certain iterative construction. In 1996, Edelman and Reiner built upon this work by introducing the *a priori* different second higher Stasheff–Tamari order. One especially beautiful facet of the higher-dimensional orders is that triangulations of $(\delta+1)$-dimensional cyclic polytopes are assembled from maximal chains of triangulations of δ-dimensional cyclic polytopes in the first higher Stasheff–Tamari order [Ram97]. In particular, the objects of the three-dimensional first higher Stasheff–Tamari order correspond to equivalence classes of maximal chains in the Tamari lattice, and the objects of the four-dimensional first higher Stasheff–Tamari order correspond to equivalence classes of maximal chains in the three-dimensional order, and so on.

Generalisations of, and variations on, the Tamari lattice is a large subject in itself, and includes Tamari lattices in other Dynkin types [Tho06], Cambrian lattices [Rea06], lattices of torsion classes of cluster-tilted algebras [GM19], m-Tamari lattices [BP12; BFP11], ν-Tamari lattices [PV17], Dyck lattices [Knu11; Dis+12], generalised Tamari orders [Ron12], and Grassmann–Tamari orders [SSW17]. However, the higher Stasheff–Tamari orders hold a particularly special position amongst these because, as we have seen, they encode higher-dimensional information hidden in the Tamari lattice itself, rather than being only variations on the Tamari lattice. This furthermore shows the virtues of viewing the Tamari lattice in terms of triangulations of convex polygons: it brings out these latent higher-dimensional structures which are obscured by other combinatorial interpretations.

Edelman and Reiner further conjectured the two higher Stasheff–Tamari orders to coincide with each other, a problem that has remained open despite several papers on the orders [ERR00; Tho02; Tho03; Ram97; RS00; RR12]. The main result of this thesis is that the Edelman–Reiner conjecture is true. The first higher Stasheff–Tamari order ($\leqslant_1$) is equal to the second higher Stasheff–Tamari order ($\leqslant_2$) [Wil21e].

Theorem 1.2.1 (Theorem 3.3.9 and Theorem 3.3.14)**.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of the cyclic polytope. Then $\mathcal{T} \leqslant_1 \mathcal{T}'$ if and only if $\mathcal{T} \leqslant_2 \mathcal{T}'$.*

The two higher Stasheff–Tamari orders are quite different in nature and each has its own advantages. The first order is more combinatorial and is defined by means of its covering relations, which are given by "increasing bistellar flips". A bistellar flip is an operation generalising the two-dimensional operation of flipping a diagonal inside a quadrilateral; these operations can be oriented, so that some bistellar flips are "increasing" and some are "decreasing". The second order is more geometric and was originally defined by comparing the heights of sections induced by triangulations. The second order allows direct comparison between triangulations, whereas comparing triangulations in the first order requires one to find a sequence of increasing bistellar flips. On the other hand, the local structure of the second poset is not clear, because the covering relations are not immediate from the definition. It is also easier to compute the entire first poset than to compute the entire second poset. Computing either poset requires computing all the triangulations of a given cyclic polytope. The most efficient algorithm for doing this is to start at the minimal triangulation and iteratively compute increasing bistellar flips, which is tantamount to computing the first order [JK18]. To construct the second order then requires additional computations on top of this.

It is clear that whenever the first higher Stasheff–Tamari order holds between a pair of triangulations, then the second order must hold too, as was noted in [ER96]. This is because if a triangulation $\mathcal{T}'$ is an increasing bistellar flip of another triangulation $\mathcal{T}$, then the section of $\mathcal{T}'$ certainly lies above the section of $\mathcal{T}$. But it is not clear whether the first order should hold whenever the second one does. Indeed, an analogous statement for the higher Bruhat orders has been known to be false since 1993 [Zie93]. *A priori* it might be possible for there to exist a pair of pathological triangulations $\mathcal{T}$ and $\mathcal{T}'$ where the section of $\mathcal{T}$ lay below the section of $\mathcal{T}'$ whilst $\mathcal{T}'$ could not be reached by a sequence of increasing bistellar flips from $\mathcal{T}$. However, Theorem 1.2.1 rules this possibility out.

The key step in proving the equivalence of the orders is to give new combinatorial interpretations of them. These new interpretations are laid out in Theorem 3.2.13. These new combinatorial interpretations of the orders allow them to be compared more easily, and this is essential in the proof of Theorem 1.2.1. The new combinatorial interpretations of the orders also enable the connection with representation theory to be forged, as we now discuss.

1.3 Representation theory

In Chapter 4, we show that the higher Stasheff–Tamari orders arise naturally in the representation theory of algebras. The connection between triangulations of cyclic polytopes and representation theory was first discovered in [OT12]. Oppermann and Thomas show in this paper that triangulations of even-dimensional cyclic polytopes are in bijection with tilting modules and cluster-tilting objects for A^d_n, the higher Auslander algebras of type A. For the definition of the algebras A^d_n, the higher Auslander algebra of type A, see Section 4.1. These algebras were introduced by Iyama within the programme of higher Auslander–Reiten theory [Iya07a; Iya07b; Iya11]—a new and active area of research within representation theory which has found connections with non-commutative algebraic geometry [Her+20] and homological mirror symmetry [DJL21]. The relation between triangulations of even-dimensional cyclic polytopes and the representation theory of A^d_n is a higher-dimensional version of the relation between triangulations of convex polygons and cluster categories of type A which was mentioned in Section 1.2. Cluster categories were introduced in [Bua+06] as a categorification of the cluster algebras of Fomin and Zelevinsky [FZ02a] and are powerful tools that have been used to solve open problems in mathematical physics [Kel13].

Our work reveals that the connection between triangulations of cyclic polytopes and higher Auslander–Reiten theory is richer than previously known. Not only do triangulations of odd-dimensional cyclic polytopes play a role, but the higher Stasheff–Tamari orders arise naturally on the algebraic side. This provides new insights into the combinatorial structure of higher Auslander–Reiten theory.

In [OT12], it was shown that triangulations of the $2d$-dimensional cyclic polytope with $n + 2d + 1$ vertices correspond to tilting A^d_{n-1}-modules or cluster-tilting objects for A^d_n. For reasons we explain in Section 4.2.2, we find it most convenient to work with a slightly different bijection, which we show holds between triangulations of the $2d$-dimensional cyclic polytope with $n+2d+1$ vertices and d-silting objects for A_n^d.

In even dimensions, we show that the higher Stasheff–Tamari orders induce classical orders on d-silting objects [Wil21a; Wil] introduced in [AI12], following the introduction of similar orders on tilting modules by [RS91]. This result was already known for the special case of the Tamari lattice [BK04; Tho12], which corresponds to $d = 1$. For the definition of left mutation and of ${}^{\perp}T$ see Section 4.1.

Theorem 1.3.1 (Theorem 4.3.1 and Theorem 4.3.4)**.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of a $2d$-dimensional cyclic polytope corresponding to d-silting objects T and T' over A_n^d. We then have that*

(1) $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if T' is a left mutation of T; and

(2) $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if ${}^{\perp}T \subseteq {}^{\perp}T'$.

It is this theorem that allows us to show how triangulations of odd-dimensional cyclic polytopes appear in the representation theory of A_n^d. It is known that odd-dimensional triangulations can be given by maximal chains of even-dimensional triangulations in the first higher Stasheff–Tamari order, modulo an equivalence relation [Ram97]. Hence, in Section 4.4.1 we define higher-dimensional "d-maximal green sequences" as sequences of mutations of d-silting complexes from the projectives to the shifted projectives. This is because this is what Theorem 1.3.1 shows maximal chains in the first higher Stasheff–Tamari order correspond to algebraically. Maximal green sequences were originally introduced in the context of Donaldson–Thomas invariants in mathematical physics [Kel11]. We thus obtain the following theorem. For the exact nature of the equivalence relation on d-maximal green sequences, see Section 4.4.1.

Theorem 1.3.2 (Theorem 4.4.2)**.** *There is a bijection between triangulations of the* $(2d+1)$*-dimensional cyclic polytope with* $n+2d+1$ *vertices and equivalence classes of* d*-maximal green sequences of* A_n^d*.*

It is natural then to ask whether this theorem can be used to give an algebraic description of the higher Stasheff–Tamari orders in odd dimensions. Indeed, one can obtain such a description, which is as follows [Wil21a; Wil]. For the definition of an increasing elementary polygonal deformation, see Section 4.4.2. This theorem originates in the combinatorial interpretations of the higher Stasheff–Tamari orders we give in Chapter 3.

Theorem 1.3.3 (Theorem 4.4.4 and Theorem 4.4.6)**.** *Let* $\mathcal{T}$ *and* $\mathcal{T}'$ *be triangulations of a* $(2d+1)$*-dimensional cyclic polytope corresponding to equivalence classes of* d*-maximal green sequences* $[G]$ *and* $[G']$ *of* A_n^d*. We then have that*

(1) $\mathcal{T} \lessdot_1 \mathcal{T}'$ *if and only if* $[G']$ *is an increasing elementary polygonal deformation of* $[G]$*; and*

(2) $\mathcal{T} \leqslant_2 \mathcal{T}'$ *if and only if the set of summands of* $[G]$ *contains the set of summands of* $[G']$*.*

These orders induced on d-maximal green sequences by the higher Stasheff–Tamari orders are very natural, but have not previously been considered. The Edelman–Reiner conjecture here corresponds to a stronger form of the "no-gap" conjecture made in [BDP14], cases of which were proven in [GM19; HI19].

An application of Theorem 1.2.1 is that the algebraic orders from Theorem 1.3.1 and Theorem 1.3.3 are equal for the higher Auslander algebras of type A. We also obtain a corollary from Theorem 1.3.3, namely that the set of equivalence classes of maximal green sequences of linearly oriented A_n is a lattice (Corollary 4.4.11). This is because in dimension 3 the two orders are known to be equivalent and known to be lattices [ER96].

A final result we prove in Chapter 4 is a criterion for mutating cluster-tilting objects in higher cluster categories. Indeed, if $\mathcal{O}_\Lambda$ is the higher cluster category of an algebra Λ containing a cluster-tilting object T, then it is shown in [OT12] that $\Gamma := \operatorname{End}_{\mathcal{O}_\Lambda} T$ has a d-cluster-tilting subcategory $\mathcal{M}$ in its module category. In the case where $\Lambda = A_n^d$, we show that a summand of T is mutable if and only if the corresponding simple Γ-module lies in $\mathcal{M}$.

1.4 Quiver combinatorics for higher-dimensional triangulations

In Chapter 5, we go on to prove further combinatorial results concerning triangulations of cyclic polytopes. In particular, we investigate the combinatorics of quivers associated to triangulations of even-dimensional cyclic polytopes. As shown by Oppermann and Thomas [OT12], such quivers provide the prototype for higher-dimensional cluster theory.

A cluster algebra can be given by choosing a quiver with a variable assigned to each vertex, and subsequently generating new quivers by a process of mutation, with new variables given from old variables via "exchange relations". A remarkable result is that the cluster algebras of finite cluster type are precisely those coming from Dynkin diagrams, which parallels the Cartan–Killing classification [FZ03c]. Since their introduction [FZ02a], cluster algebras have generated substantial amounts of fruitful research touching many areas of mathematics, including dynamical systems [FZ02b; CS04; Spe07], Poisson geometry [GSV03; GSV05], and Teichmüller theory [FG06; FG09]. Two particular topics which are connected with cluster algebras are surfaces [FST08] and representation theory of algebras [CCS06; Bua+06]. Surfaces can be used to produce cluster algebras via triangulations, whilst in representation theory cluster algebras are categorified via cluster categories. As remarked in [OT12], both these phenomena possess a two-dimensional quality, namely, the two-dimensionality of the surface and the 2-Calabi–Yau property of the cluster category.

A particularly simple example of a cluster algebra coming from a surface is the cluster algebra of type A_n, where the clusters are in bijection with triangulations of a convex $(n+3)$-gon [FZ02a]. Mutation of clusters corresponds to flipping a diagonal inside a quadrilateral. The quiver of a cluster can be easily constructed from the triangulation by drawing arrows between neighbouring arcs. In the type A cluster category, these quivers are the Gabriel quivers of the endomorphism algebras of the corresponding cluster-tilting objects.

The work of Oppermann and Thomas relating cluster-tilting objects in higher-dimensional cluster categories to triangulations of even-dimensional cyclic polytopes can be seen as discovering higher-dimensional cluster phenomena [OT12]. These cluster phenomena occur in all even dimensions, rather than only two dimensions. Higher-dimensional cluster theory remains poorly understood. A higher cluster algebra has yet to be defined—if such a definition is indeed possible. It would be remarkable if cluster algebras were the two-dimensional instance of a more general phenomenon. The necessary ingredients for a higher cluster algebra would be a rule for quiver mutation and an exchange relation to produce new cluster variables after mutation. Whilst higher tropical exchange relations were exhibited in [OT12], it is known that naïvely detropicalising these relations does not work.

Triangulations of even-dimensional cyclic polytopes present themselves as the guide for how the higher-dimensional quiver combinatorics ought to work. Just as in the classical type A case, the quiver of a cluster arises both from the endomorphism algebra of the corresponding cluster-tilting object, and may also be constructed from the corresponding triangulation: the vertices of the quiver correspond to the internal d-simplices of the triangulation, which we refer to as 'd-arcs', with arrows between the d-arcs that are closest to each other.

We investigate the information encoded by the quiver associated to a triangulation of a $2d$-dimensional cyclic polytope. The best-understood quivers are those known as 'cut quivers', which were introduced in [IO11]. These quivers have a rule for mutation at sinks and sources [IO11]. For $d = 1$, these cut quivers are precisely orientations of the A_n Dynkin diagram. Our first result shows that cut quivers correspond precisely to triangulations with no interior $(d + 1)$-simplices and that these are also exactly the triangulations whose quivers are acyclic. Hence, this is a higher-dimensional generalisation of the fact that a triangulation of a convex polygon has no internal triangles if and only if its quiver is acyclic, and in this case the quiver is an orientation of the A_n Dynkin diagram [Wil21b].

Theorem 1.4.1 (Theorem 5.2.12 and Proposition 5.2.10)**.** *A triangulation of a $2d$-dimensional cyclic polytope has no interior $(d + 1)$-simplices if and only if its quiver is acyclic, in which case its quiver is a cut quiver of type A.*

An application of Theorem 1.4.1 is that the set of triangulations of a $2d$-dimensional cyclic polytope without internal $(d + 1)$-simplices is connected via bistellar flips. The analogous fact for cluster algebras—that the acyclic seeds form a connected subgraph of the mutation graph—also holds, but no elementary proof is known.

Unlike for the two-dimensional case, for $d > 1$ it is not possible to perform a bistellar flip at every internal d-simplex of a $2d$-dimensional triangulation, or, equivalently, at every vertex of its quiver. This is an important difference with classical cluster theory, where a key property is that one can mutate a given cluster at every vertex of its quiver. This feature makes higher-dimensional cluster theory much more difficult to work with. Our second result uses the quiver of a triangulation to give a combinatorial criterion for identifying which d-simplices are mutable—that is, admit a bistellar flip. We show how the arrows in the quiver can be partitioned into paths which we call 'maximal retrograde paths', and prove the following theorem [Wil21b].

Theorem 1.4.2 (Theorem 5.3.9)**.** *Let $\mathcal{T}$ be a triangulation of a $2d$-dimensional cyclic polytope. An internal d-simplex of $\mathcal{T}$ is mutable if and only if it is not in the middle of a maximal retrograde path.*

This theorem gives a quiver-theoretic criterion for mutability, and hence points towards what a theory of higher-dimensional quiver mutation [FZ03c] could look like. Other extensions of quiver mutation have been of interest in the literature, such as to ice quivers [Pre20]. Moreover, this theorem provides a visual way of understanding mutability for higher-dimensional triangulations, and makes it easier to compute bistellar flips of higher-dimensional triangulations by hand. In the case of polygon triangulations, all retrograde paths are of length one, so that the criterion imposes no restriction and all arcs are mutable. As an application of this theorem, we give a rule for mutating cut quivers at vertices which are not necessarily sinks or sources.

1.5 The higher Bruhat orders

In the final chapter, we resolve another question concerning the higher Stasheff–Tamari orders. In work analysing the combinatorics of a class of KP solitons, Dimakis and Müller-Hoissen define a family of partial orders which they call 'the higher Tamari orders' [DM12]. KP solitons are solutions to a differential equation called the Kadomtsev–Petviashvili equation, which describes solitary waves [KP70; Kod10]. The authors conjectured the higher Tamari orders to coincide with the higher Stasheff–Tamari orders; we prove their conjecture.

Dimakis and Müller-Hoissen construct the higher Tamari orders from the higher Bruhat orders. These are a family of partial orders introduced by Manin and Schechtman [MS89] which generalise the weak Bruhat order on the symmetric group to higher dimensions just as the higher Stasheff–Tamari orders generalise the Tamari lattice to higher dimensions. The higher Bruhat orders were originally introduced to study hyperplane arrangements [MS89] and have found application in the theories of Soergel bimodules [Eli16], quasi-commuting Plücker coordinates [LZ98], and social choice [GR08]. They are also tightly connected with the quantum Yang–Baxter equation and its generalisations [DM15, and references therein].

We show how the construction in [DM12] amounts to defining the higher Tamari orders as the image of a certain order-preserving map from the higher Bruhat orders to the higher Stasheff–Tamari orders. We provide a new proof that this map is surjective, which was originally shown in [RS00] using a different framework. We furthermore show that the map is full [Wil21c; Wil21d]. This implies that the image of the map is in fact the entire higher Stasheff–Tamari orders themselves, and so we obtain the following theorem.

Theorem 1.5.1 (Corollary 6.4.4)**.** *The higher Tamari orders are equal to the higher Stasheff–Tamari orders.*

Our treatment of this problem also involves a new approach to the subject of quotients of posets which is more general than those that have previously been considered. Namely, we explain why maps which are surjective and full should be considered as quotient maps of posets. We have been unable to find this treatment of quotients of posets elsewhere in the literature.

1.6 Notation and conventions

Unless clearly stated otherwise, these conventions will apply throughout the thesis. We use $[m]$ to denote the set $\{1, 2, \ldots, m\}$. By $\binom{[m]}{k}$ we mean the set of subsets of $[m]$ of size k. When we display the elements of a subset of $[m]$, we shall always display the elements in order. Hence, if we write $S = \{a, b, c, \ldots, x, y, z\}$, we always mean that $a < b < c < \cdots < x < y < z$. Furthermore, if $A \in \binom{[m]}{k+1}$, then, unless indicated otherwise, we shall find it convenient to denote the elements of A by $A = \{a_0, a_1, \ldots, a_k\}$. The same applies to other letters of the alphabet: the upper-case letter denotes the subset; the lower-case letter is used for the elements, which are ordered according to their index starting from 0. In an effort to make the notation lighter, we often omit braces around sets, writing $A \cup x$ for $A \cup \{x\}$ and $A \setminus x$ for $A \setminus \{x\}$.

We use the convention that the symbol '$\subseteq$' denotes inclusion of subsets, whereas '$\subset$' denotes strict inclusion. Hence, for any set A, we have $A \subseteq A$ but $A \not\subset A$. Of course, if we have $B \subset A$, then we also have $B \subseteq A$.

For $l \in [m]$, using the notation of [OPS15], we use $<_l$ to denote the cyclically shifted order on $[m]$ given by

$$l <_l l+1 <_l \cdots <_l m-1 <_l m <_l 1 <_l \cdots <_l l-1.$$

For $r \geqslant 3$, $a_1 < \cdots < a_r$ is a *cyclic ordering* if there is an $l \in [m]$ such that $a_1 <_l \cdots <_l a_r$. In this thesis, it is convenient for us to consider both the linear and cyclic orderings of $[m]$. Unless stated otherwise, it should be assumed that we refer to the linear ordering on this set.

We denote by $(a, b), [a, b] \subseteq [m]$ respectively the *open* and *closed cyclic inter-*

vals. That is,

$$(a, b) := \{\, i \in [m] : a < i < b \text{ is a cyclic ordering} \,\},$$
$$[a, b] := (a, b) \cup \{a, b\}.$$

The one exception to this is that we will find it convenient to set $[a, a-1] := \varnothing$. When we have $a < b$ in the linear ordering on $[m]$, we say that $[a, b]$ and (a, b) are *intervals*. We call $I \subseteq [m]$ an *l-ple interval* if it can be written as a union of l intervals, but cannot be written as a union of fewer than l intervals. We similarly define *cyclic l-ple intervals*.

When we refer to the elements a_i of a subset $A \subseteq [m]$ with $\#A = d+1$, we will sometimes write $i \in \mathbb{Z}/(d+1)\mathbb{Z}$ to indicate that one should interpret a_{d+1} as meaning a_0. That is, if $A = \{1, 3, 5\}$, then $a_0 = 1, a_1 = 3, a_2 = 5, a_3 = 1$.

Chapter 2

Cyclic polytopes and their triangulations

In this chapter, we lay out our framework for convex polytopes and their triangulations, focusing on the specific case of cyclic polytopes. We explain various operations on triangulations of cyclic polytopes that we will need to consider. We detail the combinatorial characterisations of triangulations that we will use. In even dimensions this characterisation is due to [OT12], but the odd-dimensional characterisation is new [Wil21a]. The largest portion of the chapter is concerned with proving a technical result describing the possible triangulations that can contract to a particular triangulation. This is the subject of Section 2.3. We shall need this result to prove the main result of Chapter 3.

2.1 Background

Our framework for cyclic polytopes and their triangulations maintains a sharp distinction between the combinatorial and the geometric.

2.1.1 Convex polytopes

A subset $\mathfrak{X} \subset \mathbb{R}^\delta$ is *convex* if for any $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}$, the line segment $\overline{\mathbf{x}\mathbf{x}'}$ connecting $\mathbf{x}$ and $\mathbf{x}'$ is contained in $\mathfrak{X}$. The *convex hull* $\operatorname{conv}(\mathfrak{X})$ of $\mathfrak{X}$ is the smallest convex set containing $\mathfrak{X}$ or, equivalently, the intersection of all convex sets containing $\mathfrak{X}$.

Let $V \subseteq \mathbb{Z}_{>0}$ be a finite set and $|-|\colon V \to \mathbb{R}^\delta$ be an injective function, which we call the *geometric realisation*. For subsets $A \subseteq V$ we also write $|A| = \operatorname{conv}\{\, |a| : a \in A \,\}$. We let $\mathfrak{P} = |V|$ and suppose that the affine span of $\mathfrak{P}$ is $\mathbb{R}^\delta$. A subset $\mathfrak{P} \subseteq \mathbb{R}^\delta$ of this form is called a (*geometric*) *convex polytope*.

A *face* of a polytope $\mathfrak{P}$ is a subset on which some linear functional is maximised. That is, $\mathfrak{F} \subseteq \mathfrak{P}$ is a face of $\mathfrak{P}$ if there is a vector $\mathbf{a} \in \mathbb{R}^\delta$ such that

$$\mathfrak{F} = \{\, \mathbf{x} \in \mathfrak{P} : \langle \mathbf{a}, \mathbf{x} \rangle \geqslant \langle \mathbf{a}, \mathbf{y} \rangle,\ \forall \mathbf{y} \in \mathfrak{P} \,\},$$

where '$\langle -, - \rangle$' denotes the standard inner product. A (*geometric*) *facet* of $\mathfrak{P}$ is a face of codimension one. A (*combinatorial*) *facet* of $\mathfrak{P}$ is a subset $F \subseteq V$ such that $|F|$ is a geometric facet of $\mathfrak{P}$.

Let $v \in V$ be such that $|v|$ is the face of $\mathfrak{P}$ given by maximising a functional $\langle \mathbf{a}, - \rangle$. Further, let $\varepsilon > 0$ be sufficiently small that, for all $w \in V \setminus v$, we have that $\langle \mathbf{a}, |w| \rangle < \langle \mathbf{a}, |v| \rangle - \varepsilon$. The *vertex figure of* $\mathfrak{P}$ *at* v is then the intersection

$$\mathfrak{P}\backslash v := \mathfrak{P} \cap \{\, \mathbf{x} \in \mathbb{R}^\delta : \langle \mathbf{a}, \mathbf{x} \rangle = \langle \mathbf{a}, |v| \rangle - \varepsilon \,\},$$

that is, the intersection of $\mathfrak{P}$ with the hyperplane $\langle \mathbf{a}, \mathbf{x} \rangle = \langle \mathbf{a}, |v| \rangle - \varepsilon$.

A *circuit* of a polytope $\mathfrak{P}$ realised geometrically via $|-|\colon V \to \mathbb{R}^\delta$ is a pair, (Z_+, Z_-), of disjoint subsets of V which are inclusion-minimal with the property that $|Z_+| \cap |Z_-| \neq \varnothing$. In this case, $|Z_+|$ and $|Z_-|$ intersect in a unique point.

We are interested in the combinatorial properties of the polytope, as comprised by the facets and circuits given by its geometric realisation $|-|\colon V \to \mathbb{R}^\delta$. If we let $\mathcal{F}_{\mathfrak{P}}$ and $\mathcal{Z}_{\mathfrak{P}}$ be respectively the set of combinatorial facets of $\mathfrak{P}$ and the set of combinatorial circuits of $\mathfrak{P}$, then we say that the triple $P = (V, \mathcal{F}_{\mathfrak{P}}, \mathcal{Z}_{\mathfrak{P}})$ is a *combinatorial polytope*. We often refer to the collection of facets of a combinatorial polytope as the *boundary* of the polytope.

Remark 2.1.1. Note that there might exist $v \in V$ such that $|v|$ is not a face of $\mathfrak{P}$, since we may have $|v| \in |V \setminus v|$. Allowing such elements of V is necessary for considering one-dimensional cyclic polytopes. Hence, strictly, the data we consider comprise a point configuration rather than a polytope.

2.1.2 Cyclic polytopes

Cyclic polytopes are the higher-dimensional analogues of convex polygons. General introductions to this class of polytopes can be found in [Zie95, Lecture 0] and [Grü03, Section 4.7]. Grünbaum writes that the construction of cyclic polytopes in current use is due to Gale [Gal63] and Klee [Kle63], and that they were introduced and studied in the 1950s by Gale [Gal55] and Motzkin [Mot57]. The earlier work of Carathéodory [Car07; Car11] is related, but the convex bodies studied in these papers are not cyclic polytopes: they are the continuous analogues of even-dimensional cyclic polytopes.

The (*geometric*) *cyclic polytope* $\mathfrak{C}(V, \delta)$ is the polytope with geometric realisation

$$\begin{aligned} |-|_\delta \colon V &\to \mathbb{R}^\delta \\ v &\mapsto |v|_\delta = p_\delta(t_v) := (t_v, t_v^2, \dots, t_v^\delta), \end{aligned}$$

where $\{t_{v_0}, t_{v_1}, \dots, t_{v_k}\} \subseteq \mathbb{R}$ and $k + 1 = \#V$. (Recall our convention that $V = \{v_0, v_1, \dots, v_k\}$ and that by writing $\{t_{v_0}, t_{v_1}, \dots, t_{v_k}\}$, we indicate that $t_{v_0} < t_{v_1} < \dots < t_{v_k}$.) When the dimension of the geometric realisation is clear from the context, we will drop the subscript and write $|-|$ instead of $|-|_\delta$. In the case where $V = [m]$, we write $\mathfrak{C}(m,\delta) := \mathfrak{C}([m],\delta)$. The curve defined by $p_\delta(t) := (t, t^2, \ldots, t^\delta) \subseteq \mathbb{R}^\delta$ is called the *moment curve.*

The facets of $\mathfrak{C}(m,\delta)$ come in two different types. A facet $\mathfrak{F}$ of $\mathfrak{C}(m,\delta)$ is an *upper facet* if, for any $\mathbf{a} \in \mathbb{R}^\delta$ such that $\langle \mathbf{a}, - \rangle$ is maximised on $\mathfrak{F}$, we have that $\mathbf{a}_\delta > 0$, where $\mathbf{a}_\delta$ is the δ-th coordinate of $\mathbf{a}$. Dually, a facet $\mathfrak{F}$ of $\mathfrak{C}(m,\delta,)$ is a *lower facet* if $\mathbf{a}_\delta$ is negative for any $\mathbf{a}$ such that $\mathfrak{F}$ maximises $\langle \mathbf{a}, - \rangle$. Equivalently, a facet $\mathfrak{F}$ of $\mathfrak{C}(m,\delta)$ is an upper (lower) facet if any normal vector to $\mathfrak{F}$ which points out of the polytope has a positive (negative) δ-coordinate. Or, more informally, $\mathfrak{F}$ is an upper (lower) facet if it can be seen from a very large positive (negative) δ-th coordinate.

The upper and lower facets of a cyclic polytope can be described combinatorially. Given a subset $F \subset V$, we say that an element $v \in V \setminus F$ is an *even gap* in F if $\#\{\, x \in F : x > v \,\}$ is even. Otherwise, it is an *odd gap.* A subset $F \subset V$ is *even* if every $v \in V \setminus F$ is an even gap. A subset $F \subset V$ is *odd* if every $v \in [m] \setminus F$ is an odd gap. *Gale's Evenness Criterion* [Gal63, Theorem 3][ER96, Lemma 2.3] states that, given a δ-subset $F \subset V$, we have that $|F|$ is an upper facet of $\mathfrak{C}(V,\delta)$ if and only if F is an odd subset, and that $|F|$ is a lower facet of $\mathfrak{C}(V,\delta)$ if and only if F is an even subset. We write

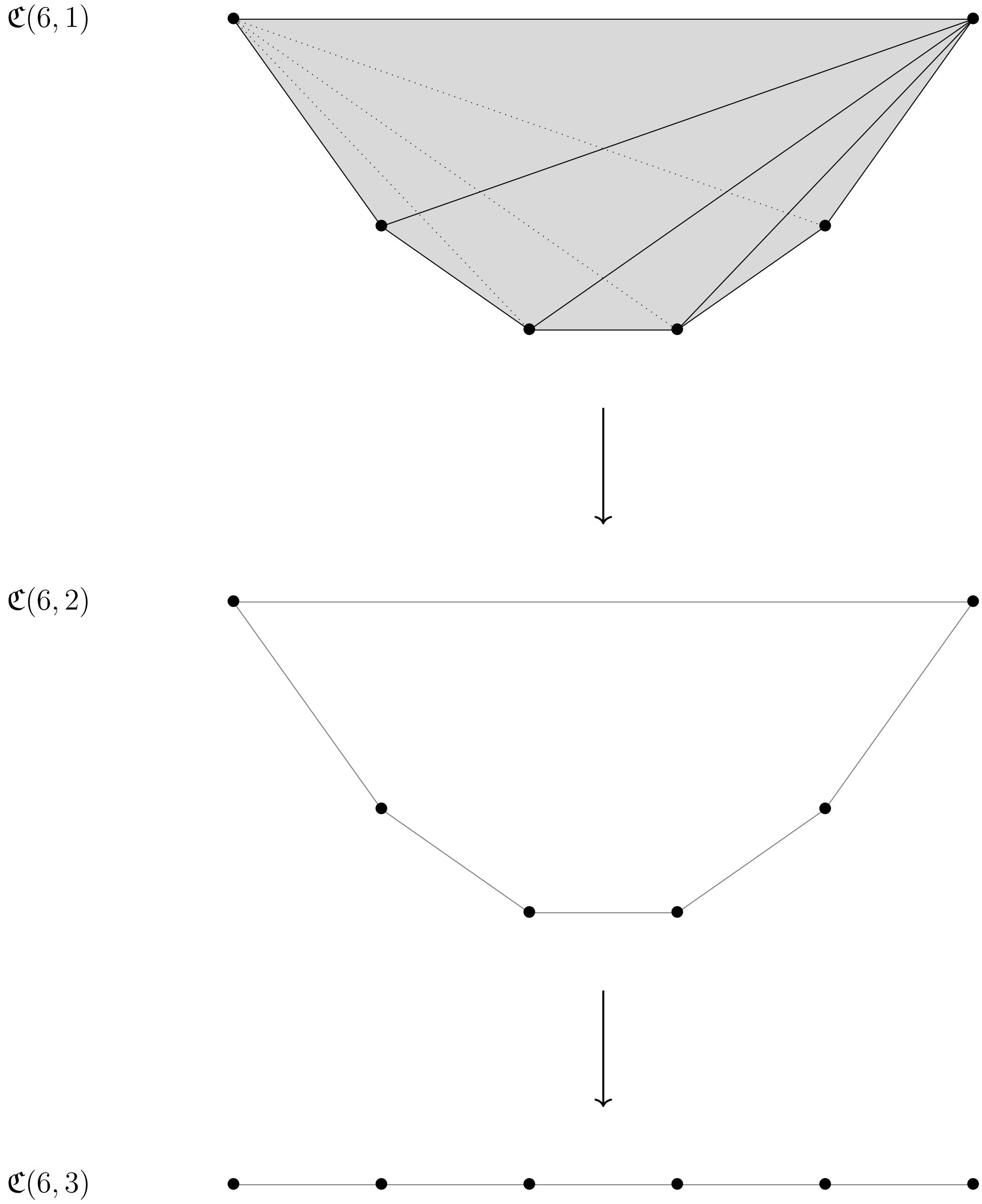
$$\mathcal{F}^l(V,\delta) := \{\, F \subseteq V : |F|_\delta \text{ is a lower facet of } |V|_\delta \,\}, \text{ and}$$
$$\mathcal{F}^u(V,\delta) := \{\, F \subseteq V : |F|_\delta \text{ is an upper facet of } |V|_\delta \,\}.$$

The circuits of a cyclic polytope can also be described combinatorially. Following [OT12, Definition 2.2], if $A, B \subseteq V$ are d-simplices, then we say that A *intertwines* B, and write $A \wr B$, if

$$a_0 < b_0 < a_1 < b_1 < \cdots < a_d < b_d.$$

If either $A \wr B$ or $B \wr A$, then we say that A and B are *intertwining* and write $A \between B$. (That is, we use 'are intertwining' and '$\between$' to refer to the symmetric closure of the

Figure 2.1: Cyclic polytopes [ER96, Figure 2]



relation 'intertwines'.) A collection of d-simplices is called *non-intertwining* if no pair of the elements are intertwining. Following [Wil21a], if A is a $(d-1)$-simplex and B is a d-simplex, then we also say that A *intertwines* B, and write $A \wr B$, if

$$b_0 < a_0 < b_1 < \cdots < a_{d-1} < b_d.$$

The circuits of $\mathfrak{C}(V, \delta)$ are then the pairs (A, B) and (B, A) such that A is a $(\lfloor \frac{\delta}{2} \rfloor)$-simplex, B is a $(\lceil \frac{\delta}{2} \rceil)$-simplex, and A intertwines B. This result is originally due to Breen [Bre73], but it is well-known from the description of the oriented matroid given by a cyclic polytope [BL78; Stu88a; CD00].

These combinatorial characterisations of facets and circuits show that these notions are independent of the particular geometric realisation of $\mathfrak{C}(V, \delta)$. Hence, we will write $\mathcal{F}(V, \delta)$ for the combinatorial facets of $\mathfrak{C}(V, \delta)$, $\mathcal{Z}(V, \delta)$ for the combinatorial circuits of $\mathfrak{C}(V, \delta)$, and $C(V, \delta)$ for the *combinatorial cyclic polytope* consisting of the triple $(V, \mathcal{F}(V, \delta), \mathcal{Z}(V, \delta))$. Facets of $C(V, \delta)$ will also be designated as either upper facets or lower facets, as dictated by Gale's Evenness Criterion. As for geometric cyclic polytopes, we write $C(m, \delta) := C([m], \delta)$. If $\#V = m$, then we say that $C(V, \delta)$ and $C(m, \delta)$ are *congruent*—they only differ by the labels of the vertices.

2.1.3 Triangulations

We now explain our framework for triangulations. We maintain our set-up, where $V \subseteq \mathbb{Z}_{>0}$ is a finite subset, with $|-|\colon V \to \mathbb{R}^\delta$ a geometric realisation giving a geometric polytope $\mathfrak{P} = |V|$, with corresponding combinatorial polytope $P = (V, \mathcal{F}_{\mathfrak{P}}, \mathcal{Z}_{\mathfrak{P}})$.

A *combinatorial δ-simplex* in V is a $(\delta + 1)$-subset $S \subseteq V$. The *k-faces* of S are the subsets of S of size $k + 1$. An *abstract simplicial complex* is a set $\mathcal{A}$ of combinatorial simplices in V such that if $S, S' \in \mathcal{A}$, then $S \not\subseteq S'$. The k-simplices of $\mathcal{A}$ are the k-faces of elements of $\mathcal{A}$. An abstract simplicial complex $\mathcal{A}'$ is an *abstract simplicial subcomplex* of $\mathcal{A}$ if every simplex of $\mathcal{A}'$ is a face of a simplex of $\mathcal{A}$. Hence, we are considering abstract simplicial complexes in terms of their maximal simplices, which is contrary to the more usual approach as having an abstract simplicial complex as the set of all its simplices.

Given a combinatorial δ-simplex $S \subseteq V$, if $\{\, |s| : s \in S \,\}$ is an affinely independent set, then $|S|$ is a *geometric δ-simplex*. A collection $\mathcal{G}$ of geometric simplices is a *geometric simplicial complex* if

$$|S \cap R| = |S| \cap |R|$$

for all $|S|, |R| \in \mathcal{G}$, and if there exist no $|S|, |R| \in \mathcal{G}$ such that $|S|$ is a face of $|R|$. *Geometric simplicial subcomplexes* are defined analogously to abstract simplicial subcomplexes. If $|S \cap R|$ is a proper subset of $|S| \cap |R|$, then we say that $|S|$ and $|R|$ *intersect transversely*. Note that it follows from the definition of a circuit that if (Z_+, Z_-) is a circuit then $|Z_+|$ and $|Z_-|$ intersect transversely, since we have that $Z_+ \cap Z_- = \varnothing$. Conversely, every pair of sets which intersect transversely must contain a circuit. We also remark that our notion of transverse intersection is distinct from the one found in differential topology.

A (*geometric*) *triangulation* of the geometric polytope $\mathfrak{P}$ is a geometric simplicial complex $\mathcal{G}$ such that $\mathfrak{P} = \bigcup_{|S| \in \mathcal{G}} |S|$. A (*combinatorial*) *triangulation* of the combinatorial polytope P is an abstract simplicial complex $\mathcal{T}$ such that

- for all $S \in \mathcal{T}$ and all facets F of S, we either have that F is contained in a facet of P, or there exists $R \in \mathcal{T} \setminus \{S\}$ such that $F \subset R$,
- there is no circuit (Z_+, Z_-) of P such that $Z_+ \subseteq S$ and $Z_- \subseteq R$ for some $S, R \in \mathcal{T}$.

We use $|\mathcal{T}|$ to refer to the geometric simplicial complex corresponding to $\mathcal{T}$. We have that $\mathcal{T}$ is a combinatorial triangulation of P if and only if $|\mathcal{T}|$ is a geometric triangulation of $\mathfrak{P}$. A proof of this can be found in [Ram97, Proposition 2.2].

Remark 2.1.2. In this framework is it possible for two different combinatorial triangulations to correspond to the same geometric triangulation. Namely, this happens when there exist $v, v' \in V$ such that $|v| = |v'|$. When this is the case, there exist different combinatorial simplices which correspond to the geometric simplex. In Chapter 6, we shall take a different approach to geometric triangulations, which allows them to be as fine-grained as combinatorial triangulations. But, until then, the framework we have laid out above is more suited to our purposes.

Triangulations of cyclic polytopes

In this thesis we are usually concerned with combinatorial triangulations of cyclic polytopes, but sometimes we shall need to consider geometric triangulations of cyclic polytopes. When we do so we shall implicitly pick a geometric realisation of $C(m, \delta)$ given by an arbitrary tuple of points on the moment curve, since the properties we will be considering will be independent of the precise choice.

One can use the descriptions of the facets and circuits of $C(m, \delta)$ to determine whether an abstract simplicial complex $\mathcal{T}$ gives a triangulation of $C(m, \delta)$. We denote the set of triangulations of the cyclic polytope $C(m, \delta)$ by $\mathcal{S}(m, \delta)$. There are two triangulations of $C(m, \delta)$ which are of particular note. Namely, the lower facets $\mathcal{F}^l([m], \delta{+}1)$ of $C(m, \delta{+}1)$ give a triangulation of $C(m, \delta)$, which is known as the *lower triangulation*. Similarly, $\mathcal{F}^u([m], \delta + 1)$ gives a triangulation of $C(m, \delta)$, which is known as the *upper triangulation*. Indeed, every triangulation $|\mathcal{T}|$ of $\mathfrak{C}(m, \delta)$ determines a unique piecewise-linear section

$$\sigma_{|\mathcal{T}|} \colon \mathfrak{C}(m, \delta) \to \mathfrak{C}(m, \delta + 1)$$

of $\mathfrak{C}(m, \delta + 1)$ by sending each δ-simplex $|S|_\delta$ of $|\mathcal{T}|$ to $|S|_{\delta+1}$ in $\mathfrak{C}(m, \delta + 1)$, in the natural way. Similarly, a δ-simplex $|S|$ in $\mathfrak{C}(m, \delta)$ defines a map $\sigma_{|A|} \colon |A|_\delta \to \mathfrak{C}(m, \delta + 1)$.

Bistellar flips

Given a triangulation $\mathcal{T}$ of $C(m, \delta)$ and $H \subseteq [m]$, we say that $C(H, \delta)$ is a *subpolytope* of $\mathcal{T}$ if the facets $\mathcal{F}(H, \delta)$ of $C(H, \delta)$ are a simplicial subcomplex of $\mathcal{T}$. Equivalently, we have that $C(H, \delta)$ is a subpolytope of $\mathcal{T}$ if and only if $\{ S \in \mathcal{T} : S \subseteq H \}$ is a triangulation of $C(H, \delta)$. We refer to this triangulation as the *induced triangulation* of $C(H, \delta)$.

Consider the cyclic polytope $C(\delta + 2, \delta)$. Any triangulation $\mathcal{T}$ of $C(\delta + 2, \delta)$ determines a section $\sigma_{|\mathcal{T}|} \colon \mathfrak{C}(\delta + 2, \delta) \to \mathfrak{C}(\delta + 2, \delta + 1)$. But $\mathfrak{C}(\delta + 2, \delta + 1)$ is a simplex. It therefore has only one triangulation and only two sections: one corresponding to its upper facets and one corresponding to its lower facets. Hence the only two triangulations of $C(\delta+2, \delta)$ are the upper triangulation and the lower triangulation. For example, when $\delta = 2$ the polytope $\mathfrak{C}(\delta+2, \delta)$ is a quadrilateral. This has two triangulations, corresponding to the two possible diagonals.

This observation can be used to define an important operation on triangulations. Let $\mathcal{T} \in \mathcal{S}(m, \delta)$. Suppose that there exists a $(\delta + 2)$-subset $H \subseteq [m]$ such that the induced triangulation of $C(H, \delta)$ is the lower triangulation. Let $\mathcal{T}'$ be the triangulation obtained by replacing the portion of $\mathcal{T}$ inside $C(H, \delta)$ with the upper triangulation of $C(H, \delta)$. We then say that $\mathcal{T}'$ is an *increasing bistellar flip* of $\mathcal{T}$ and that $\mathcal{T}$ is a *decreasing bistellar flip* of $\mathcal{T}'$. We also talk about the increasing bistellar flip being *induced by the* $(\delta + 1)$*-simplex* H: the increasing bistellar flip replaces the lower facets of H with the upper facets of H. Here when we refer to the upper and lower facets of the $(\delta+1)$-simplex H, we mean the upper and lower facets of $C(H, \delta + 1)$. We shall often talk about the upper and lower facets of a simplex in this way.

Remark 2.1.3. The simplices of a triangulation $\mathcal{T}$ of $C(m, \delta)$ possess a partial order. Following [Ram97, Definition 5.7], for δ-simplices R, S of a triangulation $\mathcal{T} \in \mathcal{S}(m, \delta)$ with δ vertices in common, we write that $R \lessdot S$ if and only if $R \cap S$ lies in the upper facets of R and the lower facets of S. The relation $\prec$ is defined as the transitive closure of $\lessdot$, so that $R \lessdot S$ implies that $R \prec S$. This is a partial order by [Ram97, Corollary 5.9].

2.1.4 Operations on triangulations

We consider the following operations on triangulations, which were introduced in [Ram97] based on the corresponding operations on oriented matroids from [BL78]. However, note that our notation and terminology is swapped from [BL78; Ram97], following [OT12], since this fits better with how the operations behave on cyclic polytopes.

Operations at the first or last vertex

If $S \subseteq [m]$ is a k-simplex, we define the *contraction* $S[m-1 \leftarrow m]$ of S by

$$S[m-1 \leftarrow m] := \begin{cases} S & \text{if } m \notin S, \\ (S \setminus m) \cup m-1 & \text{otherwise.} \end{cases}$$

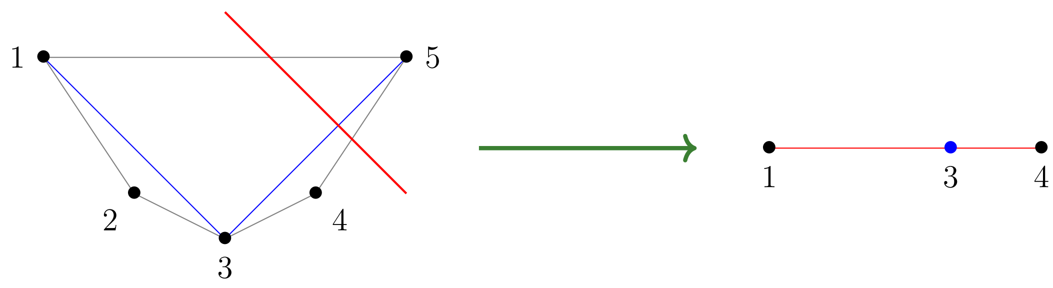
Note that $S[m-1 \leftarrow m]$ is a $(k-1)$-simplex if $S \supseteq \{m-1, m\}$. Given a triangulation $\mathcal{T}$ of $C(m, \delta)$, we define the *contraction* $\mathcal{T}[m-1 \leftarrow m]$ to be the triangulation of $C(m-1, \delta)$ given by

$$\mathcal{T}[m-1 \leftarrow m] := \left\{ S \in \tbinom{[m-1]}{\delta+1} : S = R[m-1 \leftarrow m] \text{ for } R \in \mathcal{T} \right\}.$$

This is indeed a triangulation of $C(m-1, \delta)$ by [Ram97, Theorem 4.2(iii)]. This corresponds to the triangulation obtained from $|\mathcal{T}|$ by moving vertex $|m|$ along the moment curve until it coincides with vertex $|m-1|$, as illustrated in Figure 2.2.

Figure 2.2: The contraction operation $[4 \leftarrow 5]$

Figure 2.3: The deletion operation $-\backslash 5$



Note that some simplices degenerate in this process. There is an analogous operation $-[1 \to 2]$.

Given a triangulation $\mathcal{T}$ of $C(m, \delta)$, we define the *deletion* $\mathcal{T}\backslash m$ to be the triangulation of $C(m-1, \delta-1)$ given by

$$\mathcal{T}\backslash m := \{\, S \setminus m : S \in \mathcal{T},\, m \in S \,\}.$$

This is indeed a triangulation of $C(m-1, \delta-1)$ by [Ram97, Theorem 4.2(ii)]. This is the triangulation induced by $|\mathcal{T}|$ on the vertex figure of $\mathfrak{C}(m, \delta)$ at $|m|$, as illustrated in Figure 2.3. There is an analogous operation $-\backslash 1$.

We will also use the *extension* operation from [Ram97]. Given a triangulation

$\mathcal{T}$ of $C(m,\delta)$, we define $\hat{\mathcal{T}}$ to be the triangulation of $C(m+1,\delta+1)$ given by

$$\hat{\mathcal{T}} := \mathcal{T} * (m+1) \cup \{\, (S \setminus s_\delta) \cup \{l, l+1\} : S \in \mathcal{T},\, s_{\delta-1} < l < s_\delta \,\},$$

where

$$\mathcal{T} * (m+1) := \{\, S \cup \{m+1\} : S \in \mathcal{T} \,\}.$$

More generally, given a set of simplices $\mathcal{U}$ in V and a set $N \subseteq [m] \setminus V$, then $\mathcal{U} * N := \{\, S \cup N : S \in \mathcal{U} \,\}$. One can verify easily that $\hat{\mathcal{T}} \backslash (m+1) = \mathcal{T}$.

Operations at middle vertices

In this thesis we also consider contractions of triangulations at other pairs of vertices besides $[1 \to 2]$ and $[m-1 \leftarrow m]$. For this purpose we let $[m-1]_{v+} := \{1, 2, \ldots, v-1, x, y, v+1, v+2, \ldots, m-1\}$ for $v \in [m-1]$. We also extend this notation in a natural way to subsets $H \subseteq [m-1]$, so that $H_{v+} = (H \setminus v) \cup \{x, y\}$ if $v \in H$, and $H = H$ otherwise.

Given a k-simplex $S \subseteq [m-1]_{v+}$, we define

$$S[x \to v \leftarrow y] := \begin{cases} S & \text{if } \{x, y\} \cap S = \varnothing, \\ (S \setminus \{x, y\}) \cup v & \text{otherwise.} \end{cases}$$

Given a triangulation $\mathcal{T}$ of $C([m-1]_{v+}, \delta)$, we then define $\mathcal{T}[x \to v \leftarrow y]$ to be the triangulation of $C(m-1, \delta)$ given by

$$\mathcal{T}[x \to v \leftarrow y] := \left\{ S \in \tbinom{[m-1]}{\delta+1} : S = R[x \to v \leftarrow y] \text{ for } R \in \mathcal{T} \right\}.$$

This is indeed a triangulation of $C(m-1, \delta)$ by [RS00, Theorem 3.3]. Geometrically, it is obtained from $|\mathcal{T}|$ by moving $|x|$ and $|y|$ along the moment curve towards each other until they coincide with each other at a new vertex, which we label $|v|$. We choose to relabel $[m]$ as $[m-1]_{v+}$ here so that there does not appear to be a missing vertex after contraction. It is also useful to distinguish between the two vertices before contraction and the vertex after contraction.

In order to understand how the contractions $[x \to v \leftarrow y]$ behave, we will consider the deletion operation at other vertices too. Indeed, given $C(m,\delta)$ and $v \in [m]$, define the *combinatorial vertex figure at* v to be the combinatorial polytope $C(m,\delta)\backslash v = ([m] \setminus v, \mathcal{F}_v([m] \setminus v, \delta), \mathcal{Z}_v([m] \setminus v, \delta))$, where

$$\mathcal{F}_v([m] \setminus v, \delta) = \{\, F \subseteq [m] \setminus v : F \cup v \in \mathcal{F}([m], \delta) \,\},$$

and

$$\mathcal{Z}_v([m] \setminus v, \delta) = \{\, (Z_-, Z_+) : (Z_- \cup v, Z_+) \in \mathcal{Z}([m], \delta) \text{ or } (Z_-, Z_+ \cup v) \in \mathcal{Z}([m], \delta) \,\}.$$

Given a triangulation $\mathcal{T}$ of $C(m,\delta)$, we define the deletion $\mathcal{T}\backslash v$ to be the triangulation

$$\mathcal{T}\backslash v := \{\, S \setminus v : S \in \mathcal{T}, v \in S \,\}.$$

It follows straightforwardly from the definition of $C(m,\delta)\backslash v$ and the definition of a combinatorial triangulation that $\mathcal{T}\backslash v$ is a triangulation of $C(m,\delta)\backslash v$. Note also that $|\mathcal{T}\backslash v|$ may be realised geometrically as the triangulation induced by $|\mathcal{T}|$ on the vertex figure of $\mathfrak{C}(m,\delta)$ at $|v|$.

Finally, we shall also consider triangulations given by deleting multiple vertices. Given a triangulation $\mathcal{T}$ of $C(m,\delta)$ and $V \subseteq [m]$, we define the simplicial complex

$$\mathcal{T}\backslash V := \{\, S \setminus V : S \in \mathcal{T},\, V \subseteq S \,\}.$$

In the examples we consider here, V will always be a pair of consecutive vertices, such as $\{1,2\}$ or $\{m-1,m\}$ in $[m]$, or $\{x,y\}$ in $[m]_{v+}$.

2.1.5 Describing expansion at the first or last vertex

One can understand the different triangulations $\widetilde{\mathcal{T}}$ which may contract to a given triangulation $\mathcal{T}$ under the operation $[m-1 \leftarrow m]$ by considering the vertex figure $\mathcal{T}\backslash m-1$. The result [RS00, Lemma 4.7(i)] states that, given a triangulation

$\mathcal{T} \in \mathcal{S}(m-1, \delta)$, triangulations $\widetilde{\mathcal{T}}$ of $C(m, \delta)$ such that $\widetilde{\mathcal{T}}[m-1 \leftarrow m] = \mathcal{T}$ are in bijection with sections of the vertex figure $\mathcal{T}\backslash(m-1)$. Here a *section* of $\mathcal{T}\backslash(m-1)$ is a triangulation $\mathcal{W}$ of $C(m-2, \delta-2)$ which is contained in $\mathcal{T}\backslash(m-1)$ as a simplicial subcomplex. The bijection operates as follows.

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \widetilde{\mathcal{T}} \in \mathcal{S}(m, \delta), \\ \widetilde{\mathcal{T}}[m-1 \leftarrow m] = \mathcal{T} \end{array} \right\} & \longleftrightarrow & \{ \text{Sections } \mathcal{W} \text{ of } \mathcal{T}\backslash(m-1) \} \\
\\
\widetilde{\mathcal{T}} & \longmapsto & \widetilde{\mathcal{T}}\backslash\{m-1, m\} \\
\\
\begin{array}{c} \mathcal{T}^\circ \cup \mathcal{W} * \{m-1, m\} \\ \cup \mathcal{T}\backslash(m-1)^+ * (m-1) \\ \cup \mathcal{T}\backslash(m-1)^- * m \end{array} & \longleftarrow\!\shortmid & \mathcal{W}
\end{array}$$

Here

- $\mathcal{T}^\circ$ is the set of δ-simplices of $\mathcal{T}$ which contain neither $m-1$ nor m as a vertex;
- $\mathcal{T}\backslash(m-1)^+$ is the set of $(\delta-1)$-simplices S of $\mathcal{T}\backslash(m-1)$ such that $|S|_{\delta-1}$ is above $|\mathcal{W}|_{\delta-1}$ with respect to the $(\delta-1)$-th coordinate;
- $\mathcal{T}\backslash(m-1)^-$ is the set of $(\delta-1)$-simplices S of $\mathcal{T}\backslash(m-1)$ such that $|S|_{\delta-1}$ is below $|\mathcal{W}|_{\delta-1}$ with respect to the $(\delta-1)$-th coordinate.

It is this result that we will generalise in Section 2.3. We will show that an analogous statement is true for the expansions at middle vertices $[x \to v \leftarrow y]$.

Note that the sets $\mathcal{T}\backslash(m-1)^+$ and $\mathcal{T}\backslash(m-1)^-$ are defined with respect to a geometric realisation. Recall that $\mathcal{T}\backslash(m-1)$ is a triangulation of $C(m-2, \delta-1)$, so our geometric realisation is $|-|_{\delta-1}$. The sets $\mathcal{T}\backslash(m-1)^+$ and $\mathcal{T}\backslash(m-1)^-$, of course, do not depend upon the particular geometric realisation of $C(m-2, \delta-1)$ given by the choice of points on the moment curve.

We often adopt the opposite perspective to contraction, where we think of the triangulation $\mathcal{T}$ as expanding to $\widetilde{\mathcal{T}}$. Under this perspective, we are trying to understand the triangulations $\widetilde{\mathcal{T}}$ of $C(m, \delta)$ to which $\mathcal{T}$ can expand when one expands the cyclic polytope $C(m-1, \delta)$ at a particular vertex.

We now demonstrate how the result [RS00, Lemma 4.7(i)] works, using an example.

Example 2.1.4. We consider the triangulation $\mathcal{T}$ of $C(5,3)$ with 3-simplices $\{1234, 1245, 2345\}$. Here we abbreviate by writing the simplices as strings, so that $1234 = \{1,2,3,4\}$. The triangulations $\widetilde{\mathcal{T}}$ of $C(6,3)$ such that $\widetilde{\mathcal{T}}[5 \leftarrow 6] = \mathcal{T}$ are in bijection with the sections of the triangulation $\mathcal{T} \backslash 5$. We have that $\mathcal{T} \backslash 5$ is a triangulation of $C(4,2)$ which can be realised geometrically as the triangulation of the vertex figure of $C(5,3)$ at $|5|$. The triangulations $|\mathcal{T}|$ and $|\mathcal{T} \backslash 5|$ are illustrated in Figure 2.4.

The triangulation $\mathcal{T} \backslash 5$ has three sections $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$, which are illustrated in Figure 2.5. By [RS00, Lemma 4.7(i)], these sections correspond to triangulations $\widetilde{\mathcal{T}}_1, \widetilde{\mathcal{T}}_2, \widetilde{\mathcal{T}}_3$ of $C(6,3)$ such that $\widetilde{\mathcal{T}}_i[5 \leftarrow 6] = \mathcal{T}$, and

$$\widetilde{\mathcal{T}}_i = \mathcal{T}^\circ \cup (\mathcal{W}_i * \{5,6\}) \cup (\mathcal{T} \backslash 5^+ * 5) \cup (\mathcal{T} \backslash 5^- * 6).$$

Hence one may compute that

$$\begin{aligned}
\widetilde{\mathcal{T}}_1 &= \{1234\} \cup \{1256, 2356, 3456\} \cup \{1245, 2345\} \cup \varnothing, \\
\widetilde{\mathcal{T}}_2 &= \{1234\} \cup \{1256, 2456\} \cup \{1245\} \cup \{2346\}, \\
\widetilde{\mathcal{T}}_3 &= \{1234\} \cup \{1456\} \cup \varnothing \cup \{1246, 2346\}.
\end{aligned}$$

Figure 2.4: The triangulation $|\mathcal{T}|$ of $\mathfrak{C}(5,3)$ and the triangulation $|\mathcal{T}\backslash 5|$

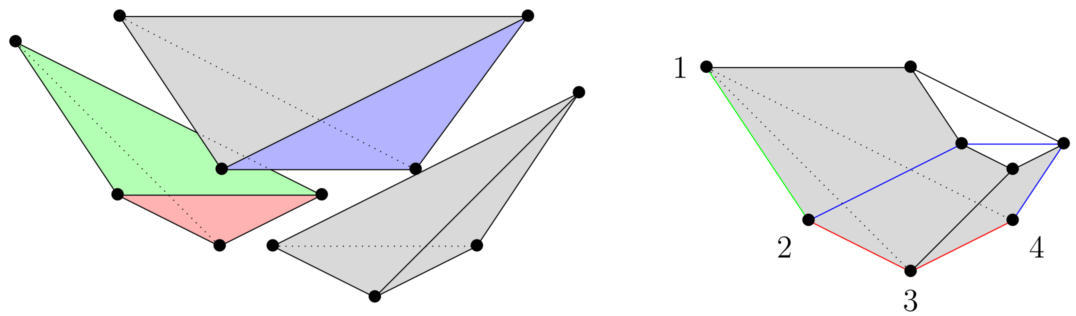
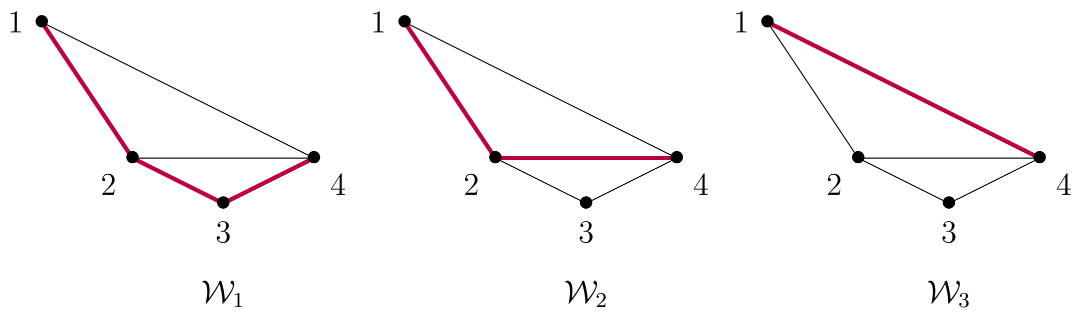


Figure 2.5: Sections of $|\mathcal{T}\backslash 5|$



2.2 Combinatorial description of triangulations

The description of a triangulation of $C(m,\delta)$ as a set of δ-simplices contains redundant information. For instance, to describe a triangulation of a polygon, it suffices to specify the arcs of the triangulation, rather than the triangles. Likewise, one can determine higher-dimensional triangulations by only specifying lower-dimensional simplices. Indeed, it follows from [Dey93] that a triangulation of $C(m,\delta)$ is determined by its internal $\lfloor \delta/2 \rfloor$-simplices, where a simplex $A \subseteq [m]$ is an *internal simplex of* $C(m,\delta)$ if A does not lie within any facet of $C(m,\delta)$. Internal geometric simplices are defined analogously. For a triangulation $\mathcal{T}$ of $C(m,\delta)$, we write $\mathring{e}(\mathcal{T})$ for its set of internal $\lfloor \delta/2 \rfloor$-simplices. In Section 2.2.1, we give the description of triangulations of $2d$-dimensional cyclic polytopes in terms of their d-simplices from [OT12]. We then show in Section 2.2.2 how $(2d+1)$-dimensional triangulations may be described in terms of their d-simplices, giving the other half of the picture.

2.2.1 Even dimensions

In even dimensions, A is an internal d-simplex of $C(m,2d)$ if and only if

$$A \in {}^{\circlearrowleft}\mathbf{I}_m^d := \left\{\, B \in \tbinom{[m]}{d+1} : b_i \leqslant b_{i+1} - 2 \;\forall i \in [d], \text{ and } b_d \leqslant b_0 + m - 2 \,\right\},$$

by [OT12, Lemma 2.1]. This can also be seen by applying Gale's Evenness Criterion.

In even dimensions, given $\mathbf{X} \subseteq {}^{\circlearrowleft}\mathbf{I}_m^d$, we have that $\mathbf{X} = \mathring{e}(\mathcal{T})$ for some triangulation $\mathcal{T}$ of $C(m,2d)$ if and only if $\#\mathbf{X} = \binom{m-d-2}{d}$ and $\mathbf{X}$ is non-intertwining [OT12, Theorem 2.3 and Theorem 2.4]. Moreover, $\mathring{e}(\mathcal{T})$ is a maximal non-intertwining collection with respect to inclusion, and $\binom{m-d-2}{d}$ is the maximal size of a non-intertwining collection.

The contraction operation can be interpreted in this framework, namely

$$\mathring{e}(\mathcal{T}[m-1 \leftarrow m]) = \{\, A \in {}^{\circlearrowleft}\mathbf{I}^d_{m-1} : A = B[m-1 \leftarrow m] \text{ for } B \in \mathring{e}(\mathcal{T}) \,\}$$

by [OT12, Lemma 2.23]. An analogous statement holds for the contraction $[1 \to 2]$.

Remark 2.2.1. Note that our sets $\mathring{e}(\mathcal{T})$ are different from the sets $e(\mathcal{T})$ from [OT12], since the latter contain d-simplices which do not lie in any lower facets of $C(m, 2d)$, but may contain d-simplices lying in upper facets of $C(m, 2d)$. Given a triangulation $\mathcal{T}$ of $C(m, 2d)$, we have that $e(\mathcal{T}) \setminus \mathring{e}(\mathcal{T})$ consists of all the d-simplices lying in the upper facets of $C(m, 2d)$ which do not lie in any lower facets of $C(m, 2d)$. One can verify using [OT12, Lemma 2.1] or Gale's Evenness Criterion that this set has cardinality $\binom{m-d-2}{d-1}$, and hence $\#e(\mathcal{T}) = \binom{m-d-2}{d} + \binom{m-d-2}{d-1} = \binom{m-d-1}{d}$, as stated in [OT12, Theorem 2.3 and Theorem 2.4].

The following lemma describing the intersections of the lower and upper facets of a $(2d+1)$-simplex will be useful later.

Lemma 2.2.2. *Let $A, B \in \binom{[m]}{d+1}$. Then $A \wr B$ if and only if we have that $A \cup B$ is a $(2d+1)$-simplex with A the intersection of its lower facets, and B the intersection of its upper facets.*

Proof. Let $A \cup B =: S$. If $\#S < 2d+2$, then A and B cannot be intertwining. Hence suppose that $\#S = 2d+2$, so that $A \cup B$ is a $(2d+1)$-simplex. We then apply Gale's Evenness Criterion. The vertices of a lower facet of S have an even gap. The intersection of these subsets is $\{s_0, s_2, \dots, s_{2d}\}$. The vertices of an upper facet of S have an odd gap. Their intersection is $\{s_1, s_3, \dots, s_{2d+1}\}$. Therefore $A \wr B$ if and only if A is the intersection of the lower facets of S and B is the intersection of its upper facets. $\square$

2.2.2 Odd dimensions

Having detailed how even-dimensional triangulations may be defined combinatorially, it is far from obvious what the counterpart description for odd dimensions should look like. In a $(2d+1)$-dimensional cyclic polytope, d-simplices do not intersect each other, and numbers of simplices vary between triangulations.

One appealing way of solving these problems might be to describe triangulations of $(2d+1)$-dimensional cyclic polytopes as inclusion-maximal sets of non-intersecting d-simplices and $(d+1)$-simplices. This approach attempts to mimic the even-dimensional description. However, there are two issues here. The first issue is that a non-intersecting collection of simplices which is maximal with respect to adding more simplices does not necessarily give a triangulation of a cyclic polytope, as first shown in [Ram97, Example 4.5]. This is why maximality of size is required in even dimensions, rather than simply maximality with respect to inclusion. But, as discussed above, numbers of simplices vary between triangulations in odd dimensions. The second issue is that, by [Dey93], a triangulation of a $(2d+1)$-dimensional cyclic polytope is determined by its d-simplices, so that including the $(d+1)$-simplices in a description is redundant.

We solve these problems by taking an approach which is distinctive to odd dimensions, rather than trying to simulate the even-dimensional description. We describe triangulations of $(2d+1)$-dimensional cyclic polytopes in terms of their d-simplices by defining two new properties (*supporting* and *bridging*) which imply that a given set of d-simplices arises from a triangulation. See Definition 2.2.11 for the definition of being supporting, and Definition 2.2.13 for the definition of being bridging. Indeed, we prove the following theorem, which we build up to using a series of lemmas. We define the set

$$\mathbf{J}_m^d := \{\, \{a_0, a_1, \ldots, a_d\} \in {}^{\circlearrowleft}\mathbf{I}_m^d : a_0 \neq 1,\, a_d \neq m \,\}.$$

Theorem 2.2.3. *There is a bijection via $\mathcal{T} \mapsto \mathring{e}(\mathcal{T})$ between triangulations of $C(m, 2d+1)$ and subcollections of $\mathbf{J}_m^d$ which are supporting and bridging.*

This result is new and completes the characterisation of triangulations of δ-dimensional cyclic polytopes in terms of their $\lfloor \delta/2 \rfloor$-simplices.

Preliminary lemmas

We begin by characterising the internal d-simplices of $C(m, 2d+1)$.

Lemma 2.2.4. *A simplex $A \in \binom{[m]}{d+1}$ is an internal d-simplex of $C(m, 2d+1)$ if and only if $A \in \mathbf{J}_m^d$.*

Proof. One way to see this is to note that A is an internal d-simplex if and only if there is a $(d+1)$-simplex B in $C(m, 2d+1)$ such that $A \wr B$, so that $|A|$ and $|B|$ intersect transversely. But such a B exists if and only if $A \in \mathbf{J}_m^d$.

Alternatively, one can apply Gale's Eveness Criterion, which entails that the vertices of an upper facet of $C(m, 2d+1)$ consist of m together with d disjoint pairs of consecutive integers, and the vertices of a lower facet consist of 1 together with d disjoint pairs of consecutive integers. Hence A is not contained in a facet if and only if $A \in \mathbf{J}_m^d$. $\square$

We can describe the effect of a bistellar flip on internal d-simplices.

Lemma 2.2.5. *Let $A \in \binom{[m]}{d+1}$ and $B \in \binom{[m]}{d+2}$. Then $A \wr B$ if and only if we have that $A \cup B$ is a $(2d+2)$-simplex with A the intersection of its lower facets, and B the intersection of its upper facets.*

Proof. Let $A \cup B =: S$. If $\#S < 2d+3$, then A and B cannot be intertwining. Hence suppose that $\#S = 2d+3$, so that $A \cup B$ is a $(2d+2)$-simplex. We then apply Gale's Evenness Criterion. The vertices of a lower facet of S miss out an even entry of S. The intersection of these subsets is $\{s_1, s_3, \ldots, s_{2d+1}\}$. The vertices of an upper facet of S miss out an odd entry of S. Their intersection is $\{s_0, s_2, \ldots, s_{2d+2}\}$. Therefore $A \wr B$ if and only if A is the intersection of the lower facets of S and B is the intersection of its upper facets. $\square$

Corollary 2.2.6. *Let $A \in \binom{[m]}{d+1}$ and $B \in \binom{[m]}{d+2}$ such that $A \wr B$. Then the only internal d-simplex of the lower triangulation of $C(A \cup B, 2d+1)$ is A, and the upper triangulation of $C(A \cup B, 2d+1)$ has no internal d-simplices.*

Proof. This follows from Lemma 2.2.4 and Lemma 2.2.5. $\square$

Hence we think of bistellar flips in odd dimensions as replacing a d-simplex with a $(d+1)$-simplex which it intertwines. Note that these simplices form two halves of a circuit. This perspective is, of course, a simplification, since removing the d-simplex also involves removing simplices of higher dimension. Likewise, adding the $(d+1)$-simplex involves adding simplices of higher dimension too.

Remark 2.2.7. Lemma 2.2.2 and Lemma 2.2.5 allow us to think of one half of a circuit as being on top and the other half of the circuit being on bottom.

The even-dimensional counterpart of the following result was shown in [OT12, Proposition 2.13]. This lemma describes the $(2d+1)$-simplex which lies below an internal d-simplex in a $(2d+1)$-dimensional triangulation.

Lemma 2.2.8. *Let $\mathcal{T} \in \mathcal{S}(m, 2d+1)$. Let A be an internal d-simplex of $\mathcal{T}$. Then there is a unique $(2d+1)$-simplex $A \cup B$ of $\mathcal{T}$ such that B is a d-simplex with $B \wr A$.*

Proof. We argue in terms of the geometric realisation. Since $|A|$ is an internal d-simplex of $|\mathcal{T}|$, the points immediately below $|A|$ must lie in a unique $(2d+1)$-simplex $|S|$. Then $|A|$ is a d-face of $|S|$, and hence is the intersection of $d+1$ facets of $|S|$. The simplex $|S|$ has $d+1$ upper facets and $d+1$ lower facets, by Gale's Evenness Criterion. Then $|A|$ must be the intersection of the upper facets of $|S|$, otherwise $|A|$ lies in a lower facet of $|S|$, and so $|S|$ cannot contain the points immediately below $|A|$. But then, by Lemma 2.2.2, we must have that $S = A \cup B$, where $B \wr A$. $\square$

By [Dey93], we know that it is possible to reconstruct a triangulation of a point configuration in $\mathbb{R}^\delta$ on the basis of knowing only its $\lfloor \frac{\delta}{2} \rfloor$-faces. Hence we can reconstruct a triangulation of $C(m, 2d+1)$ from its d-simplices alone—in particular, its internal d-simplices. However, in the manner of [OT12, Lemma 2.15], we affirm this result by showing what the reconstructed triangulation looks like.

Lemma 2.2.9. *Let $\mathcal{T} \in \mathcal{S}(m, 2d+1)$. Then $\mathcal{T}$ is determined by $\mathring{e}(\mathcal{T})$. In particular,*

(1) the $(d+1)$-simplices of $\mathcal{T}$ are those

(a) whose d-faces A are either such that $A \in \mathring{e}(\mathcal{T})$ or such that $A \notin \mathbf{J}^d_m$, and

(b) which are such that there is no d-simplex of $\mathring{e}(\mathcal{T})$ which forms a circuit with them;

(2) the k-simplices of $\mathcal{T}$ for $k > d+1$ are those whose $(d+1)$-faces satisfy (1).

Proof. It follows from Gale's Evenness Criterion that for $k < d$, every k-simplex lies on the boundary of $C(m, 2d+1)$, and hence can be ignored.

Let A be a $(d+1)$-simplex of $\mathcal{T}$. Then clearly A cannot form a circuit with any d-simplices of $\mathcal{T}$. Moreover, every d-face B of A is either internal, so that $B \in \mathring{e}(\mathcal{T})$, or not internal, so that $B \notin \mathbf{J}^d_m$. If A is a k-simplex of $\mathcal{T}$ for $k > d+1$, then all of the $(d+1)$-faces of A must satisfy (1) for these reasons.

Conversely, if A is not a k-simplex of $\mathcal{T}$ for some $k \geqslant d+1$, then $|A|$ must intersect a $(2d+1)$-simplex of $|\mathcal{T}|$ transversely. By the description of the circuits of $C(m, 2d+1)$, either A has a d-face A_d which forms a circuit with a $(d+1)$-simplex B_{d+1} of $\mathcal{T}$, or A has a $(d+1)$-face A_{d+1} which forms a circuit with a d-simplex B_d

of $\mathcal{T}$. In the first case, A_d cannot be in $\mathring{e}(\mathcal{T})$, so any $(d+1)$-face of A containing A_d does not satisfy (1). In the second case, A_{d+1} does not satisfy (1). □

The supporting and bridging conditions

We now derive the properties which characterise triangulations of odd-dimensional cyclic polytopes. The following lemma is shown for $d = 1$ in [ER96, Lemma 4.3].

Lemma 2.2.10. *Let $\mathcal{T} \in \mathcal{S}(m, 2d+1)$. Suppose that A is an internal d-simplex of $\mathcal{T}$. Then there is a $(d-1)$-simplex E such that $E \wr A$ and for every d-simplex $B \subset A \cup E$ we have that B is a d-simplex of $\mathcal{T}$.*

Proof. By Lemma 2.2.8, there is a d-simplex J of $\mathcal{T}$ such that $J \wr A$. If we let $E = \{j_1, j_2, \ldots, j_d\}$, then we have that $E \wr A$. Moreover $A \cup E$ is a face of $A \cup J$, which is a $2d$-simplex of $\mathcal{T}$. Hence every d-simplex B such that $B \subset A \cup E$ is a d-simplex of $\mathcal{T}$. □

Hence, we define the following property.

Definition 2.2.11. Let $\mathbf{X} \subseteq \mathbf{J}_m^d$. Given $A \in \mathbf{X}$ and $E \in \binom{[m]}{d}$ with $E \wr A$, we say that E is a *support for A in* $\mathbf{X}$ if for every internal d-simplex $B \subset A \cup E$, we have that $B \in \mathbf{X}$. We then say that $\mathbf{X}$ is *supporting* if every $A \in \mathbf{X}$ has a support.

The inspiration for the following lemma comes from [ER96, Proposition 3.3, D3, and Proposition 4.2, T3], which concern simpler versions of the property for dimensions 2 and 3.

Lemma 2.2.12. *Let $\mathcal{T} \in \mathcal{S}(m, 2d+1)$. Let*

$$A := \{q_0, q_1, \ldots, q_{i-1}, a_i, a_{i+1}, \ldots, a_j, q_{j+1}, q_{j+2}, \ldots, q_d\},$$
$$B := \{q_0, q_1, \ldots, q_{i-1}, b_i, b_{i+1} \ldots, b_j, q_{j+1}, q_{j+2}, \ldots, q_d\}$$

be internal d-simplices of $\mathcal{T}$, where possibly $i = 0$ or $j = d$, or both. Suppose these are such that $\{a_i, a_{i+1}, \dots, a_j\} \wr \{b_i, b_{i+1}, \dots, b_j\}$. Then

$$S_k := \{q_0, q_1, \dots, q_{i-1}, a_i, a_{i+1}, \dots, a_{k-1}, b_k, b_{k+1}, \dots, b_j, q_{j+1}, q_{j+2}, \dots, q_d\}$$

is an internal d-simplex of $\mathcal{T}$ for all $i < k < j + 1$.

Proof. First, note that if $A, B \in \mathbf{J}_m^d$, then $S_k \in \mathbf{J}_m^d$.

We use induction on increasing bistellar flips of the triangulation: all triangulations of $C(m, 2d + 1)$ can be reached via increasing bistellar flips from the lower triangulation by [Ram97, Theorem 1.1]. In the base case, which is the lower triangulation of $C(m, 2d+1)$, all d-simplices are simplices of the triangulation $\mathcal{T}$. Hence the result holds trivially in this case.

We use contradiction to show the inductive step. Suppose that we perform an increasing bistellar flip on $\mathcal{T}$ by removing the d-simplex

$$S_k = \{q_0, q_1, \dots, q_{i-1}, a_i, a_{i+1} \dots, a_{k-1}, b_k, b_{k+1} \dots, b_j, q_{j+1}, q_{j+2}, \dots, q_d\}$$

for k such that $i < k < j+1$, so that we replace it with a $(d+1)$-simplex F such that $S_k \wr F$. Then F cannot form a circuit with $S_{k'}$ for $k' \neq k$, since, by the induction hypothesis, these are d-simplices of $\mathcal{T}$. Thus we must have $a_k \leqslant f_k$ and $f_k \leqslant b_{k-1}$, otherwise $S_{k+1} \wr F$ or $S_{k-1} \wr F$, respectively. But this is a contradiction, since $b_{k-1} < a_k$. Hence we can never perform an increasing bistellar flip by removing S_k, which means that the above property must be preserved by increasing bistellar flips. $\square$

Hence, we define the following property.

Definition 2.2.13. Let $\mathbf{X} \subseteq \mathbf{J}_m^d$. We say that $\mathbf{X}$ is *bridging* if whenever

$$\{q_0, q_1, \dots, q_{i-1}, a_i, a_{i+1} \dots, a_j, q_{j+1}, q_{j+2}, \dots, q_d\} \in \mathbf{X}, \text{ and}$$
$$\{q_0, q_1, \dots, q_{i-1}, b_i, b_{i+1}, \dots, b_j, q_{j+1}, q_{j+2}, \dots, q_d\} \in \mathbf{X},$$

where possibly $i = 0$ or $j = d$, or both, such that $\{a_i, a_{i+1}, \ldots, a_j\} \wr \{b_i, b_{i+1}, \ldots, b_j\}$, we have that

$$S_k := \{q_0, q_1, \ldots, q_{i-1}, a_i, a_{i+1}, \ldots, a_{k-1}, b_k, b_{k+1}, \ldots, b_j, q_{j+1}, q_{j+2}, \ldots, q_d\} \in \mathbf{X}$$

for all $i < k < j + 1$.

Supporting and bridging under contraction and deletion

Our strategy is now to interpret the operations from Section 2.1.4 in terms of internal d-simplices, and then to use this to show that the properties of being supporting and bridging are preserved by the operations. This will allow us to inductively construct a triangulation from a collection of simplices which is supporting and bridging. This is the same as the strategy used in [OT12, Section 2].

Definition 2.2.14. Let $\mathbf{X} \subseteq \mathbf{J}_m^d$. We define

$$\mathbf{X}[1 \to 2] := \{\, \{a_0, a_1, \ldots, a_d\} \in \mathbf{X} : a_0 \neq 2 \,\},$$
$$\mathbf{X} \backslash \{1, 2\} := \{\, \{a_1, a_2, \ldots, a_d\} : \{2, a_1, a_2, \ldots, a_d\} \in \mathbf{X} \,\}.$$

Note here that our notation $\mathbf{X}\backslash\{1,2\}$ is the same as [OT12, Definition 2.17], but our operation is different. This is because we want these operations on collections of simplices to accord with the corresponding operations on triangulations. We now show that this is indeed the case.

Lemma 2.2.15. *Let $\mathcal{T} \in \mathcal{S}(m, 2d+1)$. Then $\mathring{e}(\mathcal{T}[1 \to 2]) = \mathring{e}(\mathcal{T})[1 \to 2]$.*

Proof. Let $A \in \mathring{e}(\mathcal{T}[1 \to 2])$. Then we cannot have $a_0 = 2$, otherwise A is a boundary d-simplex. Hence $a_0 > 2$. But then the pre-image of A under the contraction $[1 \to 2]$ must be A. Therefore $A \in \mathring{e}(\mathcal{T})$, and so $A \in \mathring{e}(\mathcal{T})[1 \to 2]$, since $a_0 \neq 2$.

Conversely, let A be a d-simplex in $\mathring{e}(\mathcal{T})[1 \to 2]$. Then $A \in \mathring{e}(\mathcal{T})$ and $a_0 > 2$. Hence A is unaffected by the contraction $[1 \to 2]$, and so $A \in \mathring{e}(\mathcal{T}[1 \to 2])$. $\square$

Lemma 2.2.16. *Let $\mathcal{T} \in \mathcal{S}(m, 2d+1)$. Then $\mathring{e}(\mathcal{T}\backslash\{1,2\}) = \mathring{e}(\mathcal{T})\backslash\{1,2\}$.*

Proof. Let A be a $(d-1)$-simplex in $\mathring{e}(\mathcal{T}\backslash\{1,2\})$. Then $1 \cup 2 \cup A$ must be a $(d+1)$-simplex of $\mathcal{T}$. We must have $a_0 > 3$, since A is internal in $C([3,m], 2d+1)$. Therefore $2 \cup A \in \mathring{e}(\mathcal{T})$, and so $A \in \mathring{e}(\mathcal{T})\backslash\{1,2\}$.

Conversely, let $A \in \mathring{e}(\mathcal{T})\backslash\{1,2\}$, so that $2 \cup A \in \mathring{e}(\mathcal{T})$. Therefore $\mathcal{T}$ contains every d-face of the $(d+1)$-simplex $1 \cup 2 \cup A$, since all the other d-faces lie on the boundary of $C(m, 2d+1)$. Moreover, since there cannot be a d-simplex B of $\mathcal{T}$ such that $B \wr 1 \cup 2 \cup A$, we must have that $1 \cup 2 \cup A$ is a $(d+1)$-simplex of $\mathcal{T}$ by Lemma 2.2.9. Hence, A is a d-simplex of $\mathcal{T}\backslash\{1,2\}$. Furthermore, $a_0 > 3$ because $2 \cup A \in \mathring{e}(\mathcal{T})$. Hence $A \in \mathring{e}(\mathcal{T}\backslash\{1,2\})$. $\square$

Having described the effects of contraction and deletion on the sets $\mathring{e}(\mathcal{T})$, we now show that these operations preserve the supporting and bridging conditions.

Lemma 2.2.17. *Suppose that $\mathbf{X} \subseteq \mathbf{J}^d_m$ is supporting and bridging. Then $\mathbf{X}[1 \to 2]$ is also supporting and bridging.*

Proof. We first show that $\mathbf{X}[1 \to 2]$ must be supporting. Let $A \in \mathbf{X}[1 \to 2]$. Then $A \in \mathbf{X}$. Since $\mathbf{X}$ is supporting, there must be B such that $B \wr A$ and every internal d-simplex contained in $A \cup B$ is in $\mathbf{X}$. But these d-simplices will also be contained in $\mathbf{X}[1 \to 2]$, which is therefore also supporting.

Now we show that $\mathbf{X}[1 \to 2]$ must be bridging. Let

$$A := \{q_0, q_1, \dots, q_{i-1}, a_i, a_{i+1} \dots, a_j, q_{j+1}, q_{j+2}, \dots, q_d\},$$
$$B := \{q_0, q_1, \dots, q_{i-1}, b_i, b_{i+1}, \dots, b_j, q_{j+1}, q_{j+2}, \dots, q_d\}$$

$\in \mathbf{X}[1 \to 2]$, where possibly $i = 0$ or $j = d$, or both. Suppose these are such that $\{a_i, a_{i+1}, \dots, a_j\} \wr \{b_i, b_{i+1}, \dots, b_j\}$. Then $A, B \in \mathbf{X}$. Since $\mathbf{X}$ is bridging, we must have

$$S_k := \{q_0, q_1, \dots, q_{i-1}, a_i, a_{i+1}, \dots, a_{k-1}, b_k, b_{k+1}, \dots, b_j, q_{j+1}, q_{j+2}, \dots, q_d\} \in \mathbf{X}$$

for all $i \leqslant k \leqslant j+1$. But then $S_k \in \mathbf{X}[1 \to 2]$ for all $i \leqslant k \leqslant j+1$ since $q_0 \neq 2$ by assumption. □

Lemma 2.2.18. *Suppose that $\mathbf{X} \subseteq \mathbf{J}_m^d$ is supporting and bridging. Then $\mathbf{X} \backslash \{1, 2\}$ is also supporting and bridging.*

Proof. We first show that $\mathbf{X} \backslash \{1, 2\}$ must be supporting. Let $A \in \mathbf{X} \backslash \{1, 2\}$. Then $A' := 2 \cup A \in \mathbf{X}$. Thus, since $\mathbf{X}$ is supporting, there is a $(d-1)$-simplex B' such that $B' \wr A'$ and every internal d-simplex contained in $A' \cup B'$ is in $\mathbf{X}$. Then $B = \{b'_1, b'_2, \dots, b'_{d-1}\}$ is such that $B \wr A$. Let C be a an internal $(d-1)$-simplex contained in $A \cup B$. Then $2 \cup C$ is contained in $A' \cup B'$, and so is in $\mathbf{X}$, since $c_0 \geqslant a_0 > 3$. This implies that $C \in \mathbf{X} \backslash \{1, 2\}$, which gives that $\mathbf{X}$ is supporting.

We now show that $\mathbf{X} \backslash \{1, 2\}$ must be bridging. Let

$$A := \{q_1, q_2, \dots, q_{i-1}, a_i, a_{i+1}, \dots, a_j, q_{j+1}, q_{j+2}, \dots, q_d\},$$
$$B := \{q_1, q_2, \dots, q_{i-1}, b_i, b_{i+1}, \dots, b_j, q_{j+1}, q_{j+2}, \dots, q_d\}$$

$\in \mathbf{X} \backslash \{1, 2\}$, where possibly $i = 0$ or $j = d$, or both. Suppose that these are such that $\{a_i, a_{i+1}, \dots, a_j\} \wr \{b_i, b_{i+1}, \dots, b_j\}$. Then $A' := 2 \cup A$, $B' := 2 \cup B \in \mathbf{X}$. Since $\mathbf{X}$ is bridging, we must have

$$S'_k := \{2, q_1, q_2, \dots, q_{i-1}, a_i, a_{i+1}, \dots, a_{k-1}, b_k, b_{k+1}, \dots, b_j, q_{j+1}, q_{j+2}, \dots, q_d\} \in \mathbf{X}$$

for all $i \leqslant k \leqslant j+1$. But then

$$S_k := \{q_1, \dots, q_{i-1}, a_i, a_{i+1}, \dots, a_{k-1}, b_k, b_{k+1}, \dots, b_j, q_{j+1}, \dots, q_d\} \in \mathbf{X} \backslash \{1, 2\}$$

for all $i \leqslant k \leqslant j+1$. □

Characterising odd-dimensional triangulations

The following technical proposition is key to proving our characterisation of triangulations of odd-dimensional cyclic polytopes. We shall use it in the subsequent proposition to construct a triangulation whose internal d-simplices are given by a particular supporting and bridging collection of simplices. This is the most difficult step in the proof comprised by this section.

Proposition 2.2.19. *Let* $\mathbf{X} \subseteq \mathbf{J}_m^d$ *be supporting and bridging for* $C(m, 2d+1)$. *We suppose that* $\mathbf{X}$ *is such that there are triangulations* $\mathcal{U} \in \mathcal{S}([2,m], 2d+1)$ *and* $\mathcal{W} \in \mathcal{S}([3,m], 2d-1)$ *such that* $\mathring{e}(\mathcal{U}) = \mathbf{X}[1 \to 2]$ *and* $\mathring{e}(\mathcal{W}) = \mathbf{X}\backslash\{1,2\}$. *Then* $\mathcal{W}$ *is a section of* $\mathcal{U}\backslash 2$.

Proof. For this it suffices to show that any d-simplex of $\mathcal{W}$ is a d-simplex of $\mathcal{U}\backslash 2$. This is because $\mathcal{U}\backslash 2$ is a triangulation of a $2d$-dimensional cyclic polytope, and hence is determined by its d-simplices by [OT12, Lemma 2.15]. For $k > d$, a k-simplex A is a k-simplex of $\mathcal{W}$ if and only if all its d-faces are d-simplices of $\mathcal{W}$, by Lemma 2.2.9. Moreover, A is a k-simplex of $\mathcal{U}\backslash 2$ if and only if all its d-faces are d-simplices of $\mathcal{U}\backslash 2$, by [OT12, Lemma 2.15]. Hence if $\mathcal{U}\backslash 2$ contains all the d-simplices of $\mathcal{W}$, it must contain all the higher-dimensional simplices of $\mathcal{W}$ as well.

Note that d-simplices of $\mathcal{U}\backslash 2$ result from $(d+1)$-simplices of $\mathcal{U}$ with 2 as a vertex. Hence one can show that every d-simplex of $\mathcal{W}$ is a d-simplex of $\mathcal{U}\backslash 2$ by showing that every $(d+1)$-simplex of $2 * \mathcal{W}$ is a $(d+1)$-simplex of $\mathcal{U}$. In turn, by Lemma 2.2.9, one can show this by showing that every d-simplex of $2 * \mathcal{W}$ is a d-simplex of $\mathcal{U}$ and that no d-simplex of $\mathcal{U}$ forms a circuit with a $(d+1)$-simplex of $2 * \mathcal{W}$.

We first show that no d-simplex of $\mathcal{U}$ forms a circuit with a $(d+1)$-simplex of $2 * \mathcal{W}$. Suppose that A is a d-simplex of $\mathcal{U}$ such that $A \wr B$, where B is a $(d+1)$-simplex of $2 * \mathcal{W}$. We then have that $\{b_1, b_2, \dots, b_d\} \in \mathring{e}(\mathcal{W})$, since $2 \leqslant b_0 < a_0 < b_1$, so that $b_1 > 3$, and $b_d < b_{d+1} \leqslant m$. This means that $\{2, b_1, b_2, \dots, b_d\} \in \mathbf{X}$. Since $A \in \mathring{e}(\mathcal{U}) \subseteq \mathbf{X}$, we have that $\{2, a_1, a_2, \dots, a_d\} \in \mathbf{X}$ by applying the bridging condition to $\{2, b_1, b_2, \dots, b_d\}$ and A. This implies that $\{a_1, a_2, \dots, a_d\} \in \mathring{e}(\mathcal{W})$.

But $\{a_1, a_2, \ldots, a_d\} \wr \{b_1, b_2, \ldots, b_{d+1}\}$, which is a d-simplex of $\mathcal{W}$, a contradiction.

Hence, we now show that every d-simplex of $2 * \mathcal{W}$ is a d-simplex of $\mathcal{U}$. It is clear that if $2 \cup E$ is a d-simplex of $2 * \mathcal{W}$, then $2 \cup E$ is a d-simplex of $\mathcal{U}$. Indeed, $2 \cup E$ is on the boundary of $C([2, m], 2d + 1)$.

Therefore, let J be a d-simplex of $\mathcal{W}$, and hence of $2 * \mathcal{W}$. We must show that J is a d-simplex of $\mathcal{U}$. If J is not a d-simplex of $\mathcal{U}$, then there must be a $(d + 1)$-simplex K of $\mathcal{U}$ such that $J \wr K$. Hence $K_i := \{k_0, k_1, \ldots, k_{i-1}, k_{i+1}, k_{i+2} \ldots, k_{d+1}\}$ is a d-simplex of $\mathcal{U}$ for all $i \in \{0, 1, \ldots, d + 1\}$.

Suppose first that $k_0 \neq 2$. Then we must have that $K_{d+1} \in \mathring{e}(\mathcal{U}) \subseteq \mathbf{X}$, since this is an internal d-simplex with $k_0 > 2$ and $k_d < k_{d+1} \leqslant m$. We know that $\{2, j_0, j_1, \ldots, j_{d-1}\} \in \mathbf{X}$, since $\{j_0, j_1, \ldots, j_{d-1}\}$ must be an internal $(d-1)$-simplex of $\mathcal{W}$. This is because $j_{d-1} < j_d \leqslant m$ and $j_0 > k_0 \geqslant 3$. Then, since $\mathbf{X}$ is bridging, we must have that $\{2, k_1, k_2, \ldots, k_d\} \in \mathbf{X}$ by applying the bridging condition to K_{d+1} and $\{2, j_0, j_1, \ldots, j_{d-1}\}$. But then $\{k_1, k_2, \ldots, k_d\}$ is a $(d - 1)$-simplex of $\mathcal{W}$ which is intertwining with J, a contradiction.

If $k_0 = 2$, then consider the following. We know that $\{2, j_1, j_2, \ldots, j_d\} \in \mathbf{X}$, since $\{j_1, j_2, \ldots, j_d\}$ is an internal $(d - 1)$-simplex of $\mathcal{W}$. This is because $j_1 > j_0 \geqslant 3 > 2 = k_0$ and $j_d < k_{d+1} \leqslant m$. Since $\mathbf{X}$ is supporting, there must exist a $(d - 1)$-simplex $E \wr \{2, j_1, j_2, \ldots, j_d\}$, such that every internal d-simplex in $E \cup \{2, j_1, j_2, \ldots, j_d\}$ is an element of $\mathbf{X}$. If $e_0 > j_0$, then note that $2 \cup E$ must be an element of $\mathbf{X}$, so that E is a $(d - 1)$-simplex of $\mathcal{W}$ with $E \wr J$, a contradiction. If $e_0 \leqslant j_0$, then note that $\{e_0, j_1, j_2, \ldots, j_d\}$ must be an element of $\mathbf{X}$. We then have that $\{e_0, j_1, j_2, \ldots, j_d\}$ is a d-simplex of $\mathcal{U}$ which intertwines K, since $e_0 > 2$ and $k_0 = 2$, with $e_0 \leqslant j_0 < k_1$ as well. This is another contradiction.

Therefore, every d-simplex of $2*\mathcal{W}$ is a d-simplex of $\mathcal{U}$. Combined with the fact that no d-simplex of $\mathcal{U}$ is intertwining with a $(d+1)$-simplex of $2*\mathcal{W}$, and applying Lemma 2.2.9, we obtain that every $(d + 1)$-simplex of $2 * \mathcal{W}$ is a $(d + 1)$-simplex of $\mathcal{U}$. Hence, every d-simplex of $\mathcal{W}$ is a d-simplex of $\mathcal{U}\backslash 2$, which gives us that $\mathcal{W}$ is indeed a section of $\mathcal{U}\backslash 2$, as desired. $\square$

We can now inductively construct triangulations from supporting and bridging collections.

Proposition 2.2.20. *Let* $\mathbf{X} \subseteq \mathbf{J}_m^d$ *be supporting and bridging. Then there is a triangulation* $\mathcal{T} \in \mathcal{S}(m, 2d+1)$ *such that* $\mathbf{X} = \mathring{e}(\mathcal{T})$.

Proof. We show this by induction on m and d. The base cases consist of the case where $d = 0$ and the case where $m = 2d + 2$. For $d = 0$, triangulations of $C(m, 2d + 1)$ are given by subsets of vertices from $\{2, 3, \ldots, m - 1\}$. Since the properties of being supporting or bridging are trivial for $d = 0$, the result holds for this case. For $m = 2d + 2$, $C(m, 2d + 1)$ is a simplex and so uniquely triangulates itself. In this case, $\mathbf{J}_m^d$ is empty, and so the unique triangulation is given by the empty set. Therefore the result holds for both the base cases.

For the inductive step, we consider triangulations of $C(m, 2d + 1)$ and suppose that the claim holds for $C(m', 2d' + 1)$ whenever $m' < m$ or $d' < d$. By Lemma 2.2.17, $\mathbf{X}[1 \to 2]$ is both supporting and bridging. Hence, by the induction hypothesis, there is a triangulation $\mathcal{U} \in \mathcal{S}([2, m], 2d + 1)$ such that $\mathring{e}(\mathcal{U}) = \mathbf{X}[1 \to 2]$. By Lemma 2.2.18, $\mathbf{X}\backslash\{1, 2\}$ is also supporting and bridging. Hence, by the induction hypothesis, there is a triangulation $\mathcal{W} \in \mathcal{S}([3, m], 2d - 1)$ such that $\mathring{e}(\mathcal{W}) = \mathbf{X}\backslash\{1, 2\}$.

By Proposition 2.2.19, we have that $\mathcal{W}$ is a section of $\mathcal{U}\backslash 2$. By [RS00, Lemma 4.7(i)] as illustrated in Section 2.1.5, we therefore have a triangulation $\mathcal{T} \in \mathcal{S}(m, 2d + 1)$ such that $\mathcal{T}[1 \to 2] = \mathcal{U}$ and $\mathcal{T}\backslash\{1, 2\} = \mathcal{W}$. Hence $\mathbf{X}[1 \to 2] = \mathring{e}(\mathcal{U}) = \mathring{e}(\mathcal{T}[1 \to 2]) = \mathring{e}(\mathcal{T})[1 \to 2]$, by Lemma 2.2.15, and $\mathbf{X}\backslash\{1, 2\} = \mathring{e}(\mathcal{W}) = \mathring{e}(\mathcal{T}\backslash\{1, 2\}) = \mathring{e}(\mathcal{T})\backslash\{1, 2\}$, by Lemma 2.2.16. It is straightforward to see that this means that $\mathbf{X} = \mathring{e}(\mathcal{T})$, as desired. $\square$

Proof of Theorem 2.2.3. Lemma 2.2.12 and Lemma 2.2.10 give us that $\mathring{e}(\mathcal{T})$ is supporting and bridging for every triangulation $\mathcal{T}$. Lemma 2.2.9 tells us that the assignment $\mathcal{T} \mapsto \mathring{e}(\mathcal{T})$ is injective. Finally Proposition 2.2.20 tells us that this map is a surjection. □

Remark 2.2.21. Theorem 2.2.3 generalises the bijection obtained in [FR21] between triangulations of three-dimensional cyclic polytopes and persistent graphs. Here the supporting and bridging properties correspond to the defining properties of persistent graphs, known in [FR21] as the *bar property* and the *X-property* respectively. The problem of characterising the d-skeleton of triangulations of $(2d+1)$-dimensional cyclic polytopes was raised as an open problem in the conclusion of [FR21]. Theorem 2.2.3 solves this problem.

Example 2.2.22. Consider the cyclic polytope $C(6, 3)$. This has six triangulations, $\mathcal{T}_l, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_1', \mathcal{T}_2', \mathcal{T}_u$, where $\mathring{e}(\mathcal{T}_l) = \{24, 25, 35\}$, $\mathring{e}(\mathcal{T}_1) = \{24, 25\}$, $\mathring{e}(\mathcal{T}_1') = \{35, 25\}$, $\mathring{e}(\mathcal{T}_2) = \{24\}$, $\mathring{e}(\mathcal{T}_2') = \{35\}$, $\mathring{e}(\mathcal{T}_u) = \varnothing$. The set $\{24, 35\}$ is not obtained because it is not bridging, for which it would need to contain 25. The set $\{25\}$ is not obtained because it is not supporting. The options for the support are 3 and 4, which would require 35 and 24 respectively.

2.3 Expanding triangulations

We now change focus and work on describing the pre-images of a triangulation under an arbitrary contraction $[x \to v \leftarrow y]$. The pre-images of triangulations under the contraction $[m-1 \leftarrow m]$ are well-understood due to [RS00, Lemma 4.7(i)], which states that such triangulations $\widetilde{\mathcal{T}}$ are in bijection with sections of the vertex figure $\mathcal{T}\backslash(m-1)$, as we illustrated in Section 2.1.5. By symmetry, one can apply the same theory for contractions $[1 \to 2]$.

In this section we show how one can extend the theory to all contractions, that is, for any contraction $[x \to v \leftarrow y]$ of $C([m-1]_{v+}, \delta)$. This is a technical result which will be used to prove key lemmas in the following chapter. This general case is more challenging, because the vertex figures of $C(m-1, \delta)$ are less well-behaved at vertices which are not 1 or $m-1$. Indeed, these vertex figures are not cyclic polytopes.

We consider the following example, which suggests that a version of [RS00, Lemma 4.7(i)] ought to hold at vertices besides the first and last vertex. We spend the remainder of this chapter proving this more general version of [RS00, Lemma 4.7(i)].

Example 2.3.1. We proceed in the opposite direction to Example 2.1.4. That is, we consider the same triangulation $\mathcal{T}$ of $C(5,3)$, but now directly compute the triangulations $\widetilde{\mathcal{T}}$ of $C([5]_{2+}, 3)$ such that $\widetilde{\mathcal{T}}[x \to 2 \leftarrow y] = \mathcal{T}$. We then analyse the triangulated vertex figure $\mathcal{T}\backslash 2$ to see if there is a correspondence.

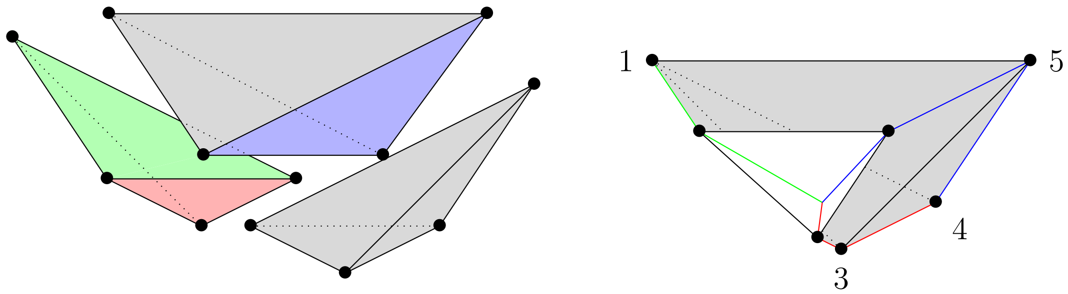
By direct computation, there are four triangulations $\widetilde{\mathcal{T}}$ of $C([5]_{2+}, 3)$ such that $\widetilde{\mathcal{T}}[x \to 2 \leftarrow y] = \mathcal{T}$, namely

$$\widetilde{\mathcal{T}}_1 = \varnothing \cup \{1xy3, xy35\} \cup \{1x34, 1x45, x345\} \cup \varnothing,$$
$$\widetilde{\mathcal{T}}_2 = \varnothing \cup \{1xy3, xy34, xy45\} \cup \{1x34, 1x45\} \cup \{y345\},$$
$$\widetilde{\mathcal{T}}_3 = \varnothing \cup \{1xy4, xy45\} \cup \{1x45\} \cup \{1y34, y345\},$$
$$\widetilde{\mathcal{T}}_4 = \varnothing \cup \{1xy5\} \cup \varnothing \cup \{1y34, 1y45, y345\}.$$

Here we have split up the simplices into sets according to whether they contain neither x nor y, both x and y, x but not y, or y but not x. Now let

$$\mathcal{W}_i = \{\, A : A \cup \{x, y\} \in \widetilde{\mathcal{T}}_i \,\},$$

Figure 2.6: The triangulation $|\mathcal{T}|$ of $\mathfrak{C}(5,3)$ and the triangulation $|\mathcal{T}\backslash 2|$



so that

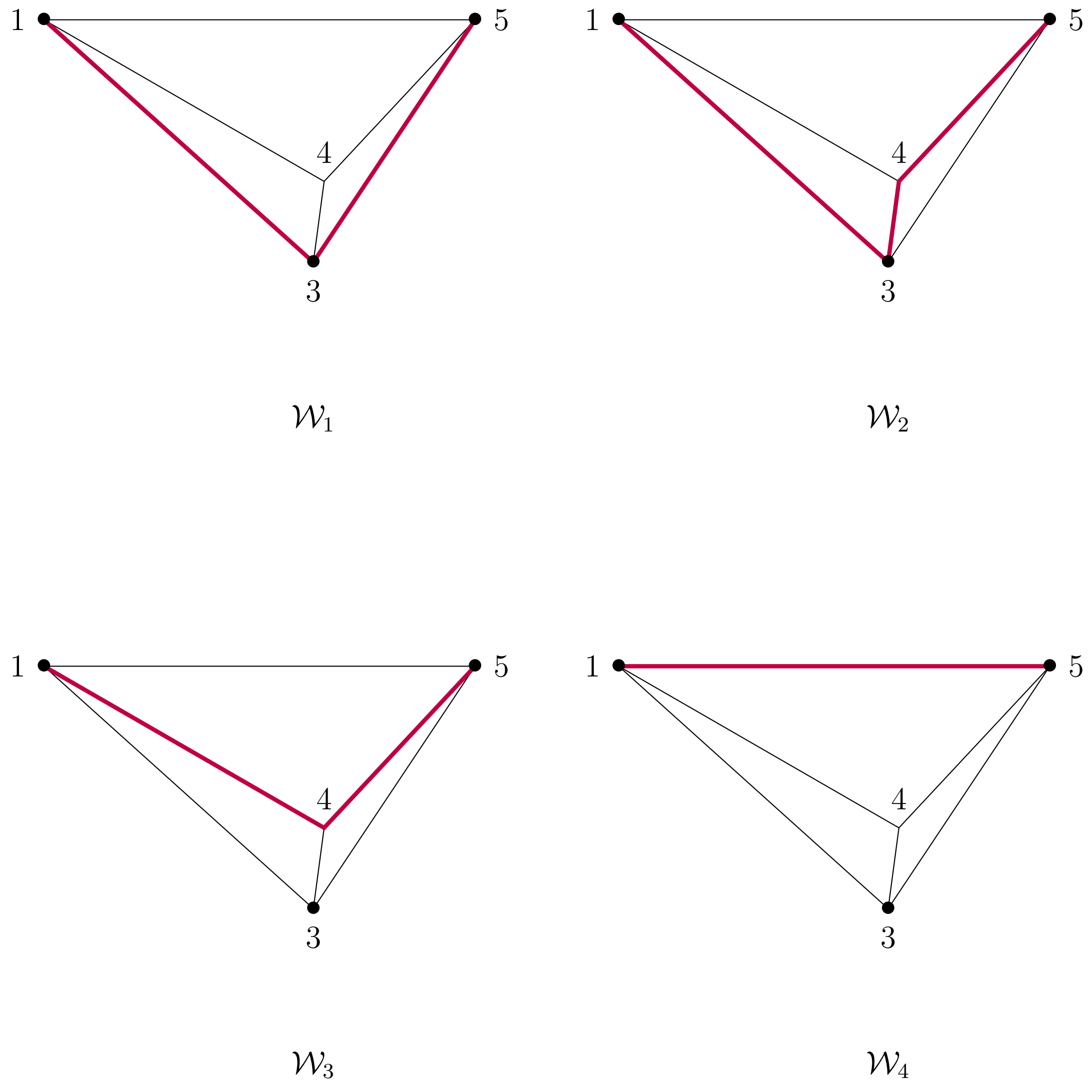
$$\mathcal{W}_1 = \{13, 35\},$$
$$\mathcal{W}_2 = \{13, 34, 45\},$$
$$\mathcal{W}_3 = \{14, 45\},$$
$$\mathcal{W}_4 = \{15\}.$$

Consider these sets of simplices as subcomplexes of $\mathcal{T}\backslash 2$. We obtain the results shown in Figure 2.7 using geometric realisations. Note further that the simplices of the triangulation $|\widetilde{\mathcal{T}_i}|$ which have $|x|$ as a vertex correspond to the simplices of $|\mathcal{T}\backslash 2|$ which are above the section, and the simplices of the triangulation $|\widetilde{\mathcal{T}_i}|$ which possess $|y|$ as a vertex correspond to the simplices of $|\mathcal{T}\backslash 2|$ which are below the section.

This suggests that there ought to be a version of [RS00, Lemma 4.7(i)] for expansion at vertices v such that $1 < v < m$. However, there are several outstanding issues.

(1) The vertex figures $C(m, \delta)\backslash v$ are not generally cyclic polytopes for $1 < v < m$, as can be seen from Figure 2.6 and Figure 2.7.

Figure 2.7: Sections of $|\mathcal{T}\backslash 2|$



(2) It is not clear how to define the orientation on the vertex figure $C(m,\delta)\backslash v$, that is, how to divide the facets of $C(m,\delta)\backslash v$ into upper and lower facets. The orientation of $\mathfrak{C}(5,3)\backslash 2$ from Figure 2.7 may look very natural, with $|13|$ and $|35|$ as lower facets and $|15|$ as the sole upper facet. But it is not clear where this comes from, because $|123|$ is a lower facet of $\mathfrak{C}(5,3)$, whereas $|125|$ and $|235|$ are upper facets.

(3) Likewise, it is not clear how to orient the simplices in the triangulation. From Figure 2.7, it seems that $|345|$ has lower facet $|35|$ and upper facets $|34|,|45|$, whereas $|134|$ has lower facets $|13|,|34|$ and upper facet $|14|$. But is not immediately obvious what the basis for this is.

(4) Finally, it is not obvious how to define sections of a triangulation of the vertex figure $C(m,\delta)\backslash v$. Moreover, if one can define the right notion of a section, it is not clear whether such sections will be triangulations of lower-dimensional cyclic polytopes, given that the vertex figures themselves are not cyclic polytopes.

Over the course of this section we shall show how to resolve all these issues and prove the analogue of [RS00, Lemma 4.7(i)] at vertices which are not 1 or m. We first show how one can orient the vertex figures $C(m,\delta)\backslash v$, that is, decide which the upper and lower facets of $C(m,\delta)\backslash v$ are. This explains the natural orientation we arrived at in Figure 2.7, and solves issue (2). Next we apply the same logic to the simplices of the triangulation, thereby answering (3).

This gives us a partial order on the simplices of $\mathcal{T}\backslash v$. Using this, we derive the relevant notion of a section within the triangulation $\mathcal{T}\backslash v$. We show that, in fact, our sections are triangulations of $C([m]\setminus v,\delta-2)$, solving issue (4). This culminates in our proving the following proposition, which is the analogue of [RS00, Lemma 4.7(i)], showing that point (1) is not a problem.

Proposition 2.3.2. *Let $\mathcal{T}$ be a triangulation of $C(m, \delta)$. There is a bijection between triangulations $\widetilde{\mathcal{T}}$ of $C([m]_{v+}, \delta)$ such that $\widetilde{\mathcal{T}}[x \to v \leftarrow y] = \mathcal{T}$ and sections of $\mathcal{T} \backslash v$, given by*

$$\left\{ \begin{matrix} \widetilde{\mathcal{T}} \in \mathcal{S}([m]_{v+}, \delta), \\ \widetilde{\mathcal{T}}[x \to v \leftarrow y] = \mathcal{T} \end{matrix} \right\} \longleftrightarrow \{ \text{ Sections } \mathcal{W} \text{ of } \mathcal{T} \backslash v \ \}$$

$$\begin{array}{ccc} \widetilde{\mathcal{T}} & \longmapsto & \widetilde{\mathcal{T}} \backslash \{x, y\} \\ \\ \begin{matrix} \mathcal{T}^\circ \cup (\mathcal{W} * \{x, y\}) \\ \cup (\mathcal{T} \backslash v^+ * x) \cup (\mathcal{T} \backslash v^- * y) \end{matrix} & \longleftarrow\!\shortmid & \mathcal{W}. \end{array}$$

Remark 2.3.3. It suffices to prove Proposition 2.3.2 for δ odd. For δ even it already follows from [RS00, Lemma 4.7(i)], since in this case the cyclic permutation $i \mapsto i + (m - v)$ defines an automorphism of $C(m, \delta)$ which sends vertex v to vertex m—see [KW03]. Hence one may apply [RS00, Lemma 4.7(i)] to the vertex v as if it were vertex m, which gives Proposition 2.3.2 in this case. But for δ odd this permutation does not define an automorphism, and so more work needs to be done. Indeed, the fact that, for δ odd and $v \in [2, m - 1]$, $C(m, \delta) \backslash v$ is not a cyclic polytope precludes this permutation from giving an automorphism.

The methods of this section may still be applied to even-dimensional cyclic polytopes. One can check that this is equivalent to considering $C(m, 2d)$ subject to the given automorphism. However, restricting our attention to δ odd allows us to simplify some proofs.

Proving Proposition 2.3.2 requires theory for working with triangulated vertex figures and their sections. Developing this theory is the task of Sections 2.3.1–2.3.4.

Remark 2.3.4. Proposition 2.3.2 and [RS00, Lemma 4.7(i)] are reminiscent of the single-element extension theorem of Las Vergnas for oriented matroids [Las78]

[Bjö+99, Section 7.1] [RZ94, Theorem 4.1.(1)]. However, it does not seem that they follow from this result in any obvious way.

2.3.1 Facets of vertex figures

Our first task is to find the correct orientation of the vertex figure $C(m, 2d+1)\backslash v$: the correct division of its facets into upper and lower facets. It is important to note that, as in Example 2.3.1, this will not generally match the orientation of $C(m, 2d+1)$. That is to say, if F is a lower facet of $C(m, 2d+1)\backslash v$ according to our orientation, then $F \cup v$ will not generally be a lower facet of $C(m, 2d+1)$.

Recall from Gale's Evenness Criterion that a facet F of $C(m, 2d+1)$ can be expressed uniquely as a union of disjoint pairs of consecutive numbers along with either 1 or m. Hence, given $v \in [2, m-1]$, and a facet F of $C(m, \delta)$ such that $v \in F$, we can talk about the pair of consecutive entries that v lies in, which must either be $\{v-1, v\}$ or $\{v, v+1\}$. We then define the upper and lower facets of $C(m, 2d+1)\backslash v$ as follows.

Definition 2.3.5. Let $v \in [2, m-1]$ and let F be a facet of $C(m, 2d+1)$ such that $v \in F$. Then $F \setminus v$ is a facet of $C(m, 2d+1)\backslash v$.

- If the other element in the pair with v in F is $v+1$, then we say that $F \setminus v$ is a *lower facet of* $C(m, 2d+1)\backslash v$.
- If the other element in the pair with v in F is $v-1$, then we say that $F \setminus v$ is an *upper facet of* $C(m, 2d+1)\backslash v$.

The following lemma indicates why our orientation of the vertex figure $C(m, 2d+1)\backslash v$ is the correct one when it comes to considering expansion. A lower facet F of $C(m, 2d+1)\backslash v$ should always lie below a section, and hence should always become a facet $F \cup y$ of $C([m]_{v+}, 2d+1)$ under the expansion given by the section.

Lemma 2.3.6. *We have that F is a lower facet of $C(m, 2d+1)\backslash v$ if and only if $F \cup y$ is a facet of $C([m]_{v+}, 2d+1)$. Dually, we have that F is an upper facet of $C(m, 2d+1)\backslash v$ if and only if $F \cup x$ is a facet of $C([m]_{v+}, 2d+1)$.*

Proof. If $F \cup v$ is either an even or an odd subset of $[m]$ and v is in a pair with $v+1$, then $F \cup y$ will respectively be either an even or an odd subset of $[m]_{v+}$. Conversely, if $F \cup y$ is either an even or an odd subset of $[m]_{v+}$ where $F \subseteq [m] \setminus v$, then y must be in a pair with $v+1$, since $x \notin F$. Consequently $F \cup v$ is either an even or an odd subset of $[m]$ with v in a pair with $v+1$. The analogous claim for upper facets follows by a similar argument. □

One can also describe the upper and lower facets of $C(m, 2d+1)\backslash v$ using the following, which can be seen as a generalisation of Gale's Evenness Criterion.

Lemma 2.3.7. *Let $F \subseteq [m] \setminus v$. Then*

- *F is a lower facet of $C(m, 2d+1)\backslash v$ if and only if $\#\{\, i \in F : j < i < v \,\}$ is even for all $j \in [v-1] \setminus F$ and $\#\{\, i \in F : v < i < j \,\}$ is odd for all $j \in [v+1, m] \setminus F$; and*
- *F is an upper facet of $C(m, 2d+1)\backslash v$ if and only if $\#\{\, i \in F : j < i < v \,\}$ is odd for all $j \in [v-1] \setminus F$ and $\#\{\, i \in F : v < i < j \,\}$ is even for all $j \in [v+1, m] \setminus F$.*

Proof. We only show the first claim, since the second claim is similar. Suppose that F is a lower facet of $C(m, 2d+1)\backslash v$. Then $F \cup v$ is a facet of $C(m, 2d+1)$ and v occurs in a pair with $v+1$. Let $j \in [v-1] \setminus F$. There are then a whole number of pairs of consecutive numbers between j and v, so $\#\{\, i \in [m] : j < i < v \,\}$ is even. Let $j \in [v+1, m] \setminus F$. Then the elements of F between j and v consist of $v+1$ and a set of pairs of consecutive numbers, so $\#\{\, i \in [m] : v < i < j \,\}$ is odd.

Suppose now that F is such that $\#\{\, i \in [m] : j < i < v \,\}$ is even for all $j \in [v-1] \setminus F$ and $\#\{\, i \in [m] : v < i < j \,\}$ is odd for all $j \in [v+1, m] \setminus F$. Then we must have $v+1 \in F$, since otherwise we can choose $j = v+1$, and $\#\{\, i \in [m] : v < i < v+1 \,\} = 0$. The remaining elements of F must consist of disjoint pairs of consecutive numbers and possibly 1 or m, otherwise we can find gaps in F which contradict our assumption. Moreover, F cannot contain both 1 and m, since F must have $2d+1$ elements. This gives that $F \cup v$ is a facet of $C(m, 2d+1)$ where v occurs in a pair with $v+1$. Hence F is a lower facet of $C(m, 2d+1)\backslash v$. □

The following lemma describes the significance of the intersections of upper and lower facets of the vertex figure $C(m, 2d+1)\backslash v$. It is analogous to the easily-verified fact that the facets of $C(m, \delta)$ correspond precisely to the $(\delta-1)$-simplices which are intersections of a lower facet of $C(m, \delta+1)$ and an upper facet of $C(m, \delta+1)$.

Lemma 2.3.8. *If $G \in \binom{[m]\backslash v}{2d-1}$ and $G = F \cap F'$, where F is an upper facet of $C(m, 2d+1)\backslash v$ and F' is a lower facet of $C(m, 2d+1)\backslash v$, then G is a facet of $C([m] \setminus v, 2d-1)$.*

Proof. Let

$$j = \max\{\, i \in [m] \setminus F : i < v \,\},$$
$$j' = \max\{\, i \in [m] \setminus F' : i < v \,\},$$
$$k = \min\{\, i \in [m] \setminus F : v < i \,\},$$
$$k' = \min\{\, i \in [m] \setminus F' : v < i \,\}.$$

We cannot have $j = j'$, since, by Lemma 2.3.7, $\#\{\, i \in F : j < i < v \,\}$ is odd, whereas $\#\{\, i \in F' : j' < i < v \,\}$ is even. Therefore, suppose that $j < j'$. This implies that $j' \in F$, so that $F = G \cup j'$. Hence, $k' \notin F$, so that $k < k'$ and $F' = G \cup k'$.

Then $\#\{ i \in G : j' < i < v \} = \#\{ i \in F' : j' < i < v \}$, which is even, and $\#\{ i \in G : v < i < k \} = \#\{ i \in F : v < i < k \}$, which is also even. Furthermore, $\#\{ i \in G : j < i < j' \}$ is even, since $\#\{ i \in F : j < i < v \}$ is odd and $\#\{ i \in F' : j' < i < v \}$ is even. Similarly, $\#\{ i \in G : k < i < k' \}$ is even.

Thus, $G \cap [1, j]$ consists of a set of disjoint pairs along with possibly 1, since this is true of F and F'; $G \cap [j, j']$ is an interval of even length; $G \cap [j', k]$ is an interval in $[m] \backslash v$ of even length; $G \cap [k, k']$ is an interval of even length; and $G \cap [k', m]$ consists of a disjoint union of pairs, along with possibly m. Consequently, G satisfies Gale's Evenness Criterion, and so is a facet of $C([m] \setminus v, 2d + 1)$. The case where $j' < j$ is similar. $\square$

Corollary 2.3.9. *If $G = F \cap F'$, where F is an upper facet of $C(m, 2d+1)\backslash v$ and F' is a lower facet of $C(m, 2d+1)\backslash v$, then $G \cup \{x, y\}$ is a facet of $C([m]_{v+}, 2d+1)$.*

Proof. This follows from Gale's Eveness Criterion and Lemma 2.3.8. If G is an even (respectively, odd) subset of $[m] \setminus v$, then $G \cup \{x, y\}$ is an even (respectively, odd) subset of $[m]_{v+}$. Adding a pair of consecutive entries cannot change the parities of any gaps. $\square$

2.3.2 Orienting the simplices

We now show how one can orient the simplices of the triangulation $\mathcal{T}\backslash v$ in a similar way to how we have oriented the vertex figure $C(m, 2d + 1)\backslash v$. This allows us to introduce a partial order on these simplices.

We first explain the logic of our orientation of the simplices of $\mathcal{T}\backslash v$. Given a triangulation $\mathcal{T}$ of $C(m - 1, 2d + 1)$, we wish to understand the different triangulations $\widetilde{\mathcal{T}}$ of $C([m - 1]_{v+}, 2d + 1)$ such that $\widetilde{\mathcal{T}}[x \to v \leftarrow y] = \mathcal{T}$. We approach this by considering the triangulated vertex figure $\mathcal{T}\backslash v$. It is clear that $\mathcal{T}\backslash v$ contains $\widetilde{\mathcal{T}}\backslash\{x, y\}$ as a simplicial subcomplex. We would like to think of these simplicial subcomplexes as sections which divide $\mathcal{T}\backslash v$ into a part where $x \leftarrow v$ under expansion and a part where $v \rightarrow y$ under expansion.

However, it is not clear how to do this geometrically. That is, it is not clear how to choose a direction to define a part of $|\mathcal{T}\backslash v|$ lying "above" $|\widetilde{\mathcal{T}}\backslash\{x, y\}|$ and a part lying "below". Hence, we look to characterise these combinatorially instead. Note that, for every $2d$-simplex S of $\mathcal{T}\backslash v$, we must have that $S \cup v$ is a simplex of $\mathcal{T}$, so that $S \cup x$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$ or that $S \cup y$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$. But we cannot have both, since $S \cup \{x, y\}$ can be decomposed into two halves of a circuit, one of which is contained in $S \cup x$, and the other of which is contained in $S \cup y$. We therefore orient the simplices of the triangulated vertex figure $\mathcal{T}\backslash v$ as follows.

Definition 2.3.10. Let S be a $2d$-simplex of $\mathcal{T}\backslash v$. Then $S \cup \{x, y\}$ consists of $2d+3$ distinct vertices, and so uniquely gives two halves of a circuit of $C([m]_{v+}, 2d+1)$, which we denote $(S_- \cup x, S_+ \cup y)$. Then we say that $S \setminus s$ is a *lower facet* of S if $s \in S_+$, and an *upper facet* of S if $s \in S_-$.

Remark 2.3.11. Recall from Remark 2.2.7 that for cyclic polytopes we can distinguish which halves of circuits are on top and which halves are on bottom. In Definition 2.3.10, we do this for circuits of the vertex figure by seeing which corresponding circuit halves contain x and y in $C([m]_{v+}, \delta)$. This then allows us to determine upper and lower facets of simplices.

One can also translate this definition of the upper and lower facets of simplices of $\mathcal{T}\backslash v$ into an evenness criterion, which can be deduced straightforwardly from the definition.

Lemma 2.3.12. *Let S be a $2d$-simplex of $\mathcal{T}\backslash v$ and let $s \in S$. Then*

(1) if $s < v$, then $S \setminus s$ is

   (a) *a lower facet of* S *if* $\#\{ i \in S : s < i < v \}$ *is even, and*

   (b) *an upper facet of* S *if* $\#\{ i \in S : s < i < v \}$ *is odd;*

(2) *if* $v < s$*, then* $S \setminus s$ *is*

   (a) *a lower facet of* S *if* $\#\{ i \in S : v < i < s \}$ *is odd, and*

   (b) *an upper facet of* S *if* $\#\{ i \in S : v < i < s \}$ *is even.*

By comparing with Lemma 2.3.7, we see that our notion of the upper and lower facets of a simplex of $\mathcal{T}\backslash v$ matches our notion of the upper and lower facets of $C(m, 2d+1)\backslash v$.

Lemma 2.3.13. *Let* $\mathcal{T}$ *be a triangulation of* $C(m, 2d+1)$*. Let* S, R *be* $2d$*-simplices of* $\mathcal{T}\backslash v$*. Then* $S \cap R$ *cannot be both a lower facet of* S *and a lower facet of* R*. Similarly,* $S \cap R$ *cannot be both an upper facet of* S *and an upper facet of* R*.*

Proof. We only show the first claim, since the second claim is similar. Suppose that F is both a lower facet of $S = F \cup s$ and a lower facet of $R = F \cup r$. Without loss of generality, assume that $s < r$. If $s < r < v$ or $v < s < r$, then are are an even number of elements $f \in F$ such that $s < f < r$, by Lemma 2.3.12. If $s < v < r$, then there are an odd number of elements $f \in F$ such that $s < f < r$, by Lemma 2.3.12.

We have $\#F \cup \{s, r, v\} = 2d+3$, and so there is a circuit (Z, Z') of $C(m, 2d+1)$ such that $Z \cup Z' = F \cup \{s, r, v\}$. Suppose, without loss of generality, that $s \in Z$. By the previous paragraph, we must then have $r \in Z'$. Hence the simplices $F \cup \{s, v\}$ and $F \cup \{r, v\}$ each contain one half of a circuit, which contradicts their both being simplices of $\mathcal{T}$. $\square$

In the manner of [Ram97, Definition 5.7], we may now define a relation on the set of $2d$-simplices of $\mathcal{T}\backslash v$. Given two $2d$-simplices S, R, we write that $S \overset{v}{\lessdot} R$ if and only if $S \cap R$ is an upper facet of S and a lower facet of R. Hence, this is the same relation as Remark 2.1.3, only applied to the simplices of the triangulated vertex figure. We show that $\overset{v}{\leqslant}$ is a partial order using the method of [Ram97, Corollary 5.8]: we define a total order on the simplices of $\mathcal{T}\backslash v$ and show that $\overset{v}{\leqslant}$ is a sub-order of it. This means that every triangulation of $C(m, 2d+1)\backslash v$ induced by a triangulation of $C(m, 2d+1)$ is *stackable*, in the sense of [RS00, Definition 2.13].

To each $S \in \binom{[m]\backslash v}{2d+1}$, we assign a unique string by

$$\Gamma \colon \binom{[m] \setminus v}{2d+1} \to \{o, *, e\}^{m-1}$$
$$\Gamma(S) := (\gamma_{v+1}(S), \gamma_{v+2}(S), \dots, \gamma_m(S), \gamma_1(S), \gamma_2(S), \dots, \gamma_{v-1}(S)),$$

where

$$\gamma_j(S) = \begin{cases} * \text{ if } j \in S \\ \text{if } j \notin S \begin{cases} \text{if } j < v \begin{cases} e \text{ if } \#\{\, b \in S : j < b < v \,\} \text{ is even,} \\ o \text{ if } \#\{\, b \in S : j < b < v \,\} \text{ is odd,} \end{cases} \\ \text{if } v < j \begin{cases} e \text{ if } \#\{\, b \in S : v < b < j \,\} \text{ is even,} \\ o \text{ if } \#\{\, b \in S : v < b < j \,\} \text{ is odd.} \end{cases} \end{cases} \end{cases}$$

We then denote by $\preceq$ the lexicographic order on $\binom{[m]\backslash v}{2d+1}$ induced by Γ and the ordering of the letters $o \prec * \prec e$.

Lemma 2.3.14. *Let $\mathcal{T}$ be a triangulation of $C(m, 2d+1)$ and consider the triangulated vertex figure $\mathcal{T}\backslash v$. Let S and R be $2d$-simplices of $\mathcal{T}\backslash v$ such that $S \overset{v}{\lessdot} R$, with $S \setminus \{s\} = R \setminus \{r\}$.*

(1) If we have $v < s < r$ in the cyclic ordering, then $\gamma_r(S) = e$ and $\gamma_s(R) = e$.

(2) If we have $v < r < s$ in the cyclic ordering, then $\gamma_r(S) = o$ and $\gamma_s(R) = o$.

(3) *For $j \notin S \cup R$, we have that $\gamma_j(S) \neq \gamma_j(R)$ if and only if j lies between s and r in the cyclically shifted order $v+1 <_v v+2 <_v \cdots <_v n <_v 1 <_v 2 <_v \cdots <_v v-1$.*

Proof. By Lemma 2.3.12, the fact that $S \overset{v}{\lessdot} R$ implies that

$$\text{if } s < v, \text{ then } \#\{\, i \in S : s < i < v \,\} \text{ is odd, and}$$
$$\text{if } v < s, \text{ then } \#\{\, i \in S : v < i < s \,\} \text{ is even,}$$

and

$$\text{if } r < v, \text{ then } \#\{\, i \in R : r < i < v \,\} \text{ is even, and}$$
$$\text{if } v < r, \text{ then } \#\{\, i \in R : v < i < r \,\} \text{ is odd.}$$

We consider the case where $v < s < r$ is a cyclic ordering.

(1) Within this set of cases, we first suppose that $v < s < r$. Then

$$\#\{\, i \in R : v < i < s \,\} = \#\{\, i \in S : v < i < s \,\},$$

which is even, and

$$\#\{\, i \in S : v < i < r \,\} = \#\{\, i \in R : v < i < r \,\} + 1,$$

which is even. Therefore $\gamma_s(R) = e$ and $\gamma_r(S) = e$. Moreover, if $j \notin S \cup R$, then $\gamma_j(S) \neq \gamma_j(R)$ if and only if $s <_v j <_v r$ in the cyclically shifted order.

(2) If $r < v < s$, then

$$\#\{\, i \in R : v < i < s \,\} = \#\{\, i \in S : v < i < s \,\},$$

which is even, and

$$\#\{\, i \in S : r < i < v \,\} = \#\{\, i \in R : r < i < v \,\},$$

which is even. Therefore $\gamma_s(R) = e$ and $\gamma_r(S) = e$. Moreover, if $j \notin S \cup R$, then $\gamma_j(S) \neq \gamma_j(R)$ if and only if $s <_v j <_v r$ in the cyclically shifted order.

(3) If $s < r < v$, then

$$\#\{\, i \in R : s < i < v \,\} = \#\{\, i \in S : s < i < v \,\} - 1,$$

which is even, and

$$\#\{\, i \in S : r < i < v \,\} = \#\{\, i \in R : r < i < v \,\},$$

which is even. Therefore $\gamma_s(R) = e$ and $\gamma_r(S) = e$. Moreover, if $j \notin S \cup R$, then $\gamma_j(S) \neq \gamma_j(R)$ if and only if $s <_v j <_v r$ in the cyclically shifted order.

The cases where $v < r < s$ is a cyclic ordering are similar. □

Corollary 2.3.15. *The relation $\overset{v}{\leqslant}$ is a partial order.*

Proof. We show that $S \overset{v}{\leqslant} R$ implies that $S \preceq R$. For this it suffices to show that $S \overset{v}{\lessdot} R$ implies that $S \preceq R$. If $v < s < r$ in the cyclic ordering, then, by Lemma 2.3.14(3), it suffices to consider $\gamma_s(S)$ and $\gamma_s(R)$ in order to compare $\Gamma(S)$ and $\Gamma(R)$ in the lexicographic order, since this is the first entry that differs. Then we have $\gamma_s(S) = *$ and $\gamma_s(R) = e$ by Lemma 2.3.14(1) so that $S \preceq R$. Similarly, if $v < r < s$ in the cyclic ordering, then we consider $\gamma_r(S) = o$ and $\gamma_r(R) = *$, so that $S \preceq R$ likewise. We conclude that $S \overset{v}{\leqslant} R$ implies that $S \preceq R$. This entails that $\overset{v}{\leqslant}$ is a partial order, since $\preceq$ is a total order. □

Recall that $\mathcal{L}$ is a *lower set* for a partial order $\leqslant$ on a set $\mathcal{P}$ if $\mathcal{L}$ is a subset of $\mathcal{P}$ such that whenever $p \in \mathcal{L}$ and $p' \leqslant p$, we also have $p' \in \mathcal{L}$. The notion of an *upper set* of a partial order is defined dually. These concepts, together with our partial order $\overset{v}{\leqslant}$, allow us to characterise the set of simplices in $\mathcal{T} \backslash v$ where $x \leftarrow v$ under expansion and the set of simplices where $v \rightarrow y$ under expansion.

Lemma 2.3.16. *Let $\mathcal{T}$ be a triangulation of $C(m, 2d+1)$ with $\widetilde{\mathcal{T}}$ a triangulation of $C([m]_{v+}, 2d+1)$ such that $\widetilde{\mathcal{T}}[x \rightarrow v \leftarrow y] = \mathcal{T}$. Let $\mathcal{L}$ be the set of $2d$-simplices*

S of $\mathcal{T}\backslash v$ such that $S \cup y$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$ and let $\mathcal{U}$ be the set of $2d$-simplices R of $\mathcal{T}\backslash v$ such that $R \cup x$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$. Then $\mathcal{L}$ is a lower set for $\overset{v}{\leqslant}$ and $\mathcal{U}$ is an upper set for $\overset{v}{\leqslant}$. Moreoever, $\mathcal{L} \cup \mathcal{U} = \mathcal{T}\backslash v$ and $\mathcal{L} \cap \mathcal{U} = \varnothing$.

Proof. It is clear that $\mathcal{L} \cup \mathcal{U}$ must comprise all of the $2d$-simplices of $\mathcal{T}\backslash v$. This is because if S is a $2d$-simplex of $\mathcal{T}\backslash v$, then $S \cup v$ is a $(2d+1)$-simplex of $\mathcal{T}$, and so either $S \cup x$ or $S \cup y$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$. Then, as we also argued earlier, we cannot have $\mathcal{L} \cap \mathcal{U} \neq \varnothing$, since then both $S \cup x$ and $S \cup y$ are $(2d+1)$-simplices of $\widetilde{\mathcal{T}}$. But this is prevented by the circuit $(S_- \cup x, S_+ \cup y)$ of Definition 2.3.10, as $S \cup x \supseteq S_- \cup x$ and $S \cup y \supseteq S_+ \cup y$.

We now show that $\mathcal{L}$ is a lower set for $\overset{v}{\leqslant}$. To show this, it suffices to consider $R \in \mathcal{L}$ and $S \in \mathcal{T}\backslash v$ such that $S \overset{v}{\lessdot} R$, and to show that $S \in \mathcal{L}$. Let $F = S \cap R$, which is an upper facet of S and a lower facet of R. Suppose for contradiction that $S \cup x$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$. Then $F \cup x$ is a $2d$-simplex of $\widetilde{\mathcal{T}}$. Since F is a lower facet of R, we have that $F \cup x \supseteq R_- \cup x$, where $(R_- \cup x, R_+ \cup y)$ is the circuit from Definition 2.3.10. Since $R \in \mathcal{L}$, we have that $R \cup y \in \widetilde{\mathcal{T}}$. But then, $R \cup y \supseteq R_+ \cup y$, so that both halves of $(R_- \cup x, R_+ \cup y)$ are contained in simplices of $\widetilde{\mathcal{T}}$, which is a contradiction. Therefore, we must have that $S \cup y$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$, which means that $S \in \mathcal{L}$. Consequently, $\mathcal{L}$ is a lower set for $\overset{v}{\leqslant}$. Since $\mathcal{U}$ is the complement of $\mathcal{L}$, it is an upper set for $\overset{v}{\leqslant}$. □

2.3.3 Sections of vertex figures

We now show how the partial order on the $2d$-simplices of $\mathcal{T}\backslash v$ allows us to define the notion of a section of $\mathcal{T}\backslash v$. We then prove fundamental properties of sections of $\mathcal{T}\backslash v$ which will enable us to prove that they are in bijection with triangulations $\widetilde{\mathcal{T}}$ of $C([m]_{v+}, 2d+1)$ such that $\widetilde{\mathcal{T}}[x \to v \leftarrow y] = \mathcal{T}$.

Definition 2.3.17. Given a lower set $\mathcal{L}$ of $(\mathcal{T}\backslash v, \overset{v}{\leqslant})$, we let $\mathcal{U} = (\mathcal{T}\backslash v) \setminus \mathcal{L}$ be the upper set which is its complement, and define the associated *section* $\mathcal{W}(\mathcal{L})$ to be the abstract simplicial complex given by the set of $(2d-1)$-simplices W of $\mathcal{T}\backslash v$ such that either

- $W = A \cap B$ where $A \in \mathcal{L}$ and $B \in \mathcal{U}$, or
- W is an upper facet of $C(m, 2d+1)\backslash v$ and an upper facet of $A \in \mathcal{L}$, or
- W is a lower facet of $C(m, 2d+1)\backslash v$ and a lower facet of $B \in \mathcal{U}$.

The following lemma shows that sections of $\mathcal{T}\backslash v$ are triangulations of $C([m] \setminus v, 2d-1)$, just as sections of $\mathcal{T}\backslash m$ are triangulations of $C(m-1, \delta-2)$ for triangulations $\mathcal{T}$ of $C(m, \delta)$.

Lemma 2.3.18. *For a triangulation $\mathcal{T}$ of $C(m, 2d+1)$, sections of $\mathcal{T}\backslash v$ are triangulations of $C([m] \setminus v, 2d-1)$.*

Proof. We prove the claim by induction on $\#\mathcal{L}$. In the base case, we have that $\mathcal{L} = \varnothing$, so that $\mathcal{W}(\mathcal{L}) = \mathcal{F}^l_v([m] \setminus v, 2d+1)$. Hence, we must show that $\mathcal{F}^l_v([m] \setminus v, 2d+1)$ is a triangulation of $C([m] \setminus v, 2d-1)$. We must first show that there is no circuit (A, B) of $C([m] \setminus v, 2d-1)$ such that A and B are both simplices in $\mathcal{F}^l_v([m] \setminus v, 2d+1)$. If this were the case, then one of $(A \cup x, B \cup y)$ or $(A \cup y, B \cup x)$ would be a circuit of $C([m]_{v+}, 2d+1)$. But this contradicts Lemma 2.3.6, which gives that $A \cup y$ and $B \cup y$ must be contained in lower facets of $C([m]_{v+}, 2d+1)$, which cannot contain halves of circuits, since halves of circuits must be internal simplices.

We now show that the facets of the $(2d-1)$-simplices in $\mathcal{F}^l_v([m] \setminus v, 2d+1)$ are either shared with other $(2d-1)$-simplices of $\mathcal{F}^l_v([m] \setminus v, 2d+1)$, or are facets of $C([m] \setminus v, 2d-1)$. Let $S \in \mathcal{F}^l_v([m] \setminus v, 2d+1)$ and let $s \in S$, so that $S \setminus s$ is a facet of S. We have that $S \cup v$ is a facet of $C(m, 2d+1)$, so we must have that $(S \setminus s) \cup v = (S \cup v) \cap (R \cup v)$ for a facet $R \cup v$ of $C(m, 2d+1)$. Hence

$S \setminus s = S \cap R$ for a facet $R \in \mathcal{F}_v([m] \setminus v, 2d+1)$. If $R \in \mathcal{F}_v^l([m] \setminus v, 2d+1)$, then we are done. Otherwise, $R \in \mathcal{F}_v^u([m] \setminus v, 2d+1)$, and so $S \setminus s = S \cap R$ is a facet of $C([m] \setminus v, 2d-1)$ by Lemma 2.3.8. This establishes that $\mathcal{F}_v^l([m] \setminus v, 2d+1)$ is a triangulation of $C([m] \setminus v, 2d-1)$, which is the base case of our induction.

Now, to show the inductive step, we suppose that we have a section $\mathcal{W}(\mathcal{L})$ such that $\#\mathcal{L} \neq \varnothing$. Choose a simplex $S \in \mathcal{L}$ which is maximal in $\mathcal{L}$ with respect to $\overset{v}{\leqslant}$. Then $\mathcal{L}' := \mathcal{L} \setminus S$ is a lower set of $\overset{v}{\leqslant}$ and, by the induction hypothesis, $\mathcal{W}(\mathcal{L}')$ is a triangulation of $C([m] \setminus v, 2d-1)$. It follows from Definition 2.3.17 and the fact that S is maximal in $\mathcal{L}$ that $\mathcal{W}(\mathcal{L}) = (\mathcal{W}(\mathcal{L}') \setminus \mathcal{F}_v^l(S, 2d+1)) \cup \mathcal{F}_v^u(S, 2d+1)$.

It can be seen that $\mathcal{W}(\mathcal{L})$ and $\mathcal{W}(\mathcal{L}')$ are bistellar flips of each other. Indeed, if we let $S_e = \{s_0, s_2, \dots, s_{2d}\}$ and $S_o = \{s_1, s_3, \dots, s_{2d-1}\}$, then either $(S_e \cup x, S_o \cup y)$ is a circuit of $C([m]_{v+}, 2d+1)$ or $(S_o \cup x, S_e \cup y)$ is a circuit of $C([m]_{v+}, 2d+1)$. Hence, either $\mathcal{F}_v^l(S, 2d+1) = \{\, S \setminus s : s \in S_o \,\} = \mathcal{F}^u(S, 2d-1)$, or $\mathcal{F}_v^l(S, 2d+1) = \{\, S \setminus s : s \in S_e \,\} = \mathcal{F}^l(S, 2d-1)$. Using this to compare $\mathcal{W}(\mathcal{L})$ and $\mathcal{W}(\mathcal{L}')$, then, respectively, we obtain that either $\mathcal{W}(\mathcal{L}) = (\mathcal{W}(\mathcal{L}') \setminus \mathcal{F}^u(S, 2d-1)) \cup \mathcal{F}^l(S, 2d-1))$ or $\mathcal{W}(\mathcal{L}) = (\mathcal{W}(\mathcal{L}') \setminus \mathcal{F}^l(S, 2d-1)) \cup \mathcal{F}^u(S, 2d-1))$. In the former case, $\mathcal{W}(\mathcal{L})$ is an increasing bistellar flip of $\mathcal{W}(\mathcal{L}')$ as a triangulation of $C([m] \setminus v, 2d-1)$; in the latter case, $\mathcal{W}(\mathcal{L})$ is a decreasing bistellar flip of $\mathcal{W}(\mathcal{L})$ as a triangulation of $C([m] \setminus v, 2d-1)$. Since bistellar flips send triangulations of $C([m] \setminus v, 2d-1)$ to triangulations of $C([m] \setminus v, 2d-1)$, we have in either case that $\mathcal{W}(\mathcal{L})$ is a triangulation of $C([m] \setminus v, 2d-1)$. The result then follows by induction. □

We obtain the following result, which will be useful in showing how sections of $\mathcal{T} \backslash v$ correspond to expanded triangulations.

Corollary 2.3.19. *Let $\mathcal{T}$ be a triangulation of $C(m, 2d+1)$ with $\mathcal{W}(\mathcal{L})$ a section of $\mathcal{T} \backslash v$. Then there exists no circuit $(A \cup x, B \cup y)$ of $C([m]_{v+}, 2d+1)$ such that A and B are both simplices in $\mathcal{W}(\mathcal{L})$.*

Proof. If there were simplices A and B of $\mathcal{W}(\mathcal{L})$, such that $(A \cup x, B \cup y)$ was a circuit of $C([m]_{v+}, 2d+1)$, then (A, B) would be a circuit of $C([m] \setminus v, 2d-1)$, which would contradict Lemma 2.3.18. □

The implication of this corollary for Proposition 2.3.2 is that we may infer that the simplices $\mathcal{W}(\mathcal{L}) * \{x, y\}$ do not contain any circuits of $C([m]_{v+}, 2d+1)$. However, we also need to show that there can be no circuits between $\mathcal{W}(\mathcal{L}) * \{x, y\}$ and $\mathcal{T}\backslash v^- * y$, and $\mathcal{W}(\mathcal{L}) * \{x, y\}$ and $\mathcal{T}\backslash v^+ * x$, for which we need the following definition and lemma, which uses Corollary 2.3.19 in its proof. Of course, we also need that there can be no circuits between $\mathcal{T}\backslash v^- * y$ and $\mathcal{T}\backslash v^+ * x$, which we subsequently deduce.

Definition 2.3.20. Let $\mathcal{T}$ be a triangulation of $C(m, 2d+1)$ with $\mathcal{W}(\mathcal{L})$ a section of $\mathcal{T}\backslash v$ and A a simplex of $\mathcal{T}\backslash v$. Then we say that A is *submerged* by $\mathcal{W}(\mathcal{L})$ if A is contained in a simplex of $\mathcal{L}$ or a simplex of $\mathcal{W}(\mathcal{L})$. Similarly, we say that A is *supermerged* by $\mathcal{W}(\mathcal{L})$ if A is contained in a simplex of $\mathcal{U}$ or a simplex of $\mathcal{W}(\mathcal{L})$.

These are analogues for vertex figures of $C(m, 2d+1)$ of the usual notions of submersion and supermersion from [ER96; Wil21a] respectively, which we will cover in Chapter 3. These usual notions are defined for $C(m, \delta)$ by comparing heights with respect to the $(\delta+1)$-th coordinate. As discussed earlier, for $C(m, \delta)\backslash v$ it is not clear what direction one should use to compare heights, so we recreate the notions combinatorially using the partial order $\overset{v}{\leqslant}$. We now prove the following lemma concerning submersion.

Lemma 2.3.21. *Let $\mathcal{T}$ be a triangulation of $C(m, 2d+1)$ with $\mathcal{W}(\mathcal{L})$ a section of $\mathcal{T}\backslash v$. Then there exists no circuit $(A \cup x, B \cup y)$ of $C([m]_{v+}, 2d+1)$ such that A is a simplex of $\mathcal{W}(\mathcal{L})$ and B is submerged by $\mathcal{W}(\mathcal{L})$.*

Proof. Suppose for contradiction that we are in the situation described and that there exists a circuit $(A \cup x, B \cup y)$ of $C([m]_{v+}, 2d+1)$ such that A is a simplex of $\mathcal{W}(\mathcal{L})$ and B is submerged by $\mathcal{W}(\mathcal{L})$.

We show the result by induction on $\#\mathcal{L}$. In the base case we have $\mathcal{L} = \varnothing$, and so both A and B must be simplices of $\mathcal{W}(\mathcal{L})$. But this contradicts Corollary 2.3.19. For the inductive step, we may assume that $\mathcal{L} \neq \varnothing$, and so choose $S \in \mathcal{L}$ which is maximal, so that $\mathcal{L}' := \mathcal{L} \setminus S$ is a lower set with associated section $\mathcal{W}(\mathcal{L}')$. By the induction hypothesis, the claim holds for $\mathcal{W}(\mathcal{L}')$, which is equal to $(\mathcal{W}(\mathcal{L}) \setminus \mathcal{F}_v^u(S, 2d+1)) \cup \mathcal{F}_v^l(S, 2d+1)$.

We have that A is a simplex of $\mathcal{W}(\mathcal{L})$ and B is submerged by $\mathcal{W}(\mathcal{L})$. However, by the induction hypothesis, we cannot have both that A is a simplex of $\mathcal{W}(\mathcal{L}')$ and that B is submerged by $\mathcal{W}(\mathcal{L}')$. Hence we must either have that A is not a simplex of $\mathcal{W}(\mathcal{L}')$ or that B is not submerged by $\mathcal{W}(\mathcal{L}')$. In the latter case, we must have that A is contained in upper facets of S but no lower facets, and in the former case we must have that B is contained in upper facets of S but no lower facets. In the first case A must contain the intersection of the upper facets of S, whereas in the second case B must contain the intersection of the upper facets of S. At most one of these cases can hold, then, since A and B are disjoint. We consider each of these cases in turn.

Suppose first that B is contained in upper facets of S but no lower facets. This means that B is a simplex of of $\mathcal{W}(\mathcal{L})$. But A is also a simplex of $\mathcal{W}(\mathcal{L})$, so that we have a circuit $(A \cup x, B \cup y)$ of $C([m]_{v+}, 2d+1)$ where A and B are both simplices of $\mathcal{W}(\mathcal{L})$. This contradicts Corollary 2.3.19.

Suppose now that A is only contained in upper facets of S. We must have that either A is the intersection of the upper facets of S, or that S has $d+1$ upper facets and A is a d-simplex contained in all but one of these facets. Note that this latter case is not possible for triangulations of cyclic polytopes, where $2d$-simplices always have d upper facets, but it is possible for triangulations of vertex figures of cyclic polytopes: see the simplex $|345|$ in Figure 2.7. Simplices of the triangulated vertex figure are sometimes upside-down, as it were.

If A is the intersection of the upper facets of S, then we have that $S = J \cup A$ where $(J \cup x, A \cup y)$ is a circuit of $C([m]_{v+}, 2d+1)$. If a, b, j are the smallest elements of the respective sets which are greater than v (or simply the smallest if no elements are greater than v), then we have that $j < a < b$ is a cyclic ordering by considering the circuits $(J \cup x, A \cup y)$ and $(A \cup x, B \cup y)$. We then obtain that $((A \setminus a) \cup \{j, x\}, B \cup y)$ is a circuit of $C([m]_{v+}, 2d+1)$. This contradicts the induction hypothesis, since B is submerged by $\mathcal{W}(\mathcal{L}')$ and $(A \setminus a) \cup j$ is a simplex of $\mathcal{W}(\mathcal{L}')$, because it lies in the lower facet $S \setminus a$ of S.

We now must consider the case where S has $d+1$ upper facets and A is a d-simplex contained in all but one of these facets. Hence, let $S = J \cup A = S_- \cup S_+$, where $(S_- \cup x, S_+ \cup y)$ is a circuit and $J \cap A = \varnothing$. By assumption, we have that $A \supseteq S_+$, and

$$\#S_+ = d, \qquad \#S_- = d+1,$$
$$\#A = d+1, \qquad \#J = d.$$

This also implies that $\#B = d$. We must have that at least one of s_0^- and s_d^- is not an element of A, since $\#A \cap S_- = 1$. Suppose that $s_0^- \notin A$; the other case behaves similarly. Here we have $s_0^- < s_0^+ = a_0 < b_0$. We then have that $((A \setminus a_0) \cup \{s_0^-, x\}, B \cup y)$ is a circuit of $C([m]_{v+}, 2d+1)$ with the lower facet $S \setminus a_0$ of S containing $(A \setminus a_0) \cup s_0^-$. Thus $(A \setminus a_0) \cup s_0^-$ a simplex of $\mathcal{W}(\mathcal{L}')$, giving a contradiction, because B is submerged by $\mathcal{W}(\mathcal{L}')$. This concludes the proof. □

We now apply Lemma 2.3.21 to prove the following lemma. The intuition here is that if we have that $(A \cup x, B \cup y)$ is a circuit of $C([m]_{v+}, 2d+1)$, then B is "above" A in the triangulated vertex figure $\mathcal{T} \backslash v$, and so there can be no section $\mathcal{W}(\mathcal{L})$ of $\mathcal{T} \backslash v$ where B is submerged by $\mathcal{W}(\mathcal{L})$ and A is supermerged by $\mathcal{W}(\mathcal{L})$.

Lemma 2.3.22. *Let $\mathcal{T}$ be a triangulation of $C(m, 2d+1)$ with $\mathcal{W}(\mathcal{L})$ a section of $\mathcal{T}\backslash v$. Then there exists no circuit $(A \cup x, B \cup y)$ of $C([m]_{v+}, 2d+1)$ such that A is supermerged by $\mathcal{W}(\mathcal{L})$ and B is submerged by $\mathcal{W}(\mathcal{L})$.*

Proof. Suppose for contradiction that we are in the situation described and that there exists a circuit $(A \cup x, B \cup y)$ of $C([m]_{v+}, 2d+1)$ such that A is supermerged by $\mathcal{W}(\mathcal{L})$ and B is submerged by $\mathcal{W}(\mathcal{L})$. Suppose that A is not a face of a simplex of $\mathcal{W}(\mathcal{L})$. Then there is a simplex $S \in \mathcal{U} := (\mathcal{T} \setminus v) \setminus \mathcal{L}$ such that the lower facets of S are all $(2d-1)$-simplices of S and none of them contain A as a face. We obtain that $\mathcal{L}' = \mathcal{L} \cup S$ is also a lower set, with A is still supermerged by $\mathcal{W}(\mathcal{L}')$ and B is still submerged by $\mathcal{W}(\mathcal{L}')$. By repeating this process, we may assume that A is a face of a simplex of $\mathcal{W}(\mathcal{L})$. But this contradicts Lemma 2.3.21. □

2.3.4 Describing expansion at other vertices

We can now derive the main result of this section, Proposition 2.3.2, which says that the different triangulations which may result from expansion at vertex v are in bijection with the sections of $\mathcal{T}\backslash v$. We prove our bijection in two halves, showing first that every expanded triangulation gives us a section.

Lemma 2.3.23. *Let $\mathcal{T}$ be a triangulation of $C(m, 2d+1)$ with $\widetilde{\mathcal{T}}$ a triangulation of $C([m]_{v+}, 2d+1)$ such that $\widetilde{\mathcal{T}}[x \to v \leftarrow y] = \mathcal{T}$. Let $\mathcal{L}$ be the set of $2d$-simplices S of $\mathcal{T}\backslash v$ such that $S \cup y$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$. Then $\mathcal{W}(\mathcal{L}) = \widetilde{\mathcal{T}}\backslash\{x, y\}$.*

Proof. To start, note that by Lemma 2.3.16, the complement $\mathcal{U}$ of $\mathcal{L}$ in $\mathcal{T}\backslash v$ consists of the $2d$-simplices S such that $S \cup x$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$.

We first prove that $\widetilde{\mathcal{T}}\backslash\{x, y\} \subseteq \mathcal{W}(\mathcal{L})$. Let W be a $(2d-1)$-simplex of $\widetilde{\mathcal{T}}\backslash\{x, y\}$. Then $W \cup \{x, y\}$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$. We have that $W \cup x$ is either a facet of $C([m]_{v+}, 2d+1)$ or a facet of $R \cup x$ for some $R \in \mathcal{U}$. Likewise, either $W \cup y$ is a facet of $C([m]_{v+}, 2d+1)$ or a facet of $S \cup y$ for some $S \in \mathcal{L}$.

Note that we cannot both have that $W \cup x$ is a facet of $C([m]_{v+}, 2d+1)$ and that $W \cup y$ is a facet of $C([m]_{v+}, 2d+1)$. To see this, suppose that $W \cup x$ is an upper facet, so that it is an odd subset. This means that x is an even gap in $W \cup y$, since, by assumption, y is an odd gap in $W \cup x$. Hence, if $W \cup y$ is a facet of $C([m]_{v+}, 2d+1)$, then x must be the only gap in $W \cup y$, otherwise $W \cup y$ would have both odd and even gaps. This means that $C([m]_{v+}, 2d+1)$ is a $(2d+1)$-simplex, and so $C(m, 2d+1)$ is degenerate. The case where $W \cup x$ is a lower facet is similar.

Hence, we either have that

- $W \cup x$ is facet of $R \cup x$ for some $R \in \mathcal{U}$ and $W \cup y$ is a facet of $S \cup y$ for some $S \in \mathcal{L}$, or

- $W \cup x$ is a facet of $R \cup x$ for some $R \in \mathcal{U}$ and $W \cup y$ is a facet of $C([m]_{v+}, 2d+1)$, giving that W is a lower facet of $C(m, 2d+1) \backslash v$ by Lemma 2.3.6, or

- $W \cup x$ is a facet of $C([m]_{v+}, 2d+1)$, giving that W is an upper facet of $C(m, 2d+1) \backslash v$ by Lemma 2.3.6, and $W \cup y$ is a facet of $S \cup y$ for some $S \in \mathcal{L}$.

Hence, in all cases $W \in \mathcal{W}(\mathcal{L})$, by comparing with Definition 2.3.17.

We now prove that $\mathcal{W}(\mathcal{L}) \subseteq \widetilde{\mathcal{T}} \backslash \{x, y\}$. Suppose that W is a $(2d-1)$-simplex of $\mathcal{W}(\mathcal{L})$. We claim that $W \cup \{x, y\}$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$. By the following reasoning, we have that both $W \cup x$ and $W \cup y$ are $2d$-simplices of $\widetilde{\mathcal{T}}$.

- If $W = R \cap S$ where $R \in \mathcal{L}$ and $S \in \mathcal{U}$, then we have that $R \cup y$ and $S \cup x$ are $(2d+1)$-simplices of $\widetilde{\mathcal{T}}$ by definition of $\mathcal{L}$ and $\mathcal{U}$.

- If W is an upper facet of $C(m, 2d+1) \backslash v$ and an upper facet of R for $R \in \mathcal{L}$, then $W \cup x$ is a $2d$-simplex of $\widetilde{\mathcal{T}}$ by Lemma 2.3.6 and $R \cup y$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$ by definition of $\mathcal{L}$.

- If W is a lower facet of $C(m, 2d+1)\backslash v$ and a lower facet of S for $S \in \mathcal{U}$, then $W \cup y$ is a $2d$-simplex of $\widetilde{\mathcal{T}}$ by Lemma 2.3.6 and $R \cup x$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$ by definition of $\mathcal{U}$.

We now show that $W \cup \{x, y\}$ is a $(2d + 1)$-simplex of $\widetilde{\mathcal{T}}$ by applying Lemma 2.2.9, which states that it suffices to check that d- and $(d + 1)$-faces of $W \cup \{x, y\}$ are in $\widetilde{\mathcal{T}}$. Let $A \subseteq W \cup \{x, y\}$ be such that $\#A = d+1$. If $x, y \in A$, then A lies on the boundary of $C([m]_{v+}, 2d + 1)$ by Gale's Evenness Criterion, so A is a d-simplex of $\widetilde{\mathcal{T}}$. If $x \notin A$ (alternatively, $y \notin A$), then A is a d-face of $W \cup y$ (alternatively, $W \cup x$), which we already know is a $(2d - 1)$-simplex of $\widetilde{\mathcal{T}}$.

Now let $B \subseteq W \cup \{x, y\}$ such that $\#B = d + 2$. Every d-face of B is a d-simplex of $\widetilde{\mathcal{T}}$, by what we have just argued. If $x, y \in B$, then B cannot be half of a circuit of $C([m]_{v+}, 2d + 1)$, since x and y are consecutive in $[m]_{v+}$. Applying Lemma 2.2.9 then gives that B is a $(d + 1)$-simplex of $\widetilde{\mathcal{T}}$. If, on the other hand, $x \notin B$ (alternatively, $y \notin B$), then B is a $(d+1)$-face of $W \cup y$ (alternatively, $W \cup x$), which we know is a $2d$-simplex of $\widetilde{\mathcal{T}}$. Therefore, by Lemma 2.2.9, $W \cup \{x, y\}$ is a $(2d + 1)$-simplex of $\widetilde{\mathcal{T}}$, and so W is a $(2d - 1)$-simplex of $\widetilde{\mathcal{T}}\backslash\{x, y\}$. □

Remark 2.3.24. Lemma 2.3.23 gives us another way of seeing Lemma 2.3.18, which tells us that $\mathcal{W}(\mathcal{L})$ is a triangulation of $C([m] \setminus v, 2d-1)$. Namely, $\widetilde{\mathcal{T}}\backslash\{x, y\}$ is the triangulation of the "line figure" of $C([m]_{v+}, 2d + 1)$ at the line given by x and y. This line figure is precisely the cyclic polytope $C([m] \setminus v, 2d - 1)$.

We now show the other half of the bijection, namely, that one can construct an expanded triangulation from every section.

Lemma 2.3.25. *Let $\mathcal{T}$ be a triangulation of $C(m, 2d+1)$ with $\mathcal{W}(\mathcal{L})$ a section of $\mathcal{T}\backslash v$. Then there is a triangulation $\widetilde{\mathcal{T}}$ of $C([m]_{v+}, 2d + 1)$ such that $\widetilde{\mathcal{T}}[x \to v \leftarrow y] = \mathcal{T}$ and $\widetilde{\mathcal{T}}\backslash\{x, y\} = \mathcal{W}(\mathcal{L})$.*

Proof. Suppose that we are in the situation described and let $\mathcal{U}$ be the complement of $\mathcal{L}$ in $\mathcal{T}\backslash v$. We define $\widetilde{\mathcal{T}}$ to consist of the $(2d+1)$-simplices

$$\widetilde{\mathcal{T}} = \mathcal{T}^\circ \cup (\mathcal{W}(\mathcal{L}) * \{x, y\}) \cup (\mathcal{U} * x) \cup (\mathcal{L} * y),$$

where $\mathcal{T}^\circ = \{ Q : Q \in \mathcal{T}, v \notin Q \}$. It is evident from the definition of $\widetilde{\mathcal{T}}$ that $\widetilde{\mathcal{T}}[x \to v \leftarrow y] = \mathcal{T}$ and $\widetilde{\mathcal{T}}\backslash\{x, y\} = \mathcal{W}(\mathcal{L})$. We now show that $\widetilde{\mathcal{T}}$ is a triangulation of $C([m]_{v+}, 2d+1)$ by explicitly verifying that it satisfies the combinatorial definition of a triangulation.

We first verify that, for any simplex Q of $\widetilde{\mathcal{T}}$ and any facet F of Q, either F is a facet of $C([m]_{v+}, 2d+1)$ or a facet of another $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$.

(1) Suppose first that $Q \in \mathcal{T}^\circ$. Then if F is a facet of $C(m, 2d+1)$ in $\mathcal{T}$, F will be a facet of $C([m]_{v+}, 2d+1)$ in $\widetilde{\mathcal{T}}$. Suppose instead that F is a facet of Q' for some simplex Q' in $\mathcal{T}$. Then, if $v \notin Q'$, then $Q' \in \mathcal{T}^\circ \subseteq \widetilde{\mathcal{T}}$. On the other hand, if $v \in Q'$, then either $(Q' \setminus v) \cup x$ or $(Q' \setminus v) \cup y$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$, and F is a facet of either of these.

(2) Suppose now that $Q \in \mathcal{U} * x$.

If $x \notin F$, then $Q = F \cup x$. Then $F \cup v$ is a $(2d+1)$-simplex of $\mathcal{T}$, where F is either a facet of $C(m, 2d+1)$, or a facet of a $(2d+1)$-simplex Q', where $v \notin Q'$. If F is a facet of $C(m, 2d+1)$, then F is a facet of $C([m]_{v+}, 2d+1)$. If F is a facet of a $(2d+1)$-simplex Q' in $\mathcal{T}$, then $Q' \in \mathcal{T}^\circ$, since $F \cup y$ cannot be a simplex of $\widetilde{\mathcal{T}}$, and F is a facet of Q' in $\widetilde{\mathcal{T}}$.

If $x \in F$, then in $\mathcal{T}\backslash v$, $F \setminus x$ is either a facet of $C(m, 2d+1)\backslash v$, or a facet of some $2d$-simplex S distinct from $Q \setminus x$. We consider these two cases in turn.

If $F \setminus x$ is a facet of $C(m, 2d+1)\backslash v$, then it is either a lower facet or an upper facet. In the latter case, by Lemma 2.3.6, F is a facet of $C([m]_{v+}, 2d+1)$. In the former case, since $F \setminus x$ is a facet of $Q \setminus x$ and $Q \setminus x \in \mathcal{U}$, we have that $F \setminus x$ is in the section $\mathcal{W}(\mathcal{L})$. This then means that F is a facet of the $(2d+1)$-simplex $(F \setminus x) \cup \{x, y\}$ in $\widetilde{\mathcal{T}}$.

If $F \setminus x = (Q \setminus x) \cap S$ for some $2d$-simplex S, then either $S \in \mathcal{U}$, or $S \in \mathcal{L}$. If $S \in \mathcal{U}$, then $S \cup x$ is a $(2d+1)$-simplex of $\mathcal{T}$ distinct from Q with F as a facet. If $S \in \mathcal{L}$, then $F \setminus x \in \mathcal{W}(\mathcal{L})$. In this case $(F \setminus x) \cup \{x, y\}$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$ and it has F as a facet.

(3) The case where $Q \in \mathcal{L} * y$ is similar to the previous case.

(4) Finally, suppose that $Q \in \mathcal{W}(\mathcal{L}) * \{x, y\}$.

If $x \notin F$, then $Q = F \cup x$. We have that $F \setminus y$ is a $(2d-1)$-simplex of $\mathcal{T}\backslash v$, and is therefore either both a facet of $C(m, 2d+1)\backslash v$ and a facet of a $2d$-simplex S of $\mathcal{T}\backslash v$, or a shared facet of two $2d$-simplices R and S of $\mathcal{T}\backslash v$. Note also that $F \setminus y \in \mathcal{W}(\mathcal{L})$, since $Q \in \mathcal{W}(\mathcal{L}) * \{x, y\}$.

If $F \setminus y$ is both a facet of $C(m, 2d+1)\backslash v$ and a facet of a $2d$-simplex S, then either $F \setminus y$ is a lower facet of $C(m, 2d+1)\backslash v$ or it is an upper facet. If it is a lower facet, then F is a facet of $C([m]_{v+}, 2d+1)$ by Lemma 2.3.6. If it is an upper facet, then we must have $S \in \mathcal{L}$, since $F \setminus y \in \mathcal{W}(\mathcal{L})$. Consequently, $S \cup y$ is a $(2d+1)$-simplex of $\widetilde{\mathcal{T}}$, and F is a facet of it.

If $F \setminus y$ is a shared facet of two $2d$-simplices S and R in $\mathcal{T}$, then we may suppose without loss of generality that $S \in \mathcal{L}$ and $R \in \mathcal{U}$, since we know that $F \setminus y \in \mathcal{W}(\mathcal{L})$. We then have that F is a shared facet of Q and $S \cup y$ in $\widetilde{\mathcal{T}}$.

The case when $y \notin F$ is similar to the case where $x \notin F$.

If $x, y \in F$, then let $W = Q \backslash \{x, y\}$, so that $W \in \mathcal{W}(\mathcal{L})$. Then $G = F \backslash \{x, y\}$ is a facet of W. Since, by Lemma 2.3.18, $\mathcal{W}(\mathcal{L})$ is a triangulation of $C([m]\backslash v, 2d-1)$, then there either exists $W' \in \mathcal{W}(\mathcal{L})$ such that $W \cap W' = G$, or that G is a facet of $C([m] \setminus v, 2d+1)$. In the second case, we are done immediately by applying

Corollary 2.3.9, which gives us that $F = G \cup \{x, y\}$ is a facet of $C([m]_{v+}, 2d+1)$. In the first case, we have that $W' \cup \{x, y\} \in \mathcal{W}(\mathcal{L}) * \{x, y\}$ and that F is a shared facet between Q and $W' \cup \{x, y\}$.

We must now show that there can be no pair of $(2d+1)$-simplices S, R in $\widetilde{\mathcal{T}}$ such that $S \supseteq Z_-$ and $R \supseteq Z_+$, where (Z_-, Z_+) is a circuit of $C([m]_{v+}, 2d+1)$. Suppose for contradiction that there does exist such a pair of $(2d+1)$-simplices S and R.

We use the fact that $\widetilde{\mathcal{T}}[x \to v \leftarrow y] = \mathcal{T}$. This implies that any such circuit (Z_-, Z_+) must degenerate under the contraction $[x \to v \leftarrow y]$, since otherwise we would obtain a circuit in $\mathcal{T}$. This means that we have $x \in Z_\pm$ and $y \in Z_\mp$. Hence we only need to consider the cases where

(1) $S \in \mathcal{U} * x$ and $R \in \mathcal{L} * y$;

(2) $S \in \mathcal{U} * x$ and $R \in \mathcal{W}(\mathcal{L}) * \{x, y\}$;

(3) $S \in \mathcal{W}(\mathcal{L}) * \{x, y\}$ and $R \in \mathcal{L} * y$; and

(4) $S, R \in \mathcal{W}(\mathcal{L}) * \{x, y\}$.

But each case gives a contradiction to Lemma 2.3.22, or the more specific instances of Lemma 2.3.21 and Corollary 2.3.19. Hence, we obtain that $\widetilde{\mathcal{T}}$ is indeed a triangulation of $C([m]_{v+}, 2d+1)$. $\square$

Putting Lemma 2.3.23, Lemma 2.3.25 and Remark 2.3.3 together finally yields Proposition 2.3.2. The main application of Proposition 2.3.2 is the following lemma, which allows us to understand how expansions affect subpolytopes. This will be a key ingredient in the proof of the main result of Chapter 3.

Lemma 2.3.26. *Let $\mathcal{T}$ be a triangulation of $C(m-1, \delta)$. Suppose that $\mathcal{T}$ contains a cyclic subpolytope $C(H, \delta)$. Let $\widetilde{\mathcal{T}}$ be a triangulation of $C([m-1]_{v+}, \delta)$ such that $\widetilde{\mathcal{T}}[x \to v \leftarrow y] = \mathcal{T}$. Then either*

(1) $C(H,\delta)$ *is a subpolytope of* $\widetilde{\mathcal{T}}$ *and* $v \notin H$*,*

(2) $C((H \setminus v) \cup x, \delta)$ *is a subpolytope of* $\widetilde{\mathcal{T}}$*, where* $v \in H$*,*

(3) $C((H \setminus v) \cup y, \delta)$ *is a subpolytope of* $\widetilde{\mathcal{T}}$*, where* $v \in H$*, or*

(4) $C(H_{v+}, \delta)$ *is a subpolytope of* $\widetilde{\mathcal{T}}$*, where* $v \in H$*.*

Proof. By Proposition 2.3.2, triangulations $\widetilde{\mathcal{T}}$ of $C([m]_{v+}, \delta)$ such that $\widetilde{\mathcal{T}}[x \to v \leftarrow y] = \mathcal{T}$ are in bijection with sections $\mathcal{W}(\mathcal{L})$ of $\mathcal{T}\backslash v$. Moreover, given a section $\mathcal{W}(\mathcal{L})$ of $\mathcal{T}\backslash v$, the corresponding triangulation $\widetilde{\mathcal{T}}$ has the set of δ-simplices

$$\mathcal{T}^{\circ} \cup (\mathcal{W}(\mathcal{L}) * \{x, y\}) \cup (\mathcal{L} * x) \cup (\mathcal{U} * y),$$

where $\mathcal{T}^{\circ}$ denotes the δ-simplices of $\mathcal{T}$ which do not contain v and $\mathcal{U}$ is the complement of $\mathcal{L}$ in $\mathcal{T}\backslash v$.

The set-up of Lemma 2.3.26 gives us that $\mathcal{T}$ contains a cyclic subpolytope $C(H,\delta)$. We let $\mathcal{T}_H$ be the induced triangulation of this subpolytope in $\mathcal{T}$. If $v \notin H$, then $\mathcal{T}_H \subseteq \mathcal{T}^{\circ}$, so $C(H,\delta)$ is a subpolytope of $\widetilde{\mathcal{T}}$, giving case (1). Hence, we assume that $v \in H$. There are then three options:

(2) $\mathcal{T}_H\backslash v \subseteq \mathcal{L}$.

(3) $\mathcal{T}_H\backslash v \subseteq \mathcal{U}$.

(4) $\mathcal{T}_H\backslash v$ has non-empty intersection with both $\mathcal{L}$ and $\mathcal{U}$.

In case (2) we have that $C((H \setminus v) \cup y, \delta)$ is a subpolytope of $\widetilde{\mathcal{T}}$. In case (3), we have that $C((H \setminus v) \cup x, \delta)$ is a subpolytope of $\widetilde{\mathcal{T}}$.

In case (4), let $\mathcal{L}_H = \mathcal{L} \cap \mathcal{T}_H$ and $\mathcal{U}_H = \mathcal{U} \cap \mathcal{T}_H$. Then $\mathcal{L}_H$ is a lower set of the restriction of $\overset{v}{\leqslant}$ to $\mathcal{T}_H\backslash v$. We then obtain a section $\mathcal{W}(\mathcal{L}_H)$ of $\mathcal{T}_H\backslash v$, and it is straightforward to see that $\mathcal{W}(\mathcal{L}_H)$ consists of the $(2d-1)$-simplices of $\mathcal{W}(\mathcal{L})$ which are also $(2d-1)$-simplices of $\mathcal{T}_H\backslash v$. By Proposition 2.3.2, we have that the section $\mathcal{W}(\mathcal{L}_H)$ of $\mathcal{T}_H \backslash v$ gives us a triangulation $\widetilde{\mathcal{T}}_H$ of $C(H_{v+}, \delta)$. Moreover, the triangulation $\widetilde{\mathcal{T}}_H$ of $C(H_{v+}, \delta)$ has simplices

$$\mathcal{T}_H^{\circ} \cup (\mathcal{W}(\mathcal{L}_H) * \{x, y\}) \cup (\mathcal{U}_H * x) \cup (\mathcal{L}_H * y).$$

It is then clear that $\mathcal{T}_H^{\circ} \subseteq \mathcal{T}^{\circ}$, $\mathcal{W}(\mathcal{L}_H) \subseteq \mathcal{W}(\mathcal{L})$, $\mathcal{U}_H \subseteq \mathcal{U}$, and $\mathcal{L}_H \subseteq \mathcal{L}$. Hence $\widetilde{\mathcal{T}}_H$ is a subtriangulation of $\widetilde{\mathcal{T}}$, which gives us that $C(H_{v+}, \delta)$ is a subpolytope of $\widetilde{\mathcal{T}}$. $\square$

Chapter 3

The higher Stasheff–Tamari orders

Recall from the introduction in Chapter 1 that the two higher Stasheff–Tamari orders are two *a priori* different orders on the set of triangulations of a cyclic polytope. The first of these orders was introduced in 1991 by Kapranov and Voevodsky, with the second order introduced in 1996 by Edelman and Reiner, who conjectured the two orders to coincide [ER96, Conjecture 2.6]. In this chapter, we prove this conjecture. The first step in the proof is to give new combinatorial interpretations of the orders which make them more comparable. This is our task in Section 3.2. We then use these new combinatorial interpretations in Section 3.3 to give an inductive argument that the two orders are equal.

3.1 Definition of the orders

The *first higher Stasheff–Tamari order* is defined by its covering relations, which are such that $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$. We write $\mathcal{S}_1(m, \delta)$ for the poset on $\mathcal{S}(m, \delta)$ this gives and $\leqslant_1$ for the partial order itself.

The first higher Stasheff–Tamari order was originally introduced by Kapranov and Voevodsky in [KV91, Definition 3.3] as the "higher Stasheff order" using a slightly different definition. Thomas showed in [Tho03, Proposition 3.3] that the higher Stasheff order of Kapranov and Voevodsky was the same as the first higher Stasheff–Tamari order of Edelman and Reiner.

Remark 3.1.1. The result [Ram97, Theorem 1.1(ii)] states that the linear extensions of the partial order $\prec$ from Remark 2.1.3 give maximal chains in $\mathcal{S}_1(m, \delta - 1)$ and that the sequences of simplices obtained are the sequences of simplices inducing the sequences of bistellar flips in the maximal chains.

Recall from the discussion of triangulations in Section 2.1 that every triangulation $|\mathcal{T}|$ of $\mathfrak{C}(m, \delta)$ determines a unique piecewise-linear section $\sigma_{|\mathcal{T}|} \colon \mathfrak{C}(m, \delta) \to \mathfrak{C}(m, \delta + 1)$ by sending each δ-simplex $|S|_\delta$ of $|\mathcal{T}|$ to $|S|_{\delta+1}$ in $\mathfrak{C}(m, \delta + 1)$ in the natural way. The *second higher Stasheff–Tamari order* on $\mathcal{S}(m, \delta)$ is defined as

$$\mathcal{T} \leqslant_2 \mathcal{T}' \iff \sigma_{|\mathcal{T}|}(\mathbf{x})_{\delta+1} \leqslant \sigma_{|\mathcal{T}'|}(\mathbf{x})_{\delta+1} \quad \forall \mathbf{x} \in \mathfrak{C}(m, \delta),$$

where $\sigma_{|\mathcal{T}|}(\mathbf{x})_{\delta+1}$ denotes the $(\delta + 1)$-th coordinate of the point $\sigma_{|\mathcal{T}|}(\mathbf{x})$. We write $\mathcal{S}_2(m, \delta)$ for the poset on $\mathcal{S}(m, \delta)$ this gives.

We also use the following different interpretation of the second higher Stasheff–Tamari order. A k-simplex A in $C(m, \delta)$ is *submerged* by the triangulation $\mathcal{T} \in \mathcal{S}(m, \delta)$ if the restriction of the piecewise linear section $\sigma_{|\mathcal{T}|}$ to the simplex $|A|$ has the property that

$$\sigma_{|A|}(\mathbf{x})_{\delta+1} \leqslant \sigma_{|\mathcal{T}|}(\mathbf{x})_{\delta+1}$$

for all points $\mathbf{x} \in |A|$. Our notion of submersion from Definition 2.3.20 is a combinatorial version of this notion for vertex figures of cyclic polytopes. For a triangulation $\mathcal{T}$ of $C(m, \delta)$, the *k-submersion set*, $\mathrm{sub}_k(\mathcal{T})$, is the set of k-simplices A which are submerged by $\mathcal{T}$. Given two triangulations $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, \delta)$, we have that $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\mathrm{sub}_{\lceil \frac{\delta}{2} \rceil}(\mathcal{T}) \subseteq \mathrm{sub}_{\lceil \frac{\delta}{2} \rceil}(\mathcal{T}')$ [ER96, Proposition 2.15].

Submersion sets are independent of the choice of geometric realisation, but this is not immediately obvious from the definition.

General introductions to the higher Stasheff–Tamari orders can be found in [RR12] and [DRS10, Section 6.1].

Operations

The deletion and contraction operations behave well with respect to the higher Stasheff–Tamari orders. By [Ram97, Proposition 5.14], the operation $[m-1 \leftarrow m]$ is order-preserving with respect to the first order, whilst the operation $-\backslash m$ is order-reversing. By [Tho02, Theorem 4.1] and [ER96, Proposition 2.11], the operation $-[m-1 \leftarrow m]$ is order-preserving with respect to the second order, whilst the operation $-\backslash m$ is order-reversing. The operations $-[1 \to 2]$ and $-\backslash 1$ are order-preserving for both orders.

3.2 Combinatorial characterisation of the orders

One can see the need for new combinatorial characterisations of the higher Stasheff–Tamari orders. In particular, the second higher Stasheff–Tamari order is defined with respect to the geometric realisation when in fact the order is independent of the chosen geometric realisation. Hence, it ought to be possible to give a combinatorial definition of the second higher Stasheff–Tamari order. Indeed, such a definition was given in [Tho02] using the different combinatorial framework of snug partitions. We will continue to work in the framework of triangulations. Just as we did in Section 2.2, it is appealing to interpret the higher Stasheff–Tamari orders on the set of triangulations of $C(m, \delta)$ in terms of the $\lfloor \delta/2 \rfloor$-simplices of the triangulation. This is indeed how we will proceed.

Furthermore, the higher Stasheff–Tamari orders behave differently in even and odd dimensions. In even dimensions, both posets are self-dual, whereas neither is self-dual in odd dimensions; likewise, in odd dimensions both posets possess an order-preserving involution which does not exist in even dimensions [ER96, Proposition 2.11]. The first order is a ranked poset in odd dimensions, but neither order is ranked in even dimensions [Ram97, Corollary 1.2]. These properties indicate that the orders are suited to different combinatorial interpretations in odd and even dimensions. This parallels how the combinatorial descriptions of triangulations from Section 2.2 differed between odd and even dimensions.

3.2.1 Even dimensions

First order

We start by showing our combinatorial interpretation of the first higher Stasheff–Tamari order. By [OT12, Theorem 4.1], triangulations $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$ are bistellar flips of each other if and only if $\mathring{e}(\mathcal{T})$ and $\mathring{e}(\mathcal{T}')$ have all but one d-simplex in common. This can then be strengthened to the following.

Theorem 3.2.1. *For $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$, we have that $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) = \mathcal{U} \cup \{A\}$ and $\mathring{e}(\mathcal{T}') = \mathcal{U} \cup \{B\}$ and $A \wr B$.*

Proof. Consider a bistellar flip between $\mathcal{T}$ and $\mathcal{T}'$ inside a subpolytope $C(H, 2d)$, where $H \in \binom{[m]}{2d+2}$. By Lemma 2.2.2, the only d-simplex contained in upper facets of the $(2d+1)$-simplex H but not any lower facets is $\{h_1, h_3, \dots, h_{2d+1}\}$. Similarly the only d-simplex contained in lower facets but not any upper facets is $\{h_0, h_2, \dots, h_{2d}\}$. Moreover, these are both internal d-simplices in $C(m, 2d)$, since $\{h_1, h_3, \dots, h_{2d+1}\}, \{h_0, h_2, \dots, h_{2d}\} \in {}^{\circlearrowleft}\mathbf{I}^d_m$. Hence an increasing bistellar flip inside $C(H, 2d)$ involves exchanging $\{h_0, h_2, \dots, h_{2d}\}$ for $\{h_1, h_3, \dots, h_{2d+1}\}$. Therefore, if $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$, we have that $\mathring{e}(\mathcal{T}) = \mathcal{U} \cup \{A\}$ and $\mathring{e}(\mathcal{T}') = \mathcal{U} \cup \{B\}$ and $A \wr B$.

Conversely, suppose that we have $\mathring{e}(\mathcal{T}) = \mathcal{U} \cup \{A\}$ and $\mathring{e}(\mathcal{T}') = \mathcal{U} \cup \{B\}$ and $A \wr B$. By [OT12, Theorem 4.1] and its proof, we have that $\mathcal{T}'$ is the result of a bistellar flip of $\mathcal{T}$ which takes place inside $C(A \cup B, 2d)$. Then, by Lemma 2.2.2, this must be an increasing bistellar flip. □

Second order

In this section we prove our combinatorial interpretation of the second order in even dimensions, which is as follows.

Theorem 3.2.2. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$. Then $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if for every $A \in \mathring{e}(\mathcal{T})$, there is no $B \in \mathring{e}(\mathcal{T}')$ such that $B \wr A$.*

To prove this theorem we use the following combinatorial characterisation of submersion in even dimensions. This was shown for $d = 1$ in [ER96, Proposition 3.2]. Lemma 2.3.21 is also analogous to this proposition.

Proposition 3.2.3. *Let $\mathcal{T} \in \mathcal{S}(m, 2d)$ and let A be an internal d-simplex in $C(m, 2d)$. Then A is submerged by $\mathcal{T}$ if and only if there is no $B \in \mathring{e}(\mathcal{T})$ such that $B \wr A$.*

Proof. We prove the backwards direction first. Suppose that A is not submerged by $\mathcal{T}$, so that there is a point $\mathbf{y}$ in $|A|$ such that $s_{|A|}(\mathbf{y})_{2d+1} > \sigma_{|\mathcal{T}|}(\mathbf{y})_{2d+1}$. We split into two cases, depending on whether $s_{|A|}(\mathbf{x})_{2d+1} > \sigma_{|\mathcal{T}|}(\mathbf{x})_{2d+1}$ for all $\mathbf{x} \in |\mathring{A}|$, or whether there are also some $\mathbf{x} \in |\mathring{A}|$ such that $s_A(\mathbf{x})_{2d+1} \leqslant \sigma_{|\mathcal{T}|}(\mathbf{x})_{2d+1}$. By $|\mathring{A}|$, we of course mean the interior of the geometric d-simplex $|A|$.

In the first case, there must exist $B \in \mathring{e}(\mathcal{T})$ such that A and B are intertwining, since $A \notin \mathring{e}(\mathcal{T})$. By Lemma 2.2.2, $|A|_{2d+1}$ must the intersection of either the lower facets or the upper facets of the $(2d + 1)$-simplex $|A \cup B|_{2d+1}$. We have that $|\mathring{A}|$ and $|\mathring{B}|$ intersect in a unique point $\mathbf{y} \in \mathfrak{C}(m, 2d)$, since (A, B) is a circuit. By assumption, $s_{|A|}(\mathbf{y})_{2d+1} > \sigma_{|\mathcal{T}|}(\mathbf{y})_{2d+1} = s_{|B|}(\mathbf{y})_{2d+1}$. This means that $|A|_{2d+1}$ must be the intersection of the upper facets of $|A \cup B|_{2d+1}$. Hence $B \wr A$ by Lemma 2.2.2.

In the second case, by continuity, we must have a point $\mathbf{z}$ in $|\mathring{A}|$ such that $s_{|A|}(\mathbf{z})_{2d+1} = \sigma_{|\mathcal{T}|}(\mathbf{z})_{2d+1}$. Hence, $\mathbf{z}$ is a point of intersection between $|A|_{2d+1}$ and the image of $|\mathcal{T}|$ under $\sigma_{|\mathcal{T}|}$. The point $\mathbf{z}$ must be contained in a $2d$-simplex $|S|_{2d+1}$ of $|\mathcal{T}|$. Then, by the description of the circuits in $C(m, 2d+1)$, there must exist $s'_i \in \{s_0, s_1, \ldots, s_{2d}\}$ such that

$$s'_0 < a_0 < s'_1 < a_1 < \cdots < s'_d < a_d < s'_{d+1}.$$

Then $B = \{s'_0, s'_1, \ldots, s'_d\}$ is a d-simplex of $\mathring{e}(\mathcal{T})$ such that $B \wr A$.

Now we prove the forwards direction by contraposition. If there is a $B \in \mathring{e}(\mathcal{T})$ such that $B \wr A$, then $|A|_{2d+1}$ is the intersection of the upper facets of the $(2d+1)$-simplex $|A \cup B|_{2d+1}$ and $|B|_{2d+1}$ is the intersection of the lower facets, by Lemma 2.2.2. We have that $|\mathring{A}|_{2d}$ and $|\mathring{B}|_{2d}$ intersect in a unique point $\mathbf{z}$. Then $s_{|B|}(\mathbf{z})_{2d+1} = \sigma_{|\mathcal{T}|}(\mathbf{z})_{2d+1} < s_{|A|}(\mathbf{z})_{2d+1}$. But this means that A is not submerged by $\mathcal{T}$. □

The following lemma shows that, in order to have $\mathcal{T} \leqslant_2 \mathcal{T}'$, it is sufficient for $\mathcal{T}'$ to submerge the d-simplices of $\mathcal{T}$.

Lemma 3.2.4. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$. Then $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if every d-simplex of $\mathcal{T}$ is submerged by $\mathcal{T}'$.*

Proof. The forwards direction is clear. Conversely, suppose that every d-simplex of $\mathcal{T}$ is submerged by $\mathcal{T}'$. Since every point in $\mathfrak{C}(m, 2d)$ lies in a $2d$-simplex of $|\mathcal{T}|$, it suffices to show that every $2d$-simplex of $\mathcal{T}$ is submerged by $\mathcal{T}'$. Suppose that there is a $2d$-simplex S of $\mathcal{T}$ such that S is not submerged by $\mathcal{T}'$. We can assume that the $2d$-simplex S has at least one face which is an internal d-simplex not belonging to $\mathcal{T}'$. Otherwise, S is a $2d$-simplex of $\mathcal{T}'$ by [OT12, Lemma 2.15].

Hence let A be a d-face of S which is an internal d-simplex not belonging to $\mathcal{T}'$. We must then have that A is intertwining with some d-simplex B of $\mathcal{T}'$. This means that $|A|$ and $|B|$ intersect in a unique point $\mathbf{x}$. Then, since A is submerged by $\mathcal{T}'$, we must have that $\sigma_{|A|}(\mathbf{x})_{2d+1} = \sigma_{|S|}(\mathbf{x})_{2d+1} < \sigma_{|\mathcal{T}'|}(\mathbf{x})$. Since S is not submerged by $\mathcal{T}'$, there must also be $\mathbf{y} \in |S|$ such that $\sigma_{|S|}(\mathbf{y})_{2d+1} > \sigma_{|\mathcal{T}'|}(\mathbf{y})$. Therefore $|S|_{2d+1}$ intersects $\sigma_{|\mathcal{T}'|}(\mathfrak{C}(m, 2d))$ by continuity. By the description of the circuits of $C(m, 2d+1)$, there must either be a d-face J of S and a $(d+1)$-simplex K of $\mathcal{T}'$ such that $J \wr K$, or a $(d+1)$-face J of S and a d-simplex K of $\mathcal{T}'$ such that $K \wr J$. In the first case, $K \setminus k_{d+1} \wr J$, so that J is not submerged by $\mathcal{T}'$ by Proposition 3.2.3, a contradiction. In the second case, $J \setminus j_0$ is a d-face of S such that $K \wr J \setminus j_0$. This gives us that $J \setminus j_0$ is not submerged by $\mathcal{T}'$ by Proposition 3.2.3, which is also a contradiction. □

Proposition 3.2.3 and Lemma 3.2.4 together prove Theorem 3.2.12, since boundary d-simplices are in, and hence submerged by, every triangulation.

3.2.2 Odd dimensions

First order

In order to prove our combinatorial characterisation of the first higher Stasheff–Tamari order in odd dimensions we use the following fact about triangulations of polytopes.

Lemma 3.2.5. *Let $\mathcal{T}$ be a triangulation of a δ-dimensional polytope P, with A an internal k-simplex of $\mathcal{T}$. Then A is the intersection of at least $\delta - k + 1$ different δ-simplices of $\mathcal{T}$.*

Proof. We prove the result by downwards induction on k. Our base case is $k = \delta - 1$. Here $(\delta - 1)$-simplices are facets of δ-simplices. A facet of a given δ-simplex must either be a shared facet with another δ-simplex, or lie within a boundary facet of P. Hence an internal $(\delta-1)$-simplex must be the intersection of at least two δ-simplices.

For the inductive step, we assume that the result holds for $k+1$. Let A be a k-simplex of $\mathcal{T}$ for $k<\delta-1$. Then A is a face of a δ-simplex, and so must be the intersection of an least two $(k+1)$-simplices. Moreover, A cannot lie in any boundary $(k+1)$-simplices, otherwise it is a boundary k-simplex. Thus, let A be the intersection of two internal $(k+1)$-simplices A^1 and A^2. By the induction hypothesis, both A^1 and A^2 are the intersection of δ-simplices $\{A_1^1, A_2^1, \ldots, A_{l_1}^1\}$ and $\{A_1^2, A_2^2, \ldots, A_{l_2}^2\}$ respectively, where $l_1, l_2 \geqslant \delta-k$. Since A^1 and A^2 are distinct, we must have that

$$\{A_1^1, A_2^1, \ldots, A_{l_1}^1\} \neq \{A_1^2, A_2^2, \ldots, A_{l_2}^2\}.$$

Thus $\#\{A_1^1, A_2^1, \ldots, A_{l_1}^1, A_1^2, A_2^2, \ldots, A_{l_2}^2\} \geqslant \delta-k+1$, and so A is the intersection of at least $\delta-k+1$ different δ-simplices of $\mathcal{T}$. $\square$

We now give a combinatorial characterisation of the first higher Stasheff–Tamari order in terms of sets internal d-simplices.

Theorem 3.2.6. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d+1)$. Then we have $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) = \mathring{e}(\mathcal{T}') \cup \{A\}$ for some $A \in \mathbf{J}_m^d \setminus \mathring{e}(\mathcal{T}')$.*

Proof. Suppose first that $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$. Then $\mathcal{T}$ and $\mathcal{T}'$ coincide everywhere but inside a copy of $C(2d+3, 2d+1)$. This direction then follows from applying Corollary 2.2.6 to this copy of $C(2d+3, 2d+1)$. That is, let the vertices of the copy of $C(2d+3, 2d+1)$ be given by $A \cup B$, where $A \wr B$. Then, since $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$, by Corollary 2.2.6 we have that in $\mathcal{T}$, the triangulation of $C(A \cup B, 2d+1)$ is given by the internal d-simplex A, whereas in $\mathcal{T}'$, there are no d-simplices which are internal in $C(A \cup B, 2d+1)$. Therefore, we have that $\mathring{e}(\mathcal{T}) = \mathring{e}(\mathcal{T}') \cup \{A\}$ where $A \in \mathbf{J}_m^d \setminus \mathring{e}(\mathcal{T}')$.

We now suppose that $\mathring{e}(\mathcal{T}) = \mathring{e}(\mathcal{T}') \cup \{A\}$ for some $A \in \mathbf{J}^d_m \setminus \mathring{e}(\mathcal{T}')$. Since A is not a d-simplex of $\mathcal{T}'$, there must be a $(d+1)$-simplex B of $\mathcal{T}'$ such that $A \wr B$. Suppose that S is a $(2d+1)$-simplex of $\mathcal{T}$ which has A as a d-face. Suppose further that S possesses a vertex $q \notin A \cup B$. If $b_{i-1} < q < b_i$ for some $i \in [d+1]$, then $(A \setminus \{a_{i-1}\}) \cup \{q\} =: A' \wr B$, which is a contradiction, since $A' \in \mathring{e}(\mathcal{T}) \setminus \{A\} = \mathring{e}(\mathcal{T}')$. Similarly, if $q > b_{d+1}$, then $\{b_1, b_2, \dots, b_{d+1}\} \wr A \cup \{q\}$, which contradicts the fact that $\{b_1, b_2, \dots, b_{d+1}\} \in \mathring{e}(\mathcal{T}') \subset \mathring{e}(\mathcal{T})$. The case $q < b_0$ can be treated in the same way.

Thus every $(2d+1)$-simplex S of $\mathcal{T}$ with A as a d-face has vertices in $A \cup B$. There are $d+2$ such $(2d+1)$-simplices, given by $S_i := (A \cup B) \setminus \{b_i\}$ for each $b_i \in B$. The triangulation $\mathcal{T}$ must contain all of these S_i, since A must be the intersection of at least $d+2$ different $(2d+1)$-simplices, by Lemma 3.2.5. The set $\{S_i\}_{i=0}^{d+1}$ gives the lower triangulation of $C(A \cup B, 2d+1)$. None of these $(2d+1)$-simplices can be contained in $\mathcal{T}'$, but every other $(2d+1)$-simplex of $\mathcal{T}$ must be contained in $\mathcal{T}'$ by Lemma 2.2.9. It then follows that $\mathcal{T}'$ must be obtained by replacing the lower triangulation of $C(A \cup B, 2d+1)$ with the upper triangulation, since these are the only two possible triangulations of $C(A \cup B, 2d+1)$. Hence $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$. $\square$

Second order

To obtain our combinatorial interpretation of the second higher Stasheff–Tamari order in odd dimensions we first give an alternative to the interpretation of Edelman and Reiner in terms of submersion sets.

Definition 3.2.7. Let A be a k-simplex in $C(m, \delta)$. Given a triangulation $\mathcal{T} \in \mathcal{S}(m, \delta)$, we say that $\mathcal{T}$ *supermerges* A if for all $\mathbf{x} \in |A|$,

$$\sigma_{|A|}(\mathbf{x})_{\delta+1} \geqslant \sigma_{|\mathcal{T}|}(\mathbf{x})_{\delta+1}.$$

We then define the *k-supermersion set of $\mathcal{T}$* to be

$$\sup_k \mathcal{T} := \left\{ A \in \tbinom{[m]}{k+1} : \mathcal{T} \text{ supermerges } A \right\}.$$

This is, of course, the geometric analogue of the combinatorial supermersion we considered in Definition 2.3.20. In this section, we will be particularly interested in the d-supermersion sets of triangulations of $C(m, 2d+1)$, which have the following significance.

Lemma 3.2.8. *If $\mathcal{T}$ is a triangulation of $C(m, 2d+1)$, then the supermersion set $\sup_d \mathcal{T}$ is precisely the set of d-simplices of $\mathcal{T}$.*

Proof. Consider a d-simplex A. Every d-simplex in $\mathfrak{C}(m, 2d+2)$ lies in a lower facet by Gale's Evenness Criterion. This is because the vertex set of a lower facet of $\mathfrak{C}(m, 2d+2)$ is a disjoint union of $d+1$ pairs of consecutive numbers from $[m]$; any subset of $[m]$ of size $d+1$ is therefore a subset of a lower facet. Therefore no points in a d-simplex $|A|_{2d+2}$ can lie strictly above the section $\sigma_{|\mathcal{T}|}(\mathfrak{C}(m, 2d+1))$. Hence, if $A \in \sup_d \mathcal{T}$, we must have that A is a d-simplex of $\mathcal{T}$. $\square$

We now prove the following theorem, which, in particular, gives us an interpretation of the second higher Stasheff–Tamari order on triangulations of $(2d+1)$-dimensional cyclic polytopes in terms of d-simplices.

Theorem 3.2.9. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, \delta)$. Then $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if*

$$\sup_{\lfloor \frac{\delta}{2} \rfloor} \mathcal{T} \supseteq \sup_{\lfloor \frac{\delta}{2} \rfloor} \mathcal{T}'.$$

Proof. We know from [ER96] that $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if

$$\mathrm{sub}_{\lceil \frac{\delta}{2} \rceil} \mathcal{T} \subseteq \mathrm{sub}_{\lceil \frac{\delta}{2} \rceil} \mathcal{T}'.$$

In the case where δ is even,

$$\left\lfloor \frac{\delta}{2} \right\rfloor = \left\lceil \frac{\delta}{2} \right\rceil,$$

so the result simply follows from the symmetry that exists in the even case via the permutation

$$\alpha := \begin{pmatrix} 1 & 2 & \dots & m-1 & m \\ m & m-1 & \dots & 2 & 1 \end{pmatrix}.$$

By [ER96, Proposition 2.11] this gives an order-reversing bijection on $\mathcal{S}_2(m, 2d)$. We write $\alpha\mathcal{T}$ and $\alpha\mathcal{T}'$ for the images of the respective triangulations under the permutation α. By Proposition 3.2.3 and its dual, $\mathrm{sub}_d\alpha\mathcal{T} = \alpha\mathrm{sup}_d\mathcal{T}$. Hence

$$\begin{aligned}
\mathrm{sup}_d\mathcal{T} \supseteq \mathrm{sup}_d\mathcal{T}' &\iff \alpha\mathrm{sub}_d\alpha\mathcal{T} \supseteq \alpha\mathrm{sub}_d\alpha\mathcal{T}' \\
&\iff \mathrm{sub}_d\alpha\mathcal{T} \supseteq \mathrm{sub}_d\alpha\mathcal{T}' \\
&\iff \alpha\mathcal{T} \geqslant_2 \alpha\mathcal{T}' \\
&\iff \mathcal{T} \leqslant_2 \mathcal{T}'.
\end{aligned}$$

We now consider the case where δ is odd. We first suppose for contradiction that $\mathrm{sup}_d\mathcal{T} \supseteq \mathrm{sup}_d\mathcal{T}'$ and $\mathcal{T} \nleqslant_2 \mathcal{T}'$. Hence there exists $\mathbf{y} \in \mathfrak{C}(m, 2d+1)$ such that $\sigma_{|\mathcal{T}|}(\mathbf{y}) > \sigma_{|\mathcal{T}'|}(\mathbf{y})$. We split into two cases, depending upon whether $\sigma_{|\mathcal{T}|}(\mathbf{x}) \geqslant \sigma_{|\mathcal{T}'|}(\mathbf{x})$ for all $\mathbf{x} \in \mathfrak{C}(m, 2d+1)$, or whether there also exist some $\mathbf{x}$ for which $\sigma_{|\mathcal{T}|}(\mathbf{x}) < \sigma_{|\mathcal{T}'|}(\mathbf{x})$, in which case $\sigma_{|\mathcal{T}|}(\mathfrak{C}(m, 2d+1))$ and $\sigma_{|\mathcal{T}'|}(\mathfrak{C}(m, 2d+1))$ intersect each other.

Suppose we are in the case where $\sigma_{|\mathcal{T}|}(\mathbf{x}) \geqslant \sigma_{|\mathcal{T}'|}(\mathbf{x})$ for all $\mathbf{x} \in \mathfrak{C}(m, 2d+1)$. We have that $\mathcal{T}$ and $\mathcal{T}'$ must be distinct triangulations, since $\mathcal{T} \nleqslant_2 \mathcal{T}'$. Hence there must be a d-simplex A and a $(d+1)$-simplex B with $A \wr B$ such that one of A and B lies in $\mathcal{T}$ and the other lies in $\mathcal{T}'$. Since $|A|$ is the intersection of the lower facets of the $(2d+2)$-simplex $|A \cup B|$ and $|B|$ is the intersection of its upper facets, we must have that A is a d-simplex of $\mathcal{T}'$ and B is a $(d+1)$-simplex of $\mathcal{T}$, as $\sigma_{|\mathcal{T}|}(\mathfrak{C}(m, 2d+1))$ lies entirely above $\sigma_{|\mathcal{T}'|}(\mathfrak{C}(m, 2d+1))$. But then $A \in \mathrm{sup}_d\mathcal{T}' \subseteq \mathrm{sup}_d\mathcal{T}$, so that $\mathcal{T}$ contains the circuit (A, B), which is a contradiction.

Alternatively, suppose that we are in the case where $\sigma_{|\mathcal{T}|}(\mathfrak{C}(m, 2d+1))$ and

$\sigma_{|\mathcal{T}'|}(\mathfrak{C}(m, 2d+1))$ intersect each other. Hence, by the description of the circuits of $\mathfrak{C}(m, 2d+2)$, we have that there must be a pair of $(d+1)$-simplices A and B such that $A \wr B$, with one simplex in $\mathcal{T}$ and the other in $\mathcal{T}'$. We suppose that A is a $(d+1)$-simplex of $\mathcal{T}$ and that B is a $(d+1)$-simplex of $\mathcal{T}'$; the other case is similar. If we let $B' := \{b_0, b_1, \dots, b_d\}$, we then have that $B' \wr A$. But $B' \in \sup_d \mathcal{T}' \subseteq \sup_d \mathcal{T}$, which implies that $\mathcal{T}$ contains the circuit (A, B')—a contradiction as above. This establishes that if $\sup_d \mathcal{T} \supseteq \sup_d \mathcal{T}'$, then $\mathcal{T} \leqslant_2 \mathcal{T}'$.

Now we suppose that $\mathcal{T} \leqslant_2 \mathcal{T}'$. Let $A \in \sup_d \mathcal{T}'$. Then $|A|_{2d+2}$ cannot intersect $\sigma_{|\mathcal{T}|}(\mathfrak{C}(m, 2d+1))$ transversely, since it is too small: a circuit in $C(m, 2d+2)$ consists of a pair of $(d+1)$-simplices. Therefore, we suppose for contradiction that $|A| \in \operatorname{sub}_d \mathcal{T} \setminus \sup_d \mathcal{T}$. This means that there is an $\mathbf{x} \in |\mathring{A}|$ such that

$$\sigma_{|A|}(\mathbf{x})_{2d+2} < \sigma_{|\mathcal{T}|}(\mathbf{x})_{2d+2} \leqslant \sigma_{\mathcal{T}'}(\mathbf{x})_{2d+2} = \sigma_{|A|}(\mathbf{x})_{2d+2},$$

which is a contradiction. Hence $\sup_d \mathcal{T}' \subseteq \sup_d \mathcal{T}$. $\square$

Remark 3.2.10. One could, of course, consider complements of supermersion sets instead of supermersion sets. Since d-simplices in $C(m, 2d+2)$ all lie on the lower facets, these would comprise the d-simplices which are *strictly submerged* by a triangulation $\mathcal{T}$, that is: submerged by $\mathcal{T}$ without being a d-simplex of $\mathcal{T}$. The inclusion of these sets would be in the same direction as the second higher Stasheff–Tamari order, so some may have an aesthetic preference for this approach.

However, d-supermersion sets of triangulations $\mathcal{T}$ of $C(m, 2d+1)$ are more natural objects to consider. As proven in Lemma 3.2.8, these are simply the d-simplices of $\mathcal{T}$. They also fit more naturally into our algebraic description of the higher Stasheff–Tamari orders in odd dimensions in Chapter 4.

Remark 3.2.11. Edelman and Reiner describe $\mathcal{S}_2(m, \delta)$ in terms of $\lceil \frac{\delta}{2} \rceil$-submersion sets. But [Dey93] tells us that a triangulation of $C(m, \delta)$ is determined by its $\lfloor \frac{\delta}{2} \rfloor$-simplices. Logically, then, the second higher Stasheff–Tamari order ought also to be controlled by $\lfloor\frac{\delta}{2}\rfloor$-simplices. Theorem 3.2.9 shows us that this is indeed the case.

Theorem 3.2.12. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d+1)$. Then $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) \supseteq \mathring{e}(\mathcal{T}')$.*

Proof. The set $\mathring{e}(\mathcal{T})$ consists of the internal d-simplices of $\mathcal{T}$, while $\sup_d \mathcal{T}$ consists of all d-simplices of $\mathcal{T}$. It is then clear that $\sup_d \mathcal{T} \supseteq \sup_d \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) \supseteq \mathring{e}(\mathcal{T}')$, since boundary d-simplices are contained in every triangulation. □

To summarise, we obtain the following combinatorial characterisations of the higher Stasheff–Tamari orders in odd and even dimensions. Note how these results make the problem of comparing the two orders more tractable.

Theorem 3.2.13 (Theorems 3.2.1, 3.2.2, 3.2.6, and 3.2.12)**.** *Given $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$, we have that*

(1) $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) = \mathcal{R} \cup \{A\}$ and $\mathring{e}(\mathcal{T}') = \mathcal{R} \cup \{B\}$, where $A \wr B$;

(2) $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if for every $A \in \mathring{e}(\mathcal{T})$, there is no $B \in \mathring{e}(\mathcal{T}')$ such that $B \wr A$.

Given $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d+1)$, we have that

(1) $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) = \mathring{e}(\mathcal{T}') \cup \{A\}$ for some $A \in \mathbf{J}_m^d \setminus \mathring{e}(\mathcal{T}')$;

(2) $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) \supseteq \mathring{e}(\mathcal{T}')$.

3.3 Equality of the two orders

Having given combinatorial interpretations of the higher Stasheff–Tamari orders which make them more comparable, we can now prove that the higher Stasheff–Tamari orders are equal. What we need to establish is that for $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, \delta)$

with $\mathcal{T} <_2 \mathcal{T}'$, then we can find a triangulation $\mathcal{T}''$ either such that $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$, or such that $\mathcal{T} \leqslant_2 \mathcal{T}'' \lessdot_1 \mathcal{T}'$. Lemma 3.3.3 explains the detail of why this is what we need to prove. As in the previous section, we treat the odd-dimensional cases separately from the even-dimensional cases. The details of the proof are different for these two cases, but the broad outlines are similar. We explain these outlines now.

The proof is by induction on the number of vertices of the cyclic polytope $C(m, \delta)$, noting that the orders are known to be equal when $m \leqslant \delta + 3$ [RR12]. We start with triangulations $\mathcal{T}, \mathcal{T}'$ of $C(m, \delta)$ such that $\mathcal{T} <_2 \mathcal{T}'$. We perform contractions to obtain triangulations $\mathcal{T}[m-1 \leftarrow m]$ and $\mathcal{T}'[m-1 \leftarrow m]$ of $C(m-1, \delta)$. In the case that $\mathcal{T}[m-1 \leftarrow m] \neq \mathcal{T}'[m-1 \leftarrow m]$, we apply the induction hypothesis to these triangulations. This provides an increasing flip $\mathcal{U}$ of $\mathcal{T}[m-1 \leftarrow m]$ such that $\mathcal{U} \leqslant_2 \mathcal{T}'[m-1 \leftarrow m]$, and hence provides a subpolytope of $\mathcal{T}[m-1 \leftarrow m]$ congruent to $C(\delta+2, \delta)$. We consider the pre-image of this subpolytope in $\mathcal{T}$. If the pre-image of this subpolytope is congruent to $C(\delta+2, \delta)$, then we choose the increasing flip inside this subpolytope to obtain our triangulation $\mathcal{T}'$. As we showed in Lemma 2.3.26, the only other option is that the preimage of the $C(\delta+2, \delta)$ subpolytope is a subpolytope congruent to $C(\delta+3, \delta)$. This polytope is still relatively small and the triangulations of it are well-understood, as we record in Lemma 3.3.1. We can find an increasing bistellar flip $\mathcal{T}''$ of $\mathcal{T}$ which occurs within the induced triangulation of this $C(\delta+3, \delta)$ subpolytope. We then show that if we do not have $\mathcal{T}'' \leqslant_2 \mathcal{T}'$, then there is a contradiction to the existence of the increasing bistellar flip we chose using the induction hypothesis. Deriving this contradiction requires a series of lemmas, and the details differ between even and odd dimensions.

If $\mathcal{T}[m-1 \leftarrow m] = \mathcal{T}'[m-1 \leftarrow m]$, then we instead consider the contractions $\mathcal{T}[1 \to 2]$ and $\mathcal{T}'[1 \to 2]$. If $\mathcal{T}[1 \to 2] \neq \mathcal{T}'[1 \to 2]$, then we can apply symmetries of the cyclic polytope to convert to the case where $\mathcal{T}[m-1 \leftarrow m] \neq \mathcal{T}'[m-1 \leftarrow m]$, which we can deal with.

If we have that both $\mathcal{T}[1 \rightarrow 2] = \mathcal{T}'[1 \rightarrow 2]$ and $\mathcal{T}[m-1 \leftarrow m] = \mathcal{T}'[m-1 \leftarrow m]$, then one can apply the results of Section 3.2 to show that, since we have $\mathcal{T} <_2 \mathcal{T}'$, there must be a $v \in [2, m-2]$ such that if we relabel the vertices of $C(m,\delta)$ such that $\mathcal{T}, \mathcal{T}'$ are triangulations of $C([m-1]_{v+}, \delta)$, then we have that $\mathcal{T}[x \rightarrow v \leftarrow y] \neq \mathcal{T}'[x \rightarrow v \leftarrow y]$. Then one can proceed similarly to before.

Our proofs in odd and even dimensions are therefore both explicit. However, in Section 3.3.4, we note that it in fact suffices only to give an explicit proof for one parity, since the result for the other parity can be deduced from this.

3.3.1 Preliminary lemmas

We begin by proving some preliminary lemmas and recording some known results which we shall need. The following lemma records the possible triangulations of $C(\delta+3,\delta)$ and their properties. These triangulations are already well understood; for instance, see [Tho03, Proof of Proposition 9.1]. This lemma can be verified using the results of Section 3.2.

Lemma 3.3.1. *The triangulations of $C(\delta+3,\delta)$ may be described as follows.*

(1) If $\delta = 2d$, then

(a) $C(2d+3, 2d)$ has $2d+3$ triangulations $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_{2d+3}$;

(b) the triangulation $\mathcal{T}_i$ is the fan *triangulation at the vertex i, that is*

$$\mathring{e}(\mathcal{T}_i) = \{\, A \in {}^{\circlearrowleft}\mathbf{I}^d_{2d+3} : i \in A \,\};$$

(c) the posets $\mathcal{S}_1(2d+3, 2d)$ and $\mathcal{S}_2(2d+3, 2d)$ are equal and have the structure depicted in Figure 3.1.

(d) *the bistellar flips of the triangulations are as follows:*

- $\mathcal{T}_1$ *possesses two increasing bistellar flips: one which replaces* $\{1, 3, \ldots, 2d+1\}$ *with* $\{2, 4, \ldots, 2d+2\}$ *and one which replaces* $\{1, 4, \ldots, 2d+2\}$ *with* $\{3, 5, \ldots, 2d+3\}$*;*
- *for* i *even,* $\mathcal{T}_i$ *admits an increasing flip replacing* $\{1, 3, \ldots, i-3, i, i+2, \ldots, 2d+2\}$ *with* $\{2, 4, \ldots, i-2, i+1, i+3, \ldots, 2d+3\}$*;*
- *for* i *odd with* $i \notin \{1, 2d+3\}$*,* $\mathcal{T}_i$ *admits an increasing flip which replaces* $\{1, 3, \ldots, i, i+3, \ldots, 2d+2\}$ *with* $\{2, 4, \ldots, i-1, i+2, i+4, \ldots, 2d+3\}$.

(2) *If* $\delta = 2d+1$*, then*

(a) $C(2d+4, 2d+1)$ *has* $2d+4$ *triangulations* $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{2d+4}$*;*

(b) *the triangulations* $\mathcal{T}_i$ *have the following sets of internal* d*-simplices:*

$$\mathring{e}(\mathcal{T}_1) = \mathbf{J}^d_{2d+4},$$
$$\mathring{e}(\mathcal{T}_{2d+4}) = \varnothing,$$

for even $i \neq 2d+4$

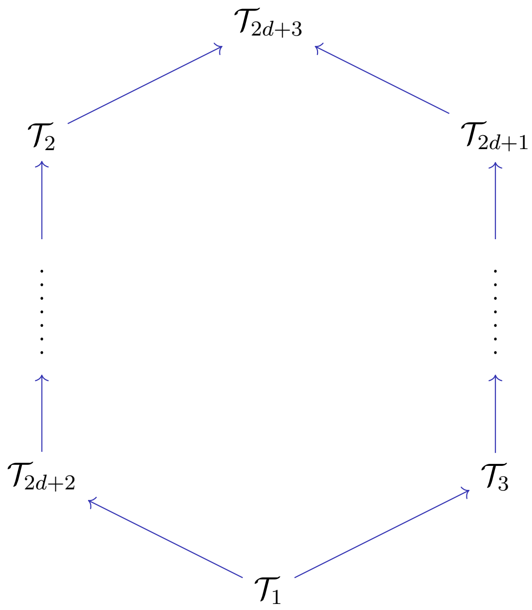
$$\begin{aligned}\mathring{e}(\mathcal{T}_i) = \bigl\{&\{2, 4, \ldots, 2d+2\}, \{2, \ldots, 2d, 2d+3\},\\ &\ldots, \{2, 4, \ldots, i, i+3, i+5, \ldots, 2d+3\}\bigr\},\end{aligned}$$

and for odd $i \neq 1$

$$\begin{aligned}\mathring{e}(\mathcal{T}_i) = \bigl\{&\{2, 4, \ldots, i-3, i, i+2, \ldots, 2d+3\},\\ &\ldots, \{2, 5, \ldots, 2d+3\}, \{3, 5, \ldots, 2d+3\}\bigr\};\end{aligned}$$

(c) *the posets* $\mathcal{S}_1(2d+4, 2d+1)$ *and* $\mathcal{S}_2(2d+4, 2d+1)$ *are equal and have the structure depicted in Figure 3.2.*

Figure 3.1: $\mathcal{S}_1(2d+3,2d) = \mathcal{S}_2(2d+3,2d)$



(d) the bistellar flips of the triangulations are as follows:

i. $\mathcal{T}_1$ *admits two increasing bistellar flips: one removes* $\{2,4,\dots,2d+2\}$ *and the other removes* $\{3,5,\dots,2d+3\}$*;*

ii. for even $i \neq 2d+4$*,* $\mathcal{T}_i$ *admits an increasing bistellar flip from removing* $\{2,4,\dots,i,i+3,i+5,\dots,2d+3\}$*;*

iii. for odd $i \neq 1$*,* $\mathcal{T}_i$ *admits an increasing bistellar flip from removing* $\{2,4,\dots,i-3,i,i+2,\dots,2d+3\}$.

Example 3.3.2. We give examples of the triangulations described in Lemma 3.3.1. We denote each triangulation $\mathcal{T}_i$ by its set of internal d-simplices $\mathring{e}(\mathcal{T})$. The poset $\mathcal{S}_1(7,4) = \mathcal{S}_2(7,4)$ is shown in Figure 3.3. The poset $\mathcal{S}_1(8,5) = \mathcal{S}_2(8,5)$ is shown in Figure 3.4.

The following lemma is straightforward, but serves to clarify what needs to be

Figure 3.2: $\mathcal{S}_1(2d+4, 2d+1) = \mathcal{S}_2(2d+4, 2d+1)$

$$\begin{tikzcd}
& \mathcal{T}_{2d+4} & \\
\mathcal{T}_{2d+2} \arrow[ur] & & \mathcal{T}_3 \arrow[ul] \\
\vdots \arrow[u] & & \vdots \arrow[u] \\
\mathcal{T}_2 \arrow[u] & & \mathcal{T}_{2d+3} \arrow[u] \\
& \mathcal{T}_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 3.3: Triangulations of $C(7,4)$

$$\begin{tikzcd}
& \{247, 257, 357\} & \\
& & \{246, 247, 257\} \arrow[ul] \\
\{257, 357, 135\} \arrow[uur] & & \\
& & \{146, 246, 247\} \arrow[uu] \\
\{357, 135, 136\} \arrow[uu] & & \\
& & \{136, 146, 246\} \arrow[uu] \\
& \{135, 136, 146\} \arrow[uul] \arrow[ur] &
\end{tikzcd}$$

Figure 3.4: Triangulations of $C(8,5)$

$$\begin{tikzcd}
 & \varnothing & \\
\{246\} \arrow[ur] & & \{357\} \arrow[ul] \\
\{246,247\} \arrow[u] & & \{257,357\} \arrow[u] \\
\{246,247,257\} \arrow[u] & & \{247,257,357\} \arrow[u] \\
 & \{246,247,257,357\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

proven in order to show that the orders are equivalent.

Lemma 3.3.3. *The following are equivalent.*

(1) For any pair of triangulations $\mathcal{T},\mathcal{T}' \in \mathcal{S}(m,\delta)$, we have that $\mathcal{T} \leqslant_1 \mathcal{T}'$ if and only if $\mathcal{T} \leqslant_2 \mathcal{T}'$.

(2) For any pair of triangulations $\mathcal{T},\mathcal{T}' \in \mathcal{S}(m,\delta)$ such that $\mathcal{T} <_2 \mathcal{T}'$, there exists a triangulation $\mathcal{T}'' \in \mathcal{S}(m,\delta)$ such that $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$.

(3) For any pair of triangulations $\mathcal{T},\mathcal{T}' \in \mathcal{S}(m,\delta)$ such that $\mathcal{T} <_2 \mathcal{T}'$, there exists a triangulation $\mathcal{T}'' \in \mathcal{S}(m,\delta)$ such that $\mathcal{T} \leqslant_2 \mathcal{T}'' \lessdot_1 \mathcal{T}'$.

Proof. First note that it is already known from [ER96, Proposition 2.5] that if $\mathcal{T} \leqslant_1 \mathcal{T}'$, then $\mathcal{T} \leqslant_2 \mathcal{T}'$. To show that (1) implies (2) and (3), suppose that we have $\mathcal{T},\mathcal{T}' \in \mathcal{S}(m,\delta)$ such that $\mathcal{T} <_2 \mathcal{T}'$. Then, from (1), it follows that $\mathcal{T} <_1 \mathcal{T}'$, so that we have

$$\mathcal{T} = \mathcal{T}_0 \lessdot_1 \mathcal{T}_1 \lessdot_1 \cdots \lessdot_1 \mathcal{T}_r = \mathcal{T}'.$$

Hence we have $\mathcal{T} \lessdot_1 \mathcal{T}_1 \leqslant_1 \mathcal{T}'$, and so $\mathcal{T} \lessdot_1 \mathcal{T}_1 \leqslant_2 \mathcal{T}'$ and (2) holds. Similarly $\mathcal{T} \leqslant_2 \mathcal{T}_{r-1} \lessdot_1 \mathcal{T}'$, and so (3) holds as well.

We now show that (2) implies (1). We can assume that $\mathcal{T} <_2 \mathcal{T}'$, since if $\mathcal{T} = \mathcal{T}'$, then it is trivial that $\mathcal{T} \leqslant_1 \mathcal{T}'$. Then, by applying (2), we obtain that there is a triangulation $\mathcal{T}_1 \in \mathcal{S}(m, \delta)$ such that

$$\mathcal{T} \lessdot_1 \mathcal{T}_1 \leqslant_2 \mathcal{T}'.$$

By applying (2) repeatedly, we obtain a chain

$$\mathcal{T} = \mathcal{T}_0 \lessdot_1 \mathcal{T}_1 \lessdot_1 \cdots \lessdot_1 \mathcal{T}_r = \mathcal{T}'.$$

This then establishes that $\mathcal{T} <_1 \mathcal{T}'$, as desired. The proof that (3) implies (1) is similar. □

We now describe the contraction operation $[x \to v \leftarrow y]$ in terms of the sets $\mathring{e}(\mathcal{T})$, which will allow us to show that this operation is order-preserving with respect to the second order.

Lemma 3.3.4. *Let $\widetilde{\mathcal{T}}$ be a triangulation of $C([m]_{v+}, \delta)$ with $v \in [2, m-1]$ and let $\mathcal{T} = \widetilde{\mathcal{T}}[x \to v \leftarrow y]$. Then*

$$\mathring{e}(\mathcal{T}) = \{\, A \in {}^{\circlearrowleft}\mathbf{I}_m^{\lfloor \delta/2 \rfloor} : A = \widetilde{A}[x \to v \leftarrow y] \text{ for some } \widetilde{A} \in \mathring{e}(\widetilde{\mathcal{T}}) \,\}.$$

Note that this lemma applies to both δ even and δ odd. If δ is odd and $A \in {}^{\circlearrowleft}\mathbf{I}_m^{\lfloor \delta/2 \rfloor}$ is such that $A = \widetilde{A}[x \to v \leftarrow y]$ for some $\widetilde{A} \in \mathring{e}(\widetilde{\mathcal{T}})$, then we automatically have that $A \in \mathbf{J}_m^{\lfloor \delta/2 \rfloor}$, since $\widetilde{A} \in \mathbf{J}_{[m]_{v+}}^{\lfloor \delta/2 \rfloor}$.

Proof. It is immediate that

$$\mathring{e}(\mathcal{T}) \supseteq \{\, A \in {}^{\circlearrowleft}\mathbf{I}_m^{\lfloor \delta/2 \rfloor} : A = \widetilde{A}[x \to v \leftarrow y] \text{ for some } \widetilde{A} \in \mathring{e}(\widetilde{\mathcal{T}}) \,\}$$

from the definitions of $\mathring{e}(\mathcal{T})$ and $\widetilde{\mathcal{T}}[x \to v \leftarrow y]$.

We now show that

$$\mathring{e}(\mathcal{T}) \subseteq \{\, A \in {}^{\circlearrowleft}\mathbf{I}_m^{\lfloor \delta/2 \rfloor} : A = \widetilde{A}[x \to v \leftarrow y] \text{ for some } \widetilde{A} \in \mathring{e}(\widetilde{\mathcal{T}}) \,\}.$$

If $A \in \mathring{e}(\mathcal{T})$, then there must exist a simplex $\widetilde{A}$ of $\widetilde{\mathcal{T}}$ such that $\widetilde{A}[x \to v \leftarrow y] = A$. Without loss of generality, we may assume that $\{x, y\} \not\subseteq \widetilde{A}$, since in this case we may remove either x or y from $\widetilde{A}$ and still have $\widetilde{A}[x \to v \leftarrow y] = A$. But then we must have that $\widetilde{A}$ is an internal $\lfloor \delta/2 \rfloor$-simplex, since A is an internal $\lfloor \delta/2 \rfloor$-simplex. Hence, $\widetilde{A} \in \mathring{e}(\widetilde{\mathcal{T}})$, as desired. □

We can now show that $[x \to v \leftarrow y]$ is order-preserving with respect to the second order.

Lemma 3.3.5. *If $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{T}}'$ are triangulations of $C([m-1]_{v+}, \delta)$, with $\widetilde{\mathcal{T}} \leqslant_2 \widetilde{\mathcal{T}}'$, then $\mathcal{T} \leqslant_2 \mathcal{T}'$, where $\mathcal{T} = \widetilde{\mathcal{T}}[x \to v \leftarrow y]$ and $\mathcal{T}' = \widetilde{\mathcal{T}}'[x \to v \leftarrow y]$.*

Proof. We split into two cases depending on whether δ is odd or even so that we can use the combinatorial interpretations of the second higher Stasheff–Tamari order.

We first let $\delta = 2d$. We show that if $\mathcal{T} \not\leqslant_2 \mathcal{T}'$, then $\widetilde{\mathcal{T}} \not\leqslant_2 \widetilde{\mathcal{T}}'$. Suppose that there exists $B \in \mathring{e}(\mathcal{T}')$ and $A \in \mathring{e}(\mathcal{T})$ such that $B \wr A$. Then we have that $\widetilde{B} \in \mathring{e}(\widetilde{\mathcal{T}}')$ and $\widetilde{A} \in \mathring{e}(\widetilde{\mathcal{T}})$, with $A = \widetilde{A}[x \to v \leftarrow y]$ and $B = \widetilde{B}[x \to v \leftarrow y]$, by Lemma 3.3.4. Then we also must have have $\widetilde{B} \wr \widetilde{A}$, since at most one of A and B can contain v. This implies that $\widetilde{\mathcal{T}} \not\leqslant_2 \widetilde{\mathcal{T}}'$, as desired.

We now let $\delta = 2d+1$. We shall show that if $\widetilde{\mathcal{T}} \leqslant_2 \widetilde{\mathcal{T}}'$, then $\mathring{e}(\mathcal{T}) \supseteq \mathring{e}(\mathcal{T}')$. Let $A \in \mathring{e}(\mathcal{T}')$. Then $A = \widetilde{A}[x \to v \leftarrow y]$ for some $\widetilde{A} \in \mathring{e}(\widetilde{\mathcal{T}}')$ by Lemma 3.3.4. Since $\mathring{e}(\widetilde{\mathcal{T}}) \supseteq \mathring{e}(\widetilde{\mathcal{T}}')$, we then have that $\widetilde{A} \in \mathring{e}(\widetilde{\mathcal{T}})$, which implies that $\widetilde{A}[x \to v \leftarrow y] = A \in \mathring{e}(\mathcal{T})$, as desired. □

3.3.2 Odd dimensions

We now prove the equivalence of the orders for odd dimensions. Our characterisation of odd-dimensional triangulations from Chapter 2 is essential to the proof. We begin by showing some preliminary lemmas which are specific to odd dimensions.

Lemma 3.3.6. *Let $\mathcal{T} \in \mathcal{S}(m, 2d+1)$ be a triangulation with a mutable d-simplex $A \in \mathring{e}(\mathcal{T})$ which is replaced by the $(d+1)$-simplex B in the increasing flip. Then $B' = \{b_1, b_2, \ldots, b_d\}$ is the unique support of A.*

Proof. First note that B' is a support of A. This follows from the fact that every d-simplex contained in $A \cup B'$, excluding A, contains consecutive entries in $A \cup B$, and is therefore a d-simplex of $\mathcal{T}$, since it lies on the boundary of $C(A \cup B, 2d+1)$, which is a subpolytope of $\mathcal{T}$.

We now suppose that A possesses a support $E = \{e_1, e_2, \ldots, e_d\}$. We show that if $e_i \neq b_i$ for any i, then $\mathcal{T}$ contains a d-simplex which forms a circuit with a $(d+1)$-simplex in the boundary of $C(A \cup B, 2d+1)$. This is a contradiction, since $C(A \cup B, 2d+1)$ is a subpolytope of $\mathcal{T}$. The internal $(d+1)$-simplices of $C(A \cup B, 2d+1)$ consist of B along with the $(d+1)$-simplices which have A as a face, by Gale's Evenness Criterion. All other $(d+1)$-simplices in $C(A \cup B, 2d+1)$ lie on the boundary.

Suppose that $e_i < b_i$ for some i. Then, since E is a support of A,

$$\mathring{e}(\mathcal{T}) \ni \{a_0, a_1, \ldots, a_{i-2}, e_i, a_i, a_{i+1} \ldots, a_d\} \wr \{b_0, b_1, \ldots, b_{i-2}, a_{i-1}, b_i, b_{i+1}, \ldots, b_{d+1}\},$$

which is a boundary $(d+1)$-simplex of $C(A \cup B, 2d+1)$. Suppose instead that $e_i > b_i$ for some i. Then, since E is a support of A,

$$\mathring{e}(\mathcal{T}) \ni \{a_0, a_1, \ldots, a_{i-1}, e_i, a_{i+1}, a_{i+2}, \ldots, a_d\} \wr \{b_0, b_1, \ldots, b_i, a_i, b_{i+2}, b_{i+3}, \ldots, b_{d+1}\},$$

which is a boundary $(d+1)$-simplex of $C(A \cup B, 2d+1)$. Therefore, we must have $E = B'$, which entails that B' is the unique support of A. $\square$

The following lemma helps us to understand what supports look like in triangulations of $C(2d+4, 2d+1)$. This is useful when we expand from $C(2d+3, 2d+1)$ subpolytopes to $C(2d+4, 2d+1)$ subpolytopes.

Lemma 3.3.7. *Let $\mathcal{T}$ be a triangulation of $C(2d+4, 2d+1)$ which is neither the upper triangulation nor the lower triangulation. Let A be the unique mutable d-simplex of $\mathcal{T}$ with E the unique support of A. Then every internal d-simplex A' of $\mathcal{T}$ has $A' \subseteq A \cup E$.*

Proof. Note first that $\mathcal{T}$ has a unique mutable d-simplex by Lemma 3.3.1. One can then proceed by direct verification. Suppose that we have the triangulation $\mathcal{T}_i$ of $C(2d+4, 2d+1)$, where i is even and $i \neq 2d+4$. Hence, as in Lemma 3.3.1, we have $\mathring{e}(\mathcal{T}_i)$ is

$$\big\{\{2, 4, \dots, 2d+2\}, \{2, 4 \dots, 2d, 2d+3\}, \dots, \{2, 4, \dots, i, i+3, i+5, \dots, 2d+3\}\big\}.$$

The mutable $(d+1)$-simplex here is $A = \{2, 4, \dots, i, i+3, i+5, \dots, 2d+3\}$ by Lemma 3.3.1. One can verify that this has support $E = \{3, 5, \dots, i-1, i+2, i+4, \dots, 2d+2\}$. Indeed, the internal d-simplices contained in $\{2, 4, \dots, i, i+3, i+5, \dots, 2d+3\} \cup \{3, 5, \dots, i-1, i+2, i+4, \dots, 2d+2\}$ are precisely $\mathring{e}(\mathcal{T})$. This establishes the claim when i is even, and the case where i is odd is the mirror image of this. $\square$

The next lemma is the inductive step of the proof of the equivalence of the orders for odd dimensions. Giving it as a separate lemma simplifies the presentation of the proof.

Lemma 3.3.8. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}([m-1]_{v+}, 2d+1)$ be triangulations such that $\mathcal{T} <_2 \mathcal{T}'$ and $\mathcal{T}[x \to v \leftarrow y] <_2 \mathcal{T}'[x \to v \leftarrow y]$. Suppose that $\mathcal{T}[x \to v \leftarrow y]$ possesses an increasing flip $\mathcal{U}$ such that $\mathcal{T}[x \to v \leftarrow y] \lessdot_1 \mathcal{U} \leqslant_2 \mathcal{T}'[x \to v \leftarrow y]$. Then $\mathcal{T}$ possesses an increasing flip $\mathcal{T}''$ such that $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$.*

Proof. Let the increasing flip from $\mathcal{T}[x \to v \leftarrow y]$ to $\mathcal{U}$ consist of replacing the d-simplex A with the $(d+1)$-simplex B inside the cyclic subpolytope $C(A\cup B, 2d+1)$. We must then have that $A \notin \mathring{e}(\mathcal{T}'[x \to v \leftarrow y])$ since $\mathcal{U} \leqslant_2 \mathcal{T}'[x \to v \leftarrow y]$ implies that $\mathring{e}(\mathcal{T}[x \to v \leftarrow y]) \setminus \{A\} = \mathring{e}(\mathcal{U}) \supseteq \mathring{e}(\mathcal{T}'[x \to v \leftarrow y])$.

By Lemma 2.3.26, we have that either

(1) $C(A \cup B, 2d+1)$ is a subpolytope of $\mathcal{T}$, where $v \notin A \cup B$,

(2) $C((A \cup B \cup x) \setminus v, 2d+1)$ is a subpolytope of $\mathcal{T}$, where $v \in A \cup B$,

(3) $C((A \cup B \cup y) \setminus v, 2d+1)$ is a subpolytope of $\mathcal{T}$, where $v \in A \cup B$, or

(4) $C((A \cup B)_{v+}, 2d+1)$ is a subpolytope of $\mathcal{T}$, where $v \in A \cup B$.

We deal with each of these cases in turn.

(1) Suppose that $C(A \cup B, 2d+1)$ is a subpolytope of $\mathcal{T}$. Then the induced triangulation of this subpolytope must contain A, since $A \in \mathring{e}(\mathcal{T}[x \to v \leftarrow y])$ and the contraction does not affect the subpolytope $C(A \cup B, 2d+1)$. Since A is contained in the subpolytope $C(A \cup B, 2d+1)$ of $\mathcal{T}$, we have that $\mathcal{T}$ admits an increasing flip $\mathcal{T}''$ where $\mathring{e}(\mathcal{T}'') = \mathring{e}(\mathcal{T}) \setminus A$. Furthermore, $A \notin \mathring{e}(\mathcal{T}')$, since $A \notin \mathring{e}(\mathcal{T}'[x \to v \leftarrow y])$. Thus, we have $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$.

(2) Suppose now that $C((A \cup B \cup x) \setminus v, 2d+1)$ is a subpolytope of $\mathcal{T}$, where $v \in A \cup B$. If $v \in A$, then let $\widetilde{A} = (A \cup x) \setminus v$. Otherwise, let $\widetilde{A} = A$. Then the induced triangulation of the subpolytope $C((A \cup B \cup x) \setminus v, 2d+1)$ must contain $\widetilde{A}$, since $A \in \mathring{e}(\mathcal{T}[x \to v \leftarrow y])$. Moreover, $\widetilde{A}$ is contained in the subpolytope $C((A \cup B \cup x) \setminus v, 2d+1)$ of $\mathcal{T}$, so that $\mathcal{T}$ admits an increasing flip $\mathcal{T}''$ where $\mathring{e}(\mathcal{T}'') = \mathring{e}(\mathcal{T}) \setminus \widetilde{A}$. We also have that $\widetilde{A} \notin \mathring{e}(\mathcal{T}')$, because $A \notin \mathring{e}(\mathcal{T}'[x \to v \leftarrow y])$. Hence $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$, as desired.

(3) The case where $C((A \cup B \cup y) \setminus v, 2d+1)$ is a subpolytope of $\mathcal{T}$ with $v \in A \cup B$ behaves similarly to the previous case.

(4) Consider now the case where $C((A \cup B \cup \{x,y\}) \setminus v, 2d+1)$ is a subpolytope of $\mathcal{T}$. Then the triangulation of this subpolytope induced by $\mathcal{T}$ must contain a d-simplex $\widetilde{A}$ such that $\widetilde{A}[x \to v \leftarrow y] = A$. This d-simplex $\widetilde{A}$ must be internal in $C((A \cup B \cup \{x,y\}) \setminus v, 2d+1)$, since A is internal in $C(A \cup B, 2d+1)$. Hence the induced triangulation of $C((A \cup B \cup \{x,y\}) \setminus v, 2d+1)$ cannot be the upper triangulation. Moreover, we cannot have $\widetilde{A} \in \mathring{e}(\mathcal{T}')$, since this implies that $A \in \mathring{e}(\mathcal{T}'[x \to v \leftarrow y])$.

Suppose first that the induced triangulation is the lower triangulation $\mathcal{T}_{b_0}$ of $C((A \cup B \cup \{x,y\}) \setminus v, 2d+1)$. Then, by Lemma 3.3.1, the lower triangulation $\mathcal{T}_{b_0}$ contains two mutable d-simplices, which we call J and K. Suppose that we have both $J, K \in \mathring{e}(\mathcal{T}')$. Then, by the bridging property from Lemma 2.2.12, $\mathcal{T}'$ must contain every internal d-simplex in $\mathcal{T}_{b_0}$, noting again the description of this triangulation from Lemma 3.3.1. But this means that $\widetilde{A} \in \mathring{e}(\mathcal{T}')$. Thus, at least one of J and K is not a d-simplex of $\mathcal{T}'$. Hence, let $\mathcal{T}''$ be the increasing flip of $\mathcal{T}$ defined by removing whichever of J and K is not a d-simplex of $\mathcal{T}'$. We therefore have $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$, as desired.

Now suppose that the induced triangulation of $C((A \cup B, \cup \{x,y\}) \setminus v, 2d+1)$ is neither the lower triangulation nor the upper triangulation. Then, by Lemma 3.3.1, the induced triangulation has a unique mutable d-simplex L. By Lemma 3.3.6, L has a unique support E in $\mathring{e}(\mathcal{T})$. We have that $\widetilde{A} \subseteq L \cup E$, by Lemma 3.3.7.

Suppose that $L \in \mathring{e}(\mathcal{T}')$. Let E' be the support of L in $\mathring{e}(\mathcal{T}')$, which we know must exist by Lemma 2.2.10. Then, since $\mathring{e}(\mathcal{T}) \supseteq \mathring{e}(\mathcal{T}')$, we have that E' is a support of L in $\mathring{e}(\mathcal{T})$. This implies that $E' = E$, $\widetilde{A} \subseteq L \cup E$, by Lemma 3.3.6. In turn, this implies that $\widetilde{A} \in \mathring{e}(\mathcal{T}')$, which is a contradiction. Therefore, if $\mathcal{T}''$ is the triangulation of $C(m, 2d+1)$ such that $\mathring{e}(\mathcal{T}'') = \mathring{e}(\mathcal{T}) \setminus \{L\}$, then we must have that $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$, as desired.

$\square$

We now have enough lemmas in place to prove the equivalence of the orders in odd dimensions.

Theorem 3.3.9. *Let $\mathcal{T}, \mathcal{T}'$ be triangulations of $C(m, 2d+1)$. Then $\mathcal{T} \leqslant_1 \mathcal{T}'$ if and only if $\mathcal{T} \leqslant_2 \mathcal{T}'$.*

Proof. We prove the result by induction on the number of vertices of the cyclic polytope. We may use the cases where $m - (2d+1) \leqslant 3$ as base cases, since the result is already known for these cases, as noted in [RR12]. Indeed, for $m = 2d+2$, the cyclic polytope $C(m, 2d+1)$ is a $(2d+1)$-simplex, so the result is trivial. The result is also clear for $m = 2d+3$, since here the cyclic polytope $C(m, 2d+1)$ only has two triangulations. Finally, the case where $m = 2d+4$ is given by Lemma 3.3.1. Hence, from now on, we assume that $m > 2d+4$.

As in Lemma 3.3.3, we seek a triangulation $\mathcal{T}''$ such that $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$. The existence of such a triangulation will establish our claim. We now split into three cases.

(1) Suppose that $\mathcal{T}[m-1 \leftarrow m] \neq \mathcal{T}'[m-1 \leftarrow m]$. Then, by the induction hypothesis, $\mathcal{T}[m-1 \leftarrow m]$ admits an increasing flip $\mathcal{U}$ such that $\mathcal{T}[m-1 \leftarrow m] \lessdot_1 \mathcal{U} \leqslant_2 \mathcal{T}'[m-1 \leftarrow m]$. By applying Lemma 3.3.8 with $m-1$ and m relabelled as x and y, we obtain that $\mathcal{T}$ possesses an increasing flip $\mathcal{T}''$ such that $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$.

(2) We now suppose that we have $\mathcal{T}[1 \to 2] \neq \mathcal{T}'[1 \to 2]$, so that $\mathcal{T}[1 \to 2] <_2 \mathcal{T}'[1 \to 2]$. By [ER96, Proposition 2.11], the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & m-1 & m \\ m & m-1 & \dots & 2 & 1 \end{pmatrix}$$

on the vertices of the cyclic polytope induces an order-preserving bijection α on both $\mathcal{S}_1(m, 2d+1)$ and $\mathcal{S}_2(m, 2d+1)$. We then have that $\alpha(\mathcal{T}[1 \to 2]) = \alpha(\mathcal{T})[m-$

$1 \leftarrow m] <_2 \alpha(\mathcal{T}')[m-1 \leftarrow m] = \alpha(\mathcal{T}'[1 \rightarrow 2])$. Hence, by applying the previous case, we obtain a triangulation $\mathcal{T}''$ such that $\alpha(\mathcal{T}') \lessdot_1 \mathcal{T}'' \leqslant_2 \alpha(\mathcal{T})$. By applying α again, we obtain that $\mathcal{T} \lessdot_1 \alpha(\mathcal{T}'') \leqslant_2 \mathcal{T}'$, which resolves this case.

(3) We may now suppose that we are in neither of the previous cases, so that $\mathcal{T}[1 \rightarrow 2] = \mathcal{T}'[1 \rightarrow 2]$ and $\mathcal{T}[m-1 \leftarrow m] = \mathcal{T}'[m-1 \leftarrow m]$. Hence, by the combinatorial interpretation of contraction from Lemma 2.2.15 and its dual, we must have that

$$\{ A \in \mathring{e}(\mathcal{T}) : 2 \notin A \} = \{ A \in \mathring{e}(\mathcal{T}') : 2 \notin A \},$$
$$\{ A \in \mathring{e}(\mathcal{T}) : m-1 \notin A \} = \{ A \in \mathring{e}(\mathcal{T}') : m-1 \notin A \},$$

and so $\mathring{e}(\mathcal{T})$ and $\mathring{e}(\mathcal{T}')$ only differ in simplices containing both 2 and $m-1$. Let A be a simplex such that $A \in \mathring{e}(\mathcal{T}) \setminus \mathring{e}(\mathcal{T}')$. Then $\{2, m-1\} \subseteq A$. There must be some $i \in [d]$ such that $a_i - a_{i-1} > 2$, otherwise $m - 1 = 2d + 2$, and we are supposing that this is not the case because $m = 2d + 3$ is a base case.

Relabel $[m]$ as $[m-1]_{v+}$, and relabel A correspondingly, such that $a_{i-1} < x < y < a_i$ and then perform the contraction $[x \rightarrow v \leftarrow y]$. Lemma 3.3.5 tells us that $\mathcal{T}[x \rightarrow v \leftarrow y] \leqslant_2 \mathcal{T}'[x \rightarrow v \leftarrow y]$. Moreover, $\mathcal{T}[x \rightarrow v \leftarrow y] <_2 \mathcal{T}'[x \rightarrow v \leftarrow y]$, since $A \in \mathring{e}(\mathcal{T}[x \rightarrow v \leftarrow y]) \setminus \mathring{e}(\mathcal{T}'[x \rightarrow v \leftarrow y])$. By the induction hypothesis, there is a increasing flip $\mathcal{U}$ of $\mathcal{T}[x \rightarrow v \leftarrow y]$ such that $\mathcal{T}[x \rightarrow v \leftarrow y] \lessdot_1 \mathcal{U} \leqslant_2 \mathcal{T}'[x \rightarrow v \leftarrow y]$. We then apply Lemma 3.3.8 to obtain that there exists an increasing flip $\mathcal{T}''$ of $\mathcal{T}$ such that $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$.

□

3.3.3 Even dimensions

We now prove the equivalence of the orders for even dimensions, beginning by proving preliminary lemmas specific to this parity.

Lemma 3.3.10. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$ such that $\mathcal{T} <_2 \mathcal{T}'$. Suppose that $\mathcal{T}$ admits an increasing flip $\mathcal{T}''$ which is the result of replacing the d-simplex A by the d-simplex B. Then, we have that $\mathcal{T}'' \nleqslant_2 \mathcal{T}'$ if and only if $\{a_0^o, a_1, \dots, a_d\} \in \mathring{e}(\mathcal{T}')$ for some a_0^o such that $a_0 \leqslant a_0^o < b_0$.*

Proof. If $\mathcal{T}'' \nleqslant_2 \mathcal{T}'$, then there must exist some $J \in \mathring{e}(\mathcal{T}')$ such that $J \wr B$, since $\mathcal{T} <_2 \mathcal{T}'$. Because we flip from A to B in $\mathcal{T}$, every internal d-simplex in $A \cup B$, excluding B, must be in $\mathring{e}(\mathcal{T})$ since they all lie in the boundary of the subpolytope $C(A \cup B, 2d)$—see [OT12, Proposition 4.6]. Indeed, they all lie in the facets of the subpolytope $C(A \cup B, 2d)$. Hence, if there is a d-simplex $K \subset A \cup B$ such that $J \wr K$, then this contradicts $\mathcal{T} \leqslant_2 \mathcal{T}'$. If we have $j_i < a_i$ for some i, then $J \wr (B \setminus \{b_i\}) \cup \{a_i\}$. Similarly, if, for $i \neq 0$, we have $j_i > a_i$, then $J \wr (B \setminus \{b_{i-1}\}) \cup \{a_i\}$. Thus, we must have $J = \{a_0^o, a_1, \dots, a_d\}$, where $a_0^o \geqslant a_0$. That $a_0^o < b_0$ follows from the fact that $J \wr B$.

Conversely, it is clear that if $J = \{a_0^o, a_1, \dots, a_d\} \in \mathring{e}(\mathcal{T}')$ such that $a_0 \leqslant a_0^o < b_0$, then $J \wr B$, so that we have $\mathcal{T}'' \nleqslant_2 \mathcal{T}'$. □

We call such a simplex $\{a_0^o, a_1, \dots, a_d\} \in \mathring{e}(\mathcal{T}')$ where $a_0 \leqslant a_0^o < b_0$ an *obstruction* to the increasing flip of $\mathcal{T}$ which replaces A with B. By Lemma 3.3.3, in order to prove the equivalence of the orders in even dimensions, we must find an increasing flip of $\mathcal{T}$ which is not obstructed by $\mathcal{T}'$. The following lemma allows us to describe the $2d$-simplex lying below the obstructing d-simplex.

Lemma 3.3.11. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$ such that $\mathcal{T} <_2 \mathcal{T}'$. Suppose that $\mathcal{T}$ admits an increasing flip via replacing the d-simplex A with the d-simplex B, and that $\{a_0^o, a_1, \dots, a_d\} \in \mathcal{T}'$, where $a_0 \leqslant a_0^o < b_0$. Then $\mathcal{T}'$ contains the $2d$-simplex $\{a_0^o, b_0, a_1, b_1, \dots, b_{d-1}, a_d\}$.*

Proof. By [OT12, Proposition 2.13], there exists a $2d$-simplex $S = \{a_0^o, q_0, a_1, q_1, \dots, q_{d-1}, a_d\}$ in $\mathcal{T}'$. We will show that we must have $q_i = b_i$ for all i by ruling out the other cases.

Suppose that $q_i < b_i$ for some i. Then $\{a_1, a_2, \ldots, a_i, b_i, b_{i+1}, \ldots, b_d\} \in \mathring{e}(\mathcal{T})$, since it is in the boundary of $C(A \cup B, 2d)$, and $\{q_0, q_1, \ldots, q_i, a_{i+1}, a_{i+2}, \ldots, a_d\} \in \mathring{e}(\mathcal{T}')$, since it is contained in S. But this contradicts $\mathcal{T} <_2 \mathcal{T}'$, since $\{q_0, q_1, \ldots, q_i, a_{i+1}, a_{i+1}, \ldots, a_d\} \wr \{a_1, a_2, \ldots, a_i, b_i, b_{i+1}, \ldots, b_d\}$.

Now suppose that $q_i > b_i$ for some i. Then $\{b_0, b_1, \ldots, b_i, a_{i+1}, a_{i+2}, \ldots, a_d\} \in \mathring{e}(\mathcal{T})$, since it is in the boundary of $C(A \cup B, 2d)$, and $\{a_0^o, a_1, \ldots, a_i, q_i, \ldots, q_{d-1}\} \in \mathring{e}(\mathcal{T}')$, since it is contained in S. But this contradicts $\mathcal{T} <_2 \mathcal{T}'$, since $\{a_0^o, a_1, \ldots, a_i, q_i, \ldots, q_{d-1}\} \wr \{b_0, b_1, \ldots, b_i, a_{i+1}, a_{i+2}, \ldots, a_d\}$, noting that $a_0^o < b_0$. Thus $q_i = b_i$ for all i, as desired, which completes the proof. □

We will use the following lemma and its corollary to find the right pair of middle vertices to contract at in the final case of our proof of the equivalence of the orders in even dimensions.

Lemma 3.3.12. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$ be triangulations such that $\mathcal{T} <_2 \mathcal{T}'$. Suppose further that both $\mathcal{T}[1 \to 2] = \mathcal{T}'[1 \to 2]$ and $\mathcal{T}[m-1 \leftarrow m] = \mathcal{T}'[m-1 \leftarrow m]$. Then, there exists $A \in \mathring{e}(\mathcal{T}) \setminus \mathring{e}(\mathcal{T}')$ and, for every such A, we have $a_0 = 1$ and $a_d = m - 1$. Dually, there exists $B \in \mathring{e}(\mathcal{T}') \setminus \mathring{e}(\mathcal{T})$ and, for every such B, we have $b_0 = 2, b_d = m$.*

Proof. Since we know that $\mathcal{T} \neq \mathcal{T}'$, there must exist $A \in \mathring{e}(\mathcal{T})$ such that $A \notin \mathring{e}(\mathcal{T}')$. We then have that A must be intertwining with some $B \in \mathring{e}(\mathcal{T}')$. Since $\mathcal{T} <_2 \mathcal{T}'$, we must in fact have that $A \wr B$. One may also arrive at this situation by first choosing $B \in \mathring{e}(\mathcal{T}')$ such that $B \notin \mathring{e}(\mathcal{T})$. If $a_0 > 1$, then $A \in \mathcal{T}[1 \to 2]$ and $B \in \mathcal{T}'[1 \to 2]$, which contradicts the fact that $\mathcal{T}[1 \to 2] = \mathcal{T}'[1 \to 2]$. Hence $a_0 = 1$, and we similarly argue that $b_d = m$. We can continue with similar deductions. If $b_0 > 2$, then $\mathcal{T}[1 \to 2] \ni \{2, a_1, a_2, \ldots, a_d\} \wr B \in \mathcal{T}'[1 \to 2]$. Therefore $b_0 = 2$, and we can likewise reason that $a_d = m - 1$. □

Corollary 3.3.13. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$, with $m > 2d+3$, be triangulations such that $\mathcal{T} <_2 \mathcal{T}'$, and both $\mathcal{T}[1 \to 2] = \mathcal{T}'[1 \to 2]$ and $\mathcal{T}[m-1 \leftarrow m] = \mathcal{T}'[m-1 \leftarrow m]$. Then there exists $v \in [3, m-2]$ such that if one relabels $[m]$ as $[m-1]_{v+}$, then we have $\mathcal{T}[x \to v \leftarrow y] <_2 \mathcal{T}'[x \to v \leftarrow y]$.*

Proof. Since $\mathcal{T} <_2 \mathcal{T}'$, there must exist $A \in \mathring{e}(\mathcal{T})$ and $B \in \mathring{e}(\mathcal{T}')$ such that $A \wr B$. By Lemma 3.3.12, we have that $a_0 = 1, b_0 = 2, a_d = m-1, b_d = m$. Because $m > 2d+3$, we must have $[m] \setminus (A \cup B) \neq \varnothing$. We may therefore choose $\{v, v+1\} \subset [m]$ such that $\#\{v, v+1\} \cap (A \cup B) \leqslant 1$ and $\{v, v+1\} \cap \{1, 2, m-1, m\} = \varnothing$. We then relabel $[m]$ as $[m-1]_{v+}$ so that $\{v, v+1\}$ becomes $\{x, y\}$. We likewise relabel $\mathcal{T}, \mathcal{T}'$, A, and B. If we let $\overline{A}, \overline{B}$ be the respective images of A and B under the contraction $[x \to v \leftarrow y]$, then, by our choice of v, we obtain that $\overline{A} \wr \overline{B}$. By Lemma 3.3.5, we obtain that $\mathcal{T}[x \to v \leftarrow y] <_2 \mathcal{T}'[x \to v \leftarrow y]$. $\square$

We now prove that the orders are equivalent in even dimensions. The structure of the proof is similar to odd dimensions, but we are not able to extract the inductive step of the proof as a separate lemma, since the details differ between the contractions $[m-1 \leftarrow m]$ and $[x \to v \leftarrow y]$.

Theorem 3.3.14. *Let $\mathcal{T}, \mathcal{T}'$ be triangulations of $C(m, 2d)$. Then $\mathcal{T} \leqslant_1 \mathcal{T}'$ if and only if $\mathcal{T} \leqslant_2 \mathcal{T}'$.*

Proof. As in the odd-dimensional case, we prove the result by induction on the number of vertices of the cyclic polytope. As noted in [RR12], the result is already known for $m \leqslant 2d+3$, so we use these as the base cases of our induction. One may also easily verify the result in these cases in the same way as explained in the proof of Theorem 3.3.9.

Hence, we suppose for induction that we have triangulations $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$, where $m > 2d+3$, such that $\mathcal{T} <_2 \mathcal{T}'$. We split into three cases, seeking a triangulation $\mathcal{T}''$ such that $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$.

(1) Suppose that $\mathcal{T}[m-1 \leftarrow m] \neq \mathcal{T}'[m-1 \leftarrow m]$, so that $\mathcal{T}[m-1 \leftarrow m] <_2 \mathcal{T}'[m-1 \leftarrow m]$. By the induction hypothesis, there exists a triangulation $\mathcal{U}$ such that $\mathcal{T}[m-1 \leftarrow m] \lessdot_1 \mathcal{U} \leqslant_2 \mathcal{T}'[m-1 \leftarrow m]$. Let this increasing flip of $\mathcal{T}[m-1 \leftarrow m]$ be given by exchanging a d-simplex A for a d-simplex B. Therefore, we have that $C(A \cup B, 2d)$ is a subpolytope of $\mathcal{T}[m-1 \leftarrow m]$. By Lemma 2.3.26, we have that either

(a) $C(A \cup B, 2d)$ is a subpolytope of $\mathcal{T}$,

(b) $C(A \cup B', 2d)$ is a subpolytope of $\mathcal{T}$, where $B' = \{b_0, b_1, \dots, b_{d-1}, m\}$ and $b_d = m-1$, or

(c) $C(A \cup B \cup m, 2d)$ is a subpolytope of $\mathcal{T}$, in which case $b_d = m-1$.

We deal with each of these cases in turn.

(a) Suppose first that $C(A \cup B, 2d)$ is a subpolytope of $\mathcal{T}$. This subpolytope therefore contains the d-simplex A, since it contains the d-simplex A in $\mathcal{T}[m-1 \leftarrow m]$. Thus $\mathcal{T}$ also admits an increasing flip by exchanging A for B to give a triangulation $\mathcal{T}''$. If $\mathcal{T}'' \nleqslant_2 \mathcal{T}'$, then $A^o = \{a_0^o, a_1, \dots, a_d\} \in \mathring{e}(\mathcal{T}')$ where $a_0 \leqslant a_0^o < b_0$, by Lemma 3.3.10. But then $A^o \in \mathring{e}(\mathcal{T}'[m-1 \leftarrow m])$, since $m \notin A^o$, which contradicts $\mathcal{U} \leqslant_2 \mathcal{T}'[m-1 \leftarrow m]$. Hence, in this case we have that $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$, as desired.

(b) If $C(A \cup B', 2d)$ is a subpolytope of $\mathcal{T}$, then we may exchange A for B'. If there is an obstruction to this, then we get a contradiction in a similar way to the previous case.

(c) Finally, suppose that $C(A \cup B \cup m, 2d)$ is a subpolytope of $\mathcal{T}$, in which case $b_d = m-1$. The triangulation of this subpolytope induced by $\mathcal{T}$ must contain the d-simplex A, since $m-1 \notin A$ but $A \in \mathring{e}(\mathcal{T}[m-1 \leftarrow m])$. Since $C(A \cup B \cup m, 2d)$ is a cyclic polytope congruent to $C(2d+3, 2d)$, all the triangulations of $C(A \cup B \cup m, 2d)$ are fan triangulations, by Lemma 3.3.1. The possible triangulations of $C(A \cup B \cup m, 2d)$ then consist of the fan triangulations determined by the elements $a_i \in A$, since we must have that A is a d-simplex of the induced triangulation of the subpolytope. By Lemma 3.3.1, the fan triangulation of $C(A \cup B \cup m, 2d)$ at a_i possesses an increasing flip at the d-simplex $J = \{a_0, a_1, \dots, a_i, b_{i+1}, b_{i+2}, \dots, b_d\}$, which is then exchanged for the d-simplex $K = \{b_0, b_1, \dots, b_{i-1}, a_{i+1}, a_{i+2}, \dots, a_d, m\}$.

Let $\mathcal{T}''$ be the triangulation resulting from performing this increasing flip on $\mathcal{T}$. If $\mathcal{T}'' \nleqslant_2 \mathcal{T}'$, then we have an obstruction $J^o = \{a_0^o, a_1, \dots, a_i, b_{i+1}, b_{i+2}, \dots, b_d\} \in \mathring{e}(\mathcal{T}')$ where $a_0 \leqslant a_0^o < b_0$, by Lemma 3.3.10. By Lemma 3.3.11, we conclude that $\{a_0^o, b_0, a_1, b_1, \dots, a_i, a_{i+1}, b_{i+1}, \dots, a_d, b_d\}$ is a $2d$-simplex of $\mathcal{T}'$. Consequently, $A^o = \{a_0^o, a_1, \dots, a_d\} \in \mathring{e}(\mathcal{T}')$, since it is a face of this $2d$-simplex. This implies that $A^o \in \mathring{e}(\mathcal{T}'[m-1 \leftarrow m])$, which obstructs the flip from $\mathcal{T}[m-1 \leftarrow m]$ to $\mathcal{U}$. But we assumed that $\mathcal{U} \leqslant_2 \mathcal{T}'[m-1 \leftarrow m]$ using the induction hypothesis. We therefore conclude that we cannot have $J^o \in \mathring{e}(\mathcal{T}')$. This means that we have $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$, as desired.

(2) We now suppose that $\mathcal{T}[1 \to 2] \neq \mathcal{T}'[1 \to 2]$. By [ER96, Proposition 2.11], the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & m-1 & m \\ m & m-1 & \dots & 2 & 1 \end{pmatrix}$$

on the vertices of the cyclic polytope induces an order-*reversing* bijection α on both $\mathcal{S}_1(m, 2d)$ and $\mathcal{S}_2(m, 2d)$. We then have that $\alpha(\mathcal{T}[1 \to 2]) = \alpha(\mathcal{T})[m-1 \leftarrow m] >_2 \alpha(\mathcal{T}')[m-1 \leftarrow m] = \alpha(\mathcal{T}'[1 \to 2])$. Hence, by applying the previous case, we obtain a triangulation $\mathcal{T}''$ such that $\alpha(\mathcal{T}') \lessdot_1 \mathcal{T}'' \leqslant_2 \alpha(\mathcal{T})$. By applying α again, we obtain that $\mathcal{T} \leqslant_2 \alpha(\mathcal{T}'') \lessdot_1 \mathcal{T}'$, which resolves this case, noting Lemma 3.3.3.

(3) We may now suppose that we have both $\mathcal{T}[m-1 \leftarrow m] = \mathcal{T}'[m-1 \leftarrow m]$ and $\mathcal{T}[1 \to 2] = \mathcal{T}'[1 \to 2]$. Since we are assuming that $m > 2d+3$, we can apply Corollary 3.3.13 and relabel $[m]$ as $[m-1]_{v+}$ such that $\mathcal{T}[x \to v \leftarrow y] <_2 \mathcal{T}'[x \to

$v \leftarrow y]$. By applying the induction hypothesis, we obtain that there exists an increasing flip $\mathcal{U}$ of $\mathcal{T}[x \to v \leftarrow y]$ such that $\mathcal{T}[x \to v \leftarrow y] \lessdot_1 \mathcal{U} \leqslant_2 \mathcal{T}'[x \to v \leftarrow y]$. Suppose that this increasing bistellar flip replaces the d-simplex A with the d-simplex B. Hence we have that $C(A \cup B, 2d)$ is a subpolytope of the triangulation $\mathcal{T}[x \to v \leftarrow y]$.

Note then that

$$\begin{aligned}(\mathcal{T}[x \to v \leftarrow y])[1 \to 2] &= (\mathcal{T}[1 \to 2])[x \to v \leftarrow y] \\ &= (\mathcal{T}'[1 \to 2])[x \to v \leftarrow y] \\ &= (\mathcal{T}'[x \to v \leftarrow y])[1 \to 2].\end{aligned}$$

This follows from [RS00, Theorem 3.4], but can also be seen directly. We similarly reason that

$$(\mathcal{T}[x \to v \leftarrow y])[m-2 \leftarrow m-1] = (\mathcal{T}'[x \to v \leftarrow y])[m-2 \leftarrow m-1].$$

Using these observations, we can deduce the values of the first and last elements of A and B. We know that $A \notin \mathring{e}(\mathcal{T}'[x \to v \leftarrow y])$, since $\mathcal{U}$ is obtained from $\mathcal{T}[x \to v \leftarrow y]$ by replacing A with B, and $\mathcal{U} \leqslant_2 \mathcal{T}'[x \to v \leftarrow y]$. By applying Lemma 3.3.12 to A, we obtain that $a_0 = 1$ and $a_d = m-2$. This implies that $b_d = m-1$. Since $A \notin \mathring{e}(\mathcal{T}'[x \to v \leftarrow y])$, but $(\mathcal{T}[x \to v \leftarrow y])[1 \to 2] = (\mathcal{T}'[x \to v \leftarrow y])[1 \to 2]$, we must have $\{2, a_1, a_2, \ldots, a_d\} \in \mathring{e}(\mathcal{T}'[x \to v \leftarrow y])$. But $\{2, a_1, a_2, \ldots, a_d\}$ is an obstruction to the flip from $\mathcal{T}[x \to v \leftarrow y]$ to $\mathcal{U}$ unless $b_0 = 2$. We thus conclude that $b_0 = 2$.

By Lemma 2.3.26, we have that either

(a) $C(A \cup B, 2d)$ is a subpolytope of $\mathcal{T}$ and $v \notin A \cup B$,

(b) $C((A \cup B \cup x) \setminus v, 2d)$ is a subpolytope of $\mathcal{T}$, where $v \in A \cup B$,

(c) $C((A \cup B \cup y) \setminus v, 2d)$ is a subpolytope of $\mathcal{T}$, where $v \in A \cup B$, or

(d) $C((A \cup B)_{v+}, 2d)$ is a subpolytope of $\mathcal{T}$, where $v \in A \cup B$.

We deal with each of these cases in turn.

(a) We suppose that $C(A \cup B, 2d)$ is a subpolytope of $\mathcal{T}$ and $v \notin A \cup B$. The induced triangulation of this subpolytope must contain the d-simplex A, since it contains the d-simplex A in $\mathcal{T}[x \to v \leftarrow y]$. We hence perform an increasing flip on $\mathcal{T}$ by replacing A with B inside this subpolytope, obtaining a triangulation $\mathcal{T}''$.

We claim that $\mathcal{T}'' \leqslant_2 \mathcal{T}'$. If not, then, by Lemma 3.3.10, there exists an obstruction $A^o = \{a_0^o, a_1, \dots, a_d\} \in \mathring{e}(\mathcal{T}')$, where $a_0 \leqslant a_0^o < b_0$. But, since $a_0 = 1$, $b_0 = 2$, we must have that $a_0^o = a_0$. This means that $A = A^o \in \mathring{e}(\mathcal{T}')$, which implies that $A \in \mathring{e}(\mathcal{T}'[x \to v \leftarrow y])$, which contradicts the fact that $\mathcal{U} \leqslant_2 \mathcal{T}'[x \to v \leftarrow y]$.

(b) The case where $C((A \cup B \cup x) \setminus v, 2d)$ is a subpolytope of $\mathcal{T}$ and $v \in A \cup B$ is largely analogous to the previous case, although there are additional details. We let $\widetilde{A}$ and $\widetilde{B}$ be such that $\widetilde{A} \cup \widetilde{B} = (A \cup B \cup x) \setminus v$, where $\widetilde{A} \wr \widetilde{B}$. Hence, we have $\widetilde{A}[x \to v \leftarrow y] = A$ and $\widetilde{B}[x \to v \leftarrow y] = B$. We must have that the induced triangulation of the subpolytope $C(\widetilde{A} \cup \widetilde{B}, 2d)$ contains the d-simplex $\widetilde{A}$, since the induced triangulation of the subpolytope $C(A \cup B, 2d)$ of $\mathcal{T}[x \to v \leftarrow y]$ contains A. We hence perform an increasing flip on $\mathcal{T}$ by replacing $\widetilde{A}$ with $\widetilde{B}$ inside this subpolytope, obtaining a triangulation $\mathcal{T}''$.

We claim that $\mathcal{T}'' \leqslant_2 \mathcal{T}$. If not, then, by Lemma 3.3.10, there exists an obstruction $A^o = \{a_0^o, \widetilde{a}_1, \dots, \widetilde{a}_d\}$, where $\widetilde{a}_0 \leqslant a_0^o < \widetilde{b}_0$. Since $1 < 2 < x < y$ in the ordering on $[m-1]_{v+}$, by our choice of x and y from Corollary 3.3.13, we must have that $\widetilde{a}_0 = a_0 = 1$ and $\widetilde{b}_0 = b_0 = 2$, and so $a_0^o = a_0 = 1$ and $A^o = \widetilde{A}$. This means that $\widetilde{A} \in \mathring{e}(\mathcal{T}')$, and so $A \in \mathring{e}(\mathcal{T}'[x \to v \leftarrow y])$. This contradicts the fact that $\mathcal{U} \leqslant_2 \mathcal{T}'[x \to v \leftarrow y]$.

(c) The case where $C((A \cup B \cup y) \setminus v, 2d+1)$ is a subpolytope of $\mathcal{T}$ and $v \in A \cup B$ is analogous to the previous case.

(d) We finally suppose that $C((A \cup B)_{v+}, 2d)$ is a subpolytope of $\mathcal{T}$, where $v \in A \cup B$. We label the vertices of this subpolytope by $H = \{h_1, h_2, \dots, h_{2d+3}\}$. From what we deduced about the first and last values of A and B, we know that $h_1 = 1, h_2 = 2, h_{2d+2} = m-2, h_{2d+3} = m-1$. We know that the triangulation of this subpolytope contains an internal d-simplex $\widetilde{A}$ such that $\widetilde{A}[x \to v \leftarrow y] = A$. Since $1 < 2 < x < y$ in the order on $[m-1]_{v+}$, we must have that $\widetilde{a}_0 = a_0 = 1$. Moreover, we must have that $\widetilde{A} \subseteq \{h_1, h_2, \dots, h_{2d+2}\}$, since $a_d < b_d$.

Using the notation of Lemma 3.3.1, there are two cases to consider, depending upon whether the triangulation of $C(H, 2d)$ is $\mathcal{T}_{h_{2i}}$ for i such that $2 \leqslant i \leqslant d+1$, or $\mathcal{T}_{h_{2i-1}}$ for i such that $1 \leqslant i \leqslant d+1$. The triangulations $\mathcal{T}_{h_2}$ and $\mathcal{T}_{h_{2d+3}}$ are excluded because they do not contain any internal simplices with 1 as a vertex, while we know that $\widetilde{a}_0 = 1$.

If the triangulation of $C(H, 2d)$ is $\mathcal{T}_{h_{2i}}$ for $i > 1$, then, by Lemma 3.3.1, there exists an increasing flip given by replacing $J = \{h_1, h_3, \dots, h_{2i-3}, h_{2i}, h_{2i+2}, \dots, h_{2d+2}\}$ with $K = \{h_2, h_4, \dots, h_{2i-2}, h_{2i+1}, h_{2i+3}, \dots, h_{2d+3}\}$. Since $h_1 = 1$ and $h_2 = 2$, if this flip is obstructed, it must be because $J \in \mathring{e}(\mathcal{T}')$. If this is the case, then by Lemma 3.3.11, we have that $J \cup K \setminus h_{2d+3} = H \setminus \{h_{2i-1}, h_{2d+3}\}$ is a $2d$-simplex of $\mathcal{T}'$. But we must have $\widetilde{A} \subseteq H \setminus \{h_{2i-1}, h_{2d+3}\}$, since $\widetilde{A} \in \mathring{e}(\mathcal{T}_{h_{2i}})$, and so $h_{2i-1} \notin \widetilde{A}$. This means that $\widetilde{A} \in \mathring{e}(\mathcal{T}')$, which implies that $A \in \mathring{e}(\mathcal{T}'[x \to v \leftarrow y])$. This contradicts the fact that $\mathcal{U} \leqslant_2 \mathcal{T}'[x \to v \leftarrow y]$, and so we conclude that the increasing flip given by replacing J with K cannot be obstructed. Hence if $\mathcal{T}''$ is the triangulation resulting from this flip, then we have $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$. The case where the triangulation of $C(H, 2d)$ is $\mathcal{T}_{h_{2i-1}}$ for $1 \leqslant i \leqslant d+1$ behaves similarly. Thus, in all cases we are able to construct a triangulation $\mathcal{T}''$ such that $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$, as desired.

□

3.3.4 One parity suffices

We conclude this section by noting that it suffices to show explicitly that the two orders are equal for one of either odd dimensions or even dimensions. The other parity can then be deduced for the parity that has been shown explicitly. This provides a shorter proof than giving an explicit proof for both parities. We submit that it is nevertheless worthwhile to have explicit proofs for both parities, as given in Section 3.3.2 and Section 3.3.3.

In particular, one can use the the extension operation from [Ram97], which was explained in Section 2.1.4. This operation produces a triangulation $\hat{\mathcal{T}}$ of $C(m+1, \delta+1)$ from a triangulation $\mathcal{T}$ of $C(m, \delta)$. This can be used to deduce that $\mathcal{S}_1(m, \delta) = \mathcal{S}_2(m, \delta)$ from $\mathcal{S}_1(m+1, \delta+1) = \mathcal{S}_2(m+1, \delta+1)$. Our first task is to show that the extension operation $(\hat{-})$ is order-reversing with respect to the second order. It is already known that this extension operation is order-reversing with respect to the first order [Ram97]. Of course, if one has already shown that the two orders are equal, then it is immediate from this that extension is order-reversing with respect to the second order. But we are supposing that one has not shown this, since we are demonstrating how to deduce the equivalence of the orders for one parity from the equivalence of the orders for the other parity. To show that extension is order-reversing for even dimensions, we explicitly describe $\mathring{e}(\hat{\mathcal{T}})$ in terms of $\mathring{e}(\mathcal{T})$.

Lemma 3.3.15. *Let $\mathcal{T} \in \mathcal{S}(m, 2d)$. Then*

$$\mathring{e}(\hat{\mathcal{T}}) = \widehat{\mathring{e}(\mathcal{T})} := \{\, A \in \mathbf{J}^d_{m+1} : \{a_0, a_1, \dots, a_{d-1}, a'_d\} \in \mathring{e}(\mathcal{T}) \text{ for } a_d \leqslant a'_d \,\}.$$

Note that, while $\mathcal{T}$ is a triangulation of $C(m, 2d)$, $\hat{\mathcal{T}}$ is a triangulation of $C(m+1, 2d+1)$, so its internal d-simplices lie in $\mathbf{J}^d_{m+1}$.

Proof. We show that the inclusion holds both ways. Let $A \in \mathring{e}(\hat{\mathcal{T}})$. Then $A \in \mathbf{J}^d_{m+1}$ and there exists a $(2d+1)$-simplex S of $\hat{\mathcal{T}}$ which possesses A as a face. If $S =$

$S' \cup \{m+1\}$ for S' a $2d$-simplex of $\mathcal{T}$, then we must have $A \subseteq S'$, since $a_d \neq m+1$, as $A \in \mathbf{J}^d_{m+1}$. Then it is immediate that $A \in \widehat{\mathring{e}(\mathcal{T})}$. Alternatively, we could have that $S = S'' \cup \{l, l+1\}$, where $S'' = \{s_0, s_1, \dots, s_{2d-1}\}$, with $S' = \{s_0, s_1, \dots, s_{2d}\}$ a $2d$-simplex of $\mathcal{T}$ for some s_{2d} such that $s_{2d-1} < l < s_{2d}$. If $a_d \notin \{l, l+1\}$, then A is a face of S' and so is straightforwardly in $\widehat{\mathring{e}(\mathcal{T})}$. If $a_d \in \{l, l+1\}$, then $\{a_0, a_1, \dots, a_{d-1}, s_{2d}\}$ is a face of S' and so $\{a_0, a_1, \dots, a_{d-1}, s_{2d}\} \in \mathring{e}(\mathcal{T})$, noting that $a_0 \geqslant 2$, which means that $\{a_0, a_1, \dots, a_{d-1}, s_{2d}\}$ is an internal d-simplex even if $s_{2d} = m$. Then $A \in \widehat{\mathring{e}(\mathcal{T})}$, since $a_d \in \{l, l+1\}$ and $l+1 \leqslant s_{2d}$.

Conversely, suppose that $A \in \widehat{\mathring{e}(\mathcal{T})}$. If $A \in \mathring{e}(\mathcal{T})$, then it is straightforwardly the case that $A \in \mathring{e}(\hat{\mathcal{T}})$, since every d-simplex of $\mathcal{T}$ is a d-simplex of $\hat{\mathcal{T}}$. Hence, suppose that $\{a_0, a_1, \dots, a_{d-1}, a'_d\} \in \mathring{e}(\mathcal{T})$, where $a_d < a'_d$. Then there exists a $2d$-simplex $S = \{a_0, q_0, a_1, q_1, a_2, \dots, a_{d-1}, q_{d-1}, a'_d\}$ of $\mathcal{T}$ by [OT12, Proposition 2.13]. We can assume that $q_{d-1} < a_d$, since otherwise we can replace $\{a_0, a_1, \dots, a_{d-1}, a'_d\}$ with $\{a_0, a_1, \dots, a_{d-1}, q_{d-1}\}$ and repeat the argument. Then, by definition of $\hat{\mathcal{T}}$, we have that

$$\{a_0, q_0, a_1, q_1, a_2, \dots, a_{d-1}, q_{d-1}, a_d, a_d + 1\}$$

is a $(2d+1)$-simplex of $\hat{\mathcal{T}}$, since $q_{d-1} < a_d < a'_d$. We then obtain that $A \in \mathring{e}(\hat{\mathcal{T}})$, since it is a face of this $(2d+1)$-simplex. $\square$

This explicit description of $\mathring{e}(\hat{\mathcal{T}})$ for an even-dimensional triangulation $\mathcal{T}$ allows us to show that the extension operation $(\hat{-})$ is order-reversing with respect to the second order for even dimensions.

Lemma 3.3.16. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$ with $\mathcal{T} \leqslant_2 \mathcal{T}'$. Then $\hat{\mathcal{T}}' \leqslant_2 \hat{\mathcal{T}}$.*

Proof. We prove the contrapositive. Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d)$ and suppose that $\hat{\mathcal{T}}' \nleqslant_2 \hat{\mathcal{T}}$. Then there exists $A \in \mathring{e}(\hat{\mathcal{T}}) \setminus \mathring{e}(\hat{\mathcal{T}}')$ by Theorem 3.2.12. Since $A \notin \mathring{e}(\hat{\mathcal{T}}')$, there exists a $(d+1)$-simplex B of $\hat{\mathcal{T}}'$ such that $A \wr B$. We claim that $B' := \{b_0, b_1, \dots, b_d\} \in \mathring{e}(\mathcal{T}')$. Indeed, B must be a face of some $(2d+1)$-simplex S. If

$S = S' \cup \{m+1\}$ for a $2d$-simplex S of $\mathcal{T}$, then the claim is clear since $B' \subseteq S$. The other option is that $S = S'' \cup \{l, l+1\}$ for $S'' = \{s_0, s_1, \dots, s_{2d-1}\}$ with $S' = \{s_0, s_1, \dots, s_{2d}\}$ a $2d$-simplex of $\mathcal{T}$ and $s_{2d-1} < l < s_{2d}$. But here we cannot have that $b_d = l$ and $b_{d+1} = l+1$, since $b_d < a_d < b_{d+1}$. Hence $B' \subseteq S'' \subseteq S'$, and so we again have that $B' \in \mathring{e}(\mathcal{T}')$. Then, by Lemma 3.3.15, we have that $A' = \{a_0, a_1, \dots, a_{d-1}, a'_d\} \in \mathring{e}(\mathcal{T})$ for some $a'_d \geqslant a_d$. Hence we have that $B' \wr A'$, which implies that $\mathcal{T} \nleqslant_2 \mathcal{T}'$ by Theorem 3.2.2. □

For odd dimensions we do not explicitly describe $\mathring{e}(\hat{\mathcal{T}})$ in terms of $\mathring{e}(\mathcal{T})$, but instead prove that the extension operation $\hat{(-)}$ is order-reversing with respect to the second order without this intermediate step.

Lemma 3.3.17. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, 2d+1)$ with $\mathcal{T} \leqslant_2 \mathcal{T}'$. Then $\hat{\mathcal{T}}' \leqslant_2 \hat{\mathcal{T}}$.*

Proof. We again prove the contrapositive. Suppose that $\hat{\mathcal{T}}' \nleqslant_2 \hat{\mathcal{T}}$. Then there exists $B \in \mathring{e}(\hat{\mathcal{T}})$ and $A \in \mathring{e}(\hat{\mathcal{T}}')$ such that $B \wr A$ by Theorem 3.2.2. We consider how A and B relate to $\mathcal{T}'$ and $\mathcal{T}$. We have that B is a face of some $(2d+2)$-simplex S of $\hat{\mathcal{T}}$. If $S = S' \cup m+1$ for S' a $(2d+1)$-simplex of $\mathcal{T}$, then $B \subseteq S'$, since $b_{d+1} \neq m+1$, as $B \wr A$. Hence, in this case B is a $(d+1)$-simplex of $\mathcal{T}$. In the other case, we have that $S = S'' \cup \{l, l+1\}$ with $\{s_0, s_1, \dots, s_{2d}, s'_{2d+1}\}$ a $(2d+1)$-simplex of $\mathcal{T}$ where $s_{2d} < l < s'_{2d+1}$ and $S'' = \{s_0, s_1, \dots, s_{2d}\}$. Here, we must have that $\{b_0, b_1, \dots, b_d\} \subseteq S''$, since $b_{d+1} \neq b_d + 1$, as $b_d < a_d < b_{d+1}$. If $b_{d+1} \in \{l, l+1\}$, then we let $b'_{d+1} = s'_{2d+1}$, noting that this means $b'_{d+1} \geqslant b_{d+1}$. Hence, in either case, we have that $\{b_0, b_1, \dots, b_d, b'_{d+1}\}$ is a $(d+1)$-simplex of $\mathcal{T}$ for some $b'_{d+1} \geqslant b_{d+1}$.

Similarly, we have that $\{a_0, a_1, \dots, a_{d+1}\}$ is a face of some $(2d+2)$-simplex U of $\hat{\mathcal{T}}'$. It can be seen that we must have $\{a_0, a_1, \dots, a_d\} \in \mathring{e}(\mathcal{T}')$, whatever form the simplex $\mathcal{U}$ takes, since a_d can neither be the final vertex of U nor the penultimate vertex, as $a_d < b_{d+1} < a_{d+1}$. But then $\{a_0, a_1, \dots, a_d\} \notin \mathring{e}(\mathcal{T})$, since we have $\{a_0, a_1, \dots, a_d\} \wr \{b_0, b_1, \dots, b_d, b'_{d+1}\}$. Hence $\mathcal{T} \nleqslant_2 \mathcal{T}'$ by Theorem 3.2.12. □

Thus, we obtain that the extension operation from $\mathcal{T}$ to $\hat{\mathcal{T}}$ is order-reversing with respect to the second order for both odd dimensions and even dimensions.

Corollary 3.3.18. *If $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, \delta)$ with $\mathcal{T} \leqslant_2 \mathcal{T}'$, then $\hat{\mathcal{T}}' \leqslant_2 \hat{\mathcal{T}}$.*

This allows us to show that the equality of the orders for δ can be deduced from the equality of the orders for $\delta + 1$.

Proposition 3.3.19. *If $\mathcal{S}_1(m, \delta + 1) = \mathcal{S}_2(m, \delta + 1)$ for all m, then $\mathcal{S}_1(m, \delta) = \mathcal{S}_2(m, \delta)$ for all m.*

Proof. Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(m, \delta)$ such that $\mathcal{T} <_2 \mathcal{T}'$. We consider the extended triangulations $\hat{\mathcal{T}}, \hat{\mathcal{T}}'$ of $C(m + 1, \delta + 1)$. These are such that $\hat{\mathcal{T}} \backslash (m + 1) = \mathcal{T}$ and $\hat{\mathcal{T}}' \backslash (m + 1) = \mathcal{T}'$. We have that $\hat{\mathcal{T}}' <_2 \hat{\mathcal{T}}$ by Corollary 3.3.18. By assumption, $\mathcal{S}_1(m, \delta + 1) = \mathcal{S}_2(m, \delta + 1)$, so we know that there exists a sequence of increasing flips

$$\hat{\mathcal{T}}' = \mathcal{U}_0 \lessdot_1 \mathcal{U}_1 \lessdot_1 \cdots \lessdot_1 \mathcal{U}_k = \hat{\mathcal{T}}.$$

We claim that, for any given i, we either have $\mathcal{U}_{i+1} \backslash (m + 1) = \mathcal{U}_i \backslash (m + 1)$ or $\mathcal{U}_{i+1} \backslash (m+1) \lessdot_1 \mathcal{U}_i \backslash (m+1)$. We know that there exists a subpolytope $C(H, \delta+1)$ of $\mathcal{U}_i$ such that the increasing flip from $\mathcal{U}_i$ to $\mathcal{U}_{i+1}$ replaces the lower triangulation of this subpolytope with its upper triangulation. If $m + 1 \notin H$, then $\mathcal{U}_{i+1} \backslash (m + 1) = \mathcal{U}_i \backslash (m + 1)$, since no $(\delta + 1)$-simplices of $\mathcal{U}_i$ with $m + 1$ as a vertex are affected by the increasing flip. If $m + 1 \in H$, then $C(H \setminus (m + 1), \delta)$ is a subpolytope of $\mathcal{U}_i \backslash (m + 1)$ with the induced triangulation given by its upper triangulation, since the operation $\backslash (m + 1)$ is order-reversing. Then $\mathcal{U}_{i+1} \backslash (m + 1)$ is the result of replacing the upper triangulation of $C(H \setminus (m + 1), \delta)$ by its lower triangulation, and so we have that $\mathcal{U}_{i+1} \backslash (m + 1) \lessdot_1 \mathcal{U}_i \backslash (m + 1)$.

Hence, we obtain a sequence of increasing flips

$$\mathcal{T} = \mathcal{T}_l \lessdot_1 \mathcal{T}_{l-1} \lessdot_1 \cdots \lessdot_1 \mathcal{T}_0 = \mathcal{T}',$$

where $\mathcal{T}_0, \dots, \mathcal{T}_l$ consist of $\mathcal{U}_0 \backslash (m+1), \dots, \mathcal{U}_k \backslash (m+1)$ with the duplicates removed. This shows that $\mathcal{T} \leqslant_1 \mathcal{T}'$, and so that $\mathcal{S}_1(m,\delta) = \mathcal{S}_2(m,\delta)$. □

This proposition allows one to deduce the equality of the higher Stasheff–Tamari orders for even dimensions from the equality for odd dimensions, and *vice versa.*

3.4 Applications

In this section we give two applications of the equivalence of the orders, before giving the main application to representation theory of algebras in Chapter 4. We may now refer to simply *the* higher Stasheff–Tamari order, rather than the first higher Stasheff–Tamari order and the second higher Stasheff–Tamari order. We denote the higher Stasheff–Tamari poset on the set of triangulations of $C(m,\delta)$ by $\mathcal{S}(m,\delta)$.

3.4.1 Minimal embeddings

As a consequence of Corollary 3.2.12, we obtain embeddings of the higher Stasheff–Tamari posets into Boolean lattices of minimal rank. The fact that the ranks we give are minimal is proven in [Tho02, Theorem 6.1]. In [Tho02] these are obtained for the second higher Stasheff–Tamari posets, but the equality of the orders gives that these are minimal embeddings for the first poset as well. Moreover, we use the results of Section 3.2 to obtain non-recursive realisations of these embeddings.

Corollary 3.4.1. *There is an embedding*

$$\iota \colon \mathcal{S}(m, 2d+1) \hookrightarrow 2^{\binom{m-d-2}{d+1}},$$

where the usual order on the Boolean lattice $2^{\binom{m-d-2}{d+1}}$ *is reversed.*

Proof. Define $\iota\colon \mathcal{S}(m, 2d+1) \hookrightarrow 2^{\mathbf{J}_m^d}$ by

$$\iota(\mathcal{T}) := \mathring{e}(\mathcal{T}).$$

This is an embedding by Lemma 2.2.9, and is full and order-preserving by Theorem 3.2.12. Then $\#\mathbf{J}_m^d = \binom{m-d-2}{d+1}$, so the result follows. $\square$

One can use a similar technique to embed $\mathcal{S}(m, 2d)$ into the smallest possible Boolean lattice. Such an embedding is realised if one restricts submersion sets to internal d-simplices which do not lie on the lower facets of $C(m, 2d+1)$.

Proposition 3.4.2. *There is an embedding*

$$\iota\colon \mathcal{S}(m, 2d) \hookrightarrow 2^{\binom{m-d-1}{d+1}},$$

where the order on the Boolean lattice $2^{\binom{m-d-1}{d+1}}$ *is as usual.*

Proof. If a d-simplex A lies in the lower facets of $C(m, 2d+1)$, then A is submerged by every triangulation of $C(m, 2d)$. Such d-simplices can therefore be ignored. The internal d-simplices of $C(m, 2d)$ which do not lie on the lower facets of $C(m, 2d+1)$ are precisely those whose first vertex is not 1. Hence define

$$\iota(\mathcal{T}) := \{\, A \in {}^{\circlearrowleft}\mathbf{I}_m^d : A \in \mathrm{sub}_d \mathcal{T}, a_0 \neq 1 \,\} = \{\, A \in \mathbf{J}_{m+1}^d : A \in \mathrm{sub}_d \mathcal{T} \,\}$$

for a triangulation $\mathcal{T}$ of $C(m, 2d)$. Then since $\#\mathbf{J}_{m+1}^d = \binom{m-d-1}{d+1}$, this gives our desired embedding. That this is a full, order-preserving injection follows, of course, from the characterisation of the second higher Stasheff–Tamari order in terms of submersion sets from [ER96, Proposition 2.15]. $\square$

3.4.2 Lattice property

For $\delta = 2$, the higher Stasheff–Tamari order $\mathcal{S}(m, \delta)$ is the Tamari lattice, which, of course, is a lattice. It was originally conjectured that both higher Stasheff–Tamari

Figure 3.5: Lattice property for $\mathcal{S}(c+\delta,\delta)$

$c\backslash\delta$	4	5	6	7	8	9	10	11	12	13	14	15	16
4	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
5	✗	✗	✗	✗	✗								
6	✗												

posets were also lattices [ER96, Conjecture 2.3], and this is also known to be true for $\delta \in \{1,3\}$. The authors of [ERR00] used computer calculations to show that $\mathcal{S}_2(9,4)$ is not a lattice, which still left open the question of whether or not the first higher Stasheff–Tamari order was a lattice. But, by the equality of the two orders, we obtain that $\mathcal{S}_2(9,4) = \mathcal{S}_1(9,4)$, which is therefore also not a lattice. We used the description of the higher Stasheff–Tamari orders in Theorem 3.2.13 to construct the posets in Sage. We tested the lattice property of the orders in several other cases, as shown in Figure 3.5.

Chapter 4

Higher Auslander–Reiten theory

In this chapter, we show how the higher Stasheff–Tamari orders arise in the higher Auslander–Reiten theory of Iyama [Iya07b; Iya07a; Iya11]. This theory studies particular subcategories of module categories of finite-dimensional algebras, which are called d-cluster-tilting subcategories. These d-cluster-tilting subcategories behave like higher-dimensional versions of abelian categories [Jas16]. If an algebra possesses a d-cluster-tilting subcategory which has finitely many indecomposable objects, then it is called 'd-representation-finite'. The canonical examples of d-representation-finite algebras are the higher Auslander algebras of type A, denoted A_n^d, which were introduced by Iyama in [Iya11]. These algebras were first shown to have a connection with triangulations of even-dimensional cyclic polytopes in [OT12]. In particular, in [OT12] it was shown that triangulations of $C(n + 2d + 1, 2d)$ were in bijection with cluster-tilting objects for A_n^d, or with tilting modules for A_{n+1}^d. We opt to introduce the slightly different framework of d-silting, showing that, under this bijection, the higher Stasheff–Tamari orders correspond to natural orders on d-silting complexes which arise from orders on tilting modules introduced by Riedtmann and Schofield [RS91]—see also [AI12; AIR14]. Furthermore, this fact allows us to show how odd-dimensional triangulations enter the picture, namely, as equivalence classes of "d-maximal green sequences". We introduce d-maximal green sequences as the higher-dimensional generalisations of maximal green sequences, which were introduced by Keller [Kel11]. Finally, we interpret the higher Stasheff–Tamari orders in terms of equivalence classes of d-maximal green sequences. These orders are very natural but have not much been studied, although there is some related work from [Gor14a; Gor14b; Gor]. These orders are related to the "no-gap" conjecture of Brüstle, Dupont, and Perotin [BDP14], as we discuss. By the results of Chapter 3, we deduce that these orders on d-silting complexes and equivalence classes of d-maximal green sequences are equal for the higher Auslander algebras of type A, which invites the question of whether this holds more generally in higher Auslander–Reiten theory.

This chapter is structured as follows. We begin in Section 4.1 by outlining the basic notions of higher Auslander–Reiten theory, defining the higher Auslander algebras of type A in particular. We explain how one may describe the representation theory of these algebras using the results of [OT12]. We then see how this leads to connections with triangulations of cyclic polytopes, which we explain using cluster-tilting and d-silting. The d-silting framework is new, and so we lay the requisite groundwork. We illustrate the results with examples. In Section 4.3, we show how the even-dimensional higher Stasheff–Tamari orders arise in the representation theory of A^d_n, using the results of Section 3.2. We then apply these results in Section 4.4 in order to describe how triangulations of odd-dimensional cyclic polytopes arise in the representation theory of A^d_n. Using the results of Section 3.2 once more, we see how the odd-dimensional higher Stasheff–Tamari orders may be interpreted algebraically. Finally, in Section 4.5, the focus changes, and we show a criterion for mutating cluster-tilting objects in higher cluster categories. This translates into an algebraic criterion for performing a bistellar flip at a given internal d-simplex in a triangulation of a $2d$-dimensional cyclic polytope.

4.1 Higher Auslander–Reiten theory

In this section let Λ be a finite-dimensional algebra over a field K. We denote by $\operatorname{mod}\Lambda$ the category of finite-dimensional right Λ-modules. We assume that all subcategories are full and replete, recalling that a subcategory is *replete* if it is closed under isomorphisms. Given a module $M \in \operatorname{mod}\Lambda$, we write $\operatorname{add} M$ for the subcategory consisting of summands of direct sums of copies of M. We denote the standard duality by $D = \operatorname{Hom}_K(-, K)\colon \operatorname{mod}\Lambda^{\mathrm{op}} \to \operatorname{mod}\Lambda$. We do not give general background on the representation theory of finite-dimensional algebras, and refer instead to the text [ASS06], and to [Hap88] for material on triangulated categories.

Given a subcategory $\mathcal{X} \subset \operatorname{mod}\Lambda$ and a map $f\colon X \to M$, where $X \in \mathcal{X}$ and $M \in \operatorname{mod}\Lambda$, we say that f is a *right $\mathcal{X}$-approximation* if for any $X' \in \mathcal{X}$, the sequence

$$\operatorname{Hom}_\Lambda(X', X) \to \operatorname{Hom}_\Lambda(X', M) \to 0$$

is exact, following [AS80]. *Left $\mathcal{X}$-approximations* are defined dually. The subcategory $\mathcal{X}$ is said to be *contravariantly finite* if every $M \in \operatorname{mod}\Lambda$ admits a right $\mathcal{X}$-approximation, and *covariantly finite* if every $M \in \operatorname{mod}\Lambda$ admits a left $\mathcal{X}$-approximation. If $\mathcal{X}$ is both contravariantly finite and covariantly finite, then $\mathcal{X}$ is *functorially finite*.

Higher Auslander–Reiten theory was introduced by Iyama in [Iya07a; Iya07b; Iya11] as a higher-dimensional generalisation of classical Auslander–Reiten theory. For more detailed background to the theory, see the papers [JK19b; Jas16; GKO13; IO11; Iya08; Jør17].

The following subcategories provide the setting for the higher theory. Let $\mathcal{M}$

be a functorially finite subcategory of $\operatorname{mod}\Lambda$. Then we call $\mathcal{M}$ *d-cluster-tilting* if

$$\begin{aligned}\mathcal{M} &= \{X \in \operatorname{mod}\Lambda : \forall i \in [d-1], \forall M \in \mathcal{M}, \operatorname{Ext}^i_\Lambda(X, M) = 0\} \\ &= \{X \in \operatorname{mod}\Lambda : \forall i \in [d-1], \forall M \in \mathcal{M}, \operatorname{Ext}^i_\Lambda(M, X) = 0\}.\end{aligned}$$

In the case $d = 1$, the conditions should be interpreted as being trivial, so that $\operatorname{mod}\Lambda$ is the unique 1-cluster-tilting subcategory of $\operatorname{mod}\Lambda$. If $\operatorname{add} M$ is a d-cluster-tilting subcategory, for $M \in \operatorname{mod}\Lambda$, then we say that M is a *d-cluster-tilting module*.

We say that Λ is *weakly d-representation-finite* if there exists a d-cluster-tilting module in $\operatorname{mod}\Lambda$, following [IO11, Definition 2.2]. If, additionally, $\operatorname{gl.dim}\Lambda \leqslant d$, we say that Λ is *d-representation-finite d-hereditary*, following [JK19a, Definition 1.25] and [HIO14]. In [IO11], d-representation-finite d-hereditary algebras simply called 'd-representation-finite'.

4.1.1 The higher Auslander algebras of type A

The canonical examples of d-representation-finite d-hereditary algebras are the higher Auslander algebras of linearly oriented A_n, introduced by Iyama in [Iya11]. The construction we give here is based on [OT12, Construction 3.4] and [IO11, Definition 5.1].

Following [OT12], we denote

$$\mathbf{I}^d_m := \{\, A \in \tbinom{[m]}{d+1} : a_i \leqslant a_{i+1} - 2 \,\forall i \in [d] \,\}.$$

The difference between $\mathbf{I}^d_m$ and ${}^{\circlearrowleft}\mathbf{I}^d_m$ is that $\mathbf{I}^d_m$ permits subsets A such that both $a_0 = 1$ and $a_d = m$.

Let $Q^{(d,n)}$ be the quiver with vertices

$$Q^{(d,n)}_0 := \mathbf{I}^{d-1}_{n+2d-2}$$

Figure 4.1: Examples of the quivers $Q^{(d,n)}$

$Q^{(1,3)}$

$$\begin{tikzcd}
 & & 3 \\
 & 2 \arrow[ur] & \\
1 \arrow[ur] & &
\end{tikzcd}$$

$Q^{(2,3)}$

$$\begin{tikzcd}
 & & 15 \arrow[dr] & & \\
 & 14 \arrow[ur] \arrow[dr] & & 25 \arrow[dr] & \\
13 \arrow[ur] & & 24 \arrow[ur] & & 35
\end{tikzcd}$$

$Q^{(3,3)}$

$$\begin{tikzcd}
 & & 137 \arrow[rr] & & 147 \arrow[rr] \arrow[dr] & & 157 \arrow[dr] & & \\
 & 136 \arrow[ur] \arrow[rr] & & 146 \arrow[ur] \arrow[dr] & & 247 \arrow[rr] & & 257 \arrow[dr] & \\
135 \arrow[ur] & & & & 246 \arrow[ur] & & & & 357
\end{tikzcd}$$

and arrows

$$Q_1^{(d,n)} := \{\, A \to \sigma_i(A) : A, \sigma_i(A) \in Q_0^{(d,n)} \,\},$$

where

$$\sigma_i(A) := \{a_0, a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_d\}.$$

We multiply arrows as if we were composing functions, so that $\xrightarrow{\alpha}\xrightarrow{\beta} = \beta\alpha$. This is the opposite convention to [ASS06].

Let A_n^d be the quotient of the path algebra $KQ^{(d,n)}$ by the relations:

$$A \to \sigma_i(A) \to \sigma_j(\sigma_i(A)) = \begin{cases} A \to \sigma_j(A) \to \sigma_j(\sigma_i(A)) & \text{if } \sigma_j(A) \in Q_0^{(d,n)} \\ 0 & \text{otherwise.} \end{cases}$$

The d-cluster-tilting subcategory of $\operatorname{mod} A_n^d$

It is shown in [Iya11] that the algebra A_n^d is d-representation-finite d-hereditary with unique basic d-cluster-tilting module $M^{(d,n)}$ and that

$$A_n^{d+1} \cong \operatorname{End}_{A_n^d} M^{(d,n)}.$$

By this result, the Auslander–Reiten quiver of $\operatorname{add} M^{(d,n)}$ is the same as the quiver of A_n^{d+1}. Hence, the indecomposable modules of $\operatorname{add} M^{(d,n)}$ are in bijection with $\mathbf{I}_{n+2d}^d$. Given $A \in \mathbf{I}_{n+2d}^d$, let M_A be the A_n^d-module in $\operatorname{add} M^{(d,n)}$ which occupies the same position in the Auslander–Reiten quiver of $\operatorname{add} M^{(d,n)}$ as A does in $Q^{(d+1,n)}$. The simple A_n^d-modules are in bijection with $\mathbf{I}_{n+2d-2}^{d-1}$ via the labelling of the vertices of $Q^{(d,n)}$. Hence, given $B \in \mathbf{I}_{n+2d-2}^{d-1}$, let S_B be the corresponding simple A_n^d-module. Given $A \in \mathbf{I}_{n+2d}^{d}$, the corresponding module $M_A \in \operatorname{add} M^{(d,n)}$ has composition factors S_B such that $B \wr A - \mathbf{1}$, where $A - \mathbf{1} = \{a_0 - 1, a_1 - 1, \ldots, a_d - 1\}$. This is due to [OT12, Theorem/Construction 3.4]. We illustrate this fact with two examples.

Example 4.1.1. (1) We first consider the algebra A_3^1, which we usually simply denote by A_3. The Auslander–Reiten quiver of $\operatorname{add} M^{(1,3)} = \operatorname{mod} A_3$ is shown in Figure 4.2. It can be seen that this is the same quiver as $Q^{(2,3)}$. The module

$$\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}$$

corresponds to the subset 15. One can see that for every $B \in \{1, 2, 3\}$, we have that $B \wr 15 - \mathbf{1} = 04$.

(2) We next consider the algebra A_3^2. In order to make the notation easier, we relabel the quiver of this algebra as shown.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & 3 \arrow[dr] & & \\
 & 2 \arrow[ur] \arrow[dr] & & 5 \arrow[dr] & \\
1 \arrow[ur] & & 4 \arrow[ur] & & 6
\end{tikzcd}$$

The Auslander–Reiten quiver of $\operatorname{add} M^{(2,3)}$ is shown in Figure 4.3. It can be seen that this is the same quiver as $Q^{(3,3)}$. The module

$$\begin{matrix} & 5 & \\ 4 & & 3 \\ & 2 & \end{matrix}$$

corresponds to the set 147. One can see that for any $B \in \{14, 24, 15, 25\}$, we have $B \wr 147 - \mathbf{1} = 036$. By our relabelling of the quiver, we have that $\{14, 24, 15, 25\}$

Figure 4.2: The Auslander–Reiten quiver of add $M^{(1,3)} = \operatorname{mod} A_3$

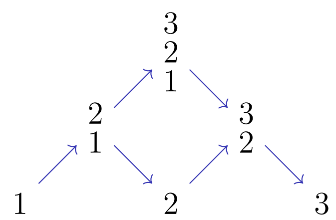
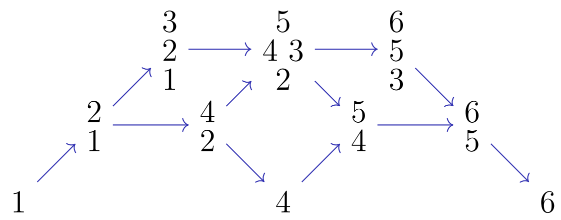


Figure 4.3: The Auslander–Reiten quiver of add $M^{(2,3)}$



corresponds to the set of vertices $\{2, 3, 4, 5\}$, which give the composition factors of the module.

Oppermann and Thomas further show that the labelling of the indecomposables of add $M^{(d,n)}$ by $\mathbf{I}^d_{n+2d}$ further encodes the homomorphisms and extensions in the category [OT12, Theorem 3.6]:

- $\operatorname{Hom}_{A^d_n}(M_A, M_B) \neq 0$ if and only if $(A - \mathbf{1}) \wr B$, and in this case the Hom-space is one-dimensional;
- $\operatorname{Ext}^d_{A^d_n}(M_A, M_B) \neq 0$ if and only if $B \wr A$, and in this case the Ext-space is one-dimensional.

Moreover, M_A is projective if and only if $a_0 = 1$.

Example 4.1.2. (1) In the case of $\operatorname{add} M^{(1,3)} = \operatorname{mod} A_3$, we have that

$$0 \neq \operatorname{Hom}_{A_3}\left(1, \begin{smallmatrix}3\\2\\1\end{smallmatrix}\right) = \operatorname{Hom}_{A_3}(M_{13}, M_{15}),$$

which is accounted for by the fact that $13 - \mathbf{1} = 02 \wr 15$. Furthermore,

$$0 \neq \operatorname{Ext}^1_{A_3}(2, 1) = \operatorname{Ext}^1_{A_3}(M_{24}, M_{13}),$$

which is accounted for by the fact that $13 \wr 24$.

(2) In the case of $\operatorname{add} M^{(2,3)}$, we have that

$$0 \neq \operatorname{Hom}_{A_3^2}\left(\begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}&5&\\4&&3\\&2&\end{smallmatrix}\right) = \operatorname{Hom}_{A_3^2}(M_{136}, M_{147}),$$

which is accounted for by the fact that $136 - \mathbf{1} = 025 \wr 147$. Furthermore

$$0 \neq \operatorname{Ext}^2_{A_3^2}\left(\begin{smallmatrix}5\\4\end{smallmatrix}, 1\right) = \operatorname{Ext}^2_{A_3^2}(M_{247}, M_{135}),$$

which is accounted for by the fact that $135 \wr 247$.

The d-cluster-tilting subcategory of $D^b(\operatorname{mod} A_n^d)$

Given a triangulated category $\mathcal{D}$, a functorially finite subcategory $\mathcal{C}$ of $\mathcal{D}$ is called *d-cluster-tilting* if

$$\begin{aligned}\mathcal{C} &= \{\, X \in \mathcal{D} : \forall i \in [d-1], \forall Y \in \mathcal{C}, \operatorname{Hom}_{\mathcal{D}}(X, Y[i]) = 0 \,\} \\ &= \{\, X \in \mathcal{D} : \forall i \in [d-1], \forall Y \in \mathcal{C}, \operatorname{Hom}_{\mathcal{D}}(Y, X[i]) = 0 \,\}.\end{aligned}$$

Geiß, Keller, and Oppermann introduce $(d+2)$-angulated categories as the analogues of triangulated categories in higher Auslander–Reiten theory and show that a d-cluster-tilting subcategory of a triangulated category which is closed under $[d]$ is a $(d+2)$-angulated category [GKO13, Theorem 1]. We refer to [GKO13] for background on $(d+2)$-angulated categories. We will often consider $(d+2)$-angles, which are to $(d+2)$-angulated categories what triangles are to triangulated categories.

Let Λ be a d-representation-finite d-hereditary algebra with d-cluster-tilting module M. Let $\mathcal{D}_\Lambda := D^b(\operatorname{mod}\Lambda)$ be the bounded derived category of finitely generated Λ-modules. We denote the shift functor in the derived category by $[1]$ and its d-th power by $[d] := [1]^d$. The subcategory

$$\mathcal{U}_\Lambda := \operatorname{add}\{\, M[id] \in \mathcal{D}_\Lambda : i \in \mathbb{Z} \,\}$$

is a d-cluster-tilting subcategory of $\mathcal{D}_\Lambda$ [Iya11, Theorem 1.21], and hence is $(d+2)$-angulated.

We denote by

$$\nu := D\Lambda \otimes^{\mathbf{L}}_{\Lambda} - \cong D\mathbf{R}\operatorname{Hom}_\Lambda \colon \mathcal{D}_\Lambda \to \mathcal{D}_\Lambda,$$

the derived Nakayama functor. By [IO13, Theorem 3.1], ν restricts to a functor $\mathcal{U}_\Lambda \to \mathcal{U}_\Lambda$.

Just as the indecomposable objects of $\operatorname{add} M^{(d,n)}$ may be labelled by subsets in a way that concords with homomorphisms and extensions, the indecomposables of $\mathcal{U}_{A^d_n}$ may also be labelled in this way. We denote

$$\tilde{\mathbf{I}}^d_m = \left\{ A \in \binom{\mathbb{Z}}{d+1} : \begin{array}{c} \forall i \in \{0,1,\dots,d-1\}, \\ a_{i+1} \geqslant a_i + 2 \text{ and } a_d + 2 \leqslant a_0 + m \end{array} \right\}.$$

By [OT12, Lemma 6.6(1)], the indecomposable objects of $\mathcal{U}_{A^d_n}$ are in bijection with $\tilde{\mathbf{I}}^d_{n+2d+1}$. Given $A \in \tilde{\mathbf{I}}^d_{n+2d+1}$, we write U_A for the corresponding indecomposable object of $\mathcal{U}_{A^d_n}$, such that if $A \in \mathbf{I}^d_{n+2d}$, then $U_A = M_A$ and, for general $A \in \tilde{\mathbf{I}}^d_{n+2d+1}$, U_A is defined such that $U_A[d] = U_{(a_1-1, a_2-1, \dots, a_d-1, a_0+n+2d)}$. Furthermore,

- $\mathrm{Hom}_{\mathcal{D}_{A_n^d}}(U_B, U_A) \neq 0$ if and only if

$$b_0 - 1 < a_0 < b_1 - 1 < a_1 < \cdots < b_d - 1 < a_d < b_0 + n + 2d.$$

  and in this case the Hom-space is one-dimensional by the proof of [OT12, Proposition 6.1];

- we have $\nu_d U_A = U_{A-\mathbf{1}}$ by [OT12, Lemma 6.6(2)].

We therefore have

- $\mathrm{Hom}_{\mathcal{D}_{A_n^d}}(U_B, U_A[d]) \neq 0$ if and only if

$$a_0 < b_0 < a_1 < b_1 < \cdots < a_d < b_d < a_0 + n + 2d + 1,$$

  with this space one-dimensional. This is also shown in the proof of [OT12, Proposition 6.1].

Example 4.1.3. We illustrate how the combinatorial labelling of the category $\mathcal{U}_{A_n^d}$ works.

(1) Figure 4.4 gives the category $\mathcal{U}_{A_3} = \mathcal{D}_{A_3}$ and Figure 4.5 gives its labelling by $\tilde{\mathbf{I}}_6^1$. The properties described above may be checked, for instance:

- $\mathrm{Hom}_{\mathcal{D}_{A_3}}(U_{14}, U_{25}) = \mathrm{Hom}_{\mathcal{D}_{A_3}}\left(\begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}\right) \neq 0$, since $1-1 < 2 < 4-1 < 5 < 1+5$;
- $U_{13}[1] = 1[1] = U_{26} = U_{(3-1,1+5)}$;
- $\nu_d U_{35} = \nu_d 3 = 2 = U_{24} = U_{(3-1,5-1)}$;
- $\mathrm{Hom}_{\mathcal{D}_{A_3}}(U_{35},\, U_{14}[1]) = \mathrm{Hom}_{\mathcal{D}_{A_3}}\left(3,\, \begin{smallmatrix}2\\1\end{smallmatrix}[1]\right) \neq 0$, since $1 < 3 < 4 < 5 < 1+6$.

(2) We further illustrate the case of A_3^2. Figure 4.6 gives the 2-cluster-tilting subcategory $\mathcal{U}_{A_3^2}$ of $\mathcal{D}_{A_3^2}$, whilst Figure 4.7 gives its combinatorial labelling by $\tilde{\mathbf{I}}_8^2$. Again, we may check that the above properties hold.

Figure 4.4: The category $\mathcal{U}_{A_3} = \mathcal{D}_{A_3}$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[dr] & & 3[-1] \arrow[dr] & & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[dr] & & 1[1] \arrow[dr] & & 2[1] \arrow[dr] & & 3[1] \arrow[dr] & \\
& \begin{smallmatrix}3\\2\end{smallmatrix}[-1] \arrow[ur] \arrow[dr] & & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}2\\1\end{smallmatrix}[1] \arrow[ur] \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix}[1] \arrow[ur] \arrow[dr] & & \cdots \\
\cdots \arrow[ur] & & 1 \arrow[ur] & & 2 \arrow[ur] & & 3 \arrow[ur] & & \begin{smallmatrix}3\\2\\1\end{smallmatrix}[1] \arrow[ur] & & 1[2] \arrow[ur] &
\end{tikzcd}$$

Figure 4.5: The combinatorial labelling of $\mathcal{U}_{A_3} = \mathcal{D}_{A_3}$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[dr] & & U_{04} \arrow[dr] & & U_{15} \arrow[dr] & & U_{26} \arrow[dr] & & U_{37} \arrow[dr] & & U_{48} \arrow[dr] & \\
& U_{03} \arrow[ur] \arrow[dr] & & U_{14} \arrow[ur] \arrow[dr] & & U_{25} \arrow[ur] \arrow[dr] & & U_{36} \arrow[ur] \arrow[dr] & & U_{47} \arrow[ur] \arrow[dr] & & \cdots \\
\cdots \arrow[ur] & & U_{13} \arrow[ur] & & U_{24} \arrow[ur] & & U_{35} \arrow[ur] & & U_{46} \arrow[ur] & & U_{57} \arrow[ur] &
\end{tikzcd}$$

- $\mathrm{Hom}_{\mathcal{D}_{A_3}}(U_{136}, U_{147}) = \mathrm{Hom}_{\mathcal{D}_{A_3}}\left(\begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix} & 5 & \\ 4 & & 3\\ & 2 & \end{smallmatrix}\right) \neq 0$, since $1-1 < 1 < 3-1 < 4 < 6-1 < 7 < 1+7$;
- $U_{135}[2] = 1[2] = U_{248} = U_{(3-1,5-1,1+7)}$;
- $\nu_d U_{257} = \nu_d \begin{smallmatrix}6\\5\end{smallmatrix} = \begin{smallmatrix}4\\2\end{smallmatrix} = U_{146} = U_{(2-1,5-1,7-1)}$;
- $\mathrm{Hom}_{\mathcal{D}_{A_3}}(U_{257}, U_{136}[2]) = \mathrm{Hom}_{\mathcal{D}_{A_3}}\left(\begin{smallmatrix}6\\5\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}[2]\right) \neq 0$, since $1 < 2 < 3 < 5 < 6 < 7 < 1+8$.

Figure 4.6: The category $\mathcal{U}_{A_3^2}$

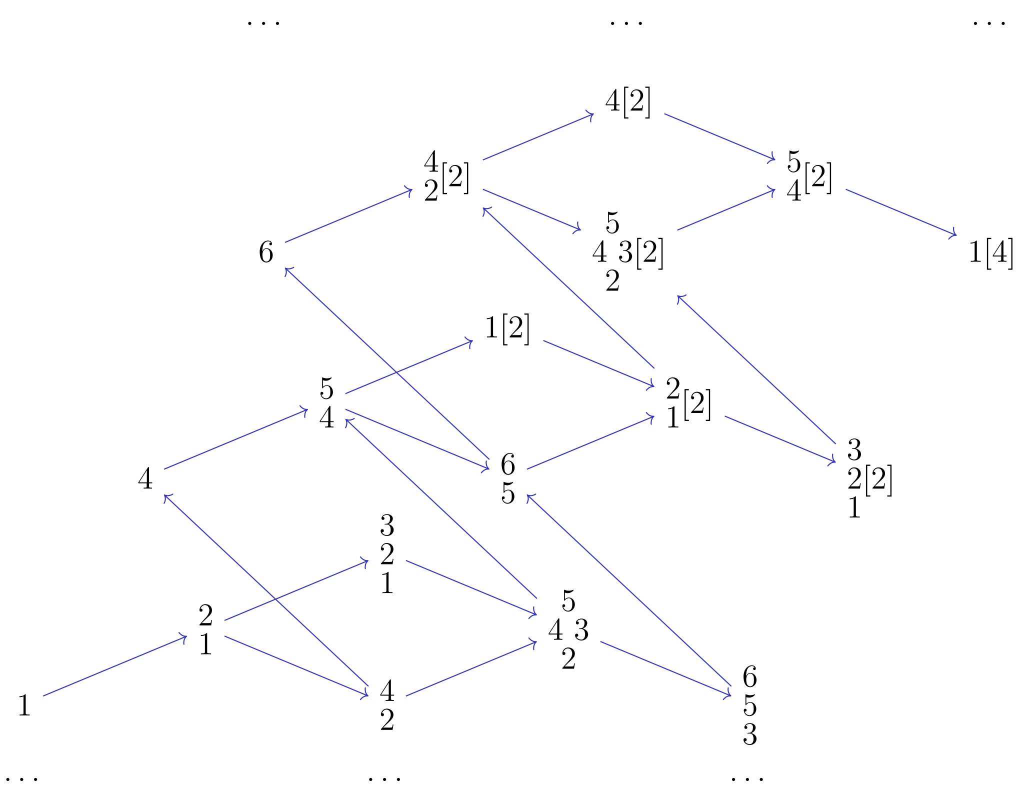
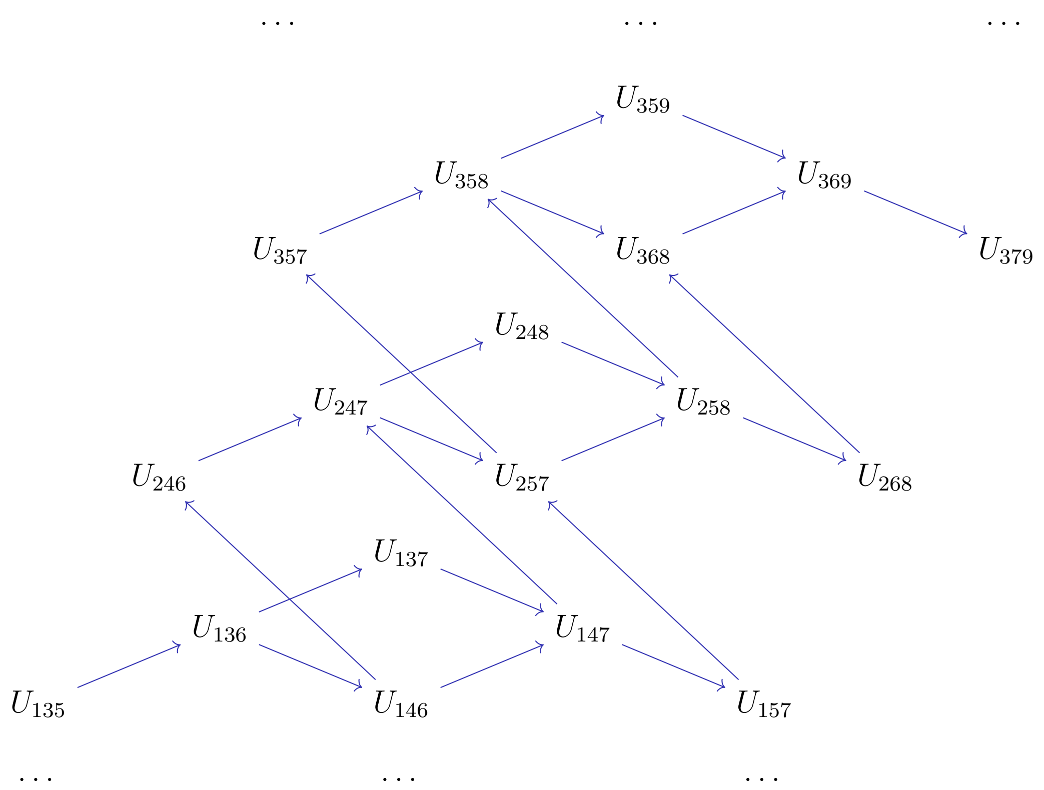


Figure 4.7: The combinatorial labelling of the category $\mathcal{U}_{A^2_3}$



4.2 Relation with triangulations of cyclic polytopes

Oppermann and Thomas give two different bijections between algebraic objects and triangulations of even-dimensional cyclic polytopes [OT12]. One uses tilting modules [OT12, Section 3, Section 4] and the other uses cluster-tilting objects [OT12, Section 5, Section 6]. We carry out our work in a yet different algebraic framework, which uses *d-silting*. This is in some ways akin to the cluster-tilting framework from [OT12], but, as we explain, it allows us to consider orders on the objects, whereas the cluster-tilting framework does not. The tilting framework also allows us to consider orders on objects, but we prefer not to use this framework since here the projective-injective modules do not correspond to internal d-simplices. All the results we prove will have analogues in the tilting framework; indeed, the paper [Wil21a] uses this framework.

4.2.1 Cluster-tilting

We first explain the cluster-tilting framework of [OT12] so that we can explain how our framework of d-silting relates to it. We shall also prove results concerning the cluster-tilting framework in Section 4.5.

Oppermann and Thomas [OT12, Definition 5.22] define the *$(d+2)$-angulated cluster category* of Λ to be the orbit category

$$\mathcal{O}_\Lambda = \frac{\mathcal{U}_\Lambda}{\nu_d[-d]} .$$

As the name suggests, this is a $(d+2)$-angulated category. By [OT12, Theorem 1.2], we have bijections between

- indecomposable objects in $\mathcal{O}_{A_n^d}$,

- elements of ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$,
- internal d-simplices of $C(n+2d+1, 2d)$.

These bijections may be explained as follows. Namely, we know from Section 2.2.1 that A is an internal d-simplex in $C(n+2d+1, 2d)$ if and only if $A \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$, which gives a bijection between ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ and the internal d-simplices of $C(n+2d+1, 2d)$. Furthermore, we know that $\nu_d[-d]U_A = U_{\{a_d-(n+2d+1),a_0,\dots,a_{d-1}\}}$. Hence, if we quotient $\mathcal{U}_{A^d_n}$ by $\nu_d[-d]$, then we can find equivalence class representatives U_A such that $a_i \in [n+2d+1]$ for all i. Since

$$\tilde{\mathbf{I}}^d_{n+2d+1} \cap \binom{[n+2d+1]}{d+1} = {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1},$$

we conclude that the indecomposables of $\mathcal{O}_{A^d_n}$ are in bijection with ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$. Given $A \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$, we write O_A for the object of $\mathcal{O}_{A^d_n}$ which is the image of U_A under the quotient $\mathcal{U}_{A^d_n} \to \mathcal{O}_{A^d_n}$. Recalling the notation $A \between B$ from Section 2.1.2, we have that

- $\operatorname{Hom}_{\mathcal{O}_{A^d_n}}(O_A, O_B[d]) = \operatorname{Hom}_{\mathcal{U}_{A^d_n}}(U_A, U_B[d]) \oplus D\operatorname{Hom}_{\mathcal{U}_{A^d_n}}(U_B, U_A[d]) \neq 0$ if and only if $A \between B$ [OT12, Theorem 5.2(3) and Proposition 6.1];
- $U_A[d] = U_{A-\mathbf{1}}$, by taking the description of $[d]$ in $\mathcal{U}_{A^d_n}$ modulo $n+2d+1$;
- hence, $\operatorname{Hom}_{\mathcal{O}_{A^d_n}}(O_A, O_B) \neq 0$ if and only if $(A-\mathbf{1}) \between B$.

An object $T \in \mathcal{O}_\Lambda$ is defined to be *cluster-tilting* if

(1) $\operatorname{Hom}_{\mathcal{O}_\Lambda}(T, T[d]) = 0$, and

(2) any $X \in \mathcal{O}_\Lambda$ occurs in a $(d+2)$-angle

$$X[-d] \to T_d \to T_{d-1} \to \cdots \to T_1 \to T_0 \to X$$

with $T_i \in \operatorname{add} T$.

Remark 4.2.1. Readers are warned that 'cluster-tilting' and 'd-cluster-tilting' are distinct terms in this thesis and are cautioned against getting confused between the two of them.

Remark 4.2.2. By [OT12, Theorem 6.4], an object $T \in \mathcal{O}_{A_n^d}$ is cluster-tilting if and only if $\mathrm{Hom}_{\mathcal{O}_{A_n^d}}(T, T[d]) = 0$ and T has as many isomorphism classes of indecomposable summands as A_n^d. For $d = 1$, this is known to hold for general Λ, by [ZZ11]. That is, for $d = 1$, an object $T \in \mathcal{O}_\Lambda$ is cluster-tilting if and only if $\mathrm{Hom}_{\mathcal{O}_\Lambda}(T, T[1]) = 0$ and T has as many isomorphism classes of indecomposable summands as Λ. For $d > 1$, it is unknown whether this always holds. The analogous problem for tilting modules is also open in general, although it is known to hold for representation-finite algebras [RS89].

By [OT12, Theorem 1.2], the bijections between ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$, indecomposable objects of $\mathcal{O}_{A_n^d}$, and the internal d-simplices in $C(n + 2d + 1, 2d)$ induce bijections between:

- non-intertwining collections of $\binom{n+d-1}{d}$ $(d + 1)$-subsets in ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$,
- triangulations of $C(n + 2d + 1, 2d)$, and
- basic cluster-tilting objects in $\mathcal{O}_{A_n^d}$.

These bijections are consequences of two facts. Firstly, we have that $\binom{n+d-1}{d}$ is both the necessary number of internal d-simplices in $C(n + 2d + 1, 2d)$ and the necessary number of isomorphism classes of indecomposable summands of a cluster-tilting object in $\mathcal{O}_{A_n^d}$. Secondly, we have that A and B being intertwining corresponds to their forming a circuit in $C(n + 2d + 1, 2d)$ and also to having $\mathrm{Hom}_{\mathcal{O}_{A_n^d}}(O_A \oplus O_B, O_A \oplus O_B[d]) \neq 0$. Hence, if $\mathcal{X}$ is a collection of non-intertwining elements of ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$, then $\mathcal{X} = \mathring{e}(\mathcal{T})$ for some triangulation $\mathcal{T}$, and $\bigoplus_{A \in \mathring{e}(\mathcal{T})} O_A$ is a cluster-tilting object by Remark 4.2.2.

Figure 4.8: The Auslander–Reiten quiver of $\mathcal{O}_{A_2}$ and ${}^{\circlearrowleft}\mathbf{I}_5^1$

$$\begin{tikzcd}[column sep=small]
& \begin{smallmatrix}2\\1\end{smallmatrix}[1] \arrow[dr] & & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[dr] & & 1[1] \arrow[dr] & & 1 \arrow[dr] & \\
\cdots \arrow[ur] & & 1 \arrow[ur] & & 2 \arrow[ur] & & \begin{smallmatrix}2\\1\end{smallmatrix}[1] \arrow[ur] & & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& O_{35} \arrow[dr] & & O_{14} \arrow[dr] & & O_{25} \arrow[dr] & & O_{13} \arrow[dr] & \\
\cdots \arrow[ur] & & O_{13} \arrow[ur] & & O_{24} \arrow[ur] & & O_{35} \arrow[ur] & & \cdots
\end{tikzcd}$$

Example 4.2.3. (1) We first consider the example where $d = 1$ and $n = 2$. The bijection between ${}^{\circlearrowleft}\mathbf{I}_5^1$ and the indecomposables in $\mathcal{O}_{A_2}$ is shown in Figure 4.8. This bijection induces a bijection between triangulations of $C(5, 2)$ and basic cluster-tilting objects in $\mathcal{O}_{A_2}$, as shown in Figure 4.9.

(2) Now consider the example where $d = 2$ and $n = 2$. The algebra A_2^2 has the Auslander–Reiten quiver of A_2 as its quiver. To make the modules of this algebra easier to denote, we relabel the quiver

$$\begin{tikzcd}[column sep=small, row sep=small]
& 2 \arrow[dr] & \\
1 \arrow[ur] & & 3.
\end{tikzcd}$$

Figure 4.10 then shows the bijection between ${}^{\circlearrowleft}\mathbf{I}_7^2$ and the indecomposables of $\mathcal{O}_{A_2^2}$. There are seven cluster-tilting objects in $\mathcal{O}_{A_2^2}$, which correspond to the seven triangulations of $C(7, 4)$. This bijection is given in Table 4.1, where the triangulations are described by their set of internal d-simplices.

We finally consider a relation between cluster-tilting objects known as 'mutation'. Let $T = E \oplus X$ be a cluster-tilting object in $\mathcal{O}_\Lambda$ where X is indecomposable. Then [OT12, Theorem 5.7] states that there is an indecomposable object Y in $\mathcal{O}_\Lambda$

Figure 4.9: Cluster-tilting objects in $\mathcal{O}_{A_2}$ and their corresponding triangulations

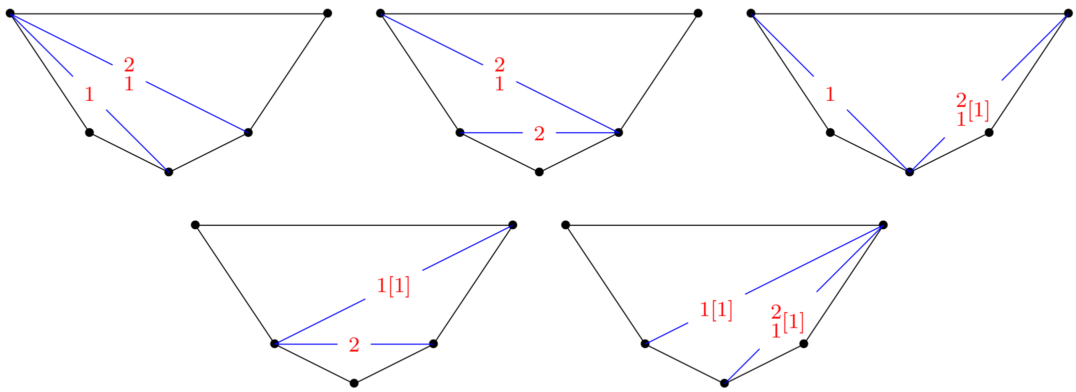


Figure 4.10: The Auslander–Reiten quiver of $\mathcal{O}_{A_2^2}$ and ${}^{\circlearrowleft}\mathbf{I}_7^2$

$$\begin{tikzcd}
\cdots \arrow[r] & \substack{3\\2}[2] \arrow[dr] & & \substack{2\\1} \arrow[r] & \substack{3\\2} \arrow[dr] & & 1[2] \arrow[r] & \substack{2\\1}[2] \arrow[dr] & & 1 \arrow[r] & \cdots \\
& & 1 \arrow[ur] & & & 3 \arrow[ur] & & & \substack{3\\2}[2] \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & O_{357} \arrow[dr] & & O_{136} \arrow[r] & O_{146} \arrow[dr] & & O_{247} \arrow[r] & O_{257} \arrow[dr] & & O_{135} \arrow[r] & \cdots \\
& & O_{135} \arrow[ur] & & & O_{246} \arrow[ur] & & & O_{357} \arrow[ur] & &
\end{tikzcd}$$

Table 4.1: Cluster-tilting objects in $\mathcal{O}_{A_2^2}$ and their corresponding triangulations

Cluster-tilting object	Triangulation
$1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix}$	$\{135, 136, 146\}$
$3 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix}$	$\{246, 136, 146\}$
$3 \oplus 1[2] \oplus \begin{smallmatrix}3\\2\end{smallmatrix}$	$\{246, 247, 146\}$
$3 \oplus 1[2] \oplus \begin{smallmatrix}2\\1\end{smallmatrix}[2]$	$\{246, 247, 257\}$
$1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix}[2]$	$\{135, 136, 357\}$
$1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix}[2] \oplus \begin{smallmatrix}3\\2\end{smallmatrix}[2]$	$\{135, 257, 357\}$
$1[2] \oplus \begin{smallmatrix}2\\1\end{smallmatrix}[2] \oplus \begin{smallmatrix}3\\2\end{smallmatrix}[2]$	$\{357, 247, 257\}$

with $Y \ncong X$ and $E \oplus Y$ cluster-tilting if and only if there exist $(d+2)$-angles

$$X \to E^1 \to \cdots \to E^d \to Y \to X[d]$$

and

$$Y \to E_1 \to \cdots \to E_d \to X \to Y[d],$$

where $E^i, E_i \in \operatorname{add} E$, the maps $X \to E^1$, $Y \to E_1$ are minimal left $\operatorname{add} E$-approximations, and the maps $E_d \to X$, $E^d \to Y$ are minimal right $\operatorname{add} E$-approximations. In this case we say that T is *mutable* at X and we call $E \oplus Y$ is the *mutation* of T at X.

By Theorem 3.2.1 or [OT12, Theorem 4.1], we have that two triangulations are related by a bistellar flip if and only if they differ by exactly one internal d-simplex. Since the internal d-simplices of a triangulation correspond to the indecomposable summands of the corresponding cluster-tilting object, we have that two cluster-tilting objects are mutations of each other if and only if the corresponding triangulations are bistellar flips of each other.

4.2.2 d-silting

We now introduce and motivate our framework of d-silting, which is the predominant framework we use in this chapter. The cluster category $\mathcal{O}_\Lambda$ is $2d$-Calabi–Yau, meaning that, for objects X, Y of $\mathcal{O}_\Lambda$, we have

$$\operatorname{Hom}_{\mathcal{O}_\Lambda}(X, Y[d]) \cong D \operatorname{Hom}_{\mathcal{O}_\Lambda}(Y, X[d]).$$

Hence $\operatorname{Hom}_{\mathcal{O}_\Lambda}(X, Y[d])$ and $\operatorname{Hom}_{\mathcal{O}_\Lambda}(Y, X[d])$ are either both zero or both non-zero. This symmetry between $\operatorname{Hom}_{\mathcal{O}_\Lambda}(X, Y[d])$ and $\operatorname{Hom}_{\mathcal{O}_\Lambda}(Y, X[d])$ prevents us from introducing partial orders on the cluster-tilting objects of $\mathcal{O}_\Lambda$. Hence, we choose a different algebraic framework. We desire a framework which does not have this

$2d$-Calabi–Yau symmetry, but which otherwise has all of the properties which connect the cluster-tilting framework with triangulations of even-dimensional cyclic polytopes.

An object T of $\mathcal{D}_\Lambda$ is *pre-silting* if $\operatorname{Hom}_{\mathcal{D}_\Lambda}(T, T[i]) = 0$ for all $i > 0$. A pre-silting complex T is *silting* if, additionally, $\operatorname{thick} T = \mathcal{D}_\Lambda$. Here $\operatorname{thick} T$ denotes the smallest full subcategory of $\mathcal{D}_\Lambda$ which contains T and is closed under cones, $[\pm 1]$, direct summands, and isomorphisms.

Recall that we are considering a d-representation-finite d-hereditary algebra with unique basic d-cluster-tilting module M and $\mathcal{U}_\Lambda$ the corresponding d-cluster-tilting subcategory of $\mathcal{D}_\Lambda$. We consider the subcategory $\mathcal{U}_\Lambda^{\{-d,0\}} := \operatorname{add}(M \oplus \Lambda[d])$ of $\mathcal{U}_\Lambda$ and call a silting complex T of $\mathcal{D}_\Lambda$ *d-silting* if, additionally, it lies in $\mathcal{U}_\Lambda^{\{-d,0\}}$. Note that for objects T, T' of $\mathcal{U}_\Lambda^{\{-d,0\}}$ we have $\operatorname{Hom}_{\mathcal{D}_\Lambda}(T, T'[i]) = 0$ if $i \notin \{-d, 0, d\}$, since $\mathcal{U}_\Lambda^{\{-d,0\}}$ is a d-cluster-tilting subcategory of $\mathcal{D}_\Lambda$ and $\operatorname{gl.dim} \Lambda \leqslant d$. Hence, for an object T of $\mathcal{U}_\Lambda^{\{-d,0\}}$ to be d-silting, it suffices that $\operatorname{Hom}_{\mathcal{D}_\Lambda}(T, T[d]) = 0$ and $\operatorname{thick} T = \mathcal{D}_\Lambda$.

Remark 4.2.4. For $d = 1$, the category $\mathcal{U}_\Lambda^{\{-d,0\}}$ is the category of *two-term complexes of projectives*: complexes of projectives concentrated in degrees -1 and 0. This category has been widely studied, for instance, in [Aih13; AIR14; DIJ19]. However, it is important to note that for $d > 1$, the category $\mathcal{U}_\Lambda^{\{-d,0\}}$ is not simply the category of $(d+1)$-term complexes of projectives: complexes of projectives concentrated in degrees $-d$ to 0. Additionally, we must have that the cohomology of the complex is concentrated in degrees $-d$ and 0, and that the cohomology must always lie in the d-cluster-tilting subcategory $\operatorname{add} M$. In general there are objects with projective resolutions of length d which do not lie in $\operatorname{add} M$, and so do not lie in $\mathcal{U}_\Lambda^{\{-d,0\}}$.

Given a d-silting complex T of $\mathcal{U}_\Lambda^{\{-d,0\}}$ with $T = E \oplus X$ and a $(d+2)$-angle

$$\begin{tikzcd}
 & & E_2 \arrow[rrrr, bend left, dotted, no head] \arrow[dr] & & & & E_{d-1} \arrow[dr] \arrow[dd] & & \\
 & E_1 \arrow[ur] \arrow[dr] & & Z_2 \arrow[dl, "1"] \arrow[rr, dotted, no head] & & Z_{d-2} \arrow[ur] & & E_d \arrow[ddr] & \\
 & & Z_1 \arrow[u] \arrow[dll, "1"] & & & & Z_{d-1} \arrow[ul, "1"] \arrow[ur] & & \\
X \arrow[uur] & & & & & & & & Y \arrow[ull, "1"] \arrow[llllllll, "d"]
\end{tikzcd}$$

such that $Y \in \mathcal{U}_\Lambda^{\{-d,0\}}$, $E_i \in \operatorname{add} E$ for all i, with $X \to E_1$ a left $(\operatorname{add} E)$-approximation, then we say that $E \oplus Y$ is a *left mutation* of $E \oplus X$. Such a $(d+2)$-angle will not necessarily exist for all indecomposable summands X of E. *Right mutation* is defined dually. Here the notation $Y \xrightarrow{i} X$ means a morphism $Y \to X[i]$.

Lemma 4.2.5. *The left mutation, and, dually, right mutation, of a d-silting complex is a d-silting complex.*

Proof. This follows similarly to [AI12, Theorem 2.3]. Note that it is clear that $\operatorname{thick} E \oplus X \subseteq \operatorname{thick} E \oplus Z_1$ from the first triangle. Repeating this argument, we obtain that $\mathcal{D}_\Lambda \subseteq \operatorname{thick} E \oplus X \subseteq \operatorname{thick} E \oplus Z_1 \subseteq \cdots \subseteq \operatorname{thick} E \oplus Y$, so $\operatorname{thick} E \oplus Y = \mathcal{D}_\Lambda$, as desired.

We now show that $\operatorname{Hom}_{\mathcal{D}_\Lambda}(E \oplus Y, (E \oplus Y)[d]) = 0$. It is immediate that $\operatorname{Hom}_{\mathcal{D}_\Lambda}(E, E[d]) = 0$. We then apply the Hom long exact sequence several times. Note that this is the Hom long exact sequence for $(d+2)$-angulated categories from [GKO13, Proposition 1.5(a)] rather than the classical Hom long exact sequence. First, we have that

$$\operatorname{Hom}(E, E_d[d]) \to \operatorname{Hom}(E, Y[d]) \to \operatorname{Hom}(E, X[2d])$$

is exact, so $\operatorname{Hom}(E, Y[d]) = 0$, since the two outer terms vanish. Next, we have that

$$\operatorname{Hom}(E_1, E) \to \operatorname{Hom}(X, E) \to \operatorname{Hom}(Y, E[d]) \to \operatorname{Hom}(E_d, E[d])$$

is exact. Since the map $X \to E_1$ is a right $(\operatorname{add} E)$-approximation, we have that the left-hand map in the sequence is a surjection. Hence, $\operatorname{Hom}_{\mathcal{D}_\Lambda}(Y, E[d]) = 0$, since $\operatorname{Hom}_{\mathcal{D}_\Lambda}(E_d, E[d]) = 0$. Finally, we have the exact sequence

$$\operatorname{Hom}(Y, E_d[d]) \to \operatorname{Hom}(Y, Y[d]) \to \operatorname{Hom}(Y, X[2d]).$$

We know that $\operatorname{Hom}_{\mathcal{D}_\Lambda}(Y, E_d[d]) = 0$, since $\operatorname{Hom}_{\mathcal{D}_\Lambda}(Y, E[d]) = 0$. Then, we have that $\operatorname{Hom}_{\mathcal{D}_\Lambda}(Y, X[2d]) = 0$, since $X, Y \in \mathcal{U}_\Lambda^{\{-d,0\}}$. Hence, $\operatorname{Hom}_{\mathcal{D}_\Lambda}(Y, Y[d]) = 0$. We conclude that $E \oplus Y$ is a d-silting complex. $\square$

We have that bistellar flips of triangulations correspond to mutations of d-silting complexes. This adapts to the d-silting framework [OT12, Theorem 4.4] and [OT12, Theorem 6.4], which are the analogous results for tilting and cluster-tilting respectively.

Proposition 4.2.6. *There is a bijection between internal d-simplices in $C(n + 2d+1, 2d)$ and indecomposable objects of $\mathcal{U}_{A_n^d}^{\{-d,0\}}$ which induces a bijection between triangulations of $C(n + 2d + 1, 2d)$ and basic d-silting complexes in $\mathcal{U}_{A_n^d}^{\{-d,0\}}$. Two triangulations are related by a bistellar flip if and only if the two corresponding d-silting complexes are related by a mutation.*

Proof. We know from Section 2.2.1 that internal d-simplices in $C(n + 2d + 1, 2d)$ are in bijection with ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$. It then follows from Section 4.1.1 that the indecomposables of $\mathcal{U}_{A_n^d}^{\{-d,0\}}$ are also in bijection with ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$, which gives the first statement of the proposition.

It furthermore follows from Section 2.2.1 and Section 4.1.1 that basic pre-silting complexes with $\binom{n+d-1}{d}$ indecomposable summands are in bijection with triangulations of $C(n + 2d + 1, 2d)$. This is because $\binom{n+d-1}{d}$ is the number of internal d-simplices of a triangulation of $C(n+2d+1, 2d)$, along with the fact that A and B are non-intertwining if and only if both $\operatorname{Hom}_{\mathcal{U}^{\{-d,0\}}_{A^d_n}}(U_A, U_B[d]) = 0$ and $\operatorname{Hom}_{\mathcal{U}^{\{-d,0\}}_{A^d_n}}(U_B, U_A) = 0$. We must show that these objects are precisely the d-silting complexes.

Note that A^d_n has $\binom{n+d-1}{d}$ indecomposable summands. Hence, by [AI12, Corollary 2.28], all d-silting complexes have $\binom{n+d-1}{d}$ indecomposable summands—and, by definition, they are all pre-silting.

We now show that bistellar flips between triangulations give exchange $(d+2)$-angles between basic pre-silting complexes with $\binom{n+d-1}{d}$ summands. We will use this to show that these objects are in fact d-silting. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are bistellar flips of each other. Then, by [OT12, Theorem 4.1], $\mathring{e}(\mathcal{T})$ and $\mathring{e}(\mathcal{T}')$ have all but one element in common. Consequently, the corresponding pre-silting complexes T and T' have all but one indecomposable summand in common, so we write $T = E \oplus X$ and $T' = E \oplus Y$. The simplices corresponding to X and Y must be intertwining, so that we either have a non-zero morphism $X \to Y[d]$ or $Y \to X[d]$. We assume the former is the case. It can then be shown, in an analogous way to [OT12, Proposition 3.19], that there is a $(d+2)$-angle

$$\begin{tikzcd}
 & & E_2 \arrow[dr] & & \cdots & & E_{d-1} \arrow[dd] \arrow[dr] & & \\
 & E_1 \arrow[ur] \arrow[dr] & & Z_2 \arrow[dl, "1"] & \cdots & Z_{d-2} \arrow[ur] & & E_d \arrow[ddr] & \\
 & & Z_1 \arrow[uu] \arrow[dll, "1"] & & & & Z_{d-1} \arrow[ul, "1"] \arrow[ur] & & \\
X \arrow[uur] & & & & & & & & Y, \arrow[ull, "1"] \arrow[llllllll, "d"]
\end{tikzcd}$$

where $E_i \in \operatorname{add} E$ for all i. Moreover, since T and T' are both pre-silting, we have exact sequences

$$\operatorname{Hom}(E, E_d) \to \operatorname{Hom}(E, Y) \to \operatorname{Hom}(E, X[d]) = 0$$

and

$$\mathrm{Hom}(E_1, E) \to \mathrm{Hom}(X, E) \to \mathrm{Hom}(Y, E[d]) = 0.$$

Hence $X \to E_1$ is a left $(\operatorname{add} E)$-approximation and $E_d \to Y$ is a right $(\operatorname{add} E)$-approximation. Therefore, if T is silting, then T' is a left mutation of T and so T' is silting. Conversely, if T' is silting, then T is a right mutation of T', and so T is silting.

Hence, we know that A_n^d is d-silting and that mutations of d-silting complexes are d-silting. We further know that A_n^d corresponds to a triangulation of $C(n+2d+1, 2d)$ and all triangulations are connected by bistellar flips [Ram97, Theorem 1.1]. Since bistellar flips of triangulations correspond to mutations, we conclude that every complex in $\mathcal{U}_{A_n^d}^{\{-d,0\}}$ corresponding to a triangulation is in fact d-silting, rather than only being pre-silting with $\binom{n+d-1}{d}$ summands. □

The relation between cluster-tilting objects and d-silting complexes is as follows. It follows from [OT12, Theorem 5.2] that the isomorphism classes of indecomposable objects of $\mathcal{O}_\Lambda$ are in bijection with the isomorphism classes of indecomposable objects of $\mathcal{U}_\Lambda^{\{-d,0\}}$. Indeed, the $(d+2)$-angulated projection

$$\widetilde{(-)} \colon \mathcal{U}_\Lambda \to \mathcal{O}_\Lambda$$

induces this bijection, since $\mathcal{U}_\Lambda^{\{-d,0\}}$ is a fundamental domain for $\nu_d[-d]$. The following lemma is similar to [JJ20, Theorem 3.5(i)], which uses a more general framework and is given in terms of τ_d-rigid pairs.

Lemma 4.2.7. *Given $X \in \mathcal{U}_\Lambda^{\{-d,0\}}$, we have that $\mathrm{Hom}_{\mathcal{U}_\Lambda}(X, X[d]) = 0$ if and only if $\mathrm{Hom}_{\mathcal{O}_\Lambda}(\widetilde{X}, \widetilde{X}[d]) = 0$.*

Proof. This is immediate from [OT12, Theorem 5.2(3)]. □

For $\Lambda = A_n^d$ we have that d-silting complexes in $\mathcal{U}_{A_n^d}^{\{-d,0\}}$ are in bijection with cluster-tilting objects in $\mathcal{O}_{A_n^d}$. This should also hold more generally.

Figure 4.11: The Auslander–Reiten quiver of $\mathcal{U}_{A_2}^{\{-d,0\}}$ and ${}^{\circlearrowleft}\mathbf{I}_5^1$

$$\begin{tikzcd}[column sep=small] & {\begin{smallmatrix}2\\1\end{smallmatrix}} \arrow[dr] & & 1[1] \arrow[dr] & \\ 1 \arrow[ur] & & 2 \arrow[ur] & & {\begin{smallmatrix}2\\1\end{smallmatrix}}[1] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & U_{14} \arrow[dr] & & U_{25} \arrow[dr] & \\ U_{13} \arrow[ur] & & U_{24} \arrow[ur] & & U_{35} \end{tikzcd}$$

Corollary 4.2.8. *Given $T \in \mathcal{U}_{A_n^d}^{\{-d,0\}}$, we have that T is d-silting if and only if $\widetilde{T}$ is cluster-tilting.*

Proof. By [OT12, Theorem 6.4], we have that $\widetilde{T}$ is cluster-tilting if and only if $\operatorname{Hom}_{\mathcal{O}_{A_n^d}}(\widetilde{T}, \widetilde{T}[d]) = 0$ and $\widetilde{T}$ has as many non-isomorphic indecomposable summands as A_n^d. Hence, it suffices to show that T is d-silting if and only if $\operatorname{Hom}_{\mathcal{U}_{A_n^d}}(T, T[d]) = 0$ and T has as many non-isomorphic indecomposable summands as A_n^d. This follows from Proposition 4.2.6. $\square$

Example 4.2.9. We illustrate the framework with d-silting complexes in $\mathcal{U}_{A_n^d}^{\{-d,0\}}$. Note the similarities with the framework with cluster-tilting objects in $\mathcal{O}_{A_n^d}$. The difference is that the category $\mathcal{U}_{A_n^d}^{\{-d,0\}}$ has fewer morphisms than $\mathcal{O}_{A_n^d}$.

(1) For $d = 1$ and $n = 2$, we have the bijection between ${}^{\circlearrowleft}\mathbf{I}_5^1$ and the indecomposables in $\mathcal{U}_{A_2}^{\{-d,0\}}$, as shown in Figure 4.11. This bijection induces a bijection between triangulations of $C(5, 2)$ and basic 1-silting complexes in $\mathcal{U}_{A_2}^{\{-d,0\}}$, as shown in Figure 4.12.

(2) For $d = 2$ and $n = 2$, Figure 4.13 shows the bijection between ${}^{\circlearrowleft}\mathbf{I}_7^2$ and the indecomposables of $\mathcal{U}_{A_2^2}^{\{-d,0\}}$. The seven 2-silting complexes in $\mathcal{U}_{A_2^2}^{\{-d,0\}}$ correspond to the seven triangulations of $C(7, 4)$, as given in Table 4.2.

Figure 4.12: 1-silting complexes in $\mathcal{U}_{A_2}^{\{-d,0\}}$ and their corresponding triangulations

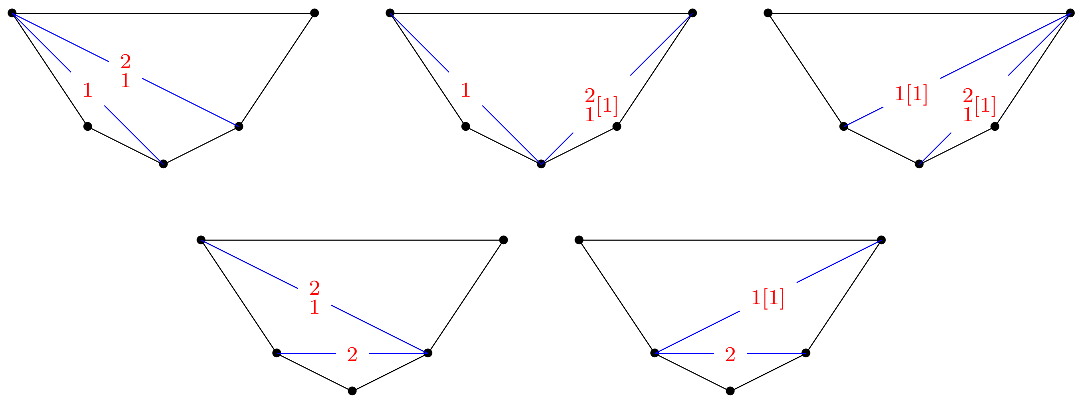


Figure 4.13: The Auslander–Reiten quiver of $\mathcal{U}_{A_2^2}^{\{-d,0\}}$ and ${}^{\circlearrowleft}\mathbf{I}_7^2$

$$\begin{tikzcd}
 & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[r] & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & & 1[2] \arrow[r] & \begin{smallmatrix}2\\1\end{smallmatrix}[2] \arrow[dr] & \\
1 \arrow[ur] & & & 3 \arrow[ur] & & & \begin{smallmatrix}3\\2\end{smallmatrix}[2]
\end{tikzcd}$$

$$\begin{tikzcd}
 & U_{136} \arrow[r] & U_{146} \arrow[dr] & & U_{247} \arrow[r] & U_{257} \arrow[dr] & \\
U_{135} \arrow[ur] & & & U_{246} \arrow[ur] & & & U_{357}
\end{tikzcd}$$

Table 4.2: 2-silting complexes in $\mathcal{U}^{\{-d,0\}}_{A^2_2}$ and their corresponding triangulations

2-silting complex	Triangulation
$1 \oplus {2 \atop 1} \oplus {3 \atop 2}$	$\{135, 136, 146\}$
$3 \oplus {2 \atop 1} \oplus {3 \atop 2}$	$\{246, 136, 146\}$
$3 \oplus 1[2] \oplus {3 \atop 2}$	$\{246, 247, 146\}$
$3 \oplus 1[2] \oplus {2 \atop 1}[2]$	$\{246, 247, 257\}$
$1 \oplus {2 \atop 1} \oplus {3 \atop 2}[2]$	$\{135, 136, 357\}$
$1 \oplus {2 \atop 1}[2] \oplus {3 \atop 2}[2]$	$\{135, 257, 357\}$
$1[2] \oplus {2 \atop 1}[2] \oplus {3 \atop 2}[2]$	$\{357, 247, 257\}$

4.3 Interpreting the higher Stasheff–Tamari orders algebraically in even dimensions

Having introduced the necessary background on the representation theory of A^d_n, we now show how the combinatorial characterisations of the orders we proved in Section 3.2 allow us to naturally interpret the higher Stasheff–Tamari orders in even dimensions in the representation theory of the higher Auslander algebras of type A.

4.3.1 First higher Stasheff–Tamari order

The first higher Stasheff–Tamari order has the following interpretation in terms of d-silting objects.

Theorem 4.3.1. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(n+2d+1, 2d)$ be triangulations with corresponding d-silting complexes T and T' in $\mathcal{U}^{\{-d,0\}}_{A^d_n}$. Then $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if T' is a left mutation of T.*

Proof. It follows from the material on the derived category in Section 4.1.1 that $\operatorname{Hom}_{\mathcal{U}_{A^d_n}}(U_B, U_A[d]) \neq 0$ if and only if $A \wr B$. Hence, as in the proof of Proposition 4.2.6, T' is a left mutation of T if and only if $T = E \oplus U_A$, $T' = E \oplus U_B$ and $A \wr B$. Therefore, T' is a left mutation of T if and only if $\mathring{e}(\mathcal{T}) = \mathcal{R} \cup \{A\}$ and $\mathring{e}(\mathcal{T}') = \mathcal{R} \cup \{B\}$ and $A \wr B$. By Theorem 3.2.1, this is true if and only if $\mathcal{T} \lessdot_1 \mathcal{T}'$. $\square$

Example 4.3.2. We illustrate this theorem with some examples, following on from Example 4.2.9.

(1) We start with the example where $n = 3$ and $d = 1$. If we take the triangulations given by $\mathring{e}(\mathcal{T}) = \{13, 14\}$ and $\mathring{e}(\mathcal{T}') = \{24, 14\}$, then $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$ via the 3-simplex 1234. The intersection of the lower facets of this simplex is 13 and the intersection of its upper facets is 24. Consulting Figure 4.11, the corresponding d-silting complexes are

$$T = 1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \quad \text{and} \quad T' = 2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix},$$

which are related by the exchange triangle

$$1 \to \begin{smallmatrix}2\\1\end{smallmatrix} \to 2 \to 1[1],$$

so that T' is a left mutation of T, as given by Theorem 4.3.1.

(2) Now consider the example where $n = 3$ and $d = 2$. Take the triangulations given by $\mathring{e}(\mathcal{T}) = \{135, 136, 146\}$ and $\mathring{e}(\mathcal{T}') = \{246, 136, 146\}$. Here $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$ via the 5-simplex 123456. The intersection of

the lower facets of this simplex is 135 and the intersection of its upper facets is 246. Consulting Figure 4.13, the corresponding d-silting complexes are then

$$T = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \quad \text{and} \quad T' = 4 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \end{smallmatrix},$$

which are related by the exchange 4-angle

$$1 \to \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \to \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \to 4 \to 1[2],$$

so that T' is again a left mutation of T, as Theorem 4.3.1 dictates.

4.3.2 Second higher Stasheff–Tamari order

We now interpret the second higher Stasheff–Tamari order algebraically in even dimensions. We first show what submersion corresponds to algebraically. Given an object T of $\mathcal{U}_\Lambda^{\{-d,0\}}$, we denote

$$\begin{aligned} {}^\perp T &= \{\, X \in \mathcal{U}_\Lambda^{\{-d,0\}} : \operatorname{Hom}_{\mathcal{D}_\Lambda}(X, T[i]) = 0 \,\forall i > 0 \,\} \\ &= \{\, X \in \mathcal{U}_\Lambda^{\{-d,0\}} : \operatorname{Hom}_{\mathcal{D}_\Lambda}(X, T[d]) = 0 \,\}. \end{aligned}$$

Proposition 4.3.3. *Let $\mathcal{T} \in \mathcal{S}(n+2d+1, 2d)$ with corresponding d-silting complex $T \in \mathcal{U}_{A_n^d}^{\{-d,0\}}$. Let A be an internal d-simplex in $C(n+2d+1, 2d)$ with corresponding indecomposable object $U_A \in \mathcal{U}_{A_n^d}^{\{-d,0\}}$. Then A is submerged by $\mathcal{T}$ if and only if $U_A \in {}^\perp T$.*

Proof. By Proposition 3.2.3, we have that A is submerged by $\mathcal{T}$ if and only if there is no $B \in \mathring{e}(\mathcal{T})$ such that $B \wr A$. We know that this is the case if and only if there is no indecomposable summand B of T such that $\operatorname{Hom}_{\mathcal{U}_{A_n^d}}(U_A, U_B[d]) \neq 0$. In turn, this is the case if and only if $U_A \in {}^\perp T$. □

We then obtain the following theorem by applying this proposition and using the interpretation of $\mathcal{S}_2(n + 2d + 1, 2d)$ in terms of d-submersion sets.

Theorem 4.3.4. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(n+2d+1, 2d)$ be triangulations with corresponding d-silting complexes $T, T' \in \mathcal{U}^{\{-d,0\}}_{A_n^d}$. Then $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if ${}^{\perp}T \subseteq {}^{\perp}T'$.*

Proof. Since d-simplices which are on the boundary of $C(n+2d+1, 2d)$ are in, and hence submerged by, every triangulation, we can restrict our attention to internal d-simplices when we consider submersion sets. Proposition 4.3.3 gives us that the complexes in ${}^{\perp}T$ correspond to internal d-simplices in $\mathrm{sub}_d\mathcal{T}$. Therefore ${}^{\perp}T \subseteq {}^{\perp}T'$ if and only if $\mathrm{sub}_d\mathcal{T} \subseteq \mathrm{sub}_d\mathcal{T}'$ if and only if $\mathcal{T} \leqslant_2 \mathcal{T}'$. □

Example 4.3.5. We illustrate Theorem 4.3.4 with examples, following on from Example 4.3.2.

(1) We first consider the case where $n = 2$ and $d = 1$. Consider the triangulations given by $\mathring{e}(\mathcal{T}) = \{13, 14\}$ and $\mathring{e}(\mathcal{T}') = \{24, 25\}$. Consulting Figure 4.11, the corresponding d-silting complexes are

$$T = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \quad \text{and} \quad T' = 2 \oplus 1[1],$$

whose orthogonal categories are

$$\begin{aligned} {}^{\perp}T &= \mathrm{add}\left\{1, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right\} \\ {}^{\perp}T' &= \mathrm{add}\left\{1, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, 2, 1[1]\right\}. \end{aligned}$$

Since ${}^{\perp}T \subseteq {}^{\perp}T'$, we have that $\mathcal{T} \leqslant_2 \mathcal{T}'$ by Theorem 4.3.4.

(2) Now consider the example where $n = 3$ and $d = 2$. Take the triangulations given by $\mathring{e}(\mathcal{T}) = \{135, 136, 146\}$ and $\mathring{e}(\mathcal{T}') = \{135, 257, 357\}$. Consulting Figure 4.13, the corresponding d-silting complexes are then

$$T = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \quad \text{and} \quad T' = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}[2] \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}[2],$$

whose orthogonal categories are

$$
\begin{aligned}
{}^{\perp}T &= \operatorname{add}\left\{1, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}\right\}, \\
{}^{\perp}T' &= \operatorname{add}\left\{1, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}[2], \begin{smallmatrix}3\\2\end{smallmatrix}[2]\right\}.
\end{aligned}
$$

Since ${}^{\perp}T \subseteq {}^{\perp}T'$, we have that $\mathcal{T} \leqslant_2 \mathcal{T}'$ by Theorem 4.3.4.

We then obtain the following by applying the result that the higher Stasheff–Tamari orders are equal (Theorem 3.3.14) to our algebraic interpretation of the higher Stasheff–Tamari orders in even dimensions (Theorem 4.3.1 and Theorem 4.3.4).

Corollary 4.3.6. *Let T, T' be d-silting complexes in $\mathcal{U}^{\{-d,0\}}_{A^d_n}$. Then there is a sequence of left mutations from T to T' if and only if ${}^{\perp}T \subseteq {}^{\perp}T'$.*

Remark 4.3.7. Orders on algebraic objects given by mutation and by inclusion of orthogonal categories occur often in the literature. Two such orders were first studied for tilting modules in [RS91]. It was shown in [HU05] that these orders had the same Hasse diagram, which implies that they are the same when there are finitely many tilting modules. Analogous orders were considered for silting complexes [AI12] and support τ-tilting modules [AIR14], and these orders were likewise shown to have the same Hasse diagram.

In Theorems 4.3.1 and 4.3.4 we see higher-dimensional analogues of these algebraic orders. Corollary 4.3.6 gives us that these two orders are equal for the higher Auslander algebras of type A. The natural question is, of course, whether the analogue of Corollary 4.3.6 holds for general d-representation-finite d-hereditary algebras. Such a result would give an algebraic proof of the equivalence of the higher Stasheff–Tamari orders.

4.4 Interpreting the higher Stasheff–Tamari orders algebraically in odd dimensions

We now give combinatorial and algebraic interpretations of the higher Stasheff–Tamari orders on triangulations of odd-dimensional cyclic polytopes. To obtain the algebraic interpretations, we first show how triangulations of odd-dimensional cyclic polytopes arise in the representation theory of A_n^d. This gives the other half of the picture from [OT12], which shows how triangulations of even-dimensional cyclic polytopes arise in the representation theory of A_n^d. As we shall see, it is precisely the interpretation of the first higher Stasheff–Tamari order in even dimensions that allows us to interpret odd-dimensional triangulations in this way.

4.4.1 Algebraic interpretation of odd-dimensional triangulations

We now explain how odd-dimensional triangulations arise in the representation theory of A_n^d, namely, as equivalence classes of d-maximal green sequences. Let Λ be a d-representation-finite d-hereditary algebra over a field K, where K is a field. We define a *d-maximal green sequence* for Λ to be a sequence $(T_0, T_1, \dots, T_r)$ of d-silting complexes in $\mathcal{U}_\Lambda^{\{-d,0\}}$ such that $T_0 = \Lambda$, $T_r = \Lambda[d]$, and, for $i \in [r]$, T_i is a left mutation of T_{i-1}. Let $\mathcal{MG}_d(\Lambda)$ denote the set of d-maximal green sequences of Λ.

Given a d-maximal green sequence G, we denote the set of indecomposable summands of d-silting complexes occurring in G by $\Sigma(G)$. We write $G \sim G'$ if and only if $\Sigma(G) = \Sigma(G')$. We use $\widetilde{\mathcal{MG}}_d(\Lambda)$ to denote the set of equivalence classes of $\mathcal{MG}_d(\Lambda)$ under the relation $\sim$.

To relate triangulations of odd-dimensional cyclic polytopes and d-maximal green sequences, we first observe the following lemma.

Lemma 4.4.1. *Let $A \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$. Then the complex U_A is neither a projective nor shifted projective if and only if A is an internal d-simplex in $C(n+2d+1, 2d+1)$.*

Proof. As we explain in Section 4.1.1, we have that U_A is a projective if and only if $a_0 = 1$. Then, by the combinatorial interpretation of $[d]$, we have that U_A is a shifted projective if and only if $a_d = n + 2d + 1$. The result then follows from Lemma 2.2.4, since $A \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ is an internal d-simplex in $C(n + 2d + 1, 2d + 1)$ if and only if $a_0 \neq 1$ and $a_d \neq n + 2d + 1$. □

Triangulations of odd-dimensional cyclic polytopes correspond to equivalence classes of d-maximal green sequences, as follows.

Theorem 4.4.2. *There is a bijection between $\mathcal{S}(n+2d+1, 2d+1)$ and $\widetilde{\mathcal{MG}}_d(A^d_n)$. Moreover, if a triangulation $\mathcal{T} \in \mathcal{S}(n+2d+1, 2d+1)$ corresponds to an equivalence class of d-maximal green sequences $[G] \in \widetilde{\mathcal{MG}}_d(A^d_n)$, then*

(1) there is a bijection between mutations in G and $(2d+1)$-simplices of $\mathcal{T}$; and

(2) there is a bijection between the internal d-simplices of $\mathcal{T}$ and elements of $\Sigma(G)$ which are neither projectives nor shifted projectives.

Proof. The key result which establishes this theorem is [Ram97, Theorem 1.1(ii)], which states that triangulations of $C(n + 2d + 1, 2d + 1)$ are in bijection with maximal chains in $\mathcal{S}_1(n+2d+1, 2d)$ under an equivalence relation of differing by a permutation of bistellar flip operations. By Theorem 4.3.1, elements of $\mathcal{MG}_d(A^d_n)$ correspond to maximal chains in $\mathcal{S}_1(n + 2d + 1, 2d)$, since, as can be checked straightforwardly, the lower triangulation of $C(n + 2d + 1, 2d)$ corresponds to A^d_n and the upper triangulation of $C(n+2d+1, 2d)$ corresponds to $A^d_n[d]$. Hence let $G = (T_0, T_1, \dots, T_r) \in \mathcal{MG}_d(A^d_n)$ correspond to a maximal chain $\mathcal{C} = (\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_r)$ of $\mathcal{S}_1(n + 2d + 1, 2d)$ which gives a triangulation $\mathcal{T}$ of $C(n + 2d + 1, 2d + 1)$.

We first establish the claims (1) and (2) for $\mathcal{T}$ and G. The claim (1) is straightforward, because the $(2d+1)$-simplices of $\mathcal{T}$ correspond to increasing bistellar flips in $\mathcal{C}$ by [Ram97, Theorem 1.1(ii)]. Then these correspond to mutations by Theorem 4.3.1.

For claim (2), let $U_A \in \Sigma(G)$ be neither a projective nor a shifted projective, so that U_A is an indecomposable summand of T_i for some i. Then implies that A is a d-simplex of $\mathcal{T}_i$, which implies that A is a d-simplex of $\mathcal{T}$. By Lemma 4.4.1, A is an internal d-simplex of $\mathcal{T}$. Conversely, if A is an internal d-simplex of $\mathcal{T}$, then by Rambau's theorem, there is a triangulation $\mathcal{T}_i$ in $\mathcal{C}$ such that A is an internal d-simplex of $\mathcal{T}_i$. This implies that U_A is an indecomposable summand of T_i, and by Lemma 4.4.1, it is neither a projective nor a shifted projective. This establishes claim (2).

Now we must show that d-maximal green sequences for A_n^d are equivalent if and only if they give the same triangulation of $C(n+2d+1, 2d+1)$. Let $G' \in \mathcal{MG}_d(A_n^d)$ correspond to a triangulation $\mathcal{T}'$ of $C(n+2d+1, 2d+1)$ and suppose that $G \sim G'$. By claim (2), since $\Sigma(G) = \Sigma(G')$, we have that $\mathring{e}(\mathcal{T}) = \mathring{e}(\mathcal{T}')$. Hence, by Lemma 2.2.9 we have that $\mathcal{T} = \mathcal{T}'$ as required.

Conversely, it is clear that if G and G' correspond to the same triangulation, then we must have $G \sim G'$. This is because if G and G' correspond to the same triangulation, then they must have in common all indecomposable summands which are neither projective nor shifted projective, by claim (2). But, since all indecomposable projectives and shifted projectives must also be summands of both G and G', we have that $\Sigma(G) = \Sigma(G')$. $\square$

Remark 4.4.3. Theorem 2.2.3 therefore also classifies d-maximal green sequences for A_n^d up to equivalence. Namely, a set $\{ U_A : A \in \mathcal{X} \}$ of complexes in $\mathcal{U}_{A_n^d}^{\{-d,0\}}$, where $\mathcal{X} \subseteq {}^{\circlearrowleft}\mathbf{I}_{n+2d+1}^d$, is the set of summands of a d-maximal green sequence if and only if $\mathcal{X}$ is supporting and bridging. Note that $\mathcal{X}$ will contain all the subsets corresponding to the projectives and shifted projectives, which are excluded from the sets $\mathring{e}(\mathcal{T})$ in Section 4.1.1. However, including these subsets does not affect whether the supporting and bridging conditions are satisfied, as can be checked.

4.4.2 First higher Stasheff–Tamari order

We now give an algebraic characterisation of the first higher Stasheff–Tamari order in terms of d-maximal green sequences. Our terminology is based on [HI19]. An *oriented polygon* is a sub-poset of $\mathcal{S}_1(m, 2d)$ formed of the union of a chain of covering relations of length $d+2$ with a chain of covering relations of length $d+1$, such that these chains intersect only at the top and bottom. (For an illustration see Figure 4.14.) Here the *length* of a chain is the number of covering relations in it. We think of an oriented polygon as being oriented from the longer side to the shorter side. If two maximal chains G, G' differ only in that G contains the longer side of an oriented polygon and G' contains the shorter side, then we say that G' is *an increasing elementary polygonal deformation* of G. Note that an increasing elementary polygonal deformation decreases the length of the chain.

Theorem 4.4.4. *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(n+2d+1, 2d+1)$ correspond to equivalence classes of d-maximal green sequences $[G], [G'] \in \widetilde{\mathcal{MG}}_d(A_n^d)$. Then $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if there are equivalence class representatives $\widehat{G} \in [G]$ and $\widehat{G'} \in [G']$ such that $\widehat{G'}$ is an increasing elementary polygonal deformation of $\widehat{G}$.*

In the proof of this theorem we shall require the partial order $\prec$ on the simplices of a triangulation from Remark 2.1.3. Recall that the covering relations of this order are denoted by $\prec\!\!\cdot$.

Proof. Let $\mathcal{T}, \mathcal{T}'$ be triangulations of $C(n+2d+1, 2d+1)$ corresponding respectively to $[G], [G'] \in \widetilde{\mathcal{MG}}_d(A_n^d)$. Suppose that $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$. Let $S \in \binom{[n+2d+1]}{2d+3}$ be the $(2d+3)$-subset of vertices giving the bistellar flip. Let

$S_i := S \backslash s_i$. The lower triangulation of $C(S, 2d+1)$ consists of the $(2d+1)$-simplices S_i for i even and the upper triangulation consists of the $(2d+1)$-simplices S_i for i odd. Then $S_{2j} \prec S_{2i}$ for $i < j$, since $S_{2i} \cap S_{2j}$ is an upper facet of S_{2j} and a lower facet of S_{2i}. Thus one can extend $\prec$ to the total order $\prec_t$ on the simplices of the lower triangulation of $C(S, 2d+1)$ by

$$S_{2d+2} \prec_t S_{2d} \prec_t \cdots \prec_t S_0.$$

This can be consistently extended to a total order on the $(2d+1)$-simplices of $\mathcal{T}$ which contains this chain as an interval. This would only be impossible if there were a $(2d+1)$-simplex R of $\mathcal{T}$ such that $S_{2j} \prec R \prec S_{2i}$, where $i < j$. But, since $S_{2j} \prec S_{2i}$ is a covering relation for $\prec$, this cannot happen.

Therefore, by [Ram97, Corollary 5.12], there is a maximal chain $\widehat{\mathcal{C}}$ of $\mathcal{S}_1(n + 2d+1, 2d)$ corresponding to $\mathcal{T}$ such that the sequence of bistellar flips in $\widehat{\mathcal{C}}$ is

$$(R_1, R_2, \ldots, R_{r-1}, S_{2d+2}, S_{2d}, \ldots, S_0, R'_1, R'_2, \ldots, R'_{s-1}).$$

A similar argument shows that there exists a maximal chain $\widehat{\mathcal{C}'}$ of $\mathcal{S}_1(n+2d+1, 2d)$ corresponding to $\mathcal{T}'$ such that the sequence of bistellar flips is

$$(R_1, R_2, \ldots, R_{r-1}, S_1, S_3, \ldots, S_{2d+1}, R'_1, R'_2, \ldots, R'_{s-1}).$$

Since the $(2d+1)$-simplices of $\mathcal{T}'$ outside $C(S, 2d+1)$ are the same as those of $\mathcal{T}$, namely $\{R_1, R_2, \ldots, R_{r-1}, R'_1, R'_2, \ldots, R'_{s-1}\}$, we may choose the same order on them in both maximal chains $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{C}'}$. It follows from the description of triangulations of $C(2d+3, 2d)$ in Lemma 3.3.1 that the chains in $\mathcal{S}_1(n+2d+1, 2d)$ given here by $(S_{2d+2}, S_{2d}, \ldots, S_0)$ and $(S_1, S_3, \ldots, S_{2d+1})$ intersect only at their top and bottom. Hence these chains form an oriented polygon.

Then, by Theorem 4.4.2, these correspond to $\widehat{G} \in \mathcal{MG}_d(A_n^d)$, where

$$\widehat{G} = (U_1, U_2, \ldots, U_r, T_1, T_2, \ldots, T_{d+1}, V_1, V_2, \ldots, V_s),$$

and $\widehat{G'} \in \mathcal{MG}_d(A_n^d)$, where

$$\widehat{G'} = (U_1, U_2, \dots, U_r, T'_1, T'_2, \dots, T'_d, V_1, V_2, \dots, V_s).$$

Thus $\widehat{G'}$ is an increasing elementary polygonal deformation of $\widehat{G}$, as required. Note that the $(2d+1)$-simplices in the sequences of bistellar flips of $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{C}}'$ come in between the respective d-silting complexes of $\widehat{G}$ and $\widehat{G'}$, which correspond to triangulations.

Conversely, suppose that we have equivalence class representatives

$$\widehat{G} = (U_1, U_2, \dots, U_r, T_1, T_2, \dots, T_{d+1}, V_1, V_2, \dots, V_s)$$

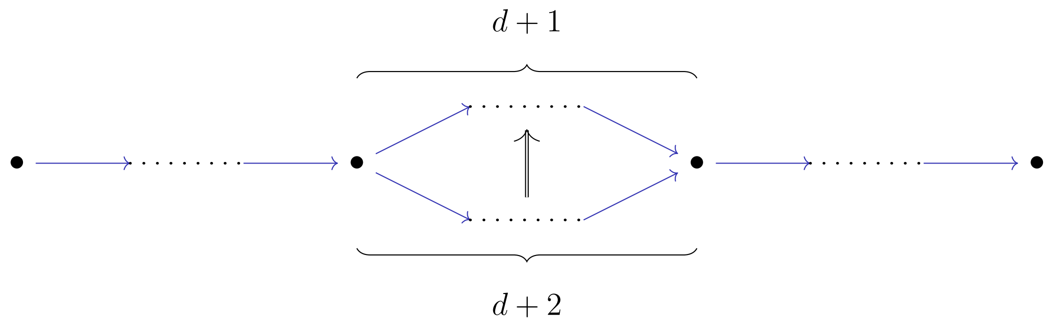
and

$$\widehat{G'} = (U_1, U_2, \dots, U_r, T'_1, T'_2, \dots, T'_d, V_1, V_2, \dots, V_s)$$

in $\mathcal{MG}_d(A_n^d)$. Here, as before, let $\widehat{G}$ give the triangulation $\mathcal{T}$ and $\widehat{G'}$ give the triangulation $\mathcal{T}'$. We claim that this implies that $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$. By Theorem 4.4.2, by transforming $\widehat{G}$ into $\widehat{G'}$, we have removed $d+2$ different $(2d+1)$-simplices $\{S_0, S_2, \dots, S_{2d+2}\}$ from $\mathcal{T}$ and replaced them by $d+1$ different $(2d+1)$-simplices $\{S_1, S_3, \dots, S_{2d+1}\}$. We can suppose that S_1 is not in the triangulation $\mathcal{T}$, since at least one of these simplices must not be. Hence $|S_1|$ must intersect a $(2d+1)$-simplex of the triangulation $|\mathcal{T}|$ transversely, and so it must intersect $|S_{2l}|$ for some l.

Hence, there is a circuit (A, B) with $A \subset S_{2l}$, $B \subset S_1$. By the description of the circuits of a cyclic polytope, one of A, B is an internal d-simplex and the other is an internal $(d+1)$-simplex. Internal d-simplices are intersections of at least $d+2$ different $(2d+1)$-simplices and internal $(d+1)$-simplices are intersections of at least $d+1$ different $(2d+1)$-simplices, by Lemma 3.2.5. Therefore, to remove an internal d-simplex, one must remove the $d+2$ different $(2d+1)$-simplices whose intersection it is. Hence, since A and B form a circuit and so cannot be in the

Figure 4.14: An increasing elementary polygonal deformation of d-maximal green sequences



same triangulation, we must have $A = \bigcap_{i=0}^{d+1} S_{2i}$ and $B = \bigcap_{j=0}^{d} S_{2j+1}$. Moreover, the only internal d-simplex we can have removed from $\mathcal{T}$ is the intersection of these $d+2$ different $(2d+1)$-simplices, which is A, and so $\mathring{e}(\mathcal{T}') = \mathring{e}(\mathcal{T}) \setminus \{A\}$. Thus $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$ by Theorem 3.2.6. $\square$

Remark 4.4.5. An *n-category* is a category enriched in $(n-1)$-categories, where an ordinary category is a 1-category. By [OT12], we have that d-silting complexes for A_n^d correspond bijectively to triangulations of $C(n+2d+1, 2d)$. By Theorem [KV91, Theorem 3.4], triangulations of $C(n+2d+1, 2d)$ form an n-category. Hence the set of d-silting complexes for A_n^d forms an n-category. Indeed, the irreducible 1-morphisms of this category are left mutations, and the irreducible 2-morphisms are the increasing elementary polygonal deformations of equivalence classes of maximal chains from Theorem 4.4.6.

4.4.3 Second higher Stasheff–Tamari order

The second order has the following interpretation on maximal green sequences.

Theorem 4.4.6. *Given two triangulations $\mathcal{T}, \mathcal{T}' \in \mathcal{S}(n+2d+1, 2d+1)$ corresponding to equivalence classes of d-maximal green sequences $[G], [G'] \in \mathcal{MG}_d(A_n^d)$, then $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\Sigma(G) \supseteq \Sigma(G')$.*

Proof. Let $\mathcal{T}$ be a triangulation of $C(n+2d+1, 2d)$ corresponding to an equivalence class $[G]$ of d-maximal green sequences of A_n^d. By Theorem 4.4.2, we have that $U_A \in \Sigma(G)$ if and only if either $A \in \mathring{e}(\mathcal{T})$, or U_A is a projective or shifted projective. Since the projectives and the shifted projectives are contained in every d-maximal green sequence, the result follows from Corollary 3.2.12. □

Example 4.4.7. We illustrate Theorem 4.4.2, Theorem 4.4.4, and Theorem 4.4.6 with the example $n = 2$, $d = 1$. The Auslander–Reiten quiver of the category $\mathcal{U}_{A_2}$ is shown in Figure 4.8. There exist five d-silting complexes in $\mathcal{U}_{A_2}$, which correspond to triangulations of $C(5, 2)$, as shown in Figure 4.9. By Theorem 4.4.2, the two maximal green sequences formed from these d-silting complexes correspond to the two possible triangulations of $C(5, 3)$, as shown in Figure 4.15. Let the longer maximal green sequence here be G and the shorter maximal green sequence be G', with $\mathcal{T}$ and $\mathcal{T}'$ the corresponding triangulations. Then it can be seen from the figure that G' is an increasing elementary polygonal deformation of G, which, by Theorem 4.4.4, corresponds to the fact that $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$. Moreover,

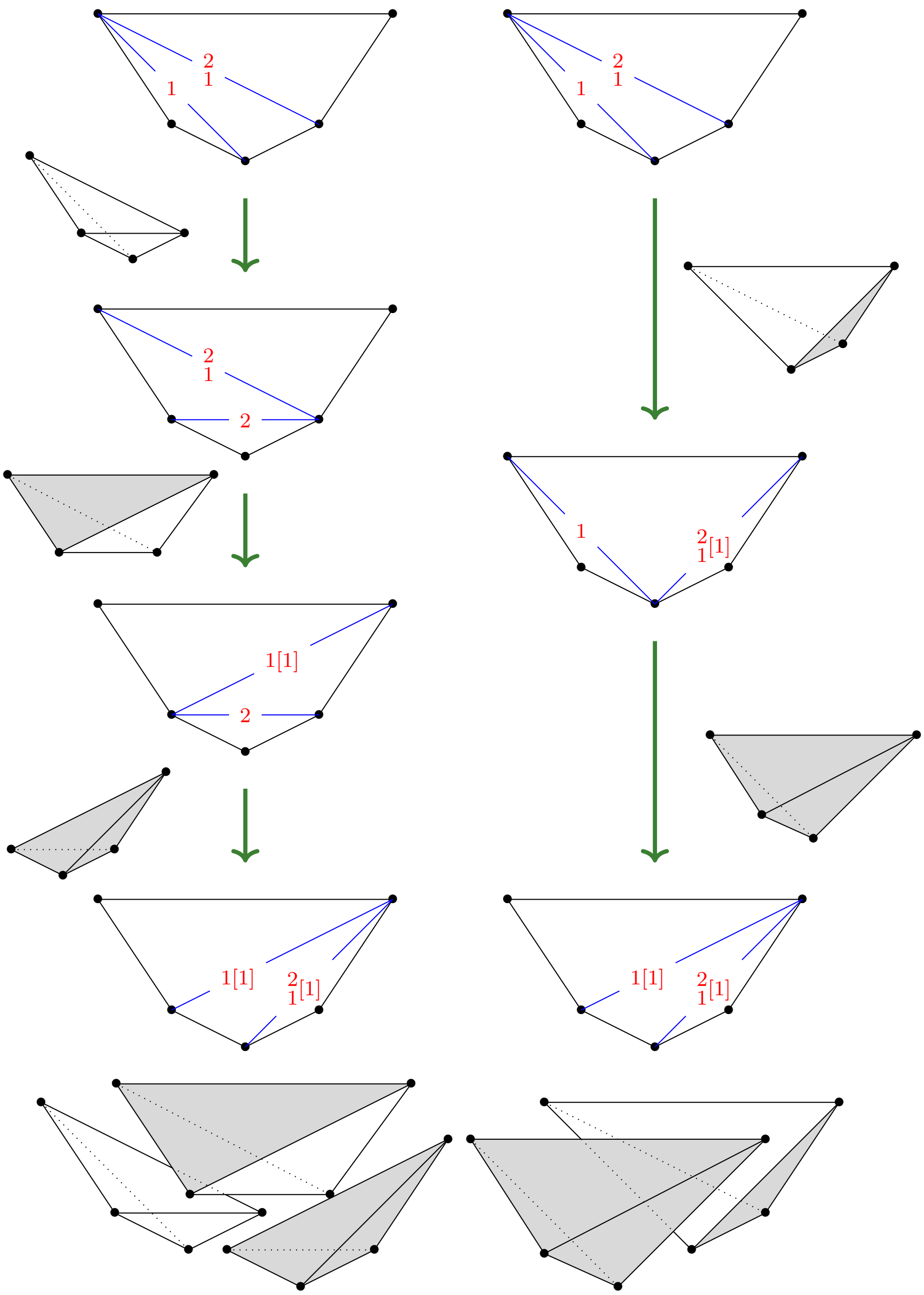
$$\Sigma(G) = \left\{1,\ {2 \atop 1},\ 2,\ 1[1],\ {2 \atop 1}[1]\right\},$$

$$\Sigma(G') = \left\{1,\ {2 \atop 1},\ 1[1],\ {2 \atop 1}[1]\right\},$$

so that $\Sigma(G) \supseteq \Sigma(G')$. Hence, by Theorem 4.4.6, we have that $\mathcal{T} \leqslant_2 \mathcal{T}'$.

By applying Theorem 3.3.9 to Theorem 4.4.4 and Theorem 4.4.6, we obtain the following result.

Figure 4.15: Maximal green sequences of A_2



Corollary 4.4.8. *Let $[G], [G']$ be two equivalence classes of d-maximal green sequences for A_n^d. Then there is a sequence of increasing elementary polygonal deformations from $[G]$ to $[G']$ if and only if $\Sigma(G) \supseteq \Sigma(G')$.*

Remark 4.4.9. In independent work, [Gor14a; Gor14b; Gor] Gorsky defines two orders on the set of equivalence classes of maximal green sequences of a Dynkin quiver, using combinatorics of the associated Coxeter group, and proves that they are the same. For type A quivers, these coincide with the two higher Stasheff–Tamari orders considered here in three dimensions. The general relationship is not altogether clear.

Remark 4.4.10. In general, there exist two orders on the set of equivalence classes of maximal green sequences, corresponding to the two different higher Stasheff–Tamari order. The general conjecture that the two orders on equivalence classes of maximal green sequences are equal can be seen as an oriented version of the "no-gap" conjecture of Brüstle, Dupont, and Perotin [BDP14]. This conjecture states that the set of lengths of maximal green sequences of an algebra contains no gaps. If the two orders on equivalence classes of maximal green sequences are equal, then if $\Sigma(G) \supseteq \Sigma(G')$, then there is no gap in the lengths of maximal green sequences between G and G'. This is because there will be a sequence of increasing elementary polygonal deformations from G to G', and each deformation will only change the length by one.

Since it is known for dimension 3 that the higher Stasheff–Tamari orders are lattices [ER96, Theorem 4.9 and Theorem 4.10], we have the following corollary.

Corollary 4.4.11. *The set $\widetilde{\mathcal{MG}}_1(A_n)$ forms a lattice under the order given by reverse inclusion of summands, or, equivalently, the order whose covering relations are given by increasing elementary polygonal deformations.*

Remark 4.4.12. It is not in general true that the set of equivalence classes of maximal green sequences of a finite-dimensional algebra is a lattice. For example, the preprojective algebra of A_2 only has two maximal green sequences. These are not equivalent to each other, and nor are they related by either of the relations described above. Hence in this case the set of maximal green sequences modulo equivalence is not a lattice.

One might wonder whether the set of equivalence classes of maximal green sequences is a lattice for other hereditary algebras. However, computer calculations reveal that the set of equivalence classes of maximal green sequences of the path algebra of Dynkin type D_4 is not a lattice.

Remark 4.4.13. A common way of considering a maximal green sequence for $d = 1$ is as a chain of torsion classes [Nag13]. A natural question to ask, therefore, is whether there exists an analogous description for $d > 1$.

For a d-silting complex T for A_n^d, the associated d-torsion class ought to be $T^{\perp} \cap \operatorname{add} M^{(d,n)}$. Indeed, this class corresponds to the internal d-simplices of the supermersion set of the associated triangulation of $C(n + 2d + 1, 2d)$, excluding internal d-simplices belonging to the upper triangulation, which are in every supermersion set.

However, the d-torsion classes that are generated in this way do not satisfy any definitions of higher torsion classes that have appeared so far in the literature, such as [Jør16; McM18; McM21].

Note also that maximal chains of torsion classes are really chains in the second order, whereas d-maximal green sequences should be chains in the first order. We know that the two orders are the same for A_n^d, but for general d-representation-finite algebras the question remains open.

4.5 An algebraic criterion for mutation

In the final section of this chapter, we return to the cluster category $\mathcal{O}_\Lambda$, where Λ is a d-representation-finite d-hereditary algebra and consider mutation of cluster-tilting objects. Recall that, given a cluster-tilting object $T \in \mathcal{O}_\Lambda$ with $T = E \oplus X$ for X indecomposable, we have that the summand X is *mutable* if and only if there is a cluster-tilting object $T' = E \oplus Y$ with $Y \ncong X$. For $d = 1$, every summand of a basic cluster-tilting object is mutable, but for $d > 1$ cluster-tilting objects generally also possess summands which are not mutable. This corresponds to the fact in terms of triangulations that one cannot perform an increasing bistellar flip at every internal d-simplex of a triangulation. In this section we provide an algebraic criterion for identifying the mutable summands of a cluster-tilting object in $\mathcal{O}_\Lambda$. In Chapter 5 we shall give a combinatorial criterion for identifying the mutable internal d-simplices of a triangulation.

We recall the theorem [OT12, Theorem 5.6]. Let T be a cluster-tilting object in $\mathcal{O}_\Lambda$ and set $\Gamma := \operatorname{End}_{\mathcal{O}_\Lambda} T$. Then the functor

$$\operatorname{Hom}_{\mathcal{O}_\Lambda}(T, -)\colon \mathcal{O}_\Lambda \to \operatorname{mod}\Gamma$$

induces a fully faithful embedding

$$\mathcal{O}_\Lambda/(T[d]) \hookrightarrow \operatorname{mod}\Gamma,$$

where $(T[d])$ denotes the ideal of all morphisms factoring through $\operatorname{add} T[d]$. The image of this functor is a d-cluster-tilting subcategory $\mathcal{M}$ of $\operatorname{mod}\Gamma$. In particular, Γ is weakly d-representation-finite.

Since $[d]$ is an automorphism of $\mathcal{O}_\Lambda$, we may restate this theorem as follows. We instead obtain that the functor

$$\operatorname{Hom}_{\mathcal{O}_\Lambda}(T, -[d])\colon \mathcal{O}_\Lambda \to \operatorname{mod}\Gamma$$

induces a fully faithful embedding

$$\mathcal{O}_\Lambda/(T) \hookrightarrow \operatorname{mod}\Gamma,$$

with the image of this functor giving a d-cluster-tilting subcategory $\mathcal{M}$ of $\operatorname{mod}\Gamma$.

The restatement of the theorem presents an interesting picture. Here we have that the indecomposables L of the d-cluster-tilting subcategory $\mathcal{M}$ of $\operatorname{mod}\Gamma$ are in bijection with the indecomposables of $\mathcal{O}_\Lambda/(T)$. Namely, for an indecomposable Y of $\mathcal{O}_\Lambda/(T)$, we have a Γ-module $\operatorname{Hom}_{\mathcal{O}_\Lambda}(T, Y[d])$. A complete set of orthogonal primitive idempotents for Γ is given by the projections of T onto its indecomposable factors. The idempotents of Γ which $\operatorname{Hom}_{\mathcal{O}_\Lambda}(T, Y[d])$ is supported on correspond to indecomposable summands X of T such that $\operatorname{Hom}_{\mathcal{O}_\Lambda}(X, Y[d]) \neq 0$. Hence the Γ-modules in $\mathcal{M}$ give information about the mutation theory of T. We crystallise this in the following theorem.

Theorem 4.5.1. *Let T be a basic cluster-tilting object in $\mathcal{O}_\Lambda$ with indecomposable summand X. Then T is mutable at X if and only if the d-cluster-tilting subcategory $\mathcal{M}$ of* $\operatorname{mod}\Gamma$ *contains a non-zero module M whose composition factors are all isomorphic to the simple* Γ*-module corresponding to X.*

Proof. Suppose that $T = E \oplus X$ is mutable at X. Hence, there is a cluster-tilting object $T' = E \oplus Y$. Then $M = \operatorname{Hom}_{\mathcal{O}_\Lambda}(T, Y[d]) \cong \operatorname{Hom}_{\mathcal{O}_\Lambda}(X, Y[d])$ is in the d-cluster-tilting subcategory $\mathcal{M}$ of $\operatorname{mod}\Gamma$ and only has composition factors given by the simple Γ-module corresponding to X.

Conversely, suppose that there is a module M in $\mathcal{M}$ whose composition factors are all isomorphic to the simple Γ-module at X. We may assume that M is indecomposable. Then there exists an object Y in $\mathcal{O}_\Lambda$ such that $M \cong \operatorname{Hom}_{\mathcal{O}_\Lambda}(T, Y[d])$. If we let $T = E \oplus X$, then we conclude from the composition factors of M that $\operatorname{Hom}_{\mathcal{O}_\Lambda}(E, Y[d]) = 0$, which means that, since $M \neq 0$, we have $\operatorname{Hom}_{\mathcal{O}_\Lambda}(X, Y[d]) \neq$

0. Therefore, $T' = E \oplus Y$ is a cluster-tilting object by [OT12, Theorem 5.7], and so T is mutable at X. $\square$

For the higher Auslander algebras of type A, we can state a slightly stronger result. To make the notation lighter, we write $\mathcal{O} := \mathcal{O}_{A_n^d}$.

Theorem 4.5.2. *Let T be a basic cluster-tilting object in $\mathcal{O}$ with indecomposable summand X. Then T is mutable at X if and only if the d-cluster-tilting subcategory $\mathcal{M}$ of $\operatorname{mod}\Gamma$ contains the simple Γ-module corresponding to X.*

Proof. Suppose that $T = E \oplus X$ is mutable at X. Hence, there is a cluster-tilting object $T' = E \oplus Y$. Then $M = \operatorname{Hom}_{\mathcal{O}}(T, Y[d]) \cong \operatorname{Hom}_{\mathcal{O}}(X, Y[d])$ is a Γ-module in $\mathcal{M}$. By [OT12, Proposition 6.1], this Hom-space is a 1-dimensional K-vector space, and hence is isomorphic to the simple Γ-module at X.

Conversely, suppose that S, the simple Γ-module at X, is in $\mathcal{M}$. Then there exists an object Y in $\mathcal{O}_{A_n^d}$ such that $S \cong \operatorname{Hom}_{\mathcal{O}}(T, Y[d])$. If we let $T = E \oplus X$, then, since S is the simple Γ-module at X, we conclude that $\operatorname{Hom}_{\mathcal{O}}(E, Y[d]) = 0$ and $\operatorname{Hom}_{\mathcal{O}}(X, Y[d]) \neq 0$. Therefore, $T' = E \oplus Y$ is a cluster-tilting object by [OT12, Theorem 5.7], and so T is mutable at X. $\square$

The following higher-dimensional version of [BMR07, Theorem 4.2] ("generalised APR tilting") also holds.

Theorem 4.5.3. *Let $T = E \oplus X$ and $T' = E \oplus Y$ be basic cluster-tilting objects in $\mathcal{O}$ with $X \ncong Y$. Let further $\Gamma = \operatorname{End}_{\mathcal{O}} T$ and $\Gamma' = \operatorname{End}_{\mathcal{O}} T'$ have respective cluster-tilting subcategories $\mathcal{M}$ and $\mathcal{M}'$, containing S_X and S_Y, the simples corresponding to X and Y. Then there is an equivalence $\mathcal{M}/\operatorname{add} S_X \cong \mathcal{M}'/\operatorname{add} S_Y$.*

Proof. This follows from noting that $\mathcal{M}/\operatorname{add} S_X$ is equivalent to $\mathcal{O}/(E \oplus X \oplus Y)$ via $\operatorname{Hom}_{\mathcal{O}}(T, -[d])$ and that $\mathcal{M}'/\operatorname{add} S_Y$ is equivalent to $\mathcal{O}/(E \oplus X \oplus Y)$ via $\operatorname{Hom}_{\mathcal{O}}(T', -[d])$. This is because, as explained above, $\operatorname{Hom}_{\mathcal{O}}(T, Y[d]) \cong S_X$ and $\operatorname{Hom}_{\mathcal{O}}(T', X[d]) \cong S_Y$. $\square$

Remark 4.5.4. Theorem 4.5.2 ought to hold for a general d-representation-finite algebra Λ. This would require a higher-dimensional generalisation of [Bua+06, Theorem 7.5], which would ensure that, in the situation of Theorem 4.5.1, $\operatorname{Hom}_{\mathcal{O}_\Lambda}(X, Y[d])$ was always a simple Γ-module.

Remark 4.5.5. Some may find it unsatisfying and artificial that, in order to conduct higher Auslander–Reiten theory, we have to restrict to a subcategory of the module category. Theorem 4.5.1 indicates why one ought not to take this perspective. In higher dimensions, it ought not to be possible to mutate cluster-tilting objects at every summand, just as it is not possible to mutate triangulations at every d-simplex. However, the presence of a given module in the d-cluster-tilting subcategory of the algebra will imply that certain mutations of the corresponding cluster-tilting object are possible. Hence, in order to reflect the fact that mutation is not everywhere possible in higher dimensions, we must restrict our attention to a proper subcategory.

Remark 4.5.6. The picture we extract from the restatement of [OT12, Theorem 5.6] above in terms of triangulations is as follows. Given a triangulation $\mathcal{T}$ corresponding to a cluster-tilting object T, we have an associated algebra $\Gamma := \operatorname{End}_{\mathcal{O}} T$. The indecomposables of the d-cluster-tilting subcategory $\mathcal{M}$ of $\operatorname{mod}\Gamma$ correspond to the internal d-simplices which are not in $\mathcal{T}$. Given an internal d-simplex A which is not in $\mathcal{T}$, the corresponding indecomposable module has composition factors according to the internal d-simplices of $\mathcal{T}$ which A intertwines with. In particular, if the indecomposable corresponding to A is simple, then one can perform a bistellar flip on $\mathcal{T}$ by exchanging some internal d-simplex B of $\mathcal{T}$ for A. Hence, the d-cluster-tilting subcategory gives information about how the d-simplices outside the triangulation interact with the triangulation.

Example 4.5.7. We illustrate how the mutation theory in this section works. We start with the cluster-tilting object T in $\mathcal{O} = \mathcal{O}_{A_3^2}$ shown in Figure 4.16. Let the endomorphism algebra endomorphism algebra of this cluster-tilting object be $\Gamma = \operatorname{End}_{\mathcal{O}} T$. The Gabriel quiver of Γ is also illustrated in Figure 4.16. Using the description of homomorphisms in $\mathcal{O}$ from Section 4.2, it can be seen that the relations for this algebra are given by the paths

$$1 \to 2 \to 3 \to 4,$$
$$2 \to 3 \to 4 \to 5,$$
$$4 \to 5 \to 6,$$
$$5 \to 6 \to 1,$$
$$6 \to 1 \to 2.$$

We can compute the 2-cluster-tilting subcategory of $\operatorname{mod}\Gamma$ using the restatement of [OT12, Theorem 5.6]. The result is illustrated in Figure 4.17. We take the indecomposable objects in $\mathcal{O}$ which do not lie in $\operatorname{add} T$ and then compute the composition factors of the corresponding module by seeing summands of T they have extensions with. For example, the object O_{247} has extensions with O_{135} and O_{136} and so corresponds to the module $\begin{smallmatrix}2\\1\end{smallmatrix}$. As explained in Theorem 4.5.2 the simple modules in the 2-cluster-tilting subcategory correspond to mutable summands of the corresponding cluster-tilting object. For example, the simple module 2 corresponds to the indecomposable object O_{257}, which only has extensions with O_{136} amongst the summands of T. Hence the summand O_{136} is mutable, since it can be exchanged for O_{257} to give a new cluster-tilting object.

Figure 4.16: A cluster-tilting object in $\mathcal{O}_{A_3^2}$ and the labelling of the corresponding quiver

$$\begin{tikzcd}[column sep=small, row sep=small]
& & O_{137} \arrow[dr] & & \\
& O_{136} \arrow[ur] & & O_{147} \arrow[dr] & \\
O_{135} \arrow[ur] & & O_{357} \arrow[ll] & & O_{157} \arrow[ll]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
& & 3 \arrow[dr] & & \\
& 2 \arrow[ur] & & 4 \arrow[dr] & \\
1 \arrow[ur] & & 6 \arrow[ll] & & 5 \arrow[ll]
\end{tikzcd}$$

Figure 4.17: The indecomposables of $\mathcal{O}/\operatorname{add} T$ and the 2-cluster-tilting subcategory of $\operatorname{mod}\Gamma$

$$\begin{tikzcd}[column sep=small, row sep=tiny]
& & O_{248} \arrow[ddr] & & \\
& & & & \\
& O_{247} \arrow[uur] \arrow[ddr] & & O_{258} \arrow[ddr] \arrow[ddddl] & \\
& & & & \\
O_{246} \arrow[uur] & & O_{257} \arrow[uur] & & O_{268} \arrow[ddddl] \\
& & & & \\
& & O_{358} \arrow[ddr] & & \\
& O_{146} \arrow[uuul] & & & \\
& & & O_{368} \arrow[ddl] & \\
& & & & \\
& & O_{468} \arrow[uuul] & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=tiny]
& & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[ddr] & & \\
& & & & \\
& \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[uur] \arrow[ddr] & & \begin{smallmatrix}4\\3\\2\end{smallmatrix} \arrow[ddr] \arrow[ddddl] & \\
& & & & \\
\begin{smallmatrix}1\\6\end{smallmatrix} \arrow[uur] & & 2 \arrow[uur] & & \begin{smallmatrix}5\\4\\3\end{smallmatrix} \arrow[ddddl] \\
& & & & \\
& & 4 \arrow[ddr] & & \\
& 6 \arrow[uuul] & & & \\
& & & \begin{smallmatrix}5\\4\end{smallmatrix} \arrow[ddl] & \\
& & & & \\
& & \begin{smallmatrix}6\\5\end{smallmatrix} \arrow[uuul] & &
\end{tikzcd}$$

Chapter 5

Quiver combinatorics for higher-dimensional triangulations

In this chapter, we return to study triangulations of cyclic polytopes themselves—in particular, triangulations of even-dimensional cyclic polytopes. We use tools that arise from Chapter 4, where we looked at the relation between triangulations of cyclic polytopes and the representation theory of algebras. Indeed, we associate quivers to even-dimensional cyclic polytopes via the endomorphism algebras of the corresponding cluster-tilting objects. We consider the information that these quivers encode about the triangulation. In Section 5.2, we show how one may describe the triangulations of even-dimensional cyclic polytopes which have no interior $(d+1)$-simplices. The upshot is that these triangulations correspond to the iterated d-APR tilts of [IO11], and an application is that the set of triangulations with no interior $(d+1)$-simplices is connected via bistellar flips. We define what an interior $(d+1)$-simplex is in Section 5.2. In Section 5.3, we explain how one may use the quivers associated to triangulations to identify the mutable d-arcs. This points towards what a theory of higher-dimensional quiver mutation could look like. Just as the quivers associated to polygon triangulations provide a prototype for classical cluster combinatorics, so the quivers associated to triangulations of even-dimensional cyclic polytopes ought to provide a prototype for higher cluster combinatorics.

5.1 The quiver of a triangulation

We first define the quiver of a triangulation, which is the higher-dimensional version of the quiver of a polygon triangulation—see, for instance, [FZ03b, Section 3] and [Wil14, Definition 2.12]. In this chapter we like to refer to internal d-simplices as *d-arcs*, since in $2d$ dimensions we think of internal d-simplices as the higher-dimensional analogues of the arcs of a triangulation.

Definition 5.1.1. Let $\mathcal{T}$ be a triangulation of $C(n+2d+1, 2d)$. We define the *quiver* $Q(\mathcal{T})$ of $\mathcal{T}$ to be the directed graph with vertices

$$Q_0(\mathcal{T}) = \mathring{e}(\mathcal{T})$$

and arrows

$$Q_1(\mathcal{T}) = \left\{ A \to B : \begin{array}{c} A \neq B, (A-\mathbf{1}) \between B, \text{ and } \nexists A' \in \mathring{e}(\mathcal{T}) \setminus \{A, B\} \\ \text{with } (A-\mathbf{1}) \between A' \text{ and } (A'-\mathbf{1}) \between B \end{array} \right\}.$$

We define the quiver in this way so that the arrows mirror the description of the homomorphisms in the cluster category $\mathcal{O}_{A_n^d}$, as discussed in Section 4.2.1. Hence the quiver coincides with the Gabriel quiver of the endmorphism algebra of the cluster-tilting object corresponding to the triangulation. However, in due course, we obtain a simpler description of the quiver, given in Corollary 5.1.4. In order to prove this, we first make some observations about cyclic polytopes.

Recalling the cyclically shifted order $<_l$ from Section 1.6, one may re-orientate the cyclic polytope $C(n+2d+1, 2d)$ by changing the ordering on $[n+2d+1]$

from $<_1$ to $<_l$ for some l [KW03], recalling these cyclically shifted orders from Section 1.6. Generalising [OT12], given a $2d$-simplex S of $C(n+2d+1, 2d)$ and an ordering $<_l$ of $[n+2d+1]$ under which S is ordered $S = \{s_0, s_1, \ldots, s_{2d}\}$, we write $e_l(S) = \{s_0, s_2, \ldots, s_{2d-2}, s_{2d}\}$. If the ordering is determined by the context, we simply write $e(S)$.

Lemma 5.1.2. *Given an ordering $<_l$ of $[n+2d+1]$ and a d-arc A of a triangulation $\mathcal{T}$ of $C(n+2d+1, 2d)$, there is a $2d$-simplex S of $\mathcal{T}$ such that $e_l(S) = A$.*

Proof. This follows from applying [OT12, Proposition 2.13] in the orientation given by $<_l$. □

Proposition 5.1.3. *Suppose $A \to B$ is an arrow in $Q(\mathcal{T})$. Then A and B share all but one entry, and there is a $(d+1)$-simplex J of $\mathcal{T}$ such that A and B are both faces of J.*

Proof. We have that $(A - \mathbf{1}) \between B$. It must be the case that $A = \{a_0, a_1, \ldots, a_d\}$ and $B = \{b_0, b_1, \ldots, b_d\}$ have at least one common vertex, otherwise $A \between B$. Hence, by re-ordering, we may assume that $a_0 = b_0$ and $a_{d-1} < a_d < b_d$. There must be a $2d$-simplex S of $\mathcal{T}$ such that $e(S) = B$ by Lemma 5.1.2. Let $S = \{b_0, q_1, b_1, q_2, \ldots, b_{d-1}, q_d, b_d\}$.

We cannot have that $b_{i-1} < q_i \leq a_i - 1$ for all $i \in [d]$, otherwise we have $A \between \{q_1, q_2, \ldots, q_d, b_d\}$, which is impossible since they are both d-arcs of $\mathcal{T}$. Hence there is an $i \in [d]$ such that $b_{i-1} < a_i - 1 < q_i$. Then we have that $\{b_0, b_1, \ldots, b_{i-1}, q_i, b_{i+1}, b_{i+2}, \ldots, b_d\}$ is a d-arc of $\mathcal{T}$, since $b_{i-1} < a_i - 1 < q_i$ and $q_i < b_i < b_{i+1}$. Moreover,

$$(A - \mathbf{1}) \between \{b_0, b_1, \ldots, b_{i-1}, q_i, b_{i+1}, b_{i+2}, \ldots, b_d\}$$

and

$$\{b_0, b_1, \ldots, b_{i-1}, q_i, b_{i+1}, b_{i+2}, \ldots, b_d\} - \mathbf{1} \between B.$$

Since $A \to B$ is an arrow in $Q(\mathcal{T})$, we must have that $A = \{b_0, b_1, \dots, b_{i-1}, q_i, b_{i+1}, \dots, b_d\}$. (In fact, by the ordering we have chosen, we must have $i = d$.) Therefore A and B are both faces of the $(d+1)$-simplex $J = \{b_0, b_1, \dots, b_{i-1}, q_i, b_i, b_{i+1}, \dots, b_d\}$, which is a $(d{+}1)$-face of S, and they share all but one entry, as desired. □

Recall the notation $\sigma_i(A)$ from Section 4.1.1. Note that this notation implicitly assumes an ordering $<_l$ of $[n+2d+1]$.

Corollary 5.1.4. *The arrows of $Q(\mathcal{T})$ are*

$$Q_1(\mathcal{T}) = \left\{ A \to \sigma_i^r(A) : \begin{array}{c} A, \sigma_i^r(A) \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}, \\ \nexists r' \in [r-1] \text{ such that } \sigma_i^{r'}(A) \in Q_0(\mathcal{T}) \end{array} \right\}.$$

For an arrow α, we denote the *head* $h(\alpha)$ and the *tail* $t(\alpha)$ such that $t(\alpha) \xrightarrow{\alpha} h(\alpha)$. We say that α is *incident* at $t(\alpha)$ and $h(\alpha)$. Given a quiver Q with vertices $A, B \in Q_0$, by a *path* in Q from A to B we mean a finite sequence of arrows $\alpha_1\alpha_2\dots\alpha_r$ such that $t(\alpha_1) = A$, $h(\alpha_r) = B$ and $h(\alpha_{i-1}) = t(\alpha_i)$ for all $i \in \{2, 3, \dots, s\}$. If there is a path from A to B, then we write $A \rightsquigarrow B$. The following property will also be useful later.

Lemma 5.1.5. *Given a triangulation $\mathcal{T}$ of $C(n+2d+1, 2d)$ and $A, B \in Q_0(\mathcal{T})$ with $(A - \mathbf{1}) \wr B$, there is a path from A to B in $Q(\mathcal{T})$.*

Proof. This is clear from Definition 5.1.1, using induction. □

The following property will be useful in Section 5.3.

Lemma 5.1.6. *Let $\mathcal{T}$ be a triangulation of $C(n+2d+1, 2d)$ with $A \in \mathring{e}(\mathcal{T})$. If $a_i + 2 < a_{i+1}$ for some i, then there is either an arrow*

$$\{a_0, a_1, \dots, a_i, a_{i+1}^-, a_{i+2}, a_{i+3}, \dots, a_d\} \to A$$

or an arrow

$$A \to \{a_0, a_1, \dots, a_{i-1}, a_i^+, a_{i+1}, a_{i+2}, \dots, a_d\},$$

where a_{i+1}^- and a_i^+ are some undetermined entries.

Proof. We use the fact that, by Lemma 5.1.2, there is a $2d$-simplex S of $\mathcal{T}$ such that $e(S) = A$. If we let $S = \{a_0, q_1, a_1, q_2, \dots, a_{d-1}, q_d, a_d\}$, then we must have either $q_{i+1} > a_i + 1$ or $q_{i+1} < a_{i+1} - 1$. In the former case, we must have a path $\{a_0, a_1, \dots, a_i, q_{i+1}, a_{i+2}, a_{i+3}, \dots, a_d\} \rightsquigarrow A$ comprised of arrows which increase the element in the position of q_{i+1}; in the latter case, we must have a path $A \rightsquigarrow \{a_0, a_1, \dots, a_{i-1}, q_i, a_{i+1}, a_{i+2}, \dots, a_d\}$ comprised of arrows which increase the element in the position of q_i. This establishes the claim. $\square$

5.2 Description of triangulations without interior $(d+1)$-simplices

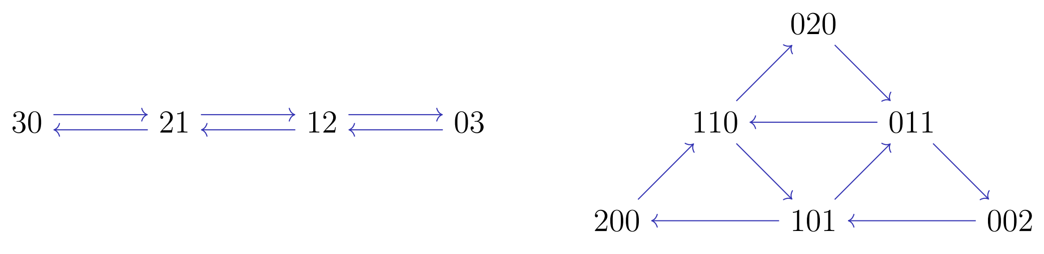
In this section we prove our combinatorial description of triangulations of $C(n + 2d + 1, 2d)$ without interior $(d+1)$-simplices and use this description to show that this class of triangulations is connected by bistellar flips. We call a $(d+1)$-simplex of a triangulation $\mathcal{T}$ of $C(n + 2d + 1, 2d)$ *interior* if all of its facets are internal d-simplices.

5.2.1 Cuts and slices

We now define the quivers which are higher analogues of orientations of the A_n Dynkin diagram, following [IO11]. Let $\overline{Q}^{(d,n)}$ be the quiver with vertices

$$\overline{Q}_0^{(d,n)} := \{ (a_0, a_1, \dots, a_d) \in \mathbb{Z}_{\geqslant 0}^{d+1} : \sum_{i=0}^{d} a_i = n - 1 \}$$

Figure 5.1: Examples of the quivers $\overline{Q}^{(d,n)}$



$\overline{Q}^{(1,4)}$ $\overline{Q}^{(2,3)}$

and arrows

$$\overline{Q}_1^{(d,n)} := \{\, A \to A + f_i : A, A + f_i \in \overline{Q}_0^{(d,n)} \,\},$$

where

$$f_i = (\ldots, 0, \overset{i}{-1}, \overset{i+1}{1}, 0, \ldots),$$

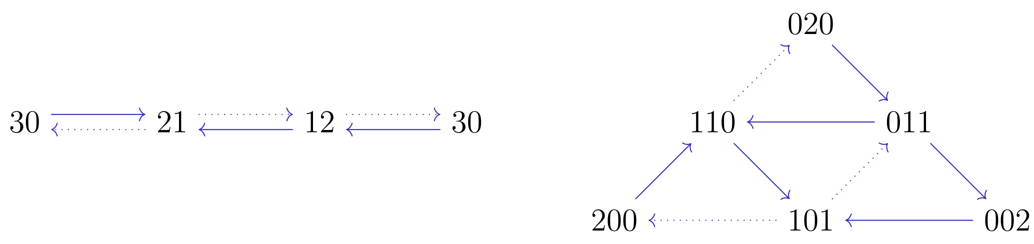
with

$$f_d = (1, 0, \ldots, 0, -1).$$

See Figure 5.1 for illustrations of these quivers. A subset $C \subseteq \overline{Q}_1^{(d,n)}$ is called a *cut* if it contains exactly one arrow from each $(d+1)$-cycle. Given a cut C, we write $\overline{Q}_C^{(d,n)}$ for the quiver with arrows $\overline{Q}_1^{(d,n)} \setminus C$ and refer to this as the *cut quiver*. Examples of cut quivers can be seen in Figure 5.2. Note that the cut quivers of $\overline{Q}^{(1,n)}$ are precisely the orientations of the A_n Dynkin diagram. Hence, for $d > 1$, we think of cut quivers of $\overline{Q}^{(d,n)}$ as higher analogues of orientations of the A_n Dynkin diagram. Note also that the arrows given by f_i give a cut C_i of $\overline{Q}^{(d,n)}$ and the cut quivers $\overline{Q}_{C_i}^{(d,n)}$ are isomorphic to the quivers $Q^{(d,n)}$ from Section 4.1.1.

Iyama and Oppermann show that cut quivers of $\overline{Q}^{(d,n)}$ are precisely the quivers than can be realised as "slices" of another family of quivers, denoted $\widetilde{Q}^{(d,n)}$, which

Figure 5.2: Cuts of the quivers $\overline{Q}^{(d,n)}$



$\overline{Q}_C^{(1,4)}$ $\overline{Q}_C^{(2,3)}$

we now define. Let $\widetilde{Q}^{(d,n)}$ be the quiver with vertices

$$\widetilde{Q}_0^{(d,n)} := \tilde{\mathbf{I}}_{n+2d+1}^d$$

and arrows

$$\widetilde{Q}_1^{(d,n)} := \{\, A \to \sigma_i(A) : A, \sigma_i(A) \in \widetilde{Q}_0^{(d,n)} \,\}.$$

Given two $(d+1)$-tuples $A, B \in \mathbb{Z}^{d+1}$, we write $A \leqslant B$ if $a_i \leqslant b_i$ for all $i \in \{0, 1, \ldots, d\}$. Note that, given $A, B \in \widetilde{Q}_0^{(d,n)}$, there is a path $A \rightsquigarrow B$ in $\widetilde{Q}^{(d,n)}$ if and only if $A \leqslant B$.

We denote by ν_d the automorphism of $\widetilde{Q}^{(d,n)}$ given by $A \mapsto A - \mathbf{1}$. We denote by $\pi \colon \widetilde{Q}_0^{(d,n)} \to {}^{\circlearrowleft}\mathbf{I}_{n+2d+1}^d$ the map given by

$$\widetilde{A} \mapsto \{\pi(\widetilde{a}_0), \pi(\widetilde{a}_1), \ldots, \pi(\widetilde{a}_d)\},$$

where $\pi(\widetilde{a}_i) := \widetilde{a}_i \mod n + 2d + 1$.

Remark 5.2.1. It can be seen from Section 4.1.1 that $\widetilde{Q}^{(d,n)}$ corresponds to the Auslander–Reiten quiver of the d-cluster-tilting subcategory $\mathcal{U}_{A_n^d}$ of the derived category $\mathcal{D}_{A_n^d}$. Moreover, the automorphism ν_d of $\widetilde{Q}^{(d,n)}$ corresponds to the derived Nakayama functor ν_d of $\mathcal{U}_{A_n^d}$. The projection π corresponds to the projection to the cluster category $\mathcal{O}_{A_n^d}$, as explained in Section 4.2.1.

Figure 5.3: $\widetilde{Q}^{(1,3)}$

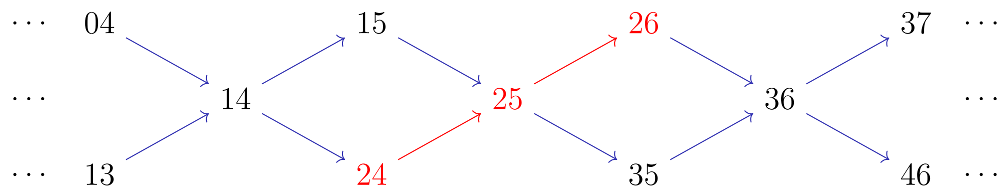
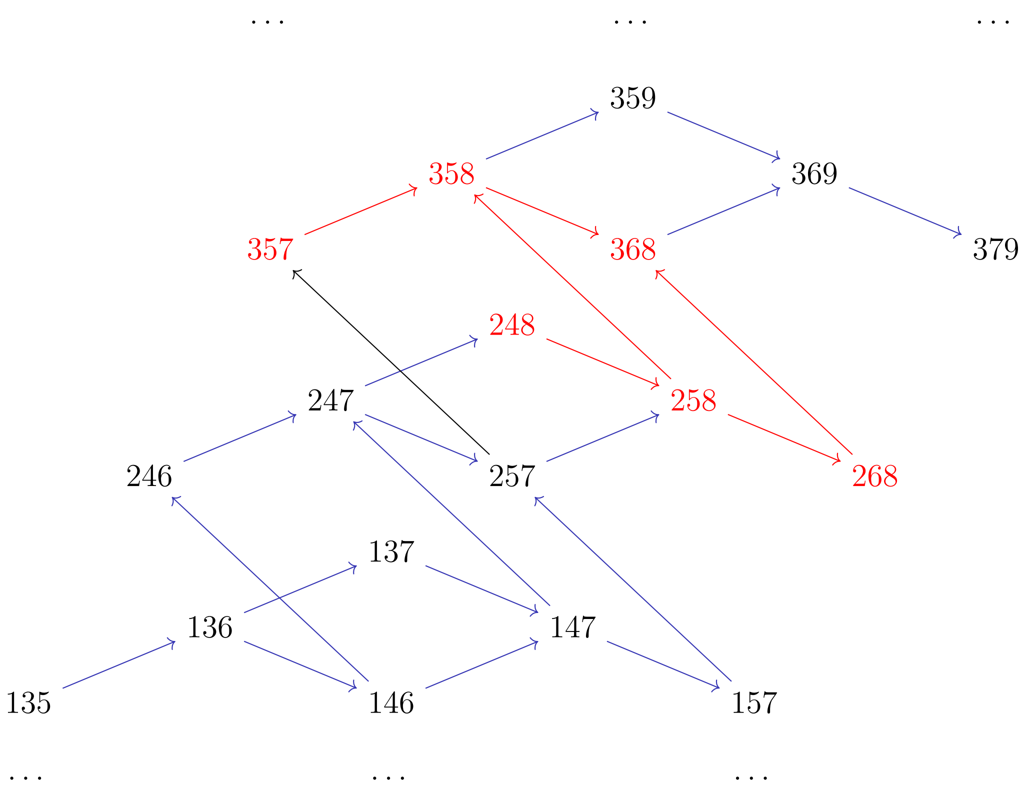


Figure 5.4: $\widetilde{Q}^{(2,3)}$



Following [IO11, Definition 5.20], we define a *slice* of $\widetilde{Q}^{(d,n)}$ to be a full subquiver L of $\widetilde{Q}^{(d,n)}$ such that:

(1) Any ν_d-orbit in $\widetilde{Q}^{(d,n)}$ contains precisely one vertex which belongs to L.

(2) L is convex, i.e., for any path p in $\widetilde{Q}^{(d,n)}$ connecting two vertices in L, all vertices appearing in p belong to L.

Slices are show in red in Figure 5.3 and Figure 5.4.

The ν_d-orbits of $\widetilde{Q}^{(d,n)}$ are in bijection with the vertices of $\overline{Q}^{(d,n)}$. Given a slice L of $\widetilde{Q}^{(d,n)}$, one can find a cut C_L of $\overline{Q}^{(d,n)}$ such that $Q^{(d,n)}_{C_L}$ is isomorphic to L with the arrows f_i of $Q^{(d,n)}_{C_L}$ corresponding to arrows $A \to \sigma_i(A)$ in L [IO11, Proposition 5.22]. For example, the cut quiver on the right of Figure 5.2 corresponds to the slice in Figure 5.4.

Slices correspond to iterated d-APR tilts [IO11, Theorem 4.15], which implies that projecting to the cluster category will give a triangulation [OT12, Theorem 6.4]. However, one can also give a direct combinatorial proof of this fact.

Proposition 5.2.2. *If L is a slice of $\widetilde{Q}^{(d,n)}$, then the vertices $\pi(L_0)$ give a triangulation of $C(n+2d+1, 2d)$.*

Proof. There are as many ν_d-orbits as there are elements of ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ containing 1, namely $\binom{n+d-1}{d}$. Suppose that there exist $\pi(A)$ and $\pi(B)$ in $\pi(L_0)$ with $\pi(A) \Taurus \pi(B)$. We assume without loss of generality that $a_0 < b_0$, noting that $\pi(A) \Taurus \pi(B)$ implies that $a_i \neq b_i$ for all i. We claim that $A < B$. Suppose for contradiction that $b_i < a_i$ for some i. We can choose the minimal i such that this is the case. Then $a_{i-1} < b_{i-1} < b_i < a_i$. Since we must have the cyclic ordering

$$\pi(a_{i-1}) < \pi(b_{i-1}) < \pi(a_i) < \pi(b_i),$$

we must have $a_i - a_{i-1} > n+2d+1$, and hence $a_d - a_0 > n+2d+1 > n+2d-1$, which contradicts $A \in \tilde{\mathbf{I}}^d_{n+2d+1}$.

Hence there is either a path $A \rightsquigarrow B$ in $\widetilde{Q}^{(d,n)}$ or a path $B \rightsquigarrow A$. Without loss of generality, we suppose that the path is $A \rightsquigarrow B$. Due to the intertwining, there is then a path $A \rightsquigarrow A + \mathbf{1} \rightsquigarrow B$, so $A + \mathbf{1} \in \pi(L_0)$ by convexity. But this contradicts the fact that L contains one vertex from every ν_d-orbit. Hence $\pi(L_0)$ is a non-intertwining subset of ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ of size $\binom{n+d-1}{d}$, and so gives a triangulation of $C(n + 2d + 1, 2d)$. □

A similar argument also shows the following lemma, which will be useful later.

Lemma 5.2.3. *If L is a convex subquiver of $\widetilde{Q}^{(d,n)}$ such that $\pi(L_0) = \mathring{e}(\mathcal{T})$ for a triangulation $\mathcal{T}$ of $C(n + 2d + 1, 2d)$, then L is a slice.*

Proof. Suppose that L is a convex subquiver such that $\pi(L_0)$ is a triangulation of $C(n + 2d + 1, 2d)$. Suppose for contradiction that L possesses two vertices A and B which are in the same ν_d-orbit. There is then either a path $A \rightsquigarrow B$ or a path $B \rightsquigarrow A$. Without loss of generality, we suppose the former. But then there is a path $A \rightsquigarrow A + \mathbf{1} \rightsquigarrow B$, so we must have $A + \mathbf{1} \in L_0$ by convexity. This is a contradiction, since $\pi(A)$ and $\pi(A + \mathbf{1})$ are intertwining. Note finally that since there are as many d-arcs of $\mathcal{T}$ as there are ν_d-orbits of $\widetilde{Q}^{(d,n)}$, there must be exactly one vertex of L per orbit, as required. □

Mutation of cuts and slices

Cuts and slices can be mutated, as was defined in [IO11].

- Let C be a cut of $\overline{Q}^{(d,n)}$ and let x be a source of $\overline{Q}^{(d,n)}_C$. Define a subset $\mu^+_x(C)$ of $\overline{Q}^{(d,n)}_1$ by removing all arrows in C which end at x and adding all arrows in $\overline{Q}^{(d,n)}_1$ which begin at x. Dually, if x is a sink of $\overline{Q}^{(d,n)}_C$, define $\mu^-_x(C)$ by removing all arrows in C which begin at x and adding all arrows in $\overline{Q}^{(d,n)}_1$ which end at x. By [IO11, Proposition 5.14], we have that $\mu^+_x(C)$ and $\mu^-_x(C)$ are also cuts of $\overline{Q}^{(d,n)}$.

- Let L be a slice of $\widetilde{Q}^{(d,n)}$. If x is a source of L, then define a full subquiver $\mu_x^+(S)$ of $\widetilde{Q}^{(d,n)}$ by removing x from L and adding $\nu_d^{-1}x$ [IO11, Definition 5.25]. Dually, if x is a sink of L, define a full subquiver $\mu_x^-(S)$ by removing x and adding $\nu_d x$.

If C_L is the cut corresponding to a slice L then $C_{\mu_x^+ L} = \mu_x^+(C_L)$ and $C_{\mu_x^- L} = \mu_x^-(C_L)$, provided x is a source or sink, respectively. Here we abuse notation by using x to refer both to the relevant vertex of L and to the relevant vertex of $\overline{Q}^{(d,n)}_{C_L}$.

Remark 5.2.4. Mutation of cut quivers is the higher-dimensional analogue of the quiver mutation from [FZ03c], but is here only defined at sinks and sources. However, it should also be possible to mutate these quivers at some vertices which are not sinks and sources. Mutation ought not to be possible at every single vertex though, just as mutation is not possible at every summand of a cluster-tilting object in a higher cluster category, as we saw in Section 4.5.

5.2.2 Quiver description

The first main result of this chapter is that a triangulation $\mathcal{T}$ of $C(n+2d+1, 2d)$ has no interior $(d+1)$-simplices if and only if its quiver $Q(\mathcal{T})$ is a cut of $\overline{Q}^{(d,n)}$, and this is the case if and only if its quiver $Q(\mathcal{T})$ has no cycle. We prove several properties of the quivers of triangulations without interior $(d+1)$-simplices. The eventual aim is to show that we can realise such a quiver as a slice of $\widetilde{Q}^{(d,n)}$.

Lemma 5.2.5. *If $\mathcal{T}$ is a triangulation of $C(n+2d+1, 2d)$ with no interior $(d+1)$-simplices, then the arrows in $Q(\mathcal{T})$ are all of the form $A \to \sigma_i(A)$.*

Proof. By Proposition 5.1.3, every arrow is of the form $A \to \sigma_i^r(A)$ for some $r > 0$ and for each such arrow, we have that $\{a_0, a_1, \ldots, a_i, a_i + r, a_{i+1}, a_{i+2}, \ldots, a_d\}$ is a face of a $2d$-simplex of $\mathcal{T}$. If $r > 1$, then this is an interior $(d+1)$-simplex. $\square$

Lemma 5.2.6. *Suppose that $\mathcal{T}$ is a triangulation of $C(n+2d+1, 2d)$ with no interior $(d+1)$-simplices. If $A \to \sigma_i(A) \to \sigma_j(\sigma_i(A))$ is a sequence of arrows in $Q(\mathcal{T})$, then, if $\sigma_j(A) \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$, the sequence $A \to \sigma_j(A) \to \sigma_j(\sigma_i(A))$ is also in $Q(\mathcal{T})$.*

Proof. We assume $\sigma_j(A) \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$. By re-ordering, we may assume that $i = d$. Then there is a $2d$-simplex S of $\mathcal{T}$ such that $e(S) = \sigma_d(\sigma_j(A))$ by Lemma 5.1.2.

Let $S = \{a_0, q_1, a_1, \dots, q_j, a_j + 1, q_{j+1}, a_{j+1}, \dots, q_d, a_d + 1\}$. If $q_d = a_d$, then $\sigma_j(A)$ is a d-face of S and hence a d-arc of $\mathcal{T}$. Hence, suppose for contradiction that $a_d \neq q_d$, so that $a_{d-1} < q_d \leqslant a_d - 1$. Note that if $q_j = a_{j-1} + 1$, then $\{a_0, a_1, \dots, a_d\} \wr \{q_1, q_2, \dots, q_d, a_d + 1\}$, which is a d-face of S. Hence $a_{j-1} + 2 \leqslant q_j < a_j + 1$. Then $\{a_0, a_1, \dots, a_{j-1}, q_j, q_{j+1} \dots, q_d, a_d + 1\}$ is a $(d+1)$-face of S and an interior $(d+1)$-simplex of $\mathcal{T}$, a contradiction. □

We now prove some properties of the quivers $\widetilde{Q}^{(d,n)}$, which will be useful in proving the main theorem of this section. We say that a full subquiver P of $\widetilde{Q}^{(d,n)}$ is *switching-closed* if whenever $A \to \sigma_i(A) \to \sigma_j(\sigma_i(A))$ is a sequence of arrows in P and $\sigma_j(A) \in \widetilde{Q}^{(d,n)}_0$, then the sequence of arrows $A \to \sigma_j(A) \to \sigma_j(\sigma_i(A))$ is also in P.

Lemma 5.2.7. *Let P be a switching-closed full subquiver of $\widetilde{Q}^{(d,n)}$. If there is a path $A \rightsquigarrow B$ in P, all other paths $A \rightsquigarrow B$ in $\widetilde{Q}^{(d,n)}$ must also lie in P.*

Proof. Suppose that we have a path $A \rightsquigarrow B$ in P. The length of all such paths is $\sum_{i=0}^{d}(b_i - a_i)$. We prove the claim by induction on this quantity. The base case, where the length is 1, follows from the fact that P is a full subquiver of $\widetilde{Q}^{(d,n)}$.

For the inductive step, we assume that the claim holds for all F and G with $\sum_{i=0}^{d}(g_i - f_i) < \sum_{i=0}^{d}(b_i - a_i)$. We have that B is the head of up to $d+1$ arrows, namely the ones with tails $\sigma_0^{-1}(B), \sigma_1^{-1}(B), \dots, \sigma_d^{-1}(B)$, provided these are vertices of $\widetilde{Q}^{(d,n)}$. Suppose that $\sigma_i^{-1}(B)$ is the penultimate vertex of our path $A \rightsquigarrow B$.

Choose one of the vertices $\sigma_j^{-1}(B) \in \widetilde{Q}_0^{(d,n)}$, where $i \neq j$. If $A \nleqslant \sigma_j^{-1}(B)$, then we may ignore this vertex, since there can be no paths from A to B through it. Hence we assume that $A \leqslant \sigma_j^{-1}(B)$. This implies that $A \leqslant \sigma_j^{-1}(\sigma_i^{-1}(B)) \leqslant \sigma_i^{-1}(B)$, so that $\sigma_j^{-1}(\sigma_i^{-1}(B)) \in P_0$ by the induction hypothesis applied to A and $\sigma_i^{-1}(B)$.

By the condition on P, we have that $\sigma_j^{-1}(B) \in P_0$, since $\sigma_j^{-1}(\sigma_i^{-1}(B))$, $\sigma_i^{-1}(B)$, $B \in P_0$. By the induction hypothesis, all paths $A \rightsquigarrow \sigma_j^{-1}(B)$ lie in P, and hence all paths $A \rightsquigarrow B$ passing through $\sigma_j^{-1}(B)$ lie in P. The result follows. $\square$

By a *walk* in P from A to B we mean a finite sequence of arrows $\beta_1\beta_2\dots\beta_s$ such that β_1 is incident at A, β_s is incident at B, and β_{i-1} and β_i are incident at a common vertex for all $i \in \{2, 3, \dots, s\}$. In this case, we write $A \dashrightarrow B$. That is, a path only consists of forwards arrows, but a walk may contain backwards arrows as well.

Lemma 5.2.8. *Let P be a connected switching-closed full subquiver of $\widetilde{Q}^{(d,n)}$. If $A, B \in P_0$ are such that there is a path $A \rightsquigarrow B$ in $\widetilde{Q}^{(d,n)}$, then there is a path $A \rightsquigarrow B$ in P.*

Proof. Let $A, B \in P_0$. Suppose that there is a path $A \rightsquigarrow B$ in $\widetilde{Q}^{(d,n)}$. There is certainly a walk $W \colon A \dashrightarrow B$ in P, since P is connected. We prove that there is also a path by induction on the number of backwards arrows in this walk. The base case, in which there are zero backwards arrows in the walk, is immediate.

Hence we suppose for induction that the claim holds for walks with fewer backwards arrows than W. We may assume that the final arrow in W is a backwards one, otherwise we may remove the final arrow and consider instead the walk $A \dashrightarrow B'$, where $A \dashrightarrow B' \to B$ is the original walk W. If there exists a path $A \rightsquigarrow B'$, then this gives a path $A \rightsquigarrow B$.

Therefore we can assume that our walk is of the form $A \dashrightarrow C \leftarrow B$, where $C \in P$. By the induction hypothesis, we can replace this with a walk of the form

$A \rightsquigarrow C \leftarrow B$ in P. Since we have a path $A \rightsquigarrow B$ in $\widetilde{Q}^{(d,n)}$, we have $A \leqslant B$ and, moreover, $A \leqslant B \leqslant C$. There is therefore a path $A \rightsquigarrow B \rightsquigarrow C$ in $\widetilde{Q}^{(d,n)}$. By Lemma 5.2.7, this path is in P, since every path $A \rightsquigarrow C$ is in P, which gives the desired path $A \rightsquigarrow B$ in P. $\square$

These two lemmas imply the following corollary.

Corollary 5.2.9. *Let P be a connected switching-closed full subquiver of $\widetilde{Q}^{(d,n)}$. Then P is convex in $\widetilde{Q}^{(d,n)}$.*

Proof. Suppose that P is a connected switching-closed full subquiver of $\widetilde{Q}^{(d,n)}$. Let $A, B \in P_0$ be such that there is a path $A \rightsquigarrow B$ in $\widetilde{Q}^{(d,n)}$. By Lemma 5.2.8, there is a path $A \rightsquigarrow B$ in P. Then, by Lemma 5.2.7, we have that all paths $A \rightsquigarrow B$ lie in P, and so P is convex. $\square$

This corollary is useful because it is easier to check the property of being connected and switching-closed than the property of being convex. Given a full subquiver P of $\widetilde{Q}^{(d,n)}$, we write $\overline{P}$ for the smallest switching-closed subquiver containing P. The subquiver $\overline{P}$ is well-defined since the intersection of a set of switching-closed full subquivers is switching-closed, so $\overline{P}$ may be constructed as the intersection of all switching-closed subquivers containing P.

Proposition 5.2.10. *A triangulation $\mathcal{T}$ contains an interior $(d+1)$-simplex if and only if $Q(\mathcal{T})$ contains a cycle.*

Proof. We first suppose for contradiction that $\mathcal{T}$ contains no interior $(d+1)$-simplices and that $Q(\mathcal{T})$ does contain a cycle. We can realise this cycle as a path $P\colon A \rightsquigarrow B$ in $\widetilde{Q}^{(d,n)}$, where A is some vertex in the cycle in $Q(\mathcal{T})$ and $\pi(B) = A$ with $A \neq B$. That is, we choose a vertex A in the cycle, and construct the path P by choosing A in $\widetilde{Q}^{(d,n)}$ and traversing the arrows in $\widetilde{Q}^{(d,n)}$ which correspond to the arrows of the cycle. Note that we know that the arrows of $Q(\mathcal{T})$ all correspond to arrows of $Q(\mathcal{T})$ by Lemma 5.2.5.

We consider this path P as a subquiver of $\widetilde{Q}^{(d,n)}$. By Lemma 5.2.7, every path $A \rightsquigarrow B$ in $\widetilde{Q}^{(d,n)}$ must lie in $\overline{P}$. There is a path $A \rightsquigarrow A+\mathbf{1} \rightsquigarrow B$ in $\widetilde{Q}^{(d,n)}$, since a cycle which starts and ends of A must end up increasing every entry of A. Hence $A+\mathbf{1}$ is a vertex of $\overline{P}$. By Lemma 5.2.6, if C is a vertex of $\overline{P}$, then $\pi(C)$ is a vertex of $Q(\mathcal{T})$. Therefore $\pi(A+\mathbf{1})$ is a vertex of $Q(\mathcal{T})$, but this is a contradiction, since $\pi(A+\mathbf{1})$ and A are intertwining.

We now suppose that $\mathcal{T}$ is a triangulation with $\{a_0, a_1, \dots, a_{d+1}\}$ an interior $(d+1)$-simplex of $\mathcal{T}$. Then $Q(\mathcal{T})$ has a cycle given by concatenating the paths $\{a_0, a_1, \dots, a_{d-1}, a_d\} \rightsquigarrow \{a_0, a_1, \dots, a_{d-1}, a_{d+1}\} \rightsquigarrow \{a_0, a_1, \dots, a_{d-2}, a_d, a_{d+1}\} \rightsquigarrow \cdots \rightsquigarrow \{a_1, a_2, \dots, a_d, a_{d+1}\} \rightsquigarrow \{a_0, a_1, \dots, a_d\}$, noting Lemma 5.1.5. □

Remark 5.2.11. For $d = 1$, an interior triangle gives a 3-cycle in the quiver where all the vertices of the cycle are edges of the triangle. For $d > 1$, the cycle obtained in Proposition 5.2.10 may not exclusively have facets of the interior $(d+1)$-simplex as its vertices. An example of this can be seen in Figure 5.6, where the interior 3-simplex is 1357, but the cycle is $135 \to 136 \to 137 \to 147 \to 157 \to 357 \to 135$. Here 136 and 147 are not faces of the interior 3-simplex 1357.

We are now ready to prove the main result of this section.

Theorem 5.2.12. *A triangulation $\mathcal{T}$ of $C(n+2d+1, 2d)$ has no interior $(d+1)$-simplices if and only if its quiver is a cut of $\overline{Q}^{(d,n)}$.*

Proof. First suppose that $\mathcal{T}$ has no interior $(d+1)$-simplices. We consider the full subquiver R of $\widetilde{Q}^{(d,n)}$ with vertices

$$R_0 = \{\, B \in \widetilde{Q}^{(d,n)} : \pi(B) \in Q_0(\mathcal{T}) \,\}.$$

We claim that this is disconnected and that each connected component gives $Q(\mathcal{T})$ by applying π. Let $A \in Q_0(\mathcal{T})$. If R is connected then it contains a walk $W\colon A \dashrightarrow$

$\{a_1, a_2, \dots, a_d, a_0+n+2d+1\}$. We consider W as a subquiver of $\widetilde{Q}^{(d,n)}$ and consider $\overline{W}$. By Lemma 5.2.8, $\overline{W}$ contains a path $P\colon A \rightsquigarrow \{a_1, a_2, \dots, a_d, a_0+n+2d+1\}$. By Lemma 5.2.6, if B is a vertex of $\overline{W}$, then $\pi(B)$ is a vertex of $Q(\mathcal{T})$. Hence all vertices of $\pi(P)$ are vertices of $Q(\mathcal{T})$, which therefore contains a cycle. But this contradicts Proposition 5.2.10.

By this argument, R is disconnected and, moreover, each connected component contains exactly one vertex which projects to each vertex of $Q(\mathcal{T})$ via π, since $Q(\mathcal{T})$ is connected. Moreover, the arrows in each connected component of R are the same as the arrows in $Q(\mathcal{T})$, by Lemma 5.2.5. Hence, by choosing one of the connected components, we obtain a full subquiver L of $\widetilde{Q}^{(d,n)}$ such that $\pi(L) = Q(\mathcal{T})$. We then have that L is a switching-closed connected subquiver of $\widetilde{Q}^{(d,n)}$ by Lemma 5.2.6, so L is convex by Lemma 5.2.9. Since $\pi(L)$ is a triangulation, it then follows from Lemma 5.2.3 that L is a slice. Hence $Q(\mathcal{T})$ is a cut of $\overline{Q}^{(d,n)}$ by [IO11, Theorem 5.24].

Now suppose that $Q(\mathcal{T})$ is a cut of $\overline{Q}^{(d,n)}$. Then $Q(\mathcal{T})$ cannot contain any cycles. This can be seen from [IO11, Theorem 5.24], which gives that cut quivers can be realised as slices. Since slices are full subquivers of $\widetilde{Q}^{(d,n)}$, which does not contain any cycles, $Q(\mathcal{T})$ cannot contain any cycles. Then we obtain that $\mathcal{T}$ contains no interior $(d+1)$-simplices by Proposition 5.2.10. □

Theorem 5.2.12 implies that the set of triangulations of $C(n+2d+1, 2d)$ without interior $(d+1)$-simplices is connected by bistellar flips.

Corollary 5.2.13. *The set of triangulations of $C(n+2d+1, 2d)$ without interior $(d+1)$-simplices is connected by bistellar flips.*

Proof. Slice mutation involves replacing one vertex of a slice L with another to obtain a new slice L'. Hence, if one considers the triangulations $\pi(L_0)$ and $\pi(L_0')$, these have all but one d-arc in common. Since two triangulations are related by a bistellar flip if and only if they have all but one d-arc in common by [OT12, Theorem 4.1] or Theorem 3.2.1, it follows that triangulations related by slice mutation are related by a bistellar flip. Iyama and Oppermann then show that all slices are connected by slice mutation [IO11, Theorem 5.27]. Hence, by Theorem 5.2.12, this implies that the set of triangulations without interior $(d+1)$-simplices is connected by bistellar flips. $\square$

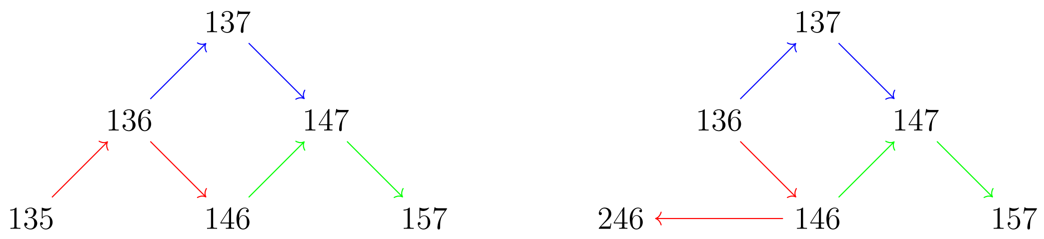
5.3 A combinatorial criterion for mutation

Given a triangulation $\mathcal{T}$ of $C(n+2d+1, 2d)$, we say that *a d-arc A of $\mathcal{T}$ is mutable* if there is a bistellar flip of $\mathcal{T}$ which replaces A with another d-arc A'. It is clear that here A and A' must intertwine, since the fact that A' is not in $\mathcal{T}$ means that it must form a circuit with a d-simplex of $\mathcal{T}$; but A' cannot intertwine any d-simplex of $\mathcal{T}$ apart from A, since it lies in a triangulation with them. For $d = 1$, where triangulations of $C(n + 2d + 1, 2d)$ are triangulations of convex m-gons, all d-arcs are mutable. But this is not true for $d > 1$. In this section, we prove a criterion for identifying the mutable d-arcs of a triangulation $\mathcal{T}$ from its quiver $Q(\mathcal{T})$. Just as mutating two-dimensional triangulations of polygons corresponds to Fomin–Zelevinsky quiver mutation [FZ03c], this ought to be related to a higher-dimensional version of quiver mutation.

5.3.1 Cuts and mutation

We begin with some motivating observations concerning cuts. We explain how a cut quiver may be decomposed into distinguished cut cycles, and observe that an arc of the triangulation is mutable if and only if it does not occur in the middle of a distinguished cut cycle. This will follow from the main result of the subsequent section.

Figure 5.5: Mutability via distinguished cut $(d+1)$-cycles



It is clear that the arrows of each $(d+1)$-cycle in $\overline{Q}^{(d,n)}$ must be labelled exactly once by each element of $\{f_0, f_1, \dots, f_d\}$. We call a $(d+1)$-cycle of $\overline{Q}^{(d,n)}$ *distinguished* if the arrows are labelled in the cyclic order $f_d < f_{d-1} < \cdots < f_0$. Given a cut C, the distinguished cut $(d+1)$-cycles of Q_C are the paths that result from removing the arrows of C from the distinguished cycles of $\overline{Q}^{(d,n)}$. Our observation is that, given a triangulation $\mathcal{T}$ whose quiver $Q(\mathcal{T})$ is a cut of $\overline{Q}^{(d,n)}$, the mutable d-arcs of $\mathcal{T}$ are precisely the d-arcs which do not lie in the middle of a distinguished cut $(d+1)$-cycle.

Example 5.3.1. The reader can check that in the left-hand triangulation in Figure 5.5 the mutable 2-arcs are 135, 146, and 157, whilst in the right-hand triangulation the mutable 2-arcs are 246, 136, and 157. In these figures we draw each distinguished cut 3-cycle in a different colour.

5.3.2 General triangulations

Cut quivers have a very particular form and it is this that allows us to determine the distinguished cut $(d+1)$-cycles of the quiver, and then to use these to determine the mutable d-arcs of the triangulation. In general, quivers may be much more complicated than cut quivers. Nevertheless, we may generalise our observation in the preceding section to arbitrary triangulations of even-dimensional cyclic polytopes using the following notion. We let $\mathcal{T}$ be a triangulation of $C(n+2d+1,2d)$.

Definition 5.3.2. Let A be a d-arc of $\mathcal{T}$. We call a path of the form

$$A \to \sigma_i^r(A) \to \sigma_{i-1}^s(\sigma_i^r(A))$$

retrograde at $\sigma_i^r(A)$ if $a_{i-1} < a_{i-1}+s < a_i$ cyclically.

A path

$$A_0 \to A_1 \to \cdots \to A_l \to A_{l+1}$$

is *retrograde* if $A_{i-1} \to A_i \to A_{i+1}$ is retrograde at i for all $i \in [l]$. We say that A_i is *in the middle* of this retrograde path for $i \in [l]$. We consider paths consisting of a single arrow to be trivially retrograde. We say that a retrograde path is *maximal* if it is not contained in any longer retrograde paths.

Remark 5.3.3. It is hence clear that the distinguished cut $(d+1)$-cycles from Section 5.3.1 are maximal retrograde paths. Arrows in this cut cycle are labelled $f_{l+1}, f_{l+2}, \ldots, f_d, f_0, f_1, \ldots, f_{l-1}$ for some l. Since arrows in $Q(\mathcal{T})$ labelled by f_i are of the form $A \to \sigma_i(A)$, we obtain that these paths are retrograde. These paths are, futhermore, maximally retrograde, since the next arrow in the path at either end would have to be labelled by f_l, but this is precisely the arrow that has been cut out.

Lemma 5.3.4. *Every arrow in $Q(\mathcal{T})$ is contained in a unique maximal retrograde path.*

Proof. If an arrow follows $A \to \sigma_i^r(A)$, then it must be of the form $\sigma_i^r(A) \to \sigma_j^s(\sigma_i^r(A))$ with $j \equiv i-1 \mod n+2d+1$ and if an arrow precedes it, then it must be of the form $\sigma_j^{-s}(A) \to A$ with $j \equiv i+1 \mod n+2d+1$. Both such arrows must be unique, by Corollary 5.1.4. Hence, there is only one way to extend an arrow to a maximal retrograde path. □

Proposition 5.3.5. *The maximal length of a retrograde path in $Q(\mathcal{T})$ is d.*

Proof. Suppose for contradiction that we have a retrograde path in $Q(\mathcal{T})$ of length $d+1$. By re-ordering, we can represent this in the form

$$A \to \{a_0, a_1, \dots, a_{d-1}, b_d\} \to \cdots \to \{a_0, b_1, b_2, \dots, b_d\} \to B.$$

Then we clearly have $a_i < b_i$ for all i. Furthermore, since this path is retrograde, we also have $b_{i-1} < a_i$ for all $i \in [d]$. But this implies that A and B are intertwining. $\square$

A d-arc A is mutable precisely if there exists a d-arc B which intertwines with it but which does not intertwine with any other d-arc in the triangulation. If such a d-arc B does not exist, then A is not mutable. Hence, we consider the collection of d-arcs of a triangulation which intertwine with a given d-arc outside the triangulation.

Lemma 5.3.6. *Let $B \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1} \setminus \mathring{e}(\mathcal{T})$. Let $Q_B(\mathcal{T})$ be the full subquiver of $Q(\mathcal{T})$ with vertex set*

$$Q_B(\mathcal{T})_0 = \{\, A \in \mathring{e}(\mathcal{T}) : A \wr B \,\}.$$

Then $Q_B(\mathcal{T})$ is connected.

Proof. Let $\mathcal{T}_B$ be the collection of $2d$-simplices of $\mathcal{T}$ which have a d-face intertwining with the d-arc B. Let $S \in \mathcal{T}_B$. We first show that the set of d-faces of S contained in $Q_B(\mathcal{T})$ is connected in $Q_B(\mathcal{T})$. Hence, let A and A' be two d-faces of S which intertwine with B. Then A and A' must have a common vertex, since they are both faces of the same $2d$-simplex, so without loss of generality we can assume that $a_0 = a'_0$. We know that A and B must be intertwining, so we may also assume that

$$a_0 < b_0 < a_1 < b_1 < \cdots < b_{d-1} < a_d < b_d.$$

Since $a_0 = a'_0$ and A' also intersects B, we also have that

$$a'_0 < b_0 < a'_1 < b_1 < \cdots < b_{d-1} < a'_d < b_d.$$

Let $c_i, c'_i \in \{a'_i, a_i\}$ be such that $b_{i-1} < c_i < c'_i < b_i$. Then $C, C' \in \mathring{e}(\mathcal{T})$ since they are both d-faces of S. Moreover, they are both in $Q_B(\mathcal{T})$. There is a path $C \rightsquigarrow A$ in $Q(\mathcal{T})$ by Lemma 5.1.5 since, by construction, $(C - \mathbf{1}) \wr A$. Moreover, this path must lie in $Q_B(\mathcal{T})$, since every d-arc between C and A must also be intertwining with B. There is likewise a path $C \rightsquigarrow A'$ in $Q_B(\mathcal{T})$. Therefore, A and A' are connected to each other in $Q_B(\mathcal{T})$. Hence, any two d-arcs lying in a common $2d$-simplex are connected by a walk in $Q_B(\mathcal{T})$.

We now show that the d-arcs in $Q_B(\mathcal{T})$ which lie in different $2d$-simplices are connected to each other. Let $S, S' \in \mathcal{T}_B$. If one chooses points $\mathbf{x} \in |S| \cap |B|$ and $\mathbf{x}' \in |S'| \cap |B|$, then the line segment $\overline{\mathbf{x}\mathbf{x}'}$ connecting $\mathbf{x}$ and $\mathbf{x}'$ must lie entirely within $|B|$, since $|B|$ is convex. If one travels from $|S|$ to $|S'|$ along $\overline{\mathbf{x}\mathbf{x}'}$, then one runs through a series of $2d$-simplices $|S| = |S_0|, |S_1|, \ldots, |S_r| = |S'|$ where each pair of $2d$-simplices $|S_{l-1}|$ and $|S_l|$ shares a common face $|U_l|$ which must also intersect $|B|$. Then, by the description of the circuits of $C(n + 2d + 1, 2d)$, there must be a d-arc J_l within U_l such that $B \wr J_l$. Therefore, the vertices of $Q_B(\mathcal{T})$ which are d-faces of S_{l-1} and the vertices of $Q_B(\mathcal{T})$ which are d-faces of S_l share the d-arc J_l. Since we know that the set of d-faces of a given $2d$-simplex is connected in $Q_B(\mathcal{T})$, we therefore obtain that $Q_B(\mathcal{T})$ itself is connected. □

Remark 5.3.7. Lemma 5.3.6 may also be seen quickly using an algebraic argument. The result [OT12, Theorem 5.6], which we discussed in Section 4.5, implies that $Q_B(\mathcal{T})$ must be the support of an indecomposable module, and so must be connected.

This gives the following useful corollary, which implies that in order to check whether a d-arc A is mutable to a d-arc B, it suffices only to check whether the

d-arcs adjacent to A in the quiver intertwine with B, rather than checking all d-arcs for whether they intertwine with B.

Corollary 5.3.8. *Let $A \in \mathcal{T}$ and $B \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ with $A \wr B$. If there is an $A' \in \mathring{e}(\mathcal{T})$ with $A' \neq A$ and $A' \wr B$, then there is an $A'' \in \mathring{e}(\mathcal{T})$ with $A \neq A''$ and $A'' \wr B$, such that A and A'' are adjacent in $Q(\mathcal{T})$.*

Proof. Suppose that we are in the situation described. We know from Lemma 5.3.6 that $Q_B(\mathcal{T})$ is connected, and the set-up gives us that it contains at least two vertices, one of which is A. Hence there is a vertex of $Q_B(\mathcal{T})$ which is adjacent to A. □

We may now prove the main theorem of this section.

Theorem 5.3.9. *Let $\mathcal{T}$ be a triangulation of $C(n+2d+1, 2d)$. Then, a d-arc of $\mathcal{T}$ is mutable if and only if it is not in the middle of a maximal retrograde path in $Q(\mathcal{T})$.*

Proof. For each entry a_i in A, let $Z_i = \{a_0, a_1, \dots, a_{i-1}, z_i, a_{i+1}, a_{i+2}, \dots, a_d\}$ be the d-arc of $\mathcal{T}$ such that there is an arrow $Z_i \to A$, if it exists. Similarly, let $B_i = \{a_0, a_1, \dots, a_{i-2}, b_{i-1}, a_i, a_{i+1}, \dots, a_d\}$ be the d-arc of $\mathcal{T}$ such that there is an arrow $A \to B_i$, if it exists.

A is mutable if and only if there exists $C \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ such that $A \wr C$ but such that there is no $A' \in \mathcal{T}$ with $A' \neq A$ with $A' \wr C$. By Corollary 5.3.8, it is necessary and sufficient that we do not have $Z_i \wr C$ or $B_i \wr C$ for any $i \in \{0, 1, \dots, d\}$. If $A \wr C$, then, since

$$a_0 < c_0 < a_1 < c_1 < \cdots < a_d < c_d,$$

we have that if Z_i and C do not intertwine, then $z_i \leqslant c_{i-1} < a_i$. Similarly, since B_i and C do not intertwine, we must have $a_{i-1} < c_{i-1} \leqslant b_{i-1}$.

Let $z_i \equiv a_{i-1} + 1 \mod n + 2d + 1$ if Z_i does not exist and let $b_{i-1} \equiv a_i - 1 \mod n + 2d + 1$ if B_i does not exist. Then, by the above reasoning, A is mutable if and only if $a_{i-1} < z_i \leqslant b_{i-1} < a_i$ for all i, which is precisely the condition that none of the $Z_i \to A \to B_i$ are retrograde paths. $\square$

Corollary 5.3.10. *Let $\mathcal{T}$ be a triangulation of $C(n + 2d + 1, 2d)$ and $A \in \mathring{e}(\mathcal{T})$ such that A is not in the middle of any maximal retrograde paths. Let Z_i and B_i be as in the proof of Theorem 5.3.9. Then $z_{i+1} = b_i$ for all i and $\{b_0, b_1, \ldots, b_d\}$ replaces A in the bistellar flip at A.*

Proof. We know from the proof of Theorem 5.3.9 that any element of

$$[z_1, b_0] \times [z_2, b_1] \times \cdots \times [z_d, b_{d-1}] \times [z_0, b_d]$$

may replace A in a bistellar flip. But, we have that the d-arc which can replace A in a bistellar flip must be unique, since it is determined by the subpolytope $C(2d + 2, 2d)$ which A lies in. Hence $z_{i+1} = b_i$ for all i and the unique element of the product must replace A in the bistellar flip. $\square$

Example 5.3.11. We provide examples of how one may use this criterion to identify the mutable d-arcs of a triangulation. We represent maximal retrograde paths using consecutive arrows of the same colour.

Considering the triangulation of $C(8, 4)$ given in Figure 5.6, the mutable 2-arcs are 136, 147, and 357. Compare Example 4.5.7.

Note that maximal retrograde paths are not always of length d. This is shown by the triangulation of $C(10, 6)$ given in Figure 5.7. (We use 'A' to denote 10.) The mutable d-arcs of this triangulation are 357A, 1368, and 1479.

One can also illustrate Corollary 5.3.10. Consider the d-arc 1368 in Figure 5.7. This is mutable by Theorem 5.3.9, so we can compute what d-arc it is exchanged for. Between 1 and 3 we must have 2 and between 6 and 8 we must

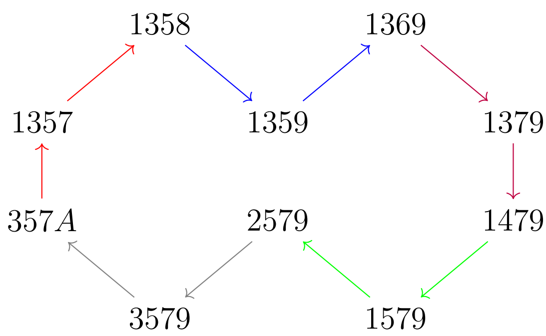
Figure 5.6: Triangulation of $C(8,4)$

$$\begin{tikzcd}
 & & 137 \arrow[dr] & & \\
 & 136 \arrow[ur] & & 147 \arrow[dr] & \\
135 \arrow[ur] & & 357 \arrow[ll] & & 157 \arrow[ll]
\end{tikzcd}$$

Figure 5.7: Triangulation of $C(10,6)$

$$\begin{tikzcd}
 & & 1359 \arrow[dr] & & \\
 & 1358 \arrow[ur] \arrow[dr] & & 1369 \arrow[dr] & \\
1357 \arrow[ur] & & 1368 \arrow[ur] & & 1379 \arrow[d] \\
357A \arrow[u] & & & & 1479 \arrow[dl] \\
 & 3579 \arrow[ul] & & 1579 \arrow[ll] &
\end{tikzcd}$$

Figure 5.8: Triangulation of $C(10, 6)$



have 7. Then, between 3 and 6 we must have 5 since 1368 is adjacent to 1358. Similarly, between 8 and 1 we must have 9, because 1368 is adjacent to 1369. Hence performing a bistellar flip at 1368 exchanges this d-arc for 2579. Observe that the retrograde-path analysis makes it easier to compute the bistellar flips of the triangulation. In the resulting triangulation, shown in Figure 5.8, none of the retrograde paths are of length d; they are all of length $d-1$.

5.3.3 Mutating cut quivers

There is a rule for mutating cut quivers at sinks and sources [IO11], as described in Section 5.2.1. In this section, we extend this rule to allow mutation at vertices which are not in the middle of retrograde paths, but which are not necessarily sinks or sources. The motivation here is to work towards a fully-fledged notion of higher-dimensional quiver mutation. In the case where the cut quiver is $Q(\mathcal{T})$ for a triangulation $\mathcal{T}$, we also describe the effect of the mutation on the triangulation $\mathcal{T}$.

For the following lemmas, we let C be a cut of $\overline{Q}^{(d,n)}$. The purpose of these lemmas is to describe the local structure of a cut quiver around a vertex which is not in the middle of a distinguished cut $(d+1)$-cycle. We then use our knowledge of this local structure to describe the effect of mutation at that vertex.

Convention 5.3.12. Given some fixed cyclically shifted ordering given by the context, we refer to arrows $\{a_0, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_d\} \to \{a_0, a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_d\}$ of some quiver $Q(\mathcal{T})$ and arrows of $\overline{Q}^{(d,n)}$ given by f_i as *arrows of type* i.

Lemma 5.3.13. *If a vertex* x *of* $\overline{Q}_C^{(d,n)}$ *is the source of one distinguished cut* $(d+1)$*-cycle and the sink of another distinguished cut* $(d+1)$*-cycle, then the arrows cut out of the two cycles are of the same type* i*.*

Proof. If the arrows cut out are of different types, then they form two consecutive arrows of a $(d+1)$-cycle, which therefore has two arrows cut out of it. But this is a contradiction, since a cut removes precisely one arrow from each $(d+1)$-cycle. □

Lemma 5.3.14. *If a vertex* x *of* $\overline{Q}_C^{(d,n)}$ *is neither a source nor a sink, nor in the middle of a distinguished cut* $(d+1)$*-cycle, then it is the head of precisely one arrow and the tail of precisely one arrow.*

Proof. Suppose that x is neither a source nor a sink, nor in the middle of a distinguished cut $(d+1)$-cycle. Then x is the head of at least one arrow α and the tail of at least one arrow β. Since x is not in the middle of a distinguished cut $(d+1)$-cycle, the arrows α' and β' succeeding α and preceding β in their distinguished cut $(d+1)$-cycles must respectively be cut out. By Lemma 5.3.13, α' and β' have the same type.

Suppose that we have another arrow γ such that x is the head of γ. Then the arrow γ' which succeeds it in the distinguished cut $(d+1)$-cycle must be cut out, since x is not in the middle of a distinguished cut $(d+1)$-cycle. But then by Lemma 5.3.13, γ', α', and β' all have the same type, so $\gamma = \alpha$. A similar argument can be made for an arrow δ with tail x. □

Lemma 5.3.15. *Let* $\mathcal{T}$ *be a triangulation of* $C(n + 2d + 1, 2d)$*. If a vertex* A *of* $Q(\mathcal{T})$ *is the head of precisely one arrow* α_d *and the tail of precisely one arrow* β_1*,* *and not in the middle of a distinguished cut $(d+1)$-cycle, then there is an a_i such that for $j \notin \{i, i-1\}$, $a_j + 2 = a_{j+1}$.*

Proof. The arrow α_d must be the final arrow in a distinguished cut $(d+1)$-cycle $\alpha_1\alpha_2\ldots\alpha_d$. The arrow β_1 must the first arrow in a distinguished cut $(d+1)$-cycle $\beta_1\beta_2\ldots\beta_d$. Let α_0 and β_0 be the respective arrows cut out of these $(d+1)$-cycles.

We know from Lemma 5.3.13 that α_0 and β_0 are of the same type. We let this type be i. Then β_1 has type $i-1$. The result then follows from Lemma 5.1.6, which implies that there would be more arrows incident to A if we didn't have $a_j + 2 = a_{j+1}$ for $j \notin \{i, i-1\}$. □

With these lemmas in place, we can now describe the effect of mutation on a cut quiver at a vertex which is not in the middle of a distinguished cut $(d+1)$-cycle, but is not a sink or a source. Since the description of mutation at sinks and sources is covered in [IO11], as described in Section 5.2.1, and we know that mutation at vertices in the middle of distinguished cut $(d+1)$-cycles is not possible by Theorem 5.3.9, this completes the description of mutation of cut quivers.

Proposition 5.3.16. *Let $\mathcal{T}$ be a triangulation of $C(n+2d+1, 2d)$ such that $Q(\mathcal{T})$ is isomorphic to $\overline{Q}_C^{(d,n)}$ for a cut C. Let A be a vertex of $Q(\mathcal{T})$ which is neither a sink nor a source, but is still not in the middle of any retrograde paths. Let $\mathcal{T}'$ be the result of performing a bistellar flip at A in $\mathcal{T}$. Then we have the following.*

(1) In the bistellar flip A is replaced by $\{a_0+1, a_1+1, \ldots, a_{i-1}+1, a_{i+1}-1, a_{i+1}+1, \ldots, a_d+1\}$, where i is the type of the arrow cut out of the distinguished cut $(d+1)$-cycles at A.

(2) $Q(\mathcal{T}')$ is obtained from $\mathcal{T}$ by removing from C the arrows beginning or ending at A and adding the arrows of $Q_1(\mathcal{T}) \setminus C$ which begin or end at A.

Proof. By Lemma 5.3.14 we have that A is the source of precisely one arrow and the sink of precisely one arrow. Moreover, by Lemma 5.3.15, the d-arc A is such that there is an a_i such that for $j \notin \{i, i-1\}$, we have $a_j + 2 = a_{j+1}$, where i is the type of the arrows cut out of the distinguished cut $(d+1)$-cycles incident at A. We assume, by re-ordering, that $i = 0$. Furthermore, the distinguished cut $(d+1)$-cycle beginning at A must look like

$$\{a_0, a_1, \dots, a_d\} \to \{a_0, a_1, \dots, a_{d-1}, a_d + 1\} \to \dots \to \{a_0, a_1 + 1, a_2 + 1, \dots, a_d + 1\}.$$

and the distinguished cut $(d+1)$-cycle ending at A must look like

$$\{a_0, a_1 - 1, a_2 - 1, \dots, a_d - 1\} \to \dots \to \{a_0, a_1 - 1, a_2, a_3, \dots, a_d\} \to \{a_0, a_1, \dots, a_d\}.$$

It follows from Corollary 5.3.10 that A mutates to

$$\{a_1 - 1, a_2 - 1, \dots, a_d - 1, a_d + 1\} = \{a_1 - 1, a_1 + 1, \dots, a_{d-1} + 1, a_d + 1\},$$

settling (1).

Then we have arrows in $Q(\mathcal{T}')$ given by

$$\{a_1 - 1, a_1 + 1, a_2 + 1, \dots, a_d + 1\} \leftarrow \{a_0, a_1 + 1, a_2 + 1, \dots, a_d + 1\},$$

and

$$\{a_0, a_1 - 1, a_2 - 1, \dots, a_d - 1\} \leftarrow \{a_d + 1, a_1 - 1, a_2 - 1, \dots, a_d - 1\},$$

since, by assumption, there are no arrows of type 0 ending at $\{a_0, a_1 + 1, a_2 + 1, \dots, a_d + 1\}$ or $\{a_d + 1, a_1 - 1, a_2 - 1, \dots, a_d - 1\}$. Indeed, these new arrows are precisely the arrows of C which begin or end at A. On the other hand, the arrows in $Q_1(\mathcal{T}) \setminus C$ beginning or ending at A are absent from $Q_1(\mathcal{T}')$. Since $a_j + 2 = a_{j+1}$ for $j \notin \{i, i-1\}$, there can be no other new arrows in $Q(\mathcal{T}')$, thus settling (2). □

Example 5.3.17. Proposition 5.3.16 can be verified by mutating the left-hand triangulation in Figure 5.5 to obtain the triangulation in Figure 5.6.

Chapter 6

The higher Bruhat orders

In this chapter, we consider the relation between the higher Stasheff–Tamari orders and the higher Bruhat orders of Manin and Schechtman [MS89]. Just as the higher Stasheff–Tamari orders are higher-dimensional versions of the Tamari lattice, the higher Bruhat orders are higher-dimensional versions of the weak Bruhat order on the symmetric group. In the paper [DM12], a poset was constructed as a quotient of the higher Bruhat orders, which the authors called the "higher Tamari orders". The authors conjectured that this poset coincided with the first higher Stasheff–Tamari orders. In this chapter, we prove this conjecture, showing how it can be understood in terms of a map g from the higher Bruhat orders to the higher Stasheff–Tamari orders. We explain how the conjecture that the higher Tamari orders are the same as the first higher Stasheff–Tamari orders is equivalent to the statement that this map g is surjective and full. The surjectivity of the map g is already known from [RS00, Theorem 3.5], but we give a new proof of this fact. We furthermore show that g is full, which proves the conjecture. These two facts about the map g mean that it is a quotient map of posets, as we explain in Section 6.6.

6.1 Another definition of the higher Stasheff–Tamari orders

It is easiest to describe the results of this chapter if we give a different conception of geometric triangulation to that of Section 2.1.3. As we discussed in Remark 2.1.2, under the conception of geometric triangulation used there, two different combinatorial triangulations may correspond to the same geometric triangulation. In this chapter, we give a conception of geometric triangulation which is as fine-grained as combinatorial triangulations. We need this since the relation between the higher Stasheff–Tamari orders and higher Bruhat orders we are concerned with is best expressed geometrically. We do not introduce this notion of geometric triangulation earlier since it is not the intuitive definition.

Recall that the cyclic polytope $\mathfrak{C}(m, \delta)$ is the convex hull of m points $p(t_1)$, $p(t_2)$, $\dots$, $p(t_m)$ on the moment curve $p(t) = (t, t^2, \dots, t^\delta)$ in $\mathbb{R}^\delta$, where $\{t_1, t_2, \dots, t_m\} \subseteq \mathbb{R}$. We fix the geometric realisation of the cyclic polytope $\mathfrak{C}(m, \delta)$ on the moment curve given by $t_i = i$. For $k \geqslant l$, we have a canonical projection map

$$\begin{aligned} \pi_{k,l} \colon \mathbb{R}^k &\to \mathbb{R}^l \\ (x_1, x_2, \dots, x_k) &\mapsto (x_1, x_2, \dots, x_l) \end{aligned}$$

which maps $\mathfrak{C}(m, k)$ to $\mathfrak{C}(m, l)$. In this chapter, a *(geometric) triangulation* of the cyclic polytope $\mathfrak{C}(m, \delta)$ is a union $\mathfrak{T}$ of a set of faces of $\mathfrak{C}(m, m-1)$ such that $\pi_{m-1,\delta} \colon \mathfrak{T} \to \mathfrak{C}(m, \delta)$ is a bijection.

These triangulations of $\mathfrak{C}(m, \delta)$ are in bijection with the combinatorial triangulations of $C(m, \delta)$ from Chapter 2. Namely, if $\mathcal{T}$ is a combinatorial triangulation of $C(m, \delta)$ with $|\mathcal{T}|_{m-1}$ the corresponding geometric simplicial complex in dimension

$m-1$, then

$$\bigcup |\mathcal{T}|_{m-1} = \bigcup_{|S|_{m-1} \in |\mathcal{T}|_{m-1}} |S|_{m-1}$$

is a geometric triangulation in the sense of this chapter. Conversely, given a triangulation $\mathfrak{T}$ of $\mathfrak{C}(m, \delta)$ in the sense of this chapter, we can let $\mathcal{T}$ be the set of combinatorial δ-faces F of $C(m, m-1)$ such that $|F|_{m-1} \subseteq \mathfrak{T}$. We extend notation and terminology to our new conception of triangulation in an intuitive way. For instance, if $\mathfrak{T} = \bigcup |\mathcal{T}|_{m-1}$ is a triangulation of $\mathfrak{C}(m, \delta)$, then

$$\mathring{e}(\mathfrak{T}) = \{ A \in \tbinom{[m]}{\lfloor \delta/2 \rfloor} : A \text{ is an internal } \lfloor \delta/2 \rfloor\text{-simplex of } \mathcal{T} \}$$

and

$$\hat{\mathfrak{T}} = \bigcup |\hat{\mathcal{T}}|_{m}.$$

When we refer to simplices of $\mathfrak{T}$, we mean faces of $\mathfrak{C}(m, m-1)$ contained in $\mathfrak{T}$.

Given two triangulations $\mathfrak{T}$ and $\mathfrak{T}'$ of $\mathfrak{C}(m, \delta)$, we say that $\mathfrak{T}'$ is an *increasing bistellar flip* of $\mathfrak{T}$ if there is a $(\delta+1)$-face $|S|$ of $\mathfrak{C}(m, m-1)$ such that $\mathfrak{T} \setminus |S| = \mathfrak{T}' \setminus |S|$ and $\pi_{m-1,\delta+1}(\mathfrak{T})$ contains the lower facets of $|S|_{\delta+1}$, whereas $\pi_{m-1,\delta+1}(\mathfrak{T}')$ contains the upper facets of $|S|_{\delta+1}$. One can then, similar to before, define the *higher Stasheff–Tamari poset* $\mathcal{S}(m, \delta)$ as having elements geometric triangulations of $\mathfrak{C}(m, \delta)$ with covering relations $\mathfrak{T} \lessdot \mathfrak{T}'$ whenever $\mathfrak{T}'$ is an increasing bistellar flip of $\mathfrak{T}$. This is how the higher Stasheff–Tamari orders are treated in [KV91; Tho03].

6.2 Three definitions of the higher Bruhat orders

We now give the definition of the higher Bruhat orders. The fundamental definition of the higher Bruhat orders for our purposes is the description in terms of cubillages of cyclic zonotopes given in [KV91] and formalised in [Tho03]. After giving this definition, we give the characterisation of cubillages of cyclic zonotopes established in [GP21] and studied in [DKK18a], and show how the higher Bruhat orders may be defined in these terms. Finally, we explain the original definition of the higher Bruhat orders from [MS89], which we will also need.

6.2.1 Cubillages

We first give the geometric description of the higher Bruhat orders due to [KV91; Tho03]. Consider the *Veronese curve* $\xi^\delta\colon \mathbb{R} \to \mathbb{R}^{\delta+1}$, given by $\xi^\delta_t = (1, t, \ldots, t^\delta)$. When δ is given by the context, we will write ξ instead of ξ^δ. Let $\{t_1, t_2, \ldots, t_m\} \subset \mathbb{R}$ with $t_1 < t_2 < \cdots < t_m$ and $m \geqslant \delta + 1$. The *cyclic zonotope* $\mathfrak{Z}(m, \delta + 1)$ is defined to be the Minkowski sum of the line segments

$$\overline{\mathbf{0}\xi_{t_1}} + \overline{\mathbf{0}\xi_{t_2}} + \cdots + \overline{\mathbf{0}\xi_{t_m}},$$

where $\mathbf{0}$ is the origin. Recall that for $\mathfrak{X}, \mathfrak{Y} \subseteq \mathbb{R}^\delta$, the *Minkowski sum* of $\mathfrak{X}$ and $\mathfrak{Y}$ is defined to be

$$\mathfrak{X} + \mathfrak{Y} = \{\, \mathbf{x} + \mathbf{y} : \mathbf{x} \in \mathfrak{X},\ \mathbf{y} \in \mathfrak{Y} \,\}.$$

The properties of the zonotope do not depend on the exact choice of the points $\{t_1, t_2, \ldots, t_m\} \subset \mathbb{R}$. Hence, for ease we set $t_i = i$, just as with our geometric realisation of the cyclic polytope in this chapter. As also with cyclic polytopes, we have that $\pi_{k,l}$ maps $\mathfrak{Z}(m, k)$ to $\mathfrak{Z}(m, l)$.

A *cubillage* $\mathfrak{Q}$ of $\mathfrak{Z}(m, \delta+1)$ is a union of faces of $\mathfrak{Z}(m, m)$ such that $\pi_{m,\delta+1}\colon \mathfrak{Q} \to \mathfrak{Z}(m, \delta + 1)$ is a bijection. Note that $\mathfrak{Q}$ therefore contains faces of $\mathfrak{Z}(m, m)$ of dimension at most $\delta + 1$. We call these $(\delta + 1)$-dimensional faces of $\mathfrak{Q}$ the *cubes* of the cubillage. In the literature, cubillages are often called *fine zonotopal tilings*—for example, in [GP21].

After [KV91, Theorem 4.4] and [Tho03, Theorem 2.1, Proposition 2.1] one may define the *higher Bruhat poset* $\mathcal{B}(m, \delta + 1)$ as follows. The elements of $\mathcal{B}(m, \delta + 1)$ consist of cubillages of $\mathfrak{Z}(m, \delta + 1)$. The covering relations of $\mathcal{B}(m, \delta + 1)$ are given by pairs of cubillages $\mathfrak{Q} \lessdot \mathfrak{Q}'$ where there is a $(\delta + 2)$-face $\mathfrak{F}$ of $\mathfrak{Z}(m, m)$

such that $\mathfrak{Q} \setminus \mathfrak{F} = \mathfrak{Q}' \setminus \mathfrak{F}$ and $\pi_{m,\delta+2}(\mathfrak{Q})$ contains the lower facets of $\pi_{m,\delta+2}(\mathfrak{F})$, whereas $\pi_{m,\delta+2}(\mathfrak{Q}')$ contains the upper facets of $\pi_{m,\delta+2}(\mathfrak{F})$. Here we say that $\mathfrak{Q}'$ is an *increasing flip* of $\mathfrak{Q}$.

The cyclic zonotope $\mathfrak{Z}(m, \delta + 1)$ possesses two canonical cubillages, one given by the union of faces $\mathfrak{Q}_l$ of $\mathfrak{Z}(m, m)$ such that $\pi_{m,\delta+2}(\mathfrak{Q}_l)$ consists of the lower facets of $\mathfrak{Z}(m, \delta + 2)$, which we call the *lower cubillage*, and the other given by the union of faces $\mathfrak{Q}_u$ of $\mathfrak{Z}(m, m)$ such that $\pi_{m,\delta+2}(\mathfrak{Q}_u)$ consists of the upper facets of $\mathfrak{Z}(m, \delta + 2)$, which we call the *upper cubillage*. The lower cubillage of $\mathfrak{Z}(m, \delta + 1)$ gives the unique minimum of the poset $\mathcal{B}(m, \delta + 1)$, and the upper cubillage gives the unique maximum. Upper and lower facets here are, of course, determined by the $(\delta + 2)$-th coordinate, in an analogous way to Section 2.1.2.

6.2.2 Separated collections

We now explain how one may characterise cubillages as separated collections of subsets, as shown in [GP21].

The subsets $E \subseteq [m]$ are naturally identified with the corresponding points $\xi_E := \sum_{e\in E} \xi_e$ in $\mathfrak{Z}(m, m)$, where $\xi_\varnothing := \mathbf{0}$. This represents each vertex of a cubillage $\mathfrak{Q}$ as a subset of $[m]$. For a cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m, \delta + 1)$, the collection of subsets corresponding to its vertices is called the *spectrum* of $\mathfrak{Q}$ and is denoted by $\mathrm{Sp}(\mathfrak{Q})$. Each cube in $\mathfrak{Q}$ is viewed as the Minkowski sum of line segments

$$\xi_E + \sum_{i=0}^{\delta} \overline{\xi_\varnothing \xi_{a_i}}$$

for some set A with $\#A = \delta + 1$ and $E \subseteq [m] \setminus A$. Here we call ξ_E the *initial vertex* of the cube, $\xi_{E\cup A}$ the *final vertex*, and A the set of *generating vectors*.

We say that, given two sets $A, B \subseteq [m]$, A *δ-interweaves* B if there exist $J \subseteq A \setminus B$ and $K \subseteq B \setminus A$ such that $J \wr K$. Here we say that J and K *witness* that A δ-interweaves B. If either A δ-interweaves B or B δ-interweaves A, then we say that A and B are *δ-interweaving*. If A δ-interweaves B as above and $A \setminus B = J$ and $B \setminus A = K$, then we say that A *tightly δ-interweaves* B, in the manner of [BBG20]. If A and B are not δ-interweaving then we say that A and B are *δ-separated*, following [GP21; DKK18a]. We call a collection $\mathcal{C} \subseteq 2^{[m]}$ δ-separated if it is pairwise δ-separated. If $\delta = 2d$, then being δ-interweaving is the same as being $(d+1)$-interlacing in the terminology of [BBG20] and $(d+1)$-intertwining in the terminology of [MW21].

It follows from [GP21, Theorem 2.7] that the correspondence $\mathfrak{Q} \mapsto \mathrm{Sp}(\mathfrak{Q})$ gives a bijection between the set of cubillages on $\mathfrak{Z}(m, \delta+1)$ and the set of δ-separated collections of maximal size in $2^{[m]}$. In particular, for any cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m, \delta+1)$, we have that $\#\mathrm{Sp}(\mathfrak{Q}) = \Sigma_{i=0}^{\delta+1}\binom{m}{i}$, which is the maximal size of a δ-separated collection in $2^{[m]}$.

Boundary vertices and internal vertices

For $A \subseteq [m]$, if $\xi_A^{\delta+1}$ is a boundary vertex of the zonotope $\mathfrak{Z}(m, \delta+1)$, then ξ_A^m is a vertex of every cubillage of $\mathfrak{Z}(m, \delta+1)$, and hence A is in every δ-separated collection in $2^{[m]}$ of maximal size. Moreover, the subsets $A \subseteq [m]$ such that $\xi_A^{\delta+1}$ is a boundary vertex of the zonotope $\mathfrak{Z}(m, \delta+1)$ are precisely those subsets which are δ-separated from every other subset of $[m]$. Hence the subsets of interest are those which project to the interior of the zonotope $\mathfrak{Z}(m, \delta+1)$, since these are the subsets that may be present in some cubillages but not others.

Lemma 6.2.1. *The number internal vertices in a cubillage of $\mathfrak{Z}(m, \delta+1)$ is $\binom{m-1}{\delta+1}$ if $m > \delta+1$, and 0 otherwise.*

Proof. The vertices of the zonotope $\mathfrak{Z}(m, \delta+1)$ are known to be in bijection with the number of regions of the arrangement of $(\delta-1)$-spheres associated with the set of points $\mathfrak{X} = \{\xi_1, \xi_2, \ldots, \xi_m\}$ on the Veronese curve—see [Bjö+99, Proposition 2.2.2]. Since no set of δ points of $\mathfrak{X}$ lie in a linear hyperplane, the number of regions of this arrangement of $(\delta-1)$-spheres is the maximal number of

$$\binom{m-1}{\delta}+\sum_{i=0}^{\delta}\binom{m}{i}.$$

(For instance, see [Com74, Problem 4, p.73].) Hence a cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m,\delta+1)$ has

$$\sum_{i=0}^{\delta+1}\binom{m}{i}-\left(\binom{m-1}{\delta}+\sum_{i=0}^{\delta}\binom{m}{i}\right)=\binom{m-1}{\delta+1}$$

vertices which project to the interior of $\mathfrak{Z}(m,\delta+1)$ if $m>\delta+1$, and 0 otherwise. $\square$

We call a point $\xi_A^{\delta+1}\in\mathbb{R}^{\delta+1}$ an *internal point in* $\mathfrak{Z}(m,\delta+1)$ if $\xi_A^{\delta+1}$ lies in the interior of $\mathfrak{Z}(m,\delta+1)$. We call a vertex ξ_A^m of a cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m,\delta+1)$ *internal* if $\xi_A^{\delta+1}$ is an internal point in $\mathfrak{Z}(m,\delta+1)$. Given a cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m,\delta+1)$, we define its *internal spectrum* $\mathrm{ISp}(\mathfrak{Q})$ to consist of the elements of $\mathrm{Sp}(\mathfrak{Q})$ which correspond to internal vertices of $\mathfrak{Q}$.

By [DKK18a, (2.7)], $\xi_A^{\delta+1}$ is an internal point in $\mathfrak{Z}(m,\delta+1)$ if and only if

- $\delta=2d$ and A is a cyclic l-ple interval for $l\geqslant d+1$, or
- $\delta=2d+1$ and A is an l-ple interval for $l\geqslant d+2$, or a $(d+1)$-ple interval containing neither 1 nor m.

Lemma 6.2.2. *Given $A\in\binom{[m]}{\lfloor\delta/2\rfloor+1}$, we have that $|A|$ is an internal $\lfloor\delta/2\rfloor$-simplex in $\mathfrak{C}(m,\delta)$ if and only if ξ_A is an internal point in $\mathfrak{Z}(m,\delta+1)$.*

Proof. By Section 2.2.1 and Lemma 2.2.4, given $A\in\binom{[m]}{\lfloor\delta/2\rfloor+1}$, we have that A is an internal $\lfloor\delta/2\rfloor$-simplex in $\mathfrak{C}(m,\delta)$ if

- $\delta=2d$ and A is a cyclic $(d+1)$-ple interval, or
- $\delta=2d+1$ and A is a $(d+1)$-ple interval containing neither 1 nor m.

Recall that $(d+1)$-ple intervals and cyclic $(d+1)$-ple intervals were defined in Section 1.6. The result then follows from comparing this with the criterion for ξ_A being an internal point. $\square$

The higher Bruhat orders in terms of separated collections

We now show how to interpret the covering relations of the higher Bruhat orders in this framework of separated collections.

Theorem 6.2.3. *Given cubillages $\mathcal{Q}, \mathcal{Q}'$ of $\mathcal{Z}(m, \delta+1)$ we have that $\mathcal{Q} \lessdot \mathcal{Q}'$ if and only if* $\mathrm{Sp}(\mathcal{Q}') = (\mathrm{Sp}(\mathcal{Q}) \setminus \{A\}) \cup \{B\}$, *where A δ-interweaves B. Moreover, in this case A tightly δ-interweaves B.*

Proof. The forwards direction follows from [DKK19a, Proposition 8.1]. Namely, if the increasing flip from $\mathcal{Q}$ to $\mathcal{Q}'$ is induced by the face $\mathfrak{F}$ of $\mathcal{Z}(m, m)$, then $\mathfrak{F}$ has a vertex ξ_A and a vertex ξ_B such that A tightly δ-interweaves B, the vertex $\xi_A^{\delta+2}$ is only contained in the lower facets of $\pi_{m,\delta+2}(\mathfrak{F})$, the vertex $\xi_B^{\delta+2}$ is only contained in the upper facets of $\pi_{m,\delta+2}(\mathfrak{F})$, and every other vertex of $\pi_{m,\delta+2}(\mathfrak{F})$ is contained in at least one lower facet and at least one upper facet. Hence, $\mathrm{Sp}(\mathcal{Q}') = (\mathrm{Sp}(\mathcal{Q}) \setminus \{A\}) \cup \{B\}$, where A tightly δ-interweaves B.

We now prove the backwards direction, supposing that $\mathrm{Sp}(\mathcal{Q}') = (\mathrm{Sp}(\mathcal{Q}) \setminus \{A\}) \cup \{B\}$, where A δ-interweaves B. Let $A' \subseteq A \setminus B$ and $B' \subseteq B \setminus A$ witness the fact that A δ-interweaves B.

We consider first the case where $\delta = 2d$. We begin by proving that $A' = A \setminus B$ and $B' = B \setminus A$, so that A tightly $2d$-interweaves B. The vertex ξ_A must be an internal vertex in the cubillage $\mathcal{Q}$, since subsets corresponding to boundary vertices are contained in every $2d$-separated collection. Therefore, ξ_A must be a vertex of at least two cubes in $\mathcal{Q}$, and so must have at least $2d+2$ edges emanating from it. The subsets at the other end of each of these edges must be $2d$-separated from B, so the edges must either add elements of B' or remove elements of A'. Since $\#A' \cup B' = 2d+2$, the edges emanating from ξ_A in $\mathcal{Q}$ must be precisely the edges which remove elements of A' and add elements of B'. Now suppose that there exists $a \in A \setminus (A' \cup B)$. Then $b'_{i-1} < a < b'_i$ for some $i \in \mathbb{Z}/(d+1)\mathbb{Z}$. But this implies that $A \setminus \{a'_i\}$ δ-interweaves B, which contradicts the fact that the edge from ξ_A to $\xi_{A\setminus\{a'_i\}}$ is in the cubillage $\mathfrak{Q}$. Hence $A' = A \setminus B$. The argument that $B' = B \setminus A$ is similar.

Therefore ξ_A is incident to $2d+2$ edges in the cubillage, where $d+1$ of the edges add elements of B' and $d+1$ of the edges remove elements of A'. The cubes with ξ_A as a vertex are generated by a choice of $2d+1$ of these edges. Hence, if $\mathfrak{P}$ is the set of cubes in $\mathfrak{Q}$ with ξ_A as a vertex, then $\mathfrak{P}$ is a set consisting of facets of a $(2d+2)$-face $\mathfrak{F}$ of $\mathfrak{Z}(m,m)$ which has initial vertex $\xi_{A\cap B}$ and which is generated by $A' \cup B'$. By [DKK19a, Proposition 8.1], $\pi_{m,\delta+2}(\mathfrak{P})$ gives the lower facets of $\pi_{m,\delta+2}(\mathfrak{F})$, since $\pi_{m,\delta+2}(\mathfrak{P})$ consists of all the facets of $\pi_{m,\delta+2}(\mathfrak{F})$ which contain $\xi_A^{\delta+2}$. If we let $\mathfrak{Q}''$ be the cubillage obtained by taking the increasing flip of $\mathfrak{Q}$ across $\mathfrak{F}$, then we obtain that $\mathrm{Sp}(\mathfrak{Q}'') = (\mathrm{Sp}(\mathfrak{Q}) \setminus \{A\}) \cup \{B\}$, since, likewise, the upper facets of $\pi_{m,\delta+2}(\mathfrak{F})$ are precisely those containing $\xi_B^{\delta+2}$. Since a cubillage is uniquely determined by its spectrum, we obtain that $\mathfrak{Q}'' = \mathfrak{Q}'$, and so we conclude that $\mathfrak{Q}'$ is an increasing flip of $\mathfrak{Q}$.

For $\delta = 2d+1$, the argument is similar. We deduce that ξ_A has $2d+3$ edges emanating from it in $\mathfrak{Q}$, $d+1$ of which remove elements of A' and $d+2$ of which add elements of B'. To show that $A' = A \setminus B$ and $B' = B \setminus A$, the only extra thing to consider is the possibility that we have $a \in A \setminus (A' \cup B)$ such that $a < b'_0$ or $a > b'_{d+1}$. But in the first instance here, we have that B δ-interweaves $A \cup \{b'_{d+1}\}$, since

$$a < b'_0 < a'_0 < b'_1 < \cdots < b'_d < a'_d.$$

But this is a contradiction, since we know that the edge from ξ_A to $\xi_{A\cup\{b'_{d+1}\}}$ is in $\mathfrak{Q}$. In the second instance, we have that B δ-interweaves $A \cup \{b'_0\}$, when we know that the edge from ξ_A to $\xi_{A\cup\{b'_0\}}$ is in $\mathfrak{Q}$. The remainder of the case where $\delta = 2d-1$ is similar. $\square$

In the setting of the above theorem, we say that (A, B) is the *exchange pair* of the flip and that we *exchange* A for B.

Operations on cubillages

We will also need the following concepts from [DKK18b]. Given a cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m, \delta+1)$ and a subset $\mathfrak{M}$ of $\mathfrak{Q}$, we say that $\mathfrak{M}$ is a *section* of $\mathfrak{Q}$ if $\mathfrak{M}$ is a cubillage of $\mathfrak{Z}(m, \delta)$. We say that an edge in a cubillage $\mathfrak{Q}$ from ξ_E to $\xi_{E\cup\{i\}}$ is an edge *of colour* i, where $E \subseteq [m]$ is any subset. For a cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m, \delta+1)$ and $i \in [m]$, we define the *i-pie* $\Pi_i(\mathfrak{Q})$ to be the subset of $\mathfrak{Q}$ given by all the cubes which have an edge of colour i.

By [DKK18a; GP21], we can obtain a cubillage $\mathfrak{Q}/i$ from $\mathfrak{Q}$ by contracting the edges of colour i until they have length zero. The cubillage $\mathfrak{Q}/i$ is known as the i-contraction of $\mathfrak{Q}$. The image of the m-pie $\Pi_m(\mathfrak{Q})$ is a section of $\mathfrak{Q}/m$, but this is not in general true for $1 < i < m$, by [DKK18a, (4.4)]. An example of 4-contraction is shown in Figure 6.1. Here the 4-pie is shown in red on the left-hand cubillage, and this is contracted to zero in the right-hand cubillage, where its image is a section. Note that here we are illustrating cubillages of $\mathfrak{Z}(4, 2)$ and $\mathfrak{Z}(3, 2)$ by their images under the projection maps $\pi_{4,2}$ and $\pi_{3,2}$ respectively. We will always illustrate cubillages in this way.

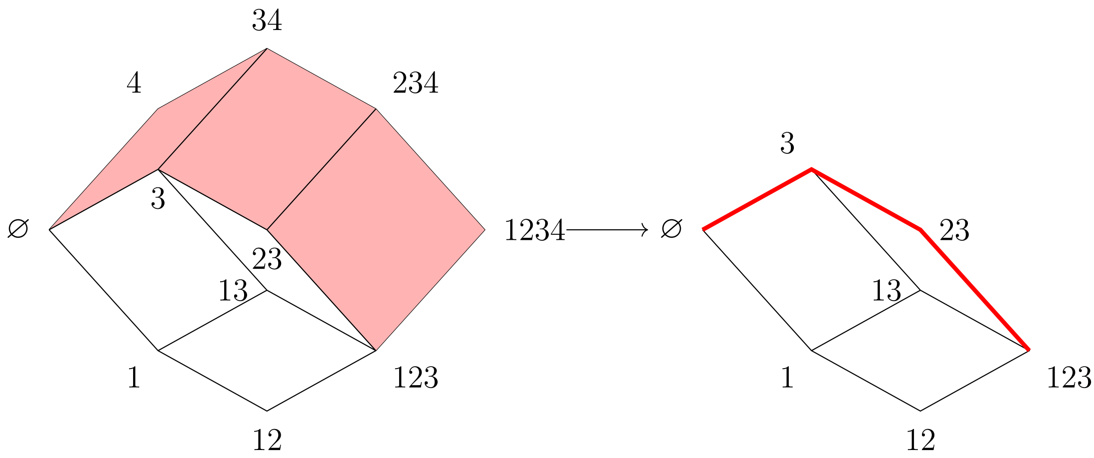
6.2.3 Admissible orders

The original definition of the higher Bruhat orders from [MS89] is as follows. Given $A \in \binom{[m]}{\delta+2}$, the set

$$P(A) = \left\{ B : B \in \binom{[m]}{\delta+1}, B \subset A \right\}$$

is called the *packet* of A. The set $P(A)$ is naturally ordered by the *lexicographic order*, where $A \setminus a_i < A \setminus a_j$ if and only if $j < i$.

Figure 6.1: 4-contraction



An ordering α of $\binom{[m]}{\delta+1}$ is *admissible* if the elements of any packet appear in either lexicographic or reverse-lexicographic order under α. Two orderings α and α' of $\binom{[m]}{\delta+1}$ are *equivalent* if they differ by a sequence of interchanges of pairs of adjacent elements that do not lie in a common packet. Note that these interchanges preserve admissibility. We use $[\alpha]$ to denote the equivalence class of α.

The *inversion set* $\mathrm{inv}(\alpha)$ of an admissible order α is the set of all elements of $\binom{[m]}{\delta+2}$ whose packets appear in reverse-lexicographic order in α. Note that inversion sets are well-defined on equivalence classes of admissible orders.

The higher Bruhat poset $\mathcal{B}(m, \delta+1)$ is the partial order on equivalence classes of admissible orders of $\binom{[m]}{\delta+1}$ with covering relations given by $[\alpha] \lessdot [\alpha']$ for $\mathrm{inv}(\alpha') = \mathrm{inv}(\alpha) \cup \{A\}$, where $A \in \binom{[m]}{\delta+2} \setminus \mathrm{inv}(\alpha)$.

Relation between admissible orders and cubillages

One can explain the bijection between cubillages of $\mathfrak{Z}(m, \delta+1)$ and admissible orders on $\binom{[m]}{\delta+1}$ as follows. Let $\mathfrak{Q}$ be a cubillage of $\mathfrak{Z}(m, \delta+1)$ corresponding to an equivalence class $[\alpha]$ of admissible orders on $\binom{[m]}{\delta+1}$. It follows from [Tho03] that the cubes of $\mathfrak{Q}$ are in bijection with the elements of $\binom{[m]}{\delta+1}$ via sending a cube to its set of generating vectors. A packet which can be inverted corresponds to a set of lower facets of $\pi_{m,\delta+2}(\mathfrak{F})$, where $\mathfrak{F}$ is a $(\delta+2)$-face $\mathfrak{F}$ of $\mathfrak{Z}(m,m)$. Inverting the packet corresponds to an increasing flip: exchanging the lower facets of $\pi_{m,\delta+2}(\mathfrak{F})$ for its upper facets.

Hence, a cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m,\delta+1)$ is determined once, for every element of $\binom{[m]}{\delta+1}$, one knows the initial vertex of the cube with that set of generating vectors. Let α be an admissible order of $\binom{[m]}{\delta+1}$ corresponding to a cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m,\delta+1)$ and let $\mathfrak{U}$ be the cube of $\mathfrak{Q}$ with set of generating vectors I and initial vertex ξ_E. Then, given $e \in [m] \setminus I$, we have that $e \in E$ if and only if either

- $I \cup \{e\} \notin \mathrm{inv}(\alpha)$ and e is an odd gap in I, or
- $I \cup \{e\} \in \mathrm{inv}(\alpha)$ and e is an even gap in I.

This follows from [Tho03, Theorem 2.1]. However, for $\delta+1$ odd, we use the opposite sign convention to the one used there. This makes the statement simpler and reveals connections with the paper [DM12], as we explain in Section 6.3.3.

Conversely, given a cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m,\delta+1)$, one can determine an equivalence class of admissible orders of $\binom{[m]}{\delta+1}$. Define a partial order on the cubes of the cubillage $\mathfrak{Q}$ by $\mathfrak{U} \lessdot \mathfrak{U}'$ if $\pi_{m,\delta+1}(\mathfrak{U}) \cap \pi_{m,\delta+1}(\mathfrak{U}')$ is an upper facet of $\pi_{m,\delta+1}(\mathfrak{U})$ and a lower facet of $\pi_{m,\delta+1}(\mathfrak{U}')$. The linear extensions of this partial order then comprise the admissible orders in the equivalence class $[\alpha]$ corresponding to $\mathfrak{Q}$, by [Tho03; MS89]. Compare this with the partial order on the simplices of a triangulation from Remark 2.1.3.

6.3 Interpretations of the map

In this section we define the map

$$g \colon \mathcal{B}(m, \delta + 1) \to \mathcal{S}(m, \delta)$$

and study it. We give three different definitions of this map, corresponding to the three different ways of defining the higher Bruhat orders.

6.3.1 Cubillages

Here we give our principal definition of the map g. This definition is geometric and uses the interpretation of $\mathcal{B}(m, \delta + 1)$ in terms of cubillages. This was how the map was considered in [DKK19a, Appendix B], where Lemma 6.3.1 and Proposition 6.3.3 were both noted.

Lemma 6.3.1. *If $\mathfrak{Q}$ is a cubillage of $\mathfrak{Z}(m, \delta + 1)$, then the vertex figure of $\mathfrak{Q}$ at $\xi_\varnothing$ gives a triangulation of $\mathfrak{C}(m, \delta)$.*

Proof. Let $\mathfrak{H}_k$ denote the affine hyperplane

$$\mathfrak{H}_k := \{\, (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : x_1 = 1 \,\}.$$

The vertex figure of the zonotope $\mathfrak{Z}(m, \delta+1)$ at the vertex $\xi_\varnothing$ can be given by the intersection $\mathfrak{Z}(m, \delta + 1) \cap \mathfrak{H}_{\delta+1}$. It is clear from the definitions of $\mathfrak{Z}(m, \delta + 1)$ and $\mathfrak{C}(m, \delta)$ that this intersection is the cyclic polytope $\mathfrak{C}(m, \delta)$. The vertex figure of the cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m, \delta + 1)$ at $\xi_\varnothing$ then induces a union of faces $\mathfrak{T} = \mathfrak{Q} \cap \mathfrak{H}_m$ of $\mathfrak{C}(m, m - 1)$. This subset $\mathfrak{T}$ is a triangulation of $\mathfrak{C}(m, \delta)$ because we have that $\pi_{m,\delta+1} \colon \mathfrak{Q} \to \mathfrak{Z}(m, \delta + 1)$ is a bijection, which then restricts to a bijection from $\mathfrak{Q} \cap \mathfrak{H}_m = \mathfrak{T}$ to $\mathfrak{Z}(m, \delta + 1) \cap \mathfrak{H}_{\delta+1} = \mathfrak{C}(m, \delta)$. $\square$

Hence we define the map

$$\begin{aligned} g\colon \mathcal{B}(m,\delta+1) &\to \mathcal{S}(m,\delta) \\ \mathfrak{Q} &\mapsto \mathfrak{Q}\cap\mathfrak{H}_m. \end{aligned}$$

For the purposes of this chapter, this is the definition of the map g, and the characterisations in Section 6.3.2 and Section 6.3.3 are simply other interpretations.

Remark 6.3.2. The intersections of cubillages with the hyperplanes given by $x_1 = l$ for $l \in [m-1]$ have been the objects of significant study in the literature. For three-dimensional zonotopes, such cross-sections are dual to plabic graphs [Gal18], which arise in the combinatorics associated to Grassmannians [Pos06; Pos18]. When the cubillage is regular, such graphs arise in the study of KP solitons [Hua15; KK21; GPW19], and it is this connection that lies behind the definition of the higher Tamari orders in [DM12]. The paper [OS19] studies hypersimplicial subdivisions and shows that, in general, only a subset of these come from cross-sections of subdivisions of zonotopes. This means that the analogues of the map g for cross-sections of cubillages given by $x_1 = l$ for $l \in \{2, 3, \dots, m-2\}$ are not generally surjective. In [DKK18b; DKK19b; DKK20], rather than studying the intersection of a cubillage with these hyperplanes, the fragmentation of a cubillage into different pieces cut by these hyperplanes is studied.

We identify the hyperplane $\mathfrak{H}_m$ with the space $\mathbb{R}^{m-1}$, so that we can consider $\mathfrak{C}(m, m-1)$ sitting inside it as usual.

Proposition 6.3.3. *If $\mathfrak{Q}, \mathfrak{Q}'$ are cubillages of $\mathfrak{Z}(m, \delta+1)$ such that $\mathfrak{Q} \lessdot \mathfrak{Q}'$, then either $g(\mathfrak{Q}) = g(\mathfrak{Q}')$ or $g(\mathfrak{Q}) \lessdot g(\mathfrak{Q}')$.*

Proof. Let $\mathfrak{Q}$ and $\mathfrak{Q}'$ be two cubillages such that $\mathfrak{Q} \lessdot \mathfrak{Q}'$. Let $\mathfrak{F}$ be the $(\delta+2)$-face of $\mathfrak{Z}(m,m)$ which induces the increasing flip, and let the initial vertex of $\mathfrak{F}$ be $\xi_E = (y_1, y_2, \dots, y_m)$. Then $\mathfrak{Q}$ and $\mathfrak{Q}'$ differ only in that $\pi_{m,\delta+2}(\mathfrak{Q})$ contains the lower facets of $\pi_{m,\delta+2}(\mathfrak{F})$ and $\pi_{m,\delta+2}(\mathfrak{Q}')$ contains the upper facets of $\pi_{m,\delta+2}(\mathfrak{F})$.

The intersection $\mathfrak{F} \cap \mathfrak{H}_m$ consists of more than a single point if and only if $E = \varnothing$. This is because, given $(x_1, x_2, \dots, x_m) \in \mathfrak{F}$, we have $x_1 \geqslant y_1 = \#E$. Hence if $\#E > 1$, then $\mathfrak{F} \cap \mathfrak{H}_m = \varnothing$; and if $\#E = 1$, then $\mathfrak{F} \cap \mathfrak{H}_m = \xi_E$. Thus if $E \neq \varnothing$, then $\mathfrak{Q}$ and $\mathfrak{Q}'$ both have the same intersection with the hyperplane $\mathfrak{H}_m$, so that $g(\mathfrak{Q}) = g(\mathfrak{Q}')$.

If $E = \varnothing$, then $\pi_{m,\delta+2}(\mathfrak{F}) \cap \mathfrak{H}_{\delta+2}$ is the $(\delta+1)$-simplex $|A|_{\delta+1}$, where A is the generating set of $\mathfrak{F}$. We then have that $g(\mathfrak{Q}) = \mathfrak{T}$ and $g(\mathfrak{Q}') = \mathfrak{T}'$ differ only in that $\pi_{m-1,\delta+1}(\mathfrak{T})$ contains the lower facets of $|A|_{\delta+1}$, whereas $\pi_{m-1,\delta+1}(\mathfrak{T}')$ contains the upper facets of $|A|_{\delta+1}$. Hence $g(\mathfrak{Q}) \lessdot g(\mathfrak{Q}')$. $\square$

Corollary 6.3.4. *The map $g\colon \mathcal{B}(m, \delta+1) \to \mathcal{S}(m, \delta)$ is order-preserving.*

Example 6.3.5. We now give two examples of taking the vertex figure of a cubillage of $\mathfrak{Z}(m, \delta+1)$ at $\xi_\varnothing$.

(1) First, consider the cubillage $\mathfrak{Q}_1$ of $\mathfrak{Z}(4, 2)$ shown in Figure 6.2. As we did above, we can find the vertex figure of $\mathfrak{Q}_1$ at $\xi_\varnothing$ by intersecting with the hyperplane $\mathfrak{H}_4$, as shown. We thus obtain the triangulation $g(\mathfrak{Q}_1) = \mathfrak{T}_1$ of $\mathfrak{C}(4, 1)$ shown in Figure 6.3.

(2) Secondly, consider the cubillage $\mathfrak{Q}_2$ of $\mathfrak{Z}(4, 3)$ illustrated in Figure 6.4. This cubillage possesses four cubes, two of which share the face highlighted in blue. The hyperplane $\mathfrak{H}_4$ is shown here in red. The intersection gives the triangulation $g(\mathfrak{Q}_2) = \mathfrak{T}_2$ of $\mathfrak{C}(4, 2)$ shown in Figure 6.5.

Remark 6.3.6. There is a dual version of the map g, given by

$$\begin{aligned} \overline{g}\colon \mathcal{B}(m, \delta+1) &\to \mathcal{S}(m, \delta) \\ \mathfrak{Q} &\mapsto \mathfrak{Q} \cap \overline{\mathfrak{H}}_m, \end{aligned}$$

Figure 6.2: The cubillage $\mathfrak{Q}_1$ of $\mathfrak{Z}(4,2)$ intersected with $\mathfrak{H}_4$

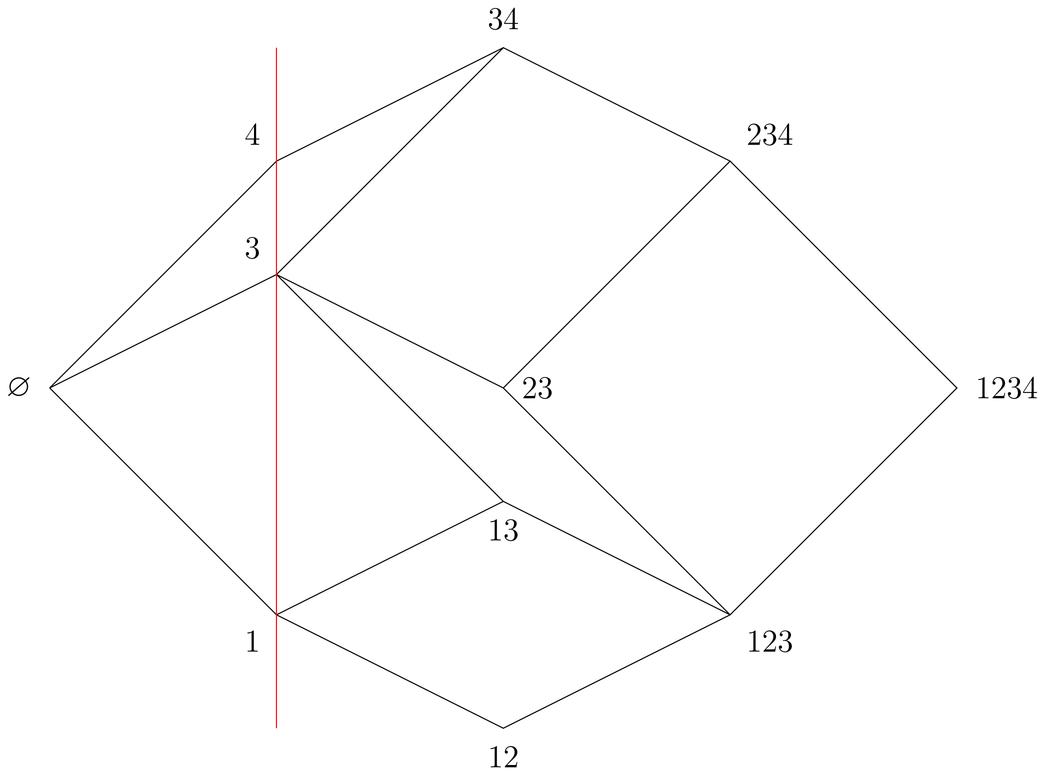


Figure 6.3: The triangulation $g(\mathfrak{Q}_1) = \mathfrak{T}_1$ of $\mathfrak{C}(4,1)$

Figure 6.4: The cubillage $\mathfrak{Q}_2$ of $\mathfrak{Z}(4,3)$

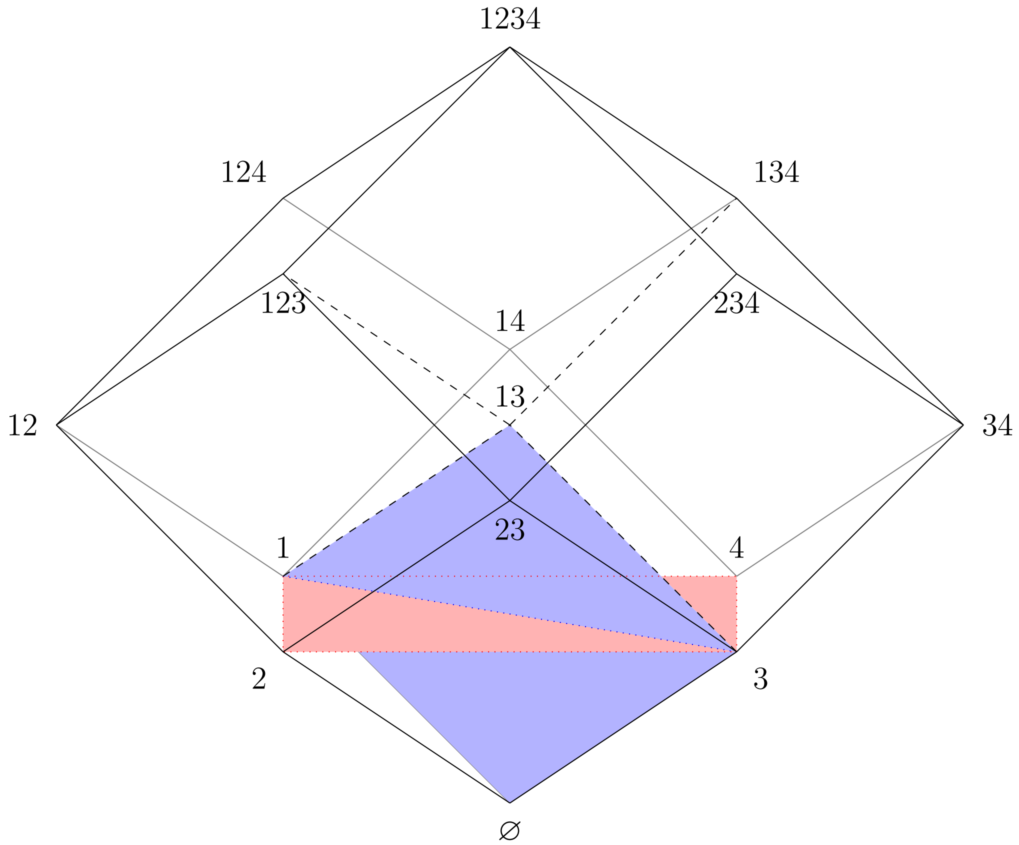
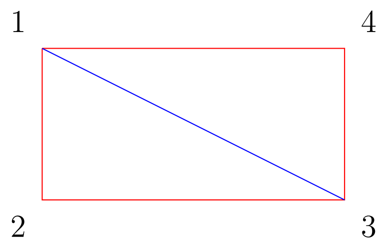


Figure 6.5: The triangulation $g(\mathfrak{Q}_2) = \mathfrak{T}_2$ of $\mathfrak{C}(4,2)$



where $\overline{\mathfrak{H}}_m = \{ (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : x_1 = m-1 \}$. Given a cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m, \delta+1)$, the triangulation $\overline{g}(\mathfrak{Q})$ is induced by taking the vertex figure of $\mathfrak{Z}(m, m)$ at the vertex $\xi_{[m]}$. This map was considered in [Tho03, Proposition 7.1]. The dual of Proposition 6.3.8 gives that if $\mathfrak{Q} \lessdot \mathfrak{Q}'$, then either $\overline{g}(\mathfrak{Q}) = \overline{g}(\mathfrak{Q}')$ or $\overline{g}(\mathfrak{Q}) \gtrdot \overline{g}(\mathfrak{Q}')$. Hence $\overline{g}$ is order-reversing. That is, if $\mathfrak{Q} \leqslant \mathfrak{Q}'$, then $g(\mathfrak{Q}) \geqslant g(\mathfrak{Q}')$.

6.3.2 Separated collections

Our second definition of the map uses the characterisation of cubillages in terms of separated collections and the combinatorial framework for triangulations of cyclic polytopes from Section 2.2. This is the framework we use to prove that g is a quotient map of posets in Section 6.5 and Section 6.6.

Given a triangulation $\mathfrak{T}$ of $\mathfrak{C}(m, \delta)$, let

$$\mathrm{Simp}(\mathfrak{T}) := \{ A \subseteq [m] : |A| \text{ is a simplex of } \mathfrak{T} \}.$$

The following lemma tells us how the value of $g(\mathfrak{Q})$ is determined by $\mathrm{Sp}(\mathfrak{Q})$.

Lemma 6.3.7. *Let $\mathfrak{Q}$ be a cubillage of $\mathfrak{Z}(m, \delta+1)$ and $\mathfrak{T}$ be a triangulation of $\mathfrak{C}(m, \delta)$. Then $g(\mathfrak{Q}) = \mathfrak{T}$ if and only if* $\mathrm{Sp}(\mathfrak{Q}) \supseteq \mathrm{Simp}(\mathfrak{T})$.

Proof. Suppose that $g(\mathfrak{Q}) = \mathfrak{T}$. Let $|A|$ be a δ-simplex of $\mathfrak{T}$. Then there is a cube $\mathfrak{U}$ of $\mathfrak{Q}$ such that $|A| = \mathfrak{U} \cap \mathfrak{H}_m$. We must have that the initial vertex of $\mathfrak{U}$ is $\xi_{\varnothing}$ and that the set of generating vectors is A. Thus if $|B|$ is a face of $|A|$, then ξ_B is a vertex of $\mathfrak{U}$, and hence $B \in \mathrm{Sp}(\mathfrak{Q})$. Since every simplex of the triangulation $\mathfrak{T}$ is a face of a δ-simplex, we have that $\mathrm{Sp}(\mathfrak{Q}) \supseteq \mathrm{Simp}(\mathfrak{T})$.

Conversely, suppose that $\mathrm{Sp}(\mathfrak{Q}) \supseteq \mathrm{Simp}(\mathfrak{T})$. Let $|A|$ be a δ-simplex of $\mathfrak{T}$. Then $2^A \subseteq \mathrm{Simp}(\mathfrak{T}) \subseteq \mathrm{Sp}(\mathfrak{Q})$. By [DKK18a, (2.5)], the cube $\mathfrak{U}$ with initial vertex $\varnothing$ and generating vectors A is therefore a cube of $\mathfrak{Q}$. This means that $|A|$ is a δ-simplex of $g(\mathfrak{Q})$, since $|A| = \mathfrak{U} \cap \mathfrak{H}_m$. Since this is true for any δ-simplex of $\mathfrak{T}$, we must have $g(\mathfrak{Q}) = \mathfrak{T}$. □

In fact, as the following proposition shows, we need only consider $\mathrm{ISp}(\mathfrak{Q}) \cap \binom{[m]}{\lfloor \delta/2 \rfloor+1}$ to know the value of $g(\mathfrak{Q})$.

Proposition 6.3.8. *Given a cubillage $\mathfrak{Q} \in \mathcal{B}(m, \delta+1)$, we have that $\mathring{e}(g(\mathfrak{Q})) = \mathrm{ISp}(\mathfrak{Q}) \cap \binom{[m]}{\lfloor \delta/2 \rfloor+1}$.*

Proof. It follows immediately from Lemma 6.3.7 that $\mathring{e}(g(\mathfrak{Q})) \subseteq \mathrm{ISp}(\mathfrak{Q}) \cap \binom{[m]}{\lfloor \delta/2 \rfloor+1}$, since if $\#A = \lfloor \delta/2 \rfloor + 1$, then $|A|$ is an internal $\lfloor \delta/2 \rfloor$-simplex in $\mathfrak{C}(m, \delta)$ if and only if ξ_A is an internal point in $\mathfrak{Z}(m, \delta+1)$, by Lemma 6.2.2.

To show that $\mathring{e}(g(\mathfrak{Q})) \supseteq \mathrm{ISp}(\mathfrak{Q}) \cap \binom{[m]}{\lfloor \delta/2 \rfloor+1}$, suppose that we have $A \in \left(\mathrm{ISp}(\mathfrak{Q}) \cap \binom{[m]}{\lfloor \delta/2 \rfloor+1}\right) \setminus \mathring{e}(g(\mathfrak{Q}))$. Then note that $|A|$ must be an internal $\lfloor \delta/2 \rfloor$-simplex in $\mathfrak{C}(m, \delta)$, since ξ_A is an internal point in $\mathfrak{Z}(m, \delta+1)$. However, $|A|$ is not a $\lfloor \delta/2 \rfloor$-simplex of $\mathfrak{T} = g(\mathfrak{Q})$, so there must be a simplex $|B|$ of $\mathfrak{T}$ such that (A, B) forms a circuit. This gives that A and B are δ-interweaving, which is a contradiction, since $B \in \mathrm{Sp}(\mathfrak{Q})$ by Lemma 6.3.7. □

Proposition 6.3.8 gives an interpretation of the map g in terms of separated collections. We know that a cubillage $\mathfrak{Q}$ of $\mathfrak{Z}(m, \delta+1)$ is determined by $\mathrm{ISp}(\mathfrak{Q})$, and likewise a triangulation $\mathfrak{T}$ of $\mathfrak{C}(m, \delta)$ is determined by $\mathring{e}(\mathfrak{T})$. Hence one could also define $g(\mathfrak{Q})$ to be the triangulation $\mathfrak{T}$ such that $\mathring{e}(\mathfrak{T}) = \mathrm{ISp}(\mathfrak{Q}) \cap \binom{[m]}{\lfloor \delta/2 \rfloor+1}$.

Example 6.3.9. We illustrate how to apply the interpretation of g from Proposition 6.3.8 to the cubillages from Example 6.3.5.

(1) Consider the internal spectrum of $\mathfrak{Q}_1$, as shown in Figure 6.2. We have $\mathrm{ISp}(\mathfrak{Q}_1) = \{3, 13, 23\}$, so $\mathrm{ISp}(\mathfrak{Q}_1) \cap \binom{[4]}{1} = \{3\}$. This implies that $\{3\} = \mathring{e}(g(\mathfrak{Q}_1)) = \mathring{e}(\mathfrak{T}_1)$, which is indeed the case. Note that having $\mathring{e}(\mathfrak{T}_1) = \{3\}$ defines $\mathfrak{T}_1$.

(2) Next, consider the internal spectrum of $\mathfrak{Q}_2$, as shown in Figure 6.4. We have $\mathrm{ISp}(\mathfrak{Q}_2) = \{13\}$, so $\mathrm{ISp}(\mathfrak{Q}_2) \cap \binom{[4]}{2} = \{13\}$. This implies that $\{13\} =$

$\mathring{e}(g(\mathfrak{Q}_2)) = \mathring{e}(\mathfrak{T}_2)$, which is indeed the case. Note that having $\mathring{e}(\mathfrak{T}_2) = \{13\}$ defines $\mathfrak{T}_2$.

Remark 6.3.10. The interpretation of $\overline{g}$ for separated collections is as follows. We have that $\overline{g}(\mathfrak{Q})$ is the triangulation $\mathfrak{T}$ such that

$$\mathring{e}(\mathfrak{T}) = \left\{ [m] \setminus A : A \in \mathrm{ISp}(\mathfrak{Q}) \cap \tbinom{[m]}{m-\lfloor \delta/2 \rfloor -1} \right\}.$$

6.3.3 Admissible orders

In this section we give a way of defining the map g while interpreting the elements of the higher Bruhat orders as equivalence classes of admissible orders. We use the following notions, which were used in [DM12] to define the higher Tamari orders.

Let α be an admissible order of $\tbinom{[m]}{\delta+1}$ and $I \in \tbinom{[m]}{\delta+1}$. Given $e \in [m] \setminus I$, we say that I is *invisible in* $P(I \cup \{e\})$ if either

- $I \cup \{e\} \notin \mathrm{inv}(\alpha)$ and e is an odd gap in I, or
- $I \cup \{e\} \in \mathrm{inv}(\alpha)$ and e is an even gap in I.

Otherwise, we say that I is *coinvisible in* $P(I \cup \{e\})$. (We note that I being invisible in $P(I \cup \{e\})$ is equivalent to e being *externally semi-active* with respect to I, in the terminology of [GPW19], which applies to more general matroids.)

Then:

- We say that I is *invisible in* α if there is a $e \in [m] \setminus I$ such that I is invisible in $P(I \cup \{e\})$.
- We say that I is *coinvisible in* α if there is a $e \in [m] \setminus I$ such that I is coinvisible in $P(I \cup \{e\})$.

- We say that I is *visible in* α if there is no $e \in [m] \setminus I$ such that I is invisible in $P(I \cup \{e\})$. (Note that this is not the same notion of visibility as in [DKK19a, Section 9].)

- We say that I is *covisible in* α if there is no $e \in [m] \setminus I$ such that I is coinvisible in $P(I \cup \{e\})$.

Given an admissible order α of $\binom{[m]}{\delta+1}$, we use $\mathcal{V}(\alpha)$ to denote the elements of $\binom{[m]}{\delta+1}$ which are visible in α and $\overline{\mathcal{V}}(\alpha)$ to denote the elements of $\binom{[m]}{\delta+1}$ which are covisible in α. (In [DM15], visible elements are labelled in blue; covisible elements are labelled in red; and elements which are neither visible nor covisible are labelled in green.)

Given an admissible order α of $\binom{[m]}{\delta+1}$, we write $\mathfrak{Q}_\alpha$ for the corresponding cubillage of $\mathfrak{Z}(m, \delta + 1)$.

Proposition 6.3.11. *Let α be an admissible order of $\binom{[m]}{\delta+1}$ and $I \in \binom{[m]}{\delta+1}$. Then the cube in $\mathfrak{Q}_\alpha$ with generating set I has initial vertex ξ_E, where*

$$E = \{\, e \in [m] \setminus I : I \text{ is invisible in } P(I \cup \{e\}) \,\}.$$

Proof. This follows immediately from the correspondence between admissible orders and cubillages in [Tho03], as described in Section 6.2.3. □

The following result was noted in [DKK19a, Appendix B].

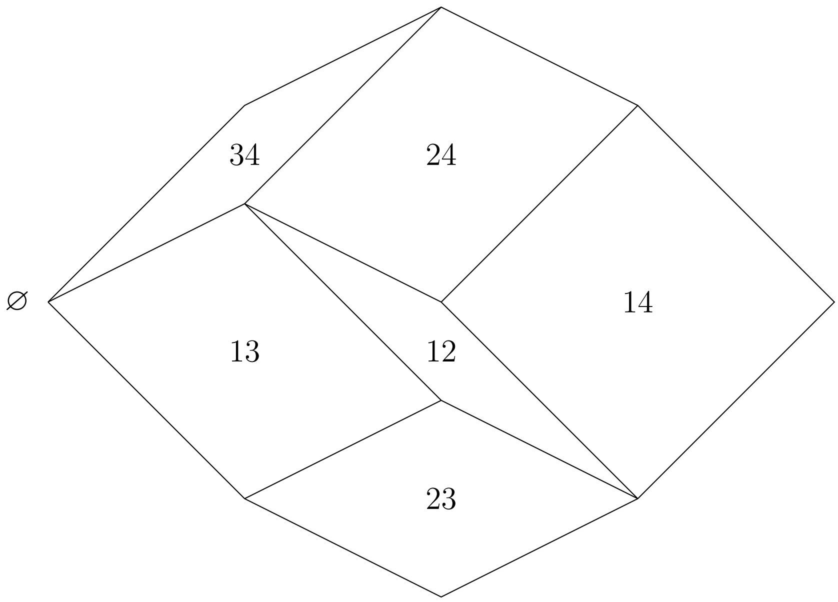
Corollary 6.3.12. *Let α be an admissible order of $\binom{[m]}{\delta+1}$ and $I \in \binom{[m]}{\delta+1}$. Then $I \in \mathcal{V}(\alpha)$ if and only if the cube in $\mathfrak{Q}_\alpha$ with generating set I has initial vertex $\xi_\varnothing$.*

This gives us yet another interpretation of the map g.

Corollary 6.3.13. *Given $[\alpha] \in \mathcal{B}(m, \delta+1)$, we have that $g(\mathfrak{Q}_\alpha)$ is the triangulation with*

$$\{\, |A| : A \in \mathcal{V}(\alpha) \,\}$$

Figure 6.6: $\mathfrak{Q}_1$ with its cubes labelled



as its set of δ-simplices.

Example 6.3.14. We continue from Example 6.3.5 and Example 6.3.9 and illustrate how the map g can also be characterised using visibility.

(1) We consider $\mathfrak{Q}_1$ first. By labelling the cubes of $\mathfrak{Q}_1$ with the elements of $\binom{[4]}{2}$, as shown in Figure 6.6, it can be seen that the admissible order corresponding to $\mathfrak{Q}_1$ is

$$\alpha_1 = \{23 < 13 < 12 < 14 < 24 < 34\}.$$

We compute that $\mathrm{inv}(\alpha_1) = \{123\}$.

We can then analyse which elements of $\binom{[4]}{2}$ are visible in α_1:

- 23: invisible because $123 \in \mathrm{inv}(\alpha_1)$ and 1 is an even gap in 23;
- 13: visible;
- 12: invisible because $123 \in \mathrm{inv}(\alpha_1)$ and 3 is an even gap in 12;

- 14: invisible because $124 \notin \mathrm{inv}(\alpha_1)$ and 2 is an odd gap in 14;
- 24: invisible because $234 \notin \mathrm{inv}(\alpha_1)$ and 3 is an odd gap in 24;
- 34: visible.

Note that, as Corollary 6.3.12 shows, 13 and 34 are precisely the cubes with $\xi_\varnothing$ as their initial vertex. Furthermore, as Corollary 6.3.13 shows, $g(\mathfrak{Q}_1) = \mathfrak{T}_1$ is the triangulation with 1-simplices $|13|$ and $|34|$.

(2) We now conduct the same analysis of $\mathfrak{Q}_2$. The admissible order corresponding to $\mathfrak{Q}_2$ is

$$\alpha_2 = \{123 < 124 < 134 < 234\}.$$

It is easy to see that $\mathrm{inv}(\alpha_2) = \varnothing$. Hence the visible elements of $\binom{[4]}{3}$ in α_2 are as follows:

- 123: visible;
- 124: invisible because $1234 \notin \mathrm{inv}(\alpha_2)$ and 3 is an odd gap in 124;
- 134: visible;
- 234: invisible because $1234 \notin \mathrm{inv}(\alpha_2)$ and 1 is an odd gap in 234.

Again, it can be seen in Figure 6.4 that 123 and 134 are precisely the cubes with $\xi_\varnothing$ as their initial vertex, as shown by Corollary 6.3.12. Moreover, as Corollary 6.3.13 shows, $g(\mathfrak{Q}_2) = \mathfrak{T}_2$ is the triangulation with 2-simplices $|123|$ and $|134|$.

The dual statements to Proposition 6.3.11, Corollary 6.3.12, and Corollay 6.3.13 are as follows.

Proposition 6.3.15. *Let α be an admissible order of $\binom{[m]}{\delta+1}$ and $I \in \binom{[m]}{\delta+1}$. Then the cube in $\mathfrak{Q}_\alpha$ with generating set I has final vertex ξ_F where*

$$F = [m] \setminus \{\, e \in [m] \setminus I : I \textit{ is coinvisible in } P(I \cup \{e\}) \,\}.$$

Corollary 6.3.16. *Let α be an admissible order of $\binom{[m]}{\delta+1}$ and $I \in \binom{[m]}{\delta+1}$. Then $I \in \overline{\mathcal{V}}(\alpha)$ if and only if the cube in $\mathfrak{Q}_\alpha$ with generating set I has final vertex $\xi_{[m]}$.*

Corollary 6.3.17. *Given $[\alpha] \in \mathcal{B}(m, \delta{+}1)$, we have that $\overline{g}(\mathfrak{Q}_\alpha)$ is the triangulation with*

$$\{\, |A| : A \in \overline{\mathcal{V}}(\alpha) \,\}$$

as its set of δ-simplices.

6.4 Quotient maps of posets

Dimakis and Müller-Hoissen use the definition of the map g from Section 6.3.3 to define the higher Tamari orders. We restate their definition in the framework of quotient posets. In this section, we explain our approach to this notion.

Given a poset $(X, \leqslant)$ subject to an equivalence relation $\sim$, the *quotient* $(X/{\sim}, R)$ is defined to be the set of $\sim$-equivalence classes $[x]$ of X, with the binary relation R defined by $[x]R[y]$ if and only if there exist $x' \in [x]$ and $y' \in [y]$ such that $x' \leqslant y'$. The quotient of a poset is in general only a reflexive binary relation, not necessarily a partial order, since the relation R may not be transitive or anti-symmetric.

Previous authors have considered various different conditions on the equivalence relation $\sim$ which are sufficient to guarantee that the quotient $X/{\sim}$ is a poset. Stanley considers the case where $\sim$ is given by the orbits of a group of automorphisms [Sta84; Sta91]. Two similar notions of congruence which also preserve lattice-theoretic properties are considered by Chajda and Snášel, and Reading [CS98; Rea02]. Most recently, Hallam and Sagan [HS15; Hal17] consider another notion of quotient in order to study the characteristic polynomials of lattices.

Whilst these conditions are sufficient to guarantee that the quotient poset is well-defined, none of them are necessary. In this chapter we are interested only in the minimal conditions which provide that the quotient poset is well-defined, and not in whether the quotient also preserves other properties. These necessary and sufficient conditions are as follows.

Proposition 6.4.1. *The quotient $X/{\sim}$ is a poset if and only if*

(1) if there exist $x_1 \sim x$ and $y_1 \sim y$ such that $x_1 \leqslant y_1$, and $x_2 \sim x$ and $y_2 \sim y$ such that $x_2 \geqslant y_2$, then $x \sim y$, and

(2) given $x, y, z \in X$ such that there exist $x_1 \sim x$ and $y_1 \sim y$ such that $x_1 \leqslant y_1$, and $y_2 \sim y$ and $z_2 \sim z$ such that $y_2 \leqslant z_2$, then there exist $x_3 \sim x$ and $z_3 \sim z$ such that $x_3 \leqslant z_3$.

Proof. Condition (1) is equivalent to the relation R being anti-symmetric. Condition (2) is equivalent to the relation R being transitive. □

If both condition (1) and condition (2) hold, then we write $\leqslant$ instead of R, to acknowledge that the relation gives us a partial order. In this case, we say that $\sim$ is a *weak order congruence* on the poset X. Note that, in particular, the congruences considered in [Rea02; CS98; HS15; Hal17] are weak order congruences.

If $\sim$ is a weak order congruence, so that $X/{\sim}$ is a poset, then we have a canonical order-preserving map

$$X \to X/{\sim}$$
$$x \mapsto [x].$$

Indeed, for any order-preserving map of posets $f \colon X \to Y$, one can consider the equivalence relation on X defined by $x \sim x'$ if and only if $f(x) = f(x')$. We then define the *image* of f to be the quotient $f(X) = X/{\sim}$. We identify the $\sim$-equivalence class $[x]$ of X with the element $f(x) \in Y$, so that $f(X) \subseteq Y$ and the quotient relation on $f(X)$ is a subrelation of the partial order on Y. If the equivalence relation $\sim$ on X is a weak order congruence, so that the image $f(X)$ is a well-defined poset, then we say that the map f is *photogenic*.

We say that a map $f\colon X \to Y$ is *full* if whenever $f(x_1) \leqslant f(x_2)$ in Y, there exist $x_1', x_2' \in X$ such that $x_1' \leqslant x_2'$, with $f(x_1') = f(x_1)$ and $f(x_2') = f(x_2)$. (In [CS98], maps which are full and order-preserving are called *strong*.)

Proposition 6.4.2. *Let X and Y be posets with $f\colon X \to Y$ an order-preserving map. Then the relation on $f(X)$ is anti-symmetric. Furthermore, if f is full, then the relation on $f(X)$ is transitive, and so f is photogenic. Finally, $f(X) = Y$ as posets if and only if f is surjective and full.*

Proof. Suppose that $x_1, x_2 \in X$ are such that $[x_1]R[x_2]$ and $[x_2]R[x_1]$, where, as before, we use R to denote the relation on the quotient poset. Since f is order-preserving, this implies that $f(x_1) \leqslant f(x_2)$ and $f(x_2) \leqslant f(x_1)$. Hence $f(x_1) = f(x_2)$ and so $x_1 \sim x_2$. Thus R is anti-symmetric.

Now suppose that f is full. Let $x_1, x_2, x_3 \in X$ be such that $[x_1]R[x_2]$ and $[x_2]R[x_3]$. This implies that $f(x_1) \leqslant f(x_2)$ and $f(x_2) \leqslant f(x_3)$, since f is order-preserving. Hence $f(x_1) \leqslant f(x_3)$. Since f is full, there exist $x_1', x_3' \in X$ such that $x_1' \leqslant x_3'$, with $f(x_1') = f(x_1)$ and $f(x_3') = f(x_3)$. Hence $[x_1]R[x_3]$, and so R is transitive.

Finally, it is clear that $f(X) = Y$ as sets if and only if f is surjective. Then f being full and order-preserving is equivalent to having $[x_1] \leqslant [x_2]$ in $f(X)$ if and only if $f(x_1) \leqslant f(x_2)$ in Y. We conclude that $f(X) = Y$ as posets if and only if f is surjective and full. $\square$

Therefore, every quotient of a poset by a weak order congruence gives an order-preserving map which is surjective and full, and, conversely, every order-preserving map which is surjective and full gives a quotient by a weak order congruence. Hence, if an order-preserving map f is surjective and full, then we say that f is a *quotient map of posets.*

With this framework in mind, the *higher Tamari order* $\mathcal{T}(m, \delta + 1)$ [DM12] is defined to be the image of the map $g\colon \mathcal{B}(m, \delta + 1) \to \mathcal{S}(m, \delta)$, or, explicitly, the quotient of $\mathcal{B}(m, \delta + 1)$ by the relation defined by $\mathfrak{Q} \sim \mathfrak{Q}'$ if and only if $g(\mathfrak{Q}) = g(\mathfrak{Q}')$. That this is equivalent to [DM12, Definition 4.7] follows from Corollary 6.3.13. Note that it is not evident that $\mathcal{T}(m, \delta + 1)$ is a well-defined poset, since it is not clear that the map g is photogenic. However, in Section 6.6 we shall prove that g is full, which implies that g is photogenic by Proposition 6.4.2, since we already know that g is order-preserving by Corollary 6.3.4. In Section 6.5, we give a new proof of the fact that g is surjective, originally known from [RS00, Theorem 3.5]. Therefore, the results of the two subsequent sections entail the following theorem.

Theorem 6.4.3. *The map $g\colon \mathcal{B}(m, \delta + 1) \to \mathcal{S}(m, \delta)$ is a quotient map of posets.*

Hence, we obtain by Proposition 6.4.2 that the higher Tamari orders are indeed the same posets as the first higher Stasheff–Tamari orders.

Corollary 6.4.4. $\mathcal{T}(m, \delta + 1) \cong \mathcal{S}(m, \delta)$.

6.5 Surjectivity

We now give a new construction showing that the map g is a surjection. Our strategy is to explicitly show that g is a surjection when δ is even, and then to use this to deduce the case where δ is odd. This parallels the strategy in Section 3.3.4. Given a triangulation $\mathfrak{T}$ of $\mathfrak{C}(m, 2d)$, we will construct a cubillage $\mathfrak{Q}_{\mathfrak{T}}$ of $\mathfrak{Z}(m, 2d+1)$ such that $g(\mathfrak{Q}_{\mathfrak{T}}) = \mathfrak{T}$. We will define $\mathfrak{Q}_{\mathfrak{T}}$ by specifying its internal spectrum.

Convention 6.5.1. In this section and in Section 6.6, we will frequently be using arithmetic modulo m. In particular, given a set $S \in \binom{[m]}{2d+2}$, we have $s_0 - s_{2d+1} \equiv$

$s_0 - s_{2d+1} + m \mod m$, which is an element of $[m]$. For instance, if $d = 1$, $m = 6$, and $\{s_0, s_1, s_2, s_3\} = \{1, 2, 4, 5\}$, then we write $s_0 - s_3 = 2$.

For $I \subseteq [m]$, we write $I = J \sqcup J'$ if $I = J \cup J'$ and there are no $j \in J, j' \in J'$ such that j, j' are cyclically consecutive. Given a cyclic l-ple interval $I = [i_0, i'_0] \sqcup [i_1, i'_1] \sqcup \cdots \sqcup [i_{l-1}, i'_{l-1}]$, we use the notation $\widehat{I} := \{i_0, i_1, \ldots, i_{l-1}\}$ from [MW21]. We claim that the collection of subsets

$$\mathcal{U}(\mathfrak{T}) = \left\{ I \subseteq [m] : |\widehat{I}| \text{ is a } d'\text{-simplex of } \mathfrak{T} \text{ for } d' \geqslant d \right\}$$

defines the internal spectrum of a cubillage on $\mathfrak{Z}(m, 2d+1)$. This is similar to the construction in [MW21, Theorem 3.8]. In order to show that $\mathcal{U}(\mathfrak{T})$ is the internal spectrum of a cubillage, we must show that it is $2d$-separated and that $\#\mathcal{U}(\mathfrak{T}) = \binom{m-1}{2d+1}$. We begin by showing that $\mathcal{U}(\mathfrak{T})$ is $2d$-separated, for which we need the following lemma. This generalises one direction of [MW21, Lemma 3.7], although the proof in *op. cit.* requires only minor changes.

Lemma 6.5.2. *Let $I, J \subseteq [m]$. Then I δ-interweaves J only if there exist subsets $X \subseteq \widehat{I}$ and $Y \subseteq \widehat{J}$ such that $\#X = \lfloor \delta/2 \rfloor$ and $\#Y = \lceil \delta/2 \rceil$, and X intertwines Y.*

Proof. We let $\delta = 2d$, since the case $\delta = 2d + 1$ behaves similarly.

Let $I = [i_0, i'_0] \sqcup [i_1, i'_1] \sqcup \cdots \sqcup [i_r, i'_r]$ and $J = [j_0, j'_0] \sqcup [j_1, j'_1] \sqcup \cdots \sqcup [j_s, j'_s]$. Suppose that I $2d$-interweaves J, and let $A \subseteq I \setminus J$ and $B \subseteq J \setminus I$ witness this. For any $0 \leqslant p < q \leqslant d$ we cannot have both $a_p \in [i_t, i'_t]$ and $a_q \in [i_t, i'_t]$, since this implies that $b_p, b_{p+1}, \ldots, b_{q-1} \in [i_t, i'_t] \subseteq I$, which contradicts $B \cap I = \varnothing$. Hence, for all $0 \leqslant k \leqslant d$, let t_k be such that $a_k \in [i_{t_k}, i'_{t_k}]$ and let u_k be such that $b_k \in [j_{u_k}, j'_{u_k}]$. Moreover, since $B \cap I = \varnothing$, we have $b_k \in (i'_{t_k}, i_{t_{k+1}})$, and similarly $a_k \in (j'_{u_{k-1}}, j_{u_k})$ for $k \in \mathbb{Z}/(d+1)\mathbb{Z}$. Then

$$i_{t_0} \leqslant a_0 < j_{u_0} \leqslant b_0 < i_{t_1} \leqslant a_1 < \cdots < i_{t_d} \leqslant a_d < j_{u_d} \leqslant b_d,$$

and so

$$i_{t_0} < j_{u_0} < i_{t_1} < \cdots < i_{t_d} < j_{u_d}.$$

Letting $X = \{i_{t_0}, i_{t_1}, \dots, i_{t_d}\}$ and $Y = \{j_{u_0}, j_{u_1}, \dots, j_{u_d}\}$ gives us the desired result. $\square$

Lemma 6.5.3. *The collection $\mathcal{U}(\mathfrak{T})$ is $2d$-separated.*

Proof. Suppose that there exist $I, J \in \mathcal{U}(\mathfrak{T})$ such that I and J are $2d$-interweaving. By Lemma 6.5.2, we have $X \subseteq \widehat{I}$ and $Y \subseteq \widehat{J}$ such that X and Y are intertwining. But this implies that $\widehat{I}$ and $\widehat{J}$ each contain one half of a circuit (X, Y) for $\mathfrak{C}(m, 2d)$. This is a contradiction, since, by construction of $\mathcal{U}(\mathfrak{T})$, $|\widehat{I}|$ and $|\widehat{J}|$ are both simplices of the triangulation $\mathfrak{T}$ of $\mathfrak{C}(m, 2d)$. $\square$

We must now show that $\#\mathcal{U}(\mathfrak{T}) = \binom{m-1}{2d+1}$. We use induction for this, showing that the size of $\mathcal{U}(\mathfrak{T})$ is preserved by increasing flips of $\mathfrak{T}$, which requires the following lemma.

Lemma 6.5.4. *Let $|S|$ be a $(2d+1)$-simplex inducing an increasing flip of a triangulation $\mathfrak{T}$ of $\mathfrak{C}(m, 2d)$ and denote $S_l = \{s_0, s_2, \dots, s_{2d}\}$ and $S_u = \{s_1, s_3, \dots, s_{2d+1}\}$. Then the following two sets have the same cardinality:*

$$\mathcal{I}_l(S, m) = \left\{ I \subseteq [m] : S_l \subseteq \widehat{I} \subset S \right\},$$
$$\mathcal{I}_u(S, m) = \left\{ I \subseteq [m] : S_u \subseteq \widehat{I} \subset S \right\}.$$

Recall from Section 1.6 that we use the symbol '$\subset$' to denote proper subsets.

Proof. Note that we may instead consider

$$\mathcal{I}'_l(S, m) := \left\{ I \subseteq [m] : S_l \subseteq \widehat{I} \subseteq S \right\},$$
$$\mathcal{I}'_u(S, m) := \left\{ I \subseteq [m] : S_u \subseteq \widehat{I} \subseteq S \right\}.$$

This is because

$$\mathcal{I}'_l(S,m) \setminus \mathcal{I}_l(S,m) = \mathcal{I}'_u(S,m) \setminus \mathcal{I}_u(S,m) = \left\{ I \subseteq [m] : \widehat{I} = S \right\}.$$

Hence if $\#\mathcal{I}'_l(S,m) = \#\mathcal{I}'_u(S,m)$, then $\#\mathcal{I}_l(S,m) = \#\mathcal{I}_u(S,m)$.

We prove the claim by explicit enumeration. Let

$$I = [s_0, s'_0] \cup [s_1, s'_1] \cup [s_2, s'_2] \cup \cdots \cup [s_{2d}, s'_{2d}] \cup [s_{2d+1}, s'_{2d+1}].$$

Then $I \in \mathcal{I}'_l(S,m)$ if and only if, for all $i \in \mathbb{Z}/(d+1)\mathbb{Z}$,

$$s'_{2i} \in [s_{2i}, s_{2i+1} - 1] \text{ and } s'_{2i+1} \in [s_{2i+1} - 1, s_{2i+2} - 2].$$

Recall that our convention here is that if $s'_j = s_j - 1$, then $[s_j, s'_j] = \varnothing$. Similarly, $I \in \mathcal{I}'_u(S,m)$ if and only if, for all $i \in \mathbb{Z}/(d+1)\mathbb{Z}$,

$$s'_{2i} \in [s_{2i} - 1, s_{2i+1} - 2] \text{ and } s'_{2i+1} \in [s_{2i+1}, s_{2i+2} - 1].$$

Therefore,

$$\begin{aligned}\#\mathcal{I}'_l(S,m) = \#\mathcal{I}'_u(S,m) &= \prod_{i \in \mathbb{Z}/(d+1)\mathbb{Z}} (s_{2i+1} - s_{2i})(s_{2i+2} - s_{2i+2}) \\ &= \prod_{j \in \mathbb{Z}/(2d+2)\mathbb{Z}} (s_{j+1} - s_j).\end{aligned}$$

□

This allows us to prove that our $2d$-separated collection $\mathcal{U}(\mathfrak{T})$ is the right size to be the internal spectrum of a cubillage.

Lemma 6.5.5. *For a triangulation $\mathfrak{T}$ of $\mathfrak{C}(m, 2d)$, we have that $\#\mathcal{U}(\mathfrak{T}) = \binom{m-1}{2d+1}$.*

Proof. We prove the claim by induction on increasing flips of the triangulation. This is valid since every triangulation of a cyclic polytope can be reached via a sequence of increasing flips from the lower triangulation by [Ram97, Theorem 1.1(i)].

For the base case, let $\mathfrak{T}_l$ be the lower triangulation of $\mathfrak{C}(m, 2d)$. By Gale's Evenness Criterion, the vertex sets of the $2d$-simplices of $\mathfrak{T}_l$ are given by 1 together with d disjoint pairs of consecutive numbers. Therefore, the only d'-simplices of $\mathfrak{T}_l$ with $d' \geqslant d$ which have no cyclically consecutive vertices are the internal d-simplices. Hence if $I \in \mathcal{U}(\mathfrak{T}_l)$, then $|\widehat{I}|$ is an internal d-simplex of $\mathfrak{T}_l$. Moreover, the internal d-simplices of $\mathfrak{T}_l$ are given by $(d+1)$-subsets which are cyclic $(d+1)$-ple intervals and contain 1.

By [DKK18a, (4.2)(ii)], the internal spectrum of the lower cubillage of the zonotope $\mathfrak{Z}(m, 2d+1)$ consists of all cyclic $(d+1)$-ple intervals which contain 1. It is then straightforward to see that $\mathcal{U}(\mathfrak{T})$ is indeed the internal spectrum of the lower cubillage of $\mathfrak{Z}(m, 2d+1)$ when $\mathfrak{T}$ is the lower triangulation of $\mathfrak{C}(m, 2d)$. Therefore, we have in this case that $\#\mathcal{U}(\mathfrak{T}) = \binom{m-1}{2d+1}$.

For the inductive step, we suppose that we have a triangulation $\mathfrak{T}'$ obtained by performing an increasing flip induced by a $(2d+1)$-simplex $|S|$ on a triangulation $\mathfrak{T}$ for which the induction hypothesis holds. Then $\mathcal{I}_l(S, m)$ contains precisely the subsets I such that $|\widehat{I}|_{2d+1}$ is contained in a lower facet of $|S|_{2d+1}$ but not any upper facets, by Gale's Evenness Criterion. Similarly, $\mathcal{I}_u(S, m)$ contains precisely the subsets I such that $|\widehat{I}|_{2d+1}$ is contained in an upper facet of $|S|_{2d+1}$ but not any lower facets. Hence

$$\mathcal{U}(\mathfrak{T}') = (\mathcal{U}(\mathfrak{T}) \setminus \mathcal{I}_l(S, m)) \cup \mathcal{I}_u(S, m),$$

and so $\#\mathcal{U}(\mathfrak{T}) = \#\mathcal{U}(\mathfrak{T}')$ by Lemma 6.5.4. The result then follows by induction. □

Hence we obtain that g is a surjection in even dimensions.

Theorem 6.5.6. *The map $g\colon \mathcal{B}(m, \delta+1) \to \mathcal{S}(m, \delta)$ is a surjection for even δ.*

Proof. Let $\delta = 2d$ and let $\mathfrak{T}$ be a triangulation of $\mathfrak{C}(m, 2d)$. By Lemma 6.5.3, Lemma 6.5.5, and the correspondence between cubillages and separated collections from [GP21], we have that the collection $\mathcal{U}(\mathfrak{T})$ is the internal spectrum of a cubillage $\mathfrak{Q}_{\mathfrak{T}}$ of $\mathfrak{Z}(m, 2d+1)$. Moreover, $g(\mathfrak{Q}_{\mathfrak{T}}) = \mathfrak{T}$ by Proposition 6.3.8, since if $\#A = d+1$, then $A \in \mathcal{U}(\mathfrak{T})$ if and only if $|A|$ is an internal d-simplex of $\mathfrak{T}$. □

Example 6.5.7. We give an example of the construction used to prove Theorem 6.5.6. Consider the triangulation $\mathfrak{T}$ of the hexagon $\mathfrak{C}(6, 2)$ which has arcs $\mathring{e}(\mathfrak{T}) = \{13, 15, 35\}$.

Then we have

$$\begin{aligned} \mathcal{U}(\mathfrak{T}) = \{&13, 15, 35, \\ &134, 125, 356, 135, \\ &1345, 1235, 1356\}. \end{aligned}$$

Note the presence of $135 \in \mathcal{U}(\mathfrak{T})$, since $|135|$ is a 2-simplex of $\mathfrak{T}$. One can check that $\mathcal{U}(\mathfrak{T})$ is 2-separated. Furthermore, $\#\mathcal{U}(\mathfrak{T}) = 10 = \binom{5}{3} = \binom{6-1}{2+1}$, as desired.

We thus obtain the cubillage $\mathfrak{Q}_{\mathfrak{T}}$ which is defined by $\mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}}) = \mathcal{U}(\mathfrak{T})$. It then follows from Proposition 6.3.8 that $g(\mathfrak{Q}_{\mathfrak{T}}) = \mathfrak{T}$; compare Example 6.3.9. Hence $\mathfrak{T}$ has a pre-image under g.

We now deduce from Theorem 6.5.6 that the map g must be a surjection for odd δ.

Theorem 6.5.8. *The map $g\colon \mathcal{B}(m, \delta+1) \to \mathcal{S}(m, \delta)$ is a surjection for odd δ.*

Proof. Let $\delta = 2d+1$ and let $\mathfrak{T}$ be a triangulation of $\mathfrak{C}(m, 2d+1)$. We show that there exists a cubillage $\mathfrak{Q}_{\mathfrak{T}}$ of $\mathfrak{Z}(m, 2d+2)$ such that $\mathrm{Sp}(\mathfrak{Q}_{\mathfrak{T}}) \supseteq \mathrm{Simp}(\mathfrak{T})$. Consider the triangulation $\hat{\mathfrak{T}}$ of $\mathfrak{C}(m+1, 2d+2)$ defined in Section 2.1.4. By Theorem 6.5.6, there is a cubillage $\mathfrak{Q}'$ of $\mathfrak{Z}(m+1, 2d+3)$ such that $g(\mathfrak{Q}') = \hat{\mathfrak{T}}$. By definition of $\hat{\mathfrak{T}}$, we have that $\mathrm{Simp}(\mathfrak{T}) \cup \mathrm{Simp}(\mathfrak{T}) * (m+1) \subseteq \mathrm{Simp}(\hat{\mathfrak{T}}) \subseteq \mathrm{Sp}(\mathfrak{Q}')$. By [DKK18a, Lemma 5.2], if we take the $(m+1)$-contraction of $\mathfrak{Q}'$ then we get a section $\mathfrak{M}$ of $\mathfrak{Q}'/(m+1)$ as the image of the $(m+1)$-pie, and we have that $\mathrm{Sp}(\mathfrak{M}) \supseteq \mathrm{Simp}(\mathfrak{T})$.

We therefore define $\mathfrak{Q}_{\mathfrak{T}} = \mathfrak{M}$, recalling that $\mathfrak{M}$ is a cubillage of $\mathfrak{Z}(m, 2d+2)$. By Lemma 6.3.7, we must have that $g(\mathfrak{Q}_{\mathfrak{T}}) = \mathfrak{T}$. $\square$

Hence, we recover the result [RS00, Theorem 3.5].

Corollary 6.5.9. *The map $g \colon \mathcal{B}(m, \delta+1) \to \mathcal{S}(m, \delta)$ is a surjection.*

Remark 6.5.10. In [KV91, Theorem 4.10], Kapranov and Voevodsky gave a map $f \colon \mathcal{B}(m, \delta) \to \mathcal{S}(m+2, \delta+1)$ which they stated was a surjection. A proof of this statement remains unfound. It was shown in [Tho03, Proposition 7.1] that there is a factorisation

$$\begin{tikzcd}
\mathcal{B}(m,\delta) \arrow[rr, "\overline{g}"] \arrow[dr, "f"'] & & \mathcal{S}(m,\delta-1), \\
& \mathcal{S}(m+2,\delta+1) \arrow[ur, dashed, two heads] &
\end{tikzcd}$$

where $\overline{g}$ is the dual map to g from Remark 6.3.6 and the dotted map is a surjection by [Ram97, Corollary 4.3].

The map f should not only be a surjection, but also a quotient map of posets, as we show is true of the map g in this chapter. This was shown for $\delta = 1$ by Reading [Rea06], drawing upon [BW97]. However, note that f cannot in general realise $\mathcal{S}(m+2, \delta+1)$ as a quotient of $\mathcal{B}(m, \delta)$ by an order congruence in the sense of [Rea02; Rea06]. This is because the equivalence classes of an order congruence must be intervals, but [Tho03, Section 6] shows that the fibres of the map f are not always intervals. Hence, f can only be a quotient map of posets in a more general sense, such as that considered in this chapter.

6.6 Fullness

We now show that the map g is full, and hence is a quotient map of posets. To do this, we must show that if $\mathfrak{T} \leqslant \mathfrak{T}'$ for triangulations $\mathfrak{T}, \mathfrak{T}'$ of $\mathfrak{C}(m, \delta)$, then there are cubillages $\mathfrak{Q}, \mathfrak{Q}'$ of $\mathfrak{Z}(m, \delta+1)$ such that $g(\mathfrak{Q}) = \mathfrak{T}$, $g(\mathfrak{Q}') = \mathfrak{T}'$, and $\mathfrak{Q} \leqslant \mathfrak{Q}'$. We follow the approach of Sections 3.3.4 and 6.5, whereby we work explicitly for even-dimensional triangulations, and then use this to show the result for odd dimensions. Indeed, we show that for triangulations $\mathfrak{T}, \mathfrak{T}'$ of $\mathfrak{C}(m, 2d)$ with $\mathfrak{T} \leqslant \mathfrak{T}'$, we have $\mathfrak{Q}_{\mathfrak{T}} \leqslant \mathfrak{Q}_{\mathfrak{T}'}$. For this, it suffices to show that if $\mathfrak{T} \lessdot \mathfrak{T}'$, then $\mathfrak{Q}_{\mathfrak{T}} < \mathfrak{Q}_{\mathfrak{T}'}$. To do this, we find a sequence of increasing flips from $\mathfrak{Q}_{\mathfrak{T}}$ to $\mathfrak{Q}_{\mathfrak{T}'}$.

Using the characterisation of increasing flips in terms of separated collections from Theorem 6.2.3, it can be seen that, in order to show that $\mathfrak{Q}_{\mathfrak{T}} \leqslant \mathfrak{Q}_{\mathfrak{T}'}$, we must show that we can gradually exchange the elements of $\mathrm{Sp}(\mathfrak{Q}_{\mathfrak{T}}) \setminus \mathrm{Sp}(\mathfrak{Q}_{\mathfrak{T}'})$ for the elements of $\mathrm{Sp}(\mathfrak{Q}_{\mathfrak{T}'}) \setminus \mathrm{Sp}(\mathfrak{Q}_{\mathfrak{T}})$. If $|S|$ is the simplex inducing the increasing flip from $\mathfrak{T}$ to $\mathfrak{T}'$, then $\mathrm{Sp}(\mathfrak{Q}_{\mathfrak{T}}) \setminus \mathrm{Sp}(\mathfrak{Q}_{\mathfrak{T}'}) = \mathcal{I}_l(S, m)$ and $\mathrm{Sp}(\mathfrak{Q}_{\mathfrak{T}'}) \setminus \mathrm{Sp}(\mathfrak{Q}_{\mathfrak{T}}) = \mathcal{I}_u(S, m)$, as in Lemma 6.5.5. Hence, we will define a sequence of exchanges which replaces $\mathcal{I}_l(S, m)$ with $\mathcal{I}_u(S, m)$. To show that our sequence of exchanges works, we will need the following lemma.

Lemma 6.6.1. *Let*

$$I = [s_0, s_0^i] \cup [s_1, s_1^i] \cup \cdots \cup [s_{2d}, s_{2d}^i] \cup [s_{2d+1}, s_{2d+1}^i]$$

and

$$J = [s_0, s_0^j] \cup [s_1, s_1^j] \cup \cdots \cup [s_{2d}, s_{2d}^j] \cup [s_{2d+1}, s_{2d+1}^j].$$

Then I $2d$-interweaves J if and only if, for all r,

$$s_{2r}^j < s_{2r}^i \text{ and } s_{2r+1}^j > s_{2r+1}^i.$$

Proof. If we have that, for all r, $s_{2r}^j < s_{2r}^i$ and $s_{2r+1}^j > s_{2r+1}^i$, then we have that $\{s_0^i, s_2^i, \ldots, s_{2d}^i\} \subseteq I \setminus J$ and $\{s_1^j, s_3^j, \ldots, s_{2d+1}^j\} \subseteq J \setminus I$ with

$$s_0^i < s_1^j < s_2^i < s_3^j < \cdots < s_{2d}^i < s_{2d+1}^j.$$

Hence I $2d$-interweaves J.

Conversely, suppose that I $2d$-interweaves J, and let $X \subseteq I \setminus J$ and $Y \subseteq J \setminus I$ witness this. We cannot have both $x_p, x_q \in [s_t, s_t^i]$ for $p \neq q$, since this implies that $y_r \in [s_t, s_t^i]$ for $p \leqslant r < q$. Furthermore, we cannot have both $x_p \in [s_t, s_t^i]$ and $y_p \in [s_t, s_t^j]$, since we must have either $[s_t, s_t^i] \subseteq [s_t, s_t^j]$ or $[s_t, s_t^j] \subseteq [s_t, s_t^i]$. By the pigeonhole principle and the fact that $x_0 < y_0$, we deduce that $x_r \in [s_{2r}, s_{2r}^i]$ and $y_r \in [s_{2r+1}, s_{2r+1}^j]$ for all r. But this implies that $s_{2r}^j < s_{2r}^i$ and $s_{2r+1}^i < s_{2r+1}^j$ for all r. □

It is now useful for us to obtain an explicit map for the bijection from Lemma 6.5.4. This allows us to construct the sequence of exchanges which replaces $\mathcal{I}_l(S,m)$ with $\mathcal{I}_u(S,m)$.

Construction 6.6.2. Given $S \in \binom{[m]}{2d+2}$, we define

$$\mathcal{I}(S,m) = \mathcal{I}_l(S,m) \cup \mathcal{I}_u(S,m),$$
$$\mathcal{I}'(S,m) = \mathcal{I}'_l(S,m) \cup \mathcal{I}'_u(S,m).$$

In order to get a convenient parametrisation of these sets, we define a map

$$\begin{aligned} \phi\colon \prod_{i \in \mathbb{Z}/(2d+2)\mathbb{Z}} [0, s_{i+1} - s_i] &\to 2^{[m]} \\ (n_0, n_1, \dots, n_{2d+1}) &\mapsto \bigcup_{i \in \mathbb{Z}/(2d+2)\mathbb{Z}} [s_i, s_i + n_i - 1]. \end{aligned}$$

We abbreviate $\mathbf{n} = (n_0, n_1, \dots, n_{2d+1})$. Then

- $\phi(\mathbf{n}) \in \mathcal{I}'_l(S,m)$ if and only if $n_{2i-1} < s_{2i} - s_{2i-1}$ and $n_{2i} > 0$ for all $i \in \mathbb{Z}/(d+1)\mathbb{Z}$;
- $\phi(\mathbf{n}) \in \mathcal{I}'_u(S,m)$ if and only if $n_{2i} < s_{2i+1} - s_{2i}$ and $n_{2i+1} > 0$ for all $i \in \mathbb{Z}/(d+1)\mathbb{Z}$;
- $\phi(\mathbf{n}) \in \mathcal{I}_l(S,m)$ if and only if $n_{2i-1} < s_{2i} - s_{2i-1}$ and $n_{2i} > 0$ for all $i \in \mathbb{Z}/(d+1)\mathbb{Z}$, and there exists a $j \in \mathbb{Z}/(d+1)\mathbb{Z}$ such that either $n_{2j+1} = 0$ or $n_{2j} = s_{2j+1} - s_{2j}$;

- $\phi(\mathbf{n}) \in \mathcal{I}_u(S,m)$ if and only if $n_{2i} < s_{2i+1} - s_{2i}$ and $n_{2i+1} > 0$ for all $i \in \mathbb{Z}/(d+1)\mathbb{Z}$, and there exists a $j \in \mathbb{Z}/(d+1)\mathbb{Z}$ such that either $n_{2j} = 0$, or $n_{2j-1} = s_{2j} - s_{2j-1}$.

We then obtain an explicit bijection by defining a map

$$\psi\colon \mathcal{I}_l(S,m) \to \mathcal{I}_u(S,m)$$

as follows. Let $I \in \mathcal{I}_l(S,m)$ such that $I = \phi(\mathbf{n})$ and let $\mathbf{t} = (-1, 1, -1, 1, \ldots, -1, 1)$. Further, define

$$\lambda_I = \max\left\{ \lambda \in \mathbb{Z}_{>0} : \mathbf{n} + \lambda\mathbf{t} \in \prod_{i \in \mathbb{Z}/(2d+2)\mathbb{Z}} [0, s_{i+1} - s_i] \right\}.$$

By construction,

$$\phi(\mathbf{n} + \lambda_I \mathbf{t}) \in \mathcal{I}_u(S,m),$$

since we must either have some $j \in \mathbb{Z}/(d+1)\mathbb{Z}$ such that $s_{2j} - \lambda_I = 0$, or some $j \in \mathbb{Z}/(d+1)\mathbb{Z}$ such that $s_{2j-1} + \lambda_I = s_{2j} - s_{2j-1}$, otherwise λ_I would not be maximal. Therefore define

$$\psi(I) = \phi(\mathbf{n} + \lambda_I \mathbf{t}).$$

It can be seen that the map ψ is a bijection because one may define its inverse as follows. Let $J \in \mathcal{I}_u(S,m)$ such that $J = \phi(\mathbf{n})$. Then let

$$\mu_J = \max\left\{ \mu \in \mathbb{Z}_{>0} : \mathbf{n} - \mu\mathbf{t} \in \prod_{i \in \mathbb{Z}/(2d+2)\mathbb{Z}} [0, s_{i+1} - s_i] \right\}.$$

By construction,

$$\phi(\mathbf{n} - \mu_J \mathbf{t}) \in \mathcal{I}_l(S,m),$$

since we must either have some $j \in \mathbb{Z}/(d+1)\mathbb{Z}$ such that $n_{2j+1} - \mu_J = 0$, or some $j \in \mathbb{Z}/(d+1)\mathbb{Z}$ such that $n_{2j} + \mu_J = s_{2j+1} - s_{2j}$. It is then clear that

$$\psi^{-1}(J) = \phi(\mathbf{n} - \mu_J \mathbf{t}).$$

Theorem 6.6.3. *Given triangulations $\mathfrak{T}, \mathfrak{T}'$ of $\mathfrak{C}(m, 2d)$ such that $\mathfrak{T} \lessdot \mathfrak{T}'$, there exist cubillages $\mathfrak{Q}_0, \mathfrak{Q}_1, \dots, \mathfrak{Q}_r$ of $\mathfrak{Z}(m, 2d+1)$ such that $\mathfrak{Q}_0 = \mathfrak{Q}_{\mathfrak{T}}$, $\mathfrak{Q}_r = \mathfrak{Q}_{\mathfrak{T}'}$ and*

$$\mathfrak{Q}_0 \lessdot \mathfrak{Q}_1 \lessdot \cdots \lessdot \mathfrak{Q}_r,$$

so that $\mathfrak{Q}_{\mathfrak{T}} \leqslant \mathfrak{Q}_{\mathfrak{T}'}$.

Proof. Suppose that the increasing flip of $\mathfrak{T}$ which gives $\mathfrak{T}'$ is induced by the $(2d+1)$-face $|S|$ of $\mathfrak{C}(m, m-1)$. Then $\mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}}) \setminus \mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}'}) = \mathcal{I}_l(S, m)$ and $\mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}'}) \setminus \mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}}) = \mathcal{I}_u(S, m)$. Let $\mathcal{R} = \mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}}) \setminus \mathcal{I}_l(S, m) = \mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}'}) \setminus \mathcal{I}_u(S, m)$. Hence we must find a sequence of flips starting at $\mathfrak{Q}_{\mathfrak{T}}$ which gradually replaces $\mathcal{I}_l(S, m)$ with $\mathcal{I}_u(S, m)$.

The flips of cubillages we wish to perform are as follows. Given $\phi(\mathbf{n}) \in \mathcal{I}_l(S, m)$, we make the sequence of exchanges given by the pairs

$$(\phi(\mathbf{n}), \phi(\mathbf{n}+\mathbf{t})),\ (\phi(\mathbf{n}+\mathbf{t}), \phi(\mathbf{n}+2\mathbf{t})), \dots, (\phi(\mathbf{n}+(\lambda_{\phi(\mathbf{n})}-1)\mathbf{t}), \phi(\mathbf{n}+\lambda_{\phi(\mathbf{n})}\mathbf{t})).$$

We must show that there is an order in which we can make these exchanges such that after each exchange we still have a $2d$-separated collection. Here each exchange gives an increasing flip by Theorem 6.2.3. Note further that $\phi(\mathbf{n}+r\mathbf{t})$ and $\phi(\mathbf{n}+(r+1)\mathbf{t})$ are tightly $2d$-interweaving, as we know must be the case from Theorem 6.2.3.

Our exchanges give a bijection

$$\begin{aligned} \mathcal{I}'(S, m) \setminus \mathcal{I}_u(S, m) &\to \mathcal{I}'(S, m) \setminus \mathcal{I}_l(S, m) \\ \phi(\mathbf{n}) &\mapsto \phi(\mathbf{n}+\mathbf{t}). \end{aligned}$$

Hence, we have one exchange per element of $\mathcal{I}'(S, m) \setminus \mathcal{I}_u(S, m)$. By Construction 6.6.2, we have that ϕ is a bijection between $[1, s_1 - s_0] \times [0, s_2 - s_1 - 1] \times \cdots \times [1, s_{2d+1} - s_{2d}] \times [0, s_0 - s_{2d+1} - 1]$ and $\mathcal{I}'(S, m) \setminus \mathcal{I}_u(S, m)$. The set $[1, s_1 - s_0] \times$

$[0, s_2 - s_1 - 1] \times \cdots \times [1, s_{2d+1} - s_{2d}] \times [0, s_0 - s_{2d+1} - 1]$ is a lattice under the order given by

$$(n_0, n_1, \ldots, n_{2d+1}) \leqslant (n'_0, n'_1, \ldots, n'_{2d+1})$$

if and only if for all j

$$n_{2j} \geqslant n'_{2j} \text{ and } n_{2j+1} \leqslant n'_{2j+1},$$

since this is just the usual product order, but reversed on coordinates with even index.

We claim that any linear extension $\mathbf{n}^1 < \mathbf{n}^2 < \cdots < \mathbf{n}^r$ of this lattice gives an order on $\mathcal{I}'(S, m) \setminus \mathcal{I}_u(S, m)$ such that if $\mathcal{C}_0 := \mathrm{Sp}(\mathfrak{Q}_{\mathfrak{T}})$ and $\mathcal{C}_i := (\mathcal{C}_{i-1} \setminus \{\phi(\mathbf{n}^i)\}) \cup \{\phi(\mathbf{n}^i + \mathbf{t})\}$, then $\mathcal{C}_i$ is $2d$-separated for all i. Note first that we always must have $\phi(\mathbf{n}^i) \in \mathcal{C}_{i-1}$. This is because either $\phi(\mathbf{n}^i) \in \mathcal{I}_l(S, m)$ or $\phi(\mathbf{n}^i - \mathbf{t}) \in \mathcal{I}'(S, m) \setminus \mathcal{I}_u(S, m)$. Hence, either $\phi(\mathbf{n}^i) \in \mathcal{C}_0$, or $\phi(\mathbf{n}^i)$ is the result of an earlier exchange, since $\mathbf{n}^i - \mathbf{t} < \mathbf{n}^i$ in our order.

Now suppose that $\mathcal{C}_i$ is not $2d$-separated for some i. We may choose the minimal i for which this is the case. We first show that no element of $\mathcal{I}'(S, m)$ is $2d$-interweaving with any element of $\mathcal{R}$. Suppose, on the contrary, that there exist $I \in \mathcal{I}'(S, m)$ and $J \in \mathcal{R}$ such that I and J are $2d$-interweaving. Then, by Lemma 6.5.2, we have $X \subseteq \widehat{I}$ and $Y \subseteq \widehat{J}$ such that $\#X = \#Y = d + 1$ and X and Y are intertwining. We have that $X \subseteq \widehat{I} \subseteq S$, and since $\#X = d + 1$, we must have either $X \not\supseteq S_u := \{s_1, s_3, \ldots, s_{2d+1}\}$, or $X \not\supseteq S_l := \{s_0, s_2, \ldots, s_{2d}\}$. If $X \not\supseteq S_u$, then $X \subseteq R$ for a $2d$-simplex $|R|$ of $\mathfrak{T}$, by Gale's Evenness Criterion. This gives a contradiction, since $|R|$ and $|\widehat{J}|$ are both simplices of $\mathfrak{T}$ and (X, Y) is a circuit. One can derive a similar contradiction using $\mathfrak{T}'$ when $X \not\supseteq S_l$.

Therefore, if $\mathcal{C}_i$ is not $2d$-separated, it must be because

$$\begin{aligned}\phi(\mathbf{n}^i + \mathbf{t}) = [s_0, s'_0 + (n^i_0 - 1) - 1] \cup [s_1, s'_1 + (n^i_1 + 1) - 1] \cup \ldots \\ \cup [s_{2d+1}, s'_{2d+1} + (n^i_{2d+1} + 1) - 1]\end{aligned}$$

is $2d$-interweaving with an element $I \in \mathcal{I}(S,m) \cap \mathcal{C}_i$. By Lemma 6.6.1, we must have

$$I = [s_0, s_0'] \cup [s_1, s_1'] \cup \cdots \cup [s_{2d+1}, s_{2d+1}'] \in \mathcal{C}_i \setminus \{\phi(\mathbf{n}^i + \mathbf{t})\} = \mathcal{C}_{i-1} \setminus \{\phi(\mathbf{n}^i)\}$$

such that either $s_{2j} + (n_{2j}^i - 1) - 1 < s_{2j}'$ and $s_{2j+1}' < s_{2j+1} + (n_{2j+1}^i + 1) - 1$ for all j, or $s_{2j}' < s_{2j} + (n_{2j}^i - 1) - 1$ and $s_{2j+1} + (n_{2j+1}^i + 1) - 1 < s_{2j+1}'$ for all j. In the latter case, we also have that $s_{2j}' < s_{2j} + n_{2j}^i - 1$ and $s_{2j+1} + n_{2j+1}^i - 1 < s_{2j+1}'$, so that $\phi(\mathbf{n}^i)$ also $2d$-interweaves I, which means that $\mathcal{C}_{i-1}$ is not $2d$-separated. This contradicts i being the minimal index such that this was the case. In the former case, we have that I precedes $\phi(\mathbf{n}^i)$ in our chosen order on $\mathcal{I}'(S,m) \setminus \mathcal{I}_u(S,m)$. This means that I must have already been exchanged, which is also a contradiction.

Therefore, we have cubillages $\mathfrak{Q}_0, \mathfrak{Q}_1, \ldots, \mathfrak{Q}_r$ such that $\mathcal{C}_i = \mathrm{Sp}(\mathfrak{Q}_i)$ for each i. By Theorem 6.2.3, we have

$$\mathfrak{Q}_0 \lessdot \mathfrak{Q}_1 \lessdot \cdots \lessdot \mathfrak{Q}_r.$$

By construction, we have that $\mathfrak{Q}_0 = \mathfrak{Q}_{\mathfrak{T}}$ and $\mathfrak{Q}_r = \mathfrak{Q}_{\mathfrak{T}'}$. □

Example 6.6.4. We give examples of the construction used to prove Theorem 6.6.3.

(1) Consider the triangulation $\mathfrak{T}$ of the heptagon $\mathfrak{C}(7,2)$ given by $\mathring{e}(\mathfrak{T}) = \{13, 16, 35, 36\}$. We perform the increasing flip on this triangulation induced by the simplex $|1236|$, thereby obtaining the triangulation $\mathfrak{T}'$ of $\mathfrak{C}(7,2)$ with $\mathring{e}(\mathfrak{T}') = \{16, 26, 35, 36\}$.

We have

$$\begin{aligned} \mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}}) = \{&13, 16, 35, 36, \\ &126, 134, 136, 346, 356, 367, \\ &1236, 1345, 1346, 1367, 3467, 3567, \\ &12346, 13456, 13467, 13567\} \end{aligned}$$

and

$$\begin{aligned}\mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}'}) = \{&16, 26, 35, 36\\ &126, 236, 267, 346, 356, 367,\\ &1236, 1367, 2346, 2367, 3467, 3567,\\ &12346, 13467, 13567, 23467\}.\end{aligned}$$

Moreover,

$$\mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}}) \setminus \mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}'}) = \mathcal{I}_l(1236, 7) = \{13, 134, 136, 1345, 1346, 13456\}$$

and

$$\mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}'}) \setminus \mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}}) = \mathcal{I}_u(1236, 7) = \{26, 236, 267, 2346, 2367, 23467\}.$$

We illustrate how we can gradually replace elements of $\mathcal{I}_l(1236, 7)$ in $\mathrm{ISp}(\mathfrak{Q}_{\mathfrak{T}})$ with the elements of $\mathcal{I}_u(1236, 7)$, whilst ensuring that the collection remains 2-separated.

The coordinate parameterisation of $\mathcal{I}'(1236, 7)$ by ϕ gives

$$\begin{aligned}\phi(1, 0, 1, 0) &= 13,\\ \phi(1, 0, 2, 0) &= 134,\\ \phi(1, 0, 1, 1) &= 136,\\ \phi(1, 0, 3, 0) &= 1345,\\ \phi(1, 0, 2, 1) &= 1346,\\ \phi(1, 0, 3, 1) &= 13456,\end{aligned}$$

$$\phi(0,1,0,1) = 26,$$
$$\phi(0,1,1,1) = 236,$$
$$\phi(0,1,0,2) = 267,$$
$$\phi(0,1,2,1) = 2346,$$
$$\phi(0,1,1,2) = 2367,$$
$$\phi(0,1,2,2) = 23467.$$

The bijection $\psi\colon \mathcal{I}_l(1236,7) \to \mathcal{I}_u(1236,7)$ in this case gives

$$\begin{array}{rcccccl}
13 & = & \phi(1,0,1,0) & \mapsto & \phi(0,1,0,1) & = & 26, \\
134 & = & \phi(1,0,2,0) & \mapsto & \phi(0,1,1,1) & = & 236, \\
136 & = & \phi(1,0,1,1) & \mapsto & \phi(0,1,0,2) & = & 267, \\
1345 & = & \phi(1,0,3,0) & \mapsto & \phi(0,1,2,1) & = & 2346, \\
1346 & = & \phi(1,0,2,1) & \mapsto & \phi(0,1,1,2) & = & 2367, \\
13456 & = & \phi(1,0,3,1) & \mapsto & \phi(0,1,2,2) & = & 23467.
\end{array}$$

Note that in this example, we have that $\mathcal{I}_l'(1236,7) = \mathcal{I}_l(1236,7)$ and $\mathcal{I}_u'(1236,7) = \mathcal{I}_u(1236,7)$, since we cannot have $\widehat{I} = 1236$ for any subset I. Thus we consider the lattice on $\mathcal{I}'(1237,6) \setminus \mathcal{I}_u(1237,6) = \mathcal{I}_l(1236,7)$ given by

$$\begin{tikzcd}
& (1,0,1,1) & & \\
(1,0,1,0) \arrow[ur] & & (1,0,2,1) \arrow[ul] & \\
& (1,0,2,0) \arrow[ul] \arrow[ur] & & (1,0,3,1) \arrow[ul] \\
& & (1,0,3,0), \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

which is

$$\begin{tikzcd}
& 136 & & \\
13 \arrow[ur] & & 1346 \arrow[ul] & \\
& 134 \arrow[ul] \arrow[ur] & & 13456 \arrow[ul] \\
& & 1345. \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Note that we place minimal element of the lattice at the bottom. Therefore, by Theorem 6.6.3, we may perform the exchanges replacing $\phi(\mathbf{n})$ by $\phi(\mathbf{n}+\mathbf{t})$ in an order given by any linear extension of

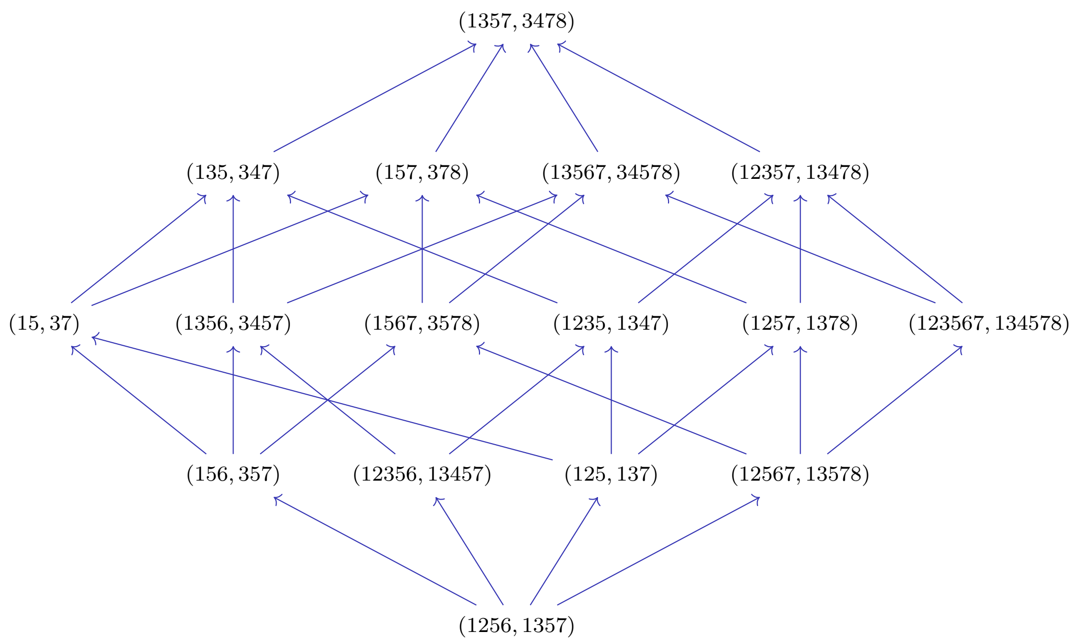
$$\begin{tikzcd}
& (136, 267) & & \\
(13, 26) \arrow[ur] & & (1346, 2367) \arrow[ul] & \\
& (134, 236) \arrow[ul] \arrow[ur] & & (13456, 23467) \arrow[ul] \\
& & (1345, 2346). \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

We first make the exchange at the bottom of the lattice, and then move up.

(2) We now give an example where we do not have $\mathcal{I}(S, m) = \mathcal{I}'(S, m)$. This example is somewhat larger than the previous example, so we do not go through it in the same level of detail.

Indeed, we do not consider full triangulations, but only the set $\mathcal{I}_l(1357, 8)$, which we wish to replace with the set $\mathcal{I}_u(1357, 8)$. Here we have $\mathcal{I}'_l(1357, 8) = \mathcal{I}_l(1357, 8) \cup \{1357\}$ and $\mathcal{I}'_u(1357, 8) = \mathcal{I}_u(1357, 8) \cup \{1357\}$. The sequence of exchanges from $\mathcal{I}_l(1357, 8)$ to $\mathcal{I}_u(1357, 8)$ is given by the bijection $\phi(\mathbf{n}) \mapsto \phi(\mathbf{n}+\mathbf{t})$ from $\mathcal{I}'(1357, 8) \setminus \mathcal{I}_u(1357, 8)$ to $\mathcal{I}'(1357, 8) \setminus \mathcal{I}_l(1357, 8)$.

Any sequence of exchanges done in the order of any linear extension of the following lattice will preserve 2-separatedness. One can check that this is the lattice from the proof of Theorem 6.6.3.



Note that here, since $1357 \in \mathcal{I}'(1357, 8) \setminus \mathcal{I}_l(1357, 8)$, but $1357 \notin \mathcal{I}_u(1357, 8)$, we have two exchange pairs containing 1357, namely $(1256, 1357)$ and $(1357, 3478)$. That is, 1357 is only an intermediate subset in the sequence of exchanges from $\mathcal{I}_l(1357, 8)$ to $\mathcal{I}_u(1357, 8)$.

We now show the result for odd dimensions. The structure of the proof here is similar to that of Proposition 3.3.19.

Theorem 6.6.5. *Given triangulations $\mathfrak{T}, \mathfrak{T}'$ of $\mathfrak{C}(m, 2d+1)$ such that $\mathfrak{T} \lessdot \mathfrak{T}'$, there exist cubillages $\mathfrak{Q}_0, \mathfrak{Q}_1, \dots, \mathfrak{Q}_r$ of $\mathfrak{Z}(m, 2d+2)$ such that $\mathfrak{Q}_0 = \mathfrak{Q}_{\mathfrak{T}}$, $\mathfrak{Q}_r = \mathfrak{Q}_{\mathfrak{T}'}$ and*

$$\mathfrak{Q}_0 \lessdot \mathfrak{Q}_1 \lessdot \cdots \lessdot \mathfrak{Q}_r,$$

so that $\mathcal{Q}_{\mathfrak{T}} \leqslant \mathcal{Q}_{\mathfrak{T}'}$.

Proof. We start, as in the proof of Theorem 6.5.8, by considering the triangulations $\hat{\mathfrak{T}}, \hat{\mathfrak{T}}'$ of $\mathfrak{C}(m+1, 2d+2)$. We know that the extension operation is order-reversing, so $\hat{\mathfrak{T}}' < \hat{\mathfrak{T}}$. By Theorem 6.6.3, there exist cubillages $\mathcal{Q}'_s \lessdot \mathcal{Q}'_{s-1} \lessdot \cdots \lessdot \mathcal{Q}'_0$ of $\mathfrak{Z}(m+1, 2d+3)$ such that $\mathcal{Q}'_s = \mathcal{Q}_{\hat{\mathfrak{T}}'}$ and $\mathcal{Q}'_0 = \mathcal{Q}_{\hat{\mathfrak{T}}}$.

As in the proof of [DKK18a, Lemma 5.2], we have that the $(m+1)$-contraction of $\mathcal{Q}'_i$ gives a section $\mathfrak{M}_i$ which is a cubillage of $\mathfrak{Z}(m, 2d+2)$. As in the proof of Theorem 6.5.8, we have that $\mathfrak{M}_s = \mathcal{Q}_{\mathfrak{T}'}$ and $\mathfrak{M}_0 = \mathcal{Q}_{\mathfrak{T}}$. We claim that for each i we either have $\mathfrak{M}_i = \mathfrak{M}_{i+1}$ or $\mathfrak{M}_i \lessdot \mathfrak{M}_{i+1}$.

Consider the increasing flip which takes $\mathcal{Q}'_{i+1}$ to $\mathcal{Q}'_i$. Suppose this increasing flip is induced by a $(2d+4)$-face $\mathfrak{F}$ of $\mathfrak{Z}(m+1, m+1)$ which has A as its set of generating vectors. If $m+1 \notin A$, then the increasing flip does not affect the $(m+1)$-pie, so that $\mathfrak{M}_i = \mathfrak{M}_{i+1}$. Hence, suppose instead that $m+1 \in A$. Let the lower facets of $\pi_{m+1,2d+4}(\mathfrak{F})$ consist of the cubes $\pi_{m+1,2d+4}(\mathfrak{U}_j)$, where $\mathfrak{U}_j$ is generated by $A \setminus \{a_j\}$, noting that we must have $a_{2d+3} = m+1$. Similarly, let the upper facets of $\pi_{m+1,2d+4}(\mathfrak{F})$ consist of the cubes $\pi_{m+1,2d+4}(\mathfrak{U}'_j)$, where $\mathfrak{U}'_j$ is generated by $A \setminus \{a_j\}$.

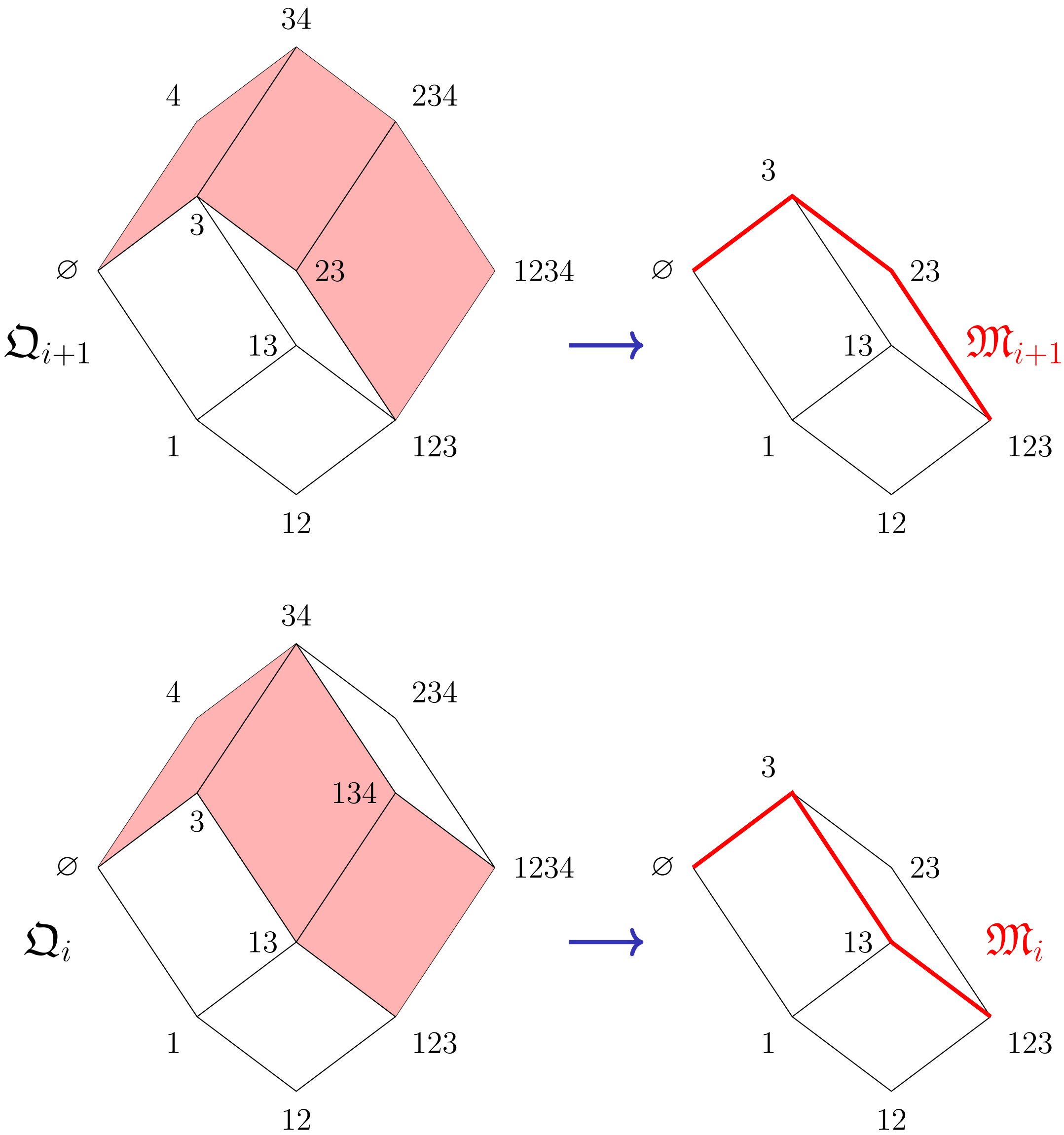
It is well-known that for $j < k$ the cubes $\pi_{m+1,2d+4}(\mathfrak{U}_j)$ and $\pi_{m+1,2d+4}(\mathfrak{U}_k)$ intersect in an upper facet of $\pi_{m+1,2d+4}(\mathfrak{U}_k)$ and a lower facet of $\pi_{m+1,2d+4}(\mathfrak{U}_j)$, while the cubes $\pi_{m+1,2d+4}(\mathfrak{U}'_j)$ and $\pi_{m+1,2d+4}(\mathfrak{U}'_k)$ intersect in an upper facet of $\pi_{m+1,2d+4}(\mathfrak{U}'_j)$ and a lower facet of $\pi_{m+1,2d+4}(\mathfrak{U}'_k)$. This is because the increasing flip corresponds to inverting the packet of A: the cubes $\mathfrak{U}_j$ and $\mathfrak{U}'_j$ correspond to the sets $A \setminus \{a_j\}$; these must be ordered lexicographically for $\mathfrak{U}_j$ and reverse-lexicographically for $\mathfrak{U}'_j$.

Contracting the $(m+1)$-pie of $\mathcal{Q}'_{i+1}$ sends the cubes $\mathfrak{U}_j$ for $j < 2d+3$ to their facet generated by $A \setminus \{a_j, m+1\}$, which is precisely the intersection $\mathfrak{U}_j \cap \mathfrak{U}_{2d+3}$. By the above paragraph, this projects to an upper facet of $\pi_{m+1,2d+4}(\mathfrak{U}_{2d+3})$. Hence the part of $\mathfrak{M}_{i+1}$ which lies within $\mathfrak{F}/(m+1)$ consists of the upper facets of $\pi_{m+1,2d+4}(\mathfrak{U}_{2d+3}/(m+1))$. Here we use $\mathfrak{F}/(m+1)$ to denote the image of $\mathfrak{F}/(m+1)$ under the $(m+1)$-contraction, and so forth. Similarly, we have that the part of $\mathfrak{M}_i$ which lies within $\mathfrak{F}/(m+1)$ consists of the lower facets of $\pi_{m+1,2d+4}(\mathfrak{U}'_{2d+3}/(m+1))$. We then have that $\mathfrak{F}/(m+1) = \mathfrak{U}_{2d+3}/(m+1) = \mathfrak{U}'_{2d+3}/(m+1)$, and so $\mathfrak{M}_i \lessdot \mathfrak{M}_{i+1}$. This is since $\mathfrak{M}_i$ and $\mathfrak{M}_{i+1}$ only differ within $\mathfrak{F}/(m+1)$, because $\mathfrak{Q}'_{i+1}$ and $\mathfrak{Q}'_i$ only differ within $\mathfrak{F}$. Moreover, $\pi_{m,2d+3}(\mathfrak{M}_{i+1})$ contains the upper facets of $\pi_{m,2d+3}(\mathfrak{F}/(m+1))$, whereas $\pi_{m,2d+3}(\mathfrak{M}_i)$ contains the lower facets of $\pi_{m,2d+3}(\mathfrak{F}/(m+1))$. This argument is illustrated in Figure 6.7; compare [DKK19a, Figure 7].

This gives a chain of cubillages $\mathfrak{Q}_{\mathfrak{T}} = \mathfrak{M}_0 = \mathfrak{Q}_0 \lessdot \mathfrak{Q}_1 \lessdot \cdots \lessdot \mathfrak{Q}_r = \mathfrak{M}_s = \mathfrak{Q}_{\mathfrak{T}'}$ by applying the result of the above paragraph to the chain $\mathfrak{Q}'_s \lessdot \mathfrak{Q}'_{s-1} \lessdot \cdots \lessdot \mathfrak{Q}'_0$. Here the cubillages $\mathfrak{Q}_0, \mathfrak{Q}_1, \ldots, \mathfrak{Q}_r$ are the cubillages $\mathfrak{M}_0, \mathfrak{M}_1, \ldots, \mathfrak{M}_s$ with the duplicates removed, corresponding to the cases above where $\mathfrak{M}_i = \mathfrak{M}_{i+1}$. $\square$

By putting together Theorem 6.5.6, Theorem 6.5.8, Theorem 6.6.3, and Theorem 6.6.5, this finally establishes Theorem 6.4.3, and hence also Corollary 6.4.4.

Figure 6.7: An illustration of the argument of Theorem 6.6.5



Bibliography

[AI12] Takuma Aihara and Osamu Iyama. "Silting mutation in triangulated categories". *J. Lond. Math. Soc. (2)* 85.3 (2012), pp. 633–668.

[Aih13] Takuma Aihara. "Tilting-connected symmetric algebras". *Algebr. Represent. Theory* 16.3 (2013), pp. 873–894.

[AIR14] Takahide Adachi, Osamu Iyama, and Idun Reiten. "τ-tilting theory". *Compos. Math.* 150.3 (2014), pp. 415–452.

[AK85] Noga Alon and Gil Kalai. "A simple proof of the upper bound theorem". *European J. Combin.* 6.3 (1985), pp. 211–214.

[And87] Tsuyoshi Ando. "Totally positive matrices". *Linear Algebra Appl.* 90 (1987), pp. 165–219.

[AS02] Miguel Azaola and Francisco Santos. "The number of triangulations of the cyclic polytope $C(n, n-4)$". Vol. 27. 1. Geometric combinatorics (San Francisco, CA/Davis, CA, 2000). 2002, pp. 29–48.

[AS80] Maurice Auslander and Sverre O. Smalø. "Preprojective modules over Artin algebras". *J. Algebra* 66.1 (1980), pp. 61–122.

[ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. Elements of the representation theory of associative algebras. Vol. 1. Vol. 65. London Mathematical Society Student Texts. Techniques of representation theory. Cambridge University Press, Cambridge, 2006, pp. x+458.

[AT14] Nima Arkani-Hamed and Jaroslav Trnka. "The Amplituhedron". *J. High Energ. Phys.* 10 (2014).

[BB80] Aage Bondesen and Arne Brøndsted. "A dual proof of the upper bound conjecture for convex polytopes". *Math. Scand.* 46.1 (1980), pp. 95–102.

[BBG20] Karin Baur, Dusko Bogdanic, and Ana Garcia Elsener. "Cluster categories from Grassmannians and root combinatorics". *Nagoya Math. J.* 240 (2020), pp. 322–354.

[BDP14] Thomas Brüstle, Grégoire Dupont, and Matthieu Pérotin. "On maximal green sequences". *Int. Math. Res. Not. IMRN* 16 (2014), pp. 4547–4586.

[BFP11] Mireille Bousquet-Mélou, Éric Fusy, and Louis-François Préville-Ratelle. "The number of intervals in the m-Tamari lattices". *Electron. J. Combin.* 18.2 (2011), Paper 31, 26.

[Bie53] Lawrence C. Biedenharn. "An identity by the Racah coefficients". *J. Math. Physics* 31 (1953), pp. 287–293.

[Bjö+99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. Oriented matroids. Second. Vol. 46. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1999, pp. xii+548.

[BK04] Aslak Bakke Buan and Henning Krause. "Tilting and cotilting for quivers of type $\tilde{A}_n$". *J. Pure Appl. Algebra* 190.1-3 (2004), pp. 1–21.

[BL78] Robert G. Bland and Michel Las Vergnas. "Orientability of matroids". *J. Combinatorial Theory Ser. B* 24.1 (1978), pp. 94–123.

[BMR07] Aslak Bakke Buan, Bethany Marsh, and Idun Reiten. “Cluster-tilted algebras”. *Trans. Amer. Math. Soc.* 359.1 (2007), pp. 323–332.

[BP12] François Bergeron and Louis-François Préville-Ratelle. “Higher trivariate diagonal harmonics via generalized Tamari posets”. *J. Comb.* 3.3 (2012), pp. 317–341.

[Bre73] Marilyn Breen. “Primitive Radon partitions for cyclic polytopes”. *Israel J. Math.* 15 (1973), pp. 156–157.

[Bri+05] Ruth Britto, Freddy Cachazo, Bo Feng, and Edward Witten. “Direct proof of the tree-level scattering amplitude recursion relation in Yang-Mills theory”. *Phys. Rev. Lett.* 94.18 (2005), pp. 181602, 4.

[BT18] Jacob Bourjaily and Hugh Thomas. “What is… the amplituhedron?” *Notices Amer. Math. Soc.* 65.2 (2018), pp. 167–169.

[Bua+06] Aslak Bakke Buan, Bethany Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov. “Tilting theory and cluster combinatorics”. *Adv. Math.* 204.2 (2006), pp. 572–618.

[BW97] Anders Björner and Michelle L. Wachs. “Shellable nonpure complexes and posets. II”. *Trans. Amer. Math. Soc.* 349.10 (1997), pp. 3945–3975.

[Car07] Constantin Carathéodory. “Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen”. *Math. Ann.* 64.1 (1907), pp. 95–115.

[Car11] Constantin Carathéodory. “Über den Variabilitätsbereich der Fourier’schen Konstanten von positiven harmonischen Funktionen”. *Rend. Circ. Mat. Palermo, I Ser.* 32.1 (1911), pp. 193–217.

[CCS06] Philippe Caldero, Frédéric Chapoton, and Ralf Schiffler. "Quivers with relations arising from clusters (A_n case)". *Trans. Amer. Math. Soc.* 358.3 (2006), pp. 1347–1364.

[CD00] Raul Cordovil and Pierre Duchet. "Cyclic polytopes and oriented matroids". Vol. 21. 1. Combinatorics of polytopes. 2000, pp. 49–64.

[Com74] Louis Comtet. Advanced combinatorics. The art of finite and infinite expansions. D. Reidel Publishing Co., Dordrecht, 1974, pp. xi+343.

[CS04] Gabriel D. Carroll and David Speyer. "The cube recurrence". *Electron. J. Combin.* 11.1 (2004), Research Paper 73, 31.

[CS98] Ivan Chajda and Václav Snášel. "Congruences in ordered sets". *Math. Bohem.* 123.1 (1998), pp. 95–100.

[Dey93] Tamal Krishna Dey. "On counting triangulations in d dimensions". *Comput. Geom.* 3.6 (1993), pp. 315–325.

[DIJ19] Laurent Demonet, Osamu Iyama, and Gustavo Jasso. "τ-tilting finite algebras, bricks, and g-vectors". *Int. Math. Res. Not. IMRN* 3 (2019), pp. 852–892.

[Dis+12] Filippo Disanto, Luca Ferrari, Renzo Pinzani, and Simone Rinaldi. "Catalan lattices on series parallel interval orders". *Associahedra, Tamari lattices and related structures.* Vol. 299. Prog. Math. Phys. Birkhäuser/Springer, Basel, 2012, pp. 323–338.

[DJL21] Tobias Dyckerhoff, Gustavo Jasso, and Yankı Lekili. "The symplectic geometry of higher Auslander algebras: symmetric products of disks". *Forum Math. Sigma* 9 (2021), Paper No. e10, 49.

[DJW19] Tobias Dyckerhoff, Gustavo Jasso, and Tashi Walde. “Simplicial structures in higher Auslander–Reiten theory”. *Adv. Math.* 355 (2019), pp. 106762, 73.

[DK19] Tobias Dyckerhoff and Mikhail Kapranov. Higher Segal spaces. Vol. 2244. Lecture Notes in Mathematics. Springer, Cham, 2019, pp. xv+218.

[DKK18a] Vladimir I. Danilov, Alexander V. Karzanov, and Gleb A. Koshevoy. Cubillages on cyclic zonotopes, membranes, and higher separatation. 2018. arXiv: `1810.05517 [math.CO]`.

[DKK18b] Vladimir I. Danilov, Alexander V. Karzanov, and Gleb A. Koshevoy. On interrelations between strongly, weakly and chord separated set-systems (a geometric approach). 2018. arXiv: `1805.09595 [math.CO]`.

[DKK19a] Vladimir I. Danilov, Alexander V. Karzanov, and Gleb A. Koshevoy. “Cubillages on cyclic zonotopes”. *Russ. Math. Surv.* 74.6 (2019), pp. 1013–1074.

[DKK19b] Vladimir I. Danilov, Alexander V. Karzanov, and Gleb A. Koshevoy. The weak separation in higher dimensions. 2019. arXiv: `1904.09798 [math.CO]`.

[DKK20] Vladimir I. Danilov, Alexander V. Karzanov, and Gleb A. Koshevoy. The purity phenomenon for symmetric separated set-systems. 2020. arXiv: `2007.02011 [math.CO]`.

[DM12] Aristophanes Dimakis and Folkert Müller-Hoissen. “KP solitons, higher Bruhat and Tamari orders”. *Associahedra, Tamari lattices and related structures.* Ed. by Folkert Müller-Hoissen, Jean Marcel Pallo, and Jim Stasheff. Vol. 299. Progress in Mathematical Physics. Tamari memorial Festschrift. Birkhäuser/Springer, Basel, 2012, pp. 391–423.

[DM15] Aristophanes Dimakis and Folkert Müller-Hoissen. "Simplex and polygon equations". *SIGMA Symmetry Integrability Geom. Methods Appl.* 11 (2015), Paper 042, 49.

[DRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. Triangulations. Vol. 25. Algorithms and Computation in Mathematics. Structures for algorithms and applications. Springer-Verlag, Berlin, 2010, pp. xiv+535.

[Eli16] Ben Elias. "Thicker Soergel calculus in type A". *Proc. Lond. Math. Soc. (3)* 112.5 (2016), pp. 924–978.

[Ell53] James Philip Elliott. "Theoretical Studies in Nuclear Structure. V. The Matrix Elements of Non-Central Forces with an Application to the $2p$-Shell". *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 218.1134 (1953), pp. 345–370.

[ER96] Paul H. Edelman and Victor Reiner. "The higher Stasheff–Tamari posets". *Mathematika* 43.1 (1996), pp. 127–154.

[ERR00] Paul H. Edelman, Jörg Rambau, and Victor Reiner. "On subdivision posets of cyclic polytopes". Vol. 21. 1. Combinatorics of polytopes. 2000, pp. 85–101.

[FG06] Vladimir Fock and Alexander Goncharov. "Moduli spaces of local systems and higher Teichmüller theory". *Publ. Math. Inst. Hautes études Sci.* 103 (2006), pp. 1–211.

[FG09] Vladimir V. Fock and Alexander B. Goncharov. "Cluster ensembles, quantization and the dilogarithm". *Ann. Sci. éc. Norm. Supér. (4)* 42.6 (2009), pp. 865–930.

[FR21] Vincent Froese and Malte Renken. "Persistent graphs and cyclic polytope triangulations". *Combinatorica* (2021).

[FST08] Sergey Fomin, Michael Shapiro, and Dylan Thurston. "Cluster algebras and triangulated surfaces. I. Cluster complexes". *Acta Math.* 201.1 (2008), pp. 83–146.

[FZ02a] Sergey Fomin and Andrei Zelevinsky. "Cluster algebras. I. Foundations". *J. Amer. Math. Soc.* 15.2 (2002), 497–529 (electronic).

[FZ02b] Sergey Fomin and Andrei Zelevinsky. "The Laurent phenomenon". *Adv. in Appl. Math.* 28.2 (2002), pp. 119–144.

[FZ03a] Sergey Fomin and Andrei Zelevinsky. "Y-systems and generalized associahedra". *Ann. of Math. (2)* 158.3 (2003), pp. 977–1018.

[FZ03b] Sergey Fomin and Andrei Zelevinsky. "Cluster algebras: notes for the CDM-03 conference". *Current developments in mathematics, 2003.* Int. Press, Somerville, MA, 2003, pp. 1–34.

[FZ03c] Sergey Fomin and Andrei Zelevinsky. "Cluster algebras. II. Finite type classification". *Invent. Math.* 154.1 (2003), pp. 63–121.

[Gal18] Pavel Galashin. "Plabic graphs and zonotopal tilings". *Proc. Lond. Math. Soc. (3)* 117.4 (2018), pp. 661–681.

[Gal55] David Gale. "On convex polyhedra". *Bull. Amer. Math. Soc.* 61.6 (1955). Abstract 794, pp. 505–574.

[Gal63] David Gale. "Neighborly and cyclic polytopes". *Proc. Sympos. Pure Math., Vol. VII.* Amer. Math. Soc., Providence, R.I., 1963, pp. 225–232.

[GKO13] Christof Geiss, Bernhard Keller, and Steffen Oppermann. "n-angulated categories". *J. Reine Angew. Math.* 675 (2013), pp. 101–120.

[GM19] Alexander Garver and Thomas McConville. "Lattice properties of oriented exchange graphs and torsion classes". *Algebr. Represent. Theory* 22.1 (2019), pp. 43–78.

[Gor] Mikhail Gorsky. "Stability associahedra, maximal green sequences, and second Cambrian orders". In preparation.

[Gor14a] Mikhail Gorsky. "Geometry and combinatorics of subword complexes and their dual polytopes". (in Russian). Available at `http://www.mi-ras.ru/dis/ref14/gorskii/gorskii_dis.pdf`. PhD thesis. Steklov Institute of Mathematics, 2014.

[Gor14b] Mikhail Gorsky. "Subword complexes and 2-truncated cubes". *Uspekhi Mat. Nauk* 69.3(417) (2014), pp. 179–180.

[GP21] Pavel Galashin and Alexander Postnikov. Purity and separation for oriented matroids. 2021. arXiv: `1708.01329 [math.CO]`.

[GPW19] Pavel Galashin, Alexander Postnikov, and Lauren Williams. Higher secondary polytopes and regular plabic graphs. 2019. arXiv: `1909.05435 [math.CO]`.

[GR08] Ádám Galambos and Victor Reiner. "Acyclic sets of linear orders via the Bruhat orders". *Soc. Choice Welf.* 30.2 (2008), pp. 245–264.

[Grü03] Branko Grünbaum. Convex polytopes. Second. Vol. 221. Graduate Texts in Mathematics. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler. Springer-Verlag, New York, 2003, pp. xvi+468.

[GSV03] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein. "Cluster algebras and Poisson geometry". *Mosc. Math. J.* 3.3 (2003), pp. 899–934, 1199.

[GSV05] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein. "Cluster algebras and Weil-Petersson forms". *Duke Math. J.* 127.2 (2005), pp. 291–311.

[Hal17] Joshua Hallam. "Applications of quotient posets". *Discrete Math.* 340.4 (2017), pp. 800–810.

[Hap88] Dieter Happel. Triangulated categories in the representation theory of finite-dimensional algebras. Vol. 119. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1988, pp. x+208.

[Hat+86] Hiroyuki Hata, Katsumi Itoh, Taichiro Kugo, Hiroshi Kunitomo, and Kaku Ogawa. "Covariant string field theory". *Phys. Rev. D Part. Fields* 34.8 (Oct. 1986), pp. 2360–2429.

[Her+20] Martin Herschend, Osamu Iyama, Hiroyuki Minamoto, and Steffen Oppermann. Representation theory of Geigle-Lenzing complete intersections. 2020. arXiv: `1409.0668 [math.RT]`.

[HI19] Stephen Hermes and Kiyoshi Igusa. "The no gap conjecture for tame hereditary algebras". *J. Pure Appl. Algebra* 223.3 (2019), pp. 1040–1053.

[HIO14] Martin Herschend, Osamu Iyama, and Steffen Oppermann. "n-representation infinite algebras". *Adv. Math.* 252 (2014), pp. 292–342.

[HM97] Lars Hesselholt and Ib Madsen. "Cyclic polytopes and the K-theory of truncated polynomial algebras". *Invent. Math.* 130.1 (1997), pp. 73–97.

[HS15] Joshua Hallam and Bruce Sagan. “Factoring the characteristic polynomial of a lattice”. *J. Combin. Theory Ser. A* 136 (2015), pp. 39–63.

[HU05] Dieter Happel and Luise Unger. “On a partial order of tilting modules”. *Algebr. Represent. Theory* 8.2 (2005), pp. 147–156.

[Hua15] Jihui Huang. “Classification of Soliton Graphs On Totally positive Grassmannian”. PhD thesis. Ohio State University, 2015.

[IO11] Osamu Iyama and Steffen Oppermann. “n-representation-finite algebras and n-APR tilting”. *Trans. Amer. Math. Soc.* 363.12 (2011), pp. 6575–6614.

[IO13] Osamu Iyama and Steffen Oppermann. “Stable categories of higher preprojective algebras”. *Adv. Math.* 244 (2013), pp. 23–68.

[Iya07a] Osamu Iyama. “Auslander correspondence”. *Adv. Math.* 210.1 (2007), pp. 51–82.

[Iya07b] Osamu Iyama. “Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories”. *Adv. Math.* 210.1 (2007), pp. 22–50.

[Iya08] Osamu Iyama. “Auslander–Reiten theory revisited”. *Trends in representation theory of algebras and related topics*. EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich, 2008, pp. 349–397.

[Iya11] Osamu Iyama. “Cluster tilting for higher Auslander algebras”. *Adv. Math.* 226.1 (2011), pp. 1–61.

[Jas16] Gustavo Jasso. “n-abelian and n-exact categories”. *Math. Z.* 283.3-4 (2016), pp. 703–759.

[JJ20] Karin M. Jacobsen and Peter Jørgensen. “Maximal τ_d-rigid pairs”. *J. Algebra* 546 (2020), pp. 119–134.

[JK18] Michael Joswig and Lars Kastner. “New counts for the number of triangulations of cyclic polytopes”. *Mathematical software—ICMS 2018.* Vol. 10931. Lecture Notes in Comput. Sci. Springer, Cham, 2018, pp. 264–271.

[JK19a] Gustavo Jasso and Julian Külshammer. “Higher Nakayama algebras I: Construction”. *Adv. Math.* 351 (2019). With an appendix by Külshammer and Chrysostomos Psaroudakis and an appendix by Sondre Kvamme, pp. 1139–1200.

[JK19b] Gustavo Jasso and Sondre Kvamme. “An introduction to higher Auslander-Reiten theory”. *Bull. Lond. Math. Soc.* 51.1 (2019), pp. 1–24.

[Jør16] Peter Jørgensen. “Torsion classes and t-structures in higher homological algebra”. *Int. Math. Res. Not. IMRN* 13 (2016), pp. 3880–3905.

[Jør17] Peter Jørgensen. Higher dimensional homological algebra. Lecture notes for LMS-CMI Research School “New trends in Representation Theory: The impact of cluster theory in representation theory”. June 2017.

[Kel11] Bernhard Keller. “On cluster theory and quantum dilogarithm identities”. *Representations of algebras and related topics.* EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich, 2011, pp. 85–116.

[Kel13] Bernhard Keller. “The periodicity conjecture for pairs of Dynkin diagrams”. *Ann. of Math. (2)* 177.1 (2013), pp. 111–170.

[KK21] Rachel Karpman and Yuji Kodama. “Triangulations and soliton graphs for totally positive Grassmannian”. *Adv. Math.* 376 (2021), Paper No. 107439, 50.

[KK74] Michio Kaku and Keiji Kikkawa. "Field theory of relativistic strings. I. Trees". *Phys. Rev. D* 10 (4 Aug. 1974), pp. 1110–1133.

[Kle63] Victor Klee. Cyclic polytopes. Lecture notes. University of Washington, Seattle, 1963.

[Knu11] Donald E. Knuth. The art of computer programming. Vol. 4A. Combinatorial algorithms. Part 1. Addison-Wesley, Upper Saddle River, NJ, 2011, pp. xv+883.

[Kod10] Yuji Kodama. "KP solitons in shallow water". *J. Phys. A* 43.43 (2010), pp. 434004, 54.

[KP70] Boris B. Kadomtsev and Vladimir I. Petviashvili. "On the stability of solitary waves in weakly dispersing media." English. *Sov. Phys., Dokl.* 15 (1970), pp. 539–541.

[KV91] Mikhail M. Kapranov and Vladimir A. Voevodsky. "Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher Bruhat orders (list of results)". *Cahiers Topologie Géom. Différentielle Catég.* 32.1 (1991). International Category Theory Meeting (Bangor, 1989 and Cambridge, 1990), pp. 11–27.

[KW03] Volker Kaibel and Arnold Wassmer. Automorphism Groups of Cyclic Polytopes. 2003.

[Las78] Michel Las Vergnas. "Extensions ponctuelles d'une géométrie combinatoire orientée". *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976).* Vol. 260. Colloq. Internat. CNRS. CNRS, Paris, 1978, pp. 265–270.

[Lus98] George Lusztig. "Introduction to total positivity". *Positivity in Lie theory: open problems.* Vol. 26. De Gruyter Exp. Math. de Gruyter, Berlin, 1998, pp. 133–145.

[LZ98] Bernard Leclerc and Andrei Zelevinsky. “Quasicommuting families of quantum Plücker coordinates” (1998), pp. 85–108.

[McM18] Jordan McMahon. Higher support tilting I: higher Auslander algebras of linearly oriented type A. 2018. arXiv: 1808.05184 [math.RT].

[McM21] Jordan McMahon. Support τ_2-tilting and 2-torsion pairs. 2021. arXiv: 2102.08254 [math.RT].

[McM70] Peter McMullen. “The maximum numbers of faces of a convex polytope”. *Mathematika* 17 (1970), pp. 179–184.

[Moo55] John C. Moore. “Le théorème de Freudenthal, la suite exacte de James et l’invariant de Hopf généralisé”. *Séminaire Henri Cartan* 7.2 (1954-1955), pp. 1–15.

[Mot57] Theodore S. Motzkin. “Comonotone curves and polyhedra”. *Bull. Amer. Math. Soc.* 63.1 (1957). Abstract 111, p. 35.

[MPS12] Folkert Müller-Hoissen, Jean Marcel Pallo, and Jim Stasheff, eds. Associahedra, Tamari lattices and related structures. Vol. 299. Progress in Mathematical Physics. Tamari memorial Festschrift. Birkhäuser/Springer, Basel, 2012, pp. xx+433.

[MS71] Peter McMullen and Geoffrey C. Shephard. Convex polytopes and the upper bound conjecture. Vol. 3. London Mathematical Society Lecture Note Series. Prepared in collaboration with J. E. Reeve and A. A. Ball. Cambridge University Press, London-New York, 1971, pp. iv+184.

[MS89] Yuriĭ I. Manin and Vadim V. Schechtman. “Arrangements of hyperplanes, higher braid groups and higher Bruhat orders”. *Algebraic number theory.* Vol. 17. Adv. Stud. Pure Math. Academic Press, Boston, MA, 1989, pp. 289–308.

[MW21] Jordan McMahon and Nicholas J. Williams. "The combinatorics of tensor products of higher Auslander algebras of type A". *Glasg. Math. J.* 63.3 (2021), pp. 526–546.

[Nag13] Kentaro Nagao. "Donaldson–Thomas theory and cluster algebras". *Duke Math. J.* 162.7 (2013), pp. 1313–1367.

[OPS15] Suho Oh, Alexander Postnikov, and David E Speyer. "Weak separation and plabic graphs". *Proc. Lond. Math. Soc. (3)* 110.3 (2015), pp. 721–754.

[OS19] Jorge Alberto Olarte and Francisco Santos. Hypersimplicial subdivisions. 2019. arXiv: `1906.05764 [math.CO]`.

[OT12] Steffen Oppermann and Hugh Thomas. "Higher-dimensional cluster combinatorics and representation theory". *J. Eur. Math. Soc. (JEMS)* 14.6 (2012), pp. 1679–1737.

[Pog17] Thomas Poguntke. Higher Segal structures in algebraic K-theory. 2017. arXiv: `1709.06510`.

[Pos06] Alexander Postnikov. Total positivity, Grassmannians, and networks. 2006. arXiv: `math/0609764`.

[Pos18] Alexander Postnikov. "Positive Grassmannian and polyhedral subdivisions". *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. IV. Invited lectures.* World Sci. Publ., Hackensack, NJ, 2018, pp. 3181–3211.

[Pre20] Matthew Pressland. "Mutation of frozen Jacobian algebras". *J. Algebra* 546 (2020), pp. 236–273.

[PV17] Louis-François Préville-Ratelle and Xavier Viennot. "The enumeration of generalized Tamari intervals". *Trans. Amer. Math. Soc.* 369.7 (2017), pp. 5219–5239.

[Ram97] Jörg Rambau. "Triangulations of cyclic polytopes and higher Bruhat orders". *Mathematika* 44.1 (1997), pp. 162–194.

[Rea02] Nathan Reading. "Order dimension, strong Bruhat order and lattice properties for posets". *Order* 19.1 (2002), pp. 73–100.

[Rea06] Nathan Reading. "Cambrian lattices". *Adv. Math.* 205.2 (2006), pp. 313–353.

[Rez01] Charles Rezk. "A model for the homotopy theory of homotopy theory". *Trans. Amer. Math. Soc.* 353.3 (2001), pp. 973–1007.

[Ron12] María Ronco. "Generalized Tamari order". *Associahedra, Tamari lattices and related structures.* Ed. by Folkert Müller-Hoissen, Jean Marcel Pallo, and Jim Stasheff. Vol. 299. Progress in Mathematical Physics. Tamari memorial Festschrift. Birkhäuser/Springer, Basel, 2012, pp. 339–350.

[RR12] Jörg Rambau and Victor Reiner. "A survey of the higher Stasheff–Tamari orders". *Associahedra, Tamari lattices and related structures.* Vol. 299. Progr. Math. Birkhäuser/Springer, Basel, 2012, pp. 351–390.

[RS00] Jörg Rambau and Francisco Santos. "The generalized Baues problem for cyclic polytopes. I". Vol. 21. 1. Combinatorics of polytopes. 2000, pp. 65–83.

[RS89] Jeremy Rickard and Aidan Schofield. "Cocovers and tilting modules". *Math. Proc. Cambridge Philos. Soc.* 106.1 (1989), pp. 1–5.

[RS91] Christine Riedtmann and Aidan Schofield. “On a simplicial complex associated with tilting modules”. *Comment. Math. Helv.* 66.1 (1991), pp. 70–78.

[RZ94] Jürgen Richter-Gebert and Günter M. Ziegler. “Zonotopal tilings and the Bohne-Dress theorem”. *Jerusalem combinatorics '93*. Vol. 178. Contemp. Math. Amer. Math. Soc., Providence, RI, 1994, pp. 211–232.

[Sch86] Alexander Schrijver. Theory of linear and integer programming. Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons, Ltd., Chichester, 1986, pp. xii+471.

[Seg74] Graeme Segal. “Categories and cohomology theories”. *Topology* 13 (1974), pp. 293–312.

[Spe07] David E Speyer. “Perfect matchings and the octahedron recurrence”. *J. Algebraic Combin.* 25.3 (2007), pp. 309–348.

[SS06] Rahul Savani and Bernhard von Stengel. “Hard-to-solve bimatrix games”. *Econometrica* 74.2 (2006), pp. 397–429.

[SS16] Rahul Savani and Bernhard von Stengel. “Unit vector games”. *Int. J. Econ. Theory* 12.1 (2016), pp. 7–27.

[SSW17] Francisco Santos, Christian Stump, and Volkmar Welker. “Noncrossing sets and a Grassmann associahedron”. *Forum Math. Sigma* 5 (2017), Paper No. e5, 49.

[Sta12] Jim Stasheff. “How I ‘met’ Dov Tamari”. *Associahedra, Tamari lattices and related structures.* Ed. by Folkert Müller-Hoissen, Jean Marcel Pallo, and Jim Stasheff. Vol. 299. Progress in Mathematical Physics. Tamari memorial Festschrift. Birkhäuser/Springer, Basel, 2012, pp. 45–63.

[Sta15] Richard P. Stanley. Catalan numbers. Cambridge University Press, New York, 2015, pp. viii+215.

[Sta63] James Dillon Stasheff. "Homotopy associativity of H-spaces. I, II". *Trans. Amer. Math. Soc.* 108 (1963), pp. 275–312.

[Sta75] Richard P. Stanley. "The upper bound conjecture and Cohen–Macaulay rings". *Studies in Appl. Math.* 54.2 (1975), pp. 135–142.

[Sta84] Richard P. Stanley. "Quotients of Peck posets". *Order* 1.1 (1984), pp. 29–34.

[Sta91] Richard P. Stanley. "Some applications of algebra to combinatorics". Vol. 34. 1-3. Combinatorics and theoretical computer science (Washington, DC, 1989). 1991, pp. 241–277.

[Ste97] Bernhard von Stengel. New lower bounds for the number of equilibria in bimatrix games. Tech. rep. 264. ETH Zürich, 1997.

[Stu87] Bernd Sturmfels. "Cyclic polytopes and d-order curves". *Geom. Dedicata* 24.1 (1987), pp. 103–107.

[Stu88a] Bernd Sturmfels. "Neighborly polytopes and oriented matroids". *European J. Combin.* 9.6 (1988), pp. 537–546.

[Stu88b] Bernd Sturmfels. "Totally positive matrices and cyclic polytopes". *Proceedings of the Victoria Conference on Combinatorial Matrix Analysis (Victoria, BC, 1987).* Vol. 107. 1988, pp. 275–281.

[Tam51] Dov Tamari. "Monoïdes préordonnés et chaînes de Malcev". PhD thesis. 1951, iv+81 pp. (mimeographed).

[Tam62] Dov Tamari. "The algebra of bracketings and their enumeration". *Nieuw Arch. Wisk. (3)* 10 (1962), pp. 131–146.

[TH96] Naoki Terai and Takayuki Hibi. “Computation of Betti numbers of monomial ideals associated with cyclic polytopes”. *Discrete Comput. Geom.* 15.3 (1996), pp. 287–295.

[Tho02] Hugh Thomas. “New combinatorial descriptions of the triangulations of cyclic polytopes and the second higher Stasheff–Tamari posets”. *Order* 19.4 (2002), pp. 327–342.

[Tho03] Hugh Thomas. “Maps between higher Bruhat orders and higher Stasheff–Tamari posets”. *Formal Power Series and Algebraic Combinatorics.* Linköping University, Sweden. 2003.

[Tho06] Hugh Thomas. “Tamari lattices and noncrossing partitions in type B”. *Discrete Math.* 306.21 (2006), pp. 2711–2723.

[Tho12] Hugh Thomas. “The Tamari lattice as it arises in quiver representations”. *Associahedra, Tamari lattices and related structures.* Vol. 299. Prog. Math. Phys. Birkhäuser/Springer, Basel, 2012, pp. 281–291.

[Wil] Nicholas J. Williams. “The higher Stasheff–Tamari orders in representation theory”. *Proceedings of the 2020 International Conference on Representations of Algebras.* Forthcoming.

[Wil14] Lauren K. Williams. “Cluster algebras: an introduction”. *Bull. Amer. Math. Soc. (N.S.)* 51.1 (2014), pp. 1–26.

[Wil21a] Nicholas J. Williams. New interpretations of the higher Stasheff–Tamari orders. 2021. arXiv: `2007.12664 [math.CO]`.

[Wil21b] Nicholas J. Williams. Quiver combinatorics for higher-dimensional triangulations. 2021. arXiv: `2112.09189 [math.CO]`.

[Wil21c] Nicholas J. Williams. The first higher Stasheff–Tamari orders are quotients of the higher Bruhat orders. 2021. arXiv: `2012.10371 [math.CO]`.

[Wil21d] Nicholas J. Williams. "The first higher Stasheff–Tamari orders are quotients of the higher Bruhat orders". *Sém. Lothar. Combin.* 85B (2021), Art. 6, 12.

[Wil21e] Nicholas J. Williams. The two higher Stasheff–Tamari orders are equal. 2021. arXiv: `2106.01050 [math.CO]`.

[Zie93] Günter M. Ziegler. "Higher Bruhat orders and cyclic hyperplane arrangements". *Topology* 32.2 (1993), pp. 259–279.

[Zie95] Günter M. Ziegler. Lectures on polytopes. Vol. 152. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. x+370.

[ZZ11] Yu Zhou and Bin Zhu. "Maximal rigid subcategories in 2-Calabi–Yau triangulated categories". *J. Algebra* 348 (2011), pp. 49–60.