Mock Modular Forms and Class Numbers of Quadratic Forms

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Kurzzusammenfassung

Die vorliegende Arbeit befasst sich im Wesentlichen mit der Frage nach möglichen Rekursionsbeziehungen zwischen den Fourier-Koeffizienten einer bestimmten Klasse von Mock-Modulformen. Die prominentesten Beispiele solcher Fourier-Koeffizienten sind die Hurwitz-Klassenzahlen binärer quadratischer Formen, für die einige Rekursionen bereits lange bekannt sind. Als Beispiele sind hier unter anderen die Kronecker-Hurwitz-Klassenzahlrelationen sowie die Eichler-Selberg-Spurformel für Hecke-Operatoren auf Räumen von Spitzenformen anzuführen.

Im Jahre 1975 vermutete H. Cohen nun eine unendliche Serie von solchen Klassenzahlrelationen, die eng verwandt sind mit der erwähnten Eichler-Selberg-Spurformel. In dieser Arbeit beweise ich Cohen's Vermutung, sowie einige ähnliche Formeln für Klassenzahlen mit Hilfe wichtiger Resultate aus der Theorie der Mock-Modulformen.

Mittels einer anderen Methode zeige ich schließlich, dass derlei Rekursionsbeziehungen ein generelles Phänomen für Fourier-Koeffizienten von Mock-Thetafunktionen und Mock-Modulformen vom Gewicht $\frac{3}{2}$ darstellen. Als Spezial-fälle erhält man aus diesem Resultat alternative Beweise für die oben erwähnten Klassenzahlrelationen.

Abstract

This thesis deals with the question for possible recurrence relations among Fourier coefficients of a certain class of mock modular forms. The most prominent examples of such Fourier coefficients are the Hurwitz class numbers of binary quadratic forms, which satisfy many well-known recurrence relations. As examples one should mention the Kronecker-Hurwitz class number relations and the famous Eichler-Selberg trace formula for Hecke operators on spaces of cusp forms.

In 1975, H. Cohen conjectured an infinite family of such class number relations which are intimately related to the aforementioned Eichler-Selberg trace formula. In this thesis, I prove Cohen's conjecture and other similar class number formulas using important results from the theory of mock modular forms.

By applying a different method I prove at the end that such recurrence relations are a quite general phenomenon for Fourier coefficients of mock theta functions and mock modular forms of weight $\frac{3}{2}$. As special cases, one gets an alternative proof for the aforementioned class number relations.

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Chapter I

Introduction

I.1 History

One classical problem in number theory is to calculate class numbers of various objects, such as quadratic forms, number fields, genera of lattices, quaternion algebras, and many more. It occurs in many different interesting questions, e.g. in elementary number theory, where it is a vital tool to answer the question which integers can be represented by a given quadratic form, or in algebraic number theory, where it "measures", how far the ring of integers of a number field, or more generally a maximal order in a division algebra over $\mathbb{Q}$, is from being a principal ideal domain. One of the first systematic treatments of class numbers of binary integral quadratic forms is given in C.F. Gauß' masterwork, his treatment *Disquisitiones Arithmeticae* from 1801 [26]. In Chapter 5 he introduces so-called reduction theory which means, in a more modern manner, to distinguish certain standard representatives of equivalence classes of quadratic forms: Binary integral quadratic forms can be viewed as matrices of the form

$$Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \quad a,\, b,\, c \in \mathbb{Z}.$$

The number $D := -\det Q$ is called the discriminant of the form Q. Now the modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on the set $\mathcal{Q}_D$ of quadratic forms with fixed discriminant D via $(Q,\gamma) \mapsto \gamma^{tr} Q \gamma$. For convenience we shall assume our forms to be primitive, i.e. $\gcd(a,b,c) = 1$.

If we view a quadratic form as a function

$$Q : \mathbb{Z}^2 \to \mathbb{Z}, \quad (x,y) \mapsto \frac{1}{2}(x,y)Q\begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + bxy + cy^2,$$

we may regard the action of $\mathrm{SL}_2(\mathbb{Z})$ as an orientation-preserving base change of the lattice $\mathbb{Z}^2$, which motivates the notion of *(properly) equivalent forms*, i.e. quadratic forms in the same $\mathrm{SL}_2(\mathbb{Z})$-orbit. The number of inequivalent forms of fixed discriminant D is called the *class number* $h(D)$. Gauß' reduction theory can be formulated as follows:

Theorem I.1.1 (C.F. Gauß, 1801). *Let $D < 0$. Then each $\mathrm{SL}_2(\mathbb{Z})$-orbit of $\mathcal{Q}_D$ contains exactly one* reduced form*. A primitive form $Q = \left(\begin{smallmatrix} 2a & b \\ b & 2c \end{smallmatrix}\right)$ is called reduced if*

$$\begin{aligned} &|b| \leq a \leq c \\ &b > 0 \text{ if } a = c \text{ or } |b| = a. \end{aligned}$$

In principle, this gives an explicit method to enumerate all reduced forms of given negative discriminant and thus determine the class number.

Another approach towards calculating class numbers was made by L. Dirichlet about 40 years after the appearance of the *Disquisitiones*. In 1839 he found a closed formula for class numbers of primitive binary quadratic forms [44, pp. 357-374 and pp. 411-496] using methods of complex analysis.

Theorem I.1.2 (L. Dirichlet, 1839). *Let D be a fundamental discriminant (i.e. either D is square-free or $D = 4m$ with $m \equiv 2, 3 \pmod 4$ and m square-free), $\chi_D = \left(\frac{D}{\cdot}\right)$ the Kronecker symbol and define the Dirichlet L-series to the character χ_D as the analytic continuation of*

$$L(s, \chi_D) := \sum_{n=1}^{\infty} \chi_D(n) n^{-s}.$$

For $D < 0$ define $w_{-3} = 6$, $w_{-4} = 4$ and $w_D = 2$ for $D < -4$. Then it holds that

$$h(D) = \frac{w_D \sqrt{|D|}}{2\pi} L(1, \chi_D)$$

A detailed proof is given in [16]. Note that the w_D is exactly the order of the unit group of $\mathbb{Z}_{\mathbb{Q}(\sqrt{D})}$. From a computational point of view, Dirichlet's class number formula is not really an improvement toward reduction theory, since evaluation of L-functions is not so easy. A faster way to produce tables of class numbers was introduced by L. Kronecker [38] and A. Hurwitz [32, 33]. They found a somewhat surprising recurrence relation for so called Hurwitz class numbers. The Hurwitz class number $H(n)$ is slight modification of the regular class number,

$$H(n) = \begin{cases} -\frac{1}{12} & \text{if } n = 0, \\ \sum\limits_{f^2 | n} \frac{2h(-n/f^2)}{w_{-n/f^2}} & \text{if } n \equiv 0, 3 \pmod 4 \text{ and } n > 0, \\ 0 & \text{otherwise} \end{cases} \tag{I.1.1}$$

with w_d as in Theorem I.1.2. The value $H(0) = -\frac{1}{12}$ is merely for convenience. Kronecker and Hurwitz relate this quantity to certain divisor sums: They prove the identity

$$\sum_{s \in \mathbb{Z}} H(4n - s^2) + 2\lambda_1(n) = 2\sigma_1(n), \tag{I.1.2}$$

where

$$\lambda_k(n) := \frac{1}{2}\sum_{d|n} \min\left(d, \frac{n}{d}\right)^k \tag{I.1.3}$$

is the k-th power *minimal-divisor sum* and

$$\sigma_k(n) := \sum_{d|n} d^k \tag{I.1.4}$$

is the usual k-th power divisor sum.

Many other such relations have been discovered since: M. Eichler found in 1955 that for all odd n the identity

$$\sum_{s\in\mathbb{Z}} H(n-s^2) + \lambda_1(n) = \frac{1}{3}\sigma_1(n) \tag{I.1.5}$$

holds [18]. Another source of such relations is the famous Eichler-Selberg trace formula [19, 20, 21, 45] which can be stated as follows (for precise definitions see Chapter II):

Theorem I.1.3 (M. Eichler, A. Selberg, 1956). *Let $n \geq 1$ be a natural number and $k \geq 4$ be an even number. Then the trace of the nth Hecke operator T_n on the space of cusp forms S_k on $\mathrm{SL}_2(\mathbb{Z})$ is given by*

$$\operatorname{trace} T_n^{(k)} = -\tfrac{1}{2}\sum_{s\in\mathbb{Z}} g_k^{(1)}(s,n)H(4n-s^2) - \lambda_{k-1}(n),$$

where $g_k^{(1)}(s,n)$ is the coefficient of X^{k-2} in the Taylor expansion of

$$(1-sX+nX^2)^{-1}.$$

Since the spaces S_4 and S_6 are 0-dimensional, we get the following class number relations from Theorem I.1.3:

$$\sum_{s\in\mathbb{Z}}(s^2-n)H(4n-s^2) + 2\lambda_3(n) = 0 \tag{I.1.6}$$

$$\sum_{s\in\mathbb{Z}}(s^4-3ns^2+n^2)H(4n-s^2) + 2\lambda_5(n) = 0. \tag{I.1.7}$$

In 1975, H. Cohen [14] and D. Zagier [30, 50] made yet another approach towards understanding class numbers. Cohen considered a slight generalization of the Hurwitz class number which is motivated by Dirichlet's class number formula (see [14, Definition 2.1, 2.2]): Let for $r, n \in \mathbb{N}$

$$h(r,n) := \begin{cases} (-1)^{\lfloor \frac{r}{2}\rfloor}\frac{(r-1)!n^{r-\frac{1}{2}}}{2^{r-1}\pi^r}L(r,\chi_{(-1)^r n}), & \text{if } (-1)^r n \equiv 1,2 \pmod 4\\ 0, & \text{otherwise}\end{cases}$$

with $L(s,\chi)$ and χ_D defined as in Theorem I.1.2 and define

$$H(r,n) := \begin{cases} \zeta(1-2r), & \text{if } n=0 \\ \sum\limits_{f^2|n} h\left(r, \frac{n}{f^2}\right), & \text{if } (-1)^r n \equiv 0,1 \pmod 4 \text{ and } n>0 \\ 0, & \textit{otherwise.} \end{cases}$$

Cohen defines the generating function of these numbers

$$\mathscr{H}_r(\tau) := \sum_{n=0}^{\infty} H(r,n)q^n, \quad \tau \in \mathbb{H},\ q := e^{2\pi i \tau} \tag{I.1.8}$$

and shows the following result (cf. [14, Theorem 3.1]):

Theorem I.1.4 (H. Cohen, 1975). *For $r \geq 2$ the function $\mathscr{H}_r$ in* (I.1.8) *defines a modular form of weight $r + \frac{1}{2}$ on $\Gamma_0(4)$.*

He proves this by writing $\mathscr{H}_r$ as a linear combination of Eisenstein series of appropriate weight.

Zagier on the other hand looked at the generating function

$$\mathscr{H}(\tau) := \mathscr{H}_1(\tau) = \sum_{n=0}^{\infty} H(n)q^n, \quad \tau \in \mathbb{H},\ q := e^{2\pi i \tau}, \tag{I.1.9}$$

which turns out *not* to be a modular form. But using an idea of Hecke (cf. [29, §2]) and analytic continuation he finds that there is a modular "completion" of $\mathscr{H}$ (cf. [30, Chapter 2,Theorem 2]). Therefore let

$$\mathscr{R}(\tau) = \frac{1+i}{16\pi} \int_{-\overline{\tau}}^{i\infty} \frac{\vartheta(z)}{(z+\tau)^{\frac{3}{2}}} dz \tag{I.1.10}$$

with ϑ as in Example II.1.4 (*iv*).

Theorem I.1.5 (D. Zagier, 1976). *Define the non-holomorphic function*

$$\widehat{\mathscr{H}}(\tau) := \mathscr{H}(\tau) + \mathscr{R}(\tau).$$

Then this function transforms like a modular form of weight $\frac{3}{2}$ on $\Gamma_0(4)$.

In later years it was recognized that $\widehat{\mathscr{H}}$ belongs to a certain class of non-holomorphic modular forms, the so-called *harmonic Maaß forms* (see Section II.3). These functions occured during the research on S. Ramanujan's *mock theta functions*: In his last letter to Hardy, Ramanujan introduced 17 functions in form of q-series which all had similar asymptotic behaviour as modular forms, but did not have a nice modular transformation property. He called these functions, which he defined in this very vague way, mock theta functions. Many attempts were made throughout the 20th century to put these mock theta functions into an appropriate context and give a mathematically precise definition of them. Finally, by work of S. Zwegers in his Ph.D. thesis [55], J.H. Bruinier and J. Funke in [11], and K. Bringmann and K. Ono in [7, 8], a proper setting for the mock theta-functions or more generally mock modular forms was obtained.

Many interesting generating functions arising from combinatorics turn out to be mock modular forms, such as the rank generating functions (see e.g. [27] and [53]) for partitions. Quite recently, in [6], K. Bringmann and J. Lovejoy related ranks of so called overpartitions again to Hurwitz class numbers.

I.2 Scope of this Thesis

In this thesis, I shall focus on relations among Fourier coefficients of mock modular forms similar to the Kronecker-Hurwitz formula (I.1.2) or the Eichler-Selberg trace formula (Theorem I.1.3).

For this we give a brief account of the needed facts about elliptic modular forms, Jacobi forms, and harmonic Maaß forms/mock modular forms as well as Appell-Lerch sums in Chapter II.

The first goal is to give a detailed proof of a conjecture of Henri Cohen from 1975. Based on his and Zagier's work on class numbers and special values of L-functions as well as computer experiments, he conjectured in [14] that the following should be true.

Conjecture I.2.1 (H. Cohen, 1975). *Let*

$$S_4^1(\tau, X) := \sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} \left[\sum_{\substack{s\in\mathbb{Z} \\ s^2\leq n}} \frac{H(n-s^2)}{1-2sX+nX^2} + \sum_{k=0}^{\infty} \lambda_{2k+1}(n) X^{2k} \right] q^n. \tag{I.2.1}$$

Then the coefficient of X^ℓ in the formal power series $S_4^1(\tau, X)$ is a (holomorphic) modular form of weight $\ell+2$ on $\Gamma_0(4)$.

The first proof of this is given in [40]. Here, I will recall this proof in greater detail. Its basic idea is as follows. The coefficient in question can essentially be realized as a so-called Rankin-Cohen bracket (see Definition II.1.8) of the functions $\mathscr{H}$ and ϑ plus a minimal divisor power sum. One can add non-holomorphic terms to each of these terms to make them transform like modular forms of the correct weight, and then it just remains to show that the non-holomorphic corrections cancel each other. This will be the content of Chapter III.

This idea has also been used by Bringmann and Kane [4] to prove other class number relations which were conjectured by Bloom et al. in [10]. In Chapter IV we shall recall and slightly extend their results.

Chapter V is dedicated to a different approach towards the previous results, namely holomorphic projection, which gives the striking observation, that

- loosely speaking - for every mock theta function and every mock modular form of weight $\frac{3}{2}$ there are class number type relations among the Fourier coefficients.

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Chapter II

Modular Forms and Generalizations

In this chapter I give a short exposé on the types of modular and automorphic objects that are going to be used in the rest of this thesis.

We fix some notation: The letter τ will always denote a variable living on the complex upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ while z, u, v may represent arbitrary (complex) variables. For brevity, we define $x := \mathrm{Re}(\tau)$, $y := \mathrm{Im}(\tau)$, and $q := e^{2\pi i\tau}$.

By $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ we usually denote the group

$$\Gamma_0(N) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod N\}, \tag{II.0.1}$$

for some $N \in \mathbb{N}$. An element of such a group is denoted by

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Sometimes, we shall also need the following special subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

$$\Gamma_1(N) := \{\gamma \in \Gamma_0(N) \mid a \equiv d \equiv 1 \pmod N\} \tag{II.0.2}$$

$$\Gamma(N) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv I_2\}, \tag{II.0.3}$$

where I_n denotes the $n \times n$ unity matrix. Note that $\Gamma(N)$ is the kernel of the canonical epimorphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and therefore a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$. It is called the *principal congruence subgroup* of *level* N, and every subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing $\Gamma(N)$ is called a *congruence subgroup*.

II.1 Elliptic Modular Forms

Here I give a short summary of the necessary theory of elliptic modular forms. The main references are [37] and [12, Chapter 1] for modular forms of integral weight, half-integral weight is treated e.g. in [47].

II.1.1 Definition and Examples

Throughout this subsection, fix $k \in \frac{1}{2}\mathbb{Z}$ and $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ (if $k \notin \mathbb{Z}$ assume $\Gamma \leq \Gamma_0(4)$). It is well-known that Γ acts on the upper half-plane via Möbius transformations,

$$(\gamma, \tau) \mapsto \gamma.\tau := \frac{a\tau + b}{c\tau + d}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

For a function $f : \mathbb{H} \to \mathbb{C}$ and $\gamma \in \Gamma$, we define the weight k *slash operator* as

$$(f|_k\gamma)(\tau) = \begin{cases} (c\tau + d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) & \text{, if } k \in \mathbb{Z} \\ \left(\frac{c}{d}\right) \varepsilon_d \left(\sqrt{c\tau + d}\right)^{-2k} f\left(\frac{a\tau+b}{c\tau+d}\right) & \text{, if } k \in \frac{1}{2} + \mathbb{Z} \end{cases},$$

where $\left(\frac{m}{n}\right)$ denotes the extended Legendre symbol in the sense of [47], $\sqrt{\tau}$ denotes the principal branch of the square root (i.e. $-\frac{\pi}{2} < \arg(\sqrt{\tau}) \leq \frac{\pi}{2}$), and

$$\varepsilon_d := \begin{cases} 1 & \text{, if } d \equiv 1 \pmod 4 \\ i & \text{, if } d \equiv 1 \pmod 4). \end{cases}.$$

Now let $\mathbb{P}_1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\}$. Defining $\frac{a}{0} := \infty$, $\frac{a}{\infty} := 0$, and $\frac{\infty + b}{d} := \infty$ for $a, b, d \neq 0$ one sees that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}_1(\mathbb{Q})$. Therefore there are only finitely many Γ-orbits on $\mathbb{P}_1(\mathbb{Q})$.

Definition II.1.1. *(i) A* cusp *of Γ is a coset representative of $\Gamma\backslash\mathbb{P}_1(\mathbb{Q})$.*

(ii) A one-periodic holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is said to be meromorphic resp. holomorphic at $i\infty$ *if the function $\hat{f} : \dot{\mathbb{E}} \to \mathbb{C}$ with $f(\tau) = \hat{f}(e^{2\pi i\tau})$ (cf. [24, Satz VI.1.4]) has a meromorphic resp. holomorphic continuation onto $\mathbb{E}$, where $\mathbb{E} := \{z \in \mathbb{C} \mid |z| < 1\}$ denotes the unit disk and $\dot{\mathbb{E}} := \mathbb{E} \setminus \{0\}$.*

(iii) Let $c \in \mathbb{P}_1(\mathbb{Q})$ and $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_0.c = i\infty$. Then a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ with $(f|_k\gamma)(\tau) = f(\tau)$ for some $k \in \frac{1}{2}\mathbb{Z}$ and all $\tau \in \mathbb{H}$ and $\gamma \in \Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ is said to be meromorphic resp. holomorphic at c, *if the function*

$$w \mapsto f(\gamma_0^{-1}.w) \left(\frac{d\tau}{dw}\right)^{\frac{k}{2}}$$

with $w := \gamma_0.\tau$ is holomorphic at $i\infty$.

Definition II.1.2. *A function $f : \mathbb{H} \to \mathbb{C}$ is called a* modular form *of* weight *$k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma = \Gamma_0(N)$ (or of* level *$N \in \mathbb{N}$) and* character *χ for χ a Dirichlet character modulo N if the following conditions are met:*

(i) f is holomorphic on $\mathbb{H}$.

(ii) f is invariant under the weight k slash operator, *i.e. for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau \in \mathbb{H}$ we have $(f|_k\gamma)(\tau) = \chi(d)f(\tau)$.*

(iii) f is holomorphic at the cusps of Γ.

If we replace (iii) by

(iii') f has at most a pole in every cusp of Γ,

then we call f a weakly holomorphic *modular form.*

A modular form that vanishes at every cusp of Γ is called a cusp form. *The $\mathbb{C}$-vector space of modular forms resp. cusp forms resp. weakly holomorphic modular forms of weight k on $\Gamma = \Gamma_0(N)$ with character χ is denoted by $M_k(N,\chi)$ resp. $S_k(N,\chi)$ resp. $M_k^!(N,\chi)$. For other groups Γ we write analogously $M_k(\Gamma)$ etc.*

Due to the fact that a (weakly holomorphic) modular form f is *per definitionem* one-periodic, it has a Fourier expansion (around ∞) of the form

$$f(\tau) = \sum_{n=m_0}^{\infty} a_f(n) q^n$$

where $m_0 \in \mathbb{Z}$ and again, $q := e^{2\pi i\tau}$.

Remark II.1.3. *Clearly it holds that products of modular forms are again modular forms which turns the graded vector space*

$$M_*(\Gamma) := \bigoplus_{k\in\mathbb{N}_0} M_k$$

into a graded $\mathbb{C}$-algebra.

Example II.1.4. *(i) Let $k \geq 4$ be an even integer and let*

$$G_k(\tau) := \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} (m\tau+n)^{-k}.$$

These so-called Eisenstein series *are absolutely convergent and therefore define holomorphic functions which are easily seen to be modular forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$ (cf. [37, Chapter III, §2]). They have the following Fourier expansion,*

$$E_k(\tau) := \frac{1}{2\zeta(k)} G_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\zeta(s)$ is the Riemann ζ-function, B_k is the kth Bernoulli number (cf. [24, p. 203]), and σ_k is defined as in (I.1.4).

(ii) In the case $k=2$, the series G_2 is only conditionally convergent and gives, fixing a certain order of summation, a Fourier development

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

which is perfectly convergent. This function defines a so-called quasi-modular form *of weight* 2*, see [12] for details. The transformation under* $\mathrm{SL}_2(\mathbb{Z})$ *is given by*

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2E_2(\tau)-\frac{6i}{\pi}c(c\tau+d). \tag{II.1.1}$$

The non-holomorphic function $\widehat{E}_2(\tau):=E_2(\tau)-\frac{3}{\pi y}$ *transforms like a modular form of weight* 2 *under* $\mathrm{SL}_2(\mathbb{Z})$*.*

(iii) The discriminant function $\Delta(\tau):=\frac{E_4^3(\tau)-E_6^2(\tau)}{1728}$ *is a cusp form of weight* 12 *on* $\mathrm{SL}_2(\mathbb{Z})$ *since the only cusp of* $\mathrm{SL}_2(\mathbb{Z})$ *is* $i\infty$ *and since the constant term of the Fourier expansion of* Δ *vanishes,* Δ *vanishes at that cusp.*

(iv) The Dedekind η*-function defined by*

$$\eta(\tau):=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

is a modular form of weight $\frac{1}{2}$ with multiplier system *on* $\mathrm{SL}_2(\mathbb{Z})$ *which satisfies* $\eta^{24}(\tau)=\Delta(\tau)$*. For details, see e.g. [37, Chapter III, §6].*

(v) The ϑ*-series of the lattice* $2\mathbb{Z}$ *defined by*

$$\vartheta(\tau):=\sum_{n\in\mathbb{Z}}q^{n^2}$$

is a modular form of weight $\frac{1}{2}$ *on* $\Gamma_0(4)$*.*

(vi) More generally, for $N\in\mathbb{N}$ *and* $s\in\mathbb{N}$ *and* χ *an even character modulo* N *of conductor* F *with* $4sF^2|N$*, the theta series*

$$\vartheta_{s,\chi}(\tau):=\sum_{n\in\mathbb{Z}}\chi(n)q^{sn^2} \tag{II.1.2}$$

is a modular form of weight $\frac{1}{2}$ *on* $\Gamma_1(4N)$*. For an odd character* χ*, the theta series*

$$\theta_{s,\chi}(\tau):=\sum_{n\in\mathbb{Z}}\chi(n)nq^{sn^2} \tag{II.1.3}$$

is a cusp form of weight $\frac{3}{2}$ *on the same group.*

An important fact about modular forms is, that they are in a sense quite rare. As B. Mazur put it:

> *"Modular forms are functions on the complex plane that are inordinately symmetric. They satisfy so many internal symmetries that their mere existence seem like accidents. But they do exist."*[1]

[1]quoted from `https://www.math.umass.edu/~weston/rs/mf.html`

Theorem II.1.5. *For every congruence subgroup Γ, the space $M_k(\Gamma)$ (and therefore $S_k(\Gamma)$) are finite-dimensional. Moreover, we have the dimension formulas*

$$\dim M_k(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} 0, & \text{if } k < 0 \text{ or } k \text{ odd} \\ \lfloor \frac{k}{12} \rfloor, & \text{if } k \geq 0 \text{ and } k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor + 1, & \text{if } k \geq 0 \text{ and } k \not\equiv 2 \pmod{12}, \end{cases}$$
$$\dim S_k(\mathrm{SL}_2(\mathbb{Z})) = \dim M_{k-12}$$

for $k \in \mathbb{Z}$ and

$$\dim M_k(\Gamma_0(4)) = \begin{cases} 0, & \text{if } k < 0 \\ \lfloor \frac{k}{2} \rfloor + 1, & \text{if } k \geq 0, \end{cases}$$
$$\dim S_k(\Gamma_0(4)) = \begin{cases} 0, & \text{if } k \leq 4 \\ \lfloor \frac{k}{2} \rfloor - 1, & \text{if } k > 2 \text{ and } k \notin 2\mathbb{Z} \\ \lfloor \frac{k}{2} \rfloor - 2, & \text{if } k > 2 \text{ and } k \in 2\mathbb{Z} \end{cases}$$

for $k \in \frac{1}{2} + \mathbb{Z} \cup 2\mathbb{Z}$.

A proof of this can be found in [17, Chapter 3].

For computational purposes one has the following result called the *Sturm bound* or *Hecke bound* due to J. Sturm (see [36, Theorem 3.13]).

Theorem II.1.6 (J. Sturm, 1987). *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of index M and $f \in M_k(\Gamma)$ with Fourier series $f(\tau) = \sum_{n=0}^{\infty} a_f(n) q^n$ where there is an $m_0 \geq 0$ such that $a_f(n) = 0$ for all $n \leq m_0$. If*

$$m_0 > M \cdot \frac{k}{12},$$

then f is identically zero.

This means that in order to decide equality of two modular forms it suffices to compare the first Fourier coefficients up to a certain bound. For special groups, there are are explicit formulas for their indices in $\mathrm{SL}_2(\mathbb{Z})$.

Proposition II.1.7. *For every $N \in \mathbb{N}$ we have*

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p \mid N} \left(1 + \frac{1}{p}\right),$$
$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)] = N^2 \prod_{p \mid N} \left(1 - \frac{1}{p^2}\right),$$

where the product is taken over the primes dividing N.

For a proof, see for example [17, pp. 13f].

Since in many cases arithmetic functions such as divisor sums occur as Fourier coefficients of modular forms, the finite dimension of the space of modular forms is the source of a vast collection of striking identities among these functions. First examples are the Hurwitz identities for divisor sums which A. Hurwitz discovered in his dissertation [31]: Since the spaces $M_8(\mathrm{SL}_2(\mathbb{Z}))$ and $M_{10}(\mathrm{SL}_2(\mathbb{Z}))$ are one-dimensional, it must hold that $E_4^2 = E_8$ and $E_4E_6 = E_{10}$ since the constant terms of all the Fourier series is 1. This yields the famous Hurwitz identities for divisor power sums

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m),$$
$$11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040\sum_{m=1}^{n-1}\sigma_3(m)\sigma_5(n-m),$$

which are extremely difficult to prove without using the theory of modular forms (or of elliptic functions).

II.1.2 Operators on Modular Forms

As we already remarked, products of modular forms are again modular forms. It turns out that by involving derivatives of modular forms (which themselves are *not* modular forms but rather *quasi-modular forms*, see [12, section 5.3]) one can define a different product on the algebra of modular forms.

Definition II.1.8. *Let f, g be smooth functions defined on the upper half plane and $k, \ell \in \mathbb{R}_{>0}$, $\nu \in \mathbb{N}_0$. Then we define the νth* Rankin-Cohen bracket *of f and g as*

$$[f,g]_\nu = \sum_{r+s=\nu}(-1)^r\binom{k+\nu-1}{s}\binom{\ell+\nu-1}{r}D^r f D^s g$$

where for non-integral entries we define

$$\binom{m}{s} := \frac{\Gamma(m+1)}{\Gamma(s+1)\Gamma(m-s+1)}.$$

Here, the letter Γ denotes the usual Gamma function. Furthermore, we set $D_t = \frac{1}{2\pi i}\frac{d}{dt}$.

Proposition II.1.9 (Theorem 7.1 in [14]). *Let f, g be (not necessarily holomorphic) modular forms of weights k and ℓ respectively on the same group Γ. Then $[f,g]_\nu$ is modular of weight $k + \ell + 2\nu$ on Γ.*

Since each Rankin-Cohen bracket is obviously a bilinear operator on $M_*(\Gamma)$, it can as well be regarded as a product on this algebra. For $\nu > 0$, it is not associative and it is commutative if and only if ν is even. The bracket $[\cdot,\cdot]_0$ coincides with the usual product, while the bracket $[\cdot,\cdot]_1$ defines a Lie-bracket on $M_*(\Gamma)$. This gives $M_*(\Gamma)$ the structure of a so called *Poisson-algebra* (cf. [12, p. 53]).

Some important unary operators on modular forms are defined as follows.

Definition II.1.10. *Let $f:\mathbb{H}\to\mathbb{C}$ be a not necessarily holomorphic, but 1-periodic function with Fourier expansion*

$$f(\tau)=\sum_{n\in\mathbb{Z}} a_f(n,y)q^n.$$

Then we define for $N\in\mathbb{N}$ and χ a generalized Dirichlet character modulo N the operators

$$(f|U(N))(\tau):=\sum_{n\in\mathbb{Z}} a_f\left(Nn,\frac{y}{N}\right)q^n, \tag{II.1.4}$$

$$(f|V(N))(\tau):=f(N\tau), \tag{II.1.5}$$

$$(f|S_{N,r})(\tau):=\sum_{\substack{n\in\mathbb{Z}\\ n\equiv r \pmod N}} a_f(n,y)q^n \qquad \text{“sieving operator”}, \tag{II.1.6}$$

$$(f\otimes\chi)(\tau):=\sum_{n\in\mathbb{Z}} a_f(n,y)\chi(n)q^n. \tag{II.1.7}$$

If f transforms like a modular form of level M, then each of the above operators sends f to a modular form of (in general) higher level. Under certain conditions, the operator $U(N)$ can also preserve or even reduce the level. Note that all these operators can be realized by extending the definition of the slash operator to $\mathrm{GL}_2(\mathbb{Q})^+$, the group of 2×2 matrices over $\mathbb{Q}$ with positive determinant. In particular, we have the following.

Proposition II.1.11. *Let $f:\mathbb{H}\to\mathbb{C}$ a function transforming like a modular form of weight k and character χ on $\Gamma_0(M)$. Then it holds that*

(i) $f|U(N)$ and $f|V(N)$ transform like modular forms of the same weight and character on $\Gamma_0(NM)$.

(ii) If $N\mid M$ then $f|U(N)$ also keeps the same level as f and if $N^2|M$ and χ is a character modulo $\frac{M}{N}$ then the level of $f|U(N)$ reduces to $\frac{M}{N}$.

(iii) For $f\in M_k(4)$ with Fourier coefficient $a_f(n)=0$ for all $n\equiv 2 \pmod 4$ we have $f|U(4)\in M_k(1)$.

(iv) For $N=p$ a prime and $r\not\equiv 0 \pmod p$, the sieving operator $S_{p,r}$ changes the group to $\Gamma_0(\mathrm{lcm}(M,p^2))\cap\Gamma_1(p)$.

(v) Let ψ be a character modulo m, then $f\otimes\psi$ transforms like a modular form on $\Gamma_0(Mm^2)$ with character $\chi\psi^2$.

Proof. Assertions (i)-(iii) are contained in Lemmas 1 and 4 of [39], for (v) we refer the reader to Proposition 2.8 in [42].

Claim (iv) is mentioned e.g. in [4], but not proven, so we give a short prove here. It is clear that

$$(f|S_{p,r})(\tau) = \frac{1}{p}\sum \ell = 0^{N-1}\zeta_p^{-r\ell}\left(f\left|\begin{pmatrix}1 & \frac{\ell}{p}\\ 0 & 1\end{pmatrix}\right.\right)(\tau)$$

where $\zeta_p := e^{\frac{2\pi i}{p}}$ is a primitive pth root of unity. For $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(M)$ we have that

$$\begin{pmatrix}1 & -\frac{\ell}{p}\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & \frac{\ell}{p}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}a-\frac{\ell}{p}c & b+\frac{\ell}{p}(a-d)-\frac{\ell^2}{p^2}c\\ c & d+\frac{\ell}{p}c\end{pmatrix},$$

thus the group $\Gamma_0(\mathrm{lcm}(M,p^2))\cap\Gamma_1(p)$ is a subgroup of $\Gamma_0(M)$ which is normalized by $\begin{pmatrix}1 & \frac{\ell}{p}\\ 0 & 1\end{pmatrix}$. This implies (iv). $\square$

Definition II.1.12. *Let $k \in \mathbb{Z}$ and $f \in M_k(\Gamma_0(N))$ with Fourier expansion $f(\tau) = \sum_{m=0}^{\infty} a_f(m)q^m$ and let $n \in \mathbb{N}$. Then the nth* Hecke operator *of level N is defined by*

$$(f|T_n^{(k)}(N))(\tau) = \sum_{m=0}^{\infty}\left(\sum_{\substack{d|\gcd(m,n)\\ \gcd(d,N)=1}} d^{k-1}a_f\left(\frac{mn}{d^2}\right)\right)q^m.$$

.

Remark II.1.13. *The operator $T_n^{(k)(N)}$ maps $M_k(\Gamma_0(N))$ to $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ to $S_k(\Gamma_0(N))$.*

II.2 Jacobi Forms

Jacobi forms appear in many different contexts of modular forms. They are more or less an amalgam of modular forms and elliptic functions. First examples, namely the Jacobi Theta function, were already studied by C.G.J. Jacobi in the 19th century, special cases of these even go back to L. Euler, but it was not until 1985, when the first systematic treatment of their theory appeared. This treatment [22] by M. Eichler and D. Zagier is still the standard reference for the theory of Jacobi forms. Here, we only need some basic properties.

Definition II.2.1. *Let $\phi : \mathbb{C}\times\mathbb{H} \to \mathbb{C}$ be a holomorphic function and $k, m \in \mathbb{N}_0$. We call ϕ a* Jacobi form *of* weight k *and* index m *on Γ, if the following conditions hold.*

(i) For all $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma$ and $(z,\tau) \in \mathbb{C}\times\mathbb{H}$ it holds that

$$(\phi|_{k,m}\gamma)(z,\tau) := (c\tau+d)^{-k}e^{2\pi i m\frac{-cz^2}{c\tau+d}}\phi\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) = \phi(z,\tau),$$

(ii) For all $(\lambda,\mu)\in\mathbb{Z}^2$ *we have*

$$(\phi|_m(\lambda,\mu))(z,\tau):=e^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(z+\lambda\tau+\mu,\tau)=\phi(z,\tau),$$

(iii) ϕ *has a Fourier development*

$$\sum_{n,r\in\mathbb{Z}} c(n,r)q^n\zeta^r, \qquad (\zeta:=e^{2\pi i z}),$$

with $c(n,r)=0$ *for* $n<\frac{r^2}{4m}$.

We call z *the* elliptic *and* τ *the* modular *variable of* ϕ*.*

This definition has been extended to include Jacobi forms of half-integral weight and index as well as Jacobi forms with several elliptic and modular variables (then indexed by symmetric matrices or lattices), see for example [13], [49], [54], and the references therein.

The probably most popular and best-known example of a Jacobi form is the *Jacobi Theta function*,

$$\Theta(v;\tau):=\sum_{\nu\in\frac{1}{2}+\mathbb{Z}} q^{\frac{\nu^2}{2}}e^{2\pi i\nu\left(v+\frac{1}{2}\right)}, \tag{II.2.1}$$

which has weight and index $\frac{1}{2}$.

Remark II.2.2. *In the case of integral weight and index, the slash operators from Definition II.2.1 (i) and (ii) define a group action of the* Jacobi group $\Gamma\ltimes\mathbb{Z}^2$ *on the space of holomorphic functions* $\mathbb{C}\times\mathbb{H}\to\mathbb{C}$*.*

Recall that for elliptic modular forms we could construct new examples of modular forms (having possibly different level) by extending the definition of the slash operator to $\mathrm{SL}_2(\mathbb{Q})$. Naively trying this here unfortunately yields some trouble, since for $(\lambda,\mu)\in\mathbb{Q}^2$ [2] the elliptic transformation property (ii) in Definition II.2.1 does not define a group action anymore. The proper extension of the slash operator for Jacobi forms is rather given by the next theorem (see Theorem 1.4, [22]).

Theorem II.2.3. *The set*

$$G^J:=\left\{(\gamma,X,\zeta)\mid\gamma\in\mathrm{SL}_2(\mathbb{R}),\ X\in\mathbb{R}^2,\ \zeta\in\mathbb{C},\ |\zeta|=1\right\}$$

is a group via the multiplication law

$$(\gamma,X,\zeta)(\gamma',X',\zeta')=(\gamma\gamma',X\gamma'+X',\zeta\zeta'\cdot e^{2\pi i\det\left(\begin{smallmatrix}X\gamma'\\X'\end{smallmatrix}\right)}.$$

[2]Throughout this thesis a vector shall always be understood as row vector, if not stated otherwise

This group acts on the space of holomorphic functions $\phi : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ *via*

$$\phi| \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), (\lambda, \mu), \zeta \right) (z, \tau) = \zeta^m (c\tau + d)^{-k} e^{2\pi i m \left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu \right)} \times \phi\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right).$$

The above result now implies a handy theorem that we make use of later on.

Theorem II.2.4 (Theorem 1.3, [22]). *Let* ϕ *be a Jacobi form on* Γ *of weight* $k \in \mathbb{N}$ *and index* $m \in \mathbb{N}$ *and let* λ *and* μ *be rational numbers. Then the function* $f(\tau) := e^{2\pi i \lambda^2 m \tau} \phi(\lambda\tau + \mu, \tau)$ *is a modular form of weight* k *on the group*

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid (a-1)\lambda + c\mu, b\lambda + (d-1)\mu, m(c\mu^2 + (d-a)\lambda\mu - b\lambda^2) \in \mathbb{Z} \right\}.$$

II.3 Harmonic Maaß-Forms and Mock Modular Forms

The first one to study non-holomorphic modular forms systematically was H. Maaß. The notion of *harmonic weak Maaß forms* was first introduced by J.H. Bruinier and J. Funke in [11]. A survey of their work and its connection to number theory is given in [43] and also in [15].

For $k \in \frac{1}{2}\mathbb{Z}$ let us first introduce the weight k *hyperbolic Laplacian*

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right). \tag{II.3.1}$$

Definition II.3.1. *We call a smooth function* $f : \mathbb{H} \to \mathbb{C}$ *a* harmonic (weak) Maaß[3] form *of weight* $k \in \frac{1}{2}\mathbb{Z}$ *on* $\Gamma = \Gamma_0(N)$ *with character* χ *if the following conditions are met.*

1. f *transforms like a modular form of weight* k*, i.e.* $(f|_k\gamma) = \chi(d)f$ *for all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$.

2. f *lies in the kernel of the hyperbolic Laplacian, i.e.* $\Delta_k f \equiv 0$.

3. f *grows at most linearly exponentially at the cusps of* Γ.

The vector space of harmonic Maaß forms of weight k *on the group* $\Gamma_0(N)$ *and character* χ *is denoted by* $\mathcal{H}_k(N, \chi)$ *(resp.* $\mathcal{H}_k(\Gamma)$ *for other groups* Γ*).*

From the Γ-equivariance it is plain that a harmonic Maaß form possesses a Fourier expansion, and from the fact that it is annihilated by Δ_k as well as the growth condition it is not hard to see the following.

[3] in the literature the spelling "Maass form" is more common

Lemma II.3.2 ([43], Lemma 7.2). *Let f be a harmonic weak Maaß form of weight $k \neq 1$. Then f has a canonical splitting into*

$$f(\tau) = f^+(\tau) + \frac{(4\pi y)^{1-k}}{k-1}\overline{c_f^-(0)} + f^-(\tau), \tag{II.3.2}$$

where for some $m_0, n_0 \in \mathbb{Z}$ we have the Fourier expansions

$$f^+(\tau) = \sum_{n=m_0}^{\infty} c_f^+(n) q^n$$

and

$$f^-(\tau) = \sum_{\substack{n=n_0\\ n\neq 0}}^{\infty} \overline{c_f^-(n)} n^{k-1} \Gamma(1-k; 4\pi n y) q^{-n}.$$

As usually we set $q := e^{2\pi i \tau}$ and

$$\Gamma(\alpha; x) := \int_x^\infty t^{\alpha-1} e^{-t} dt, \qquad x > 0, \tag{II.3.3}$$

denotes the incomplete Gamma function.

This motivates the following definition.

Definition II.3.3. *(i) The functions f^+ (resp. $\frac{(4\pi y)^{1-k}}{k-1}\overline{c_f^-(0)} + f^-(\tau)$) in Lemma II.3.2 are referred to as the* holomorphic *(resp.* non-holomorphic*)* part *of the harmonic Maaß form f.*

(ii) The holomorphic part of a harmonic weak Maaß form of weight k is called a mock modular form *of weight k.*

The non-holomorphic part of a harmonic Maaß form is now associated to a weakly holomorphic modular form.

Proposition II.3.4 ([11], Proposition 3.2). *Define for $1 \neq k \in \frac{1}{2}\mathbb{Z}$ the operator*

$$\xi_k := 2iy^k \overline{\frac{\partial}{\partial \overline{\tau}}}.$$

Then the mapping

$$\mathcal{H}_k(N, \chi) \mapsto M^!_{2-k}(N, \overline{\chi}), \ f \mapsto \xi_k f$$

is well-defined and surjective with kernel $M^!_k(N, \chi)$. With the notation from Lemma II.3.2 we have

$$(\xi_k f)(\tau) = (4\pi)^{1-k} \sum_{n=n_0}^{\infty} c_f^-(n) q^n.$$

Definition II.3.5. *Let f be a harmonic Maaß form of weight k on Γ.*

(i) We call the function $\xi_k f$ the shadow *of f or the mock modular form f^+.*

(ii) A mock modular form of weight $\frac{1}{2}$, whose shadow is a linear combination of weight $\frac{3}{2}$ theta functions as in (II.1.3)*, is called a* mock theta function*.*

(iii) The preimage of $M_{2-k}(\Gamma)$ resp. $S_{2-k}(\Gamma)$ under ξ_k is denoted by $\mathcal{M}_k(\Gamma)$ resp. $\mathcal{S}_k(\Gamma)$.

In many cases applications, mock modular forms do not occur on their own, but often combined with regular modular forms. This motivates the following definition.

Definition II.3.6. *Let f be a mock modular form of weight k and g be a holomorphic modular form of weight ℓ.*

(i) The product $f \cdot g$ is called a mixed mock modular form *of weight (k, ℓ).*

(ii) More generally, the νth Rankin-Cohen bracket $[f, g]_\nu$ of f and g is called a mixed mock modular form of weight (k, ℓ) and degree ν*.*

II.4 Appell-Lerch sums

As already mentioned in the introduction, there were originally 17 examples of mock theta functions that Ramanujan introduced in his deathbed letter to Hardy. Further examples were discovered later for example in Ramanujan's Lost Notebook. The first consistent framework for all the mock theta functions of Ramanujan was given by S. Zwegers in his 2002 Ph.D. thesis [55] written under the direction of D. Zagier. Actually, Zwegers found 3 different frameworks which the mock theta functions of Ramanujan fit into: Appell-Lerch sums , indefinite theta functions, and Fourier coefficients of meromorphic Jacobi forms. In this section, we shall give a brief account of the most important facts about Appell-Lerch sums as can be found in [55, 56] that we use in Chapter III.

Definition II.4.1. *For $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C} \setminus (\mathbb{Z} \oplus \mathbb{Z}\tau)$ we define the level ℓ (not to be confuse with the level of a modular form or group)* Appell-Lerch sum *by the expression*

$$A_\ell(u, v) = A_\ell(u, v; \tau) := e^{\pi i \ell u} \sum_{n \in \mathbb{Z}} \frac{(-1)^{\ell n} q^{\frac{\ell}{2} n(n+1)} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n}.$$

This function is holomorphic for all u, v, τ where it is defined, but does not quite transform nicely under modular inversion, see e.g. Proposition 1.5 in [55] for the case of level 1. But by adding a certain non-holomorphic, but real-analytic function which has the same "modular defect", we can complete this sum to transform like a Jacobi form. This function is defined as

(II.4.1)
$$R(u;\tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left\{\operatorname{sgn}(\nu) - E\left(\left(\nu + \frac{\operatorname{Im} u}{y}\right)\sqrt{2y}\right)\right\} (-1)^{\nu-\frac{1}{2}} q^{-\frac{\nu^2}{2}} e^{-2\pi i\nu u},$$

(II.4.2) $$E(t) := 2\int_0^t e^{-\pi u^2} du = \operatorname{sgn}(t)\left(1-\beta\left(t^2\right)\right),$$

(II.4.3) $$\beta(x) := \int_x^\infty u^{-\frac{1}{2}} e^{-\pi u} du,$$

where for the second equality in (II.4.2) we refer to [55, Lemma 1.7].

The function R itself satisfies several functional equations that we will need later.

Proposition II.4.2 ([55], Proposition 1.9). *The function R fulfills the elliptic transformation properties*

(i) $R(u+1;\tau) = -R(u;\tau)$

(ii) $R(u;\tau) + e^{-2\pi i u - \pi i \tau} R(u+\tau;\tau) = 2e^{-\pi i u - \pi i \frac{\tau}{4}}$

(iii) $R(-u) = R(u)$.

In the proof of Theorem III.2.2, the following observation is vital. It has already been mentioned in [9] (without proof), so we give one here.

Proposition II.4.3. *The function R lies in the kernel of the renormalized Heat operator $2D_\tau + D_u^2$, hence*

(II.4.4) $$D_u^2 R = -2D_\tau R.$$

Proof. We set $Z_\nu = (-1)^{\nu-1/2} q^{-\frac{\nu^2}{2}} e^{-2\pi i \nu u}$ for $\nu \in \frac{1}{2}+\mathbb{Z}$ and abbreviate

$$\Upsilon_\nu := \left\{\operatorname{sgn}(\nu) - E\left(\left(\nu + \frac{\operatorname{Im} u}{y}\right)\sqrt{2y}\right)\right\}.$$

With this we get

$$\begin{aligned}
&-2D_\tau R(u,\tau)\\
&= -2\sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left[-\frac{1}{\pi i} e^{-2\pi\left(\nu+\frac{\operatorname{Im} u}{y}\right)^2 y} \cdot \left(-\frac{\sqrt{2y}\operatorname{Im} u}{2iy^2} + \left(\nu + \frac{\operatorname{Im} u}{y}\right)\frac{1}{2i\sqrt{2y}}\right) - \Upsilon_\nu \frac{\nu^2}{2}\right] Z_\nu\\
&= \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left[\frac{1}{\sqrt{2y}\pi}\left(\frac{\operatorname{Im} u}{y} - \nu\right) e^{-2\pi\left(\nu+\frac{\operatorname{Im} u}{y}\right)^2 y} + \nu^2 \Upsilon_\nu\right] Z_\nu
\end{aligned}$$

and

$$\text{(II.4.5)}\quad \begin{aligned} D_u(R(u;\tau)) &= \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left[-\frac{1}{\pi i} e^{-2\pi\left(\nu+\frac{\operatorname{Im} u}{y}\right)^2 y} \cdot \frac{1}{2i}\frac{\sqrt{2y}}{y} - \nu\Upsilon_\nu\right] Z_\nu \\ &= \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left[\frac{1}{\sqrt{2y}\pi} e^{-2\pi\left(\nu+\frac{\operatorname{Im}(u)}{y}\right)^2 y} - \nu\Upsilon_\nu\right] Z_\nu, \end{aligned}$$

hence

$$\begin{aligned} D_u^2(R(u;\tau)) = \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \frac{1}{2\pi i} &\left[\frac{1}{\sqrt{2y}\pi} e^{-2\pi\left(\nu+\frac{\operatorname{Im} u}{y}\right)^2 y} \cdot \left(-4\pi y\left(\nu+\frac{\operatorname{Im} u}{y}\right)\frac{1}{2iy}\right)\right. \\ &\left. +2\nu e^{-2\pi\left(\nu+\frac{\operatorname{Im}(u)}{y}\right)^2 y} \cdot \frac{\sqrt{2y}}{2iy}\right] Z_\nu \\ &- \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left[\frac{1}{\sqrt{2y}\pi} e^{-2\pi(\nu+\frac{\operatorname{Im}(u)}{y})^2 y} - \nu\Upsilon_\nu\right] \nu Z_\nu \\ = \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} &\left[\frac{1}{\sqrt{2y}\pi}\left(\frac{\operatorname{Im} u}{y}-\nu\right) e^{-2\pi\left(\nu+\frac{\operatorname{Im}(u)}{y}\right)^2 y} + \nu^2\Upsilon_\nu\right] Z_\nu \end{aligned}$$

and the assertion is proven. ∎

Theorem II.4.4 (Theorem 2.2, [56]). *Define*

(II.4.6)

$$\begin{aligned} \widehat{A}_\ell(u,v;\tau) &:= A_\ell(u,v;\tau) \\ &+ \frac{i}{2}\sum_{k=0}^{\ell-1} e^{2\pi iku}\Theta\left(v+k\tau+\frac{\ell-1}{2};\ell\tau\right) R\left(\ell u - v - k\tau - \frac{\ell-1}{2};\ell\tau\right). \end{aligned}$$

with Θ *as in* (II.2.1). *Then the following equations hold true.*

- *(S)* $\widehat{A}_\ell(-u,-v) = -\widehat{A}_1(u,v)$.
- *(E)* $\widehat{A}_\ell(u+\lambda_1\tau+\mu_1, v+\lambda_2\tau+\mu_2) = (-1)^{\ell(\lambda_1+\mu_1)} e^{2\pi iu(\ell\lambda_1-\lambda_2)} e^{-2\pi iv\lambda_1} q^{\ell\frac{\lambda_1^2}{2}-\lambda_1\lambda_2}\widehat{A}_\ell(u,v)$ *for* $\lambda_i,\mu_i\in\mathbb{Z}$.
- *(M)* $\widehat{A}_\ell(\frac{u}{c\tau+d},\frac{v}{c\tau+d};\frac{a\tau+b}{c\tau+d}) = (c\tau+d)e^{\pi ic\frac{-\ell u^2+2uv}{c\tau+d}}\widehat{A}_\ell(u,v;\tau)$ *for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$.

Another way to summarize (E) and (M) is to say that $\widehat{A}_\ell$ *transforms like a Jacobi form of weight* 1 *and index* $\left(\begin{smallmatrix} -\ell & 1 \\ 1 & 0\end{smallmatrix}\right)$.

Chapter III

Cohen's Conjecture

In this chapter, we give a proof of Conjecture I.2.1 following [40], but in greater detail. For this, we need some preparatory calculations which we give in Section III.1, the proof itself is subject of Section III.2.

III.1 Some Preliminaries

III.1.1 The Gamma function

Since many of the calculations in this and the following chapters involve identities about the Gamma function, we recall them in this very short subsection. As a reference, see e.g. [25, Chapter IV.1].

For $\mathrm{Re}(s) > 0$, the Gamma function is defined by the integral

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt.$$

It has an meromorphic continuation to the entire complex plane with simple poles in $s = -n$, $n \in \mathbb{N}_0$, with residue $\frac{(-1)^n}{n!}$. It satisfies the following functional equations wherever all the expressions make sense,

$$\Gamma(s+1) = s\Gamma(s), \tag{III.1.1}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \tag{III.1.2}$$

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right). \tag{III.1.3}$$

Equation (III.1.2) dates back to Euler and is sometimes called the *reflection formula.* It yields for example immediately the special value

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Equation (III.1.3) is known as *Legendre's duplication formula.*

III.1.2 Preparatory Lemmas

We first rewrite Cohen's conjecture I.2.1 in a matter such that we see, which kinds of modular objects we are dealing with.

Remark III.1.1. *The coefficient of* $X^{2\nu}$ *in* (I.2.1) *is given by*

$$\frac{c_\nu}{2}\left([\mathscr{H},\vartheta]_\nu(\tau)-[\mathscr{H},\vartheta]_\nu(\tau+\tfrac{1}{2})\right)+\Lambda_{2\nu+1,odd}(\tau), \tag{III.1.4}$$

where $c_\nu=\nu!\frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}$ *and*

$$\Lambda_{\ell,odd}(\tau):=\sum_{n=0}^{\infty}\lambda_\ell(2n+1)q^{2n+1} \tag{III.1.5}$$

with λ_ℓ *as in* (I.1.3)*. The coefficient of* $X^{2\nu+1}$ *is identically* 0*.*

Proof. From [14, p. 283] we see that the first part of (I.2.1) equals

$$\sum_{\nu=0}^{\infty}\frac{\sqrt{\pi}\nu!F_\nu^{odd}(\vartheta,\mathscr{H})X^{2\nu}}{\Gamma(\nu+\frac{1}{2})\nu!(2\pi i)^\nu},$$

where for smooth functions f,g on $\mathbb{H}$ we define

$$F_\nu^{odd}(f,g)(\tau):=\frac{1}{2}\left(F_\nu(f,g)(\tau)-F_\nu(f,g)(\tau+\tfrac{1}{2})\right)$$

with F_ν as in [14, Theorem 7.1]. Using that in general for smooth functions f,g we have $\nu!(2\pi i)^\nu[f,g]_\nu=F_\nu(f,g)$ we see that this equals

$$\sum_{\nu=0}^{\infty}\frac{c_\nu}{2}\left([\mathscr{H},\vartheta]_\nu(\tau)-[\mathscr{H},\vartheta]_\nu(\tau+\tfrac{1}{2})\right)X^{2\nu}$$

which implies our claim. □

Thus each of the coefficients has to parts, one including the class number generating function and one including the function Λ_ℓ, which is sometimes called a *mock Eisenstein series*.

Lemma III.1.2. *For odd* $k\in\mathbb{N}$*, the function* $\Lambda_{k,odd}$ *can be written as a linear combination of derivatives of Appell-Lerch sums, more precisely*

$$\Lambda_{k,odd}=\frac{1}{2}(D_v^kA_1^{odd})(0,\tau+\tfrac{1}{2};2\tau),$$

where we define

$$\begin{aligned}A_1^{odd}(u,v;\tau):=&e^{\pi iu}\sum_{\substack{n\in\mathbb{Z}\\ n\ odd}}\frac{(-1)^nq^{\frac{n(n+1)}{2}}e^{2\pi inv}}{1-e^{2\pi iu}q^n}\\ =&\frac{1}{2}\left(A_1(u,v;\tau)-A_1(u,v+\tfrac{1}{2};\tau)\right).\end{aligned}$$

Proof. First we remark that the right-hand side of the identity to be shown is actually well-defined because as a function of u, $A_1(u,v;\tau)$ has simple poles in $\mathbb{Z}\tau+\mathbb{Z}$ (cf. [55, Proposition 1.4]) which cancel out if the sum is only taken over odd integers. Thus the equation actually makes sense.

Then we write $\Lambda_{k,odd}$ as a q-series

$$\begin{aligned}
&2\Lambda_{k,odd}(\tau)\\
=&\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\min(2\ell+1,2m+1)^k q^{(2\ell+1)(2m+1)}\\
=&2\sum_{\ell=0}^{\infty}\left((2\ell+1)^k\sum_{r=1}^{\infty}q^{(2\ell+1)(2\ell+1+2r)}\right)+\sum_{\ell=0}^{\infty}(2\ell+1)^k q^{(2\ell+1)^2}\\
=&2\sum_{\ell=0}^{\infty}\left((2\ell+1)^k q^{(2\ell+1)^2}\sum_{r=1}^{\infty}\left(q^{2(2\ell+1)}\right)^r\right)+\sum_{\ell=0}^{\infty}(2\ell+1)^k q^{(2\ell+1)^2}\\
=&2\sum_{\ell=0}^{\infty}(2\ell+1)^k q^{(2\ell+1)^2}\left(\frac{1}{1-q^{2(2\ell+1)}}-1\right)+\sum_{\ell=0}^{\infty}(2\ell+1)^k q^{(2\ell+1)^2}\\
=&\sum_{\ell=0}^{\infty}(2\ell+1)^k\left(\frac{2q^{(2\ell+1)^2}}{1-q^{2(2\ell+1)}}-q^{(2\ell+1)^2}\right)\\
=&\sum_{\ell=0}^{\infty}(2\ell+1)^k\left(\frac{q^{(2\ell+1)^2}+q^{(2\ell+1)^2+2(2\ell+1)}}{1-q^{2(2\ell+1)}}\right)
\end{aligned}$$

and compare this to

$$\begin{aligned}
&\left(D_v^k A_1^{odd}\right)\left(0,\tau+\tfrac{1}{2},2\tau\right)\\
=&\sum_{\substack{n\in\mathbb{Z}\\ n\text{ odd}}} n^k\frac{q^{n^2+2n}}{1-q^{2n}}\\
=&\sum_{n\in\mathbb{Z}}(2n+1)^k\frac{q^{(2n+1)^2+2(2n+1)}}{1-q^{2(2n+1)}}\\
=&\sum_{n=0}^{\infty}(2n+1)^k\frac{q^{(2n+1)^2+2(2n+1)}}{1-q^{2(2n+1)}}+\sum_{n=1}^{\infty}(-2n+1)^k\frac{q^{(-2n+1)^2+2(-2n+1)}}{1-q^{2(-2n+1)}}\\
=&\sum_{n=0}^{\infty}(2n+1)^k\frac{q^{(2n+1)^2+2(2n+1)}}{1-q^{2(2n+1)}}+\sum_{n=0}^{\infty}(-2n-1)^k\frac{q^{(2n+1)^2-2(2n+1)}}{1-q^{-2(2n+1)}}\\
=&\sum_{n=0}^{\infty}(2n+1)^k\frac{q^{(2n+1)^2+2(2n+1)}+q^{(2n+1)^2}}{1-q^{2(2n+1)}}\\
=&2\Lambda_{k,odd}
\end{aligned}$$

□

From this lemma we immediately see that the function

$$\frac{c_\nu}{2}\left([\widehat{\mathscr{H}},\vartheta]_\nu(\tau)-[\widehat{\mathscr{H}},\vartheta]_\nu(\tau+\tfrac{1}{2})\right)+(D_v^{2\nu+1}\widehat{A}_1^{odd})(0,\tau+\tfrac{1}{2};2\tau) \tag{III.1.6}$$

transforms like a modular form on some subgroup of $\Gamma_0(4)$. Due to the polynomial growth of the Fourier coefficients it is plain that this function doesn't explode near the cusps. Thus in order to prove Conjecture I.2.1 it is enough to make sure that the mentioned subgroup actually equals $\Gamma_0(4)$ and to look at the non-holomorphic parts

$$\frac{c_\nu}{2}\left([\mathscr{R},\vartheta]_\nu(\tau)-[\mathscr{R},\vartheta]_\nu(\tau+\tfrac{1}{2})\right) \tag{III.1.7}$$

and

$$\begin{aligned}\frac{i}{4}\Bigg(&\sum_{\ell=0}^{2\nu+1}\binom{2\nu+1}{\ell}(-1)^\ell(D_v^\ell R)(-\tau-\tfrac{1}{2};2\tau)(D_v^{2\nu-\ell+1}\Theta)(\tau+\tfrac{1}{2};2\tau)\\ &-\sum_{\ell=0}^{2k+1}\binom{2k+1}{\ell}(-1)^\ell(D_v^\ell R)(-\tau-1;2\tau)(D_v^{2\nu-\ell+1}\Theta)(\tau+1;2\tau)\Bigg)\end{aligned} \tag{III.1.8}$$

and show that these cancel each other.

From this we see the necessity to investigate the derivatives of Θ and R evaluated at the torsion points $(\pm(\tau+\frac{1}{2});2\tau)$ and $(\pm(\tau+1);2\tau)$. First we observe the following:

Remark III.1.3. *Assume that $(D_v^r\Theta)(\tau+\frac{1}{2};2\tau)=f_r(\tau)$ and $(D_u^rR)(-\tau-\frac{1}{2};2\tau)=g_r(\tau)$ for some functions f_r, g_r. From the definitions we immediately see that $(D_v^r\Theta)(v;\tau+1)=e^{\frac{\pi i}{4}}(D_v^r\Theta)(v;\tau)$ and $(D_u^rR)(u;\tau+1)=e^{-\frac{\pi i}{4}}(D_u^rR)(u;\tau)$. Then we see by shifting $\tau\mapsto\tau+\frac{1}{2}$, we get*

$$(D_v^r\Theta)(\tau+1;2\tau)=e^{-\frac{\pi i}{4}}(D_v^r\Theta)\left(\left(\tau+\tfrac{1}{2}\right)+\tfrac{1}{2};2\left(\tau+\tfrac{1}{2}\right)\right)=e^{-\frac{\pi i}{4}}f_r\left(\tau+\tfrac{1}{2}\right)$$

and similarly

$$(D_u^rR)\left(-\tau-1;2\tau\right)=e^{\frac{\pi i}{4}}g_r\left(\tau+\tfrac{1}{2}\right).$$

Hence it suffices to show only one of the identities in question.

Lemma III.1.4. *For $r\in\mathbb{N}_0$ one has*

$$(D_v^r\Theta)\left(\tau+\tfrac{1}{2};2\tau\right)=-q^{-\frac{1}{4}}\sum_{s=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2s}\left(-\frac{1}{2}\right)^{r-2s}(D_\tau^s\vartheta)(\tau) \tag{III.1.9}$$

and

$$(D_v^r\Theta)(\tau+1;2\tau)=iq^{-1/4}\sum_{s=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2s}\left(-\frac{1}{2}\right)^{r-2s}(D_\tau^s\vartheta)\left(\tau+\tfrac{1}{2}\right) \tag{III.1.10}$$

Proof. The proof is just a simple calculation:

Obviously it holds that

$$(D_v^r\Theta)(v;\tau) = \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \nu^r q^{\frac{\nu^2}{2}} e^{2\pi i\nu\left(v+\frac{1}{2}\right)}.$$

Therefore

$$\begin{aligned}(D_v^r\Theta)(\tau+\tfrac{1}{2},2\tau) &= \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \nu^r q^{\nu^2+\nu} e^{2\pi i\nu} \\ &= -q^{-\frac{1}{4}}\sum_{n\in\mathbb{Z}}(n+\tfrac{1}{2})^r q^{(n+1)^2} \\ &= -q^{-\frac{1}{4}}\sum_{n\in\mathbb{Z}}\left[\sum_{s=0}^{r}\binom{r}{s}n^s\left(-\frac{1}{2}\right)^{r-s}\right]q^{n^2} \\ &= -q^{-\frac{1}{4}}\sum_{s=0}^{r}\binom{r}{s}\left(-\frac{1}{2}\right)^{r-s}\underbrace{\sum_{n\in\mathbb{Z}}n^s q^{n^2}}_{=0\text{ for }s\text{ odd}} \\ &= -q^{-\frac{1}{4}}\sum_{s=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2s}\left(-\frac{1}{2}\right)^{r-2s}\sum_{n\in\mathbb{Z}}(n^2)^s q^{n^2} \\ &= -q^{-\frac{1}{4}}\sum_{s=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2s}\left(-\frac{1}{2}\right)^{r-2s}(D_\tau^s\vartheta)(\tau)\end{aligned}$$

By Remark III.1.3 one gets the result for plugging in $(\tau+1,2\tau)$. ☐

Lemma III.1.5. *The following identities are true:*

$$R\left(-\tau-\tfrac{1}{2};2\tau\right) = iq^{\frac{1}{4}} \tag{III.1.11}$$

$$R(-\tau-1;2\tau) = -q^{\frac{1}{4}} \tag{III.1.12}$$

$$(D_uR)\left(-\tau-\tfrac{1}{2};2\tau\right) = \frac{-1+i}{4\pi}q^{\frac{1}{4}}\int_{-\overline{\tau}}^{i\infty}\frac{\vartheta(z)}{(z+\tau)^{\frac{3}{2}}}dz - \frac{i}{2}q^{\frac{1}{4}} \tag{III.1.13}$$

$$(D_uR)(-\tau-1;2\tau) = -\frac{1+i}{4\pi}q^{\frac{1}{4}}\int_{-\overline{\tau}}^{i\infty}\frac{\vartheta\left(z+\frac{1}{2}\right)}{(z+\tau)^{\frac{3}{2}}}dz + \frac{1}{2}q^{\frac{1}{4}}. \tag{III.1.14}$$

Proof. As above, we only prove (III.1.11) and (III.1.13).

Equation (III.1.11) follows easily from the transformation properties of R as given in Proposition II.4.2 (the numbers above the equality signs give the used transformation):

$$\begin{aligned}R\left(-\tau-\tfrac{1}{2};2\tau\right) &\overset{(ii)}{=} 2e^{\pi i\left(\tau+\frac{1}{2}\right)-\frac{\pi i\tau}{2}} - e^{2\pi i\left(\tau+\frac{1}{2}\right)-2\pi i\tau}R\left(\tau-\tfrac{1}{2};2\tau\right) \\ &\overset{(iii)}{=} 2iq^{\frac{1}{4}} + R\left(-\tau+\tfrac{1}{2};2\tau\right) \overset{(i)}{=} 2iq^{\frac{1}{4}} - R\left(-\tau-\tfrac{1}{2};2\tau\right),\end{aligned}$$

which gives (III.1.11).

Let us now turn to (III.1.13). We have by (II.4.5) that

$$(D_uR)\left(-\tau-\tfrac{1}{2};2\tau\right)=iq^{\frac{1}{4}}\sum_{n\in\mathbb{Z}}\left(\frac{1}{\sqrt{4y}\pi}e^{-4\pi n^2y}-\operatorname{sgn}(n)\left(n+\frac{1}{2}\right)\beta(4n^2y)\right)q^{-n^2},$$

with β as in (II.4.3) and $\operatorname{sgn}(0):=1$. By partial integration we get for real $t\geq 0$ that

$$\beta(t)=\frac{1}{\pi}t^{-\frac{1}{2}}e^{-\pi t}-\frac{1}{2\sqrt{\pi}}\Gamma\left(-\frac{1}{2};\pi t\right) \tag{III.1.15}$$

with $\Gamma(\alpha;x)$ as in (II.3.3).

We further observe the following ($\tau\in\mathbb{H},n\in\mathbb{N}$):

$$\begin{aligned}\int_{-\overline{\tau}}^{i\infty}\frac{e^{2\pi inz}}{(-i(z+\tau))^{\frac{3}{2}}}dz&=i\int_y^\infty\frac{e^{2\pi in(-x+it)}}{(y+t)^{\frac{3}{2}}}dt\\&=ie^{-2\pi inx+2\pi ny}\int_y^\infty\frac{e^{-2\pi n(y+t)}}{(y+t)^{\frac{3}{2}}}\\&=iq^{-n}\sqrt{2\pi n}\int_{4\pi ny}^\infty e^{-u}u^{-\frac{3}{2}}du\\&=iq^{-n}\sqrt{2\pi n}\Gamma\left(-\frac{1}{2};4\pi ny\right).\end{aligned} \tag{III.1.16}$$

From here we calculate

$$\begin{aligned}(D_uR)\left(-\tau-\tfrac{1}{2},2\tau\right)=&i\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}\Bigg(\frac{1}{\sqrt{4y}\pi}e^{-4\pi\left(\nu-\frac{1}{2}\right)^2y}\\&-\nu\left\{\operatorname{sgn}(\nu)-\operatorname{sgn}\left(\nu-\frac{1}{2}\right)\cdot\left(1-\beta\left(4\left(\nu-\tfrac{1}{2}\right)^2y\right)\right)\right\}\Bigg)q^{-\nu^2+\nu}\\=&iq^{\frac{1}{4}}\sum_{n\in\mathbb{Z}}\Bigg(\frac{1}{\sqrt{4y}\pi}e^{-4\pi n^2y}\\&-\left(n+\frac{1}{2}\right)\left\{\operatorname{sgn}\left(n+\frac{1}{2}\right)-\operatorname{sgn}(n)\cdot(1-\beta(4n^2y))\right\}\Bigg)q^{-n^2}\\=&iq^{\frac{1}{4}}\sum_{n\in\mathbb{Z}}\left(\frac{1}{\sqrt{4y}\pi}e^{-4\pi n^2y}-\operatorname{sgn}(n)\left(n+\frac{1}{2}\right)\beta(4n^2y)\right)q^{-n^2}\\=&iq^{\frac{1}{4}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{2\sqrt{\pi}}\operatorname{sgn}(n)n\Gamma\left(-\frac{1}{2};4\pi n^2y\right)q^{-n^2}\\&-\frac{i}{2}q^{\frac{1}{4}}\underbrace{\beta(0)}_{=1}+\frac{iq^{\frac{1}{4}}}{\sqrt{4y}\pi}\end{aligned}$$

$$\begin{aligned}
=&iq^{\frac{1}{4}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{2\sqrt{\pi}}|n|\left(\int_{-\overline{\tau}}^{i\infty}\frac{e^{2\pi in^2z}}{(-i(z+\tau))^{3/2}}dz\right)\frac{1}{i\sqrt{2\pi n^2}}\\
&-\frac{i}{2}q^{\frac{1}{4}}+\frac{iq^{\frac{1}{4}}}{\sqrt{4y\pi}}\\
=&\frac{-1+i}{4\pi}q^{\frac{1}{4}}\int_{-\overline{\tau}}^{i\infty}\frac{\vartheta(z)}{(z+\tau)^{\frac{3}{2}}}-\frac{i}{2}q^{\frac{1}{4}},
\end{aligned}$$

which is what we claimed. $\square$

Remark III.1.6. *From* (III.1.16) *we can also deduce another representation of the function* $\mathscr{R}$*:*

$$\mathscr{R}(\tau)=\frac{1}{8\pi\sqrt{y}}+\frac{1}{4\sqrt{\pi}}\sum_{n=1}^{\infty}n\Gamma\left(-\frac{1}{2};4\pi n^2y\right)q^{-n^2}.$$

With this, we can modify (III.1.8) in the following way.

Lemma III.1.7. *For all* $\nu\in\mathbb{N}_0$ *it holds true that*

$$\begin{aligned}
&\sum_{\mu=0}^{2\nu+1}(-1)^{\mu}\binom{2\nu+1}{\mu}(D_v^{\mu}R)\left(-\tau-\tfrac{1}{2};2\tau\right)\left(D_v^{2\nu-\mu+1}\Theta\right)\left(\tau+\tfrac{1}{2};2\tau\right)\\
=&q^{-\frac{1}{4}}\sum_{\lambda=0}^{\nu}\sum_{\mu=0}^{\nu-\lambda}\left[\frac{1}{2}(D_v^{2\mu}R)\left(-\tau-\tfrac{1}{2};2\tau\right)+\frac{2(\nu-\mu-\lambda)+1}{2\mu+1}(D_v^{2\mu+1}R)\left(-\tau-\tfrac{1}{2};2\tau\right)\right]\\
&\qquad\times b_{\nu,\mu,\lambda}\left(\frac{1}{2}\right)^{2(\nu-\mu-\lambda)}(D_\tau^{\lambda}\vartheta)(\tau)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\mu=0}^{2\nu+1}(-1)^{\mu}\binom{2\nu+1}{\mu}(D_v^{\mu}R)(-\tau-1;2\tau)(D_v^{2\nu-\mu+1}\Theta)(\tau+1;2\tau)\\
=&-iq^{-\frac{1}{4}}\sum_{\lambda=0}^{\nu}\sum_{\mu=0}^{\nu-\lambda}\left[\frac{1}{2}(D_v^{2\mu}R)(-\tau-1;2\tau)+\frac{2(\nu-\mu-\lambda)+1}{2\mu+1}(D_v^{2\mu+1}R)(-\tau-1;2\tau)\right]\\
&\qquad\times b_{\nu,\mu,\lambda}\left(\frac{1}{2}\right)^{2(\nu-\mu-\lambda)}(D_\tau^{\lambda}\vartheta)(\tau+\tfrac{1}{2}),
\end{aligned}$$

where

$$b_{\nu,\mu,\lambda}:=\frac{(2\nu+1)!}{(2\mu)!(2\lambda)!(2(\nu-\mu-\lambda)+1)!}=\binom{2\nu+1}{2\mu,\ 2\lambda,\ 2(\nu-\mu-\lambda)+1}.$$

Proof. Again by Remark III.1.3 it is enough to prove the first claim. For simplicity, we omit the arguments of the functions in the calculation. We obtain

$$
\begin{aligned}
&\sum_{\mu=0}^{2\nu+1}(-1)^\mu\binom{2\nu+1}{\mu}(D_v^\mu R)(D_v^{2\nu-\mu+1}\Theta)\\
=&\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu}(D_v^{2\mu}R)(D_v^{2(\nu-\mu)+1}\Theta)-\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu+1}(D_v^{2\mu+1}R)(D_v^{2(\nu-\mu)}\Theta)\\
\overset{(\text{III.1.9})}{=}&-q^{-\frac14}\Bigg[\sum_{\mu=0}^{\nu}\sum_{\lambda=0}^{\nu-\mu}\underbrace{\binom{2\nu+1}{2\mu}\binom{2(\nu-\mu)+1}{2\lambda}}_{=b_{\nu,\mu,\lambda}}\left(-\frac12\right)^{2(\nu-\mu-\lambda)+1}(D_v^{2\mu}R)(D_\tau^\lambda\vartheta)\\
&-\sum_{\mu=0}^{\nu}\sum_{\lambda=0}^{\nu-\mu}\underbrace{\binom{2\nu+1}{2\mu+1}\binom{2(\nu-\mu)}{2\lambda}}_{=\frac{2(\nu-\mu-\lambda)+1}{2\mu+1}b_{\nu,\mu,\lambda}}\left(-\frac12\right)^{2(\nu-\mu-\lambda)}(D_v^{2\mu+1}R)(D_\tau^\lambda\vartheta)\Bigg]\\
=&q^{-\frac14}\sum_{\mu=0}^{\nu}\sum_{\lambda=0}^{\nu-\mu}\left[\frac12(D_v^{2\mu}R)+\frac{2(\nu-\mu-\lambda)+1}{2\mu+1}(D_v^{2\mu+1}R)\right]b_{\nu,\mu,\lambda}\left(\frac12\right)^{2(\nu-\mu-\lambda)}(D_\tau^\lambda\vartheta)
\end{aligned}
$$

Interchanging the sums gives the desired result. □

This yields the

Corollary III.1.8. *Up to the level, Conjecture I.2.1 is true if the identity*

(III.1.17)

$$
\begin{aligned}
D_\tau^\lambda\mathscr{R}(\tau)=-\frac{i}{4}q^{-\frac14}(-1)^\lambda\sum_{\mu=0}^{\lambda}\binom{2\lambda+1}{2\mu}\left(\frac14\right)^{\lambda-\mu}\Bigg[&\frac12(D_v^{2\mu}R)\left(-\tau-\tfrac12;2\tau\right)\\
&+\frac{2(\lambda-\mu)+1}{2\mu+1}(D_v^{2\mu+1}R)\left(-\tau-\tfrac12;2\tau\right)\Bigg]
\end{aligned}
$$

holds true for all $\lambda\in\mathbb{N}_0$.

Proof. Lemma III.1.7 gives us that Conjecture I.2.1 holds true if the identity

(III.1.18)

$$
\begin{aligned}
&c_\nu(-1)^{\nu-\lambda}\binom{\nu+\frac12}{\lambda}\binom{\nu-\frac12}{\nu-\lambda}D_\tau^{\nu-\lambda}\mathscr{R}(\tau)\\
=&-\frac{i}{4}q^{-\frac14}\sum_{\mu=0}^{\nu-\lambda}\left[\frac12(D_v^{2\mu}R)(-\tau-\tfrac12;2\tau)+\frac{2(\nu-\mu-\lambda)+1}{2\mu+1}(D_v^{2\mu+1}R)(-\tau-\tfrac12;2\tau)\right]\\
&\cdot b_{\nu,\mu,\lambda}\left(\frac12\right)^{2(\nu-\mu-\lambda)}
\end{aligned}
$$

does as well.

We can simplify this a little further: We have

$$\begin{aligned} c_\nu \binom{\nu+\frac{1}{2}}{\lambda}\binom{\nu-\frac{1}{2}}{\nu-\lambda} &= \nu! \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)} \cdot \frac{\Gamma\left(\nu+\frac{3}{2}\right)}{\Gamma\left(\nu-\lambda+\frac{3}{2}\right)\lambda!} \cdot \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)(\nu-\lambda)!} \\ &= \binom{\nu}{\lambda} \frac{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}{\Gamma\left(\nu-\lambda+\frac{3}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)} \end{aligned}$$

By Legendre's duplication formula (III.1.3) we see that

$$\begin{aligned} (2\lambda)! = \Gamma(2\lambda+1) &= \frac{1}{\sqrt{\pi}} 2^{2\lambda}\Gamma\left(\lambda+\frac{1}{2}\right)\lambda!, \\ \Gamma\left(\nu-\lambda+\frac{3}{2}\right) &= \sqrt{\pi}2^{-2(\nu-\lambda)-2)}\frac{(2(\nu-\lambda)+2)!}{(\nu-\lambda+1)!}, \\ \Gamma\left(\nu-\frac{3}{2}\right) &= \sqrt{\pi}2^{-2\nu-2}\frac{(2\nu+2)!}{(\nu+1)!}. \end{aligned}$$

Thus we get that

$$\begin{aligned} &\frac{\left(\frac{1}{2}\right)^{2(\nu-\mu-\lambda)} b_{\nu,\mu,\lambda}}{c_\nu\binom{\nu+\frac{1}{2}}{\lambda}\binom{\nu-\frac{1}{2}}{\nu-\lambda}} \\ =&\frac{\lambda!(\nu-\lambda)!}{\nu!}\frac{\Gamma(\nu-\lambda+\frac{3}{2})\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)} \cdot \frac{(2\nu+1)!}{(2\mu)!(2\lambda)!(2(\nu-\mu-\lambda)+1)!}\left(\frac{1}{4}\right)^{\nu-\mu-\lambda} \\ \overset{(III.1.3)}{=}&\frac{(\nu-\lambda)!}{\nu!}\frac{\Gamma\left(\nu+\frac{3}{2}\right)}{\Gamma\left(\nu-\lambda+\frac{3}{2}\right)}\frac{(2\nu+1)!}{(2\mu)!(2(\nu-\mu-\lambda)+1)!}\left(\frac{1}{4}\right)^{\nu-\mu} \\ \overset{(III.1.3)}{=}&\frac{(\nu-\lambda)!}{\nu!}\frac{\sqrt{\pi}2^{-2(\nu-\lambda)-2)}\frac{(2(\nu-\lambda)+2)!}{(\nu-\lambda+1)!}}{\sqrt{\pi}2^{-2\nu-2}\frac{(2\nu+2)!}{(\nu+1)!}}\frac{(2\nu+1)!}{(2\mu)!(2(\nu-\mu-\lambda)+1)!}\left(\frac{1}{4}\right)^{\nu-\mu} \\ =&\frac{1}{2(\nu-\lambda+1)}\frac{(2(\nu-\lambda)+2)!}{(2\mu)!(2(\nu-\mu-\lambda)+1)!}\left(\frac{1}{4}\right)^{\nu-\mu-\lambda} \\ =&\binom{2(\nu-\lambda)+1}{2\mu}\left(\frac{1}{4}\right)^{\nu-\mu-\lambda}. \end{aligned}$$

and hence the corollary. □

Before we conclude this section, we take care that the completed coefficient in (III.1.6) indeed transforms like a modular form on $\Gamma_0(4)$.

Remark III.1.9. *Note that we already know by Proposition II.1.11 the function*

$$\frac{c_\nu}{2}\left([\widehat{\mathscr{H}},\vartheta]_\nu(\tau) - [\widehat{\mathscr{H}},\vartheta]_\nu\left(\tau+\tfrac{1}{2}\right)\right)$$

from (III.1.6) *transforms like a modular form of weight* $2\nu+2$ *on* $\Gamma_0(4)$.

Lemma III.1.10. *The function* $(D_v^{2\nu+1}\widehat{A}_1^{odd})(0,\tau+\frac{1}{2};2\tau)$ *transforms like a modular form of weight* $2\nu+2$ *on* $\Gamma_0(4)$.

Proof. It is plain from Theorem II.4.4 that the $(2\nu+1)$st derivative of $\widehat{A}_1(0,v;\tau)$ with respect to v has the modular transformation properties of a Jacobi form of weight $2\nu+2$ and index 0. By Theorem II.2.4, we therefore see that $\mathcal{A}_\nu(\tau) := (D_v^{2\nu+1}\widehat{A}_1)\left(0,\frac{\tau+1}{2};\tau\right)$ transforms like a modular form of that weight on the group

$$\Gamma_A := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \frac{a-1}{2}+\frac{c}{2}\in\mathbb{Z} \text{ and } \frac{b}{2}+\frac{d-1}{2}\in\mathbb{Z}\right\}.$$

The function that we are interested in is

$$(D_v^{2\nu+1}\widehat{A}_1)\left(0,\tau+\frac{1}{2};2\tau\right) = \frac{1}{2^{\ell+1}}\mathcal{A}_\nu|_{2\nu+2}\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right).$$

Since one easily sees that

$$\begin{pmatrix}2&0\\0&1\end{pmatrix}\Gamma_0(4)\begin{pmatrix}\frac{1}{2}&0\\0&1\end{pmatrix} \le \Gamma_A,$$

we get that $(D_v^{2\nu+1}\widehat{A}_1)\left(0,\tau+\frac{1}{2};2\tau\right)$ transforms nicely under $\Gamma_0(4)$ and therefore, by Remark III.1.9, so does $(D_v^{2\nu+1}\widehat{A}_1^{odd})\left(0,\tau+\frac{1}{2};2\tau\right)$ ☐

III.2 The Proof

Our proof for Conjecture I.2.1 makes use of Corollary III.1.8. We show by induction that the identity stated there does indeed hold. The base case of the induction gives a new proof of Eichler's class number relation (I.1.5), so we give this as proof of a separate theorem.

Theorem III.2.1 (M. Eichler, 1955). *For odd numbers* $n\in\mathbb{N}$ *we have the class number relation*

$$\sum_{s\in\mathbb{Z}} H(n-s^2)+\lambda_1(n) = \frac{1}{3}\sigma_1(n).$$

Proof. Let

$$E_2^{odd}(\tau) = \sum_{n=0}^{\infty}\sigma_1(2n+1)q^{2n+1},$$

which is known to be a modular form of weight 2 on $\Gamma_0(4)$.

Plugging in $\lambda=0$ into (III.1.17) gives us the equation

$$\mathscr{R}(\tau) = -\frac{i}{4}q^{-\frac{1}{4}}\left[\frac{1}{2}R\left(-\tau-\tfrac{1}{2};2\tau\right)+(D_vR)\left(-\tau-\tfrac{1}{2};2\tau\right)\right].$$

This equality holds by Lemma III.1.5. Hence we know by Corollary III.1.8, Lemma III.1.2, Remark III.1.9 and Lemma III.1.10 that

$$\frac{1}{2}\left(\mathscr{H}(\tau)\vartheta(\tau) - \mathscr{H}\left(\tau+\tfrac{1}{2}\right)\vartheta\left(\tau+\tfrac{1}{2}\right)\right) + \Lambda_{1,odd}(\tau)$$

is indeed a holomorphic modular form of weight 2 on $\Gamma_0(4)$ as well.

By Theorem II.1.6 and Proposition II.1.7, a comparison of the first non-zero Fourier coefficient yields the result. □

The methods employed in [18] to prove this, are very different from these here. Eichler uses topological arguments about the action of Hecke operators on the Riemann surface associated to $\Gamma_0(2)$ and counting of ideal classes in maximal orders of quaternion algebras.

Now we can prove the main result of this chapter.

Theorem III.2.2. *Cohen's conjecture I.2.1 is true. Moreover, for $\nu > 0$ the coefficient of $X^{2\nu}$ in $S_4^1(\tau; X)$ (see (I.2.1)) is a cusp form.*

Proof. The base case of the induction in Theorem III.2.1. Thus suppose that (III.1.17) holds for one $\lambda \in \mathbb{N}_0$. Again omitting the arguments of the occurring R-derivatives for the sake of clearence of presentation, the induction hypothesis gives us

$$\begin{aligned} &D_\tau^{\lambda+1}\mathscr{R}(\tau) = D_\tau(D_\tau^\lambda \mathscr{R}(\tau)) \\ &= -\frac{i}{4}q^{-\frac{1}{4}}(-1)^\lambda \left\{ -\frac{1}{4}\sum_{\mu=0}^{\lambda}\left[\frac{1}{2}(D_v^{2\mu}R) + \frac{2(\lambda-\mu)+1}{2\mu+1}(D_v^{2\mu+1}R)\right]\cdot\binom{2\lambda+1}{2\mu}\left(\frac{1}{4}\right)^{\lambda-\mu} \right.\\ &\qquad \left. + \sum_{\mu=0}^{\lambda}\left[\frac{1}{2}(D_\tau D_v^{2\mu}R) + \frac{2(\lambda-\mu)+1}{2\mu+1}(D_\tau D_v^{2\mu+1}R)\right]\cdot\binom{2\lambda+1}{2\mu}\left(\frac{1}{4}\right)^{\lambda-\mu}\right\} \end{aligned}$$

Here we omitted again the argument $\left(-\tau-\frac{1}{2}; 2\tau\right)$ of $(D_v R)$ again for the sake of clearence of presentation.

The theorem of Schwarz now tells us that the partial derivatives interchange and therefore we have

$$D_\tau\left((D_u^\ell R)\left(-\tau-\tfrac{1}{2}; 2\tau\right)\right) = -(D_u^{\ell+1}R)\left(-\tau-\tfrac{1}{2}; 2\tau\right) + 2(D_\tau D_u^\ell R)\left(-\tau-\tfrac{1}{2}; 2\tau\right).$$

According to Proposition II.4.3 this equals

$$\begin{aligned} -\frac{i}{4}q^{-\frac{1}{4}}(-1)^{\lambda+1}&\left\{\sum_{\mu=0}^{\lambda}\left[\frac{1}{8}(D_v^{2\mu}R) + \left(\frac{1}{4}\cdot\frac{2(\lambda-\mu)+1}{2\mu+1} + \frac{1}{2}\right)(D_v^{2\mu+1}R)\right.\right.\\ &\left.\left. + \left(\frac{1}{2} + \frac{2(\lambda-\mu)+1}{2\mu+1}\right)(D_v^{2\mu+2}R) + \frac{2(\lambda-\mu)+1}{2\mu+1}(D_v^{2\mu+3}R)\right]\right. \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times \binom{2\lambda+1}{2\mu}\left(\frac{1}{4}\right)^{\lambda-\mu}\Bigg\} \\
= &-\frac{i}{4}q^{-\frac{1}{4}}(-1)^{\lambda+1}\Bigg\{\left[\frac{1}{2}R+(2\lambda+3)(D_vR)\right]\left(\frac{1}{4}\right)^{\lambda+1}+\sum_{\mu=1}^{\lambda}\binom{2\lambda+1}{2\mu}\left(\frac{1}{4}\right)^{\lambda-\mu+1} \\
&\quad\Bigg[\left(\frac{1}{2}+\frac{2(2\lambda-\mu)+5}{4\mu-2}\cdot\frac{(2\mu-1)(2\mu)}{(2(\lambda-\mu)+2)(2(\lambda-\mu)+3)}\right)(D_v^{2\mu}R) \\
&\quad+\left(\frac{2(\lambda+\mu)+3}{2\mu+1}+\frac{2(\lambda-\mu)+3}{2\mu-1}\cdot\frac{(2\mu-1)(2\mu)}{(2(\lambda-\mu)+2)(2(\lambda-\mu)+3)}\right)(D_v^{2+1}R)\Bigg] \\
&+\left[\frac{2\lambda+3}{4\lambda+2}(D_v^{2\lambda+2}R)+\frac{1}{2\lambda+1}(D_v^{2\lambda+3}R)\right]\binom{2\lambda+1}{2\lambda}\Bigg\}.
\end{aligned}
$$

The last summand of the last equation simplifies to

$$
\left[\frac{1}{2}(D_v^{2\lambda+2}R)+\frac{1}{2\lambda+3}(D_v^{2\lambda+3}R)\right]\cdot(2\lambda+3).
$$

For the rest we see by elementary computations that

$$
\begin{aligned}
&\binom{2\lambda+1}{2\mu}\left(\frac{1}{2}+\frac{(2(2\lambda-\mu)+5)(2\mu)}{(2(\lambda-\mu)+2)(2(\lambda-\mu)+3)}\right) \\
=&\frac{1}{2}\binom{2\lambda+3}{2\mu}\left(\frac{(2(\lambda-\mu)+2)(2(\lambda-\mu)+3)}{(2\lambda+2)(2\lambda+3)}+\frac{2\mu(2(2\lambda-\mu)+5)}{(2\lambda+2)(2\lambda+3)}\right) \\
=&\frac{1}{2}\binom{2\lambda+3}{2\mu}\left(\frac{4\lambda^2+10\lambda+6}{(2\lambda+2)(2\lambda+3)}\right) \\
=&\frac{1}{2}\binom{2\lambda+3}{2\mu}
\end{aligned}
$$

and in the same way that

$$
\begin{aligned}
&\binom{2\lambda+1}{2\mu}\cdot\left(\frac{2(\lambda+\mu)+3}{2\mu+1}+\frac{2(\lambda-\mu)+3}{2\mu-1}\cdot\frac{(2\mu-1)(2\mu)}{(2(\lambda-\mu)+2)(2(\lambda-\mu)+3)}\right) \\
=&\frac{2(\lambda-\mu)+3}{2\mu+1}\cdot\binom{2\lambda+3}{2\mu}.
\end{aligned}
$$

Thus in summary, we have that

$$
\begin{aligned}
D_\tau^{\lambda+1}\mathscr{R}(\tau)=&-\frac{i}{4}q^{-\frac{1}{4}}(-1)^{\lambda+1}\sum_{\mu=0}^{\lambda+1}\binom{2\lambda+3}{2\mu}\left(\frac{1}{4}\right)^{\lambda-\mu+1}\left[\frac{1}{2}(D_v^{2\mu}R)\left(-\tau-\tfrac{1}{2};2\tau\right)\right. \\
&\left.+\frac{2(\lambda-\mu)+3}{2\mu+1}(D_v^{2\mu+1}R)\left(-\tau-\tfrac{1}{2};2\tau\right)\right]
\end{aligned}
$$

which proves Conjecture I.2.1.

The additional assertion that the modular forms we get from S_4^1 are cusp forms except when $\nu = 0$ can be seen by the fact that for smooth functions $f, g : \mathbb{H} \to \mathbb{C}$ and real numbers k, ℓ we have

$$[(f|_k\gamma), (g|_\ell\gamma)]_\nu = ([f, g]_\nu)|_{k+\ell+2\nu}$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{R})$ (cf. [14, Theorem 7.1]). Now for $\nu > 0$ the function $\tau \mapsto [\mathscr{H}, \vartheta]_\nu(\tau)$ vanishes at the cusp $i\infty$ by construction and the intertwining property from above then yields that it has to vanish at all cusps. A similar intertwining property also holds for derivatives of index 0 Jacobi forms which is essentially shown in Lemma III.1.10. Since $\left(D_v^{2\nu+1}A_1\right)\left(0, \tau + \frac{1}{2}; 2\tau\right)$ also vanishes at the cusp $i\infty$ if $\nu > 0$, it has to vanish at all cusps, which finally proves the theorem.$\square$

As an easy consequence from this theorem, we get some new class number relations. The first few are worked out below.

Corollary III.2.3. *By comparing the first few Fourier coefficients of the modular forms in Theorem III.2.2 one finds for all odd $n \in \mathbb{N}$ the following class number relations*

$$\sum_{s\in\mathbb{Z}} \left(4s^2 - n\right) H\left(n - s^2\right) + \lambda_3(n) = 0,$$

$$\sum_{s\in\mathbb{Z}} g_4(s,n) H\left(n - s^2\right) + \lambda_5(n) = -\frac{1}{12} \sum_{n=x^2+y^2+z^2+t^2} \mathcal{Y}_4(x,y,z,t),$$

$$\sum_{s\in\mathbb{Z}} g_6(s,n) H\left(n - s^2\right) + \lambda_7(n) = -\frac{1}{3} \sum_{n=x^2+y^2+z^2+t^2} \mathcal{Y}_6(x,y,z,t),$$

$$\sum_{s\in\mathbb{Z}} g_8(s,n) H\left(n - s^2\right) + \lambda_9(n) = -\frac{1}{70} \sum_{n=x^2+y^2+z^2+t^2} \mathcal{Y}_8(x,y,z,t)$$

where $g_\ell(n,s)$ is the ℓ-th Taylor coefficient of $\left(1 - 2sX + nX^2\right)^{-1}$ and $\mathcal{Y}_d(x,y,z,t)$ is a certain harmonic polynomial of degree d in 4 variables. Explicitly, we have

$$\begin{aligned}
g_4(s,n) &= \left(16s^4 - 12ns^2 + n^2\right), \\
g_6(s,n) &= \left(64s^6 - 80s^4n + 24s^2n^2 - n^3\right), \\
g_8(s,n) &= \left(256s^8 - 448s^6n + 240s^4n^2 - 40s^2n^3 + n^4\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Y}_4(x,y,z,t) &= \left(x^4 - 6x^2y^2 + y^4\right), \\
\mathcal{Y}_6(x,y,z,t) &= \left(x^6 - 5x^4y^2 - 10x^4z^2 + 30x^2y^2z^2 + 5x^2z^4 - 5y^2z^4\right), \\
\mathcal{Y}_8(x,y,z,t) &= \left(13x^8 + 63x^6y^2 - 490x^6z^2 + 63x^6t^2 - 630x^4y^2z^2 - 315x^4y^2t^2\right. \\
&\quad +1435x^4z^4 - 630x^4z^2t^2 + 315x^2y^2z^4 + 1890x^2y^2z^2t^2 - 616x^2z^6 \\
&\quad \left. +315x^2z^4t^2 - 315t^2y^2z^4 + 22z^8\right).
\end{aligned}$$

The first two of the above relations were already mentioned in [14].

Remark III.2.4. *Because of its close resemblance to the Eichler-Selberg trace formula in Theorem I.1.3 one is lead to think that these class number relations somehow also encode traces of certain Hecke operators. This is indeed true as we shall see in Section V.4 since the proof requires a different method.*

Chapter IV

Other Class Number Relations

In [10], B. Brown and others have conjectured several identities for sums of class numbers of similar shape as the Kronecker-Hurwitz formula (I.1.2), e.g. Let for an integer a and an odd prime p

$$H_{a,p}(n) := \sum_{\substack{s \in \mathbb{Z} \\ s \equiv a \pmod{p}}} H(4n - s^2).$$

Then for a prime ℓ and $p = 5$ it holds that (cf. [4], Equation (4.3))

$$\text{(IV.0.1)} \qquad H_{a,5}(\ell) = \begin{cases} \frac{\ell+1}{2} & \text{if } a \equiv 0 \pmod{5} \text{ and } \ell \equiv 1 \pmod{5} \\ \frac{\ell+1}{3} & \text{if } a \equiv 0 \pmod{5} \text{ and } \ell \equiv 2, 3 \pmod{5} \\ \frac{\ell+1}{3} & \text{if } a \equiv \pm 1 \pmod{5} \text{ and } \ell \equiv 1, 2 \pmod{5} \\ \frac{5\ell+5}{12} & \text{if } a \equiv \pm 1 \pmod{5} \text{ and } \ell \equiv 4 \pmod{5} \\ \frac{5\ell-7}{12} & \text{if } a \equiv \pm 2 \pmod{5} \text{ and } \ell \equiv 1 \pmod{5} \\ \frac{\ell+1}{3} & \text{if } a \equiv \pm 2 \pmod{5} \text{ and } \ell \equiv 3, 4 \pmod{5}, \end{cases}$$

which have been proven by K. Bringmann and B. Kane in [4].

In this chapter we aim to show a slight generalization of their result using similar methods as in Chapter III, which also give a "mock-modular" proof of the Kronecker-Hurwitz class number relation (I.1.2). It is organized as follows. We shall first prove some preliminary results similar to those in Section III.1 in Section IV.1, and then in Section IV.2 we prove Theorem IV.1.1, our main result of this section.

IV.1 Completions

First define for $N, r, k \in \mathbb{N}$ the functions

$$\text{(IV.1.1)} \qquad \vartheta^{(N,r)}(\tau) := \sum_{\substack{n \in \mathbb{Z} \\ n \equiv r \pmod{N}}} q^{n^2}$$

and

$$\text{(IV.1.2)}\qquad \Lambda_k^{N,r}(\tau) := \sum_{n=1}^{\infty} \lambda_k^{N,r}(n) q^n,$$

$$\lambda_k^{N,r}(n) := \frac{1}{2}\left(\sum_{\substack{d|n,\, d\leq\sqrt{n} \\ d\equiv -r \pmod N}} d + \sum_{\substack{d|n,\, d\gneqq\sqrt{n} \\ d\equiv +r \pmod N}} d \right),$$

where for simplicity we write ϑ, Λ_k, and λ_k instead of $\vartheta^{(1,0)}$, $\Lambda_k^{1,0}$, and $\lambda_k^{1,0}$, respectively. Note that this setting is consistent with (I.1.3) and that $\vartheta^{(N,r)}$ from above is not to be confused with $\vartheta_{s,\chi}$ from (II.1.2), although they are of course closely related.

By $\mathcal{G}_k$, $k \geq 2$ we denote in this section the weight k Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$ which is normalized such that the coefficient of q in its Fourier expansion is one, so that we have for example

$$\text{(IV.1.3)}\qquad \mathcal{G}_2(\tau) := -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

Finally, let for p an odd prime $\chi_p := \left(\frac{\cdot}{p}\right)$ denote the non-trivial, real-valued character modulo p.

With these notations, we can state our main result.

Theorem IV.1.1. *We have the identities*

$$\begin{aligned}
(\mathscr{H}\vartheta^{(5,0)})|U(4) &= \frac{1}{2}\mathcal{G}_2 + \frac{1}{12}\mathcal{G}_2 \otimes \chi_5(1-\chi_5) - \mathcal{G}_2|V(5) + \frac{5}{2}\mathcal{G}_2|V(25) - 10\Lambda_1|V(25) \\
&\qquad - 4\Lambda_1^{5,1}|S_{5,4} - 4\Lambda_1^{5,2}|S_{5,4} \\
(\mathscr{H}\vartheta^{(7,0)})|U(4) &= \frac{1}{4}\mathcal{G}_2 - \frac{1}{24}\mathcal{G}_2 \otimes \chi_7(1-\chi_7) - 14\Lambda_1|V(49) \\
&\qquad - 4\Lambda_1^{7,2}|S_{7,3} - 4\Lambda_1^{7,4}|S_{7,5} - 4\Lambda_1^{7,1}|S_{7,6} + \frac{1}{4}g_7,
\end{aligned}$$

where g_7 denotes the cusp form of weight 2 associated to the elliptic curve with equation $y^2 + xy = x^3 - x^2 - 2x - 1$ defined over $\mathbb{Q}$ (its Cremona label is 49a1, cf. [23])

The plan to prove this is similar to that in Chapter III. We find non-holomorphic completions of both sides of the equations and then show that their non-holomorphic parts are equal, so that the difference of both sides is a holomorphic modular form. With Theorem II.1.6 we then show that this difference must be zero.

We start by completing the left-hand side of the identities in Theorem IV.1.1.

Lemma IV.1.2. *Let $N \in \mathbb{N}$ be an odd integer and $a \in \mathbb{Z}$. Then the function*

$$\left(\mathscr{H}\vartheta^{(N,2a)}\right)|U(4) + \sum_{r=0}^{1} \mathscr{R}|S_{2,r}\left(\frac{\tau}{4}\right)\vartheta^{(2N,2a+rN)}\left(\frac{\tau}{4}\right)$$

with $\mathscr{R}$ as in (I.1.10) *transforms like a modular form of weight* 2 *on some congruence subgroup* Γ *of* $\mathrm{SL}_2(\mathbb{Z})$*. Moreover, one can take $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ for $N = 1$ and $\Gamma = \Gamma_0(p^2) \cap \Gamma_1(p)$ for $N = p$ an odd prime number.*

Proof. For $N = p$, the level has already been determined in [4, Lemma 3.1]. For $N = 1$, one may assume $a = 0$ (which we do). Since the nth Fourier coefficient of $\mathscr{H}$ (resp. ϑ) is 0 unless $n \equiv 0, 3 \pmod 4$ (resp. $n \equiv 0, 1 \pmod 4$) one sees that necessarily the nth Fourier coefficient of their product is 0 whenever $n \equiv 2 \pmod 4$. By Theorem I.1.5, $\widehat{\mathscr{H}}$ fulfills weight $\frac{3}{2}$ modularity on $\Gamma_0(4)$ and ϑ is a modular form of weight $\frac{1}{2}$ on the same group, thus $\widehat{\mathscr{H}\vartheta}$ fulfills weight 2 modularity on $\Gamma_0(4)$. Now Proposition II.1.11 tells us that applying $U(4)$ to such a form reduces the level to 1, thus the claim about the level follows.

It remains to compute the Fourier expansion of the non-holomorphic part of $\widehat{\mathscr{H}}\vartheta^{(N,2a)}|U(4)$ explicitly. This is given by

$$\left(\sum_{k\in\mathbb{Z}} a(k,y)q^k\right)|U(4) + \left(\sum_{k\in\mathbb{Z}} b(k,y)q^k\right)|U(4),$$

where

$$a(k,y) := \frac{1}{4\sqrt{\pi}} \sum_{\substack{n\in\mathbb{N}, m\in\mathbb{Z}\\ m\equiv 2a\ (N)\\ m^2-n^2=k}} n\Gamma\left(-\frac{1}{2}; 4\pi n^2 y\right) \quad \text{and} \quad b(k,y) := \sum_{\substack{m\in\mathbb{Z}\\ m\equiv 2a\ (N)\\ m^2=k}} \frac{1}{8\pi\sqrt{y}}.$$

Obviously, the second summand equals $\frac{1}{4\pi\sqrt{y}}\vartheta^{(2N,2a)}\left(\frac{\tau}{4}\right)$, whereas for the first one, we get by splitting the sum into k even and k odd

$$\sum_{k\in\mathbb{Z}} a\left(4k, \tfrac{y}{4}\right)q^k = \frac{1}{4\sqrt{\pi}} \sum_{r=0}^{1} \left(\sum_{\substack{n\in\mathbb{N}, m\in\mathbb{Z}\\ n\equiv r \pmod 2,\\ m\equiv 2a+rN \pmod{2N}}} n\Gamma\left(-\frac{1}{2}; \pi n^2 y\right) q^{\frac{m^2-n^2}{4}} \right).$$

Hence putting these together the assertion follows ☐

Remark IV.1.3. *For $N = p$ an odd prime, Lemma IV.1.2 becomes exactly [4, Lemma 3.1] by twisting the formula therein with χ_p^2.*

We now proceed to the right-hand side of the identities to prove.

Lemma IV.1.4. *The following identity is valid for all $\tau \in \mathbb{H}$ and odd $k \in \mathbb{N}$:*

$$2\Lambda_k^{N,r}(\tau) = \frac{1}{N}\sum_{\ell=0}^{N-1} \zeta_N^{-\ell r}(D_v^k A_2)\left(0, \frac{\ell}{N}; \tau\right),$$

where $\zeta_N := e^{\frac{2\pi i}{N}}$ is a primitive Nth root of unity.

Proof. Since

$$(D_v^k A_\ell)(u, v; \tau) = e^{\pi i \ell u} \sum_{n\in\mathbb{Z}} n^k \frac{q^{\frac{\ell}{2}n(n+1)} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n},$$

we see that the limit $\lim_{u\to 0}(D_v^k A_\ell)(u, v; \tau)$ exists for $k > 0$, thus the right-hand side of the identity to show indeed makes sense.

As in Lemma III.1.2, we write $\Lambda_k^{N,r}$ as a q-series.

$$\begin{aligned}
\Lambda_k^{N,r}(\tau) &= \sum_{s=0}^{\infty}\sum_{t=1}^{\infty}(sN + r)^k q^{(sN+r)(sN+r+t)} + \sum_{s=1}^{\infty}\sum_{t=0}^{\infty}(sN - r)^k q^{(sN-r)(sN-r+t)} \\
&= \sum_{s=0}^{\infty}(sN + r)^k \frac{q^{(sN+r)+(sN+r)^2}}{1 - q^{sN+r}} + \sum_{s=1}^{\infty}(sN - r)^k \frac{q^{(sN-r)^2}}{1 - q^{sN-r}} \\
&= \sum_{s\in\mathbb{Z}}(sN + r)^k \frac{q^{(sN+r)+(sN+r)^2}}{1 - q^{sN+r}}.
\end{aligned}$$

This equals the kth v-derivative of a sieved level 2 Appell-Lerch sum evaluated at $(0, 0; \tau)$, which is what we stated. □

The following lemma is again a slight generalization of [4, Lemma 3.3].

Lemma IV.1.5. *For odd $k \in \mathbb{N}$, the function $\Lambda_k^{N,r}(\tau)$ can be completed to transform like a modular form of weight 2 on $\Gamma_1(N)$ by adding the non-holomorphic term*

(IV.1.4)

$$\begin{aligned}
&\frac{i}{2N}\sum_{\ell=0}^{N-1}\zeta_N^{-\ell r}\sum_{j=0}^{1}\left[\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}(D_v^s\Theta)\left(\frac{2\ell + N}{2N} + j\tau; 2\tau\right)\right. \\
&\qquad\qquad \left.\times(D_v^{k-s}R)\left(-\frac{2\ell + N}{2N} - j\tau; 2\tau\right)\right] + \delta_{k,1}\frac{1}{4\pi y},
\end{aligned}$$

where $\delta_{i,j}$ is Kronecker's δ function.

Proof. By differentiating the transformation law of $\widehat{A}_2$ in Theorem II.4.4 with respect to v, we obtain for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$\begin{aligned}
(D_v\widehat{A}_2)\left(\frac{u}{c\tau + d}, \frac{v}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right) =& (c\tau + d)e^{\pi i c\frac{-u^2+2uv}{c\tau+d}} cu\widehat{A}_2(u, v; \tau) \\
&+ (c\tau + d)^2 e^{\pi i c\frac{-u^2+2uv}{c\tau+d}}(D_v\widehat{A}_2)(u, v; \tau).
\end{aligned}$$

Since A_2 (viewed as a function of u) has simple poles in $\mathbb{Z}\tau+\mathbb{Z}$ and $\operatorname{Res}_{u=0} A_2 = -\frac{1}{2\pi i}$ (cf. [55, Proposition 1.4]), the limit $u \to 0$ of the above equation becomes

$$(D_v\widehat{A}_2)\left(0, \frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = -\frac{c(c\tau+d)}{2\pi i} + (c\tau+d)^2 (D_v\widehat{A}_2)(u,v;\tau).$$

Recall, that the weight 2 Eisenstein series $\mathcal{G}_2$ transforms in almost the same way (see (II.1.1)) under $\mathrm{SL}_2(\mathbb{Z})$ and that $\mathcal{G}_2(\tau)+\frac{1}{8\pi y}$ transforms like a modular form of weight 2, hence we can deduce that

$$(D_v\widehat{A}_2)^*(0,v;\tau) := (D_v\widehat{A}_2)(0,v;\tau) + \frac{1}{4\pi y}$$

has the modular transformation properties of a Jacobi form of weight 2 and index 0.

Taking the kth derivative ($k \geq 2$) always gives terms of the form $C \cdot u^\ell (D_v^{k-\ell}\widehat{A}_2)$, such that the limit $u \to 0$ is 0 unless $\ell = 0$. Thus in this case we see that $(D_v^k\widehat{A}_2)(0,v;\tau)$ without any additional term has the modular transformation properties of a Jacobi form of weight $k+1$ and index 0.

Using Lemma IV.1.4, we get the formula for the completion term that we claimed and from Theorem II.2.4 we can see, that the completed function transforms like a modular form on the group $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid N \mid c \equiv 0\,(N) \text{ and } d \equiv 1\,(N)\}$ which obviously contains $\Gamma_1(N)$. $\square$

Next, we express the derivatives of R and Θ evaluated at the necessary torsion points in terms of $\mathscr{R}$ and $\vartheta^{(N,r)}$.

Lemma IV.1.6. *For $j \in \{0,1\}$ we have that*

$$\Theta\left(j\tau+\tfrac{1}{2}; 2\tau\right) = \begin{cases} -\vartheta^{(2,1)}\left(\frac{\tau}{4}\right) & \text{for } j=0 \\ -q^{\frac{1}{4}}\vartheta\left(\frac{\tau}{4}\right) & \text{for } j=1, \end{cases} \tag{IV.1.5}$$

$$(D_v\Theta)\left(j\tau+\tfrac{1}{2}; 2\tau\right) = \begin{cases} 0 & \text{for } j=0 \\ \frac{1}{2}q^{\frac{1}{4}}\vartheta(\tau) & \text{for } j=1. \end{cases} \tag{IV.1.6}$$

Proof. The case $j=1$ is already contained in Lemma III.1.4. Thus put $j=0$. We calculate

$$\begin{aligned}\Theta\left(\tfrac{1}{2}; 2\tau\right) &= \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} q^{\nu^2} e^{2\pi i\nu} = -\sum_{n\in\mathbb{Z}} q^{\frac{1}{4}(2n+1)^2} \\ &= -\vartheta^{(2,1)}\left(\frac{\tau}{4}\right),\end{aligned}$$

and

$$(D_v\Theta)\left(\tfrac{1}{2}; 2\tau\right) = \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \nu q^{\nu^2} e^{2\pi i\nu} = 0.$$

Lemma IV.1.7. *We have that*

$$R\left(-j\tau-\tfrac{1}{2};2\tau\right)=\begin{cases}0 & \textit{for } j=0\\ iq^{\frac{1}{4}} & \textit{for } j=1,\end{cases} \tag{IV.1.7}$$

$$(D_uR)\left(-j\tau-\tfrac{1}{2};2\tau\right)=\begin{cases}2i\mathscr{R}|S_{2,1}\left(\frac{\tau}{4}\right) & \textit{for } j=0\\ 2iq^{\frac{1}{4}}\mathscr{R}|S_{2,0}\left(\frac{\tau}{4}\right) & \textit{for } j=1.\end{cases} \tag{IV.1.8}$$

Proof. Again, the identities for $j=1$ have already been proven, see Lemma III.1.5. From the properties of R in Proposition II.4.2 we get

$$R\left(-\tfrac{1}{2};2\tau\right)\overset{(i)}{=}-R\left(-\tfrac{1}{2}+1;2\tau\right)\overset{(iii)}{=}-R\left(-\tfrac{1}{2};2\tau\right),$$

hence it must be 0.

From (II.4.5) we get as in the proof of Lemma III.1.5 that

$$\begin{aligned}(D_uR)\left(-\tfrac{1}{2};2\tau\right)&=\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}\left[\frac{1}{\pi\sqrt{4y}}e^{-4\pi\nu^2y}-\nu\left\{\operatorname{sgn}(\nu)-\operatorname{sgn}(\nu\sqrt{4y})(1-\beta(4\nu^2y))\right\}\right]\\&\qquad\times(-1)^{\nu-\frac{1}{2}}q^{-\nu^2}e^{\pi i\nu}\\&=i\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}\left[\frac{1}{\pi\sqrt{4y}}e^{-4\pi\nu^2y}-|\nu|\beta(4\nu^2y)\right]q^{-\nu^2}.\end{aligned}$$

With (III.1.15) we can simplify this to

$$\begin{aligned}(D_uR)\left(-\tfrac{1}{2};2\tau\right)&=\frac{i}{4\sqrt{\pi}}\sum_{n\in\mathbb{Z}}|2n+1|\Gamma\left(-\frac{1}{2};4\pi(2n+1)^2\frac{y}{4}\right)q^{-\frac{(2n+1)^2}{4}}\\&=\frac{i}{2\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\Gamma\left(-\frac{1}{2};4\pi(2n+1)^2\frac{y}{4}\right)q^{-\frac{(2n+1)^2}{4}}.\end{aligned}$$

By Remark III.1.6, the claim follows. □

IV.2 The Results

We begin this section by the announced new proof of the Kronecker-Hurwitz class number relation.

Theorem IV.2.1. *For every $n\in\mathbb{N}$ we have that*

$$\sum_{s\in\mathbb{Z}}H(4n-s^2)+2\lambda_1(n)=2\sigma_1(n).$$

Proof. In terms of generating functions the equality to show is equivalent to the identity

$$(\mathscr{H}\vartheta)|U(4)(\tau)+2\Lambda_1(\tau)=2\mathcal{G}_2(\tau).$$

We complete both sides such that they transform like modular forms of weight 2 on $\mathrm{SL}_2(\mathbb{Z})$. By Lemma IV.1.2, Lemma IV.1.5, Lemma IV.1.6, and Lemma IV.1.7 the non-holomorphic parts on the left-hand side add up to $\frac{1}{4\pi y}$ which is exactly equal to the non-holomorphic part on the right-hand side (see Example II.1.4 (ii)). Therefore, the difference of both sides is a weight 2 cusp form on $\mathrm{SL}_2(\mathbb{Z})$, and hence indentically 0. This completes the proof. $\square$

The proof of Theorem IV.1.1 works more or less in the same way:

Proof of Theorem IV.1.1. Let

$$\begin{aligned}\mathcal{F}_5 := (\mathscr{H}\vartheta^{(5,0)})|U(4) - \frac{1}{2}\mathcal{G}_2 - \frac{1}{12}\mathcal{G}_2 \otimes \chi_5(1-\chi_5) + \mathcal{G}_2|V(5) - \frac{5}{2}\mathcal{G}_2|V(25)\\ + 5\Lambda_1|V(25) + 2\Lambda_1^{5,1}|S_{5,4} + 2\Lambda_1^{5,2}|S_{5,4},\\ \mathcal{F}_7 := (\mathscr{H}\vartheta^{(7,0)})|U(4) - \frac{1}{4}\mathcal{G}_2 + \frac{1}{24}\mathcal{G}_2 \otimes \chi_7(1-\chi_7) - \frac{7}{4}\mathcal{G}_2|V(49) + 7\Lambda_1|V(49)\\ + 2\Lambda_1^{7,2}|S_{7,3} + 2\Lambda_1^{7,4}|S_{7,5} + 2\Lambda_1^{7,1}|S_{7,6} - \frac{1}{4}g_7.\end{aligned}$$

We want to show that these functions vanish identically. To do so, we write these in the form

$$\mathcal{F}_p = \mathcal{F}_p \otimes \chi_p^2 + \mathcal{F}_p|S_{p,0}, \quad p \in \{5,7\}.$$

From Corollaries 4.3 and 4.5 in [4] we already know that $\mathcal{F}_p \otimes \chi_p^2 \equiv 0$, thus it is enough to look at $\mathcal{F}_p|S_{p,0}$, explicitly

$$\begin{aligned}\mathcal{F}_5|S_{5,0} &= (\mathscr{H}\vartheta^{(5,0)})|U(4)|S_{5,0} - \frac{1}{2}\mathcal{G}_2|S_{5,0} + \mathcal{G}_2|V(5) - \frac{5}{2}\mathcal{G}_2|V(25) + 5\Lambda_1|V(25),\\ \mathcal{F}_7|S_{7,0} &= (\mathscr{H}\vartheta^{(7,0)})|U(4)|S_{7,0} - \frac{1}{4}\mathcal{G}_2|S_{7,0} + 7\Lambda_1|V(49) - \frac{1}{4}g_7|S_{7,0}.\end{aligned}$$

As in the proof of Theorem IV.2.1, we complete all occuring functions to make them transform like modular forms and show that the non-holomorphic parts add up to be 0.

For any odd prime p we immediately get from Lemma IV.1.2 that the non-holomorphic part belonging to $(\mathscr{H}\vartheta^{(p,0)})|U(4)|S_{p,0}$ is

$$\sum_{j=0}^{1}\left[\frac{1}{4\sqrt{\pi}}\sum_{\substack{n\in\mathbb{N},m\in\mathbb{Z}\\ n,m\equiv j\ (2)}} n\Gamma\left(-\frac{1}{2};\pi n^2 y\right)q^{p^2\frac{m^2-n^2}{4}} + \delta_{j,0}\frac{1}{8\pi\sqrt{y}}\sum_{m\in\mathbb{Z}} q^{(pm)^2}\right]$$

From the Eisenstein series we get the non-holomorphic term

$$\frac{1}{4p\pi y} \quad \text{for } p \in \{5,7\},$$

and finally, using Lemma IV.1.5, Lemma IV.1.6, and Lemma IV.1.7, we get the following contribution from $p\Lambda_1|V(p^2)$ (now again for every odd prime p):

$$-\frac{1}{4p\pi y} - \sum_{j=0}^{1} \left[\theta^{(2p,jp)}\left(\frac{\tau}{4}\right)\mathscr{R}|S_{p,2p}\left(\frac{\tau}{4}\right)\right].$$

Obviously, these terms cancel each other such that $\mathcal{F}_p$ is a holomorphic modular form of weight 2 on $\Gamma := \Gamma_0(p^2) \cap \Gamma_1(p)$.

The index of this group in $\mathrm{SL}_2(\mathbb{Z})$ can be determined as follows. The index of $\Gamma_0(p^2)$ is $p(p+1)$, the index of $\Gamma_1(p)$ is p^2-1 by the index formulas in Proposition II.1.7. These two groups are obviously not equal and their intersection contains $\Gamma_1(p^2)$ as a proper subgroup, which has index p^2 in $\Gamma_1(p)$, hence $[\Gamma_1(p) : \Gamma] = p$ which implies $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = p(p^2-1)$.

Therefore we know by Sturm's theorem Theorem II.1.6 that $\mathcal{F}_p \equiv 0$ if and only if the first $\frac{p}{6}(p^2-1)$ Fourier coefficents vanish. This being the case, the proof is done. $\square$

Remark IV.2.2. *In the proof of Theorem IV.1.1, we have not really used the fact that $p \in \{5, 7\}$ except when we use the results of [4], hence we can find other such identities for every odd prime p in the very same way. For another method of finding such identities, see Section V.4.*

Chapter V

Relations for Fourier Coefficients of Mock Modular Forms

Both in Chapter III and Chapter IV we have used the same kind of calculations to prove class number relations. A natural question that arises is whether there is a general way to prove these results simultaneously. The answer to this question is, as we shall see in this chapter, yes. We even prove that recurrence relations of the type of the Cohen-conjecture, the Kronecker-Hurwitz formula or the Eichler-Selberg trace formula are in a way not even a special phenomenon for class numbers but they hold essentially for Fourier coefficients of mock modular forms of weights $\frac{3}{2}$ and mock theta functions. We begin with a section about some general results concerning holomorphic projection, the main tool of this investigation, then we deal with the special cases of weight $\frac{3}{2}$ and the mock theta functions separately, and conclude this chapter by a section with examples, which contains alternative proofs (and generalizations) for the main results of the previous chapters. This chapter is an extended version of [41].

V.1 Holomorphic Projection

Holomorphic projection has been introduced by J. Sturm in 1980 [48] and has been used and developed further in the seminal work [28] of B. Gross and D. Zagier on Heegner points and derivatives of L-functions. In the theory of mock modular forms, holomorphic projection has for example been used in [5], [34], where the definition has been extended to vector-valued forms, [3], and [1].

Here, we are interested in the action of holomorphic projection on Rankin-Cohen brackets of harmonic Maaß forms with holomorphic shadow and holomorphic modular forms. To some extent, this has been worked out in a talk [52] by D. Zagier, but since the notes of this talk are not publicly available, we recall and slightly extend his general results here.

Let us start by defining our object of interest.

Definition V.1.1. *Let $f : \mathbb{H} \to \mathbb{C}$ be a continuous function satisfying weight k modularity on $\Gamma_0(N)$ $(k \geq 2)$ with Fourier expansion*

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_f(n,y)q^n.$$

For a cusp κ_j, $j = 1,...,M$ and $\kappa_1 := i\infty$, of $\Gamma_0(N)$ fix $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma_j\kappa_j = i\infty$. Assume that for some $\delta, \varepsilon > 0$ we have

1. $f(\gamma_j^{-1}w)\left(\frac{d}{dw}\tau\right)^{\frac{k}{2}} = c_0^{(j)} + O(\mathrm{Im}(w)^{-\delta})$ *for all* $j = 1,...,M$ *and* $w = \gamma_j\tau$,
2. $a_f(n,y) = O(y^{1-k+\varepsilon})$ *as* $y \to 0$ *for all* $n > 0$

Then we define the holomorphic projection *of f by*

$$(\pi_{hol}f)(\tau) := (\pi_{hol}^k f)(\tau) := c_0 + \sum_{n=1}^{\infty} c(n)q^n,$$

with $c_0 = c_0^{(1)}$ and

$$c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!}\int_0^\infty a_f(n,y)e^{-4\pi ny}y^{k-2}dy \tag{V.1.1}$$

for $n \geq 1$. For $\ell \notin \mathbb{N}_0$ we set as usual $\ell! := \Gamma(\ell+1)$.

The holomorphic projection operator has the following properties, see Proposition 5.1 and Proposition 6.2 of [28], and Proposition 3.2 and Theorem 3.3 of [34].

Proposition V.1.2. *Let $f : \mathbb{H} \to \mathbb{C}$ be a continuous, periodic function as in Definition V.1.1.*

(i) If f is holomorphic, then $\pi_{hol}f = f$.

(ii) For $k > 2$ it holds that $\pi_{hol}f$ is a holomorphic modular form of weight k on $\Gamma_0(N)$ and for $k = 2$, $\pi_{hol}f$ is a quasi-modular form (cf. [35] for details) of weight 2 on $\Gamma_0(N)$.

Furthermore the operator π_{hol} behaves nicely under the modular operators from (II.1.4)-(II.1.7).

Lemma V.1.3. *Let $f : \mathbb{H} \to \mathbb{C}$ be a function as in Definition V.1.1, $N \in \mathbb{N}$, and $r \in \{0,\dots,N-1\}$. Then the following holds.*

(i) The operator π_{hol} commutes with all the operators $U(N)$, $V(N)$, $S_{N,r}$, and $\otimes\chi$.

(ii) If f is modular of weight $k > 2$ on $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ then we have

$$\langle f, g\rangle = \langle \pi_{hol}(f), g\rangle,$$

where $\langle \cdot, \cdot\rangle$ denotes the Petersson scalar product[1]*, for every cusp form $g \in S_k(\Gamma)$ such that the integral defining the scalar product converges.*

Proof. Due to [5, Lemma 3.4] we have that π_{hol} interchanges with the slash operator provided that everything converges nicely. Since the modular operators are all built from slash operators, the claim (i) follows. For (ii), we refer to Proposition 5.1 of [28]. $\square$

From now on for the rest of this section, let $\nu \in \mathbb{N}_0$ and $f \in \mathcal{M}_k(N)$ be a harmonic Maaß form with holomorphic shadow with a Fourier expansion as given in Lemma II.3.2 and $g \in M_\ell(N)$ be a holomorphic modular form ($k, \ell \in \frac{1}{2}\mathbb{Z}$, $k, \ell \neq 1$) with Fourier expansion

$$\sum_{n=0}^{\infty} a_g(n) q^n$$

such that $k + \ell + 2\nu \in \mathbb{N}$ and ≥ 2 and $[f, g]_\nu$ satisfies the conditions of Definition V.1.1. This is the case for example if $f \in \mathcal{S}_k$ and $f^+ \cdot g$ is holomorphic at the cusps (cf. Theorem 3.5 in [34]).

We would like to investigate

$$\pi_{hol}([f, g]_\nu) = [f^+, g]_\nu + \frac{(4\pi)^{1-k}}{k-1}\overline{c_f^-(0)}\pi_{hol}([y^{1-k}, g]_\nu) + \pi_{hol}([f^-, g]_\nu). \tag{V.1.2}$$

Let us first deal with the second of these summands.

Lemma V.1.4. *We have*

$$\frac{(4\pi)^{1-k}}{k-1}\pi_{hol}([y^{1-k}, g]_\nu) = \kappa \sum_{n=0}^{\infty} n^{k+\nu-1} a_g(n) q^n$$

where κ depends only on k, ℓ, ν. To be precise, it holds that

$$\begin{aligned} \kappa = \kappa(k, \ell, \nu) = \frac{1}{(k+\ell+2\nu-2)!(k-1)} \sum_{\mu=0}^{\nu} \Bigg[&\frac{\Gamma(2-k)\Gamma(\ell+2\nu-\mu)}{\Gamma(2-k-\mu)} \\ &\times \binom{k+\nu-1}{\nu-\mu}\binom{\ell+\nu-1}{\mu}\Bigg]. \end{aligned} \tag{V.1.3}$$

[1] this scalar product on $S_k(\Gamma)$ is defined by the integral

$$\langle f, g\rangle := \int_{\mathcal{F}(\Gamma)} f(\tau)\overline{g(\tau)} y^k \frac{dxdy}{y^2},$$

where $\mathcal{F}(\Gamma)$ denotes a fundamental domain for the action of Γ on $\mathbb{H}$. See e.g. [37, Kapitel IV, §3]

Proof. It holds that

$$D^{\mu}(y^{1-k}) = \frac{\Gamma(2-k)}{\Gamma(2-k-\mu)}\left(-\frac{1}{4\pi}\right)^{\mu} y^{1-k-\mu}$$

and

$$(D^{\nu-\mu}g)(\tau) = \sum_{n=0}^{\infty} n^{\nu-\mu} a_g(n) q^n.$$

Thus the nth coefficient of $[y^{1-k}, g]_{\nu}$ equals

$$a_g(n)\sum_{\mu=0}^{\nu}(-1)^{\mu}\binom{k+\nu-1}{\nu-\mu}\binom{\ell+\nu-1}{\mu}\frac{\Gamma(2-k)}{\Gamma(2-k-\mu)}\left(-\frac{1}{4\pi}\right)^{\mu} y^{1-k-\mu} n^{\nu-\mu}.$$

We also calculate

$$\int_0^{\infty} e^{-4\pi n y} y^{\ell+2\nu-\mu-1} = \left(\frac{1}{4\pi n}\right)^{\ell+2\nu-\mu}\Gamma(\ell+2\nu-\mu),$$

so that we get the following for the nth coefficient of $\pi_{hol}([y^{1-k}, g]_{\nu})$:

$$\begin{aligned}
&\frac{(4\pi n)^{k+\ell+2\nu-1}}{(k+\ell+2\nu-2)!}\cdot a_g(n)\sum_{\mu=0}^{\nu}\binom{k+\nu-1}{\nu-\mu}\binom{\ell+\nu-1}{\mu}\frac{\Gamma(2-k)}{\Gamma(2-k-\mu)}\left(\frac{1}{4\pi}\right)^{\mu}\\
&\qquad\times n^{\nu-\mu}\left(\frac{1}{4\pi n}\right)^{\ell+2\nu-\mu}\Gamma(\ell+2\nu-\mu)\\
=&\frac{(4\pi n)^{k-1}}{(k+\ell+2\nu-2)!}\cdot n^{\nu} a_g(n)\sum_{\mu=0}^{\nu}\binom{k+\nu-1}{\nu-\mu}\binom{\ell+\nu-1}{\mu}\frac{\Gamma(2-k)\Gamma(\ell+2\nu-\mu)}{\Gamma(2-k-\mu)},
\end{aligned}$$

which yields the asserted formula. □

For the third summand in (V.1.2), we need a little more work. Let us define for this purpose the homogeneous polynomial

$$P_{a,b}(X,Y) := \sum_{j=0}^{a-2}\binom{j+b-2}{j} X^j (X+Y)^{a-j-2} \in \mathbb{C}[X,Y] \tag{V.1.4}$$

of degree $a-2$. Then we get:

Theorem V.1.5. *Assuming that the coefficients $c_f^-(n)$ and $a_g(n)$ grow sufficiently moderately, i.e. the integral defining $\pi_{hol}([f^-, g]_{\nu})$ is absolutely convergent, the holomorphic projection of f^- and g is given by*

$$\pi_{hol}([f^-, g]_{\nu}) = \sum_{r=1}^{\infty} b(r) q^r,$$

whereas $b(r)$ is given by

$$\text{(V.1.5)}\quad \begin{aligned} b(r) = -\,\Gamma(1-k) \sum_{m-n=r} \sum_{\mu=0}^{\nu} \binom{k+\nu-1}{\nu-\mu} \binom{\ell+\nu-1}{\mu} m^{\nu-\mu} a_g(m) \overline{c_f^-(n)} \\ \times \left(m^{\mu-2\nu-\ell+1} P_{k+\ell+2\nu, 2-k-\mu}(r,n) - n^{k+\mu-1}\right) \end{aligned}$$

For the proof, we need some additional lemmas. First we compute the derivatives of f^-.

Lemma V.1.6. *It holds that*

$$(D^\mu f^-)(\tau) = (-1)^\mu \frac{\Gamma(1-k)}{\Gamma(1-k-\mu)} \sum_{n=1}^{\infty} n^{k+\mu-1} \overline{c_f^-(n)} \Gamma(1-k-\mu; 4\pi n y) q^{-n}$$

Proof. We write f^- as

$$f^-(\tau) = \sum_{n=1}^{\infty} n^{k-1} \overline{c_f^-(n)} \Gamma^*(1-k, 4\pi n y) \overline{q}^{-n},$$

where $\Gamma^*(\alpha; x) := e^x \Gamma(\alpha; x)$. We compute

$$\Gamma^*(\alpha; x) = e^x \Gamma(\alpha; x) = e^x \int_x^\infty e^{-t} t^{\alpha-1} dt = \int_0^\infty e^{-t} (t+x)^{\alpha-1} dt,$$

which makes calculating derivatives fairly easy:

$$\frac{d}{dx} \Gamma^*(\alpha; x) = (\alpha-1) \int_0^\infty e^{-t} (t+x)^{\alpha-2} dt = (\alpha-1) \Gamma^*(\alpha-1; x)$$

and hence

$$\frac{d^\mu}{dx^\mu} \Gamma^*(\alpha; x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)} \Gamma^*(\alpha-\mu; x).$$

Therefore we have

$$\begin{aligned} (D^\mu f^-)(\tau) &= \left(\frac{1}{2\pi i}\right)^\mu \sum_{n=1}^{\infty} n^{k-1} \overline{c_f^-(n)} \left(\frac{4\pi n}{2i}\right)^\mu \frac{\Gamma(1-k)}{\Gamma(1-k-\mu)} \\ &\qquad \times \Gamma^*\left(1-k-\mu; 4\pi n \frac{1}{2i}(\tau - \overline{\tau})\right) \overline{q}^{-n} \\ &= (-1)^\mu \frac{\Gamma(1-k)}{\Gamma(1-k-\mu)} \sum_{n=1}^{\infty} n^{k+\mu-1} \overline{c_f^-(n)} \Gamma(1-k-\mu; 4\pi n y) q^{-n}, \end{aligned}$$

which is what we have claimed. $\square$

Now consider the integral

$$I := \int_0^\infty \Gamma(1-k-\mu; 4\pi ny) e^{-4\pi ry} y^{k+\ell+2\nu-2} dy. \tag{V.1.6}$$

Lemma V.1.7. *The following identity holds true.*

$$I = -(4\pi)^{1-(k+\ell+2\nu)} n^{1-k-\mu} \frac{\Gamma(1-k-\mu)(k+\ell+2\nu-2)!}{r^{k+\ell+2\nu-1}} \times \left[(r+n)^{\mu-\ell-2\nu+1} P_{k+\ell+2\nu,2-k-\mu}(r,n) - n^{k+\mu-1}\right]$$

with P as in (V.1.4).

Proof. Recall the functional equation (III.1.2).

Written as double integral, I equals

$$\int_0^\infty \int_{4\pi ny}^\infty e^{-t} t^{-k-\mu} e^{4\pi ry} y^{k+\ell+2\nu-2} dt dy$$

Substituting $4\pi nyt' = t$, hence $dt = 4\pi nydt'$, this equals

$$\begin{aligned}
&(4\pi n)^{1-k-\mu} \int_0^\infty \int_1^\infty e^{-4\pi y(r+nt)} t^{-k-\mu} y^{\ell+2\nu-1} dt dy \\
=&(4\pi n)^{1-k-\mu} \int_1^\infty t^{-k-\mu} \int_0^\infty e^{-4\pi y(r+nt)} y^{\ell+2\nu-1} dt dy.
\end{aligned}$$

For the inner integral we get substituting $4\pi(r+nt) = y'$, hence $dy = (4\pi(r+nt))^{-1} dy'$, the expression

$$(4\pi(r+nt))^{\mu-\ell-2\nu} \int_0^\infty e^{-y} y^{\ell+2\nu-\mu-1} dy = (4\pi(r+nt))^{\mu-\ell-2\nu} \Gamma(\ell+2\nu-\mu).$$

Therefore we get

$$I = (4\pi)^{1-(k+\ell+2\nu)} n^{1-k-\mu} \Gamma(\ell+2\nu-\mu) \int_1^\infty t^{-k-\mu} (r+nt)^{\mu-\ell-2\nu} dt,$$

which after yet another substitution $t = \frac{1}{t'}$, hence $dt = -\frac{1}{t'^2} dt'$, simplifies to

$$\begin{aligned}
&(4\pi)^{1-(k+\ell+2\nu)} n^{1-k-\mu} \Gamma(\ell+2\nu-\mu) \int_0^1 t^{k+\mu-2} \left(r + \frac{n}{t}\right)^{\mu-\ell-2\nu} dt \\
=&(4\pi)^{1-(k+\ell+2\nu)} n^{1-k-\mu} \Gamma(\ell+2\nu-\mu) \int_0^1 t^{k+\ell+2\nu-2} (rt+n)^{\mu-\ell-2\nu} dt.
\end{aligned}$$

Since $k+\ell$ is an integer by assumption, we note

$$\frac{\partial^{k+\ell+2\nu-2}}{\partial r^{k+\ell+2\nu-2}}(rt+n)^{k+\mu-2} = \frac{\Gamma(k+\mu-1)}{\Gamma(\mu-\ell-2\nu+1)} t^{k+\ell+2\nu-2}(rt+n)^{\mu-\ell-2\nu},$$

and thus (writing $M := k+\ell+2\nu$ for brevity)

$$\begin{aligned}
&\int_0^1 t^{M-2}(rt+n)^{\mu-\ell-2\nu} dt \\
=&\frac{\Gamma(\mu-\ell-2\nu+1)}{\Gamma(k+\mu-1)} \frac{\partial^{M-2}}{\partial r^{M-2}} \int_0^1 (rt+n)^{k+\mu-2} dt \\
=&\frac{\Gamma(\mu-\ell-2\nu+1)}{\Gamma(k+\mu-1)} \frac{\partial^{M-2}}{\partial r^{M-2}} \left[\frac{1}{k+\mu-1}\frac{1}{r}\left((r+n)^{k+\mu-1} - n^{k-\mu-1}\right)\right] \\
=&\frac{\Gamma(\mu-\ell-2\nu+1)}{\Gamma(k+\mu)} \left[\sum_{j=0}^{M-2} \binom{M-2}{j} \frac{(-1)^{M-j-2}(M-j-2)!}{r^{M-j-1}}\right. \\
&\qquad \left. \times \frac{\Gamma(k+\mu)}{\Gamma(k+\mu-j)}(r+n)^{k+\mu-j-1} - \frac{(-1)^{M-2}(M-2)!}{r^{M-1}} n^{k+\mu-1}\right] \\
\overset{\text{(III.1.2)}}{=}&\frac{\Gamma(\mu-\ell-2\nu+1)(k+\ell+2\nu-2)!}{\Gamma(k+\mu)} \left[\sum_{j=0}^{M-2} \frac{1}{j!} \frac{(-1)^{M-j-2}}{r^{M-j-1}} \frac{\Gamma(1-k-\mu+j)}{\Gamma(1-k-\mu)}\right. \\
&\qquad \left. \times \underbrace{\frac{\sin((1-k-\mu)\pi)}{\sin((1-k-\mu+j)\pi)}}_{=(-1)^j}(r+n)^{k+\mu-j-1} - \frac{(-1)^{M-2}}{r^{M-1}} n^{k+\mu-1}\right] \\
=&(-1)^{M-2}\frac{\Gamma(\mu-\ell-2\nu+1)(M-2)!}{\Gamma(k+\mu)r^{M-1}} \\
&\qquad \times \left[\sum_{j=0}^{M-2} \binom{j-k-\mu}{j} r^j (r+n)^{k+\mu-j-1} - n^{k+\mu-1}\right] \\
=&(-1)^{M-2}\frac{\Gamma(\mu-\ell-2\nu+1)(M-2)!}{\Gamma(k+\mu)r^{M-1}} \\
&\qquad \times \left[(r+n)^{\mu-\ell-2\nu+1} P_{M,2-k-\mu}(r,n) - n^{k+\mu-1}\right] \\
\overset{\text{(III.1.2)}}{=}&(-1)^{k+\ell}\frac{\Gamma(1-k-\mu)(M-2)!}{\Gamma(2\nu-\mu+\ell)r^{M-1}} \underbrace{\frac{\sin((k+\mu)\pi)}{\sin((\mu-\ell-2\nu+1)\pi)}}_{=(-1)^{k+\ell-1}} \\
&\qquad \times \left[(r+n)^{\mu-\ell-2\nu+1} P_{M,2-k-\mu}(r,n) - n^{k+\mu-1}\right],
\end{aligned}$$

So in total, the assertion follows. ∎

Now we can prove the main result of this section.

Proof of Theorem V.1.5. By Lemma V.1.6 we see that

$$[f^-, g]_\nu(\tau) = \sum_{r\in\mathbb{Z}} b(r,y)q^r,$$

where

$$b(r,y) = \sum_{m-n=r}\sum_{\mu=0}^{\nu}\binom{k+\nu-1}{\nu-\mu}\binom{\ell+\nu-1}{\mu} \times \frac{\Gamma(1-k)}{\Gamma(1-k-\mu)} n^{k+\mu-1}\overline{c_f^-(n)}\Gamma(1-k-\mu;4\pi ny)a_g(m)m^{\nu-\mu}.$$

Applying π_{hol} then yields

$$\pi_{hol}([f^-,g]_\nu) = \sum_{r=1}^{\infty} b(r)q^r,$$

with

$$b(r) = \frac{(4\pi r)^{k+\ell+2\nu-1}}{(k+\ell+2\nu-2)!}\sum_{m-n=r}\sum_{\mu=0}^{\nu}\binom{k+\nu-1}{\nu-\mu}\binom{\ell+\nu-1}{\mu} \times \frac{\Gamma(1-k)}{\Gamma(1-k-\mu)} n^{k+\mu-1}\overline{c_f^-(n)}a_g(m)m^{\nu-\mu}\int_0^\infty \Gamma(1-k-\mu;4\pi ny)e^{-4\pi ry}y^{k+\ell+2\nu-2}dy.$$

The interchanging of integration and summation is justified by the assumption on the growth of the Fourier coefficients and applying the Theorem of Fubini-Tonelli.

The remaining integral is precisely the one from (V.1.6) so that the claim follows by Lemma V.1.7. □

To end this section, we state and prove some useful identities for the polynomial P which we shall need in the following two sections.

Lemma V.1.8. *For $b \neq 1, 2$, the polynomial $P_{a,b}(X,Y)$ from* (V.1.4) *fulfills the following identities.*

$$\begin{aligned}P_{a,b}(X,Y) &= \sum_{j=0}^{a-2}\binom{a+b-3}{j}X^jY^{a-2-j}\\ &= \sum_{j=0}^{a-2}\binom{a+b-3}{a-2-j}\binom{j+b-2}{j}(X+Y)^{a-2-j}(-Y)^j.\end{aligned}$$

Proof. We first prove the first identity by induction on a.

For $a = 2$ we have $1 = 1$.

Assume the claim to be true for one $a \geq 2$, then we get for $a+1$:

$$P_{a+1,b}(X,Y) = \sum_{j=0}^{a-1}\binom{j+b-2}{j}X^j(X+Y)^{a-2-j+1}$$

$$\begin{aligned}
&\overset{IV}{=} (x+y)\sum_{j=0}^{a-2}\binom{a+b-3}{j}X^jY^{a-2-j}+\binom{a+b-3}{a-1}X^{a-1}\\
&=\sum_{j=0}^{a-2}\binom{a+b-3}{j}X^{j+1}Y^{a-2-j}+\sum_{j=0}^{a-2}\binom{a+b-3}{j}X^jY^{a+1-j-2}\\
&\qquad+\binom{a+b-3}{a-1}X^{a-1}\\
&=\sum_{j=1}^{a-1}\binom{a+b-3}{j-1}X^jY^{a-1-j}+\sum_{j=0}^{a-2}\binom{a+b-3}{j}X^jY^{a-j-1}\\
&\qquad+\binom{a+b-3}{a-1}X^{a-1}\\
&=\sum_{j=1}^{a-2}\left[\binom{a+b-3}{j-1}+\binom{a+b-3}{j}\right]X^jY^{a-1-j}\\
&\qquad+Y^{a-1}+\left[\binom{a+b-3}{a-1}+\binom{a+b-3}{a-2}\right]X^{a-1}\\
&=\sum_{j=0}^{a-1}\binom{a+b-2}{j}X^jY^{a-1-j},
\end{aligned}$$

hence the first equation follows.

Now we show that

$$P_{a,b}(X,Y)=\sum_{j=0}^{a-2}\binom{a+b-3}{a-2-j}\binom{j+b-2}{j}(X+Y)^{a-2-j}(-Y)^j,$$

again by induction on a.

The case $a=2$ again yields $1=1$.

Suppose the equality to show holds for some $a\geq 2$ (Z:=X+Y):

$$\begin{aligned}
&\sum_{j=0}^{a-2}\binom{j+b-2}{j}(Z-Y)^jZ^{a-2-j}\\
=&\sum_{j=0}^{a-2}\binom{a+b-3}{a-2-j}\binom{j+b-2}{j}Z^{a-2-j}(-Y)^j.
\end{aligned}$$

Integration with respect to Y gives (C some constant to be determined later):

$$\begin{aligned}
&-\sum_{j=0}^{a-2}\binom{j+b-2}{j}\frac{1}{j+1}(Z-Y)^{j+1}Z^{a-2-j}\\
=&C+\sum_{j=0}^{a-2}\binom{a+b-3}{a-2-j}\binom{j+b-2}{j}Z^{a-2-j}\frac{(-1)^j}{j+1}Y^{j+1}.
\end{aligned}$$

Since by assumption we have $b \neq 2$, one easily sees that

$$\frac{1}{j+1}\binom{j+b-2}{j} = \frac{((j+1)+(b-1)-2)!}{(j+1)!(b-2)!} = \frac{1}{b-2}\binom{j+b-2}{j+1},$$

and therefore the above is equivalent to (replacing b by $b+1$)

$$\begin{aligned}&\sum_{j=1}^{a-1}\binom{j+b-2}{j}(Z-Y)^j Z^{a+1-j-2}\\ =&C' + \sum_{j=1}^{a-1}\binom{a+1+b-3}{a+1-2-j}\binom{j+b-2}{j} Z^{a+1-j-2}(-1)^j Y^j.\end{aligned}$$

If we let both sums start at $j=0$, then we just add constant terms in Y, thus comparison of the constant terms yields the assertion. Hence we have to prove

$$Z^{a-1}\sum_{j=0}^{a-1}\binom{j+b-2}{j} = Z^{a-1}\binom{a+b-2}{a-1},$$

which follows from the first identity we showed by plugging in $X=1$ and $Y=0$.□

V.2 Mock Modular Forms of weight $\left(\frac{3}{2}, \frac{1}{2}\right)$

In this section, we apply the results of Section V.1 to the case where $f \in \mathcal{M}_{\frac{3}{2}}(\Gamma_1(4N))$ and $g \in M_{\frac{1}{2}}(\Gamma_1(4N))$ for some $N \in \mathbb{N}$ such that $[f^+, g]_\nu$ is holomorphic at the cusps of Γ. In this case, we obtain the following main result.

Theorem V.2.1. *Fix $\nu \in \mathbb{N}_0$. Then there is a finite linear combination $L_\nu^{f,g}$ of functions of the form*

$$\Lambda_{s,t}^{\chi,\psi}(\tau;\nu) = \sum_{r=1}^{\infty}\left(2\sum_{\substack{sm^2-tn^2=r\\ m,n\geq 1}}\chi(m)\overline{\psi(n)}(\sqrt{s}m-\sqrt{t}n)^{2\nu+1}\right)q^r + \overline{\psi(0)}\sum_{r=1}^{\infty}\chi(r)(\sqrt{s}r)^{2\nu+1}q^{sr^2}$$

where $s,t \in \mathbb{N}$ and χ,ψ are even characters of conductors $F(\chi)$ and $F(\psi)$ respectively with $sF(\chi)^2, tF(\psi)^2|N$, such that

$$[f,g]_\nu + L_\nu^{f,g}$$

is a (holomorphic) quasi-modular form of weight 2 if $\nu = 0$ or otherwise a holomorphic modular form of weight $2\nu+2$.

Apart from some calculations, this result relies on the important theorem of Serre-Stark, see [46], Theorem A.

Theorem V.2.2 (J.-P. Serre and H. Stark, 1977). *Let Ω_N be defined as the set of all pairs (s,χ) of natural numbers s and characters χ of conductor $F(\chi)$ such that $4sF(\chi)^2|N$. Then the set*

$$\{\vartheta_{s,\chi}\,|\,(s,\chi)\in\Omega_N\}$$

with $\vartheta_{s,\chi}$ as in (II.1.2) *forms a basis of the space $M_{\frac{1}{2}}(\Gamma_1(N))$.*

The calculations rely on some basic facts about hypergeometric series which we recall in Section V.2.1. The calculations themselves can be found in Section V.2.2 and the proof of Theorem V.2.1 in Section V.2.3 concludes this section.

V.2.1 Hypergeometric series

Definition V.2.3. *(i) For $a\in\mathbb{C}$ and $n\in\mathbb{N}_0$ we define the* Pochhammer symbol *by*

$$(a)_n := \prod_{j=0}^{n-1}(a+j) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

(ii) For $|x|<1$ and complex numbers $a_1,\dots,a_p,b_1,\dots,b_q$ with $b_i\notin-\mathbb{N}_0$ we define the (generalized) hypergeometric series

$$ {}_pF_q\begin{pmatrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{pmatrix}; x\Big) := \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{x^n}{n!}. \tag{V.2.1}$$

Remark V.2.4. *(i) By considering the residues of the Gamma function (see Section III.1.1) one sees that also for $a\in-\mathbb{N}_0$ the description of the Pochhammer symbol by a quotient of Gamma functions makes sense.*

(ii) Every (convergent) power series $\sum\limits_{n=0}^{\infty}c_nx^n$ ($c_n\in\mathbb{C}$) where the quotient $\frac{c_{n+1}}{c_n}$ is a rational function in n can be written as a multiple of a hypergeometric series, more precisely, if

$$\frac{c_{n+1}}{c_n} = \alpha\frac{(a_1+n)\cdots(a_p+n)}{(b_1+n)\cdots(b_q+n)\cdot(1+n)}$$

then

$$\sum_{n=0}^{\infty}c_nx^n = c_0\,{}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};\alpha x\right).$$

For details, see [2, pp. 61f].

Among the vast collection of identities for hypergeometric series, we will make use primarily of the following one.

Theorem V.2.5 ([2], Theorem 2.2.6). *For complex numbers a, b, c such that $c, a+b-c \notin -\mathbb{N}_0$ and a non-negative integer $n \in \mathbb{N}_0$ the so-called* Pfaff-Saalschütz identity *holds,*

$$\text{(V.2.2)} \qquad {}_3F_2\left({-n, a, b \atop c, 1+a+b-c-n}; 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}.$$

V.2.2 Calculations

We first prove some identities for sums of binomial coefficients that we need to use the formulas in Lemma V.1.4 and Theorem V.1.5.

Lemma V.2.6. *(i) The following identity is valid for $\nu > 0$,*

$$\sum_{\mu=0}^{\nu} \frac{(-1)^\mu}{\mu - j + \frac{1}{2}} \binom{4\nu - 2\mu - 1}{2(\nu-\mu), 2\nu - \mu - 1} = 2^{4\nu}(-1)^j \frac{(2\nu - j)!j!}{(2j)!(2(\nu-j)+1)!}.$$

(ii) For κ from (V.1.3) *we have for all $\nu \geq 0$ that*

$$\kappa\left(\frac{3}{2}, \frac{1}{2}, \nu\right) = 2^{1-2\nu}\sqrt{\pi}\binom{2\nu}{\nu}.$$

Proof. We first prove (i). Write

$$\sum_{\mu=0}^{\nu} \frac{(-1)^\mu}{\mu - j + \frac{1}{2}} \binom{4\nu - 2\mu - 1}{2(\nu-\mu), 2\nu - \mu - 1} = \sum_{\mu=0}^{\nu} c_\mu$$

Then we have

$$\begin{aligned}
\frac{c_{\mu+1}}{c_\mu} &= \frac{\frac{(-1)^{\mu+1}}{\mu-j+\frac{3}{2}} \frac{(4\nu-2\mu-3)!}{(2(\nu-\mu-1))!(2\nu-\mu-2)!(\mu+1)!}}{\frac{(-1)^{\mu}}{\mu-j+\frac{1}{2}} \frac{(4\nu-2\mu-1)!}{(2(\nu-\mu))!(2\nu-\mu-1)!\mu!}} \\
&= -\frac{\mu - j + \frac{1}{2}}{\mu - j + \frac{3}{2}} \frac{(2(\nu-\mu))(2(\nu-\mu)-1)(2\nu-\mu-1)}{(4\nu-2\mu-1)(4\nu-2\mu-2)(\mu+1)} \\
&= \frac{\left(\mu - j + \frac{1}{2}\right)(\mu-\nu)\left(\mu - \nu + \frac{1}{2}\right)}{\left(\mu - j + \frac{3}{2}\right)\left(\mu - 2\nu + \frac{1}{2}\right)(\mu+1)}.
\end{aligned}$$

As we can see, the quotient $\frac{c_{\mu+1}}{c_\mu}$ is a monic rational function in μ, hence $\sum c_\mu$ is a multiple of a hypergeometric series, more precisely we have

$$\sum_{\mu=0}^{\nu} c_\mu = \frac{1}{-j+\frac{1}{2}}\binom{4\nu-1}{2\nu} {}_3F_2\left({-\nu, -j+\frac{1}{2}, -\nu+\frac{1}{2} \atop -j+\frac{3}{2}, -2\nu+\frac{1}{2}}; 1\right),$$

which by the Pfaff-Saalschütz identity (V.2.2) equals

$$\begin{aligned}
&\frac{1}{-j+\frac{1}{2}}\binom{4\nu-1}{2\nu} \frac{(1)_\nu(\nu-j+1)_\nu}{\left(-j+\frac{3}{2}\right)_\nu\left(\nu+\frac{1}{2}\right)_\nu} \\
=&\binom{4\nu-1}{2\nu}\nu! \frac{(2\nu-j)!}{(\nu-j)!} \frac{\Gamma\left(-j+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\nu-j+\frac{3}{2}\right)\Gamma\left(2\nu+\frac{1}{2}\right)}.
\end{aligned}$$

Simplifying this a little further yields the assertion:

$$\begin{aligned}\sum_{\mu=0}^{\nu} c_\mu &\overset{(\text{III.1.2})}{=} \binom{4\nu-1}{2\nu}\nu!\frac{(2\nu-j)!}{(\nu-j)!}(-1)^j\frac{\pi\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(\nu-j+\frac{3}{2}\right)\Gamma\left(2\nu+\frac{1}{2}\right)}\\ &\overset{(\text{III.1.3})}{=} \binom{4\nu-1}{2\nu}\nu!\frac{(2\nu-j)!}{(\nu-j)!}(-1)^j\frac{\pi\frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)}\sqrt{\pi}2^{-2\nu}}{\frac{\Gamma(2j+1)}{\Gamma(j+1)}\sqrt{\pi}2^{-2j}\frac{\Gamma(2(\nu-j)+3)}{\Gamma(\nu-j+2)}\sqrt{\pi}2^{-2(\nu-j+1)}}\\ &\qquad\times\frac{1}{\frac{\Gamma(4\nu+1)}{\Gamma(2\nu+1)}\sqrt{\pi}2^{-4\nu}}\\ &=\binom{4\nu-1}{2\nu}(2\nu-j)!(-1)^j2^{4\nu+2}\frac{(2\nu)!(2\nu-1)!}{(4\nu-1)!}\frac{1}{2}\frac{j!(\nu-j+1)}{(2(\nu-j)+2)!}\\ &=2^{4\nu}(-1)^j\frac{(2\nu-j)!j!}{(2j)!(2(\nu-j)+1)!}.\end{aligned}$$

The proof of (ii) works in the very same way, but is simpler:

We have

$$\kappa\left(\frac{3}{2},\frac{1}{2},\nu\right)=\frac{2\sqrt{\pi}}{(2\nu)!}\sum_{\mu=0}^{\nu}\underbrace{\frac{\Gamma\left(2\nu-\mu+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\mu\right)}\binom{\nu+\frac{1}{2}}{\nu-\mu}\binom{\nu-\frac{1}{2}}{\mu}}_{=:c_\mu}.$$

This sum can again be written as a hypergeometric series:

$$\begin{aligned}\frac{c_{\mu+1}}{c_\mu}&=\frac{\Gamma\left(2\nu-\mu-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}-\mu\right)}\frac{\Gamma\left(\nu+\frac{3}{2}\right)}{(\nu-\mu-1)!\Gamma\left(\mu+\frac{5}{2}\right)}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{(\mu+1)!\Gamma\left(\nu-\mu-\frac{1}{2}\right)}\\ &\quad\times\frac{\Gamma\left(\frac{1}{2}-\mu\right)}{\Gamma\left(2\nu-\mu+\frac{1}{2}\right)}\frac{(\nu-\mu)!\Gamma\left(\mu+\frac{3}{2}\right)}{\Gamma\left(\nu+\frac{3}{2}\right)}\frac{\mu!\Gamma\left(\nu-\mu+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\\ &=\frac{\left(\mu+\frac{1}{2}\right)(\mu-\nu)\left(\mu-\nu+\frac{1}{2}\right)}{\left(\mu-2\nu+\frac{1}{2}\right)\left(\mu+\frac{3}{2}\right)(\mu+1)},\end{aligned}$$

hence it holds that

$$\begin{aligned}\kappa\left(\frac{3}{2},\frac{1}{2},\nu\right)&=\frac{2\Gamma\left(2\nu+\frac{1}{2}\right)}{(2\nu)!}\binom{\nu+\frac{1}{2}}{\nu}{}_3F_2\left(\begin{matrix}-\nu,\frac{1}{2},-\nu+\frac{1}{2}\\ \frac{3}{2},-2\nu+\frac{1}{2}\end{matrix};1\right)\\ &\overset{(\text{V.2.2})}{=}\frac{2\Gamma\left(2\nu+\frac{1}{2}\right)}{(2\nu)!}\frac{\Gamma\left(\nu+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\nu!}\frac{(1)_\nu(\nu+1)_\nu}{\left(\frac{3}{2}\right)_\nu\left(\nu+\frac{1}{2}\right)_\nu}\\ &=\frac{2\Gamma\left(\nu+\frac{1}{2}\right)}{\nu!}\\ &\overset{(\text{III.1.3})}{=}\sqrt{\pi}2^{1-2\nu}\binom{2\nu}{\nu},\end{aligned}$$

which is what we claimed. □

With this we can prove the following.

Proposition V.2.7. *Let $r = m - n$. Then it holds that*

$$\sum_{\mu=0}^{\nu} \binom{\nu+\frac{1}{2}}{\nu-\mu}\binom{\nu-\frac{1}{2}}{\mu}\left(m^{\frac{1}{2}-\nu}P_{2\nu+2,\frac{1}{2}-\mu}(r,n) - n^{\frac{1}{2}+\mu}m^{\nu-\mu}\right)$$
$$= 2^{-2\nu}\binom{2\nu}{\nu}(m^{\frac{1}{2}} - n^{\frac{1}{2}})^{2\nu+1}.$$

Proof. For $\nu = 0$, the identity is immediate, thus assume from now on $\nu \geq 1$.

First, we can simplify the binomial coefficients on the left-hand side using Legendre's duplication formula (III.1.3) several times.

$$\begin{aligned}
\binom{\nu+\frac{1}{2}}{\nu-\mu}\binom{\nu-\frac{1}{2}}{\mu} &= \frac{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu-\mu+1)\Gamma\left(\mu+\frac{3}{2}\right)\Gamma(\mu+1)\Gamma\left(\nu-\mu+\frac{1}{2}\right)}\\
=&\frac{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}2^{1-(2(\nu-\mu)+1)}\Gamma(2(\nu-\mu)+1)\cdot\sqrt{\pi}2^{1-(2\mu+2)}\Gamma(2\mu+2)}\\
=&\frac{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{2^{-2\nu-1}\pi(2\mu+1)!(2(\nu-\mu))!}\\
=&\frac{\sqrt{\pi}2^{-2\nu-1}(2\nu+1)!\Gamma\left(\nu+\frac{1}{2}\right)}{2^{-2\nu-1}\pi\nu!(2\mu+1)!(2(\nu-\mu))!}\\
=&\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\nu!}\binom{2\nu+1}{2\mu+1}\\
=&2^{-2\nu}\binom{2\nu}{\nu}\binom{2\nu+1}{2\mu+1}.
\end{aligned}$$

On the one hand we have

$$(m^{\frac{1}{2}} - n^{\frac{1}{2}})^{2\nu+1} = \sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu}m^{\nu-\mu+\frac{1}{2}}n^{\mu} - \sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu+1}m^{\nu-\mu}n^{\mu+\frac{1}{2}},$$

on the other we see by Lemma V.1.8 that

$$\begin{aligned}
&\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu+1}\left(m^{\frac{1}{2}-\nu}P_{2\nu+2,\frac{1}{2}-\mu}(r,n) - n^{\frac{1}{2}+\mu}m^{\nu-\mu}\right)\\
=&\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu+1}\left(m^{\frac{1}{2}-\nu}\sum_{j=0}^{2\nu}\binom{2\nu-\mu-\frac{1}{2}}{2\nu-j}\binom{j-\mu-\frac{3}{2}}{j}m^{2\nu-j}(-n)^j - n^{\frac{1}{2}+\mu}m^{\nu-\mu}\right)\\
=&\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu+1}\sum_{j=0}^{2\nu}\binom{2\nu-\mu-\frac{1}{2}}{2\nu-j}\binom{j-\mu-\frac{3}{2}}{j}m^{\nu-j+\frac{1}{2}}(-n)^j\\
&\qquad -\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu+1}n^{\frac{1}{2}+\mu}m^{\nu-\mu}.
\end{aligned}$$

Thus it remains to show the following identity.

$$\text{(V.2.3)}\quad \begin{aligned}\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu+1}\sum_{j=0}^{2\nu}\binom{2\nu-\mu-\frac{1}{2}}{2\nu-j}\binom{j-\mu-\frac{3}{2}}{j}m^{\nu-j+\frac{1}{2}}(-n)^{j}\\ =\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu}m^{\nu-\mu+\frac{1}{2}}n^{\mu}.\end{aligned}$$

Again, we first simplify the product of the binomial coefficients in the inner sum. We calculate

$$\begin{aligned}&\binom{2\nu-\mu-\frac{1}{2}}{2\nu-j}\binom{j-\mu-\frac{3}{2}}{j}\\
=&\frac{\Gamma\left(2\nu-\mu+\frac{1}{2}\right)}{\Gamma(2\nu-j+1)\Gamma\left(j-\mu+\frac{1}{2}\right)}\cdot\frac{\Gamma\left(j-\mu-\frac{1}{2}\right)}{\Gamma(j+1)\Gamma\left(-\mu-\frac{1}{2}\right)}\\
=&\frac{1}{j-\mu-\frac{1}{2}}\frac{\Gamma\left(2\nu-\mu+\frac{1}{2}\right)}{\Gamma(2\nu-j+1)\Gamma(j+1)\Gamma\left(-\mu-\frac{1}{2}\right)}\\
\overset{\text{(III.1.2)}}{=}&\frac{1}{j-\mu-\frac{1}{2}}\frac{\Gamma\left(2\nu-\mu+\frac{1}{2}\right)\Gamma\left(\mu+\frac{3}{2}\right)}{\pi(2\nu-j)!j!}\underbrace{\sin\left(\left(\mu+\frac{3}{2}\right)\pi\right)}_{=(-1)^{\mu+1}}\\
\overset{\text{(III.1.3)}}{=}&\frac{(-1)^{\mu+1}}{j-\mu-\frac{1}{2}}\frac{\sqrt{\pi}2^{-4\nu+2\mu+1}\frac{\Gamma(4\nu-2\mu)}{\Gamma(2\nu-\mu)}\sqrt{\pi}2^{-2\mu-1}\frac{\Gamma(2\mu+2)}{\Gamma(\mu+1)}}{\pi(2\nu-j)!j!}\\
=&\frac{(-1)^{\mu+1}}{j-\mu-\frac{1}{2}}2^{-4\nu}\frac{(4\nu-2\mu-1)!(2\mu+1)!}{(2\nu-\mu-1)!(2\nu-j)!j!\mu!}.\end{aligned}$$

Now we have a look at the left-hand side of (V.2.3).

$$\begin{aligned}&\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu+1}\sum_{j=0}^{2\nu}\binom{2\nu-\mu-\frac{1}{2}}{2\nu-j}\binom{j-\mu-\frac{3}{2}}{j}m^{2\nu-j}(-n)^{j}\\
=&\sum_{j=0}^{2\nu}\sum_{\mu=0}^{\nu}\frac{(-1)^{\mu+1}}{j-\mu-\frac{1}{2}}2^{-4\nu}\frac{(2\nu+1)!(4\nu-2\mu-1)!}{(2(\nu-\mu))!(2\nu-\mu-1)!(2\nu-j)!j!\mu!}m^{2\nu-j}(-n)^{j}\\
=&2^{-4\nu}\sum_{j=0}^{2\nu}\binom{2\nu+1}{2j}\frac{(2j)!(2(\nu-j)+1)!}{2(\nu-j)!j!}m^{2\nu-j}(-n)^{j}\\
&\qquad\times\sum_{\mu=0}^{\nu}\frac{(-1)^{\mu+1}}{j-\mu-\frac{1}{2}}\binom{4\nu-2\mu-1}{2(\nu-\mu),2\nu-\mu-1,\mu}.\end{aligned}$$

By Lemma V.2.6(i) we see that (V.2.3) is valid and hence our Proposition. ☐

V.2.3 Proof of Theorem V.2.1

With our results from the previous subsection we can now step towards the proof of Theorem V.2.1.

Proof of Theorem V.2.1. By assumption both (ξf) and g are holomorphic modular forms of weight $\frac{1}{2}$, hence by the Theorem of Serre-Stark V.2.2 linear combinations of unary theta functions, i.e. functions of the form

$$\vartheta_{s,\chi}(\tau) = \sum_{n\in\mathbb{Z}} \chi(n) q^{sn^2},$$

where s, χ fulfill the conditions given in our Theorem. Thus we may assume without loss of generality that (ξf) and g are in fact unary theta series, say $(\xi f) = \vartheta_{t,\psi}$ and $g = \vartheta_{s,\chi}$. By formally using Proposition V.2.7, Lemma V.1.4, and Lemma V.2.6(ii) inside (V.1.5) we immediately get except for a constant factor of

$$4^{1-\nu}\binom{2\nu}{\nu}\sqrt{\pi}. \tag{V.2.4}$$

the formula for $\Lambda_{s,t}^{\chi,\psi}$ that we stated in the Theorem.

To complete the proof, we have to check that the sum

$$\sum_{\substack{sm^2-tn^2=r\\ m,n\geq 1}} \chi(m)\overline{\psi(n)}(\sqrt{s}m-\sqrt{t}n)^{2\nu+1}$$

for the coefficients converges. If s and t are both perfect squares, the sum is actually finite, since then $sm^2 - tn^2$ factors in the rational integers and thus each summand is a power of a divisor of r of which there are but finitely many. Let us now assume for simplicity that $s = 1$ and t is square-free, the general case works essentially in the same way. The Pell type equation

$$m^2 - tn^2 = r \tag{V.2.5}$$

is well-known to have at most finitely many fundamental integer solutions: if (a, b) is the fundamental solution of $a^2 - tb^2 = 1$ (i.e. $\varepsilon := a - \sqrt{t}b > 1$), then there is a solution (m_0, n_0) of (V.2.5) such that $|m_0| \leq \frac{\varepsilon+1}{2\sqrt{\varepsilon}}\sqrt{r}$ and $n_0 \leq \sqrt{\frac{m^2-r}{t}}$. In particular, there are only finitely many such so-called fundamental solutions. Then all solutions (m, n) in $\mathbb{Z}$ of (V.2.5) satisfy

$$m + \sqrt{t}n = \pm(m_0 + \sqrt{t}n_0)\cdot\varepsilon^k$$

for one $k \in \mathbb{Z}$. We are interested in solutions (m, n) in $\mathbb{N}$. It is plain that such a solution exists, provided that there is one in $\mathbb{Z}$. Furthermore, we see immediately, that the power of the fundamental unit ε has to be negative to parametrize all possible solutions. In particular this means that

$$\left|\sum_{\substack{sm^2-tn^2=r\\ m,n\geq 1}} \chi(m)\overline{\psi(n)}(\sqrt{s}m-\sqrt{t}n)^{2\nu+1}\right|$$

$$\leq \sum_{\substack{sm^2-tn^2=r\\ m,n\geq 1}} (m-\sqrt{t}n)^{2\nu+1} = \sum_{m_0,n_0}(m_0-\sqrt{t}n_0)^{2\nu+1}\sum_{k=0}^{\infty}\varepsilon^{-k(2\nu+1)} < \infty$$

because $\sum_{k=0}^{\infty} \varepsilon^{-k}$ is a geometric series and the set of possible (m_0, n_0) is finite. This completes the proof. $\square$

This theorem has the following nice corollary about the space of mixed mock modular forms of weight $\left(\frac{3}{2}, \frac{1}{2}\right)$ and degree ν.

Corollary V.2.8. *With the notation from Theorem V.2.1 the following is true. The equivalence classes $\Lambda_{s,t}^{\chi,\psi} + M_{2\nu+2}^{!}(\Gamma_1(4N))$ generate the vector space*

$$[\mathcal{M}_{\frac{3}{2}}^{mock}(\Gamma_1(4N)), M_{\frac{1}{2}}(\Gamma_1(4N))]_\nu / M_{2\nu+2}^{!}(\Gamma(4N))$$

of all νth order Rankin-Cohen brackets of mock modular forms of weight $\frac{3}{2}$ with holomorphic shadow with weight $\frac{1}{2}$ modular forms modulo the space of weakly holomorphic modular forms of weight $2\nu + 2$.

Proof. This is just another way to state Theorem V.2.1. $\square$

V.3 Mock Theta Functions

In this section, we prove an analogue to Theorem V.2.1 for mock theta functions. Note that there is no analogue for the theorem of Serre-Stark in weight $\frac{3}{2}$, so that we don't automatically get that the shadow of a weight $\frac{1}{2}$ mock modular form is a linear combination of theta functions, thus the assumption to deal with mock theta functions is really a restriction. So let $f \in \mathcal{S}_{\frac{1}{2}}(\Gamma_1(4N))$ be a completed mock theta function and $g \in S_{\frac{3}{2}}(\Gamma_1(N))$ be a (linear combination of) theta functions as in (II.1.3), such that again, for all ν the degree ν mixed mock modular form $[f^+, g]_\nu$ is holomorphic at the cusps. Then we have the following result.

Theorem V.3.1. *Let ν be a fixed non-negative integer. Then there is a finite linear combination $D_\nu^{f,g}$ of functions of the form*

$$\Delta_{s,t}^{\chi,\psi}(\tau;\nu) = 2\sum_{r=1}^{\infty} \left(\sum_{\substack{sm^2 - tn^2 = r \\ m,n \geq 1}} \chi(m)\overline{\psi(n)}(\sqrt{s}m - \sqrt{t}n)^{2\nu+1} \right) q^r,$$

where $s, t \in \mathbb{N}$ and χ, ψ are odd characters of conductors $F(\chi)$ and $F(\psi)$ respectively with $sF(\chi)^2, tF(\psi)^2 | N$, such that

$$[f, g]_\nu + D_\nu^{f,g}$$

is a (holomorphic) quasi-modular form of weight 2 if $\nu = 0$ or otherwise a holomorphic modular form of weight $2\nu + 2$.

Once we have done the necessary calculations to use the general formulas from Theorem V.1.5, the proof of Theorem V.3.1 works in exactly the same way as that of Theorem V.2.1, so we don't repeat it here. Note that because the shadow is a cusp form, the evaluation of Lemma V.1.4 is not necessary.

V.3.1 Calculations

Lemma V.3.2. *For all* $\nu \in \mathbb{N}_0$ *we have that*

$$\sum_{\mu=0}^{\nu} \frac{(-1)^\mu}{(2(j-\mu)+1} \binom{4\nu-2\mu+1}{2(\nu-\mu)+1, 2\nu-\mu} = (-1)^j 2^{4\nu} \frac{(2\nu-j)!j!}{(2(\nu-j))!(2j+1)!}.$$

Proof. As in Lemma V.2.6, we write the left hand side as a hypergeometric series where we call the summand c_μ. We get

$$\begin{aligned}
\frac{c_{\mu+1}}{c_\mu} =& \frac{\frac{(-1)^{\mu+1}}{(2(j-\mu)-1)} \frac{(4\nu-2\mu-1)!}{2(\nu-\mu)-1)!(2\nu-\mu-1)!(\mu+1)!}}{\frac{(-1)^\mu}{(2(j-\mu)+1} \frac{(4\nu-2\mu+1)!}{(2(\nu-\mu)+1)!(2\nu-\mu)!\mu!}} \\
=&(-1)\frac{2(j-\mu)+1}{2(j-\mu)-1} \frac{(2(\nu-\mu)+1)(2(\nu-\mu))(2\nu-\mu)}{(4\nu-2\mu+1)(4\nu-2\mu)(\mu+1)} \\
=&\frac{\left(\mu-j-\frac{1}{2}\right)\left(\mu-\nu-\frac{1}{2}\right)(\mu-\nu)}{\left(\mu-j+\frac{1}{2}\right)\left(\mu-2\nu-\frac{1}{2}\right)(\mu+1)},
\end{aligned}$$

hence we see that

$$\begin{aligned}
\sum_{\mu=0}^{\nu} c_\mu =& \frac{1}{2j+1}\binom{4\nu+1}{2\nu+1} {}_3F_2\left(\begin{matrix}-\nu, -j-\frac{1}{2}, -\nu-\frac{1}{2} \\ -j+\frac{1}{2}, -2\nu-\frac{1}{2}\end{matrix}; 1\right) \\
\overset{(\text{V.2.2})}{=}& \frac{1}{2j+1}\binom{4\nu+1}{2\nu+1} \frac{(1)_\nu(\nu-j+1)_\nu}{\left(-j+\frac{1}{2}\right)_\nu\left(\nu+\frac{3}{2}\right)_\nu} \\
=& \frac{1}{2j+1}\binom{4\nu+1}{2\nu+1}\nu!\frac{(2\nu-j)!}{(\nu-j)!}\frac{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(-j+\frac{1}{2}\right)}{\Gamma\left(\nu-j+\frac{1}{2}\right)\Gamma\left(2\nu+\frac{3}{2}\right)} \\
\overset{(\text{III.1.2})}{=}& \frac{1}{2j+1}\binom{4\nu+1}{2\nu+1}\nu!\frac{(2\nu-j)!}{(\nu-j)!}(-1)^j\frac{\Gamma\left(\nu+\frac{3}{2}\right)\pi}{\Gamma\left(\nu-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(2\nu+\frac{3}{2}\right)} \\
\overset{(\text{III.1.3})}{=}& \frac{1}{2j+1}\binom{4\nu+1}{2\nu+1}\nu!\frac{(2\nu-j)!}{(\nu-j)!}(-1)^j \\
&\times \frac{\Gamma(2\nu+3)\sqrt{\pi}2^{-2\nu-2}\pi\Gamma(\nu-j+1)\Gamma(j+1)\Gamma(2\nu+2)}{\Gamma(\nu+2)\Gamma(2(\nu-j)+1)\sqrt{\pi}2^{-2(\nu-j)}\Gamma(2j+1)\sqrt{\pi}2^{-2j}\Gamma(4\nu+3)\sqrt{\pi}2^{-4\nu-2}} \\
=& \frac{1}{2j+1}\binom{4\nu+1}{2\nu+1}\nu!(2\nu-j)!(-1)^j2^{4\nu}\frac{(2\nu+2)!j!(2\nu+1)!}{(\nu+1)!(2(\nu-j))!(2j)!(4\nu+2)!} \\
=& \frac{1}{2j+1}\binom{4\nu+1}{2\nu+1}\nu!(2\nu-j)!(-1)^j2^{4\nu}\frac{2(2\nu+1)!j!(2\nu)!}{\nu!(2(\nu-j))!(2j)!2(4\nu+1)!} \\
=&(-1)^j2^{4\nu}\frac{(2\nu-j)!j!}{(2(\nu-j))!(2j+1)!}
\end{aligned}$$

and the proof is finished. $\square$

Now we prove the analogue of Proposition V.2.7 in weight $\left(\frac{3}{2}, \frac{1}{2}\right)$.

Proposition V.3.3. *The following identity holds true for all $\nu \geq 0$ and $r := m - n$.*

$$\sum_{\mu=0}^{\nu} \binom{\nu-\frac{1}{2}}{\nu-\mu}\binom{\nu+\frac{1}{2}}{\mu}\left(m^{-\nu-\frac{1}{2}}P_{2\nu+2,\frac{3}{2}-\mu}(r,n) - n^{\mu-\frac{1}{2}}m^{\nu-\mu}\right) = -2^{-2\nu}\binom{2\nu}{\nu}(mn)^{-\frac{1}{2}}\left(m^{\frac{1}{2}}-n^{\frac{1}{2}}\right)^{2\nu+1} \tag{V.3.1}$$

Proof. The assertion is obvious for $\nu = 0$, thus suppose $\nu \geq 1$.

From the proof of Proposition V.2.7 we immediately get that

$$\begin{aligned}\binom{\nu-\frac{1}{2}}{\nu-\mu}\binom{\nu+\frac{1}{2}}{\mu} &= \frac{\mu+\frac{1}{2}}{\nu-\mu+\frac{1}{2}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu-\mu+1)\Gamma\left(\mu+\frac{3}{2}\right)}\frac{\Gamma\left(\nu+\frac{3}{2}\right)}{\Gamma(\mu+1)\Gamma\left(\nu-\mu+\frac{1}{2}\right)}\\ &= \frac{2\mu+1}{2(\nu-\mu)+1}2^{-2\nu}\binom{2\nu}{\nu}\binom{2\nu+1}{2\mu+1}\\ &= 2^{-2\nu}\binom{2\nu}{\nu}\binom{2\nu+1}{2\mu}.\end{aligned}$$

Thus using Lemma V.1.8 we get that the left-hand side of (V.3.1) equals

$$2^{-2\nu}\binom{2\nu}{\nu}\left(\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu}\left[\sum_{j=0}^{2\nu}(-1)^j\binom{2\nu-\mu+\frac{1}{2}}{2\nu-j}\binom{j-\mu-\frac{1}{2}}{j}m^{\nu-j-\frac{1}{2}}n^j\right] - \sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu}m^{\nu-\mu}n^{\mu-\frac{1}{2}}\right)$$

while the right-hand side is given by

$$-2^{-2\nu}\binom{2\nu}{\nu}\left(\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu}m^{\nu-\mu+\frac{1}{2}}n^{\mu-\frac{1}{2}} - \sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu+1}m^{\nu-\mu-\frac{1}{2}}n^{\mu}\right).$$

Hence we just have to show that

$$\sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu}\left[\sum_{j=0}^{2\nu}(-1)^j\binom{2\nu-\mu+\frac{1}{2}}{2\nu-j}\binom{j-\mu-\frac{1}{2}}{j}m^{\nu-j-\frac{1}{2}}n^j\right] = \sum_{\mu=0}^{\nu}\binom{2\nu+1}{2\mu+1}m^{\nu-\mu-\frac{1}{2}}n^{\mu}.$$

The product of the binomial coefficients in the inner sum can be simplified as in the proof of Lemma V.2.6:

$$\begin{aligned}\binom{2\nu-\mu+\frac{1}{2}}{2\nu-j}\binom{j-\mu-\frac{1}{2}}{j} =&\frac{\left(2\nu-\mu+\frac{1}{2}\right)\left(j-\mu-\frac{1}{2}\right)}{\left(j-\mu+\frac{1}{2}\right)\left(-\mu-\frac{1}{2}\right)}\binom{2\nu-\mu-\frac{1}{2}}{2\nu-j}\binom{j-\mu-\frac{3}{2}}{j}\\ =&\frac{(4\nu-2\mu+1)\left(j-\mu-\frac{1}{2}\right)}{(2(j-\mu)+1)(2\mu+1)}\frac{(-1)^{\mu}}{j-\mu-\frac{1}{2}}2^{-4\nu+1}\\ &\times\frac{(4\nu-2\mu-1)!(2\mu+1)!}{(2\nu-\mu-1)!(2\nu-j)!j!\mu!}\\ =&\frac{(4\nu-2\mu+1)}{(2(j-\mu)+1)}(-1)^{\mu}2^{-4\nu+1}\frac{(4\nu-2\mu-1)!(2\mu)!}{(2\nu-\mu-1)!(2\nu-j)!j!\mu!},\end{aligned}$$

such that we have for the left-hand side the following,

$$\begin{aligned}&2^{-4\nu}\sum_{\mu=0}^{\nu}\sum_{j=0}^{2\nu}(-1)^j\frac{(2\nu+1)!\,(4\nu-2\mu+1)!}{(2(\nu-\mu)+1)!(2\nu-\mu)!\,(2(j-\mu)+1)\,(2\nu-j)!j!\mu!}\\ &\qquad\times(-1)^{\mu}m^{\nu-j-\frac{1}{2}}n^j\\ =&2^{-4\nu}\sum_{j=0}^{2\nu}(-1)^j\frac{(2\nu+1)!}{(2\nu-j)!j!}m^{\nu-j-\frac{1}{2}}n^j\sum_{\mu=0}^{\nu}\frac{(-1)^{\mu}}{(2(j-\mu)+1}\binom{4\nu-2\mu+1}{2(\nu-\mu)+1,2\nu-\mu},\end{aligned}$$

thus by Lemma V.3.2 the assertion follows. □

Of course, there is also an analogue to Corollary V.2.8.

Corollary V.3.4. *With the notation from Theorem V.3.1 the following is true. The equivalence classes $\Delta_{s,t}^{\chi,\psi}+M^!_{2\nu+2}(\Gamma_1(4N))$ generate the vector space*

$$[\mathcal{S}^{mock-\vartheta}_{\frac{1}{2}}(\Gamma_1(4N)),S^{\vartheta}_{\frac{3}{2}}(\Gamma_1(4N))]_{\nu}/M^!_{2\nu+2}(\Gamma_1(4N))$$

of all νth order Rankin-Cohen brackets of mock theta functions with weight $\frac{3}{2}$ theta functions modulo the space of weakly holomorphic modular forms of weight $2\nu+2$.

To conclude this section, we make the following remark.

Remark V.3.5. *It is plain that the functions $\Lambda_{s,t}^{\chi,\psi}(\tau;\nu)$ and $\Delta_{s,t}^{\chi,\psi}(\tau;\nu)$ in this chapter's main theorems are clearly (except for a polynomial factor) indefinite theta series. Suppose from now on that s and t are coprime. In the case where s and t are both perfect squares, we see that the coefficients of $\Lambda_{s,t}^{\chi,\psi}$ and $\Delta_{s,t}^{\chi,\psi}$ are essentially minimal divisor power sums where there may be some congruence conditions on the divisors taken into account, see e.g. Proposition V.4.1 or Proposition V.4.3. From Lemma III.1.2 and Lemma IV.1.4 we know that these can be realized as derivatives of (sieved) Appell-Lerch sums.*

In the case when s and t are not both perfect squares however the coefficients still resemble minimal divisor power sums, but then in a certain subring of a real-quadratic (if exactly one of them is a square) or real-biquadratic (if both are non-squares) number field. But these can, at least to my knowledge, not be represented *as derivatives of regular Appell-Lerch sums, which brings up the question for a generalization of Appell-Lerch sums. I hope to come back to this question in my further research.*

V.4 Examples

In this section, we use Theorem V.2.1 to obtain new proofs of the main results in Chapters III and IV.

V.4.1 Trace Formulas

From Remark III.1.6 and Proposition II.3.4 we see that the shadow of the class number generating function $\mathscr{H}$ is $\frac{1}{8\sqrt{\pi}}\vartheta$. Thus Theorem V.2.1, equation (V.2.4) and Proposition V.1.2 tell us that

$$[\mathscr{H},\vartheta]_\nu + 2^{-2\nu-1}\binom{2\nu}{\nu}\Lambda'$$

with $\Lambda'(\tau;\nu) = \Lambda_{1,1}^{1,1}(\tau;\nu)$ as in Theorem V.2.1 is a quasi-modular form of weight 2 for $\nu = 0$ and a holomorphic modular form of weight $2\nu+2$ otherwise, both on the group $\Gamma_0(4)$.

With a little elementary number theory we see the following.

Proposition V.4.1. *With λ_k defined as in* (I.1.3) *we have*

$$(\Lambda'|U(4))(\tau;\nu) = 2^{2\nu+1}\sum_{n=1}^{\infty} 2\lambda_{2\nu+1}(n)q^n$$

and

$$(\Lambda'|S_{2,1})(\tau;\nu) = 2\sum_{n\ odd}\lambda_{2\nu+1}(n)q^n.$$

Proof. Let $L_r := \{(m,n)\,|\,m>n>0,\ m^2-n^2=r\}$ and $D_r := \{d|r \mid d<\sqrt{n}\}$. Then the map

$$L_{4r}\to D_r,\ (m,n)\mapsto \frac{m-n}{2}$$

defines a bijection between the two sets for all $r\in\mathbb{N}$ since in order to fulfill $m^2-n^2=4r$, it must hold that $m\equiv n \pmod 2$. The inverse map is given by

$$D_r\to L_{4r},\ d\mapsto\left(d+\frac{r}{d},\frac{r}{d}-d\right).$$

For odd r we also have the bijection

$$L_r\to D_r,\ (m,n)\mapsto m-n,$$

since with r every divisor of r is odd and thus the inverse map

$$D_r\to L_r,\ d\mapsto\left(\frac{d^2+r}{2d},\frac{r-d^2}{2d}\right)$$

is well-defined. $\square$

By [14, Theorem 6.1] and Remark III.1.1 one as the formal identity

$$S_f^1(\tau;X) := \sum_{n=0}^{\infty}\left(\sum_{s\in\mathbb{Z}}\frac{a(n-s^2)}{1-2sX+nX^2}\right)q^n = \sum_{\nu=0}^{\infty}\frac{4^\nu}{\binom{2\nu}{\nu}}[f,\vartheta]_\nu(\tau),$$

where $f(\tau)=\sum_{n=0}^{\infty}a(n)q^n$ is a modular form of weight $\frac{3}{2}$. From there we can deduce that also the following is true

$$\left(S_f^1|U(4)\right)(\tau;X) = \sum_{n=0}^{\infty}\left(\sum_{s\in\mathbb{Z}}\frac{a(4n-s^2)}{1-s(2X)+n(2X)^2}\right)q^n = \sum_{\nu=0}^{\infty}\frac{4^\nu}{\binom{2\nu}{\nu}}([f,\vartheta]_\nu|U(4))(\tau).$$

Hence the function ($N\in\{1,4\}$)

$$C^{(N)}(\tau)=\sum_{n=1}^{\infty}c_\nu^{(N)}(n)q^n$$

with

$$c_\nu^{(1)}(n) = \sum_{s\in\mathbb{Z}} g_\nu^{(1)}(s,n)H(4n-s^2)+2\lambda_{2\nu+1}(n) \tag{V.4.1}$$

$$c_\nu^{(4)}(n) = \begin{cases}\sum_{s\in\mathbb{Z}} g_\nu^{(4)}(s,n)H(n-s^2)+\lambda_{2\nu+1}(n) & \text{for } n \text{ odd}\\ 0 & \text{otherwise,}\end{cases} \tag{V.4.2}$$

where $g_\nu^{(1)}(s,n)$ (resp. $g_\nu^{(4)}$) is the coefficient of $X^{2\nu}$ in the Taylor expansion of $\frac{1}{1-sX+nX^2}$ (resp. $\frac{1}{1-2sX+nX^2}$), is a holomorphic modular form of weight $2\nu+2$ on $\Gamma_0(4)$. By Proposition II.1.11, the modular forms with coefficients (V.4.1) are in fact on the full modular group $\mathrm{SL}_2(\mathbb{Z})$. That we actually have cusp forms follows as in the second part of the proof of Theorem III.2.2.

Due to the invariance of the Petersson scalar product under holomorphic projection one can even say something about the nature of these cusp forms. For the case of the Eichler-Selberg trace formula this can be found e.g. in [51]. There this is carried out in greater detail as here.

Theorem V.4.2. *It holds that*

$$-\frac{1}{2}\sum_{s\in\mathbb{Z}} g_\nu^{(1)}(s,n)H(4n-2^2)-\lambda_{2\nu+1}(n) = \mathrm{trace}(T_n^{(2\nu+2)}(1)) \tag{V.4.3}$$

$$-3\sum_{s\in\mathbb{Z}} g_\nu^{(4)}(s,n)H(n-2^2)-3\lambda_{2\nu+1}(n) = \mathrm{trace}(T_n^{(2\nu+2)}(4)), \tag{V.4.4}$$

where $T_n^{(k)}(N)$ denotes the nth Hecke operator acting on the space $S_k(\Gamma_0(N))$. Note that (V.4.4) *is only valid for odd n.*

The equation (V.4.3) is the Eichler-Selberg trace formula from Theorem I.1.3, the equation (V.4.4) has been alluded to in Remark III.2.4.

Proof. In order to prove these two trace formulas (V.4.3) and (V.4.4) one may use the Rankin-Selberg unfolding trick to see that for any normalized Hecke eigenform f on $\mathrm{SL}_2(\mathbb{Z})$ (resp. $\Gamma_0(4)$) we get for $\nu \geq 1$ (cf. Lemma V.1.3)

$$\langle [\widehat{\mathscr{H}}, \vartheta]_\nu, f\rangle = \langle \pi_{hol}([\widehat{\mathscr{H}}, \vartheta]_\nu), f\rangle \doteq \langle f, f\rangle$$

(for $\mathrm{SL}_2(\mathbb{Z})$ one has to apply $U(4)$ to obtain a cusp form of level 1). The Hecke trace generating function

$$\mathcal{T}_{2\nu+2} = \sum_{n=1}^{\infty} \mathrm{trace}(T_n^{(2\nu+2)})q^n$$

is the sum over all normalized Hecke eigenforms, hence we also have $\langle \mathcal{T}_{2\nu+2}, f\rangle = \langle f, f\rangle$ since Hecke eigenforms are orthogonal (actually, the summation should be restricted to the n coprime to the level). Therefore $\pi_{hol}([\widehat{\mathscr{H}}, \vartheta]_\nu)) \doteq \mathcal{T}_{2\nu+2}$ which proves both trace formulas. □

V.4.2 Class Number Relations

With Theorem V.2.1 we can consider more general types of sums as in (IV.0.1) without too much more work. Let therefore

$$H_{a,p}^{(\nu)}(n) := \sum_{\substack{s\in\mathbb{Z} \\ s\equiv a \pmod p}} g_\nu^{(1)}(s,n)H(4n-s^2) \tag{V.4.5}$$

with $g_\nu^{(1)}$ as in (V.4.1), $a \in \mathbb{Z}$ and p an odd prime. This is up to a constant factor the coefficient of q^n in the Fourier expansion of the function $([\mathscr{H}, \vartheta^{(p,a)}]_\nu|U(4))$ with $\vartheta^{(p,a)}$ as in (IV.1.1). From Theorem V.2.1 we can now deduce that for

$$\Lambda_{2\nu+1}^{(p,a)}(\tau) := \sum_{\pm}\left[2\sum_{\substack{m^2-n^2>0 \\ m,n\geq 1 \\ m\equiv \pm a \pmod p}} (m-n)^{2\nu+1}q^{m^2-n^2} + \sum_{\substack{m\geq 1 \\ m\equiv \pm a \pmod p}} m^{2\nu+1}q^{m^2}\right]$$

the function $([\mathscr{H}, \vartheta^{(p,a)}]_\nu) + \Lambda^{(p,a)})|U(4)$ is a holomorphic modular form of weight $2+2\nu$ (resp. a quasi modular form of weight 2 for $\nu = 0$) on $\Gamma = \Gamma_0(p^2)\cap\Gamma_1(p)$ as described in Lemma 3.1 of [4].

We can also represent $\Lambda_{2\nu+1}^{(p,a)}$ as a minimal divisor power sum:

Proposition V.4.3. *Let*

$$D_k^{(p,a)}(\tau) := \sum_{n=1}^{\infty} \lambda_k^{(p,a)}(n)q^n,$$

where

$$\lambda_k^{(p,a)}(n) := \sum_{\substack{d|n \\ d\leq\sqrt{n} \\ d\equiv -a \pmod p}} d^k + \sum_{\substack{d|n \\ d\lneq\sqrt{n} \\ d\equiv a \pmod p}} d^k.$$

Then it holds that for $\nu \in \mathbb{N}_0$ *we have for* $a \neq 0$

$$\left((\Lambda_{2\nu+1}^{(p,a)}|U(4)\right) = 2^{2\nu+1}\left[\sum_{b\not\equiv\pm a}\left(D_{2\nu+1}^{\left(p,\frac{a-b}{2}\right)}|S_{p,\frac{a^2-b^2}{4}}\right)(\tau) + \left(\left(D_{2\nu+1}^{(p,a)} + D_{2\nu+1}^{(p,-a)}\right)|S_{p,0}\right)(\tau)\right.$$
$$\left. + p^{2\nu+1}(D_{2\nu+1}^{(1,0)}|V(p))(\tau)\right].$$

and

$$\left((\Lambda_{2\nu+1}^{(p,0)}|U(4)\right) = 2^{2\nu+1}\cdot\left[\sum_{b\not\equiv 0}\left(D_{2\nu+1}^{\left(p,\frac{b}{2}\right)}|S_{p,-\frac{b^2}{4}}\right) + p^{2\nu+1}\left(D_{2\nu+1}^{(1,0)}|V(p^2)\right)\right]$$

otherwise.

Proof. To ease up notation a little bit, we write $d| < r$ in the following if $d|r$ and $d < \sqrt{r}$. All equivalences from now on are to be understood modulo p if not stated otherwise.

We have

$$
\begin{aligned}
&\left((\Lambda_{2\nu+1}^{(p,a)}|U(4)\right)(\tau)\\
=&\sum_{\pm}\sum_{r=1}^{\infty}\left[2\sum_{\substack{m^2-n^2=4r\\ m,n\geq 1\\ m\equiv\pm a}}(m-n)^{2\nu+1}q^r+\sum_{\substack{m\geq 1\\ m\equiv\pm\frac{a}{2}}}(2m)^{2\nu+1}q^{m^2}\right]\\
=&2^{2\nu+1}\sum_{\pm}\sum_{r=1}^{\infty}2\sum_{\substack{m^2-n^2=4r\\ m,n\geq 1\\ m\equiv\pm a}}\left(\frac{m-n}{2}\right)^{2\nu+1}q^r+2^{2\nu+1}\sum_{\substack{m\geq 1\\ m\equiv\pm\frac{a}{2}}}m^{2\nu+1}q^{m^2}\\
=&2^{2\nu+1}\sum_{\pm}\sum_{b\pmod p}\sum_{r\equiv\frac{a^2-b^2}{4}}2\sum_{\substack{m^2-n^2=4r\\ m,n\geq 1\\ m\equiv\pm a,\, n\equiv b}}\left(\frac{m-n}{2}\right)^{2\nu+1}q^r\\
&\quad+2^{2\nu+1}\sum_{\substack{m\geq 1\\ m\equiv\pm\frac{a}{2}}}m^{2\nu+1}q^{m^2}\\
=&2^{2\nu+1}\sum_{\pm}\sum_{b\pmod p}\sum_{r\equiv\frac{a^2-b^2}{4}}2\left(\sum_{\substack{d|<r\\ d\equiv\frac{\pm a-b}{2}}}d^{2\nu+1}\right)q^r\\
&\quad+2^{2\nu+1}\sum_{\substack{m\geq 1\\ m\equiv\pm\frac{a}{2}}}m^{2\nu+1}q^{m^2}
\end{aligned}
$$

Now we have to distiguish two cases. First assume $a \neq 0$. Then the above equals

$$\begin{aligned}
&2^{2\nu+1} \cdot \left[\sum_{b \not\equiv \pm a, 0} \sum_{r \equiv \frac{a^2-b^2}{4}} 2 \left(\sum_{\substack{d|<r \\ d \equiv \frac{a-b}{2}}} d^{2\nu+1} + \sum_{\substack{d|<r \\ d \equiv \frac{-a-b}{2}}} d^{2\nu+1} \right) \right. \\
&\quad + \sum_{r \equiv 0} 2 \left(\sum_{\substack{d|<r \\ d \equiv 0}} d^{2\nu+1} + \sum_{\substack{d|<r \\ d \equiv -a}} d^{2\nu+1} + \sum_{\substack{d|<r \\ d \equiv a}} d^{2\nu+1} \right) q^r \\
&\quad \left. + \sum_{r \equiv \frac{a^2}{4}} 2 \left(\sum_{\substack{d|<r \\ d \equiv \frac{a}{2}}} d^{2\nu+1} + \sum_{\substack{d|<r \\ d \equiv -\frac{a}{2}}} d^{2\nu+1} \right) q^r + \sum_{m \equiv \frac{a}{2}} m^{2\nu+1} q^{m^2} + \sum_{m \equiv \frac{a}{2}} m^{2\nu+1} q^{m^2} \right] \\
=& 2^{2\nu+1} \left[\sum_{b \not\equiv \pm a} \left(D_{2\nu+1}^{\left(p, \frac{a-b}{2}\right)} | S_{p, \frac{a^2-b^2}{4}} \right)(\tau) + \left(\left(D_{2\nu+1}^{(p,a)} + D_{2\nu+1}^{(p,-a)} \right) | S_{p,0} \right)(\tau) \right. \\
&\qquad \left. + p^{2\nu+1} (D_{2\nu+1}^{(1,0)} | V(p))(\tau) \right].
\end{aligned}$$

For $a = 0$ we get that the expression equals

$$\begin{aligned}
&2^{2\nu+1} \cdot \left[\sum_{b \not\equiv 0} \sum_{r \equiv -\frac{b^2}{4}} 2 \left(\sum_{\substack{d|<r \\ d \equiv -\frac{b}{2}}} d^{2\nu+1} \right) q^r + \sum_{r \equiv 0 \pmod{p^2}} 2 \left(\sum_{\substack{d|<r \\ d \equiv 0}} d^{2\nu+1} \right) + \sum_{m \equiv 0} m^{2\nu+1} q^{m^2} \right] \\
=& 2^{2\nu+1} \cdot \left[\sum_{b \not\equiv 0} \left(D_{2\nu+1}^{\left(p, \frac{b}{2}\right)} | S_{p, -\frac{b^2}{4}} \right)(\tau) + p^{2\nu+1} \left(D_{2\nu+1}^{(1,0)} | V(p^2) \right)(\tau) \right].
\end{aligned}$$

This yields at once the following generalization on Theorem 1.4 in[4].

Theorem V.4.4. *For $\nu \in \mathbb{N}_0$, p an odd prime, and $a \neq 0$ the function*

$$\begin{aligned}
&([\mathscr{H}, \vartheta^{(p,a)}]_\nu | U(4))(\tau) + 2^{2\nu+1} \left[\sum_{b \not\equiv \pm a} \left(D_{2\nu+1}^{\left(p, \frac{a-b}{2}\right)} | S_{p, \frac{a^2-b^2}{4}} \right)(\tau) \right. \\
&\qquad \left. + \left(\left(D_{2\nu+1}^{(p,a)} + D_{2\nu+1}^{(p,-a)} \right) | S_{p,0} \right)(\tau) + p^{2\nu+1} (D_{2\nu+1}^{(1,0)} | V(p))(\tau) \right]
\end{aligned}$$

is a quasi-modular form of weight 2 for $\nu = 0$ and a cusp form of weight $2\nu + 2$ on $\Gamma_0(p^2) \cap \Gamma_1(p)$ and analogously

$$([\mathscr{H}, \vartheta^{(p,0)}]_\nu | U(4))(\tau) + 2^{2\nu+1} \cdot \left[\sum_{b \not\equiv 0} \left(D_{2\nu+1}^{\left(p, \frac{b}{2}\right)} | S_{p, -\frac{b^2}{4}} \right)(\tau) + p^{2\nu+1} \left(D_{2\nu+1}^{(1,0)} | V(p^2) \right)(\tau) \right]$$

is quasi-modular resp. cuspidal on $\Gamma_0(p^2)$.

From this general result, we get in particular an alternative proof of Theorem IV.1.1 and several class number relations conjectured in [10].

By comparing Fourier coefficients one could therefore figure out infinitely many further class number relations but we won't do this here.

V.5 Conclusion

The proofs of the results in Chapters III and IV are rather *ad hoc* proofs. Although they are similar in nature, One proof only applies to one of the results. The method of holomorphic projection now yields a general theorem which implies both Cohen's conjecture I.2.1 and a not unexpected, but also not obvious generalization of Theorem IV.1.1 to mixed mock modular forms of higher degree as simple corollaries. For instance, the identities for smallest parts functions proven in [1] can now be extended in the same way without too much more effort.

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Nomenclature

$(a)_n$	Pochhammer symbol, cf. p. 57
$[\cdot,\cdot]_\nu$	νth Rankin-Cohen bracket, cf. p. 16
$\mathbb{H}$	complex upper half-plane
$\mathcal{G}_k$	re-normalized Eisenstein-series of weight k, cf. p. 40
$\mathscr{H}$	generating function of Hurwitz class numbers, cf. 8
χ_p	non-trivial, real-valued character mod p, cf. p. 40
Δ_k	hyperbolic Laplacian of weight k, cf. p. 20
$\Gamma(N)$	principal congruence subgroup of level N of $\mathrm{SL}_2(\mathbb{Z})$, cf. p. 11
$\Gamma_0(N)$, $\Gamma_1(N)$	certain congruence subgroups of level N of $\mathrm{SL}_2(\mathbb{Z})$, cf. p. 11
γ	2×2-matrix of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, mostly contained in a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$
$\Gamma(\alpha;x)$	incomplete Gamma function, cf. p. 21
$\mathcal{H}_k(N,\chi)$	vector space of harmonic weak Maaß forms of weight k on $\Gamma_0(N)$ with character χ, cf. p. 20
$\lambda_k(n)$	k-th power minimal-divisor sum, cf. p. 7
$\mathcal{M}_k(\Gamma)$, $\mathcal{S}_k(\Gamma)$	spaces of harmonic Maaß forms with holomorphic resp. cuspidal shadow, cf. p. 22
$\sigma_k(n)$	k-th power divisor sum, cf. p. 7
$\mathrm{SL}_n(R)$	special linear group of degree n over the commutative ring R, $\{g \in R^{n\times n} \mid \det g = 1\}$
τ	variable from $\mathbb{H}$
Θ	Jacobi Theta function, cf. p. 19
ξ_k	ξ-operator of weight k, cf. p. 21

$\mathbb{Z}_K$	ring of integers in an algebraic number field K
$\zeta(s)$	Riemann ζ-function
A_ℓ	Appell-Lerch sum of level ℓ, cf. p. 22
D_t	differential operator, $\frac{1}{2\pi i}\frac{d}{dt}$
E_k	normalized Eisenstein series of weight $k \geq 4$, cf. p. 13
f^+, f^-	holomorphic and non-holomorphic part of a harmonic Maaß form f, cf. p. 21
G_k	Eisenstein series of weight $k \geq 4$, cf. p. 13
$h(D)$	class number for discriminant D, cf. p. 5
$H(n)$	Hurwitz class number, cf. p. 6
$H(r, n)$	generalization of the Hurwitz class number, cf. p. 8
$h(r, n)$	generalization of the class number, cf. p. 8
$M_*(\Gamma)$	graded algebra of modular forms on Γ
q	$e^{2\pi i\tau}$
$S_k(\Gamma)$	vector space of cusp forms of weight k on Γ, cf. p. 12
$T_n^{(k)}(N)$	nth Hecke operator on $M_k(\Gamma_0(N))$ or $S_k(\Gamma_0(N))$, cf. p. 18
$U(N), V(N), S_{N,r}, \otimes\chi$	modular operators, cf. p. 17
w_D	$\lvert\mathbb{Z}^*_{\mathbb{Q}(\sqrt{D})}\rvert$ for $D < 0$ fundamental discriminant, cf. p. 6
$M_k(\Gamma)$	vector space of modular forms of weight k on Γ, cf. p. 12
$M_k^!(\Gamma)$	vector space of weakly holomorphic modular forms of weight k on Γ, cf. p. 12

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Glossary

L-series, 6
ϑ-series
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Appell-Lerch sum, 22

congruence subgroup, 11
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Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbstständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit - einschließlich Tabellen, Karten und Abbildungen -, die anderen Werken entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen - noch nicht veröffentlicht worden ist, sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Frau Prof. Dr. Kathrin Bringmann betreut worden.

Köln, den 05. Mai 2014

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seit Oktober 2012	Mitglied des DFG Graduiertenkollegs 1269 "'Globale Strukturen in Geometrie und Analysis"'

Lehre

WS 2009/10 - SS 2012	verschiedene Positionen als studentische Hilfskraft, vorwiegend am Lehrstuhl D für Mathematik, RWTH Aachen
SS 2014	Mit-Betreuer (mit Prof. Dr. Kathrin Bringmann, Dr. Yingkun Li und Dr. Larry Rolen) des Studenten-Seminars “Asymptotische Entwicklungen von Modulformen”, Universität zu Köln

Professionelle Aktivitäten

März 2015	Mit-Organisator (mit Prof. Dr. Kathrin Bringmann, Prof. Dr. Igor Burban, Bea Schumann und Prof. Dr. Peter Littelmann) der Spring School “Modular Invariance and Characters of Representations” Max-Planck-Institut für Mathematik, Bonn

Veröffentlichungen

1. (mit Markus Kirschmer) “On an analogue of the Lucas-Lehmer-Riesel test using elliptic curves”,
   Integers (2013) 13 A15, online verfügbar unter
   `http://www.integers-ejcnt.org/vol13.html`

2. “Automorphism Groups of Hyperbolic Lattices”,
   Journal of Algebra, vol. 408, pp. 147-165

3. “Mock Modular Forms and Class Number Relations”,
   zur Veröffentlichung angenommen in *Research in the Mathematical Sciences*

4. (mit Jehanne Dousse) “Asymptotic Formulas for Partition Ranks”,
   in Vorbereitung

5. “Kronecker-Hurwitz type Identities for Mixed Mock Modular Forms”,
   in Vorbereitung

Vorträge

11. März 2013	"Automorphism Groups of Hyperbolic Lattices", 27th Automorphic Forms Workshop, University College, Dublin
11. Juni 2013	"Mock Modular Forms and Class Number Relations"*, 39. AKLS Seminar über automorphe Formen, Universität zu Köln
28. August 2013	"Mock Modular Forms and Class Number Relations"*, Number Theory Lunch Seminar, Max-Planck-Institut für Mathematik, Bonn
6. März 2014	"Class Number Type Relations for Fourier Coefficients of Mock Modular Forms"*, Konferenz "Automorphic Forms, Lie Algebra and String Theory, Université Lille 1

Ein $*$ neben dem Vortragstitel markiert einen Vortrag auf Einladung.