On the Plateau problem in metric spaces

Inaugural-Dissertation

zur

Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Fakultät

der Universität zu Köln

vorgelegt von

Paul Creutz

aus Köln

Berichterstatter:	Prof. Dr. **Alexander Lytchak**
(Gutachter)	Prof. Dr. **George Marinescu**
	Prof. Dr. **Urs Lang**

Tag der mündlichen Prüfung: **28.01.2022**

ABSTRACT

The Plateau problem asks whether every Jordan curve in $\mathbb{R}^3$ can bound a minimal surface. Its solution by Douglas and Radó dates back to the 1930s. In recent work Lytchak–Wenger have generalized the solution of Plateau's problem to singular metric ambient spaces. This thesis studies the structure of the arising metric space valued minimal surfaces. We investigate the analytical and topological regularity of these minimal surfaces, as well as their intrinsic geometry. We also provide applications of the metric theory that are new even for Euclidean space. E.g. we solve the Plateau problem (and the more general Plateau–Douglas problem) for singular boundary values where self-intersections are allowed.

Zusammenfassung

Das Plateau Problem fragt, ob jede Jordan Kurve im $\mathbb{R}^n$ eine Minimalfläche berandet. Seine Lösung durch Douglas und Radó datiert zurück in die 1930er Jahre. In einer kürzlich veröffentlichen Arbeit verallgemeinern Lytchak–Wenger die Lösung des Plateau Problems vom $\mathbb{R}^n$ auf allgemeine metrische Räume. In dieser Dissertation untersuchen wir die Struktur der sich ergebenden metrischen Minimalflächen. Wir betrachten hierbei die analytische und topologische Regularität der Minimalflächen, sowie ihre innere Geometrie. Ebenfalls geben wir Anwendungen der metrischen Theorie, die sogar für den euklidischen Raum neu sind. Beispielsweise lösen wir das Plateau Problem, sowie das allgemeinere Plateau–Douglas Problem, auch für singuläre Randwerte bei denen Selbstschnitte erlaubt sind.

ACKNOWLEDGEMENTS

First of all I want to thank Alexander for having been quite in every sense the best PhD advisor that I could have imagined. Also I am very grateful to my co-authors Teri, Martin, Matthew and Nikita for the inspiring discussions and their patience. Furthermore I want to thank Giuliano Basso, Franois Fillastre, Piotr Hajłasz, Jess Jaramillo, Mikhail Katz, Urs Lang, Christian Lange, Artem Nepechiy, Hubert Sidler, Stephan Stadler, Sophia Wagner and Stefan Wenger for helpful discussions, comments or pointing out typos. Finally, and maybe most importantly, I want to thank my family, Laura and all my friends for their support!

Funding

This thesis was partially supported by the DFG-grants SPP 2026 "Geometry at Infinity" and SFB/TRR 191 "Symplectic structures in Geometry, Algebra and Dynamics". Also the work on [I] benefited from a joint stay of Martin and mine at the IMPAN in Warsaw that was funded by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund. These supports are gratefully acknowledged.

ORGANIZATION

This cumulative thesis is comprised of the following articles and preprints.

[A] Paul Creutz and Nikita Evseev. An approach to metric space valued Sobolev maps via weak* derivatives. *Preprint*, arXiv:2106.15449, submitted, 2021.

[B] Paul Creutz. Majorization by hemispheres and quadratic isoperimetric constants. *Trans. Amer. Math. Soc.*, 373(3):1577–1596, 2020.

[C] Paul Creutz. Rigidity of the Pu inequality and quadratic isoperimetric constants of normed spaces. *Rev. Mat. Iberoam.*, online first, 2021.

[D] Paul Creutz. Plateau's problem for singular curves. *Comm. Anal. Geom.*, to appear.

[E] Paul Creutz and Matthew Romney. The branch set of minimal disks in metric spaces. *Int. Math. Res. Not. IMRN*, online first, 2022.

[F] Paul Creutz. Space of minimal discs and its compactification. *Geom. Dedicata*, 210(1):151–164, 2021.

[G] Paul Creutz and Matthew Romney. Triangulating metric surfaces. *Proc. Lond. Math. Soc. (3)*, to appear.

[H] Paul Creutz and Elefterios Soultanis. Maximal metric surfaces and the Sobolev-to-Lipschitz property. *Calc. Var. Partial Differential Equations*, 59(5):Paper No. 177, 34 pp., 2020.

[I] Paul Creutz and Martin Fitzi. The Plateau–Douglas problem for singular configurations and in general metric spaces. *Preprint*, arXiv:2008.08922, submitted, 2020.

In a wider sense all these works are related to the Plateau problem in singular metric ambient spaces. After summarizing each of the articles separately in Chapter 1 we outline these connections in detail in the introductory Chapter 2.

In all of the joint projects [A], [E], [G], [H] and [I] I have intensively collaborated with my coauthors and we have respectively both contributed an equal share of input concerning results as well as concerning presentation.

CONTENTS

CHAPTER 1

Main results

In this chapter we quickly go through the main results of the respective articles and preprints. For the a more streamlined overview the reader is referred to Chapter 2 and for more detailed summaries to the respective introductions of Chapters A–I. Note that throughout this chapter *solutions of Plateau's problem* will formally refer to infinitesimally isotropic area minimizers as defined in Chapter 2. The reader should think of these objects as analogs of minimal surfaces that make sense in metric ambient spaces.

1.A An approach to metric space valued Sobolev maps via weak* derivatives

In this chapter we propose the following definition of the first-order Sobolev space $W^{1,p}(\Omega, X)$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain, X is a complete separable metric space and $p \in [1, \infty]$.

Definition 1.1. Denote by κ the Kuratowski embedding of X into the Banach space ℓ^∞ of bounded sequences.

(i) The space $L^p(\Omega, \ell^\infty)$ consists of those maps $f\colon \Omega \to \ell^\infty$ that are Bochner measurable and for which the function $x \mapsto ||f(x)||_\infty$ lies in $L^p(\Omega)$.

(ii) The space $L^p_*(\Omega, \ell^\infty)$ consists of those maps $u\colon \Omega \to \ell^\infty$ that are weak* measurable and for which the function $x \mapsto ||u(x)||_\infty$ lies in $L^p(\Omega)$.

(iii) A map u lies in $W^{1,p}(\Omega, \ell^\infty)$ if $u \in L^p(\Omega, \ell^\infty)$ and for every $j = 1, \ldots, n$ there is a map $u_j \in L^p_*(\Omega, \ell^\infty)$ such that

$$\int_\Omega \frac{\partial \varphi}{\partial x_j}(x) \cdot u(x)\ \mathrm{d}x = -\int_\Omega \varphi(x) \cdot u_j(x)\ \mathrm{d}x \qquad \text{for every } \varphi \in C_0^\infty(\Omega)$$

in the sense of Gelfand integrals.

(iv) A map u lies in $W^{1,p}(\Omega, X)$ if $\kappa \circ u$ lies in $W^{1,p}(\Omega, \ell^\infty)$.

The main result of this chapter is the equivalence of Definition 1.1 to other definitions of metric space valued Sobolev maps proposed in the literature before. It is tempting to instead require the weak derivatives u_j in Definition 1.1 to lie in $L^p(\Omega, \ell^\infty)$. However we prove that in this case all maps in $W^{1,p}(\Omega, X)$ would be constant. This more restrictive definition has previously been used in several articles. Hence some results in the literature turn out to be not correct as stated.

1.B Majorization by hemispheres and quadratic isoperimetric constants

The main result of this chapter is the following analog of Reshetnyak's majorization theorem for CAT spaces.

Theorem 1.2. *Let V be a Banach space and $\eta\colon S^1 \to V$ be 1-Lipschitz. Then there is a 1-Lipschitz extension $m\colon S^2 \to V$ of η.*

The proof relies on explicitly constructing an isometric embedding of the hemisphere into the 1-Wasserstein space over S^1, and hence optimal transport techniques. It applies more generally for metric spaces which admit a conical geodesic bicombing as introduced by Descombes–Lang. Theorem 1.2 implies that Banach spaces support a quadratic isoperimetric inequality with the spherical isoperimetric constant.

Theorem 1.3. *Let V be a Banach space. Then V supports a quadratic isoperimetric inequality with constant $\frac{1}{2\pi}$.*

By work of Ivanov the optimal isoperimetric constant of ℓ^∞ is at least $\frac{1}{2\pi}$, and hence Theorem 1.3 is sharp. Combining Theorem 1.3 with a regularity result of Lytchak–Wenger we obtain the following corollary.

Corollary 1.4. *Let X be a compact Finsler manifold and $u\colon D^2 \to X$ be a solution of Plateau's problem. Then u is locally α-Hölder continuous on D^2 for any $\alpha < \frac{\pi}{8}$.*

1.C Rigidity of the Pu inequality and quadratic isoperimetric constants of normed spaces

In this chapter we refine and apply the results from Chapter B. The starting point is the following nontrivial observation: *if the curve η in Theorem 1.2 is not an isometric embedding then the majorization map m is area decreasing.* As consequences we derive rigidity of Pu's classical systolic inequality and the following theorem.

Theorem 1.5. *For a Banach space V we denote by $C(V)$ its optimal quadratic isoperimetric constant.*

(i) For $n \in \mathbb{N}$ with $n \geq 2$ there is a constant $C_n < \frac{1}{2\pi}$ such that

$$\{C(V) \ : \ V \text{ normed space of dimension } n\} = \left[\tfrac{1}{4\pi}, C_n\right].$$

(ii) Furthermore $C_n \to \frac{1}{2\pi}$ as $n \to \infty$ and hence

$$\{C(V) \ : \ V \text{ Banach space}\} = \{0\} \cup \left[\tfrac{1}{4\pi} \tfrac{1}{2\pi}\right].$$

Theorem 1.5 allows to improve the constant α in Corollary 1.4 beyond the threshold case of $\frac{\pi}{8}$. Further auxiliary results of independent interest are that isoperimetric curves in Banach spaces must be bi-Lipschitz embeddings and the following lemma.

Lemma 1.6. *Let V be a finite dimensional normed space. Then one cannot isometrically embed S^1 into V.*

1.D Plateau's problem for singular curves

The Plateau problem asks whether every Jordan curve Γ in $\mathbb{R}^n$ can bound a minimal surface. Its solution gained Douglas the first Fields medail back in 1936. In turn not every self-intersecting curves Γ can bound a minimal surface. Nevertheless we are able to prove the following existence result.

Theorem 1.7. *Let Γ be a closed rectifiable curve in $\mathbb{R}^n$. Then there exists a Hölder continuous disk $u\colon \bar{D}^2 \to \mathbb{R}^n$ spanning Γ that is of least area among all disks in $W^{1,2}(D^2, X)$ spanning Γ.*

The proof of Theorem 1.7 relies on the metric category's flexibility for constructions and Lytchak–Wenger's solution of the Plateau problem for general metric spaces. In particular it seems impossible to prove the result using only classical smooth techniques. More generally a variant of the theorem applies when X is proper and supports a quadratic isoperimetric inequality. Theorem 1.7 improves a previous result of Hass which shows the existence of continuous area minimizers. Indeed even for Jordan curves of low analytic regularity the existence of globally Hölder continuous area minimizers is new. Theorem 1.7 has the following technical but useful corollary.

Corollary 1.8. *Let X be a proper metric space. If X supports a quadratic isoperimetric inequality with constant C' for every $C' > C$ then X supports a quadratic isoperimetric inequality with constant C.*

E.g. Corollary 1.8 can be used to prove a geometric characterization of quadratic isoperimetric inequalities when X is a surface and that quadratic isoperimetric inequalities are stable under ultralimits.

1.E The branch set of minimal disks in metric spaces

Recent results of Lytchak–Wenger and Stadler respectively state that any space with quadratic isoperimetric constant $\frac{1}{4\pi}$ is CAT(0) and that solutions to the

Plateau problem in CAT(0) spaces are local embeddings outside a finite set of branch points. The following main result of this chapter shows that nothing similar is true for larger isoperimetric constants.

Theorem 1.9. *Let $C > \frac{1}{4\pi}$ and $E \subset \mathbb{R}^2$ be a non-degenerate cell-like set. Then there are a compact metric space X which supports a quadratic isoperimetric inequality with constant C, a solution to Plateau's problem $u\colon D^2 \to X$ and a point $x \in X$ such that $u^{-1}(x)$ is homeomorphic to E.*

Theorem 1.9 answers two questions formulated by Lytchak–Wenger in [108]. Also we prove several related results concerning energy-minimizing parametrizations of disks. An example of these is the following theorem.

Theorem 1.10. *Let Z be a metric space which is homeomorphic to $\bar{D}^2$, supports a quadratic isoperimetric inequality and has a bi-Lipschitz boundary curve ∂Z. If u is an energy-minimizing parametrization of Z and ...*

(i) ... Z is doubling then u is a quasisymmetric homeomorphism.

(ii) ... Z is quasiconformally equivalent to $\bar{D}^2$ then u is a quasiconformal homeomorphism.

1.F Space of minimal disks and its compactification

In this chapter we investigate the collection of all minimal disks as a subset of Gromov–Hausdorff space. The main result is the following theorem.

Theorem 1.11. *Let $C, L \in (0, \infty)$. Denote by $\mathcal{D}(L, C)$ the collection of geodesic metric disks Z satisfying $\ell(\partial Z) \leq L$ and a quadratic isoperimetric inequality with constant C. And by $\mathcal{E}(L, C)$ the collection of geodesic metric disk retracts satisfying the same two properties. Then $\mathcal{E}(L, C)$ is compact in the Gromov–Hausdorff topology and*

$$\overline{\mathcal{D}(L, C)} \subset \mathcal{E}(L, C) \subset \overline{\mathcal{D}(L, C + \tfrac{1}{2\pi})}.$$

1.G Triangulating metric surfaces

The main result of this chapter is the following triangulation theorem.

Theorem 1.12. *Let X be a geodesic metric space homeomorphic to a closed surface and $\varepsilon > 0$. Then X may be decomposed into finitely many non-overlapping convex triangles, each of diameter at most ε.*

Theorem 1.12 generalizes a classical result by Alexandrov–Zalgaller for surfaces of synthetically bounded integral curvature. We also simplify the proof for the bounded curvature case and correct several technical mistakes in its proof. As the classical result Theorem 1.12 allows for applications concerning the smooth approximation and uniformization of metric surfaces [123].

1.H Maximal metric surfaces and the Sobolev-to-Lipschitz property

By a recent result of Lytchak–Wenger every Ahlfors 2-regular, linearly locally connected metric sphere Z admits a unique energy-minimizing parametrization. This parametrization gives rise to a measurable almost everywhere defined pull-back Finsler structure F_Z on S^2. We call two such spheres *analytically equivalent* if they give rise to the same Finsler structure. The main result of this chapter is the following theorem.

Theorem 1.13. *Let Z be a linearly locally connected, Ahlfors 2-regular sphere. Then there is a unique sphere $\widehat{Z}$ in the analytic equivalence class of Z that is equivalently characterized by any of the following properties.*

(1) 2-Sobolev-to-Lipschitz property. *If $f \in N^{1,2}(\widehat{Z})$ has 2-weak upper gradient 1, then f has a 1-Lipschitz representative.*

(2) 2-thick geodecity. *For arbitrary measurable subsets $E, F \subset \widehat{Z}$ of positive measure and $C > 1$, one has $\mathrm{Mod}_2\,\Gamma(E, F; C) > 0$.*

(3) Maximality. *If Y is analytically equivalent to $\widehat{Z}$ then there exists a 1-Lipschitz homeomorphism $f\colon \widehat{Z} \to Y$.*

(4) Volume rigidity. *If Y is a linearly locally connected Ahlfors 2-regular sphere and $f\colon Y \to \widehat{Z}$ is a 1-Lipschitz area preserving map which is moreover cell-like, then f is an isometry.*

Variants of the seemingly unrelated properties that appear in Theorem 1.13 have before been studied by different research communities. $\widehat{Z}$ is constructed from Z using a more general construction scheme. Applying this scheme in less restrictive contexts we obtain the following result.

Theorem 1.14. *Let (X, d, μ) be a doubling metric measure space which is p-thick quasiconvex where $p \in [1, \infty]$. Then there exists a minimal metric $d_p \geq d$ such that (X, d_p, μ) is p-thick geodesic.*

Also we prove that the equivalence of Sobolev-to-Lipschitz property and thick geodecity holds almost unconditionally.

Theorem 1.15. *Let X be a doubling metric measure space and $p \in [1, \infty]$. Then X has the p-Sobolev-to-Lipschitz property if and only if X is p-thick geodesic.*

Furthermore we apply a refined version of our construction scheme to solutions of the Plateau problem and obtain a variant of the intrinsic disk studied by Lytchak–Wenger. Also here the resulting disks are characterized by a maximality condition and have the Sobolev-to-Lipschitz property.

1.I The Plateau–Douglas problem for singular configurations and in general metric spaces

The Plateau–Douglas problem is a variant of the Plateau problem. One prescribes a finite configuration of disjoint Jordan curves Γ and searches for a minimal surface of fixed topological type that bounds the given configuration Γ. It was first solved by Douglas in Euclidean space and then more generally by Jost in homogeneously regular Riemannian manifolds. In this chapter we generalize the existence result to arbitrary Riemannian ambient manifolds.

Theorem 1.16. *Let M be a compact connected orientable surface with k boundary components, $X = (\mathcal{N}, g)$ be a complete Riemannian manifold and Γ be a configuration of k rectifiable disjoint Jordan curves. If the Douglas condition holds then there is a solution to the Plateau–Douglas problem for (M, Γ, X).*

Again the proof relies on a metric construction and a quite general result of Fitzi–Wenger who solve the Plateau–Douglas problem in proper metric spaces which support a local quadratic isoperimetric inequality. Indeed we prove Theorem 1.16 not only for Riemannian manifolds but for all proper metric spaces X. Hence we also obtain a full generalization of the existence result by Fitzi–Wenger.

We also extend Theorem 1.7 from Chapter D to the Plateau–Douglas problem. That is we prove the existence of area minimizers of given topological type for singular configurations Γ of possibly non-disjoint or self-intersecting curves. We remark that Hass' method to prove the existence of continuous area minimizers seems limited to the disktype case. Hence our existence result for singular boundary values and non-disktype surfaces is completely new even in Euclidean space.

CHAPTER 2

Introduction

In this chapter we give an overview on the Plateau problem in $\mathbb{R}^3$, and discuss generalizations of the classical results concerning existence, regularity and geometric structure of solutions for singular ambient spaces. We will also see some surprising applications of the generalized singular theory to classical smooth questions.

2.1 Existence

The classical Plateau problem asks whether every simple closed curve Γ in $\mathbb{R}^3$ can bound a minimal surface. It is named after the 19th century physicist Joseph Plateau. In his experiments Plateau had made the following observation: if one takes a thin circle-shaped metal wire and arbitrarily bends it in space then it is always possible to produce a stable soap film spanned by the deformed wire [129]. Indeed the mathematical existence question is even older. It was first formulated by Lagrange [97] more than 250 years ago. However it turned out to be an extremely challenging problem that was studied intensively by leading mathematicians of the 19th and early 20th century. For example Riemann and Schwarz constructed solutions for certain quadrilaterals [140, 142], Weierstrass handled all polygonal curves [157] and Haar proved existence when Γ projects to a planar convex Jordan curve [70]. A satisfying solution for general rectifiable curves Γ was however only given by Douglas [45] and Radó [130] independently around 1930.

By definition a minimal surface is a local minimum of the area functional. Hence the existence of a minimal surface can be established by showing that the area functional achieves a global minimum among surfaces that span the given contour Γ. In the mentioned early works the topological type of the bounding surface is usually assumed to be that of the disk. This allows to think of the competing surfaces as maps with values in $\mathbb{R}^3$ that are defined on the unit disk in $\mathbb{R}^2$. Classically often the more restrictive question whether a global minimum of the area functional exists among disk-type surfaces is called the Plateau problem. From now on, throughout this thesis, we will also follow this convention. The straightforward approach to this strengthened Plateau problem is the following: First one specifies a sufficiently compact class of maps $D^2 \to \mathbb{R}^3$ for which it is possible to prove by a direct argument that the infimum of a suitable area functional is achieved. Then only a posteriori one establishes further regularity of the obtained minimizers. We will now discuss possible choices of the class of admissible maps, existence proofs for these classes and their generalizability to more general ambient spaces. The regularity question will mostly be postponed to the next section.

Naively one might consider the class of smooth embedded or immersed disks that bound the given contour. These classes however radically lack any suitable compactness condition. In the case of Douglas' solution one can instead take the class of those continuous maps $u \colon \bar{D}^2 \to \mathbb{R}^3$ that are harmonic on the interior and restrict to a monotone parametrization of Γ on S^1. The *area* of such admissible map is defined in terms of the usual formula

$$\text{Area}(u) := \int_{D^2} \sqrt{\left|\frac{\partial u}{\partial x}\right|^2 \left|\frac{\partial u}{\partial y}\right|^2 - \left(\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y}\right)^2} \, \mathrm{d}(x,y). \tag{2.1}$$

Geometrically this is justified by the area formula which states that

$$\text{Area}(u) = \int_{\mathbb{R}^3} \text{card}(u^{-1}(x)) \, \mathrm{d}\mathcal{H}^2_{\mathbb{R}^3}(x) \tag{2.2}$$

where $\mathcal{H}^2_{\mathbb{R}^3}$ denotes the Hausdorff 2-measure on $\mathbb{R}^3$. By uniqueness of harmonic extensions the admissible maps of Douglas are determined by their boundary values. This allowed Douglas to reduce the Plateau problem to the simpler minimization of an integral depending only on a single variable function. An obvious advantage of this approach is that his minimizers come with a lot of ad-hoc regularity. A disadvantage is that it is not very flexible. In particular it does not seem amenable for a generalization to other ambient spaces.

While Douglas minimizes over a relatively small class of objects, Radó takes the other extreme and minimizes over all continuous maps $u \colon \bar{D}^2 \to \mathbb{R}^3$ that bound Γ. For such general maps neither (2.1) nor (2.2) can be generalized to define an area functional with suitable properties. Instead Radó works with the so-called *Lebesgue notion of surface area*, see [134, 25]. This functional is somewhat harder to handle but it can be generalized for metric ambient spaces. And indeed in 1979 Nikolaev extended the existence result of Radó to metric spaces satisfying upper curvature bounds à la Alexandrov [118]. For more general ambient spaces X however this approach seems hardly tractable. Furthermore anyway the generalized Lebesgue surface area does not appear particularly natural when X has non-Euclidean tangent spaces.

A third intermediate choice of admissible maps essentially goes back to McShane [111]. Here the *admissible class*, denoted $\Lambda(\Gamma, \mathbb{R}^3)$, contains those Sobolev maps $u \in W^{1,2}(D^2, \mathbb{R}^3)$ for which the trace $\text{tr}(u) \in L^2(S^1, \mathbb{R}^3)$ is a monotone parametrization of Γ. Since elements of $W^{1,2}(D^2, \mathbb{R}^3)$ have square integrable weak partial derivatives, their area can be defined in terms of (2.1). We will discuss here only a more refined version of McShane's solution of the Plateau problem which relies on work of Courant [31]. Courant's idea was to solve the

Plateau problem by minimizing the so-called Dirichlet energy instead of the area functional directly. The *Dirichlet energy* of a Sobolev map $u \in W^{1,2}(D^2, \mathbb{R}^3)$ is defined as

$$E(u) := \frac{1}{2} \cdot \int_{D^2} \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 \, \mathrm{d}(x, y).$$

Hence by the AM-GM inequality

$$\operatorname{Area}(u) \leq E(u) \tag{2.3}$$

with equality if and only if f is *weakly conformal*. That is if

$$\left|\frac{\partial u}{\partial x}\right| = \left|\frac{\partial u}{\partial y}\right| \quad \text{and} \quad \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = 0$$

almost everywhere on D^2. Much less trivial, but true, is that

$$\inf_{u \in \Lambda(\Gamma, \mathbb{R}^3)} \operatorname{Area}(u) = \inf_{u \in \Lambda(\Gamma, \mathbb{R}^3)} E(u). \tag{2.4}$$

In particular the Dirichlet energy calibrates the area functional in the sense that energy minimizers are area minimizers and weakly conformal. Using the Courant–Lebesgue lemma, Courant was able to derive equicontinuity of certain energy minimizing sequences and hence conclude the existence of energy minimizers. Thus his approach allows to conclude the following theorem.

Theorem 2.1.1 ([31])**.** *Let $\Gamma \subset \mathbb{R}^3$ be a rectifiable Jordan curve. Then there is an admissible map $u \in \Lambda(\Gamma, \mathbb{R}^3)$ which is weakly conformal and such that*

$$\operatorname{Area}(u) = \inf_{v \in \Lambda(\Gamma, \mathbb{R}^3)} \operatorname{Area}(v).$$

The weak conformality guaranteed by Theorem 2.1.1 already provides some nice ad-hoc regularity. In particular it excludes degenerate area minimizers such as the so-called hairy disk, see e.g. [40, p. 248]. Beyond this ad-hoc regularity, another nice feature of Courant's approach is its generalizability for metric ambient spaces X. To obtain such generalization is however not completely straightforward. A first obstacle is to make sense of X-valued Sobolev maps as well as of the corresponding area and energy functionals.

When $X = (\mathcal{N}, g)$ is a Riemannian manifold this is relatively straightforward. Namely, by Nash's theorem, there is a Riemannian isometric embedding $\iota\colon X \to \mathbb{R}^N$. Hence one can define $W^{1,2}(D^2, X)$ as the collection of those maps $u\colon D^2 \to X$ for which $\iota \circ u$ lies in the classical Sobolev space $W^{1,2}(D^2, \mathbb{R}^N)$. For maps $u \in W^{1,2}(D^2, X)$ *area* and *energy* are then respectively given by the area and energy of $\iota \circ u$. The *admissible class* $\Lambda(\Gamma, X)$ consists of those f for which $\iota \circ u$ lies in the admissible class $\Lambda(\iota \circ \Gamma, \mathbb{R}^N)$. Certainly inequality (2.3) holds when $\mathbb{R}^3$ is replaced X and Morrey proved that also the nontrivial equality (2.4) generalizes [117]. This allowed Morrey to apply Courant's method and solve the Plateau problem for Riemannian ambient manifolds. More precisely he deduces the existence of weakly conformal area minimizers in $\Lambda(\Gamma, X)$ whenever X is a so-called homogeneously regular Riemannian manifold and the necessary condition $\Lambda(\Gamma, X) \neq \emptyset$ is satisfied. Here *homogeneous regularity* means that X is locally uniformly bi-Lipschitz equivalent to the respective Euclidean unit ball. In particular the large class of homogeneously regular Riemannian manifolds includes all compact manifolds as well as more generally all manifolds of bounded geometry. Before this quite general result of Morrey existence had only been generalized to the hyperbolic space $\mathbb{H}^3$ [103].

Morrey's paper on the Plateau problem in Riemannian manifolds dates back to the 1940's. For nonsmooth spaces X it was however only over the last 30 years that a satisfactory theory of X-valued Sobolev maps (and hence a suitable framework to attack the Plateau problem via Courant's method) have been developed. Nevertheless nowadays numerous equivalent definitions of the metric Sobolev space $W^{1,2}(D^2, X)$ are well established, see [79, 9, 139]. Here we will however promote another characterization that we provide in Chapter A. The key observations are that every separable space X admits an isometric embedding $\kappa\colon X \to \ell^\infty$ (called the *Kuratowski embedding*) and that ℓ^∞ is the dual of the separable Banach space ℓ^1. This allows to define $W^{1,2}(D^2, \ell^\infty)$ as the collection of square integrable maps that have square integrable weak partial derivatives in a weak* sense (see Definition 1.1), and then set

$$W^{1,2}(D^2, X) := \big\{u\colon D^2 \to X \mid \kappa \circ u \in W^{1,2}(D^2, \ell^\infty)\big\}.$$

Our definition has the advantage that Sobolev maps $u \in W^{1,2}(D^2, X)$ come with actual linear differentials

$$\mathrm{d}u_z\colon \mathbb{R}^2 \to \ell^\infty$$

defined at almost every $z \in D^2$. From this differentials one can almost everywhere recover the *approximate metric differential* seminorms

$$\mathrm{apmd}u_z\colon \mathbb{R}^2 \to [0, \infty) \quad \text{as} \quad \mathrm{apmd}u_z(V) := ||\mathrm{d}u_z(V)||_\infty.$$

When X is complete there is also a natural trace function $\mathrm{tr}(u) \in L^2(S^1, X)$ for given $u \in W^{1,2}(D^2, X)$. Thus for a Jordan curve $\Gamma \subset X$ the admissible class $\Lambda(\Gamma, X)$ can be defined virtually as before upon replacing $\mathbb{R}^3$ by X.

The rich variety of natural equivalent characterizations (compare [95, 71, 138, 26, 144, 74, 10] and Chapter A below) strongly indicates that $W^{1,2}(D^2, X)$ and hence also $\Lambda(\Gamma, X)$ are the right classes to consider. Unfortunately however, there does not seem to be 'the' way of defining area and energy of Sobolev maps $u \in W^{1,2}(D^2, X)$. Instead there are many non-equivalent definitions that all come with their own respective advantages and disadvantages. One example is the *Busemann area functional* Area^b. It corresponds to the Hausdorff 2-measure $\mathcal{H}^2_X$ in the sense that

$$\mathrm{Area}^b(u) = \int_X \mathrm{card}\left(u^{-1}(y)\right)\ \mathrm{d}\mathcal{H}^2_X(y).$$

for disks $u\colon D^2 \to X$ of reasonable regularity. However depending on the context sometimes other definitions seem more natural, compare e.g. [7]. Concerning energy functionals one possibility is to consider the *Reshetnyak energy* E_+ given by

$$E_+(u) := \int_{D^2} \sup_{V \in S^1} \left(\mathrm{apmd}u_{(x,y)}(V)\right)^2 \, \mathrm{d}(x,y).$$

The Reshetnyak energy satisfies

$$\mathrm{Area}^b(u) \leq E_+(u) \tag{2.5}$$

with equality if and only if u is weakly conformal. Here *weak conformality* means that at almost every $z \in D^2$ the approximate metric differential seminorm $\mathrm{apmd}u_z$ is a scalar multiple of the standard Euclidean norm on $\mathbb{R}^2$. However again there are many other reasonable energy functionals, compare e.g. [105]. Optimally to solve the Plateau problem for a given area functional one would choose an energy functional that is lower semicontinuous and calibrates in the sense of (2.3) and (2.4). However e.g. for the Busemann area functional it is not known whether an energy functional enjoying both these properties exists.

Non-uniqueness of area functionals and non-existence of calibrating lower semicontinuous energy functionals do not occur when X has *property (ET)*. This property means that for every $u \in W^{1,2}(D^2, X)$ the metric differential seminorm $\mathrm{apmd}u_z$ is Euclidean at almost every $z \in D^2$. It holds e.g. when X is a Riemannian manifold, an equiregular sub-Riemannian manifold or of curvature bounded from above/below in the Alexandrov sense, see [104, Section 11]. On the other hand it fails for non-Euclidean normed spaces. When X has property (ET) then all reasonable area functionals agree with Area^b. Hence by Theorem 1.1 in [105] one has

$$\inf_{u \in \Lambda(\Gamma,X)} \mathrm{Area}^b(u) = \inf_{u \in \Lambda(\Gamma,X)} E_+(u). \tag{2.6}$$

In particular, when X has property (ET) it might not be surprising that the Plateau problem can be solved via the energy minimization method. This had been succesfully worked out by Jost [90] and Mese–Zulkowski [116] respectively when X is compact and satisfies a weak local convexity condition, or when X is an Alexandrov space. In full generality it is a recent result of Lytchak–Wenger.

Theorem 2.1.2 ([104])**.** *Let X be a proper metric space which has property (ET), and $\Gamma \subset X$ be a Jordan curve. If $\Lambda(\Gamma, X)$ is non-empty then there is an admissible map $u \in \Lambda(\Gamma, X)$ which is weakly conformal and such that*

$$\mathrm{Area}^b(u) = \inf_{v \in \Lambda(\Gamma,X)} \mathrm{Area}^b(v).$$

Here a metric space X is called *proper* if bounded closed subsets of X are compact. This assumption implies that the Arzelà–Ascoli theorem holds for sequences of X-valued maps. Indeed it is possible to prove Theorem 2.1.2 (and Theorem 2.1.4 below) more generally for spaces satisfying much weaker local compactness conditions [69]. By [69, Example 5.1] however the theorems fail when the properness condition is completely dropped.

When X does not have property (ET) the energy minimization method does not apply directly. Hence more surprising than Theorem 2.1.2 might be that Lytchak–Wenger also provide solutions for this case. As mentioned above the choice of area functional is quite ambiguous when not assuming property (ET). For simplicity here in Chapter 2 we will restrict to discussing the Busemann area functional. Note however that many of the discussed results or at least variants of them apply in much wider generality. As a replacement for weak conformality Lytchak–Wenger introduce what they call infinitesimally isotropic maps. A map $u \in W^{1,2}(D^2, X)$ is called *infinitesimally isotropic* if at almost every point $z \in D^2$, at which $\text{apmd}u_z \colon \mathbb{R}^2 \to \mathbb{R}$ does not vanish, $\text{apmd}u_z$ is a norm and the ellipse of maximal area inscribed in its unit ball is a scalar multiple of the standard disk D^2. When X has property (ET) weak conformality is derived from the existence of Reshetnyak energy minimizers, equality (2.6) and the rigidity case of (2.5). This is not possible in the presence of non-Euclidean tangent spaces. Instead the following theorem comes into play.

Theorem 2.1.3 ([105])**.** *Let X be a complete metric space and $u \in W^{1,2}(D^2, X)$. Then u is infinitesimally isotropic if and only if*

$$E_+^2(u) = \inf \left\{ E_+^2(u \circ \varphi) \mid \varphi \colon D^2 \to D^2 \text{ biLipschitz homeomorphism} \right\}.$$

To solve the Plateau problem for the general case Lytchak–Wenger proceed as follows: First they deduce from Courant's argument that for every $u \in \Lambda(\Gamma, X)$ the infimum of the Reshetnyak energy over

$$\left\{ v \in \Lambda(\Gamma, X) \, : \text{Area}^b(v) \le \text{Area}^b(u) \right\}$$

is achieved. Thus when choosing an area minimizing sequence in $\Lambda(\Gamma, X)$ they can assume that all of its elements are infinitesimally isotropic. This implies a uniform energy bound on the sequence. Now they apply Courant's method to this sequence and derive that there is a Busemann area minimizer within $\Lambda(\Gamma, X)$. By the initial observation and Theorem 2.1.3 they can even assume that the area minimizer is infinitesimally isotropic. Hence they conclude the following theorem.

Theorem 2.1.4 ([104])**.** *Let X be a proper metric space and $\Gamma \subset X$ be a Jordan curve. If $\Lambda(\Gamma, X)$ is non-empty then there is an admissible map $u \in \Lambda(\Gamma, X)$ which is infinitesimally isotropic and such that*

$$\text{Area}^b(u) = \inf_{v \in \Lambda(\Gamma, X)} \text{Area}^b(v).$$

Before [104] the only results concerning the classical Plateau problem that do not fall into the property (ET) setting where due to Overath–von der Mosel and Pistre–von der Mosel [125, 128]. In these works the Plateau problem is solved for certain Finsler manifolds diffeomorphic and bi-Lipschitz equivalent to $\mathbb{R}^n$ [125, 128]. Their idea is to interpret the area functional on X as a so-called Cartan functional on $\mathbb{R}^n$. Then the existence of area minimizers is derived from the theory of Cartan functionals developed by Hildebrandt–von der Mosel in [84, 85]. This approach however seems to heavily rely on coordinate representations and smooth analytic methods. In particular it appears not well suitable for a generalization to nonsmooth ambient spaces.

2.2 Analytic regularity

In this section we discuss the analytic regularity theory of the solutions guaranteed by Theorems 2.1.1, 2.1.2 and 2.1.4. To this end we will distinguish between interior and boundary regularity.

2.2.1 Interior regularity. In the setting of Theorem 2.1.1 there is a simple answer to the interior regularity question: Weakly conformal area minimizers are energy minimizers and hence weakly harmonic. Thus the Weyl Lemma [159] implies that the minimizers are smooth and harmonic. Already when the ambient space $X = (\mathcal{N}, g)$ is a Riemannian manifold the situation is more delicate. In this case the weakly conformal area minimizers guaranteed by Theorem 2.1.2 are still weakly harmonic. If u is continuous, this implies that u fulfills an explicit variational equation in local coordinates. Thus, as a consequence of classical PDE results, u is smooth and harmonic, see e.g. [91]. In general however the minimizers in Theorem 2.1.2 do not need to be continuous even when X is a complete Riemannian manifold. For example in [117] Morrey describes a simple configuration (Γ, X) for which the admissible class $\Lambda(\Gamma, X)$ is non-empty but all of its elements are discontinuous. Nevertheless when $X = (\mathcal{N}, g)$ is homogeneously regular he shows that minimizers as in Theorem 2.1.2 are locally α-Hölder continuous and hence smooth. Indeed by more recent works not only weakly conformal area minimizers but all weakly harmonic surfaces in ambient manifolds X of bounded geometry are continuous and hence smooth [81, 91]. More interesting to us however is that Lytchak–Wenger generalize the regularity result of Morrey to nonsmooth ambient spaces.

Theorem 2.2.1 ([104])**.** *Let X be a complete metric space which supports a local quadratic isoperimetric inequality and $\Gamma \subset X$ be a Jordan curve. If $u \in \Lambda(\Gamma, X)$ is an infinitesimally isotropic area minimizer then u is locally α-Hölder continuous on D^2. The Hölder exponent $\alpha \in (0, 1]$ depends only on the isoperimetric constant of X.*

Here we say that a metric space X supports a *local quadratic isoperimetric inequality* if there is a constant $C \geq 0$ such that every sufficiently short Lipschitz curve $\gamma\colon S^1 \to X$ is the trace of a disk $u \in W^{1,2}(D^2, X)$ for which

$$\mathrm{Area}^b(u) \leq C \cdot \mathrm{length}(\gamma)^2.$$

The isoperimetric constant of the Euclidean plane $\mathbb{R}^2$ is $\frac{1}{4\pi}$. Hence by Reshetnyak's majorization theorem [137] every CAT(0) space X supports a quadratic isoperimetric inequality with constant $C = \frac{1}{4\pi}$. This implies that homogeneously regular Riemannian manifolds support a local quadratic isoperimetric inequality with constant C for any $C > \frac{1}{4\pi}$. On the other hand by Theorem 2.2.1 and Morrey's counterexample not every complete Riemannian manifold can support a local quadratic isoperimetric inequality.

Quadratic isoperimetric inequalities are not limited to spaces which have property (ET). Indeed in Chapters B and C we prove a variant of Reshetnyak's majorization theorem for Banach spaces and conclude the following result.

Theorem 2.2.2. *Every Banach space X supports a quadratic isoperimetric inequality with constant $C = \frac{1}{2\pi}$. If X is finite dimensional with $\dim(X) = n$ then C improves to a constant $C_n < \frac{1}{2\pi}$.*

Theorem 2.2.2 is sharp. Namely by [88] and [87] the Banach space ℓ^∞ cannot support a quadratic isoperimetric inequality with a constant smaller than $\frac{1}{2\pi}$. Furthermore by Theorem 1.4 in [C] there are finite dimensional normed spaces with optimal isoperimetric constants arbitrary close to $\frac{1}{2\pi}$. Theorem 2.2.2 implies that every homogeneously regular Finsler manifold of dimension n supports a local quadratic isoperimetric inequality with a constant $C_n < \frac{1}{2\pi}$. Combining this with Theorem 2.2.1 gives the following corollary.

Corollary 2.2.3. *Let X be a homogeneously regular Finsler manifold of dimension n and $\Gamma \subset X$ be a Jordan curve. If $u \in \Lambda(\Gamma, X)$ is an infinitesimally isotropic area minimizer then u is locally α-Hölder continuous on D^2 for some Hölder exponent $\alpha_n > \pi/8$.*

For Finsler manifolds that are diffeomorphic and bi-Lipschitz equivalent to $\mathbb{R}^n$, Corollary 2.2.3 improves regularity results from [125, 128]. Namely the Hölder exponents of the Cartan minimizers in [125, 128] depend on the bi-Lipschitz constants while the constant $\pi/8$ in Corollary 2.2.3 is uniform.

Beyond these smooth examples also many nonsmooth spaces fall into the quadratic isoperimetric inequality setting. For example quadratic isoperimetric inequalities are supported by the higher dimensional Heisenberg groups and quite degenerate surfaces, see [3] and Chapter E.

In some situations, such as when X is a CAT(κ) space, the Hölder regularity in Theorem 2.2.1 can be improved to local Lipschitz continuity [95, 143, 19]. Example 8.3 in [104] however shows that solutions to the Plateau problem in ambient spaces X which support a local quadratic isoperimetric inequality are not always locally Lipschitz continuous. In particular Theorem 2.2.1 is sharp in general. For Finsler manifolds it remains an open question whether some higher regularity in the sense of Lipschitz regularity or even smoothness can be achieved. So far the only result in this context is due to Overath–von der Mosel [125]. They find area minimizers of $C^{1,\alpha}$-regularity when the Finsler manifold is close in C^2-sense to the standard Euclidean metric on $\mathbb{R}^3$. There is the hope that the results from Chapter C can be used to prove similar results in less restrictive settings.

2.2.2 Boundary regularity. Since weakly conformal area minimizers in $\mathbb{R}^3$ are harmonic they are continuous up to the boundary. As discussed above for mere metric spaces X this need not be true even in the interior. But Lytchak–Wenger prove that infinitesimally isotropic area minimizers are continuous up to the boundary when the ambient space X supports a local quadratic inequality [104].

Concerning higher boundary regularity of weakly conformal area minimizers $u \in \Lambda(\Gamma, \mathbb{R}^3)$ the situation is more complicated. One would hope that if Γ enjoys a certain regularity then this regularity is shared by u. The first result of this spirit is already from 1951 and due to Lewy. He proves that if Γ is real analytic then u is real analytic up to the boundary [102]. About 20 years later similar results for smooth curves, curves of $C^{m,\alpha}$-regularity where $m \geq 1$ and curves with Dini continuous derivatives have been established by Hildebrandt, Nitsche, Kinderlehrer, Warschawski and Lesley [82, 119, 94, 156, 100]. These regularity theorems generalize to Riemannian ambient manifolds [80] but they cannot even be formulated properly for nonsmooth ambient spaces. Around the same time however also another boundary regularity result has been obtained by Goldhorn–Hildebrandt and Nitsche [61, 120]. They prove that if Γ is bi-Lipschitz then u is globally Hölder continuous. This theorem indeed generalizes to nonsmooth spaces.

Theorem 2.2.4 ([104]). *Let X be complete metric space which supports a local quadratic isoperimetric inequality and $\Gamma \subset X$ be a bi-Lipschitz Jordan curve. If $u \in \Lambda(\Gamma, X)$ is an infinitesimally isotropic area minimizer then u is Hölder continuous on $\bar{D}^2$.*

For Lipschitz curves in $\mathbb{R}^3$ a classical result of Tsuji from 1942 says that weakly conformal area minimizers have absolutely continuous traces [154]. However when Γ has cusps then weakly conformal area minimizers are not always globally Hölder continuous, compare e.g. [92]. In this light the following result that we derive in Chapter D from Theorem 2.2.4 might seem quite surprising.

Theorem 2.2.5. *Let $\Gamma \subset \mathbb{R}^n$ be a rectifiable Jordan curve. Then there is an admissible map $u \in \Lambda(\Gamma, X)$ which is $\frac{1}{27}$-Hölder continuous on $\bar{D}^2$ and such that*

$$\mathrm{Area}^b(u) = \inf_{v \in \Lambda(\Gamma, X)} \mathrm{Area}^b(v).$$

Furthermore, u may be chosen weakly conformal, smooth and harmonic on the open set $D^2 \setminus u^{-1}(\Gamma)$.

Note that the Hölder exponent of weakly conformal area minimizers depends on the bi-Lipschitz constant of Γ. Hence Theorem 2.2.5 is even partially new for bi-Lipschitz curves Γ. The proof of Theorem 2.2.5 relies on Theorem 2.2.4 and the metric category's flexibility for constructions. More generally it allows to conclude the following theorem.

Theorem 2.2.6. *Let X be complete metric space which supports a local quadratic isoperimetric inequality and $\Gamma \subset X$ be a rectifiable Jordan curve.*
If $\Lambda(\Gamma, X)$ is non-empty then there is an admissible map $u \in \Lambda(\Gamma, X)$ which is

α-Hölder continuous on $\bar{D}^2$ and such that

$$\mathrm{Area}^b(u) = \inf_{v \in \Lambda(\Gamma, X)} \mathrm{Area}^b(v).$$

Here the Hölder exponent $\alpha \in (0,1)$ depends only on the isoperimetric constant of X. Furthermore, u may be chosen infinitesimally isotropic on the open set $D^2 \setminus u^{-1}(\Gamma)$.

2.3 Topological regularity

In this section we discuss the topological behaviour of solutions to the Plateau problem. Optimally weakly conformal area minimzers $u \in \Lambda(\Gamma, \mathbb{R}^3)$ would be embeddings and hence correspond to actual submanifolds. This would be desirable from the physical viewpoint if one wants the minimizers to model actual soap films. In general however it cannot hold. A simple obstruction is that the boundary curve Γ most be unknotted. Nevertheless there are two known conditions that imply embeddedness. The first one is due to Meeks–Yau who show (improving results from [133, 68, 150, 4]) embeddedness when Γ is contained in the boundary of a convex body [113]. Variants of this result hold for 3-dimensional Riemannian and more generally Finsler manifolds [113, 126]. The other condition is due to Ekholm–White–Wienholtz [50] who prove embeddedness when Γ is of total curvature bounded above by 4π. This result is not restricted to dimension 3. Indeed variants of it were even proven for manifolds satisfying upper curvature bounds by Choe–Gulliver [29] and for CAT(0) spaces by Stadler [148].

For general curves $\Gamma \subset \mathbb{R}^3$ one can only hope for immersed minimizers. Indeed back in 1928 (even before the works of Douglas and Radó) Garnier published a proof that every piecewise smooth Γ bounds an immersed minimal disk [57]. The article of Garnier however seems hardly readable and contains several gaps. This is why the solution of Plateau's problem is usually credited to Douglas and Radó. Indeed a few years ago Desideri and Desideri–Jakob worked through Garnier's paper trying to fill the gaps [38, 39]. Their conclusion was that Garnier's proof seems to work only when Γ is of total curvature at most 6π. In any case Garnier neither claims that there is an area minimizing disk u nor that every such u is immersed. Indeed Douglas and Courant gave examples of non-immersed minimal disks in $\mathbb{R}^3$ which they claim to be area minimizing [46, 32]. Concerning Douglas counterexample already Radó argued why it does not seem correct [133]. Courant's counterexamples however survived until a breakthrough paper of Ossermann appeared in 1970, [124]. Ossermann proves that all weakly conformal area minimizers in $\mathbb{R}^3$ are immersed. Unfortunately also Ossermann's argument was not completely clean. But after some complementary work by Alt, Gulliver and Gulliver–Osserman–Royden the result turned out to be true [5, 6, 65, 67]. Indeed it more generally applies for all homogeneously regular Riemannian 3-manifolds.

Note that the results from the preceeding paragraph only concern the interior. Nevertheless Gulliver–Lesley proved in the '70s that when Γ is real analytic then also boundary branch points do not occur [66]. However it is not known whether boundary branch points can be excluded for all smooth curves Γ. Classifications of potential boundary branch points have been developed and by now most types of them can be excluded [162, 83]. It seems however that the remaining cases cannot be handled using the classical techniques [153]. I believe that the collar extension trick developed in Chapters D, F, I and [148] might prove useful to tackle this classical question.

The results from the preceeding two paragraphs are stated for dimension 3. This is not by accident. Indeed all complex varieties are area minimizers [53]. Hence in $\mathbb{R}^n$ with $n \geq 4$ there are many examples of weakly conformal area minimizers that do have singularities. On the other hand if $u \in \Lambda(\Gamma, \mathbb{R}^3)$ is a weakly conformal area minimizers then u is harmonic and hence real analytic. In particular the branch points of u must lie discretely within D^2. Indeed by Lewy's boundary regularity result [102] when Γ is real analytic then u can only have finitely many branch points.

For nonsmooth ambient spaces X there are no linear differentials and even apmd$u$ is defined only almost everywhere. Hence one cannot talk about branch points in an analytic sense. Nevertheless one can still study the *topological branch points* of infinitesimally isotropic area minimizers $u \in \Lambda(\Gamma, X)$. That is the points at which u is not a local embedding. In [148] Stadler proves that if X is CAT(0) then u has only finitely many topological branch points. On the other hand there is the following counterexample: Let X be the quotient of the Euclidean disk $\bar{D}^2$ where a small ball B in its interior is collapsed to a point. Then the canonical projection $u \colon \bar{D}^2 \to X$ is an infinitesimally isotropic area minimizer and every point in B is a topological branch point of u. Furthermore X supports a quadratic isoperimetric inequality with constant $\frac{1}{2\pi}$. This counterexample and the result of Stadler motivated Question 11.4 in [108]. This question by Lytchak–Wenger asks whether the set of topological branch points can be controlled when the isoperimetric constant of X is between $\frac{1}{4\pi}$ and $\frac{1}{2\pi}$. In Chapter E we provide the following negative answer.

Theorem 2.3.1. *Let $C > \frac{1}{4\pi}$. Then there are a compact metric space X which supports a quadratic isoperimetric inequality with constant C and a weakly conformal area minimizer $u \colon \bar{D}^2 \to X$ that is constant on an open set.*

Furthermore in Chapter E we answer Question 11.5 from [106]. This one concerns the topological structure of the fibers of u. More precisely it asks whether the connected components of fibers of infinitesimally isotropic area minimizers need to be contractible. Since cell-like sets can be non-contractible it is a consequence of the following theorem that such fibers may occur.

Theorem 2.3.2. *Let X be a complete metric space which supports a quadratic isoperimetric inequality, $u \colon \bar{D}^2 \to X$ be an infinitesimally isotropic area minimizer and $E \subset \mathbb{R}^2$ be a cell-like subset. If $u^{-1}(x)$ is non-discrete for some $x \in X$ then there is another infinitesimally isotropic area minimizer $v \colon \bar{D}^2 \to X$ such that some connected component of $v^{-1}(x)$ is homeomorphic to E.*

The preceeding theorem also relates to another classical question which concerns the uniqueness of solutions. If $u \in \Lambda(\Gamma, \mathbb{R}^3)$ is a conformal area minimizer and $\varphi \colon \bar{D}^2 \to \bar{D}^2$ is a conformal diffeomorphism then $u \circ \varphi \in \Lambda(\Gamma, \mathbb{R}^3)$ is another conformal area minimizer. Hence one can only ask whether conformal area minimizers or more generally minimal disks for a given curve Γ are unique up to conformal diffeomorphism of $\bar{D}^2$. This was proven when Γ projects to planar convex Jordan curves by Radó [131, 132], when Γ is C^2-close to a planar curve by Tromba [152] and when Γ is of curvature at most 4π by Nitsche [121]. A further uniqueness result for certain graphs over non-convex curves is due to Sauvigny [141]. On the other hand (building on previous work in [101, 33, 96, 114, 146]) White [160, 161] proved in 1994 that there are Jordan $\Gamma \subset \mathbb{R}^3$ which bound uncountably many minimal disks. However it is not known whether a Jordan curve in $\mathbb{R}^3$ can also bound infinitely many weakly conformal area minimizers. Indeed by work of Tomi [149] real analytic curves only bound finitely many such minimizers and by work of Böhme–Tromba [16] the same is true for generic smooth curves. At least in the metric space setting however finiteness fails in general by the following Corollary of Theorems 2.3.1 and 2.3.2.

Corollary 2.3.3. *Let $C > \frac{1}{4\pi}$. Then there are a compact metric space X which supports a quadratic isoperimetric inequality with constant C and a rectifiable Jordan curve $\Gamma \subset X$ which bounds uncountably many weakly conformal area minimizers.*

2.4 Intrinsic geometry

Equivalently minimal surfaces in $\mathbb{R}^3$ can be characterized as surfaces of vanishing mean curvature. The mean curvature however is not an intrinsic quantity of the surface. Hence the inner geometry of the surface does not determine whether it is minimal. In turn, by Gauß' theorema egregium, the Gauß curvature is indeed an intrinsic quantity. While the mean curvature is the average of the two principal curvatures, the Gauß curvature is the product of these. In particular, the Gauß curvature of minimal surfaces in $\mathbb{R}^3$ is pointwise bounded above by zero. More generally when X is a Riemannian manifold with sectional curvatures bounded above by K then the Gauß curvature of minimal surfaces in X is also bounded above by K. This imposes strong geometric constraints on the intrinsic geometry of minimal surfaces. For example by Toponogov's comparison theorem small geodesic triangles in minimal surfaces are at least as convex as the corresponding comparison triangles in the respective model space.

One has to be a bit careful about actually defining the intrinsic geometry of weakly conformal area minimizers even when $f \in \Lambda(\Gamma, \mathbb{R}^n)$. This is because f might have branch points and because Γ might be nonsmooth. When $X = (\mathcal{N}, g)$ is a homogeneously regular Riemannian manifold then the intrinsic geometry of weakly conformal area minimizers $u \in \Lambda(\Gamma, X)$ can be defined as follows: First one considers the (possibly degenerate) pullback Riemannian metric u^*g on D^2 given by

$$(u^*g)_z(V,W) := g_{u(z)}\left(\mathrm{d}u_z(V), \mathrm{d}u_z(W)\right).$$

Then one defines the *u-length* of piecewise smooth curves $\gamma\colon [a,b] \to D^2$ as

$$\ell_u(\gamma) := \int_a^b \sqrt{(u^*g)_{\gamma(t)}\left(\gamma'(t), \gamma'(t)\right)}\,\mathrm{d}t$$

and the *u-metric* on D^2 by

$$\bar{d}_u(x,y) := \inf_{\gamma\colon x \rightsquigarrow y} \ell_u(\gamma). \tag{2.7}$$

Since the branch points of u are isolated, the u-metric is indeed a metric. The *intrinsic disk* Z_u is defined as the metric completion of (D^2, d_u).

When X is nonsmooth and $u \in \Lambda(\Gamma, X)$ is an infinitesimally isotropic area minimizer then there is no pullback Riemannian metric u^*g. Nevertheless apmd$u$ defines a measurable Finsler structure on D^2. The main problem is that this Finsler structure is defined only almost everywhere. Hence it is not straighforward to define a metric $\bar{d}_u$ as in (2.7) in terms of this Finsler structure. Instead one can define a semimetric on $\bar{D}^2$ by

$$d_u(x,y) := \inf_{\gamma\colon x \rightsquigarrow y} \ell(u \circ \gamma).$$

If X is homogeneously regular then d_u defines an actual metric and $(\bar{D}^2, d_u)$ is isometric to Z_u. In general however we have to identify points in $\bar{D}^2$ at zero d_u-distance and define Z_u as the arising quotient space. This intrinsic disk Z_u comes with a canonical surjection $P_u\colon \bar{D}^2 \to Z_u$ and a 1-Lipschitz map $\bar{u}\colon Z_u \to X$ such that

$$\begin{array}{ccc} \bar{D}^2 & \xrightarrow{\quad u \quad} & X \\ & {\scriptstyle P_u}\searrow \quad \nearrow{\scriptstyle \bar{u}} & \\ & Z_u & \end{array}$$

commutes.

By work of Mese and Lytchak–Wenger the initial observations on curvature bounds generalize to the singular setting [115, 107]. More precisely, when X is a CAT(κ) space and $u \in \Lambda(\Gamma, X)$ is a weakly conformal area minimizer then the intrinsic disk Z_u is a CAT(κ) space homeomorphic to $\bar{D}^2$. Indeed more generally this holds for so-called metric minimizing disks [127, 155]. We have noted in Section 2.2.1 that CAT(0) spaces support a quadratic isoperimetric inequality with constant $\frac{1}{4\pi}$. Conversely Lytchak–Wenger prove in [107] that proper geodesic spaces supporting a quadratic isoperimetric inequality with constant $\frac{1}{4\pi}$ are CAT(0). In this light one may think of quadratic isoperimetric inequalities as very weak notions of upper curvature bounds. Thus the following structure theorem by Lytchak–Wenger may be considered a generalization of the fact that minimal surfaces inherit upper curvature bounds from their ambient spaces.

Theorem 2.4.1 ([106]). *Let X be a complete metric space which supports a quadratic isoperimetric inequality with constant C. Further let $\Gamma \subset X$ be a rectifiable Jordan curve and $u \in \Lambda(\Gamma, X)$ be an infinitesimally isotropic area minimizer. Then we have the following list of properties.*

(i) Z_u *is a geodesic metric space homeomorphic to* $\bar{D}^2$.

(ii) Z_u *supports a quadratic isoperimetric inequality with constant* C.

(iii) $P_u \in \Lambda(\partial Z_u, Z_u)$ *is an infinitesimally isotropic area minimizer. Furthermore the approximate metric differential seminorms of* u *and* P_u *agree at almost every* $z \in D^2$.

(iv) ∂Z_u *is rectifiable and* $\ell(\partial Z_u) = \ell(\Gamma)$.

2.4.1 The space of minimal disks. Denote by $\mathcal{D}$ the collection of all geodesic metric spaces Z homeomorphic to $\bar{D}^2$ which support a quadratic isoperimetric inequality and have a rectifiable boundary curve ∂Z. By Theorem 2.4.1 every intrinsic minimal disk Z_u is an element of $\mathcal{D}$. Indeed up to isometry all elements of $\mathcal{D}$ arise this way. Namely by Theorem 2.1.4 there is an infinitesimally isotropic area minimizer $u \in \Lambda(\partial Z, Z)$ for $Z \in \mathcal{D}$. Thus this is implied by the following result from Chapter F.

Theorem 2.4.2. *Let $Z \in \mathcal{D}$. If $u \in \Lambda(\partial Z, Z)$ is an infinitesimally isotropic area minimizer then $\bar{u}\colon Z_u \to Z$ is an isometry.*

In particular one may think of $\mathcal{D}$ as the space of minimal disks and study its properties as a subset of Gromov–Hausdorff space. E.g. one may hope to understand its closure or to find a dense subset of 'nice' disks. The space $\mathcal{D}$ as a whole however seems too large to hope for interesting results of this type. Compare this to the smooth situation where one wants to understand the class $\mathcal{R}(n)$ of n-dimensional Riemmanian manifolds. To obtain interesting results however one would further impose uniform bounds on parameters such as curvature, volume and diameter, see e.g. [63, 64, 22, 27]. Similarly we denote by $\mathcal{D}(L, C)$ the collection of those $Z \in \mathcal{D}$ with isoperimetric constant at most C and boundary length at most L. It follows from Theorem 2.4.2, results in [106] and Gromov's compactness criterion that $\mathcal{D}(L, C)$ is precompact in Gromov–Hausdorff space. Thus a natural task is to understand the closure of $\mathcal{D}(L, C)$.

As noted above when $C = \frac{1}{4\pi}$ then $\mathcal{D}(L, C)$ is the class of CAT(0) disks of boundary length at most L. An upper bound concerning the closure for this special case was given by Petrunin–Stadler [127]. They show that the larger class of CAT(0) disk retracts of boundary length at most L is compact in Gromov–Hausdorff space. Here Z is called a *disk retract* when there is a closed curve $\gamma\colon S^1 \to Z$ such that the mapping cylinder of γ is homeomorphic to $\bar{D}^2$. Further we call the length of γ the boundary length of Z. The following main result of Chapter F is a generalization and refinement of the compactness theorem by Petrunin–Stadler.

Theorem 2.4.3. *Let $C, L \in (0,\infty)$. Denote by $\mathcal{E}(L,C)$ the collection of geodesic disk retracts which support a quadratic isoperimetric inequality with constant C and are of boundary length at most L. Then $\mathcal{E}(L,C)$ is compact in Gromov–Hausdorff space and*

$$\overline{\mathcal{D}(L,C)} \subset \mathcal{E}(L,C) \subset \overline{\mathcal{D}(L,C+\tfrac{1}{2\pi})}.$$

Theorem 2.4.3 provides us with a quite good understanding of the closure of $\mathcal{D}(L,C)$. Very interesting however is also the dual question whether $\mathcal{D}(L,C)$ has a dense subset of 'nice', say smooth, disks. Again this restrictive question might be unreasonably hard. But at least one may hope to answer the following one.

Question 2.4.4. *Can any disk $Z \in \mathcal{D}(L,C)$ be approximated by smooth disks satisfying comparable bounds on boundary length and isoperimetric constant?*

An example of similar approximation results is a classical theorem by Alexandrov–Zalgaller. It states that every surface of synthethically bounded integral curvature is the limit of smooth surfaces with comparable curvature measure [2]. A key ingredient for the proof of their approximation theorem is a triangulation result for surfaces of bounded integral curvature. The latter says that every such surface admits a tessellation by arbitrary small convex geodesic triangles. The following main result of Chapter G generalizes this triangulation theorem of Alexandrov–Zalgaller to general metric surfaces.

Theorem 2.4.5. *Let X be a geodesic metric space homeomorphic to a closed surface and $\varepsilon > 0$. Then X may be decomposed into finitely many non-overlapping convex triangles, each of diameter at most ε.*

Theorem 2.4.5 has been successfully applied by Ntalampekos–Romney to show that every metric surface of finite area is a limit of smooth surfaces of uniformly bounded area [123]. In [123] this approximation result is then itself applied to deduce a generalization of the celebrated uniformization theorems by Bonk–Kleiner, Rajala and Lytchak–Wenger [17, 135, 108]. I believe that the techniques developed in [123] together with Theorem 2.4.5 (or more precisely the refined version Theorem 1.2 in [G]) can be used to positively answer Question 2.4.4.

2.4.2 Intrinsic geometry from analytic data. Let X be a complete metric space and $u\colon \bar{D}^2 \to X$ be an infinitesimally isotropic area minimizer. At the beginning of this section we have discussed why it is not straightforward to define a reasonable intrinsic disk for u only in terms of the analytic data. That is in terms of the Finsler structure $F_u = \operatorname{apmd} u$. Nevertheless in Chapter H we demonstrate that it is possible.

One idea could be to use the following definition proposed by De Cecco–Palmieri in the context of Lipschitz Finsler manifolds [35, 36]: An absolutely continuous curve $\gamma\colon [a,b] \to \bar{D}^2$ is called *transversal* to $E \subset \bar{D}^2$ if $\gamma^{-1}(E)$ is a

Lebesgue null set. If γ is tranversal to the set of point at which F_u is undefined then the *u-length* of γ is given by

$$\ell_u(\gamma) := \int_a^b F_u\left(\gamma(t), \gamma'(t)\right)\, \mathrm{d}t.$$

The ∞-*essential u-semimetric* on $\bar{D}^2$ is defined as

$$d_u^{\infty}(x,y) := \sup_{E \subset \bar{D}^2,\ |E|=0} \inf\left\{\ell_{F_u}(\gamma) \mid \gamma\colon x \rightsquigarrow y,\ \gamma \text{ transversal to } E\right\}.$$

This definition can be rephrased in terms of modulus. The modulus of curve families is a classical concept from complex analysis. More recently it also turned out to be a powerful tool for developing the research field nowadays called "Analysis on metric spaces", see [79]. For given coefficient $p \in [1,\infty]$ and metric measure space Y the *p-modulus* is an outer measure on the collection of all absolutely continuous curves in Y. Using the modulus language $d_u^{\infty}(x,y)$ is precisely the essential infimum of the u-length with respect to ∞-modulus among all curves from x to y. This definition works well when u is a Lipschitz map. Essentially the reason is that Lip $= W^{1,\infty}$. In our setting u is however only of $W^{1,2}$-regularity. Thus for us it would make more sense to consider something like an essential infimum with respect to 2-modulus. This does not quite work directly since the entire class of all paths from x to y is a 2-modulus nullset. Nevertheless the idea can be implemented and in Chapter H we define the 2-*essential u-semimetric* d_u^2.

We call the quotient space $\widehat{Z}_u$ corresponding to $(\bar{D}^2, d_u^2)$ the *essential intrinsic disk*. Also $\widehat{Z}_u$ comes with a canonical surjection $\widehat{P}_u\colon \bar{D}^2 \to \widehat{Z}_u$ and a 1-Lipschitz map $\widehat{u}\colon \widehat{Z}_u \to X$ such that

$$\begin{array}{ccc} \bar{D}^2 & \xrightarrow{\quad u \quad} & X \\ & {\scriptstyle \widehat{P}_u} \searrow \quad \nearrow {\scriptstyle \widehat{u}} & \\ & \widehat{Z}_u & \end{array}$$

commutes. The relevant properties of the essential intrinsic disk are summarized by the following theorem from Chapter H.

Theorem 2.4.6. *Let X be a complete metric space which supports a quadratic isoperimetric inequality with constant C. Further let $\Gamma \subset X$ be a bi-Lipschitz Jordan curve and $u \in \Lambda(\Gamma, X)$ be an infinitesimally isotropic area minimizer. Then we have the following list of properties.*

(i) *$\widehat{Z}_u$ is a geodesic metric space homeomorphic to $\bar{D}^2$.*

(ii) *$\widehat{Z}_u$ supports a quadratic isoperimetric inequality with constant C.*

(iii) *$\widehat{P}_u \in \Lambda(\partial\widehat{Z}_u, \widehat{Z}_u)$ is an infinitesimally isotropic area minimizer. Furthermore the approximate metric differential seminorms of u and $\widehat{P}_u$ agree at almost every $z \in D^2$.*

(iv) $\widehat{Z}_u$ *has the Sobolev-to-Lipschitz property.*

The factorization is uniquely characterized by the following universal property: If $u = \tilde{u} \circ \widetilde{P}$, *where* $\widetilde{P}\colon \bar{D}^2 \to \widetilde{Z}$ *and* $\tilde{u}\colon \widetilde{Z} \to X$ *satisfy (i)-(iii), then there exists a surjective* 1*-Lipschitz map* $f\colon \widehat{Z}_u \to \widetilde{Z}$ *such that* $\widetilde{P} = f \circ \widehat{P}_u$.

Propertywise the main advantages of $\widehat{Z}_u$ over Z_u are its characterization in terms of a universal property and that $\widehat{Z}_u$ satisfies the so-called *Sobolev-to-Lipschitz property*. This property was introduced by Gigli in the context of RCD spaces [59, 60]. Geometrically it essentially means that the space has a rich family of geodesics in the sense of 2-modulus, see Theorem 1.7 in [H]. This property is closely related to thick quasiconvexity and Poincaré inequalities. On the other hand the essential intrinsic disk also comes with two drawbacks. The first is that we do not know whether $\partial\widehat{Z}_u$ needs to be rectifiable. The other is that we can only prove Theorem 2.4.6 when Γ is bi-Lipschitz because then Theorem 2.2.4 grants a quantitative control on the continuity of u.

2.5 Applications: Singular boundary values and the Plateau–Douglas problem

After solving the Plateau problem Douglas moved on to tackle the following much harder question:

Question 2.5.1. *Given a compact surface* M *with* k *boundary components and a configuration of disjoint Jordan curves* $\Gamma = (\Gamma_1, \ldots, \Gamma_k)$ *in* $\mathbb{R}^3$. *When is there a minimal surface which is of the topological type of* M *and bounds the given configuration of Jordan curves?*

This question is nowadays called the Plateau–Douglas problem, the Douglas problem or also the general Plateau problem. It was first posed in a research announcement by Douglas from 1930 in which he announces its solution [43]. During the next years Douglas published many further research announcements but only two related articles [44, 46]. In these he provides answers when M is an annulus or a Möbius strip. His treatement of the general case appeared only some years later in [47, 48, 49]. The proof is a generalization of his technique developed to handle the Plateau problem. However, while his approach worked quite well to solve the Plateau problem, for the Plateau–Douglas problem it becomes extremely technical. Indeed later several people have raised doubts about the correctness of his proof [89, 151, 62]. In any case as of today his result is certainly known to be correct. It states that if the so-called *Douglas condition* holds then indeed such minimal surface exists. The Douglas condition asserts that in terms of area infima the configuration Γ cannot be filled equally well by surfaces of lower topological complexity than M. Here lower complexity refers to a lower genus or a larger number of connected components. Indeed existence cannot hold unconditionally. E.g. a planar convex curve in $\mathbb{R}^3$ cannot bound a minimal surface of positive genus [112].

Another proof of Douglas' theorem was suggested by Courant [30, 31]. Again he proposes to again attack the problem by minimizing the Dirichlet energy. This does indeed work but also needs much more care than before. The reason is that now the Dirichlet energy not only depends on the map $u\colon M \to \mathbb{R}^3$ but also on choosing a geometry on M. Thus one has to minimize the Dirichlet energy not only over all admissible maps $u \in \Lambda(M, \Gamma, \mathbb{R}^3)$ but also simultaneously over all Riemannian metrics g on M. The energy minimizing admissible map is then an area minimizer which is weakly conformal with respect to the minimizing Riemannian metric on M. In principle Courant's reasoning can be implemented. Courant himself however discussed only certain special cases and even concerning these his arguments are somewhat vague. Partly this might be due to his mathematical style [136]. But also partly a reason might be that the necessary tools to formalize his reasoning were not yet available to him. E.g. Teichmüller's theory of moduli space of conformal structures on surfaces was only developed about the same time over in Germany from where Courant had just fled.

Thus it seems that the first complete and clean proof of Douglas theorem appeared only about 50 years later in an article by Jost [89, 151, 62]. Jost's actual main intent however was to generalize Douglas' result from $\mathbb{R}^n$ to homogeneously regular ambient Riemannian manifolds. Jost limits himself to the case of orientable surfaces M. But some years later also the nonorientable case was worked out by his student Bernatzki [15]. In a recent article by Fitzi–Wenger Douglas' theorem has been further generalized to proper metric spaces which support a local quadratic isoperimetric inequality [56]. In contrast to Theorem 2.1.4 here also the existence proof seems to rely on cut-and-paste arguments and hence the quadratic isoperimetric inequality. The following first main result of Chapter I shows that this assumption can be dropped.

Theorem 2.5.2. *Let M be a compact connected orientable surface with k boundary components, X be a proper metric space and Γ be a configuration of k disjoint rectifiable Jordan curves. If the Douglas condition holds for the configuration (M, Γ, X) then there exists $u \in \Lambda(M, \Gamma, X)$ as well as a Riemannian metric g on M such that*

$$\operatorname{Area}^b(u) = \inf_{v \in \Lambda(M,\Gamma,X)} \operatorname{Area}^b(v)$$

and f is infinitesimally isotropic with respect to g.

Note in particular that Theorem 2.5.2 is new when X is a nonhomogeneously regular Riemannian manifold. The proof combines the generalized Douglas theorem of Fitzi–Wenger with Wenger's thickening trick [56]. In particular this result is new in smooth settings but proofwise relies on metric constructions.

The even more interesting contribution of Chapter I however is that we are also able to handle singular curve configurations. That is we allow for singular configurations $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ where the curves are possibly nondisjoint or self-intersection. For such situations the approaches of Douglas and Courant cannot work even in $\mathbb{R}^3$. Both methods would produce conformal area minimizers and such cannot always exist. A simple counterexample is the planar figure eight curve which does not bound a disk type conformal area minimizer, see e.g. [77].

The Plateau problem for self-intersecting curves has before been investigated by Hass [77]. His method for the figure eight curve Γ would proceed as follows:

1.) Cut Γ along the self intersection as to form two Jordan curves Γ_1 and Γ_2.

2.) Fill each of the Jordan curves Γ_1 and Γ_2 separately by energy minimizers $u_1 \in \Lambda(\Gamma_1, \mathbb{R}^3)$ and $u_2 \in \Lambda(\Gamma_2, \mathbb{R}^3)$.

3.) Paste the minimizers u_1 and u_2 together to form a single map $u \in \Lambda(\Gamma, \mathbb{R}^3)$.

When considering more complicated self-intersections the pasting process can mess up the regularity. In particular the constructed map u would no longer be of Sobolev regularity. Hence Hass has to work with the Lebesgue notion of surface area mentioned in Section 2.1. In this light the following main result of Chapter I is interesting even for the Plateau problem and $\mathbb{R}^3$.

Theorem 2.5.3. *Let M be a compact connected orientable surface with k boundary components, X be a proper metric space and Γ be a configuration of k rectifiable closed curves. If the Douglas condition holds for the configuration (M, Γ, X) then there exists $u \in \Lambda(M, \Gamma, X)$ as well as a Riemannian metric g on M such that*

$$\operatorname{Area}^b(u) = \inf_{v \in \Lambda(M,\Gamma,X)} \operatorname{Area}^b(v)$$

and f is infinitesimally isotropic with respect to g on $M \setminus f^{-1}(\Gamma)$.

By the figure eight counterexample in general a map u as in Theorem 2.5.3 cannot be infinitesimally isotropic everywhere on M. When X supports a quadratic isoperimetric inequality one can as in Theorem 2.2.6 assume that u is globally Hölder continuous. In the case of $\mathbb{R}^3$ and smooth curves Γ_i one can furthermore achieve local Lipschitz continuity on $M \setminus \partial M$.

Again the proof exploits the metric category's flexibility for constructions. The idea is the following: Say we start with a singular configuration Γ in a regular space such as $X = \mathbb{R}^n$. Then one constructs a new singular metric space $\widetilde{X}$ and a system of disjoint Jordan curves $\widetilde{\Gamma} \subset \widetilde{X}$ such that the Plateau–Douglas problem for (M, Γ, X) is equivalent to the Plateau–Douglas problem for $(M, \widetilde{\Gamma}, \widetilde{X})$. Now Theorem 2.5.2 solves the Plateau–Douglas problem for $(M, \widetilde{\Gamma}, \widetilde{X})$ and hence concludes the proof. Thus the idea is to trade the singularity from the curve side to the space side, and to then apply the generalization of a classical result for nonsmooth ambient spaces. A sketch of the construction for a simple example is illustrated in Figure 2.1.

For regular configurations $\Gamma \subset \mathbb{R}^3$ of disjoint Jordan curves there is a weaker condition than the one by Douglas which still implies existence. This condition is called the *condition of cohesion*. It asserts that the energy infimum over $\Lambda(M, \Gamma, X)$ can be approximated by a sequence of fillings which satisfy a geometric non-degeneracy condition. That the condition of cohesion implies

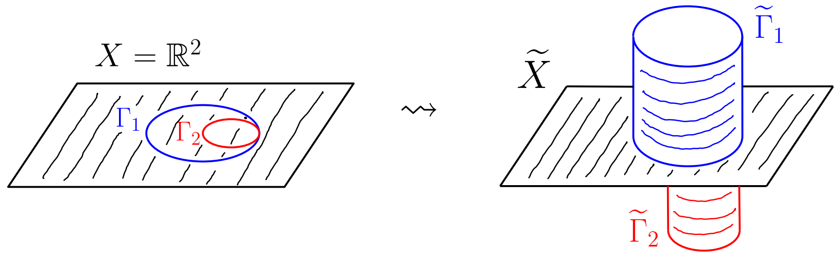

Figure 2.1: Construction of $(\widetilde{X}, \widetilde{\Gamma})$

the existence of weakly conformal area minimizers was already noted by Shiffman [145] in 1939. However only about 50 years later in [151] Tomi–Tromba proved that the Douglas condition implies the condition of cohesion. In [56] Fitzi–Wenger also generalize the condition of cohesion to nonsmooth ambient spaces and prove that it implies the existence of energy minimizers. Note however that when X does not have property (ET) then neither the Douglas condition implies the condition of cohesion nor the condition of cohesion the existence of area minimizers.

Also concerning the condition of cohesion additional difficulties arise for singular curve configurations $\Gamma \subset \mathbb{R}^n$. Nevertheless imposing an additional so-called *condition of adhesion* Iseri was able to prove the existence of energy minimizers [86]. Note however that the Douglas condition does in general not imply that adhesion. Hence his result can only be applied to obtain existence for very particular configurations, cf. [86]. In Chapter I we also generalize the condition of adhesion and Iseri's result to metric ambient spaces.

2.6 Further remarks

We refer to [122, 136, 62, 40, 41] for more detailed accounts and references concerning the history of the classical Plateau problem.

Admittedly there are some serious drawbacks concerning the classical formalizations and solutions of Plateau's problem as discussed here and in [122, 40, 41]. For example these seem limited to dimension two and from the physical viewpoint one would also not count multiplicities as in (2.4). Thus over the years many other frameworks to formalize and solve the Plateau problem have been developed. To discuss all these approaches is much beyond the scope of this introduction. Instead we refer to the overview article [34] for further reading. Here we only remark that the nowadays most standard formalization and solution of the Plateau problem in terms of currents has also been generalized to metric ambient spaces, see [11, 98]. Note that the proof of the existence of minimizing currents is quite straightforward once the machinery has been built up. However as of today a satisfactory regularity theory for the obtained minimizing currents in nonsmooth ambient spaces seems still quite out of reach. As we saw in Section 2.2 this is in contrast to the 'classical' solutions guaranteed by Theorem 2.1.4.

CHAPTER A

An approach to metric space valued Sobolev maps via weak* derivatives

with Nikita Evseev

A.1 Introduction

A.1.1 Objective. The present article concerns possible definitions of the first-order Sobolev space $W^{1,p}(\Omega; X)$ for an open subset $\Omega \subset \mathbb{R}^n$, a metric space X and a coefficient $p \in (1, \infty)$. Since the early 1990's several definitions of such Sobolev spaces have been proposed in [95, 71, 138, 26, 144, 74, 10]. Many of these make sense when Ω is an arbitrary metric measure space and, in such generality, the arising Sobolev space may depend on the chosen definition. However, for bounded domains $\Omega \subset \mathbb{R}^n$, all of these definitions are equivalent, see [139, 9, 79]. The mentioned characterizations of $W^{1,p}(\Omega; X)$ take very different approaches that mostly involve slightly advanced concepts such as energy, modulus of curve families or Poincar inequalities. Hence, from the point of view of classical analysis, all these characterizations might either seem a bit complicated or at least not very straightforward. Another definition of the Sobolev space $W^{1,p}(\Omega; X)$ was proposed in [76] which is more similar to the traditional definition of classical Sobolev spaces in terms of weak derivatives. Our first main result, Theorem A.1.2 below, however shows that for technical reasons the space $W^{1,p}(\Omega; X)$ as introduced in [76] is essentially empty. The main objective of this article is then to propose a variation on the definition from [76] and show that this new definition indeed gives an equivalent characterization of the Sobolev spaces introduced in [95, 71, 138, 26, 144, 74, 10].

A.1.2 Definitions and main results. If X is a Riemannian manifold then, by Nash's theorem, there is a Riemannian isometric embedding $\iota\colon X \to \mathbb{R}^N$. In this case $W^{1,p}(\Omega; X)$ can be defined as the set of those functions $f\colon \Omega \to X$ for which the composition $\iota \circ f$ lies in the classical Sobolev space $W^{1,p}(\Omega; \mathbb{R}^N)$. Similarly one can embed any metric space X isometrically into some Banach space V as to force a linear structure on the target space. For example every separable metric space embeds isometrically into ℓ^∞ by means of the Kuratowski embedding. Thus it is natural to first define Sobolev functions with values in the Banach space V and then $W^{1,p}(\Omega; X)$ as the subspace of those functions in $W^{1,p}(\Omega; V)$ that take values in X with respect to the fixed embedding. The following definition of Banach space valued Sobolev functions goes back to [147].

Definition A.1.1. Let V be a Banach space and $p \in [1, \infty)$. The space $L^p(\Omega; V)$ consists of those functions $f \colon \Omega \to V$ that are measurable and essentially separably valued, and for which the function $x \mapsto ||f(x)||$ lies in $L^p(\Omega)$. A function f lies in the Sobolev space $W^{1,p}(\Omega; V)$ if $f \in L^p(\Omega; V)$ and for every $j = 1, \dots, n$ there is a function $f_j \in L^p(\Omega; V)$ such that

$$\int_\Omega \frac{\partial \varphi}{\partial x_j}(x) \cdot f(x) \, \mathrm{d}x = -\int_\Omega \varphi(x) \cdot f_j(x) \, \mathrm{d}x \quad \text{for every } \varphi \in C_0^\infty(\Omega)$$

in the sense of Bochner integrals.

It was claimed in [76] that if Y is separable then $W^{1,p}(\Omega; Y^*)$ is equal to the Reshetnyak–Sobolev space $R^{1,p}(\Omega; Y^*)$ introduced in [138]. This would imply that the Sobolev space $W^{1,p}(\Omega; X)$, defined in terms of Definition A.1.1 and the Kuratowski embedding $\kappa \colon X \to \ell^\infty$, is the same as the Sobolev spaces introduced in [138, 95, 144, 71, 74, 26, 10]. Unfortunately, it has recently been observed in [23] that there is a subtle measurability-related mistake in the proof of the equality and indeed $W^{1,p}(\Omega; Y^*)$ equals $R^{1,p}(\Omega; Y^*)$ only if Y^* has the Radon–Nikodým property. For the sake of defining metric space valued Sobolev maps this is potentially problematic because many spaces of geometric interest, such as the Heisenberg group or even $\mathbb{S}^1$ (equipped with the angular metric), do not isometrically embed into a Banach space which has the Radon–Nikodým property, see [28] and Remark 4.2 in [C]. Our first main result shows that indeed $W^{1,p}(\Omega; X)$, as defined in [76] in terms of Definition A.1.1 and the Kuratowski embedding, is always trivial, and hence $W^{1,p}(\Omega; X)$ is not equal to $R^{1,p}(\Omega; X)$ for any geometrically interesting space X.

Theorem A.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, X be a complete separable metric space and $p \in [1, \infty)$. Denote by $\kappa \colon X \to \ell^\infty$ the Kuratowski emdedding of X. Then every function in*

$$W^{1,p}(\Omega; X) := \{f \colon \Omega \to X \mid \kappa \circ f \in W^{1,p}(\Omega; \ell^\infty)\} \tag{A.1}$$

is almost everywhere constant.

Note that, by Theorem A.1.2, if X is a separable Banach space then the definition of $W^{1,p}(\Omega; X)$ given in (A.1) is not compatible with the one given in Definition A.1.1. For example, most trivially, one may consider the case $X = \mathbb{R}$ where Definition A.1.1 gives the classical Sobolev space $W^{1,p}(\Omega)$.

There is a number of articles subsequent to [76] that have worked with (A.1) as definition of metric space valued Sobolev maps, see [72, 163, 73, 14, 13, 75, 37]. In particular, important results such as [163, Theorem 1.2], [73, Theorem 1.4] or [75, Theorem 1.9] are formally not correct as stated. To fix this technical problem, instead of Definition A.1.1, we suggest the following one.

Definition A.1.3. Let V^* be a dual Banach space and $p \in [1, \infty)$. The space $L^p_*(\Omega; V^*)$ consists of those functions $f \colon \Omega \to V^*$ that are weak* measurable and for which the function $x \mapsto ||f(x)||$ lies in $L^p(\Omega)$.
A function f lies in the Sobolev space $W^{1,p}_*(\Omega; V^*)$ if $f \in L^p(\Omega; V^*)$ and for every $j = 1, \dots, n$ there is a function $f_j \in L^p_*(\Omega; V^*)$ such that

$$\int_\Omega \frac{\partial \varphi}{\partial x_j}(x) \cdot f(x) \, \mathrm{d}x = -\int_\Omega \varphi(x) \cdot f_j(x) \, \mathrm{d}x \quad \text{for every } \varphi \in C_0^\infty(\Omega)$$

in the sense of Gelfand integrals.

The main difference between $W^{1,p}_*$ and $W^{1,p}$ is that for $W^{1,p}_*$ the weak derivatives do not need to be measurable and instead one only assumes weak* measurability. In particular, the functions f_j in Definition A.1.3 do not need to be Bochner integrable. Our second main result shows that $W^{1,p}_*$ indeed gives the right Sobolev space.

Theorem A.1.4. *Let $\Omega \subset \mathbb{R}^n$ be open, Y be a separable Banach space, and $p \in [1, \infty)$. Then*

$$W^{1,p}_*(\Omega; Y^*) = R^{1,p}(\Omega; Y^*).$$

Thus, for a bounded Ω and a separable metric space X, one can define $W^{1,p}_*(\Omega; X)$ as the set of those functions $f \colon \Omega \to X$ such that $\kappa \circ f \in W^{1,p}_*(\Omega; \ell^\infty)$ and deduce that

$$W^{1,p}_*(\Omega; X) = R^{1,p}(\Omega; X).$$

We believe that essentially all results in the articles [76, 72, 163, 73, 14, 13, 75, 37] become true if one respectively replaces $W^{1,p}(\Omega; X)$ by $W^{1,p}_*(\Omega; X)$ and that the proofs apply up to straightforward adjustments.

An advantage of our definition of $W^{1,p}_*(\Omega; X)$ over the other equivalent definitions of metric space valued Sobolev maps is that it gives a characterization in terms of actual linear differentials and not just upper gradients, metric differential seminorms or alike. It might seem that such linear differentials are somewhat artificial in the context of general metric target spaces. However, indeed there are some nice arguments and constructions that heavily rely on this sort of objects, see e.g. [73, 37, 12, 87].

A.1.3 Organization. First in Section A.2 we will go through some auxiliary results and definitions concerning the calculus of functions with values in Banach spaces. More precisely, in Sections A.2.1 and A.2.2 we discuss different notions concerning measurability and integrals of Banach space valued functions. Then in Section A.2.3 we study some basic properties of the weak* derivatives of absolutely continuous curves in dual-to-separable Banach spaces. Section A.3 is dedicated to Sobolev maps with values in Banach spaces and more particularly the proof of Theorem A.1.4. To this end we will consider an auxiliary space $R^{1,p}_*(\Omega; Y^*)$ whose definition interpolates between the definitions of $R^{1,p}(\Omega; Y^*)$ and $W^{1,p}_*(\Omega; Y^*)$. In Sections A.3.1 and A.3.2 we then respectively prove the equalities $R^{1,p}_* = R^{1,p}$ and $R^{1,p}_* = W^{1,p}_*$. The more original part here is the proof of the equality $R^{1,p}_* = R^{1,p}$ since the proof of $R^{1,p}_* = W^{1,p}_*$ is very much along the lines of the intended proof of $W^{1,p} = R^{1,p}$ in [76]. In the final Section A.4 we discuss Sobolev functions with values in a metric space X. First in Section A.4.1 we shortly introduce the Sobolev spaces $W^{1,p}_*(\Omega; X)$. Then in Section A.4.2 we focus on $W^{1,p}(\Omega; X)$ and prove Theorem A.1.2. The proof here is a slightly involved argument that exploits the strange analytic properties of the Kuratowski embedding.

A.2 Calculus of Banach space valued functions

During this section let $E \subset \mathbb{R}^n$ be Lebesgue measurable and V be a Banach space.

A.2.1 Measurability of Banach space valued functions. We call a function $f\colon E \to V$ *measurable* if it is measurable with respect to the Borel σ-algebra on V and the σ-algebra of Lebesgue measurable subsets on E. It is called *weakly measurable* if $x \mapsto \langle v^*, f(x)\rangle$ defines a measurable function $E \to \mathbb{R}$ for every $v^* \in V^*$ and *essentially separably valued* if there is a null set $N \subset E$ such that $f(E \setminus N)$ is separable. Trivially measurability implies weak measurability. If additionally one assumes that f is essentially separably valued then, by Pettis' measurability theorem, also the converse implication holds, see e.g. [79, Section 3.1]. In general however, weakly measurable functions do not need to be measurable, see [79, Remark 3.1.3].

A function $f\colon E \to V$ is called *approximately continuous* at $x \in E$ if for every $\varepsilon > 0$ one has

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^n\left(\{y \in B(x,r) \cap E \,:\, ||f(y) - f(x)|| \geq \varepsilon\}\right)}{\mathcal{L}^n\left(B(x,r)\right)} = 0.$$

The following characterization of measurability will be important in the proof of Theorem A.1.2.

Theorem A.2.1 ([53], Theorem 2.9.13)**.** *Let $f\colon E \to V$ be essentially separably valued. Then f is measurable if and only if f is approximately continuous at a.e. $x \in E$.*

A function $f\colon E \to V^*$ is called *weak* measurable* if $x \mapsto \langle v, f(x)\rangle$ defines a measurable function $E \to \mathbb{R}$ for every $v \in V$. We will need the following slight strengthening of Pettis' theorem.

Lemma A.2.2. *Let $f\colon E \to V^*$ be essentially separably valued. Then f is measurable if and only if f is weak* measurable.*

Proof. Clearly measurable functions are weak* measurable. So we only prove the other implication.

By assumption there is a null set $N \subset E$ such that $f(E \setminus N)$ is separable. Let $D = \{v_1^*, v_2^*, \dots\}$ be a countable dense subset in $f(E \setminus N)$. Then $D - D$ is a countable dense subset of the difference set $f(E \setminus N) - f(E \setminus N)$. By definition of the dual norm for every $i, j \in \mathbb{N}$ there is a sequence $(v_k^{ij})_{k\in\mathbb{N}}$ of unit vectors in V such that

$$\langle v_k^{ij}, v_i^* - v_j^* \rangle \to ||v_i^* - v_j^*|| \quad \text{as } k \to \infty.$$

Thus, it follows from the weak* measurability of f that for every $i \in \mathbb{N}$ the function

$$x \mapsto ||f(x) - v_i^*|| = \sup_{j,k\in\mathbb{N}} \langle v_k^{ij}, v_i^* - f(x) \rangle$$

is measurable. In particular, $f^{-1}(B)$ is measurable for every open ball $B \subset V^*$ with center in D.

Let $U \subset V^*$ be open. Then there is a countable collection $(B_i)_{i\in\mathbb{N}}$ of balls in V^* with centers in D such that

$$f(E \setminus N) \cap U = f(E \setminus N) \cap \left(\bigcup_{i\in\mathbb{N}} B_i \right)$$

and hence

$$f^{-1}(U) \cup N = \left(\bigcup_{i\in\mathbb{N}} f^{-1}(B_i) \right) \cup N. \tag{A.2}$$

Since $\bigcup_{i\in\mathbb{N}} f^{-1}(B_i)$ is Lebesgue measurable and N is a null set, (A.2) implies that $f^{-1}(U)$ is Lebesgue measurable. The open subsets generate the Borel σ-algebra of V, so we conclude that f is measurable. □

A.2.2 Integrals of Banach space valued functions. A function $f\colon E \to V$ is called *simple* if there are measurable subsets $E_1, \dots, E_k$ of E and vectors $v_1, \dots, v_k$ in V such that $f = \sum_{i=1}^k \chi_{E_i} \cdot v_i$. If f is simple and all the subsets E_i are of finite $\mathcal{L}^n$-measure, then f is called *integrable* and one defines the *integral* of f as

$$\int_E f(x)\, \mathrm{d}x := \sum_{i=1}^k \mathcal{L}^n(E_i) \cdot v_i.$$

A function $f\colon E \to V$ is called *Bochner integrable* if there are integrable simple functions $(f_k\colon E \to V)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \int_E ||f_k(x) - f(x)||\, \mathrm{d}x = 0$$

The *Bochner integral* of such Bochner integrable function f is defined as

$$\int_E f(x)\, \mathrm{d}x := \lim_{k\to\infty} \int_E f_k(x)\, \mathrm{d}x.$$

Indeed, a function f is Bochner integrable if and only it lies in the space $L^1(E; V)$ introduced in Definition A.1.1, see [79, Proposition 3.2.7]. Furthermore, if f is Bochner integrable and $v^* \in V^*$ then $x \mapsto \langle v^*, f(x) \rangle$ is integrable and

$$\left\langle v^*, \int_E f(x)\, \mathrm{d}x \right\rangle = \int_E \langle v^*, f(x) \rangle\, \mathrm{d}x. \tag{A.3}$$

The Bochner integral is arguably the most popular notion concerning integrals of Banach space valued functions. However, its limitation to essentially separably valued measurable functions is somewhat to rigid for our purposes. Instead we will often work with the so-called Gelfand integral which is a weak* variant of the more well-known Pettis integral that is defined for weakly measurable functions. It goes back to [58] and can be defined in terms of the following lemma. See also [42, p. 53].

Lemma A.2.3. *Let $f\colon E \to V^*$ be a weak* measurable function such that for every $v \in V$ the function $x \mapsto \langle v, f(x)\rangle$ lies in $L^1(E)$. Then there is a unique vector $v_f^* \in V^*$ such that*

$$\langle v, v_f^* \rangle = \int_E \langle v, f(x)\rangle \; dx \qquad \textit{for every } v \in V.$$

Proof. First we claim that the operator $T\colon V \to L^1(E)$ defined by $Tv = \langle v, f\rangle$ is continuous. To this end let $(v_k, Tv_k)_{k\in\mathbb{N}}$ belong to the graph of T. Suppose that $v_k \to v$ in V and $Tv_k \to g$ in $L^1(E)$. Then there is a subsequence $(Tv_{k_m})_{m\in\mathbb{N}}$ which converges a.e. on E to g. In particular

$$g(x) = \lim_{m\to\infty} Tv_{k_m}(x) = \lim_{m\to\infty} \langle v_{k_m}, f(x)\rangle = \langle v, f(x)\rangle = (Tv)(x)$$

for a.e. $x \in \Omega$. Hence the linear operator T has a closed graph and the closed graph theorem implies that T is continuous.

Thus for every $v \in V$ one has

$$\left| \int_E \langle v, f(x)\rangle \; \mathrm{d}x \right| \leq \|Tv\| \leq \|T\| \cdot \|v\|.$$

This shows that the functional v_f^* given by $v_f^*(v) := \int_E \langle v, f(x)\rangle \; \mathrm{d}x$ is continuous and hence completes the proof. □

Functions $f\colon E \to V^*$ that meet the assumptions of Lemma A.2.3 are called *Gelfand integrable* and for such f the arising functional v_f^* is called the *Gelfand integral* of f. By (A.3) and Lemma A.2.3, if $f\colon E \to V^*$ is Bochner integrable then f is Gelfand integrable and $\int_E f(x) \; \mathrm{d}x = v_f^*$. Hence we will not create ambiguity when we also denote Gelfand integrals by $\int_E f(x) \; \mathrm{d}x$ instead of v_f^*. Note that if $\Omega \subset \mathbb{R}^n$ is open and $f \in L^p_*(\Omega; V^*)$, then $\varphi \cdot f$ is Gelfand integrable for every $\varphi \in C_0^\infty(\Omega)$ and hence the Gelfand integrals that appear in Definition A.1.3 are well-defined.

A.2.3 Absolutely continuous curves in Banach spaces. Recall that a function $f\colon [a, b] \to \mathbb{R}$ is called *absolutely continuous* when it satisfies the fundamental theorem of calculus. That is when f is differentiable almost everywhere, the derivative f' is Lebesgue integrable and

$$f(t) - f(a) = \int_a^t f'(s) \; \mathrm{d}s$$

for every $t \in [a,b]$. The *length* of a continuous curve $\gamma\colon [a,b] \to V$ is defined as

$$l(\gamma) := \sup \sum_{i=1}^{n} ||\gamma(t_i) - \gamma(t_{i-1})||$$

where the supremum ranges over all $n \in \mathbb{N}$ and all $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$. The curve γ is called *rectifiable* if $l(\gamma)$ is finite. For a rectifiable curve γ we define its *length function* $s_\gamma\colon [a,b] \to [0, l(\gamma)]$ by

$$s_\gamma(t) = l(\gamma|_{[a,t]}).$$

The length function gives rise to a unique curve $\bar{\gamma}\colon [0, l(\gamma)] \to V$ such that

$$\bar{\gamma} \circ s_\gamma = \gamma.$$

The curve $\bar{\gamma}$ is called the *unit-speed parametrization* of γ because one has for every $t \in [0, l(\gamma)]$ that

$$l(\bar{\gamma}|_{[a,t]}) = t - a.$$

A curve $\gamma\colon [a,b] \to V$ is called *absolutely continuous* if it is rectifiable and the length function s_γ is absolutely continuous. Absolutely continuous curves in a Banach space V do not need to be differentiable almost everywhere unless V has the Radon–Nikodým property. Nevertheless, if V is dual to a separable Banach space then absolutely continuous curves in V are weak* differentiable almost everywhere in the sense of the following lemma.

Lemma A.2.4 ([76, Lemma 2.8])**.** *Let Y be a separable Banach space. Then for every absolutely continuous curve $\gamma\colon [a,b] \to Y^*$ there is a weak* measurable function $\gamma'\colon [a,b] \to Y^*$ such that for almost every $t \in [a,b]$ and every $y \in Y$ one has*

$$\left\langle y, \frac{\gamma(t+h) - \gamma(t)}{h} \right\rangle \to \langle y, \gamma'(t) \rangle \qquad \textit{as } h \to 0. \tag{A.4}$$

If $t \in [a,b]$ is such that (A.4) holds for every $y \in Y$ then γ is called *weak* differentiable* at t and $\gamma'(t)$ is called the *weak* derivative* of γ at t. By the next two lemmas weak* derivatives have desirable analytical and metric properties.

Lemma A.2.5. *Let Y be a separable Banach space and $\gamma\colon [a,b] \to Y^*$ be absolutely continuous. Then for every $\varphi \in C_0^\infty((a,b))$ one has*

$$\int_a^b \frac{\partial \varphi}{\partial t}(t) \cdot \gamma(t) \, \mathrm{d}t = -\int_a^b \varphi(t) \cdot \gamma'(t) \, \mathrm{d}t \tag{A.5}$$

in the sense of Gelfand integrals.

Lemma 2.11 in [76] claims that the equality (A.5) holds in the sense of Bochner integrals. In general however, as the subsequent example shows, the weak* derivative of an absolutely continuous curve in Y^* does not need to be essentially separably valued and hence the Bochner integral $\int_a^b \varphi(t) \cdot \gamma'(t) \, \mathrm{d}t$ may not be defined.

Example A.2.6. Consider the curve $\gamma\colon [0,1] \to L^\infty([0,1])$ given by $(\gamma(t))(s) = |t-s|$. Then γ is an isometric embedding and hence in particular absolutely continuous. Further γ is weak* differentiable at every $t \in [0,1]$ with weak* derivative

$$\gamma'(t) = -\chi_{(0,t)} + \chi_{(t,1)}.$$

Thus

$$||\gamma'(s) - \gamma'(t)||_\infty = 2$$

for every $t \neq s$ and hence $\gamma'\colon [0,1] \to L^\infty([0,1])$ cannot be essentially separably valued.

Proof of Lemma A.2.5. Let $\varphi \in C_0^\infty((a,b))$ and $y \in Y$. For $t \in [a,b]$ we will denote $\gamma_y(t) := \langle y, \gamma(t)\rangle$. Then $\gamma_y\colon [a,b] \to \mathbb{R}$ is absolutely continuous and, by the classical product rule,

$$\int_a^b \frac{\partial \varphi}{\partial t}(t) \cdot \gamma_y(t)\ \mathrm{d}t = -\int_a^b \varphi(t) \cdot \gamma_y'(t)\ \mathrm{d}t. \tag{A.6}$$

Furthermore by (A.4) for almost every $t \in [a,b]$ one has

$$\langle y, \gamma'(t)\rangle = \gamma_y'(t). \tag{A.7}$$

By (A.6) and (A.7), and because $y \in Y$ was arbitrary, we conclude equality (A.5). ☐

Lemma A.2.7. *Let Y be a separable Banach space. If $\gamma\colon [a,b] \to Y^*$ is absolutely continuous then*

$$||\gamma'(t)|| = \lim_{h\to 0} \frac{||\gamma(t+h) - \gamma(t)||}{|h|} = s_\gamma'(t)$$

for almost every $t \in [a,b]$.

Proof. Assume $t \in [a,b]$ is such that s_γ is differentiable at t and that γ is weak* differentiable at t. Then for every $y \in Y$ with $||y|| \leq 1$ one has

$$\langle y, \gamma'(t)\rangle = \lim_{h\to 0} \left\langle y, \frac{\gamma(t+h)-\gamma(t)}{h} \right\rangle \leq \liminf_{h\to 0} \frac{||\gamma(t+h) - \gamma(t)||}{|h|}$$

and hence

$$||\gamma'(t)|| \leq \liminf_{h\to 0} \frac{||\gamma(t+h) - \gamma(t)||}{|h|}. \tag{A.8}$$

Furthermore

$$\limsup_{h\to 0} \frac{||\gamma(t+h) - \gamma(t)||}{|h|} \leq \limsup_{h\to 0} \frac{l(\gamma|_{[t,t+h]})}{|h|} = s_\gamma'(t). \tag{A.9}$$

To prove the reverse inequalities, let $t, \bar{t} \in [a,b]$ with $t < \bar{t}$. Then for every $y \in Y$ with $||y|| \leq 1$ one has

$$\langle y, \gamma(\bar{t}) - \gamma(t)\rangle = \int_t^{\bar{t}} \langle y, \gamma'(r)\rangle\ \mathrm{d}r \leq \int_t^{\bar{t}} ||\gamma'(r)||\ \mathrm{d}r$$

and thus

$$||\gamma(\bar{t}) - \gamma(t)|| \leq \int_t^{\bar{t}} ||\gamma'(r)|| \, \mathrm{d}r.$$

Since t and $\bar{t}$ were arbitrary, we conclude that

$$\int_a^b s_\gamma'(r) \, \mathrm{d}r = l(\gamma) \leq \int_a^b ||\gamma'(r)|| \, \mathrm{d}r. \tag{A.10}$$

(A.8), (A.9) and (A.10) together imply the claim. □

A.3 Banach space valued Sobolev maps

Throughout this section let $\Omega \subset \mathbb{R}^n$ be open, V be a Banach space, Y be a separable Banach space and $p \in [1, \infty)$.

A.3.1 The Reshetnyak–Sobolev space. The following definition of first-order Sobolev functions with values in Banach spaces goes back to [138].

Definition A.3.1. The *Reshetnyak–Sobolev space* $R^{1,p}(\Omega; V)$ consists of those functions $f \in L^p(\Omega; V)$ such that:

(i) for every $v^* \in V^*$ the function $x \mapsto \langle v^*, f(x)\rangle$ lies in the classical Sobolev space $W^{1,p}(\Omega) := W^{1,p}(\Omega; \mathbb{R})$;

(ii) there is a function $g \in L^p(\Omega)$ such that for every $v^* \in V^*$ one has

$$|\nabla\langle v^*, f(x)\rangle| \leq ||v^*|| \cdot g(x) \quad \text{for a.e. } x \in \Omega.$$

A function g as in (ii) will be called a *weak upper gradient* of f. A seminorm is defined on $R^{1,p}(\Omega; V)$ by

$$||f||_{R^{1,p}} := \left(\int_\Omega ||f(x)||^p \, \mathrm{d}x\right)^{1/p} + \inf_g ||g||_{L^p}$$

where g ranges over all weak upper gradients of f.

Indeed, Definition A.3.1 is a variation on the original definition by Reshetnyak. The reason for the present choice of definition is that, in contrast to the definition in [138], it also allows for unbounded domains Ω. This extension is possible because we limit ourselves here to maps with values in Banach spaces while Reshetnyak considers general metric target spaces. In any case the two definitions are equivalent if Ω is a bounded domain, see [76, Lemma 2.16] and [138, Theorem 5.1].

To prove that $R^{1,p}(\Omega; Y^*)$ equals $W_*^{1,p}(\Omega; Y^*)$, we will work with the following auxiliary definition that interpolates between the two spaces.

Definition A.3.2. The Sobolev space $R_*^{1,p}(\Omega; V^*)$ consists of those functions $f \in L^p(\Omega; V^*)$ such that:

(i*) for every $v \in V$ the function $x \mapsto \langle v, f(x) \rangle$ lies in $W^{1,p}(\Omega)$;

(ii*) there is a function $g \in L^p(\Omega)$ such that for every $v \in V$ one has

$$|\nabla \langle v, f(x) \rangle| \leq ||v|| \cdot g(x) \quad \text{for a.e. } x \in \Omega.$$

A function g as in (ii*) will be called a *weak* upper gradient* of f. A seminorm is defined on $R^{1,p}_*(\Omega; V^*)$ by

$$||f||_{R^{1,p}_*} := \left(\int_\Omega ||f(x)||^p \, \mathrm{d}x \right)^{1/p} + \inf_g \ ||g||_{L^p}$$

where g ranges over all weak* upper gradients of f.

We will denote by ACL(Ω) the collection of all functions $f : \Omega \to \mathbb{R}$ for which the restriction of f to almost every compact line segment, that is contained in Ω and parallel to some coordinate axis, is absolutely continuous. Recall that every real valued Sobolev function in $f \in W^{1,p}(\Omega)$ has a representative $\widetilde{f} \in \mathrm{ACL}(\Omega)$. The following lemma shows that similar is true for functions in $R^{1,p}_*(\Omega; Y^*)$.

Lemma A.3.3. *Let V be a Banach space and $f \in R^{1,p}_*(\Omega; V^*)$. Then for every $j \in \{1, \ldots, n\}$ the function f has a representative $\widetilde{f}^j$ that is absolutely continuous on almost every compact line segment which is contained in Ω and parallel to the x_j-axis. Moreover, for every weak* upper gradient g of f one has*

$$\lim_{h \to 0} \frac{\|\tilde{f}^j(x + he_j) - \tilde{f}^j(x)\|}{|h|} \leq g(x) \quad \text{for a.e. } x \in \Omega. \tag{A.11}$$

Lemma A.3.3 generalizes Lemma 2.13 in [76] from $R^{1,p}$ to $R^{1,p}_*$. A posteriori Proposition A.3.4 will show that this is not a proper generalization.

Proof. Fix $j \in \{1, \ldots, n\}$ and a weak* upper gradient g of f. Since we have $f \in L^p(\Omega; V^*)$, there is a nullset $\Sigma_0 \subset \Omega$ such that $f(\Omega \setminus \Sigma_0)$ is separable. Let $(v_i^*)_{i \in \mathbb{N}}$ be a dense sequence in the difference set $f(\Omega \setminus \Sigma_0) - f(\Omega \setminus \Sigma_0)$. For each $i \in \mathbb{N}$ let $(v_{ik})_{k \in \mathbb{N}}$ be a sequence of unit vectors in V such that

$$\|v_i^*\| = \lim_{k \to \infty} \langle v_{ik}, v_i^* \rangle.$$

Then for every $i, k \in \mathbb{N}$ one has $\langle v_{ik}, f \rangle \in W^{1,p}(\Omega)$ and

$$|\nabla \langle v_{ik}, f(x) \rangle| \leq g(x) \quad \text{for a.e. } x \in \Omega. \tag{A.12}$$

Denote by f_{ik} a representative of $\langle v_{ik}, f \rangle$ that is in ACL(Ω) and by Σ_{ik} the null set on which f_{ik} differs from $\langle v_{ik}, f \rangle$. Then for almost every line segment $l : [a, b] \to \Omega$ that is parallel to the x_j-axis one has:

(i) g is integrable over l;

(ii) $\mathcal{H}^1(l \cap \Sigma) = 0$ where $\Sigma = \Sigma_0 \cup \bigcup_{i,k} \Sigma_{ik}$;

(iii) for every $i, k \in \mathbb{N}$ and every $a \leq s \leq t \leq b$

$$|f_{ik}(l(t)) - f_{ik}(l(s))| \leq \int_s^t g(l(\tau)) \, \mathrm{d}\tau.$$

The Fubini theorem ensures (i) and (ii), while (iii) follows by (A.12).

Let $l \colon [a, b] \to \Omega$ be a line segment parallel to the x_j-axis for which the properties (i), (ii) and (iii) are satisfied. For given $s, t \in l^{-1}(\Omega \setminus \Sigma)$ with $s \leq t$ there is a subsequence $(v^*_{i_m})$ that converges to $f(l(t)) - f(l(s))$ in V^*. Thus, we have

$$\begin{aligned}
\text{(A.13)} \qquad & \|f(l(t)) - f(l(s))\| \\
&= \lim_{m\to\infty} \|v^*_{i_m}\| = \lim_{m\to\infty} \lim_{k\to\infty} \langle v_{i_m k}, v^*_{i_m} \rangle \\
&= \limsup_{m\to\infty} \limsup_{k\to\infty} \big(\langle v_{i_m k}, v^*_{i_m} - (f(l(t)) - f(l(s))) \rangle \\
&\qquad\qquad + \langle v_{i_m k}, f(l(t)) - f(l(s)) \rangle \big) \\
&\leq \limsup_{m\to\infty} \limsup_{k\to\infty} \big(\|v^*_{i_m} - (f(l(t)) - f(l(s)))\| \\
&\qquad\qquad + |f_{i_m k}(l(t)) - f_{i_m k}(l(s))| \big) \\
&\leq \int_s^t g(l(\tau)) \, \mathrm{d}\tau.
\end{aligned}$$

In particular, by properties (i) and (ii), and inequality (A.13) the restriction of f to l has a unique $\mathcal{H}^1$-representative that is absolutely continuous. The uniqueness implies that these representatives coincide where different line segments overlap. Hence we conclude that f has a representative $\widetilde{f}^j$ that is absolutely continuous on every compact line segment l that satisfies the properties (i), (ii) and (iii). Furthermore, by (A.13) for every such l one has

$$\|\widetilde{f}^j(l(t)) - \widetilde{f}^j(l(s))\| \leq \int_s^t g(l(\tau)) \, \mathrm{d}\tau$$

and hence we conclude that (A.11) is satisfied. □

Given that in general $W^{1,p}_*(\Omega; V^*)$ does not equal $W^{1,p}(\Omega; V^*)$ the following proposition might be a bit surprising.

Proposition A.3.4. *Let V be a Banach space. Then*

$$R^{1,p}_*(\Omega; V^*) = R^{1,p}(\Omega; V^*)$$

with $|| \cdot ||_{R^{1,p}_*} \leq || \cdot ||_{R^{1,p}} \leq \sqrt{n} || \cdot ||_{R^{1,p}_*}$.

Proof. Trivially $R^{1,p}(\Omega; V^*) \subseteq R^{1,p}_*(\Omega; V^*)$, and $||f||_{R^{1,p}_*} \leq ||f||_{R^{1,p}}$ for functions $f \in R^{1,p}(\Omega; V^*)$. For the other inclusion let $f \in R^{1,p}_*(\Omega; V^*)$ and g be a weak* upper gradient of f. Since $f \in L^p(\Omega; V^*)$, for $v^{**} \in V^{**}$ the function $f_{v^{**}} = \langle v^{**}, f \rangle$ lies in $L^p(\Omega)$. For $j \in \{1, \ldots, n\}$ let $\widetilde{f}^j$ be a representative of f as in Lemma A.3.3. Then $\widetilde{f}^j_{v^{**}} := \langle v^{**}, \widetilde{f}^j \rangle$ is a representative of $f_{v^{**}}$ that is absolutely continuous on almost every compact line segment parallel to the x_j-axis. Thus $\widetilde{f}^j_{v^{**}}$ is almost everywhere partial differentiable in the x_j-direction. By the product rule and the Fubini theorem it follows that

$$\int_\Omega \frac{\partial \varphi}{\partial x_j}(x) \cdot f_{v^{**}}(x)\ \mathrm{d}x = \int_\Omega \frac{\partial \varphi}{\partial x_j}(x) \cdot \widetilde{f}^j_{v^{**}}(x)\ \mathrm{d}x = \int_\Omega \varphi(x) \cdot \frac{\partial \widetilde{f}^j_{v^{**}}}{\partial x_j}(x)\ \mathrm{d}x$$

for every $\varphi \in C_0^\infty(\Omega)$. In particular $\frac{\partial \widetilde{f}^j_{v^{**}}}{\partial x_j}$ is a j-th weak partial derivative of $f_{v^{**}}$. Furthermore, by Lemma A.3.3 at almost every $x \in \Omega$ one has

$$\left| \frac{\partial \widetilde{f}^j_{v^{**}}}{\partial x_j}(x) \right| \leq \lim_{h \to 0} \frac{\|\tilde{f}^j(x + he_j) - \tilde{f}^j(x)\|}{|h|} \leq g(x)$$

and hence

$$|\nabla f_{v^{**}}(x)| = \left(\sum_{j=1}^n \left(\frac{\partial \widetilde{f}^j_{v^{**}}}{\partial x_j}(x) \right)^2 \right)^{1/2} \leq \sqrt{n} \cdot g(x).$$

Since $v^{**} \in V^{**}$ and the weak* upper gradient $g \in L^p(\Omega)$ were arbitrary, we conclude that $f \in R^{1,p}(\Omega; V^*)$ and

$$\|f\|_{R^{1,p}} \leq \sqrt{n} \cdot \|f\|_{R^{1,p}_*}.$$

This completes the proof. □

A.3.2 The Sobolev space $W^{1,p}_*$. Let $f \in W^{1,p}_*(\Omega; V^*)$. We will denote by $\partial_j f$ the function f_j as in Definition A.1.1 and call the vector

$$\nabla f(x) = (\partial_1 f(x), \dots, \partial_n f(x))$$

the *weak* gradient* of f at $x \in \Omega$. Further we define

$$|\nabla f(x)| := \left(\sum_{i=0}^n ||\partial_i f(x)||^2 \right)^{1/2}$$

and a seminorm on $W^{1,p}_*(\Omega; V)$ by

$$||f||_{W^{1,p}_*} := \left(\int_\Omega ||f(x)||^p\ \mathrm{d}x \right)^{1/p} + \left(\int_\Omega |\nabla f(x)|^p\ \mathrm{d}x \right)^{1/p}$$

Proposition A.3.5. *Let Y be a separable Banach space. Then*

$$W^{1,p}_*(\Omega; Y^*) = R^{1,p}_*(\Omega; Y^*)$$

with $|| \cdot ||_{R^{1,p}_*} \leq || \cdot ||_{W^{1,p}_*} \leq \sqrt{n} || \cdot ||_{R^{1,p}_*}$.

Proof. Let $f \in W^{1,p}_*(\Omega; Y^*)$. Since $f \in L^p(\Omega; Y^*)$ we know that for $y \in Y$ the function $f_y := \langle y, f \rangle$ lies in $L^p(\Omega)$. Further, by definition of the Gelfand integral, for $j \in \{1, \dots, n\}$ the function $\langle y, \partial_j f\rangle$ is a j-th weak partial derivative of f_j. Hence $f_y \in R^{1,p}_*(\Omega; Y^*)$ and

$$|\nabla f_y(x)| = \left(\sum_{j=1}^{n} \langle y, \partial_j f(x)\rangle^2\right)^{1/2} \leq \left(\sum_{j=1}^{n} ||\partial_j f(x)||^2\right)^{1/2} = |\nabla f(x)|$$

for a.e. $x \in \Omega$. In particular $f \in R^{1,p}_*(\Omega; Y^*)$ and $|\nabla f|$ is a weak* upper gradient of f. The latter also implies $||f||_{R^{1,p}_*} \leq ||f||_{W^{1,p}_*}$.

Now, for the other inclusion, let $f \in R^{1,p}_*(\Omega; Y^*)$ and g be a weak* upper gradient of f. For $j \in \{1, \dots, n\}$ let $\widetilde{f^j}$ be a representative of f as in Lemma A.3.3. Define $f_j(x)$ as the weak* partial derivative $\frac{\partial \widetilde{f^j}}{\partial x_j}(x)$, which is defined almost everywhere due to Lemma A.2.4. Then the function $f_j : \Omega \to Y^*$ is weak* measurable. Furthermore, by Lemma A.2.5 and the Fubini theorem, for every $\varphi \in C_0^\infty(\Omega)$ one has

$$\int_\Omega \frac{\partial \varphi}{\partial x_j}(x) \cdot f(x)\, \mathrm{d}x = \int_\Omega \frac{\partial \varphi}{\partial x_j}(x) \cdot \widetilde{f^j}(x)\, \mathrm{d}x = \int_\Omega \varphi(x) \cdot f_j(x)\, \mathrm{d}x \tag{A.14}$$

in the sense of Gelfand integrals. Also, by Lemmas A.2.7 and A.3.3,

$$\|f_j(x)\| \leq g(x) \quad \text{for a.e. } x \in \Omega.$$

In particular, since $g \in L^p(\Omega)$, we conclude that $f_j \in L^p_*(\Omega; Y^*)$ and hence by (A.14) that $f \in W^{1,p}(\Omega; Y^*)$ with

$$|\nabla f(x)| = \left(\sum_{j=1}^{n} \|f_j(x)\|^2\right)^{1/2} \leq \sqrt{n} \cdot g(x)$$

for almost every $x \in \Omega$. Since g was an arbitrary weak* upper gradient of f it also follows that $||f||_{W^{1,p}_*} \leq \sqrt{n}||f||_{R^{1,p}_*}$. □

Propositions A.3.4 and A.3.5 together imply the following quantitative version of Theorem A.1.4.

Theorem A.3.6. *Let Y be a separable Banach space. Then*

$$W^{1,p}_*(\Omega; Y^*) = R^{1,p}(\Omega; Y^*)$$

with $\frac{1}{\sqrt{n}}|| \cdot ||_{R^{1,p}} \leq || \cdot ||_{W^{1,p}_*} \leq \sqrt{n}|| \cdot ||_{R^{1,p}}$.

It has been shown in [23, 52] that $W^{1,p}(\Omega; V) = R^{1,p}(\Omega; V)$ if and only if V has the Radon–Nikodm property. Concerning Theorem A.3.6, it seems to be a natural conjecture that conversely the equality $W^{1,p}_*(\Omega; V^*) = R^{1,p}(\Omega; V^*)$ implies that V is separable.

A.4 Metric space valued Sobolev maps

Throughout this section let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $X = (X, d)$ be a complete metric space and $p \in [1, \infty)$.

A.4.1 The Sobolev space $W^{1,p}_*(\Omega; X)$. The *Reshetnyak–Sobolev space* can be defined as

$$R^{1,p}(\Omega; X) := \{f: \Omega \to X \mid \iota \circ f \in R^{1,p}(\Omega; V)\}$$

where $\iota: X \to V$ is any fixed isometric embedding of X into a Banach space V. By the following example such embedding ι always exists.

Example A.4.1. Let X be a metric space. Denote by $\ell^\infty(X)$ the Banach space of bounded functions $f: X \to \mathbb{R}$ with norm given by

$$||f||_\infty := \sup_{z \in X} |f(z)|.$$

Then for given $z_0 \in X$ the function $\bar{\kappa}: X \to \ell^\infty(X)$ given by

$$(\bar{\kappa}(z))(w) := d(z, w) - d(w, z_0)$$

defines an isometric embedding, see e.g. [78, p. 5].

Furthermore, under the present assumption that Ω is bounded, the definition of $R^{1,p}(\Omega; X)$ does not depend on the chosen embedding ι and is equivalent to the original definition by Reshetnyak, see [76, Lemma 2.16] and [138, Theorem 5.1]. Thus Theorem A.1.4 has the following consequence.

Theorem A.4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, X be a complete metric space, Y be a separable Banach space and $\iota: X \to Y^*$ be an isometric embedding. Then*

$$R^{1,p}(\Omega; X) = \left\{f: \Omega \to X \mid \iota \circ f \in W^{1,p}_*(\Omega; Y^*)\right\}.$$

Certainly not every metric space X isometrically embeds into the dual of a separable Banach space. A simple obstruction is the cardinality of X which must be bounded above by 2^{2^ω}. For a separable metric space X however, due to the following example, there is always an isometric embedding as in Theorem A.4.2.

Example A.4.3. Let X be a separable metric space and $(z_i)_{i \in \mathbb{N}}$ be a dense sequence of points in X. Denote $\ell^\infty := \ell^\infty(\mathbb{N})$. Then ℓ^∞ is the dual of the separable Banach space $\ell^1 := \ell^1(\mathbb{N})$. The function $\kappa: X \to \ell^\infty$ given by

$$\kappa(z) := (d(z, z_i) - d(z_i, z_1))_{i \in \mathbb{N}}$$

is called the *Kuratowski embedding* of X. It is not hard to check that κ defines an isometric embedding, see e.g. [78, p. 11].

Thus, for a bounded domain Ω and a complete separable metric space X, one can define

$$W^{1,p}_*(\Omega; X) := \left\{f: \Omega \to X \mid \kappa \circ f \in W^{1,p}_*(\Omega; \ell^\infty)\right\} \tag{A.15}$$

and deduce from Theorem A.4.2 that $W^{1,p}_*(\Omega; X) = R^{1,p}(\Omega; X)$. The assumption that Ω is bounded is needed to ensure that $W^{1,p}_*(\Omega; X)$ is well-defined by means of (A.15) and does not depend on the concrete choice of Kuratowski embedding. For a non-separable complete metric space X one can define $W^{1,p}_*(\Omega; X)$ as the union of the spaces $W^{1,p}_*(\Omega; S)$ where S ranges over all separable closed subsets of X. Since Sobolev functions are essentially separably valued, also for such non-separable X, Theorem A.4.2 implies that

$$W^{1,p}_*(\Omega; X) \subset R^{1,p}(\Omega; X)$$

and that every function in $R^{1,p}(\Omega; X)$ has a representative in $W^{1,p}_*(\Omega; X)$.

A.4.2 The Sobolev space $W^{1,p}(\Omega, X)$. The aim of this subsection is to prove Theorem A.1.2. To this end let X be a complete separable metric space, $(x_i)_{i\in\mathbb{N}}$ be a dense sequence of points in X and $\kappa\colon X \to \ell^\infty$ be the corresponding Kuratowski embedding. The key step for the proof is the following lemma.

Lemma A.4.4. *If $\gamma\colon [a,b] \to \kappa(X) \subset \ell^\infty$ is a non-constant absolutely continuous curve then the weak* derivative $\gamma'\colon [a,b] \to \ell^\infty$ is not essentially separably valued.*

Proof. Since γ is non-constant we have $l := l(\gamma) > 0$. As in Section A.2.3 we factorize $\gamma = \bar\gamma \circ s_\gamma$ where $\bar\gamma\colon [0,l] \to \kappa(X)$ is the unit-speed parametrization of γ and $s_\gamma\colon [a,b] \to [0,l]$ is the length function of γ. First we show that $\bar\gamma'\colon [0,l] \to \ell^\infty$ is not essentially separably valued.

By Lemma A.2.7 for a.e. $t \in [0,l]$ one has that

$$||\bar\gamma'(t)||_\infty = \lim_{h\to 0} \frac{||\bar\gamma(t+h) - \bar\gamma(t)||_\infty}{|h|} = 1. \tag{A.16}$$

Let E be the set of points $t_0 \in (0,l)$ at which $\bar\gamma$ is weak* differentiable and (A.16) holds. By Theorem A.2.1, to show that $\bar\gamma'$ is not essentially separably valued, it suffices to prove that $\bar\gamma'$ is not approximately continuous at every $t_0 \in E$.

So fix $t_0 \in E$ and let $h_0 > 0$ be so small that for any $h \in \mathbb{R}$ with $|h| \leq h_0$ one has

$$\frac{1}{2} \cdot |h| < ||\gamma(t_0+h) - \gamma(t_0)||_\infty. \tag{A.17}$$

Further fix some arbitrary $0 < h < h_0$ and accordingly choose $i \in \mathbb{N}$ such that

$$||\kappa(x_i) - \gamma(t_0)||_\infty \leq \frac{1}{4} \cdot h. \tag{A.18}$$

By Lemma A.2.4 for every point $t \in [0,l]$ at which $\bar\gamma$ is weak* differentiable one has

$$\bar\gamma'(t) = (\bar\gamma_i'(t))_{i\in\mathbb{N}} \quad \text{where } \bar\gamma(t) = (\bar\gamma_i(t))_{i\in\mathbb{N}}$$

is the coordinate representation of $\bar\gamma$. From the fundamental theorem of calculus, the definition of the Kuratowski embedding, (A.17) and (A.18) it follows that

$$\begin{aligned}\int_{t_0}^{t_0+h} \bar{\gamma}_i'(t)\ \mathrm{d}t &= \bar{\gamma}_i(t_0+h) - \bar{\gamma}_i(t_0) \\ &= ||\bar{\gamma}(t_0+h) - \kappa(x_i)||_\infty - ||\bar{\gamma}(t_0) - \kappa(x_i)||_\infty \\ &\geq \frac{1}{4} \cdot h\end{aligned}$$

Since $|\bar{\gamma}_i'(t)| \leq 1$ for a.e. t, this implies that

$$\mathcal{L}^1\left(F_h^+\right) \geq \frac{1}{8} \cdot h \quad \text{where } F_h^+ := \left\{t \in (t_0, t_0+h)\ :\ \bar{\gamma}_i'(t) \geq \frac{1}{8}\right\}. \tag{A.19}$$

Similarly

$$\int_{t_0-h}^{t_0} \bar{\gamma}_i'(t)\ \mathrm{d}t \leq -\frac{1}{4} \cdot h$$

and hence

$$\mathcal{L}^1\left(F_h^-\right) \geq \frac{1}{8} \cdot h \quad \text{where } F_h^- := \left\{t \in (t_0-h, t_0)\ :\ \bar{\gamma}_i'(t) \leq -\frac{1}{8}\right\}. \tag{A.20}$$

Note that for every $t^+ \in F_h^+ \cap E$ and $t^- \in F_h^- \cap E$ one has

$$||\bar{\gamma}'(t^+) - \bar{\gamma}'(t^-)||_\infty \geq |\bar{\gamma}_i'(t^+) - \bar{\gamma}_i'(t^-)| \geq \frac{1}{4}. \tag{A.21}$$

Since $0 < h < h_0$ was arbitrary, (A.19), (A.20) and (A.21) together imply that $\bar{\gamma}'$ cannot be approximately continuous at t_0. In turn, because $t_0 \in E$ was arbitrary, we conclude from Theorem A.2.1 that $\bar{\gamma}'$ is not essentially separably valued.

Now let $N \subset [a,b]$ be an arbitrary nullset. We need to show that $\gamma'([a,b] \setminus N)$ is not separable. By Lemma A.2.4, after possibly passing to a larger null set, we may assume that for every $t \in [a,b] \setminus N$ the curve γ is weak* differentiable at t and the function s_γ is differentiable at t. Note that

$$\mathcal{L}^1(s_\gamma(A)) = \int_A s_\gamma'(t)\ \mathrm{d}t \tag{A.22}$$

for every measurable subset $A \subset [a,b]$. Thus, we may further assume that for every $t \in [a,b] \setminus N$ either $\bar{\gamma}$ is weak* differentiable at $s_\gamma(t)$ or $s_\gamma'(t) = 0$. In particular, it follows that

$$(\bar{\gamma}' \circ s_\gamma)(t) \cdot s_\gamma'(t) = \gamma'(t) \tag{A.23}$$

on $[a,b] \setminus N$. By (A.22) one has that $M := s_\gamma(N) \cup s_\gamma(\{s_\gamma' = 0\})$ is a null set and hence $\bar{\gamma}'([0,l] \setminus M)$ is not separable. On the other hand, s_γ is surjective and hence by (A.23) it follows that

$$\bar{\gamma}'([0,l] \setminus M) \subset \langle \gamma'([a,b] \setminus N) \rangle_{\mathbb{R}}$$

where $\langle \gamma'([a,b] \setminus N) \rangle_{\mathbb{R}}$ denotes the linear span of $\gamma'([a,b] \setminus N)$ in ℓ^∞. In particular, the linear span of $\gamma'([a,b] \setminus N)$ is not separable and hence also $\gamma'([a,b] \setminus N)$ itself cannot be separable □

Proof of Theorem A.1.2. Let $f \in W^{1,p}(\Omega; X)$. Then, by definition, $h := \kappa \circ f$ lies in $W^{1,p}(\Omega; \ell^\infty)$. Trivially this implies that $h \in W_*^{1,p}(\Omega; \ell^\infty)$ and that $\partial_j h$ lies in $L^p(\Omega; X) \subset L_*^p(\Omega; X)$ for each j. Since $W_*^{1,p}(\Omega; \ell^\infty)$ equals $R_*^{1,p}(\Omega; \ell^\infty)$, Lemma A.3.3 implies that for each j the function h has a representative $\widetilde{h}^j$ that is absolutely continuous on almost every compact line segment parallel to the x_j-axis. In particular, there is a nullset $N \subset \Omega$ such that $\partial_j h(\Omega \setminus N)$ is separable for every j. Note that, since X is complete, for almost every compact line segment $l \colon [a, b] \to \Omega$ parallel to the x_j-axis the image $\widetilde{h}^j \circ l([a, b])$ must be contained in $\kappa(X)$. Further the proof of Proposition A.3.5 shows that, possibly enlarging N, we can assume that for each j one has

$$\partial_j h(x) = \frac{\partial \widetilde{h}^j}{\partial x_j}(x)$$

for every $x \in \Omega \setminus N$.

Assume f was not almost everywhere constant. Since Ω is connected, this implies that there is some j such that not for almost every line segment parallel to the x_j-axis the restriction of f to the line segment is constant. Hence we can find a line segment $l \colon [a, b] \to \Omega$ such that

(i) $\mathcal{H}^1(l([a, b] \cap N)) = 0$,

(ii) $\widetilde{h}^j \circ l([a, b]) \subset \kappa(X)$, and

(iii) $\widetilde{h}^j \circ l$ is absolutely continuous and non-constant.

By Lemma A.4.4, $(\widetilde{h}^j \circ l)' \colon [a, b] \to X$ cannot be essentially separably valued. This gives a contradiction because

$$(\widetilde{h}^j \circ l)'(t) = \partial_j h(l(t))$$

for every $t \in l([a, b]) \setminus N$ and $\partial_j h(\Omega \setminus N)$ is separable. □

CHAPTER B

Majorization by hemispheres and quadratic isoperimetric constants

This chapter was published in 'Transactions of the AMS' and is therefore left out here as to avoid copyright infringement. Bibliographical information can be found under [B].

CHAPTER C

Rigidity of the Pu inequality and quadratic isoperimetric constants of normed spaces

This chapter has been accepted for publication in 'Revista Matemática Iberoamericana' and is therefore left out here as to avoid copyright infringement. Bibliographical information can be found under [C].

CHAPTER D

Plateau's problem for singular curves

This chapter has been accepted for publication in 'Communications in Analysis and Geometry' and is therefore left out here as to avoid copyright infringement. Bibliographical information can be found under [D].

CHAPTER E

The branch set of minimal disks in metric spaces

with Matthew Romney

This chapter has been accepted for publication in 'International Mathematics Research Notices' and is therefore left out here as to avoid copyright infringement. Bibliographical information can be found under [E].

CHAPTER F

Space of minimal disks and its compactification

This chapter was published in 'Geometriae Dedicata' and is therefore left out here as to avoid copyright infringement. Bibliographical information can be found under [F].

CHAPTER G

Triangulating metric surfaces

with Matthew Romney

This chapter has been accepted for publication in 'Proceedings of the LMS' and is therefore left out here as to avoid copyright infringement. Bibliographical information can be found under [G].

CHAPTER H

Maximal metric surfaces and the Sobolev-to-Lipschitz property

with Elefterios Soultanis

This chapter was published in 'Calculus of Variations and PDE' and is therefore left out here as to avoid copyright infringement. Bibliographical information can be found under [H].

CHAPTER I

The Plateau–Douglas problem for singular configurations and in general metric spaces

with Martin Fitzi

I.1 Introduction and statement of main results

I.1.1 Introduction. The classical Plateau problem asked whether any given rectifiable Jordan curve Γ in $\mathbb{R}^n$ bounds a Sobolev disk of least area. The positive answer was obtained independently by Douglas and Radó in the early 1930's, [130, 45]. Over the years their result was generalized from $\mathbb{R}^n$ to so-called homogeneously regular Riemannian manifolds, metric spaces satisfying different synthetic notions of curvature bounds and particular classes of homogeneously regular Finsler manifolds, [117, 118, 90, 116, 125, 128]. The solution of Plateau's problem in proper metric spaces given by Lytchak–Wenger in [104] covers all these settings. However, even in $\mathbb{R}^n$, the arguments break down if Γ is allowed to self-intersect. Still the generality of [104] and a simple extension trick allowed the first author to solve the Plateau problem for possibly self-intersecting curves in proper metric spaces which satisfy a local quadratic isoperimetric inequality, see Chapter D. In $\mathbb{R}^n$ this improved a previous existence result due to Hass, [77].

The Plateau–Douglas problem is a variation of the Plateau problem, where one allows for various boundary components and surfaces of nontrivial topology. One way to state the solution obtained by Douglas in [49] is the following: assume you are given a finite configuration of disjoint rectifiable Jordan curves Γ in $\mathbb{R}^n$ and a natural number $p \geq 0$. Then there exists an area minimizer among all compact surfaces which have genus at most p and span Γ. Douglas' result has since been extended by Jost to homogeneously regular Riemannian manifolds (closing also a gap in the original proof of Douglas), and recently even further by the second author together with Stefan Wenger to proper metric spaces admitting a local quadratic isoperimetric inequality, [89, 56]. Again, the machinery fails if one allows for singular, possibly non-disjoint or self-intersecting configurations. Our main result, Theorem I.1.2 below, solves the Plateau–Douglas problem for such possibly singular configurations and in general proper metric spaces. The solution for singular configurations is new even in $\mathbb{R}^n$. Theorem I.1.2 also generalizes the main results of [56] and Chapter D as we are able to drop the assumption that X admits a local quadratic isoperimetric inequality. In particular, existence is new for regular configurations in complete Riemannian manifolds which might not be homogeneously regular. It is not surprising that existence in this case is harder to obtain, since already for such a setting discontinuous solutions can only be excluded under additional geometric assumptions, cf. [117].

Note that the somewhat more modern approach to Plateau's problem via currents as in [54, 11] does not allow for bounding the topology of solutions, and for singular configurations currents would consider the boundary curves rather as unparametrized objects and could not keep track of the order in which they are traversed, in contrast to our approach. Moreover, beyond the Riemannian setting, there is no appropriate regularity theory available.

I.1.2 Main result. Simple examples show that, without additional assumptions, one cannot hope for reasonably regular area minimizers of prescribed topological type to bound a given contour Γ. For example, a Jordan curve in $\mathbb{R}^n$ which is convex and contained in a plane does not span a minimal surface of genus $p > 0$, see [112]. There are two ways to handle this issue. As in [49, 89] we will state our result in terms of the so-called Douglas condition. It is however not hard to see that that this formulation, which we discuss below, is equivalent to the one via (possibly disconnected) surfaces of bounded topology promoted in Section I.1.1, cf. [56].

For the convenience of a reader who might not be familiar with the theory of metric space valued Sobolev maps, we first state our main result in the smooth context before moving to the more general setting. To this end, let X be a smooth complete Riemannian manifold and M be a smooth, orientable, compact surface (which might be disconnected). Assume furthermore that all connected components of M have non-empty boundary. For a map u in the Sobolev space $W^{1,2}(M, X)$ we denote by $\text{Area}(u)$ the parametrized Riemannian area of u.

Assume now that M has $k \geq 1$ boundary components ∂M_i and Γ is a collection of k rectifiable closed curves Γ_j in X. By a *rectifiable closed curve* we mean an equivalence class of parametrized rectifiable curves $\gamma \colon S^1 \to X$. We identify two such parametrized curves if they are reparametrizations of each other, meaning more precisely that their constant speed parametrizations agree up to a homeomorphism of S^1. We say that a map $u \in W^{1,2}(M, X)$ *spans* Γ if for each curve Γ_j there exists a boundary component ∂M_i such that the trace $u|_{\partial M_i}$ is a parametrization of Γ_j. Let $\Lambda(M, \Gamma, X)$ be the family of Sobolev maps $u \in W^{1,2}(M, X)$ which span Γ. We define

$$a(M, \Gamma, X) := \inf\{\text{Area}(u) \colon u \in \Lambda(M, \Gamma, X)\}$$

and $a_p(\Gamma, X) := a(M, \Gamma, X)$ if M is the (up to a diffeomorphism) unique connected surface of genus p with k boundary components. We say that the *Douglas* *condition* holds for p, Γ and X if $a_p(\Gamma, X)$ is finite and

$$a_p(\Gamma, X) < a(M, \Gamma, X) \tag{I.1}$$

for every M as in the previous paragraph and of one of the following types. Either M is connected and of genus strictly smaller than p, or M is disconnected and of total genus at most p. Note that in the case where Γ is a single curve and $p = 0$, which corresponds to the classical Plateau problem, the Douglas condition is equivalent to the assumption that there is at least one Sobolev disk spanning Γ.

Theorem I.1.1. *Let X be a smooth complete Riemannian manifold and $\Gamma \subset X$ be a configuration of $k \geq 1$ rectifiable closed curves. Let M be a compact, connected and orientable surface with k boundary components and of genus $p \geq 0$. If the Douglas condition holds for p, Γ and X, then there exists $u \in \Lambda(M, \Gamma, X)$ as well as a Riemannian metric g on M such that*

$$\text{Area}(u) = a_p(\Gamma, X)$$

and u is weakly conformal with respect to g on $M \setminus u^{-1}(\Gamma)$. Furthermore, if...

(i) ...X is homogeneously regular, then u may be chosen Hölder continuous on M and smooth on $M \setminus u^{-1}(\Gamma)$.

(ii) ...X is homogeneously regular and Γ is C^2, then u may be chosen locally Lipschitz on $M \setminus \partial M$.

(iii) ...Γ is a union of disjoint Jordan curves, then u and g may be chosen such that u is weakly conformal with respect to g on M.

Here, by weakly conformal we mean that almost everywhere the weak differential of u either vanishes or is angle preserving. Already the most simple example of a figure eight curve in $\mathbb{R}^2$ shows that self-intersecting curves need not always bound globally weakly conformal area minimizing disks, cf. [77]. So the assumption of (iii) seems quite sharp. Note that the existence of globally Hölder continuous area minimizers guaranteed by (i) is new already for topologically regular configurations in $\mathbb{R}^n$ which potentially are of low analytic regularity. Compare the respective discussion for the Plateau problem in Chapter D. Without geometric assumptions one cannot hope for the conclusion of (i) to be true. See [117, p. 809] for a complete Riemannian manifold X and a Jordan curve $\Gamma \subset X$ which only bounds discontinuous area minimizers. Parts (i) and (ii), respectively (ii) and (iii), are compatible in the sense that when both respective assumptions are satisfied then one can achieve the conclusion simultaneously for a single map u, compare Remark I.4.4. However, if both the assumptions in (i) and (iii) hold, we can only cook up a single area minimizer which is simultaneously weakly conformal and globally Hölder continuous in the previously known case where all the curves of Γ satisfy a chord-arc condition.

We sketch the main ideas entering in the proof of Theorem I.1.2. For (i), the procedure is conceptually similar to the respective disk type result obtained in

Chapter D. Namely, we attach a cylinder to each of the curves in Γ. This way we obtain a singular metric space X_Γ, which admits a local quadratic isoperimetric inequality and contains X isometrically, as well as a regular configuration $\tilde{\Gamma}$ in X_Γ. Now we apply [56] to solve the Plateau–Douglas problem for the new pair $(X_\Gamma, \tilde{\Gamma})$ and project the obtained solution down to X. This gives the desired solution for (X, Γ). For (ii), the proof follows essentially the same lines. However, the construction is now performed in a way that is more sensitive to the concrete geometric situation. The construction scheme, which is a generalization of the funnel extensions introduced by Stadler in [148], allows us to obtain an extension space $\hat{X}_\Gamma$ which admits a local quadratic isoperimetric inequality *and* is locally of curvature bounded above in the sense of Alexandrov. This latter feature allows to apply the regularity theory for harmonic maps into spaces of curvature bounded above as developed e.g. in [95, 143, 19], and hence derive the desired Lipschitz regularity. For the special case (iii), we use ε-thickenings as introduced in [158] to approximate X by metric spaces $(X_n)_{n\in\mathbb{N}}$ which admit local quadratic isoperimetric inequalities and contain X isometrically. Then we apply again [56] to obtain solutions $(u_n)_{n\in\mathbb{N}}$ for the pairs (X_n, Γ) respectively. A variant of the Rellich-Kondrachov compactness theorem allows us to pass to a limit surface in X which is our desired solution. The proof of the remaining general case involves a mix of the arguments discussed for (i) and (iii).

At this point, we would like to emphasize the following remarkable feature of Theorem I.1.1 and its proof: despite major additional complications that arise, the results and methods developed in [56] for the Plateau–Douglas problem in metric spaces are in principle adaptations of respective ones developed for the classical Plateau–Douglas problem in smooth ambient spaces. However, the flexibility of the metric setting therein allows us to draw new conclusions in the smooth setting that seem out of reach within the classical methods.

A theory of metric space valued Sobolev maps has been developed over the last 30 years. With this language at hand, one can generalize all the introduced terminology to the setting where X is a complete metric space, see Sections I.2 and I.3 below. Recall that a metric space X is called *proper* if all closed and bounded subsets of X are compact. In fact, Theorem I.1.1 is a special case of the following very general result.

Theorem I.1.2. *Let X be a proper metric space and $\Gamma \subset X$ be a configuration of $k \geq 1$ rectifiable closed curves. Let M be a compact, connected and orientable surface with k boundary components and of genus $p \geq 0$. If the Douglas condition holds for p, Γ and X, then there exists $u \in \Lambda(M, \Gamma, X)$ as well as a Riemannian metric g on M such that*

$$\operatorname{Area}(u) = a_p(\Gamma, X)$$

and u is infinitesimally isotropic with respect to g on $M \setminus u^{-1}(\Gamma)$. Furthermore, if...

(i) ...X admits a local quadratic isoperimetric inequality, then u may be chosen Hölder continuous on M and to satisfy Lusin's property (N).

(ii) *...X is geodesic, admits a local quadratic isoperimetric inequality and is locally of curvature bounded above, and* Γ *is of finite total curvature, then* u *may be chosen locally Lipschitz on* $M \setminus \partial M$*.*

(iii) *...*Γ *is a union of disjoint Jordan curves, then* u *and* g *may be chosen such that* u *is infinitesimally isotropic with respect to* g *on* M*.*

The respective assumptions and conclusions in Theorem I.1.2 are natural metric generalizations of the respective smooth ones in Theorem I.1.1. For example homogeneously regular Riemannian manifolds admit a local quadratic isoperimetric inequality. In fact, the huge class of metric spaces admitting a local quadratic isoperimetric inequality includes also homogeneously regular Finsler manifolds, CAT(κ) spaces, compact Alexandrov spaces as well as more exotic examples such as higher dimensional Heisenberg groups, cf. [104]. In particular, the assumption on X in Theorem I.1.2.(ii) is satisfied if X is a CAT(κ) space.

We would also like to remark that, despite the fact that we exclusively restrict our discussion to the parametrized Hausdorff area (see Definition I.2.3), an appropriate variant of Theorem I.1.2 holds for any area functional which induces quasi-convex 2-volume densities in the sense of [105, 8] such as the Holmes–Thompson area functional. In order to obtain the respective results, only minor modifications in the proof of the theorem are needed.

I.1.3 Conditions of cohesion and adhesion. As discussed above, in general one cannot hope for a given configuration Γ of disjoint Jordan curves to bound a minimal surface of prescribed topological type if the Douglas condition for p, Γ and X fails. However, there are still situations where the Douglas condition fails but one can show the existence of such a desired surface. Namely, if the area infimum may be approximated by a sequence of surfaces which satisfies a geometric nondegeneracy condition, called *condition of cohesion*. In increasingly more general settings this has been shown to hold true in [31, 145, 151, 56]. Additional difficulties arise if one allows for singular configurations Γ. Imposing an additional so-called *condition of adhesion*, Iseri was able to show a statement of similar spirit for singular configurations in $\mathbb{R}^n$, [86]. In Section I.6 we generalize the definition of adhesion and Iseri's result to the setting of metric spaces. For regular configurations in sufficiently nice ambient spaces, the Douglas condition implies the condition of cohesion for any sequence of surfaces approaching the energy infimum. Note however that nothing similar is true for singular configurations and the condition of adhesion. Hence these results can only be applied to obtain existence for very particular configurations, cf. [86].

I.1.4 Organization. After recalling some basic notions in Section I.2, we discuss the proof of Theorem I.1.2.(i) in Section I.3, where we first recall some terminology and the main result of [56] in Subsection I.3.1 before giving the actual proof of (i) in Subsection I.3.2. Moving forward, we discuss a generalization of the Cartan–Hadamard theorem due to Bowditch and a gluing result due to Stadler in Subsection I.4.1, and the proof of Theorem I.1.2.(ii) is performed in

Subsection I.4.2. Section I.5 is then dedicated to the proofs of Theorems I.1.2 and I.1.1 in the general case. In Subsection I.5.1, we first discuss how general proper metric spaces X can be approximated by more regular spaces admitting local quadratic isoperimetric inequalities and when one can pass from a sequence of fillings within the approximating spaces to a limit filling in X. Then in Subsection I.5.2, we recall two devices from [56] that allow, in spaces admitting a local quadratic isoperimetric inequality, to lower the topological type of an area minimizing sequence whenever this sequence degenerates. These devices are combined in Section I.5.3 with the approximating spaces discussed before. The proof of Theorem I.1.2 is then completed in Section I.5.4. In Section I.5.5 we briefly discuss how Theorem I.1.1 follows from Theorem I.1.2. Finally in Section I.6, we discuss the method using minimizing sequences satisfying conditions of cohesion and adhesion.

I.2 Preliminaries

I.2.1 Basic notation. We write $|v|$ for the Euclidean norm of a vector $v \in \mathbb{R}^2$,

$$D := \{z \in \mathbb{R}^2 \colon |z| < 1\}$$

for the open unit disk in $\mathbb{R}^2$ and $\bar{D}$ for its closure. The differential at z of a (weakly) differentiable map φ between smooth manifolds is denoted $D\varphi_z$.

For a subset $A \subset \mathbb{R}^2$, $|A|$ denotes its Lebesgue measure. If (X, d) is a metric space then we use the notation $\mathcal{H}^2_X(A)$ for the 2-dimensional Hausdorff measure of a subset $A \subset X$. The normalizing constant is chosen such that $\mathcal{H}^2_X$ coincides with the 2–dimensional Lebesgue measure when X is Euclidean $\mathbb{R}^2$. Thus, the Hausdorff 2–measure $\mathcal{H}^2_g := \mathcal{H}^2_{(M,g)}$ on a 2–dimensional Riemannian manifold (M, g) coincides with the Riemannian area.

I.2.2 Seminorms. The *(Reshetnyak) energy* of a seminorm s on $\mathbb{R}^2$ is defined by

$$\mathbf{I}^2_+(s) := \max\{s(v)^2 \colon v \in \mathbb{R}^2, |v| = 1\}.$$

If s is a norm on $\mathbb{R}^2$, then the *Jacobian* of s is defined as the unique number $\mathbf{J}(s)$ satisfying

$$\mathcal{H}^2_{(\mathbb{R}^2,s)}(A) = \mathbf{J}(s) \cdot |A|$$

for some and thus every subset $A \subset \mathbb{R}^2$ such that $|A| > 0$. For a degenerate seminorm s we set $\mathbf{J}(s) := 0$. A seminorm s on $\mathbb{R}^2$ is *isotropic* if $s = 0$ or if it is a norm and the ellipse of maximal area contained in $\{v \in \mathbb{R}^2 \colon s(v) \leq 1\}$ is a Euclidean ball. If s is a Euclidean seminorm, i.e. if s is induced by a (potentially degenerate) inner product, then s is isotropic precisely if it is a scalar multiple of the standard Euclidean norm $|\cdot|$.

If s is a seminorm on a 2-dimensional Euclidean vector space V then we define the concepts of Jacobian, energy, and isotropy by identifying V with Euclidean $(\mathbb{R}^2, |\cdot|)$ via a linear isometry.

I.2.3 Metric space valued Sobolev maps. Let (X, d) be a proper metric space and let M be a smooth, compact, orientable 2–dimensional manifold, possibly disconnected and with non-empty boundary. We fix a Riemannian metric g on M and let $\Omega \subset M$ be an open set.

Definition I.2.1. A measurable $u\colon \Omega \to X$ belongs to the Sobolev space $W^{1,2}(\Omega, X)$ if there exists $h \in L^2(\Omega)$ with the following property. For every real-valued 1–Lipschitz function f on X the composition $f \circ u$ belongs to the classical Sobolev space $H^{1,2}(\Omega \setminus \partial M)$ and

$$|D(f \circ u)_z|_g \leq h(z)$$

for almost every $z \in \Omega$.

If $u \in W^{1,2}(\Omega, X)$ then for almost every $z \in \Omega$ there exists a seminorm $\operatorname{ap\,md} u_z$ on T_zM, called *approximate metric derivative*, such that

$$\operatorname{ap}\lim_{v \to 0} \frac{d(u(\exp_z(v)), u(z)) - \operatorname{ap\,md} u_z(v)}{|v|_g} = 0,$$

where the approximate limit is taken within T_zM and $\exp_z$ denotes the exponential map of g at z. See [51] for the definition of approximate limits.

Assume $N = (N, h)$ is a smooth complete Riemannian manifolds. Then, by Nash's theorem, there is an isometric embedding $\iota\colon N \to \mathbb{R}^m$ (in the Riemannian sense). Equivalently one may define $W^{1,2}(\Omega, N)$ as the set of measurable mappings $u\colon \Omega \to N$ such that $\iota \circ u$ lies in the classical Sobolev space $H^{1,2}(\Omega \setminus \partial M, \mathbb{R}^m)$; compare e.g. Lemma 9.3.3 and Exercise 2 in Section 9 of [91]. In particular, for every Sobolev map $u \in W^{1,2}(\Omega, N)$ there is a measurable weak differential $Du\colon T\Omega \to TN \subset N \times \mathbb{R}^m$. At almost every $z \in \Omega$ the approximate metric derivative is given by

$$\operatorname{ap\,md} u_z(v) = |Du_z(v)|_h \quad \text{for all } v \in T_z\Omega, \tag{I.2}$$

compare Theorem 6.4 and the subsequent remark in [51].

The approximate metric derivative allows one to define the Reshetnyak energy and the parametrized Hausdorff area of a Sobolev map using the pointwise quantities introduced in Section I.2.2 above.

Definition I.2.2. The (Reshetnyak) energy of $u \in W^{1,2}(\Omega, X)$ with respect to g is defined by

$$E_+^2(u, g) := \int_\Omega \mathbf{I}_+^2(\operatorname{ap\,md} u_z)\, d\mathcal{H}_g^2(z).$$

The energy E_+^2 is conformally invariant in the sense that

$$E_+^2(u \circ \varphi, g') = E_+^2(u, g)$$

whenever $\varphi\colon (M', g') \to (M, g)$ is a conformal diffeomorphism.

Definition I.2.3. The parametrized (Hausdorff) area of $u \in W^{1,2}(\Omega, X)$ is defined by

$$\mathrm{Area}(u) := \int_\Omega \mathbf{J}(\mathrm{ap}\,\mathrm{md}\, u_z)\; d\mathcal{H}^2_g(z).$$

If $A \subset \Omega$ is measurable, then the area of the restriction $u|_A$ is defined analogously.

It is easy to see that

$$\mathrm{Area}(u \circ \varphi) = \mathrm{Area}(u)$$

for any biLipschitz homeomorphism $\varphi\colon \Omega' \to \Omega$. In particular, $\mathrm{Area}(u)$ is independent of the choice of the Riemannian metric g. A measurable map $u\colon \Omega \to X$ satisfies *Lusin's property (N)* if $\mathcal{H}^2_X(u(A)) = 0$ for every null set $A \subset \Omega$. If $u \in W^{1,2}(\Omega, X)$, then by the area formula

$$\mathrm{Area}(u) \leq \int_X \#\{z \in \Omega\colon u(z) = x\}\; d\mathcal{H}^2_X(x),$$

with equality if u satisfies Lusin's property (N); see [93].

Definition I.2.4. A map $u \in W^{1,2}(M, X)$ is infinitesimally isotropic with respect to the metric g on a measurable subset $A \subset M$ if for almost every $z \in A$ the approximate metric derivative $\mathrm{ap}\,\mathrm{md}\, u_z$ is isotropic with respect to $g(z)$. If no subset $A \subset M$ is specified, it is understood that u is infinitesimally isotropic with respect to g on M.

It is not hard to see that

$$\mathrm{Area}(u) \leq E^2_+(u, g),$$

where equality holds precisely if u is infinitesimally isotropic and the approximate metric derivative of u at almost every $z \in M$ is a Euclidean seminorm, compare [105].

If $\Omega \subset M \setminus \partial M$ is a Lipschitz domain, then for every $u \in W^{1,2}(\Omega, X)$ there is a well defined *trace* $\mathrm{tr}(u) \in L^2(\partial\Omega, X)$. If u extends to a continuous map $\bar{u}$ on $\bar{\Omega}$, then the trace is simply given by $\bar{u}|_{\partial\Omega}$. Hence, in abuse of notation, we also denote the trace of u by $u|_{\partial\Omega}$. If no continuous extension exists, define $\mathrm{tr}(u)$ locally around $p \in \partial\Omega$ in the following way. Choose an open neighborhood U of p and a biLipschitz map $\psi\colon (0,1) \times [0,1) \to M$ such that $\psi((0,1) \times (0,1)) = U \cap \Omega$ and $\psi((0,1) \times \{0\}) = U \cap \partial\Omega$. Then for almost every $s \in (0,1)$ the trace at $\psi(s,0)$ is given by $\lim_{t \searrow 0}(u \circ \psi)(s,t)$, compare [95].

I.3 Proof for regular metric spaces

I.3.1 The Plateau–Douglas problem for regular configurations. Let $\mathcal{M}(k)$ be the family of compact, orientable, smooth surfaces M with k boundary components and such that each connected component of M has non-empty boundary. Denote by $M_{k,p}$ the, up to a diffeomorphism, unique connected surface in $\mathcal{M}(k)$ of genus p. A *reduction* of $M_{k,p}$ is a surface $M^* \in \mathcal{M}(k)$ with one of the following properties. Either M^* is connected and has genus at most $p-1$ or M^* has several connected components and the total genus of M^* is at most p. Since the Euler characteristic of $M_{k,p}$ is given by

$$\chi(M_{k,p}) = 2 - 2p - k,$$

it follows that $\chi(M^*) > \chi(M_{k,p})$ for any reduction M^* of $M_{k,p}$, and hence $\chi(M^*) = k$ if and only if M^* is the union of k smooth disks. For $M \in \mathcal{M}(k)$ with $n > 1$ connected components, we say that M^* is a reduction of M if there exists a partition $M^* = M_1^* \cup \cdots \cup M_n^*$ such that each M_l^* is the reduction of exactly one connected component of M. Notice that for any $M \in \mathcal{M}(k)$ there are only finitely many reductions M^* up to diffeomorphism, and that any reduction M^{**} of such M^* is also a reduction of M.

Let $\Gamma = \bigcup \Gamma_j$ be a configuration of $k \geq 1$ rectifiable closed curves in a complete metric space X and $p \geq 0$. By defining

$$a_p^*(\Gamma, X) := \min\{a(M^*, \Gamma, X)\colon M^* \text{ is a reduction of } M_{k,p}\},$$

the Douglas condition (I.1) can be rewritten as

$$a_p(\Gamma, X) < a_p^*(\Gamma, X).$$

We would like to point out that the notion of reduction used here is broader than the one given in [56], where a reduction of the second type consists of *exactly* two connected components. Consequently, the Douglas condition used in [56] is à priori a weaker assumption than the respective one in this article, which turns out to be more convenient for us. However, the two conditions are in fact equivalent. This follows since $a_p(\Gamma, X) < \infty$ implies that all curves Γ_j lie in the same component of rectifiable connectedness of X, i.e. the curves can be joined pairwise by paths of finite length, and using this fact one can show that $a(M^*, \Gamma, X) \leq a(M^{**}, \Gamma, X)$ whenever M^{**} is a reduction of a reduction M^* of $M_{k,p}$.

The basis for our proof of Theorem I.1.2 in the special cases (i) and (ii) will be the existence results [56, Theorem 1.2] and [56, Theorem 1.4.(iii)] for Jordan curves, which we now state as a combined theorem for convenience of the reader.

Theorem I.3.1. *Let X be a proper metric space admitting a local quadratic isoperimetric inequality, $\Gamma \subset X$ be the disjoint union of $k \geq 1$ rectifiable Jordan curves and $p \geq 0$. If the Douglas condition* (I.1) *holds for p, Γ and X, then there exists a continuous $u \in \Lambda(M_{k,p}, \Gamma, X)$ and a Riemannian metric g on $M_{k,p}$ such that*

$$\operatorname{Area}(u) = a_p(\Gamma, X)$$

and u is infinitesimally isotropic with respect to g. Furthermore, if every Jordan curve in Γ is chord-arc, then any such u is Hölder continuous on $M_{k,p}$ and satisfies Lusin's property (N).

Here, a metric space X is said to admit a (C, ℓ_0)-*quadratic isoperimetric inequality* if every closed Lipschitz curve $c\colon S^1 \to X$ of length $\ell(c) \leq \ell_0$ is the trace of a Sobolev disk $u \in W^{1,2}(D, X)$ satisfying

$$\operatorname{Area}(u) \leq C \cdot \ell(c)^2.$$

If there is no need to specify the constants $C, \ell_0 > 0$, we simply say that X admits a *local quadratic isoperimetric inequality.* A Jordan curve Γ is called *chord-arc* if it is biLipschitz equivalent to S^1.

The following replacement lemma will be used in the proof of Lemma I.3.4. It follows from the proof of [106, Lemma 4.8] and the gluing result [95, Theorem 1.12.3]. While [106, Lemma 4.8] is stated for disk-type surfaces, the arguments in the proof thereof are local around the boundary curve and can be applied without changes to the present situation.

Lemma I.3.2. *Let X be a complete metric space admitting a local quadratic isoperimetric inequality, $\Gamma \subset X$ be a configuration of $k \geq 1$ rectifiable closed curves and $M \in \mathcal{M}(k)$. Then for every $u \in \Lambda(M, \Gamma, X)$ and $\varepsilon > 0$ there is $v \in \Lambda(M, \Gamma, X)$ such that*

$$\operatorname{Area}(v) \leq \operatorname{Area}(u) + \varepsilon$$

and the continuous representative of $\operatorname{tr}(v)|_{\partial M_i}$ *is a constant speed parametrization for each $i \in \{1, \ldots, k\}$.*

Lemma 3.2 is applied in the proofs of Propositions 5.1 and 6.1 in [56]. It is one of the implications in [56] making use of the assumption of a local quadratic isoperimetric inequality. In fact the only implications needing this assumption and used in the proof of the existence result therein may be phrased as Lemmas I.5.3 and I.5.4 below. While these lemmas seem to heavily rely on the assumption, it is an open question whether Lemma I.3.2, which enters in their proofs, holds true without it or not.

I.3.2 Proof of Theorem I.1.2.(i). Let X be a complete metric space and Γ be a configuration of $k \geq 1$ rectifiable closed curves Γ_j in X. Since the Douglas condition fails as soon as $k > 1$ and one of the curves Γ_j is constant, and since the minimization problem is trivial for a single constant curve Γ, we may assume without loss of generality that $\Gamma_1, \ldots, \Gamma_k$ are all nonconstant. For each j, let S_j be a geodesic circle of circumference $\ell(\Gamma_j)$, let $\gamma_j\colon S_j \to X$ be a unit speed parametrization of Γ_j and $Z_j := S_j \times [0, 1]$ be the cylinder equipped with the product metric. We define the quotient space X_Γ as the disjoint union $X \sqcup Z_1 \sqcup \cdots \sqcup Z_k$ under the identification $\gamma_j(p) \sim (p, 0)$ for every $p \in Z_j$, and we equip this space with the quotient metric, see for example [20]. Furthermore, let $P_\Gamma\colon X_\Gamma \to X$ be the projection given by

$$P_\Gamma(x) := \begin{cases} x & x \in X, \\ \gamma_j(p) & x = (p, t) \in Z_j. \end{cases}$$

The proof of Lemma 4.1 in [D] shows that $X \subset X_\Gamma$ isometrically and that $P_\Gamma \colon X_\Gamma \to X$ is a 1-Lipschitz retraction. Lastly, we define $\tilde\Gamma_j$ as the (equivalence class of the) rectifiable curve $p \mapsto (p,1) \in Z_j$, $p \in S_j$, and $\tilde\Gamma$ as the configuration consisting of the curves $\tilde\Gamma_1, \dots, \tilde\Gamma_k$. Then $\tilde\Gamma$ is a configuration of disjoint chord-arc curves and $P_\Gamma \circ \tilde\Gamma_j = \Gamma_j$ for each j.

Lemma I.3.3. *Let X be a complete metric space, $\Gamma \subset X$ be a configuration of $k \geq 1$ rectifiable closed curves and $M \in \mathcal{M}(k)$. Then for every $u \in \Lambda(M, \tilde\Gamma, X_\Gamma)$ one has $P_\Gamma \circ u \in \Lambda(M, \Gamma, X)$ and*

$$\operatorname{Area}(u) \geq \operatorname{Area}(P_\Gamma \circ u) + \sum_{j=1}^{k} \mathcal{H}^2(Z_j).$$

In particular, one has the inequality

$$a(M, \tilde\Gamma, X_\Gamma) \geq a(M, \Gamma, X) + \sum_{j=1}^{k} \mathcal{H}^2(Z_j).$$

Proof. Let $u \in \Lambda(M, \tilde\Gamma, X_\Gamma)$. Without loss of generality, we may assume that M is connected. By the 1-Lipschitz continuity of P_Γ, we have $P_\Gamma \circ u \in \Lambda(M, \Gamma, X)$. Since $P_\Gamma(Z_j)$ is contained in the rectifiable curve Γ_j, the area formula in Section I.2.3 implies that

$$\operatorname{Area}\left((P_\Gamma \circ u)|_{u^{-1}(Z_j)}\right) = 0.$$

Thus, since the restriction $P_\Gamma|_X$ is an isometry, we obtain

$$\begin{aligned} \operatorname{Area}(u) &= \operatorname{Area}(u|_{u^{-1}(X)}) + \sum_j \operatorname{Area}\left(u|_{u^{-1}(Z_j)}\right) \\ &= \operatorname{Area}(P_\Gamma \circ u) + \sum_j \operatorname{Area}\left(u|_{u^{-1}(Z_j)}\right). \end{aligned}$$

To complete the proof, it therefore suffices to show that

$$\operatorname{Area}\left(u|_{u^{-1}(Z_j)}\right) \geq \mathcal{H}^2(Z_j) \tag{I.3}$$

for each j. In order to see this, fix j and define Y_j as the quotient space X_Γ/A, where $A := X \cup \bigcup_{i \neq j} Z_i$. Then Y_j is isometric to $Z_j/(S_j \times \{0\})$. Hence Y_j is homeomorphic to $\bar D$ and, by Theorem 3.2 in [F], admits a local quadratic isoperimetric inequality. Furthermore, let $Q_j \colon X_\Gamma \to Y_j$ be the 1-Lipschitz map given by $Q_j(x) := [x]$. Then the composition $Q_j \circ u$ is an element in $\Lambda(M, Q_j \circ \tilde\Gamma, Y_j)$ with

$$\operatorname{Area}(Q_j \circ u) = \operatorname{Area}\left(u|_{u^{-1}(Z_j)}\right). \tag{I.4}$$

Let ∂M_i be the boundary component of M such that $\operatorname{tr}(u)|_{\partial M_i}$ is an element of Γ_j, and consider M embedded into a smooth compact surface $\tilde M \in \mathcal{M}(1)$ of same genus as that of M such that each boundary component ∂M_l bounds a topological disk in $\tilde{M}$ except for ∂M_i, which agrees with the boundary component of $\tilde{M}$. The map $Q_j \circ u$ extends naturally onto $\tilde{M}$ by setting its value on $\tilde{M} \setminus M$ to be $[x]$ for any $x \in X$, yielding a map $v_j \in \Lambda(\tilde{M}, Q_j \circ \tilde{\Gamma}_j, Y_j)$ satisfying

$$\text{Area}(v_j) = \text{Area}(Q_j \circ u). \tag{I.5}$$

Apparently, there exists a surface M^*, either being equal to $\tilde{M}$ or else being a reduction of it, such that

$$a(\tilde{M}, Q_j \circ \tilde{\Gamma}_j, Y_j) = a(M^*, Q_j \circ \tilde{\Gamma}_j, Y_j)$$

and the Douglas condition holds for $M^*, Q_j \circ \tilde{\Gamma}_j$ and Y_j. Hence by Theorem I.3.1 there exists a continuous map $w_j \in \Lambda(M^*, Q_j \circ \tilde{\Gamma}_j, Y_j)$ satisfying Lusin's property (N) and

$$\text{Area}(w_j) \leq \text{Area}(v_j). \tag{I.6}$$

Since Y_j is homeomorphic to $\bar{D}$ with boundary curve $Q_j \circ \tilde{\Gamma}_j$, it follows that w_j is surjective. Otherwise assume $p \in Y_j \setminus w_j(M^*)$. Then $Q_j \circ \tilde{\Gamma}_j$, considered as a 1-cycle, would be a generator of $H_1(Y_j \setminus \{p\}) \cong H_1(\bar{D} \setminus \{0\}) \cong \mathbb{Z}$ and at the same time would bound the 2-chain defined in $Y_j \setminus \{p\}$ by w_j, which is a clear contradiction. Hence, by the area formula, we have

$$\text{Area}(w_j) = \int_{Y_j} \# \left\{ w_j^{-1}(x) \right\} \, d\mathcal{H}^2(x) \geq \mathcal{H}^2(Y_j) = \mathcal{H}^2(Z_j). \tag{I.7}$$

Combining (I.4), (I.5), (I.6) and (I.7), we finally obtain (I.3). □

While we did not need to assume a local quadratic isoperimetric inequality on X in the previous lemma, this assumption is required in the proof of the upcoming reverse inequality.

Lemma I.3.4. *Let X be a complete metric space admitting a local quadratic isoperimetric inequality, $\Gamma \subset X$ be a configuration of $k \geq 1$ rectifiable closed curves and $M \in \mathcal{M}(k)$. Then one has*

$$a(M, \tilde{\Gamma}, X_\Gamma) \leq a(M, \Gamma, X) + \sum_{i=1}^{k} \mathcal{H}^2(Z_j).$$

Proof. Let $\varepsilon > 0$. By Lemma I.3.2 there exists $v \in \Lambda(M, \Gamma, X)$ such that

$$\text{Area}(v) \leq a(M, \Gamma, X) + \varepsilon$$

and such that $\text{tr}(v)|_{\partial M_i}$ is a constant speed parametrization for each i. We relabel the boundary components of M such that $\text{tr}(v)|_{\partial M_j}$ is an element of Γ_j for each j. Embed M diffeomorphically into a smooth compact surface $\tilde{M} \in \mathcal{M}(k)$ such that $\tilde{M} \setminus \text{int}(M)$ is the disjoint union of k smooth cylinders Ω_j with boundary, each Ω_j having ∂M_j as one boundary component. Notice that $\tilde{M}$ is diffeomorphic to M. Now if $\tilde{\gamma}_j \colon S_j \to X_\Gamma$ is a constant speed parametrization of $\tilde{\Gamma}_j$, then the inclusion $\iota_j \colon Z_j \to X_\Gamma$ is a Lipschitz homotopy between $\tilde{\gamma}_j$

and γ_j of area $\mathcal{H}^2(Z_j)$. Thus, by identifying Ω_j with Z_j via a biLipschitz homeomorphism, there exist maps $w_j \in W^{1,2}(\Omega_j, X_\Gamma)$ with trace $\tilde{\gamma}_j$ respectively $\gamma_j = \operatorname{tr}(v)|_{\partial M_j}$ and of area $\mathcal{H}^2(Z_j)$. Let $w\colon \tilde{M} \to X_\Gamma$ be the mapping obtained by stitching v together with every w_j along ∂M_j, which is a well-defined element in $W^{1,2}(\tilde{M}, X_\Gamma) = W^{1,2}(M, X_\Gamma)$ by [95, Thm. 1.12.3]. Then w spans $\tilde{\Gamma}$ and satisfies

$$a(M, \tilde{\Gamma}, X_\Gamma) \leq \operatorname{Area}(w) = \operatorname{Area}(v) + \sum_{j=1}^{k} \operatorname{Area}(w_j) \leq a(M, \Gamma, X) + \sum_{j=1}^{k} \mathcal{H}^2(Z_j) + \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrary, the assertion in the lemma follows and the proof is complete. □

With these preparations at hand, it is now not hard to give a proof of Theorem I.1.2.(i).

Proof of Theorem I.1.2.(i). Since X admits a local quadratic isoperimetric inequality, it follows from the proof of Theorem 3.2 in [F] that X_Γ admits a local quadratic isoperimetric inequality as well. Lemmas I.3.3 and I.3.4 imply that one has the equality

$$a(\tilde{M}, \tilde{\Gamma}, X_\Gamma) = a(\tilde{M}, \Gamma, X) + \sum_{j=1}^{k} \mathcal{H}^2(Z_j) \tag{I.8}$$

for every $\tilde{M} \in \mathcal{M}(k)$. Hence the Douglas condition

$$a_p(\tilde{\Gamma}, X_\Gamma) < a_p^*(\tilde{\Gamma}, X_\Gamma)$$

holds for p, $\tilde{\Gamma}$ and X_Γ. Since $\tilde{\Gamma}$ is a disjoint configuration of chord-arc curves, we have by Theorem I.3.1 that there is a Hölder continuous $v \in \Lambda(M, \tilde{\Gamma}, X_\Gamma)$ satisfying Lusin's property (N) and a Riemannian metric g on M such that

$$\operatorname{Area}(v) = a_p(\tilde{\Gamma}, X_\Gamma)$$

and v is infinitesimally isotropic with respect to g. By Lemma I.3.3 and equation (I.8) the projection $u := P_\Gamma \circ v \in \Lambda(M, \Gamma, X)$ then satisfies

$$\operatorname{Area}(u) = a_p(\Gamma, X).$$

Moreover, since P_Γ is isometric on X, the map u is infinitesimally isotropic with respect to g on $M \setminus u^{-1}(\Gamma) \subset M \setminus v^{-1}(X_\Gamma \setminus X)$. Thus the proof of (i) is complete. □

I.4 Interior Lipschitz regularity

I.4.1 Upper curvature bounds. Let X be a metric space. Closed piecewise geodesic curves in X will be denoted $\overline{x_0 x_1 \ldots x_m}$, where $x_i \in X$ indicate the endpoints of the geodesic segments. For $\kappa \in \mathbb{R}$, let D_κ be the diameter of the model space M^2_κ of constant curvature κ. That is, $D_\kappa = \pi/\sqrt{\kappa}$ for $\kappa > 0$ and $D_\kappa = \infty$ for $\kappa \leq 0$. A geodesic triangle $\overline{xyz}$ will be called *κ-admissible* if $\ell(\overline{xyz}) < 2D_\kappa$. For every κ-admissible triangle $\overline{xyz}$, there is a (up to isometry) unique comparison triangle $\overline{x_\kappa y_\kappa z_\kappa}$ in M^2_κ which has the same side lengths. A κ-admissible triangle $\overline{xyz}$ is called *CAT(κ)* if there is a 1-Lipschitz map $f\colon \overline{x_\kappa y_\kappa z_\kappa} \to \overline{xyz}$ such that $f(x_\kappa) = x$, $f(y_\kappa) = y$ and $f(z_\kappa) = z$. We say that X is a *CAT(κ) space* if X is geodesic and every κ-admissible triangle in X is CAT(κ), and call X *locally CAT(κ)* if every point in X has a neighbourhood which is a CAT(κ) space. Two standard facts are that CAT(κ) spaces are also CAT(κ') for any $\kappa' \geq \kappa$, and that balls of radius at most $D_\kappa/2$ in CAT(κ) spaces are themselves CAT(κ) spaces. Finally, we say that X is *locally of curvature bounded above* if every point $p \in X$ has a neighbourhood U_p which is a CAT(κ_p) space for some $\kappa_p \in \mathbb{R}$. By the preceeding observations, we may always assume that $\kappa_p > 0$ and U_p is a small ball.

If X is geodesic and locally CAT(0), then the Cartan–Hadamard theorem states that X is a CAT(0) space if and only if X is simply connected. Aiming to handle also spaces satisfying positive upper curvature bounds, we discuss a variant of this result due to Bowditch. For Lipschitz curves $\gamma_0, \gamma_1 \colon S^1 \to A \subset X$, we say that γ_0 is *monotonically homotopic to γ_1 in A* if there exists a continuous homotopy $h\colon [0,1] \times S^1 \to A$ such that $h(0,\cdot) = \gamma_0$, $h(1,\cdot) = \gamma_1$ and $\ell(h(t,\cdot)) \leq \ell(\gamma_0)$ for all $t \in [0,1]$. We say that γ is *monotonically nullhomotopic in A* if γ is monotonically homotopic to a constant curve in A. If X is a CAT(κ) space, then Reshetnyak's majorization theorem (see for example [1]) implies that every closed Lipschitz curve in X of length smaller than $2D_\kappa$ is monotonically nullhomotopic. Dually, the following holds by Theorem 3.1.2 in [18].

Theorem I.4.1. *Let X be a proper geodesic metric space, $\kappa \in \mathbb{R}$ and $A \subset X$ be compact such that the D_κ-neighbourhood of A is locally* CAT(κ)*. If a κ-admissible triangle $\Delta \subset A$ is monotonically nullhomotopic in A, then Δ is* CAT(κ)*.*

Theorem 3.1.2 in [18] is stated under the assumption that the entire space X is locally CAT(κ). However, as discussed in Section 3.6 of [18], the argument is local in the D_κ-neighbourhood of any set in which Δ is monotonically nullhomotopic, and hence the proof readily gives Theorem I.4.1. As a corollary of Theorem I.4.1, we obtain the following result allowing to derive quantitatively controlled "local globalizations".

Corollary I.4.2. *Let X be a proper geodesic metric space, $\kappa \in \mathbb{R}$ and $B(p,r)$ be a ball in X which is locally* CAT(κ)*. If every triangle $\Delta \subset \bar{B}(p, r/2)$ is monotonically nullhomotopic in $\bar{B}(p,r/2)$, then $\bar{B}(p,\bar{r})$ is a* CAT($\bar{\kappa}$) *space, where $\bar{\kappa} = \bar{\kappa}(\kappa, r)$ and $\bar{r} = \bar{r}(\kappa, r)$ only depend on κ and r.*

Proof. Set $\bar{\kappa} := \max\{\kappa, 4\pi^2 r^{-2}\}$ and $\bar{r} := D_{\bar{\kappa}}/4$. Note that $\bar{\kappa}$ is chosen such that $D_{\bar{\kappa}} \leq r/2$. To see that $\bar{B}(p,\bar{r})$ is convex, let $x, y \in \bar{B}(p, \bar{r})$ and observe that any geodesic triangle $\overline{pxy}$ is $\bar{\kappa}$-admissible and contained in $\bar{B}(p, 2\bar{r}) \subset \bar{B}(p, r/2)$, and hence by assumption monotonically nullhomotopic within $\bar{B}(p, r/2)$. Then

Theorem I.4.1 implies that $\overline{pxy}$ is CAT($\bar{\kappa}$). Since $\bar{r} < D_{\bar{\kappa}}/2$, it follows that $\overline{pxy} \subset \bar{B}(p, \bar{r})$, and we conclude that $\bar{B}(p, \bar{r})$ is convex. Now let $\overline{xyz} \subset \bar{B}(p, \bar{r})$. Then $\overline{xyz}$ is $\bar{\kappa}$-admissible and monotonically nullhomotopic in $\bar{B}(p, r/2)$. Again Theorem I.4.1 implies that $\overline{xyz}$ is CAT($\bar{\kappa}$). □

For $\alpha \geq 0$ and $r > 0$, we let $S_{\alpha,r}$ be the ball of radius r around the vertex in the cone over a compact interval of length α (see [21] for the definition of cones), and call $S_{\alpha,r}$ the *sector* of radius r and angle α. On any sector, we fix an orientation so that the left leg and the right leg of $S_{\alpha,r}$ are defined. The following lemma generalizes [148, Lemma 21] to spaces satisfying positive upper curvature bounds.

Lemma I.4.3. *Let $\kappa \geq 0$, $0 < r \leq D_\kappa/2$, X be a proper* CAT(κ) *space, $p \in X$ and $\eta_1, \dots, \eta_l, \nu_1, \dots, \nu_l \subset X$ be geodesic segments all of length r and starting at p. For $i = 1, \dots, l$, let $\alpha_i \in [0, \pi]$ be the angle at p between η_i and ν_i, and let S_i be the sector of angle $2\pi - \alpha_i$ and radius r. Then the space Z, obtained by gluing each sector S_i to X via isometric identifications of its left leg with η_i and its right leg with ν_i, is a* CAT(κ) *space.*

In the lemma, the isometric identifications are chosen such that p corresponds to the vertex point in S_i. In the following, we assume without further mentioning that the orientations of isometric identifcations are chosen in such a natural way.

Proof. By induction, it is sufficient to prove the statement for $l = 1$, and hence we set $\eta := \eta_1$, $\nu := \nu_1$ and $\alpha := \alpha_1$. Reshetnyak's gluing theorem (see for example [20]) implies that the space Y, obtained by gluing $S_{\pi-\alpha,r}$ to X via an isometric identification of the left leg of $S_{\pi-\alpha,r}$ and η, is a CAT(κ) space. Observe that the angle in Y between the right leg η' of $S_{\pi-\alpha,r}$ and ν equals π and that the length of the concatenation $\eta' \cup \nu$ is at most D_k. Hence the curve $\eta' \cup \nu$ is a geodesic in Y and in particular a convex subset of Y, see [20, Proposition 1.7]. Thus the claim follows from another application of Reshetnyak's theorem upon noting that Z may be constructed alternatively by gluing the sector $S_{\pi,r}$ to Y via isometric identifications of its left leg with η' and its right leg with ν. □

I.4.2 Proof of Theorem I.1.2.(ii). Let X be a metric space which is locally of curvature bounded above. The *total curvature* of a closed piecewise geodesic curve $\overline{x_0 x_1 \dots x_m}$ in X is defined by

$$\sigma(\overline{x_0 x_1 \dots x_m}) := \sum_{i=0}^{m} (\pi - \beta_i),$$

where β_i denotes the angle at x_i between the geodesic segments $\overline{x_i x_{i-1}}$ and $\overline{x_i x_{i+1}}$. Let L be a closed rectifiable curve. The curve $\overline{x_0 x_1 \dots x_m}$ is called *inscribed to* L if the points $x_0, x_1, \dots, x_m$ lie on L and are traversed by L in cyclic order. The *total curvature of* L, denoted $\sigma(L)$, may be defined as $\lim_{n\to\infty} \sigma(L_n)$, where (L_n) is a sequence of closed piecewise geodesic curves which are inscribed to L and converge uniformly to L, see [110, Proposition 2.4].

Proof of Theorem I.1.2.(ii). Let X be as in the statement of the theorem. Assume first $L = \overline{x_0 x_1 \dots x_m}$ is a closed piecewise geodesic curve in X. For $i = 0, \dots, m$, we set $S_i := S_{\pi-\beta_i,1}$ and $Q_i := I_i \times [0,1]$, where $I_i \subset \mathbb{R}$ is a compact interval of length $d(x_i, x_{i+1})$. We define a geodesic metric cylinder $\hat{Z}_L$ by gluing the left end interval of each Q_i isometrically to the right leg of S_i and the right end interval of each Q_i to the left leg of S_{i+1}. Then, by Reshetnyak's gluing theorem, balls of radius at most $\ell(L)/4$ in $\hat{Z}_L$ are CAT(0) spaces. Denote the inner boundary curve of $\hat{Z}_L$ by $\bar{L}$ and the outer boundary curve of $\hat{Z}_L$ by $\hat{L}$. There exist a 1-Lipschitz retraction $\hat{P}_L \colon \hat{Z}_L \to \bar{L}$ such that $\hat{P}_L \circ \hat{L} = \bar{L}$, as well as a $(\ell(L) + \sigma(L))$-Lipschitz homotopy $h_L \colon S^1 \times [0,1] \to \hat{Z}_L$ between $\bar{L}$ and $\hat{L}$ such that $\text{Area}(h) = \mathcal{H}^2(\hat{Z}_L)$. In particular, $\bar{L}$ is a geodesic circle of circumference $\ell(L)$ and there is a canonical unit-speed parametrization $c_L \colon \bar{L} \to L$. Now let L be any closed rectifiable curve of finite total curvature. All the properties discussed for piecewise geodesic curve are quantitative and hence stable under ultralimits; see e.g. [1] for the definition and properties of ultralimits. Thus we may approximate L by a sequence (L_n) of L-inscribed piecewise geodesic curves, perform the construction for each L_n, pass to an ultralimit and obtain that there exist $\hat{Z}_L$, $\hat{L}$, $\bar{L}$, c_L, h_L, $\hat{P}_L$ as above, all enjoying the very same properties.

Let $\hat{Z}_j := \hat{Z}_{\Gamma_j}$ for $j = 1, \dots, k$. We define the quotient space $\hat{X}_\Gamma$ as the disjoint union $X \sqcup \hat{Z}_1 \sqcup \dots \sqcup \hat{Z}_k$ under the identification $c_{\Gamma_j}(p) \sim p$ for $p \in \bar{\Gamma}_j$, and we equip this space with the quotient metric. Also, we let $\hat{P}_\Gamma \colon \hat{X}_\Gamma \to X$ be the 1-Lipschitz retraction given by $\hat{P}_\Gamma(x) := x$ for $x \in X$ and $\hat{P}_\Gamma(x) = \hat{P}_{\Gamma_j}(x)$ for $x \in \hat{Z}_j$. By Reshetnyak's majorization theorem each $\hat{Z}_j$ admits a local quadratic isoperimetric inequality. This, together with the facts that $\hat{P}_\Gamma$ is 1-Lipschitz and X admits a local quadratic isoperimetric inequality, makes it straightforward to modify the proof of Theorem 3.2 in [F] and derive that the space $\hat{X}_\Gamma$ admits a local quadratic isoperimetric inequality. Let $\hat{\Gamma}$ be the configuration formed by $\hat{\Gamma}_1, \dots, \hat{\Gamma}_k$. The properties discussed above allow us to imitate the proofs of Lemmas I.3.3 and I.3.4 for the configuration $\hat{\Gamma} \subset \hat{X}_\Gamma$, and hence derive that

$$a(\tilde{M}, \hat{\Gamma}, \hat{X}_\Gamma) = a(\tilde{M}, \Gamma, X) + \sum_{i=1}^{k} \mathcal{H}^2(\hat{Z}_i) \tag{I.9}$$

for every $\tilde{M} \in \mathcal{M}(k)$.

So far we have not achieved any advantage from our more complicated construction over the one in Section I.3.2. However, and this is the crucial difference, now we claim that $\hat{X}_\Gamma$ is locally of curvature bounded above. Since $\hat{X}_\Gamma \setminus X$ is locally CAT(0), it suffices to show that every $p \in X$ has a CAT neighbourhood within $\hat{X}_\Gamma$. So let p in X and choose $\kappa > 0$ as well as $0 < r < D_\kappa/2$ such that $B_X(p,r)$ is a CAT(κ) space. The proof that X is locally of curvature bounded above will be completed by showing that $\bar{B}_{\hat{X}_\Gamma}(p, \bar{r})$ is a CAT($\bar{\kappa}$) space, where $\bar{\kappa}$ and $\bar{r}$ are as in the statement of Corollary I.4.2. Since $\bar{\kappa}$ and $\bar{r}$ are independent of Γ and the CAT($\bar{\kappa}$) condition is stable under ultralimits, we lose no generality in assuming that $\Gamma_1, \dots, \Gamma_k$ are piecewise geodesic curves. Thus it remains to verify the assumptions of Corollary I.4.2. Clearly, $B_{\hat{X}_\Gamma}(p, r) \setminus \Gamma$ is locally CAT(κ). Since we assumed Γ consists of piecewise geodesic curves, for

$q \in B_{\hat{X}_\Gamma}(p, r) \cap \Gamma$ and $s > 0$ sufficiently small the ball $\bar{B}_{\hat{X}_\Gamma}(q, s)$ is obtained from $\bar{B}_X(q, s)$ as the space Z is obtained from X in Lemma I.4.3. Thus the lemma states that $\bar{B}_{\hat{X}_\Gamma}(q, s)$ is a CAT(κ) space and hence we conclude that $B_{\hat{X}_\Gamma}(p, r)$ is locally CAT(κ). To verify the other assumption of Corollary I.4.2, let $\Delta \subset \bar{B}_{\hat{X}_\Gamma}(p, r/2)$ be a geodesic triangle. Sliding Δ down to X we see that Δ is monotonically homotopic in $\bar{B}_{\hat{X}_\Gamma}(p, r/2)$ to a curve $\eta \subset X$. Since $\bar{B}_X(p, r/2)$ is a CAT(κ) space and $\ell(\eta) < 2D_\kappa$, Reshetnyak's majorization theorem implies in turn that η is monotonically nullhomotopic in $\bar{B}_X(p, r/2)$. Hence we may apply Corollary I.4.2 and conclude the claim.

Departing from (I.9) and the fact that $\hat{X}_\Gamma$ admits a local quadratic isoperimetric inequality, we can proceed as we did when proving (i) in the last section. The advantage is now that by [143], see also [19, Theorem 1.3], the minimizer $v \in \Lambda(M, \hat{X}_\Gamma, \hat{\Gamma})$ is locally Lipschitz on $M \setminus \partial M$, and hence so is our final solution $u = \hat{P}_\Gamma \circ v$. In order to apply these regularity results, note that v is a continuous harmonic map into a space which is locally of curvature bounded above. Harmonicity of v follows since v is infinitesimally isotropic and $\hat{X}_\Gamma$ is locally of curvature bounded from above and hence has property (ET), see [104, Section 11]. □

Remark I.4.4. The map u we produce in the proof of Theorem I.1.2.(ii) is also globally Hölder continuous on M. This follows as in the proof of Theorem I.1.2.(i) upon noting that the configuration $\hat{\Gamma}$ we construct consists of chord-arc curves.

I.5 General case

Throughout this section, we use the terminology introduced in the beginning of Section I.3.

I.5.1 Approximating sequences. Let X be a complete metric space. We call a metric space Y an *ε-thickening of* X if Y contains X isometrically and X is ε-dense in Y. We will need the following variant of the thickening results obtained in [158] and [109].

Lemma I.5.1. *There is a universal constant $C \geq 0$ such that for every proper metric space X and $\varepsilon > 0$, there exists a $(C\varepsilon)$-thickening Y of X such that Y is proper and admits a (C, ε)-quadratic isoperimetric inequality.*

If X is geodesic, then Lemma I.5.1 follows readily from [109, Lemma 3.3] and in this case, the space Y may also be chosen geodesic. This version suffices to obtain Theorem I.1.2 in the special case that X is geodesic, and hence in particular to obtain Theorem I.1.1. Thus for the convenience of a reader who is only interested in Theorem I.1.2 for geodesic target spaces, the general proof of Lemma I.5.1 is postponed to the appendix.

Let X be a proper metric space and $(Y_n)_{n\in\mathbb{N}}$ be a sequence of proper ε_n-thickenings of X. We call (Y_n) an *X-approximating sequence* if $\varepsilon_n \to 0$. The following consequence of the generalized Rellich-Kondrachov compactness theorem, [95, Theorem 1.13], allows to pass from a sequence of maps in approximating spaces to a limit map in X.

Proposition I.5.2. *Let X be a proper space and Γ be a configuration of $k \geq 1$ disjoint rectifiable Jordan curves in X. Let $M \in \mathcal{M}(k)$ be connected and be endowed with a Riemannian metric g. Assume there exist an X-approximating sequence $(Y_n)_{n\in\mathbb{N}}$ and mappings $u_n \in \Lambda(M, \Gamma, Y_n)$ of uniformly bounded energies $E_+^2(u_n, g)$ and such that the traces $\operatorname{tr}(u_n)\colon \partial M \to \Gamma$ are equicontinuous with respect to g. Then there is $u \in \Lambda(M, \Gamma, X)$ such that*

$$\text{(I.10)} \qquad \operatorname{Area}(u) \leq \limsup_{n\to\infty} \operatorname{Area}(u_n) \quad \& \quad E_+^2(u, g) \leq \limsup_{n\to\infty} E_+^2(u_n, g).$$

The proof is the following standard argument, which is similar to respective steps e.g. in the proofs of [69, Theorem 1.5] and [109, Theorem 5.1].

Proof. Let Z be the proper metric space obtained by gluing all the spaces Y_n along X. Note that $Y_n \subset Z$ isometrically and hence $\Lambda(M, \Gamma, Y_n) \subset \Lambda(M, \Gamma, Z)$ for each $n \in \mathbb{N}$. For fixed $p \in \Gamma$, [56, Lemma 2.4] implies that there is a constant C such that

$$\int_M d^2(p, u_n(z))\, d\mathcal{H}_g^2(z) \leq C \cdot \left(\operatorname{diam}(\Gamma)^2 + E_+^2(u_n, g)\right)$$

for all $n \in \mathbb{N}$. In particular,

$$\sup_{n\in\mathbb{N}} \left[\int_M d^2(p, u_n(z))\, d\mathcal{H}_g^2(z) + E_+^2(u_n, g)\right] < \infty.$$

Thus by the metric space version of the Rellich-Kondrachov compactness theorem, [95, Theorem 1.13], there is $v \in W^{1,2}(M, Z)$ such that $v_j \to v$ in $L^2(M, Z)$. In fact, since $(Y_n)_{n\in\mathbb{N}}$ is an approximating sequence, we may assume that v takes values in $X \subset Z$ and hence $v \in W^{1,2}(M, X)$. By lower semicontinuity of area and energy, see e.g. [104], the inequalities (I.10) are satisfied for u. Finally, the Arzelà-Ascoli theorem and [95, Theorem 1.12.2] imply that $v \in \Lambda(M, \Gamma, X)$. □

I.5.2 Reductions of fillings. Let X be a complete metric space, $p \geq 0$ and $\Gamma \subset X$ be a configuration of $k \geq 1$ disjoint rectifiable Jordan curves Γ_j. The two following results are needed for the proof of Lemma I.5.6 and can be extracted from the proofs of [56, Proposition 6.1] and [56, Proposition 5.1] respectively. For the first lemma, we assume that $k + p > 2$, which is equivalent to the assumption that the surface $M_{k,p}$ is neither of disk- nor of cylindrical type. In this case $M_{k,p}$ may be endowed with a *hyperbolic* metric, which we define to be a Riemannian metric g of constant sectional curvature -1 and such that the boundary $\partial M_{k,p}$ is geodesic with respect to g. By a *relative geodesic* in $(M_{k,p}, g)$ we mean either a simple closed geodesic in $M_{k,p}$ or a geodesic arc with endpoints on $\partial M_{k,p}$ that is non-contractible via a homotopy of curves of the same type. We define $\operatorname{sys}_{\mathrm{rel}}(M_{k,p}, g)$ as the infimal length of relative geodesics in $(M_{k,p}, g)$.

Furthermore, we choose for each $\rho > 0$ a parameter $\rho'_\Gamma = \rho'_\Gamma(\rho)$ as in the first paragraph in the proof of [56, Proposition 6.1]. That is, for each $\rho > 0$ we choose $0 < \rho'_\Gamma < \rho$ such that whenever two points $x, x' \in \Gamma$ satisfy $d_X(x, x') \leq \rho'_\Gamma$, then they lie on the same Jordan curve Γ_j and the shorter segment of Γ_j between x and x' has length at most ρ. The notation emphasizes that ρ'_Γ only depends on the induced metric on $\Gamma \subset X$.

Lemma I.5.3. *Let $C, K, \rho > 0$. Assume X admits a $(C, 2\rho)$-quadratic isoperimetric inequality and g is a hyperbolic metric on $M_{k,p}$ such that*

$$\mathrm{sys}_{\mathrm{rel}}(M_{k,p}, g) < \min\left\{ \frac{\rho'^2_\Gamma(\rho)}{4K}, \operatorname{arsinh}\left(\frac{1}{\sinh(2)}\right)\right\}.$$

Then for every $u \in \Lambda(M_{k,p}, \Gamma, X)$ with $E^2_+(u, g) \leq K$, there exist a reduction M^* of $M_{k,p}$ and a map $u^* \in \Lambda(M^*, \Gamma, Y)$ such that

$$\mathrm{Area}(u^*) \leq \mathrm{Area}(u) + 8C\rho^2.$$

An analogue of the above lemma holds for cylindrical $M_{k,p}$ endowed with a *flat* metric, which we define as a Riemannian metric with vanishing sectional curvature and such that the Riemannian area of $(M_{k,p}, g)$ is equal to 1 and the boundary $\partial M_{k,p}$ geodesic. The analogue follows by using a basic flat collar (instead of a hyperbolic one) in the proof of [56, Proposition 6.1]. Compare also the respective remark in the proof of [56, Theorem 1.2].

For the second lemma, we assume that $k+p \geq 2$, hence we only exclude that $M_{k,p}$ is of disk-type. Let g be a Riemannian metric on $M_{k,p}$ and $0 < \delta_g < 1$ be so small that every point $z_0 \in \partial M_{k,p}$ has a neighbourhood in $(M_{k,p}, g)$ which is the image of the set

$$B := \{z \in \mathbb{C} \colon |z| \leq 1 \text{ and } |z - 1| < \sqrt{\delta_g}\}$$

under a 2-biLipschitz diffeomorphism ψ with $z_0 = \psi(1)$.

Lemma I.5.4. *Let $C, K, \rho > 0$. Assume that X admits a $(C, 2\rho)$-quadratic isoperimetric inequality and $0 < \delta \leq \delta_g$ is so small that*

$$\pi \cdot \left(\frac{8K}{|\log(\delta)|}\right)^{\frac{1}{2}} < \rho'_\Gamma(\rho).$$

If there exist $u \in \Lambda(M_{k,p}, \Gamma, Y)$ with $E^2_+(u, g) \leq K$ and a subarc $\gamma^- \subset \partial M_{k,p}$ satisfying

$$\ell_g(\gamma^-) \leq \delta \quad \& \quad \ell_X(\mathrm{tr}(u) \circ \gamma^-) > \rho,$$

then there exist a reduction M^* of $M_{k,p}$ and a map $u^* \in \Lambda(M^*, \Gamma, X)$ such that

$$\mathrm{Area}(u^*) \leq \mathrm{Area}(u) + 8C\rho^2.$$

I.5.3 Reductions of approximating sequences. Let X be a proper metric space, Γ be a configuration of $k \geq 1$ disjoint rectifiable Jordan curves in X and $p \geq 0$. The next proposition is going to be important in the proof of Theorem I.1.2.

Proposition I.5.5. *Let* (Y_n) *be an* X*-approximating sequence. If there exist maps* $u_n \in \Lambda(M_{k,p}, \Gamma, Y_n)$ *satisfying*

$$a := \limsup_{n\to\infty} \operatorname{Area}(u_n) < a_p^*(\Gamma, X),$$

then there exists $u \in \Lambda(M_{k,p}, \Gamma, X)$ *such that* $\operatorname{Area}(u) \leq a$. *Moreover, for any sequence* (g_n) *of Riemannian metrics on* $M_{k,p}$, *there exists* u *as above and a Riemannian metric* g *on* $M_{k,p}$ *such that*

$$E_+^2(u, g) \leq \limsup_{n\to\infty} E_+^2(u_n, g_n).$$

The proposition follows by repeatedly applying the next lemma.

Lemma I.5.6. *Let* (Y_n) *be an* X*-approximating sequence,* $M \in \mathcal{M}(k)$, (g_n) *be a sequence of Riemannian metrics on* M *and* $u_n \in \Lambda(M, \Gamma, Y_n)$ *be fillings such that* $\operatorname{Area}(u_n)$ *is uniformly bounded. Then one of the following two options holds. Either there is* $u \in \Lambda(M, \Gamma, X)$ *and a Riemannian metric* g *on* M *such that*

$$\operatorname{Area}(u) \leq \limsup_{n\to\infty} \operatorname{Area}(u_n) \quad \& \quad E_+^2(u, g) \leq \limsup_{n\to\infty} E_+^2(u_n, g_n),$$

or there exist a reduction M^* *of* M, *an* X*-approximating sequence* (Y_n^*) *and maps* $u_n^* \in \Lambda(M^*, \Gamma, Y_n^*)$ *such that*

$$\limsup_{n\to\infty} \operatorname{Area}(u_n^*) \leq \limsup_{n\to\infty} \operatorname{Area}(u_n).$$

Proof of Proposition I.5.5. Let M, Y_n, u_n and g_n be as in the proposition. If the first possibility in Lemma I.5.6 when applied to these elements is true, i.e. if the existence of $u \in \Lambda(M, \Gamma, X)$ and a metric g on M as in this lemma is given, then the proposition follows immediately. We claim that the second possibility in the lemma cannot occur. Otherwise, we could iteratedly apply Lemma I.5.6 to M^*, the sequences (Y_n^*) and (u_n^*) given by the lemma and arbitrarily chosen metrics g_n^* on M^*, as well as their respective successors, until eventually the first possibility holds. This has to be the case after finitely many iterations, since the Euler characteristic strictly increases when passing to a reduction, but is also bounded from above by k in our setting. Thus we would obtain a reduction M^* of M and a map $u \in \Lambda(M^*, \Gamma, X)$ such that

$$\operatorname{Area}(u) \leq \limsup_{n\to\infty} \operatorname{Area}(u_n) < a_p^*(\Gamma, X),$$

which gives a contradiction. $\square$

At the end of this section, we give a proof for Lemma I.5.6. It is based on Proposition I.5.2 as well as Lemmas I.5.3 and I.5.4.

Proof of Lemma I.5.6. Without loss of generality, we may assume that M is connected. Define

$$a := \limsup_{n\to\infty} \operatorname{Area}(u_n) < \infty \quad \& \quad e := \limsup_{n\to\infty} E_+^2(u_n, g_n).$$

If e is infinite, we choose a sequence of auxiliary metrics g'_n on M satisfying

$$E_+^2(u_n, g'_n) \leq \frac{4}{\pi} \operatorname{Area}(u_n) + 1,$$

which exist by [55, Theorem 1.2] and [55, Section 5]. Thus, after potentially redefining $g_n := g'_n$, we may assume that e is finite.

We first address the special setting where Γ is a single Jordan curve and M a disk-type surface. We may assume that $M = \bar{D}$ and, since all Riemannian metrics on $\bar{D}$ are conformally equivalent, that each g_n is equal to the standard Euclidean metric g_{Eucl}. Now precompose each u_n with a conformal diffeomorphism φ_n of $\bar{D}$ such that $\operatorname{tr}(u_n \circ \varphi_n)$ satisfies for each n the same prefixed three-point condition on ∂D and Γ, see p. 1149 in [104]. Note that the maps $v_n := u_n \circ \varphi_n$ satisfy $\operatorname{Area}(v_n) = \operatorname{Area}(u_n)$ and $E_+^2(v_n, g_{\text{Eucl}}) = E_+^2(u_n, g_{\text{Eucl}})$. It then follows by [104, Proposition 7.4] that the family $\{\operatorname{tr}(v_n) \colon n \in \mathbb{N}\}$ is equicontinuous, and therefore by Proposition I.5.2 that there exists $u \in \Lambda(\bar{D}, \Gamma, X)$ with

$$\operatorname{Area}(u) \leq \limsup_{n\to\infty} \operatorname{Area}(v_n) = a \quad \& \quad E_+^2(u, g_{\text{Eucl}}) \leq \limsup_{n\to\infty} E_+^2(v_n, g_{\text{Eucl}}) = e$$

as in the first option proposed by the lemma.

From now on, we assume that M is a connected surface which is not of disk-type. Since every conformal class of Riemannian metrics on M has a hyperbolic representative (respectively a flat one if M is of cylindrical type), we lose no generality in assuming that all the metrics g_n are hyperbolic (respectively flat). In the rest of the proof, we discuss three different cases of outcomes in which ultimately either Lemma I.5.3, Lemma I.5.4 or Proposition I.5.2 is used to deduce one of the options stated in the lemma itself.

First assume that

$$\inf\{\operatorname{sys}_{\text{rel}}(M, g_n) \colon n \in \mathbb{N}\} > 0. \tag{I.11}$$

Then by [56, Theorem 3.3] (respectively its analogue for flat metrics) there exist diffeomorphisms φ_n of M and a metric g on M such that the pullback-metrics $\varphi_n^* g_n$ converge (up to a subsequence) smoothly to g. This convergence implies for the maps $v_n := u_n \circ \varphi_n \in \Lambda(M, \Gamma, Y_n)$ that

$$E_+^2(v_n, g) \leq C_n \cdot E_+^2(u_n, g_n),$$

where $C_n \geq 1$ tends to 1 as $n \to \infty$. In particular, the energies $E_+^2(v_n, g)$ are uniformly bounded. Now assume furthermore that the family

$$\{\operatorname{tr}(v_n) \colon n \in \mathbb{N}\} \quad \text{is equicontinuous} \tag{I.12}$$

with respect to the metric g. Then by Proposition I.5.2 there exists a filling $u \in \Lambda(M, \Gamma, X)$ with

$$\text{Area}(u) \leq a \quad \& \quad E_+^2(u, g) \leq e$$

as in the first option of the lemma.

In the remaining two cases, we discuss the outcomes if either the bound (I.11) does not hold; or if it does indeed, but property (I.12) fails for the traces of the constructed maps $v_n \in \Lambda(M, \Gamma, Y_n)$. Let

$$\rho_j := \frac{1}{\sqrt{C 2^{j+3}}},$$

where $C \geq 0$ is the universal constant from Lemma I.5.1, and $\rho_j' := \rho_\Gamma'(\rho_j)$ for each $j \in \mathbb{N}$. We claim that in either of these subcases, there exist a sequence of reductions M_j^* of M, a subsequence $(u_{n_j}) \subset \Lambda(M, \Gamma, Y_{n_j})$, $(2C\rho_j)$-thickenings Y_j^* of Y_{n_j} and fillings

$$u_j^* \in \Lambda(M_j^*, \Gamma, Y_j^*)$$

such that

$$\text{Area}(u_j^*) \leq \text{Area}(u_{n_j}) + 2^{-j}.$$

The existence of a sequence as implied in the lemma is then true by the following two observations. Firstly, there are only finitely many reductions of M up to diffeomorphism, hence we may assume that each M_j^* is equal to the same reduction M^* of M by passing to a subsequence of M_j^*. Secondly, the spaces Y_j^* are $(\varepsilon_{n_j} + 2C\rho_j)$-thickenings of X, where ε_n is the thickening parameter of Y_n, and thus (Y_j^*) an X-approximating sequence.

We continue by showing the claim and first suppose that (I.11) is violated. We only discuss the case for hyperbolic metrics, the situation for flat metrics being analogous. The assumption on the systoles of g_n implies that there exists a subsequence (g_{n_j}) such that

$$\text{sys}_{\text{rel}}(M, g_{n_j}) =: \lambda_j \to 0.$$

Choosing this subsequence appropriately, we may assume that

$$\lambda_j < \min\left\{\frac{{\rho_j'}^2}{4K}, \text{arsinh}\left(\frac{1}{\sinh(2)}\right)\right\},$$

where we define $K := \sup_n E_+^2(u_n, g_n) < \infty$. By Lemma I.5.1, for each j there exists a $(2C\rho_j)$-thickening Y_j^* of Y_{n_j} admitting a $(C, 2\rho_j)$-quadratic isoperimetric inequality. Since the spaces Y_j^* contain X (and hence Γ) isometrically and since the metrics g_n are all hyperbolic, we have by Lemma I.5.3 that there exist reductions M_j^* of M and maps $u_j^* \in \Lambda(M_j^*, \Gamma, Y_j^*)$ with

$$\text{Area}(u_j^*) \leq \text{Area}(u_{n_j}) + 8C\rho_j^2 \leq \text{Area}(u_{n_j}) + 2^{-j}.$$

This shows the claim in the first subcase.

Lastly, we address the case where (I.11) is true, but (I.12) is violated for the obtained metric g. Choose for each $j \in \mathbb{N}$ a number $0 < \delta_j \leq \delta_g$ such that

$$\pi \cdot \left(\frac{8K}{|\log(\delta_j)|} \right)^{\frac{1}{2}} \leq \rho_j'.$$

From the assumption of nonequicontinuity of $\{\mathrm{tr}(v_n)\}$, it follows that there exists $\varepsilon > 0$ such that for every j there exists a map $\mathrm{tr}(v_{n_j}) \colon M \to Y_{n_j}$ and a segment $\gamma_j^- \subset \partial M$ satisfying

$$\ell_g(\gamma_j^-) \leq \delta_j \quad \& \quad \ell_X(\mathrm{tr}(v_{n_j}) \circ \gamma_j^-) > \varepsilon.$$

Notice that for all j big enough we have that $\rho_j \leq \varepsilon$, so in particular

$$\ell_X(\mathrm{tr}(v_{n_j}) \circ \gamma_j^-) > \rho_j.$$

Let Y_j^* be given analogously as in the previous subcase. Then by Lemma I.5.4 there exist reductions M_j^* of M and mappings $u_j^* \in \Lambda(M_j^*, \Gamma, Y_j^*)$ satisfying

$$\mathrm{Area}(u_j^*) \leq \mathrm{Area}(v_{n_j}) + 8C\rho_j^2 \leq \mathrm{Area}(u_{n_j}) + 2^{-j}.$$

This shows the claim in the second subcase and completes the proof of the lemma. □

I.5.4 Proof of the main result. Finally, we are able to complete the proof of Theorem I.1.2.

Proof of Theorem I.1.2. The statements (i) and (ii) of the theorem have already been proved in Sections I.3.2 and I.4.2. Thus it remains to show (iii) as well as existence in the general case, where X might not admit a local quadratic isoperimetric inequality and Γ might be a configuration of overlapping or self-intersecting curves.

We begin with the proof of part (iii) and assume that Γ is a collection of disjoint rectifiable Jordan curves. For $n \in \mathbb{N}$ we set $Y_n := X$ and choose maps $u_n \in \Lambda(M, \Gamma, X)$ such that

$$\mathrm{Area}(u_n) \leq a_p(\Gamma, X) + 2^{-n}.$$

Since we assumed that the Douglas condition holds for p, Γ and X, we may apply Proposition I.5.5 to the sequences (Y_n) and (u_n). This shows that

$$\Lambda_{\min} := \{u \in \Lambda(M, \Gamma, X) : \mathrm{Area}(u) = a_p(\Gamma, X)\}$$

is non-empty. Choose sequences of maps $u_n \in \Lambda_{\min}$ and Riemannian metrics g_n on M such that

$$\lim_{n\to\infty} E_+^2(u_n, g_n) = \underbrace{\inf\{E_+^2(w, h) \colon w \in \Lambda_{\min},\, h \text{ a Riemannian metric on } M\}}_{=:e}.$$

Applying Proposition I.5.5 to the sequences (Y_n), (g_n) and (u_n), one sees that there exist $u \in \Lambda_{\min}$ and a Riemannian metric g on M such that $E_+^2(u, g) = e$.

Then by [55, Corollary 1.3] u is infinitesimally isotropic with respect to g. This completes the proof in the special case that the configuration is assumed to consist of disjoint Jordan curves.

We move on to the general case. Let (X_n) be an X-approximating sequence, where every X_n admits some local quadratic isoperimetric inequality: such an approximating sequence exists by Proposition I.5.1. Then $(Y_n) := ((X_n)_\Gamma)$ defines an X_Γ-approximating sequence, where the collar extensions are performed as defined in Section I.3.2. By Lemma I.3.4, there exist maps $u_n \in \Lambda(M, \tilde{\Gamma}, Y_n)$ such that

$$\mathrm{Area}(u_n) \leq a_p(\Gamma, X_n) + \sum_{j=1}^{k} \mathcal{H}^2(Z_j) + 2^{-n} \leq a_p(\Gamma, X) + \sum_{j=1}^{k} \mathcal{H}^2(Z_j) + 2^{-n}.$$

Then by Lemma I.3.3, and since the Douglas condition holds for p, Γ and X, one has

$$\limsup_{n\to\infty} \mathrm{Area}(u_n) \leq a_p(\Gamma, X) + \sum_{j=1}^{k} \mathcal{H}^2(Z_j) < a_p^*(\Gamma, X) + \sum_{j=1}^{k} \mathcal{H}^2(Z_j) \leq a_p^*(\tilde{\Gamma}, X_\Gamma).$$

Thus applying Proposition I.5.5 to the sequences (Y_n) and (u_n) shows that the Douglas condition holds for p, $\tilde{\Gamma}$ and X_Γ and that

$$a_p(\tilde{\Gamma}, X_\Gamma) \leq a_p(\Gamma, X) + \sum_{j=1}^{k} \mathcal{H}^2(Z_j).$$

Since $\tilde{\Gamma}$ is a configuration of disjoint Jordan curves, the Douglas condition and the first part of the proof imply that there exist $v \in \Lambda(M, \tilde{\Gamma}, X_\Gamma)$ and a Riemannian metric g on M such that $\mathrm{Area}(v) = a_p(\tilde{\Gamma}, X_\Gamma)$ and v is infinitesimally isotropic with respect to g. For the projection $u := P_\Gamma \circ v$ Lemma I.3.3 implies that $u \in \Lambda(M, \Gamma, X)$ with

$$\mathrm{Area}(u) \leq \mathrm{Area}(v) - \sum_{j=1}^{k} \mathcal{H}^2(Z_j) \leq a_p(\Gamma, X),$$

and thus $\mathrm{Area}(u) = a_p(\Gamma, X)$. Furthermore, the composition $P_\Gamma \circ v$ agrees with v on the complement of $v^{-1}(Z) = u^{-1}(\Gamma)$, hence u is infinitesimally isotropic on $M \setminus u^{-1}(\Gamma)$ with respect to g. This concludes the proof of the theorem in the general case. □

I.5.5 Translation to the smooth setting. To obtain Theorem I.1.1, we make the following observations for $M \in \mathcal{M}(k)$, a complete Riemannian manifold (X, h) and $u \in W^{1,2}(M, X)$.

- By the Hopf-Rinow theorem, X defines a proper geodesic metric space.
- Homogeneously regular Riemannian manifolds admit a local quadratic isoperimetric inequality. See [89] for the definition and compare Section 4.3 in [B] for the simple argument.

- Smooth Riemannian manifolds are locally of curvature bounded above, compare for example [20, Theorem II.1A.6].
- Compact C^2 curves in smooth Riemannian manifolds have finite total curvature, see [24].
- As a consequence of (I.2), for almost every $z \in M$ the approximate metric derivative $\operatorname{ap}\operatorname{md} u_z$ defines a Euclidean seminorm on T_zM, and hence u is infinitesimally isotropic if and only if it is weakly conformal.
- Weakly conformal area minimizers in X are minimizers of the Dirichlet energy, and thus weakly harmonic in the classical sense. Continuous weakly harmonic maps between Riemannian manifolds are however smooth by [91, Theorem 9.4.1].

With these observations at hand, Theorem I.1.2 is easily seen to imply Theorem I.1.1.

I.6 Minimizers under the conditions of cohesion and adhesion

Let X be a complete metric space, M a smooth compact and connected surface and $\eta > 0$. A mapping $u\colon M \to X$ is said to be *η-cohesive* if u is continuous and

$$\ell(u(c)) \geq \eta$$

for every non-contractible closed curve c in M.

Definition I.6.1. A family $\mathcal{F}$ of maps from M to X is said to satisfy the condition of cohesion if there exists $\eta > 0$ such that every map in $\mathcal{F}$ is η-cohesive.

Now let $c \subset M$ be an embedded arc such that the endpoints of c lie on ∂M and let $u\colon M \to X$ be continuous. If the endpoints of c lie on a single component ∂M_j, then they divide ∂M_j into two components c^- and c^+, where the notation is chosen such that $\ell(u(c^-)) \leq \ell(u(c^+))$. Let $\bar{\rho}\colon (0,\infty) \to (0,\infty)$ be a function such that $\bar{\rho}(\rho) \leq \rho$ for every $\rho \in (0,\infty)$. We say that $u\colon M \to X$ is *$\bar{\rho}$-adhesive* if u is continuous and for every arc c with endpoints in ∂M and of image-length $\ell(u(c)) \leq \bar{\rho}(\rho)$, one has that the endpoints lie in the same connected component of ∂M and

$$\ell(u(c^-)) < \rho.$$

Definition I.6.2. A family $\mathcal{F}$ of maps from M to X is said to satisfy the condition of adhesion if there exists a function $\bar{\rho}\colon (0,\infty) \to (0,\infty)$ as above such that every map in $\mathcal{F}$ is $\bar{\rho}$-adhesive.

Let Γ be a configuration of $k \geq 1$ rectifiable closed curves in X and let $M \in \mathcal{M}(k)$. Set

$$e(M,\Gamma,X) := \inf\{E_+^2(u,g)\colon u \in \Lambda(M,\Gamma,X),\ g \text{ a Riemannian metric on } M\}.$$

An *energy minimizing sequence in* $\Lambda(M,\Gamma,X)$ is a sequence of pairs (u_n, g_n) of mappings $u_n \in \Lambda(M,\Gamma,X)$ and Riemannian metrics g_n on M such that

$$E_+^2(u_n, g_n) \to e(M,\Gamma,X)$$

as n tends to infinity.

Theorem I.6.3. *Let X be a proper metric space and $\Gamma \subset X$ be a configuration of $k \geq 1$ rectifiable closed curves. Let $M \in \mathcal{M}(k)$ be connected. If there exist an energy minimizing sequence in $\Lambda(M,\Gamma,X)$ satisfying the conditions of cohesion and adhesion, then there exist $u \in \Lambda(M,\Gamma,X)$ and a Riemannian metric g on M such that*

$$E_+^2(u,g) = e(M,\Gamma,X).$$

For any such u and g the map u is infinitesimally isotropic with respect to g.

If X is a complete Riemannian manifold, then energy minimizers are precisely weakly conformal area minimizers. For more general spaces X however, the relation is more complicated and energy minimizers need not be area minimizers, see for example [105, 104]. Nevertheless, one can obtain existence of area minimizers for singular configurations in proper metric spaces if there exists an area minimizing sequence satisfying the conditions of cohesion and adhesion by modifying the proofs of [55, Theorem 1.6] and [55, Proposition 5.3] accordingly. However, as in [55, Theorem 1.6] and [55, Proposition 5.3], either the obtained area minimizers are potentially not infinitesimally isotropic, or one has to choose a somewhat different interpretation of the term 'area'.

Proof of Theorem I.6.3. It follows from [55, Corollary 1.3] that any energy minimizing pair (u,g) is infinitesimally isotropic. Thus it remains to show existence of such a pair.

First assume that M is not of disk-type. If Γ is a configuration of disjoint Jordan curves, then any continuous $u \in \Lambda(M,\Gamma,X)$ satisfies a ρ'_Γ-condition of adhesion, where ρ'_Γ is as in Section I.5.2. In fact, under this observation, the proof of Theorem I.6.3 for such M is a straightforward generalization of the proof of [56, Theorem 8.2]. Namely, if one replaces in the statements of Propositions 8.3 and 8.4 in [56] the assumption that Γ consists of disjoint Jordan curves by the assumption that u is $\bar{\rho}$-adhesive, the proofs become virtually identical upon replacing $\rho' = \rho'_\Gamma$ by $\bar{\rho}$. With these modified propositions at hand, the proof of Theorem I.6.3 is completed as is that of [56, Theorem 8.2].

Finally assume that Γ is a single curve and that $M = \bar{D}$. If Γ is constant, the result is trivial. Otherwise we may represent Γ as a composition of 3 curves $\Gamma_1, \Gamma_2, \Gamma_3$ of equal length. We also decompose S^1 into three consecutive arcs $\bar{\Gamma}_1$, $\bar{\Gamma}_2, \bar{\Gamma}_3$ of equal length. We say that a continuous map $u \in \Lambda(M,\Gamma,X)$ satisfies the 3-*arc condition* if $u|_{\bar{\Gamma}_i}$ is a parametrization of Γ_i for every $i = 1,2,3$. Fix $K \geq 0$ and adhesiveness function $\bar{\rho}\colon (0,\infty) \to (0,\infty)$. Let $\mathcal{F}$ be the family of maps $u \in \Lambda(M,\Gamma,X)$ which are $\bar{\rho}$-adhesive, satisfy the 3-arc condition and have energy $E_+^2(u, g_{\text{Eucl}}) \leq K$. We claim that the trace family $\{u|_{S^1} \colon u \in \mathcal{F}\}$ is equicontinuous. To prove this claim, we fix $0 < \varepsilon < \ell(\Gamma)/3$, $p \in S^1$ and $u \in \mathcal{F}$. Let $0 < \delta < 1$ be so small that

$$\pi \left(\frac{2K}{|\log \delta|} \right)^{\frac{1}{2}} < \bar{\rho}(\varepsilon).$$

For $0 < r < 1$, denote by c_r the arc $\{z \in \bar{D} \colon |z - p| = r\}$. By the Courant-Lebesgue lemma, [104, Lemma 7.3], there is $r \in (\delta, \sqrt{\delta})$ such that $\ell(u \circ c_r) \leq \bar{\rho}(\varepsilon)$. The $\bar{\rho}$-adhesiveness then implies that $\ell(u \circ c_r^-) \leq \varepsilon$, and hence it follows from the 3-arc condition together with the choice of ε that $c_r^- = B(p, r) \cap S^1$. Thus, for any $x \in B(p, \delta) \cap S^1$, one has $d(u(x), u(p)) \leq \varepsilon$. Since the choice of δ was independent of u and p, the claimed equicontinuity follows.

Now let (u_n, g_n) be an energy minimizing sequence which is $\bar{\rho}$-adhesive. Since all metrics on the disk are conformally equivalent, we may assume that $g_n = g_{\mathrm{Eucl}}$ for each $n \in \mathbb{N}$. Furthermore, after precomposing with Moebius transforms, one has that all u_n satisfy the 3-arc condition. Thus by the claim the sequence $(u_n|_{S^1})$ is equicontinuous and hence Proposition I.5.2 implies the existence of the desired energy minimizer. □

I.7 Appendix

In this section we discuss the proof of Lemma I.5.1. A metric space X will be called *δ-geodesic*, where $\delta > 0$, if for all $x, y \in X$ satisfying $d(x, y) < \delta$ there is a curve γ in X joining x to y such that $\ell(\gamma) = d(x, y)$. Lemma I.5.1 is only a slight strengthening of the following consequence of [109, Lemma 3.3].

Lemma I.7.1. *There is a universal constant $C \geq 0$ such that for every proper, δ-geodesic metric space X and $0 < \varepsilon \leq \delta$, there exists an ε-thickening Y of X such that Y is proper and satisfies a $(C, \varepsilon/C)$-quadratic isoperimetric inequality.*

[109, Lemma 3.3] is stated for spaces which are globally geodesic, though the proof readily gives the claimed result for δ-geodesic spaces. Namely, in the proof the assumption only comes into play when estimating the diameter of the small ball B_z with respect to its induced intrinsic metric by twice the radius. This estimate holds in a δ-geodesic space as soon as the radius of the ball is bounded from above by δ. More precisely, this estimate is used twice: on p. 241 of [158] to estimate the diameter of X_z and on p. 242 to find the curves $\bar{\gamma}_j$.

For the proof of Lemma I.5.1, recall that the injective hull $E(X)$ of a compact metric space X is a compact geodesic metric space. Furthermore, $X \subset E(X)$ isometrically and $\mathrm{diam}(E(X)) = \mathrm{diam}(X)$, see for example [99].

Proof of Lemma I.5.1. We claim that for any $\delta > 0$, there is an (8δ)-thickening Z of X such that Z is proper and δ-geodesic. Lemma I.5.1 then follows by first applying the claim to X, yielding a $(8C\varepsilon)$-thickening Z of X which is proper and $(C\varepsilon)$-geodesic, where C is as in Lemma I.7.1; and then applying Lemma I.7.1 to Z to obtain a $(C\varepsilon)$-thickening Y of Z which is proper and admits a (C, ε)-quadratic isoperimetric inequality. It remains to note that Y is a $(9C\varepsilon)$-thickening of X and redefine C.

In order to prove the claim, we perform a variation of the construction discussed in [158] and [109]. Let S be a maximal δ-separated subset in X. For $z \in S$ set $B_z := B(z, 2\delta)$ and $X_z := E(B_z)$. Then $\operatorname{diam}(B_z) \leq 4\delta$ and hence $\operatorname{diam}(X_z) \leq 4\delta$. We set

$$Z := \Big(\bigsqcup_{z \in S} X_z \Big)_{/\sim},$$

where $x \sim y$ if $x \in B_z \subset X_z$, $y \in B_w \subset X_w$ and $x = y$. The space Z is endowed with the quotient metric. It follows from the construction that Z is proper and a (4δ)-thickening of X, compare also [109].

It remains to show that Z is δ-geodesic. To this end, let $x, y \in Z$ such that $d(x, y) < \delta$. Then either x and y lie in a common X_z and $d(x, y) = d_{X_z}(x, y)$ or there are $z, w \in S$, $u \in X_z \cap X$ and $v \in X_w \cap X$ such that

$$d(x, y) = d_{X_z}(x, u) + d_X(u, v) + d_{X_w}(v, y).$$

In the former case, the distance is realized by a curve because X_z is geodesic. By the same reasoning, it suffices to show that $d(u, v)$ is realized by the length of a curve in Z in the latter case. By maximality of S there exists $s \in S$ such that $d_X(s, u) \leq \delta$ and hence $u, v \in X_s$. As $X_s \subset Z$ is a geodesic subset, the claim follows. □

BIBLIOGRAPHY

[1] Stephanie Alexander, Vitali Kapovitch, and Anton Petrunin. Alexandrov geometry: preliminary version no. 1. *Preprint*, arXiv:1903.08539, 2019.

[2] Alexander D. Alexandrov and Victor A. Zalgaller. *Intrinsic geometry of surfaces*. Translations of Mathematical Monographs, Vol. 15. American Mathematical Society, Providence (RI), 1967. Translated from the Russian by J. M. Danskin.

[3] Daniel Allcock. An isoperimetric inequality for the Heisenberg groups. *Geom. Funct. Anal.*, 8(2):219–233, 1998.

[4] Frederick J. Almgren Jr. and Leon Simon. Existence of embedded solutions of Plateau's problem. *Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4)*, 6(3):447–495, 1979.

[5] Hans W. Alt. Verzweigungspunkte von h-flächen. i. *Math. Z.*, 127:333–362, 1972.

[6] Hans W. Alt. Verzweigungspunkte von H-Flächen. ii. *Math. Ann.*, 201:33–55, 1973.

[7] Juan-Carlos Álvarez Paiva and Gautier Berck. What is wrong with the Hausdorff measure in Finsler spaces. *Adv. Math.*, 204(2):647–663, 2006.

[8] Juan-Carlos Álvarez Paiva and Anthony C. Thompson. Volumes on normed and Finsler spaces. In *A sampler of Riemann–Finsler geometry*, Mathematical Sciences Research Institute Publications, Vol. 50, pages 1–48. Cambridge University Press, Cambridge, 2004.

[9] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces. *Rev. Mat. Iberoam.*, 29(3):969–996, 2013.

[10] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. *Invent. Math.*, 195(2):289–391, 2014.

[11] Luigi Ambrosio and Bernd Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.

[12] Luigi Ambrosio and Bernd Kirchheim. Rectifiable sets in metric and Banach spaces. *Math. Ann.*, 318(3):527–555, 2000.

[13] Zoltán M. Balogh, Piotr Hajłasz, and Kevin Wildrick. Weak contact equations for mappings into Heisenberg groups. *Indiana Univ. Math. J.*, 63(6):1839–1873, 2014.

[14] Zoltán M. Balogh, Roberto Monti, and Jeremy T. Tyson. Frequency of Sobolev and quasiconformal dimension distortion. *J. Math. Pures Appl. (9)*, 99(2):125–149, 2013.

[15] Felicia Bernatzki. The Plateau-Douglas problem for nonorientable minimal surfaces. *Manuscripta Math.*, 79(1):73–80, 1993.

[16] Reinhold Böhme and Anthony J. Tromba. The index theorem for classical minimal surfaces. *Ann. of Math. (2)*, 113(3):447–499, 1981.

[17] Mario Bonk and Bruce Kleiner. Quasisymmetric parametrizations of two-dimensional metric spheres. *Invent. Math.*, 150(1):127–183, 2002.

[18] Brian H. Bowditch. Notes on locally CAT(1) spaces. In *Geometric group theory. Proceedings of a Special Research Quarter at The Ohio State University, Spring 1992*, Ohio State University Mathematical Research Institute Publications, Vol. 3, pages 1–48. de Gruyter, Berlin, 1995.

[19] Christine Breiner, Ailana Fraser, Lan-Hsuan Huang, Chikako Mese, Pam Sargent, and Yingying Zhang. Regularity of harmonic maps from polyhedra to CAT(1) spaces. *Calc. Var. Partial Differential Equations*, 57(1):Paper No. 12, 35 pp., 2018.

[20] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, Vol. 319. Springer, Berlin, 1999.

[21] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry.* Graduate Studies in Mathematics, Vol. 33. American Mathematical Society, Providence (RI), 2001.

[22] Yuri Burago, Michail L. Gromov, and Grigori Y. Perelman. A.D. Alexandrov spaces with curvature bounded below. *Russian Math. Surveys*, 47(2):1–58, 1992.

[23] Iván Caamaño, Jesús Á. Jaramillo, Ángeles Prieto, and Alberto Ruiz de Alarcón. Sobolev spaces of vector-valued functions. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 115(1):Paper No. 19, 14 pp., 2021.

[24] Marco Castrillón López, Victor Fernández Mateos, and Jaime Muñoz Masqué. Total curvature of curves in Riemannian manifolds. *Differential Geom. Appl.*, 28(2):140–147, 2010.

[25] Lamberto Cesari. *Surface area.* Annals of Mathematics Studies, Vol. 35. Princeton University Press, Princeton (NJ), 1956.

[26] Jeff Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9(3):428–517, 1999.

[27] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.*, 46(3):406–480, 1997.

[28] Jeff Cheeger and Bruce Kleiner. On the differentiability of Lipschitz maps from metric measure spaces to Banach spaces. In *Inspired by S. S. Chern. A Memorial Volume in Honor of A Great Mathematician*, Nankai Tracts in Mathematics, Vol. 11, pages 129–152. World Scientific, Hackensack (NJ), 2006.

[29] Jaigyoung Choe and Robert D. Gulliver, II. Embedded minimal surfaces and total curvature of curves in a manifold. *Math. Res. Lett.*, 10(2-3):343–362, 2003.

[30] Richard Courant. On the problem of Plateau. *Proc. Nat. Acad. Sci. USA*, 22:367–372, 1936.

[31] Richard Courant. Plateau's problem and Dirichlet's principle. *Ann. of Math. (2)*, 38(3):679–724, 1937.

[32] Richard Courant. On a generalized form of Plateau's problem. *Trans. Amer. Math. Soc.*, 50:40–47, 1941.

[33] Richard Courant. *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces.* Interscience Publishers, Inc., New York (NY), 1950. Appendix by M. Schiffer.

[34] Guy David. Should we solve Plateau's problem again? In *Advances in analysis: the legacy of Elias M. Stein*, Princeton Mathematical Series, Vol. 50, pages 108–145. Princeton University Press, Princeton (NJ), 2014.

[35] Giuseppe De Cecco and Giuliana Palmieri. Length of curves on Lip manifolds. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 1(3):215–221, 1990.

[36] Giuseppe De Cecco and Giuliana Palmieri. Intrinsic distance on a LIP Finslerian manifold. *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)*, 17:129–151, 1993.

[37] Noel Dejarnette, Piotr Hajłasz, Anton Lukyanenko, and Jeremy T. Tyson. On the lack of density of Lipschitz mappings in Sobolev spaces with Heisenberg target. *Conform. Geom. Dyn.*, 18:119–156, 2014.

[38] Laura Desideri. Problème de Plateau, équations fuchsiennes et problème de Riemann-Hilbert. *Mém. Soc. Math. Fr. (N.S.)*, 133:116 pp., 2013.

[39] Laura Desideri and Ruben Jakob. Immersed solutions of Plateau's problem for piecewise smooth boundary curves with small total curvature. *Results Math.*, 63(3-4):891–901, 2013.

[40] Ulrich Dierkes, Stefan Hildebrandt, and Friedrich Sauvigny. *Minimal surfaces.* Grundlehren der mathematischen Wissenschaften, Vol. 339. Springer, Berlin–Heidelberg, second edition, 2010.

[41] Ulrich Dierkes, Stefan Hildebrandt, and Anthony J. Tromba. *Regularity of minimal surfaces.* Grundlehren der mathematischen Wissenschaften, Vol. 340. Springer, Berlin–Heidelberg, second edition, 2010.

[42] Joseph Diestel and John J. Uhl Jr. *Vector measures.* Mathematical Surveys, Vol. 15. American Mathematical Society, Providence (RI), 1977. With a foreword by B. J. Pettis.

[43] Jesse Douglas. A general formulation of the problem of Plateau. *Bull. Amer. Math. Soc.*, 36(1):50, 1930.

[44] Jesse Douglas. The problem of Plateau for two contours. *Journal of Math. and Physics*, 10:310–359, 1931.

[45] Jesse Douglas. Solution of the problem of Plateau. *Trans. Amer. Math. Soc.*, 33(1):263–321, 1931.

[46] Jesse Douglas. One-sided minimal surfaces with a given boundary. *Trans. Amer. Math. Soc.*, 34(4):731–756, 1932.

[47] Jesse Douglas. Minimal surfaces of general topological structure with any finite number of assigned boundaries. *Journal of Math. and Physics*, 15:106–123, 1936.

[48] Jesse Douglas. Some new results in the problem of Plateau. *Journal of Math. and Physics*, 15:55–64, 1936.

[49] Jesse Douglas. Minimal surfaces of higher topological structure. *Ann. of Math. (2)*, 40(1):205–298, 1939.

[50] Tobias Ekholm, Brian White, and Daniel Wienholtz. Embeddedness of minimal surfaces with total boundary curvature at most 4π. *Ann. of Math. (2)*, 155(1):209–234, 2002.

[51] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions.* Textbooks in Mathematics. CRC Press, Boca Raton (FL), revised edition, 2015.

[52] Nikita Evseev. Vector-valued Sobolev spaces based on Banach function spaces. *Nonlinear Anal.*, 211:Paper No. 112479, 15 pp., 2021.

[53] Herbert Federer. Some theorems on integral currents. *Trans. Amer. Math. Soc.*, 117:43–67, 1965.

[54] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.

[55] Martin Fitzi and Stefan Wenger. Morrey's ε-conformality lemma in metric spaces. *Proc. Amer. Math. Soc.*, 148(10):4285–4298, 2020.

[56] Martin Fitzi and Stefan Wenger. Area minimizing surfaces of bounded genus in metric spaces. *J. Reine Angew. Math.*, 770:87–112, 2021.

[57] René Garnier. Le problème de Plateau. *Ann. Sci. École Norm. Sup. (3)*, 45:53–144, 1928.

[58] Israel M. Gelfand. Sur un lemme de la théorie des espaces linéaires. *Comm. Inst. Sci. Math. Méc. Univ. de Kharkoff et Soc. Math. Kharkoff*, 13(4):35–40, 1936.

[59] Nicola Gigli. The splitting theorem in non-smooth context. *Preprint*, arXiv:1302.5555, 2013.

[60] Nicola Gigli. An overview of the proof of the splitting theorem in spaces with non-negative Ricci curvature. *Anal. Geom. Metr. Spaces*, 2(1):169–213, 2014.

[61] Karlheinz Goldhorn and Stefan Hildebrandt. Zum Randverhalten der Lösungen gewisser zweidimensionaler Variationsprobleme mit freien Randbedingungen. *Math. Z.*, 118:241–253, 1970.

[62] Jeremy Gray and Mario Micallef. The work of jesse douglas on minimal surfaces. *Preprint*, arXiv:0710.5478, 2007.

[63] Michail L. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* Progress in Mathematics, Vol. 152. Birkhäuser, Boston (MA), 1999. With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[64] Karsten Grove, Peter V. Petersen, and Jyh-Yang Wu. Geometric finiteness theorems via controlled topology. *Invent. Math.*, 99(1):205–214, 1990.

[65] Robert D. Gulliver, II. Regularity of minimizing surfaces of prescribed mean curvature. *Ann. of Math. (2)*, 97:275–305, 1973.

[66] Robert D. Gulliver, II. and Frank D. Lesley. On boundary branch points of minimizing surfaces. *Arch. Ration. Mech. Anal.*, 52:20–25, 1973.

[67] Robert D. Gulliver, II., Robert Osserman, and Halsey L. Royden. A theory of branched immersions of surfaces. *Amer. J. Math.*, 95:750–812, 1973.

[68] Robert D. Gulliver, II. and Joel Spruck. On embedded minimal surfaces. *Ann. of Math. (2)*, 103(2):331–347, 1976.

[69] Chang-Yu Guo and Stefan Wenger. Area minimizing discs in locally non-compact metric spaces. *Comm. Anal. Geom.*, 28(1):89–112, 2020.

[70] Alfred Haar. Über das Plateausche Problem. *Math. Ann.*, 97(1):124–158, 1927.

[71] Piotr Hajłasz. Sobolev spaces on an arbitrary metric space. *Potential Anal.*, 5(4):403–415, 1996.

[72] Piotr Hajłasz. Sobolev mappings between manifolds and metric spaces. In *Sobolev spaces in mathematics. I*, International Mathematical Series, Vol. 8, pages 185–222. Springer, New York (NY), 2009.

[73] Piotr Hajłasz. Sobolev mappings: Lipschitz density is not an isometric invariant of the target. *Int. Math. Res. Not. IMRN*, 12:2794–2809, 2011.

[74] Piotr Hajłasz and Pekka Koskela. Sobolev met Poincaré. *Mem. Amer. Math. Soc.*, 145(688):101 pp., 2000.

[75] Piotr Hajłasz, Armin Schikorra, and Jeremy T. Tyson. Homotopy groups of spheres and Lipschitz homotopy groups of Heisenberg groups. *Geom. Funct. Anal.*, 24(1):245–268, 2014.

[76] Piotr Hajłasz and Jeremy T. Tyson. Sobolev Peano cubes. *Michigan Math. J.*, 56(3):687–702, 2008.

[77] Joel Hass. Singular curves and the Plateau problem. *Internat. J. Math.*, 2(1):1–16, 1991.

[78] Juha Heinonen. *Geometric embeddings of metric spaces.* Reports, University of Jyväskylä, Department of Mathematics and Statistics, Vol. 90. University of Jyväskylä, Jyväskylä, 2003.

[79] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T. Tyson. *Sobolev Spaces on Metric Measure Spaces. An Approach Based on Upper Gradients.* New Mathematical Monographs, Vol. 27. Cambridge University Press, Cambridge, 2015.

[80] Erhard Heinz and Stefan Hildebrandt. Some remarks on minimal surfaces in Riemannian manifolds. *Comm. Pure Appl. Math.*, 23:371–377, 1970.

[81] Frédéric Hélein. Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne. *C. R. Math. Acad. Sci. Paris Sér. I Math.*, 312(8):591–596, 1991.

[82] Stefan Hildebrandt. Boundary behavior of minimal surfaces. *Arch. Ration. Mech. Anal.*, 35:47–82, 1969.

[83] Stefan Hildebrandt and Anthony J. Tromba. On the branch point index of minimal surfaces. *Arch. Math. (Basel)*, 92(5):493–500, 2009.

[84] Stefan Hildebrandt and Heiko von der Mosel. Plateau's problem for parametric double integrals. I. Existence and regularity in the interior. *Comm. Pure Appl. Math.*, 56(7):926–955, 2003.

[85] Stefan Hildebrandt and Heiko von der Mosel. Plateau's problem for parametric double integrals. II. Regularity at the boundary. *J. Reine Angew. Math.*, 565:207–233, 2003.

[86] Howard Iseri. On the existence of minimal surfaces with singular boundaries. *Proc. Amer. Math. Soc.*, 124(11):3493–3500, 1996.

[87] Sergei V. Ivanov. Volumes and areas of Lipschitz metrics. *Algebra i Analiz*, 20(3):74–111, 2008.

[88] Sergei V. Ivanov. Filling minimality of Finslerian 2-discs. *Tr. Mat. Inst. Steklova*, 273:192–206, 2011.

[89] Jürgen Jost. Conformal mappings and the Plateau-Douglas problem in Riemannian manifolds. *J. Reine Angew. Math.*, 359:37–54, 1985.

[90] Jürgen Jost. Equilibrium maps between metric spaces. *Calc. Var. Partial Differential Equations*, 2(2):173–204, 1994.

[91] Jürgen Jost. *Riemannian geometry and geometric analysis*. Universitext. Springer, Cham, seventh edition, 2017.

[92] Tobias Kaiser and Sabrina Lehner. Asymptotic behaviour of the Riemann mapping function at analytic cusps. *Ann. Acad. Sci. Fenn. Math.*, 42(1):3–15, 2017.

[93] Maria B. Karmanova. Area and co-area formulas for mappings of the Sobolev classes with values in a metric space. *Sib. Math. J.*, 48(4):778–788, 2007.

[94] David Kinderlehrer. The boundary regularity of minimal surfaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (3)*, 23:711–744, 1969.

[95] Nicholas J. Korevaar and Richard M. Schoen. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, 1(3-4):561–659, 1993.

[96] Martin Kruskal. The bridge theorem for minimal surfaces. *Comm. Pure Appl. Math.*, 7:297–316, 1954.

[97] Joseph-Louis Lagrange. Essai dune nouvelle méthode pour determiner les maxima et les minima des formules intégrales indéfinies. *Misc. Philos.-Math. Soc. Priv. Taurinensis*, 2:173–195, 1761.

[98] Urs Lang. Local currents in metric spaces. *J. Geom. Anal.*, 21(3):683–742, 2011.

[99] Urs Lang. Injective hulls of certain discrete metric spaces and groups. *J. Topol. Anal.*, 5(3):297–331, 2013.

[100] Frank D. Lesley. Differentiability of minimal surfaces at the boundary. *Pacific J. Math.*, 37:123–139, 1971.

[101] Paul Lévy. Exemples de contours pour lesquels le problème de Plateau a 3 ou $2p+1$ solutions. *C. R. Acad. Sci. Paris*, 224:325–327, 1947.

[102] Hans Lewy. On the boundary behavior of minimal surfaces. *Proc. Natl. Acad. Sci. USA*, 37:103–110, 1951.

[103] Arvid T. Lonseth. The problem of Plateau in hyperbolic space. *Amer. J. Math.*, 64:229–259, 1942.

[104] Alexander Lytchak and Stefan Wenger. Area minimizing discs in metric spaces. *Arch. Ration. Mech. Anal.*, 223(3):1123–1182, 2017.

[105] Alexander Lytchak and Stefan Wenger. Energy and area minimizers in metric spaces. *Adv. Calc. Var.*, 10(4):407–421, 2017.

[106] Alexander Lytchak and Stefan Wenger. Intrinsic structure of minimal discs in metric spaces. *Geom. Topol.*, 22(1):591–644, 2018.

[107] Alexander Lytchak and Stefan Wenger. Isoperimetric characterization of upper curvature bounds. *Acta Math.*, 221(1):159–202, 2018.

[108] Alexander Lytchak and Stefan Wenger. Canonical parameterizations of metric disks. *Duke Math. J.*, 169(4):761–797, 2020.

[109] Alexander Lytchak, Stefan Wenger, and Robert Young. Dehn functions and Hölder extensions in asymptotic cones. *J. Reine Angew. Math.*, 763:79–109, 2020.

[110] Chaiwat Maneesawarng and Yongwimon Lenbury. Total curvature and length estimate for curves in CAT(K) spaces. *Differential Geom. Appl.*, 19(2):211–222, 2003.

[111] Edward J. McShane. Parametrizations of Saddle Surfaces, with Application to the Problem of Plateau. *Trans. Amer. Math. Soc.*, 35(3):716–733, 1933.

[112] William H. Meeks, III. Uniqueness theorems for minimal surfaces. *Illinois J. Math.*, 25(2):318–336, 1981.

[113] William H. Meeks, III. and Shing-Tung Yau. The classical Plateau problem and the topology of three-dimensional manifolds. The embedding of the solution given by Douglas-Morrey and an analytic proof of Dehn's lemma. *Topology*, 21(4):409–442, 1982.

[114] William H. Meeks, III. and Shing-Tung Yau. The existence of embedded minimal surfaces and the problem of uniqueness. *Math. Z.*, 179(2):151–168, 1982.

[115] Chikako Mese. The curvature of minimal surfaces in singular spaces. *Comm. Anal. Geom.*, 9(1):3–34, 2001.

[116] Chikako Mese and Patrick R. Zulkowski. The Plateau problem in Alexandrov spaces. *J. Differential Geom.*, 85(2):315–356, 2010.

[117] Charles B. Morrey, Jr. The problem of Plateau on a Riemannian manifold. *Ann. of Math. (2)*, 49:807–851, 1948.

[118] Igor G. Nikolaev. Solution of the Plateau problem in spaces of curvature at most K. *Sib. Math. J.*, 20(2):345–353, 1979.

[119] Johannes C. C. Nitsche. The boundary behavior of minimal surfaces. Kellogg's theorem and branch points on the boundary. *Invent. Math.*, 8(4):313–333, 1969.

[120] Johannes C. C. Nitsche. Minimal surfaces with partially free boundary. Least area property and Hölder continuity for boundaries satisfying a chord-arc condition. *Arch. Ration. Mech. Anal.*, 39:131–145, 1970.

[121] Johannes C. C. Nitsche. A new uniqueness theorem for minimal surfaces. *Arch. Ration. Mech. Anal.*, 52:319–329, 1973.

[122] Johannes C. C. Nitsche. *Vorlesungen über Minimalflächen*. Grundlehren der mathematischen Wissenschaften, Vol. 199. Springer-Verlag, Berlin-New York, 1975.

[123] Dimitrios Ntalampekos and Matthew Romney. Polyhedral approximation of metric surfaces and applications to uniformization. *Preprint*, arXiv:2107.07422, 2021.

[124] Robert Osserman. A proof of the regularity everywhere of the classical solution to Plateau's problem. *Ann. of Math. (2)*, 91:550–569, 1970.

[125] Patrick Overath and Heiko von der Mosel. Plateau's problem in Finsler 3-space. *Manuscripta Math.*, 143(3-4):273–316, 2014.

[126] Patrick Overath and Heiko von der Mosel. On minimal immersions in Finsler space. *Ann. Global Anal. Geom.*, 48(4):397–422, 2015.

[127] Anton Petrunin and Stephan Stadler. Metric-minimizing surfaces revisited. *Geom. Topol.*, 23(6):3111–3139, 2019.

[128] Sven Pistre and Heiko von der Mosel. The Plateau problem for the Busemann-Hausdorff area in arbitrary codimension. *Eur. J. Math.*, 3(4):953–973, 2017.

[129] Joseph A. F. Plateau. *Statique expérimentale et théoretique des liquides soumis aux seule forces moléculaires.* Gauthier-Villars, Paris, 1873.

[130] Tibor Radó. On Plateau's problem. *Ann. of Math. (2)*, 31(3):457–469, 1930.

[131] Tibor Radó. Some remarks on the problem of Plateau. *Proc. Natl. Acad. Sci. USA*, 16:242–248, 1930.

[132] Tibor Radó. Contributions to the theory of minimal surfaces. *Acta Sci. Math. Univ. Szeged.*, 6:1–20, 1932.

[133] Tibor Radó. *On the problem of Plateau.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 2. Springer, Berlin, 1933.

[134] Tibor Radó. *Length and Area.* American Mathematical Society Colloquium Publications, Vol. 30. American Mathematical Society, New York (NY), 1948.

[135] Kai Rajala. Uniformization of two-dimensional metric surfaces. *Invent. Math.*, 207(3):1301–1375, 2017.

[136] Constance Reid. *Courant.* Copernicus, New York (NY), 1996. Reprint of the 1976 original.

[137] Yurii G. Reshetnyak. Non-expansive maps in a space of curvature no greater than K. *Sib. Math. J.*, 9:918–927, 1968.

[138] Yurii G. Reshetnyak. Sobolev classes of functions with values in a metric space. *Sib. Math. J.*, 38(3):657–675, 1997.

[139] Yurii G. Reshetnyak. Sobolev classes of functions with values in a metric space. II. *Sib. Math. J.*, 45(4):855–870, 2004.

[140] Bernd Riemann. Über die Fläche vom kleinsten Inhalt bei gegebener Begrenzung. *Abh. K. Ges. Wiss. Gött., Math. Kl.*, 13:3–52, 1867.

[141] Friedrich Sauvigny. Uniqueness of Plateau's problem for certain contours with a one-to-one, nonconvex projection onto a plane. In *Geometric analysis and the calculus of variations*, pages 297–314. International Press, Cambridge (MA), 1996.

[142] Hermann A. Schwarz. Ueber die Minimalfläche, deren Begrenzung als ein von vier Kanten eines regulären Tetraeders gebildetes räumliches Vierseit gegeben ist. *Monatsberichte der Königlichen Akad. Wiss. Berlin*, pages 149–153, 1865.

[143] Tomasz Serbinowski. *Harmonic maps into metric spaces with curvature bounded above.* ProQuest LLC, Ann Arbor(MI), 1995. Ph. D. Thesis–The University of Utah.

[144] Nageswari Shanmugalingam. Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoam.*, 16(2):243–279, 2000.

[145] Max Shiffman. The Plateau problem for minimal surfaces of arbitrary topological structure. *Amer. J. Math.*, 61:853–882, 1939.

[146] Nathan Smale. A bridge principle for minimal and constant mean curvature submanifolds of $\mathbf{R}^N$. *Invent. Math.*, 90(3):505–549, 1987.

[147] Sergei L. Sobolev. Some generalizations of imbedding theorems. *Amer. Math. Soc. Transl. (2)*, 30:295–344, 1963.

[148] Stephan Stadler. The structure of minimal surfaces in CAT(0) spaces. *J. Eur. Math. Soc. (JEMS)*, 23(11):3521–3554, 2021.

[149] Friedrich Tomi. On the local uniqueness of the problem of least area. *Arch. Ration. Mech. Anal.*, 52:312–318, 1973.

[150] Friedrich Tomi and Anthony J. Tromba. Extreme curves bound embedded minimal surfaces of the type of the disc. *Math. Z.*, 158(2):137–145, 1978.

[151] Friedrich Tomi and Anthony J. Tromba. Existence theorems for minimal surfaces of nonzero genus spanning a contour. *Mem. Amer. Math. Soc.*, 71(382):83 pp., 1988.

[152] Anthony J. Tromba. On the number of solutions to Plateau's problem. *Bull. Amer. Math. Soc.*, 82(1):66–68, 1976.

[153] Anthony J. Tromba. On the nonexistence of boundary branch points for minimal surfaces spanning smooth contours II. *J. Fixed Point Theory Appl.*, 10(2):253–277, 2011.

[154] Masatsugu Tsuji. On a theorem of F. and M. Riesz. *Proc. Imp. Acad. Tokyo*, 18:172–175, 1942.

[155] Sophia Wagner. Metric minimizing discs in CAT(κ) spaces. to appear.

[156] Stefan E. Warschawski. Boundary derivatives of minimal surfaces. *Arch. Ration. Mech. Anal.*, 38(4):241–256, 1970.

[157] Karl Weierstrass. Analytische Bestimmung einfach zusammenhängender Minimalflächenstcke, deren Begrenzung aus geradlinigen, ganz im Endlichen liegenden Strecken besteht. In *Mathematische Werke von Karl Weierstrass, Dritter Band*, pages 221–240. Mayer & Müller, Berlin, 1903.

[158] Stefan Wenger. Gromov hyperbolic spaces and the sharp isoperimetric constant. *Invent. Math.*, 171(1):227–255, 2008.

[159] Hermann Weyl. The method of orthogonal projection in potential theory. *Duke Math. J.*, 7:411–444, 1940.

[160] Brian White. The bridge principle for stable minimal surfaces. *Calc. Var. Partial Differential Equations*, 2(4):405–425, 1994.

[161] Brian White. The bridge principle for unstable and for singular minimal surfaces. *Comm. Anal. Geom.*, 2(4):513–532, 1994.

[162] Daniel Wienholtz. A method to exclude branch points of minimal surfaces. *Calc. Var. Partial Differential Equations*, 7(3):219–247, 1998.

[163] Kevin Wildrick and Thomas Zürcher. Peano cubes with derivatives in a Lorentz space. *Illinois J. Math.*, 53(2):365–378, 2009.

STATUTORY DECLARATION

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne die Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Werken dem Wortlaut oder dem Sinn nach entnommen wurden, sind als solche kenntlich gemacht. Ich versichere an Eides statt, dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen und eingebundenen Artikeln und Manuskripten - noch nicht veröffentlicht worden ist sowie, dass ich eine Veröffentlichung der Dissertation vor Abschluss der Promotion nicht ohne Genehmigung des Promotionsausschusses vornehmen werde. Die Bestimmungen dieser Ordnung sind mir bekannt. Darüber hinaus erkläre ich hiermit, dass ich die Ordnung zur Sicherung guter wissenschaftlicher Praxis und zum Umgang mit wissenschaftlichem Fehlverhalten der Universität zu Köln gelesen und sie bei der Durchführung der Dissertation zugrundeliegenden Arbeiten und der schriftlich verfassten Dissertation beachtet habe und verpflichte mich hiermit, die dort genannten Vorgaben bei allen wissenschaftlichen Tätigkeiten zu beachten und umzusetzen. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.

Teilpublikationen: siehe Seite 3.

1.4.2022, Paul Creutz, Paul Creutz