Elliptic and Siegel Theta Series for Indefinite Quadratic Forms

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Kurzfassung

Die vorliegende Dissertation basiert auf vier Artikeln zu elliptischen und Siegelschen Thetareihen für indefinite quadratische Formen. In den ersten beiden Artikeln werden bekannte Aussagen für elliptische Thetareihen auf Siegelsche Thetareihen, das heißt Thetareihen von höherem Geschlecht $n \in \mathbb{N}$, verallgemeinert. Im dritten und vierten Artikel betrachten wir elliptische Thetareihen zu quadratischen Formen der Signatur $(m-1,1)$, um holomorphe und fast holomorphe Modulformen zu konstruieren.

In der Theorie der elliptischen Thetareihen ist Vignéras' Modularitäts-Kriterium ein wichtiges Werkzeug, um modulare Thetareihen zu konstruieren oder Modularität nachzuprüfen: Betrachtet man Funktionen mit einer bestimmten Wachstumsbedingung, die eine Differentialgleichung zweiter Ordnung erfüllen, so kann man mit diesen Funktionen modulare Thetareihen zu indefiniten quadratischen Formen konstruieren. Wir zeigen, dass es für Siegelsche Thetareihen ein System von partiellen Differentialgleichungen gibt, die dieselbe Eigenschaft erfüllen, und wir somit eine naheliegende Verallgemeinerung erhalten.

Die explizite Konstruktion von Thetareihen, die holomorph sind und modular transformieren, birgt für indefinite quadratische Formen im Gegensatz zur Betrachtung positiv definiter Formen zusätzliche Schwierigkeiten. Für elliptische Thetareihen existieren verschiedene Ansätze, in denen einerseits eine nicht-holomorphe modulare Thetareihe und andererseits eine holomorphe (aber nicht-modulare) Version definiert werden. Wir beschäftigen uns insbesondere mit Zwegers' Konstruktion für quadratische Formen mit Signatur $(m-1,1)$ und zeigen, dass wir auf eine ähnliche Weise modulare Siegelsche Thetareihen zu quadratischen Formen dieser Signatur konstruieren können. Darüberhinaus bestimmen wir den holomorphen Anteil dieser Thetareihen.

In den letzten beiden Artikeln erweitern wir Zwegers' Konstruktion in eine andere Richtung, indem wir die Definition der Thetareihen um homogene und sphärische Polynome erweitern. Die Definition dieser Thetareihen hängt von zwei Vektoren $\boldsymbol{c_1}$ und $\boldsymbol{c_2}$ ab, die in einem festgelegten Kegel C_Q liegen. Zunächst betrachten wir den Fall, dass $\boldsymbol{c_1}$ und $\boldsymbol{c_2}$ im Inneren von C_Q liegen und definieren eine holomorphe, eine fast holomorphe und eine modulare Version einer Thetareihe. Wir untersuchen auch, unter welcher Bedingung diese übereinstimmen, um fast holomorphe und holomorphe Modulformen zu konstruieren und letztendlich viele explizite Beispiele zu generieren. Anschließend betrachten wir auch den Fall, dass einer oder beide Vektoren $\boldsymbol{c_i}$ auf dem Rand von C_Q liegen. Diese Erweiterung ermöglicht die Betrachtung von Beispielen einer ganz anderen Art: Zum einen bietet dies eine Möglichkeit, das modulare Transformationsverhalten der Eisensteinreihen herzuleiten. Außerdem fügen sich auf diese Art Modulformen auf $\Gamma_0(4)$, die bei der Betrachtung von Potenzen quadratischer Polynome mit ganzzahligen Koeffizienten entstehen, in die Theorie der Thetareihen zu indefiniten quadratischen Formen ein. Zuletzt kann so auch die Modularität einer Maass–Form vom Gewicht $3/2$, deren holomorpher Anteil durch die erzeugende Funktion der Hurwitzschen Klassenzahlen $H(8n+7)$ gegeben ist, gezeigt werden.

Abstract

In this thesis, we embed four research articles on elliptic and Siegel theta series for indefinite quadratic forms. In the first two articles, we generalize results on elliptic theta series to Siegel theta series, i. e. theta series for arbitrary genus $n \in \mathbb{N}$. In the third and fourth article we consider elliptic theta series for quadratic forms of signature $(m-1,1)$ in order to construct holomorphic and almost holomorphic modular forms.

A useful tool in the theory of elliptic theta series is Vignéras' modularity criterion: considering functions with a certain growth condition, which satisfy a particular second order differential equation, one can construct modular theta series for indefinite quadratic forms. Furthermore, this result can be applied to check whether a given theta series transforms as a modular form. We show that for Siegel theta series we can define an analogous system of partial differential equations and thus obtain a straightforward generalization.

If one does not consider positive definite quadratic forms but indefinite ones, the explicit construction of theta series that are holomorphic and modular, poses some difficulties. For elliptic theta series, there exist various approaches in which a modular (non-holomorphic) theta series is defined on the one hand and a holomorphic (but not modular) version on the other hand. We examine Zwegers' construction for quadratic forms of signature $(m-1,1)$ and show that in a similar way non-holomorphic modular Siegel theta series for quadratic forms of this particular signature can be constructed. Moreover, we describe the holomorphic part of these Siegel theta series.

In the last two articles, we generalize Zwegers' construction in a different direction by including homogeneous as well as spherical polynomials in the definition of the theta functions. The definition depends on two vectors $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$, which are located in a special cone C_Q. In the first article on this matter, we consider the case that $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ are located in the inner part of C_Q and give a general construction of a holomorphic, an almost holomorphic, and a modular theta series. We state under which condition these versions agree and thus obtain almost holomorphic and holomorphic modular forms. This turns out to be a rich source of explicit examples, which we can often identify as eta products or eta quotients. In a sequel to this project, we also allow $\boldsymbol{c_1}$ or $\boldsymbol{c_2}$ to be located on the boundary of C_Q. This extension provides examples of a slightly different flavor. In particular, we can recover the modular transformation behavior of the Eisenstein series, of certain modular forms on $\Gamma_0(4)$ that appear during the investigation of sums of powers of quadratic polynomials of a fixed discriminant and of a Maass form of weight $3/2$, whose holomorphic part is determined by the generating function of the Hurwitz class numbers $H(8n+7)$.

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Contents

1 Introduction

In this introductory chapter, we provide an overview of the theory of modular forms with a special focus on elliptic and Siegel theta series for indefinite quadratic forms. First, we wish to motivate the study of theta series by investigating an example that makes use of the characteristic properties of modular forms. Instead of giving precise definitions at this point, we focus on general connections and ideas. We then make a short remark on the notation used throughout this thesis, give a description of the main objects that form the subject of this work and present important previous work related to our contributions. After these preliminaries, we conclude this chapter by giving a summary of the main results of this thesis.

1.1 Theta series and their applications

The main motivation to get involved with theta series is the fact that they provide nice examples of modular forms, i. e. holomorphic functions on the complex upper half plane $\mathbb{H} = \{x + iy = z \in \mathbb{C} \mid y > 0\}$ that behave "nicely" when we consider a certain group action of $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ (or certain subgroups thereof) on $\mathbb{H}$. We call these modular forms *elliptic* to distinguish them from *Siegel* modular forms of genus $n \in \mathbb{N}$. These form a generalization of this concept in the sense that we replace $\mathbb{H}$ by the Siegel upper half-space $\mathbb{H}_n$, which is the space of complex symmetric $n \times n$-matrices with positive definite imaginary part, and that we find an appropriate generalization Γ_n of Γ_1.

First, we consider the number theoretic problem of describing the number of ways an integer can be represented by a fixed positive definite quadratic form. Here, we will see that the Fourier coefficients of elliptic and Siegel modular forms contain interesting arithmetic information and that the theory of modular forms provides a very elegant way to approach this problem.

Let $Q : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a positive definite quadratic form, which is integer-valued on the lattice $\mathbb{Z}^m$, and $P : \mathbb{R}^m \longrightarrow \mathbb{C}$ a spherical polynomial of degree α. Then the elliptic theta series

$$\Theta_{Q,P}(z) := \sum_{\boldsymbol{u} \in \mathbb{Z}^m} P(\boldsymbol{u})\, q^{Q(\boldsymbol{u})} \qquad (q = e^{2\pi i z},\ z \in \mathbb{H})$$

is a holomorphic modular form of weight $m/2 + \alpha$ (on some subgroup of Γ_1 and with some character, which both depend on the quadratic form Q).

If for example we take $m = 8$, $Q = \|\cdot\|^2$ and $P \equiv 1$, then the theta function $\Theta_{Q,P}$ is a modular form of weight 4 on the congruence subgroup $\Gamma_0(4)$. The Fourier expansion of this theta series is

$$\sum_{n=0}^{\infty} r_8(n)\, q^n \quad \text{with } r_8(n) := \left|\left\{\boldsymbol{u} \in \mathbb{Z}^8 \mid \sum_{i=1}^{8} u_i^2 = n\right\}\right|.$$

On the other hand, one can show that for even integers $k \geq 4$ the Eisenstein series

$$G_k(z) := -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n, \quad \text{where } \sigma_k(n) := \sum_{\substack{1 \leq d \leq n \\ d \mid n}} d^k$$

and B_k the k-th Bernoulli number, are modular forms of weight k on Γ_1 and that a basis of the three-dimensional vector space of modular forms of weight 4 on $\Gamma_0(4)$ is given by $\{G_4(z), G_4(2z), G_4(4z)\}$. Thus, we can write the theta series as a linear combination of these three functions and by comparing the Fourier coefficients we get for any $n \in \mathbb{N}$ the formula:

$$r_8(n) = 16 \sum_{\substack{1 \leq d \leq n \\ d \mid n}} (-1)^{n-d}\, d^3$$

Naturally, one may consider other positive definite quadratic forms Q and obtain similar results for the number of representations of an integer n by Q, which is more conveniently denoted by $r_Q(n)$ and is always finite.

Taking this problem to a more general setting, we consider an $n \times n$-matrix T instead of an integer and ask how many ways there are to represent this matrix with regard to the map $U \mapsto U^{\mathrm{t}}AU$ (for $U \in \mathbb{R}^{m \times n}$) induced by a positive definite symmetric matrix $A \in \mathbb{Z}^{m \times m}$. In other words, we want to investigate the properties of

$$r(A,T) := |\{U \in \mathbb{Z}^{m \times n} \mid U^{\mathrm{t}}AU = T\}|.$$

First of all, we notice that we can only have $r(A,T) \neq 0$ if T is symmetric, positive semi-definite and even. By considering the diagonal entries of T, for which $r_Q(T_{ii}) < \infty$ holds for all $1 \leq i \leq n$, we deduce that $r(A,T)$ is finite. In order to investigate this number a bit further, we consider the Siegel theta series

$$\vartheta_A(Z) := \sum_{U \in \mathbb{Z}^{m \times n}} e^{\pi i \operatorname{tr}(U^{\mathrm{t}}AUZ)},$$

which is a Siegel modular form of weight $m/2$ (on the full Siegel modular group Γ_n if we additionally assume that A is unimodular) and has the Fourier expansion

$$\vartheta_A(Z) = \sum_{\substack{T = T^{\mathrm{t}} \geq 0 \\ T \text{ even}}} r(A,T)\, e^{\pi i \operatorname{tr}(TZ)}.$$

Further, we define the Siegel Eisenstein series of weight k and genus n as

$$E_k(Z) := \sum_{M \in \Gamma_{n,0} \backslash \Gamma_n} \det(CZ+D)^{-k} \quad \text{with } M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right),$$

where we sum over a complete set of left coset representatives of $\Gamma_{n,0} = \{M \in \Gamma_n \mid C = \mathrm{O}\}$ in Γ_n. One can show that E_k is a (non-vanishing) Siegel modular form of weight k on Γ_n for even integers k with $k > n+1$. Let us now assume that $8 \mid m$ and denote by $A_1, \ldots, A_h$ a complete set of representatives of all positive definite unimodular even symmetric $m \times m$-matrices. A first result is given by the analytic version of Siegel's Hauptsatz:

Theorem. *For $m = 2k > 2(n+1)$, we have*

$$E_k(Z) = \sum_{\nu=1}^{h} m_\nu \vartheta_{A_\nu}(Z) \quad \textit{with } m_\nu = \frac{r(A_\nu, A_\nu)^{-1}}{r(A_1, A_1)^{-1} + \ldots + r(A_h, A_h)^{-1}}.$$

One can now determine the Fourier coefficients of the Eisenstein series to describe the weighted sum $\sum_{\nu=1}^{h} m_\nu r(A_\nu, T)$ appearing as the Fourier coefficients of the right-hand side. For $N \in \mathbb{N}$, we introduce the notation $r_N(A, T)$ for the number of solutions $U \in \mathbb{Z}^{m\times n}$ of $U^{\mathrm{t}}AU \equiv T \,(\mathrm{mod}\, N)$, which are distinct modulo N. Further, $\lim_{N\to\infty}$ means here that N runs through a suitable sequence of numbers, for example the sequence $\{2!, 3!, 4!, \ldots\}$, and Γ denotes the usual Gamma function. Beyond that we keep the same assumptions as before and obtain Siegel's Hauptsatz in its original version:

Theorem. *For any even symmetric positive definite matrix $T \in \mathbb{Z}^{n\times n}$, we have*

$$\sum_{\nu=1}^{h} m_\nu r(A_\nu, T) = K_{mn} \det T^{(m-n-1)/2} \lim_{N\to\infty} N^{n((n+1)/2-m)} r_N(A, T)$$

$$\textit{with } K_{mn} := 2^{-n}\pi^{-mn/2} \prod_{\nu=0}^{n-1} \pi^{(m-\nu)/2} \Gamma\Big(\frac{m-\nu}{2}\Big)^{-1}.$$

Note that Siegel [Sie35] proves this theorem under slightly more general assumptions, whereas we followed Freitag's description [Fre83, p. 285–297] here. As one might guess from the right-hand side of the formula above, the proof of the last theorem requires various non-trivial calculations. Then the result follows from the analytic version stated before, which is established by using the theory of Hecke eigenforms and thus involves a profound understanding of these objects itself. However, here it suffices to note the parallels in the approach for fixed genus 1 and for arbitrary genus n and the fact that in both cases we can apply results on elliptic and Siegel theta series to solve number theoretic problems.

Apart from that, there are numerous connections to other areas that motivate the study of theta series. In this Ph. D. dissertation, we focus on the construction of modular forms by considering Siegel and elliptic theta series. In contrast to the examples we have given here, we will consider indefinite quadratic forms and study series that are (in general) non-holomorphic.

1.2 General remarks on notation

Except for Chapters 4 and 5, we use bold letters for vectors, whereas the vector entries just occur in light print, i. e. we write $\boldsymbol{v} = (v_1, \ldots, v_m)$ for $\boldsymbol{v} \in \mathbb{R}^m$. As usual, matrices are represented by capital letters. When we consider a matrix $M \in \mathbb{R}^{m\times n}$ in terms of its column vectors $\boldsymbol{m_i} \in \mathbb{R}^m$, we use the notation $M = (\boldsymbol{m_1} \ldots \boldsymbol{m_n})$. We denote by $\mathbb{N}$ the set of positive integers, whereas $\mathbb{N}_0$ indicates that zero is included. For a better readability, we often write $\exp(z)$ instead of e^z , especially when the exponent is a very long term or contains matrices.

We explain general definitions and notation throughout the next sections. Non-standard definitions are also repeated within the main chapters to enhance readability. Moreover, recurring terms are collected in the index.

1.3 Definitions and previous results

In this section, we give a comprised review of the theory of Siegel and elliptic modular forms with due regard to theta series for positive definite and (non-degenerate) indefinite quadratic forms. In Section 1.3.1, we introduce elliptic modular forms and variations of this definition. First examples of holomorphic modular forms are then provided by theta series attached to positive definite quadratic forms. We have a closer look at the case where the quadratic form is indefinite. In order to do so, we review a widely used modularity criterion, which also allows a quite general construction, in Section 1.3.2. In Section 1.3.3, we give an overview of recent constructions of theta series for indefinite quadratic forms with a special emphasis on signature $(m-1,1)$. In Section 1.3.4, we shortly address the concept of a (harmonic) Maass form. The last part of this introductory section is dedicated to the notion of Siegel modular forms, and again we conclude this part by addressing the construction of explicit examples via Siegel theta series.

1.3.1 Elliptic modular forms and the role of theta series

An interesting survey on elliptic modular forms is, for example, given by Zagier [Zag08], where many connections to other topics in number theory are explained. We follow his exposition here, but focus on the most important definitions and put a special emphasis on theta series.

Let $\mathbb{H}$ denote the *complex upper half plane*

$$\mathbb{H} = \{x + iy = z \in \mathbb{C} \mid y > 0\}.$$

Further, define the *full modular group*

$$\Gamma_1 = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} \,\middle|\, ad - bc = 1 \right\}$$

that acts on $\mathbb{H}$ by the Möbius transformation

$$(\gamma, z) \mapsto \gamma z := \frac{az + b}{cz + d}.$$

Now we can already state the definition of a modular form:

Definition 1.1. Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ and $k \in \mathbb{Z}$. We call f a *(holomorphic) modular form of weight k* if it satisfies the following three conditions:

(a) The function f is holomorphic on $\mathbb{H}$.

(b) For every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$ we have $f(\gamma z) = (cz + d)^k f(z)$.

(c) The function f is holomorphic at infinity, i. e. $f(x + iy) = \mathcal{O}(1)$ for $y \to \infty$.

We often refer to these forms as *elliptic modular forms* as opposed to the notion of Siegel modular forms of higher genus n. When it is clear from the context that we deal with holomorphic elliptic modular forms, we just speak of modular forms. The modular forms above build the finite-dimensional vector space $M_k(\Gamma_1)$ over $\mathbb{C}$. Thus, we may establish several useful identities among them, but we also see that these properties are very restrictive: only if $k \geq 4$ is even, we obtain non-trivial functions satisfying Definition 1.1 at all. Relaxing these conditions allows us to study a bigger variety of functions.

There are numerous variations of condition (a). In general we might just consider real analytic functions on $\mathbb{H}$. One notion that will appear throughout this thesis is that of an *almost holomorphic modular form*. We use this term for functions that transform like modular forms and can be written as a polynomial in $1/y$ with holomorphic coefficients f_i, that is $f(z) = \sum_{i=0}^{d} f_i(z) y^{-i}$. We call d the *depth* of f. The constant term f_0 plays a special role, as this holomorphic function, which we call a *quasimodular form*, determines the almost holomorphic modular form.

There are several possibilities to relax condition (b). First of all, one can restrict oneself to considering the action of a discrete subgroup Γ of Γ_1. (Then we also reformulate condition (c) and require that f is holomorphic at all the cusps of Γ.) Often, one considers subgroups that are defined by certain congruence relations. In this thesis, we only encounter the *congruence subgroup* $\Gamma_0(N)$ of level $N \in \mathbb{N}$, defined as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \,\middle|\, c \equiv 0 \pmod{N} \right\}.$$

Moreover, we can replace $(cz+d)^k$ by an automorphic factor $\varepsilon : \Gamma \times \mathbb{H} \longrightarrow \mathbb{C}$, which is holomorphic in $z \in \mathbb{H}$, and satisfies $|\varepsilon(\gamma, z)| = |cz+d|^k$ and $\varepsilon(\gamma\gamma', z) = \varepsilon(\gamma, \gamma' z)\,\varepsilon(\gamma', z)$ for all $\gamma, \gamma' \in \Gamma$, see for example Rankin's book [Ran77, p. 70–87] for details. In the course of this work, we mostly substitute (b) by the condition

$$f(\gamma z) = \chi(\gamma)\,(cz+d)^k\, f(z) \quad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N),$$

where χ is a character, and we also consider modular forms of half-integral weight k. In this context we define for a non-integer exponent x, as usual: let $z \in \mathbb{C}$, then $z^x := \exp(x \log z)$, where $\log z = \log|z| + i \arg(z)$, $-\pi < \arg(z) \leq \pi$. For non-holomorphic modular forms it is sometimes convenient to consider the automorphic factor as $(cz+d)^{k_1}\,(c\overline{z}+d)^{k_2}$ with $k_1, k_2 \in \frac{1}{2}\mathbb{Z}$ and then give the weight in the form of the pair (k_1, k_2). Note that in our constructions of modular forms, we rather modify the function itself by multiplying by the imaginary part of z to avoid this.

Further, the automorphic factor does not need to be scalar-valued. We can also consider *vector-valued modular forms*, i. e. holomorphic functions $F : \mathbb{H} \longrightarrow \mathbb{C}^r$ that satisfy $F(\gamma z) = M(\gamma)\,(cz+d)^k\, F(z)$ with $M(\gamma) \in \mathbb{C}^{r \times r}$ for $\gamma \in \Gamma_1$. The components of F are scalar-valued modular forms on a congruence subgroup of Γ_1, so this is a different way to represent modular forms on congruence subgroups.

Note that the dimension of $M_k(\Gamma)$ is finite and that we can use the explicit *Sturm bound* (given by Sturm [Stu87] for integral and by Kumar and Purkait [KP14] for half-integral weight) to establish identities among modular forms of weight k on $\Gamma_0(N)$ as is done in Chapter 4.

Before we address a reasonable relaxation of condition (c), we note that a modular form that satisfies the stronger condition of vanishing at the cusp (at all the cusps if we consider a subgroup of Γ_1) is called a *cusp form*.

It is quite common to weaken (c), such that f only needs to be meromorphic at ∞. Then we call f a *weakly holomorphic modular form* and for a fixed weight k we denote the vector space over $\mathbb{C}$ formed by these functions as $M_k^!(\Gamma_1)$. Note that this vector space is already infinite-dimensional for all even k. Again, we can transfer this to modular forms on subgroups.

The concept of a modular form in general seems to be interesting and, as we have seen in the introduction, one can establish number theoretic and arithmetic relations by considering explicit examples. One of the richest sources of examples is the theory of theta series. We fix some notation in order to define these objects.

Definition 1.2. We let $Q : \mathbb{R}^m \longrightarrow \mathbb{R}$ denote a non-degenerate quadratic form of signature (r, s) that is integer-valued (or half-integer-valued) on $\mathbb{Z}^m$. More often, we directly use the matrix A that defines Q, i. e. $Q(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}^{\mathrm{t}} A\boldsymbol{u}$. So $A \in \mathbb{Z}^{m\times m}$ is a non-degenerate symmetric matrix of signature (r, s). We call the smallest $N \in \mathbb{N}$ such that NA^{-1} is an integer matrix the *level* of A. If additionally the diagonal entries of A are even, we call A an *even matrix* (in this case, we also define the level of A as the smallest $N \in \mathbb{N}$ such that NA^{-1} is an even matrix). The associated bilinear form is, as usual, defined by

$$B : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R},\ B(\boldsymbol{u}, \boldsymbol{v}) = Q(\boldsymbol{u} + \boldsymbol{v}) - Q(\boldsymbol{u}) - Q(\boldsymbol{v}).$$

Remark 1.3. In Chapter 4, we assume that the matrix A is even, whereas in Chapter 5, A is not necessarily even.

We take the quadratic form to be fixed and define the elliptic theta series.

Definition 1.4. Let $\lambda \in \mathbb{Z}$. Further, let $p : \mathbb{R}^m \longrightarrow \mathbb{C}$ such that the following series is absolutely convergent. We define the *theta series associated to p with characteristics* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$ as

$$\Theta_{\boldsymbol{a},\boldsymbol{b},p}(z) := y^{-\lambda/2} \sum_{\boldsymbol{\ell}\in\boldsymbol{a}+\mathbb{Z}^m} p(\boldsymbol{\ell} y^{1/2})\, q^{Q(\boldsymbol{\ell})}\, e^{2\pi i B(\boldsymbol{\ell},\boldsymbol{b})} \qquad (q = e^{2\pi i z}).$$

Let Q denote a positive definite quadratic form and P a spherical polynomial of degree d. Further, set $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$. This type of theta series has been studied by Schoeneberg [Sch39] and Ogg [Ogg69] for the case that m is even and by Shimura [Shi73] for the case that m is odd. They showed that $\Theta_P(z) := \sum_{\boldsymbol{\ell}\in\mathbb{Z}^m} P(\boldsymbol{\ell})\, q^{Q(\boldsymbol{\ell})}$ is a holomorphic modular form of weight $m/2 + d$ on some subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and with some character. We have already seen an example of this construction and its connection to representation numbers in Section 1.1.

The construction of modular forms via theta series becomes more intriguing when we consider indefinite quadratic forms. In the next section, we review a result by Vignéras that describes a useful tool to determine the modular transformation behavior of theta series for indefinite quadratic forms and also provides an explicit construction.

1.3.2 Vignéras' modularity criterion for elliptic theta series

In order to check whether a theta series transforms like a modular form, one considers the generators of Γ_1, namely the transformations $z \mapsto z + 1$ and $z \mapsto -1/z$. Under the translation $z \mapsto z + 1$, the form of the theta series usually allows us to check by a straightforward calculation that this transformation preserves the theta series up to a constant factor depending on the characteristic and the quadratic form. The second case is a bit more complicated: one calculates the Fourier transform of the summand in the theta series and checks whether it is more or less an eigenfunction with regard to the Fourier transform. If that is the case, one can apply the Poisson summation formula to determine the transformation behavior of the theta series. Then $\Theta_{\boldsymbol{a},\boldsymbol{b},p}(-1/z)$ can be written as a finite linear combination of theta series $\Theta_{\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{b}},p}(z)$ with altered characteristics $\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}}$ and we say that it "transforms like a modular form" – the precise transformation behavior can then be determined depending on the explicit form of the given theta series.

Following these steps, Vignéras [Vig75, Vig77] gives a quite simple criterion to determine modularity, which is widely used in the construction of modular theta series. This criterion is applicable to theta series for non-degenerate quadratic forms of arbitrary signature (r,s). We apply this result in Chapters 4 and 5 and we also use a part of Vignéras' argument in Chapter 2 when deriving an analogous result for Siegel theta series, so we recapitulate the assumptions and the main ingredients of the proof within this section.

We use the usual multi-index notation on $\mathbb{R}^m$: for $\boldsymbol{\alpha} \in \mathbb{N}_0^m$ we have $|\boldsymbol{\alpha}| := \sum_{i=1}^m \alpha_i$ and

$$\partial^{\boldsymbol{\alpha}} := \prod_{i=1}^{m} \Big(\frac{\partial}{\partial u_i}\Big)^{\alpha_i}.$$

For $p \in [1,\infty)$ let $\mathcal{L}^p(\mathbb{R}^m)$ denote the Lebesgue space of functions $f : \mathbb{R}^m \longrightarrow \mathbb{C}$ for which

$$\|f\|_p := \Bigg(\int\limits_{\mathbb{R}^m} |f(\boldsymbol{x})|^p \, d\boldsymbol{x}\Bigg)^{1/p}$$

is finite. Further, we define the following differential operators:

Definition 1.5. Let $\boldsymbol{u} \in \mathbb{R}^m$, then we define the Euler operator

$$\mathcal{E} := \boldsymbol{u}^{\mathrm{t}} \frac{\partial}{\partial \boldsymbol{u}} = \sum_{d=1}^{m} u_d \frac{\partial}{\partial u_d}$$

and the Laplace operator associated to A

$$\Delta_A := \Big(\frac{\partial}{\partial \boldsymbol{u}}\Big)^{\mathrm{t}} A^{-1} \frac{\partial}{\partial \boldsymbol{u}} = \sum_{a,b=1}^{m} \frac{\partial}{\partial u_a} (A^{-1})_{ab} \frac{\partial}{\partial u_b}.$$

Further, we set $D := \mathcal{E} - \Delta_A/4\pi$.

According to Vignéras, we then have the following theorem to characterize modular forms:

Theorem 1.6 ([Vig75, Theorem 1])**.** *Let $\lambda \in \mathbb{Z}$. We set $\widetilde{p}(\boldsymbol{u}) := p(\boldsymbol{u}) \exp\big(-2\pi Q(\boldsymbol{u})\big)$. Assume that for any polynomial $R : \mathbb{R}^m \longrightarrow \mathbb{R}$ with $\deg(R) \le 2$ and any partial derivative $\partial^{\boldsymbol{\alpha}}$ with $|\boldsymbol{\alpha}| \le 2$, the function $\widetilde{p}$ satisfies the growth condition*

$$R \cdot \widetilde{p} \in \mathcal{L}^1(\mathbb{R}^m) \cap \mathcal{L}^2(\mathbb{R}^m) \quad \textit{and} \quad \partial^{\boldsymbol{\alpha}} \widetilde{p} \in \mathcal{L}^1(\mathbb{R}^m) \cap \mathcal{L}^2(\mathbb{R}^m). \tag{1.1}$$

Further assume that p is a solution of the differential equation $Dp = \lambda p$. Then $\Theta_{\boldsymbol{a},\boldsymbol{b},p}$ transforms like a modular form of weight $m/2 + \lambda$ for some character χ and on some subgroup of Γ_1 both depending on the level of the quadratic form.

The proof can be summarized as follows: By writing $\boldsymbol{u} = (u_1, \dots, u_m)$ in a suitable basis of $\mathbb{R}^m$, we can replace the non-degenerate quadratic form Q of signature (r,s) by the normalized quadratic form $\boldsymbol{u} \mapsto \frac{1}{2}(u_1^2 + \ldots + u_r^2 - u_{r+1}^2 - \ldots - u_m^2)$. For $\boldsymbol{k} = (k_1, \dots, k_m) \in \mathbb{N}_0^m$, the multi-dimensional *Hermite functions*

$$H_{\boldsymbol{k}}(\boldsymbol{u}) := \prod_{\mu=1}^{m} H_{k_\mu}(u_\mu) \quad \text{with } H_{k_\mu}(u_\mu) = e^{\pi u_\mu^2} \frac{\partial^{k_\mu}}{\partial u_\mu^{k_\mu}} \Big(e^{-2\pi u_\mu^2}\Big) \tag{1.2}$$

form an orthogonal basis of $\mathcal{L}^2(\mathbb{R}^m)$ with regard to the inner product

$$\langle f, g\rangle := \int_{\mathbb{R}^m} f(\boldsymbol{u})g(\boldsymbol{u})\, d\boldsymbol{u} \quad \text{for } f, g \in \mathcal{L}^2(\mathbb{R}^m)$$

so we can write any $\widetilde{p}$ that satisfies the growth condition (1.1) as $\widetilde{p} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} c_{\boldsymbol{k}} H_{\boldsymbol{k}}$ with $c_{\boldsymbol{k}} \in \mathbb{R}$. Considering the Hermite function $H_{\boldsymbol{k}}$ for a fixed $\boldsymbol{k}$, $H_{\boldsymbol{k}}$ itself is an eigenfunction with respect to the Fourier transform and it satisfies a certain second order differential equation depending on $\varepsilon(\boldsymbol{k}) := k_1 + \ldots + k_r - k_{r+1} - \ldots - k_m$.

The first property is used by Vignéras to construct modular theta series. We do not need the explicit construction in the further course of this thesis and thus omit further details here. However, from the second property it follows that if $p(\boldsymbol{u}) = \widetilde{p}(\boldsymbol{u}) \exp\bigl(2\pi Q(\boldsymbol{u})\bigr)$ satisfies $Dp = \lambda p$ for a fixed $\lambda \in \mathbb{Z}$, the indices $\boldsymbol{k}$ with $c_{\boldsymbol{k}} \neq 0$ are determined by λ, namely $\boldsymbol{k}$ has to satisfy $\lambda = \varepsilon(\boldsymbol{k}) - s$. If the quadratic form is positive definite, we immediately deduce that there remain only finitely many possibilities to choose $\boldsymbol{k}$, which gives us a finite basis for the vector space of solutions of $Dp = \lambda p$. Moreover, we only have polynomial solutions p, which follows immediately by considering the form of the Hermite functions in (1.2).

If the quadratic form is indefinite, we use the fact that the differential operator D (for the diagonalized quadratic form as we consider it here) as well as the Hermite functions can be separated into parts that depend on subspaces of $\mathbb{R}^m$ where the quadratic form is positive or negative definite, respectively. For $n = 1$, this is already sufficient to argue that $\widetilde{p}$ and thus also all possible solutions $p(\boldsymbol{u}) = \widetilde{p}(\boldsymbol{u}) \exp\bigl(2\pi Q(\boldsymbol{u})\bigr)$ of $Dp = \lambda p$ are described by a (now possibly infinite) linear combination of functions $H_{\boldsymbol{k}}(\boldsymbol{u}) \exp\bigl(2\pi Q(\boldsymbol{u})\bigr)$ with $\boldsymbol{k}$ such that $\lambda = \varepsilon(\boldsymbol{k}) - s$.

For Siegel theta series of higher genus n, namely in the proofs of Proposition 2.19 and 2.23, we use the same argument to describe the polynomial part of the solution. However, the final conclusion that a basis of all solutions is provided by functions for which the associated theta series transforms like a modular form requires additional argumentation.

Remark 1.7. (a) If $\widetilde{p}$ is a Schwartz function, it satisfies in particular the growth condition (1.1). Throughout this thesis, we only consider Schwartz functions.
(b) Vignéras also gives a formula for the modular transformation behavior of $\Theta_{\boldsymbol{a},\boldsymbol{b},p}$ on the congruence subgroup $\Gamma_0(N)$. We review this result in Theorem 4.6.

1.3.3 Explicit constructions of theta series for indefinite quadratic forms

Vignéras' result does not only provide a modularity criterion but also gives a very nice construction of modular theta series. In particular, the construction of a theta series for a positive definite quadratic form that we have seen at the end of Section 1.3.1 is a special case thereof. Moreover, Siegel [Sie51] constructs non-holomorphic theta series for non-degenerate quadratic forms of arbitrary signature and ensures the convergence by using a majorant associated to the quadratic form. These series can be seen as an explicit realization of Vignéras' result as well.

Borcherds [Bor98] provides a similar construction but also includes polynomials in the definition of the theta series. We examine his result a bit further, as we will see some parallels in our construction of Siegel theta series.

Borcherds considers the non-degenerate quadratic form Q with signature (r, s), an even lattice $L \subset \mathbb{R}^m$ with the associated dual lattice L' and an isometry v mapping $L \otimes \mathbb{R}$ to $\mathbb{R}^{r,s}$. Considering the inverse images v^+ and v^- of $\mathbb{R}^{r,0}$ and $\mathbb{R}^{0,s}$ under v, one decomposes $L\otimes\mathbb{R}$ in the orthogonal direct sum of a positive definite subspace v^+ and a negative definite subspace v^-. For the projection of $\boldsymbol{\lambda} \in L \otimes \mathbb{R}$ into $v^\pm$ one writes $\boldsymbol{\lambda}_{v^\pm}$ and obtains the positive definite quadratic form $Q_v(\boldsymbol{\lambda}) = Q(\boldsymbol{\lambda}_{v^+}) - Q(\boldsymbol{\lambda}_{v^-})$. As the decomposition into the subspaces v^+ and v^- is not unique, Borcherds' theta series include an additional parameter to indicate the choice of $v^+ \in G(M)$, where the Grassmannian $G(M)$ denotes the set of positive definite r-dimensional subspaces of $L \otimes \mathbb{R}$. For $z \in \mathbb{H}, \boldsymbol{h}, \boldsymbol{k} \in L \otimes \mathbb{R}, \boldsymbol{\gamma} \in L'/L$, Δ the Laplacian on $\mathbb{R}^m$, and $p : \mathbb{R}^m \longrightarrow \mathbb{R}$ a polynomial that is homogeneous of degree α in the first r variables and homogeneous of degree β in the last s variables, he defines

$$\theta_{L+\boldsymbol{\gamma}}(z, \boldsymbol{h}, \boldsymbol{k}; v, p) := \sum_{\boldsymbol{\lambda} \in L+\boldsymbol{\gamma}} \left(e^{-\Delta/8\pi y} p\right)\left(v(\boldsymbol{\lambda} + \boldsymbol{h})\right) \cdot e^{2\pi i \left(z(\boldsymbol{\lambda}+\boldsymbol{h})^2_{v^+}/2 + \overline{z}(\boldsymbol{\lambda}+\boldsymbol{h})^2_{v^-}/2 - (\boldsymbol{\lambda}+\boldsymbol{h}/2, \boldsymbol{k})\right)}$$

and shows that this is a non-holomorphic modular form of weight $(r/2 + \alpha, s/2 + \beta)$.

So we can obtain modular forms via theta series, but we have to ensure the convergence of the sum that defines the theta series and, in general, we will obtain non-holomorphic series. A different approach opposed to using majorants, is the following: For a quadratic form of signature $(m-1, 1)$, Göttsche and Zagier [GZ98] construct indefinite theta functions by restricting the summation in the series to a cone where the quadratic form can be bounded by a positive definite quadratic form. We can sum up the construction as follows: let Q be an integer-valued quadratic form of signature $(m-1, 1)$ on $\mathbb{Z}^m$, let B be the bilinear form associated to Q and fix a vector $\boldsymbol{c_0} \in \mathbb{R}^m$ with $Q(\boldsymbol{c_0}) < 0$. Then let

$$C_Q := \{\boldsymbol{c} \in \mathbb{R}^m \mid Q(\boldsymbol{c}) < 0,\ B(\boldsymbol{c}, \boldsymbol{c_0}) < 0\}$$

denote one component of the light cone in $\mathbb{R}^m$, where the quadratic form is negative, and

$$S_Q := \{\boldsymbol{c} \in \mathbb{Q}^m \mid Q(\boldsymbol{c}) = 0,\ B(\boldsymbol{c}, \boldsymbol{c_0}) < 0\}$$

the set of cusps of C_Q. Let $\boldsymbol{c_1}, \boldsymbol{c_2} \in \overline{C}_Q := C_Q \cup S_Q$ and let $\boldsymbol{a} \in R(\boldsymbol{c_1}) \cap R(\boldsymbol{c_2})$, $\boldsymbol{b} \in \mathbb{R}^m$, where $R(\boldsymbol{c})$ is $\mathbb{R}^m$ if $\boldsymbol{c} \in C_Q$ and $\{\boldsymbol{a} \in \mathbb{R}^m \mid B(\boldsymbol{c}, \boldsymbol{a}) \notin \mathbb{Z}\}$ if $\boldsymbol{c} \in S_Q$, then

$$\Theta^{\boldsymbol{c_1},\boldsymbol{c_2}}_{\boldsymbol{a},\boldsymbol{b}}(\tau) := \sum_{\boldsymbol{\ell} \in \boldsymbol{a}+\mathbb{Z}^m} \left\{\operatorname{sgn}(B(\boldsymbol{c_1}, \boldsymbol{\ell})) - \operatorname{sgn}(B(\boldsymbol{c_2}, \boldsymbol{\ell}))\right\} q^{Q(\boldsymbol{\ell})} e^{2\pi i B(\boldsymbol{\ell}, \boldsymbol{b})}. \tag{1.3}$$

These theta series are holomorphic and for the case $\boldsymbol{c_1}, \boldsymbol{c_2} \in S_Q$ it is shown in [GZ98] that they are also modular. For $\boldsymbol{c_1}, \boldsymbol{c_2} \in C_Q$ the function $\Theta^{\boldsymbol{c_1},\boldsymbol{c_2}}_{\boldsymbol{a},\boldsymbol{b}}$ is in general not modular. However, special choices of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c_1}, \boldsymbol{c_2} \in C_Q$ (see the articles by Andrews [And84] and Polishchuk [Pol01] for examples) give modular theta series. Note that for signature $(1, 1)$ these theta series are related to the indefinite theta functions constructed by Hecke [Hec25, Hec27].

Zwegers [Zwe02] defines a modular completion of the holomorphic theta series in (1.3). Replacing the sign function by the error function

$$E(z) := 2\int_0^z e^{-\pi u^2} du = \operatorname{sgn}(z) - \operatorname{sgn}(z) \int_{z^2}^\infty u^{-1/2} e^{-\pi u} du$$

he constructs a modular version of this theta series as follows: for $\boldsymbol{c_1}, \boldsymbol{c_2} \in \overline{C}_Q$ set

$$\widehat{\Theta}^{\boldsymbol{c_1},\boldsymbol{c_2}}_{\boldsymbol{a},\boldsymbol{b}}(\tau) := \sum_{\boldsymbol{\ell} \in \boldsymbol{a}+\mathbb{Z}^m} \left\{p^{\boldsymbol{c_1}}(\boldsymbol{\ell} y^{1/2}) - p^{\boldsymbol{c_2}}(\boldsymbol{\ell} y^{1/2})\right\} q^{Q(\ell)} e^{2\pi i B(\ell, b)},$$

where

$$p^{\boldsymbol{c}}(\boldsymbol{v}) := \begin{cases} E\Big(\frac{B(\boldsymbol{c},\boldsymbol{v})}{\sqrt{-Q(\boldsymbol{c})}}\Big) & \text{if } \boldsymbol{c} \in C_Q, \\ \operatorname{sgn}(B(\boldsymbol{c},\boldsymbol{v})) & \text{if } \boldsymbol{c} \in S_Q. \end{cases}$$

In [Zwe02] it is shown that $\widehat{\Theta}^{\boldsymbol{c_1},\boldsymbol{c_2}}_{\boldsymbol{a},\boldsymbol{b}}$ is modular of weight $m/2$. As these theta series are in general not holomorphic, he further investigates when the modular and the holomorphic version agree in order to obtain a holomorphic modular form. Let

$$\operatorname{Aut}^+(Q,\mathbb{Z}^m) := \big\{ g \in \operatorname{GL}_m(\mathbb{Z}) \mid g^{\mathrm{t}} A g = A,\ B(g\boldsymbol{c},\boldsymbol{c}) < 0 \text{ for all } \boldsymbol{c} \in C_Q \big\}$$

denote the automorphism group that leaves the quadratic form Q, the lattice $\mathbb{Z}^m$ and the choice of C_Q unchanged. Then

$$\widehat{\Theta}^{g\boldsymbol{c_1},g\boldsymbol{c_2}}_{g\boldsymbol{a},g\boldsymbol{b}} = \widehat{\Theta}^{\boldsymbol{c_1},\boldsymbol{c_2}}_{\boldsymbol{a},\boldsymbol{b}}$$

holds and for $\boldsymbol{c_1}, \boldsymbol{c_2}$ and $\boldsymbol{a}, \boldsymbol{b}$ such that the conditions $g\boldsymbol{c_i} = \boldsymbol{c_i}$ for $i \in \{1,2\}$, and $g^{-1}\boldsymbol{a} \in \boldsymbol{a} + \mathbb{Z}^m$, $g^{-1}\boldsymbol{b} \in \boldsymbol{b} + \mathbb{Z}^m$ are satisfied, it is shown in [Zwe02] that the non-holomorphic part of the theta series vanishes. We will extend this result in Chapters 4 and 5 by including homogeneous and spherical polynomials in the definition of the theta series.

Analogous constructions can be found for quadratic forms of general signature: Alexandrov, Banerjee, Manschot, and Pioline [ABMP18a] give a construction for signature $(m-2,2)$ by considering two pairs $\{\boldsymbol{c_1}, \boldsymbol{c_1'}\}, \{\boldsymbol{c_2}, \boldsymbol{c_2'}\}$ of vectors in $\mathbb{R}^m$. By imposing suitable conditions on these vectors they construct holomorphic theta series and modular completions thereof (for the latter, they introduce generalized error functions). They also outline a generalization to arbitrary signature (r,s), which is then explicitly realized by Nazaroglu [Naz18]: here, s pairs $\{\boldsymbol{c_i}, \boldsymbol{c_i'}\}$ are needed and the construction requires slightly stricter prerequisites to ensure the convergence of the theta series.

We emphasize these two constructions since they extend the methods used in [Zwe02]. However, one should also mention Westerholt-Raum's more abstract approach [WR16] and the connection to the theta forms introduced by Kudla and Millson [KM86, KM87], which is described by Funke and Kudla [FK17, FK19] and in the special cases $s = 1, 2$ by Kudla [Kud13, Kud18] and Livinsky [Liv16]. Furthermore, Alexandrov, Banerjee, Manschot, and Pioline [ABMP18b] suggest (for signature $(m-2,2)$) a construction of theta series for N-gons. In a recent preprint, Funke and Kudla [FK21] explicitly realize this construction and outline a generalization to arbitrary signature (r,s).

We will not give any further details concerning these interesting constructions of theta series since we primarily work with Zwegers' construction for signature $(m-1,1)$.

1.3.4 Harmonic Maass forms and mock modular forms

Among real analytic modular forms (that we can, for example, construct via theta series as in the last two sections), we address the concept of a *Maass form*. We only provide the most important definitions here and refer to the expositions by Ono [Ono08] and Bringmann, Folsom, Ono, and Rolen [BFOR17] for a more detailed description.

For $k \in \frac{1}{2}\mathbb{Z}$, we introduce the *hyperbolic Laplacian of weight k* that is defined as

$$\Delta_k := -4y^2 \frac{\partial^2}{\partial z \partial \overline{z}} + 2iky\frac{\partial}{\partial \overline{z}} = -y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) + iky\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) \quad (z = x+iy).$$

Maass forms are real analytic functions that transform as modular forms of weight k and are eigenfunctions of Δ_k. We will focus on harmonic Maass forms here, i. e. forms that vanish under the hypberbolic Laplacian:

Definition 1.8. We call a real analytic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ a *harmonic Maass form of weight* k (on Γ with character χ) if the following conditions are satisfied:

(a) We have $f(\gamma z) = \chi(\gamma)(cz+d)^k f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

(b) We have $\Delta_k f = 0$.

(c) There exists a polynomial $P_f(z) \in \mathbb{C}[q^{-1}]$ such that

$$f(z) - P_f(z) = \mathcal{O}(e^{-\varepsilon y}) \quad \text{for } y \to \infty$$

for some $\varepsilon > 0$. At all the other cusps of Γ we require analogous growth conditions.

The differential operator $\xi_k(f) := 2iy^k \overline{\frac{\partial f}{\partial \overline{z}}}$ maps harmonic Maass forms of weight k to weakly holomorphic modular forms of weight $2-k$ (see Bruinier's and Funke's work [BF04]). It is common to call the image of a harmonic Maass form f under the ξ_k-operator *shadow* of f. Further, one can show that a harmonic Maass form has a unique decomposition $f = f^+ + f^-$ into a holomorphic part f^+ and a non-holomorphic part f^-. If f^- is zero, we just have a (weakly holomorphic) modular form of weight k. Otherwise we refer to the holomorphic part f^+ as *mock modular form*. If the shadow of f has the form of a theta series, we say that f^+ is a *mock theta function*.

1.3.5 Siegel modular forms and Siegel theta series

From the end of the 19th century on, Picard [Pic82], Blumenthal [Blu03, Blu04], Hecke [Hec11, Hec13, Hec24], and Braun [Bra38, Bra39] (among others) investigated generalizations of elliptic modular forms. We focus on functions that are defined on the space of complex $n \times n$-matrices with positive definite imaginary part. These are nowadays called Siegel modular forms since a first systematic description was given by Siegel [Sie35, Sie39], including the substantial result known as "Siegel's Hauptsatz" that we have stated in the introduction. Concerning the theory of Siegel modular forms, the books by Andrianov [And09] and Freitag [Fre83] and v. d. Geer's exposition [vdG08] serve as good introductory texts emphasizing different aspects of this subject. We follow these references here.

First of all, let us fix a generalization of the complex upper half plane, the so called *Siegel upper half-space* that is defined as

$$\mathbb{H}_n := \{Z = X + iY \mid X, Y \in \mathbb{R}^{n\times n} \text{ symmetric}, Y > 0\},$$

where we use the notation $Y > 0$ for symmetric matrices that are positive definite and consequently $Y \geq 0$ for positive semi-definite matrices. Further, for $Y > 0$ we let $Y^{1/2}$ denote the uniquely determined symmetric positive definite matrix, for which $Y^{1/2} \cdot Y^{1/2} = Y$ holds.
We define modular forms on $\mathbb{H}_n$ for the *full Siegel modular group*

$$\Gamma_n := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{Z}^{2n\times 2n} \,\middle|\, M^{\mathrm{t}} J M = J \right\}, \quad \text{where } J = \begin{pmatrix} \mathrm{O} & I_n \\ -I_n & \mathrm{O} \end{pmatrix},$$

which operates on $\mathbb{H}_n$ by

$$Z \mapsto M\langle Z\rangle = (AZ+B)(CZ+D)^{-1}.$$

One can show that this is indeed a group action on $\mathbb{H}_n$ by verifying that this map is well-defined, maps $\mathbb{H}_n$ into itself and satisfies $(MM')\langle Z\rangle = M\langle M'\langle Z\rangle\rangle$ for $M, M' \in \Gamma_n$. We mention one relation that is obtained while showing the second point: the imaginary part Y of Z and the imaginary part $\widetilde{Y}$ of $M\langle Z\rangle$ satisfy

$$(C\overline{Z} + D)^{\mathrm{t}}\widetilde{Y}(CZ + D) = Y.$$

In particular, $\widetilde{Y}$ is positive definite and symmetric.

Definition 1.9. We call $f : \mathbb{H}_n \longrightarrow \mathbb{C}$ a *(classical) Siegel modular form of genus n and weight k* if the following conditions hold:

(a) The function f is holomorphic on $\mathbb{H}_n$.

(b) For every $M \in \Gamma_n$ we have $f(M\langle Z\rangle) = \det(CZ + D)^k f(Z)$.

(c) $|F(Z)|$ is bounded on domains in $\mathbb{H}_n$ of the form $\mathbb{H}_n^\varepsilon := \{X + iY \in \mathbb{H}_n \mid Y \geq \varepsilon \cdot I\}$ with $\varepsilon > 0$.

For $n > 1$, condition (c) is automatically satisfied due to the *Koecher principle*, see [Koe53]. In order to state this principle we recall that an even matrix is a symmetric integer-valued matrix with even diagonal entries.

Proposition 1.10 (Koecher principle)**.** *Let $n > 1$ and let $f : \mathbb{H}_n \longrightarrow \mathbb{C}$ satisfy (a) and (b), then f has a Fourier expansion of the form*

$$f(Z) = \sum_{V \ \text{even}, V \geq 0} a(V) \exp\bigl(\pi i \operatorname{tr}(VZ)\bigr).$$

Moreover, $f(Z)$ is bounded on any subset $\{Z \in \mathbb{H}_n \mid \operatorname{Im}(Z) > \varepsilon \cdot I\}$ for $\varepsilon > 0$.

Remark 1.11. For non-holomorphic modular forms, the Koecher principle does not necessarily hold anymore. In our case, we build Siegel theta series by using Schwartz functions and obtain absolutely convergent series. These series also admit a Fourier expansion and thus we can still apply the Koecher principle. Note that the Koecher principle still holds on congruence subgroups of Γ_n, see for example [And09, p. 22f.].

As for elliptic modular forms, the conditions in Definition 1.9 might be weakened to obtain a bigger variety of functions. Most concepts that were described above for elliptic forms have a straightforward generalization for higher genus n. For example, one can define an almost holomorphic Siegel modular form as a function that satisfies condition (b) and can be written as a polynomial in $1/\det Y$ with holomorphic coefficients.

When we consider congruence subgroups $\Gamma_0(N)$ of Γ_1, the congruence condition applies to the lower left entry of the matrices in $\Gamma_0(N)$. The analogue for the Siegel modular group is the subgroup that consists of matrices $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n$ with $C \equiv \mathrm{O} \pmod N$. In the vector-valued setting, we can replace the automorphic factor $\det(CZ + D)^k$ by $\rho(CZ + D)$, where $\rho : \mathrm{SL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}(V)$ is a representation in the automorphisms of a $\mathbb{C}$-vector space V.

Remark 1.12. In this thesis, we generally obtain real analytic Siegel theta series that can be seen as entries of vector-valued Siegel modular forms, so we just say that the series we consider have modular transformation properties. We do not compute the automorphic factor for general modular substitutions of Γ_n here but restrict ourselves to describing the transformation behavior for the generators of Γ_n. For an explicit description of the modular transformation behavior of Siegel theta series for positive definite quadratic forms on congruence subgroups see for example the work by Andrianov and Maloletkin [AM75].

As in the last section, we construct Siegel theta series as examples of Siegel modular forms.

Definition 1.13. For $A \in \mathbb{Z}^{m\times m}$ a non-degenerate symmetric matrix of signature (r, s), we define the (indefinite) quadratic form $\boldsymbol{Q}(U) := \frac{1}{2}\operatorname{tr}(U^{\mathrm{t}}AU)$ on $\mathbb{R}^{m\times n}$.

Then we define the Siegel theta series for an indefinite quadratic form as follows:

Definition 1.14. Let $\mathcal{H}, \mathcal{K} \in \mathbb{R}^{m\times n}$ and let $\lambda \in \mathbb{Z}$. Further, let $p : \mathbb{R}^{m\times n} \longrightarrow \mathbb{C}$ such that the following series is absolutely convergent. The *theta series with characteristics $\mathcal{H}$ and $\mathcal{K}$ associated to p and A* is

$$\vartheta_{\mathcal{H},\mathcal{K},p,A}(Z) := \det Y^{-\lambda/2} \sum_{U\in\mathcal{H}+\mathbb{Z}^{m\times n}} p(UY^{1/2}) \exp\bigl(\pi i \operatorname{tr}(U^{\mathrm{t}}AUZ) + 2\pi i \operatorname{tr}(\mathcal{K}^{\mathrm{t}}AU)\bigr).$$

Remark 1.15. As in the elliptic case, the absolute convergence will become apparent in the explicit construction of the Siegel theta series. Depending on the context we drop some of the parameters:
(a) In Chapter 2, we do *not* assume that A is even as we also consider matrices with 1's and -1's on the diagonal to simplify certain proofs. Further, we take A and p to be fixed in the definition of the theta series and these parameters do not change when we consider modular substitutions of Z (in contrast to the characteristics $\mathcal{H}, \mathcal{K}$), so we write $\vartheta_{\mathcal{H},\mathcal{K}}$ instead of $\vartheta_{\mathcal{H},\mathcal{K},p,A}$.
(b) In Chapter 3, we construct two versions of Siegel theta series, which have different properties according to the functions p that we associate. In both cases we have $\lambda = 0$. Again, we take A to be fixed, so we drop this parameter in the index. Besides, we do not determine the explicit transformation behavior (as we do that in Chapter 2 already), so we also disregard the characteristics $\mathcal{H}, \mathcal{K}$. Thus we write ϑ_p instead of $\vartheta_{\mathcal{H},\mathcal{K},p,A}$.

If A is positive definite, one can construct a holomorphic Siegel modular form as follows: Let P be a spherical polynomial of degree $\alpha \in \mathbb{N}_0$, i. e. $P : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ satisfies the *homogeneity property* $P(UN) = \det N^{\alpha} P(U)$ for all $N \in \mathbb{C}^{n\times n}$ and P vanishes when we apply the operator $\operatorname{tr}\boldsymbol{\Delta}_A$, where

$$\boldsymbol{\Delta}_A := \Bigl(\frac{\partial}{\partial U}\Bigr)^{\mathrm{t}} A^{-1} \frac{\partial}{\partial U}.$$

If we additionally require that the matrix A is unimodular (which implies $8|m$), we get that $\vartheta_{\mathrm{O},\mathrm{O},P,A}$ is a holomorphic Siegel modular form of genus n and weight $m/2 + \alpha$ on the full Siegel modular group. We keep this construction in mind, when we describe Siegel theta series for quadratic forms of arbitrary signature (r, s).

We will shortly address the notion of a Siegel–Maass form in Chapter 6. We refrain here from giving an exact definition but just note that the concept is a natural generalization of the definition of a (harmonic) Maass–form in Section 1.3.4.

1.4 Outline and summary of the main results

The review of the known results in this area provides a good framework for the four research articles that we present in the following chapters. Up to some minor changes concerning typographical errors and notational adjustments, these chapters contain the articles in its original form. Therefore, each chapter starts with an introduction into the topic and a section about notation and preliminaries and ends with a list of references. For this reason, each chapter can be read independently. However, we use results from Chapter 2 in Chapter 3, and Chapter 5 can be seen as an extension of Chapter 4, so it seems advisable to maintain this order. Here, we briefly summarize the main results.

1.4.1 A modularity criterion for Siegel theta series

In Chapter 2, we give a generalization of Vignéras' modularity criterion [Vig77] to Siegel theta series of arbitrary genus $n \in \mathbb{N}$. First, we find an analogue to the differential equation of second order that Vignéras considers by introducing the $n \times n$-system of partial differential equations of second order

$$\Big(\mathbf{E} - \frac{\boldsymbol{\Delta}_A}{4\pi}\Big)p = \lambda I_n p \quad (\lambda \in \mathbb{Z}), \tag{1.4}$$

where $\mathbf{E}$ and $\boldsymbol{\Delta}_A$ are generalized versions of the Euler operator and the Laplacian associated to an indefinite matrix $A \in \mathbb{Z}^{m\times m}$, respectively. In the second part, we explicitly construct Siegel theta series: For a positive definite quadratic form, it is sufficient to slightly generalize the known holomorphic constructions that we have seen in Section 1.3.5 to the construction of non-holomorphic theta series with modular transformation behavior. For indefinite quadratic forms, we provide a new construction, which is very similar to Borcherds' construction of elliptic theta series in [Bor98]. We choose homogeneous polynomials in the construction of the theta series and ensure convergence by using the majorant matrix associated to the indefinite matrix, which gives us non-holomorphic Siegel theta series with modular transformation properties. Finally, we come back to the first part and show that the functions that we use in the construction of the theta series satisfy the system of partial differential equations we have set up before. In particular, they describe a basis of all solutions – thus we have established a modularity criterion for Siegel theta series that proves to be an analogue to Vignéras' result. In a simplified version, choosing the characteristics $\mathcal{H} = \mathcal{K} = \mathrm{O}$, we can state this result as follows: if p satisfies a certain growth condition and is a solution of (1.4), the associated Siegel theta series

$$\vartheta_p(Z) := \det Y^{-\lambda/2} \sum_{U\in\mathbb{Z}^{m\times n}} p(UY^{1/2}) \exp\big(\pi i \operatorname{tr}(U^{\mathrm{t}}AUZ)\big) \quad (\mathbb{H}_n \ni Z = X + iY)$$

transforms like a Siegel modular form of genus n and weight $m/2 + \lambda$.

1.4.2 Siegel theta series for quadratic forms of signature $(m-1, 1)$

Since the Siegel theta series for indefinite quadratic forms that are constructed in Chapter 2 are in general non-holomorphic, the question arises, whether one can describe the holomorphic part of such theta series. Investigating this problem is the subject of Chapter 3. As the construction above depends on the signature (r, s) of the quadratic form, we just consider quadratic forms of signature $(m-1, 1)$ and build upon Zwegers' description [Zwe02] of elliptic theta series for this specific signature. We show that in a similar way we can construct holomorphic Siegel theta series and an associated modular version of this series. For the holomorphic series, it is quite straightforward to generalize Zwegers' result to arbitrary genus $n \in \mathbb{N}$: We restrict the summation to a subset of the lattice on which the quadratic form is bounded by a positive definite quadratic form to ensure the convergence of the series and, as we take locally constant functions, we obtain a holomorphic theta series. For our purposes, we can consider the quadratic form Q on $\mathbb{R}^m$ instead of $\boldsymbol{Q}$ on $\mathbb{R}^{m\times n}$ (both are defined by the same matrix A, though). We fix $n+1$ vectors in $\mathbb{R}^m$ that lie in C_Q and collect them in a matrix

$$C := (\boldsymbol{c_0}\,\boldsymbol{c_1}\ldots\boldsymbol{c_n}) \in \mathbb{R}^{m\times(n+1)} \quad \text{with } \boldsymbol{c_i} \in C_Q \subset \mathbb{R}^m.$$

Then we consider the matrices

$$\widetilde{C}_i := (\boldsymbol{c_0}\ldots\widehat{\boldsymbol{c_i}}\ldots\boldsymbol{c_n}) \in \mathbb{R}^{m\times n} \quad (0 \le i \le n),$$

where $\widehat{\cdot}$ means that the respective column is omitted. Further, let $\mathbb{R}^n \ni \boldsymbol{x_i} := U^{\mathrm{t}}A\boldsymbol{c_i}$ for $0 \le i \le n$ and define $\widetilde{x}_i$ by setting

$$\widetilde{x}_i := (-1)^i \det\bigl(\boldsymbol{x_0}\ldots\widehat{\boldsymbol{x_i}}\ldots\boldsymbol{x_n}\bigr) = (-1)^i \det\bigl(U^{\mathrm{t}}A\widetilde{C}_i\bigr) \quad (0 \le i \le n).$$

If we then set

$$f(U) := \prod_{i=0}^{n} \frac{\operatorname{sgn}(\widetilde{x}_i)+1}{2} - \prod_{i=0}^{n} \frac{\operatorname{sgn}(\widetilde{x}_i)-1}{2},$$

the theta series ϑ_f is defined by an absolutely convergent series and is holomorphic in $Z \in \mathbb{H}_n$.

In order to describe a modular Siegel theta series, we have to introduce some more notation. Let B denote the bilinear form associated to Q and let A^- denote the negative semi-definite part of A. Further, $\boldsymbol{u}^\perp$ is the projection of $\boldsymbol{u} \in \mathbb{R}^m$ onto a subspace where the quadratic form defined by A is positive definite, and S_n is the n-simplex whose vertices lie in one of the components of $\mathbb{R}^m$, where Q is negative. For

$$g(U) := \int_{S_n} \bigl(-Q(\boldsymbol{c})\bigr)^{n/2} \exp\bigl(2\pi \operatorname{tr}(U^{\mathrm{t}}A^-U)\bigr) \bigwedge_{j=1}^{n} B(\boldsymbol{u_j}^\perp, d\boldsymbol{c}),$$

the theta series ϑ_g (for $\lambda = 0$) transforms like a Siegel modular form of genus n and weight $m/2$. It is easy to deduce modularity and determine the weight, when we observe that g can be written as one of the functions that we already know from the construction in Chapter 2. Finally, we show that the series ϑ_f describes almost everywhere the holomorphic part of ϑ_g.

1.4.3 Elliptic theta series with homogeneous and spherical polynomials

In Chapters 4 and 5, we present two research articles that are joint work with Sander Zwegers, in which we consider elliptic theta series for quadratic forms of signature $(m-1,1)$. We generalize Zwegers' construction [Zwe02] that we have recapitulated in Section 1.3.3 by including homogeneous and spherical polynomials in the definition of the theta series. (We call a polynomial $f : \mathbb{R}^m \longrightarrow \mathbb{C}$ *spherical* (of degree d) if it is homogeneous (of degree d) and vanishes under the Laplacian $\Delta = \Delta_A$, i. e. $\Delta f = 0$.)

We set

$$e^{-\Delta/8\pi} := \sum_{k=0}^{\infty} \frac{(-1)^k}{(8\pi)^k k!}\Delta^k \quad \text{and} \quad \partial_{\boldsymbol{c}} := \frac{1}{\sqrt{-Q(\boldsymbol{c})}}\boldsymbol{c}^{\mathrm{t}}\frac{\partial}{\partial \boldsymbol{v}} = \frac{1}{\sqrt{-Q(\boldsymbol{c})}}\sum_{i=1}^{n} c_i \frac{\partial}{\partial v_i},$$

and for a homogeneous polynomial $f : \mathbb{R}^m \longrightarrow \mathbb{C}$ of degree d we define $\widehat{f} := e^{-\Delta/8\pi} f$. Further we set

$$p^{\boldsymbol{c}}[f](\boldsymbol{v}) := \begin{cases} \sum_{k=0}^{d} \frac{(-1)^k}{(4\pi)^k k!} E^{(k)}\left(\frac{B(\boldsymbol{c},\boldsymbol{v})}{\sqrt{-Q(\boldsymbol{c})}}\right) \cdot \partial_{\boldsymbol{c}}^k \widehat{f}(\boldsymbol{v}) & \text{if } \boldsymbol{c} \in C_Q, \\ \operatorname{sgn}(B(\boldsymbol{c},\boldsymbol{v})) \cdot \widehat{f}(\boldsymbol{v}) & \text{if } \boldsymbol{c} \in S_Q. \end{cases}$$

We define the following three versions of a theta series associated to Q and f with characteristics $\boldsymbol{a} \in R(\boldsymbol{c_1}) \cap R(\boldsymbol{c_2})$ and $\boldsymbol{b} \in \mathbb{R}^m$: we have the holomorphic series

$$\Theta_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{c_1},\boldsymbol{c_2}}[f](\tau) := \sum_{\boldsymbol{\ell} \in \boldsymbol{a}+\mathbb{Z}^m} \left\{\operatorname{sgn}(B(\boldsymbol{c_1},\boldsymbol{\ell})) - \operatorname{sgn}(B(\boldsymbol{c_2},\boldsymbol{\ell}))\right\} f(\boldsymbol{\ell}) q^{Q(\boldsymbol{\ell})} e^{2\pi i B(\boldsymbol{\ell},\boldsymbol{b})},$$

the almost holomorphic series

$$\widehat{\Theta}_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{c_1},\boldsymbol{c_2}}[f](\tau) := y^{-d/2} \sum_{\boldsymbol{\ell} \in \boldsymbol{a}+\mathbb{Z}^m} \left\{\operatorname{sgn}(B(\boldsymbol{c_1},\boldsymbol{\ell})) - \operatorname{sgn}(B(\boldsymbol{c_2},\boldsymbol{\ell}))\right\} \widehat{f}(\boldsymbol{\ell} y^{1/2}) q^{Q(\boldsymbol{\ell})} e^{2\pi i B(\boldsymbol{\ell},\boldsymbol{b})},$$

and the non-holomorphic modular series

$$\widehat{\widehat{\Theta}}_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{c_1},\boldsymbol{c_2}}[f](\tau) := y^{-d/2} \sum_{\boldsymbol{\ell} \in \boldsymbol{a}+\mathbb{Z}^m} \left\{p^{\boldsymbol{c_1}}[f](\boldsymbol{\ell} y^{1/2}) - p^{\boldsymbol{c_2}}[f](\boldsymbol{\ell} y^{1/2})\right\} q^{Q(\boldsymbol{\ell})} e^{2\pi i B(\boldsymbol{\ell},\boldsymbol{b})}.$$

If f is spherical of degree d, the almost holomorphic theta series $\widehat{\Theta}^{\boldsymbol{c_1},\boldsymbol{c_2}}[f]$ and the holomorphic theta series $\Theta^{\boldsymbol{c_1},\boldsymbol{c_2}}[f]$ agree.

In Chapter 4, we consider theta series with respect to $\boldsymbol{c_1}, \boldsymbol{c_2} \in C_Q$. In this case, we may just consider $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$. We determine a criterion according to which $\widehat{\widehat{\Theta}}^{\boldsymbol{c_1},\boldsymbol{c_2}}$ and $\widehat{\Theta}^{\boldsymbol{c_1},\boldsymbol{c_2}}$ agree: we consider elements g_i from the automorphism group $\operatorname{Aut}^+(Q,\mathbb{Z}^m) \subset \mathrm{GL}_m(\mathbb{Z})$, which leaves Q, $\mathbb{Z}^m$ and the choice of the component C_Q unchanged, and homogeneous polynomials f_i of degree d that satisfy

$$\sum_{i \in I} (f_i - f_i \circ g_i) = 0 \quad \text{for } I \text{ a finite set of indices.}$$

We show that the theta function

$$\sum_{i \in I} \widehat{\widehat{\Theta}}^{\boldsymbol{c},g_i\boldsymbol{c}}[f_i] = \sum_{i \in I} \widehat{\Theta}^{\boldsymbol{c},g_i\boldsymbol{c}}[f_i]$$

is an almost holomorphic cusp form of weight $m/2+d$, depth $\leq d/2$ and a certain character χ on $\Gamma_0(N)$. We also note that the function does not depend on the choice of $\boldsymbol{c} \in C_Q$. This construction provides numerous explicit examples. We provide some for quadratic forms of signature $(1,1)$ and $(2,1)$, taking homogeneous and spherical polynomials f_i to obtain almost holomorphic and holomorphic cusp forms. We also introduce additional periodic functions in the definition of the theta series to obtain a greater variety of examples. Often, we can identify these functions as eta products or eta quotients.

Since there are numerous interesting modular forms that appear in the form of theta series with homogeneous or spherical polynomials but require at least one $\boldsymbol{c_i}$ to be located on the boundary of the cone C_Q, we extend this result to the case $\boldsymbol{c_i} \in S_Q$ in Chapter 5. We then recover the modular transformation behavior of the Eisenstein series and we discuss modular forms on $\Gamma_0(4)$ that were discovered by Zagier [Zag99] when he considered the sum of powers of quadratic polynomials with integer coefficients. In both cases we have to take into account that we need to relax the condition $\boldsymbol{a} \in R(\boldsymbol{c_1}) \cap R(\boldsymbol{c_2})$ and thus get additional meromorphic terms. It is not difficult though to find functions with the same transformation behavior as the theta series, which exactly describe these meromorphic parts. A slightly different application is the construction of a harmonic Maass form of weight $3/2$ on $\Gamma_0(2)$, whose holomorphic part is (up to a rational power of q) the generating function of the Hurwitz class numbers $H(8n+7)$.

2 Siegel theta series for indefinite quadratic forms

This chapter is based on the manuscript [Roe21a] of the same name published in the journal *Research in Number Theory*.

2.1 Introduction

In the course of his work on the Minkowski-Hasse principle for quadratic forms over the rationals, Siegel introduced a natural generalization of elliptic modular forms of higher genus n [Sie35]. Among those functions, nowadays called Siegel modular forms, Siegel theta series play a similarly important role as do theta series do in the context of elliptic modular forms. In a recently published article by Dittmann, Salvati Manni, and Scheithauer [DSMS21], a basis of the space of Siegel cusp forms of degree 6 and weight 14 is given by harmonic Siegel theta series. By considering one of these basis elements, the authors deduce that the Kodaira dimension of the Siegel modular variety $\mathcal{A}_6 = \mathrm{Sp}_{12}(\mathbb{Z}) \setminus \mathbb{H}_6$ is non-negative.

In order to give more examples of Siegel theta series and make use of the connection to various topics – such as algebraic geometry and number theory – it is desirable to give a general framework for the description of holomorphic and non-holomorphic Siegel theta series analogous to what is already known for elliptic theta series owing to the work of Vignéras [Vig77]. If theta series are built from functions that satisfy a certain second-order differential equation, the modularity of these series immediately follows. For the (generalized) error functions, which are employed in the recent discussions of theta series for indefinite quadratic forms, this criterion is used to derive the modular transformation behavior of the emerging theta series. Namely, these are the results by Zwegers [Zwe02] for quadratic forms of signature $(m-1,1)$, by Alexandrov, Banerjee, Manschot, and Pioline [ABMP18a] for signature $(m-2,2)$ and for arbitrary signature by Nazaroglu [Naz18] and Westerholt-Raum [WR16], which are brought together by Kudla [Kud18] and Funke and Kudla [FK19]. Even before that, Kudla and Millson [KM86, KM87] considered a certain class of Schwartz functions to define modular forms in terms of theta functions and obtain holomorphic modular forms valued in the set of cohomology classes.

In most of these examples the criterion given by Vignéras plays an important role in order to deduce modularity, so the question arises whether a similar result holds for more general types of theta series. Vignéras herself derives the result of [Vig77] in a second paper by considering the Weil representation and mentions that the result is expected to hold for Hilbert and Siegel theta series as well, see [Vig75].

In the following, we prove this for the latter case by describing Siegel theta series for indefinite quadratic forms and deriving a generalization of Vignéras' result for generic genus n. We adopt an elementary approach similar to the one in [Vig77], which has the advantage that we explicitly construct a basis of suitable functions. This construction also embeds the known results for positive definite quadratic forms, which is for instance described by Freitag [Fre83]. In this case, these "suitable functions" are harmonic polynomials and one obtains holomorphic series. However, the Siegel theta series that are constructed in the present paper are in general non-holomorphic. In a sequel to this paper, we will investigate the special case where the quadratic form has signature $(m-1,1)$ and, by applying the result shown here, deduce the modularity of non-holomorphic Siegel theta series, which are related to holomorphic (non-modular) Siegel theta series.

We give a short overview on the main results. We use standard conventions concerning the notation, so $\mathrm{e}(z) := \exp(2\pi i z)$ and multiplication is hierarchically higher than division, for example $1/8\pi$ means $1/(8\pi)$.

Definition 2.1. Throughout this paper, let $A \in \mathbb{Z}^{m\times m}$ denote a non-degenerate symmetric matrix of signature (r,s).

Remark 2.2. Note that we do not generally assume that A is even. Also, in some sections we explicitly set $s=0$ and thus employ properties of the then positive definite matrix A.

We construct modular forms on the Siegel upper half-space

$$\mathbb{H}_n := \{Z = X + iY \mid X, Y \in \mathbb{R}^{n\times n} \text{ symmetric}, Y \text{ positive definite}\}$$

in the form of Siegel theta series. We denote by $\mathcal{S}(\mathbb{R}^{m\times n})$ the space of Schwartz functions on $\mathbb{R}^{m\times n}$ and then choose $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ such that

$$f(U)\exp\bigl(-\pi \operatorname{tr}(U^{\mathrm{t}}AU)\bigr) \in \mathcal{S}(\mathbb{R}^{m\times n}).$$

This ensures the absolute convergence of the theta series that we define in the following.

Definition 2.3. Let $\mathcal{H}, \mathcal{K} \in \mathbb{R}^{m\times n}$ and let $\lambda \in \mathbb{Z}$. The theta series with characteristics $\mathcal{H}$ and $\mathcal{K}$ associated with f and A is

$$\vartheta_{\mathcal{H},\mathcal{K}}(Z) = \vartheta_{\mathcal{H},\mathcal{K},f,A}(Z) := \det Y^{-\lambda/2} \sum_{U \in \mathcal{H}+\mathbb{Z}^{m\times n}} f(UY^{1/2})\, \mathrm{e}\bigl(\operatorname{tr}(U^{\mathrm{t}}AUZ)/2 + \operatorname{tr}(\mathcal{K}^{\mathrm{t}}AU)\bigr).$$

Remark 2.4. We drop the parameters f and A in the index, when the transformation of $\vartheta_{\mathcal{H},\mathcal{K}}$ leaves them invariant. In the following, it becomes clear that the choice of λ depends on f, so we do not include it as additional parameter in the definition.

For a positive definite matrix A, we consider polynomials $P : \mathbb{C}^{m\times n} \longrightarrow \mathbb{C}$ that satisfy $P(UN) = \det N^{\alpha} P(U)$ for all $N \in \mathbb{C}^{n\times n}$ and a fixed $\alpha \in \mathbb{N}_0$. These polynomials form a complex vector space, which we denote by $\mathcal{P}_{\alpha}^{m,n}$. For a modified polynomial

$$p(U) = \exp\bigl(-\operatorname{tr}\boldsymbol{\Delta}_A/8\pi\bigr)\bigl(P(U)\bigr) \quad \text{where } \boldsymbol{\Delta}_A := \Bigl(\frac{\partial}{\partial U}\Bigr)^{\mathrm{t}} A^{-1} \frac{\partial}{\partial U},$$

and when we take A to be even and set $\lambda = \alpha$, the theta series $\vartheta_{\mathrm{O},\mathrm{O},p,A}$ transforms like a Siegel modular form of weight $m/2+\alpha$ on a congruence subgroup of Γ_n and with some character, where both depend on the level of A. If $P \in \mathcal{P}_{\alpha}^{m,n}$ is annihilated by the Laplacian $\operatorname{tr}\boldsymbol{\Delta}_A$, we obtain the holomorphic theta series considered by Freitag [Fre83].

When A denotes an indefinite quadratic form of signature (r,s), we write $A = A^+ + A^-$ with a positive semi-definite matrix A^+ and a negative semi-definite matrix A^- and denote by $M = A^+ - A^-$ the positive definite majorant matrix of A (see Remark 2.9). We consider the function

$$g(U) = \exp\bigl(-\operatorname{tr}\boldsymbol{\Delta}_M/8\pi\bigr)\bigl(P(U)\bigr)\exp\bigl(2\pi\operatorname{tr}(U^{\mathrm{t}}A^-U)\bigr),$$

assuming that $P \in \mathcal{P}^{m,n}_{\alpha+\beta}$ factorizes as $P(U) = P_\alpha(U^+) \cdot P_\beta(U^-)$ with $P_\alpha \in \mathcal{P}^{m,n}_\alpha$, $P_\beta \in \mathcal{P}^{m,n}_\beta$ and $U = U^+ + U^-$, where U^+ denotes the part of U that belongs to the subspace on which A is positive semi-definite, i. e. $\operatorname{tr}\big((U^+)^{\mathrm{t}} A U^+\big) = \operatorname{tr}\big(U^{\mathrm{t}} A^+ U\big)$ and similarly $\operatorname{tr}\big((U^-)^{\mathrm{t}} A U^-\big) = \operatorname{tr}\big(U^{\mathrm{t}} A^- U\big)$. For this choice of g and considering an even matrix A and setting $\lambda = \alpha - \beta - s$, the theta series $\vartheta_{\mathrm{O},\mathrm{O},g,A}$ transforms like a non-holomorphic Siegel modular form of weight $m/2 + \lambda$ on a congruence subgroup of Γ_n and with some character, where both depend on the level of A.

These explicit constructions do not only give examples of Siegel modular forms, but by applying Vignéras' result for genus $n = 1$, we show that we obtain a similar criterion as in [Vig77] to determine whether a Siegel theta series transforms like a modular form:

Theorem 2.5. *Let $\lambda \in \mathbb{Z}$ and let $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ such that*

$$f(U) \exp\big(-\pi \operatorname{tr}(U^{\mathrm{t}} A U)\big) \in \mathcal{S}(\mathbb{R}^{m\times n})$$

and f is a solution of the $n \times n$-system of partial differential equations

$$\Big(\boldsymbol{E} - \frac{\boldsymbol{\Delta}_A}{4\pi}\Big) f = \lambda \cdot I \cdot f \qquad \text{with } \boldsymbol{E} := U^{\mathrm{t}} \frac{\partial}{\partial U} \text{ and } \boldsymbol{\Delta}_A \text{ as defined above.}$$

For $\mathcal{H} = \mathcal{K} = \mathrm{O}$ and A even, the theta series $\vartheta_{\mathcal{H},\mathcal{K},f,A}$ in Definition 2.3 transforms like a Siegel modular form of genus n and weight $m/2 + \lambda$, where the level and character depend on A.

Remark 2.6. In this paper, we determine the transformation behavior of $\vartheta_{\mathcal{H},\mathcal{K},f,A}$ with respect to the transformations $Z \mapsto Z + S$ for a symmetric matrix $S \in \mathbb{Z}^{n\times n}$ (see Lemma 2.25) and $Z \mapsto -Z^{-1}$ (see Proposition 2.33). The results hold for any $\mathcal{H}, \mathcal{K} \in \mathbb{R}^{m\times n}$ and we do not generally assume that A is even. By setting further preconditions for $\mathcal{H}, \mathcal{K}$ and A, one can then construct vector-valued Siegel modular forms of genus n and weight $m/2 + \lambda$ on the full Siegel modular group or scalar-valued modular forms on congruence subgroups, see also Remark 2.8. However, we will not explicitly elaborate on that here.

The outline of the paper is as follows: In Section 2.2, we briefly summarize the most important notions about Siegel modular forms that are relevant for this paper. In the next section, we examine the complex vector space formed by the solutions of the $n \times n$-system of partial differential equations from Theorem 2.5. Under the additional assumption that a solution f must satisfy the growth condition $f(U) \exp(-\pi \operatorname{tr}(U^{\mathrm{t}} A U)) \in \mathcal{S}(\mathbb{R}^{m\times n})$, we explicitly determine a basis (which is finite if A is positive or negative definite and infinite otherwise) of this vector space. In Section 2.4, we show that these basis elements can be used to construct theta series of genus n that transform like Siegel modular forms of weight $m/2 + \lambda$. In order to do so, we first construct non-holomorphic theta series for positive definite quadratic forms. With some modifications, this can be generalized to theta series associated with indefinite quadratic forms.

2.2 Notation and preliminaries

We fix notation and summarize standard results about Siegel modular forms and refer to Andrianov [And09, p. 1–25] and Freitag [Fre83] for further details. For convenience, we replicate some definitions of the last section. We also comment on results by Borcherds [Bor98] and Vignéras [Vig75] and point out differences to our set-up.

Denote the Siegel upper half-space by

$$\mathbb{H}_n := \{Z = X + iY \mid X, Y \in \mathbb{R}^{n\times n} \text{ symmetric}, Y \text{ positive definite}\}.$$

We let $Y^{1/2}$ denote the uniquely determined symmetric positive definite matrix that satisfies $Y^{1/2} \cdot Y^{1/2} = Y$. The same holds for the square root of A, when A is a positive definite matrix.

We define modular forms on $\mathbb{H}_n$ for the full Siegel modular group

$$\Gamma_n := \left\{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{Z}^{2n\times 2n} \mid M^{\mathrm{t}} J M = J\right\}, \quad \text{where } J = \left(\begin{smallmatrix} \mathrm{O} & I_n \\ -I_n & \mathrm{O} \end{smallmatrix}\right),$$

which operates on $\mathbb{H}_n$ by

$$Z \mapsto M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}.$$

The imaginary part Y of Z and the imaginary part $\widetilde{Y}$ of $M\langle Z\rangle$ satisfy the relation

$$(C\overline{Z} + D)^{\mathrm{t}}\widetilde{Y}(CZ + D) = Y. \tag{2.1}$$

In particular, $\widetilde{Y}$ is positive definite and symmetric.

Definition 2.7. We call $F : \mathbb{H}_n \longrightarrow \mathbb{C}$ a (classical) Siegel modular form of genus n and weight k if the following conditions hold:

(a) The function F is holomorphic on $\mathbb{H}_n$,

(b) For every $M \in \Gamma_n$ we have $F(M\langle Z\rangle) = \det(CZ + D)^k F(Z)$,

(c) $|F(Z)|$ is bounded on domains in $\mathbb{H}_n$ of the form $\mathbb{H}_n^\varepsilon := \{X + iY \in \mathbb{H}_n \mid Y \geq \varepsilon \cdot I\}$ with $\varepsilon > 0$.

Note that the weight is not necessarily an integral number. In this context, we define – as usual – for $z \in \mathbb{C}$ and any non-integer exponent r that $z^r := \exp(r \log z)$, where $\log z = \log|z| + i\arg(z)$, $-\pi < \arg(z) \leq \pi$.

Due to the Koecher principle (cf. [Fre83, p. 44f.]), which holds for $n > 1$, all functions satisfying (a) and (b) admit a Fourier expansion over positive semi-definite even symmetric matrices and are in particular bounded on $\mathbb{H}_n^\varepsilon$ for any $\varepsilon > 0$. So we do not need to impose an analogue of (c) as condition. If we consider non-holomorphic modular forms, the Koecher principle does not necessarily hold anymore. In our case, we build Siegel theta series by using Schwartz functions and obtain absolutely convergent series, so these functions also satisfy condition (c).

Remark 2.8. The full Siegel modular group Γ_n is generated by the matrices $\left(\begin{smallmatrix} I_n & S \\ \mathrm{O} & I_n \end{smallmatrix}\right)$ with $S = S^{\mathrm{t}}$ and $\left(\begin{smallmatrix} \mathrm{O} & -I_n \\ I_n & \mathrm{O} \end{smallmatrix}\right)$ (cf. [Fre83, p. 322–328]), so any function F with $F(Z + S) = F(Z)$ for symmetric matrices $S \in \mathbb{Z}^{n\times n}$ and $F(-Z^{-1}) = \det Z^k F(Z)$ satisfies condition (b). For the theta series with characteristics $\mathcal{H}, \mathcal{K}$ that we construct here, we observe the following: Up to a factor depending on $\mathcal{H}$, A and S, we can write $\vartheta_{\mathcal{H},\mathcal{K}}(Z + S)$ as a theta series of the same form but with a slightly changed characteristic $\mathcal{H}, \widetilde{\mathcal{K}}$, see Lemma 2.25. We can express $\vartheta_{\mathcal{H},\mathcal{K}}(-Z^{-1})$ as a linear combination of theta series $\vartheta_{\mathcal{J}+\mathcal{K},-\mathcal{H}}(Z)$, where $\mathcal{J} \in A^{-1}\mathbb{Z}^{m\times n} \bmod \mathbb{Z}^{m\times n}$, see Proposition 2.33.

If A is an even unimodular matrix and $\mathcal{H} = \mathcal{K} = \mathrm{O}$, the theta series transforms like a modular form on the full group Γ_n, see Example 2.31 when A is positive definite and

Example 2.34 when A is indefinite (in the last case we might obtain a character of Γ_n as an additional automorphic factor).

If $\mathcal{H}$ and $\mathcal{K}$ are rational matrices, we can take the series $\vartheta_{\mathcal{H},\mathcal{K}}$ as entries of vector-valued functions, which then define modular forms on the full Siegel modular group. In another approach (see for example Andrianov and Maloletkin [AM75]), one could consider suitable congruence subgroups of finite index in Γ_n.

In Section 2.3 as well as Section 2.4, we will consider a fixed decomposition of the non-degenerate matrix A of signature (r, s), so we give a precise description here.

Remark 2.9. Let $\boldsymbol{v_1}, \dots, \boldsymbol{v_r}$ denote the eigenvectors that correspond to the positive eigenvalues of A and $\boldsymbol{v_{r+1}}, \dots, \boldsymbol{v_m}$ the ones that correspond to the negative eigenvalues. We normalize these eigenvectors in a suitable way so that for $S = (\boldsymbol{v_1}, \dots, \boldsymbol{v_m}) \in \mathbb{R}^{m\times m}$

$$S^{\mathrm{t}} A S = \mathcal{I} \quad \text{with } \mathcal{I} := \begin{pmatrix} I_r & \mathrm{O} \\ \mathrm{O} & -I_s \end{pmatrix}.$$

As $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_m}\}$ forms a basis of $\mathbb{R}^m$, we write any vector $\boldsymbol{u} \in \mathbb{R}^m$ as $\boldsymbol{u} = \sum_{i=1}^{r} \lambda_i \boldsymbol{v_i} + \sum_{i=r+1}^{m} \lambda_i \boldsymbol{v_i}$ and define $\boldsymbol{u}^+ := \sum_{i=1}^{r} \lambda_i \boldsymbol{v_i}$ and $\boldsymbol{u}^- := \sum_{i=r+1}^{m} \lambda_i \boldsymbol{v_i}$.

So for the inverse of S, we have $A = (S^{-1})^{\mathrm{t}} \mathcal{I} S^{-1}$. This enables us to write A as the sum of the positive semi-definite respectively negative semi-definite matrices

$$A^+ := (S^{-1})^{\mathrm{t}} \begin{pmatrix} I_r & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{pmatrix} S^{-1} \quad \text{and} \quad A^- := (S^{-1})^{\mathrm{t}} \begin{pmatrix} \mathrm{O} & \mathrm{O} \\ \mathrm{O} & -I_s \end{pmatrix} S^{-1}.$$

We also associate the positive definite matrix $M := (S^{-1})^{\mathrm{t}} S^{-1} = A^+ - A^-$. If we write $U \in \mathbb{R}^{m\times n}$ as $U = U^+ + U^-$, where $U^+ := (\boldsymbol{u_1^+}, \dots, \boldsymbol{u_n^+})$ and $U^- := (\boldsymbol{u_1^-}, \dots, \boldsymbol{u_n^-})$, it is straightforward to check that

$$\operatorname{tr}\big((U^+)^{\mathrm{t}} A U^+\big) = \operatorname{tr}\big(U^{\mathrm{t}} A^+ U\big) \quad \text{and} \quad \operatorname{tr}\big((U^-)^{\mathrm{t}} A U^-\big) = \operatorname{tr}\big(U^{\mathrm{t}} A^- U\big).$$

As our construction of Siegel theta series in Section 2.4 is very similar to Borcherds' set-up [Bor98] for $n = 1$, we briefly recall his result and point out the main differences.

Remark 2.10. Borcherds considers a non-degenerate quadratic form Q with signature (r, s), an even lattice $L \subset \mathbb{R}^m$ with the associated dual lattice L' and an isometry v mapping $L \otimes \mathbb{R}$ to $\mathbb{R}^{r,s}$. Considering the inverse images v^+ and v^- of $\mathbb{R}^{r,0}$ and $\mathbb{R}^{0,s}$ under v, one decomposes $L \otimes \mathbb{R}$ in the orthogonal direct sum of a positive definite subspace v^+ and a negative definite subspace v^-. For the projection of $\boldsymbol{\lambda} \in L \otimes \mathbb{R}$ into $v^\pm$ one writes $\boldsymbol{\lambda}_{v^\pm}$ and obtains the positive definite quadratic form $Q_v(\boldsymbol{\lambda}) = Q(\boldsymbol{\lambda}_{v^+}) - Q(\boldsymbol{\lambda}_{v^-})$. As the decomposition into the subspaces v^+ and v^- is not unique, Borcherds' theta series include an additional parameter to indicate the choice of $v^+ \in G(M)$, where the Grassmannian $G(M)$ denotes the set of positive definite r-dimensional subspaces of $L \otimes \mathbb{R}$. For $z \in \mathbb{H}_1, \boldsymbol{h}, \boldsymbol{k} \in L \otimes \mathbb{R}, \boldsymbol{\gamma} \in L'/L$, Δ the Laplacian on $\mathbb{R}^m$, and $p : \mathbb{R}^m \longrightarrow \mathbb{R}$ a polynomial that is homogeneous of degree α in the first r variables and homogeneous of degree β in the last s variables, he defines

$$\begin{aligned}\theta_{L+\boldsymbol{\gamma}}(z, \boldsymbol{h}, \boldsymbol{k}; v, p) := \sum_{\boldsymbol{\lambda} \in L+\boldsymbol{\gamma}} & \exp(-\Delta/8\pi y)(p)\big(v(\boldsymbol{\lambda} + \boldsymbol{h})\big) \\ & \cdot e\big(z(\boldsymbol{\lambda} + \boldsymbol{h})^2_{v^+}/2 + \overline{z}(\boldsymbol{\lambda} + \boldsymbol{h})^2_{v^-}/2 - (\boldsymbol{\lambda} + \boldsymbol{h}/2, \boldsymbol{k})\big)\end{aligned}$$

and shows that this is a non-holomorphic modular form of weight $(r/2 + \alpha, s/2 + \beta)$.

In the present paper, we fix the decomposition $A = A^+ + A^-$ and the majorant matrix $M = A^+ - A^-$ by taking the eigenvectors of A as a basis in $\mathbb{R}^m$. Then $U \in \mathbb{R}^{m\times n}$ is projected onto U^+ in the positive definite subspace and U^- in the negative definite subspace. However, choosing any other decomposition of A into a negative and a positive definite part leads to an analogous construction.

In Definition 2.3, we represented $\vartheta_{\mathcal{H},\mathcal{K}}$ such that the analogy with Vignéras' construction (see Remark 2.11) is visible. We can also write these theta series as

$$\vartheta_{\mathcal{H},\mathcal{K}}(Z) = \sum_{U\in\mathcal{H}+\mathbb{Z}^{m\times n}} \exp(-\operatorname{tr}(\boldsymbol{\Delta}_M Y^{-1})/8\pi)\big(P(U)\big) \cdot e\big(\operatorname{tr}(U^{\mathrm{t}}A^+UZ)/2 + \operatorname{tr}(U^{\mathrm{t}}A^-U\overline{Z})/2 + \operatorname{tr}(\mathcal{K}^{\mathrm{t}}AU)\big),$$

which resembles Borcherds' construction. Note that we can multiply the series by the factor $\det Y^{s/2+\beta}$ to obtain the weight $m/2+\lambda$ (where $\lambda = \alpha-\beta-s$) instead of $(r/2+\alpha, s/2+\beta)$.

We conclude this section by reviewing Vignéras' construction [Vig77] and addressing essential differences.

Remark 2.11. Vignéras considers theta series of genus 1

$$\vartheta_{\mathbf{0},\mathbf{0}}(z) = y^{-\lambda/2}\sum_{\boldsymbol{u}\in L} f(\boldsymbol{u}\sqrt{y})\,\mathrm{e}\big(Q(\boldsymbol{u})z\big),$$

where $L \subset \mathbb{R}^m$ denotes a lattice, $Q(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}^{\mathrm{t}}A\boldsymbol{u}$ a quadratic form of signature (r,s) and $z = x+iy$ an element of the upper half-plane $\mathbb{H}_1$. The following two requirements are imposed on the function f: Set $\widetilde{f}(\boldsymbol{u}) = f(\boldsymbol{u})\exp\big(-2\pi Q(\boldsymbol{u})\big)$. Then for any polynomial $p : \mathbb{R}^m \longrightarrow \mathbb{R}$ with $\deg(p) \le 2$ and any partial derivative ∂^α with $|\alpha| \le 2$,

$$p\cdot\widetilde{f} \in \mathcal{L}^1(\mathbb{R}^m)\cap\mathcal{L}^2(\mathbb{R}^m) \quad\text{and}\quad \partial^\alpha\widetilde{f} \in \mathcal{L}^1(\mathbb{R}^m)\cap\mathcal{L}^2(\mathbb{R}^m).$$

Furthermore, f satisfies the differential equation of second order

$$\Big(E - \frac{\Delta_A}{4\pi}\Big)f = \lambda\cdot f \qquad \text{with } E := \sum_{d=1}^m u_d\frac{\partial}{\partial u_d} \text{ and } \Delta_A := \sum_{a,b=1}^m \frac{\partial}{\partial u_a}(A^{-1})_{ab}\frac{\partial}{\partial u_b}.$$

Then $\vartheta_{\mathbf{0},\mathbf{0}}$ transforms like a modular form of weight $m/2+\lambda$.

For higher genus $n \in \mathbb{N}$, we introduce some notation to formulate an analogous growth condition. For $p \in [1,\infty)$ let $\mathcal{L}^p(\mathbb{R}^{m\times n})$ denote the Lebesgue space of functions $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{C}$ for which

$$\|f\|_p := \Bigg(\int_{\mathbb{R}^{m\times n}} |f(U)|^p\,dU\Bigg)^{1/p}$$

is finite. We use the usual multi-index notation on $\mathbb{R}^{m\times n}$, where $\alpha \in \mathbb{N}_0^{m\times n}$ with $|\alpha| = \sum_{i=1}^m\sum_{j=1}^n \alpha_{ij}$, so

$$U^\alpha = \prod_{\substack{1\le i\le m\\ 1\le j\le n}} U_{ij}^{\alpha_{ij}} \quad\text{and}\quad \partial^\alpha = \prod_{\substack{1\le i\le m\\ 1\le j\le n}} \Big(\frac{\partial}{\partial U_{ij}}\Big)^{\alpha_{ij}}.$$

For $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$, one sets $\widetilde{f}(U) := f(U)\exp\bigl(-\pi\operatorname{tr}(U^{\mathrm{t}}AU)\bigr)$ and – analogously to Vignéras – assumes that for any polynomial $p : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ with $\deg(p) \leq 2$ and any partial derivative ∂^α with $|\alpha| \leq 2$,

$$p \cdot \widetilde{f} \in \mathcal{L}^1(\mathbb{R}^{m\times n}) \cap \mathcal{L}^2(\mathbb{R}^{m\times n}) \quad \text{and} \quad \partial^\alpha \widetilde{f} \in \mathcal{L}^1(\mathbb{R}^{m\times n}) \cap \mathcal{L}^2(\mathbb{R}^{m\times n}). \tag{2.2}$$

This allows us to apply Vignéras' result for theta series of genus 1 (as we make use of the fact that Hermite functions build an orthogonal basis of $\mathcal{L}^2$-functions) and the Poisson summation formula.

However, for simplification, we replace assumption (2.2) by the more restrictive assumption that $\widetilde{f}$ is a Schwartz function.

2.3 A generalization of Vignéras' differential equation

To derive an analogue of Vignéras' result for Siegel modular forms of higher genus $n \in \mathbb{N}$, we introduce matrix-valued operators generalizing E and Δ_A.

Definition 2.12. For $U \in \mathbb{R}^{m\times n}$ let $\partial/\partial U = \bigl(\partial/\partial U_{\mu\nu}\bigr)_{1\leq\mu\leq m,\, 1\leq\nu\leq n}$. We define the generalized Euler operator

$$\mathbf{E} := U^{\mathrm{t}}\frac{\partial}{\partial U} \quad \text{with } \mathbf{E}_{ij} = \sum_{d=1}^{m} U_{di}\frac{\partial}{\partial U_{dj}} \quad (1 \leq i \leq n,\ 1 \leq j \leq n)$$

and the generalized Laplace operator associated with A

$$\mathbf{\Delta}_A := \Bigl(\frac{\partial}{\partial U}\Bigr)^{\mathrm{t}} A^{-1}\frac{\partial}{\partial U} \quad \text{with } (\mathbf{\Delta}_A)_{ij} = \sum_{a,b=1}^{m}\frac{\partial}{\partial U_{ai}}(A^{-1})_{ab}\frac{\partial}{\partial U_{bj}}.$$

For the normalized Laplacian $\mathbf{\Delta}_I$ we simply write $\mathbf{\Delta}$. Further, we set

$$\mathcal{D}_A := \mathbf{E} - \frac{\mathbf{\Delta}_A}{4\pi}.$$

The $n \times n$-system of partial differential equations

$$\mathcal{D}_A f = \lambda \cdot I \cdot f \quad \text{for } \lambda \in \mathbb{Z} \text{ and } A \text{ indefinite of signature } (r,s) \tag{2.3}$$

is a direct generalization of the set-up in [Vig77]. In this section, we examine the complex vector space formed by the solutions f of (2.3) that additionally satisfy the growth condition

$$f(U)\exp(-\pi\operatorname{tr}(U^{\mathrm{t}}AU)) \in \mathcal{S}(\mathbb{R}^{m\times n}).$$

We explicitly determine a basis (which is finite if A is positive or negative definite and infinite otherwise) of this vector space.

2.3.1 Functions with a homogeneity property

As mentioned in the introduction, we employ polynomials with a certain homogeneity property to construct Siegel theta series. In the following, we introduce the complex vector space of all functions with this homogeneity property. For a *differentiable* function f, we show in Proposition 2.15 that f is homogeneous of degree α if and only if f solves the system of partial differential equations $\mathbf{E}f = \alpha \cdot I \cdot f$. Further, we show in Lemma

2.16 that for a *polynomial* function p it is already sufficient that $\mathbf{E}p = C \cdot p$ holds for some $C \in \mathbb{C}^{n\times n}$ to deduce that p is a homogeneous function.

Definition 2.13. For $\alpha \in \mathbb{N}_0$, $m, n \in \mathbb{N}$, we define the complex vector space

$$\mathcal{F}_\alpha^{m,n} := \{f : \mathbb{C}^{m\times n} \longrightarrow \mathbb{C} \mid f \text{ continuous, } f(UN) = \det N^\alpha f(U) \text{ for all } N \in \mathbb{C}^{n\times n}\}.$$

For $n = 1$, this is the usual definition of a homogeneous function of non-negative degree. As a subspace, we consider all polynomials of this class, which is the space $\mathcal{P}_\alpha^{m,n}$ from the introduction.

Remark 2.14. The vector space $\mathcal{P}_\alpha^{m,n}$ is described by Maass [Maa59]. He determines the structure of $\mathcal{P}_\alpha^{m,n}$, shows that it has finite dimension and even gives an explicit formula for the dimension. In the following, $\mathcal{B}_\alpha^{m,n}$ denotes a finite basis of $\mathcal{P}_\alpha^{m,n}$. We state some observations to show that we obtain non-trivial examples.

- For $m < n$, we have $\mathcal{F}_\alpha^{m,n} = \mathbb{C}$: We take $U \in \mathbb{R}^{m\times n}$ such that $f(U) \neq 0$. One can multiply elementary matrices from the right such that U is in reduced column echelon form. If U has less rows than columns, at least the last column is a zero column. Setting $N = \operatorname{diag}(1, \dots, 1, \lambda)$ with $\lambda \notin \{0, 1\}$, leads to the identity $f(U) = \lambda^\alpha f(U)$, which is only satisfied for $\alpha = 0$. The orbit of the right action of invertible matrices on $U \in \mathbb{C}^{m\times n}$ is dense and f is continuous, so f is a constant function.

- Note that $f \cdot g \in \mathcal{F}_{\alpha+\beta}^{m,n}$ for $f \in \mathcal{F}_\alpha^{m,n}$, $g \in \mathcal{F}_\beta^{m,n}$ and $f + g \in \mathcal{F}_\alpha^{m,n}$ for $f, g \in \mathcal{F}_\alpha^{m,n}$.

- For $m \geq n$ let $\widetilde{U} \in \mathbb{C}^{n\times n}$ be a square submatrix of maximal size of $U \in \mathbb{C}^{m\times n}$. Clearly, we have $\det \widetilde{U}^\alpha \in \mathcal{P}_\alpha^{m,n}$. Due to this and by picking up on the previous point, we obtain all functions in $\mathcal{P}_\alpha^{m,n}$ by taking the product of α (possibly different) $n \times n$-minors $\det \widetilde{U}$ and linear combinations thereof.

Homogeneous functions that are also differentiable are characterized by the identity $Ef = \alpha \cdot f$. We observe that this statement can be generalized.

Proposition 2.15. *Let $f : \mathbb{C}^{m\times n} \longrightarrow \mathbb{C}$ be a differentiable function. We have $f \in \mathcal{F}_\alpha^{m,n}$ if and only if $\boldsymbol{E}f = \alpha \cdot I \cdot f$.*

Proof. For $U \in \mathbb{C}^{m\times n}$ and $N \in \mathbb{C}^{n\times n}$, the derivative of the entry $(UN)_{k\ell} = \sum_{\nu=1}^{n} U_{k\nu} N_{\nu\ell}$ with $1 \leq k \leq m$, $1 \leq \ell \leq n$ is

$$\frac{\partial (UN)_{k\ell}}{\partial N_{ij}} = \begin{cases} 0 & \text{if } j \neq \ell, \\ U_{ki} & \text{if } j = \ell. \end{cases}$$

Therefore,

$$\frac{\partial}{\partial N_{ij}}\bigl(f(UN)\bigr) = \sum_{k=1}^{m}\sum_{\ell=1}^{n} \frac{\partial f}{\partial U_{k\ell}}(UN)\,\frac{\partial (UN)_{k\ell}}{\partial N_{ij}} = \sum_{k=1}^{m} U_{ki}\,\frac{\partial f}{\partial U_{kj}}(UN).$$

Hence we obtain for the derivative of $f(UN)$ with respect to N that

$$\frac{\partial}{\partial N}\bigl(f(UN)\bigr) = U^{\mathrm{t}}\,\frac{\partial f}{\partial U}(UN)$$

and by the definition of the generalized Euler operator $\mathbf{E}$ with respect to U we have

$$(\mathbf{E}f)(UN) = (UN)^{\mathrm{t}} \frac{\partial f}{\partial U}(UN) = N^{\mathrm{t}} \frac{\partial}{\partial N}\big(f(UN)\big). \tag{2.4}$$

The adjugate matrix $\operatorname{adj}(N) \in \mathbb{C}^{n\times n}$ is defined as $\big(\operatorname{adj}(N)\big)_{ij} := (-1)^{i+j} \det \widetilde{N}_{ji}$, where $\widetilde{N}_{ji}$ denotes the $(n-1)\times(n-1)$-matrix obtained by deleting the j-th row and i-th column. Laplace expansion of the determinant gives

$$\frac{\partial}{\partial N_{ij}}(\det N) = \frac{\partial}{\partial N_{ij}}\left(\sum_{k=1}^{n}(-1)^{i+k} N_{ik} \det \widetilde{N}_{ik}\right) = (-1)^{i+j} \det \widetilde{N}_{ij}.$$

Hence the derivative of the determinant of N is the transpose of the adjugate matrix:

$$\frac{\partial}{\partial N}(\det N) = \operatorname{adj}(N)^{\mathrm{t}} \tag{2.5}$$

For $f \in \mathcal{F}_{\alpha}^{m,n}$, the identity $f(UN) = \det N^{\alpha} f(U)$ holds for all $N \in \mathbb{C}^{n\times n}$. From equations (2.4) and (2.5) we obtain

$$\begin{aligned}\big(\mathbf{E}f\big)(UN) = N^{\mathrm{t}}\frac{\partial}{\partial N}\big(f(UN)\big) &= N^{\mathrm{t}}\frac{\partial}{\partial N}\big(\det N^{\alpha} f(U)\big)\\ &= \alpha \det N^{\alpha-1}\cdot N^{\mathrm{t}} \operatorname{adj}(N)^{\mathrm{t}} \cdot f(U) = \alpha \det N^{\alpha}\cdot I \cdot f(U),\end{aligned}$$

since $\operatorname{adj}(N)N = \det N \cdot I$. We set $N = I$ and get the identity $\mathbf{E}f = \alpha \cdot I \cdot f$.
To show the other implication, notice that $f(UN)(\det N)^{-\alpha}$ is constant with respect to N if f satisfies $\mathbf{E}f = \alpha \cdot I \cdot f$: using the identities (2.4) and (2.5), we obtain

$$N^{\mathrm{t}}\frac{\partial}{\partial N}\big(f(UN)\det N^{-\alpha}\big) = \det N^{-\alpha}\cdot \big(\mathbf{E}f\big)(UN) - \alpha \det N^{-\alpha}\cdot I \cdot f(UN) = \mathrm{O}.$$

Thus we have $f(UN)\det N^{-\alpha} = C(U)$, where C is independent of N. For $N = I$, this is $f(U)$, and hence we conclude $f(UN) = \det N^{\alpha} f(U)$. ☐

Later we will only consider polynomial solutions. In this case, we can state the following lemma, which can be left aside for now, but will be used in the proof of Proposition 2.23.

Lemma 2.16. *Let $p : \mathbb{C}^{m\times n} \longrightarrow \mathbb{C}$ be a polynomial that solves the system of partial differential equations*

$$\boldsymbol{E}p = C \cdot p \quad (C \in \mathbb{C}^{n\times n}). \tag{2.6}$$

If p is not the zero function, the matrix C has the form $C = \alpha \cdot I$ for some $\alpha \in \mathbb{N}_0$.

Proof. First, we examine the case $m = n = 2$ and write $U = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ and

$$p(a,b,c,d) = \sum_{\alpha,\beta,\gamma,\delta\in\mathbb{N}_0} c_{\alpha,\beta,\gamma,\delta}\, a^{\alpha} b^{\beta} c^{\gamma} d^{\delta} \quad \text{with } c_{\alpha,\beta,\gamma,\delta} \in \mathbb{C}.$$

By assumption, p satisfies the 2×2-system of partial differential equations

$$\begin{pmatrix} a & c\\ b & d\end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial a} & \frac{\partial}{\partial b}\\ \frac{\partial}{\partial c} & \frac{\partial}{\partial d}\end{pmatrix} p = \begin{pmatrix} C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}\cdot p. \tag{2.7}$$

Considering the upper left equation, we have

$$\Big(a\frac{\partial}{\partial a}+c\frac{\partial}{\partial c}\Big)p=\sum_{\alpha,\beta,\gamma,\delta\in\mathbb{N}_0}(\alpha+\gamma)\,c_{\alpha,\beta,\gamma,\delta}\,a^\alpha b^\beta c^\gamma d^\delta=C_{11}\cdot p,$$

thus $\alpha+\gamma=C_{11}$. Analogously, we deduce by the bottom right equation that $\beta+\delta=C_{22}$ holds. As p is a polynomial, C_{11} and C_{22} denote non-negative integers. We write $C_{11}=k$ and $C_{22}=\ell$ from now on. We have shown that p is homogeneous (in the original sense) of degree k in the variables of the first column a,c and homogeneous of degree ℓ in the variables of the last column b,d. It is easy to see that $C_{12}=C_{21}=0$ holds: By assumption, the polynomial p satisfies the upper right equation

$$\Big(a\frac{\partial}{\partial b}+c\frac{\partial}{\partial d}\Big)p=C_{12}\cdot p. \tag{2.8}$$

As the left-hand side is a polynomial, homogeneous of degree $k+1$ in a,c and of degree $\ell-1$ in b,d, and the right-hand side is a multiple of p, i.e. homogeneous of degree k and ℓ, we deduce that C_{12} must equal zero. Analogously, we conclude by the bottom left equation of (2.7) that $C_{21}=0$.

It remains to be shown that $k=\ell$ holds. We write

$$p(a,b,c,d)=\sum_{\alpha+\gamma=k}a^\alpha c^\gamma\,p_{\alpha,\gamma}(b,d),$$

where $p_{\alpha,\gamma}$ denote homogeneous polynomials in b,d of degree ℓ. Then equation (2.8) with $C_{12}=0$ has the form

$$\sum_{\alpha+\gamma=k}a^{\alpha+1}c^\gamma\,\frac{\partial}{\partial b}p_{\alpha,\gamma}(b,d)+\sum_{\alpha+\gamma=k}a^\alpha c^{\gamma+1}\,\frac{\partial}{\partial d}p_{\alpha,\gamma}(b,d)\equiv 0.$$

We obtain by comparison of the coefficients of $a^\nu c^\mu$, $0\le\nu\le k+1$, $\mu=k+1-\nu$:

$$\frac{\partial}{\partial d}p_{0,k}(b,d)\equiv 0\quad\text{and}\quad\frac{\partial}{\partial d}p_{\alpha,\gamma}(b,d)=-\frac{\partial}{\partial b}p_{\alpha-1,\gamma+1}(b,d)\quad\text{for } 1\le\alpha\le k,\ \gamma=k-\alpha$$

Thus, we recursively determine the structure of $p_{\alpha,\gamma}$ to be $p_{\alpha,\gamma}(b,d)=\sum_{r=0}^{\alpha}e_r\,b^{\ell-r}d^r$ with $e_r\in\mathbb{R}$. In particular, we see that the exponent of d does not exceed α, i.e. $\delta\le\alpha$.

We make use of the symmetric structure of the polynomial p and exchange a and c and also b and d in the equations above. Then we obtain $\beta\le\gamma$. By interchanging a and d along with their exponents α and δ as well as b and c along with their exponents β and γ and using the bottom left equation of (2.7), we obtain $\alpha\le\delta$ and $\gamma\le\beta$. We have shown $\alpha=\delta$ and $\gamma=\beta$, and in particular, $k=\ell$ holds.

For generic $m,n\in\mathbb{N}$, we reduce the $n\times n$-system $\mathbf{E}p=C\cdot p$ to the case $m=n=2$. We write $U=(\boldsymbol{u_1},\dots,\boldsymbol{u_n})$ with $\boldsymbol{u_i}\in\mathbb{C}^m$ and choose $N\in\mathbb{C}^{n\times n}$ such that the i-th column of U is substituted by $a\boldsymbol{u_i}+c\boldsymbol{u_j}$ and the j-th column by $b\boldsymbol{u_i}+d\boldsymbol{u_j}$, where we assume that $i<j$, i.e. we have

$$UN=(\boldsymbol{u_1},\dots,a\boldsymbol{u_i}+c\boldsymbol{u_j},\dots,b\boldsymbol{u_i}+d\boldsymbol{u_j},\dots,\boldsymbol{u_n}).$$

A simple calculation yields

$$\begin{pmatrix} a\frac{\partial}{\partial a} + c\frac{\partial}{\partial c} & a\frac{\partial}{\partial b} + c\frac{\partial}{\partial d} \\ b\frac{\partial}{\partial a} + d\frac{\partial}{\partial c} & b\frac{\partial}{\partial b} + d\frac{\partial}{\partial d} \end{pmatrix} (p(UN)) = \left(\begin{pmatrix} \mathbf{E}_{ii} & \mathbf{E}_{ij} \\ \mathbf{E}_{ji} & \mathbf{E}_{jj} \end{pmatrix} p \right) (UN).$$

As p solves (2.6) by assumption, we have

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial a} & \frac{\partial}{\partial b} \\ \frac{\partial}{\partial c} & \frac{\partial}{\partial d} \end{pmatrix} (p(UN)) = \begin{pmatrix} C_{ii} & C_{ij} \\ C_{ji} & C_{jj} \end{pmatrix} \cdot p(UN).$$

For 2×2-systems of this form we have shown above that $C_{ii} = C_{jj} = \alpha$ for some $\alpha \in \mathbb{N}_0$ and $C_{ij} = C_{ji} = 0$. As we can choose any $i, j \in \{1, \ldots, n\}$ with $i < j$, we deduce the claim. □

2.3.2 Description of theta series with modular transformation behavior by partial differential equations

In this section, we show the connection between the functions with the homogeneity property that was described in the last section and the functions that are employed in Section 2.4 to construct modular Siegel theta series. Moreover, we apply Vignéras' result for $n = 1$ to explicitly give a basis for the vector space of solutions of (2.3) under the additional growth condition.

First, we state a lemma that holds for any symmetric non-degenerate matrix A of signature (r, s). Namely, we compute the commutator of the k-th power of the Laplacian $(\operatorname{tr} \mathbf{\Delta}_A)^k$ (we will drop the brackets and write $\operatorname{tr} \mathbf{\Delta}_A^k$ for simplicity) and the Euler operator.

Lemma 2.17. *The commutator of $\boldsymbol{E}_{ij}$ $(1 \le i \le n,\ 1 \le j \le n)$ and $\operatorname{tr} \mathbf{\Delta}_A^k$ is*

$$\left[\boldsymbol{E}_{ij}, \operatorname{tr} \mathbf{\Delta}_A^k\right] := \boldsymbol{E}_{ij} \cdot \operatorname{tr} \mathbf{\Delta}_A^k - \operatorname{tr} \mathbf{\Delta}_A^k \cdot \boldsymbol{E}_{ij} = -2k \cdot (\mathbf{\Delta}_A)_{ij} \cdot \operatorname{tr} \mathbf{\Delta}_A^{k-1} \quad (k \in \mathbb{N}).$$

Proof. We show the claim by induction on k. For $k = 1$ one calculates the commutator of $\operatorname{tr} \mathbf{\Delta}_A$ and $\mathbf{E}_{ij}$. By definition we have

$$\operatorname{tr} \mathbf{\Delta}_A \cdot \mathbf{E}_{ij} = \sum_{c=1}^{n} \sum_{a,b,d=1}^{m} \frac{\partial}{\partial U_{ac}} (A^{-1})_{ab} \frac{\partial}{\partial U_{bc}} U_{di} \frac{\partial}{\partial U_{dj}},$$

which we can write – denoting by δ_{ij} the Kronecker delta – as

$$\sum_{c=1}^{n} \sum_{a,b,d=1}^{m} (A^{-1})_{ab} \left(U_{di} \frac{\partial^3}{\partial U_{ac} \partial U_{bc} \partial U_{dj}} + \delta_{ad}\delta_{ci} \frac{\partial^2}{\partial U_{bc} \partial U_{dj}} + \delta_{bd}\delta_{ci} \frac{\partial^2}{\partial U_{ac} \partial U_{dj}} \right)$$
$$= \sum_{c=1}^{n} \sum_{a,b,d=1}^{m} U_{di} \frac{\partial}{\partial U_{dj}} (A^{-1})_{ab} \frac{\partial^2}{\partial U_{ac} \partial U_{bc}} + \sum_{a,b=1}^{m} (A^{-1})_{ab} \left(\frac{\partial^2}{\partial U_{aj} \partial U_{bi}} + \frac{\partial^2}{\partial U_{bj} \partial U_{ai}} \right).$$

Since A^{-1} is symmetric, this is $\mathbf{E}_{ij} \cdot \operatorname{tr} \mathbf{\Delta}_A + 2 \cdot (\mathbf{\Delta}_A)_{ij}$. The operators $(\mathbf{\Delta}_A)_{ij}$ and $\operatorname{tr} \mathbf{\Delta}_A$ commute; thus we deduce for $k \mapsto k+1$

$$\begin{aligned} \operatorname{tr} \mathbf{\Delta}_A^{k+1} \cdot \mathbf{E}_{ij} &= \operatorname{tr} \mathbf{\Delta}_A \cdot \left(\mathbf{E}_{ij} \cdot \operatorname{tr} \mathbf{\Delta}_A^k + 2k \cdot (\mathbf{\Delta}_A)_{ij} \cdot \operatorname{tr} \mathbf{\Delta}_A^{k-1} \right) \\ &= \left(\mathbf{E}_{ij} \cdot \operatorname{tr} \mathbf{\Delta}_A + 2 \cdot (\mathbf{\Delta}_A)_{ij} \right) \cdot \operatorname{tr} \mathbf{\Delta}_A^k + 2k \cdot (\mathbf{\Delta}_A)_{ij} \cdot \operatorname{tr} \mathbf{\Delta}_A^k \\ &= \mathbf{E}_{ij} \cdot \operatorname{tr} \mathbf{\Delta}_A^{k+1} + 2(k+1) \cdot (\mathbf{\Delta}_A)_{ij} \cdot \operatorname{tr} \mathbf{\Delta}_A^k. \end{aligned}$$

□

We can now conclude that all solutions of (2.3) can be ascribed to functions that have the homogeneity property of degree λ by applying the previous lemma and Proposition 2.15.

Lemma 2.18. *Let $f, g : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ denote functions for which $\exp(c_1 \operatorname{tr} \boldsymbol{\Delta}_A)f$ and $\exp(c_2 \operatorname{tr} \boldsymbol{\Delta}_A)g$ are well-defined for any $c_1, c_2 \in \mathbb{R}$ (we apply this result to polynomials f and g in the following, hence these conditions make sense). Moreover, we assume that f and g are related by $f = \exp(-\operatorname{tr} \boldsymbol{\Delta}_A/8\pi)g$. Then f is a solution of (2.3) if and only if g satisfies $\boldsymbol{E}g = \lambda \cdot I \cdot g$, i. e. $g \in \mathcal{F}_\lambda^{m,n}$.*

Proof. We set $c := -1/8\pi$ to shorten notation. For

$$f(U) = \exp(c \operatorname{tr} \boldsymbol{\Delta}_A)\big(g(U)\big) = \sum_{k=0}^{\infty} \frac{c^k}{k!} \operatorname{tr} \boldsymbol{\Delta}_A^k\big(g(U)\big) \quad (U \in \mathbb{R}^{m\times n})$$

we consider the entry (i, j) for $1 \le i \le n$, $1 \le j \le n$ of the system of partial differential equations (2.3):

$$\begin{aligned}\mathbf{E}_{ij} f + 2c \cdot (\boldsymbol{\Delta}_A)_{ij} f &= \sum_{k=0}^{\infty} \frac{c^k}{k!} \big(\mathbf{E}_{ij} \cdot \operatorname{tr} \boldsymbol{\Delta}_A^k\big)g + 2\sum_{k=0}^{\infty} \frac{c^{k+1}}{k!} \big((\boldsymbol{\Delta}_A)_{ij} \cdot \operatorname{tr} \boldsymbol{\Delta}_A^k\big)g \\ &= \mathbf{E}_{ij} g + \sum_{k=1}^{\infty} \frac{c^k}{k!} \big(\mathbf{E}_{ij} \cdot \operatorname{tr} \boldsymbol{\Delta}_A^k\big)g + 2\sum_{k=1}^{\infty} \frac{c^k}{(k-1)!} \big((\boldsymbol{\Delta}_A)_{ij} \cdot \operatorname{tr} \boldsymbol{\Delta}_A^{k-1}\big)g \\ &= \mathbf{E}_{ij} g + \sum_{k=1}^{\infty} \frac{c^k}{k!} \big(\mathbf{E}_{ij} \cdot \operatorname{tr} \boldsymbol{\Delta}_A^k + 2k \cdot (\boldsymbol{\Delta}_A)_{ij} \cdot \operatorname{tr} \boldsymbol{\Delta}_A^{k-1}\big)g\end{aligned}$$

Due to Lemma 2.17, we have

$$\big(\mathbf{E}_{ij} \cdot \operatorname{tr} \boldsymbol{\Delta}_A^k + 2k \cdot (\boldsymbol{\Delta}_A)_{ij} \cdot \operatorname{tr} \boldsymbol{\Delta}_A^{k-1}\big)g = \big(\operatorname{tr} \boldsymbol{\Delta}_A^k \cdot \mathbf{E}_{ij}\big)g$$

and therefore obtain

$$\mathbf{E}_{ij} f + 2c \cdot (\boldsymbol{\Delta}_A)_{ij} f = \sum_{k=0}^{\infty} \frac{c^k}{k!} \operatorname{tr} \boldsymbol{\Delta}_A^k\big(\mathbf{E}_{ij} g\big) = \exp\big(c \operatorname{tr} \boldsymbol{\Delta}_A\big)(\mathbf{E}_{ij} g).$$

If $\mathbf{E}_{ij} g = \lambda \cdot \delta_{ij} \cdot g$ holds, the right-hand side equals $\lambda \cdot \delta_{ij} \cdot f$. As we have $g = \exp(-c \operatorname{tr} \boldsymbol{\Delta}_A)f$ (see Property 2.20), we deduce that $\exp(c \operatorname{tr} \boldsymbol{\Delta}_A)(\mathbf{E}_{ij} g) = \lambda \cdot \delta_{ij} \cdot f$ implies $\mathbf{E}_{ij} g = \lambda \cdot \delta_{ij} \cdot g$. By Proposition 2.15, this is equivalent to $g \in \mathcal{F}_\lambda^{m,n}$. $\square$

In the next proposition, we consider (2.3) for *positive definite* matrices A, namely the system of partial differential equations

$$\mathcal{D}_A p = \alpha \cdot I \cdot p \quad \text{for } \alpha \in \mathbb{N}_0 \text{ and } A \text{ positive definite.} \tag{2.9}$$

We determine a finite basis of all solutions of (2.9) by additionally imposing a certain growth condition. Together with Proposition 2.30, where it is shown that theta series associated with these functions transform like modular forms, we obtain Theorem 2.5 for positive definite matrices A. In the proof, we employ Vignéras' result [Vig77] and the fact that we can explicitly construct a finite basis $\mathcal{B}_\alpha^{m,n}$ of $\mathcal{P}_\alpha^{m,n}$ due to Maass' result [Maa59].

Proposition 2.19. *Let $\mathcal{B}_\alpha^{m,n}$ denote a finite basis of $\mathcal{P}_\alpha^{m,n}$ and let $A \in \mathbb{Z}^{m\times m}$ denote a positive definite symmetric matrix. Every solution $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ of (2.9) that addi-*

tionally satisfies the growth condition $\widetilde{f}(U) := f(U)\exp\bigl(-\pi\operatorname{tr}(U^{\mathrm{t}}AU)\bigr) \in \mathcal{S}(\mathbb{R}^{m\times n})$ *is a polynomial. Moreover, a finite basis of this space of solutions is given by*

$$p = \exp\bigl(-\operatorname{tr}\boldsymbol{\Delta}_A/8\pi\bigr)P \quad \textit{with } P \in \mathcal{B}_\alpha^{m,n}.$$

Proof. We give a short review of Vignéras' reasoning and apply it to functions with matrix variables. We identify $\mathbb{R}^{m\times n}$ with $\mathbb{R}^{mn}$ by writing $\mathbb{R}^{m\times n} \ni U = (\boldsymbol{u_1},\dots,\boldsymbol{u_n})$, $\boldsymbol{u_i} \in \mathbb{R}^m$, as column vector

$$\boldsymbol{u} = \begin{pmatrix} \boldsymbol{u_1} \\ \vdots \\ \boldsymbol{u_n} \end{pmatrix} \in \mathbb{R}^{mn}.$$

If f satisfies the system of differential equations (2.9) for $c = -1/8\pi$, it follows in particular that

$$\operatorname{tr}(\mathcal{D}_A)f = \alpha n \cdot f \tag{2.10}$$

holds. We have

$$\operatorname{tr}\mathbf{E} = \sum_{i=1}^{n}\sum_{d=1}^{m} U_{di}\frac{\partial}{\partial U_{di}} \quad \text{and} \quad \operatorname{tr}\boldsymbol{\Delta}_A = \sum_{\nu=1}^{n}\sum_{\mu,\rho=1}^{m}\frac{\partial}{\partial U_{\mu\nu}}(A^{-1})_{\mu\rho}\frac{\partial}{\partial U_{\rho\nu}},$$

which are the usual Euler operator on $\mathbb{R}^{mn}$ and the Laplacian associated with the positive definite $mn\times mn$- matrix that consists of blocks of $m\times m$-matrices that are zero except for n copies of A on the diagonal. We write U in a suitable basis such that we consider the quadratic form $(S^{-1})^{\mathrm{t}}AS^{-1} = I$ to express $\widetilde{f}$ in an orthogonal basis of Hermite functions $H_{\boldsymbol{k}}$ in mn variables as

$$\widetilde{f} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^{mn}} c_{\boldsymbol{k}}H_{\boldsymbol{k}} \quad \text{with } c_{\boldsymbol{k}} \in \mathbb{R} \text{ and } \boldsymbol{k} = (k_{\mu\nu})_{1\le\mu\le m, 1\le\nu\le n},$$

where the Hermite functions in several variables are defined in terms of Hermite functions in one dimension:

$$H_{\boldsymbol{k}}(U) = \prod_{\mu=1}^{m}\prod_{\nu=1}^{n} H_{k_{\mu\nu}}(U_{\mu\nu}) \quad \text{with } H_{k_{\mu\nu}}(U_{\mu\nu}) = \exp\bigl(\pi U_{\mu\nu}^2\bigr)\frac{d^{k_{\mu\nu}}}{dU_{\mu\nu}^{k_{\mu\nu}}}\exp\bigl(-2\pi U_{\mu\nu}^2\bigr)$$

Since f is a solution of (2.10), a basis of all functions $\widetilde{f}$ is determined by the finite set of Hermite functions $H_{\boldsymbol{k}}$ with $|\boldsymbol{k}| = \sum_{\mu=1}^{m}\sum_{\nu=1}^{n}k_{\mu\nu} = \alpha n$ (this is Vignéras' argument, see [Vig77]), where we can rewrite $H_{\boldsymbol{k}}(U) = p_{\boldsymbol{k}}(U)\exp\bigl(-\pi\operatorname{tr}(U^{\mathrm{t}}AU)\bigr)$ with the Hermite polynomial $p_{\boldsymbol{k}}$. So $f(U) = \widetilde{f}(U)\exp\bigl(\pi\operatorname{tr}(U^{\mathrm{t}}AU)\bigr)$ can be expanded in terms of finitely many orthogonal Hermite polynomials $p_{\boldsymbol{k}}$ and thus is a polynomial itself.

Thus, $g := \exp\bigl(\operatorname{tr}\boldsymbol{\Delta}_A/8\pi\bigr)f$ is a polynomial that satisfies $\mathbf{E}g = \alpha\cdot I\cdot g$ by Lemma 2.18. We can choose any basis $\mathcal{B}_\alpha^{m,n}$ of $\mathcal{P}_\alpha^{m,n}$ to describe these homogeneous polynomials. Hence we also obtain a basis of the solutions of (2.9). As $\mathcal{P}_\alpha^{m,n}$ is a finite-dimensional vector space, the basis $\mathcal{B}_\alpha^{m,n}$ is finite. $\square$

Now we let A denote an indefinite matrix of signature (r,s) again. When we consider the associated system of partial differential equations (2.3), the solutions, which we describe in Proposition 2.23, can be traced to functions that are defined on $U^\pm$ respectively, where U^+

is the projection of U onto the subspace where A is positive definite and U^- the projection into the subspace where A is negative definite. So we first consider $\mathcal{I} = \left(\begin{smallmatrix} I_r & \mathrm{O} \\ \mathrm{O} & -I_s \end{smallmatrix}\right)$ instead of A and the corresponding system of partial differential equations

$$\mathcal{D}_{\mathcal{I}} f = \lambda \cdot I \cdot f \quad (\lambda \in \mathbb{Z}), \tag{2.11}$$

which can easily be split up into one part that depends on the first r rows of U and another part depending on the last s rows of U. We write U_r and U_s for these projections of U. Here, we have $M = I$ and thus $\mathbf{\Delta}_M = \mathbf{\Delta}$, and we show that a basis of all solutions is given by the functions

$$\exp\bigl(-\operatorname{tr} \mathbf{\Delta}/8\pi\bigr)\bigl(P(U)\bigr) \exp\bigl(-2\pi \operatorname{tr}(U_s^{\mathrm{t}} U_s)\bigr), \tag{2.12}$$

where P splits as $P(U) = P_r(U_r) \cdot P_s(U_s)$ with $P_r \in \mathcal{B}_\alpha^{m,n} \subset \mathcal{P}_\alpha^{m,n}$ and $P_s \in \mathcal{B}_\beta^{m,n} \subset \mathcal{P}_\beta^{m,n}$ with $(\alpha, \beta) \in \mathbb{N}_0^2$ such that $\alpha - \beta = \lambda + s$.

Lemma 2.20. *If one applies the Laplacian $\mathbf{\Delta}_{\mathcal{I}}$ and the Euler operator on a product of functions $g, h : \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}$, then the following rules hold:*

$$\begin{aligned} \mathbf{\Delta}_{\mathcal{I}}(g \cdot h) &= g \cdot \mathbf{\Delta}_{\mathcal{I}} h + h \cdot \mathbf{\Delta}_{\mathcal{I}} g + \Bigl(\frac{\partial}{\partial U} g\Bigr)^{\mathrm{t}} \cdot \mathcal{I} \cdot \Bigl(\frac{\partial}{\partial U} h\Bigr) + \Bigl(\frac{\partial}{\partial U} h\Bigr)^{\mathrm{t}} \cdot \mathcal{I} \cdot \Bigl(\frac{\partial}{\partial U} g\Bigr) \\ \boldsymbol{E}(g \cdot h) &= g \cdot \boldsymbol{E} h + h \cdot \boldsymbol{E} g \end{aligned}$$

We omit the proof as the claim follows by a straightforward calculation. The part of (2.12) that depends on the subspace of $\mathbb{R}^{m \times n}$, on which the quadratic form is negative definite, satisfies a slightly different system of partial differential equations than the one given in Lemma 2.18, as an additional exponential factor occurs.

Lemma 2.21. *Let $\mathcal{B}_\beta^{m,n}$ denote a basis of $\mathcal{P}_\beta^{m,n}$. We consider the system of partial differential equations*

$$\mathcal{D}_{-I} f = -(\beta + m) \cdot I \cdot f. \tag{2.13}$$

A finite basis of all solutions of (2.13) *that additionally satisfy the growth condition $f(U) \exp(\pi \operatorname{tr}(U^{\mathrm{t}} U)) \in \mathcal{S}(\mathbb{R}^{m \times n})$ is given by the functions*

$$f_P(U) := \exp\bigl(-\operatorname{tr} \mathbf{\Delta}/8\pi\bigr)\bigl(P(U)\bigr) \exp\bigl(-2\pi \operatorname{tr}(U^{\mathrm{t}} U)\bigr) \quad \textit{with } P \in \mathcal{B}_\beta^{m,n}.$$

Proof. We define $g(U) := \exp(-2\pi \operatorname{tr}(U^{\mathrm{t}} U))$ and $h_P(U) := \exp(-\operatorname{tr} \mathbf{\Delta}/8\pi)(P(U))$. Both functions satisfy systems of partial differential equations similar to (2.13): we check that

$$\bigl(\mathbf{E} g\bigr)(U) = -4\pi g(U) \cdot U^{\mathrm{t}} U$$

and

$$\bigl(\mathbf{\Delta} g\bigr)(U) = -4\pi m \cdot I \cdot g(U) + 16\pi^2 g(U) \cdot U^{\mathrm{t}} U$$

hold. Hence we have

$$\frac{1}{4\pi} \mathbf{\Delta} g = -\mathbf{E} g - m \cdot I \cdot g. \tag{2.14}$$

Due to Proposition 2.19 for $A = I$, the identity

$$\frac{1}{4\pi} \mathbf{\Delta} h_P = \mathbf{E} h_P - \beta \cdot I \cdot h_P \tag{2.15}$$

holds if and only if $P \in \mathcal{P}_{\beta}^{m,n}$. Using the multiplication rules from Lemma 2.20 and applying (2.14) and (2.15) in the calculation of $\Delta f_P = \Delta(g \cdot h_P)$, we obtain

$$\begin{aligned}\frac{1}{4\pi}\mathbf{\Delta} f_P &= \frac{1}{4\pi}\Big(g \cdot \mathbf{\Delta} h_P + h_P \cdot \mathbf{\Delta} g + \Big(\frac{\partial}{\partial U} g\Big)^{\mathrm{t}}\Big(\frac{\partial}{\partial U} h_P\Big) + \Big(\frac{\partial}{\partial U} h_P\Big)^{\mathrm{t}}\Big(\frac{\partial}{\partial U} g\Big)\Big)\\ &= g \cdot \big(\mathbf{E} h_P - \beta \cdot I \cdot h_P\big) + h_P \cdot \big(-\mathbf{E} g - m \cdot I \cdot g\big) - 2g \cdot \mathbf{E} h_P\\ &= -(\beta + m) \cdot I \cdot f_P - \mathbf{E} f_P,\end{aligned}$$

where we use in the second step that $\mathbf{E}^{\mathrm{t}} h_P = \mathbf{E} h_P$ holds, since h_P satisfies (2.15) and the Laplacian is symmetric.

Analogously, one can show that for any solution f of the system (2.13) of partial differential equations, the function $h(U) = f(U)\exp\big(2\pi \operatorname{tr}(U^{\mathrm{t}}U)\big)$ satisfies $\mathcal{D}_I h = \beta \cdot I \cdot h$. Since

$$h(U)\exp\big(-\pi \operatorname{tr}(U^{\mathrm{t}}U)\big) = f(U)\exp(\pi \operatorname{tr}(U^{\mathrm{t}}U)) \in \mathcal{S}(\mathbb{R}^{m\times n})$$

by assumption, we can apply Proposition 2.19, which states that we can describe a finite basis for all functions h by $h_P = \exp(-\operatorname{tr}\mathbf{\Delta}/8\pi)(P(U))$ with $P \in \mathcal{B}_{\beta}^{m,n}$. Thus, the functions f_P form a finite basis of the solutions of (2.13) that satisfy the aforementioned growth condition. □

In the next lemma, we show that the substitution of U by $S^{-1}U$ leads to the desired system of partial differential equations that is associated with A.

Lemma 2.22. *Let $S \in \mathbb{R}^{m\times m}$ such that $A = (S^{-1})^{\mathrm{t}}\mathcal{I}S^{-1}$ and consider the functions $f, f[S^{-1}] : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$, where $f[S^{-1}](U) = f(S^{-1}U)$. The function f satisfies (2.11) if and only if $f[S^{-1}]$ satisfies (2.3).*

Proof. Let $i, j \in \{1, \ldots, n\}$. It suffices to calculate

$$\begin{aligned}(\mathbf{\Delta}_A)_{ij}\big(f[S^{-1}](U)\big) &= \sum_{a,b=1}^{m} \frac{\partial}{\partial U_{ai}}(A^{-1})_{ab}\frac{\partial}{\partial U_{bj}}\big(f(S^{-1}U)\big)\\ &= \sum_{a,b,\mu,\nu=1}^{m} (S^{-1})_{\mu a}(A^{-1})_{ab}(S^{-1})_{\nu b}\frac{\partial^2 f}{\partial U_{\mu i}\partial U_{\nu j}}(S^{-1}U)\\ &= \sum_{\mu=1}^{m} \mathcal{I}_{\mu\mu}\frac{\partial^2 f}{\partial U_{\mu i}\partial U_{\mu j}}(S^{-1}U) = \big((\mathbf{\Delta}_{\mathcal{I}})_{ij} f\big)(S^{-1}U)\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_{ij}\big(f[S^{-1}](U)\big) &= \sum_{d=1}^{m} U_{di}\frac{\partial}{\partial U_{dj}}\big(f(S^{-1}U)\big) = \sum_{d,\nu=1}^{m} U_{di}(S^{-1})_{\nu d}\frac{\partial f}{\partial U_{\nu j}}(S^{-1}U)\\ &= \sum_{\nu=1}^{m}(S^{-1}U)_{\nu i}\frac{\partial f}{\partial U_{\nu j}}(S^{-1}U) = (\mathbf{E}_{ij} f)(S^{-1}U)\end{aligned}$$

to deduce the claim. □

Proposition 2.23. *Let $\mathcal{B}_{\alpha}^{m,n}$ denote a basis of $\mathcal{P}_{\alpha}^{m,n}$ and let $A \in \mathbb{Z}^{m\times m}$ denote a non-degenerate symmetric matrix of signature (r, s). As in Remark 2.9, we write A as the sum of a positive semi-definite matrix A^{+} and a negative semi-definite matrix A^{-} and define*

$M := A^+ - A^-$. The functions

$$f(U) = f_{\alpha,\beta}(U) = \exp\bigl(-\operatorname{tr}\mathbf{\Delta}_M/8\pi\bigr)\bigl(P(U)\bigr)\exp\bigl(2\pi\operatorname{tr}(U^{\mathrm{t}}A^-U)\bigr),$$

where $P \in \mathcal{P}^{m,n}_{\alpha+\beta}$ is given as the product $P(U) = P_r(U^+) \cdot P_s(U^-)$ with $P_r \in \mathcal{B}^{m,n}_\alpha \subset \mathcal{P}^{m,n}_\alpha$ and $P_s \in \mathcal{B}^{m,n}_\beta \subset \mathcal{P}^{m,n}_\beta$ for $(\alpha,\beta) \in \mathbb{N}_0^2$ such that $\alpha - \beta = \lambda + s$, form a (possibly infinite) basis for the space of solutions $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ of (2.3) that additionally satisfy the growth condition

$$f(U)\exp\bigl(-\pi\operatorname{tr}(U^{\mathrm{t}}AU)\bigr) \in \mathcal{S}(\mathbb{R}^{m\times n}).$$

Proof. We consider the case $A = \mathcal{I}$. First we take $(\alpha,\beta) \in \mathbb{N}_0^2$ with $\alpha - \beta = \lambda + s$ to be fixed and show that $f = f_{\alpha,\beta}$ solves (2.11). As the eigenvectors of A form the canonical basis of $\mathbb{R}^m$, the polynomial P splits as $P(U) = P_r(U_r) \cdot P_s(U_s)$, where $U_r \in \mathbb{R}^{r\times n}$ consists of the first r rows of U and $U_s \in \mathbb{R}^{s\times n}$ of the last s rows of U. The exponential part of f has the form $\exp\bigl(-2\pi\operatorname{tr}(U_s^{\mathrm{t}}U_s)\bigr)$, so we can write $f := f_r \cdot f_s$, where f_r denotes the part dependent on U_r and f_s the part dependent on U_s. By Lemma 2.20, we have

$$\mathbf{E}f = f_r \cdot \mathbf{E}f_s + f_s \cdot \mathbf{E}f_r.$$

The expression

$$\begin{aligned}\mathbf{\Delta}_{\mathcal{I}}f = \mathbf{\Delta}_{\mathcal{I}}(f_r \cdot f_s) = f_r \cdot \mathbf{\Delta}_{\mathcal{I}}f_s + f_s \cdot \mathbf{\Delta}_{\mathcal{I}}f_r \\ + \Bigl(\frac{\partial}{\partial U}f_r\Bigr)^{\mathrm{t}} \cdot \mathcal{I} \cdot \Bigl(\frac{\partial}{\partial U}f_s\Bigr) + \Bigl(\frac{\partial}{\partial U}f_r\Bigr)^{\mathrm{t}} \cdot \mathcal{I} \cdot \Bigl(\frac{\partial}{\partial U}f_s\Bigr)\end{aligned}$$

simplifies to

$$\mathbf{\Delta}_{\mathcal{I}}f = f_r \cdot \mathbf{\Delta}_{\mathcal{I}}f_s + f_s \cdot \mathbf{\Delta}_{\mathcal{I}}f_r,$$

since

$$\frac{\partial}{\partial U}f_r = \begin{pmatrix}\frac{\partial}{\partial U_r}f_r \\ \mathrm{O}\end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial U}f_s = \begin{pmatrix}\mathrm{O} \\ \frac{\partial}{\partial U_s}f_s\end{pmatrix}.$$

These relations also show that we can write $\mathbf{\Delta}_{\mathcal{I}}f_r = \mathbf{\Delta}_{I_r}f_r$ and $\mathbf{\Delta}_{\mathcal{I}}f_s = -\mathbf{\Delta}_{I_s}f_s$. Then we consider the system of partial differential equations depending on the first r rows of U, where f_r corresponds to the function f from Lemma 2.18. Independent from that, consider the part depending on the last s rows of U and apply Lemma 2.21 for f_s. Putting these results together, we obtain

$$\begin{aligned}\mathbf{\Delta}_{\mathcal{I}}f &= 4\pi \cdot \Bigl(f_r \cdot \bigl(\mathbf{E}f_s + (\beta + s) \cdot I \cdot f_s\bigr) + f_s \cdot \bigl(\mathbf{E}f_r - \alpha \cdot I \cdot f_r\bigr)\Bigr) \\ &= 4\pi \cdot \bigl(\mathbf{E}(f_r \cdot f_s) + (-\alpha + \beta + s) \cdot I \cdot (f_r \cdot f_s)\bigr) \\ &= 4\pi \cdot \bigl(\mathbf{E}f - (\alpha - \beta - s) \cdot I \cdot f\bigr),\end{aligned}$$

where $\alpha - \beta - s = \lambda$.

To show that these functions form a basis of all solutions, we employ a similar argument as in the proof of Proposition 2.19. Again, we use Vignéras' result to show that the solutions f of (2.11) have a certain form: We define the function $\widetilde{f}(U) := f(U)\exp\bigl(-\pi\operatorname{tr}(U^{\mathrm{t}}\mathcal{I}U)\bigr)$, which is a Schwartz function by assumption. Furthermore, identify $\mathbb{R}^{m\times n}$ with $\mathbb{R}^{mn}$ by writing $U \in \mathbb{R}^{m\times n}$ as a column vector in $\mathbb{R}^{mn}$. As we have

$$\operatorname{tr}(U^{\mathrm{t}}\mathcal{I}U) = \sum_{\nu=1}^{n}\Big(\sum_{\mu=1}^{r} U_{\mu\nu}^2 - \sum_{\mu=r+1}^{r+s} U_{\mu\nu}^2\Big),$$

which equals the normalized quadratic form of signature (rn, sn) on $\mathbb{R}^{mn}$, we write $\widetilde{f}$ as

$$\widetilde{f}(U) = f(U)\exp\Big(-\pi\sum_{\nu=1}^{n}\Big(\sum_{\mu=1}^{r} U_{\mu\nu}^2 - \sum_{\mu=r+1}^{r+s} U_{\mu\nu}^2\Big)\Big).$$

As an $\mathcal{L}^2(\mathbb{R}^{mn})$-function, $\widetilde{f}$ is given in an orthogonal basis of Hermite functions $H_{\boldsymbol{k}}$ in mn variables in the form of $\widetilde{f} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^{mn}} c_{\boldsymbol{k}} H_{\boldsymbol{k}}$ with $c_{\boldsymbol{k}} \in \mathbb{R}$. Since f is a solution of

$$\operatorname{tr}(\mathcal{D}_{\mathcal{I}})f = \lambda n \cdot f \quad (\lambda \in \mathbb{Z}), \tag{2.16}$$

we restrict the possible basis elements that appear in the expansion of $\widetilde{f}$:

$$\widetilde{f} = \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^{mn}\\ \varepsilon(\boldsymbol{k})=n(\lambda+s)}} c_{\boldsymbol{k}} H_{\boldsymbol{k}}, \quad \text{where } \varepsilon(\boldsymbol{k}) := \sum_{\nu=1}^{n}\Big(\sum_{\mu=1}^{r} k_{\mu\nu} - \sum_{\mu=r+1}^{r+s} k_{\mu\nu}\Big)$$

Thus, as a consequence of Vignéras' result for genus 1, any solution of (2.16) is given as a (possibly infinite) linear combination of functions

$$f_{\boldsymbol{k}}(U) := H_{\boldsymbol{k}}(U)\exp\Big(\pi\sum_{\nu=1}^{n}\Big(\sum_{\mu=1}^{r} U_{\mu\nu}^2 - \sum_{\mu=r+1}^{r+s} U_{\mu\nu}^2\Big)\Big),$$

where the Hermite functions on $\mathbb{R}^{mn}$ (respectively $\mathbb{R}^{m\times n}$) are given as product of one-dimensional Hermite functions:

$$H_{\boldsymbol{k}}(U) = \prod_{\mu=1}^{m}\prod_{\nu=1}^{n} H_{k_{\mu\nu}}(U_{\mu\nu}) = p(U_r)\,q(U_s)\exp\Big(-\pi\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} U_{\mu\nu}^2\Big)$$

with polynomials p, q, which are defined on U_r, U_s respectively. Rewriting $f_{\boldsymbol{k}}$ as

$$f_{\boldsymbol{k}}(U) = p(U_r)\,q(U_s)\exp\Big(-2\pi\sum_{\mu=r+1}^{r+s}\sum_{\nu=1}^{n} U_{\mu\nu}^2\Big) = p(U_r)\,q(U_s)\exp\big(-2\pi\operatorname{tr}(U_s^{\mathrm{t}}U_s)\big), \tag{2.17}$$

each solution of (2.11) is given as a linear combination of functions of the form (2.17). The system of partial differential equations (2.11) is separable, i. e. can be broken into the part that depends on U_r and the part that depends on U_s. Likewise, $f_{\boldsymbol{k}}$ is given by a polynomial factor p depending on U_r and a factor of the form $q(U_s)\exp\big(-2\pi\operatorname{tr}(U_s^{\mathrm{t}}U_s)\big)$, where q also denotes a polynomial. We can write $\mathcal{D}_{\mathcal{I}} = \mathcal{D}^r + \mathcal{D}^s$ such that the differential operator $\mathcal{D}^r$ vanishes if we apply it to a function on U_s, and has the form $\mathcal{D}_{I_r}$ when applying it to a function that is defined on U_r. Analogously, $\mathcal{D}^s$ only depends on U_s and is of the form $\mathcal{D}_{-I_s}$ when applying it to functions on U_s. So we have $\mathcal{D}_{\mathcal{I}} f_{\boldsymbol{k}} = \lambda \cdot I \cdot f_{\boldsymbol{k}}$ with

$$(\mathcal{D}_{\mathcal{I}} f_{\boldsymbol{k}})(U) = q(U_s)\exp\big(-2\pi\operatorname{tr}(U_s^{\mathrm{t}}U_s)\big)\,\mathcal{D}^r\big(p(U_r)\big) + p(U_r)\,\mathcal{D}^s\Big(q(U_s)\exp\big(-2\pi\operatorname{tr}(U_s^{\mathrm{t}}U_s)\big)\Big).$$

For $f_{\boldsymbol{k}}(U) \neq 0$ we divide by $f_{\boldsymbol{k}}$ and obtain for each entry of the system of partial differential equations a sum of two partial differential equations that depend on different variables and therefore have to admit constant solutions. It follows that a function $f_{\boldsymbol{k}}$ solving (2.11) is given as the product described in (2.17) with the additional restriction that

$$\frac{\mathcal{D}^r\big(p(U_r)\big)}{p(U_r)} = C_r \quad \text{and} \quad \frac{\mathcal{D}^s\big(q(U_s)\exp\big(-2\pi\,\mathrm{tr}(U_s^{\mathrm{t}}U_s)\big)\big)}{q(U_s)\exp\big(-2\pi\,\mathrm{tr}(U_s^{\mathrm{t}}U_s)\big)} = C_s$$
$$\text{with } C_r, C_s \in \mathbb{R}^{n\times n} \text{ and } C_r + C_s = \lambda \cdot I.$$

We show that $C_r = \alpha \cdot I$ for some $\alpha \in \mathbb{N}_0$ holds and thus $C_s = (\lambda - \alpha) \cdot I$. By applying the operator $\exp\big(\mathrm{tr}\,\boldsymbol{\Delta}/8\pi\big)$ to $p(U_r)$, we can deduce analogously to the proof of Lemma 2.18 that $p(U_r)$ satisfies

$$\mathcal{D}^r\big(p(U_r)\big) = C_r \cdot p(U_r)$$

if and only if the polynomial $P_r(U_r) := \exp\big(\mathrm{tr}\,\boldsymbol{\Delta}/8\pi\big)\big(p(U_r)\big)$ satisfies $\mathbf{E}P_r = C_r \cdot P_r$. We have shown in Lemma 2.16 that this system of partial differential equations admits polynomial solutions only if $C_r = \alpha \cdot I$ with $\alpha \in \mathbb{N}_0$.

Thus, every solution f of (2.11) is described by basis elements $f_{\boldsymbol{k}}$ that consist of two factors that depend on different variables: p solves the system of partial differential equations in Proposition 2.19, where we have shown that these solutions can be described by a basis of homogeneous polynomials of degree α. Similarly, the function $q(U_s)\exp\big(-2\pi\,\mathrm{tr}(U_s^{\mathrm{t}}U_s)\big)$ solves the system of equations in Lemma 2.21, where we also described a basis of solutions using homogeneous polynomials of degree β. We conclude that all solutions of (2.11) are described by the functions $f_{\alpha,\beta}$ defined above with $\alpha, \beta \in \mathbb{N}_0$ such that $\alpha - \beta = \lambda + s$. Thus, the basis consists of infinitely many elements if $r, s > 0$ (i. e. when A is indefinite) and finitely many otherwise (i. e. when A is positive or negative definite).

We substitute $U \mapsto S^{-1}U$ and apply Lemma 2.22 to obtain the result for the system of partial differential equations (2.3). Note that for every basis element $f_{\alpha,\beta}$ the polynomial P splits as $P(U) = P_r(U^+) \cdot P_s(U^-)$ by assumption. □

2.4 Construction of theta series with modular transformation behavior

In this section, we construct Siegel theta series, which transform like modular forms of weight $m/2+\lambda$, arising from the functions that we considered in the last section as solutions of $\mathcal{D}_A f = \lambda \cdot I \cdot f$. We explicitly determine the transformation behavior of the theta series with respect to $Z \mapsto Z + S$ (for a symmetric matrix $S \in \mathbb{Z}^{n\times n}$) and $Z \mapsto -Z^{-1}$. To state the next lemma, in which we describe the transformation behavior of $\vartheta_{\mathcal{H},\mathcal{K}}$ with respect to the first-mentioned transformation, we introduce the following notation for matrices:

Definition 2.24. (a) For $M \in \mathbb{Z}^{\mu\times\mu}$ we define $M_0 \in \mathbb{Z}^{\mu\times\mu}$ by $(M_0)_{ij} = M_{ii}$ for $i = j$ and zero otherwise.
(b) We write $1_{\mu\nu}$ for a matrix with μ rows and ν columns, whose entries are all equal to 1.

Lemma 2.25. *Let $S \in \mathbb{Z}^{n\times n}$ denote a symmetric matrix. With respect to $Z \mapsto Z + S$, the theta series from Definition 2.3 transforms as follows:*

$$\vartheta_{\mathcal{H},\mathcal{K}}(Z+S) = \mathrm{e}\big(-\operatorname{tr}(\mathcal{H}^{\mathrm{t}} A\mathcal{H}S)/2 - \operatorname{tr}(S_0 1_{nm} A_0 \mathcal{H})/2\big)\, \vartheta_{\mathcal{H},\widetilde{\mathcal{K}}}(Z)$$
$$\text{with } \widetilde{\mathcal{K}} := \mathcal{K} + \mathcal{H}S + \frac{1}{2}A^{-1}A_0 1_{mn} S_0$$

Proof. Write $U = \mathcal{H} + R$ with $R \in \mathbb{Z}^{m\times n}$ such that

$$\operatorname{tr}(U^{\mathrm{t}}AUS) = \operatorname{tr}\big(\mathcal{H}^{\mathrm{t}}A\mathcal{H}S\big) + 2\operatorname{tr}\big((\mathcal{H}S)^{\mathrm{t}}AR\big) + \operatorname{tr}\big(R^{\mathrm{t}}ARS\big).$$

It is straightforward to see that

$$\operatorname{tr}\big(\mathcal{H}^{\mathrm{t}}A\mathcal{H}S\big) + 2\operatorname{tr}\big((\mathcal{H}S)^{\mathrm{t}}AR\big) = -\operatorname{tr}\big(\mathcal{H}^{\mathrm{t}}A\mathcal{H}S\big) + 2\operatorname{tr}\big((\mathcal{H}S)^{\mathrm{t}}AU\big).$$

As A and S both denote symmetric matrices and $x^2 \equiv x \bmod 2$ for any $x \in \mathbb{Z}$, we have

$$\operatorname{tr}\big(R^{\mathrm{t}}ARS\big) \equiv \sum_{\nu=1}^{n}\sum_{\mu=1}^{m} R_{\mu\nu}A_{\mu\mu}S_{\nu\nu} \bmod 2.$$

To rewrite the expression on the right-hand side in terms of matrices, we introduce the matrix $1_{nm} \in \mathbb{Z}^{n\times m}$ that only contains 1's as entries and obtain

$$\begin{aligned}\mathrm{e}\big(\operatorname{tr}(R^{\mathrm{t}}ARS)/2\big) &= \mathrm{e}\big(\operatorname{tr}(S_0 1_{nm} A_0 R)/2\big)\\ &= \mathrm{e}\Big(\operatorname{tr}\big((A^{-1}A_0 1_{mn} S_0)^{\mathrm{t}}AU\big)/2 - \operatorname{tr}\big(S_0 1_{nm} A_0 \mathcal{H}\big)/2\Big).\end{aligned}$$ □

In the following two sections, we determine how the theta series behaves under $Z \mapsto -Z^{-1}$. To put it briefly, we calculate the Fourier transform of the summand and then apply the Poisson summation formula. We define the Fourier transform associated with the matrix A:

Definition 2.26. Let $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{C}$ such that $f \in \mathcal{S}(\mathbb{R}^{m\times n})$. Then $\widehat{f} \in \mathcal{S}(\mathbb{R}^{m\times n})$ denotes the Fourier transform

$$\widehat{f}(V) := \int\limits_{\mathbb{R}^{m\times n}} f(U)\,\mathrm{e}\big(\operatorname{tr}(V^{\mathrm{t}}AU)\big)\,dU$$

with dU the Euclidean volume element.

Note that we do not take the standard definition of the Fourier transform as a unitary operator here, but rather we obtain the additional normalizing factor $|\det A|^{-n/2}$. Consequently, the Poisson summation formula has the form

$$\sum_{U\in\mathbb{Z}^{m\times n}} f(U) = \sum_{V\in A^{-1}\mathbb{Z}^{m\times n}} \widehat{f}(V). \tag{2.18}$$

In Section 2.4.1, we consider theta series associated with positive definite quadratic forms and give a set of examples of non-holomorphic Siegel modular forms. We obtain those results by generalizing the set-up of Freitag [Fre83]. In Section 2.4.2, we see that a similar construction also yields theta series associated with indefinite quadratic forms that transform like Siegel modular forms.

2.4.1 Theta series for positive definite quadratic forms

In this section, $p : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ is a polynomial and $A \in \mathbb{Z}^{m\times m}$ is a symmetric positive definite matrix. Following Freitag [Fre83], we first examine the series

$$\sum_{U\in\mathbb{Z}^{m\times n}} p(U)\,\mathrm{e}\big(\mathrm{tr}(U^{\mathrm{t}}AUZ)/2\big) \quad (Z \in \mathbb{H}_n). \tag{2.19}$$

We consider the operator

$$\mathrm{tr}\,\mathbf{\Delta}_A = \sum_{\nu=1}^{n}\sum_{\mu,\rho=1}^{m} \frac{\partial}{\partial U_{\mu\nu}}(A^{-1})_{\mu\rho}\frac{\partial}{\partial U_{\rho\nu}}$$

and define

$$\exp(c\,\mathrm{tr}\,\mathbf{\Delta}_A)\big(p(U)\big) := \sum_{k=0}^{\infty}\frac{c^k}{k!}(\mathrm{tr}\,\mathbf{\Delta}_A)^k\big(p(U)\big) \quad (c\in\mathbb{C}).$$

Since we are assuming that p is a polynomial, this sum is finite.

Lemma 2.27. *The following rules hold for $a, b, c \in \mathbb{C}$ and $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$:*

$$\exp(a\,\mathrm{tr}\,\mathbf{\Delta}_A)\big(\exp(b\,\mathrm{tr}\,\mathbf{\Delta}_A)(p(U))\big) = \exp\big((a+b)\,\mathrm{tr}\,\mathbf{\Delta}_A\big)\big(p(U)\big) \tag{2.20}$$

$$\exp(c\,\mathrm{tr}\,\mathbf{\Delta}_A)\big(p(aU)\big) = \big(\exp(a^2c\,\mathrm{tr}\,\mathbf{\Delta}_A)p\big)(aU) \tag{2.21}$$

$$\exp(c\,\mathrm{tr}\,\mathbf{\Delta}_A)\big(p(UN)\big) = \big(\exp\big(c\,\mathrm{tr}(N\,\mathbf{\Delta}_A N^{\mathrm{t}})\big)p\big)(UN) \tag{2.22}$$

$$\exp(c\,\mathrm{tr}\,\mathbf{\Delta}_A)\big(p(MU)\big) = \bigg(\exp\Big(c\,\mathrm{tr}\Big(\big(\frac{\partial}{\partial U}\big)^{\mathrm{t}} MA^{-1}M^{\mathrm{t}}\frac{\partial}{\partial U}\Big)\Big)p\bigg)(MU) \tag{2.23}$$

Proof. We derive Property (2.20) by considering the Cauchy product for the absolutely convergent series $\sum_{k=0}^{\infty}\frac{1}{k!}(a\,\mathrm{tr}\,\mathbf{\Delta}_A)^k$ and $\sum_{k=0}^{\infty}\frac{1}{k!}(b\,\mathrm{tr}\,\mathbf{\Delta}_A)^k$. The identity (2.21) follows immediately from (2.22), when we set $N := a\cdot I \in \mathbb{C}^{n\times n}$. To show (2.22) we consider $p(UN)$ and apply the Laplacian. We have

$$\frac{\partial}{\partial U_{\mu\nu}}\big(p(UN)\big) = \sum_{\ell=1}^{m}\sum_{i=1}^{n}\frac{\partial p}{\partial U_{\ell i}}(UN)\frac{\partial (UN)_{\ell i}}{\partial U_{\mu\nu}} = \sum_{i=1}^{n} N_{\nu i}\frac{\partial p}{\partial U_{\mu i}}(UN),$$

since

$$\frac{\partial (UN)_{\ell i}}{\partial U_{\mu\nu}} = \begin{cases} N_{\nu i} & \text{if } \ell = \mu, \\ 0 & \text{otherwise.}\end{cases}$$

With the same argument we obtain

$$\frac{\partial^2}{\partial U_{\mu\nu}\partial U_{\rho\nu}}\big(p(UN)\big) = \sum_{d,e=1}^{n} N_{\nu d}N_{\nu e}\frac{\partial^2 p}{\partial U_{\rho d}\partial U_{\mu e}}(UN)$$

and therefore

$$\begin{aligned}
\operatorname{tr} \boldsymbol{\Delta}_A\bigl(p(UN)\bigr) &= \sum_{\nu=1}^{n} \sum_{\mu,\rho=1}^{m} \frac{\partial}{\partial U_{\mu\nu}} (A^{-1})_{\mu\rho} \frac{\partial}{\partial U_{\rho\nu}} \bigl(p(UN)\bigr) \\
&= \sum_{\mu,\rho=1}^{m} \sum_{\nu,d,e=1}^{n} N_{\nu d} N_{\nu e} (A^{-1})_{\mu\rho} \frac{\partial^2 p}{\partial U_{\rho d} \partial U_{\mu e}} (UN) \\
&= \Biggl(\sum_{\nu=1}^{n} \sum_{\mu,\rho=1}^{m} \Bigl(N \bigl(\frac{\partial}{\partial U}\bigr)^{\mathrm{t}} \Bigr)_{\nu\mu} (A^{-1})_{\mu\rho} \Bigl(\frac{\partial}{\partial U} N^{\mathrm{t}} \Bigr)_{\rho\nu} p \Biggr) (UN) \\
&= \bigl(\operatorname{tr}(N \boldsymbol{\Delta}_A N^{\mathrm{t}}) p\bigr)(UN).
\end{aligned}$$

Rewriting the Laplacian in the sum then gives (2.22). Analogously we obtain (2.23). □

We calculate the Fourier transform of the summands in the series (2.19). To shorten the calculation, we apply the following result by Freitag [Fre83, p. 158f.], who considers Gauss transforms: we have

$$\int_{\mathbb{R}^{m\times n}} p(U+V) \exp\bigl(-\pi \operatorname{tr}(U^{\mathrm{t}}U)\bigr)\, dU = \exp\bigl(\operatorname{tr} \boldsymbol{\Delta}/4\pi\bigr)\bigl(p(V)\bigr). \tag{2.24}$$

Note that Freitag uses the normalized Laplace operator $\operatorname{tr} \boldsymbol{\Delta} = \operatorname{tr} \boldsymbol{\Delta}_I$. In the next lemma, we see that for arbitrary polynomials p, the functions $p(U)\, \mathrm{e}\bigl(\operatorname{tr}(U^{\mathrm{t}}AUZ)/2\bigr)$ are not necessarily eigenfunctions with regard to the Fourier transform:

Lemma 2.28. *Let $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{C}$, $f(U) := p(U)\, \mathrm{e}\bigl(\operatorname{tr}(U^{\mathrm{t}}AUZ)/2\bigr)$. The Fourier transform of f is*

$$\begin{aligned}
\widehat{f}(V) = \det A^{-n/2} \det(-iZ)^{-m/2}\, \mathrm{e}\bigl(-\operatorname{tr}(V^{\mathrm{t}}AVZ^{-1})/2\bigr) \\
\cdot \Bigl(\exp\bigl(i \operatorname{tr}(\boldsymbol{\Delta}_A Z^{-1})/4\pi\bigr)p\Bigr)(-VZ^{-1}).
\end{aligned}$$

Proof. We rewrite Freitag's result (2.24) to obtain a form that is suitable for the calculation of the Fourier transform: We substitute U by $U+iV$, and as we examine a holomorphic integrand in several complex variables, we apply the global residue theorem (i. e. instead of integrating over U one can integrate over $U+iV$ without changing the integral) and obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{m\times n}} p(U) \exp\bigl(-\pi \operatorname{tr}(U^{\mathrm{t}}U) + 2\pi i \operatorname{tr}(V^{\mathrm{t}}U)\bigr)\, dU \\
&\qquad = \exp\bigl(-\pi \operatorname{tr}(V^{\mathrm{t}}V)\bigr) \int_{\mathbb{R}^{m\times n}} p(U+iV) \exp\bigl(-\pi \operatorname{tr}(U^{\mathrm{t}}U)\bigr)\, dU \\
&\qquad = \exp\bigl(-\pi \operatorname{tr}(V^{\mathrm{t}}V)\bigr) \bigl(\exp(\operatorname{tr} \boldsymbol{\Delta}/4\pi)p\bigr)(iV).
\end{aligned} \tag{2.25}$$

To determine

$$\widehat{f}(V) = \int_{\mathbb{R}^{m\times n}} p(U)\, \mathrm{e}\bigl(\operatorname{tr}(U^{\mathrm{t}}AUZ)/2 + \operatorname{tr}(V^{\mathrm{t}}AU)\bigr)\, dU,$$

we set $Z = iY$ and substitute U by $A^{-1/2}UY^{-1/2}$ (A and Y are positive definite symmetric matrices, so the same holds for the inverses and uniquely determined square roots):

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{m\times n}} p(U)\exp\bigl(-\pi\operatorname{tr}(U^{\mathrm{t}}AUY)+2\pi i\operatorname{tr}(V^{\mathrm{t}}AU)\bigr)\,dU \\
&\qquad = \det A^{-n/2}\det Y^{-m/2}\int\limits_{\mathbb{R}^{m\times n}} p(A^{-1/2}UY^{-1/2}) \\
&\qquad\qquad \cdot\exp\Bigl(-\pi\operatorname{tr}(U^{\mathrm{t}}U)+2\pi i\operatorname{tr}\bigl((A^{1/2}VY^{-1/2})^{\mathrm{t}}U\bigr)\Bigr)\,dU
\end{aligned}$$

This is (2.25) evaluated at $A^{1/2}VY^{-1/2}$ with a slightly changed argument in the polynomial p. We apply (2.22) and (2.23) and use that Y is symmetric to write $\widehat{f}$ as

$$\det A^{-n/2}\det Y^{-m/2}\exp\bigl(-\pi\operatorname{tr}(V^{\mathrm{t}}AVY^{-1})\bigr)\Bigl(\exp\bigl(\operatorname{tr}(\mathbf{\Delta}_A Y^{-1})/4\pi\bigr)p\Bigr)(iVY^{-1}).$$

As the integrand is a holomorphic function, we resubstitute $Y=-iZ$ (for the inverse we have $Y^{-1}=iZ^{-1}$) and deduce the claim by analytic continuation. □

In order to obtain an eigenfunction under the Fourier transformation, Freitag [Fre83] chooses p to be a harmonic polynomial, i. e. $(\operatorname{tr}\mathbf{\Delta})p=0$ and $p(UN)=\det N^{\alpha}p(U)$ holds for all $N\in\mathbb{C}^{n\times n}$. We consider the more general class of polynomials

$$p_Z(U):=\exp\bigl(-\operatorname{tr}(\mathbf{\Delta}_A Y^{-1})/8\pi\bigr)\bigl(P(U)\bigr)\quad\text{with } P\in\mathcal{P}^{m,n}_{\alpha}. \tag{2.26}$$

We described the vector space $P\in\mathcal{P}^{m,n}_{\alpha}$ in Section 2.3.1, where we have also seen that the functions $p(U):=\exp(-\operatorname{tr}(\mathbf{\Delta}_A)/8\pi)\bigl(P(U)\bigr)$ with $P\in\mathcal{P}^{m,n}_{\alpha}$ form a basis for the vector space of the solutions of $\mathcal{D}_A f=\alpha\cdot I\cdot f$. The slightly modified functions in (2.26) depend on the imaginary part Y of Z, which means that we lose holomorphicity in the construction of the theta series. However, for harmonic polynomials P, we obtain the holomorphic theta series considered by Freitag. Note that this is basically a generalization of Borcherds' construction for $n=1$ in [Bor98], see Remark 2.10 for a more detailed explanation.

Lemma 2.29. *Let p_Z denote a polynomial from (2.26) and define*

$$f_Z(U):=p_Z(U)\,\mathrm{e}\bigl(\operatorname{tr}(U^{\mathrm{t}}AUZ)/2\bigr).$$

The Fourier transform of f_Z is

$$\widehat{f_Z}(V)=i^{-mn/2}\det A^{-n/2}\det(-Z^{-1})^{m/2+\alpha}f_{-Z^{-1}}(V).$$

Proof. We apply Lemma 2.28 and then use the linearity of the trace and Property (2.20):

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{m\times n}} p_Z(U)\,\mathrm{e}\bigl(\operatorname{tr}(U^{\mathrm{t}}AUZ)/2+\operatorname{tr}(V^{\mathrm{t}}AU)\bigr)\,dU \\
&\qquad = \det A^{-n/2}\det(-iZ)^{-m/2}\,\mathrm{e}\bigl(-\operatorname{tr}(V^{\mathrm{t}}AVZ^{-1})/2\bigr) \\
&\qquad\qquad \cdot\bigl(\exp\bigl(i\operatorname{tr}\bigl(\mathbf{\Delta}_A Z^{-1}\bigr)/4\pi-\operatorname{tr}(\mathbf{\Delta}_A Y^{-1})/8\pi\bigr)P\bigr)\,(-VZ^{-1}) \qquad (2.27)\\
&\qquad = \det A^{-n/2}\det(-iZ)^{-m/2}\,\mathrm{e}\bigl(-\operatorname{tr}(V^{\mathrm{t}}AVZ^{-1})/2\bigr) \\
&\qquad\qquad \cdot\bigl(\exp\bigl(-\operatorname{tr}(\mathbf{\Delta}_A(Y^{-1}-2iZ^{-1}))/8\pi\bigr)P\bigr)\,(-VZ^{-1})
\end{aligned}$$

If $\widetilde{Y}$ denotes the imaginary part of $-Z^{-1}$, the identity $\widetilde{Y}=\overline{Z}^{-1}YZ^{-1}$ holds by (2.1).

Hence

$$Y^{-1} - 2iZ^{-1} = Y^{-1}(Z - 2iY)Z^{-1} = Y^{-1}\overline{Z}Z^{-1} = Z^{-1}(ZY^{-1}\overline{Z})Z^{-1} = Z^{-1}\widetilde{Y}^{-1}Z^{-1}.$$

The matrix Z is symmetric and therefore also its inverse Z^{-1}, which means that we can rewrite (2.27) as follows:

$$\det A^{-n/2} \det(-iZ)^{-m/2} \operatorname{e}\bigl(-\operatorname{tr}(V^{\mathrm{t}}AVZ^{-1})/2\bigr) \\ \cdot \Bigl(\exp\Bigl(-\operatorname{tr}\bigl((-Z^{-1})\mathbf{\Delta}_A(-Z^{-1})^{\mathrm{t}}\widetilde{Y}^{-1}\bigr)/8\pi\Bigr)P\Bigr)\,(-VZ^{-1})$$

Using Property (2.22) and the homogeneity of $P \in \mathcal{P}_\alpha^{m,n}$, we conclude that the Fourier transform of f_Z has the form

$$\begin{aligned}\widehat{f_Z}(V) &= \det A^{-n/2} \det(-iZ)^{-m/2} \det(-Z^{-1})^\alpha \operatorname{e}\bigl(-\operatorname{tr}(V^{\mathrm{t}}AVZ^{-1})/2\bigr) \\ &\qquad \cdot \Bigl(\exp\bigl(-\operatorname{tr}(\mathbf{\Delta}_A\widetilde{Y}^{-1})/8\pi\bigr)P\Bigr)(V) \\ &= \det A^{-n/2} \det(-iZ)^{-m/2} \det(-Z^{-1})^\alpha f_{-Z^{-1}}(V).\end{aligned}$$

Separating constant factors and factors depending on the determinants of A and Z, we deduce the claim. $\square$

This construction yields theta series that transform like Siegel modular forms:

Proposition 2.30. *Let $A \in \mathbb{Z}^{m\times m}$ denote a positive definite symmetric matrix and p the polynomial defined as $p(U) = \exp\bigl(-\operatorname{tr}\mathbf{\Delta}_A/8\pi\bigr)\bigl(P(U)\bigr)$ with $P \in \mathcal{P}_\alpha^{m,n}$. For the corresponding theta series $\vartheta_{\mathcal{H},\mathcal{K}}$ given in Definition 2.3 we have*

$$\vartheta_{\mathcal{H},\mathcal{K}}(-Z^{-1}) = i^{-mn/2} \det A^{-n/2} \det Z^{m/2+\alpha} \operatorname{e}\bigl(\operatorname{tr}(\mathcal{H}^{\mathrm{t}}A\mathcal{K})\bigr) \\ \cdot \sum_{\mathcal{J}\in A^{-1}\mathbb{Z}^{m\times n} \bmod \mathbb{Z}^{m\times n}} \vartheta_{\mathcal{J}+\mathcal{K},-\mathcal{H}}(Z).$$

Proof. We recall the definition of $\vartheta_{\mathcal{H},\mathcal{K}}$, which is

$$\vartheta_{\mathcal{H},\mathcal{K}}(Z) = \det Y^{-\alpha/2} \sum_{U\in\mathcal{H}+\mathbb{Z}^{m\times n}} p(UY^{1/2}) \operatorname{e}\bigl(\operatorname{tr}(U^{\mathrm{t}}AUZ)/2 + \operatorname{tr}(\mathcal{K}^{\mathrm{t}}AU)\bigr).$$

We use Property (2.22) and the homogeneity property of P to rewrite

$$\det Y^{-\alpha/2} p(UY^{1/2}) = \exp\bigl(-\operatorname{tr}(\mathbf{\Delta}_A Y^{-1})/8\pi\bigr)\bigl(P(U)\bigr) = p_Z(U),$$

and analogously $p_{-Z^{-1}}(U) = \det\widetilde{Y}^{-\alpha/2}p(U\widetilde{Y}^{1/2})$. That means the theta series has the form

$$\vartheta_{\mathcal{H},\mathcal{K}}(-Z^{-1}) = \sum_{U\in\mathbb{Z}^{m\times n}} \Bigl\{p_{-Z^{-1}}(U+\mathcal{H}) \\ \cdot \operatorname{e}\bigl(-\operatorname{tr}((U+\mathcal{H})^{\mathrm{t}}A(U+\mathcal{H})Z^{-1})/2 + \operatorname{tr}(\mathcal{K}^{\mathrm{t}}A(U+\mathcal{H}))\bigr)\Bigr\}.$$

By Lemma 2.29, the Fourier transform of the summand equals

$$i^{-mn/2} \det A^{-n/2} \det Z^{m/2+\alpha} p_Z(V+\mathcal{K}) \operatorname{e}\bigl(\operatorname{tr}((V+\mathcal{K})^{\mathrm{t}}A(V+\mathcal{K})Z)/2 - \operatorname{tr}(\mathcal{H}^{\mathrm{t}}AV)\bigr).$$

The summands in the theta series are Schwartz functions as A denotes a positive definite quadratic form. Hence we apply the Poisson summation formula (2.18), and obtain

$$\begin{aligned}\vartheta_{\mathcal{H},\mathcal{K}}(-Z^{-1}) &= i^{-mn/2}\det A^{-n/2}\det Z^{m/2+\alpha}\,\mathrm{e}\big(\mathrm{tr}(\mathcal{H}^{\mathrm{t}}A\mathcal{K})\big)\\ &\quad\cdot\sum_{V\in\mathcal{K}+A^{-1}\mathbb{Z}^{m\times n}} p_Z(V)\,\mathrm{e}\big(\mathrm{tr}(V^{\mathrm{t}}AVZ)/2-\mathrm{tr}(\mathcal{H}^{\mathrm{t}}AV)\big)\\ &= i^{-mn/2}\det A^{-n/2}\det Z^{m/2+\alpha}\,\mathrm{e}\big(\mathrm{tr}(\mathcal{H}^{\mathrm{t}}A\mathcal{K})\big)\\ &\quad\cdot\sum_{\mathcal{J}\in A^{-1}\mathbb{Z}^{m\times n}\bmod\mathbb{Z}^{m\times n}}\vartheta_{\mathcal{J}+\mathcal{K},-\mathcal{H}}(Z),\end{aligned}$$

which completes the proof. □

Example 2.31. For $m\equiv 0\bmod 8$, we choose an even unimodular matrix $A\in\mathbb{Z}^{m\times m}$, which means in particular that $\det A=1$ and $A^{-1}\in\mathbb{Z}^{m\times m}$. Considering the theta series

$$\vartheta_{\mathrm{O},\mathrm{O}}(Z)=\det Y^{-\alpha/2}\sum_{U\in\mathbb{Z}^{m\times n}} p(UY^{1/2})\,\mathrm{e}\big(\mathrm{tr}(U^{\mathrm{t}}AUZ)/2\big),$$

we have $\vartheta_{\mathrm{O},\mathrm{O}}(Z+S)=\vartheta_{\mathrm{O},\mathrm{O}}(Z)$ for any symmetric matrix $S\in\mathbb{Z}^{n\times n}$ by Lemma 2.25 and $\vartheta_{\mathrm{O},\mathrm{O}}(-Z^{-1})=\det Z^{m/2+\alpha}\vartheta_{\mathrm{O},\mathrm{O}}(Z)$ for a polynomial p as chosen in Proposition 2.30. Thus, $\vartheta_{\mathrm{O},\mathrm{O}}$ is a non-holomorphic Siegel modular form of weight $m/2+\alpha$ on the full Siegel modular group Γ_n.

2.4.2 Theta series for indefinite quadratic forms

In this section, we consider theta series associated with non-degenerate symmetric matrices $A\in\mathbb{Z}^{m\times m}$ with signature (r,s), where $s\geq 0$. As described in Remark 2.9, we decompose $A=A^{+}+A^{-}$ by employing the matrix of normalized eigenvectors $S\in\mathbb{R}^{m\times m}$ so that we obtain the associated majorant matrix $M=A^{+}-A^{-}$ and the projections $U^{\pm}$ of U into the positive and negative subspaces of $\mathbb{R}^{m\times n}$ respectively. We replace the polynomials p that were defined in Proposition 2.30 by functions of the form

$$g(U):=\exp\big(-\,\mathrm{tr}\,\boldsymbol{\Delta}_M/8\pi\big)\big(P(U)\big)\exp\big(2\pi\,\mathrm{tr}(U^{\mathrm{t}}A^{-}U)\big),\tag{2.28}$$

where $P\in\mathcal{P}^{m,n}_{\alpha+\beta}$ is given as the product $P(U)=P_\alpha(U^{+})\cdot P_\beta(U^{-})$ with $P_\alpha\in\mathcal{P}^{m,n}_{\alpha}$ and $P_\beta\in\mathcal{P}^{m,n}_{\beta}$. For $\alpha-\beta=\lambda+s$, we know from Section 2.3.2 that these functions are solutions of $\mathcal{D}_A f=\lambda\cdot I\cdot f$. Of course, we can also replace g by a linear combination of functions of this type, under the assumption that $(\alpha,\beta)\in\mathbb{N}_0^2$ such that $\alpha-\beta=\lambda+s$, to construct modular Siegel theta series. However, it is sufficient for the proof of Theorem 2.5 and simplifies the following calculations just to consider g as defined above, since these functions in particular include the basis elements of the vector space of solutions of $\mathcal{D}_A f=\lambda\cdot I\cdot f$.

In analogy with the last section, we define

$$g_Z(U):=\exp\big(-\,\mathrm{tr}(\boldsymbol{\Delta}_M Y^{-1})/8\pi\big)\big(P(U)\big)\exp\big(2\pi\,\mathrm{tr}(U^{\mathrm{t}}A^{-}UY)\big)$$

and

$$f_Z(U):=g_Z(U)\,\mathrm{e}\big(\mathrm{tr}(U^{\mathrm{t}}AUZ)/2\big).$$

For $s=0$, we get back the functions from Lemma 2.29, so we use the same notation.

Lemma 2.32. *The Fourier transform of* $f_Z(U) = g_Z(U)\,\mathrm{e}\big(\mathrm{tr}(U^{\mathrm{t}}AUZ)/2\big)$ *is*

$$\widehat{f_Z}(V) = i^{-mn/2}(-1)^{\beta s}|\det A|^{-n/2}\det(-Z^{-1})^{r/2+\alpha}\det\overline{Z}^{-(s/2+\beta)} f_{-Z^{-1}}(V).$$

Proof. We change the basis of $\mathbb{R}^m$ by the substitution of $U \mapsto SU$ to obtain a part that depends on the first r rows of U (again, we denote this part of the matrix by U_r) and one part that depends on the last s rows of U (analogously, we denote this part by U_s):

$$\begin{aligned}
&\int_{\mathbb{R}^{m\times n}} g_Z(U)\,\mathrm{e}\big(\mathrm{tr}(U^{\mathrm{t}}AUZ)/2 + \mathrm{tr}(U^{\mathrm{t}}AV)\big)\,dU \\
&\qquad = \int_{\mathbb{R}^{m\times n}} \exp\big(-\mathrm{tr}(\boldsymbol{\Delta}_M Y^{-1})/8\pi\big)\big(P(U)\big) \\
&\qquad\qquad \cdot \exp\big(2\pi\,\mathrm{tr}(U^{\mathrm{t}}A^{-}UY) + \pi i\,\mathrm{tr}(U^{\mathrm{t}}AUZ) + 2\pi i\,\mathrm{tr}(U^{\mathrm{t}}AV)\big)\,dU \\
&\qquad = \det S^n \int_{\mathbb{R}^{m\times n}} \exp\big(-\mathrm{tr}(\boldsymbol{\Delta} Y^{-1})/8\pi\big)\big(P(SU)\big) \\
&\qquad\qquad \cdot \exp\big(-2\pi\,\mathrm{tr}(U_s^{\mathrm{t}}U_sY) + \pi i\,\mathrm{tr}(U^{\mathrm{t}}\mathcal{I}UZ) + 2\pi i\,\mathrm{tr}(U^{\mathrm{t}}\mathcal{I}S^{-1}V)\big)\,dU
\end{aligned}$$

We can split up the integral, as the polynomial P factors as a polynomial dependent on U_r and U_s respectively. We now apply the results for positive definite quadratic forms. By Lemma 2.29, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{r\times n}} \exp\Big(-\frac{1}{8\pi}\,\mathrm{tr}\Big(\Big(\frac{\partial}{\partial U_r}\Big)^{\mathrm{t}}\frac{\partial}{\partial U_r}Y^{-1}\Big)\Big)\big(P_\alpha(U_r)\big)\,\mathrm{e}\big(\mathrm{tr}(U_r^{\mathrm{t}}U_rZ)/2 + \mathrm{tr}(V_r^{\mathrm{t}}U_r)\big)\,dU_r \\
&\qquad = i^{-rn/2}\det(-Z^{-1})^{r/2+\alpha}\,\mathrm{e}\big(-\mathrm{tr}(V_r^{\mathrm{t}}V_rZ^{-1})/2\big) \\
&\qquad\qquad \cdot \exp\Big(-\frac{1}{8\pi}\,\mathrm{tr}\Big(\Big(\frac{\partial}{\partial V_r}\Big)^{\mathrm{t}}\frac{\partial}{\partial V_r}\widetilde{Y}^{-1}\Big)\Big)\big(P_\alpha(V_r)\big).
\end{aligned} \tag{2.29}$$

We treat the part that depends on the negative definite subspace like an expression that is associated with a positive definite quadratic form given by I_s and consider $-\overline{Z} \in \mathbb{H}_n$ as variable in the Siegel upper half-space. Also note that by (2.1) we have

$$\overline{Z}^{-1} = \overline{Z}^{-1}ZZ^{-1} = \overline{Z}^{-1}(\overline{Z} + 2iY)Z^{-1} = Z^{-1} + 2i\overline{Z}^{-1}YZ^{-1} = Z^{-1} + 2i\widetilde{Y}. \tag{2.30}$$

In particular, $\mathrm{Im}\big(\overline{Z}^{-1}\big) = \mathrm{Im}(-Z^{-1}) = \widetilde{Y}$ and thus we have

$$\begin{aligned}
&\int_{\mathbb{R}^{s\times n}} \exp\Big(-\frac{1}{8\pi}\,\mathrm{tr}\Big(\Big(\frac{\partial}{\partial U_s}\Big)^{\mathrm{t}}\frac{\partial}{\partial U_s}Y^{-1}\Big)\Big)\big(P_\beta(U_s)\big)\,\mathrm{e}\big(-\mathrm{tr}(U_s^{\mathrm{t}}U_s\overline{Z})/2 - \mathrm{tr}(V_s^{\mathrm{t}}U_s)\big)\,dU_s \\
&\qquad = i^{-sn/2}(-1)^{\beta s}\det(\overline{Z}^{-1})^{s/2+\beta}\,\mathrm{e}\big(\mathrm{tr}(V_s^{\mathrm{t}}V_s\overline{Z}^{-1})/2\big) \\
&\qquad\qquad \cdot \exp\Big(-\frac{1}{8\pi}\,\mathrm{tr}\Big(\Big(\frac{\partial}{\partial V_s}\Big)^{\mathrm{t}}\frac{\partial}{\partial V_s}\widetilde{Y}^{-1}\Big)\Big)\big(P_\beta(V_s)\big),
\end{aligned} \tag{2.31}$$

where we evaluate the Fourier transform for $-V_s$, and use Property (2.22) and the identity $P_\beta(-V_s) = (-1)^{\beta s}P_\beta(V_s)$ to rewrite the expression. We now consider the product of (2.29) and (2.31) (we make use of (2.30) again to rewrite the exponential factor) and obtain:

$$\begin{aligned}\int\limits_{\mathbb{R}^{m\times n}} &\exp\big(-\operatorname{tr}(\mathbf{\Delta} Y^{-1})/8\pi\big)(P(SU))\\ &\quad\cdot \exp\big(-2\pi \operatorname{tr}(U_s^{\mathrm{t}} U_s Y) + \pi i \operatorname{tr}(U^{\mathrm{t}} \mathcal{I} U Z) + 2\pi i \operatorname{tr}(U^{\mathrm{t}} \mathcal{I} V)\big)\, dU\\ &= i^{-mn/2}(-1)^{\beta s} \det(-Z^{-1})^{r/2+\alpha} \det \overline{Z}^{-(s/2+\beta)} \exp\big(-(\operatorname{tr} \mathbf{\Delta} \widetilde{Y}^{-1})/8\pi\big)(P(SV))\\ &\qquad\qquad \cdot \exp\big(-\pi i \operatorname{tr}(V^{\mathrm{t}} \mathcal{I} V Z^{-1}) - 2\pi \operatorname{tr}(V_s^{\mathrm{t}} V_s \widetilde{Y})\big)\end{aligned}$$

We evaluate this integral at $S^{-1}V$ to complete the proof. Without loss of generality, we can assume that $\det S > 0$ and therefore write $\det S^n$ as $|\det A|^{n/2}$. □

Thus, we can state a more general version of Proposition 2.30 for Siegel theta series for indefinite quadratic forms.

Proposition 2.33. *Let $\lambda = \alpha - \beta - s$ and let $g : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ define a function from (2.28). The theta series of the form*

$$\vartheta_{\mathcal{H},\mathcal{K}}(Z) = \det Y^{-\lambda/2} \sum_{U \in \mathcal{H} + \mathbb{Z}^{m\times n}} g(UY^{1/2})\, \mathrm{e}\big(\operatorname{tr}(U^{\mathrm{t}} AUZ)/2 + \operatorname{tr}(\mathcal{K}^{\mathrm{t}} AU)\big)$$

transforms as follows:

$$\begin{aligned}\vartheta_{\mathcal{H},\mathcal{K}}(-Z^{-1}) = i^{-mn/2}(-1)^{(s/2+\beta)n+\beta s} |\det A|^{-n/2} \det Z^{(r-s)/2+\alpha-\beta}\, \mathrm{e}\big(\operatorname{tr}(\mathcal{H}^{\mathrm{t}} A\mathcal{K})\big)&\\ \cdot \sum_{\mathcal{J} \in A^{-1}\mathbb{Z}^{m\times n} \bmod \mathbb{Z}^{m\times n}} \vartheta_{\mathcal{J}+\mathcal{K},-\mathcal{H}}(Z)&\end{aligned}$$

Proof. We use the same approach as in the proof of Proposition 2.30. By Property (2.22), we have $g_{-Z^{-1}}(U) = \det \widetilde{Y}^{-(\alpha+\beta)/2} g(U\widetilde{Y}^{1/2})$, and thus we rewrite the theta series as

$$\begin{aligned}\vartheta_{\mathcal{H},\mathcal{K}}(-Z^{-1}) = \det \widetilde{Y}^{s/2+\beta} \sum_{U \in \mathbb{Z}^{m\times n}} \Big\{ & g_{-Z^{-1}}(U + \mathcal{H})\\ & \cdot \mathrm{e}\big(-\operatorname{tr}((U+\mathcal{H})^{\mathrm{t}} A(U+\mathcal{H})Z^{-1})/2 + \operatorname{tr}(\mathcal{K}^{\mathrm{t}} A(U+\mathcal{H}))\big)\Big\}.\end{aligned}$$

By Lemma 2.32, the Fourier transform of the summand equals

$$\begin{aligned}i^{-mn/2}(-1)^{\beta s} |\det A|^{-n/2} \det Z^{r/2+\alpha} \det(-\overline{Z})^{s/2+\beta} g_Z(V+\mathcal{K})&\\ \cdot \mathrm{e}\big(\operatorname{tr}((V+\mathcal{K})^{\mathrm{t}} A(V+\mathcal{K})Z)/2 - \operatorname{tr}(\mathcal{H}^{\mathrm{t}} AV)\big).&\end{aligned}$$

By (2.30), we have $\widetilde{Y} = \overline{Z}^{-1} Y Z^{-1}$ and thus rewrite

$$\det \widetilde{Y}^{s/2+\beta} \det Z^{r/2+\alpha} \det(-\overline{Z})^{s/2+\beta} = (-1)^{(s/2+\beta)n} \det Z^{(r-s)/2+\alpha-\beta} \det Y^{s/2+\beta}.$$

As $\det Y^{s/2+\beta} g_Z(U) = \det Y^{-\lambda/2} g(UY^{1/2})$ by Property (2.22), we have

$$\begin{aligned}\vartheta_{\mathcal{H},\mathcal{K}}(-Z^{-1}) &= i^{-mn/2}(-1)^{(s/2+\beta)n+\beta s} |\det A|^{-n/2} \det Z^{(r-s)/2+\alpha-\beta}\, \mathrm{e}\big(\operatorname{tr}(\mathcal{H}^{\mathrm{t}} A\mathcal{K})\big)\\ &\qquad \cdot \det Y^{-\lambda/2} \sum_{V \in \mathcal{K} + A^{-1}\mathbb{Z}^{m\times n}} g(VY^{1/2})\, \mathrm{e}\big(\operatorname{tr}(V^{\mathrm{t}} AVZ)/2 - \operatorname{tr}(V^{\mathrm{t}} A\mathcal{H})\big).\end{aligned}$$

Again, we write

$$\det Y^{-\lambda/2} \sum_{V\in\mathcal{K}+A^{-1}\mathbb{Z}^{m\times n}} g(VY^{1/2})\, \mathrm{e}\big(\mathrm{tr}(V^{\mathrm{t}}AVZ)/2 - \mathrm{tr}(V^{\mathrm{t}}A\mathcal{H})\big)$$
$$= \sum_{\mathcal{J}\in A^{-1}\mathbb{Z}^{m\times n} \bmod \mathbb{Z}^{m\times n}} \vartheta_{\mathcal{J}+\mathcal{K},-\mathcal{H}}(Z),$$

which completes the proof. □

Example 2.34. We obtain examples of non-holomorphic Siegel modular forms on the full Siegel modular group if $\mathcal{H} = \mathcal{K} = \mathrm{O}$ and A is an even unimodular matrix and additionally $i^{mn/2}(-1)^{(s/2+\beta)n+\beta s} = 1$ holds. Note that an even symmetric unimodular matrix of indefinite signature (r, s) only exists when $r - s \equiv 0 \bmod 8$ and is isomorphic to $H_2^k \oplus (\pm E_8)^\ell$ with $k = \min\{r, s\}$ and $\ell = |r - s|/8$, where $H_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and E_8 represents the equivalence class of all even unimodular positive definite matrices of rank 8 (note that we take E_8 if $r > s$ and $-E_8$ if $r < s$), see for example Husemoller and Milnor [HM73, p. 24–26] for more details.

2.4.3 Proof of Theorem 2.5

In Section 2.3, we introduced the $n \times n$-system of partial differential equations $\mathcal{D}_A f = \lambda \cdot I \cdot f$ and determined a basis for all the solutions f that additionally satisfy the growth condition $f(U) \exp(-\pi \,\mathrm{tr}(U^{\mathrm{t}}AU)) \in \mathcal{S}(\mathbb{R}^{m\times n})$ (see Proposition 2.23). In this section, we have determined the modular transformation behavior of the associated Siegel theta series $\vartheta_{\mathcal{H},\mathcal{K},f,A}$ by explicitly calculating the transformation formulas for the generators $Z \mapsto Z + S$ (see Lemma 2.25) and $Z \mapsto -Z^{-1}$ (see Proposition 2.33) of the Siegel modular group. For an even matrix A and $\lambda = \alpha - \beta - s$, the theta series $\vartheta_{\mathrm{O},\mathrm{O},f,A}$ transforms like a Siegel modular form of genus n and weight $m/2 + \lambda$ on some congruence subgroup of Γ_n. This proves Theorem 2.5.

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3 Siegel theta series for quadratic forms of signature $(m-1,1)$

This chapter is based on the preprint [Roe21b] of the same name, which is available under arXiv:2106.05703. We use the same notation as in the previous chapter. However, the matrix A is even now and has fixed signature $(m-1,1)$.

3.1 Introduction

While Siegel modular forms play an important role in various areas of mathematics, such as number theory and algebraic geometry, the number of explicit constructions is rather limited. We can obtain interesting examples by considering Siegel theta series that are associated with quadratic forms $\boldsymbol{Q}(U) = \frac{1}{2}\operatorname{tr}(U^{\mathrm{t}}AU)$ on $\mathbb{R}^{m\times n}$, where $A \in \mathbb{Z}^{m\times m}$ is an even symmetric and non-degenerate matrix with signature (r,s).

If A is positive definite, we can construct Siegel theta series as follows (see [Roe21a] for a more detailed description). We consider functions of the form $p = \exp(-\operatorname{tr}\boldsymbol{\Delta}_A/8\pi)P$, where

$$\exp\Bigl(-\frac{\operatorname{tr}\boldsymbol{\Delta}_A}{8\pi}\Bigr) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(8\pi)^k\,k!}(\operatorname{tr}\boldsymbol{\Delta}_A)^k \quad \text{with } \boldsymbol{\Delta}_A = \Bigl(\frac{\partial}{\partial U}\Bigr)^{\mathrm{t}} A^{-1}\frac{\partial}{\partial U},$$

and $P : \mathbb{C}^{m\times n} \longrightarrow \mathbb{C}$ is a polynomial that satisfies the homogeneity property $P(UN) = \det N^{\alpha}P(U)$ for all $N \in \mathbb{C}^{n\times n}$ and a fixed $\alpha \in \mathbb{N}_0$. We define the Siegel theta series associated with p as

$$\vartheta_p(Z) = \det Y^{-\alpha/2}\sum_{U\in\mathbb{Z}^{m\times n}} p(UY^{1/2})\exp\bigl(\pi i\operatorname{tr}(U^{\mathrm{t}}AUZ)\bigr) \quad (\mathbb{H}_n \ni Z = X+iY),$$

where $\mathbb{H}_n$ is the Siegel upper half-space and $Y^{1/2}$ is the square root of the positive definite matrix Y. Then ϑ_p transforms like a Siegel modular form of weight $m/2+\alpha$. As the homogeneity property of P is not maintained when we apply the operator $\exp(-\operatorname{tr}\boldsymbol{\Delta}_A/8\pi)$, these examples are in general non-holomorphic. Only when we consider harmonic polynomials, i. e. $(\operatorname{tr}\boldsymbol{\Delta}_A)P = 0$, we obtain holomorphic functions (see for example Freitag's exposition [Fre83]).

For indefinite quadratic forms it is generally difficult to construct Siegel theta series that are holomorphic *and* modular. Kudla [Kud81], however, considered quadratic forms of signature $(n,1)$ to construct holomorphic Siegel modular forms of genus n and weight $(n+1)/2$ as the integrals of non-holomorphic theta series. We employ a different approach and recall that we can generalize the construction above to obtain Siegel theta series for indefinite quadratic forms that transform like modular forms (see Borcherds' construction [Bor98] for $n=1$ and our own recent result [Roe21a] for higher n). In general, these are non-holomorphic and choosing a harmonic polynomial P does not suffice to establish holomorphicity.

In the present paper, we will deal with quadratic forms of signature $(m-1,1)$ and find for arbitrary genus $n \in \mathbb{N}$ holomorphic Siegel theta series that are related to non-holomorphic modular Siegel theta series. We obtain this construction by generalizing the result on elliptic modular forms by Zwegers [Zwe02]. So we review some results on elliptic theta series first.

For positive definite quadratic forms, we have Schoeneberg's description [Sch39] for the case that m is even and the result by Shimura [Shi73] for the case that m is odd. For $Q(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}^{\mathrm{t}} A \boldsymbol{u}$, where A has signature $(m-1,1)$, Göttsche and Zagier [GZ98] introduced holomorphic theta series, which were then modified by Zwegers [Zwe02] to construct theta series with modular transformation behavior. Instead of defining the theta series as a series over a full lattice in $\mathbb{R}^m$, one sums over a suitable cone in $\mathbb{R}^m$ to ensure the absolute convergence of the series. We introduce some notation to make this more explicit. We refer to one of the components of $\{\boldsymbol{c} \in \mathbb{R}^m \mid Q(\boldsymbol{c}) < 0\}$ as C_Q and define the theta function depending on two vectors $\boldsymbol{c_0}, \boldsymbol{c_1} \in C_Q$. Further, let $\boldsymbol{h}, \boldsymbol{k} \in \mathbb{R}^m$, and let B denote the bilinear form associated with Q. Then the holomorphic theta series is given by

$$\vartheta_{\boldsymbol{c_0},\boldsymbol{c_1}}(z) = \sum_{\boldsymbol{u} \in \boldsymbol{h}+\mathbb{Z}^m} \big\{\operatorname{sgn}\big(B(\boldsymbol{c_0},\boldsymbol{u})\big) - \operatorname{sgn}\big(B(\boldsymbol{c_1},\boldsymbol{u})\big)\big\} \exp\big(2\pi i Q(\boldsymbol{u}) z + 2\pi i B(\boldsymbol{u},\boldsymbol{k})\big) \quad (z \in \mathbb{H})$$

and the modular theta series by

$$\widehat{\vartheta}_{\boldsymbol{c_0},\boldsymbol{c_1}}(z) = \sum_{\boldsymbol{u} \in \boldsymbol{h}+\mathbb{Z}^m} \Big\{E\Big(\frac{B(\boldsymbol{c_0},\boldsymbol{u})}{\sqrt{-Q(\boldsymbol{c_0})}} y^{1/2}\Big) - E\Big(\frac{B(\boldsymbol{c_1},\boldsymbol{u})}{\sqrt{-Q(\boldsymbol{c_1})}} y^{1/2}\Big)\Big\} \cdot \exp\big(2\pi i Q(\boldsymbol{u}) z + 2\pi i B(\boldsymbol{u},\boldsymbol{k})\big),$$

where $y = \operatorname{Im} z > 0$ and

$$E(x) = 2\int_0^x \exp(-\pi v^2)\, dv = \operatorname{sgn}(x) - \operatorname{sgn}(x) \int_{x^2}^\infty v^{-1/2} \exp(-\pi v)\, dv \quad (x \in \mathbb{R}).$$

Since $E(xy^{1/2}) \to \operatorname{sgn}(x)$ for $y \to \infty$, the theta series $\vartheta_{\boldsymbol{c_0},\boldsymbol{c_1}}$ describes the holomorphic part of $\widehat{\vartheta}_{\boldsymbol{c_0},\boldsymbol{c_1}}$ and therefore we call $\widehat{\vartheta}_{\boldsymbol{c_0},\boldsymbol{c_1}}$ the modular completion of $\vartheta_{\boldsymbol{c_0},\boldsymbol{c_1}}$.

For signature $(m-2,2)$, holomorphic theta series and their modular completions were constructed in the work of Alexandrov, Banerjee, Manschot, and Pioline [ABMP18a]. Their suggestion for a generalization to generic signature (r,s) was then explicitly carried out by Nazaroglu [Naz18]. In a slightly different setting, considering positive polyhedral cones, similar results were presented by Raum [WR16]. Also Funke and Kudla [FK17, FK19] gave a general framework for the construction of non-holomorphic theta series for indefinite quadratic forms.

From a geometric point of view, an analogue of these theta series can be constructed as integrals of the theta forms introduced by Kudla and Millson [KM86, KM87, KM90] (in the latter, Siegel modular forms of higher genus are also discussed). The connection between the geometric and the classical approach was established in the aforementioned works by Funke and Kudla [FK17, FK19] and in the special cases where $s = 1, 2$ by Kudla [Kud13, Kud18].

We will employ this connection in the following when we describe modular Siegel theta series for quadratic forms of signature $(m-1,1)$. This has also been done by Livinsky [Liv16]. He deduces the connection to Zwegers' theta series, gives a construction for arbitrary $n \in \mathbb{N}$ and furthermore a very explicit description of these theta series for the case $n = 2$. We obtain almost the same construction for the modular version of the theta series. However, a connection to holomorphic series is not given in [Liv16]. In contrast to that, we determine an explicit construction of the associated holomorphic Siegel theta series and establish a connection between the holomorphic and the modular version. Note that we have to assume $m > n$ to obtain non-vanishing series.

We state the main results in more detail in the next section. There we also introduce the notation and several important definitions that will be used throughout the rest of the paper. Besides, we shortly recapitulate a result of [Roe21a], which states that theta series arising from certain functions transform like Siegel modular forms. In Section 3.3, we construct holomorphic Siegel theta series and thus prove part (i) of the main theorem. In Section 3.4, we consider Siegel theta series with modular transformation behavior (this is part (ii) of the main theorem). To conclude this section, we show that the function used in the construction of the modular series asymptotes to the function employed in the construction of the holomorphic one, which proves part (iii). We will also see there that for $n = 1$ we get back Zwegers' result [Zwe02].

3.2 Definitions, previous results and statement of the main results

We define the Siegel upper half-space as

$$\mathbb{H}_n := \{Z = X + iY \mid X, Y \in \mathbb{R}^{n\times n} \text{ symmetric and } Y \text{ positive definite}\}$$

and the full Siegel modular group

$$\Gamma_n := \left\{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{Z}^{2n\times 2n} \mid M^{\mathrm{t}} J M = J\right\}, \quad \text{where } J = \left(\begin{smallmatrix} \mathrm{O} & I_n \\ -I_n & \mathrm{O} \end{smallmatrix}\right),$$

which operates on $\mathbb{H}_n$ by

$$Z \mapsto M\langle Z\rangle = (AZ+B)(CZ+D)^{-1}.$$

The group Γ_n is generated by the matrices $\left(\begin{smallmatrix} I_n & S \\ \mathrm{O} & I_n \end{smallmatrix}\right)$, where $S \in \mathbb{Z}^{n\times n}$ is symmetric, and the matrix $\left(\begin{smallmatrix} \mathrm{O} & -I_n \\ I_n & \mathrm{O} \end{smallmatrix}\right)$, cf. [Fre83, p. 322–328]. We define Siegel theta series of the following form:

Definition 3.1. Let $\mathbb{H}_n \ni Z = X + iY$ and let $A \in \mathbb{Z}^{m\times m}$ be an even symmetric and non-degenerate matrix with signature (r, s). The theta series with characteristics $\mathcal{H}, \mathcal{K} \in \mathbb{R}^{m\times n}$, associated with $p : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ and A is defined as

$$\vartheta_p(Z) = \vartheta_{\mathcal{H},\mathcal{K},p,A}(Z) := \sum_{U\in\mathcal{H}+\mathbb{Z}^{m\times n}} p(UY^{1/2}) \exp\bigl(\pi i \operatorname{tr}(U^{\mathrm{t}}AUZ) + 2\pi i \operatorname{tr}(\mathcal{K}^{\mathrm{t}}AU)\bigr).$$

Remark 3.2. (a) We have to ensure by the choice of p that the theta series ϑ_p is defined by an absolutely convergent series (which is not obvious as A is indefinite). For the modular as well as for the holomorphic version, we will see that this is satisfied (see Remark 3.5 and Proposition 3.16).
(b) Note that we normally also have the factor $\det Y^{-\lambda/2}$ in the definition of ϑ_p, where the choice of $\lambda \in \mathbb{Z}$ depends on p, but here we will consider theta series with $\lambda = 0$.
(c) As we take A to be fixed, we usually drop this parameter in the index. We do the same for the characteristics $\mathcal{H}, \mathcal{K}$, as they only play a role when we determine the explicit modular transformation behavior.

In [Roe21a] we specifically constructed modular Siegel theta series of this form. In order to do so, we split up A into a positive semi-definite part A^+ and a negative semi-definite part A^-. Further, we define M as a positive definite majorant matrix associated with A, i. e. $M = A^+ - A^-$. Then we can also write $U \in \mathbb{R}^{m\times n}$ as $U = U^+ + U^-$, where U^+ lies in a subspace of $\mathbb{R}^{m\times n}$ on which the quadratic form is positive semi-definite and U^- in a subspace of $\mathbb{R}^{m\times n}$ on which the quadratic form is negative semi-definite. In [Roe21a] we obtained the decomposition of A by considering the eigenvectors of A. However, we can consider any decomposition of this form, see (3.2) for the one used in the present paper.

For $\alpha \in \mathbb{N}_0$, let $\mathcal{P}_\alpha^{m,n}$ denote the vector space of polynomials $P : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ with the homogeneity property $P(UN) = \det N^\alpha P(U)$ for all $N \in \mathbb{R}^{n\times n}$. Further, for $M \in \mathbb{Z}^{\mu\times\mu}$ we define $M_0 \in \mathbb{Z}^{\mu\times\mu}$ by $(M_0)_{ij} = M_{ii}$ for $i = j$ and zero otherwise and we use the notation 1_{nm} for a matrix with n rows and m columns, whose entries are all equal to 1.

A special case of Lemma 4.2 and Proposition 4.10 in [Roe21a] is the following result, where we choose $\alpha, \beta \in \mathbb{N}_0$ with $\alpha - \beta = s$ so that the weight is $m/2$. Also note that in contrast to [Roe21a] we choose A to be even in the present paper.

Theorem 3.3. *Let $\alpha, \beta \in \mathbb{N}_0$ with $\alpha - \beta = s$. Further, let P be defined as the product $P(U) = P_r(U^+) \cdot P_s(U^-)$ with $P_r \in \mathcal{P}_\alpha^{m,n}$ and $P_s \in \mathcal{P}_\beta^{m,n}$ and set*

$$p(U) = \exp\Big(-\frac{\operatorname{tr} \mathbf{\Delta}_M}{8\pi}\Big)\big(P(U)\big)\exp\big(2\pi \operatorname{tr}(U^{\mathrm{t}} A^- U)\big). \tag{3.1}$$

Then the transformation behavior of ϑ_p is as follows. For any symmetric matrix $S \in \mathbb{Z}^{n\times n}$ and $\widetilde{\mathcal{K}} := \mathcal{K} + \mathcal{H}S$, we have

$$\vartheta_{\mathcal{H},\mathcal{K},p,A}(Z+S) = \exp\big(-\pi i \operatorname{tr}(\mathcal{H}^{\mathrm{t}} A\mathcal{H}S) - \pi i \operatorname{tr}(S_0 1_{nm} A_0 \mathcal{H})\big)\,\vartheta_{\mathcal{H},\widetilde{\mathcal{K}},p,A}(Z),$$

and we have

$$\begin{aligned}\vartheta_{\mathcal{H},\mathcal{K},p,A}(-Z^{-1}) = i^{-mn/2}(-1)^{(s/2+\beta)n+\beta s}|\det A|^{-n/2}\det Z^{m/2}\exp\big(2\pi i \operatorname{tr}(\mathcal{H}^{\mathrm{t}} A\mathcal{K})\big)\\ \cdot \sum_{\mathcal{J}\in A^{-1}\mathbb{Z}^{m\times n} \bmod \mathbb{Z}^{m\times n}} \vartheta_{\mathcal{J}+\mathcal{K},-\mathcal{H},p,A}(Z).\end{aligned}$$

Remark 3.4. We can then either take rational matrices $\mathcal{H}$ and $\mathcal{K}$ and consider these theta series as entries of vector-valued Siegel modular forms or we set $\mathcal{H} = \mathcal{K} = \mathrm{O}$ and obtain scalar-valued modular forms for a certain character and on a suitable congruence subgroup of level N in Γ_n (where N is the level of A, i. e. the smallest $N \in \mathbb{N}$ such that NA^{-1} is an even matrix). However, we restrict ourselves to the description for the generating matrices of Γ_n. To put it short, we just say that a Siegel theta series of this kind *transforms like a (Siegel) modular form.*

Remark 3.5. For $p : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ as in (3.1), the series that defines ϑ_p is absolutely convergent, because the part of the expression that determines the growth is

$$\exp\big(2\pi \operatorname{tr}(U^{\mathrm{t}} A^- UY) - \pi \operatorname{tr}(U^{\mathrm{t}} AUY)\big) = \exp\big(-\pi \operatorname{tr}(U^{\mathrm{t}} MUY)\big),$$

where $M = A^+ - A^-$ is the positive definite majorant associated with A described above and Y is positive definite.

From now on we take the signature of A to be $(m-1,1)$. This and more notational conventions are fixed in the following definition.

Definition 3.6. Let A be an even symmetric and non-degenerate matrix of signature $(m-1,1)$. Then we define the quadratic forms $\boldsymbol{Q} : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$, $\boldsymbol{Q}(U) := \frac{1}{2}\operatorname{tr}(U^{\mathrm{t}} AU)$

and $Q : \mathbb{R}^m \longrightarrow \mathbb{R}$, $Q(\boldsymbol{u}) := \frac{1}{2}\boldsymbol{u}^{\mathrm{t}} A\boldsymbol{u}$ with the associated bilinear form $B(\boldsymbol{u},\boldsymbol{v}) = Q(\boldsymbol{u}+\boldsymbol{v}) - Q(\boldsymbol{u}) - Q(\boldsymbol{v})$.

We write henceforth $\boldsymbol{u_j}$ for the j-th column vector of $U = (\boldsymbol{u_1}\dots\boldsymbol{u_n}) \in \mathbb{R}^{m\times n}$. We can thus write $\boldsymbol{Q}$ as $\boldsymbol{Q}(U) = \sum_{j=1}^{n} Q(\boldsymbol{u_j})$ to consider the quadratic forms on the column vectors of U.

Further, we fix an element $\boldsymbol{c} \in \mathbb{R}^m$ with $Q(\boldsymbol{c}) < 0$ to split A into a negative semi-definite and a positive semi-definite part. In order to do so, we set

$$A^- := \frac{A\boldsymbol{c}\boldsymbol{c}^{\mathrm{t}}A}{2Q(\boldsymbol{c})} \quad \text{and} \quad A^+ := A - A^- \tag{3.2}$$

and we define the corresponding quadratic forms $\boldsymbol{Q}^-(U) := \frac{1}{2}\operatorname{tr}(U^{\mathrm{t}}A^-U)$ and $\boldsymbol{Q}^+(U) := \frac{1}{2}\operatorname{tr}(U^{\mathrm{t}}A^+U)$. In Lemma 3.18, we will see that indeed $\boldsymbol{Q}^-$ is a negative semi-definite and $\boldsymbol{Q}^+$ a positive semi-definite quadratic form. We will also show there that if we write $U = U^\perp + U^c$ by setting $\boldsymbol{u_j^c} := \frac{B(\boldsymbol{c},\boldsymbol{u_j})}{2Q(\boldsymbol{c})}\boldsymbol{c}$ and $\boldsymbol{u_j^\perp} := \boldsymbol{u_j} - \boldsymbol{u_j^c}$, the part $U^\perp$ lies in the subspace where $\boldsymbol{Q}$ is positive semi-definite and U^c in the subspace where $\boldsymbol{Q}$ is negative semi-definite.

We recall the construction of elliptic theta series in [Zwe02] for quadratic forms of signature $(m-1,1)$. One fixes one of the components in $\mathbb{R}^m$, where Q is negative, by taking a vector $\boldsymbol{c_0} \in \mathbb{R}^m$ with $Q(\boldsymbol{c_0}) < 0$ and setting

$$C_Q := \{\boldsymbol{u} \in \mathbb{R}^m \mid Q(\boldsymbol{u}) < 0,\, B(\boldsymbol{u},\boldsymbol{c_0}) < 0\}.$$

The theta series in [Zwe02] then depend on two vectors $\boldsymbol{c_0},\boldsymbol{c_1} \in C_Q$. To generalize this construction to higher genus n we need to define several similar objects:

Definition 3.7. We fix $n+1$ vectors in $\mathbb{R}^m$, which lie in C_Q, and collect them in a matrix

$$C := (\boldsymbol{c_0}\,\boldsymbol{c_1}\dots\boldsymbol{c_n}) \in \mathbb{R}^{m\times(n+1)} \quad \text{with } \boldsymbol{c_i} \in C_Q \subset \mathbb{R}^m.$$

We will also consider the matrices

$$\widetilde{C}_i := (\boldsymbol{c_0}\dots\widehat{\boldsymbol{c_i}}\dots\boldsymbol{c_n}) \in \mathbb{R}^{m\times n} \quad (0 \le i \le n),$$

where $\widehat{\cdot}$ means that the respective column is omitted. For $U \in \mathbb{R}^{m\times n}$, let $\mathbb{R}^n \ni \boldsymbol{x_i} := U^{\mathrm{t}}A\boldsymbol{c_i}$ for $0 \le i \le n$ and define $\widetilde{x}_i$ by setting

$$\widetilde{x}_i := (-1)^i \det\bigl(\boldsymbol{x_0}\dots\widehat{\boldsymbol{x_i}}\dots\boldsymbol{x_n}\bigr) = (-1)^i\det\bigl(U^{\mathrm{t}}A\widetilde{C}_i\bigr) \quad (0 \le i \le n). \tag{3.3}$$

We use these $\widetilde{x}_i$ to define an absolutely convergent and holomorphic theta series. For the definition of a modular version we consider an n-simplex in $\mathbb{R}^m$ that is defined as follows:

Definition 3.8. We define the n-simplex

$$S_n := \left\{ \sum_{i=0}^{n} t_i\boldsymbol{c_i} \,\middle|\, t_i \ge 0 \text{ for all } 0 \le i \le n \quad \text{and} \quad \sum_{i=0}^{n} t_i = 1 \right\}.$$

Remark 3.9. (a) Up to a sign the integral of a certain n-form over S_n is independent of the explicit parameterization of S_n. So we will only fix a parameterization when we explicitly evaluate the integral.

(b) We have $S_n \subset C_Q$: By definition we can write any $\boldsymbol{c} \in S_n$ as $\boldsymbol{c} = \sum_{i=0}^{n} t_i \boldsymbol{c_i}$ with $t_i \geq 0$ for all $0 \leq i \leq n$ and $\sum_{i=0}^{n} t_i = 1$. As not all t_i vanish, we have

$$Q(\boldsymbol{c}) = \frac{1}{2} \sum_{i,j=0}^{n} t_i t_j B(\boldsymbol{c_i}, \boldsymbol{c_j}) < 0 \quad \text{and} \quad B(\boldsymbol{c}, \boldsymbol{c_0}) = \sum_{i=0}^{n} t_i B(\boldsymbol{c_i}, \boldsymbol{c_0}) < 0,$$

thus $\boldsymbol{c} \in C_Q$.
(c) S_n is a compact set in $\mathbb{R}^m$, because it is closed and bounded.

We establish a connection between the holomorphic and the modular versions of the theta series. For this purpose we need to transfer the concept of a modular completion that we know for elliptic modular forms to higher genus n.

Definition 3.10. Let $Y \in \mathbb{R}^{n \times n}$ denote a positive definite symmetric matrix. Then all diagonal entries of Y are positive. When all the entries on the diagonal simultaneously go to infinity, we define this as $Y \to \infty$. If we have a modular theta series ϑ_g and a holomorphic theta series ϑ_f for which $g(UY^{1/2}) \to f(U)$ for $Y \to \infty$ holds, we then say that ϑ_f describes the holomorphic part of ϑ_g and on the other hand ϑ_g is referred to as the modular completion of ϑ_f.

The main theorem we are going to prove is the following:

Theorem 3.11. *(i) For*

$$f(U) = f^C(U) := \prod_{i=0}^{n} \frac{1 + \operatorname{sgn}(\widetilde{x}_i)}{2} - \prod_{i=0}^{n} \frac{1 - \operatorname{sgn}(\widetilde{x}_i)}{2},$$

the theta series ϑ_f is absolutely convergent and holomorphic in $Z \in \mathbb{H}_n$.
(ii) Let

$$g(U) = g^C(U) := \int_{S_n} (-Q(\boldsymbol{c}))^{-n/2} \exp\bigl(2\pi \operatorname{tr}(U^{\mathrm{t}} A^{-} U)\bigr) \bigwedge_{j=1}^{n} B(\boldsymbol{u_j^{\perp}}, d\boldsymbol{c}).$$

Then the theta series ϑ_g transforms like a Siegel modular form of weight $m/2$.
(iii) We have $g(UY^{1/2}) \to f(U)$ almost everywhere for $Y \to \infty$.

Remark 3.12. (a) We define the function f in order to describe the holomorphic part of g as good as possible with a relatively simple function. By definition, f is locally constant and evaluates to ± 1 or 0 almost everywhere. We see in Proposition 3.22 that the same holds for $g(UY^{1/2})$ for $Y \to \infty$. Further, $f(-U) = (-1)^n f(U)$, as $\operatorname{sgn}\bigl(\det((-U)^{\mathrm{t}} A \widetilde{C}_i)\bigr) = (-1)^n \operatorname{sgn}\bigl(\det(U^{\mathrm{t}} A \widetilde{C}_i)\bigr)$ for $U \in \mathbb{R}^{m \times n}$, and we also have $g(-U) = (-1)^n g(U)$, which can be deduced immediately from the definition of g. Although f and g occur to have similar properties, ϑ_f does not exactly describe the holomorphic part of $\vartheta_g(Z)$ for $Y \to \infty$. In Remark 3.23, we outline a more precise description of the holomorphic part of ϑ_g.
(b) In order to obtain non-vanishing functions f and g, we must assume $m > n$, so we make this assumption throughout the rest of this paper. From Lemma 3.13, it immediately follows that this is a necessary condition so that f does not vanish identically. The same holds for g: the column vectors $\boldsymbol{u_j^{\perp}}$ of $U^{\perp}$ lie in an $(m-1)$ - dimensional subspace of $\mathbb{R}^m$, so for $n \geq m$ we have for all $U \in \mathbb{R}^{m \times n}$ linear dependencies among the n vectors $\boldsymbol{u_j^{\perp}}$. Using the fact that the wedge product is a distributive and alternating map, we deduce that the n-form $\bigwedge_{j=1}^{n} B(\boldsymbol{u_j^{\perp}}, d\boldsymbol{c})$ vanishes identically and thus in particular g.

3.3 Holomorphic Siegel theta series

In this section, we construct theta series of genus n associated with indefinite quadratic forms of signature $(m-1,1)$ that are holomorphic. For this purpose, we consider the locally constant functions f described in Theorem 3.11(i). As in Zwegers' work [Zwe02], we show that the choice of f restricts the summation in the definition of the theta function ϑ_f to a component in $\mathbb{R}^{m\times n}$ on which the indefinite form is bounded from below by a positive definite quadratic form.

For $\widetilde{x}_i$ as in (3.3), we define this component as

$$\mathcal{C}_A := \{U \in \mathbb{R}^{m\times n} \mid \widetilde{x}_i \geq 0 \text{ for all } 0 \leq i \leq n \quad \text{or} \quad \widetilde{x}_i \leq 0 \text{ for all } 0 \leq i \leq n, \\ \text{where in both cases not all } \widetilde{x}_i \text{ vanish}\}.$$

By definition, $f(U) = 0$ for $U \notin \mathcal{C}_A$, as either $\widetilde{x}_i = 0$ for all $0 \leq i \leq n$, or both summands vanish since there exist $i, j \in \{0, \dots, n\}$ with $\widetilde{x}_i > 0$ and $\widetilde{x}_j < 0$, so the support of f lies in $\mathcal{C}_A$. Even if $\mathcal{C}_A$ is large, the corresponding theta series might vanish but at least we can exclude choices of C for which $\mathcal{C}_A$ is the empty set and thus $f \equiv 0$. For $n = 1$, this is done by choosing two linearly independent vectors $\boldsymbol{c_0}$ and $\boldsymbol{c_1}$. For higher genus n, we show in the next lemma that the matrix C should have full rank, i. e. the $n+1$ column vectors are linearly independent (as we assume $m > n$, we can always choose that many linearly independent vectors).

Lemma 3.13. *If $C = (\boldsymbol{c_0}\, \boldsymbol{c_1} \dots \boldsymbol{c_n})$ does not have full rank, $\mathcal{C}_A$ is empty.*

Proof. If $\boldsymbol{c_0}, \boldsymbol{c_1}, \dots, \boldsymbol{c_n} \in \mathbb{R}^m$ are linearly dependent, we can write without loss of generality $\boldsymbol{c_0} = \sum_{i=1}^n \lambda_i \boldsymbol{c_i}$ for $\lambda_i \in \mathbb{R}$. We determine $\widetilde{x}_k$ for $k \in \{1, \dots, n\}$, substituting $\boldsymbol{x_0} = \sum_{i=1}^n \lambda_i \boldsymbol{x_i}$:

$$\widetilde{x}_k = (-1)^k \sum_{i=1}^n \lambda_i \det\bigl(\boldsymbol{x_i}\, \boldsymbol{x_1} \dots \widehat{\boldsymbol{x_k}} \dots \boldsymbol{x_n}\bigr) = (-1)^k \lambda_k \det\bigl(\boldsymbol{x_k}\, \boldsymbol{x_1} \dots \widehat{\boldsymbol{x_k}} \dots \boldsymbol{x_n}\bigr) = -\lambda_k \widetilde{x}_0.$$

For $U \in \mathcal{C}_A$, $\widetilde{x}_0$ and $\widetilde{x}_k$ are both non-positive or both non-negative, so we either have $\lambda_k \leq 0$ or $\widetilde{x}_0 = \widetilde{x}_k = 0$. If the latter case holds for any k, all $\widetilde{x}_i$ $(0 \leq i \leq n)$ vanish, which contradicts our definition of $\mathcal{C}_A$. So we have $\lambda_k \leq 0$ for all $k \in \{1, \dots, n\}$. But since

$$0 > B(\boldsymbol{c_0}, \boldsymbol{c_0}) = \sum_{i=1}^n \lambda_i B(\boldsymbol{c_0}, \boldsymbol{c_i}), \quad \text{where } B(\boldsymbol{c_0}, \boldsymbol{c_i}) < 0 \text{ for all } 1 \leq i \leq n,$$

not all λ_k can be non-positive. Hence $\mathcal{C}_A$ is the empty set. □

We use the following statement to construct positive definite quadratic forms:

Lemma 3.14 ([Zwe02, Lemma 2.6])**.** *Let $\boldsymbol{c_k}, \boldsymbol{c_\ell} \in C_Q$ be linearly independent. The quadratic form*

$$Q^+_{k,\ell} : \mathbb{R}^m \longrightarrow \mathbb{R}, \quad Q^+_{k,\ell}(\boldsymbol{v}) := Q(\boldsymbol{v}) + \frac{B(\boldsymbol{c_k}, \boldsymbol{c_\ell}) B(\boldsymbol{c_k}, \boldsymbol{v}) B(\boldsymbol{c_\ell}, \boldsymbol{v})}{4Q(\boldsymbol{c_k})Q(\boldsymbol{c_\ell}) - B(\boldsymbol{c_k}, \boldsymbol{c_\ell})^2}$$

is positive definite.

For $U \in \mathcal{C}_A$ we have:

Lemma 3.15. *For any column vector $\boldsymbol{u_j}$ of $U \in \mathcal{C}_A$, there exist $k, \ell \in \{0, \dots, n\}$ such that*

$$\operatorname{sgn}\big(B(\boldsymbol{c_k}, \boldsymbol{u_j})\big) \neq \operatorname{sgn}\big(B(\boldsymbol{c_\ell}, \boldsymbol{u_j})\big).$$

Moreover, $\boldsymbol{c_k}$ and $\boldsymbol{c_\ell}$ are linearly independent.

Proof. Let $\boldsymbol{v} \in \mathbb{R}^m$. We calculate the determinant of the $(n+1) \times (n+1)$-matrix

$$\begin{pmatrix} C^{\mathrm{t}} A \boldsymbol{v} & C^{\mathrm{t}} A U \end{pmatrix} = \begin{pmatrix} C^{\mathrm{t}} A \boldsymbol{v} & C^{\mathrm{t}} A \boldsymbol{u_1} & \dots & C^{\mathrm{t}} A \boldsymbol{u_n} \end{pmatrix} \tag{3.4}$$

by expanding along the first column and obtain

$$\det \begin{pmatrix} C^{\mathrm{t}} A \boldsymbol{v} & C^{\mathrm{t}} A U \end{pmatrix} = \sum_{i=0}^{n} \big(C^{\mathrm{t}} A \boldsymbol{v}\big)_{i+1} \widetilde{x}_i = \sum_{i=0}^{n} B(\boldsymbol{c_i}, \boldsymbol{v}) \widetilde{x}_i.$$

For $\boldsymbol{v} = \boldsymbol{u_j}$ the determinant vanishes, as (3.4) has two identical columns, i. e.

$$\sum_{i=0}^{n} B(\boldsymbol{c_i}, \boldsymbol{u_j}) \widetilde{x}_i = 0.$$

For any $U \in \mathcal{C}_A$, we cannot have $B(\boldsymbol{c_i}, \boldsymbol{u_j}) = 0$ for all $i \in \{0, \dots, n\}$, since otherwise the j-th row of $U^{\mathrm{t}} A \widetilde{C_k}$ is zero, which implies $\widetilde{x}_k = 0$ for all $k \in \{0, \dots, n\}$. If $B(\boldsymbol{c_i}, \boldsymbol{u_j}) > 0$ for all $i \in \{0, \dots, n\}$, the expression $\sum_{i=0}^{n} B(\boldsymbol{c_i}, \boldsymbol{u_j}) \widetilde{x}_i$ would be strictly positive (resp. negative) since $\widetilde{x}_k \geq 0$ (resp. $\widetilde{x}_k \leq 0$) for all $k \in \{0, \dots, n\}$, where at least one inequality is strict. With the same argument, we exclude the case $B(\boldsymbol{c_i}, \boldsymbol{u_j}) < 0$ for all $i \in \{0, \dots, n\}$. Hence there exist $k, \ell \in \{0, \dots, n\}$ with $\operatorname{sgn}\big(B(\boldsymbol{c_k}, \boldsymbol{u_j})\big) \neq \operatorname{sgn}\big(B(\boldsymbol{c_\ell}, \boldsymbol{u_j})\big)$.

We note that any linearly dependent column vectors $\boldsymbol{c_k}$ and $\boldsymbol{c_\ell}$ admit the same sign: Let $\lambda \in \mathbb{R}$ such that $\boldsymbol{c_\ell} = \lambda \boldsymbol{c_k}$. Then $B(\boldsymbol{c_k}, \boldsymbol{c_\ell}) = \lambda B(\boldsymbol{c_k}, \boldsymbol{c_k})$ holds. As we have $B(\boldsymbol{c_k}, \boldsymbol{c_\ell}) < 0$ and $B(\boldsymbol{c_k}, \boldsymbol{c_k}) < 0$ for $\boldsymbol{c_k}, \boldsymbol{c_\ell} \in C_Q$, the factor λ is strictly positive, hence $\operatorname{sgn}\big(B(\boldsymbol{c_\ell}, \boldsymbol{u_j})\big) = \operatorname{sgn}\big(\lambda B(\boldsymbol{c_k}, \boldsymbol{u_j})\big) = \operatorname{sgn}\big(B(\boldsymbol{c_k}, \boldsymbol{u_j})\big)$. □

We use Lemma 3.14 and Lemma 3.15 to construct holomorphic theta series.

Proposition 3.16. *The series defining ϑ_f is absolutely convergent. Moreover, ϑ_f is a holomorphic function in $Z \in \mathbb{H}_n$.*

Proof. We have $f(U) = 0$ for $U \notin \mathcal{C}_A$ as noted above. For $U \in \mathcal{C}_A$ we obtain:

$$f(U) = \begin{cases} 1 & \text{if } \widetilde{x}_i > 0 \quad \text{for all } 0 \leq i \leq n, \\ -1 & \text{if } \widetilde{x}_i < 0 \quad \text{for all } 0 \leq i \leq n, \\ 2^{k-n-1} & \text{if } \widetilde{x}_i > 0 \quad \text{for } k \text{ values in } 0 \leq i \leq n \text{ and } \widetilde{x}_i = 0 \text{ otherwise}, \\ -2^{k-n-1} & \text{if } \widetilde{x}_i < 0 \quad \text{for } k \text{ values in } 0 \leq i \leq n \text{ and } \widetilde{x}_i = 0 \text{ otherwise}. \end{cases}$$

Since only the values $U \in \mathcal{C}_A$ contribute non-vanishing terms, we split up $\mathcal{C}_A$ in smaller components. In each component, the expression $\operatorname{tr}(U^{\mathrm{t}} A U)$ is bounded from below by a positive definite quadratic form built from the quadratic forms $Q^{+}_{k,\ell}$ that were introduced in Lemma 3.14. For a fixed $\boldsymbol{u_j}$, we apply Lemma 3.15 and take k, ℓ with $\operatorname{sgn}\big(B(\boldsymbol{c_k}, \boldsymbol{u_j})\big) \neq \operatorname{sgn}\big(B(\boldsymbol{c_\ell}, \boldsymbol{u_j})\big)$, so $B(\boldsymbol{c_k}, \boldsymbol{u_j}) B(\boldsymbol{c_\ell}, \boldsymbol{u_j}) \leq 0$. Since $\boldsymbol{c_k}, \boldsymbol{c_\ell} \in C_Q$ and these vectors are linearly independent, we have

$$\frac{B(\boldsymbol{c_k}, \boldsymbol{c_\ell})}{4Q(\boldsymbol{c_k})Q(\boldsymbol{c_\ell}) - B(\boldsymbol{c_k}, \boldsymbol{c_\ell})^2} > 0$$

and thus $Q(\boldsymbol{u_j}) \geq Q^+_{k,\ell}(\boldsymbol{u_j})$. Considering $\boldsymbol{Q}(U) = \sum_{j=1}^n Q(\boldsymbol{u_j})$, the quadratic forms $Q^+_{k,\ell}$ give a lower bound for each $\boldsymbol{u_j}$ and thus also a bound for $U \in \mathcal{C}_A$. Note that in general we have to take different forms $Q^+_{k,\ell}$ for each column vector. As $Y \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, the square root $Y^{1/2} \in \mathbb{R}^{n\times n}$ is uniquely determined and positive definite. The set $\mathcal{C}_A$ is invariant under the substitution $U \mapsto \breve{U} = UY^{1/2}$ and we find for every column $\breve{\boldsymbol{u}}_{\boldsymbol{j}}$ of $\breve{U}$ a lower bound in terms of a positive definite quadratic form as before. Hence,

$$\left|\exp\bigl(\pi i \operatorname{tr}(U^{\mathrm{t}}AUZ)\bigr)\right| = \exp\bigl(-\pi \operatorname{tr}(U^{\mathrm{t}}AUY)\bigr) = \exp\Bigl(-2\pi \sum_{j=1}^n Q(\breve{\boldsymbol{u}}_{\boldsymbol{j}})\Bigr)$$

is bounded from above by the sum of positive definite quadratic forms $Q^+_{k,\ell}$ that we choose for all column vectors $\breve{\boldsymbol{u}}_{\boldsymbol{j}}$ independently. We split up the sum over $U \in \mathcal{C}_A \cap \mathbb{Z}^{m\times n}$ in finitely many sets, according to the quadratic forms $Q^+_{k,\ell}$ that give a lower bound for $\boldsymbol{Q}(U)$. Thus, the series is absolutely convergent. Since f is locally constant (the points of discontinuity are given by the matrices in $\mathcal{H} + \mathbb{Z}^{m\times n}$ with $\widetilde{x}_i = 0$ for some $i \in \{0,\ldots,n\}$), the series ϑ_f is holomorphic in Z. □

Remark 3.17. We give a specific formula for f here, but we can replace f by any locally constant function that is zero for $U \notin \mathcal{C}_A$ to obtain a holomorphic theta series.

This shows part (i) of Theorem 3.11. In the following section, we construct certain functions g (depending on the choice of C) such that ϑ_g has modular transformation properties. We will see that $g(UY^{1/2}) \to f(U)$ almost everywhere for $Y \to \infty$.

3.4 Siegel theta series with modular transformation behavior

In [Roe21a] we have constructed theta series ϑ_g that transform like modular forms by considering a certain family of functions g, which we described here in Theorem 3.3. A crucial attribute of these functions is that we can split up g in two factors where one depends on a subspace of $\mathbb{R}^{m\times n}$, on which the quadratic form is positive semi-definite, and the other on a subspace where the form is negative semi-definite. In the following, we first determine explicitly how we split up the quadratic form for matrices of signature $(m-1,1)$. Then we show that we can apply the result of [Roe21a] to deduce the modular transformation behavior of the theta series.

In the next lemma, we show that $\boldsymbol{Q}$ is positive semi-definite on $U^\perp = (\boldsymbol{u_1^\perp} \ldots \boldsymbol{u_n^\perp})$ and negative semi-definite on $U^c = (\boldsymbol{u_1^c} \ldots \boldsymbol{u_n^c})$.

Lemma 3.18. *For $\boldsymbol{Q} : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$, we have the decomposition $\boldsymbol{Q} = \boldsymbol{Q}^+ + \boldsymbol{Q}^-$, where $\boldsymbol{Q}^+$ is positive semi-definite and $\boldsymbol{Q}^-$ negative semi-definite. Moreover, $\boldsymbol{Q}^+(U) = \boldsymbol{Q}(U^\perp)$ and $\boldsymbol{Q}^-(U) = \boldsymbol{Q}(U^c)$.*

Proof. By the definition of A^- in (3.2), we immediately obtain

$$\boldsymbol{Q}^-(U) = \operatorname{tr}\left(\frac{U^{\mathrm{t}}A\boldsymbol{c}\boldsymbol{c}^{\mathrm{t}}AU}{4Q(\boldsymbol{c})}\right) = \frac{1}{4Q(\boldsymbol{c})}\sum_{j=1}^n B(\boldsymbol{c},\boldsymbol{u_j})^2 \leq 0 \quad \text{for all } U \in \mathbb{R}^{m\times n}. \tag{3.5}$$

Since $A^+ = A - A^-$, we have

$$\begin{aligned}\boldsymbol{Q}^+(U) &= \frac{1}{2}\operatorname{tr}(U^{\mathrm{t}}AU) - \operatorname{tr}\left(\frac{U^{\mathrm{t}}A\boldsymbol{c}\boldsymbol{c}^{\mathrm{t}}AU}{4Q(\boldsymbol{c})}\right)\\ &= \sum_{j=1}^{n} Q(\boldsymbol{u_j}) - \frac{1}{4Q(\boldsymbol{c})}\sum_{j=1}^{n} B(\boldsymbol{c},\boldsymbol{u_j})^2\\ &= \sum_{j=1}^{n} \frac{4Q(\boldsymbol{c})Q(\boldsymbol{u_j}) - B(\boldsymbol{c},\boldsymbol{u_j})^2}{4Q(\boldsymbol{c})}.\end{aligned} \tag{3.6}$$

The numerator of each summand represents the determinant of the Gram matrix

$$\begin{pmatrix} 2Q(\boldsymbol{u_j}) & B(\boldsymbol{c},\boldsymbol{u_j}) \\ B(\boldsymbol{c},\boldsymbol{u_j}) & 2Q(\boldsymbol{c}) \end{pmatrix}.$$

For linearly independent vectors $\boldsymbol{c}$ and $\boldsymbol{u_j}$, the quadratic form Q has signature $(1,1)$ on $\operatorname{span}_{\mathbb{R}}\{\boldsymbol{c},\boldsymbol{u_j}\}$, i. e. the Gram matrix has negative determinant. For linearly dependent vectors $\boldsymbol{c}$ and $\boldsymbol{u_j}$, we obtain zero. As $Q(\boldsymbol{c}) < 0$, we thus see, using (3.6), that $\boldsymbol{Q}^+(U) \geq 0$ for all $U \in \mathbb{R}^{m\times n}$. Note that $\boldsymbol{Q}^+(U) = 0$ holds if and only if every column of U is a multiple of $\boldsymbol{c}$.

The negative semi-definite part $\boldsymbol{Q}^-$ only depends on U^c. This follows immediately when we use the identity $Q(\boldsymbol{u_j^c}) = \frac{B(\boldsymbol{c},\boldsymbol{u_j})^2}{4Q(\boldsymbol{c})}$:

$$\boldsymbol{Q}^-(U) = \frac{1}{4Q(\boldsymbol{c})}\sum_{j=1}^{n} B(\boldsymbol{c},\boldsymbol{u_j})^2 = \sum_{j=1}^{n} Q(\boldsymbol{u_j^c}) = \frac{1}{2}\operatorname{tr}\big((U^c)^{\mathrm{t}}AU^c\big) = \boldsymbol{Q}(U^c)$$

Then the positive semi-definite quadratic form $\boldsymbol{Q}^+$ only depends on $U^\perp$, i. e. the part of U that is perpendicular to $\boldsymbol{c}$, because $Q(\boldsymbol{u_j}) = Q(\boldsymbol{u_j^\perp}) + Q(\boldsymbol{u_j^c})$. Thus we have $\boldsymbol{Q} = \boldsymbol{Q}^+ + \boldsymbol{Q}^-$ with $\boldsymbol{Q}^+(U) = \boldsymbol{Q}(U^\perp)$ and $\boldsymbol{Q}^-(U) = \boldsymbol{Q}(U^c)$. □

Before we prove the remaining parts (ii) and (iii) of Theorem 3.11, we recall Zwegers' construction [Zwe02] for the case $n = 1$, as that makes clear how we choose the set-up for higher genus n. We merely give the function h that defines ϑ_h here:

Definition 3.19 ([Zwe02, Definition 2.1]). Let $\boldsymbol{c_0}, \boldsymbol{c_1} \in C_Q \subset \mathbb{R}^m$ and define

$$h(\boldsymbol{u}) = h^{\boldsymbol{c_0},\boldsymbol{c_1}}(\boldsymbol{u}) := E\Big(\frac{B(\boldsymbol{c_0},\boldsymbol{u})}{\sqrt{-Q(\boldsymbol{c_0})}}\Big) - E\Big(\frac{B(\boldsymbol{c_1},\boldsymbol{u})}{\sqrt{-Q(\boldsymbol{c_1})}}\Big).$$

Kudla [Kud13] and Livinsky [Liv16] showed that the corresponding theta series ϑ_h can be constructed as integrals of the theta forms introduced by Kudla and Millson [KM86, KM87]. We do something similar here, that is we show in the next lemma that h is obtained by integrating a certain 1-form over S_1. To this end, we have to choose an explicit parameterization of S_1, here we consider

$$S_1 = \{t\,\boldsymbol{c_0} + (1-t)\,\boldsymbol{c_1} \mid t \in [0,1]\}.$$

Lemma 3.20. *We can write h as*

$$h(\boldsymbol{u}) = 2\int_{S_1} \exp\Big(\pi\frac{B(\boldsymbol{c},\boldsymbol{u})^2}{Q(\boldsymbol{c})}\Big)\frac{B(\boldsymbol{u}^\perp, d\boldsymbol{c})}{\sqrt{-Q(\boldsymbol{c})}} \quad \textit{with } d\boldsymbol{c} = (dc_1,\dots,dc_m)^{\mathrm{t}}.$$

Proof. Note that
$$\frac{\partial}{\partial \boldsymbol{c}} B(\boldsymbol{c},\boldsymbol{u}) = A\boldsymbol{u} \quad\text{and}\quad \frac{\partial}{\partial \boldsymbol{c}} Q(\boldsymbol{c}) = A\boldsymbol{c}.$$
Since $\boldsymbol{u} = \boldsymbol{u}^\perp + \frac{B(\boldsymbol{c},\boldsymbol{u})}{2Q(\boldsymbol{c})}\boldsymbol{c}$, we obtain
$$\frac{\partial}{\partial \boldsymbol{c}}\Big(\frac{B(\boldsymbol{c},\boldsymbol{u})}{\sqrt{-Q(\boldsymbol{c})}}\Big) = \frac{A\boldsymbol{u}\sqrt{-Q(\boldsymbol{c})} + A\boldsymbol{c}B(\boldsymbol{c},\boldsymbol{u})\big(2\sqrt{-Q(\boldsymbol{c})}\big)^{-1}}{-Q(\boldsymbol{c})} = \frac{A\boldsymbol{u}^\perp}{\sqrt{-Q(\boldsymbol{c})}}.$$
Hence
$$\frac{\partial}{\partial \boldsymbol{c}} E\Big(\frac{B(\boldsymbol{c},\boldsymbol{u})}{\sqrt{-Q(\boldsymbol{c})}}\Big) = \frac{A\boldsymbol{u}^\perp}{\sqrt{-Q(\boldsymbol{c})}} E'\Big(\frac{B(\boldsymbol{c},\boldsymbol{u})}{\sqrt{-Q(\boldsymbol{c})}}\Big) = 2\,\frac{A\boldsymbol{u}^\perp}{\sqrt{-Q(\boldsymbol{c})}}\exp\Big(\pi\frac{B(\boldsymbol{c},\boldsymbol{u})^2}{Q(\boldsymbol{c})}\Big),$$
so the total differential of E with regard to $\boldsymbol{c}$ is the exact 1-form
$$dE\Big(\frac{B(\boldsymbol{c},\boldsymbol{u})}{\sqrt{-Q(\boldsymbol{c})}}\Big) = 2\exp\Big(\pi\frac{B(\boldsymbol{c},\boldsymbol{u})^2}{Q(\boldsymbol{c})}\Big)\frac{B(\boldsymbol{u}^\perp, d\boldsymbol{c})}{\sqrt{-Q(\boldsymbol{c})}}. \tag{3.7}$$
We integrate both sides of (3.7) over S_1 to finish the proof. □

We transfer this construction to Siegel theta series of generic genus $n \in \mathbb{N}$ by considering the integrand of the function h from Lemma 3.20 for each column vector $\boldsymbol{u_j}$ of U and taking the wedge product over all $j = 1,\dots,n$ to obtain an (exact) n-form that is integrated over the n-simplex S_n.

Note that this is an explicit realization of the theta forms θ_{KM} for arbitrary signature valued in closed differential forms that were constructed by Kudla and Millson [KM86, KM87, KM90]. The next proposition is a result that was also shown by Livinsky [Liv16] in his Ph. D. thesis (based on an unpublished manuscript by Kudla [Kud13]): he defines Θ^{Δ}_{KM} as the integral of the closed n-form θ_{KM} over the simplex Δ (which is S_n in our notation) and thus constructs a non-holomorphic Siegel modular form.

We make a similar construction but instead of using the connection to the theta forms θ_{KM}, we show that we obtain functions that we already know from [Roe21a], which also shows that the Siegel theta series that we obtain are modular.

Proposition 3.21. *The theta series ϑ_g transforms like a Siegel modular form of weight $m/2$.*

Proof. We recall that the integrand of g is
$$\big(-Q(\boldsymbol{c})\big)^{-n/2}\exp\big(2\pi\,\mathrm{tr}(U^{\mathrm{t}}A^{-}U)\big)\bigwedge_{j=1}^{n} B(\boldsymbol{u_j}^\perp, d\boldsymbol{c}).$$

Now we first consider the wedge product and write the bilinear forms as sums. Using the distributivity of the wedge product, we obtain
$$\bigwedge_{j=1}^{n}\Big(\sum_{k_j=1}^{m}(A\boldsymbol{u_j}^\perp)_{k_j} dc_{k_j}\Big) = \sum_{k_1,\dots,k_n=1}^{m}(A\boldsymbol{u_1}^\perp)_{k_1}\cdots(A\boldsymbol{u_n}^\perp)_{k_n} dc_{k_1}\wedge\ldots\wedge dc_{k_n}.$$

As we have $dc_{k_1}\wedge\ldots\wedge dc_{k_n} = 0$ if $k_i = k_j$ for any $i \neq j$ and $dc_{k_1}\wedge\ldots\wedge dc_{k_n} = \mathrm{sgn}(\sigma) dc_{k_{\sigma(1)}}\wedge\ldots\wedge dc_{k_{\sigma(n)}}$ for any permutation in the symmetric group $\sigma \in S_n$, this expression equals

$$\sum_{1\le k_1<k_2<\ldots<k_n\le m}\left(\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{j=1}^{n}(A\boldsymbol{u}_{\boldsymbol{j}}^{\perp})_{k_{\sigma(j)}}\right)dc_{k_1}\wedge\ldots\wedge dc_{k_n}.$$

We observe that the part in brackets is the determinant of the matrix

$$\big((A\boldsymbol{u}_{\boldsymbol{j}}^{\perp})_{k_i}\big)_{ij}=\begin{pmatrix}(A\boldsymbol{u}_{\boldsymbol{1}}^{\perp})_{k_1} & (A\boldsymbol{u}_{\boldsymbol{2}}^{\perp})_{k_1} & \cdots & (A\boldsymbol{u}_{\boldsymbol{n}}^{\perp})_{k_1}\\ (A\boldsymbol{u}_{\boldsymbol{1}}^{\perp})_{k_2} & (A\boldsymbol{u}_{\boldsymbol{2}}^{\perp})_{k_2} & \cdots & (A\boldsymbol{u}_{\boldsymbol{n}}^{\perp})_{k_2}\\ \vdots & \vdots & & \vdots\\ (A\boldsymbol{u}_{\boldsymbol{1}}^{\perp})_{k_n} & (A\boldsymbol{u}_{\boldsymbol{2}}^{\perp})_{k_n} & \cdots & (A\boldsymbol{u}_{\boldsymbol{n}}^{\perp})_{k_n}\end{pmatrix}\quad(1\le i\le n,\,1\le j\le n),\tag{3.8}$$

which is a square submatrix of maximal size of $AU^{\perp}\in\mathbb{R}^{m\times n}$. We use multi-index notation to rewrite this. Let

$$K:=\{\boldsymbol{k}=(k_1,\ldots,k_n)\in\mathbb{N}^n\mid 1\le k_1<k_2<\ldots<k_n\le m\}$$

and for $\mathcal{A}\in\mathbb{R}^{m\times n}$ let us denote by $\mathcal{A}_{\boldsymbol{k}}\in\mathbb{R}^{n\times n}$ the square submatrix that consists of the rows determined by $\boldsymbol{k}$. So (3.8) can be written as $(AU^{\perp})_{\boldsymbol{k}}$ for $\boldsymbol{k}\in K$. Obviously, $P_{\boldsymbol{k}}(U):=\det\big((AU^{\perp})_{\boldsymbol{k}}\big)$ has the homogeneity property $P_{\boldsymbol{k}}(UN)=\det N\cdot P_{\boldsymbol{k}}(U)$ for $N\in\mathbb{C}^{n\times n}$, with the degree of homogeneity being 1. Applying the Laplacian $\operatorname{tr}\boldsymbol{\Delta}_M$ means differentiating twice with regard to every row, so a function with the determinant-like structure of $P_{\boldsymbol{k}}$ and degree 1 will necessarily vanish under this operator, i. e. $(\operatorname{tr}\boldsymbol{\Delta}_M)P_{\boldsymbol{k}}=0$. So $P_{\boldsymbol{k}}$ is harmonic and we simply have $\exp\big(-\operatorname{tr}\boldsymbol{\Delta}_M/8\pi\big)\big(P_{\boldsymbol{k}}(U)\big)=P_{\boldsymbol{k}}(U)$. Further, we note that $P_{\boldsymbol{k}}$ only depends on a subspace of $\mathbb{R}^{m\times n}$ where the quadratic form $\boldsymbol{Q}$ is positive semi-definite.

By Lemma 3.18, we have

$$\exp\big(2\pi\operatorname{tr}(U^{\mathrm{t}}A^{-}U)\big)=\exp\big(4\pi\boldsymbol{Q}^{-}(U)\big)=\exp\big(4\pi\boldsymbol{Q}(U^{c})\big),$$

so the exponential factor solely depends on a subspace of $\mathbb{R}^{m\times n}$ on which $\boldsymbol{Q}$ is negative semi-definite. Thus the integrand of g has the form

$$\big(-Q(\boldsymbol{c})\big)^{-n/2}\exp\big(4\pi\boldsymbol{Q}(U^{c})\big)\sum_{\boldsymbol{k}\in K}\det\big((AU^{\perp})_{\boldsymbol{k}}\big)\,d\boldsymbol{c}_{\boldsymbol{k}}\quad\text{with } d\boldsymbol{c}_{\boldsymbol{k}}=dc_{k_1}\wedge\ldots\wedge dc_{k_n},\tag{3.9}$$

which is a function as described in Theorem 3.3. Using the notation of this theorem, we have a function where P_r is a harmonic polynomial of degree $\alpha=1$ and $P_s\equiv 1$ (and so has degree $\beta=0$). So the associated theta series transforms like a Siegel modular form of weight $m/2$.

In Remark 3.9, we observed that $S_n\subset C_Q$ holds and that S_n is compact in $\mathbb{R}^m$. Hence the points in S_n do not accumulate near the boundary of C_Q, i. e. where $Q(\boldsymbol{c})$ is almost zero. So the summands of the theta series associated with (3.9) are rapidly decaying functions and we can integrate termwise over S_n to obtain ϑ_g. The modular transformation properties are preserved as they are independent of the choice of C. Thus ϑ_g is well-defined and transforms as a modular Siegel theta series of weight $m/2$. □

In the definition of ϑ_g we consider $g(UY^{1/2})$ instead of $g(U)$, where Y denotes the imaginary part of Z. We are interested in the behavior of the theta series for large values of Y. For $n=1$, it is clear that we consider the imaginary part $y\in\mathbb{R}_{>0}$ as large, when $y \to \infty$. We recall that in Definition 3.19 the error function E was considered, where $E(xy^{1/2}) \to \operatorname{sgn}(x)$ for $y \to \infty$. For arbitrary genus $n \in \mathbb{N}$, we gave a generalizing definition of $Y \to \infty$ in Definition 3.10. We show in the next proposition that $g(UY^{1/2})$ asymptotes to the locally constant function $f(U)$ for $Y \to \infty$.

Proposition 3.22. *We have $g(UY^{1/2}) \to f(U)$ almost everywhere for $Y \to \infty$.*

Proof. Using identity (3.5) we can write the integrand of g as

$$\exp\Big(\frac{\pi}{Q(\boldsymbol{c})}\sum_{j=1}^{n} B(\boldsymbol{c},\boldsymbol{u_j})^2\Big)\bigwedge_{j=1}^{n}\frac{B(\boldsymbol{u_j}^{\perp},d\boldsymbol{c})}{\sqrt{-Q(\boldsymbol{c})}}.$$

We substitute

$$\boldsymbol{v} := \frac{U^{\mathrm{t}}A\boldsymbol{c}}{\sqrt{-Q(\boldsymbol{c})}} = \frac{1}{\sqrt{-Q(\boldsymbol{c})}}\begin{pmatrix} B(\boldsymbol{c},\boldsymbol{u_1}) \\ \vdots \\ B(\boldsymbol{c},\boldsymbol{u_n}) \end{pmatrix}.$$

In the proof of Lemma 3.20 we have shown that

$$\frac{\partial}{\partial \boldsymbol{c}}\Big(\frac{B(\boldsymbol{c},\boldsymbol{u})}{\sqrt{-Q(\boldsymbol{c})}}\Big) = \frac{A\boldsymbol{u}^{\perp}}{\sqrt{-Q(\boldsymbol{c})}},$$

so the differential of $v_j = \frac{B(\boldsymbol{c},\boldsymbol{u_j})}{\sqrt{-Q(\boldsymbol{c})}}$ with respect to $\boldsymbol{c}$ is easily seen to be $dv_j = \frac{B(\boldsymbol{u_j}^{\perp},d\boldsymbol{c})}{\sqrt{-Q(\boldsymbol{c})}}$. We can thus write g as

$$\int\limits_{X_n} \exp\big(-\pi(v_1^2+\ldots+v_n^2)\big)\,dv_1\wedge\ldots\wedge dv_n,$$

where (for $\boldsymbol{x_i} = U^{\mathrm{t}}A\boldsymbol{c_i}$)

$$X_n := \left\{ \frac{t_0\boldsymbol{x_0}+t_1\boldsymbol{x_1}+\ldots+t_n\boldsymbol{x_n}}{\sqrt{-Q(t_0\boldsymbol{c_0}+t_1\boldsymbol{c_1}+\ldots+t_n\boldsymbol{c_n})}} \;\middle|\; t_i \geq 0 \text{ for all } 0\leq i\leq n \quad\text{and}\quad \sum_{i=0}^{n} t_i = 1 \right\}.$$

We describe this integral depending on whether U is in $\mathcal{C}_A$ or not. If

$$\det\begin{pmatrix} 1 & 1 & \ldots & 1 \\ \boldsymbol{x_0} & \boldsymbol{x_1} & \ldots & \boldsymbol{x_n} \end{pmatrix} = \sum_{i=0}^{n}\widetilde{x}_i \tag{3.10}$$

vanishes, we have $U \notin \mathcal{C}_A$, as either all $\widetilde{x}_i$ vanish or there occur $\widetilde{x}_i$ and $\widetilde{x}_j$ that have different signs. Moreover, considering the left-hand side of (3.10), the vectors $\boldsymbol{x_0},\ldots,\boldsymbol{x_n}$ are linearly dependent, and so the vertices $\boldsymbol{x_i}/\sqrt{-Q(\boldsymbol{c_i})}$ $(0\leq i\leq n)$ form a simplex whose dimension is strictly lower than n. But when integrating an n-form over this simplex, the integral takes the value zero.

If (3.10) does not vanish, we have $\boldsymbol{0} \in X_n$ if and only if $U \in \mathcal{C}_A$, which is a direct consequence of Cramer's rule: the system of $n+1$ linear equations

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \boldsymbol{x_0} & \boldsymbol{x_1} & \dots & \boldsymbol{x_n} \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a unique solution $(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1}$ with $t_i \geq 0$ for all $0 \leq i \leq n$ if and only if

$$\widetilde{x}_i = (-1)^i \det\bigl(\boldsymbol{x_0} \dots \widehat{\boldsymbol{x_i}} \dots \boldsymbol{x_n}\bigr) \quad (0 \leq i \leq n)$$

are all non-negative or all non-positive. If $\widetilde{x}_i = 0$ for any $i \in \{0, \dots, n\}$, the zero vector is located on the boundary of X_n.

For $U \in \mathcal{C}_A$, we also determine the orientation of X_n in $\mathbb{R}^n$. Let the canonical basis $\{\boldsymbol{e_1}, \dots, \boldsymbol{e_n}\}$ of $\mathbb{R}^n$ represent the equivalence class of positive orientations in $\mathbb{R}^n$. We consider the unit simplex with the vertices $\boldsymbol{0}, \boldsymbol{e_1}, \dots, \boldsymbol{e_n}$ that we can also write as

$$T_n := \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n \;\middle|\; t_i \geq 0 \text{ for all } 1 \leq i \leq n \quad \text{and} \quad \sum_{i=1}^n t_i \leq 1 \right\}.$$

This simplex has a positive orientation in $\mathbb{R}^n$. When we consider a diffeomorphism between T_n and any other n-simplex in $\mathbb{R}^n$, we can thus determine whether this orientation is preserved (then this simplex also carries the positive orientation in $\mathbb{R}^n$) or reversed. Setting

$$X_n' = \left\{ \boldsymbol{x_0} + \sum_{i=1}^n t_i(\boldsymbol{x_i} - \boldsymbol{x_0}) \;\middle|\; t_i \geq 0 \text{ for all } 1 \leq i \leq n \quad \text{and} \quad \sum_{i=1}^n t_i \leq 1 \right\},$$

we consider the diffeomorphism

$$\varphi : T_n \longrightarrow X_n', \quad \boldsymbol{t} = (t_1, \dots, t_n) \mapsto \boldsymbol{x_0} + \sum_{i=1}^n t_i(\boldsymbol{x_i} - \boldsymbol{x_0}).$$

We denote the differential of φ by $D\varphi$ and observe that $\det\bigl(D\varphi(\boldsymbol{t})\bigr)$ is independent of $\boldsymbol{t}$ and – adding an extra column and row – equals

$$\det\bigl(\boldsymbol{x_1} - \boldsymbol{x_0} \quad \boldsymbol{x_2} - \boldsymbol{x_0} \quad \dots \quad \boldsymbol{x_n} - \boldsymbol{x_0}\bigr) = \det\begin{pmatrix} 1 & 0 & \dots & 0 \\ \boldsymbol{x_0} & \boldsymbol{x_1} - \boldsymbol{x_0} & \dots & \boldsymbol{x_n} - \boldsymbol{x_0} \end{pmatrix}.$$

When adding the first column to each of the other columns, we observe that we obtain the expression in (3.10), so the determinant of $D\varphi$ is strictly positive if $\widetilde{x}_i \geq 0$ for all $i \in \{0, \dots, n\}$ and strictly negative if $\widetilde{x}_i \leq 0$ (in both cases at least one of the inequalities is strict, as $U \in \mathcal{C}_A$). Up to a normalization, this is X_n, which we can write as

$$X_n = \left\{ \frac{\boldsymbol{x_0} + \sum_{i=1}^n t_i(\boldsymbol{x_i} - \boldsymbol{x_0})}{\sqrt{-Q\bigl(\boldsymbol{c_0} + \sum_{i=1}^n t_i(\boldsymbol{c_i} - \boldsymbol{c_0})\bigr)}} \;\middle|\; t_i \geq 0 \text{ for all } 1 \leq i \leq n \quad \text{and} \quad \sum_{i=1}^n t_i \leq 1 \right\}.$$

Since $S_n \subset C_Q$ as observed in Remark 3.9, we have $\sqrt{-Q\bigl(\boldsymbol{c_0} + \sum_{i=1}^n t_i(\boldsymbol{c_i} - \boldsymbol{c_0})\bigr)} > 0$, so the orientation of X_n and X_n' agree. We make a final substitution, which does not change the orientation: we substitute $U \mapsto UY^{1/2}$ for $Y = \operatorname{Im} Z$, which means that we set

$\breve{\boldsymbol{x}}_{\boldsymbol{i}} = Y^{1/2}\boldsymbol{x}_{\boldsymbol{i}}$ and integrate over

$$\breve{X}_n = \left\{ \frac{t_0\breve{\boldsymbol{x}}_{\boldsymbol{0}} + t_1\breve{\boldsymbol{x}}_{\boldsymbol{1}} + \ldots + t_n\breve{\boldsymbol{x}}_{\boldsymbol{n}}}{\sqrt{-Q(t_0\boldsymbol{c}_{\boldsymbol{0}} + t_1\boldsymbol{c}_{\boldsymbol{1}} + \ldots + t_n\boldsymbol{c}_{\boldsymbol{n}})}} \,\middle|\, t_i \geq 0 \text{ for all } 0 \leq i \leq n \quad \text{and} \quad \sum_{i=0}^{n} t_i = 1 \right\}$$

instead of X_n. As Y is positive definite and symmetric, so is its square root $Y^{1/2}$. Since $\det Y^{1/2} > 0$, the property that $\mathbf{0} \in \breve{X}_n$ if and only if $U \in \mathcal{C}_A$ is still maintained. While the integrand is still the same as before, the set $\breve{X}_n$ also depends on $Y^{1/2}$. Now we determine how $\breve{X}_n$ changes when we consider $Y \to \infty$, so first we observe which property the square root of Y necessarily satisfies for large Y.

We have defined $Y \to \infty$ as $Y_{jj} \to \infty$ for all $j \in \{1, \ldots, n\}$. By definition of $Y^{1/2}$ we have $Y = Y^{1/2} \cdot Y^{1/2}$, which means that the j-th diagonal entry is $Y_{jj} = \sum_{\nu=1}^{n} \big((Y^{1/2})_{j\nu}\big)^2$. So when $Y \to \infty$, in each row of $Y^{1/2}$ the absolute value of at least one entry tends to infinity. Thus, every entry of $\breve{\boldsymbol{x}}_{\boldsymbol{i}}$ tends to $\pm\infty$ and as the vertices of $\breve{X}_n$ expand outwards, the object we obtain depends on the location of $\mathbf{0}$ in relation to this simplex.

For any U such that $\mathbf{0} \notin \breve{X}_n$, the set is shifted away from $\mathbf{0}$. As the integrand decays fast for large values of $\boldsymbol{v} \in \mathbb{R}^n$, the value of the integral and thus the whole expression $g(UY^{1/2})$ tends to zero. By the definition of f, we also have $f(U) = 0$ for $U \notin \mathcal{C}_A$, so $g(UY^{1/2}) \to f(U)$ for $Y \to \infty$ here.

If $\mathbf{0}$ is an interior point of $\breve{X}_n$, the simplex asymptotes to $\mathbb{R}^n$. But we know that we have

$$\int_{\mathbb{R}^n} \exp\big(-\pi(v_1^2 + \ldots + v_n^2)\big)\, dv_1 \wedge \ldots \wedge dv_n = 1,$$

when the canonical basis $\{\boldsymbol{e}_{\boldsymbol{1}}, \ldots, \boldsymbol{e}_{\boldsymbol{n}}\}$ fixes a positive orientation in $\mathbb{R}^n$. Thus $g(UY^{1/2}) \to \pm 1$ for $Y \to \infty$, where the sign depends on the orientation of the original n-simplex X_n in $\mathbb{R}^n$ that we described above: the sign is positive if $\widetilde{x}_i \geq 0$ for all $1 \leq i \leq n$ and negative if $\widetilde{x}_i \leq 0$ for all $1 \leq i \leq n$. When $\widetilde{x}_i$ are strictly positive or negative for all $i \in \{0, \ldots, n\}$, this is exactly the definition of f for $U \in \mathcal{C}_A$. So $g(UY^{1/2}) \to f(U)$ for $Y \to \infty$ almost everywhere. The limit of $g(UY^{1/2})$ may differ from $f(U)$ when $\mathbf{0}$ is a boundary point, but the values of U for which this holds form a null set in $\mathbb{R}^{m \times n}$. □

In the following remark, we give a short description of $g(UY^{1/2})$ for $Y \to \infty$ for the case that $\mathbf{0}$ is a boundary point of $\breve{X}_n$.

Remark 3.23. Let $1 \leq n' \leq n$. When for exactly n' values $\widetilde{x}_i = 0$ holds, the zero vector is located on an $(n - n')$-face of $\breve{X}_n$. If $n' = 1$, this is a facet of $\breve{X}_n$, so the value of g approaches $\pm 1/2$. In this case, we actually see that this agrees with the value of $f(U)$, as $\widetilde{x}_i = 0$ holds for exactly one $i \in \{0, \ldots, n\}$.

For $n' \geq 2$, the area that we obtain by intersecting $\breve{X}_n$ with the $(n'-1)$-dimensional unit-sphere is called the solid angle Ω. Then g asymptotes to $\pm\Omega/A_{n'}$, where $A_{n'}$ is the surface area of the unit-sphere. Again, the sign reflects the orientation of X_n (resp. $\breve{X}_n$) in $\mathbb{R}^n$, as described in the previous proof.

Considering the solid angle that depends on the exact position of $\mathbf{0}$ in the simplex, one could determine an exact formula for the holomorphic part of ϑ_g. However, the resulting function will look extremely complicated (for $n = 2$ one could for example use the result in [Liv16]), so we used the holomorphic function f, which has a much simpler form.

In this section, we have thus shown the second and third part of the main theorem: By Proposition 3.21, the theta series in Theorem 3.11(ii) transforms like a Siegel modular form of genus n and weight $m/2$. Part (iii), giving us the connection between the holomorphic version and the modular version of the theta series, follows by Proposition 3.22.

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4 Theta series for quadratic forms of signature $(n-1,1)$ with (spherical) polynomials

This chapter is based on the manuscript [RZ21] of the same name, which is available under arXiv:2102.09329 and accepted for publication in the *International Journal of Number Theory*. This project is joint work with Prof. Dr. Sander Zwegers and my share of the work amounted to 50%.

Note that the notation differs from the previous chapters, as we deal with elliptic modular forms, so we fix the genus $n = 1$, and from now on n is the dimension of the matrix A, which defines the quadratic form. Also, we do not use bold print for vectors as it will be more obvious from the context when we consider vectors.

4.1 Introduction

One of the few known general constructions of holomorphic modular forms is via theta series: if $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a positive definite quadratic form, which is integer-valued on the lattice $\mathbb{Z}^n$, and $P : \mathbb{R}^n \longrightarrow \mathbb{C}$ a spherical polynomial of degree d, then the theta series

$$\Theta_{Q,P}(\tau) := \sum_{\ell \in \mathbb{Z}^n} P(\ell)\, q^{Q(\ell)} \qquad (q = e^{2\pi i \tau},\ \mathrm{Im}(\tau) > 0)$$

is a (holomorphic) modular form of weight $n/2 + d$ (on some subgroup of $\mathrm{SL}_2(\mathbb{Z})$, with some character; see [Sch39, Ogg69] for the case that n is even and [Shi73] for the case that n is odd). If for example we take $n = 4$, $Q = \|\cdot\|^2$ and $P \equiv 1$, then the theta function $\Theta_{Q,P}$ is modular of weight 2 on $\Gamma_0(4)$. Writing this theta series as a linear combination of Eisenstein series we obtain Jacobi's four-square theorem, which gives a formula for the number of ways that a given positive integer m can be represented as the sum of four squares:

$$|\{\ell \in \mathbb{Z}^4 \mid \ell_1^2 + \ell_2^2 + \ell_3^2 + \ell_4^2 = m\}| = 8 \sum_{d \mid m,\, 4 \nmid d} d$$

For indefinite quadratic forms the situation is more complicated, because the sum $\sum_{\ell \in \mathbb{Z}^n} P(\ell)\, q^{Q(\ell)}$ doesn't converge. However, there are several ways to remedy this and attach theta functions to indefinite quadratic forms. For example one can use majorants (see [Sie51]) to obtain non-holomorphic modular forms.

In [Vig75, Vig77] Vignéras gives a nice general construction for indefinite theta functions: if Q is a non-degenerate integer-valued quadratic form on $\mathbb{Z}^n$, and $p : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a function that satisfies certain growth conditions and the differential equation $Dp = dp$, where $d \in \mathbb{Z}$ and

$$D := \mathcal{E} - \frac{1}{4\pi}\Delta,$$

with $\mathcal{E} := \sum_{i=1}^n v_i \frac{\partial}{\partial v_i}$ the Euler operator and $\Delta = \Delta_Q$ the Laplace operator for Q, then the theta series

$$\Theta_{Q,p}(\tau) := y^{-d/2} \sum_{\ell\in\mathbb{Z}^n} p(\ell y^{1/2})\, q^{Q(\ell)} \qquad (y = \mathrm{Im}(\tau))$$

is a (non-holomorphic) modular form of weight $n/2 + d$. Note that both the construction for positive definite quadratic forms and Siegel's construction ([Sie51]) are a special case of Vignéras' result.

Another way to obtain indefinite theta functions is to restrict the sum over the full lattice to the sum over a cone. In [GZ98] Göttsche and Zagier construct such indefinite theta functions for the case that the signature of Q is $(n-1,1)$. The slightly modified definition of these functions is: let Q be an integer-valued quadratic form of signature $(n-1,1)$ on $\mathbb{Z}^n$, let B be the bilinear form associated to Q, let $c_1, c_2 \in \overline{C}_Q := C_Q \cup S_Q$ with C_Q one of the components of $\{c \in \mathbb{R}^n \mid Q(c) < 0\}$ and S_Q the set of cusps of C_Q, and let $a \in R(c_1) \cap R(c_2)$, $b \in \mathbb{R}^n$, where $R(c)$ is $\mathbb{R}^n$ if $c \in C_Q$ and $\{a \in \mathbb{R}^n \mid B(c,a) \notin \mathbb{Z}\}$ if $c \in S_Q$, then

$$\Theta_{a,b}^{c_1,c_2}(\tau) := \sum_{\ell\in a+\mathbb{Z}^n} \big\{\mathrm{sgn}(B(c_1,\ell)) - \mathrm{sgn}(B(c_2,\ell))\big\}\, q^{Q(\ell)} e^{2\pi i B(\ell,b)}.$$

These functions are holomorphic and their Fourier coefficients can easily be computed. In the case that $c_1, c_2 \in S_Q$ it is shown in [GZ98] that they are in fact modular. For $c_1, c_2 \in C_Q$ the function $\Theta_{a,b}^{c_1,c_2}$ is in general *not* modular, only for special choices of a, b and $c_1, c_2 \in C_Q$ (see [And84] and [Pol01] for examples). Note that for signature $(1,1)$ such modular $\Theta_{a,b}^{c_1,c_2}$ are related to the indefinite theta functions constructed by Hecke in [Hec25, Hec27].

In [Zwe02] the second author showed that we can remedy the non-modularity of $\Theta_{a,b}^{c_1,c_2}$ by considering a slightly modified version: for $c_1, c_2 \in C_Q$ define

$$\widehat{\Theta}_{a,b}^{c_1,c_2}(\tau) := \sum_{\ell\in a+\mathbb{Z}^n} \Big\{E\Big(\frac{B(c_1,\ell)}{\sqrt{-Q(c_1)}} y^{1/2}\Big) - E\Big(\frac{B(c_2,\ell)}{\sqrt{-Q(c_2)}} y^{1/2}\Big)\Big\}\, q^{Q(\ell)} e^{2\pi i B(\ell,b)},$$

where

$$E(z) := 2\int_0^z e^{-\pi u^2} du = \mathrm{sgn}(z) - \mathrm{sgn}(z)\int_{z^2}^\infty u^{-1/2} e^{-\pi u} du. \tag{4.1}$$

In [Zwe02] it is shown that $\widehat{\Theta}_{a,b}^{c_1,c_2}$ is modular of weight $n/2$ (alternatively, one could use the methods from [Vig75, Vig77] to simplify the proof), but in general it is *not* holomorphic. We can view $\widehat{\Theta}_{a,b}^{c_1,c_2}$ as the modular "completion" of $\Theta_{a,b}^{c_1,c_2}$. One application of these indefinite theta functions is that one can use them to study the modular behavior of Ramanujan's mock theta functions (see [Zwe02]). Further, in certain special cases one has $\Theta_{a,b}^{c_1,c_2} = \widehat{\Theta}_{a,b}^{c_1,c_2}$, which explains the modularity of $\Theta_{a,b}^{c_1,c_2}$ in these cases. Note that recently, analogous constructions have been found for quadratic forms of general signature: see [ABMP18a] and [Kud18] for signature $(n-2,2)$, and [WR16], [Naz18] and [FK19] (in chronological order) for the general case.

The aim of this paper is to generalize the results from [Zwe02] for quadratic forms of signature $(n-1,1)$ to include (spherical) polynomials. For this we actually construct three versions for the theta function attached to a homogeneous polynomial: a holomorphic, an almost holomorphic, and a modular version. By giving a criterion for the almost holomorphic and the modular version to agree, we obtain a construction of theta functions for quadratic forms of signature $(n-1,1)$ which are almost holomorphic modular forms. The restriction to spherical polynomials then yields holomorphic modular forms.

4.2 Definitions and statement of the main results

For the rest of the paper, we assume that the quadratic form Q has signature $(n-1,1)$ and is integer-valued on $\mathbb{Z}^n$. We let A denote the corresponding even symmetric matrix (so $Q(v)=\frac{1}{2}v^{\mathrm{t}}Av$) and let B be the bilinear form associated to Q: $B(u,v)=u^{\mathrm{t}}Av=Q(u+v)-Q(u)-Q(v)$. Since Q has signature $(n-1,1)$, the set of vectors $c\in\mathbb{R}^n$ with $Q(c)<0$ has two components. If $B(c_1,c_2)<0$, then c_1 and c_2 belong to the same component, while if $B(c_1,c_2)>0$ then c_1 and c_2 belong to opposite components. Let C_Q be one of those components. If c_0 is in that component, then C_Q is given by:

$$C_Q:=\{c\in\mathbb{R}^n \mid Q(c)<0,\ B(c,c_0)<0\}$$

We normalize the elements of C_Q such that $Q(c)=-1$ and set

$$\mathcal{C}_Q:=\{c\in\mathbb{R}^n \mid Q(c)=-1,\ B(c,c_0)<0\}.$$

Definition 4.1. Let $f:\mathbb{R}^n\longrightarrow\mathbb{C}$ be a homogeneous polynomial of degree d and let $c_1,c_2\in\mathcal{C}_Q$. We define the holomorphic theta series associated to Q and f by

$$\Theta^{c_1,c_2}[f](\tau):=\sum_{\ell\in\mathbb{Z}^n}\big\{\operatorname{sgn}(B(c_1,\ell))-\operatorname{sgn}(B(c_2,\ell))\big\}\,f(\ell)\,q^{Q(\ell)}.$$

For the corresponding almost holomorphic and non-holomorphic versions we also need:

Definition 4.2. Let $\Delta=\Delta_Q:=\left(\frac{\partial}{\partial v}\right)^{\mathrm{t}}A^{-1}\frac{\partial}{\partial v}$ denote the Laplacian associated to Q (we omit Q in the notation, as we take it to be fixed). We set

$$e^{-\Delta/8\pi}:=\sum_{k=0}^{\infty}\frac{(-1)^k}{(8\pi)^k k!}\Delta^k,\qquad \partial_c:=c^{\mathrm{t}}\frac{\partial}{\partial v}=\sum_{i=1}^{n}c_i\frac{\partial}{\partial v_i}$$

and for a homogeneous polynomial $f:\mathbb{R}^n\longrightarrow\mathbb{C}$ of degree d we define $\widehat{f}:=e^{-\Delta/8\pi}f$ and

$$p^c[f](v):=\sum_{k=0}^{d}\frac{(-1)^k}{(4\pi)^k k!}E^{(k)}(B(c,v))\cdot\partial_c^k\widehat{f}(v).$$

Definition 4.3. Let $f:\mathbb{R}^n\longrightarrow\mathbb{C}$ be a homogeneous polynomial of degree d and let $c_1,c_2\in\mathcal{C}_Q$. We define the almost holomorphic theta series associated to Q and f by

$$\widehat{\Theta}^{c_1,c_2}[f](\tau):=y^{-d/2}\sum_{\ell\in\mathbb{Z}^n}\big\{\operatorname{sgn}(B(c_1,\ell))-\operatorname{sgn}(B(c_2,\ell))\big\}\,\widehat{f}(\ell y^{1/2})\,q^{Q(\ell)}.$$

Further, we define the corresponding non-holomorphic theta series by

$$\widehat{\widehat{\Theta}}^{c_1,c_2}[f](\tau):=y^{-d/2}\sum_{\ell\in\mathbb{Z}^n}\big\{p^{c_1}[f](\ell y^{1/2})-p^{c_2}[f](\ell y^{1/2})\big\}\,q^{Q(\ell)}.$$

Remark 4.4. (a) According to Lemma 4.14, $v\mapsto e^{-2\pi Q(v)}(p^{c_1}[f](v)-p^{c_2}[f](v))$ is a Schwartz function. This ensures the absolute convergence of the sum defining $\widehat{\widehat{\Theta}}^{c_1,c_2}[f]$.

In the proof of Lemma 4.14 it is further shown that for any polynomial P we have

$$\big|P(v)\,e^{-2\pi Q(v)}\big(\operatorname{sgn}(B(c_1,v))-\operatorname{sgn}(B(c_2,v))\big)\big|\le 2\,\big|P(v)\big|\,e^{-2\pi Q^+(v)},$$

where Q^+ is a positive definite quadratic from. This directly gives us the absolute convergence of the sums in the definition of $\Theta^{c_1,c_2}[f]$ and of $\widehat{\Theta}^{c_1,c_2}[f]$.
(b) For a homogeneous polynomial f of degree d, $\Delta^k f$ is zero (in particular for $k>d/2$) or a homogeneous polynomial of degree $d-2k$. Therefore, the degrees of the monomials in $\widehat{f}$ have the same parity as the degree of f, and $y^{-d/2}\widehat{f}(\ell y^{1/2})$ is a polynomial of degree $\le d/2$ in $1/y$. Thus we can view $\widehat{\Theta}^{c_1,c_2}[f]$ as a polynomial of degree $\le d/2$ in $1/y$ with holomorphic coefficients. Such functions are called almost holomorphic of depth $\le d/2$, so $\widehat{\Theta}^{c_1,c_2}[f]$ denotes an almost holomorphic theta series of depth $\le d/2$. Further, $\Theta^{c_1,c_2}[f]$ is the "constant term" of $\widehat{\Theta}^{c_1,c_2}[f]$ (viewed as a polynomial in $1/y$).

We denote by N the level of A, thus the smallest $N\in\mathbb{N}$ such that NA^{-1} is an even matrix. To describe the modular transformation behavior of $\widehat{\widehat{\Theta}}^{c_1,c_2}[f]$ for a congruence subgroup of level N, we introduce the character χ which is defined as follows (see Theorem 2 in [Vig77]):

Definition 4.5. Let $\left(\frac{\cdot}{\cdot}\right)$ denote the Kronecker symbol. For $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(N)$ we set

$$\chi(\gamma):=\left(\frac{D}{d}\right)\cdot v(\gamma)\quad\text{with}\quad\begin{cases} D=(-1)^{n/2}\det A \text{ and } v(\gamma)=1 & \text{for } n \text{ even},\\ D=2\det A \text{ and } v(\gamma)=\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-n/2} & \text{for } n \text{ odd}.\end{cases}$$

Theorem 4.6. *The theta function $\widehat{\widehat{\Theta}}^{c_1,c_2}[f]$ transforms as a modular form of weight $n/2+d$ and character χ on $\Gamma_0(N)$: we have*

$$\widehat{\widehat{\Theta}}^{c_1,c_2}[f]\Big(\frac{a\tau+b}{c\tau+d}\Big)=\chi(\gamma)\,(c\tau+d)^{n/2+d}\,\widehat{\widehat{\Theta}}^{c_1,c_2}[f](\tau)$$

for all $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(N)$.

In order to construct almost holomorphic and holomorphic modular forms, we consider the automorphism group, which leaves the quadratic form Q, the lattice $\mathbb{Z}^n$ and the choice of the component $\mathcal{C}_Q$ unchanged:

Definition 4.7. Let

$$\operatorname{Aut}^+(Q,\mathbb{Z}^n):=\big\{g\in\operatorname{GL}_n(\mathbb{Z})\mid g^{\mathrm{t}}Ag=A,\ B(gc,c)<0\text{ for all }c\in\mathcal{C}_Q\big\}.$$

Theorem 4.8. *Let I be a finite set of indices. For all $i\in I$ let f_i be a homogeneous polynomial of degree d and let $g_i\in\operatorname{Aut}^+(Q,\mathbb{Z}^n)$. Under the assumption that*

$$\sum_{i\in I}(f_i-f_i\circ g_i)=0,$$

the theta function

$$\sum_{i\in I}\widehat{\widehat{\Theta}}^{c,g_ic}[f_i]=\sum_{i\in I}\widehat{\Theta}^{c,g_ic}[f_i]$$

is an almost holomorphic cusp form of weight $n/2+d$, depth $\le d/2$ and character χ on $\Gamma_0(N)$. Further, it doesn't depend on the choice of $c\in\mathcal{C}_Q$.

Remark 4.9. Since $\Theta^{c_1,c_2}[f]$ is the "constant term" of $\widehat{\Theta}^{c_1,c_2}[f]$, we get that the corresponding holomorphic theta function $\sum_{i\in I}\Theta^{c,g_ic}[f_i]$ is a quasimodular form (with the given weight, depth, character and subgroup). For more details on almost holomorphic modular forms and quasimodular forms see for example Section 5.3 in [Zag08].

Definition 4.10. We call a polynomial $f:\mathbb{R}^n\longrightarrow\mathbb{C}$ *spherical* (of degree d) if it is homogeneous (of degree d) and vanishes under the Laplacian, i. e. $\Delta f=0$.

Remark 4.11. If f is spherical of degree d, then we have $\widehat{f}=e^{-\Delta/8\pi}f=f$ and $y^{-d/2}\widehat{f}(\ell y^{1/2})=y^{-d/2}f(\ell y^{1/2})=f(\ell)$. Hence the holomorphic theta function $\Theta^{c_1,c_2}[f]$ and the almost holomorphic theta function $\widehat{\Theta}^{c_1,c_2}[f]$ agree. This observation immediately leads to the following corollary to Theorem 4.8.

Corollary 4.12. *Let I, f_i and g_i be as in Theorem 4.8, with the additional condition that f_i is spherical for all $i\in I$. Under the assumption that*

$$\sum_{i\in I}(f_i-f_i\circ g_i)=0,$$

the theta function

$$\sum_{i\in I}\widehat{\widehat{\Theta}}^{c,g_ic}[f_i]=\sum_{i\in I}\Theta^{c,g_ic}[f_i]$$

is a (holomorphic) cusp form of weight $n/2+d$ and character χ on $\Gamma_0(N)$. Further, it doesn't depend on the choice of $c\in\mathcal{C}_Q$.

Remark 4.13. Since E is odd, we get $p^c[f](-v)=(-1)^{d+1}p^c[f](v)$. Hence if the degree d of f is even, we trivially have that $\widehat{\widehat{\Theta}}^{c_1,c_2}[f]$ is identically zero. Similarly, $\widehat{\Theta}^{c_1,c_2}[f]$ and $\Theta^{c_1,c_2}[f]$ also vanish. In this case, non-trivial results can still be obtained by introducing characteristics $a,b\in\mathbb{Q}^n$ and setting

$$\widehat{\widehat{\Theta}}^{c_1,c_2}_{a,b}[f](\tau):=y^{-d/2}\sum_{\ell\in a+\mathbb{Z}^n}\left\{p^{c_1}[f](\ell y^{1/2})-p^{c_2}[f](\ell y^{1/2})\right\}q^{Q(\ell)}\,e^{2\pi iB(\ell,b)},$$

and similarly for the holomorphic and the almost holomorphic versions (as was done in [Zwe02], where the case $d=0$ is considered). Analogously, one can include periodic functions on $\mathbb{Z}^n$, that is, functions $m:\mathbb{Z}^n\longrightarrow\mathbb{C}$ such that there is an $L\in\mathbb{N}$ for which we have $m(\ell+\ell')=m(\ell)$ for all $\ell\in\mathbb{Z}^n$ and all $\ell'\in L\mathbb{Z}^n$, and consider

$$\begin{aligned}
\Theta^{c_1,c_2}[m,f](\tau)&:=\sum_{\ell\in\mathbb{Z}^n}\left\{\operatorname{sgn}(B(c_1,\ell))-\operatorname{sgn}(B(c_2,\ell))\right\}m(\ell)\,f(\ell)\,q^{Q(\ell)},\\
\widehat{\Theta}^{c_1,c_2}[m,f](\tau)&:=y^{-d/2}\sum_{\ell\in\mathbb{Z}^n}\left\{\operatorname{sgn}(B(c_1,\ell))-\operatorname{sgn}(B(c_2,\ell))\right\}m(\ell)\,\widehat{f}(\ell y^{1/2})\,q^{Q(\ell)},\\
\widehat{\widehat{\Theta}}^{c_1,c_2}[m,f](\tau)&:=y^{-d/2}\sum_{\ell\in\mathbb{Z}^n}\left\{p^{c_1}[f](\ell y^{1/2})-p^{c_2}[f](\ell y^{1/2})\right\}m(\ell)\,q^{Q(\ell)}.
\end{aligned}$$

With a slight generalization of Vignéras' result one can show that $\widehat{\widehat{\Theta}}^{c_1,c_2}[m,f]$ again transforms as a modular form of weight $n/2+d$, but the subgroup and the character now also depend on the choice of m. We omit the details. Theorem 4.8 and Corollary 4.12 then generalize to: for all $i\in I$ let m_i be a periodic function on $\mathbb{Z}^n$, let f_i be a homogeneous polynomial of degree d, let $\widetilde{f}_i=m_i\cdot f_i$ and let $g_i\in\operatorname{Aut}^+(Q,\mathbb{Z}^n)$. Under the assumption that $\sum_{i\in I}(\widetilde{f_i} - \widetilde{f_i} \circ g_i) = 0$, the theta function

$$\sum_{i\in I} \widehat{\widehat{\Theta}}^{c,g_i c}[m_i, f_i] = \sum_{i\in I} \widehat{\Theta}^{c,g_i c}[m_i, f_i]$$

is an almost holomorphic cusp form of weight $n/2 + d$ and depth $\leq d/2$. If we further assume that f_i is spherical for all $i \in I$, then

$$\sum_{i\in I} \widehat{\widehat{\Theta}}^{c,g_i c}[m_i, f_i] = \sum_{i\in I} \Theta^{c,g_i c}[m_i, f_i] = \sum_{i\in I} \Theta^{c,g_i c}[\widetilde{f_i}]$$

is a (holomorphic) cusp form of weight $n/2 + d$.

4.3 Proof of Theorems 4.6 and 4.8, and Corollary 4.12

To prove Theorem 4.6 we'll use the results from [Vig75, Vig77], for which we have to show that the function $p := p^{c_1}[f] - p^{c_2}[f]$ satisfies the necessary growth conditions (Lemma 4.14) and the differential equation $Dp = dp$ (Lemma 4.15).

Lemma 4.14. *For $c_1, c_2 \in \mathcal{C}_Q$ and f, $p^c[f]$ as in Definition 4.2*

$$v \mapsto e^{-2\pi Q(v)}\big(p^{c_1}[f](v) - p^{c_2}[f](v)\big)$$

is a Schwartz function.

Proof. If c_1 and c_2 are linearly dependent, the expression $p^{c_1}[f](v) - p^{c_2}[f](v)$ vanishes, thus we assume that they are linearly independent.

Since $p^c[f]$ is the finite sum of C^∞-functions, $v \mapsto e^{-2\pi Q(v)}(p^{c_1}[f](v) - p^{c_2}[f](v))$ is also a C^∞-function. To show that it is a Schwartz function we begin by splitting $p^c[f](v)$ as the sum of $E(B(c, v))\,\widehat{f}(v)$ and

$$\widetilde{p}^c[f](v) := \sum_{k=1}^{d} \frac{(-1)^k}{(4\pi)^k k!} E^{(k)}(B(c, v)) \cdot \partial_c^k \widehat{f}(v). \tag{4.2}$$

We have $E'(B(c, v)) = 2e^{-\pi B(c,v)^2}$ and with induction we can easily see that for all $k \in \mathbb{N}$ we can write $E^{(k)}(B(c, v))$ as a polynomial (in v) times $e^{-\pi B(c,v)^2}$. Hence we can write $e^{-2\pi Q(v)}\widetilde{p}^c[f](v)$ as a polynomial times $e^{-2\pi Q_c(v)}$, where $Q_c(v) := Q(v) + \frac{1}{2}B(c, v)^2$. Since Q_c is a positive definite quadratic form (Lemma 2.5 in [Zwe02]), $v \mapsto e^{-2\pi Q_c(v)}$ is a Schwartz function and hence so is $v \mapsto e^{-2\pi Q(v)}\widetilde{p}^c[f](v)$. What remains to be shown is that $v \mapsto e^{-2\pi Q(v)}[E(B(c_1, v)) - E(B(c_2, v))]\,\widehat{f}(v)$ is a Schwartz function. Since $\widehat{f}$ is a polynomial, we actually only need to show that $v \mapsto e^{-2\pi Q(v)}[E(B(c_1, v)) - E(B(c_2, v))]$ is one. By induction on the total number of derivatives we can easily see that any higher-order partial derivative of $e^{-2\pi Q(v)}[E(B(c_1, v)) - E(B(c_2, v))]$ is of the form

$$P_0(v)\,e^{-2\pi Q(v)}\big(E(B(c_1, v)) - E(B(c_2, v))\big) \\ + P_1(v)\,e^{-2\pi Q(v)}e^{-\pi B(c_1,v)^2} - P_2(v)\,e^{-2\pi Q(v)}e^{-\pi B(c_2,v)^2},$$

where P_0, P_1 and P_2 are polynomials. By the same argument as before, the expression $P_i(v)\,e^{-2\pi Q(v)}e^{-\pi B(c_i,v)^2}$ is a Schwartz function, so it suffices to show that for any polynomial P

$$\left|P(v)\,e^{-2\pi Q(v)}\big(E(B(c_1,v))-E(B(c_2,v))\big)\right| \tag{4.3}$$

is bounded on $\mathbb{R}^n$. For this we use more or less the same arguments as in [Zwe02]: rewriting E as in (4.1), an upper bound for the expression in (4.3) is given by the sum of the three expressions

$$\left|P(v)\,e^{-2\pi Q(v)}\,\mathrm{sgn}(B(c_i,v))\,\beta(B(c_i,v)^2)\right|, \qquad (i=1,2) \tag{4.4}$$

where $\beta(x) := \int_x^\infty u^{-1/2}e^{-\pi u}du$, and

$$\left|P(v)\,e^{-2\pi Q(v)}\big(\mathrm{sgn}(B(c_1,v))-\mathrm{sgn}(B(c_2,v))\big)\right|. \tag{4.5}$$

Considering (4.4), we use $0 \le \beta(x) \le e^{-\pi x}$ for $x \in \mathbb{R}_{\ge 0}$ to find

$$\left|P(v)\,e^{-2\pi Q(v)}\,\mathrm{sgn}(B(c_i,v))\,\beta(B(c_i,v)^2)\right| \le \left|P(v)\right|e^{-2\pi Q_{c_i}(v)}, \tag{4.6}$$

where the last expression is bounded on $\mathbb{R}^n$ (as before).

Obviously, (4.5) vanishes if $\mathrm{sgn}(B(c_1,v))\,\mathrm{sgn}(B(c_2,v)) > 0$ holds. For linearly independent vectors $c_1, c_2 \in \mathcal{C}_Q$ we can check (Lemma 2.6 in [Zwe02]) that $Q^+(v) := Q(v) + \frac{B(c_1,c_2)}{4-B(c_1,c_2)^2}B(c_1,v)\,B(c_2,v)$ is a positive definite quadratic form. As $c_1, c_2 \in \mathcal{C}_Q$, we have $B(c_1,c_2) < 0$ and c_1, c_2 span a subspace where Q is of signature $(1,1)$, thus the determinant of $\left(\begin{smallmatrix} 2Q(c_1) & B(c_1,c_2) \\ B(c_1,c_2) & 2Q(c_2)\end{smallmatrix}\right)$ is negative, i. e. $4 - B(c_1,c_2)^2 < 0$. Hence, for $\mathrm{sgn}(B(c_1,v))\,\mathrm{sgn}(B(c_2,v)) \le 0$ we have $Q(v) \ge Q^+(v)$ and so

$$\left|P(v)\,e^{-2\pi Q(v)}\big(\mathrm{sgn}(B(c_1,v))-\mathrm{sgn}(B(c_2,v))\big)\right| \le 2\left|P(v)\right|e^{-2\pi Q^+(v)},$$

where $2|P(v)|\,e^{-2\pi Q^+(v)}$ is bounded on $\mathbb{R}^n$.

Thus we have shown that the expression in (4.3) is bounded, which completes the proof. $\square$

Lemma 4.15. *For f, $\widehat{f}$ and $p^c[f]$ as in Definition 4.2, we have $D\widehat{f} = d\widehat{f}$ and $Dp^c[f] = dp^c[f]$.*

For the proof we need:

Lemma 4.16. *We have $De^{-\Delta/8\pi} = e^{-\Delta/8\pi}\mathcal{E}$. Further, the differential operators D and ∂_c^k ($k \in \mathbb{N}_0$) satisfy the commutator relation $[D, \partial_c^k] = -k\partial_c^k$.*

Proof. One can easily check that $[\mathcal{E}, \Delta] = -2\Delta$ and $[D, \partial_c] = -\partial_c$ hold, which by induction generalize directly to $[\mathcal{E}, \Delta^k] = -2k\Delta^k$ and $[D, \partial_c^k] = -k\partial_c^k$ for all $k \in \mathbb{N}_0$. Further,

$$\begin{aligned}[\mathcal{E}, e^{-\Delta/8\pi}] &= \sum_{k=0}^\infty \frac{(-1)^k}{(8\pi)^k k!}[\mathcal{E}, \Delta^k] \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{(8\pi)^k k!}(-2k\Delta^k) = \frac{1}{4\pi}\sum_{k=0}^\infty \frac{(-1)^k}{(8\pi)^k k!}\Delta^{k+1} = \frac{\Delta}{4\pi}e^{-\Delta/8\pi}\end{aligned}$$

and so $De^{-\Delta/8\pi} = e^{-\Delta/8\pi}\mathcal{E}$. $\square$

Proof of Lemma 4.15. Since f is homogeneous of degree d, it satisfies $\mathcal{E}f = df$ and so

Lemma 4.16 gives

$$D\widehat{f} = D\,e^{-\Delta/8\pi} f = e^{-\Delta/8\pi}\mathcal{E} f = d\,e^{-\Delta/8\pi} f = d\,\widehat{f}.$$

Further, we have

$$D\,\partial_c^k \widehat{f} = \partial_c^k D\widehat{f} - k\,\partial_c^k \widehat{f} = (d-k)\,\partial_c^k \widehat{f}$$

and a direct computation gives

$$\mathcal{E}\big(E(B(c,v))\big) = 2B(c,v)\,e^{-\pi B(c,v)^2} \qquad \text{and} \qquad \Delta\big(E(B(c,v))\big) = 8\pi B(c,v)\,e^{-\pi B(c,v)^2}.$$

Hence we find $D(E(B(c,v))) = 0$ and

$$D\,\partial_c^k\big(E(B(c,v))\big) = \partial_c^k D\big(E(B(c,v))\big) - k\,\partial_c^k\big(E(B(c,v))\big) = -k\,\partial_c^k\big(E(B(c,v))\big).$$

Since $\partial_c^k\big(E(B(c,v))\big) = (-2)^k E^{(k)}(B(c,v))$, this yields

$$D\big(E^{(k)}(B(c,v))\big) = -kE^{(k)}(B(c,v)).$$

For the product of two functions we have

$$D(f_1 \cdot f_2) = Df_1 \cdot f_2 + f_1 \cdot Df_2 - \frac{1}{2\pi}\sum_{i=1}^n \sum_{j=1}^n (A^{-1})_{ij} \frac{\partial f_1}{\partial v_i}\frac{\partial f_2}{\partial v_j}.$$

Setting $f_1(v) = E^{(k)}(B(c,v))$ and $f_2 = \partial_c^k \widehat{f}$ gives

$$\begin{aligned}\sum_{i=1}^n \sum_{j=1}^n (A^{-1})_{ij} \frac{\partial f_1}{\partial v_i}\frac{\partial f_2}{\partial v_j} &= \sum_{i=1}^n \sum_{j=1}^n (A^{-1})_{ij}(Ac)_i\, E^{(k+1)}(B(c,v))\,\frac{\partial}{\partial v_j}\,\partial_c^k \widehat{f}(v)\\ &= \sum_{j=1}^n E^{(k+1)}(B(c,v))\,c_j \frac{\partial}{\partial v_j}\,\partial_c^k \widehat{f}(v) = E^{(k+1)}(B(c,v)) \cdot \partial_c^{k+1}\widehat{f}(v)\end{aligned}$$

and so

$$D\big(E^{(k)}(B(c,v)) \cdot \partial_c^k \widehat{f}(v)\big) = (d-2k)\,E^{(k)}(B(c,v)) \cdot \partial_c^k \widehat{f}(v) - \frac{1}{2\pi}E^{(k+1)}(B(c,v)) \cdot \partial_c^{k+1}\widehat{f}(v).$$

Hence

$$\begin{aligned}Dp^c[f](v) &= \sum_{k=0}^d \frac{(-1)^k}{(4\pi)^k k!}\Big\{(d-2k)\,E^{(k)}(B(c,v)) \cdot \partial_c^k \widehat{f}(v) - \frac{1}{2\pi}E^{(k+1)}(B(c,v)) \cdot \partial_c^{k+1}\widehat{f}(v)\Big\}\\ &= \sum_{k=0}^d (d-2k)\frac{(-1)^k}{(4\pi)^k k!}\,E^{(k)}(B(c,v)) \cdot \partial_c^k \widehat{f}(v)\\ &\qquad + \frac{1}{2\pi}\sum_{k=1}^d \frac{(-1)^k}{(4\pi)^{k-1}(k-1)!}\,E^{(k)}(B(c,v)) \cdot \partial_c^k \widehat{f}(v)\\ &= d\sum_{k=0}^d \frac{(-1)^k}{(4\pi)^k k!}\,E^{(k)}(B(c,v)) \cdot \partial_c^k \widehat{f}(v) = dp^c[f](v),\end{aligned}$$

where we have used that $\partial_c^{d+1}\widehat{f}(v) = 0$. To prove this last identity we observe that if f is homogeneous of degree d, then $\partial_c f$ is homogeneous of degree $d-1$. Hence $\partial_c^{d+1} f = 0$ and so

$$\partial_c^{d+1}\widehat{f} = \partial_c^{d+1} e^{-\Delta/8\pi} f = e^{-\Delta/8\pi}\partial_c^{d+1} f = 0.$$

This finishes the proof. □

Proof of Theorem 4.6. We set $p := p^{c_1}[f] - p^{c_2}[f]$. According to Lemma 4.14, $v \mapsto e^{-2\pi Q(v)}p(v)$ is a Schwartz function. Further, it follows directly from Lemma 4.15 that p satisfies the differential equation $Dp = dp$. Hence we can apply Theorems 1 and 2 from [Vig77] to get the desired result. □

As is usual in the theory of theta functions, we additionally introduce the characteristic λ in the dual lattice $(\mathbb{Z}^n)^* = A^{-1}\mathbb{Z}^n$ to be able to study the modular transformation properties of $\widehat{\widehat{\Theta}}^{c_1,c_2}[f]$ in more detail.

Definition 4.17. Let $\lambda \in A^{-1}\mathbb{Z}^n$ and let f and c_1, c_2 be as in Definitions 4.1 and 4.3. We define

$$\begin{aligned}
\Theta_\lambda^{c_1,c_2}[f](\tau) &:= \sum_{\ell\in\lambda+\mathbb{Z}^n} \left\{\operatorname{sgn}(B(c_1,\ell)) - \operatorname{sgn}(B(c_2,\ell))\right\} f(\ell)\, q^{Q(\ell)},\\
\widehat{\Theta}_\lambda^{c_1,c_2}[f](\tau) &:= y^{-d/2} \sum_{\ell\in\lambda+\mathbb{Z}^n} \left\{\operatorname{sgn}(B(c_1,\ell)) - \operatorname{sgn}(B(c_2,\ell))\right\} \widehat{f}(\ell y^{1/2})\, q^{Q(\ell)},\\
\widehat{\widehat{\Theta}}_\lambda^{c_1,c_2}[f](\tau) &:= y^{-d/2} \sum_{\ell\in\lambda+\mathbb{Z}^n} \left\{p^{c_1}[f](\ell y^{1/2}) - p^{c_2}[f](\ell y^{1/2})\right\} q^{Q(\ell)}.
\end{aligned}$$

Remark 4.18. Since these definitions depend only on λ modulo $\mathbb{Z}^n$, we will consider λ to be in the finite set $A^{-1}\mathbb{Z}^n/\mathbb{Z}^n$.

The modular transformation properties for the generators T and S of $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$ we immediately get from (1) and (3) in [Vig77]. Note that there is a typo in (3), which we have corrected here ($e^{2\pi i B(\lambda,\mu)}$ is missing).

Lemma 4.19 (Vignéras [Vig77])**.** *The theta functions with characteristic satisfy*

$$\begin{aligned}
\widehat{\widehat{\Theta}}_\lambda^{c_1,c_2}[f](\tau+1) &= e^{2\pi i Q(\lambda)}\,\widehat{\widehat{\Theta}}_\lambda^{c_1,c_2}[f](\tau),\\
\widehat{\widehat{\Theta}}_\lambda^{c_1,c_2}[f](-1/\tau) &= (-i\tau)^{n/2+d}\,\frac{(-i)^{d+1}}{\sqrt{|\det A|}} \sum_{\mu\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} e^{2\pi i B(\lambda,\mu)}\,\widehat{\widehat{\Theta}}_\mu^{c_1,c_2}[f](\tau).
\end{aligned}$$

Using these we'll show:

Lemma 4.20. *For all $\gamma \in \Gamma_1$ we can write $\widehat{\widehat{\Theta}}^{c_1,c_2}[f]\big|_{n/2+d}\gamma$ as*

$$\sum_{\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi_\gamma(\lambda)\,\widehat{\widehat{\Theta}}_\lambda^{c_1,c_2}[f],$$

where $\varphi_\gamma : A^{-1}\mathbb{Z}^n/\mathbb{Z}^n \longrightarrow \mathbb{C}$ satisfies $\varphi_\gamma \circ g = \varphi_\gamma$ for all $g \in \mathrm{Aut}^+(Q,\mathbb{Z}^n)$.

Remark 4.21. (a) We can easily check that left-multiplication with $g \in \mathrm{Aut}^+(Q,\mathbb{Z}^n)$ is a bijection from $A^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ into itself. Hence for a function φ on $A^{-1}\mathbb{Z}^n/\mathbb{Z}^n$, $\varphi\circ g$ is also a function on $A^{-1}\mathbb{Z}^n/\mathbb{Z}^n$.
(b) In [Sch39] and [Shi73] explicit formulas are given for the coefficients $\varphi_\gamma(\lambda)$ for the case that Q is positive definite. Similar formulas hold for the indefinite case, but these are not

given explicitly in [Vig77]. Considering the pair $(A^{-1}\mathbb{Z}^n/\mathbb{Z}^n, Q)$ as a finite quadratic module, one could give an exact formula for the matrix coefficients of the Weil representation associated with $(A^{-1}\mathbb{Z}^n/\mathbb{Z}^n, Q)$ by applying Strömberg's result (Theorem 6.4 in [Str13]). This gives an explicit formula for $\varphi_\gamma(\lambda)$ for $d = 0$, which can then be modified for arbitrary d. However, for our purposes it suffices to know that the coefficients φ_γ satisfy $\varphi_\gamma \circ g = \varphi_\gamma$ for all $g \in \mathrm{Aut}^+(Q, \mathbb{Z}^n)$.

Proof of Lemma 4.20. We say that a function is of the right form if we can write it as

$$\sum_{\lambda \in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi(\lambda)\, \widehat{\widehat{\Theta}}{}^{c_1,c_2}_\lambda[f],$$

where $\varphi \circ g = \varphi$ holds for all $g \in \mathrm{Aut}^+(Q, \mathbb{Z}^n)$. We first observe that $\widehat{\widehat{\Theta}}{}^{c_1,c_2}[f]$ is of the right form: we have

$$\widehat{\widehat{\Theta}}{}^{c_1,c_2}[f] = \sum_{\lambda \in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi(\lambda)\, \widehat{\widehat{\Theta}}{}^{c_1,c_2}_\lambda[f],$$

with $\varphi(\lambda) = 1$ if $\lambda \equiv 0 \bmod \mathbb{Z}^n$ and 0 otherwise. Indeed we have $\varphi \circ g = \varphi$ for all $g \in \mathrm{Aut}^+(Q, \mathbb{Z}^n)$. Further, it follows from Lemma 4.19 that if h is of the right form, then so are $h\big|_{n/2+d}T$ and $h\big|_{n/2+d}S$:

$$\begin{aligned}
h\big|_{n/2+d}T &= \sum_{\lambda \in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi(\lambda)\, \widehat{\widehat{\Theta}}{}^{c_1,c_2}_\lambda[f]\big|_{n/2+d}T \\
&= \sum_{\lambda \in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi(\lambda)\, e^{2\pi i Q(\lambda)}\, \widehat{\widehat{\Theta}}{}^{c_1,c_2}_\lambda[f] = \sum_{\lambda \in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi_1(\lambda)\, \widehat{\widehat{\Theta}}{}^{c_1,c_2}_\lambda[f], \\
h\big|_{n/2+d}S &= \sum_{\lambda \in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi(\lambda)\, \widehat{\widehat{\Theta}}{}^{c_1,c_2}_\lambda[f]\big|_{n/2+d}S \\
&= \sum_{\lambda \in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi(\lambda) \frac{(-i)^{n/2+2d+1}}{\sqrt{|\det A|}} \sum_{\mu \in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} e^{2\pi i B(\lambda,\mu)}\, \widehat{\widehat{\Theta}}{}^{c_1,c_2}_\mu[f] \\
&= \sum_{\mu \in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi_2(\mu)\, \widehat{\widehat{\Theta}}{}^{c_1,c_2}_\mu[f],
\end{aligned}$$

with $\varphi_1(\lambda) := \varphi(\lambda)\, e^{2\pi i Q(\lambda)}$ and

$$\varphi_2(\mu) := \frac{(-i)^{n/2+2d+1}}{\sqrt{|\det A|}} \sum_{\lambda \in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} e^{2\pi i B(\lambda,\mu)}\, \varphi(\lambda).$$

Since we assume $\varphi \circ g = \varphi$ for all $g \in \mathrm{Aut}^+(Q, \mathbb{Z}^n)$, it follows directly that we also have $\varphi_i \circ g = \varphi_i$ $(i = 1, 2)$ for all $g \in \mathrm{Aut}^+(Q, \mathbb{Z}^n)$ (for $i = 2$ replace (λ, μ) by $(g\lambda, g\mu)$ and use part (a) of Remark 4.21).

Since the group Γ_1 is generated by T and S, and $\widehat{\widehat{\Theta}}{}^{c_1,c_2}[f]$ is of the right form, it now follows that $\widehat{\widehat{\Theta}}{}^{c_1,c_2}[f]\big|_{n/2+d}\gamma$ is of the right form for all $\gamma \in \Gamma_1$. □

Remark 4.22. From these computations we also see that the coefficients $\varphi_\gamma(\lambda)$ don't depend on the choice of c_1, c_2. Further, they do depend on the degree d of f, but not on f itself.

Proof of Theorem 4.8. The key to the proof is that under the assumption that $\sum_{i \in I}(f_i - f_i \circ g_i) = 0$ is satisfied, the modular theta series $\sum_{i \in I} \widehat{\widehat{\Theta}}{}^{c, g_i c}[f_i]$ and the almost holomorphic theta series $\sum_{i\in I}\widehat{\Theta}^{c,g_ic}[f_i]$ coincide. In fact, we will show that more generally we have

$$\sum_{i\in I,\,\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n}\varphi(\lambda)\,\widehat{\widehat{\Theta}}{}^{c,g_ic}_{\lambda}[f_i]=\sum_{i\in I,\,\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n}\varphi(\lambda)\,\widehat{\Theta}^{c,g_ic}_{\lambda}[f_i] \tag{4.7}$$

if $\varphi\circ g=\varphi$ holds for all $g\in\operatorname{Aut}^+(Q,\mathbb{Z}^n)$. As in the proof of Lemma 4.14, we split $p^c[f](v)$ as the sum of $E(B(c,v))\,\widehat{f}(v)$ and $\widetilde{p}^c[f](v)$. Further, we split E by using (4.1). Together this gives

$$p^{c_1}[f](v)-p^{c_2}[f](v)=\big\{\operatorname{sgn}(B(c_1,v))-\operatorname{sgn}(B(c_2,v))\big\}\,\widehat{f}(v)+\breve{p}^{c_1}[f](v)-\breve{p}^{c_2}[f](v), \tag{4.8}$$

where

$$\breve{p}^{c}[f](v)=\widetilde{p}^{c}[f](v)-\operatorname{sgn}(B(c,v))\,\beta(B(c,v)^2)\,\widehat{f}(v).$$

In the proof of Lemma 4.14 we have seen that $v\mapsto e^{-2\pi Q(v)}\widetilde{p}^c[f](v)$ is a Schwartz function and from (4.6) we get

$$\big|e^{-2\pi Q(v)}\operatorname{sgn}(B(c,v))\,\beta(B(c,v)^2)\,\widehat{f}(v)\big|\le\big|\widehat{f}(v)\big|\,e^{-2\pi Q_c(v)},$$

where Q_c is a positive definite quadratic form. Hence for $c\in\mathcal{C}_Q$

$$\vartheta^c_\lambda[f](\tau):=y^{-d/2}\sum_{\ell\in\lambda+\mathbb{Z}^n}\breve{p}^c[f](\ell y^{1/2})\,q^{Q(\ell)}$$

converges absolutely. With (4.8) we thus obtain

$$\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{\lambda}[f]=\widehat{\Theta}^{c_1,c_2}_{\lambda}[f]+\vartheta^{c_1}_\lambda[f]-\vartheta^{c_2}_\lambda[f]. \tag{4.9}$$

Now let $g\in\operatorname{Aut}^+(Q,\mathbb{Z}^n)$. We can easily check $\Delta(f\circ g)=(\Delta f)\circ g$, which gives $\widehat{f}\circ g=\widehat{f\circ g}$, and $(\partial^k_{gc}\widehat{f})\circ g=\partial^k_c(\widehat{f}\circ g)$. Using these in the definition of $\widetilde{p}^c[f]$ (equation (4.2)) and of $\breve{p}^c[f]$ we find $\widetilde{p}^{gc}[f](gv)=\widetilde{p}^c[f\circ g](v)$ and $\breve{p}^{gc}[f](gv)=\breve{p}^c[f\circ g](v)$. Replacing (c,λ,ℓ) by $(gc,g\lambda,g\ell)$ in the definition of $\vartheta^c_\lambda[f]$ then gives

$$\vartheta^{gc}_{g\lambda}[f]=\vartheta^c_\lambda[f\circ g]. \tag{4.10}$$

Using (4.9) we find

$$\begin{aligned}\sum_{i\in I,\,\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n}\varphi(\lambda)\,\widehat{\widehat{\Theta}}{}^{c,g_ic}_{\lambda}[f_i]&=\sum_{i\in I,\,\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n}\varphi(\lambda)\,\widehat{\Theta}^{c,g_ic}_{\lambda}[f_i]\\&\quad+\sum_{i\in I,\,\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n}\varphi(\lambda)\,\vartheta^c_\lambda[f_i]-\sum_{i\in I,\,\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n}\varphi(\lambda)\,\vartheta^{g_ic}_\lambda[f_i].\end{aligned}$$

Replacing λ by $g_i\lambda$, assuming $\varphi\circ g=\varphi$ for all $g\in\operatorname{Aut}^+(Q,\mathbb{Z}^n)$, and using (4.10) we get

$$\sum_{\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n}\varphi(\lambda)\,\vartheta^{g_ic}_\lambda[f_i]=\sum_{\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n}\varphi(g_i\lambda)\,\vartheta^{g_ic}_{g_i\lambda}[f_i]=\sum_{\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n}\varphi(\lambda)\,\vartheta^c_\lambda[f_i\circ g_i]$$

and so we obtain

$$\sum_{i\in I,\,\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi(\lambda)\,\vartheta^c_\lambda[f_i] - \sum_{i\in I,\,\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi(\lambda)\,\vartheta^{g_i c}_\lambda[f_i]$$
$$= \sum_{\lambda\in A^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \varphi(\lambda)\,\vartheta^c_\lambda\Big[\sum_{i\in I}(f_i - f_i\circ g_i)\Big] = 0,$$

which proves (4.7).

We have already seen in the proof of Lemma 4.20 that we can choose φ as $\varphi(\lambda) = 1$ if $\lambda \equiv 0 \bmod \mathbb{Z}^n$ and 0 otherwise, which gives that the modular theta series $\sum_{i\in I} \widehat{\widehat{\Theta}}{}^{c,g_i c}[f_i]$ and the almost holomorphic theta series $\sum_{i\in I} \widehat{\Theta}^{c,g_i c}[f_i]$ agree, thus we obtain an almost holomorphic theta series of depth $\leq d/2$ with the desired transformation behavior.

In the Fourier expansion of $\widehat{\Theta}^{c_1,c_2}_\lambda[f]$ only positive powers of q occur, as the quadratic form Q is bounded from below by a positive definite quadratic form Q^+ on the support of $\big\{\mathrm{sgn}(B(c_1,\ell)) - \mathrm{sgn}(B(c_2,\ell))\big\}\,\widehat{f}(\ell y^{1/2})$ (as shown in the proof of Lemma 4.14). Using (4.7), Lemma 4.20, Remark 4.22, and again (4.7) we can write for any $\gamma \in \Gamma_1$ the function $\sum_{i\in I} \widehat{\Theta}^{c,g_i c}[f_i]\big|_{n/2+d}\gamma$ as a linear combination of $\widehat{\Theta}^{c,g_i c}_\lambda[f_i]$ and thus we have a Fourier expansion with positive powers of q in any cusp (where the Fourier coefficients are polynomials of degree $\leq d/2$ in $1/y$). This shows that $\sum_{i\in I} \widehat{\widehat{\Theta}}{}^{c,g_i c}[f_i] = \sum_{i\in I} \widehat{\Theta}^{c,g_i c}[f_i]$ is an almost holomorphic cusp form of depth $d/2$ with the given modular transformation properties.

Exactly as in the proof of $\widetilde{p}^{gc}[f](gv) = \widetilde{p}^c[f\circ g](v)$ we also have $p^{gc}[f](gv) = p^c[f\circ g](v)$. Replacing (c_1, c_2, ℓ) by $(gc_1, gc_2, g\ell)$ in the definition of $\widehat{\widehat{\Theta}}{}^{c_1,c_2}[f]$ hence gives

$$\widehat{\widehat{\Theta}}{}^{gc_1,gc_2}[f] = \widehat{\widehat{\Theta}}{}^{c_1,c_2}[f\circ g]$$

for all $g \in \mathrm{Aut}^+(Q,\mathbb{Z}^n)$, and so for $c, c' \in \mathcal{C}_Q$ we have

$$\widehat{\widehat{\Theta}}{}^{c,g_i c}[f_i] - \widehat{\widehat{\Theta}}{}^{c',g_i c'}[f_i] = \widehat{\widehat{\Theta}}{}^{c,c'}[f_i] - \widehat{\widehat{\Theta}}{}^{g_i c,g_i c'}[f_i] = \widehat{\widehat{\Theta}}{}^{c,c'}[f_i] - \widehat{\widehat{\Theta}}{}^{c,c'}[f_i\circ g_i] = \widehat{\widehat{\Theta}}{}^{c,c'}[f_i - f_i\circ g_i],$$

where in the first step we used the trivial identity

$$\big(p^c[f_i] - p^{g_i c}[f_i]\big) - \big(p^{c'}[f_i] - p^{g_i c'}[f_i]\big) = \big(p^c[f_i] - p^{c'}[f_i]\big) - \big(p^{g_i c}[f_i] - p^{g_i c'}[f_i]\big).$$

Summing over all $i \in I$ then gives

$$\sum_{i\in I} \widehat{\widehat{\Theta}}{}^{c,g_i c}[f_i] = \sum_{i\in I} \widehat{\widehat{\Theta}}{}^{c',g_i c'}[f_i],$$

which proves the last part of the theorem. □

Proof of Corollary 4.12. The result follows immediately from Theorem 4.8 and Remark 4.11. □

4.4 Explicit examples

Finally, we give explicit examples which can be constructed by using the main results of this work. We already obtain a vast variety of nice examples for low dimensions. Considering some quadratic forms more thoroughly, we can also make statements about the number of different modular forms we might get (see Example 4.25) and give a general construction for specific quadratic forms of level $4N$ (see Examples 4.27 and 4.28). If possible, we identify the theta series as eta quotients and define as usual $\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ and $\eta_M(\tau) := \eta(M\tau)$. Note that we don't actually prove these identities, we only verified them by computing the first 1000 coefficients in the Fourier series. For the proof one would have to use results on eta quotients (as for example in [GH93]) to determine the exact modular transformation behavior, as well as determine the corresponding Sturm bound (see [Stu87] for modular forms of integral weight and [KP14] for modular forms of half-integral weight).

Studying binary quadratic forms of signature $(1,1)$, we do not seem to obtain any interesting examples when we include spherical polynomials of degree $d > 0$. For $d = 0$ though, we can introduce for $i \in I$ periodic functions m_i and choose $g_i \in \mathrm{Aut}^+(Q,\mathbb{Z}^2)$ and polynomials f_i such that the functions $\widetilde{f_i} = m_i \cdot f_i$ satisfy the assumption $\sum_{i\in I}(\widetilde{f_i} - \widetilde{f_i}\circ g_i) = 0$ in Remark 4.13. We begin this section by giving some very simple examples for this case choosing $I = \{1\}$ and $f_1 \equiv 1$.

Example 4.23. Let $Q(v) = v_1^2 + 5v_1v_2 + v_2^2$. We take $g = \left(\begin{smallmatrix} 5 & 1 \\ -1 & 0\end{smallmatrix}\right) \in \mathrm{Aut}^+(Q,\mathbb{Z}^2)$ and $c = \frac{1}{\sqrt{21}}\left(\begin{smallmatrix} -2 \\ 5\end{smallmatrix}\right) \in \mathcal{C}_Q$. As a periodic function $m : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ with $m \circ g = m$ we choose

$$m(v) = \left(\frac{-3}{v_1 + v_2}\right),$$

where $\left(\frac{-3}{\cdot}\right)$ is odd and has period 3. In this way we find the modular theta series

$$\sum_{\ell\in\mathbb{Z}^2}\{\mathrm{sgn}(\ell_1) + \mathrm{sgn}(\ell_2)\}\left(\frac{-3}{\ell_1+\ell_2}\right) q^{\ell_1^2 + 5\ell_1\ell_2 + \ell_2^2},$$

which we identify as $4\eta_3\eta_{21}$. Note that this is also an example given in [Pol01], where a similar construction yields a big number of modular forms associated to indefinite binary quadratic forms. We can also transfer our construction to other quadratic forms, for instance $Q(v) = v_1^2 + 6v_1v_2 + v_2^2$, so that we obtain the theta series

$$\sum_{\ell\in\mathbb{Z}^2}\{\mathrm{sgn}(\ell_1) + \mathrm{sgn}(\ell_2)\}\left(\frac{-4}{\ell_1+\ell_2}\right) q^{\ell_1^2 + 6\ell_1\ell_2 + \ell_2^2},$$

which equals $4\eta_8\eta_{16}$.

So for signature $(1,1)$ we obtain examples (some of which are already known) by employing periodic functions and the homogeneous polynomial $f \equiv 1$. However, we will now have a look at quadratic forms of signature $(2,1)$, and here we will take (spherical) polynomials of higher degree to obtain a new interesting set of examples. We'll first give two examples, where we can immediately apply Theorem 4.8 and Corollary 4.12. In the first of those we will basically determine all pairs of homogeneous polynomials satisfying the condition of Theorem 4.8, hence providing many cases where we obtain almost holomorphic cusp forms of weight $3/2 + d$. To make this more precise we consider the following lemma.

Lemma 4.24. *Let $Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a quadratic form of signature $(1,1)$ and let $g \in \mathrm{Aut}(Q,\mathbb{R}^2)$ with $\det g = 1$ and $g \neq \pm I$. Let $d \in \mathbb{N}$ be odd and let $U_d \subset \mathbb{R}[x_1,x_2]$ be the vector space of homogeneous polynomials of degree d. Further, let Ψ_g be the endomorphism of U_d given by $\Psi_g(f) := f - f\circ g$. Then Ψ_g is an automorphism of U_d.*

Proof. Over $\mathbb{R}$ we can split Q as the product of two linear factors: $Q = h_1 \cdot h_2$, where this decomposition is unique up to the order of the factors and multiplication by a scalar.

Since $Q \circ g = Q$, we thus have

$$\begin{cases} h_1 \circ g &= \lambda h_2, \\ h_2 \circ g &= \lambda^{-1} h_1, \end{cases} \qquad \text{or} \qquad \begin{cases} h_1 \circ g &= \lambda h_1, \\ h_2 \circ g &= \lambda^{-1} h_2, \end{cases}$$

with $\lambda \in \mathbb{R}^*$. The first situation cannot occur, since then the matrix of the linear map $U_1 \longrightarrow U_1$, $h \mapsto h \circ g$ with respect to the basis $\{h_1, h_2\}$ of U_1 would be $\left(\begin{smallmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{smallmatrix}\right)$, which has determinant -1. However, with respect to the canonical basis the matrix is g^{t}, which has determinant 1. Further, in the second situation we have $\lambda \neq \pm 1$, since $\lambda = \pm 1$ would imply $g = \pm I$, which we have excluded. So we have

$$h_1 \circ g = \lambda h_1 \qquad \text{and} \qquad h_2 \circ g = \lambda^{-1} h_2,$$

with $\lambda \in \mathbb{R}^*$ and $\lambda \neq \pm 1$. With a suitable change of variables we can write any $f \in U_d$ as a homogeneous polynomial of degree d in h_1 and h_2, that is as a linear combination of the monomials $h_1^{\alpha_1} h_2^{\alpha_2}$ with $\alpha_1 + \alpha_2 = d$. For such monomials we then have

$$\Psi_g(h_1^{\alpha_1} h_2^{\alpha_2}) = (1 - \lambda^{\alpha_1 - \alpha_2}) h_1^{\alpha_1} h_2^{\alpha_2},$$

where $1 - \lambda^{\alpha_1 - \alpha_2} \neq 0$, since $\lambda \neq \pm 1$ and $\alpha_1 \neq \alpha_2$ ($\alpha_1 + \alpha_2 = d$ is odd). For the inverse we have

$$\Psi_g^{-1}(h_1^{\alpha_1} h_2^{\alpha_2}) = \frac{1}{1 - \lambda^{\alpha_1 - \alpha_2}} h_1^{\alpha_1} h_2^{\alpha_2},$$

so it follows directly that Ψ_g is an isomorphism. □

Example 4.25. Let $Q(v) = v_1^2 + 4v_2^2 - 2v_3^2$ and fix $c = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 0 \\ 0 \\ -1 \end{smallmatrix}\right) \in \mathcal{C}_Q$. Further, we pick the matrices

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 4 & 3 \end{pmatrix} \qquad \text{and} \qquad g_2 = \begin{pmatrix} 3 & 0 & 4 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

from the automorphism group $\mathrm{Aut}^+(Q, \mathbb{Z}^3)$. For arbitrary odd degree d we construct homogeneous polynomials f_1 and f_2 such that the condition $\sum_{i=1}^2 (f_i - f_i \circ g_i) = 0$ from Theorem 4.8 is fulfilled. We start by noting that we can directly eliminate certain polynomials which will necessarily give vanishing theta series: if f_1 is odd in the first variable v_1, then we have

$$\{\mathrm{sgn}(v_3) - \mathrm{sgn}(-4v_2 + 3v_3)\} f_1(v) = -\{\mathrm{sgn}(v_3) - \mathrm{sgn}(-4v_2 + 3v_3)\} f_1(v)$$

under the substitution $v_1 \to -v_1$. Therefore, we assume that f_1 is even in v_1 and hence so is $f_1 - f_1 \circ g_1$. Similarly, we assume that f_2 and $f_2 - f_2 \circ g_2$ are even in v_2. Thus we are looking for polynomials f_1 and f_2 for which $f_1 - f_1 \circ g_1 = -(f_2 - f_2 \circ g_2)$ is in the vector space V_d of homogeneous polynomials of degree d that are even in both v_1 and v_2 (and odd in v_3). We note that $\dim V_d = (d+1)(d+3)/8$.

The idea now is that we can start with any polynomial $f \in V_d$ and use Lemma 4.24 to construct a unique polynomial f_1, which is even in v_1 and satisfies $f_1 - f_1 \circ g_1 = f$: we write $f(v)$ as $\sum_k v_1^{2k} p_k(v_2, v_3)$, where p_k is a homogeneous polynomial of degree $d - 2k$. Then $\sum_k v_1^{2k} (\Psi_{\widetilde{g}_1}^{-1}(p_k))(v_2, v_3)$, where $\widetilde{g}_1 = \left(\begin{smallmatrix} 3 & 2 \\ 4 & 3 \end{smallmatrix}\right)$, satisfies $f_1 - f_1 \circ g_1 = f$. Similarly, there exists a unique polynomial f_2, which is even in v_2 and satisfies $f_2 - f_2 \circ g_2 = -f$. This way we obtain a vector space of dimension $(d+1)(d+3)/8$ of solutions (f_1, f_2) satisfying the condition $\sum_{i=1}^2 (f_i - f_i \circ g_i) = 0$ from Theorem 4.8.

We now go a step further and assume that f is spherical and consider the corresponding polynomials f_1 and f_2. Since f_1 is even in v_1, so is Δf_1. Further, we have $\Delta(f_1 \circ g_1) = (\Delta f_1) \circ g_1$ and so we get

$$(\Delta f_1) - (\Delta f_1) \circ g_1 = \Delta(f_1 - f_1 \circ g_1) = \Delta f = 0,$$

which by Lemma 4.24 and our previous construction yields $\Delta f_1 = 0$. Similarly, we also have $\Delta f_2 = 0$. Hence we have shown that if f is spherical, then so are f_1 and f_2. Further, the map $\Delta|_{V_d} : V_d \longrightarrow V_{d-2}$ is surjective, which one can easily check by using that $\Delta(v_1^{\alpha_1} v_2^{\alpha_2} v_3^{\alpha_3})$ is a linear combination of $v_1^{\alpha_1-2} v_2^{\alpha_2} v_3^{\alpha_3}$, $v_1^{\alpha_1} v_2^{\alpha_2-2} v_3^{\alpha_3}$ and $v_1^{\alpha_1} v_2^{\alpha_2} v_3^{\alpha_3-2}$, together with induction on $\alpha_1 + \alpha_2$. Therefore, the kernel of $\Delta|_{V_d}$ has dimension $\dim V_d - \dim V_{d-2} = (d+1)/2$, and so we obtain a vector space of dimension $(d+1)/2$ of spherical solutions (f_1, f_2) satisfying the condition $\sum_{i=1}^2 (f_i - f_i \circ g_i) = 0$ from Corollary 4.12.

For $d = 1$ the vector space V_1 is one-dimensional and is spanned by $f(v) = 2v_3$. The corresponding polynomials f_1 and f_2 are easily determined to be $f_1(v) = -2v_2 + v_3$ and $f_2(v) = v_1 - v_3$. Since they are spherical, we obtain the following holomorphic cusp form of weight $5/2$ on $\Gamma_0(16)$:

$$\begin{aligned}
\sum_{i=1}^{2} \widehat{\widehat{\Theta}}^{c,g_i c}[f_i](\tau) &= \sum_{i=1}^{2} \Theta^{c,g_i c}[f_i](\tau) \\
&= \sum_{\ell \in \mathbb{Z}^3} \{\operatorname{sgn}\bigl(B(c,\ell)\bigr) - \operatorname{sgn}\bigl(B(g_1 c,\ell)\bigr)\} f_1(\ell)\, q^{Q(\ell)} \\
&\qquad + \sum_{\ell \in \mathbb{Z}^3} \{\operatorname{sgn}\bigl(B(c,\ell)\bigr) - \operatorname{sgn}\bigl(B(g_2 c,\ell)\bigr)\} f_2(\ell)\, q^{Q(\ell)} \\
&= \sum_{\ell \in \mathbb{Z}^3} \bigl\{ \bigl(\operatorname{sgn}(\ell_3) - \operatorname{sgn}(-4\ell_2 + 3\ell_3)\bigr)(-2\ell_2 + \ell_3) \\
&\qquad + \bigl(\operatorname{sgn}(\ell_3) - \operatorname{sgn}(-2\ell_1 + 3\ell_3)\bigr)(\ell_1 - \ell_3) \bigr\}\, q^{Q(\ell)}
\end{aligned}$$

We identify this theta function as the eta product $4\eta_2^2 \eta_4 \eta_8^2$.

For $d = 3$ the vector space V_3 has dimension three. In Table 4.1 we list a possible basis of polynomials f, together with Δf and the corresponding polynomials f_1 and f_2.

f	Δf	f_1	f_2
$v_1^2 v_3$	v_3	$\frac{1}{2} v_1^2(-2v_2 + v_3)$	$\frac{1}{14} v_1(3v_1 - 4v_3)(v_1 - v_3)$
$v_2^2 v_3$	$\frac{1}{4} v_3$	$\frac{1}{14} v_2(3v_2 - 2v_3)(-2v_2 + v_3)$	$\frac{1}{2} v_2^2(v_1 - v_3)$
v_3^3	$-\frac{3}{2} v_3$	$-\frac{1}{14}(8v_2^2 + 4v_2 v_3 - 7v_3^2)(-2v_2 + v_3)$	$-\frac{1}{14}(2v_1^2 + 2v_1 v_3 - 7v_3^2)(v_1 - v_3)$

Table 4.1: Basis elements f of V_3 and the corresponding polynomials f_1 and f_2

In all three cases the corresponding theta function $\sum_{i=1}^2 \widehat{\widehat{\Theta}}^{c,g_i c}[f_i] = \sum_{i=1}^2 \widehat{\Theta}^{c,g_i c}[f_i]$ is an almost holomorphic cusp form and the holomorphic theta function $\sum_{i=1}^2 \Theta^{c,g_i c}[f_i]$ is a quasimodular form of weight $9/2$ and depth 1 on $\Gamma_0(16)$. We can identify these quasimodular forms as

$$\frac{8}{7}\eta(2\tau)^2\eta(4\tau)\eta(8\tau)^2\Big(G_2(\tau)-5G_2(2\tau)+10G_2(8\tau)-24G_2(16\tau)+4\frac{\eta(8\tau)^2\eta(16\tau)^4}{\eta(4\tau)^2}\Big),$$

$$-\frac{2}{7}\eta(2\tau)^2\eta(4\tau)\eta(8\tau)^2$$
$$\cdot\Big(G_2(\tau)-G_2(2\tau)-2G_2(4\tau)+26G_2(8\tau)-24G_2(16\tau)+4\frac{\eta(8\tau)^2\eta(16\tau)^4}{\eta(4\tau)^2}\Big)$$

and

$$\frac{12}{7}\eta(2\tau)^2\eta(4\tau)\eta(8\tau)^2\big(G_2(2\tau)+3G_2(4\tau)+4G_2(8\tau)\big),$$

where $G_2(\tau) := -\frac{1}{24}+\sum_{n=1}^{\infty}\sigma_1(n)\,q^n$ is the quasimodular Eisenstein series of weight 2 and depth 1.

From Table 4.1 we can directly construct two linearly independent spherical solutions: for the spherical polynomial $f(v)=(v_1^2-4v_2^2)v_3$ we have

$$f_1(v)=\frac{1}{14}(7v_1^2-12v_2^2+8v_2v_3)(-2v_2+v_3)\quad\text{and}\quad f_2(v)=\frac{1}{14}(3v_1^2-4v_1v_3-28v_2^2)(v_1-v_3).$$

The corresponding holomorphic theta function $\sum_{i=1}^{2}\Theta^{c,g_ic}[f_i]$ is a modular form of weight 9/2 on $\Gamma_0(16)$ and equals

$$\frac{16}{7}\eta(2\tau)^2\eta(4\tau)\eta(8\tau)^2$$
$$\cdot\Big(G_2(\tau)-3G_2(2\tau)-G_2(4\tau)+18G_2(8\tau)-24G_2(16\tau)+4\frac{\eta(8\tau)^2\eta(16\tau)^4}{\eta(4\tau)^2}\Big).$$

For the spherical polynomial $f(v)=(3v_1^2+12v_2^2+4v_3^2)v_3$ we have

$$f_1(v)=\frac{1}{14}(21v_1^2+4v_2^2-40v_2v_3+28v_3^2)(-2v_2+v_3),$$
$$f_2(v)=\frac{1}{14}(v_1^2+84v_2^2+28v_3^2-20v_1v_3)(v_1-v_3).$$

The corresponding holomorphic theta function is again modular and equals

$$-\frac{48}{7}\eta(2\tau)^2\eta(4\tau)\eta(8\tau)^2\big(G_2(2\tau)-4G_2(4\tau)+4G_2(8\tau)\big).$$

Example 4.26. Considering the twelve diagonalized quadratic forms of signature $(2,1)$ and level 24 and choosing spherical polynomials of degree 1, we already obtain eight different eta quotients of weight 5/2. We list possible choices of Q, g_i and f_i and the corresponding eta quotients that $\sum_{i=1}^{2}\widehat{\widehat{\Theta}}^{c,g_ic}[f_i]$ evaluate to in Table 4.2 (we omit the quadratic forms which lead to the same eta quotients). As in Example 4.25 one could generate many (almost) holomorphic cusp forms of weight $3/2+d$ by constructing suitable homogeneous polynomials of higher degree d.

Just like for quadratic forms of signature $(1,1)$, we obtain further examples for quadratic forms of signature $(2,1)$ if we modify the polynomials in the theta series by introducing an additional periodic factor as described in Remark 4.13.

Example 4.27. Let $Q(v)=v_1^2+v_2^2-Nv_3^2$, where $N\in\mathbb{N}$ is such that $2N$ is not a perfect

Q	$g_i \in \mathrm{Aut}^+(Q,\mathbb{Z}^3)$	f_i spherical of degree 1	$\frac{1}{4}\sum_{i=1}^{2}\widehat{\widehat{\Theta}}^{c,g_ic}[f_i]$ $=$ $\frac{1}{4}\sum_{i=1}^{2}\Theta^{c,g_ic}[f_i]$
$v_1^2+3v_2^2-2v_3^2$	$g_1=\left(\begin{smallmatrix}1&0&0\\0&5&4\\0&6&5\end{smallmatrix}\right)$, $g_2=\left(\begin{smallmatrix}3&0&4\\0&1&0\\2&0&3\end{smallmatrix}\right)$	$f_1(v)=-3v_2+2v_3$, $f_2(v)=2v_1-2v_3$	$\eta_2^3\eta_4^2\eta_6\eta_{24}/(\eta_8\eta_{12})$
$v_1^2+2v_2^2-3v_3^2$	$g_1=\left(\begin{smallmatrix}1&0&0\\0&5&6\\0&4&5\end{smallmatrix}\right)$, $g_2=\left(\begin{smallmatrix}2&0&3\\0&1&0\\1&0&2\end{smallmatrix}\right)$	$f_1(v)=-v_2+v_3$, $f_2(v)=v_1-v_3$	$\eta\eta_6^9\eta_8^2/(\eta_2\eta_3^3\eta_{12}^3)$
$v_1^2+6v_2^2-2v_3^2$	$g_1=\left(\begin{smallmatrix}1&0&0\\0&2&1\\0&3&2\end{smallmatrix}\right)$, $g_2=\left(\begin{smallmatrix}3&0&4\\0&1&0\\2&0&3\end{smallmatrix}\right)$	$f_1(v)=-3v_2+v_3$, $f_2(v)=v_1-v_3$	$\eta_2^2\eta_3\eta_4^3\eta_{12}/(\eta\eta_6)$
$v_1^2+2v_2^2-6v_3^2$	$g_1=\left(\begin{smallmatrix}1&0&0\\0&2&3\\0&1&2\end{smallmatrix}\right)$, $g_2=\left(\begin{smallmatrix}5&0&12\\0&1&0\\2&0&5\end{smallmatrix}\right)$	$f_1(v)=-2v_2+2v_3$, $f_2(v)=v_1-2v_3$	$\eta^2\eta_8\eta_{12}^9/(\eta_4\eta_6^3\eta_{24}^3)$
$v_1^2+6v_2^2-3v_3^2$	$g_1=\left(\begin{smallmatrix}1&0&0\\0&3&2\\0&4&3\end{smallmatrix}\right)$, $g_2=\left(\begin{smallmatrix}2&0&3\\0&1&0\\1&0&2\end{smallmatrix}\right)$	$f_1(v)=-2v_2+v_3$, $f_2(v)=v_1-v_3$	$\eta_2\eta_6^3\eta_8\eta_{12}^2/(\eta_4\eta_{24})$
$3v_1^2+6v_2^2-v_3^2$	$g_1=\left(\begin{smallmatrix}1&0&0\\0&5&2\\0&12&5\end{smallmatrix}\right)$, $g_2=\left(\begin{smallmatrix}2&0&1\\0&1&0\\3&0&2\end{smallmatrix}\right)$	$f_1(v)=-3v_2+v_3$, $f_2(v)=3v_1-v_3$	$\eta_2^9\eta_3\eta_{24}^2/(\eta^3\eta_4^3\eta_6)$
$2v_1^2+6v_2^2-3v_3^2$	$g_1=\left(\begin{smallmatrix}1&0&0\\0&3&2\\0&4&3\end{smallmatrix}\right)$, $g_2=\left(\begin{smallmatrix}5&0&6\\0&1&0\\4&0&5\end{smallmatrix}\right)$	$f_1(v)=-2v_2+v_3$, $f_2(v)=v_1-v_3$	$\eta\eta_4\eta_6^2\eta_{12}^3/(\eta_2\eta_3)$
$3v_1^2+6v_2^2-2v_3^2$	$g_1=\left(\begin{smallmatrix}1&0&0\\0&2&1\\0&3&2\end{smallmatrix}\right)$, $g_2=\left(\begin{smallmatrix}5&0&4\\0&1&0\\6&0&5\end{smallmatrix}\right)$	$f_1(v)=-6v_2+2v_3$, $f_2(v)=3v_1-2v_3$	$\eta_3^2\eta_4^9\eta_{24}/(\eta_2^3\eta_8^3\eta_{12})$

Table 4.2: Eight different cusp forms of weight 5/2 on $\Gamma_0(24)$

square. As matrices in the automorphism group $\mathrm{Aut}^+(Q,\mathbb{Z}^3)$ we choose

$$g_1=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix},\qquad g_2=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\qquad\text{and}\qquad g_3=\begin{pmatrix}\frac{x+1}{2}&\frac{x-1}{2}&Ny\\\frac{x-1}{2}&\frac{x+1}{2}&Ny\\y&y&x\end{pmatrix},$$

where (x,y) is an integer solution to the Pell equation $x^2-2Ny^2=1$. We take the spherical polynomials $f_1(v)=\frac{1}{2}yv_2+\frac{x-1}{4}v_3$, $f_2(v)=\frac{1}{2}yv_1+\frac{x-1}{4}v_3$ and $f_3(v)=v_3$ and consider the periodic function

$$m(v)=\left(\frac{-4}{v_1}\right)\left(\frac{-4}{v_2}\right),$$

where the Dirichlet character $\left(\frac{-4}{\cdot}\right)$ is given by

$$\left(\frac{-4}{n}\right)=\begin{cases}1&\text{if } n\equiv 1 \pmod 4,\\-1&\text{if } n\equiv -1 \pmod 4,\\0&\text{otherwise.}\end{cases}$$

We set $\widetilde{f_i}:=m\cdot f_i$ and observe that $m\circ g_1=m\circ g_2=-m$ and $m\circ g_3=m$. Hence $\sum_{i=1}^{3}(\widetilde{f_i}-\widetilde{f_i}\circ g_i)=0$ is equivalent to $f_1+f_1\circ g_1+f_2+f_2\circ g_2+f_3-f_3\circ g_3=0$, which we can easily verify for the polynomials f_1, f_2 and f_3 above. Thus, $\sum_{i=1}^{3}\widehat{\widehat{\Theta}}^{c,g_ic}[m,f_i]=\sum_{i=1}^{3}\Theta^{c,g_ic}[m,f_i]$ is a cusp form of weight 5/2. For $c=\frac{1}{\sqrt{N}}\left(\begin{smallmatrix}0\\0\\-1\end{smallmatrix}\right)$ we have $g_1c=g_2c=c$, so for this choice of $c\in\mathcal{C}_Q$ the first two theta functions $\Theta^{c,g_1c}[m,f_1]$ and $\Theta^{c,g_2c}[m,f_2]$

vanish. Hence we obtain the modular theta series

$$\sum_{i=1}^{3} \widehat{\widehat{\Theta}}^{c,g_i c}[m,f_i] = \Theta^{c,g_3 c}[m,f_3](\tau)$$
$$= \sum_{\ell\in\mathbb{Z}^3} \{\operatorname{sgn}(\ell_3) + \operatorname{sgn}(y\ell_1 + y\ell_2 - x\ell_3)\} \left(\frac{-4}{\ell_1}\right)\left(\frac{-4}{\ell_2}\right) \ell_3\, q^{\ell_1^2+\ell_2^2-N\ell_3^2}.$$

For $N = 1$ we take $(x, y) = (3, 2)$ as our solution to the equation $x^2 - 2y^2 = 1$. The corresponding theta series is $4\eta_2^5\eta_8^2/\eta^2$. For $N = 3$ we take $(x, y) = (5, 2)$ and identify the theta series as

$$8\,\frac{\eta_8^4\eta_{24}^9}{\eta_4\eta_{12}^3\eta_{16}\eta_{48}^3} - 32\,\eta_8\eta_{16}\eta_{48}^3.$$

For $N = 4$ we take $(x, y) = (3, 1)$ and the theta series equals $4\eta_4^2\eta_8\eta_{16}^2$. For $N = 6$, $x = 7$ and $y = 2$ we find $8\,\eta_2\eta_8^4\eta_{12}^2/(\eta_4\eta_6)$.

For $N = 1$ and $d = 3$ we could for example take the spherical polynomials

$$f_1(v) = 7(v_2 + v_3)^3,$$
$$f_2(v) = 7(v_1 + v_3)^3,$$
$$f_3(v) = 4v_1^3 + 4v_2^3 + 3(v_1 + v_2)v_3^2 - 9v_1v_2(v_1 + v_2).$$

Again, the condition $f_1 + f_1\circ g_1 + f_2 + f_2\circ g_2 + f_3 - f_3\circ g_3 = 0$ is satisfied, so

$$\sum_{\ell\in\mathbb{Z}^3} \{\operatorname{sgn}(\ell_3) + \operatorname{sgn}(2\ell_1 + 2\ell_2 - 3\ell_3)\} \left(\frac{-4}{\ell_1}\right)\left(\frac{-4}{\ell_2}\right) f_3(\ell)\, q^{\ell_1^2+\ell_2^2-\ell_3^2}$$

is a cusp form of weight $9/2$, which we can identify as

$$48\,\frac{\eta(2\tau)^5\eta(8\tau)^2}{\eta(\tau)^2}\big(G_2(\tau) - 5\,G_2(2\tau) + 12\,G_2(8\tau)\big).$$

In the next example, we give a similar construction for quadratic forms of level $4N$. Here we assume that N itself is not a perfect square, and consider $I = \{1, 2\}$ and a different periodic function.

Example 4.28. Let $Q(v) = v_1^2 + v_2^2 - Nv_3^2$, where $N \in \mathbb{N}$ is not a perfect square. As matrices in the automorphism group we choose

$$g_1 = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \qquad \text{and} \qquad g_2 = \begin{pmatrix} 1 & 0 & 0\\ 0 & x & Ny\\ 0 & y & x\end{pmatrix},$$

where (x, y) denotes an integer solution of the Pell equation $x^2 - Ny^2 = 1$ for which x is odd (if for the fundamental solution (x_1, y_1) the integer x_1 is even, we take the solution (x_2, y_2) given by $x_2 = x_1^2 + Ny_1^2 = 1 + 2Ny_1^2$, $y_2 = 2x_1y_1$). Further, we choose $c = \frac{1}{\sqrt{N}}\left(\begin{smallmatrix} 0\\ 0\\ -1\end{smallmatrix}\right) \in \mathcal{C}_Q$. As $g_1c = c$ holds, the theta series $\widehat{\widehat{\Theta}}^{c,g_1 c}$ vanishes, thus we only need to determine $\operatorname{sgn}\big(B(c, v)\big) = \operatorname{sgn}(v_3)$ and $\operatorname{sgn}\big(B(g_2c, v)\big) = -\operatorname{sgn}(yv_2 - xv_3)$ to write out the theta series later. For the periodic function

$$m(v) = \begin{cases} (-1)^{v_2} & \text{if } v_1 \not\equiv v_2 \pmod 2, \\ 0 & \text{if } v_1 \equiv v_2 \pmod 2 \end{cases}$$

we have $m \circ g_1 = -m$ and $m \circ g_2 = m$, as x is odd and y is even. For a polynomial f_i, we again set $\widetilde{f_i} = m \cdot f_i$ and observe that $\sum_{i=1}^{2}(\widetilde{f_i} - \widetilde{f_i} \circ g_i) = 0$ is equivalent to $f_1 + f_1 \circ g_1 + f_2 - f_2 \circ g_2 = 0$.

Let $\alpha := y/\gcd(x-1,y)$ and $\beta := (x-1)/\gcd(x-1,y)$. For $0 \le r \le (d-1)/2$, we define the homogeneous polynomials of degree d

$$f_1(v) := f_1^r(v) = h_1^r(v)\,Q(v)^r \qquad \text{and} \qquad f_2(v) := f_2^r(v) = h_2^r(v)\,Q(v)^r,$$

where

$$\begin{aligned} h_1^r(v) &= \frac{1}{4}\Big\{(\alpha(v_1+v_2)+\beta v_3)^{d-2r} + (\alpha(-v_1+v_2)+\beta v_3)^{d-2r} \\ &\qquad + (\alpha(v_1-v_2)+\beta v_3)^{d-2r} + (\alpha(-v_1-v_2)+\beta v_3)^{d-2r}\Big\}, \\ h_2^r(v) &= \frac{1}{2}\Big\{(\alpha(v_1+v_2)-\beta v_3)^{d-2r} + (\alpha(-v_1+v_2)-\beta v_3)^{d-2r}\Big\}. \end{aligned}$$

By a simple calculation, where we use that $f_1 = f_1 \circ g_1$, d is odd and g_2 leaves Q invariant, we check that the polynomials above fulfill the equation $f_1 + f_1 \circ g_1 + f_2 - f_2 \circ g_2 = 0$. We use these to construct the almost holomorphic cusp forms

$$\begin{aligned} \sum_{i=1}^{2} \widehat{\widehat{\Theta}}{}^{c,g_i c}[m, f_i](\tau) &= \widehat{\Theta}^{c,g_2 c}[m, f_2](\tau) \\ &= y^{-d/2} \sum_{\ell \in \mathbb{Z}^3} \big\{\operatorname{sgn}(\ell_3) + \operatorname{sgn}(y\ell_2 - x\ell_3)\big\}\, m(\ell)\, \widehat{f_2}(\ell y^{1/2})\, q^{Q(\ell)} \end{aligned}$$

of weight $3/2 + d$. In general, f_1 and f_2 are not spherical, but applying the Laplacian, we see that for $i = 1, 2$

$$\Delta\big(f_i^r(v)\big) = r\,(2d-2r+1)\,h_i^r(v)\,Q(v)^{r-1} + \frac{1}{2}(2\alpha^2 - \beta^2/N)\,(d-2r)\,(d-2r-1)\,h_i^{r+1}(v)\,Q(v)^r$$

holds. So we can choose a linear combination of these homogeneous polynomials to construct the spherical polynomials

$$F_i := \sum_{r=0}^{(d-1)/2} \big(\beta^2/N - 2\alpha^2\big)^r \binom{2(d-r)}{r, d-r, d-2r} f_i^r \quad (i = 1, 2),$$

using the notation of trinomial coefficients $\binom{n}{j,k,\ell} := \frac{n!}{j!\,k!\,\ell!}$ with $n = j+k+\ell$ for $j, k, \ell \in \mathbb{N}_0$. Hence we obtain the holomorphic cusp form

$$\sum_{i=1}^{2} \widehat{\widehat{\Theta}}{}^{c,g_i c}[m, F_i](\tau) = \Theta^{c,g_2 c}[m, F_2](\tau) = \sum_{\ell \in \mathbb{Z}^3} \big\{\operatorname{sgn}(\ell_3) + \operatorname{sgn}(y\ell_2 - x\ell_3)\big\}\, m(\ell)\, F_2(\ell)\, q^{Q(\ell)}.$$

For $N = 2$ we take $(x, y) = (3, 2)$, thus $\alpha = \beta = 1$. Then we have for $d = 1$ the spherical polynomials $f_1(v) = v_3$ and $f_2(v) = v_2 - v_3$ and the theta series equals $(-4)\,\eta_2^2\eta_4\eta_8^2$ (we obtained the same eta product in Example 4.25).

For $d = 3$ we obtain for $r = 0$ the pair of homogeneous polynomials

$$f_1^0(v) = v_3\bigl(3v_1^2 + 3v_2^2 + v_3^2\bigr),$$
$$f_2^0(v) = 3v_1^2(v_2 - v_3) + (v_2 - v_3)^3.$$

The corresponding almost holomorphic theta series can be identified as

$$24\,\eta(2\tau)^2\eta(4\tau)\eta(8\tau)^2\bigl(G_2^*(2\tau) - G_2^*(4\tau) + 4G_2^*(8\tau)\bigr),$$

where G_2^* is the almost holomorphic modular form of weight 2 given by $G_2^*(\tau) := G_2(\tau) + \frac{1}{8\pi y}$. For $r = 1$ we have

$$f_1^1(v) = v_3\bigl(v_1^2 + v_2^2 - 2v_3^2\bigr),$$
$$f_2^1(v) = (v_2 - v_3)(v_1^2 + v_2^2 - 2v_3^2)$$

and the corresponding theta series is

$$16\,\eta(2\tau)^2\eta(4\tau)\eta(8\tau)^2\bigl(G_2^*(2\tau) + G_2^*(4\tau) + 4G_2^*(8\tau)\bigr).$$

Thus, we have constructed two almost holomorphic cusp forms of weight 9/2 and depth 1.

Now we set $F_i = 10f_i^0 - 9f_i^1$ (we take the formula for F_i given above, but divide by the greatest common divisor of the integer coefficients in the linear combination) and get the holomorphic cusp form

$$96\,\eta(2\tau)^2\eta(4\tau)\eta(8\tau)^2\bigl(G_2(2\tau) - 4G_2(4\tau) + 4G_2(8\tau)\bigr).$$

For $N = 3$ we take $(x, y) = (7, 4)$, thus $\alpha = 2$ and $\beta = 3$. For $d = 1$ we have the spherical polynomials $f_1(v) = 3v_3$ and $f_2(v) = 2v_2 - 3v_3$ and the theta series equals $(-8)\,\eta\eta_4^4\eta_6^2/(\eta_2\eta_3)$.

For $d = 3$ we obtain for $r = 0$ the pair of homogeneous polynomials

$$f_1^0(v) = 9v_3\bigl(4v_1^2 + 4v_2^2 + 3v_3^2\bigr),$$
$$f_2^0(v) = 12v_1^2(2v_2 - 3v_3) + (2v_2 - 3v_3)^3.$$

The corresponding almost holomorphic theta series can be identified as

$$24\,\frac{\eta(\tau)\eta(4\tau)^4\eta(6\tau)^2}{\eta(2\tau)\eta(3\tau)}\bigl(3G_2^*(\tau) - 6G_2^*(2\tau) - 9G_2^*(3\tau) + 8G_2^*(4\tau) + 36G_2^*(6\tau)\bigr).$$

For $r = 1$ we have

$$f_1^1(v) = 3v_3\bigl(v_1^2 + v_2^2 - 3v_3^2\bigr),$$
$$f_2^1(v) = (2v_2 - 3v_3)\bigl(v_1^2 + v_2^2 - 3v_3^2\bigr)$$

and the corresponding theta series is

$$8\,\frac{\eta(\tau)\eta(4\tau)^4\eta(6\tau)^2}{\eta(2\tau)\eta(3\tau)}\bigl(G_2^*(\tau) - 2G_2^*(2\tau) - 3G_2^*(3\tau) + 16G_2^*(4\tau) + 12G_2^*(6\tau)\bigr).$$

We set $F_i = f_i^0 - 3f_i^1$ and get the holomorphic cusp form of weight $9/2$

$$48\,\frac{\eta(\tau)\eta(4\tau)^4\eta(6\tau)^2}{\eta(2\tau)\eta(3\tau)}\big(G_2(\tau)-2G_2(2\tau)-3G_2(3\tau)-4G_2(4\tau)+12G_2(6\tau)\big).$$

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5 Theta series for quadratic forms of signature $(n-1,1)$ with (spherical) polynomials II

This chapter is based on an article in preparation that is joint work with Prof. Dr. Sander Zwegers. My share of the work amounted to 50%.

5.1 Introduction

Theta series for positive definite quadratic forms $Q:\mathbb{R}^n \longrightarrow \mathbb{R}$ associated to a function $f:\mathbb{R}^n \longrightarrow \mathbb{C}$, i. e. series of the form

$$\Theta_{Q,f}(\tau)=\sum_{\ell\in\mathbb{Z}^n} f(\ell)\, q^{Q(\ell)} \quad (q=e^{2\pi i\tau},\, \mathrm{Im}(\tau)>0)$$

play an important role in the construction of modular forms. In particular, if f a spherical polynomial of degree d, we obtain a holomorphic modular form of weight $n/2+d$ on some subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and with some character (see [Sch39, Ogg69, Shi73]). For indefinite quadratic forms one has to ensure the convergence of the series $\Theta_{Q,f}(\tau)=\sum_{\ell\in\mathbb{Z}^n} f(\ell)\, q^{Q(\ell)}$. One way to do this is to include majorants as was done by Siegel [Sie51]. Further, Vignéras [Vig75, Vig77] gave a general construction for indefinite theta functions. However, note that both constructions give modular forms that are in general non-holomorphic.

Modular forms also arise from different contexts. Zagier [Zag99] studied sums of powers of quadratic polynomials with integer coefficients and discovered for an even integer k and a fixed real number x a modular form of weight $k+1/2$ on $\Gamma_0(4)$ that has the following expansion:

$$T_x(\tau)=\sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ ax^2+bx+c>0>a}} (ax^2+bx+c)^{k-1}\, q^{b^2-4ac} - \frac{1}{2k}\sum_{m=-\infty}^{\infty} \overline{B}_k(mx)\, q^{m^2} + \delta_{k,2}\frac{\kappa(x)}{2}\sum_{m=1}^{\infty} m^2\, q^{m^2}$$

(See Section 5.4.2 for the precise definitions of $\overline{B}_k$ and κ.) The first sum has the form of a theta series, where $(ax^2+bx+c)^{k-1}$ is a spherical polynomial with respect to the binary quadratic form b^2-4ac of signature $(2,1)$. In contrast to the definition for positive definite quadratic forms, here one restricts the summation to a cone in $\mathbb{Z}^3$ determined by $ax^2+bx+c>0>a$. Zagier remarks that this observation "suggests that there may be an arithmetic theory of theta series attached to indefinite quadratic forms in which the summation runs over the intersection of the lattice with some simplicial cone on which the quadratic form is positive and the result is still a modular form of the expected weight and level".

Later, this theory was developed by Göttsche and Zagier [GZ98] and the second author [Zwe02] for quadratic forms of signature $(n-1,1)$. In these constructions it is crucial to employ vectors $c_1,c_2\in\mathbb{R}^n$ with $Q(c_i)\le 0$ and $B(c_1,c_2)<0$, which restrict the summation to a part of the lattice where the quadratic form can be bounded by a positive definite quadratic form, which ensures convergence.

However, these constructions do not include spherical polynomials of higher degree, so it is not possible to recover, for example, the function T_x in the scope of this theory. In a previous project [RZ21], we extended the definition of the theta functions from [Zwe02] to include homogeneous and spherical polynomials. In [RZ21] we focused on the case $Q(c) < 0$, constructed a holomorphic theta series which is not modular and a corresponding non-holomorphic modular theta series. Further, we gave a criterion to determine when these two versions coincide in order to construct holomorphic and almost holomorphic modular forms.

Since there are many interesting modular forms, which can be realized as theta series with spherical polynomials, but require $c \in \mathbb{R}^n$ to be located on the boundary of the cone, the aim of the present paper is to extend the results from [RZ21] to include vectors c with $Q(c) = 0$. Most of the reasoning to establish the absolute convergence of the theta series and its modularity properties are quite analogous to the reasoning in [RZ21] and [Zwe02]. More interestingly, we show in Section 5.4 how we can use the results for $Q(c) = 0$ to establish the (mock) modularity in certain special cases: we consider the usual Eisenstein series, embed the aforementioned function T_x in this theory, and show that the generating function of the Hurwitz class numbers $H(8n + 7)$ is a mock theta function.

5.2 Definitions and statement of the main results

For the rest of the paper, we assume that the quadratic form Q has signature $(n-1, 1)$. We let A denote the corresponding symmetric matrix (so $Q(v) = \frac{1}{2}v^{\mathrm{t}}Av$), where we assume $A \in \mathbb{Z}^{n\times n}$. Further, let B be the bilinear form associated to Q: $B(u, v) = u^{\mathrm{t}}Av = Q(u + v) - Q(u) - Q(v)$. Since Q has signature $(n - 1, 1)$, the set of vectors $c \in \mathbb{R}^n$ with $Q(c) < 0$ has two components. If $B(c_1, c_2) < 0$, then c_1 and c_2 belong to the same component, while if $B(c_1, c_2) > 0$ then c_1 and c_2 belong to opposite components. Let C_Q be one of those components. If c_0 is in that component, then C_Q is given by:

$$C_Q := \{c \in \mathbb{R}^n \mid Q(c) < 0,\ B(c, c_0) < 0\}$$

Here we also consider the corresponding set of cusps

$$S_Q := \{c \in \mathbb{Q}^n \mid Q(c) = 0,\ B(c, c_0) < 0\}$$

and let $\overline{C}_Q := C_Q \cup S_Q$. For $c \in \overline{C}_Q$ we set

$$R(c) := \begin{cases} \mathbb{R}^n & \text{if } c \in C_Q, \\ \{a \in \mathbb{R}^n \mid B(c, a) \notin \mathbb{Z}\} & \text{if } c \in S_Q. \end{cases}$$

In [Zwe02], the second author used the error function

$$E(z) := 2\int_0^z e^{-\pi u^2} du = \operatorname{sgn}(z) - \operatorname{sgn}(z)\int_{z^2}^{\infty} u^{-1/2}e^{-\pi u} du$$

to define a non-holomorphic modular theta series and determine its holomorphic part. We generalize this construction as follows (note that this is a slightly more general definition as in [RZ21] since we do not necessarily normalize $c \in C_Q$ and include $c \in S_Q$):

Definition 5.1. Let $\Delta = \Delta_Q := \left(\frac{\partial}{\partial v}\right)^{\mathrm{t}} A^{-1} \frac{\partial}{\partial v}$ denote the Laplacian associated to Q (we often omit Q in the notation, as we take it to be fixed). We set

$$e^{-\Delta/8\pi} := \sum_{k=0}^{\infty} \frac{(-1)^k}{(8\pi)^k k!} \Delta^k, \qquad \partial_c := \frac{1}{\sqrt{-Q(c)}}\, c^{\mathrm{t}} \frac{\partial}{\partial v} = \frac{1}{\sqrt{-Q(c)}} \sum_{i=1}^{n} c_i \frac{\partial}{\partial v_i}$$

and for a homogeneous polynomial $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ of degree d we define $\widehat{f} := e^{-\Delta/8\pi} f$. Further, we set

$$p^c[f](v) := \begin{cases} \sum_{k=0}^{d} \frac{(-1)^k}{(4\pi)^k k!} E^{(k)}\left(\frac{B(c,v)}{\sqrt{-Q(c)}}\right) \cdot \partial_c^k \widehat{f}(v) & \text{if } c \in C_Q, \\ \operatorname{sgn}(B(c,v)) \cdot \widehat{f}(v) & \text{if } c \in S_Q. \end{cases}$$

Definition 5.2. Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a homogeneous polynomial of degree d and let $c_1, c_2 \in \overline{C}_Q$. We define the holomorphic theta series associated to Q and f with characteristics $a \in R(c_1) \cap R(c_2)$ and $b \in \mathbb{R}^n$ by

$$\Theta_{a,b}^{c_1,c_2}[f](\tau) := \sum_{\ell \in a+\mathbb{Z}^n} \left\{ \operatorname{sgn}(B(c_1,\ell)) - \operatorname{sgn}(B(c_2,\ell)) \right\} f(\ell)\, q^{Q(\ell)}\, e^{2\pi i B(\ell,b)},$$

the almost holomorphic theta series by

$$\widehat{\Theta}_{a,b}^{c_1,c_2}[f](\tau) := y^{-d/2} \sum_{\ell \in a+\mathbb{Z}^n} \left\{ \operatorname{sgn}(B(c_1,\ell)) - \operatorname{sgn}(B(c_2,\ell)) \right\} \widehat{f}(\ell y^{1/2})\, q^{Q(\ell)}\, e^{2\pi i B(\ell,b)},$$

and the non-holomorphic theta series by

$$\widehat{\widehat{\Theta}}_{a,b}^{c_1,c_2}[f](\tau) := y^{-d/2} \sum_{\ell \in a+\mathbb{Z}^n} \left\{ p^{c_1}[f](\ell y^{1/2}) - p^{c_2}[f](\ell y^{1/2}) \right\} q^{Q(\ell)}\, e^{2\pi i B(\ell,b)}.$$

Remark 5.3. (a) We show in Lemma 5.6 that all three theta series $\Theta_{a,b}^{c_1,c_2}[f]$, $\widehat{\Theta}_{a,b}^{c_1,c_2}[f]$, $\widehat{\widehat{\Theta}}_{a,b}^{c_1,c_2}[f]$ are absolutely convergent. In [RZ21] this was already shown for the case $c_1, c_2 \in C_Q$.
(b) In [RZ21] we derived certain conditions under which the non-holomorphic theta series agrees with the almost holomorphic theta series. For $c_1, c_2 \in S_Q$ it immediately follows from the definition that $\widehat{\widehat{\Theta}}_{a,b}^{c_1,c_2}$ agrees with $\widehat{\Theta}_{a,b}^{c_1,c_2}$, which is an almost holomorphic theta series of depth $\leq d/2$ since f is a homogeneous polynomial of degree d.
(c) As usual we call a polynomial $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ spherical (of degree d) if it is homogeneous (of degree d) and vanishes under the Laplacian, i. e. $\Delta f = 0$. If f is spherical of degree d, we have $\widehat{f} = e^{-\Delta/8\pi} f = f$ and $y^{-d/2} \widehat{f}(\ell y^{1/2}) = y^{-d/2} f(\ell y^{1/2}) = f(\ell)$. Hence in this case $\widehat{\Theta}_{a,b}^{c_1,c_2}[f]$ and the holomorphic theta series $\Theta_{a,b}^{c_1,c_2}[f]$ agree.

In the next section, we'll show that the following two theorems hold for the modular theta series $\widehat{\widehat{\Theta}}_{a,b}^{c_1,c_2}$.

Theorem 5.4. *Let $c_1, c_3 \in C_Q$, $c_2 \in S_Q$ and $a \in R(c_2)$, $b \in \mathbb{R}^n$. For $c(t) = c_2 + tc_3$, we have $c(t) \in C_Q$ for all $t \in (0, \infty)$ and $\lim_{t \downarrow 0} \widehat{\widehat{\Theta}}_{a,b}^{c_1,c(t)} = \widehat{\widehat{\Theta}}_{a,b}^{c_1,c_2}$.*

In the next theorem, we collect the (modular) properties of $\widehat{\widehat{\Theta}}_{a,b}^{c_1,c_2}$:

Theorem 5.5. *Let $c_1, c_2 \in \overline{C}_Q$ and $a \in R(c_1), b \in \mathbb{R}^n$. Further, let $A \in \mathbb{Z}^{n\times n}$ be a symmetric matrix. The theta function with respect to (c_1, c_2) satisfies*

1. $\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a+\lambda,b}[f](\tau) = \widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,b}[f](\tau)$ *for* $\lambda \in \mathbb{Z}^n$,
2. $\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,b+\mu}[f](\tau) = e^{2\pi i B(a,\mu)}\,\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,b}[f](\tau)$ *for* $\mu \in A^{-1}\mathbb{Z}^n$,
3. $\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{-a,-b}[f](\tau) = (-1)^{d+1}\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,b}[f](\tau)$,
4. $\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,b}[f](\tau+1) = e^{-2\pi i Q(a)-\pi i B(A^{-1}A^*,a)}\,\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,a+b+\frac{1}{2}A^{-1}A^*}[f](\tau)$ *with A^* the vector of diagonal elements of A, and* $\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,b}[f](\tau+1) = e^{-2\pi i Q(a)}\,\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,a+b}[f](\tau)$ *if A is even,*
5. *For $a, b \in R(c_1) \cap R(c_2)$ we have*
$$\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,b}[f]\big(-\frac{1}{\tau}\big) = (-i\tau)^{n/2+d}\,\frac{i^{d+1}}{\sqrt{|\det A|}}\,e^{2\pi i B(a,b)} \sum_{p\in A^{-1}\mathbb{Z}^n \bmod \mathbb{Z}^n} \widehat{\widehat{\Theta}}{}^{c_1,c_2}_{b+p,-a}[f](\tau).$$

5.3 Convergence of the theta series and proof of the main results

We first show the convergence of the theta series.

Lemma 5.6. *For $c_1, c_2 \in \overline{C}_Q$ the series defining the theta functions $\Theta^{c_1,c_2}_{a,b}[f]$, $\widehat{\Theta}^{c_1,c_2}_{a,b}[f]$ and $\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,b}[f]$ are absolutely convergent.*

Proof. In Remark 2.4(a) and Lemma 3.1 in [RZ21] this is shown for the case $c_1, c_2 \in C_Q$. Note that we slightly changed the definition of the theta series, as we introduced the characteristics a, b. However, this does not change the convergence properties. Using the same argumentation we find that it suffices to show that the series

$$\sum_{\ell\in a+\mathbb{Z}^n} \{\operatorname{sgn}\big(B(c_1,\ell)\big) - \operatorname{sgn}\big(B(c_2,\ell)\big)\}\,P(\ell)\,q^{Q(\ell)}\,e^{2\pi i B(\ell,b)} \tag{5.1}$$

is absolutely convergent for any polynomial P and $c_1, c_2 \in \overline{C}_Q$. In [RZ21] we have already treated the case $c_1, c_2 \in C_Q$. To obtain the other cases, we note that by the cocycle condition $\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,b} + \widehat{\widehat{\Theta}}{}^{c_2,c_3}_{a,b} + \widehat{\widehat{\Theta}}{}^{c_3,c_1}_{a,b} = 0$ for $c_1, c_2, c_3 \in \overline{C}_Q$ it suffices to consider the case $c_1 \in C_Q$ and $c_2 \in S_Q$ (the claim then also follows for $c_1 \in S_Q$, $c_2 \in C_Q$ and $c_1, c_2 \in S_Q$). For this, we use more or less the same arguments as in Case 2 of the proof of Proposition 2.4 in [Zwe02]: first of all, we can assume that $c_1 \in \mathbb{Z}^n \cap C_Q$ and $c_2 \in \mathbb{Z}^n \cap S_Q$. We also choose the same decomposition $\ell = \mu + mc_2$ with $\mu \in a + \mathbb{Z}^n$ and $m \in \mathbb{Z}$, such that $\frac{B(c_1,\mu)}{B(c_1,c_2)} \in [0,1)$. Then we can write the series in (5.1) as

$$-\sum_{\substack{\mu\in a+\mathbb{Z}^n\\ \frac{B(c_1,\mu)}{B(c_1,c_2)}\in[0,1)}} \sum_{m\in\mathbb{Z}} \Big\{\operatorname{sgn}\big(B(c_2,\mu)\big) + \operatorname{sgn}\Big(m + \frac{B(c_1,\mu)}{B(c_1,c_2)}\Big)\Big\}$$
$$\cdot P(\mu+mc_2)\,q^{Q(\mu)+B(c_2,\mu)m}\,e^{2\pi i B(\mu,b)+2\pi i B(c_2,b)m}.$$

We have $P(\mu+mc_2) = \sum_{k=0}^{d} P_k(\mu)\,m^k$ for some polynomials $P_k : \mathbb{R}^n \longrightarrow \mathbb{C}$. It suffices to show that the series above converges absolutely for one of these summands with a fixed

$k \in \{0, \dots, d\}$. Thus we consider the inner series

$$\sum_{m \in \mathbb{Z}} \Big\{ \operatorname{sgn}\big(B(c_2, \mu)\big) + \operatorname{sgn}\Big(m + \frac{B(c_1, \mu)}{B(c_1, c_2)}\Big) \Big\} m^k \, q^{B(c_2,\mu)m} \, e^{2\pi i B(c_2,b)m}. \tag{5.2}$$

For $B(c_2, \mu) > 0$, this equation has the form

$$2 \sum_{m \geq 0} m^k \, x^m - \delta_{k,0} \delta_{B(c_1,\mu),0} \quad \text{with } |x| = |q^{B(c_2,\mu)} \, e^{2\pi i B(c_2,b)}| = e^{-2\pi B(c_2,\mu) y} < 1,$$

and for $B(c_2, \mu) < 0$, equation (5.2) has the form

$$-2 \sum_{m \leq -1} m^k \, x^m \quad \text{with } |x| = |q^{B(c_2,\mu)} \, e^{2\pi i B(c_2,b)}| = e^{-2\pi B(c_2,\mu) y} > 1.$$

Let $E_{k,j}$ be the Eulerian number, then we have the identities

$$\frac{x}{(1-x)^{k+1}} \sum_{j=0}^{k-1} E_{k,j} \, x^j = \begin{cases} \sum_{m \geq 0} m^k \, x^m & \text{if } |x| < 1, \\ -\sum_{m \leq -1} m^k \, x^m & \text{if } |x| > 1. \end{cases}$$

For $k = 0$ this is just the usual geometric series. The second identity follows from the first one for x^{-1} using the fact that $E_{k,j} = E_{k,k-1-j}$ holds. Thus, equation (5.2) has the form

$$\frac{x}{(1-x)^{k+1}} \sum_{j=0}^{k-1} E_{k,j} \, x^j \quad \text{with } x = q^{B(c_2,\mu)} \, e^{2\pi i B(c_2,b)}.$$

The expression $B(c_2, \mu)$ does not become arbitrarily small since $a \in R(c_2)$, so the term above is bounded. Then one can proceed exactly as in [Zwe02] to conclude the proof. □

Now we can prove the two main theorems, which generalize the results in [RZ21] to the case $c \in \overline{C}_Q$.

Proof of Theorem 5.4. Again, we can make use of the results on theta series, for which we do not include polynomials in the definition. We proceed as in [Zwe02]: from Proposition 2.7(5) we know that $c(t) \in C_Q$ for all $t \in (0, \infty)$ and that for the second part of the theorem it is sufficient to show $\lim_{t \downarrow 0} \widehat{\Theta}_{a,b}^{c_2, c(t)} = 0$ for $c(t) = c_2 + t c_1$. For $\big(p^{c_2}[f] - p^{c(t)}[f]\big)(v)$ we have the decomposition

$$\big(\operatorname{sgn}(B(c_2, v)) - \operatorname{sgn}(B(c(t), v))\big) \widehat{f}(v) + \operatorname{sgn}(B(c(t), v)) \, \beta\Big(\frac{B(c(t), v)^2}{-Q(c(t))}\Big) \widehat{f}(v) - \widetilde{p}^{c(t)}[f](v), \tag{5.3}$$

where $\widetilde{p}^c[f](v) := \sum_{k=1}^d \frac{(-1)^k}{(4\pi)^k k!} E^{(k)}\Big(\frac{B(c,v)}{\sqrt{-Q(c)}}\Big) \cdot \partial_c^k \widehat{f}(v)$ for $c \in C_Q$.

The first two summands are the terms that were also considered in [Zwe02] except for the polynomial factor $\widehat{f}$. However, since $\widehat{f}$ is independent of t, we can use the same argumentation as in [Zwe02] and use Lemma 4.14 to get

$$\lim_{t \downarrow 0} \sum_{\ell \in a + \mathbb{Z}^n} \big\{ \operatorname{sgn}(B(c_2, \ell)) - \operatorname{sgn}(B(c(t), \ell)) \big\} \widehat{f}(\ell) \, q^{Q(\ell)} e^{2\pi i B(\ell, b)} = 0,$$

and

$$\lim_{t\downarrow 0} \sum_{\ell \in a+\mathbb{Z}^n} \operatorname{sgn}(B(c(t),\ell))\,\beta\Big(\frac{B(c(t),\ell)^2}{-Q(c(t))}\Big)\,\widehat{f}(\ell)\,q^{Q(\ell)}e^{2\pi i B(\ell,b)} = 0,$$

since both series are uniformly convergent for $t \in (0,\infty)$.

As $c(t) \in C_Q$ for $t \in (0,\infty)$, we observe (exactly as in the proof of Lemma 3.1 in [RZ21]) that $\widetilde{p}^{c(t)}[f](v)$ can be written as a polynomial in $c(t)/\sqrt{-Q(c(t))}$ and v times the non-polynomial factor $e^{\pi \frac{B(c(t),v)^2}{Q(c(t))}}$. Hence, we have to show that for any polynomial P there exists a majorant for

$$P\Big(\frac{c(t)}{\sqrt{-Q(c(t))}}, v\Big)\, e^{-2\pi Q(v) + \pi \frac{B(c(t),v)^2}{Q(c(t))}}$$

that is independent of t and for which the sum is absolutely convergent. We have to consider what happens if $t \downarrow 0$, so we now assume $t \in (0,t_0)$ for some $t_0 > 0$. We follow the proof of Proposition 2.7(5) in [Zwe02] and use the same decomposition of $a+\mathbb{Z}^n$ into the three subsets

$$\begin{aligned}
P_1 &:= \{v \in a+\mathbb{Z}^n \mid \operatorname{sgn}(B(c_2,v)) = -\operatorname{sgn}(B(c_1,v))\},\\
P_2 &:= \{v \in a+\mathbb{Z}^n \mid B(c_1,v)\big(B(c_1,c_2)B(c_1,v) - 2Q(c_1)B(c_2,v)\big) \ge 0\},\\
P_3 &:= \{v \in a+\mathbb{Z}^n \mid \operatorname{sgn}(B(c_2,v)) = -\operatorname{sgn}\big(B(c_1,c_2)B(c_1,v) - 2Q(c_1)B(c_2,v)\big)\}.
\end{aligned}$$

We then determine a majorant on each subset P_i separately. We show that there exists a polynomial $\widetilde{P}$ such that

$$\Big|P\Big(\frac{c(t)}{\sqrt{-Q(c(t))}}, v\Big)\Big| \le \widetilde{P}\Big(\Big|\frac{B(c(t),v)}{\sqrt{-Q(c(t))}}\Big|, |v|\Big) \tag{5.4}$$

holds for all $v \in \mathbb{R}^n$ ($|v|$ stands for $|v_1|,\dots,|v_n|$ here) and $t \in (0,t_0)$: We use induction on the degree of P as a polynomial in $c(t)$ and first note that we can write P as a finite linear combination of terms of the form $B\big(\xi, \frac{c(t)}{\sqrt{-Q(c(t))}}\big)\,P_\xi\Big(\frac{B(c(t),v)}{\sqrt{-Q(c(t))}}, v\Big)$ for suitable $\xi \in \mathbb{C}$, where the degree of P_ξ is strictly lower than the degree of P in $c(t)$. So we can use the induction hypothesis on P_ξ, and we also observe

$$\begin{aligned}
&\Big|B\Big(\xi, \frac{c(t)}{\sqrt{-Q(c(t))}}\Big)\Big|\\
&\qquad \le \Big|\frac{B(\xi,c_2)}{B(c_2,v)}\,\frac{B(c(t),v)}{\sqrt{-Q(c(t))}}\Big| + \frac{t}{\sqrt{-Q(c(t))}}\,\Big|\frac{B(\xi,c_1)B(c_2,v) - B(\xi,c_2)B(c_1,v)}{B(c_2,v)}\Big|\\
&\qquad \le \alpha\Big|\frac{B(c(t),v)}{\sqrt{-Q(c(t))}}\Big| + R(|v|)
\end{aligned}$$

for a constant $\alpha > 0$ and some polynomial R. In the last step, we use that $|B(c_2,v)|$ does not become arbitrarily small for $v \in a+\mathbb{Z}^n$ since we assume $a \in R(c_2)$, and that $t/\sqrt{-Q(c(t))} \to 0$ for $t \downarrow 0$, so that this estimate holds for $t \in (0,t_0)$.

Using (5.4) we can now consider

$$\widetilde{P}\Big(\Big|\frac{B(c(t),v)}{\sqrt{-Q(c(t))}}\Big|, |v|\Big)\, e^{\pi \frac{B(c(t),v)^2}{Q(c(t))}}.$$

This expression has polynomial growth in v for $t \downarrow 0$ since it is clear that the term

$\left|\frac{B(c(t),v)}{\sqrt{-Q(c(t))}}\right|^{\alpha} e^{\pi\frac{B(c(t),v)^2}{Q(c(t))}}$ is bounded for any $\alpha \in \mathbb{N}_0$. So we find a polynomial S as an upper bound that is independent of t. In Lemma 4.14 we have seen that the sum

$$\sum_{v\in P_1} S(v)\, e^{-2\pi Q(v)}$$

is absolutely convergent.

On P_2 we observe that

$$\frac{B(c(t),v)^2}{Q(c(t))} \leq \frac{B(c_1,v)^2}{Q(c_1)} + \frac{B(c_2,v)^2}{Q(c(t))}$$

holds and thus

$$\begin{aligned}&\left|P\Big(\frac{c(t)}{\sqrt{-Q(c(t))}},v\Big)\right| e^{-2\pi Q(v)+\pi\frac{B(c(t),v)^2}{Q(c(t))}}\\ &\qquad\qquad \leq \left|P\Big(\frac{c(t)}{\sqrt{-Q(c(t))}},v\Big)\right| e^{\pi\frac{B(c_2,v)^2}{2Q(c(t))}}\, e^{-2\pi\left(Q(v)-\frac{B(c_1,v)^2}{2Q(c_1)}\right)}.\end{aligned}$$

Similarly as above, we have

$$\left|P\Big(\frac{c(t)}{\sqrt{-Q(c(t))}},v\Big)\right| \leq \widetilde{P}\Big(\Big|\frac{B(c_2,v)}{\sqrt{-Q(c(t))}}\Big|,|v|\Big)$$

for some polynomial $\widetilde{P}$ and conclude that

$$\widetilde{P}\Big(\Big|\frac{B(c_2,v)}{\sqrt{-Q(c(t))}}\Big|,|v|\Big)\, e^{\pi\frac{B(c_2,v)^2}{2Q(c(t))}}$$

has polynomial growth in v for $t \downarrow 0$. Since the quadratic form $v \mapsto Q(v) - \frac{B(c_1,v)^2}{2Q(c_1)}$ is positive definite (see Lemma 2.5 in [Zwe02]), we have constructed a suitable majorant on P_2.

On P_3 we consider the quadratic form $\widetilde{Q}$ of signature $(n-1,1)$ that is defined in [Zwe02] as follows:

$$\widetilde{Q}(v) := Q(v) - \frac{2B(c_2,v)}{B(c_1,c_2)^2}\big(B(c_1,c_2)B(c_1,v) - Q(c_1)B(c_2,v)\big)$$

We denote by $\widetilde{B}$ the associated bilinear form. Setting

$$\widetilde{c}(\widetilde{t}) = \widetilde{c}_2 + \widetilde{t}\,\widetilde{c}_1 \qquad \text{with} \quad \widetilde{c}_1 = \frac{B(c_1,c_2)}{2Q(c_1)}c_1 - c_2, \quad \widetilde{c}_2 = -c_2 \quad \text{and} \quad \widetilde{t} = \frac{2Q(c_1)}{B(c_1,c_2)}t,$$

we have

$$Q(v) - \frac{B(c(t),v)^2}{2Q(c(t))} = \widetilde{Q}(v) - \frac{\widetilde{B}(\widetilde{c}(\widetilde{t}),v)^2}{2\widetilde{Q}(\widetilde{c}(\widetilde{t}))}.$$

(This identity follows quite easily using the relations shown in [Zwe02].) Again, we can find a polynomial $\widetilde{P}$ with

$$\left|P\Big(\frac{c(t)}{\sqrt{-Q(c(t))}},v\Big)\right| \leq \widetilde{P}\left(\left|\frac{\widetilde{B}(\widetilde{c}(\widetilde{t}),v)}{\sqrt{-\widetilde{Q}(\widetilde{c}(\widetilde{t}))}}\right|,|v|\right)$$

and then conclude analogously as for P_1 that it is sufficient to show the absolute convergence of

$$\sum_{v\in P_3} S(v)\, e^{-2\pi\widetilde{Q}(v)}$$

for some polynomial S. This follows for $S\equiv 1$ as in [Zwe02] and then also for non-constant polynomials by Lemma 4.14.

Due to the uniform convergence in $t\in(0,t_0)$, we can then consider $\lim_{t\downarrow 0}\widetilde{p}^{c(t)}[f](v)$ and see that the convergence is dominated by the part $e^{\pi\frac{B(c(t),v)^2}{Q(c(t))}}$, which goes to zero for $t\downarrow 0$. We have thus shown

$$\lim_{t\downarrow 0}\sum_{\ell\in a+\mathbb{Z}^n}\widetilde{p}^{c(t)}[f](\ell)\, q^{Q(\ell)} e^{2\pi i B(\ell,b)} = 0.$$

Combining the results for the three separate series that we have obtained by the decomposition (5.3), we conclude that $\lim_{t\downarrow 0}\widehat{\widehat{\Theta}}{}^{c_2,c(t)}_{a,b} = 0$ holds. □

To describe the modular transformation behavior of $\widehat{\widehat{\Theta}}{}^{c_1,c_2}_{a,b}$ we can employ Vignéras' result [Vig77]. For the proof, we consider the Fourier transform

$$\big(\mathcal{F}f\big)(v) := \int_{\mathbb{R}^n} f(u)\, e^{-2\pi i u\cdot v}\, du.$$

Proof of Theorem 5.5. (1), (2) and (4) immediately follow by the same calculations as in [Zwe02]. The third identity holds since $p^c[f](-v) = (-1)^{d+1}p^c[f](v)$ for $c\in\overline{C}_Q$. To show part (5) we first consider $c_1,c_2\in C_Q$ and make use of Vignéras' result [Vig77]: let $g := p^{c_1}[f]-p^{c_2}[f]$. From Lemma 3.2 of [RZ21] we know that $Dg = dg$, so from [Vig77] we get that the following identity holds for $g_\tau(u) := y^{-d/2}\, g(uy^{1/2})\, e^{2\pi i Q(u)\tau}$:

$$\big(\mathcal{F}g_{-1/\tau}\big)(v) = (-i\tau)^{n/2+d}\,\frac{(-i)^{d+1}}{\sqrt{|\det A|}}\, g_\tau(A^{-1}v)$$

Hence the Fourier transform of $v\mapsto g_{-1/\tau}(v+a)\, e^{2\pi i B(v+a,b)}$ is

$$\int_{\mathbb{R}^n} g_{-1/\tau}(u+a)\, e^{2\pi i B(u+a,b)-2\pi i u\cdot v}\, du = (-i\tau)^{n/2+d}\,\frac{(-i)^{d+1}}{\sqrt{|\det A|}}\, e^{2\pi i a\cdot v} g_\tau(A^{-1}v-b)$$

By applying the Poisson summation formula, which states

$$\sum_{v\in\mathbb{Z}^n} f(v) = \sum_{v\in\mathbb{Z}^n}\big(\mathcal{F}f\big)(v),$$

we thus obtain:

$$\begin{aligned}
\widehat{\Theta}^{c_1,c_2}_{a,b}[f](-\frac{1}{\tau}) &= \sum_{v\in\mathbb{Z}^n} g_{-1/\tau}(v+a)\, e^{2\pi i B(v+a,b)} \\
&= (-i\tau)^{n/2+d}\,\frac{(-i)^{d+1}}{\sqrt{|\det A|}} \sum_{v\in\mathbb{Z}^n} e^{2\pi i a\cdot v} g_\tau(A^{-1}v-b) \\
&= (-i\tau)^{n/2+d}\,\frac{(-i)^{d+1}}{\sqrt{|\det A|}}\, e^{2\pi i B(a,b)} \sum_{u\in -b+A^{-1}\mathbb{Z}^n} e^{2\pi i B(a,u)} g_\tau(u) \\
&= (-i\tau)^{n/2+d}\,\frac{(-i)^{d+1}}{\sqrt{|\det A|}}\, e^{2\pi i B(a,b)} \sum_{p\in A^{-1}\mathbb{Z}^n \bmod \mathbb{Z}^n} \widehat{\Theta}^{c_1,c_2}_{-b-p,a}[f](\tau)
\end{aligned}$$

With part (3) we then get the desired result for the case $c_1, c_2 \in C_Q$. Using Lemma 5.4 we get that the formula also holds for the case $c_1 \in C_Q$ and $c_2 \in S_Q$. By the cocycle condition the remaining two cases then follow immediately. □

5.4 Examples

The next lemma proves to be helpful in the construction of the following examples.

Lemma 5.7. *Let $k \in \mathbb{N}$ and let B_k denote the usual k-th Bernoulli polynomial. For all $\alpha, \beta \in \mathbb{R}$, and $z \in \mathbb{C}$ with $|z| < 2\pi$ we have*

$$\frac{(-1)^k\,(k-1)!}{z^k} - \sum_{m=0}^{\infty} \frac{B_{m+k}(\alpha+\beta-\lfloor\beta\rfloor)}{m+k}\frac{z^m}{m!} = \begin{cases} \sum_{n+\beta\ge 0}(n+\alpha+\beta)^{k-1}\, e^{(n+\alpha+\beta)z} & \text{if } \mathrm{Re}(z)<0, \\ -\sum_{n+\beta\le -1}(n+\alpha+\beta)^{k-1}\, e^{(n+\alpha+\beta)z} & \text{if } \mathrm{Re}(z)>0. \end{cases}$$

Proof. We note that

$$(n+\alpha+\beta)^{k-1}\, e^{(n+\alpha+\beta)z} = \Big(\frac{\partial}{\partial z}\Big)^{k-1} e^{(n+\alpha+\beta)z}$$

holds. Well-known identities for the geometric series yield for any $x \in \mathbb{C}$ and $\beta \in \mathbb{R}$

$$\frac{x^{-\lfloor\beta\rfloor}}{1-x} = \begin{cases} \sum_{n+\beta\ge 0} x^n & \text{if } |x|<1, \\ -\sum_{n+\beta\le -1} x^n & \text{if } |x|>1. \end{cases}$$

As these series are uniformly convergent, we can interchange summation and differentiation and obtain for $\mathrm{Re}(z) < 0$

$$\sum_{n+\beta\ge 0} (n+\alpha+\beta)^{k-1}\, e^{(n+\alpha+\beta)z} = \Big(\frac{\partial}{\partial z}\Big)^{k-1}\Big(e^{(\alpha+\beta)z}\sum_{n+\beta\ge 0} e^{nz}\Big) = \Big(\frac{\partial}{\partial z}\Big)^{k-1}\Big(\frac{e^{(\alpha+\beta-\lfloor\beta\rfloor)z}}{1-e^z}\Big)$$

and for $\mathrm{Re}(z) > 0$

$$-\sum_{n+\beta\le -1} (n+\alpha+\beta)^{k-1}\, e^{(n+\alpha+\beta)z} = \Big(\frac{\partial}{\partial z}\Big)^{k-1}\Big(\frac{e^{(\alpha+\beta-\lfloor\beta\rfloor)z}}{1-e^z}\Big).$$

For $|z| < 2\pi$, we know the generating function for the Bernoulli polynomials B_m:

$$\frac{e^{(\alpha+\beta-\lfloor\beta\rfloor)z}}{1-e^z} = -\sum_{m=0}^{\infty} B_m(\alpha+\beta-\lfloor\beta\rfloor)\,\frac{z^{m-1}}{m!}$$

From

$$\Big(\frac{\partial}{\partial z}\Big)^{k-1}\Big(-\sum_{m=0}^{\infty} B_m(\alpha+\beta-\lfloor\beta\rfloor)\,\frac{z^{m-1}}{m!}\Big) = \frac{(-1)^k\,(k-1)!}{z^k} - \sum_{m=k}^{\infty}\frac{B_m(\alpha+\beta-\lfloor\beta\rfloor)}{m}\,\frac{z^{m-k}}{(m-k)!}$$

the claim follows by shifting m to $m+k$. □

5.4.1 Eisenstein Series

For the Eisenstein series of positive even weight k we use the normalized version

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{n\in\mathbb{N}^2} n_1^{k-1} q^{n_1 n_2}$$

(with B_k the k-th Bernoulli number) as definition.

The matrix $A = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ of signature $(1,1)$ induces the quadratic form $Q(v) = v_1 v_2$ and the associated bilinear form $B(u,v) = u_1 v_2 + u_2 v_1$. For $c_1 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $c_2 = \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)$ both in S_Q, $a,b \in R(c_1)\cap R(c_2)$ and f the spherical polynomial $f(v) = v_1^{k-1}$, we obtain the holomorphic theta series

$$\Theta_{a,b}^{c_1,c_2}(\tau) = \widehat{\widehat{\Theta}}{}_{a,b}^{c_1,c_2}(\tau) = \sum_{\ell\in a+\mathbb{Z}^2} \{\operatorname{sgn}(\ell_1)+\operatorname{sgn}(\ell_2)\}\,\ell_1^{k-1}\,q^{\ell_1\ell_2}\,e^{2\pi i(\ell_1 b_2+\ell_2 b_1)}.$$

By Theorem 5.5 this theta series transforms as follows (A is an even unimodular matrix):

$$\Theta_{a,b}^{c_1,c_2}(\tau+1) = e^{-2\pi i Q(a)}\,\Theta_{a,a+b}^{c_1,c_2}(\tau) \quad\text{and}\quad \Theta_{a,b}^{c_1,c_2}\Big(-\frac{1}{\tau}\Big) = \tau^k\,e^{2\pi i B(a,b)}\,\Theta_{b,-a}^{c_1,c_2}(\tau)$$

Next we consider what happens if we let $a,b\to 0$ (note that $0\notin R(c_1)\cap R(c_2)$).

Lemma 5.8. *The meromorphic function*

$$f_{a,b}^G(\tau) := e^{2\pi i a_2 b_1}\,\frac{(k-1)!}{(2\pi i(a_2\tau+b_2))^k}$$

has the same modular transformation behavior as $\Theta_{a,b}^{c_1,c_2}$ on Γ_1.

Proof. We derive the modular transformation behavior by the following two straightforward calculations: we have

$$f_{a,b}^G(\tau+1) = e^{-2\pi i a_1 a_2}\,e^{2\pi i a_2(a_1+b_1)}\,\frac{(k-1)!}{(2\pi i(a_2\tau+a_2+b_2))^k} = e^{-2\pi i Q(a)}\,f_{a,a+b}^G(\tau)$$

and

$$f_{a,b}^G\Big(-\frac{1}{\tau}\Big) = \tau^k\,e^{2\pi i(a_1 b_2+a_2 b_1)}\,e^{-2\pi i a_1 b_2}\,\frac{(k-1)!}{(2\pi i(b_2\tau-a_2))^k} = \tau^k\,e^{2\pi i B(a,b)}\,f_{b,-a}^G(\tau).$$ □

Lemma 5.9. *We have*

$$G_k(\tau) = \lim_{a,b\to 0}\Big(\frac{1}{4}\Theta_{a,b}^{c_1,c_2}(\tau) - \frac{1}{2}f_{a,b}^G(\tau)\Big) \quad (k \geq 4),$$

and for $k = 2$

$$G_2(\tau) - \frac{1}{4\pi i\tau} = \lim_{a\to 0}\Big(\lim_{b\to 0}\Big(\frac{1}{4}\Theta_{a,b}^{c_1,c_2}(\tau) - \frac{1}{2}f_{a,b}^G(\tau)\Big)\Big),$$
$$G_2(\tau) = \lim_{b\to 0}\Big(\lim_{a\to 0}\Big(\frac{1}{4}\Theta_{a,b}^{c_1,c_2}(\tau) - \frac{1}{2}f_{a,b}^G(\tau)\Big)\Big).$$

Proof. To ensure $a, b \in R(c_1) \cap R(c_2)$, we assume $a_i, b_i \in (-1,0)\cup(0,1)$ for $i = 1,2$. Further, we write $a + \mathbb{Z}^2$ as the disjoint union of L_1, L_2, L_3 with

$$L_1 := \big\{a + \big(\begin{smallmatrix}\ell_1\\ \ell_2\end{smallmatrix}\big)\,|\,\ell_1,\ell_2 \in \mathbb{Z}\setminus\{0\}\big\}, \quad L_2 := \big\{\big(\begin{smallmatrix}a_1+\ell_1\\ a_2\end{smallmatrix}\big)\,|\,\ell_1 \in \mathbb{Z}\setminus\{0\}\big\}$$
$$\text{and} \quad L_3 := \big\{\big(\begin{smallmatrix}a_1\\ a_2+\ell_2\end{smallmatrix}\big)\,|\,\ell_2 \in \mathbb{Z}\big\}.$$

We consider $\frac{1}{4}\Theta_{a,b}^{c_1,c_2}$, where we first restrict the summation to L_1. Letting $a, b \to 0$ we get (for $k \geq 2$ even):

$$\frac{1}{4}\sum_{\substack{\ell\in\mathbb{Z}^2\\ \ell_1,\ell_2\neq 0}} \{\mathrm{sgn}(\ell_1) + \mathrm{sgn}(\ell_2)\}\,\ell_1^{k-1}q^{\ell_1\ell_2} = \sum_{\ell\in\mathbb{N}^2}\ell_1^{k-1}q^{\ell_1\ell_2}$$

Restricting the summation to L_2 we obtain the expression

$$\frac{e^{2\pi i a_2 b_1}}{4}\sum_{\ell_1\in\mathbb{Z}\setminus\{0\}}\{\mathrm{sgn}(a_1+\ell_1) + \mathrm{sgn}(a_2)\}\,(a_1+\ell_1)^{k-1}q^{(a_1+\ell_1)a_2}\,e^{2\pi i(a_1+\ell_1)b_2}.$$

Using Lemma 5.7 (for even k) with $\alpha = a_1$, $\beta = 0$ and $z = 2\pi i(a_2\tau + b_2)$ this equals

$$\frac{e^{2\pi i a_2 b_1}}{2}\Big(\frac{(k-1)!}{(2\pi i(a_2\tau+b_2))^k} - \sum_{m=0}^{\infty}\frac{B_{m+k}(a_1)}{m+k}\frac{(2\pi i(a_2\tau+b_2))^m}{m!}\Big).$$

Note that we have to add the summand for $\ell_1 = 0$ if a_2 is positive in order to apply Lemma 5.7, but as this extra term goes to zero if a goes to zero, we immediately neglect it here. We subtract the non-holomorphic part $\frac{1}{2}f_{a,b}^G$, and for the remaining part we obtain

$$\lim_{a,b\to 0}\Big(\frac{-e^{2\pi i a_2 b_1}}{2}\sum_{m=0}^{\infty}\frac{B_{m+k}(a_1)}{m+k}\frac{(2\pi i(a_2\tau+b_2))^m}{m!}\Big) = -\frac{B_k}{2k}.$$

On L_3 we just use the usual identity for the geometric series and thus have

$$\lim_{a,b\to 0}\Big(\frac{1}{4}q^{a_1a_2}e^{2\pi i(a_1b_2+a_2b_1)}a_1^{k-1}\sum_{\ell_2\in\mathbb{Z}}\{\mathrm{sgn}(a_1)+\mathrm{sgn}(a_2+\ell_2)\}\,e^{2\pi i(a_1\tau+b_1)\ell_2}\Big)$$
$$= \lim_{a,b\to 0}\Big(\frac{1}{2}q^{a_1a_2}e^{2\pi i(a_1b_2+a_2b_1)}a_1^{k-1}\frac{1}{1-q^{a_1}e^{2\pi i b_1}}\Big). \quad (5.5)$$

If a_2 is negative, we again have to add an extra term for $\ell_2 = 0$, which also goes to zero if a goes to zero. If $k \geq 4$, we immediately see that (5.5) is zero, as a_1^{k-1} has a zero of order higher or equal to three and we only have a simple pole. If $k = 2$, it plays a role in which order we take the limit: we have

$$\lim_{b\to 0}\Big(\lim_{a\to 0}\Big(\frac{1}{2}q^{a_1a_2}e^{2\pi i(a_1b_2+a_2b_1)}a_1\frac{1}{1-q^{a_1}e^{2\pi i b_1}}\Big)\Big) = 0$$

since $1 - e^{2\pi i b_1} \neq 0$. On the other hand, we have

$$\lim_{a\to 0}\Big(\lim_{b\to 0}\Big(\frac{1}{2}q^{a_1a_2}e^{2\pi i(a_1b_2+a_2b_1)}a_1\frac{1}{1-q^{a_1}e^{2\pi i b_1}}\Big)\Big) = \frac{1}{2}\lim_{a\to 0}\frac{a_1 q^{a_1a_2}}{1-q^{a_1}} = -\frac{1}{4\pi i\tau}$$

by L'Hospital's rule. Combining the different parts then gives the desired result. □

Using the identities in Lemma 5.9 we can now easily recover the well-known modular transformation properties of G_k.

Theorem 5.10. *For $k \geq 4$ the Eisenstein series G_k is a modular form of weight k on Γ_1: it satisfies $G_k(\tau+1) = G_k(\tau)$ and $G_k(-1/\tau) = \tau^k G_k(\tau)$. Further, G_2 satisfies $G_2(\tau+1) = G_2(\tau)$ and*

$$G_2(\tau) - \frac{1}{\tau^2}G_2\Big(-\frac{1}{\tau}\Big) = \frac{1}{4\pi i\tau}.$$

Proof. For $k \geq 4$ we have

$$\begin{aligned}G_k(\tau+1) &= \lim_{a,b\to 0}\Big(\Big(\frac{1}{4}\Theta^{c_1,c_2}_{a,b} - \frac{1}{2}f^G_{a,b}\Big)(\tau+1)\Big)\\ &= \lim_{a,b\to 0}\Big(e^{-2\pi i Q(a)}\Big(\frac{1}{4}\Theta^{c_1,c_2}_{a,a+b} - \frac{1}{2}f^G_{a,a+b}\Big)(\tau)\Big) = G_k(\tau)\end{aligned}$$

and

$$\begin{aligned}G_k\Big(-\frac{1}{\tau}\Big) &= \lim_{a,b\to 0}\Big(\Big(\frac{1}{4}\Theta^{c_1,c_2}_{a,b} - \frac{1}{2}f^G_{a,b}\Big)\Big(-\frac{1}{\tau}\Big)\Big)\\ &= \lim_{a,b\to 0}\Big(\tau^k e^{2\pi i B(a,b)}\Big(\frac{1}{4}\Theta^{c_1,c_2}_{b,-a} - \frac{1}{2}f^G_{b,-a}\Big)(\tau)\Big) = \tau^k G_k(\tau),\end{aligned}$$

which shows that G_k is a holomorphic modular form of weight k on Γ_1.

For $k = 2$ we get $G_2(\tau+1) = G_2(\tau)$ using the same identity as for $k \geq 4$, when we obey the order of the limits and take $\lim_{b\to 0}\lim_{a\to 0}$. Further,

$$\begin{aligned}\frac{1}{\tau^2}G_2\Big(-\frac{1}{\tau}\Big) &= \frac{1}{\tau^2}\lim_{b\to 0}\Big(\lim_{a\to 0}\Big(\Big(\frac{1}{4}\Theta^{c_1,c_2}_{a,b} - \frac{1}{2}f^G_{a,b}\Big)\Big(-\frac{1}{\tau}\Big)\Big)\Big)\\ &= \lim_{b\to 0}\Big(\lim_{a\to 0}\Big(e^{2\pi i B(a,b)}\Big(\frac{1}{4}\Theta^{c_1,c_2}_{b,-a} - \frac{1}{2}f^G_{b,-a}\Big)(\tau)\Big)\Big) = G_2(\tau) - \frac{1}{4\pi i\tau}.\end{aligned}$$ □

5.4.2 Quadratic Polynomials

In this example, we consider two modular forms of weight $k + 1/2$ that were discussed by Zagier in [Zag99]. One considers the quadratic polynomial ax^2+bx+c with $a, b, c \in \mathbb{Z}$ and discriminant $D := b^2 - 4ac$. In the following, B_k denotes the k-th Bernoulli polynomial and $\overline{B}_k(x) := B_k(x - \lfloor x \rfloor)$ the periodic version of the Bernoulli polynomial.

For $k \in \mathbb{N}$ even and D not a square, let

$$P_{k,D}(x) := \sum_{\substack{b^2-4ac=D \\ a>0>c}} (ax^2+bx+c)^{k-1} \in \mathbb{Z}[x].$$

For $D = m^2 > 0$, we define the right-hand side as P^*_{k,m^2} and set

$$P_{k,m^2}(x) = P^*_{k,m^2}(x) + \frac{1}{k}\Big(B_k(mx) - x^{2k-2}B_k\big(\frac{m}{x}\big)\Big).$$

For $D = 0$ we simply define

$$P_{k,0}(x) = (1-x^{2k-2})\frac{B_k}{2k}.$$

Analogously, we define

$$F_{k,D}(x) := \sum_{\substack{b^2-4ac=D \\ ax^2+bx+c>0>a}} (ax^2+bx+c)^{k-1} \in \mathbb{Z}[x]$$

and

$$\begin{aligned} F_{k,m^2}(x) &= F^*_{k,m^2}(x) - \frac{1}{k}\overline{B}_k(mx) + \delta_{k,2}\frac{m^2\kappa(x)}{2} \\ &\qquad \text{with } \kappa(x) = \begin{cases} \frac{1}{s^2} & \text{for } x = \frac{r}{s} \in \mathbb{Q}\,(\gcd(r,s)=1), \\ 0 & \text{for } x \in \mathbb{R}\setminus\mathbb{Q}, \end{cases} \\ F_{k,0}(x) &= -\frac{B_k}{2k}. \end{aligned}$$

Summing over all positive discriminants, we consider the generating functions

$$S_x(\tau) := \sum_{D\geq 0} P_{k,D}(x)\,q^D \quad \text{and} \quad T_x(\tau) := \sum_{D\geq 0} F_{k,D}(x)\,q^D \qquad (x\in\mathbb{R})$$

and by plugging in the definition of $P_{k,D}$ and $F_{k,D}$ we obtain the expansions

$$S_x(\tau) = \sum_{\substack{(a,b,c)\in\mathbb{Z}^3 \\ a>0>c}} (ax^2+bx+c)^{k-1}\,q^{b^2-4ac} + \frac{1}{2k}\sum_{m=-\infty}^{\infty}\Big(B_k(mx) - x^{2k-2}B_k\big(\frac{m}{x}\big)\Big)\,q^{m^2}$$

and

$$T_x(\tau) = \sum_{\substack{(a,b,c)\in\mathbb{Z}^3 \\ ax^2+bx+c>0>a}} (ax^2+bx+c)^{k-1}\,q^{b^2-4ac} - \frac{1}{2k}\sum_{m=-\infty}^{\infty}\overline{B}_k(mx)\,q^{m^2} + \delta_{k,2}\frac{\kappa(x)}{2}\sum_{m=1}^{\infty} m^2\,q^{m^2}.$$

The goal here is to recover the results of Zagier [Zag99] on the modularity of these functions by using these expansions and the theory of theta series for quadratic forms of signature $(n-1,1)$.

Theorem 5.11. *We have:*

1. *For $x \in \mathbb{R}$, S_x is a modular form of weight $k + 1/2$ on $\Gamma_0(4)$;*

2. *For $x \in \mathbb{Q}$, T_x is a modular form of weight $k + 1/2$ on $\Gamma_0(4)$.*

The indefinite theta function that is associated to these two functions is constructed as follows. The matrix $A = \left(\begin{smallmatrix} 0 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 0 \end{smallmatrix}\right)$ defines the quadratic form $Q(n) = n_2^2 - 4n_1n_3$ of signature $(2, 1)$. Further, let $k \in \mathbb{N}$ be even and choose the polynomial $f(n) = (n_1x^2 + n_2x + n_3)^{k-1}$, which is spherical of degree $k-1$ with respect to Q. While these parameters stay the same when considering S_x and T_x, we choose different elements $c_1, c_2 \in S_Q$. Note that the definition of the cone is independent of the choice of x for S_x, while it plays a role when we consider T_x (this is also the reason why we only allow rational parameters x in the last case). For the characteristics $a, b \in R(c_1) \cap R(c_2)$, we introduce the holomorphic theta series

$$\Theta_{a,b}^{c_1,c_2}[f](\tau) = \sum_{n' \in a+\mathbb{Z}^3} \{\operatorname{sgn}\big(B(n', c_1)\big) - \operatorname{sgn}\big(B(n', c_2)\big)\} \cdot (n_1'x^2 + n_2'x + n_3')^{k-1}\, q^{Q(n')}\, e^{2\pi i B(n', b)}. \quad (5.6)$$

Further, we consider (a, b) as the 3×2-matrix with the two column vectors a and b and define the meromorphic function

$$f_{(a,b)}(\tau) := -\frac{(k-1)!}{(8\pi i)^k} \frac{1}{(a_1\tau + b_1)^k}\, e^{-8\pi i a_1 b_3} \sum_{m \in a_2+\mathbb{Z}} q^{m^2}\, e^{4\pi i b_2 m}.$$

Also we consider the unary theta function

$$\vartheta(\tau) := \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 \tau}. \quad (5.7)$$

The connection with S_x and T_x is then given by:

Lemma 5.12. *Let*

$$\widetilde{a} = (a_1, 2a_1x + a_2, a_1x^2 + a_2x + a_3), \quad \widetilde{b} = (b_1, 2b_1x + b_2, b_1x^2 + b_2x + b_3)$$

and

$$\widehat{a} = (a_3, 2a_3x^{-1} + a_2, a_3x^{-2} + a_2x^{-1} + a_1), \quad \widehat{b} = (b_3, 2b_3x^{-1} + b_2, b_3x^{-2} + b_2x^{-1} + b_1).$$

(a) Let $c_1 = -\frac{1}{4}\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$ and $c_2 = -\frac{1}{4}\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, both in S_Q. Then we have

$$S_x(\tau) = \frac{1}{2} \lim_{a,b \to 0} \Big(\frac{1}{2}\Theta_{a,b}^{c_1,c_2}[f](\tau) - f_{(\widetilde{a},\widetilde{b})}(\tau) + x^{2k-2}\, f_{(\widehat{a},\widehat{b})}(\tau)\Big).$$

(b) Let $c_1 = -\frac{1}{4}\left(\begin{smallmatrix} 1 \\ -2x \\ x^2 \end{smallmatrix}\right)$ and $c_2 = -\frac{1}{4}\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$, both in S_Q. Then we have

$$T_x(\tau) = \frac{1}{2} \lim_{a,b \to 0} \Big(\frac{1}{2}\Theta_{a,b}^{c_1,c_2}[f](\tau) + f_{(\widetilde{a},\widetilde{b})}(\tau)\Big)$$

for $k \geq 4$. Let $(a', b') = (a, b)g$, with $g = \left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right)$ (g_{21} and g_{22} not both zero). Then we *have*

$$T_x(\tau) = \frac{1}{2}\lim_{b\to 0}\lim_{a\to 0}\Big(\frac{1}{2}\Theta^{c_1,c_2}_{a',b'}[f](\tau) + f_{(\widetilde{a}',\widetilde{b}')}(\tau)\Big) + \frac{\kappa(x)}{8\pi i}\left(\frac{1}{2}\frac{g_{21}}{g_{21}\tau+g_{22}}\,\vartheta(\tau)+\vartheta'(\tau)\right)$$

for $k=2$.

To establish the modular behavior of S_x and T_x we first consider the modular behavior of $\Theta^{c_1,c_2}_{a,b}[f]$ and $f_{(a,b)}$. For this we consider modular substitutions on $\Gamma_0(4)$ and make use of Shimura's definition (see [Shi73]) of the automorphic factor $j(\gamma,\tau) := \vartheta(\gamma\tau)/\vartheta(\tau)$ for any $\gamma\in\Gamma_0(4)$. As the modular substitutions alter the characteristics, we introduce the modified characteristics a', b' as the one satisfying $(a',b') := (a,b)\gamma$.

In the next lemma we show that $f_{(a,b)}$ and the theta function $\Theta^{c_1,c_2}_{a,b}[f]$ defined in (5.6) have the same modular transformation behavior on $\Gamma_0(4)$:

Lemma 5.13. *We have*

$$\Theta^{c_1,c_2}_{a,b}[f](\gamma\tau) = j(\gamma,\tau)^{2k+1}\,e^{\pi i B(a,b)-\pi i B(a',b')}\,\Theta^{c_1,c_2}_{a',b'}[f](\tau) \tag{5.8}$$

and

$$f_{(a,b)}(\gamma\tau) = j(\gamma,\tau)^{2k+1}\,e^{\pi i B(a,b)-\pi i B(a',b')}\,f_{(a',b')}(\tau) \tag{5.9}$$

for all $\gamma\in\Gamma_0(4)$.

Proof of Theorem 5.11. We note that the maps $(a,b)\mapsto(a',b')=(a,b)\gamma$ and $(a,b)\mapsto(\widetilde{a},\widetilde{b})$ commute and that $B(\widetilde{a}',\widetilde{b}') = B(a',b')$ holds. Similarly this holds for $\widehat{a}$ and $\widehat{b}$. Hence from Lemma 5.12 and Lemma 5.13 we directly get

$$\begin{aligned}
S_x(\gamma\tau) &= \frac{1}{2}\lim_{a,b\to 0}\Big(\frac{1}{2}\Theta^{c_1,c_2}_{a,b}[f](\gamma\tau) - f_{(\widetilde{a},\widetilde{b})}(\gamma\tau) + x^{2k-2}\,f_{(\widehat{a},\widehat{b})}(\gamma\tau)\Big)\\
&= j(\gamma,\tau)^{2k+1}\,\frac{1}{2}\lim_{a,b\to 0} e^{\pi i B(a,b)-\pi i B(a',b')}\Big(\frac{1}{2}\Theta^{c_1,c_2}_{a',b'}[f](\tau) - f_{(\widetilde{a}',\widetilde{b}')}(\tau) + x^{2k-2}\,f_{(\widehat{a}',\widehat{b}')}(\tau)\Big)\\
&= j(\gamma,\tau)^{2k+1}\,S_x(\tau)
\end{aligned}$$

for all $\gamma\in\Gamma_0(4)$. In exactly the same way we obtain that for $k\geq 4$

$$T_x(\gamma\tau) = j(\gamma,\tau)^{2k+1}\,T_x(\tau)$$

holds for all $\gamma\in\Gamma_0(4)$. For $k=2$ we use Lemma 5.12 twice (first with $g=I$ and then with $g=\gamma=\left(\begin{smallmatrix}\gamma_{11}&\gamma_{12}\\ \gamma_{21}&\gamma_{22}\end{smallmatrix}\right)\in\Gamma_0(4)$) and find

$$\begin{aligned}
T_x(\gamma\tau) &= \frac{1}{2}\lim_{b\to 0}\lim_{a\to 0}\Big(\frac{1}{2}\Theta^{c_1,c_2}_{a,b}[f](\gamma\tau) + f_{(\widetilde{a},\widetilde{b})}(\gamma\tau)\Big) + \frac{\kappa(x)}{8\pi i}\,\vartheta'(\gamma\tau)\\
&= j(\gamma,\tau)^5\,\frac{1}{2}\lim_{b\to 0}\lim_{a\to 0} e^{\pi i B(a,b)-\pi i B(a',b')}\Big(\frac{1}{2}\Theta^{c_1,c_2}_{a',b'}[f](\tau) + f_{(\widetilde{a}',\widetilde{b}')}(\tau)\Big) + \frac{\kappa(x)}{8\pi i}\,\vartheta'(\gamma\tau)\\
&= j(\gamma,\tau)^5\,T_x(\tau) + \frac{\kappa(x)}{8\pi i}\left(\vartheta'(\gamma\tau) - j(\gamma,\tau)^5\left(\frac{1}{2}\frac{\gamma_{21}}{\gamma_{21}\tau+\gamma_{22}}\,\vartheta(\tau)+\vartheta'(\tau)\right)\right).
\end{aligned}$$

It is known that

$$j(\gamma,\tau)^2 = \Big(\frac{-1}{\gamma_{22}}\Big)(\gamma_{21}\tau+\gamma_{22}),$$

from which we get

$$\frac{j'(\gamma,\tau)}{j(\gamma,\tau)} = \frac{1}{2}\frac{\gamma_{21}}{\gamma_{21}\tau+\gamma_{22}},$$

where j' denotes the derivative of j with respect to τ. Using $\vartheta(\gamma\tau) = j(\gamma,\tau)\vartheta(\tau)$ and $j(\gamma,\tau)^4\frac{\partial}{\partial\tau}(\gamma\tau) = 1$, we then obtain

$$\begin{aligned} T_x(\gamma\tau) &= j(\gamma,\tau)^5 T_x(\tau) + \frac{\kappa(x)}{8\pi i}\left(\vartheta'(\gamma\tau) - j(\gamma,\tau)^5\left(\frac{j'(\gamma,\tau)}{j(\gamma,\tau)}\vartheta(\tau) + \vartheta'(\tau)\right)\right) \\ &= j(\gamma,\tau)^5 T_x(\tau) + \frac{\kappa(x)}{8\pi i}\left(\vartheta'(\gamma\tau) - j(\gamma,\tau)^4\frac{\partial}{\partial\tau}\vartheta(\gamma\tau)\right) = j(\gamma,\tau)^5 T_x(\tau), \end{aligned}$$

as desired.

Also note that $S_x|\gamma$ and $T_x|\gamma$ are holomorphic at ∞ for all $\gamma\in\mathrm{SL}_2(\mathbb{Z})$: First of all, we know that this holds for ϑ and ϑ'. Since we can deduce from Theorem 2.5(4) and (5) (analogously to Lemma 3.7 in [RZ21]) that this also holds for $\Theta_{a,b}^{c_1,c_2}[f]$ if $a,b\in R(c_1)\cap R(c_2)$ and one can show a similar relation as in the aforementioned lemma for $f_{(a,b)}$, we conclude that S_x and T_x satisfy the desired growth conditions at the cusps of $\Gamma_0(2)$. □

Proof of Lemma 5.12. We consider $\frac{1}{2}\Theta_{a,b}^{c_1,c_2}[f]$, with the theta series as in (5.6). To obtain the formula for S_x we set $c_1 = -\frac{1}{4}\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$ and $c_2 = -\frac{1}{4}\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$. Writing $n' = a + \left(\begin{smallmatrix}n_1\\n_2\\n_3\end{smallmatrix}\right)$ with $n_i\in\mathbb{Z}$, this series still converges absolutely if we restrict the summation to the part of the lattice $\mathbb{Z}^3$ for which $n_1\neq 0$ and $n_3\neq 0$ holds and let $a,b\to 0$. It doesn't play a role whether we consider the limit of a or of b first, in any case the partial sum asymptotes to

$$\left(\sum_{\substack{n\in\mathbb{Z}^3\\ n_1>0>n_3}} - \sum_{\substack{n\in\mathbb{Z}^3\\ n_1<0<n_3}}\right)(n_1x^2+n_2x+n_3)^{k-1}q^{Q(n)} = 2\sum_{\substack{n\in\mathbb{Z}^3\\ n_1>0>n_3}}(n_1x^2+n_2x+n_3)^{k-1}q^{Q(n)},$$

substituting $n\mapsto -n$ in the second sum and using the fact that $k-1$ is odd.

We fix $a = \left(\begin{smallmatrix}a_1\\a_2\\a_3\end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix}b_1\\b_2\\b_3\end{smallmatrix}\right)$, where for simplicity we will only consider the case that $a_1, a_3\in(0,1)$. The other cases can be dealt with in a similar manner. In our case we then have $B(c_i,a)>0$. We first investigate the series in (5.6) on the part where $n_1=0$ holds: note that

$$\mathrm{sgn}\big(B(n',c_1)\big) - \mathrm{sgn}\big(B(n',c_2)\big) = \mathrm{sgn}(a_1) - \mathrm{sgn}(a_3+n_3)$$

equals 2 for strictly negative n_3 and vanishes otherwise, so this partial sum equals

$$\sum_{n_2\in\mathbb{Z}}\sum_{n_3=-\infty}^{-1}(a_1x^2+a_3+(a_2+n_2)x+n_3)^{k-1}q^{(a_2+n_2)^2-4a_1(a_3+n_3)} \cdot e^{-8\pi i(b_1(a_3+n_3)+a_1b_3)+4\pi i b_2(a_2+n_2)}. \tag{5.10}$$

We set $z_1 := a_1\tau+b_1$ and write (5.10) as

$$e^{-8\pi i a_1b_3+8\pi i z_1a_1x^2}\sum_{n_2\in\mathbb{Z}}q^{(a_2+n_2)^2}e^{8\pi i z_1(a_2+n_2)x}e^{4\pi i b_2(a_2+n_2)} \cdot\sum_{n_3=-\infty}^{-1}(a_1x^2+a_3+(a_2+n_2)x+n_3)^{k-1}e^{-8\pi i z_1(a_1x^2+a_3+(a_2+n_2)x+n_3)}.$$

We make use of Lemma 5.7 with $\alpha = a_1x^2 + a_3 + (a_2+n_2)x$, $\beta = 0$ and $z = -8\pi i z_1$ to rewrite the sum over n_3 as

$$\frac{(-1)^{k-1}(k-1)!}{(-8\pi i z_1)^k} + \sum_{m=0}^{\infty} \frac{B_{m+k}(a_1x^2 + a_3 + (a_2+n_2)x)}{m+k} \frac{(-8\pi i z_1)^m}{m!}.$$

First, we consider the second summand, which contains no poles for $a, b \to 0$: if we let $a, b \to 0$, this part goes to $B_k(n_2x)/k$, as $(-8\pi i z_1)^m \to 0$ whenever $m > 0$. So for the "regular" part in (5.10), we obtain $\frac{1}{k}\sum_{m\in\mathbb{Z}} B_k(mx)\, q^{m^2}$ for $a, b \to 0$.

We define the remaining part of (5.10), using that k is even, as

$$\begin{aligned} f(a,b,x;\tau) &:= -e^{-8\pi i a_1 b_3 + 8\pi i z_1 a_1 x^2} \frac{(k-1)!}{(8\pi i z_1)^k} \sum_{m\in a_2+\mathbb{Z}} q^{m^2} e^{8\pi i m x z_1 + 4\pi i b_2 m} \\ &= -e^{-8\pi i a_1(b_1x^2+b_2x+b_3)} \frac{(k-1)!}{(8\pi i z_1)^k} \sum_{m\in 2a_1x+a_2+\mathbb{Z}} q^{m^2} e^{4\pi i(2b_1x+b_2)m}. \end{aligned}$$

If we now consider the series in (5.6) on the part of the sum where $n_3 = 0$ holds we have

$$\begin{aligned} -\sum_{n_2\in\mathbb{Z}} \sum_{n_1=-\infty}^{-1} ((a_1+n_1)x^2 + a_3 + (a_2+n_2)x)^{k-1} q^{(a_2+n_2)^2 - 4a_3(a_1+n_1)} \\ \cdot e^{-8\pi i(b_1a_3 + b_3(a_1+n_1)) + 4\pi i b_2(a_2+n_2)}, \end{aligned}$$

which equals

$$\begin{aligned} -x^{2k-2} \sum_{n_2\in\mathbb{Z}} \sum_{n_1=-\infty}^{-1} \Big(a_1 + n_1 + \frac{a_3}{x^2} + \frac{a_2+n_2}{x}\Big)^{k-1} q^{(a_2+n_2)^2 - 4a_3(a_1+n_1)} \\ \cdot e^{-8\pi i(b_1a_3 + b_3(a_1+n_1)) + 4\pi i b_2(a_2+n_2)}. \end{aligned}$$

As this series is (up to the factor $-x^{2k-2}$) just (5.10) for $1/x$ instead of x interchanging a_1 and a_3 and b_1 and b_3, respectively, we obtain the function $-x^{2k-2} f\Big(\Big(\begin{smallmatrix} a_3 \\ a_2 \\ a_1 \end{smallmatrix}\Big), \Big(\begin{smallmatrix} b_3 \\ b_2 \\ b_1 \end{smallmatrix}\Big), \frac{1}{x}; \tau\Big)$ and as "regular" part

$$-x^{2k-2} \frac{1}{k} \sum_{m=-\infty}^{\infty} B_k\Big(\frac{m}{x}\Big) q^{m^2}.$$

Combining these results for the different parts of the lattice $\mathbb{Z}^3$ we then get

$$\begin{aligned} &\lim_{a,b\to 0} \Big(\frac{1}{2}\Theta_{a,b}^{c_1,c_2}(\tau) - f(a,b,x;\tau) + x^{2k-2} f\Big(\Big(\begin{smallmatrix} a_3 \\ a_2 \\ a_1 \end{smallmatrix}\Big), \Big(\begin{smallmatrix} b_3 \\ b_2 \\ b_1 \end{smallmatrix}\Big), \frac{1}{x}; \tau\Big)\Big) \\ &\quad = 2 \sum_{\substack{n\in\mathbb{Z}^3 \\ n_1>0>n_3}} (n_1x^2 + n_2x + n_3)^{k-1} q^{Q(n)} + \frac{1}{k} \sum_{m=-\infty}^{\infty} \Big(B_k(mx) - x^{2k-2} B_k\Big(\frac{m}{x}\Big)\Big) q^{m^2} \\ &\quad = 2S_x(\tau). \end{aligned}$$

In terms of $\widetilde{a}$ and $\widetilde{b}$ we have $f(a,b,x;\tau) = f_{(\widetilde{a},\widetilde{b})}(\tau)$ and similarly $f\Big(\Big(\begin{smallmatrix} a_3 \\ a_2 \\ a_1 \end{smallmatrix}\Big), \Big(\begin{smallmatrix} b_3 \\ b_2 \\ b_1 \end{smallmatrix}\Big), \frac{1}{x}; \tau\Big) = f_{(\widehat{a},\widehat{b})}(\tau)$. This gives the desired result for S_x.

To get the formula for T_x we now set $c_1 = -\frac{1}{4}\left(\begin{smallmatrix} 1 \\ -2x \\ x^2 \end{smallmatrix}\right)$ and $c_2 = -\frac{1}{4}\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$. We then have

$$\begin{aligned}\frac{1}{2}\Theta_{a,b}^{c_1,c_2}[f](\tau) = \frac{1}{2}\sum_{n\in\mathbb{Z}^3}\big\{\operatorname{sgn}(n_1x^2+n_2x+n_3+\widetilde{a}_3) - \operatorname{sgn}(n_1+a_1)\big\}\\ \cdot (n_1x^2+n_2x+n_3+\widetilde{a}_3)^{k-1}\, q^{(n_2+a_2)^2-4(n_1+a_1)(n_3+a_3)}\\ \cdot e^{4\pi i((n_2+a_2)b_2-2(n_1+a_1)b_3-2(n_3+a_3)b_1)}.\end{aligned}$$

Similar to the previous case we assume $\widetilde{a}_3, a_1 \in (0,1)$ for simplicity and first consider the part of $\mathbb{Z}^3$ where $n_1x^2+n_2x+n_3 \neq 0$ and $n_1 \neq 0$ hold. Letting $a, b \to 0$ we then obtain

$$\begin{aligned}\Bigg(\sum_{\substack{n\in\mathbb{Z}^3\\ n_1x^2+n_2x+n_3>0>n_1}} - \sum_{\substack{n\in\mathbb{Z}^3\\ n_1x^2+n_2x+n_3<0<n_1}}\Bigg)(n_1x^2+n_2x+n_3)^{k-1}\, q^{Q(n)}\\ = 2\sum_{\substack{n\in\mathbb{Z}^3\\ n_1x^2+n_2x+n_3>0>n_1}}(n_1x^2+n_2x+n_3)^{k-1}\, q^{Q(n)}.\end{aligned}$$

For the part of $\mathbb{Z}^3$ where $n_1 = 0$ holds we get

$$\begin{aligned}&-\sum_{\substack{n_2,n_3\in\mathbb{Z}\\ n_2x+n_3<0}}(n_2x+n_3+\widetilde{a}_3)^{k-1}\, q^{(n_2+a_2)^2-4a_1(n_3+a_3)}\, e^{4\pi i((n_2+a_2)b_2-2a_1b_3-2(n_3+a_3)b_1)}\\ &\quad= -e^{-8\pi i a_1\widetilde{b}_3}\sum_{\substack{n_2,n_3\in\mathbb{Z}\\ n_2x+n_3<0}}(n_2x+n_3+\widetilde{a}_3)^{k-1}\, q^{(n_2+\widetilde{a}_2)^2}\, e^{4\pi i(n_2+\widetilde{a}_2)\widetilde{b}_2}\, e^{-8\pi i z_1(n_2x+n_3+\widetilde{a}_3)},\end{aligned}$$

where again $z_1 = a_1\tau + b_1$. Using Lemma 5.7 with $\alpha = \widetilde{a}_3$, $\beta = n_2x$ and $z = -8\pi i z_1$, we get

$$\begin{aligned}-\sum_{\substack{n_3\in\mathbb{Z}\\ n_2x+n_3<0}}(n_2x+n_3+\widetilde{a}_3)^{k-1}\, e^{-8\pi i z_1(n_2x+n_3+\widetilde{a}_3)}\\ = \frac{(k-1)!}{(8\pi i z_1)^k} - \sum_{m=0}^{\infty}\frac{B_{m+k}(\widetilde{a}_3+n_2x-\lfloor n_2x\rfloor)}{m+k}\frac{(-8\pi i z_1)^m}{m!}\end{aligned}$$

so for the “regular” part with $a, b \to 0$ we have

$$-\sum_{n_2\in\mathbb{Z}} q^{n_2^2}\frac{B_k(n_2x-\lfloor n_2x\rfloor)}{k} = -\frac{1}{k}\sum_{m\in\mathbb{Z}}\overline{B}_k(mx)\, q^{m^2}.$$

The remaining part is

$$e^{-8\pi i a_1\widetilde{b}_3}\frac{(k-1)!}{(8\pi i z_1)^k}\sum_{n_2\in\mathbb{Z}} q^{(n_2+\widetilde{a}_2)^2}\, e^{4\pi i(n_2+\widetilde{a}_2)\widetilde{b}_2} = -f(a,b,x;\tau) = -f_{(\widetilde{a},\widetilde{b})}(\tau).$$

Finally, we consider the part of $\mathbb{Z}^3$ where $n_1x^2+n_2x+n_3=0$ holds and get

$$\begin{aligned}\widetilde{a}_3^{k-1}\sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}\\ n_1x^2+n_2x+n_3=0>n_1}} q^{Q(n+a)}\, e^{2\pi i B(n+a,b)} \\ = \widetilde{a}_3^{k-1} q^{Q(a)}\, e^{2\pi i B(a,b)} \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}\\ n_1x^2+n_2x+n_3=0>n_1}} q^{Q(n)}\, e^{2\pi i B(n,z)},\end{aligned}$$

with $z := a\tau+b$. We write $x=\frac{r}{s}$ with $s\in\mathbb{N}$, $\gcd(r,s)=1$ and consider the substitution

$$(m_1,m_2,m_3) = (\frac{1}{s}n_1, 2\frac{r}{s}n_1+n_2, \frac{r^2}{s}n_1+rn_2+sn_3).$$

It gives a bijection between the sets $\{n\in\mathbb{Z}^3 \mid n_1\equiv 0 \bmod s\}$ and $\{m\in\mathbb{Z}^3 \mid r^2m_1 - rm_2+m_3\equiv 0 \bmod s\}$. We note that if $n_1x^2+n_2x+n_3=0$ holds, then we have $n_1\equiv 0 \bmod s$. Further, we have $Q(n)=Q(m)$ and $B(n,z) = 2\widetilde{z}_2m_2 - 4\frac{1}{s}\widetilde{z}_1m_3 - 4s\widetilde{z}_3m_1$, with $\widetilde{z} = (z_1, 2z_1x+z_2, z_1x^2+z_2x+z_3)$, so with this substitution we obtain

$$\begin{aligned}&\widetilde{a}_3^{k-1}q^{Q(a)}\, e^{2\pi i B(a,b)} \sum_{\substack{m_1,m_2,m_3\in\mathbb{Z}\\ m_3=0>m_1,\ r^2m_1-rm_2+m_3\equiv 0 \bmod s}} q^{Q(m)}\, e^{4\pi i(\widetilde{z}_2m_2-2\frac{1}{s}\widetilde{z}_1m_3-2s\widetilde{z}_3m_1)} \\ &= \widetilde{a}_3^{k-1}q^{Q(a)}\, e^{2\pi i B(a,b)} \sum_{m_2\in\mathbb{Z}} q^{m_2^2}\, e^{4\pi i\widetilde{z}_2m_2} \sum_{\substack{m_1\in\mathbb{Z}\\ m_1<0,\ m_1\equiv r^*m_2 \bmod s}} e^{-8\pi i s\widetilde{z}_3m_1} \qquad (5.11)\\ &= \frac{\widetilde{a}_3^{k-1}q^{Q(a)}\, e^{2\pi i B(a,b)}}{e^{-8\pi i\widetilde{z}_3s^2}-1} \sum_{m_2\in\mathbb{Z}} q^{m_2^2}\, e^{4\pi i\widetilde{z}_2m_2 - 8\pi i\widetilde{z}_3s^2(\frac{r^*}{s}m_2-\lfloor\frac{r^*}{s}m_2\rfloor)},\end{aligned}$$

where r^* is the multiplicative inverse of r modulo s and we used the geometric series in the last step. As in the proof of Lemma 5.9 this part vanishes for $k\geq 4$ if we let $a,b\to 0$. Combining the results for the different parts we thus find for $k\geq 4$:

$$\begin{aligned}&\lim_{a,b\to 0}\Big(\frac{1}{2}\Theta_{a,b}^{c_1,c_2}[f](\tau) + f_{(\widetilde{a},\widetilde{b})}(\tau)\Big) \\ &= 2\sum_{\substack{n\in\mathbb{Z}^3\\ n_1x^2+n_2x+n_3>0>n_1}} (n_1x^2+n_2x+n_3)^{k-1}\, q^{Q(n)} - \frac{1}{k}\sum_{m=-\infty}^{\infty} \overline{B}_k(mx)\, q^{m^2} = 2T_x(\tau),\end{aligned}$$

which gives the desired result. For $k=2$ we have to be careful about the order of the limits. We denote the last expression in (5.11) by $g_{a,b}(\tau)$ and consider

$$\lim_{b\to 0}\lim_{a\to 0} g_{a',b'}(\tau),$$

where $(a', b') = (a, b)g$, with $g = \left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right)$ (g_{21} and g_{22} not both zero). We find

$$\begin{aligned}
\lim_{b\to 0}\lim_{a\to 0} g_{a',b'}(\tau) &= \sum_{m_2\in\mathbb{Z}} q^{m_2^2} \cdot \lim_{b\to 0}\lim_{a\to 0} \frac{\widetilde{a}'_3}{e^{-8\pi i \widetilde{z}'_3 s^2} - 1} \\
&= \sum_{m_2\in\mathbb{Z}} q^{m_2^2} \cdot \lim_{b\to 0}\lim_{a\to 0} \frac{g_{11}\widetilde{a}_3 + g_{21}\widetilde{b}_3}{e^{-8\pi i((g_{11}\widetilde{a}_3+g_{21}\widetilde{b}_3)\tau + g_{12}\widetilde{a}_3 + g_{22}\widetilde{b}_3)s^2} - 1} \\
&= \sum_{m_2\in\mathbb{Z}} q^{m_2^2} \cdot \lim_{b\to 0} \frac{g_{21}\widetilde{b}_3}{e^{-8\pi i(g_{21}\tau+g_{22})\widetilde{b}_3 s^2} - 1} \\
&= -\frac{1}{8\pi i}\frac{1}{s^2}\frac{g_{21}}{g_{21}\tau + g_{22}} \sum_{m_2\in\mathbb{Z}} q^{m_2^2} = -\frac{\kappa(x)}{8\pi i}\frac{g_{21}}{g_{21}\tau + g_{22}}\vartheta(\tau).
\end{aligned}$$

Combining the results for the different parts we thus find for $k = 2$:

$$\begin{aligned}
&\lim_{b\to 0}\lim_{a\to 0}\left(\frac{1}{2}\Theta^{c_1,c_2}_{a',b'}[f](\tau) + f_{(\widetilde{a}',\widetilde{b}')}(\tau)\right) \\
&\quad = 2 \sum_{\substack{n\in\mathbb{Z}^3 \\ n_1x^2+n_2x+n_3>0>n_1}} (n_1x^2 + n_2x + n_3)\, q^{Q(n)} \\
&\qquad\qquad - \frac{1}{2}\sum_{m=-\infty}^{\infty} \overline{B}_2(mx)\, q^{m^2} - \frac{\kappa(x)}{8\pi i}\frac{g_{21}}{g_{21}\tau + g_{22}}\vartheta(\tau) \\
&\quad = 2T_x(\tau) - \frac{\kappa(x)}{8\pi i}\frac{g_{21}}{g_{21}\tau + g_{22}}\vartheta(\tau) - \kappa(x)\sum_{m=1}^{\infty} m^2 q^{m^2} \\
&\quad = 2T_x(\tau) - \frac{\kappa(x)}{4\pi i}\left(\frac{1}{2}\frac{g_{21}}{g_{21}\tau + g_{22}}\vartheta(\tau) + \vartheta'(\tau)\right),
\end{aligned}$$

which gives the desired result for this case. □

Proof of Lemma 5.13. We can easily see that (5.8) and (5.9) are compatible with matrix multiplication. Hence we only have to show them for the generators $-I, T$ and $\gamma' = \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ of $\Gamma_0(4)$. Note that $j(-I, \tau) = j(T, \tau) = 1$ and $j(\gamma', \tau)^2 = 4\tau + 1$. We also note that $(a, b)(-I) = -(a, b)$, $(a, b)T = (a, a + b)$ and $(a, b)\gamma' = (a + 4b, b)$.

For the sake of better readability we write $\Theta_{(a,b)}$ instead of $\Theta^{c_1,c_2}_{a,b}[f]$. By Theorem 5.5(3), $\Theta_{-(a,b)} = \Theta_{(a,b)}$ holds since the degree of f is odd. Thus we have

$$\Theta_{(a,b)}(-I\tau) = \Theta_{(a,b)}(\tau) = \Theta_{-(a,b)}(\tau),$$

which is (5.8) for $\gamma = -I$. From Theorem 5.5(4) we obtain

$$\Theta_{(a,b)}(\tau + 1) = e^{-2\pi i Q(a)}\,\Theta_{(a,a+b)}(\tau),$$

which is (5.8) for $\gamma = T$. As a result, we also have

$$\Theta_{(a,b)}(\tau - 4) = e^{8\pi i Q(a)}\,\Theta_{(a,-4a+b)}(\tau). \tag{5.12}$$

By Theorem 5.5(5) we have

$$\Theta_{(a,b)}\Big(-\frac{1}{\tau}\Big) = (-i\tau)^{k+1/2}\,\frac{i^k}{4\sqrt{2}}\,e^{2\pi i B(a,b)} \sum_{p\in A^{-1}\mathbb{Z}^3/\mathbb{Z}^3} \Theta_{(b+p,-a)}(\tau).$$

To keep track of the choice of the square root, we write $\sqrt{-i\tau}$ as

$$\sqrt{2}\,\frac{\vartheta(-1/\tau)}{\vartheta(\tau/4)},$$

where ϑ is defined as in (5.7). Hence we have

$$\frac{\Theta_{(a,b)}}{\vartheta^{2k+1}}\Big(-\frac{1}{\tau}\Big) = 2^{k-2} i^k\, e^{2\pi i B(a,b)}\,\frac{1}{\vartheta(\tau/4)^{2k+1}} \sum_{p\in A^{-1}\mathbb{Z}^3/\mathbb{Z}^3} \Theta_{(b+p,-a)}(\tau). \tag{5.13}$$

Using (5.12) we obtain

$$\Theta_{(b+p,-a)}(\tau-4) = e^{8\pi i Q(b+p)}\,\Theta_{(b+p,-4(b+p)-a)}(\tau).$$

Next we use Theorem 5.5(2) with $\mu = -4p$. Note that $4A^{-1}$ is even, so we have $\mu \in \mathbb{Z}^3$ and $4Q(p) \in \mathbb{Z}$. Hence we get

$$\begin{aligned}\Theta_{(b+p,-a)}(\tau-4) &= e^{8\pi i Q(b+p)-8\pi i B(b+p,p)}\,\Theta_{(b+p,-a-4b)}(\tau)\\ &= e^{8\pi i Q(b)-8\pi i Q(p)}\,\Theta_{(b+p,-a-4b)}(\tau)\\ &= e^{8\pi i Q(b)}\,\Theta_{(b+p,-a-4b)}(\tau).\end{aligned}$$

We replace τ by $\tau-4$ in (5.13), use the identity above and then (5.13) again with a replaced by $a+4b$ to obtain

$$\frac{\Theta_{(a,b)}}{\vartheta^{2k+1}}\Big(-\frac{1}{\tau-4}\Big) = e^{-8\pi i Q(b)}\,\frac{\Theta_{(a+4b,b)}}{\vartheta^{2k+1}}\Big(-\frac{1}{\tau}\Big).$$

Replacing τ by $-1/\tau$ then gives (5.8) for $\gamma = \gamma' = \left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)$.

The function $f_{(a,b)}$ is the product of a function g_{a_1,b_1} that transforms like a modular form of weight k and a function ϑ_{a_2,b_2} that transforms like a modular form of weight 1/2, as we show in the following. We define

$$g_{a_1,b_1}(\tau) := \frac{1}{z_1^k} = \frac{1}{(a_1\tau+b_1)^k} \quad\text{and}\quad \vartheta_{a_2,b_2}(\tau) := \sum_{m\in a_2+\mathbb{Z}} q^{m^2}\,e^{4\pi i b_2 m}.$$

We can easily check that $\vartheta_{-a_2,-b_2} = \vartheta_{a_2,b_2}$ holds, which implies $f_{-(a,b)} = f_{(a,b)}$. Thus we have

$$f_{(a,b)}(-I\tau) = f_{(a,b)}(\tau) = f_{-(a,b)}(\tau),$$

which is (5.9) for $\gamma = -I$.

By straightforward calculations the modular transformation behavior of g_{a_1,b_1} for T and

γ' follows:

$$g_{a_1,b_1}(\tau+1) = \frac{1}{(a_1\tau + a_1 + b_1)^k} = g_{a_1,a_1+b_1}(\tau),$$
$$g_{a_1,b_1}\Big(\frac{\tau}{4\tau+1}\Big) = \frac{(4\tau+1)^k}{\big((a_1+4b_1)\tau + b_1\big)^k} = j(\gamma',\tau)^{2k} g_{a_1+4b_1,b_1}(\tau),$$

where we use in the last equation $j(\gamma',\tau)^2 = 4\tau+1$.

For ϑ_{a_2,b_2} we can directly compute for any $N \in \mathbb{Z}$

$$\vartheta_{a_2,b_2}(\tau+N) = \sum_{m\in a_2+\mathbb{Z}} q^{m^2} e^{4\pi i b_2 m + 2\pi i m^2 N} = e^{-2\pi i N a_2^2}\vartheta_{a_2,b_2+Na_2}(\tau), \tag{5.14}$$

as $m^2N \equiv 2Na_2m - Na_2^2 \,(\mathrm{mod}\,\mathbb{Z})$.

Since we know that the classical Jacobi theta function $\theta_{a,b}(\tau) := \sum_{m\in a+\mathbb{Z}} e^{\pi i m^2\tau + 2\pi i b m}$ satisfies

$$\theta_{a,b}\Big(-\frac{1}{\tau}\Big) = e^{2\pi i ab}\sqrt{-i\tau}\,\theta_{b,-a}(\tau),$$

we derive the functional equation

$$\vartheta_{a_2,b_2}\Big(-\frac{1}{\tau}\Big) = e^{4\pi i a_2 b_2}\sqrt{\frac{-i\tau}{2}}\,\vartheta_{-2b_2,\frac{a_2}{2}}\Big(\frac{\tau}{4}\Big)$$

and thus

$$\frac{\vartheta_{a_2,b_2}}{\vartheta}\Big(-\frac{1}{\tau}\Big) = e^{4\pi i a_2 b_2}\frac{\vartheta_{-2b_2,\frac{a_2}{2}}}{\vartheta}\Big(\frac{\tau}{4}\Big). \tag{5.15}$$

Replacing τ by $\tau-4$ in this equation and applying (5.14) for $N=-1$, we obtain

$$\frac{\vartheta_{a_2,b_2}}{\vartheta}\Big(-\frac{1}{\tau-4}\Big) = e^{4\pi i a_2 b_2 + 8\pi i b_2^2}\frac{\vartheta_{-2b_2,\frac{a_2}{2}+2b_2}}{\vartheta}\Big(\frac{\tau}{4}\Big).$$

We use (5.15) again with a_2 replaced by a_2+4b_2 and get

$$\frac{\vartheta_{a_2,b_2}}{\vartheta}\Big(-\frac{1}{\tau-4}\Big) = e^{-8\pi i b_2^2}\frac{\vartheta_{a_2+4b_2,b_2}}{\vartheta}\Big(-\frac{1}{\tau}\Big).$$

Replacing τ by $-1/\tau$ then gives

$$\vartheta_{a_2,b_2}(\gamma'\tau) = e^{-8\pi i b_2^2} j(\gamma',\tau)\,\vartheta_{a_2+4b_2,b_2}(\tau).$$

Now it just remains to be shown that for $(a',b') = (a,b)T = (a,a+b)$ we have

$$\pi i B(a,b) - \pi i B(a',b') - 8\pi i a_1' b_3' = -8\pi i a_1 b_3 - 2\pi i a_2^2$$

and for $(a',b') = (a,b)\gamma' = (a+4b,b)$ we have

$$\pi i B(a,b) - \pi i B(a',b') - 8\pi i a_1' b_3' = -8\pi i a_1 b_3 - 8\pi i b_2^2.$$

These identities follow by straightforward calculations and thus (5.9) holds for $-I, T$ and γ' and therefore on the whole group $\Gamma_0(4)$. $\square$

5.4.3 Hurwitz class numbers

Chen and Garvan [CG20] investigate generating functions $\mathcal{H}_{a,b}$ for Hurwitz class numbers $H(an+b)$ with $a|24$ and $(a,b)=1$ and show, inter alia, that $\mathcal{H}_{8,7}(q) \equiv \frac{A(-q)}{-q} \pmod 4$, where A is a certain second order mock theta function. They use the following identity shown by Humbert [Hum07]:

$$\mathcal{H}_{8,7}(q) = \frac{1}{q(q)_\infty^3} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} m^2 q^{m(m+1)/2}}{1+q^m} \quad \text{with } (q)_\infty := \prod_{m=1}^{\infty} (1-q^m),$$

which suggests that we can interpret the sum that defines $\mathcal{H}_{8,7}$ as the holomorphic part of a theta series of exactly the type that we investigated throughout this paper. Indeed, taking $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $f(n) = n_1^2$ (which is a spherical polynomial with regard to Q), $a = \frac{1}{2}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $b = \frac{1}{2}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and $c_1 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \in S_Q$, $c_2 = \sqrt{2}\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right) \in C_Q$, we obtain the theta series

$$\widehat{\Theta}_{a,b}^{c_1,c_2}[f](\tau) = y^{-1} \sum_{n \in \frac{1}{2}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) + \mathbb{Z}^2} \left\{ \operatorname{sgn}(n_1) f(ny^{1/2}) - p^{c_2}[f](ny^{1/2}) \right\} q^{\frac{1}{2}n_1^2 + n_1 n_2} e^{\pi i (n_1+n_2)}. \tag{5.16}$$

Note that $a \notin R(c_1)$, but the series is still well-defined, as the summand for $n_1 = 0$ vanishes. In the following theorem we give the connection between $\mathcal{H}_{8,7}$ and $\widehat{\Theta}_{a,b}^{c_1,c_2}[f]$, and give their modular behavior in terms of the classical Jacobi theta function

$$\theta_2(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n+\frac{1}{2})^2 \tau}.$$

Theorem 5.14. *The function $\tau \mapsto q^{7/8} \mathcal{H}_{8,7}(q)$ is a mock theta function of weight $3/2$ with shadow proportional to θ_2: it is the holomorphic part of the harmonic Maass form*

$$\mathcal{F} := \frac{i}{4} \frac{\widehat{\Theta}_{a,b}^{c_1,c_2}[f]}{\eta^3} \quad \textit{with } \widehat{\Theta}_{a,b}^{c_1,c_2}[f] \textit{ as in (5.16) and } \eta(\tau) := q^{1/24}(q)_\infty.$$

In particular, we have

$$\mathcal{F}(\tau) = q^{7/8} \mathcal{H}_{8,7}(q) + \frac{1}{4\pi i} \int_{-\overline{\tau}}^{i\infty} \frac{\theta_2(z)}{(-i(z+\tau))^{3/2}} \, dz,$$

where $\mathcal{F}$ is a harmonic Maass form of weight $3/2$ on the congruence subgroup $\Gamma_0(2)$. The transformation properties of $\mathcal{F}$ are such that $\theta_2 \mathcal{F}$ transforms as a modular form of weight 2 on $\Gamma_0(2)$ (without a character).

Proof. First we consider the modular transformation behavior of $\mathcal{F}$: from Theorem 5.5 we get

$$\widehat{\Theta}_{a,b}^{c_1,c_2}[f](\tau+1) = \widehat{\Theta}_{a,b}^{c_1,c_2}[f](\tau) \qquad \text{and} \qquad \widehat{\Theta}_{a,b}^{c_1,c_2}[f]\left(-\frac{1}{\tau}\right) = i\tau^3 \, \widehat{\Theta}_{b,-a}^{c_1,c_2}[f](\tau).$$

Using the last equation (twice) we also find

$$\widehat{\Theta}_{a,b}^{c_1,c_2}[f]\left(\frac{\tau}{2\tau+1}\right) = -i(2\tau+1)^3 \, \widehat{\Theta}_{a,b}^{c_1,c_2}[f](\tau).$$

Using the well-known transformation behavior of η and θ_2 we get

$$\frac{\theta_2}{\eta^3}(\tau+1) = \frac{\theta_2}{\eta^3}(\tau) \qquad \text{and} \qquad \frac{\theta_2}{\eta^3}\Big(\frac{\tau}{2\tau+1}\Big) = \frac{i}{2\tau+1}\,\frac{\theta_2}{\eta^3}(\tau)$$

and so $\theta_2\mathcal{F} = (i/4)\,\theta_2/\eta^3 \cdot \widehat{\Theta}_{a,b}^{c_1,c_2}[f]$ satisfies

$$(\theta_2\mathcal{F})(\tau+1) = (\theta_2\mathcal{F})(\tau) \qquad \text{and} \qquad (\theta_2\mathcal{F})\Big(\frac{\tau}{2\tau+1}\Big) = (2\tau+1)^2(\theta_2\mathcal{F})(\tau).$$

Since $\Gamma_0(2)$ is generated by $-I$, T and $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ we thus get that $\theta_2\mathcal{F}$ transforms as a modular form of weight 2 on $\Gamma_0(2)$ (without character). Hence

$$\mathcal{F}\Big(\frac{a\tau+b}{c\tau+d}\Big) = \zeta(\gamma)\,(c\tau+d)^{3/2}\,\mathcal{F}(\tau) \quad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2),$$

where $\zeta(\gamma) = \sqrt{c\tau+d}\,\theta_2(\tau)/\theta_2(\gamma\tau)$ is an eighth root of unity.

By the definition of $p^c[f]$ and using

$$E(z) = \operatorname{sgn}(z)(1-\beta(z^2)) \quad \text{with } \beta(x) = \int_x^\infty u^{-1/2}e^{-\pi u}du,$$

we have

$$\begin{aligned} y^{-1}\{\operatorname{sgn}(v_1)f(vy^{1/2}) - p^{c_2}[f](vy^{1/2})\} &= \{\operatorname{sgn}(v_1)+\operatorname{sgn}(v_2)\}v_1^2 \\ &+ \Big\{ -\operatorname{sgn}(v_2)\beta(2yv_2^2)v_1^2 - \frac{1}{\pi\sqrt{2y}}v_1E'\big(v_2\sqrt{2y}\big) + \frac{1}{8\pi^2 y}E''\big(v_2\sqrt{2y}\big)\Big\}. \end{aligned}$$

This already shows how we can decompose the theta function into its holomorphic and non-holomorphic part. For the holomorphic part we immediately derive:

$$\begin{aligned} i\sum_{m,n\in\mathbb{Z}} &\{\operatorname{sgn}(m)+\operatorname{sgn}(n+1/2)\}\,(-1)^{m+n}\,m^2\,q^{\frac{1}{2}m^2+m(n+\frac{1}{2})} \\ &= i\sum_{m\in\mathbb{Z}\smallsetminus\{0\}}(-1)^m\,m^2\,q^{m(m+1)/2}\sum_{n\in\mathbb{Z}}\{\operatorname{sgn}(m)+\operatorname{sgn}(n+1/2)\}\,(-q^m)^n \\ &= 2i\sum_{m\in\mathbb{Z}\smallsetminus\{0\}}\frac{(-1)^m\,m^2\,q^{m(m+1)/2}}{1+q^m} = -4i\sum_{m=1}^\infty\frac{(-1)^{m+1}\,m^2\,q^{m(m+1)/2}}{1+q^m} \end{aligned}$$

So $q^{7/8}\,\mathcal{H}_{8,7}(q)$ is the holomorphic part of $\mathcal{F}$ since $q(q)_\infty^3 = q^{7/8}\eta^3(\tau)$.

For the non-holomorphic part of $\widehat{\Theta}_{a,b}^{c_1,c_2}[f]$ we have

$$\begin{aligned} i\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\Big\{ &-\operatorname{sgn}\big(n+\tfrac{1}{2}\big)\beta\big(2y(n+\tfrac{1}{2})^2\big)m^2 - \frac{1}{\pi\sqrt{2y}}E'\big((n+\tfrac{1}{2})\sqrt{2y}\big)m \\ &+ \frac{1}{8\pi^2 y}E''\big((n+\tfrac{1}{2})\sqrt{2y}\big)\Big\}\,(-1)^{m+n}q^{m(m+1)/2+mn}. \end{aligned} \tag{5.17}$$

When we substitute $m-n$ for m, (5.17) can be written as the sum of the three series

$$-i\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\operatorname{sgn}\big(n+\frac{1}{2}\big)\beta\big(2y(n+\frac{1}{2})^2\big)(m-n)^2\,(-1)^m q^{m(m+1)/2-n(n+1)/2}, \tag{5.18}$$

$$\frac{-i}{\pi\sqrt{2y}}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}E'\big((n+\frac{1}{2})\sqrt{2y}\big)\,(m-n)\,(-1)^m q^{m(m+1)/2-n(n+1)/2}, \tag{5.19}$$

$$\frac{i}{8\pi^2 y}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}E''\big((n+\frac{1}{2})\sqrt{2y}\big)\,(-1)^m q^{m(m+1)/2-n(n+1)/2}. \tag{5.20}$$

From the identities

$$\sum_{m\in\mathbb{Z}}(-1)^m q^{m(m+1)/2}=0,\quad \sum_{m\in\mathbb{Z}}(-1)^m m q^{m(m+1)/2}=(q)_\infty^3$$
$$\text{and}\quad \sum_{m\in\mathbb{Z}}(-1)^m m^2 q^{m(m+1)/2}=-(q)_\infty^3,$$

we get that (5.18) equals

$$i(q)_\infty^3\sum_{n\in\mathbb{Z}}|2n+1|\,\beta\big(2(n+\frac{1}{2})^2y\big)q^{-n(n+1)/2}.$$

For (5.19) we also use $E'(z)=2e^{-\pi z^2}$ and obtain

$$\frac{-i}{\pi}(q)_\infty^3\sqrt{\frac{2}{y}}\sum_{n\in\mathbb{Z}}e^{-2\pi(n+\frac{1}{2})^2y}\,q^{-n(n+1)/2}.$$

Furthermore, we see that (5.20) vanishes. As β is related to the incomplete gamma function, we define $\beta(\alpha;x):=\int_x^\infty u^{\alpha-1}e^{-\pi u}\,du$, i. e. $\beta(x)=\beta(\frac{1}{2};x)$. By partial integration we then see that

$$\beta\big(\frac{1}{2};x\big)=\frac{1}{\pi\sqrt{x}}e^{-\pi x}-\frac{1}{2\pi}\beta\big(-\frac{1}{2};x\big).$$

So (5.17) can be simplified to

$$\frac{-i}{2\pi}(q)_\infty^3\sum_{n\in\mathbb{Z}}|2n+1|\beta\big(-\frac{1}{2};2(n+\frac{1}{2})^2y\big)\,q^{-n(n+1)/2}.$$

By the substitution of u by $(n+\frac{1}{2})^2u$, we have

$$|2n+1|\beta\big(-\frac{1}{2};2(n+\frac{1}{2})^2y\big)=|2n+1|\int_{2(n+\frac{1}{2})^2y}^\infty u^{-3/2}e^{-\pi u}\,du=2\int_{2y}^\infty u^{-3/2}e^{-\pi(n+\frac{1}{2})^2u}\,du.$$

Now we can write the non-holomorphic part of the theta function as

$$\frac{-i}{\pi}q^{1/8}(q)_\infty^3\int_{2y}^\infty\frac{\sum_{n\in\mathbb{Z}}e^{-\pi(n+\frac{1}{2})^2u}q^{-(n+\frac{1}{2})^2/2}}{u^{3/2}}\,du=-\frac{1}{\pi}\eta^3(\tau)\int_{-\overline{\tau}}^{i\infty}\frac{\sum_{n\in\mathbb{Z}}e^{\pi i(n+\frac{1}{2})^2z}}{(-i(z+\tau))^{3/2}}\,dz,$$

where in the last step we have substituted $u=-i(z+\tau)$. Combining the holomorphic and non-holomorphic parts we obtain the desired result:

$$\mathcal{F}(\tau) = q^{7/8}\,\mathcal{H}_{8,7}(q) + \frac{1}{4\pi i}\int_{-\overline{\tau}}^{i\infty} \frac{\theta_2(z)}{(-i(z+\tau))^{3/2}}\,dz$$

If we apply the usual ξ-operator, given by $\xi_k(f)(\tau) = 2iy^k\overline{\frac{\partial f}{\partial \overline{\tau}}}$, in weight $k = 3/2$ we directly get

$$\xi_{3/2}(\mathcal{F})(\tau) = -\frac{1}{4\pi\sqrt{2}}\,\theta_2(\tau),$$

which is a holomorphic function. Hence it is annihilated by the ξ-operator, so we have $\xi_{1/2}(\xi_{3/2}(\mathcal{F})) = 0$. Since the weight k hyperbolic Laplacian

$$\Delta_k = -y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) + iky\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

splits as $\Delta_k = -\xi_{2-k}\xi_k$, we thus have $\Delta_{3/2}\mathcal{F} = 0$ and so $\mathcal{F}$ is a harmonic Maass form. Further, we have seen that the image under $\xi_{3/2}$, and hence the shadow of $q^{7/8}\,\mathcal{H}_{8,7}(q)$, is $-1/(4\pi\sqrt{2})\,\theta_2$.

Applying Lemma 3.7 in [RZ21] one can immediately deduce that $\widehat{\widehat{\Theta}}{}_{a,b}^{c_1,c_2}[f]|\gamma$ is holomorphic at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Since the order of the pole of η^{-3} at ∞ is bounded, we get that $\mathcal{F}$ also satisfies the growth condition of a harmonic Maass form. $\square$

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6 Summary and Discussion

In this chapter, we summarize the main results of this thesis and present some related questions, which might be interesting for future research. For the notation used and the detailed results we refer to the summary in the introductory chapter.

6.1 Siegel theta series for indefinite quadratic forms as examples of Siegel–Maass forms

In Chapter 2, we have shown the following generalization of Vignéras' result [Vig77]: if p satisfies a certain growth condition and is a solution of

$$\Big(\mathbf{E} - \frac{\mathbf{\Delta}_A}{4\pi}\Big)p = \lambda I_n p \quad (\lambda \in \mathbb{Z}), \tag{6.1}$$

the associated Siegel theta series

$$\vartheta_p(Z) = \det Y^{-\lambda/2} \sum_{U \in \mathbb{Z}^{m\times n}} p(UY^{1/2}) \exp\big(\pi i \operatorname{tr}(U^{\mathrm{t}} AUZ)\big) \quad (\mathbb{H}_n \ni Z = X + iY)$$

transforms like a Siegel modular form of genus n and weight $m/2+\lambda$ (for a character and a subgroup that depend on A).

Vignéras concludes her article [Vig77] by additionally giving another differential equation of second order that determines whether a real analytic theta series constructed as above is a Maass form. This result is obtained by studying the action of the Maass lowering and raising operators on the theta series and showing that mapping a weight k theta series ϑ_f to a theta series ϑ_g of weight $k \pm 2$ under these operators can be reduced to a relation among the functions f and g. For example, we take a solution p of (6.1) for $n = 1$ to define the (elliptic) theta series ϑ_p of weight $k = m/2 + \lambda$, and introduce the operators

$$\mathcal{L}_k := iy^2 \frac{\partial}{\partial \overline{z}} \quad \text{and} \quad \Lambda_\lambda := -\frac{1}{2}(E - \lambda).$$

We observe that $\mathcal{L}_k$ lowers the weight of the theta series by 2 and Λ_λ maps a solution of (6.1) to a solution of almost the same differential equation, where we just replace λ by $\lambda - 2$. Moreover, one can show that the relation $\mathcal{L}_k(\vartheta_p) = \vartheta_{\Lambda_\lambda p}$ is satisfied.

Maass [Maa53, Maa71] introduced differential operators for Siegel modular forms of genus n. Here, we only recall the definition of the operator

$$M_{\frac{n-1}{2}} := \det(Z - \overline{Z}) \det(\partial_Z), \quad \text{where } (\partial_Z)_{ij} = \frac{1}{2}(1+\delta_{ij})\partial_{Z_{ij}} \text{ with } \partial_{Z_{ij}} = \frac{1}{2}(\partial_{X_{ij}} - i\partial_{Y_{ij}}).$$

Imamoḡlu and Richter [IR10] then defined Siegel–Maass forms as real analytic functions that vanish under $M_{(n-1)/2}$ (see [IR10] for the precise definition).

For $n = 2$, Bringmann, Raum, and Richter [BRR11] considered a slightly different differential operator, which allowed them to describe the Fourier expansions of these forms and establish a connection to so-called harmonic skew-Maass-Jacobi forms. In particular, they introduced higher-dimensional generalizations of the ξ_k-operator defined by Bruinier and Funke [BF04], inter alia, $\xi^{(2)}_{1/2,k-1/2} := \det(Y)^{k-3/2} M_{1/2}$. Since the functions considered in [IR10] vanish under this operator, one can see them as “holomorphic” Siegel–Maass forms.

One should also mention Westerholt-Raum’s [WR16a] more abstract approach for the case $n = 2$: as a generalization of Bruinier’s and Funke’s ξ_k-operator he considers a vector-valued lowering operator L and defines harmonic weak Siegel–Maass forms as real analytic functions that transform like vector-valued Siegel modular forms and are preimages of non-holomorphic Saito–Kurokawa lifts under L.

These various definitions of Siegel–Maass forms raise the question of how a generalization of Vignéras’ result to higher genus n fits into this picture. When we act with Λ_λ on a solution of (6.1), we essentially apply (up to a constant) the Laplacian Δ_A. Thus a natural generalization of this operator seems to be $\widetilde{\Lambda}_\lambda := \det \boldsymbol{\Delta}_A$, which acts on functions $p : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$. Indeed, one can show that $\widetilde{\Lambda}_\lambda$ maps a solution of (6.1) to a solution of almost the same system of differential equations, where we again just replace λ by $\lambda - 2$. It remains to be shown that this action on p is compatible with the action of a suitable Siegel lowering operator on ϑ_p. Finding a similar relation for a Siegel raising operator would then allow a generalization of Vignéras’ criterion to define Siegel–Maass forms among the modular Siegel theta series that were constructed in Chapter 2.

6.2 Generalizing the construction of Siegel theta series for quadratic forms of signature $(m-1,1)$

In Chapter 3, we constructed Siegel theta series for quadratic forms of signature $(m-1,1)$. Again, we considered a result in genus $n = 1$ due to Zwegers [Zwe02], where for $\boldsymbol{c_0}, \boldsymbol{c_1} \in \overline{C}_Q = C_Q \cup S_Q$, a holomorphic theta series is constructed using the function

$$\Phi(\boldsymbol{u}) := \operatorname{sgn}(B(\boldsymbol{c_0}, \boldsymbol{u})) - \operatorname{sgn}(B(\boldsymbol{c_1}, \boldsymbol{u})).$$

This ensures the holomorphicity of the associated theta series since the summation is restricted to a part of the lattice, where the quadratic form can be bounded by a positive definite quadratic form. For higher genus $n \in \mathbb{N}$, we fixed $n + 1$ vectors $\boldsymbol{c_i} \in C_Q$ and replaced Φ by

$$f(U) := \prod_{i=0}^{n} \frac{\operatorname{sgn}(\widetilde{x}_i) + 1}{2} - \prod_{i=0}^{n} \frac{\operatorname{sgn}(\widetilde{x}_i) - 1}{2},$$

where

$$\widetilde{x}_i = (-1)^i \det\bigl(U^{\mathrm{t}} A \widetilde{C}_i\bigr) \quad \text{with } \widetilde{C}_i := (\boldsymbol{c_0} \dots \widehat{\boldsymbol{c_i}} \dots \boldsymbol{c_n}).$$

Further, we constructed a non-holomorphic Siegel theta series that transforms like a Siegel modular form of genus n and weight $m/2$ and showed that its holomorphic part is described (almost everywhere) by the previously defined holomorphic series ϑ_f.

In Remark 3.23, we shortly addressed what has to be done to drop the restriction “almost everywhere” and find a complete description of the holomorphic part. For the case $n = 2$, there is an explicit description in Livinsky’s thesis [Liv16]. In a similar way one might approach the case $n \in \mathbb{N}$, although the result is probably given by a quite complicated formula.

Zwegers considers $\boldsymbol{c_0}, \boldsymbol{c_1} \in \overline{C}_Q$, i. e. vectors that can as well be located on the boundary of the cone C_Q. The fact that we assume the stricter condition $\boldsymbol{c_i} \in C_Q$ simplifies the structures of the proofs, as we do not have to consider the case $\boldsymbol{c_i} \in S_Q$ separately. In order to obtain a full generalization, one might investigate how the proofs have to be modified in order to include these vectors.

Another question that arises in the context of this construction is the generalization to Siegel theta series for quadratic forms of a more general signature, namely whether one can find an analogue to the constructions by Alexandrov, Banerjee, Manschot, and Pioline [ABMP18a] of theta series for quadratic forms of signature $(m-2,2)$ and Nazaroglu's generalization [Naz18] to arbitrary signature.

We shortly address the first case, that is, we take a quadratic form Q of signature $(m-2,2)$ and the associated bilinear form B. Further, we fix two pairs of vectors $\{\boldsymbol{c_0}, \boldsymbol{c_1}\}, \{\boldsymbol{c'_0}, \boldsymbol{c'_1}\}$ that are negative and satisfy certain additional incidence relations (compare Kudla's description in Section 3 of [Kud18] here, as his description fits better into the setting of Chapter 3).

A holomorphic theta series is then constructed via the function

$$\Phi_2(\boldsymbol{u}) := \frac{1}{4}\big[\operatorname{sgn}(B(\boldsymbol{c_0}, \boldsymbol{u})) - \operatorname{sgn}(B(\boldsymbol{c_1}, \boldsymbol{u}))\big]\big[\operatorname{sgn}(B(\boldsymbol{c'_0}, \boldsymbol{u})) - \operatorname{sgn}(B(\boldsymbol{c'_1}, \boldsymbol{u}))\big].$$

The convergence of the associated theta series follows by observing that on the support of Φ_2 the quadratic form Q can be bounded from below by a positive definite quadratic form Q^+ that depends on the vectors $\{\boldsymbol{c_0}, \boldsymbol{c_1}, \boldsymbol{c'_0}, \boldsymbol{c'_1}\}$.

In a somewhat naive generalization, one might choose two $n+1$-tuples of vectors in C_Q, that is $\{\boldsymbol{c_0}, \dots, \boldsymbol{c_n}\}$ and $\{\boldsymbol{c'_0}, \dots, \boldsymbol{c'_n}\}$, and replace Φ_2 by

$$f_2(U) := \left(\prod_{i=0}^{n} \frac{\operatorname{sgn}(\widetilde{x}_i)+1}{2} - \prod_{i=0}^{n} \frac{\operatorname{sgn}(\widetilde{x}_i)-1}{2}\right)\left(\prod_{i=0}^{n} \frac{\operatorname{sgn}(\widetilde{x}'_i)+1}{2} - \prod_{i=0}^{n} \frac{\operatorname{sgn}(\widetilde{x}'_i)-1}{2}\right),$$

where $\widetilde{x}'_i$ is defined analogously as above for $\{\boldsymbol{c'_0}, \dots, \boldsymbol{c'_n}\}$.

In Chapter 3, we have seen that for $U \in \mathbb{Z}^{m\times n}$ with $f_2(U) \neq 0$ the quadratic form $\boldsymbol{Q}(U) = \frac{1}{2}\operatorname{tr}(U^{\mathrm{t}}AU)$ can be bounded from below, as we can find for any column vector $\boldsymbol{u_j}$ of U a positive definite quadratic form $Q^+_{k,\ell}$ with $Q(\boldsymbol{u_j}) \geq Q^+_{k,\ell}(\boldsymbol{u_j})$. Here, we can apply the same argument and thus take a positive definite quadratic form depending on $\{\boldsymbol{c_k}, \boldsymbol{c_\ell}, \boldsymbol{c'_i}, \boldsymbol{c'_j}\}$ for each column vector of U. However, this construction requires all vectors $\boldsymbol{c_i}$ to be linearly independent to obtain non-vanishing theta series, which means that m has to be sufficiently large. This might be resolved by a modification of the construction above. Moreover, it remains to investigate whether a suitable modular completion can be obtained following the approach for signature $(m-1,1)$.

6.3 Indefinite theta series with (spherical) polynomials

In Chapters 4 and 5, we extended Zwegers' construction [Zwe02] of elliptic theta series for quadratic forms of signature $(m-1,1)$ by including homogeneous and spherical polynomials in the definition of the theta series. The definition depends on two vectors $\boldsymbol{c_1}, \boldsymbol{c_2} \in \overline{C}_Q$. For $\boldsymbol{c_1}, \boldsymbol{c_2} \in C_Q$, we gave a general construction of a holomorphic, an almost holomorphic, and a modular theta series and determined under which condition these versions agree. Thus we obtained a construction of almost holomorphic and holomorphic modular forms and gave numerous explicit examples that could often be identified as eta products or eta quotients. Further considering the case $\boldsymbol{c_i} \in S_Q$ for one or both $i \in \{1,2\}$, we obtained examples of a different type.

As these constructions of holomorphic modular theta series strongly exploit the fact that the quadratic form has signature $(m-1,1)$, the natural question arises whether there is also an extension to other signatures – a first step would certainly be to generalize Zwegers' building block in [Zwe02] to signature $(m-2,2)$ and then also to arbitrary signature. If this is possible, the inclusion of polynomials should again provide a more general construction of examples.

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07/2013–09/2013	Praktikum bei der *Deutschen Aktuarvereinigung e.V.* in Köln

Lehre

Seit 04/2019	Koordination des Übungsbetriebs und Tutorentätigkeit in den Vorlesungen *Lineare Algebra II*, *Algebra*, *Elementare Zahlentheorie*, *Modulformen*, *Thetafunktionen* und *Lineare Algebra I*
04/2018 – 03/2019	Koordination des Übungsbetriebs in den Vorlesungen *Elementare Zahlentheorie*, *Lineare Algebra I*
10/2017 – 02/2019	Betreuung der Seminare *Zahlentheorie und Kryptologie*, *Spezielle Funktionen* und *Modulformen*
2018–2021	Zweitgutachterin für Bachelorarbeiten in verschiedenen Themengebieten der Zahlentheorie

Vorträge

09/2021	Women in Automorphic Forms, Konferenz in Bielefeld (Speed Talk)
06/2021	Number Theory Lunch Seminar des MPIM Bonn (Online)
02/2021	Heilbronn Number Theory Seminar der University of Bristol (Online)
02/2021	Algebra and Number Theory Seminar des University College of Dublin (Online)
11/2020	International Seminar on Automorphic Forms (Online)
09/2019	DMV Jahrestagung, Konferenz in Karlsruhe (Speed Talk)
09/2018	Women in Automorphic Forms, Konferenz in Darmstadt (Speed Talk)

Weitere Tätigkeiten

12/2019	Gutachterin für *The Ramanujan Journal*
05/2018–12/2019	Mitglied des Advisory Board im Gleichstellungs-Netzwerk *Hypatia* des Departments Mathematik/Informatik der Universität zu Köln
12/2011–01/2015	Mitglied der Fachschaft Mathematik: Studentische Vertretung in diversen Gremien (Fachschaftsrat, Fachausschuss, Berufungskommission), Tagesgeschäft, Erstsemesterbetreuung