The Value of Supply Chain Visibility when Yield is Random

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To my lovely wife, Astrid,
for her love and patience.

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List of Abbreviations

dist.	distribution
e.g.	Latin: exempli gratia – English: for example
et al.	Latin: et alii – English: and others
GPS	global positioning system
i.e.	Latin: id est – meaning „that is to say“
i.i.d.	independent and identically distributed
PC	personal computer
Prod. Qty.	production quantity
pdf	probability density function
RFID	radio frequency identification
s.t.	such that
VRTYI	value of real time yield information

List of Symbols

b	cost for backordered units per unit per period (> 0)
c	Chapter 3: fixed cost for tracking an order, Chapter 5: per unit production cost
d_t	demand in period t
D	random variable for demand
h	inventory holding cost per unit per period (> 0)
I_t	Chapter 5: on hand inventory at the beginning of period t before replenishment arrives
IL_t	Chapter 2-4: on hand inventory at the beginning of period t before replenishment arrives
$IL^{\min}$	upper bound for backorders
$IL^{\max}$	upper bound for inventory level
k	fixed order cost
O_t	order quantity in period t
$O^{\max}$	upper bound for order quantity
p_i	revenue for one unit of quality i
Q	Chapter 1-4: yield of an order (replenishment quantity), Chapter 5: production quantity (decision variable)

Q_t	yield of order placed in period t
r_i	cost for testing one unit with test process i
s	state variable without yield information
S_t	set of all feasible states s in period t
T	number of considered periods for the finite horizon problem and for simulations
$u_{r,t}$	random yield rate in period t for lead time period $r; r = 1, \dots, \lambda$
$\bar{u}_{r,t}$	expected yield rate in period t for lead time period $r; r = 1, \dots, \lambda$
$u_{1 \cdots \lambda,t}$	random yield rate over all lead time periods in period t
$\bar{u}_{1 \cdots \lambda,t}$	expected random yield rate over all lead time periods in period t
u_{ikj}	Chapter 5: random yield rate of product k for input from soft bin i tested with test process j
v_i	capacity consumption for testing one unit with test process i
V	available capacity per period
X_{ij}	input quantity for test process j from soft bin i
y_t	on hand inventory in period t after replenishment and before demand is satisfied
Y	Chapter 1-4: On hand inventory after replenishment. Chapter 5: Yield matrix for first stage production.
z	state variable with yield information
Z_t	set of all feasible states z in period t
β	Chapter 3: inflation factor
γ	discount factor with $0 \leq \gamma < 1$

θ	order threshold
λ	lead time
μ	mean
ρ	coefficient of variation
σ^2	variance
Ψ	binary decision variable, whether or not to track an order
$C(x)$	inventory cost of the current period
$V_t(z_t)$	minimal expected cost from period t to period T, given the current state.
$H_t(z_t, O_t)$	total cost function in period t depending on the current state and the order quantity
$\mathbb{I}[*]$	Indicator function. Returns 1 if the expression in brackets is true and 0 else.

Chapter 1

Introduction

1.1. Motivation - Supply Chain Visibility and Random Yield

This dissertation focuses on inventory systems with supply uncertainty due to random yields. A common assumption in such systems is that the yield of an order is observable only upon arrival of this order. The rapid dissemination of sensor, communication, and control technologies, such as RFID and GPS tracking, throughout all stages of the supply chain provides an increasing amount of data in real time. Our main interest is the determination of the value of real time information in the context of random yields as well as in the development of algorithms for its proper use. We want to contribute with our work to make real time control of supply chains possible in the near future.

A significant portion of freight (e.g. food and pharmaceutical products) is perishable and therefore subject to yield uncertainty. Damage in transit and pilferage can also contribute to a reduction of the replenishment quantity. Estimates vary between 20 % and 30 % of food lost, wasted or discarded in the supply chain (Dobbs et al., 2011, p. 93, Green and Johnston, 2004, p. 35, Gustavsson et al., 2011, p. 6). These numbers exclude food waste at consumer stage. About 25 % of all vaccine products spoil before reaching their destinations. This loss is most often due to broken cold chains during distribution (White and Cheong 2012).

Pharmaceutical regulatory officials in the UK state that 36 % of all major and critical findings during inspections in 2003/2004 are related to the control and monitoring of temperature during shipping and distribution (Bishara 2006). We show in Chapter 3 and 4 of this thesis that the ability to observe the yield of in-transit products in real time leads to significant cost reductions.

There are also a variety of production processes (e.g. in semiconductor manufacturing and chemical production) that produce a significant and random percentage of unusable products. A semiconductor supply chain produces 30-50 % unusable items. In addition, usable items have random quality levels. Long lead times (3-12 months), short product life cycles, and restricted capacities add further to the complexity of the semiconductor production system (Gavirneni 2004; Han, Dong, and Shao 2011; Taouil et al. 2012). Other examples for production processes with random yield include the production of pharmaceuticals where the concentration of the active ingredients follows a random distribution or the coating of metal where the thickness of different layers is subject to uncertainty. It is challenging to make lot sizing decisions that take into account that the output of useable products is only randomly related to the input quantity. In Chapter 5 we model a semiconductor production system in such a context and focus on the value of in-process inspections. In-process inspections can reveal the current yield of products and are in that sense another application of real time yield information.

Many new information technologies suffer from the so called "credibility gap" (Lee and Özer 2007). This is the gap between the proclaimed benefits of the new information and the understanding of how these benefits can actually be realized. We are motivated to close this gap for real time yield information by developing optimal and heuristic inventory policies and identifying conditions under which real time yield information is particularly beneficial.

1.2. Outline

This section describes the general structure and the research focus of this thesis. The thesis consists of three main chapters which are preceded by an introduction and a chapter on the foundations of inventory management under random yield. In the last chapter we conclude and summarize the key results.

The three main chapters share the overall common topic of real time yield information. However, they represent mostly independent research and can be read independently. The structure of these chapters is similar. The first section consists of an abstracts that provides an overview of the chapter. The second section in each chapter introduces the problem, motivates its relevance and provides the connection to existing literature. The proceeding section develops the mathematical model, which is followed by optimal and heuristic solution approaches if applicable. The next section provides a numerical analysis and a discussion of the main findings. Each chapter is completed by a conclusion that provides a summary of the results. A brief overview of each chapter follows.

Chapter 2 provides the basic knowledge to understand the key issues of inventory planning under random yield. Different types of yield risk require different yield models which are addressed in Chapter 2.1. The main challenge when dealing with inventory systems under random yield is that myopic solutions are no longer optimal. This key insight is elaborated in Chapter 2.2. The effects on cost modeling in inventory system under random yield are discussed in Chapter 2.3.

Chapter 3 analyzes the value of real time yield information in a periodic review inventory system.* The inventory model is developed as a dynamic program and structural

* This chapter is joint work with Prof. Ulrich W. Thonemann and was published as:
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properties are provided. Using these properties an optimal and two heuristic solution approaches are developed. Numerical results evaluate the performance of the heuristics and provided insight on the dependency between the value of real time yield information and the system's parameters. In an extension the effect of fixed order costs is analyzed.

Chapter 4 extends the base case of Chapter 3 by introducing costs for real time yield information and by modeling tracking as a decision variable on an order-by-order basis.[†] The mathematical model is developed and the structure of the objective function is evaluated. An optimal solution approach is used to elaborate on conditions under which real time yield information with flexible tracking is particularly beneficial and to identify the key drivers for the tracking decision.

Chapter 5 is motivated by the production system of a global semiconductor manufacturer.[‡] Semiconductor production has two main features: the first feature is random yield, the second feature is co-production. Products of different quality levels are produced simultaneously at random yield rates. Products must be tested in dedicated test processes to observe their quality level. These test processes are capacity restricted. This led to the idea of pre-testing products to make more efficient use of limited test capacities. The pre-test is an inexpensive and quick process that discloses preliminary yield information. That is, the final quality of products is detected with some probability. This makes it, for example, possible to avoid testing products for a high quality level that most likely fail this test. Building up on structural results of the mathematical model, an optimal and a heuristic solution approach are developed. They are used to evaluate the value of preliminary yield information and its dependencies.

[†] This chapter benefited from discussions with Prof. Ulrich W. Thonemann and Michael Völkel, M.Sc.

[‡] This chapter benefited from discussion with Prof. Candace A. Yano and Prof. Ulrich W. Thonemann

Chapter 6 summarizes the key results of this thesis and provides a general outlook for future research in the field of inventory management under random yield.

1.3. Contribution

A piece of research can contribute to the literature, for instance, because it deals with a new and innovative problem, models and solves a real problem, or contains an innovative solution approach. The topic of Chapter 3 and 4, real time yield information in inventory management, is rather new and connected to recent advances in information technology. Especially the RFID technology and the progress of concepts like “the internet of things” are enabler for the kind of supply chain visibility that is considered in this thesis. Much research has been done on real time information in different contexts but to the best of our knowledge White and Cheong (2012) are the only authors that address real time yield information in the same context as we do. Their work is mainly focused on vehicle routing decisions, whereas our work focuses on inventory management.

Another driver for real time yield information is supply chain collaboration and, in this context, information sharing between supplier and customer. A supplier that shares in-process production yields in real time with its customer can significantly improve supply chain performance. Upstream information sharing has received much less attention in literature than downstream information sharing (e.g. a retailer shares point-of-sale data with its supplier). To our best knowledge, Hyun-cheol Paul Choi et al. (2008) is the only paper that considers sharing of real time yield information in inventory management. However, their analysis is limited to one heuristic solution and does not consider the cost of information sharing.

The problem in Chapter 5 is motivated by a production process of a global semiconductor manufacturer. We add to the existing literature on semiconductor manufacturing by developing a model that differentiates between the production process and the test process. Because we treat both processes individually, our model allows for multiple test runs while in parallel a production run is in progress. Our model reflects the industry practice more accurately and enables the analysis of parameter and process changes with higher precision than existing models that consider production and testing as a single step. In this context we introduce and analyze the concept of preliminary yield information as a mean to enable more efficient use of limited test capacity. Although the semiconductor production problem is not new, we model it in a new way, solve a real world problem and analyze the potential advantages of (preliminary) real time yield information.

For all models we develop optimal solution approaches and implement them to perform numerical analysis. To solve larger problems we propose several new heuristics and elaborate on their performances. These heuristics facilitate the transfer of our research into real world applications.

Chapter 2

Foundations of Inventory Management under Random Yield

2.1. Modeling of Random Yield

The way random yield is modeled depends on the analyzed random yield process. Next we describe the six most prominent approaches: Bernoulli process, proportional yield, additive yield, decreasing yield, increasing yield, and random capacity. Most research focuses on one of the first two processes. These processes are therefore discussed in greater detail.

The most intuitive yield process is a Bernoulli process. Each unit has an all-or-nothing yield rate of u. The yield rate coefficient u is independent of the order quantity O. Placing an order of O units yields in expectation $E[Q] = uO$ with a variance of $Var[Q] = Ou(1-u)$. The assumption for this process is that the yield of each unit is independent of the yield of all other units, i.e. the yield rates are not correlated between units within one batch (and also not between batches). Note, that the variance of the fraction of good units decreases in O. Due to the law of large numbers, the more units ordered the more likely it is to get a result close to the expected value $E[Q]$.

Examples for this process are the transportation of containers with cooling units or raw materials with imperfections. Under the assumption that the risk for a failure of an individual cooling unit is only depending on the state of this cooling unit, like age or time since last inspection, the number of containers with working cooling units follows a Bernoulli process. Another yield process that resembles a Bernoulli process is the processing of raw materials with imperfections, like a cavity in a metal, stone or wood. If the cavity becomes observable during production, e.g. during cutting, it makes the product useless. The risk for cavities is not correlated among raw material units. Articles that feature this kind of yield process include Hadjinicola (2010), Shang, Tsung, and Zou (2013), and Tang (1990).

The yield model, most commonly analyzed in literature on inventory management under random yield, is the proportional yield model. A yield process with proportional yield is described by a random yield rate u that is independent of batch size O. The yield expectation is $E[Q] = E[u]O$ and the variance is $Var[Q] = Var[u]O^2$. Note, that the variance is quadratic in the order quantity which is kind of a worst case scenario. This reflects the underlying assumption that the yield of units within on order is perfectly correlated. That means each unit is spoiled or good to the same degree. E.g. the observation that 80 % of the units within an order are fresh, means that each individual unit is 80 % fresh and 20 % spoiled.

Examples for this process are all processes that endure a systemic risk. Recall the example of transporting containers with cooling units. This time we consider the units within one container. These units have all the same temperature risk. If the temperature deviates from the optimal range, all units decay to the same degree. A production process with proportional yield can be found e.g. in semiconductor manufacturing. Due to the complex production processes and high quality requirements in semiconductor production systems, the resulting yield is often a random fraction of the input quantity. Articles that feature this kind of yield process include Federgruen and Yang (2014), Huang and Song (2010), and White and Cheong (2012).

The expected output under additive yield is $E[Q] = O + E[u]$ with variance $Var[Q] = Var[u]$, where u is a random variable independent of batch size O. Note, that in contrast to the previous yield processes the variability of the replenishment quantity Q is independent of batch size. Because $E[u] \neq 0$ results in a simple shift of the order quantity, $E[u]$ can be set to zero, without loss of generality. Examples for this yield process are handling errors that lead to inventory inaccuracies. Imagine, for instance, a supplier that ships 9 instead of 6 units, because of a type error in the order document. Theft of units during transportation can also be modeled with such a yield process. Additive yield is similar to modeling a system with two random demand streams. The articles by Graves, Meal, Dasu, and Qui (1986) and Rekik, Sahin, and Dallery (2009) are examples for considering an additive yield process.

The next three yield processes are no longer independent of the batch size. For decreasing yield rates the fraction of expected good units decreases in batch size. This pattern is applicable for production processes that have an increasing failure rate the longer they run. Examples are processes with deteriorating production equipment, like tools that get less precise or have a higher probability to break the longer they are used. Sample articles for this yield process include Glock and Jaber (2013), Lee (1992), and Zhu, Zhang, and Tsung (2007).

The opposite of the aforementioned is a process with increasing yield rates. This pattern can be observed when the production process needs to be calibrated in the beginning, e.g. in a trial-and-error fashion. Until the process is calibrated the risk for defective units is higher compared to the risk after the calibration is completed. Examples are finding the correct setting for milling machines or the correct temperature to provoke a chemical reaction. Learning curve effects can also be modeled by increasing yield rates. The modeling of settings with increasing yield rates can be done similar to models with decreasing yield rate. Literature on models with increasing yield rates is extremely scarce.

In processes with random capacity the output is the minimum of the input quantity and the random capacity. The chosen input quantity influences the yield distribution by defining an upper limit for the yield. Examples are production processes where some machinery is not available for a random period of time due to unplanned maintenance. Articles that feature this kind of yield process include Fu, Sun, Lai, and Leung (2014), Hwang and Singh (1998), and Iida (2002).

2.2. Implications of Random Yield on Inventory Management

Our objective is to explain the implications of random yield as intuitively as possible. We refer the reader to the cited references for a more thorough analysis of this topic. We start by recalling the principals of periodic inventory management with perfect yield, random demand and zero lead time. Then we point out the differences that are caused by the presence of random yield. We proceed from one period models to multi period models.

Under certain yield the sequence of events is as follows. At the beginning of each period an order of quantity O is placed. This order arrives instantaneously. After order arrival demand d is realized. Demand is satisfied from on hand inventory $Y = IL + O$ that is the sum of inventory level IL and order quantity O. Unsatisfied demand is backordered. Based on the net inventory at the end of the period, backorder cost b or inventory holding costs h are charged.

The first observation is that the optimal policy can be found by focusing solely on one period. The key sufficient condition for a myopic optimum is that given the current action, the current state has no influence on the next state (Heyman and Sobel, 1984, p. 84). At the beginning of the current period the system is in state IL. The decision about the order quantity O is in fact a decision about the on hand inventory Y for the current period, since $Y = IL + O$.

Because an arbitrary order can be placed, Y depends only on O. In theory even negative orders are possible, but not needed in this case.[§] Therefore the decision about O can be substituted by a decision about Y. The inventory level in the next period depends only on this decision (and an independent demand distribution). This myopic nature of the problem simplifies the analysis significantly. The objective function for one period is

$$\begin{aligned}\min_{O\geq 0} E_d[C(IL+O-d)] &= E_d[h[IL+O-d]^+ + b[d-IL-O]^+] \\ &= \min_{Y\geq 0} E_d[C(Y-d)]\end{aligned} \tag{1}$$

with $[x]^+ = \max(0,x)$. It is well established that this resembles the basic newsvendor model with $\theta^* = IL + O^* = F^{-1}(b/(b+h))$ as optimal solution, where F^{-1} denotes the inverse cumulative probability distribution of the demand over one period. The resulting optimal policy is an order-up-to policy: If IL is below the order threshold θ^*, then an order is placed with $O^* = \theta^* - IL$. Else O^*= 0.

Next, we focus on the implications of random yield on a single period model. We analyze the case of proportional yield, because this case is most relevant in literature and in this thesis. The conclusions also hold for all other yield models introduced in the previous section. We focus on the order threshold first. Using the order threshold under certain yield θ^*, we analyze if it can still be applied under uncertain yield. Assume $IL \geq \theta^*$. Using the optimal results from the certain yield model we know that for $IL \geq \theta^*$, $E_d[C(IL+uO-d)] \geq E_d[C(IL+0-d)]$ for any yield realization of u. It is more costly to order than not to order. Therefore, it cannot be optimal to order when $IL \geq \theta^*$.

Next we consider $IL < \theta^*$ and compare the decision to order the order-up-to quantity $O = \theta^* - IL$ with the decision not to order, $O = 0$. Using the same logic as before we

§ Intuitively, the inventory level at the beginning of the period will not be higher than the optimal value for Y, because this would cause unnecessary holding costs.

get $E_d[C(IL+0-d)] \geq E_d[C(IL+u(\theta^*-IL)-d)]$, because $(\theta^*-IL) \geq u(\theta^*-IL) \geq 0$ for any realization of u. From this reasoning we can conclude that in the one period model with random yield an order threshold θ^* exists and that it is the same as under certain yield. A more formal proof of this fact is provided in Corollary 1 from Henig and Gerchak (1990).

It is intuitive that if θ^* is the same for certain and uncertain yield the optimal policy cannot be of an order-up-to type. Under random yield there is an additional source of uncertainty and using an order-up-to policy that ignores this fact cannot be optimal. Therefore the order quantity under random yield must be larger than the order quantity under certain yield (see also Henig and Gerchak, 1990, Corollary 2). The computation of O is rather complex and there exists no closed form solution. To summarize the single period case: The optimal policy is not of an order-up-to type as for certain yield. The order threshold is the same as for certain yield but the order quantity is a complex function that cannot be solved analytically.

In the multi period case we lose another important feature. In the case of certain yield we observed a myopic optimum. Under random yield the stochastic independence between states in consecutive periods is lost. Given the state IL and the decision about the order quantity O, the available inventory level for the current period is defined as: $Y = IL + uO$. Y is a random variable with mean $E[y] = IL + E[u]O$ and variance $Var[y] = Var[u]O^2$. Because the uncertainty is caused only by the order quantity (and not by $IL+O$) we can no longer substitute the decision about O by a decision about Y. The inventory level of the next period depends stochastically on both, the order quantity and the current inventory level. Example 2-1 illustrates this fact for normal distributed random variables. This kind of problem typically requires dynamic programming. The recursive cost function V for the infinite horizon under random yield is

$$V(IL) = \min_{O\geq 0}\{E_u E_d[C(IL+uO-d) + \gamma V(IL+uO-d)]\} \qquad (2)$$

γ denotes the discount factor. Henig and Gerchak (1990) show that for this infinite horizon problem a stable optimal policy exists. This policy consists of an order threshold (which is larger than θ^*) and of an order quantity $O(IL)$ that depends on the inventory level. For both values an analytical solution is not available.

Example 2-1 Illustration of loss of condition for myopic optimum under random yield

Idea: The state of the system is defined by the current inventory level IL. We consider two different states for the current period: $IL_a \neq IL_b$ and analyze their impact on the probability distribution for the state in the next period IL^+. If a myopic optimum exists, the probability distribution for the state in the next period $N\left(\mu_{IL^+},\ \sigma_{IL^+}\right)$ is not depending on the state in the current period.

Given: $IL_a \neq IL_b,\ O_a \neq O_b,$ random demand $D = N\left(\mu_d,\ \sigma_d\right),$ random yield rate $U = N\left(\mu_u,\ \sigma_u\right)$

Wanted: Probability distribution for the inventory level in the next period $IL^+ = N\left(\mu_{IL^+},\ \sigma_{IL^+}\right)$

For certain yield we get

$$\text{Current state} = IL_a\text{: } IL^+ = IL_a + O_a - N\left(\mu_d,\ \sigma_d\right) = Y - N\left(\mu_d,\ \sigma_d\right) = N\left(Y - \mu_d,\ \sigma_d\right) = N_1\left(\mu_{IL^+},\ \sigma_{IL^+}\right)$$

$$\text{Current state} = IL_b\text{: } IL^+ = IL_b + O_b - N\left(\mu_d,\ \sigma_d\right) = Y - N\left(\mu_d,\ \sigma_d\right) = N\left(Y - \mu_d,\ \sigma_d\right) = N_2\left(\mu_{IL^+},\ \sigma_{IL^+}\right)$$

$$\boldsymbol{N_1\left(\mu_{IL^+},\ \sigma_{IL^+}\right) = N_2\left(\mu_{IL^+},\ \sigma_{IL^+}\right)}$$

For uncertain yield we get

$$\text{Current state} = IL_a\text{: } IL^+ = IL_a + UO_a - N\left(\mu_d,\ \sigma_d\right) = N(IL_a + \mu_u O_a, \sigma_u O_a) - N\left(\mu_d,\ \sigma_d\right)$$
$$= N\left(IL_a + \mu_u O_a - \mu_d, \sqrt{\sigma_d^2 + \sigma_u^2 O_a}\right) = N_3\left(\mu_{IL^+},\ \sigma_{IL^+}\right)$$

$$\text{Current state} = IL_b\text{: } IL^+ = IL_b + UO_b - N\left(\mu_d,\ \sigma_d\right) = N(IL_b + \mu_u O_b, \sigma_u O_b) - N\left(\mu_d,\ \sigma_d\right)$$
$$= N\left(IL_b + \mu_u O_b - \mu_d, \sqrt{\sigma_d^2 + \sigma_u^2 O_b}\right) = N_4\left(\mu_{IL^+},\ \sigma_{IL^+}\right)$$

$$\boldsymbol{N_3\left(\mu_{IL^+},\ \sigma_{IL^+}\right) \neq N_4\left(\mu_{IL^+},\ \sigma_{IL^+}\right)}$$

Conclusion: For the certain yield case the probability distribution for the next state is the same for both current states, IL_a and IL_b. This is not true for the uncertain yield case.

2.3. Implications of Random Yield on Cost Modeling

With very few exceptions (e.g. Bitran and Leong 1992) the minimization of expected cost is the objective in literature on inventory management with random yields. We therefore discuss in this section the cost modeling under random yield. The influence of random yield on costs depends on the analyzed process and the underlying yield model. As each process and each yield model requires a special cost modeling approach, we cannot discuss all possibilities in detail and concentrate on the commonly used cost modeling approaches that are relevant in the context of this thesis. Typically variable unit cost, cost of handling defective units, inspection cost and inventory holding cost are effected by random yield.

The modeling of variable unit cost depends on whether or not the decision maker has to pay the cost for defective units. In inventory settings where the quantity received is not equal to the quantity ordered, the customer usually pays for good units only and rejects all defective units. This assumes that at some point in time the customer can observe the yield of each unit. Examples for this type of cost modeling are the models in Chapter 3 and 4 and models studied by Yigal Gerchak (1992) and Huh and Nagarajan (2010). In production processes variable unit costs are typically charged on the input quantity. The assumption is that raw material costs and production costs have to be paid independently of the yield outcome. Examples for such cost modeling can be found in Chapter 5, Federgruen and Yang (2014), and Han, Dong, and Shao (2012).

The costs for handling defective units include scraping costs and costs for rework. In some cases defective units might have a salvage value. These costs can either be modeled explicitly or can be included in the calculation for costs of producing one good unit. For a detailed discussion see e.g. Hadjinicola (2010). In this thesis we follow the commonly used assumption that defective units are discarded at no cost (e.g. Bollapragada and Morton 1999; Choi, Blocher, and Gavirneni 2008; Ferrer and Ketzenberg 2004).

In some cases determining the quality of a unit can be costly. These costs are referred to as inspection costs. The modeling of inspection costs depends on the yield model and the production process. For instance, under binomial yield inspection costs are modeled as cost per inspected unit. The correct modeling approach under proportional yield is a fixed cost for inspecting an entire batch independent of its quantity (see e.g. Chapter 4 or White and Cheong 2012).

The possibility to inspect units influences the modeling of holding costs. If the yield of all units is observable defective units can be returned to the supplier or discarded. In this case holding cost is only charged on good units (Chapters 3 and 4). If inspection is not possible or the inspection process is imperfect, holding costs are also charged on defective units until they are identified and discarded. An example for such an imperfect inspection process is the pre-test, implemented in the semiconductor production process that is analyzed in Chapter 5.

Chapter 3

The Value of Real Time Yield Information in Multi-State Inventory Systems

Chapter 3 was published as:

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3.1. Abstract

We consider a random yield inventory system, where a company has access to real time information about the actual yield realizations. To contribute to a better understanding of the value of this information, we develop a mathematical model of the inventory system and derive structural properties. We build on these properties to develop an optimal solution approach that can be used to solve small to medium sized problems. To solve large problems, we develop two heuristics. We conduct numerical experiments to test the performances of our approaches and to identify conditions under which real time yield information is particularly beneficial. Our research provides the approaches that are necessary to implement inventory control policies that utilize real time yield information. The results can also be used to estimate the cost savings that can be achieved by using real time yield information. The cost savings can then be compared against the required investments to decide if such an investment is profitable.

Keywords: inventory management; random yield; value of information

3.2. Introduction and Literature Review

We consider an inventory system, where replenishment orders are subject to random yield. Random yields are an important issue in many procurement, production, and assembly processes (Yano and Lee 1995). In the food or chemical cold chain, for instance, products are shipped over long distances in refrigerated containers. If the temperature of the product leaves a certain range, the product is spoiled and must be re-ordered. Another example is the semiconductor industry, where production steps are subject to random yield (Wang 2009).

Recently, technologies have been developed that collect and transmit data about the state of a product in the order pipeline. In cold chains, smart sensors are used to monitor the temperature of products and to inform customers immediately if the temperature leaves a pre-defined range (Zacharewicz et al. 2011). White and Cheong (2012), for instance, consider a food supply chain that requires this type of supply chain visibility. They quantify the benefit of observing the quality of a perishable product that is processed in multiple steps from origin to destination. At each step during the journey the decision has to be made whether or not to inspect the quality of the product at a certain cost and whether or not to continue the transport. More application examples of technologies that enable real time yield information sharing in this context can be found in Hsueh and Chang (2010).

Real time yield information is also relevant in production processes. Consider a supplier that manufactures a product in several production steps, where each step has random yields. The customer of the supplier considers this risk when placing orders with the supplier and therefore determines the input quantity for the supplier’s first production step. The supplier holds no inventory (except work in progress) and shares yield information after each production step with the customer. Gavirneni (2004), Inderfurth and Vogelgesang (2013), and Wang (2009) provide details of such a process in the semiconductor industry. Choi et al. (2008), for instance, consider real time yield information sharing in such a context. However, collecting and transmitting real time yield information requires investments in information technology. To decide whether or not such investments are profitable, the value of using real time yield information must be quantified and we address this topic in this paper.

Research on random yield inventory models can be traced back to Karlin (1958). Karlin (1958) considers a single period inventory system where the yield of an order is a random variable with a known distribution and where order decisions are binary. The structure of the *optimal random yield policy* for inventory systems with zero lead time has been derived by Gerchack et al. (1988) and Henig and Gerchak (1990). Gerchack et al. (1988) analyze a finite horizon periodic review problem and show that the optimal policy is complex and not myopic. They determine the optimal solution by dynamic programming. Henig and Gerchak (1990) derive structural results for the finite and infinite horizon problems and show that there exists a threshold for each period, such that an order is placed if and only if the on-hand inventory is below the threshold value. They show that the threshold is higher under stochastic yield than under deterministic yield. An overview of periodic review systems with random yield can be found in Yano and Lee (1995).

Because large problems cannot be solved optimally in reasonable time, research has also addressed the development of *random yield heuristics*. Many of these heuristics rely on myopic linear inflation policies (Huh and Nagarajan 2010). These policies use an order threshold and an inflation factor: If the inventory level is below the order threshold, then the difference between the order threshold and the inventory level multiplied by an inflation factor is ordered. A seminal article in this area is by Bollapragada and Morton (1999). They develop three myopic heuristics that are based on the solution of a newsvendor model with random yield. For a discounted cost model, Li et al. (2008) develop upper and lower bounds for the optimal order threshold and the order quantity. They use these bounds in a heuristic that outperforms the heuristics of Bollapragada and Morton (1999). Huh and Nagarajan (2010) show how the optimal order threshold of a linear inflation policy can be computed for a given inflation factor.

The existing literature on optimal and heuristic solutions considers models with zero lead time, an assumption under which real time yield information sharing is not an issue. In inventory systems with positive lead times, real time yield information sharing can improve performance. To our best knowledge, Choi et al. (2008) is the only article that analyzes the value of real time yield information sharing in settings with positive lead times. They consider a supply chain with a single supplier and a single manufacturer. The supplier uses a manufacturing process with two processing steps with random yields. Translated to a supply chain setting, their model corresponds to an inventory model with a lead time of three periods, where the first two periods are subject to random yield. To solve the model, Choi et al. (2008) modify one of the heuristics of Bollapragada and Morton (1999).

We also consider a model with positive lead time and allow for an arbitrarily long lead time. Unlike previous research, we derive structural properties of the objective function and prove the existence of a stationary optimal policy for the infinite horizon problem. We show that the objective function is convex and build on this property to optimally solve small and medium sized problems. To solve large problems, we develop two heuristic solution approaches based on linear inflation policies. The first heuristic builds on the MULT-heuristic that was first proposed by Ehrhardt and Taube (1987). The second heuristic is based on the work of Huh and Nagarajan (2010). We provide numerical results that indicate that our heuristics perform well in a variety of settings and we identify conditions under which real time yield information is particularly beneficial.

Related to our research is the research on RFID. For a comprehensive literature review we refer to Lee and Özer (2007), Ngai et al. (2008), and Sarac et al. (2010). For a literature review on applications of RFID technology we refer to Zhu et al. (2012). To analyze the value of increased supply chain transparency few analytical models have been developed. Our paper derives an analytical model and quantifies the value of real time yield information and we contribute to the filling of the frequently cited credibility gap of the value of RFID (H. Lee and Özer 2007; Sari 2010).

The remainder of the paper is organized as follows. In Section 3.3, we develop a dynamic program for a periodic review inventory system with random yields. In Section 3.4, we discretize the state space and use a Markov decision process to compute the optimal solution. In Section 3.5, we develop heuristic solution approaches. In Section 3.6, we provide numerical results. In Section 3.7, we discuss the value of real time yield information in detail. In Section 3.8, we extend our analysis for the case where fixed order cost is charged. In Section 3.9, we conclude. All proofs can be found in the Appendix.

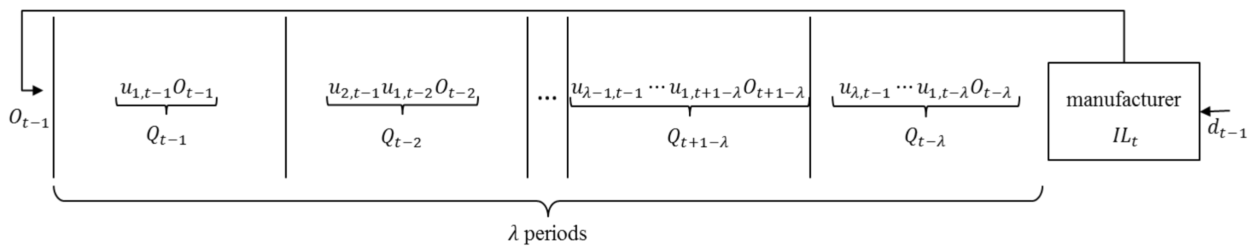
3.3. Model Formulation

We first consider a supply chain with real time yield information sharing (Subsection 3.3.1) and analyze the finite horizon version and the infinite horizon of the problem. We consider both versions of the problem, because each version has properties beneficial in our analyses. For the finite horizon version, we prove the convexity of the value function. We build upon this property to derive the stationary optimal policy for the infinite horizon version, which allows us to compute the optimal expected cost with arbitrary accuracy. One of our objectives is to analyze the value of using real time yield information, which requires us to compare the cost of a supply chain that utilizes real time yield information with the cost of a supply chain that does not utilize this information. Therefore, we also analyze a supply chain without real time yield information (Subsection 3.3.2), again for both the finite horizon version and the infinite horizon version of the problem.

3.3.1. Model with Real Time Yield Information

Consider a single manufacturer who places orders with a single supplier. The demand d_t of the product is stochastic and i.i.d. across periods. We denote the order quantity in period t by O_t and orders arrive after a lead time of λ periods. In each lead time period, orders are subject to random yields. The yield rate of lead time period r ($r = 1, \dots, \lambda$) in period t is $u_{r,t}$. Order $O_{t-\lambda}$ placed in period $t-\lambda$ experiences λ random yields and the replenishment quantity $Q_{t-\lambda}$ in period t is $Q_{t-\lambda} = u_{\lambda,t-1}\, u_{\lambda-1,t-2} \cdots u_{1,t-\lambda} O_{t-\lambda}$. The yield rates $u_{r,t}$ are i.i.d. over time and can be arbitrarily distributed. For ease of presentation, we will drop the index t in $u_{r,t}$ whenever it is appropriate. This yield model is commonly used to analyze the random yield inventory problem, e.g. Choi et al. (2008), Ehrhardt and Taube (1987), and Gerchack et al. (1988).

Figure 3-1 Information set at the beginning of period t with real time yield information



The sequence of events in each period is as follows: First, the manufacturer observes the current state of the inventory system $z_t = (IL_t, Q_{t-\lambda}, \dots, Q_{t-1})$, which consists of the inventory level IL_t and the current yield of the λ outstanding orders (Figure 3-1). Then, the manufacturer decides on the order quantity of the current period and orders, O_t. Next, the manufacturer receives the order that was placed λ periods ago, $Q_{t-\lambda}$. The manufacturer satisfies demand and backorders excess demand. Based on the net inventory IL_{t+1} at the end of period

t, backorder or inventory holding costs are charged. With this sequence of events, there are $\lambda+1$ state variables. The intricate dynamics make it impossible to reduce the state to a single variable. All notation is summarized in Appendix 3.A.

Finite horizon model

We formulate the finite horizon version of the optimization problem as a dynamic program. Given the current state z_t, the objective is to determine the order quantities for the current and all future periods, such that the sum of expected inventory holding and backorder costs is minimized:

$$V_t(z_t) = \min_{O_t \geq 0} H_t(z_t, O_t), \tag{3}$$

$$\text{with } H_t(z_t, O_t) = E_{d_t}[C(IL_t + Q_{t-\lambda} - d_t)] + \gamma E_{u_{1,t}} \cdots E_{u_{\lambda,t}} E_{d_t}[V_{t+1}(z_{t+1})].$$

γ denotes the discount factor. Without loss of generality, we assume that $V_{T+1}(z_{T+1}) = 0$. The total cost function $H_t(z_t, O_t)$ is the sum of the expected cost of the current period $E_{d_t}[C(IL_t + Q_{t-\lambda} - d_t)]$ and the minimum expected cost from periods $t+1$ to T, $E_{u_{1,t}} \cdots E_{u_{\lambda,t}} E_{d_t}[V_{t+1}(z_{t+1})]$. The cost of the current period is $C(x) = h[x]^+ + b[-x]^+$, with $[x]^+ = \max(0, x)$. Because $E_{d_t}[C(IL_t + Q_{t-\lambda} - d_t)]$ is not affected by the current order decision, the dynamic program can be written as

$$V_t(z_t) = E_{d_t}[C(IL_t + Q_{t-\lambda} - d_t)] + \min_{O_t \geq 0} \gamma E_{u_{1,t}} \cdots E_{u_{\lambda,t}} E_{d_t}[V_{t+1}(z_{t+1})]. \tag{4}$$

The transition function is

$$z_{t+1} = f_z(z_t, O_t, d_t, u_{1,t}, \cdots, u_{\lambda,t}) = (IL_t + Q_{t-\lambda} - d_t, u_{\lambda,t} Q_{t+1-\lambda}, \cdots, u_{1,t} O_t). \tag{5}$$

Theorem 3-1 states that the total cost function $H_t(z_t, O_t)$ and the minimal cost function $V_t(z_t)$ are convex. The proof is by induction and can be found in Appendix 3.B.

Theorem 3-1. $H_t(IL_t, Q_{t-\lambda}, \cdots, Q_{t-1}, O_t)$ is convex in IL_t, Q_{t-r}, $r = 1, \cdots, \lambda$, and O_t. $V_t(IL_t, Q_{t-\lambda}, \cdots, Q_{t-1})$ is convex in IL_t and Q_{t-r}, $r = 1, \cdots, \lambda$.

In the next subsection, we use the results of Theorem 3-1 to show that there exists a stationary optimal policy for the infinite horizon model and we utilize the convexity in the solution algorithm.

Infinite horizon model

We analyze the infinite horizon model by analyzing the finite horizon problem for $T \rightarrow \infty$. For the infinite horizon model, the state variable is $z = (IL, Q_{-\lambda}, \dots, Q_{-1})$, where Q_{-i} is the current yield of the order placed i periods ago. The recursive cost function V(z) is the limiting function of Equation (3) as $T \rightarrow \infty$. $V(z)$ is defined as

$$V(z) = V(IL, Q_{-\lambda}, \dots, Q_{-1}) =$$
$$\min_{O\geq 0}\{E_d[C(IL + Q_{-\lambda} - d)] + \gamma E_{u_1} \cdots E_{u_\lambda} E_d[V(IL + Q_{-\lambda} - d, u_\lambda Q_{-\lambda+1}, \cdots, u_1 O)]\}. \tag{6}$$

We next show in Lemma 3-1 that for each $z \epsilon Z$, $V_t(z)$ converges to $V(z)$ as $T \rightarrow \infty$. Building on this result, we prove in Theorem 3-2 that the limit function is the unique solution of Equation (6).

Lemma 3-1. For each $z \epsilon Z$ there exists a limit function $V(z) = \lim_{T\rightarrow\infty} V_t(z)$.

To prove the existence of an optimal stationary policy in Theorem 3-3, we require the limit function of Lemma 3-1 to solve the functional equation of the dynamic program. This is proven in Theorem 3-2.

Theorem 3-2. For each $z \epsilon Z$ the limit function $V(z) = \lim_{T\rightarrow\infty} V_t(z)$ satisfies Equation (6).

Theorem 3-3. For each $z \epsilon Z$ there exists an optimal stationary policy $O^*(z) = \lim_{T\rightarrow\infty} O_t(z)$ and $V(z)$ is its return function.

We have shown that a stationary optimal policy exists and in Section 3.4 we demonstrate how it can be computed. Because of the stochastic nature of the demand and the yield rate distribution, the structure of the optimal policy is too complex to obtain further analytical results.

3.3.2. Model without Real Time Yield Information

Without real time yield information, the manufacturer observes in period t the current inventory level $\mathrm{IL_t}$ and the open order quantities $O\_(t-\lambda), \dots, O_{t-1}$. We denote the state by $s_t = (IL_t, O_{t-\lambda}, \dots, O_{t-1})$. The objective function of the finite horizon model is

$$V_t(s_t) = \min_{O_t \geq 0} H_t(s_t, O_t) =$$
$$\min_{O_t \geq 0}\{E_{u_{1\cdots\lambda,t}} E_{d_t}[C\big(IL_t + u_{1\cdots\lambda,t} O_{t-\lambda} - d_t\big)] + \gamma E_{u_{1\cdots\lambda,t}} E_{d_t}[V_{t+1}(s_{t+1})]\}, \tag{7}$$

where $u_{1\cdots\lambda,t}$ denotes the yield rate over the lead time that is observed in period t with pdf $v(u_{1\cdots\lambda})$. We assume that $V_{T+1}(s_{T+1}) = 0$.

The transition function is

$$s_{t+1} = f_s\big(s_t, O_t, d_t, u_{1\cdots\lambda,t}\big) = (IL_t + u_{1\cdots\lambda,t} O_{t-\lambda} - d_t, O_{t+1-\lambda}, \cdots, O_t). \tag{8}$$

For the infinite horizon model, the state is $s = (IL, O_{-\lambda}, \dots, O_{-1})$, where O_{-i} is the order placed i periods ago. Then,

$$V(s) = V(IL, O_{-\lambda}, \dots, O_{-1}) =$$
$$\min_{O \geq 0}\{E_{u_{1\cdots\lambda}} E_d[C(IL + u_{1\cdots\lambda} O_{-\lambda} - d)] + \gamma E_{u_{1\cdots\lambda}} E_d[V(IL + u_{1\cdots\lambda} O_{-\lambda} - d, O_{-\lambda+1}, \cdots, O)]\}. \tag{9}$$

The results of Lemma 3-1 and Theorems 3-1 to 3-3 also hold for the model without real time yield information, because it features the same structure as the model with real time yield information. The proofs are very similar and therefore omitted. We note that this model derives the optimal policy for the setting without real time yield information. If the decision maker chooses a conservative approach and takes an expectation over the yield rates conditionally, for example over the 95th percentile of the yield rates to hedge against the uncertainty in the yield rates, this would result in increased order quantities and inventory holding costs and reduced backorder costs. However, it would increase expected total cost.

3.4. Optimal Solution Approach for the Infinite Horizon Model with Discrete and Finite State Space

To obtain the optimal policy and minimal expected cost for the infinite horizon model, we model the system as a discounted Markov decision process. Because Lemma 3-1 and Theorems 3-2 and 3-3 also hold for finite state spaces (Heyman and Sobel 1984, Proposition 8-2), we can determine the optimal policy and the minimal expected cost for each state via value iteration combined with MacQueen extrapolation (MacQueen 1966).

We next show how the optimal solution can be computed for the model that utilizes real time yield information (Subsection 3.4.1). The approach is very similar for the model without real time yield information and we will only describe the differences between the solution approaches in Subsection 3.4.2.

3.4.1. Solution Approach with Real Time Yield Information

Let the state space of the Markov decision process be defined by Z with truncated inventory level ($IL^{min} \leq IL \leq IL^{max}$) and order quantity ($0 \leq O \leq O^{\max}$). Z has $(IL^{\max} - IL^{\min} + 1)(O^{\max} + 1)^{\lambda}$ states. The action space is $A = \{0, 1, 2, \dots, O^{\max}\}$.

Given state $z \epsilon Z$, we compute for every order decision O the transition probabilities from z to $\tilde{z} \epsilon Z, p_{z,\tilde{z}}(O)$. For state z, let Q_z^r be the current yield of an order after lead time period r, with $r = 0, \dots, \lambda$ and $Q_z^0 \equiv O$. The transition probability from $z = (IL_z, Q_z^\lambda, \dots, Q_z^1)$ to $\tilde{z} = (IL_{\tilde{z}}, Q_{\tilde{z}}^\lambda, \dots, Q_{\tilde{z}}^1)$ is

$$p_{z,\tilde{z}}(O) = P(\tilde{z}|z, O) = P\big(D = IL_z + Q_z^\lambda - IL_{\tilde{z}}\big) \cdot \prod_{r=1}^{\lambda} g(Q_z^{r-1}),$$

$$\text{with } g(Q_z^{r-1}) = \begin{cases} 1, & \text{for } Q_z^{r-1} = 0 \text{ and } Q_{\tilde{z}}^r = 0 \\ P(u_r = Q_{\tilde{z}}^r / Q_z^{r-1}), & \text{for } Q_z^{r-1} > 0 \\ 0, & \text{for } Q_z^{r-1} = 0 \text{ and } Q_{\tilde{z}}^r \neq 0. \end{cases} \quad (10)$$

The value iteration algorithm can be found in Appendix 3.F. It can be used to calculate the optimal policy with and without real time yield information and provides the minimal expected cost $V(z)$ and the optimal order decision $O(z)$ for each state.

Using the optimal policy $O(z)$, we calculate the steady state distribution and then determine the minimal expected cost. Proposition 3-1 states that the state probabilities have a unique stationary distribution. Define $Y^{\min}(O)$ as the minimal yield over lead time of order quantity O.

Proposition 3-1. For any discrete i.i.d. demand distribution with $P(D_t = d_t) > 0$ for $Y^{min}(O^{max}) + 1 \geq d_t \geq 0$ and any λ discrete i.i.d yield distributions, the finite state space Z has a unique essential class and therefore has a unique stationary distribution.

The steady state probabilities $p(z)$ can be calculated using power iteration. The minimal expected cost for the infinite horizon model is $\sum_z p(z)V(z)$.

3.4.2. Solution Approach without Real Time Yield Information

Without real time yield information, the transition probabilities differ and the expected one period cost is calculated according to Equation (7). Beside these differences, the approach described above can be used. To calculate the transition probability from $s \in S$ to state $\tilde{s} \in S$, it is sufficient to know the probability distribution of demand and of the yield rate over the lead time $v(u_{1\cdots\lambda})$. The state variable s contains the information about the order O_s^λ placed λ periods ago. This order quantity is used to calculate a set of potential replenishment quantities Q and their probabilities $p_Q = P(u_{1\cdots\lambda} = Q/O_s^\lambda)$. Given the order quantity O, the transition probabilities are calculated as

$$p_{s,\tilde{s}}(O) = P(\tilde{s}|s,O) = \begin{cases} P(D = IL_s + Q - \mathrm{I}L_{\tilde{s}}) \cdot p_Q, \text{ for } O_{\tilde{s}}^r = O_s^{r-1} \text{ with } r = 1, \dots, \lambda \\ 0, \text{ else.} \end{cases} \tag{11}$$

The optimal solution approaches can be used to solve small and medium sized problems. To solve large problems, heuristics can be used. We introduce two heuristics in the next section.

3.5. Heuristic Solution Approaches

Heuristics that belong to the class of linear inflation policies have proven to perform well for solving problems such as the one that we consider (Bollapragada and Morton 1999; Hsueh and Chang 2010; Inderfurth and Transchel 2007; Li, Xu, and Zheng 2008; Zipkin 2000). These policies require the specification of two parameters, θ and β. θ is the order threshold value that triggers an order as soon as the inventory position is below θ. β is an inflation factor by which the difference between the order threshold and the inventory position $(\theta - IP_t)$ is multiplied.

The resulting order quantity is

$$O_t(IP_t) = \begin{cases} \beta(\theta - IP_t), \text{ for } IP_t < \theta \\ 0, \text{ else.} \end{cases} \tag{12}$$

Because not all yield realizations are known when the order quantity is determined, the inventory position must be estimated as the sum of the current inventory level IL_t and the expected replenishment quantities. **With real time yield information,** the expected inventory position is estimated by

$$E[IP_t(z_t)] \ = \ IL_t + Q_{t-\lambda} + \bar{u}_\lambda Q_{t+1-\lambda} + \bar{u}_\lambda \bar{u}_{\lambda-1} Q_{t+2-\lambda} + \cdots + \prod_{r=2}^{\lambda} \bar{u}_r \cdot Q_{t-1}, \tag{13}$$

where $\bar{u}_r$ denotes the expected yield rate for lead time period r.

Without real time yield information, the current inventory position $IP_t(s_t)$ cannot be estimated as accurately as with real time yield information, because the estimates for the expected replenishment quantities are based solely on expected yield rates rather than on a mix of expected and observed yield rates. Therefore the expected inventory position is estimated by

$$E[IP_t(s_t)] \ = \ IL_t + (O_{t-\lambda} + \cdots + O_{t-1}) \prod_{r=1}^{\lambda} \bar{u}_r = IL_t + (O_{t-\lambda} + \cdots + O_{t-1}) \bar{u}_{1\cdots\lambda}. \tag{14}$$

Next, we introduce two heuristics that differ by how they determine the inflation factor β and the order threshold θ. The first heuristic is the MULT-heuristic. This heuristic is most often applied in practice. The idea for this heuristic was first mentioned in Ehrhardt and Taube (1987). The second heuristic is the OPT-heuristic. This heuristic is based on the work of Huh and Nagarajan (2010) and currently one of the best perfoming heuristics for the random yield problem. We modify the Huh and Nagarajan (2010) approach to allow for non-zero lead times and different yield rate distributions in different lead time periods.

3.5.1. MULT-Heuristic

The MULT-heuristic assumes perfect yield and calculates the order threshold value θ as

$$\theta = F_{\lambda+1}^{-1}(b/(b+h)), \tag{15}$$

where $F_{\lambda+1}^{-1}$ denotes the inverse cumulative probability distribution of the demand over $\lambda + 1$ periods. β is set equal to the reciprocal of the total expected yield rate:

$$\beta = \frac{1}{\prod_{r=1}^{\lambda} \bar{u}_r} = \frac{1}{\bar{u}_{1\cdots\lambda}}. \tag{16}$$

3.5.2. OPT-Heuristic

Huh and Nagarajan (2010) show how an optimal order threshold $\theta^*(\beta)$ can be computed for a given inflation factor β. Consequently, the choice of β defines the performance of the heuristic. Huh and Nagarajan (2010) test several choices of β for an inventory system with zero lead time and show that the approach outperforms existing linear inflation policies.

We propose a heuristic that allows for non-zero lead times. We set

$$\beta = \frac{1}{2}\left(\frac{1}{\bar{u}_{1\cdots\lambda}} + \sup\left\{n : E\left[u_{1\cdots\lambda} \cdot \mathbb{I}\left[\frac{1}{n} \leq u_{1\cdots\lambda}\right]\right] \leq \frac{b}{b+h}\bar{u}_{1\cdots\lambda}\right\}\right). \tag{17}$$

The first term corresponds to the choice of β for the MULT-heuristic and ignores yield variability. It is the best choice when the yield rate is deterministic. Then, the first term has the same value as the second term. The second term considers the yield rate distribution over lead time and the ratio of b and h. It is motivated by the fact, that for the single period random yield problem with deterministic demand and zero lead time, the optimal order quantity is determined by $\max\{n(D - IL), 0\}$ (Huh and Nagarajan 2010, Proposition 4), when n is computed according to the second term of Equation (17). Huh and Nagarajan (2010) show that β, computed according to Equation (17), delivers better results than other choices of β under the zero lead time assumption and over a wide range of parameter settings.

$\theta^*(\beta)$ can be computed by minimizing the cost function $C(\theta,\beta)$ for a given β. It is difficult to calculate $\theta^*(\beta)$ analytically, because an analytic expression for the cost function is not available. However, $\theta^*(\beta)$ can be computed efficiently by first simulating the system over T periods for $\theta = 0$ and then computing $\theta^*(\beta)$ as (Huh and Nagarajan 2010)

$$\theta^*(\beta) = \inf\left\{\theta : \frac{1}{T}\sum_{t=1}^{T} P\left[IL_{t+1}^{(0,\beta)} + \theta \leq 0\right] \leq \frac{h}{b+h}\right\}, \tag{18}$$

where $IL_{t+1}^{(0,\beta)}$ denote the simulated inventory levels.

3.6. Computational Results

In Subsection 3.6.1, we use the optimal solution approach from Section 3.4 and provide numerical results on the value of real time yield information for small and medium sized problems. In Subsection 3.6.2, we analyze the heuristics. We use the MULT- and the OPT-heuristic to solve the same test cases that we solved by the optimal solution approach. The main purpose is to analyze the performance gap between optimal and heuristic solutions. Then, we apply the MULT- and the OPT-heuristic to larger problems to analyze the difference in performance for a wider range of parameter settings.

3.6.1. Optimal Solutions

For our numerical analysis of the optimal solutions, we use Poisson distributed demand with a mean of 2, Geometric distributed demand with a mean of 2 and a variance of 6, and Binomial distributed demand with a mean of 12 and variance of 6 (Table 3-1). We use lead times of $\lambda = 1$, 2, 3, and 4 periods and a Bernoulli distributed yield rate with expected yields of $u = 0.9$, 0.94, and 0.98. Unless stated otherwise, only the first lead time period has random yield. We vary the lead time period with random yield and discuss the effects in Section 6. Without loss of generality, we set unit inventory holding cost to $h = 1$ and choose unit backorder costs b that result in critical ratios ($CR = b/(b + h)$) of 0.85, 0.90, 0.95, and 0.99. We use a discount factor γ of 0.9. This results in 288 test cases. The value iteration is conducted with an accuracy of $\varepsilon = 0.001$.

Table 3-1 Discrete demand distributions

D	Parameter	μ_D	σ^2
Poisson	$\mu = 2.0$	2	2
Geometric	$p = 1/3$	2	6
Binomial	$n = 24, q = 0.5$	12	6

We truncate demand at 6, 12, and 18, for Poisson, Geometric, and Binomial distributions, respectively. The truncated probability mass is at most 0.5 %. For Poisson and Geometric distributed demand we limit inventory levels to ±50 and order quantities to 15, limits that are essentially never binding. This creates a state space with $101 \cdot 16^{\lambda}$ states. For Binomial distributed demand we set the limit to ±120 for the inventory level and 36 for the order quantity, creating a state space of $241 \cdot 37^{\lambda}$ states.

Table 3-2 Average run times of optimal solution approach (minutes)

D	yield information	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$
Poisson	with	< 0.1	2.0	48	1126
	without	< 0.1	1.7	34	823
Geometric	with	< 0.1	3.1	75	1710
	without	< 0.1	2.3	51	1206
Binomial	with	4.7	92.5	4192	257,000*
	without	3.5	72.2	3186	203,000*

* estimated based on duration of the first ten iterations and the average number of iterations for lead time = 3

All algorithms were implemented in C++ and all experiments were conducted on a PC with eight Intel 3.06 GHz processors and 8 GB of RAM. Table 3-2 shows average run times using all eight processors. Run times increase exponentially in lead time, because the state space increases exponentially in lead time, which limits the applicability of the optimal solution approach to small and medium size problems.

The costs of the solutions are shown in Table 3-3. The column labeled as value of real time yield information shows the relative cost difference of a system with real time yield information versus a system without real time yield information. The results show that substantial savings can be achieved if yield information can be utilized. Over all test cases, yield information reduces cost by 6.8 % for Poisson demands, 2.8 % for Geometric demands, and 22.9 % for Binomial demands. The results indicate that savings are particularly high when yield variability is high, demand variability is low, and lead time is long. When average demand and, as a consequence average order quantity are high, yield information is particularly beneficial, because yield variability increases over-proportionally in the order quantity. Because we consider small, discrete problems, some values in Table 3-3 do not follow a monotone trend. We discuss the effect of the parameters on the value of real time yield information in more detail in Section 3.7.

Table 3-3 Cost with and without real time yield information for optimal solution

			Cost with real time yield information				Cost without real time yield information				Value of real time yield information (Percent)			
			CR				*CR*				*CR*			
D	u	λ	0.85	0.9	0.95	0.99	0.85	0.9	0.95	0.99	0.85	0.9	0.95	0.99
Poisson	0.9	1	35.6	40.7	49.6	69.3	38.5	44.4	54.2	75.1	7.6	8.4	8.5	7.7
		2	41.6	48.2	58.0	78.5	46.2	53.2	64.3	87.5	9.9	9.3	9.7	10.4
		3	46.9	53.6	64.3	87.4	52.8	60.5	72.8	98.2	11.2	11.4	11.7	11.0
		4	52.0	59.2	71.0	94.3	58.5	67.0	80.3	107.7	11.0	11.6	11.6	12.5
	0.94	1	33.8	39.0	47.2	64.9	35.9	41.1	49.9	68.5	5.8	5.2	5.4	5.2
		2	40.6	46.3	55.6	74.7	43.2	49.7	59.8	80.8	5.9	6.9	7.1	7.5
		3	45.8	52.5	62.5	83.2	49.6	56.7	68.1	91.2	7.7	7.6	8.2	8.8
		4	50.9	57.8	69.4	91.4	55.0	62.9	75.3	100.5	7.5	8.0	7.8	9.1
	0.98	1	32.4	37.6	45.2	58.8	33.0	38.2	46.1	61.0	1.9	1.5	2.0	3.7
		2	39.8	44.7	53.3	70.3	40.5	46.1	55.2	73.5	1.7	3.1	3.5	4.3
		3	44.9	51.5	61.1	80.2	46.1	52.8	62.9	83.7	2.6	2.3	2.9	4.2
		4	49.9	56.8	67.7	89.3	51.6	58.7	70.2	92.6	3.3	3.3	3.6	3.6
Geometric	0.9	1	62.5	73.1	89.9	125.5	64.6	75.7	92.9	129.2	3.3	3.4	3.3	2.9
		2	73.2	85.1	104.0	143.7	76.6	89.1	108.9	150.2	4.4	4.5	4.5	4.3
		3	82.3	95.6	116.3	160.1	86.8	100.5	122.3	167.7	5.2	4.8	4.9	4.5
		4	90.7	104.7	127.4	174.0	95.7	110.6	134.2	183.0	5.2	5.3	5.0	4.9
	0.94	1	61.4	71.7	88.2	121.4	62.6	73.2	89.9	124.5	1.9	2.1	1.9	2.5
		2	72.1	84.0	102.5	141.1	74.4	86.3	105.5	145.4	3.0	2.7	2.8	3.0
		3	81.4	94.4	115.1	157.5	84.3	97.6	118.8	162.7	3.4	3.2	3.1	3.2
		4	89.8	103.7	126.2	172.2	93.0	107.5	130.5	177.9	3.4	3.5	3.3	3.2
	0.98	1	60.1	70.6	86.5	118.2	60.7	71.1	87.2	119.4	0.9	0.6	0.8	1.0
		2	71.2	83.0	101.3	139.0	72.0	83.8	102.2	140.5	1.2	1.0	0.9	1.0
		3	80.7	93.4	113.9	155.5	81.6	94.7	115.3	157.7	1.1	1.4	1.1	1.4
		4	89.0	102.9	125.0	170.3	90.3	104.3	126.8	172.8	1.4	1.3	1.4	1.4
Binomial	0.9	1	99.2	124.5	164.0	251.2	133.3	157.0	193.3	283.8	25.5	20.7	15.2	11.5
		2	105.6	129.1	168.9	255.7	155.4	180.7	223.0	320.6	32.0	28.6	24.3	20.2
		3	111.6	134.2	174.2	260.4	174.0	202.4	250.1	351.0	35.8	33.7	30.3	25.8
	0.94	1	79.8	98.8	135.2	205.1	104.2	129.2	163.2	231.5	23.4	23.5	17.2	11.4
		2	88.5	106.3	140.1	212.0	127.1	150.9	185.8	265.4	30.4	29.5	24.6	20.1
		3	96.1	114.2	145.9	218.5	143.8	168.2	206.4	293.0	33.2	32.1	29.3	25.4
	0.98	1	61.4	72.1	92.2	156.1	69.8	84.1	113.1	175.6	12.0	14.3	18.5	11.1
		2	72.6	83.9	104.5	161.4	87.0	104.2	135.5	196.5	16.5	19.5	22.9	17.9
		3	81.8	94.3	115.2	168.7	101.0	119.8	152.2	215.7	19.0	21.2	24.3	21.8

3.6.2. Heuristic Solutions

The heuristic solutions have negligible run times and can be used to solve large problems. However, they are not necessarily optimal. The performance and the effect of real time yield information on this performance are analyzed before we focus on the performance comparison of the MULT- and the OPT-heuristics when applied to large problems.

Heuristics vs. Optimal solution

We evaluate the accuracy of the MULT- and the OPT-heuristic by applying them to the same problems that we solved optimally. For both heuristics, we compute β according to Equation (16), because for Bernoulli distributed yield rates the second term of Equation (17) is not meaningful. The threshold θ for the MULT-heuristic is calculated by Equation (15). To compute the threshold $\theta^*(\beta)$ for the OPT-heuristic, we follow Huh and Nagarajan (2010) and simulate the inventory system over T = 7000 periods (with an initial transient of T^0 = 2000 periods) and replicate the simulation N = 2000 times. We select the threshold value $\theta^*(\beta)$ according to Equation (18) using 10 million simulated inventory levels. We compute the expected cost of the heuristics, using the same approach as for the optimal solution: We first calculate the order quantities $O(z)$ for every state and then use the order quantities to calculate the steady state probabilities and expected cost for every state using power iteration.

Table 3-4 reports the accuracy of the heuristics and shows the percentage error of the heuristics versus the optimal solution. On average, the error of the OPT-heuristic is 1.6 % for treatments without real time yield information and 0.03 % for treatments with real time yield information. The corresponding errors of the MULT-heuristic are 14.8 % and 4.3 %. As mentioned in Section 3.5.1, the MULT-heuristic ignores yield variability and therefore performs worse for larger order quantities under Binomial demand with a mean of 12, due to the fact that yield variability increases quadratic in order quantity. In these cases yield information is particularly beneficial.

We have compared the performances of the heuristics with the optimal solution for small and medium sized problems and discrete demand distributions, for which we know the optimal solution. Real time yield information largely improves the accuracy of heuristics. For the relatively small test problems analyzed so far the OPT-heuristic clearly outperforms the MULT-heuristic. For larger problems, the optimal solution is not known, but we can still compare the performances of the heuristics.

Table 3-4 Cost of heuristics above optimal solution (percent)

			MULT-heuristic vs. optimal solution								OPT-heuristic vs. optimal solution							
			with real time yield information				without real time yield information				with real time yield information				without real time yield information			
			CR				*CR*				*CR*				*CR*			
D	u	λ	0.85	0.9	0.95	0.99	0.85	0.9	0.95	0.99	0.85	0.9	0.95	0.99	0.85	0.9	0.95	0.99
Poisson	0.9	1	0.2	0.7	6.6	5.4	2.1	2.0	11.6	18.8	0.1	0.1	0.1	0.1	0.8	1.7	2.8	4.4
		2	3.3	0.0	0.7	2.6	4.9	4.0	6.9	17.4	0.1	0.0	0.0	0.0	1.9	2.0	3.5	5.3
		3	0.3	0.5	0.3	0.0	2.1	2.5	4.5	14.6	0.1	0.1	0.1	0.0	1.9	2.4	3.7	5.8
		4	0.1	0.2	0.8	0.2	3.3	3.9	8.3	12.1	0.0	0.1	0.0	0.1	2.5	2.9	3.7	5.8
	0.94	1	0.0	0.0	3.3	0.0	0.5	0.6	8.3	9.2	0.0	0.0	0.0	0.0	0.4	0.1	0.4	2.0
		2	1.7	0.0	0.0	0.0	4.7	2.6	4.2	9.1	0.0	0.0	0.0	0.0	0.2	0.6	0.9	2.9
		3	0.0	0.0	0.0	0.0	1.3	1.6	2.1	6.3	0.0	0.0	0.0	0.0	1.2	1.2	1.5	3.1
		4	0.0	0.0	0.0	0.0	1.4	1.5	3.4	4.1	0.0	0.0	0.0	0.0	0.6	0.9	1.9	3.0
	0.98	1	0.0	0.0	0.0	0.0	0.0	0.0	1.4	0.4	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4
		2	0.0	0.0	0.0	0.0	1.7	0.0	0.1	0.7	0.0	0.0	0.0	0.0	0.1	0.0	0.1	0.7
		3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5
		4	0.0	0.0	0.0	0.0	0.1	0.0	1.0	0.4	0.0	0.0	0.0	0.0	0.1	0.0	0.5	0.3
Geometric	0.9	1	0.5	1.4	1.2	0.0	1.8	2.8	3.1	3.4	0.2	0.2	0.2	0.0	0.8	1.1	1.3	2.1
		2	0.1	0.7	1.0	1.6	1.7	2.7	3.7	6.9	0.1	0.1	0.1	0.1	1.4	1.6	2.0	2.7
		3	0.3	0.0	0.6	0.0	1.7	2.3	3.8	4.7	0.1	0.0	0.1	0.0	1.7	1.8	2.2	3.0
		4	0.0	0.2	0.2	0.3	2.2	2.2	3.6	5.6	0.0	0.1	0.1	0.0	1.9	2.1	2.5	3.2
	0.94	1	0.0	0.9	0.5	0.1	0.9	1.9	1.8	1.4	0.0	0.0	0.0	0.0	0.3	0.3	0.5	1.1
		2	0.1	0.3	0.5	1.0	0.7	1.6	2.1	3.8	0.0	0.0	0.0	0.0	0.7	0.6	0.8	1.5
		3	0.1	0.0	0.2	0.0	0.9	1.1	2.0	2.2	0.0	0.0	0.0	0.0	0.7	0.8	1.0	1.5
		4	0.0	0.1	0.0	0.0	1.0	1.0	1.7	2.6	0.0	0.0	0.0	0.0	0.8	1.0	1.1	1.7
	0.98	1	0.0	0.2	0.0	0.0	0.0	0.6	0.3	0.1	0.0	0.0	0.0	0.0	0.0	0.1	0.2	0.1
		2	0.0	0.0	0.0	0.3	0.0	0.4	0.6	1.3	0.0	0.0	0.0	0.0	0.0	0.2	0.2	0.2
		3	0.0	0.0	0.0	0.0	0.1	0.1	0.6	0.3	0.0	0.0	0.0	0.0	0.1	0.1	0.2	0.3
		4	0.0	0.0	0.0	0.0	0.1	0.1	0.4	0.6	0.0	0.0	0.0	0.0	0.1	0.1	0.3	0.4
Binomial	0.9	1	0.3	3.9	15.6	118.5	3.1	18.4	45.7	216.5	0.2	0.1	0.0	0.0	0.4	1.4	3.9	5.3
		2	1.3	3.6	11.6	99.6	11.0	24.5	51.8	245.1	0.2	0.1	0.0	0.0	1.8	3.3	5.6	5.7
		3	0.8	2.7	8.4	64.7	12.0	24.9	52.6	225.4	0.3	0.2	0.0	0.0	2.9	4.1	5.4	6.7
	0.94	1	0.5	0.3	2.2	67.4	0.0	5.5	22.4	150.4	0.0	0.0	0.0	0.0	0.0	0.1	1.3	6.0
		2	0.2	0.9	2.3	54.5	3.3	11.1	30.7	168.5	0.0	0.1	0.0	0.0	0.3	1.1	2.8	5.8
		3	0.2	0.2	2.0	31.0	4.3	12.7	31.9	142.0	0.1	0.0	0.0	0.1	0.9	1.8	3.5	5.4
	0.98	1	0.0	1.3	0.0	12.6	0.9	1.8	1.1	45.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.4
		2	0.4	0.9	0.0	14.5	0.0	0.5	2.3	55.4	0.0	0.0	0.0	0.0	0.0	0.0	0.3	2.4
		3	0.0	0.0	0.0	6.5	0.1	1.3	5.1	48.2	0.0	0.0	0.0	0.0	0.1	0.1	0.6	2.7

MULT-heuristic vs. OPT-heuristic

Our test cases are based on Bollapragada and Morton (1999), Huh and Nagarajan (2010), and Inderfurth and Transchel (2007). Demand and yield rates are both Normal distributed. We use demand distributions with mean 20 and coefficients of variation ρ_D = 0.2 and 0.4. The left tail of the demand distribution is truncated below 0, which slightly increases the mean and slightly decreases the standard deviation. For the yield rate distributions, we use a mean of 0.5 and coefficients of variation of ρ_u = 0.1, 0.2, 0.3, and 0.4. We limit the yield rates, such that they are in the range between 0 and 1. The set of critical ratios is 0.85, 0.9, 0.95, and 0.99. We analyze lead times of λ = 1, 5, 10, and 30 periods. This results in 128 test cases for the heuristics. We use both heuristics to determine order thresholds and inflation factors. Then, we use the threshold levels and inflation factors and determine the actual cost of this solution using the same approaches as Huh and Nagarajan (2010, page 248) and Choi et al. (2008, page 619). The numerical results are calculated with T = 7000 and T^0 = 2000 and averaged over all N = 2000 simulation runs. The average half-width of the 95% confidence interval over all simulations is 0.3 % with a maximum at 2 %. Optimal solutions are not available for these large problems.

Results are shown in Table 3-5. They show that the OPT-heuristic clearly outperforms the MULT-heuristic. In 122 of 128 cases with real time yield information and in all 128 cases without real time yield information, the OPT-heuristic has lower cost than the MULT-heuristic. However, real time yield information largely reduces the distance between the heuristics and therefore the disadvantage of using less sophisticated heuristics, like the MULT-heuristic, which is often applied in practice.

Both results are as expected. Note, that both heuristics would be optimal solutions if the yield rate would be deterministic. As discussed in Section 3.5, the OPT-heuristic considers yield randomness far more efficient than the MULT-heuristic which explains the better performance. However, real time yield information mitigates the effects of improper consideration of random yield. Over all test cases without real time yield information the MULT-heuristic has 12.7 % higher cost than the OPT-heuristic. With real time yield information the cost difference is reduced to 2.6 %. Real time yield information reduces the yield risk and therefore the negative effect of ignoring yield variability for the MULT-heuristic. The OPT-heuristic still performs better than the MULT-heuristic, because real time yield information does not completely mitigate the yield risk.

Table 3-5 Cost of MULT-heuristic above OPT-heuristic (percent)

		$\rho_D = 0.2$								$\rho_D = 0.4$							
		with yield information				without yield information				with yield information				without yield information			
		CR				*CR*				*CR*				*CR*			
ρ_u	λ	0.85	0.9	0.95	0.99	0.85	0.9	0.95	0.99	0.85	0.9	0.95	0.99	0.85	0.9	0.95	0.99
0.1	1	0.2	0.4	0.8	2.3	0.7	1.1	2.1	5.5	0.0	0.0	0.1	0.6	0.1	0.1	0.3	1.1
	5	0.0	0.1	0.1	0.4	0.7	1.2	2.1	5.0	0.1	0.1	0.1	0.3	0.3	0.3	0.5	1.0
	10	0.0	0.0	0.1	0.2	0.8	1.2	2.1	4.9	0.3	0.3	0.3	0.4	0.5	0.6	0.8	1.3
	30	0.0	0.0	0.0	0.1	0.8	1.3	2.2	4.9	0.9	0.9	0.9	1.0	1.3	1.5	1.7	2.4
0.2	1	2.2	3.6	6.5	16.2	4.9	7.9	13.9	30.5	0.3	0.4	0.8	2.2	0.7	1.2	2.1	4.7
	5	0.4	0.6	1.1	2.5	5.1	8.1	14.1	30.7	0.2	0.2	0.2	0.1	1.2	1.6	2.6	5.3
	10	0.1	0.2	0.4	0.7	5.2	8.2	14.3	31.0	0.3	0.3	0.3	0.2	1.6	2.1	3.2	6.1
	30	0.0	0.0	0.1	0.1	5.3	8.3	14.3	31.0	0.9	0.9	0.9	0.9	2.8	3.6	4.9	8.5
0.3	1	5.5	9.0	16.5	38.2	10.1	15.9	26.5	48.5	0.9	1.6	3.0	7.6	2.3	3.6	6.2	11.8
	5	1.4	2.2	4.1	9.5	10.8	16.7	27.9	52.1	0.3	0.3	0.4	-0.1	3.0	4.5	7.2	13.8
	10	0.5	0.8	1.4	2.6	11.0	17.1	28.4	53.1	0.3	0.3	0.2	-0.4	3.6	5.2	8.3	15.6
	30	0.0	0.1	0.1	-0.4	11.2	17.4	28.8	53.9	0.9	0.9	0.8	0.5	5.3	7.3	10.8	19.6
0.4	1	8.2	13.5	24.2	51.5	13.5	20.8	33.1	58.0	1.7	3.0	5.8	14.8	3.9	6.1	10.1	21.2
	5	2.5	4.2	7.8	18.7	14.5	22.1	35.4	59.7	0.4	0.5	0.7	0.2	4.9	7.4	11.9	20.9
	10	0.9	1.6	2.9	6.2	14.9	22.7	36.2	60.7	0.3	0.3	0.2	-1.1	5.8	8.4	13.3	22.9
	30	0.1	0.1	0.1	-0.8	15.3	23.2	36.9	62.4	0.9	0.9	0.7	-0.1	7.7	10.8	16.3	28.2

3.7. Value of Real Time Yield Information

One of the key objectives of our research is to quantify the monetary benefits that can be achieved by utilizing real time yield information as opposed to relying on average yield rates. We are also interested in identifying conditions under which real time yield information is particularly beneficial. In this section, we use the OPT-heuristic to address these issues. We use the same test cases as in Subsection 3.6.2 and add test cases for $\rho_D = 0.1$ and $\rho_D = 0.3$ to cover a wider range of parameter values.

Figure 3-2 Value of real time yield information (VRTYI)

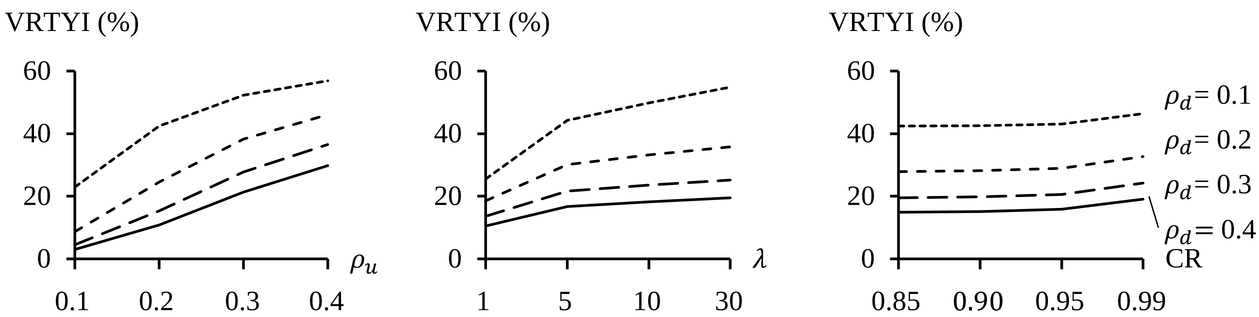


Figure 3-2 shows how the value of real time yield information (VRTYI) depends on the key problem parameters. The results show that substantial savings can be achieved for most parameter settings, but that the magnitude of the savings depends on the values of the parameters: The value of real time yield information is increasing in yield variability (ρ_u), decreasing in demand variability (ρ_d), increasing in the lead time (λ), and increasing in the critical ratio (CR).

Yield variability. The left graph of Figure 3-2 shows how the value of real time yield information is increasing in yield variability ρ_u. This result is intuitive, because real time yield information reduces uncertainty about the state of the open orders more if yield variability is high than if it is low.

Figure 3-3 Effect of yield variability on heuristic parameters and performance measures for test case

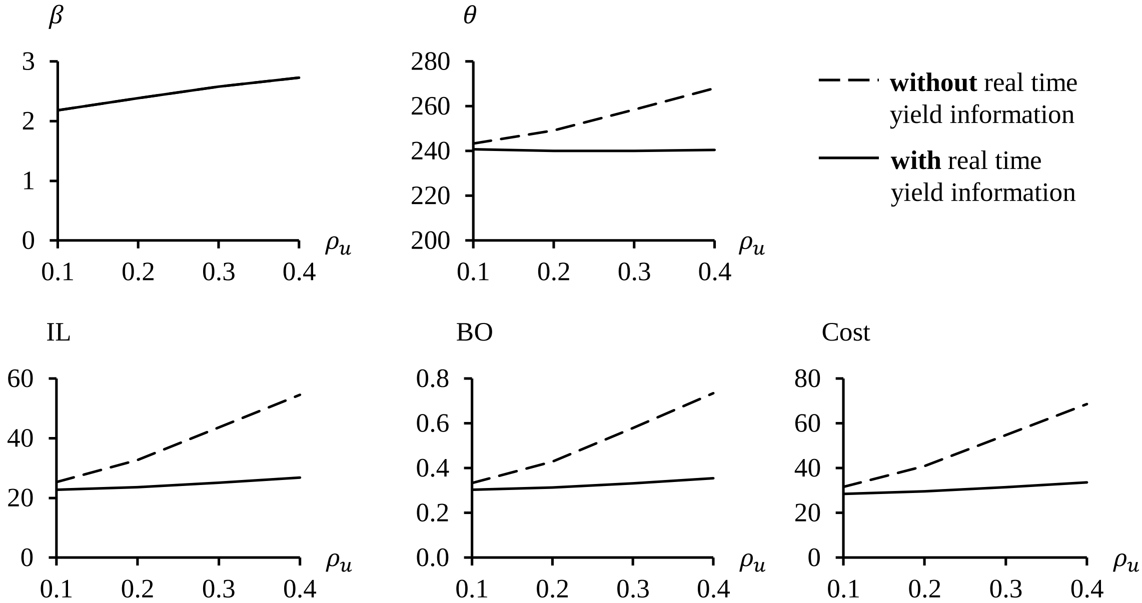


Figure 3-3 provides more details and shows the effect of yield variability on the inflation factor (β), the order threshold (θ), the average inventory level (IL), the average backorder level (BO), and the average cost per period for the test case λ = 10, CR = 0.9, and ρ_d= 0.2. The results are similar to the other test cases that we analyzed.

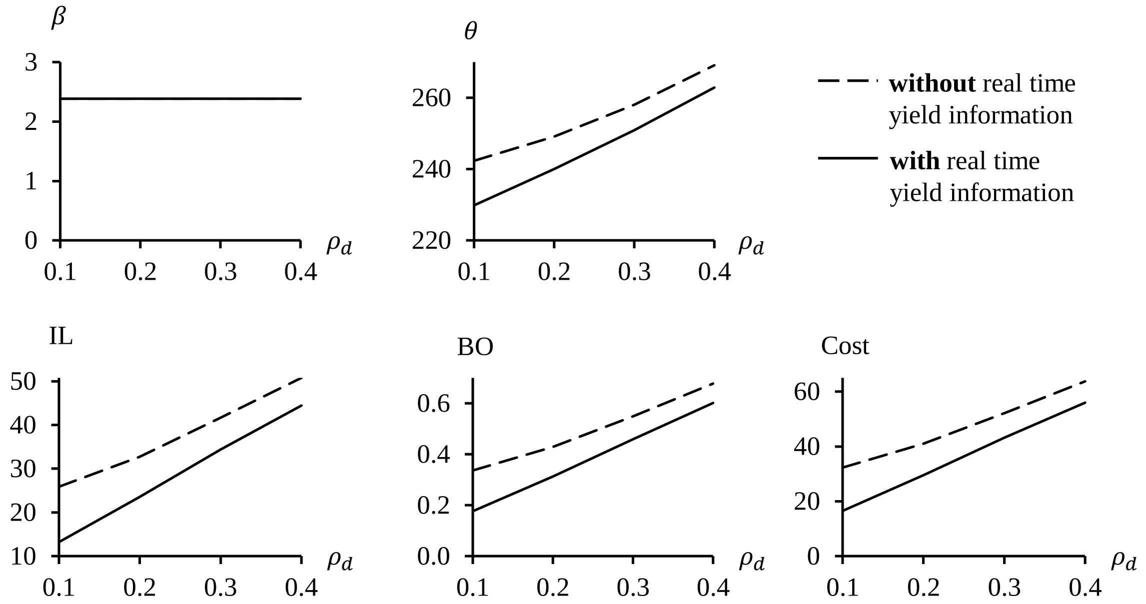
With and without real time yield information, the inflation factor β is increasing in yield uncertainty (see Equation (17)). For a given inflation factor β, the OPT-heuristic computes the optimal threshold value θ. With real time yield information, increases in the inflation factor β are sufficient to compensate increases in yield uncertainty and the order threshold θ is essentially unaffected by yield uncertainty. However, without real time yield information, increasing yield variability is not sufficiently compensated by an increase in the inflation factor β, and an increase in the order threshold θ is required.

With real time yield information, the actual state of the inventory system can be more accurately evaluated and supply can be better matched with demand than without real time yield information. Consequently, average inventory levels, average backorder levels, and average cost increase less in yield variability and are lower with real time yield information than without real time yield information.

Demand variability. The graphs of Figure 3-2 show that the value of real time yield information decreases as demand variability increases. At a first glance, this result might seem surprising, but it can be explained by risk pooling. In our model, inventory is used for hedging against demand uncertainty and against yield uncertainty. The inventory used for hedging against yield uncertainty is higher without real time yield information than with real time yield information. If demand uncertainty increases, there is a larger pool of existing inventory available to hedge against the increased uncertainty in an inventory system without real time yield information than in an inventory system with real time yield information. Therefore, inventory increases less in demand uncertainty in an inventory system without real time yield information than in an inventory system with real time yield information.

Figure 3-4 provides details for the same test case that we used for Figure 3-3 for $\rho_u = 0.2$. Obviously the inflation factor β is unaffected by demand uncertainty and increases in demand uncertainty are compensated by increases in the order threshold θ. Inventory and backorder levels are also increasing in demand uncertainty and consequently expected costs are increasing in demand uncertainty. The increase in the performance measures is higher with real time yield information than without.

Figure 3-4 Effect of demand variability on heuristic parameters and performance measures for test case

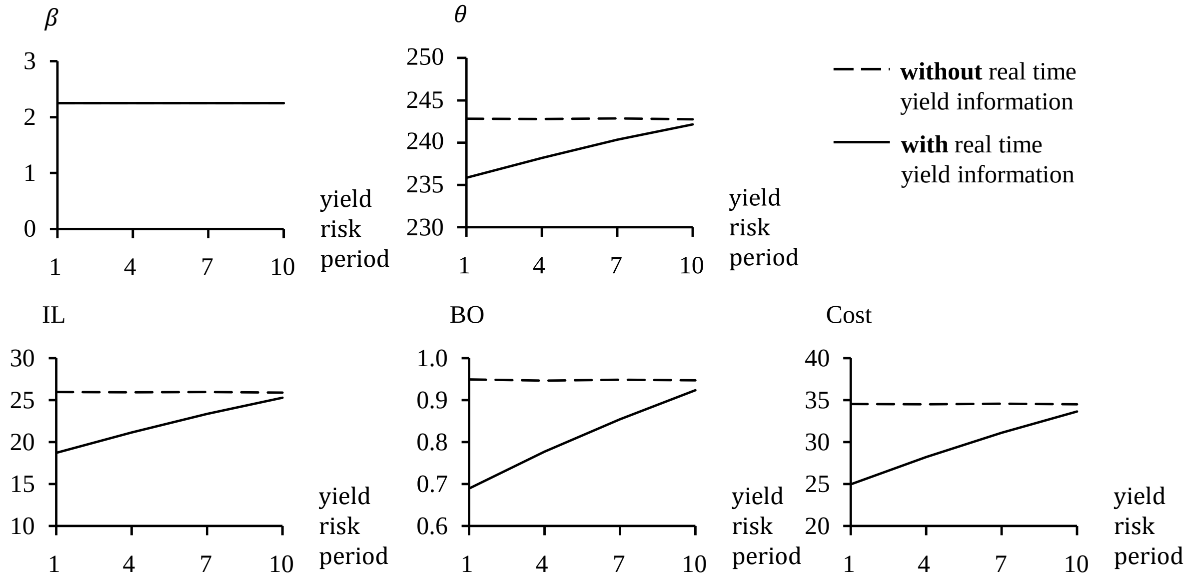


Lead time. The center graph of Figure 3-2 shows that the value of real time yield information is increasing in the lead time. Without real time yield information, longer lead time results in higher transit stock and consequently in higher uncertainty about the state of the open orders. Therefore, the value of resolving this uncertainty by using real time yield information is higher for longer lead time than for shorter.

In our main numerical experiments, only the first lead time period is exposed to yield uncertainty and the remaining periods have no yield uncertainty. Figure 3-5 shows numerical results for our test case (λ = 10), where we shifted the yield risk to later lead time periods. The figure shows that the closer the yield risk is to the delivery period, the lower is the value of real time yield information.

If the yield uncertainty is in lead time period 1, then the risk materializes in the first period after an order has been placed. If an order is placed to compensate a yield loss, this order arrives only one period later than the original order. Additionally, at the time an order is placed, all previous orders have passed the yield risk and their final yield is known. Therefore, real time yield information is very valuable. In our example, the expected cost is 25.0 with real time yield information and 34.6 without real time yield information.

Figure 3-5 Effect of lead time period with yield risk on heuristic parameters and performance measures for test case



If the yield uncertainty is in yield risk period 10, then the risk materializes one period before the order arrives. If an order is placed to compensate a yield loss, this order arrives only one period faster than the regular lead time. In addition, at the time an order is placed, only one order has passed the yield risk and revealed its final yield and the final yields of all other orders are unknown. Therefore, real time yield information is not very valuable. In our example, the expected cost is 33.4 with real time yield information and 34.6 without real time yield information.

Our analyses show that the value of real time yield information is substantial for a wide range of parameter values, but that it is particularly high in settings with high yield uncertainty, long lead times with yield risk in early periods and low demand variability. We note that our model can also be used to analyze production systems, where different production stages have different production times and different yield rates and where the yield realization can only be observed at the end of production stage. For production systems with two production stages, the yield risk period of Figure 3-5 corresponds to the duration of the first production stage. Other production settings can be analyzed analogously. For instance, assume we have a three stage production system with production times of 1, 2, and 3 periods (total production time of 6 periods) and Bernoulli distributed yield rates with expected yields of 0.80, 0.90, and 0.95. Using our test case we set λ = 6, u_1 = 0.80, u_3 = 0.90, and u_6 = 0.95. For this example, we obtain expected cost per period of 96.5 without real time yield information and of 67.2 with real time yield information, which results in a value of real time yield information of 30.4 %.

3.8. Extension: Fixed Order Cost

In this section we extend our analysis for the case where fixed order cost is charged independent of the order size. From Scarf (1960) and Iglehart (1963), we know that an (s,S) policy is optimal under the perfect yield assumption. An order of $S - IP$ is placed whenever $IP \leq s$. The optimal determination of the reorder point s and the order-up-to level S is by dynamic programming and many approximately optimal policies have been discussed in the literature (e.g. Schneider and Ringuest 1990; Tijms and Groenevelt 1984; Zied Babai et al. 2010). A heuristic that performs quite well is the Modified Continuous Review (MRC)-heuristic introduced by Porteus (1985). The MCR-heuristic is a modification of the continuous review method of Hadley and Whitin (1963). Through approximations it avoids iteration and adapts the parameters to the periodic review inventory model.

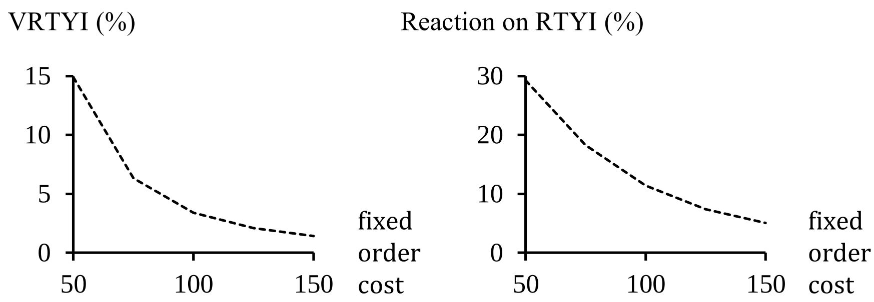
To analyze periodic review inventory systems with random yield and fixed order cost, we introduce the MCR-MULT-heuristic that combines the MCR-heuristic and the MULT-heuristic. The parameters s and S are determined by the MCR-heuristic, assuming perfect yield. The order quantity is calculated as

$$O_t(IP_t) = \begin{cases} \beta(S - IP_t), \text{ for } IP_t \leq s \\ 0, \text{ else,} \end{cases} \tag{19}$$

where the inflation factor β is set equal to the reciprocal of the total expected yield rate (Equation (16)).

To analyze the effect of fixed order cost, we use the simulation approach described in Section 3.6.2 and apply the MCR-MULT-heuristic to test cases with Normal distributed demand, (mean = 20, ρ_D = 0.2) and Normal distributed yield rates (mean = 0.5, ρ_u = 0.3). We set λ = 1, h = 1, b = 19, and fixed order cost k = 50, 75, 100, 125, and 150.

Figure 3-6 Effect of fixed order cost for test case



The left graph in Figure 3-6 shows the effect of fixed order cost on the value of real time yield information. It shows that the value of real time information is decreasing in fixed order cost. Real time yield information leads to a quantifiable benefit if it results in an action before the order arrives. After the order arrived the advantage of early information is elapsed. The right graph in Figure 3-6 shows the percentage of periods for which real time yield information caused a different action than without real time yield information. If order cost and thus order quantities are large and sufficient for filling the demand of various periods it is less likely that yield rates are so low that their observation has an immediate effect. Potential future stock outs can often be avoided by placing an order after the current order has arrived. Real time yield information allows for faster responses to observed yield realizations, but its value is smaller if order quantities are large than if they are small.

3.9. Conclusion

We analyzed a periodic review, random yield problem with stochastic demand and positive lead time. We modeled the problem with and without real time yield information. We proved that the cost function is convex and that a stationary optimal solution for the infinite horizon problem exists. Based on these properties, we developed an optimal solution approach. The algorithm is applicable for discrete state spaces and small to medium sized problems. To solve large problems, we developed two heuristics. We conducted numerical experiments that show that real time yield information is of significant value for a wide range of problem parameter values.

Our research provides the algorithms that are necessary to utilize real time yield information. Companies who decide to use real time yield information can use our heuristics to compute close-to-optimal solutions with low computation times. Companies can also use the results of our research to determine whether or not it is beneficial to invest in using real time yield information. They can use our algorithms to quantify the cost savings that can be achieved by using real time yield information and compare these with the necessary investments.

Our model is built on two assumptions. We have assumed that yield information is perfect and free. In some applications, these assumptions might not hold. If yield information is not perfect, for instance, because of noisy sensor signals, then the stochastic nature of the information must be taken into account. If yield information is not free, the cost of collecting the information can be incorporated in the model by implementing a second decision variable beside order quantity. The second decision that has to be made is whether or not to receive real time yield information. The decision can be made for the whole order or for each order item individually. We leave the analysis of both extensions to future research.

Appendix 3.A Summary of Notation

d_t	demand in period t
D	random variable for demand
O_t	order quantity in period t
Q_t	yield of order placed in period t
$O^{\max}$	upper bound for order quantity
IL_t	on hand inventory at the beginning of period t before replenishment arrives
$IL^{\min}$	upper bound for backorders
$IL^{\max}$	upper bound for inventory level
y_t	on hand inventory in period t after replenishment and before demand is satisfied
$u_{r,t}$	random yield rate in period t for lead time period $r; r = 1, \dots, \lambda$
$\bar{u}_{r,t}$	expected yield rate in period t for lead time period $r; r = 1, \dots, \lambda$
$u_{1\cdots\lambda,t}$	random yield rate over all lead time periods in period t
$\bar{u}_{1\cdots\lambda,t}$	expected random yield rate over all lead time periods in period t
h	inventory holding cost per unit per period (> 0)
b	cost for backordered units per unit per period (> 0)
λ	lead time
γ	discount factor with $0 \le \gamma < 1$
μ	mean
σ^2	variance
ρ	coefficient of variation
θ	order threshold
β	inflation factor
z	state variable with yield information
s	state variable without yield information
Z_t	set of all feasible states z in period t
S_t	set of all feasible states s in period t
T	Number of considered periods for the finite horizon problem and for simulations
$C(x)$	inventory cost of the current period
$V_t(z_t)$	minimal expected cost from period t to period T, given the current state.
$H_t(z_t, O_t)$	total cost function in period t depending on the current state and the order quantity
$\mathbb{I}[*]$	Indicator function. Returns 1 if the expression in brackets is true and 0 else.

Appendix 3.B Proof of Theorem 3-1

Proof Because $E_{d_t}[C(x-d_t)]$ is convex in x (Heyman and Sobel 1984, Proposition B-2), $E_{d_t}[C(IL_t+Q_{t-\lambda}-d_t)]$ is convex in IL_t and $Q_{t-\lambda}$ (Rockafellar 1970, Theorem 5.7). By assumption, $V_{T+1}(IL_{T+1},Q_{T+1-\lambda},\cdots,Q_T)=0$ is convex. We continue with induction. Suppose that $V_t(IL_t,Q_{t-\lambda},\cdots,Q_{t-1})$ is convex in IL_t and Q_{t-r}, $r=1,\cdots,\lambda$. From Theorem 5.7 of Rockafellar (1970) it follows that $V_t\left(IL_t+Q_{t-\lambda}-d_{t-1},u_{\lambda,t-1}Q_{t+1-\lambda},\cdots,u_{1,t-1}O_t\right)$ is convex in IL_t, Q_{t-r}, $r=1,\cdots,\lambda$, and O_t. The expectation conserves convexity (Heyman and Sobel 1984, Proposition B-2).

Thus, $E_{u_{1,t-1}}\cdots E_{u_{\lambda,t-1}}E_{d_{t-1}}\left[V_t\left(IL_t+Q_{t-\lambda}-d_{t-1},u_{\lambda,t-1}Q_{t+1-\lambda},\cdots,u_{1,t-1}O_t\right)\right]$ is convex in IL_t, Q_{t-r}, $r=1,\cdots,\lambda$, and O_t. Because the sum of convex functions is convex, $H_{t-1}(IL_t,Q_{t-\lambda},\cdots,Q_{t-1},O_{t-1})=E_{d_{t-1}}[C(IL_t+Q_{t-\lambda}-d_{t-1})]+$ $\gamma E_{u_{1,t-1}}\cdots E_{u_{\lambda,t-1}}E_{d_{t-1}}[V_t(IL_t+Q_{t-\lambda}-d_{t-1},u_{\lambda,t-1}Q_{t+1-\lambda},\cdots,u_{1,t-1}O_t)]$ is convex in IL_t, Q_{t-r}, $r=1,\cdots,\lambda$, and O_t. The minimization also conserves convexity and therefore $V_{t-1}(IL_t,Q_{t-\lambda},\cdots,Q_{t-1})=\min_{O_t\geq 0}H_{t-1}(IL_t,Q_{t-\lambda},\cdots,Q_{t-1},O_t)$ is convex in IL_t and Q_{t-r}, $r=1,\cdots,\lambda$. ■

Appendix 3.C Proof of Lemma 3-1

Proof. $V_t(z)$ is monotone increasing in T, because the single-period cost function is non-negative and $V_{T+1}(z)=0$ for all z. Therefore, it suffices to show that $V_t(z)$ is bounded from above. Denote the current period by $t=1$ and consider a stationary policy where nothing is ordered, i.e. $O^{\infty}(z)=0$ for all z. Denote the initial inventory level by I_1 and the cumulative demand from period 1 to period t by x_t. Without loss of generality, we assume that no order is outstanding in period 1. The inventory at the end of period t is I_1-x_t and the expected cost in period t is $E_{x_t}[C(I_1-x_t)]\leq E_{x_t}[(b+h)|I_1-x_t|]\leq(b+h)E_{x_t}[|I_1|+x_t]=(b+h)[|I_1|+\mu t]$. Where μ is the expected one period demand. If we apply our stationary policy for an infinite number of periods, the total discounted expected cost is bounded by $\sum_{t=1}^{\infty}\gamma^{t-1}[(b+h)[|I_1|+\mu t]]=(b+h)\left(\frac{|I_1|}{1-\gamma}+\frac{\mu}{(1-\gamma)^2}\right)<\infty$. We have shown that there exists a stationary policy for which the expected present value is $<\infty$ for $T\rightarrow\infty$ and for all initial values of z. This proves the existence of the limit function $V(z)$ for each $z\in Z$, according to Theorem 8.13 of Heyman and Sobel (1984). ■

Appendix 3.D Proof of Theorem 3-2

Proof. Similarly as Henig and Gerchak (1990), we show that the four conditions of Theorem 8-14 of Heyman and Sobel (1984) are satisfied to prove Theorem 3-2**.**

Condition a) requires that for each z, there exists a limit function $V(z) = \lim_{T\to\infty} V_t(z)$, which is proven by Lemma 3-1.

Condition b) requires that the reward function is non-negative, which is obviously the case in our setting, with $C(x) = h[x]^+ + b[-x]^+$.

Condition c) requires that for all z the action space is a compact set. From the proof of Lemma 3-1, we know that for O = 0 the discounted expected cost can be bounded from above for each z. If the order quantity goes to infinity, for all z the discounted expected one period cost goes to infinity, i.e. $\gamma^\lambda E[C(I_{+\lambda} + u_{1\cdots\lambda}O - d_{+\lambda})] \to \infty$, for $O \to \infty$. $E[C(I_{+\lambda} + u_{1\cdots\lambda}O - d_{+\lambda})]$ is convex in O. Therefore we can restrict the search of the optimal order quantity O to values that correspond to cost that is below the bound for the discounted expected cost of $O^\infty(z) = 0$ for $T \to \infty$. Thus there exists for each z a finite $\tilde{O}(z)$, such that the search for O is limited to compact interval $[0, \tilde{O}(z)]$.

Condition d) requires that $H_t(z, O)$ is continuous on the action space for each z. From Theorem 3-1, we know that $H_t(z, O)$ is convex in O for all z. Therefore $H_t(z, O)$ is continuous in O which implies the continuity on the actions space for all z. (Rockafellar 1970, Theorem 10.1). ■

Appendix 3.E Proof of Theorem 3-3

Proof. The limit functions $V(z) = \lim_{T\to\infty} V_t(z)$ and $H(z, O) = \lim_{T\to\infty} H_t(z, O)$ are limits over convex functions as proven in Theorem 3-1 and therefore convex (Rockafellar 1970, Theorem 10.8). The convexity of $H(z, O)$ together with the satisfaction of the conditions of Theorem 8-14 satisfy the conditions of Theorem 8-15 of Heyman and Sobel (1984). ■

Algorithm for value iteration

The following value iteration algorithm can be applied to calculate the optimal policy with and without real time yield information:

1. Select accuracy $\varepsilon > 0$. For each $z \epsilon Z$, calculate the expected one period cost $E[C(z)]$ according to Equations (3). Set $n = 1$ and compute the minimal expected cost $V_0(z) = E[C(z)]$, $\forall z \epsilon Z$.
2. Compute the cost vector $V_n(z) = \min_{0 \le O \le O^{max}} E[C(z)] + \gamma \sum_{\tilde{z} \epsilon Z} p_{z,\tilde{z}}(O) V_{n-1}(\tilde{z})$ and store the optimal order decision $O(z)$, $\forall z \epsilon Z$.
3. If $[\gamma/(1-\gamma)] \left[\sup_{z \epsilon Z} \{V_n(z) - V_{n-1}(z)\} - \inf_{z \epsilon Z} \{V_n(z) - V_{n-1}(z)\} \right] < 2\varepsilon$, then stop. Else, repeat step 2 with $n = n+1$.
4. For each $z \epsilon Z$, compute the minimal expected cost

$$V(z) = V_n(z) + \frac{1}{2}[\gamma/(1-\gamma)] \left[\sup_{z \epsilon Z} \{V_n(z) - V_{n-1}(z)\} + \inf_{z \epsilon Z} \{V_n(z) - V_{n-1}(z)\} \right].$$

After termination of the algorithm, the optimal policy $O(z)$ has been computed in the last iteration of step 2.

Appendix 3.F Proof of Proposition 3-1

Proof. Every finite Markov chain has at least one essential class (Levine et al. 2009, p. 16). If there would be more than one essential class on Z it would be possible to separate the state space into at least two disjoint essential classes, each of which consists of a communicating set of essential states but with the property that passage between different classes is impossible (Shiryaev 1996, p. 570).

We demonstrate that this separation of essential states on Z is not possible. To do this we focus on the inventory level dimension. Let the system be in an arbitrary state $z^a \epsilon Z$ with inventory level IL^a. There is a positive probability to reach a state with IL^{min} as inventory level, because all demands $Y^{min}(O^{max}) + 1 \ge d \ge 0$ have a strict positive probability.

We have shown that from any inventory level the minimal inventory level can be reached. Next we show that there is an interval $[IL^{min}, \overline{IL}]$ for which all inventory levels are communicating and that all states with an inventory level greater than $\overline{IL}$ are inessential.

Applying the optimal policy and starting from a state with IL^{min} as inventory level the inventory level can be increased up to a certain limit $IL^{min} \le \overline{IL} \le IL^{max}$. E.g. demand is 0 and yield rates are maximal for a sufficiently long time. Starting from $\overline{IL}$, all inventory levels between $\overline{IL}$ and IL^{min} can be reached, because there is a positive possibility to reduce the inventory level by 1 every period until IL^{min} is reached.

Starting in any state with inventory level less or equal to $\overline{IL}$, there is no possibility to reach states with an inventory level greater than $\overline{IL}$. Starting from any state with an inventory level greater than $\overline{IL}$ there is positive probability to reduce the inventory level to $\overline{IL}$. Once this happened there is no possibility to return to a state with inventory level greater than $\overline{IL}$.

Therefore, all essential states have an inventory level in $[IL^{min}, \overline{IL}]$ and the state space cannot be separated into two non-communicating classes of essential states. Therefore a unique essential class exists which leads the existence of a unique steady state distribution (Levin et al. 2009, p. 17). ■

Chapter 4

The Value of Supply Chain Visibility when Visibility is Costly

4.1. Abstract

We consider a random yield inventory system, where items are exposed to yield risk during transit or in production. Order batches can be tracked to get access to real time information about the actual yield realizations. Tracking induces fixed costs per order and the decision maker can decide for each order whether or not to obtain yield information. To contribute to a better understanding of the value of this information and its use, we develop a mathematical model of the inventory systems. We derive structural properties and derive the optimal policy. We conduct numerical experiments to quantify the benefits of a flexible tracking system vs. systems that track all orders or do not track any order. We identify conditions under which real time yield information with flexible tracking is particularly beneficial and identify the key drivers for the tracking decision. Our research provides the approaches that are necessary to implement inventory control policies that utilize real time yield information on an order-by-order basis.

Keywords: inventory management; random yield; value of information; RFID

4.2. Introduction and Literature Review

The practical importance of considering random yields in inventory management has been highlighted by many authors. Random yields can be found in procurement processes as well as in production and assembly processes (Grosfeld-Nir and Gerchak 2004; Inderfurth and Clemens 2012; Yano and Lee 1995). A prominent field where random yields is applicable is the semi-conductor industry (Gavirneni 2004; Uzsoy, Lee, and Marting-Vega 1992; Wang 2009). Random yields are also characteristic of many other processes, e.g., electronics manufacturing and chemical production processes (Choi et al. 2008).

Usually the yield is observable upon arrival of an order or after the production process is finished. In this article we focus on real time yield information. Real time yield information is available prior to order arrival or end of production. It becomes available, e.g., by tracking orders while in transit or by accessing information on current production yields while production is still in progress. Analyzing the proper use and the value of this information has attracted attention in recent years (e.g. Choi et al. 2008; Dettenbach and Thonemann 2015; White and Cheong 2012). We contribute to this stream of literature by analyzing an inventory system where tracking the yield of an order incurs a fixed cost per order and a decision about whether or not to track an order can be made when an order is placed. This allows for an order pipeline with tracked and not tracked orders, which is applicable to transportation processes where each order must be equipped with a sensor or to production processes where order inspection is costly. We consider the tradeoff between incurring the tracking cost for an order versus accepting the additional risk that a not tracked order adds to the inventory system.

The most common approach to model yield risk is the proportional yield model (Bollapragada and Morton 1999; Henig and Gerchak 1990; Huh and Nagarajan 2010; Yano and Lee 1995). Under proportional yield, all items of an order are affected in the same way by realized yield rates, i.e., the yield of all items is perfectly correlated. Information on the yield of one item reveals the yield of all items. Examples for these systemic risks include products in transit that are affected by a temperature change. It reduces the tracking decision to whether or not to track an order.

This work builds on the literature on inventory systems with random yield and information sharing. For a review of random yield literature, we refer to Dettenbach and Thonemann (2015) and focus on discussing literature on information sharing. When yield is random, it can be reasonable to acquire yield information, so that yield uncertainty is reduced. Choi (2010) presents a brief summary of papers that examine upstream information sharing as well as downstream information sharing. Upstream information sharing means that upstream members of a supply chain share their information with downstream members. Choi (2010) concludes in his review on information sharing that upstream information sharing is rarely examined. Furthermore, to our best knowledge there are only two papers that consider sharing of yield information (Choi, Blocher, and Gavirneni 2008; Dettenbach and Thonemann 2015) in inventory management. All of these papers do not consider the costs of yield information. Without including these costs, the decision on whether or not to acquire yield information is trivial.

To the best of our knowledge White and Cheong (2012) are the only authors that consider the cost of tracking and model tracking as a decision variable. Their modeling approach of yield risk and yield information is similar to ours. They consider a single order that is transported from origin to destination through multiple stages. Items can deteriorate during transit. At each stage the decision maker can chose to continue or abort the transportation process. If the transportation is continued, the decision maker can decide whether or not the order is inspected upon arrival at the next stage. An inspection reveals the current yield of the order and induces inspection cost. Their system is modeled as a partially observed Markov decision process and they derive the optimal inspection policy. The focus of their model is on transportation decisions. Our focus is on inventory control. We contribute to the existing research by analyzing the value of real time yield information when acquiring this information is costly. In addition, we model access to this information as a decision variable for each individual order.

The remainder of the paper is organized as follows. In Section 4.3, we develop the inventory models and derive structural properties. In Section 4.4, we provide numerical results on the value of a flexible tracking policy and elaborate on the influencing factors for the tracking decision. In Section 4.5, we conclude.

4.3. Model

Consider a single manufacturer who places orders with a single supplier. The manufacturer uses a periodic review inventory policy. The decision variables in each period are the order quantity O and the decision Ψ whether or not to track this order. Tracking an order causes tracking cost c that is independent of the ordered quantity. The yield of all items in a tracked order is observable in real time. The lead time is λ periods and each lead time period r has a yield risk of u_r. The yield rates u_r and the demand rate d are i.i.d. across periods and can be arbitrarily distributed. The state of the inventory system in period t is defined by the inventory level IL and λ orders in transit and modeled as $z = \big(IL, R_\lambda(O_\lambda, \Psi_\lambda),\ \cdots, R_1(O_1, \Psi_1)\big)$, where

$$R_i(O_i, \Psi_i) = \begin{cases} O_i & \text{, for } \Psi_i = 0 \\ Q_i = u_1 u_2 \cdots u_i O_i & \text{, for } \Psi_i = 1. \end{cases} \tag{20}$$

The state space consists of a mix of tracked and not tracked orders. For a tracked order $(\Psi_i = 1)$ all yield rate realization $u_1, \dots, u_i$ up to period i are observable and the current yield Q_i is known. For not tracked orders only the order quantity O_i is known.

The sequence of events in each period is as follows: First, the manufacturer observes the current state of the inventory system z. Then, the manufacturer decides on O and Ψ. Next, the manufacturer receives the order O_λ. Demand is satisfied from on-hand inventory and any unsatisfied demand is backordered. Based on the net inventory at the end of period, backorder costs b or holding costs h are charged per unit and period.

Our objective is to minimize total expected cost over an infinite horizon. To obtain the optimal policy, we model the system as a dynamic program:

$$\begin{aligned} V(z) &= \min_{O \geq 0, \Psi \in \{0,1\}} H(z, O, \Psi) \\ &= h[J(z)]^+ + b[-J(z)]^+ + \min_{O \geq 0, \Psi \in \{0,1\}} \{\Psi c + \gamma[V(\tilde{z})]\}, \end{aligned} \tag{21}$$

where

$$J(z) = \begin{cases} E_{\bar{u}_\lambda} E_d[IL + \bar{u}_\lambda O_\lambda - d] & \text{, for } \Psi_\lambda = 0 \\ E_d[IL + Q_\lambda - d] & \text{, for } \Psi_\lambda = 1 \end{cases} \tag{22}$$

is the function for the net inventory at the end of the period and $[x]^+ = \max(0, x)$. $\gamma < 1$ denotes the discount factor. In case $\Psi_\lambda = 1$ the yield of the arriving order is known. For $\Psi_\lambda = 0$ the total yield risk over lead time $\bar{u}_\lambda = u_1 u_2 \cdots u_\lambda$ has to be considered to estimate the replenishment quantity for the current period.

The transition function from state z to state $\tilde{z}$ is

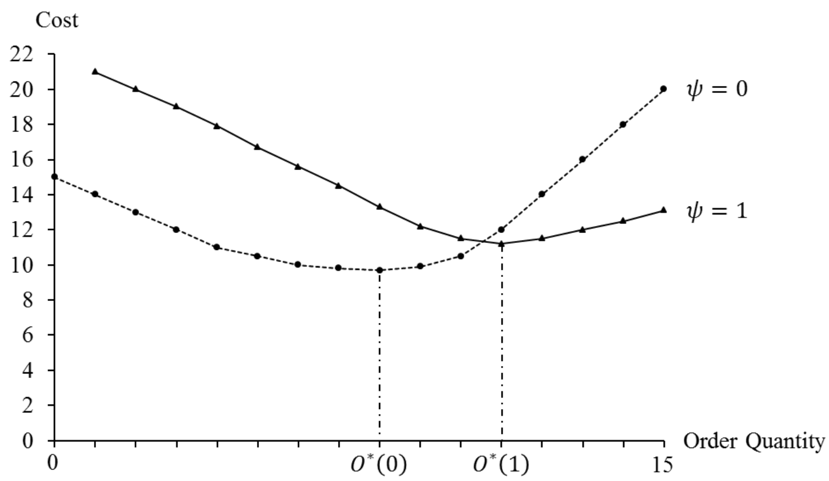
$$\tilde{z} = f(z, O, \Psi, d, u_1, \cdots, u_\lambda) = \left(J(z), E_{u_\lambda}[R_\lambda(O_{\lambda-1}, \Psi_{\lambda-1})], \cdots, E_{u_1}[R_1(O, \Psi)]\right). \tag{23}$$

Theorem 4-1 states that for a given tracking decision Ψ the objective function $V(z)$ is convex in the order decision O.

Theorem 4-1. $H(z, O, \Psi)$ is convex in z and O and $V(z)$ is convex in z for fixed Ψ and any convex terminal function.

The proof can be found in Appendix 4.A. Theorem 4-1 reduces the effort to find the optimal decision $[O^*, \Psi^*]$ because the optimal order quantity $O^*(\Psi)$ can be determined by convex optimization. This reduces the decision to the two options: $[O^*(0), \Psi = 0]$ and $[O^*(1), \Psi = 1]$. The optimal decision $[O^*, \Psi^*]$ is found by selecting the option with lower expected costs. These costs have already been obtained when $O^*(0)$ and $O^*(1)$ were determined. Figure 4-1 provides an illustrative example for the convexity by plotting $V(z)$ for different order quantities and fixed tracking decision. For this example $O^*(0)$ has lower cost than $O^*(1)$ resulting in the optimal decision $[O^* = 8, \Psi^* = 0]$.

Figure 4-1 Illustrative sample plot of $V(z)$ as a function of order quantity for fixed tracking decision



Next, we show that a solution for the infinite-horizon problem exists.

Theorem 4-2. $V(z) = \lim_{t\to\infty} V_t(z)$ exists for every z.

The proof can be found in Appendix 4.B. Applying standard Markov decision process arguments, an optimal stationary policy exists for any system with a finite number of states and actions (Heyman and Sobel 1984). Any Markov decision process solution method can be applied to solve Equation (21) for the optimal policy. We use MacQueen extrapolation (MacQueen 1966) and exploit the results from Theorem 4-1.

4.4. Computational Results

4.4.1. The Value of a Flexible Tracking Policy

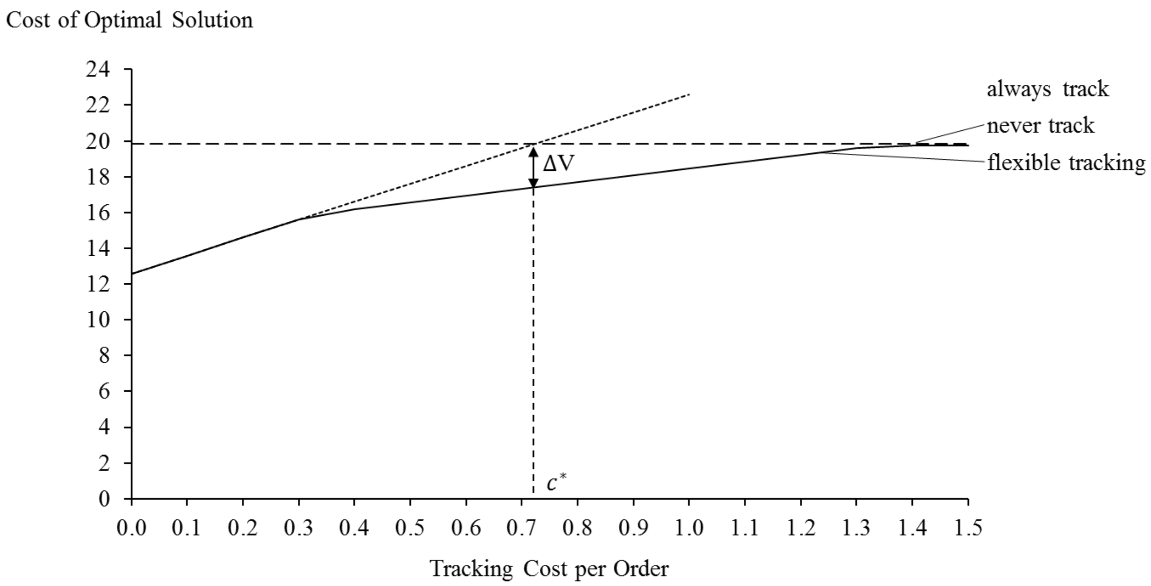
A very important parameter in our model is tracking cost c. Intuitively, for tracking cost of zero the optimal decision is to always track all orders. From the same reasoning follows that if tracking cost is very high, the optimal decision is to never track an order. To enable a comparison between test cases we evaluate all test cases at tracking cost for which the decision maker would be indifferent between the tracking policies: “always track all orders” and “never track any order”. Figure 4-2 uses a sample case[**] to illustrate how the cost of the optimal solution depends on the tracking cost for three tracking policies: always track, never track, and flexible tracking. The optimal cost for the flexible tracking model behaves as expected and can never be higher than the lower cost of the other two policies. Interesting test cases arise at tracking cost of c^*. At this point the benefit ΔV of a flexible tracking policy reaches its maximum compared to the two static policies. In our numerical results we analyze c^* and ΔV for varying lead times, yield risks and critical ratios.

For our numerical analysis of the optimal solutions, we use lead times of $\lambda = 1$, 2, 3, and 4 periods. We use a Bernoulli distributed yield rate with expected yields of $u = 0.8$, 0.85, and 0.9. Unless stated otherwise, only the first lead time period has random yield. To concentrate all yield risk in one lead time period reduces the computational effort. For a discussion on the effect of varying the yield risk positions over different lead time periods we refer to Dettenbach and Thonemann (2015, Section 6). Without loss of generality, we set unit

[**] Test case: $\lambda = 1$, $u = 0.9$ (Bernoulli distributed), CR = 0.85

inventory holding cost to $h = 1$ and choose unit backorder costs b that result in critical ratios ($CR = b/(b+h)$) of 0.85, 0.90, 0.95, and 0.99. We use a discount factor γ of 0.9 and deterministic demand of 2. This results in 48 test cases. To compute the optimal solution we use McQueen extrapolation and the value iteration is conducted with an accuracy of $\varepsilon = 0.001$.

Figure 4-2 Example for cost of optimal solution for different tracking policies under proportional yield



We consider a discrete state space and limit inventory levels to ±30 and order quantities to 10, limits that are essentially never binding. This creates a state space with $61 \cdot 21^{\lambda}$ states. All algorithms were implemented in C++ and all experiments were conducted on a PC with eight Intel 3.06 GHz processors and 8 GB of RAM. Table 4-1 shows average run times in minutes using all eight processors.

Table 4-1 Average run times of optimal solution approach (minutes)

yield model	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$
proportional	< 0.1	0.6	16	425

Run times increase exponentially in lead time, because the state space increases exponentially in lead time, which limits the applicability of the optimal solution approach to small and medium size problems. Note that computations times would be much longer if demand was stochastic. The results for the test cases are shown in Table 4-2. To facilitate the analysis Figure 4-3 and Figure 4-4 provide averaged results for the three parameters: yield risk, critical ratio and lead time.

Table 4-2 Analysis of optimal solution for proportional yield model

		c^*				ΔV in %			
		CR				CR			
u	λ	0.85	0.9	0.95	0.99	0.85	0.9	0.95	0.99
0.80	1	0.6	0.8	0.7	0.7	13.4	9.1	6.6	4.4
	2	1.2	1.4	1.4	1.4	14.2	13.0	13.8	9.1
	3	1.6	1.9	2.0	2.3	16.9	14.1	15.0	13.4
	4	2.0	2.4	2.5	2.8	17.1	16.5	17.2	10.6
0.85	1	0.3	0.5	0.8	0.7	7.7	11.9	5.7	5.2
	2	0.7	1.1	1.4	1.6	15.9	12.2	10.7	7.3
	3	1.2	1.6	1.9	2.1	15.9	15.9	13.9	10.7
	4	1.5	1.9	2.4	2.7	18.1	14.1	15.2	14.4
0.90	1	0.7	0.2	0.5	0.2	12.3	5.6	9.0	3.1
	2	0.9	0.6	1.1	0.8	8.3	14.0	9.1	11.1
	3	1.2	0.9	1.6	1.3	12.1	13.6	14.7	12.2
	4	1.5	1.4	1.8	1.9	14.1	13.2	9.9	16.7

Figure 4-3 c^* for proportional and binomial yield model

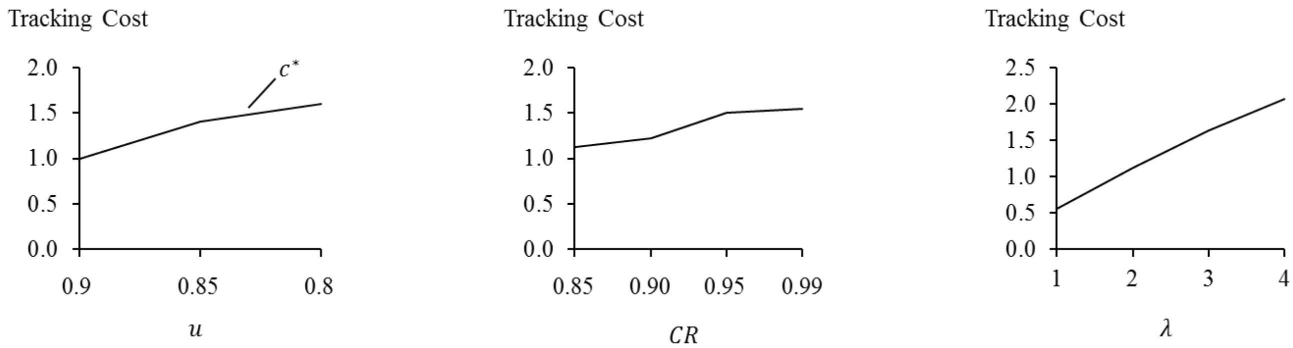
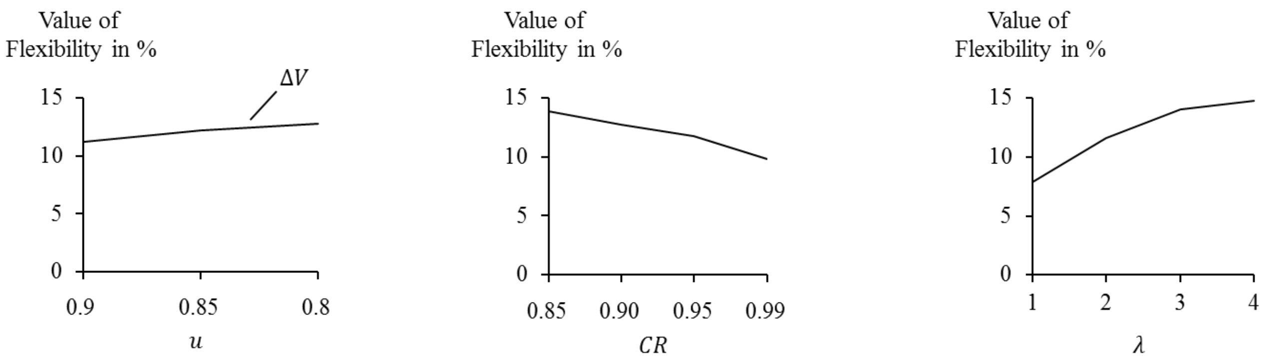


Figure 4-4 Value of flexibility (ΔV) for proportional yield model



The results show that flexible tracking results in significant cost savings. The optimal policy at tracking cost c^* is a mix of tracked and not tracked orders. The flexible tracking policy is superior to the two strict policies (see also Figure 4-2 for an illustration).

As the yield rate decreases, the tracking cost threshold c^* increases. Using Bernoulli distributed yield rates; a decreasing yield rate increases the yield variability which peaks at a yield rate of 0.5. This result is intuitive, because real time yield information reduces uncertainty about the state of the open orders more if yield variability is high than if it is low. Therefore a higher price for order tracking can be established. Changing yield risks have minor effects on the value of flexibility ΔV.

The tracking cost threshold c^* increases in lead time. As lead time increases, a not tracked order adds more uncertainty to the system. Therefore the value of tracking increases and so does the tracking cost threshold c^*. The value of flexibility ΔV increases in lead time. Multiple lead time periods create states with multiple outstanding orders. The flexible tracking policy makes states with a mix of tracked and not tracked orders accessible which results in cost savings. These states are not accessible when a static policy is applied which is increasingly disadvantageous as lead time increases.

Increasing critical ratios have a minor effect on the tracking cost threshold c^*. Costs under the always track and the never track policy increase by a similar amount. This results in an upward shift of both cost functions (see also Figure 4-2) which has no effect on the tracking cost threshold c^*. An interesting result is the decreasing value of flexibility ΔV as critical ratios increase. Higher critical ratios bring the flexible policy closer to the two strict policies. If critical ratios are higher it is optimal to track every order for higher tracking cost values. As the flexible tracking theory deviates later (in the sense of higher tracking costs) from the always track policy the difference at the tracking cost threshold c^* decreases.

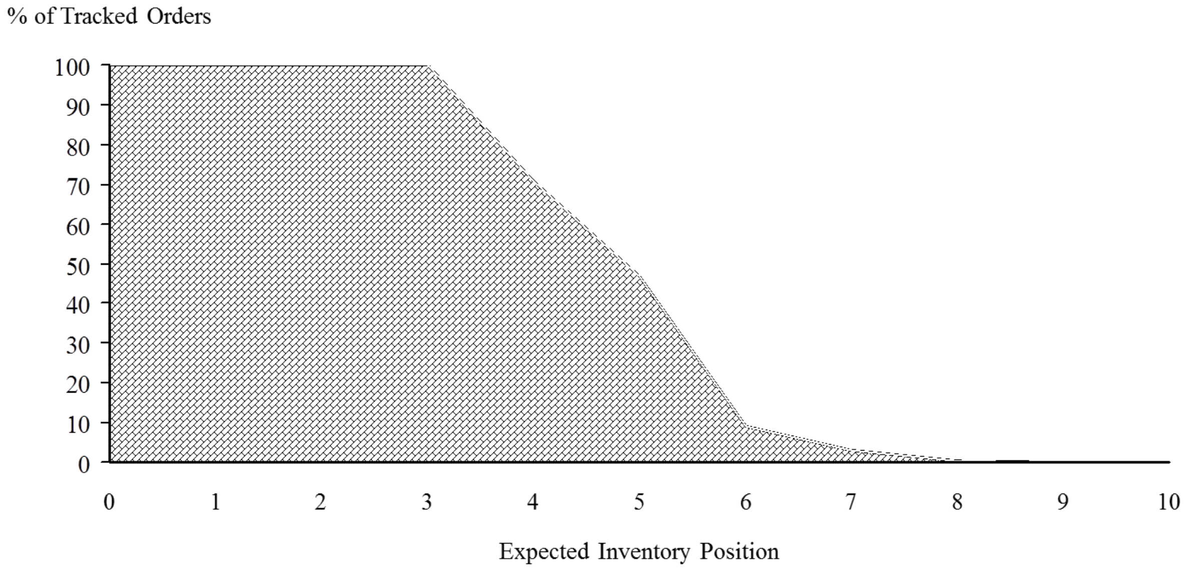
4.4.2. Influences on Tracking Decision

To develop heuristics it is useful to understand the key factors that influence the tracking decision. There exist multiple heuristics on order quantities under random yield (e.g. Bollapragada and Morton 1999; Dettenbach and Thonemann 2015). These heuristics cannot be applied directly to our model because the tracking decision and the order decision have to be made simultaneously. Our intention is to understand the connection between current state of the inventory system, order quantity and tracking decision. The current state of the system can be quantified by the expected inventory position and the variance of the expected inventory position.

The analysis of the effects of expected inventory position and order quantity on tracking decision is connected because higher expected inventory positions result in lower order quantities (see also Figure 4-6). To make the effects observable, we have to analyze the optimal decisions for individual states. We focus on the test case with $\lambda = 2$, CR = 85 and $u = 0.8$. The results are similar to the other test cases that we analyzed.

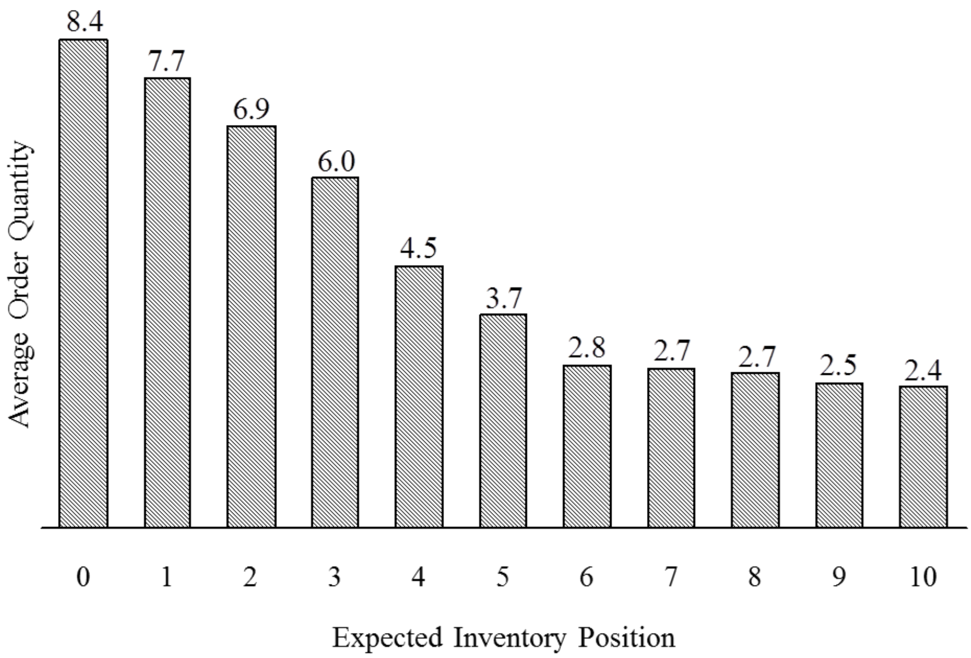
For each state we compute the expected inventory position. Figure 4-5 shows the percentage of tracked orders depending on the expected inventory position of the current state. Up to an expected inventory position of 3 all orders are tracked. For expected inventory positions larger than 3 the percentage of tracked orders is decreasing.

Figure 4-5 Percentage of tracked orders depending on the expected inventory position



To show the connection between expected inventory position and order quantity we analyze the average order quantity per inventory position in Figure 4-6. Not surprisingly the order quantity is decreasing in inventory position.

Figure 4-6 Average Order Quantity per expected inventory position.

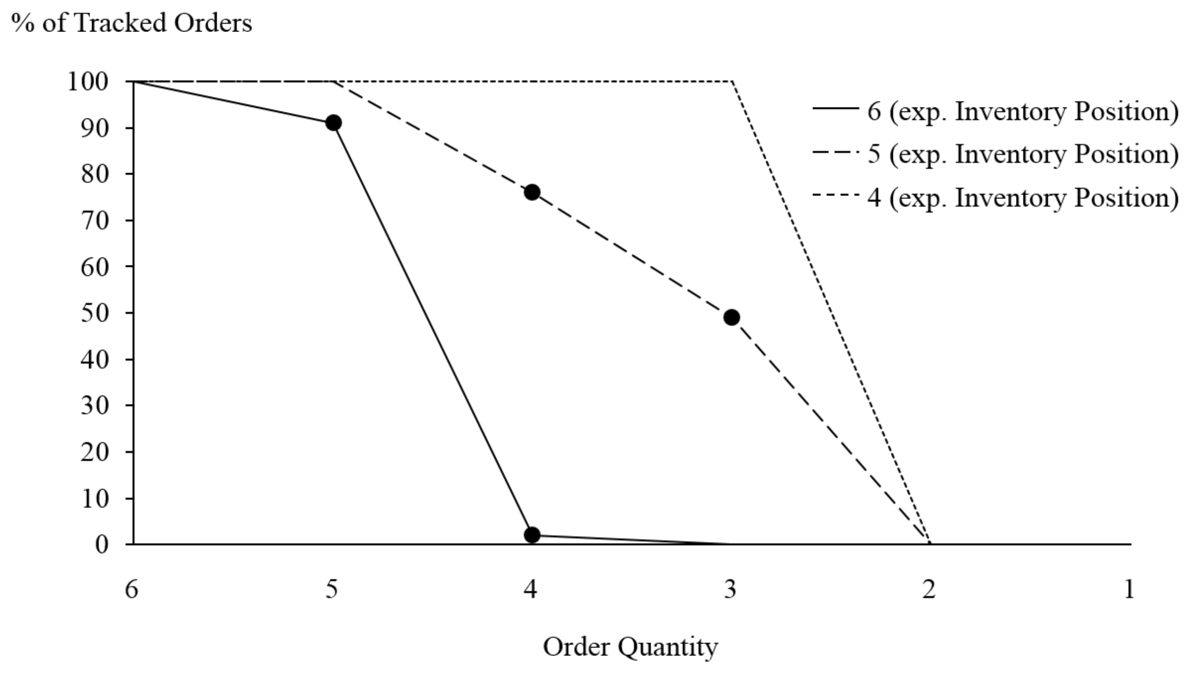


To test whether or not the decrease in tracking is due to the increase in expected inventory position and/or due to decreasing order quantities we control for the expected inventory position and analyze the tracking decision for different order quantities. According to Figure 4-5 the expected inventory positions of interest are 4, 5, and 6. In Figure 4-7 we see that smaller order quantities result in less tracking. At the same time the expected inventory position influences the tracking decision, too. When the inventory position is decreasing smaller order quantities are tracked with a higher probability. Concluding from the analyzed data we make the following observations.

Observation 1: Tracking is less beneficial when the expected inventory position is higher.

Observation 2: Tracking is more beneficial for higher order quantities.

Figure 4-7 Percentage of tracked orders depending on order quantity for different inventory positions



To analyze the effect of the variance of the inventory position we have to control for expected inventory position and order quantity. This results in four interesting test cases that are indicated by black circles in Figure 4-7. Analyzing the expected inventory position's standard deviation for each state we get the following data as shown in Table 4-3.

Table 4-3 Percentage of tracked orders depending on standard deviation of inventory position

exp. inventory position	5						6					
order quantity	3			4			4			5		
std. dev. of inv. pos.	1	2	4	2	3	4	2	3	4	2	3	4
number of orders	37	2	6	26	3	4	12	21	27	9	1	1
nbr. of tracked orders	22	0	0	25	1	0	1	0	0	9	1	0
% of tracked orders	59%	0%	0%	96%	33%	0%	8%	0%	0%	100%	100%	0%

Table 4-3 can be read in following way. For an expected inventory position of 5 and an order quantity of 3 we found test cases with standard deviations of 1, 2, and 4. 37 test cases with standard deviation of 1, order quantity of 3 and expected inventory position of 5 exist. For 22 (59 %) of these 37 test cases the order has been tracked. The data shows that the percentage of tracked orders is decreasing in variability of the inventory position. To add a not tracked order to a state with low variability is causing more cost than adding the same not tracked order to a state that already has a high variability. The data leads to the following observation.

Observation 3: Tracking is more beneficial when the variance of the inventory position is lower.

This discussion provides first insights on the influential factors of the tracking decision. The observations can be used to develop heuristic solutions approaches.

4.5. Conclusion

We analyzed a periodic review, random yield problem with positive lead time and tracking cost. We modeled the problem with two decision variables: order quantity and tracking. We proved that the cost function is convex for a given tracking decision and that a solution for the infinite horizon problem exists. Based on these properties we applied an optimal solution approach for discrete state spaces. We conducted numerical experiments that show that a flexible tracking policy can create significant value. Further analysis provided insights on the influential factors for the tracking decision.

Since the optimal solution approach can only be applied to small and medium problem instances there is a need for heuristics. This would be the intuitive extension to this work.

Appendix 4.A Proof of Theorem 4-1

Proof Clearly $R_i(O_i, \Psi_i)$, $J(z)$, and $h[J(z)]^+ + b[-J(z)]^+$ are convex in IL and $R_i(O_i, \Psi_i)$ for $i = 1, \cdots, \lambda$ (Rockafellar 1970, Theorem 5.7, Heyman and Sobel 1984, Proposition B-2). For fixed Ψ the only decision variable is the order quantity O and the term Ψc becomes a constant. The proof then follows the reasoning of the proof of Theorem 3-1. By assumption, the terminal function $V_{T+1}(z)$ is convex. We continue with induction. Suppose that $V_t(z)$ is convex in z. From Theorem 5.7 of Rockafellar (1970) it follows that $V_t\big(J(z), R_\lambda(O_{\lambda-1}, \Psi_{\lambda-1}), \cdots, R_1(O, \Psi)\big)$ is convex in IL, $R_i(O_i, \Psi_i)$, $i = 1, \cdots, \lambda$, O, and Ψ. Because the expectation conserves convexity and the sum of convex functions is a convex function $H_{t-1}(z, O, \Psi) = E_d, E_{u_1}, \dots, E_{u_\lambda}\Big[h[J(z)]^+ + b[-J(z)]^+ + \min\limits_{O \geq 0}\big\{\Psi c + \gamma V_t\big(J(z), R_\lambda(O_{\lambda-1}, \Psi_{\lambda-1}), \cdots, R_1(O, \Psi)\big)\big\}\Big]$ is convex in z and O for fixed Ψ. The minimization also conserves convexity and therefore $V_{t-1}(z) = \min\limits_{O \geq 0} H_{t-1}(z, O, \Psi)$ is convex in z and O for fixed Ψ. ■

Appendix 4.B Proof of Theorem 4-2

Proof Let the zero order policy: $[O = 0, \Psi = 0]^\infty(z)$ for all z denote a stationary policy and $V^\infty(z)$ be its expected present value. We apply Theorem 8.13 of Heyman and Sobel (1984), which states that if $V^\infty(z) < \infty$ for all z, then $V(z) = \lim\limits_{t \to \infty} V_t(z)$ exists. Clearly the single period costs are bounded, and since $\gamma < 1$ we have $V^\infty(z) < \infty$ for all z, and the result follows. ■

Chapter 5

Co-Production and Partial Supply Chain Visibility in Semiconductor Manufacturing

5.1. Abstract

We consider a two-stage production system which produces a hierarchy of multiple grades of outputs. In the first stage, a single type of input is used to produce products of different quality levels with random yield rates. In the second stage, products are tested for their quality level. Test capacity is limited. This setting is motivated by the production process of a global semiconductor manufacturer. We develop a mathematical model of the production system and derive structural properties for the one and two-period case. Building up on these properties we provide two solution approaches that are close to optimal. In addition we analyze the value of implementing a pre-test that partially reveals a product's quality level after first stage production is completed. We show how this preliminary yield information can be used to make more efficient use of limited test capacities at the second stage. We conduct numerical experiments to evaluate the accuracy of our solution approaches and to identify conditions under which preliminary yield information at the first stage is particularly beneficial.

Keywords: inventory management; co-production; random yield; value of information

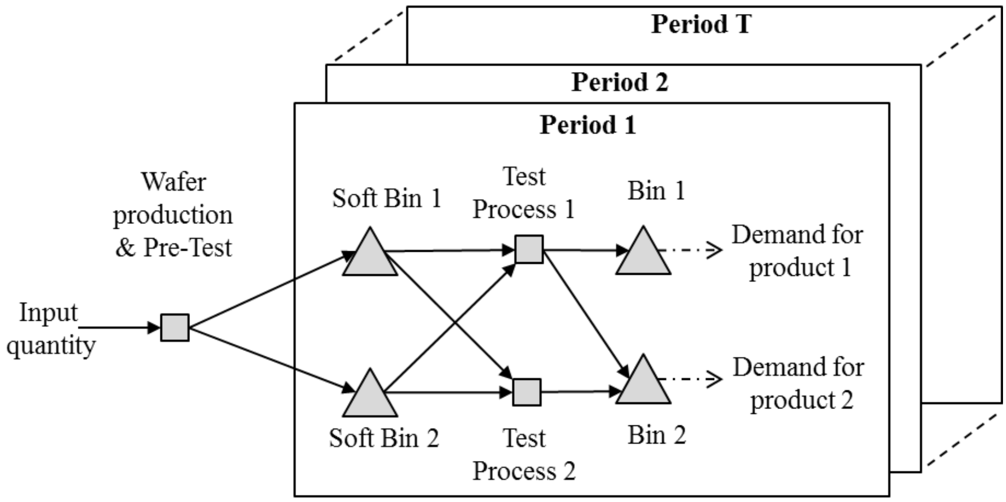
5.2. Introduction

Semiconductor production processes are subject to random yield and co-production. A single input is used to produce simultaneously products of different quality levels at random yield rates. Most previous research models the semiconductor production process as a one step process. The production process of the big semiconductor manufacturing company that motivated our research consists of multiple steps and is typical for the whole industry. In an aggregated and simplified form the process can be described as follows. After initial wafer production, wafers are sliced into chips. Then, chips go through a test process to determine their quality level. A chip's quality level refers to features like speed, memory capacity and heat resistant. A detailed overview of the semiconductor production process can be found in Gavirneni (2004), Han, Dong, and Shao (2012) and Taouil and Hamdioui (2012).

We develop a model that differentiates between the production process and the test process. This is motivated by two observations. First, the bottleneck in semiconductor manufacturing and in the researched company is often test capacity (Freed et al. 2007; Lin et al. 2004; Tai et al. 2012). By differentiation between production and testing we can model the test capacity constraint explicitly and without affecting the production process. Second, the production process takes much longer (several weeks) than the test process (one week) (Freed, Doerr, and Chang 2007; Gavirneni 2004; Han, Dong, and Shao 2012). Because we treat both processes individually, our model allows for multiple test runs while in parallel a production run is in progress. Our decision model reflects the industry practice more accurately and enables the analysis of parameter and process changes with higher precision than a one step process model could achieve.

In the described process products would enter the test stage with unknown quality. We introduce and analyze the concept of preliminary yield information as a mean to enable more efficient use of limited test capacity. At the end of the production stage chips are pre-tested for their quality levels and sorted into so called “soft bins”. The pre-test is a fast and inexpensive process that indicates the quality of each chip to a certain extent, i.e., the test result is not a prediction of final quality and the quality level of a chip after pre-testing can differ from the quality level after final testing. Figure 5-1 illustrates the two stage process for two quality levels. The process consist of a single production opportunity that is followed by T periods of testing and demand fulfilment.

Figure 5-1 Overview of the two-stage semiconductor production system



The results of our research can be used to improve the efficiency of semiconductor manufacturing. We extend previous research by providing a model that distinguishes between production and test processes. We allow for up- and downgrading at the test stage and include random yield rates and co-production in the model. Unlike previous research, we consider limited test capacity. We develop a finite horizon model and provide an approach that finds solutions that are close to optimal. Using this solution approach, we quantify the value of preliminary yield information. This information is provided by a pre-test after first stage production is completed. We also introduce a heuristic, which builds on structural properties of the one- and two-period problem and can be applied to solve larger problems efficiently. Our numerical results indicate that the heuristic performs well for a variety of parameter settings. The results also indicate that substantial profit improvements can be achieved by taking advantage of preliminary yield information.

The remainder of the paper is organized as follows. In Section 5.3, we review the related literature. In Section 5.4, we develop a two-stage dynamic program for a periodic review inventory system with random yields and co-production. In Section 5.5, we provide structural results for the one- and two-period model. In Section 5.6, we develop an arbitrary close to optimal solution approach and use the analysis of the one- and two-period model as building blocks to introduce a heuristic solution approach. In Section 5.7, we provide numerical results. In Section 5.8, we conclude.

5.3. Related Literature

Our problem belongs to the class of random yield problems with co-production. Bitran and Dasu (1992) were among the first to address random yield problems with co-production. They formulate a two-stage dynamic program where the first stage determines the production quantity and the second stage the allocation quantities for products of different quality to customer demands. Because the optimal solution is computational intractable, a decomposition heuristic is introduced. Bitran and Leong (1992) provide deterministic approximations for the finite horizon version of the same problem and develop a heuristic. Bitran and Gilbert (1994) formulate a nested dynamic program for a finite horizon problem with deterministic demand and derive a lower bound on the cost of the optimal solution. They develop a production quantity heuristic that is designed to satisfy the demand for a given number of periods with a certain probability.

Gerchak, Tripathy, and Wang (1996) consider a single period problem with deterministic demand and two quality levels. They prove joint concavity of the objective functions and derive optimality conditions. Hsu and Bassok (1999) also consider the single period case and provide a decomposition approach to calculate optimal solutions. Gerchak and Grosfeld-Nir (1999) consider a single period make-to-order process. Production happens in multiple sequential production runs until all demand is satisfied. The objective is to determine lot sizes that minimize total set up and production cost. Numerical results are provided for the two-product case. Murr and Prékopa (2000) consider the process of manufacturing optical fibers which features the same characteristics as semiconductor production. They develop a chance constraint stochastic program with the objective to minimize production cost. Programs of this class are still largely intractable because they require multidimensional integration and have a non-convex feasible region. The authors apply a two-period solution on a rolling horizon basis to plan multi-period scenarios. Duenyas and Tsai (2000) model the same system as Bitran and Dasu (1992) as a queueing system with the extension of uncertain demands and production times. In the case of two product classes, they characterize the structure of the optimal policy and develop a heuristic that can be applied to problems with an arbitrary number of product classes. Gallego, Katircioglu, and Ramachandran (2006) develop an infinite horizon cost minimization model with service level constraints. They propose two heuristics that use the concept of a critical part as decision driver and show that a single period allocation scheme does not result in inventory performance deterioration when applied to the stationary infinite horizon case. Han et al. (2011, 2012) introduce a model that has a single production opportunity prior to the first period, which is followed by multiple periods of demand allocation decisions. This is motivated by the fact that wafer production at the first production stage has a much longer lead time than demand periods. We also cover this fact in our article. Han et al. (2012) show that the objective function is concave in the production quantity.

The above literature considers co-production systems with direct upward substitution of demands. More recently, semiconductor manufacturers adopted the approach of downgrade production where higher quality products are intentionally disabled to resemble a lower quality product and satisfy lower quality demand. Downgrade production is done to protect profit margins and to prevent opportunistic customer behavior. Hsu, Li, and Xiao (2005) compare direct substitution with downgrade production for a generic setting with downgrading cost, deterministic demand, independent yield rates and finite planning horizon. They state that the problem is NP-hard and provide algorithms that find optimal solutions in polynomial time if the number of products is fixed. Ng and Fowler (2007) use robust optimization to solve a finite horizon problem under service level constraints. In their problem setting, inventory can be held either as semi-finished products before downgrade production or as finished products after downgrade production. Huang and Song (2010) consider a semiconductor production problem similar to ours. The first stage of their two-stage production system is equal to the first stage of our model, where a single input quantity is determined. At the second stage semi-finished products are transformed to finished products. At this stage downgrading is allowed and all yield rates are deterministic which as a consequence eliminates co-production. They show that some parts of the model follow a renewal process and use this property to develop two heuristics.

Unlike previous literature, we consider a setting that differentiates between production and testing, which allows for modeling different lead times for production and testing as well as limited test capacity. Our model also allows for up- and downgrades and considers random yield and co-production for all processes. Motivated by the production system of a global semiconductor manufacturer, we introduce and analyze the concept of preliminary yield information.

5.4. Model

We consider a model with a single production opportunity that is followed by T testing periods. In the first stage, the decision variable is the production quantity Q. After the products have been produced, they are pre-tested and assigned to soft bins. The second stage consists of T periods of testing from soft bin inventory into bin inventory.

The per unit production cost is c. The first stage production process yields two quality levels with yield coefficients $Y = \{Y_1, Y_2\}$. Quality level 1 is superior to quality level 2. Y_1 and Y_2 are correlated random variables with $0 \leq Y_1 + Y_2 \leq 1$. Products with unusable quality are produced with a yield rate of $1 - Y_1 - Y_2$ and discarded at no cost. Products that are not discarded are sorted into soft bins according to their quality level revealed by the pre-test.

At the second stage, products from a soft bin are selected as inputs for the two testing processes. The decision variables for the second stage are summarized in the following matrix: $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$, where X_{ij} is the input quantity from soft bin i for test process j. Test process j reveals if the product is at least of quality j or if it is of a specific lower quality level $k < j$ (co-production). U_{ikj} is the yield rate coefficient of product k from soft bin i that is tested with test process j. The yield rate coefficients matrixes are $U_1 = \begin{pmatrix} U_{111} & U_{121} \\ U_{211} & U_{221} \end{pmatrix}$ and $U_2 = \begin{pmatrix} U_{122} \\ U_{222} \end{pmatrix}$. There are multiple dependencies between these yield rate coefficients and we will discuss in Section 5.5.1 how the distributions of Y, U_1 and U_2 can be determined.

The unit costs for testing depend on the test processes. It is more costly to test for a higher quality than for a lower quality. E.g., it consumes more time and energy to test for a higher heat resistance than for a lower heat resistance. The unit testing cost for quality level j is r_j, with $r_1 \geq r_2$ and unit capacity consumption for quality level j is v_j, with $v_1 \geq v_2$. We denote the total test capacity per period by V.

Inventory in soft bin i is denoted by S_i and inventory in bin i by I_i. Unit inventory holding cost is h_{Si} for soft bin inventory and h_{Ii} for bin inventory, with $h_{Ii} \geq h_{Si}$. Unsatisfied demand is backordered at backorder cost b_i for quality level i with $b_1 \geq b_2$. Products from bins with higher quality are sold at a higher price, i.e., $p_1 \geq p_2$. The demands of the products $D = \{D_1, D_2\}$ are stochastic and i.i.d. across products and periods.

The sequence of events is as follows: In the beginning of the first stage, the manufacturer observes the current state of the inventory system, which consists of the bin inventory levels $I_t = \{I_{1,t}, I_{2,t}\}$ and soft bin inventory levels $S_t = \{S_{1,t}, S_{2,t}\}$. The manufacturer decides on the input quantity Q. The production output replenishes the soft bins according to the pre-test results. At the beginning of each period t, $t \in \{1, \ldots, T\}$ of the second stage, the manufacturer observes the demands $d_t = \{d_{1,t}, d_{2,t}\}$ for the current period together with the current state of the inventory system and decides on the input quantities X for the two test processes. Test outputs replenish bin inventories which are used to satisfy demand.

The model is formulated as a nested dynamic program. The objective is to maximize the profit over a finite horizon of T periods.

1st Stage

$$F(I_0, S_0) = \max_{Q} E_D\, E_Y E_{U_1} E_{U_2}[G_1(I_0, S_1)] - cQ \tag{24}$$

s.t.

$$S_1 = S_0 + yQ \tag{25}$$

$$Q \geq 0 \tag{26}$$

2nd Stage

$$G_t(I_t, S_t) = \max_X \; E_D E_{U_1} E_{U_2} \left[G_{t+1}(I_{t+1}, S_{t+1})\right] - \sum_{i=1}^{2}\sum_{j=1}^{2} r_j X_{ij}$$

$$- \sum_{i=1}^{2} \left[h_{Si} S_{i,t+1} + h_{Ii}\left[I_{i,t+1}\right]^+ + b_{Ii}\left[-I_{i,t+1}\right]^+\right] \tag{27}$$

$$+ \sum_{i=1}^{2} p_i \min\left(d_i + \left[-I_{i,t}\right]^+, \left[I_{i,t}\right]^+ + \sum_{j=1}^{2}\sum_{k=1}^{i} u_{jik} X_{jk}\right)$$

s.t.

$$S_{i,t+1} = S_{i,t} - \textstyle\sum_{j=1}^{2} X_{ij} \quad \forall\, i = 1, 2 \tag{28}$$

$$I_{i,t+1} = I_{i,t} + \textstyle\sum_{j=1}^{2}\sum_{k=1}^{i} u_{jik} X_{jk} - d_i \quad \forall\, i = 1, 2 \tag{29}$$

$$\textstyle\sum_{j=1}^{2}\left(v_j \sum_{i=1}^{2} X_{ij}\right) \le V \tag{30}$$

$$S_{i,t+1},\; X_{ij} \ge 0 \quad \forall\, i = 1, 2 \text{ and } \forall\, j = 1, 2 \tag{31}$$

with $[x]^+ = \max(0, x)$. Equations (25), (28), and (29) are inventory balancing constraints. Equation (30) is the test capacity constraint. Without loss of generality, we assume that $G_{T+1}(I_{T+1}, S_{T+1})$ = 0. After period T the whole planning cycle starts again and any excess inventories or back orders will be carried over to the next cycle. Since the first stage production quantity Q is not constraint and all parameters are kept constant between cycles there is no advantage from early production for use in future cycles. We must specify two first stage production yield distributions and six second stage test yield distributions that are correlated between products. The randomness of demands and the nested nature of the problem further complicate solving the problem optimally.

5.5. Structural Results

In Sections 5.5.1 and 5.5.2 we analyze the one- and two-period models analytically. We obtain structural results that are of some interest in their own rights and that we use as building blocks for the general heuristic introduced in Section 5.6.2.

5.5.1. Single-Period Analysis

Consider a single period problem for the second stage. Our goal is to derive equations for the optimal input quantities for the two test processes from each soft bin and to understand the structure of the optimal policy.

We follow the approach by Gerchak et al. (1996) and Bitran and Gilbert (1994) to model the yield rate coefficients and their dependencies as a combination of independent random variables. Consider soft bin i. A random fraction β_{i2} of the products have at least quality 2. Products that achieve quality level 1 constitute a random fraction β_{i1} of products that have at least quality level 2. The yield rate coefficients for both test processes can be calculated as

$$U_1 = \begin{pmatrix} U_{111} & U_{121} \\ U_{211} & U_{221} \end{pmatrix} = \begin{pmatrix} \beta_{11}\beta_{12} & \beta_{12}(1-\beta_{11}) \\ \beta_{21}\beta_{22} & \beta_{22}(1-\beta_{21}) \end{pmatrix} \text{ and } U_2 = \begin{pmatrix} U_{122} \\ U_{222} \end{pmatrix} = \begin{pmatrix} \beta_{12} \\ \beta_{22} \end{pmatrix}.$$

The fractions β_{ij} are independent and arbitrary distributed random variables over [0, 1] with known pdf. The yield rate coefficients for the first stage can be modeled analogously. The total yields of quality level 1 and 2 products are $P_1 = X_{11}\beta_{11}\beta_{12} + X_{21}\beta_{21}\beta_{22}$ and $P_2 = X_{11}\beta_{12}(1-\beta_{11}) + X_{12}\beta_{12} + X_{21}\beta_{22}(1-\beta_{21}) + X_{22}\beta_{22}$, respectively. For the profit function π_1, we have to distinguish four mutually exclusive and collectively exhaustive cases.

$$\pi_1(X) = -r_1(X_{11}+X_{21}) - r_2(X_{12}+X_{22}) + h_{S1}(X_{11}+X_{12}) + h_{S2}(X_{21}+X_{22})$$
$$+\begin{cases} p_1 d_1 + p_2 d_2 - h_1(P_1 - d_1) - h_2(P_2 - d_2) \\ p_1 d_1 + p_2 P_2 - h_1(P_1 - d_1) - b_2(d_2 - P_2) \\ p_1 P_1 + p_2 d_2 - b_1(d_1 - P_1) - h_2(P_2 - d_2) \\ p_1 P_1 + p_2 P_2 - b_1(d_1 - P_1) - b_2(d_2 - P_2) \end{cases} \tag{32}$$

if $(d_1 - X_{21}\beta_{21}\beta_{22})/X_{11}\beta_{12} \le \beta_{11}$ and $(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})/(X_{11}(1-\beta_{11}) + X_{12}) \le \beta_{12}$

if $(d_1 - X_{21}\beta_{21}\beta_{22})/X_{11}\beta_{12} \le \beta_{11}$ and $(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})/(X_{11}(1-\beta_{11}) + X_{12}) > \beta_{12}$

if $(d_1 - X_{21}\beta_{21}\beta_{22})/X_{11}\beta_{12} > \beta_{11}$ and $(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})/(X_{11}(1-\beta_{11}) + X_{12}) \le \beta_{12}$

if $(d_1 - X_{21}\beta_{21}\beta_{22})/X_{11}\beta_{12} > \beta_{11}$ and $(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})/(X_{11}(1-\beta_{11}) + X_{12}) > \beta_{12}$

s.t. $X_{11} + X_{12} \le S_1$ (33)

$X_{21} + X_{22} \le S_2$ (34)

$v_1(X_{11} + X_{12}) + v_2(X_{21} + X_{22}) \le V$ (35)

Non-negativity constraints

For notational convenience, we define $W_1 = (d_1 - X_{21}\beta_{21}\beta_{22})/X_{11}\beta_{12}$ and $W_2 = (d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})/(X_{11}(1-\beta_{11}) + X_{12})$. From Equations (32), (33), (34), (35) we obtain the expected profit function

$$\hat{\pi}_1(X) = -r_1(X_{11} + X_{21}) - r_2(X_{12} + X_{22}) + h_{S1}(X_{11} + X_{12}) + h_{S2}(X_{21} + X_{22}) \int_0^1 h(\beta_{22}) \int_0^1 k(\beta_{21}) \tilde{\pi}_1(X_{11}, X_{12}, X_{21}, X_{22}, \beta_{22}, \beta_{21}) d\beta_{21} d\beta_{22} \quad (36)$$

s.t. Equations (33)-(35) and non-negativity constraints with

$$\begin{aligned}
&\tilde{\pi}_1\,(X, \beta_{22}, \beta_{21}) = \\
&+ \int_0^{W_2} f(\beta_{12}) \int_0^{W_1} g(\beta_{11})[p_1P_1 + p_2P_2 - b_1(d_1 - P_1) - b_2(d_2 - P_2)]\, d\beta_{11} \\
&+ \int_{W_1}^1 g(\beta_{11})[p_1d_1 + p_2P_2 - h_1(P_1 - d_1) - b_2(d_2 - P_2)]\ d\beta_{11}d\beta_{21} \\
&+ \int_{W_2}^1 f(\beta_{12}) \int_0^{W_1} g(\beta_{11})[p_1P_1 + p_2d_2 - b_1(d_1 - P_1) - h_2(P_2 - d_2)]\, d\beta_{11} \\
&+ \int_{W_1}^1 g(\beta_{11})[p_1d_1 + p_2d_2 - h_1(P_1 - d_1) - h_2(P_2 - d_2)]\ d\beta_{11}d\beta_{21}.
\end{aligned} \quad (37)$$

Our objective is finding the optimal quantities $X^* = \begin{pmatrix} X_{11}^* & X_{12}^* \\ X_{21}^* & X_{22}^* \end{pmatrix}$ that maximize the expected profit $\hat{\pi}_1(X)$.

Theorem 5-1. $\hat{\pi}_1(X)$ is jointly concave in $X_{11} \geq 0, X_{12} \geq 0, X_{21} \geq 0$, and $X_{22} \geq 0$.

The proof can be found in Appendix 5.A. From the proof of Theorem 5-1 it is easy to see that the use of soft bin inventory follows a greedy approach. Soft bin 1 is fully utilized before soft bin 2 is used because the expected profits of units from soft bin 1 are always higher than those of units from soft bin 2. This myopic behavior is not necessarily optimal in the multi-period setting that we analyze next.

5.5.2. Two-Period Analysis

We next analyze a static two period setting. The problem is static, because we compute the optimal solution for period one and two at the beginning of period one. This is of course not the optimal policy for a dynamic two period problem. However, the solution for the first period is the same as the solution of the dynamic program and hence optimal. In the subsequent heuristic we use the two-period model on a rolling horizon basis and apply only the optimal solution for the first period. The main benefit is that this solution considers expectations about the future and hence avoids myopic behavior.

In the two-period setting we have eight decision variables. The decision variables are $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ and $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$, where X is the decision matrix for period 1 and Y the decision matrix for period 2. To model the profit function $\pi_2(X, Y)$ we have to distinguish 16 cases that follow the same logic as for the one-period model. The formulation of $\pi_2(X, Y)$ is provided in Appendix 5.B.

From Equations (54)-(65) in Appendix 5.B we get the expected profit function

$$\begin{aligned} &\hat{\pi}_2(X, Y) = \\ &-r_1(X_{11} + X_{21} + Y_{11} + Y_{21}) - r_2(X_{12} + X_{22} + Y_{12} + Y_{22}) \\ &+h_{S1}(2X_{11} + 2X_{12} + Y_{11} + Y_{12}) + h_{S2}(2X_{21} + 2X_{22} + Y_{21} + Y_{22}) \\ &+ \int_{D_1=0}^{\infty} m(D_1) \int_{D_2=0}^{\infty} n(D_2) \int_{\beta_{22,1}=0}^{1} h(\beta_{22}) \int_{\beta_{21,1}=0}^{1} k(\beta_{21}) \int_{\beta_{22,2}=0}^{1} h(\beta_{22}) \int_{\beta_{21,2}=0}^{1} k(\beta_{21}) \end{aligned} \tag{38}$$

$\tilde{\pi}_2\big(X, Y, D_1, D_2, \beta_{21,1}, \beta_{22,1}, \beta_{21,2}, \beta_{22,2}\big) d\beta_{22} d\beta_{21} d\beta_{22} d\beta_{21} d_{D_1} d_{D_2}$

s. t. $\quad v_1(X_{11} + X_{21}) + v_2(X_{12} + X_{22}) \leq V$ (39)

$v_1(Y_{11} + Y_{21}) + v_2(Y_{12} + Y_{22}) \leq V$ (40)

$X_{11} + X_{12} + Y_{11} + Y_{12} \leq S_1$ (41)

$X_{21} + X_{22} + Y_{21} + Y_{22} \leq S_2$ (42)

non-negativity constraints

Let P_{it} be the test yield of quality level i in period t (see Appendix 5.B) and $W_1 = \frac{d_1 - X_{21}\beta_{21,1}\beta_{22,1}}{X_{11}\beta_{12,1}}$, $W_2 = \frac{d_2 - X_{21}\beta_{22,1}(1-\beta_{21,1}) - X_{22}\beta_{22,1}}{X_{11}(1-\beta_{11,1}) + X_{12,1}}$, $W_3 = \frac{d_1 + D_1 - Y_{21}\beta_{21,2}\beta_{22,2} - P_{11}}{Y_{11}\beta_{12,2}}$ and $W_4 = \frac{d_2 + D_2 - Y_{21}\beta_{22,2}(1-\beta_{21,2}) - Y_{22}\beta_{22,2} - P_{21}}{Y_{11}(1-\beta_{11,2}) + Y_{12}}$. After some algebraic transformations we get

$$\begin{aligned}
&\tilde{\pi}_2\big(X, Y, D_1, D_2, \beta_{21,1}, \beta_{22,1}, \beta_{21,2}, \beta_{22,2}\big) = \\
&\int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \int_{\beta_{11,1}=0}^{W_1} g(\beta_{11}) [(h_1 + b_1)(P_{11} - d_1)]\, d\beta_{11} d\beta_{12} \\
&+ \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{W_2} f(\beta_{12})\, [(h_2 + b_2)(P_{21} - d_2)] d\beta_{12} d\beta_{11} \\
&+ \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{1} f(\beta_{12}) [h_1(d_1 - P_{11}) + h_2(d_2 - P_{21})] \\
&+ \int_{\beta_{12,2}=0}^{1} f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11})\, [(p_1 + h_1 + b_1)(P_{12} + P_{11} - D_1 - d_1)] d\beta_{11} d\beta_{12} \\
&+ \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12})\, [(p_2 + h_2 + b_2)(P_{22} + P_{21} - D_2 - d_2)] d\beta_{12} d\beta_{11} \\
&+ \int_{\beta_{12,2}=0}^{1} f(\beta_{12}) \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) [h_1(D_1 + d_1 - P_{12} - P_{11}) \\
&\qquad\qquad + h_2(D_2 + d_2 - P_{22} - P_{21})]\, d\beta_{11} d\beta_{12} d\beta_{12} d\beta_{11} \\
&+ p_1(D_1 + d_1) + p_2(D_2 + d_2)
\end{aligned} \tag{43}$$

The solution is the optimal static test decision for the first and the second period.

Theorem 5-2. $\hat{\pi}_2(X, Y)$ is jointly concave in $X \geq 0$ and $Y \geq 0$.

The proof of Theorem 5-2 is provided in Appendix 5.D. For the one and two-period model we can conclude that, the optimal test quantities can be obtained by standard solution approaches for concave optimization problems. Those approaches require the first order derivatives that are provided in Appendix 5.A and 5.C.

5.6. Solution Approaches

Solving the dynamic program is computational intractable even for small problem sizes. In Section 5.6.1, we use a linear programming approach that determines approximate solutions that are arbitrary close to the optimal solution. This solution approach can be applied to small and medium size problems. In Section 5.6.2, we use the structural results to propose a heuristic that reduces computation time and can be used to solve larger problems.

5.6.1. ε-optimal Solution Approach

We formulate a stochastic LP for each stage, where the yield and demand distributions are represented by a collection of random scenarios. The objective is to maximize the expected profit over these scenarios. This approach of solving a stochastic LP is equivalent to Monte Carlo sampling. By the law of large numbers, the LP solution converges to the optimal solution as the number of scenarios goes to infinity. We use statistical techniques to determine the gap between the LP solution and the optimal solution (Bayraksan and Morton 2006). By adjusting the number of scenarios this approach can compute an ε-optimal solution for arbitrary ε.

Figure 5-2 Illustration of ε -optimal solution approach for the second stage problem

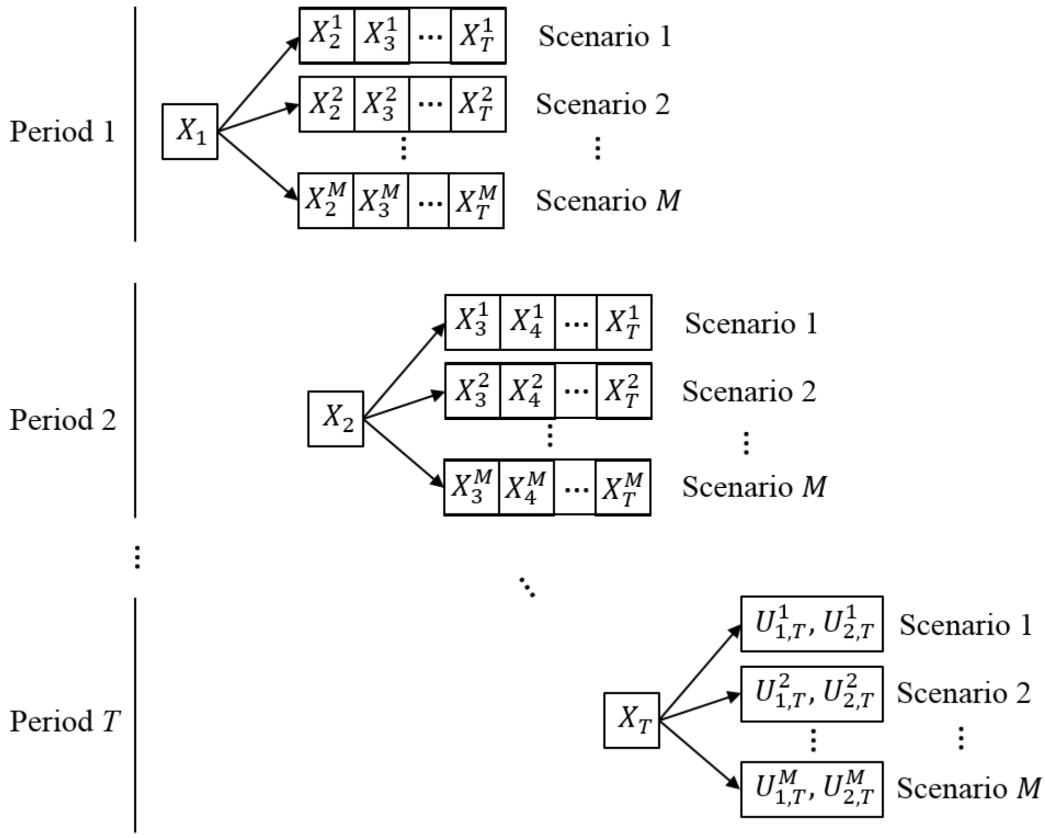


Figure 5-2 illustrates the solution approach using the second stage problem. To compute the decision matrix X_1 in period 1, M scenarios are created. The future decisions for periods 2 to T are optimized for each scenario. The decision for period 1 is optimized over all scenarios and hence the same for all scenarios. The test decisions matrix for the current period X_1 is the only applicable output of this solution approach. To compute X_2 in period 2 a similar LP is solved with one period less for each scenario. In period T no future decisions have to be made and X_T is optimized over scenarios that resemble the yield realizations $U_{1,T}$ and $U_{2,T}$ in period T.

Next, we provide the LP for the first stage. Each scenario has the probability weight $a = 1/M$, where M is the number of scenarios.

$$\max_{Q} \quad -cQ + \sum_{m=1}^{M} a\left[\sum_{t=1}^{T}\left[\sum_{i=1}^{2} p_i \min\left(d_{i,t}^m + \left[-I_{i,t}^m\right]^+, \left[I_{i,t-1}^m\right]^+ + \sum_{j=1}^{2}\sum_{r=1}^{i} u_{jir,t}^m X_{jr,t}^m\right) - \sum_{j=1}^{2} r_j X_{ij,t}^m - h_{Si} S_{i,t}^m - h_{Ii}\left[I_{i,t}^m\right]^+ - b_i\left[-I_{i,t}^m\right]^+\right]\right] \tag{44}$$

s.t.

$$S_{i,1}^m = S_{i,0} + y_i^m Q - \textstyle\sum_{j=1}^{2} X_{ij,1}^m \qquad \forall\, m = 1, M \text{ and } \forall\, i = 1, 2 \tag{45}$$

$$S_{i,t}^m = S_{i,t-1}^m - \textstyle\sum_{j=1}^{n} X_{ij,t}^m \qquad \forall\, m = 1, M \text{ and } \forall\, i = 1, 2 \text{ and } \forall\, t = 2, T \tag{46}$$

$$I_{i,1}^m = I_{i,0} - d_{i,1}^m + \textstyle\sum_{j=1}^{2}\sum_{r=1}^{i} u_{jir,1}^m X_{jr,1}^m \quad \forall\, m = 1, M \text{ and } \forall\, i = 1, 2 \tag{47}$$

$$I_{i,t}^m = I_{i,t-1}^m - d_{i,t}^m + \textstyle\sum_{j=1}^{2}\sum_{r=1}^{i} u_{jir,t}^m X_{jr,t}^m \quad \forall\, m = 1, M, \forall\, i = 1, 2, \forall\, t = 2, T \tag{48}$$

$$\textstyle\sum_{i=1}^{2}\sum_{j=1}^{n} v_j X_{ij,t}^m \le V \qquad \forall\, m = 1, M \text{ and } \forall\, t = 1, T \tag{49}$$

Non-negativity constraints.

Constraints (45) increase the initial soft bin inventory by the realized production yields. Constraints (46) are the inventory balancing constraints for soft bin inventory. Constraints (47) and (48) are the inventory balancing constraints for finished products. The first period requires a special constraint, because the initial inventory $I_{i,0}$ is the same for all scenarios. Constraints (49) capture the test capacities.

The second stage LP has the same reasoning and is given as

$$
\begin{aligned}
\max_X \quad & \sum_{m=1}^{M} a\left[\sum_{i=1}^{2} p_i \min\left(d_{i,i} + [-I_{i,0}]^+, [-I_{i,0}]^+ + \sum_{j=1}^{2}\sum_{r=1}^{i} u_{jir,t}^m X_{jr,1}\right)\right] \\
& - \sum_{j=1}^{2}\sum_{i=1}^{2} r_j X_{ij,1} \\
& + \sum_{m=1}^{M} a\left[\sum_{t=2}^{T}\left[\sum_{i=1}^{2} p_i \min\left(d_{i,t}^m + [-I_{i,t}^m]^+, [I_{i,t-1}^m]^+ \right.\right.\right. \\
& \left.\left. + \sum_{j=1}^{2}\sum_{r=1}^{i} u_{jir,t}^m X_{jr,t}^m \right) - \sum_{j=1}^{2}\sum_{i=1}^{2} r_j X_{ij,t}^m \right] \\
& \left. - \sum_{t=1}^{T}\left[\sum_{i=1}^{2} h_{Si} S_{i,t}^m + h_{Ii}[I_{i,t}^m]^+ + b_i[-I_{i,t}^m]^+\right]\right]
\end{aligned} \tag{50}
$$

s.t. $S_{i,1}^m = S_{i,0} - \sum_{j=1}^{2} X_{ij,1}$ $\quad \forall\, m = 1, M$ and $\forall\, i = 1, 2$ (51)

$I_{i,1}^m = I_{i,0} - d_{i,1} + \sum_{j=1}^{2}\sum_{r=1}^{i} u_{jir,t}^m X_{ij,1}$ $\quad \forall\, m = 1, M$ and $\forall\, i = 1, 2$

$\sum_{i=1}^{2}\sum_{j=1}^{2} v_j X_{ij,1} \leq V$ (52)

$\sum_{i=1}^{2}\sum_{j=1}^{n} v_j X_{ij,t}^m \leq V$ $\quad \forall\, m = 1, M$ and $\forall\, t = 2, T$ (53)

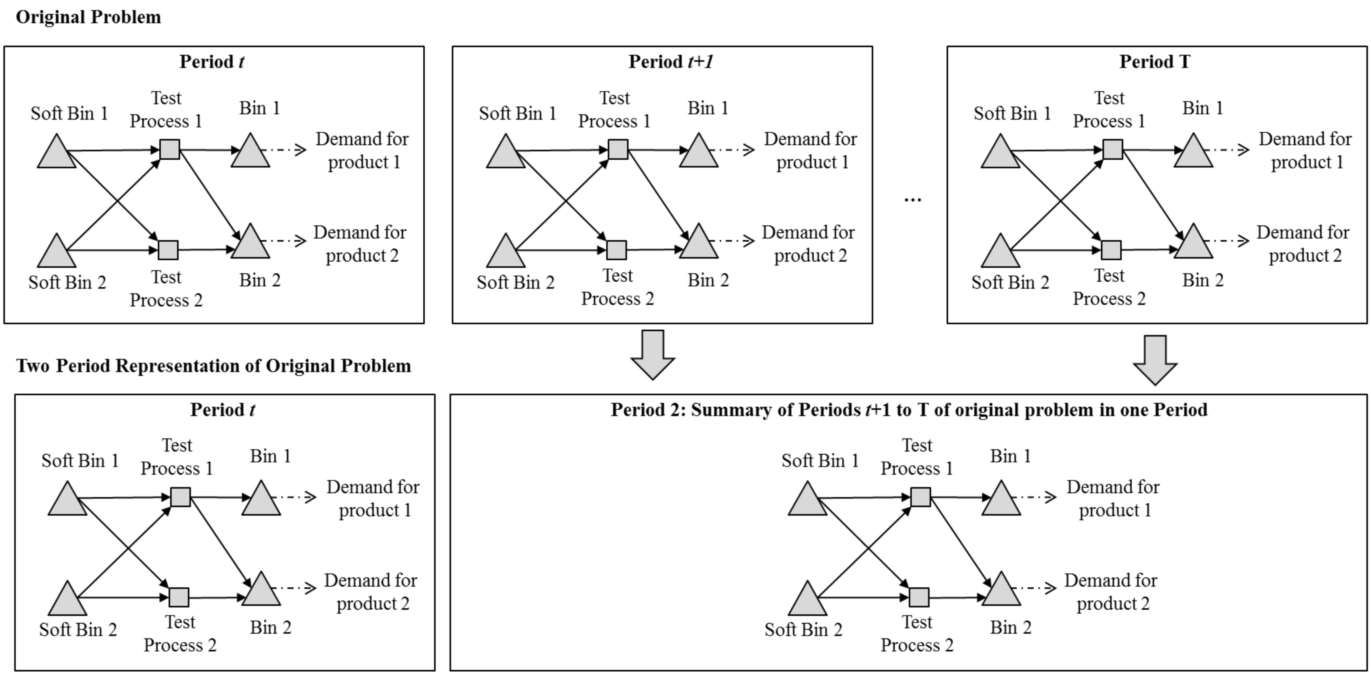
Constraints (46), (48) and non-negativity constraints.

5.6.2. Heuristic Solution Approach

Using the results from Section 5.5 the optimal test quantities for the one and two-period problem can be computed very efficiently. For problems with $T > 2$ we introduce the two-opt-heuristic. Problems with longer planning horizons are transformed into a two-period problem to apply the two-period solution from Section 5.5.2.

Assume a problem with $T > 2$ periods. For the two-period representation of this problem periods $t+1$ to T are aggregated into a single period. Period t, for which the test quantities must be calculated, remains unchanged. Figure 5-3 illustrates the concept.

Figure 5-3 Two-period representation of multi period problem



The capacity in the second period is adjusted to $V(T-t)$ and the demand distribution for each product is the convolution of the demand distributions for periods $t+1$ to T. The two-opt-heuristic is applied on a rolling horizon basis. The main advantages of the two-opt heuristic are the efficient computation and the ability to avoid myopic use of soft bin inventory as it would be the case for any one-period solution approach. The accuracy and the run time performance of the two-opt heuristic are evaluated in Section 5.7.2.

5.7. Numerical Results

In Subsection 5.7.1, we state our test cases and provide results for the ε-optimal solution approach from Section 5.5.1. In Subsection 5.7.2, we use the same test cases to analyze the accuracy of the two-opt-heuristic and elaborate on run times. In Subsection 5.7.3, we provide numerical results and a discussion on the value of preliminary yield information.

5.7.1. ε-optimal Solution Approach

For our numerical analysis we use three scenarios with low, medium, and high coefficients of variation for the second stage yield rates (ρ). The yield rate distributions are derived as explained in Section 5.5.1. The random yield fractions β_{ij} follow a beta distribution. To vary the coefficients of variation we multiply the parameters p and q of the beta distributions by 10. This results in lower variability without affecting the mean. Table 5-1 shows the parameters for the beta distributions and the corresponding mean values and coefficients of variation.

Table 5-1 Yield distributions for second stage test processes

ρ level	i	j	Parameter for beta dist. p	q	$E[\beta_{ij}]$	$\rho[\beta_{ij}]$
high	1	1	8.42	1.58	0.842	0.131
	1	2	9.5	0.5	0.95	0.069
	2	1	1.58	8.42	0.176	0.625
	2	2	8.5	1.5	0.85	0.127
medium	1	1	84.2	15.8	0.842	0.043
	1	2	95	5	0.95	0.023
	2	1	15.8	84.2	0.176	0.206
	2	2	85	15	0.85	0.042
low	1	1	842	158	0.842	0.014
	1	2	950	50	0.95	0.007
	2	1	158	842	0.176	0.066
	2	2	850	150	0.85	0.013

We vary test capacity from low over medium to unlimited. Our base capacity level is calculated as the capacity needed if yield rates were deterministic and only same level testing was allowed. We define the low capacity as 110 % of the base capacity level and medium test capacity is 120 % of the base capacity level.

We analyze planning horizons of T = 1, 2, 5, and 10 periods. This results in 36 test cases. The remaining parameters are kept constant and set as follows: The demand for each grade follows a normal distribution with mean 20 and coefficient of variation = 0.2. Unit inventory holding cost, unit backorder cost, and unit test cost are h_{Si}={0.8; 0.8}, h_{Ii}={1; 1}, h_{bi}={9; 9}, and r_i={1; 1}, respectively. Unit revenue is p_i={30; 25} and test capacity consumption is v_i={1.2; 1}. For the first stage production we set a per unit production cost of c = 5. The yield rate coefficients for the first stage production process are derived from two beta distributed variables β_1(50; 50) and β_2(90; 10), which result in $E[y_i]$ ={0.45; 0.45}.

The optimality gap of the ε-optimal solution approach depends on the number of considered scenarios M. M can be selected sufficiently large to achieve any desired accuracy. For details on how the optimality gap is obtained we refer to Bayraksan and Morton (2006). We choose the number of scenarios M sufficiently large to achieve for all test cases an optimality gap of less than 0.01 % with 95 % probability.

To evaluate the performance we simulate the entire planning cycle of T periods 1,000 times. For each simulation run we collect the cycle profit. We use all simulated cycle profits to estimate the average cycle profit for a test case. Over all test cases, the average half-width of the 95 % confidence interval for this estimate is 0.8 %. Table 5-2 reports the average profit per period (cycle profit divided by cycle length T) for the test cases.

Table 5-2 Average profit per period for ε-optimal solution

ρ level	V	T = 1	2	5	10
low	low	721	721	679	597
	medium	730	731	691	607
	unlimited	731	734	694	610
medium	low	712	716	676	595
	medium	722	727	688	604
	unlimited	723	730	692	608
high	low	671	686	657	579
	medium	683	699	669	586
	unlimited	683	710	680	599

As the yield rate variability increases from low over medium to high, profit decreases on average by 0.6 % and 4.1 % compared to low yield rate variability. The yield variability at the second stage is an indicator of how much information about the true quality of the products is revealed by preliminary yield information. Higher yield rate coefficients of variation at the second stage represent poorer preliminary yield information. The results show that profits increase in the quality of preliminary yield information.

Profits increase in available capacity and decrease in cycle length T. An increasing cycle length increases the uncertainty for the first stage production decision. In addition inventory for later periods must be kept in stock from the beginning causing higher inventory holding costs. The effects of capacity and cycle length become more relevant in Section 5.7.2 and Section 5.7.3 when we discuss the accuracy of the two-opt-heuristic and the value of preliminary yield information. In the next section we also elaborate on the run times.

5.7.2. Comparison of Heuristic with ε-optimal Solution

We apply the heuristic to the same test cases that we solved with the ε-optimal solution. We calculate the heuristic solutions by using the Lagrangian Penalty Method (Quarteroni, Sacco, and Saleri 2007, Chapter 7.2). The optimal solution for each iteration of the Lagrangian Penalty Method is calculated by the Gradient Search Method. Because there is no closed form solution for the integrals they must be evaluated numerically. To enable this, we represent all probability distributions by the corresponding five-point distributions. We discuss the loss of accuracy caused due to this numeric evaluation of the integrals together with the results.

Table 5-3 shows average run times for the ε-optimal solution approach and the two-opt-heuristic. Run times are averaged over the parameters test capacity and yield variability because these parameters have only marginal influence on run time. For the two-opt-heuristic the run time increase is linear because for $T > 1$ each additional period increases run time by the time it takes to solve another two-period problem. Run times increase exponentially for the ε-optimal solution because each additional period increases the complexity of the LP significantly.

Table 5-3 Average run time for one cycle in seconds

	T = 1	2	5	10
ε-optimal solution	1	15	152	968
two-opt-heuristic	0.03	5	20	45

All algorithms were implemented in C++ and all experiments were conducted on a PC with eight Intel 3.06 GHz processors and 8 GB of RAM. The LPs for the ε-optimal solution are solved by CPLEX solver 12.1 with default settings.

To calculate the average profit per period for the two-opt-heuristic we follow the same simulation approach as in Section 5.7.1 and use common random numbers to compare solutions. Table 5-4 reports the accuracy of the two-opt-heuristic as the percentage profit loss of the heuristic solution versus the ε-optimal solution. Longer planning horizons T result in larger optimality gaps. This is due to the nature of the two-opt heuristic to transform any multi period problem into a two-period problem. The average optimality gaps are 0.1 %, 0.8 %, 1.8 %, and 3.1 % for T = 1, 2, 5, and 10, respectively. The two-opt heuristic is the optimal solution for T = 1 and T = 2 but loses some accuracy due to the numerical evaluation of the integrals. The effect is larger for T = 2 because the number of probability distributions that must be evaluated numerically increases from 4 to 10.

The performance of the two-opt-heuristic is increasing in capacity level for low and medium yield variability. With higher capacity levels it is easier to reduce the effects of less precise planning from the beginning in later periods, e.g. by reducing possible backorders from earlier periods.

The numerical evaluation of the integrals also explains the decrease of accuracy with increasing ρ level. For high yield variability the gap is the greatest. The heuristic can be adjusted to higher variability by using more than five fulcrums for the probability distributions. However, this is a trade-off between accuracy and runtime. Even for the rather long planning horizon of 10 periods and high yield variability the gap is less than 5 %. Considering the large runtime improvement and the intuitive appeal of the heuristic the results are encouraging.

Table 5-4 Profit of two-opt-heuristic below ε-optimal solution (percent)

ρ level	V	T = 1	2	5	10
low	low	0.2	1.1	1.5	3.3
	medium	0.1	0.4	1.1	2.1
	unlimited	0.0	0.4	1.1	1.7
medium	low	0.2	1.1	1.6	3.6
	medium	0.1	0.5	1.4	2.4
	unlimited	0.0	0.6	1.4	2.4
high	low	0.3	0.7	2.0	4.7
	medium	0.3	0.5	2.0	3.0
	unlimited	0.1	1.6	3.4	5.0

5.7.3. The Value of Preliminary Yield Information

To compute the value of preliminary yield information, we compare the setting with preliminary yield information with a setting without this information. The setting without preliminary yield information can be modeled as a special case of our model by setting the number of soft bins to one and assuming deterministic replenishment in stage one. The yield rate distributions of the second stage have to be adapted so that they incorporate the unobserved risk of the first stage.

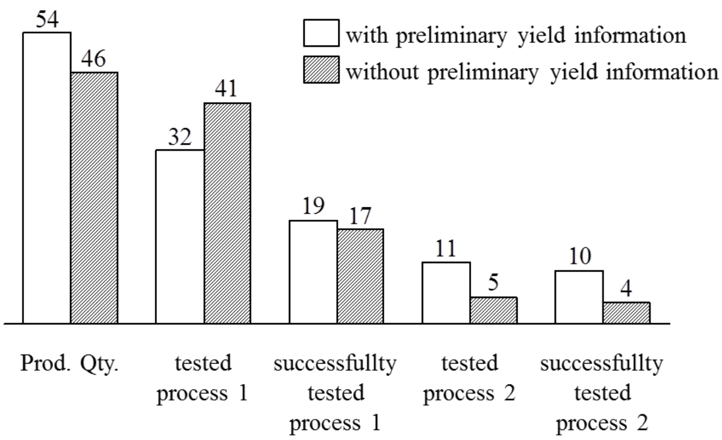
We apply the setting without preliminary yield information to the same test cases used in Section 5.7.1. We use the ε-optimal solution approach for $T \leq 10$ and the two-opt-heuristic to solve additional test cases with planning horizons of 15 and 20 periods. These larger problems cannot be solved by the ε-optimal solution approach with reasonable effort. Table 5-5 shows the relative profit difference of a system with preliminary yield information versus a system without preliminary yield information.

Table 5-5 Profit of system with preliminary yield information above system without (in percent)

ρ level	V	T=1	2	5	10	15	20
low	low	10	12	21	45	95	321
	medium	5	5	7	12	21	49
	unlimited	2	1	1	2	2	4
medium	low	10	12	21	45	94	328
	medium	5	5	7	12	20	44
	unlimited	1	1	1	2	1	3
high	low	9	12	21	45	95	365
	medium	5	5	8	13	20	40
	unlimited	1	1	1	2	1	2

The parameter yield variability (ρ level) is not relevant in this context because changes in this parameter affect both settings in the same way. Focusing on the capacity parameter V, results show that substantial profit increases can be achieved for low and medium test capacity. These results indicate that preliminary yield information is an effective mean to make more efficient use of scarce test capacities. The effect of preliminary yield information on test decisions under limited capacity becomes more visible when we analyze a single test case on a more detailed level. Figure 5-4 considers the test case with $T = 1$, low capacity and low yield variability. It shows the first stage production quantity, the average number of units tested with test processes 1 and 2 and the average number of units that complete this test successfully. The reported results are similar for all test cases with limited capacity.

Figure 5-4 Effect of preliminary yield information on measures for test case



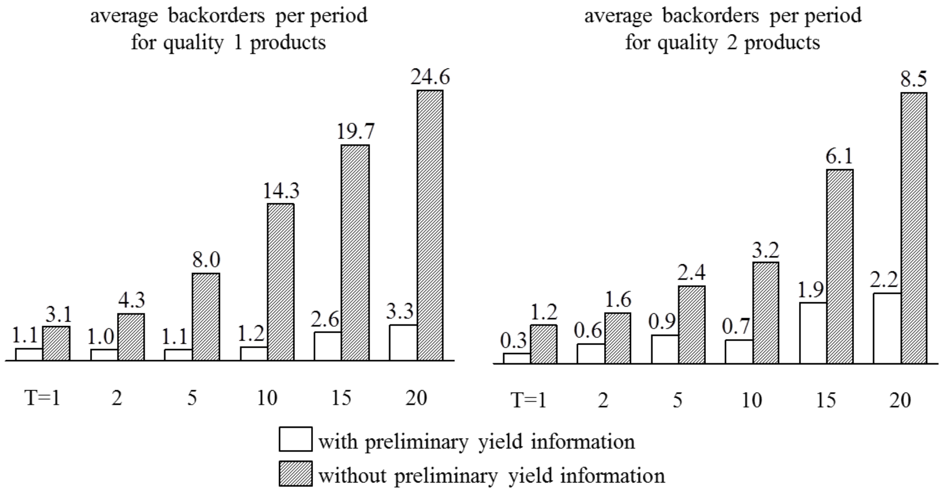
The first observation is that without preliminary yield information the production quantity is smaller. To understand the reasoning we have to look at stage two first. Assuming unlimited soft bin inventory, testing 41 units with test process one and 5 units with test process two is the optimal solution which *utilizes all of the available capacity*. Because only 46 units can be tested only 46 units need to be produced. With preliminary yield information, pre-testing yields on average 90 % usable units. This results in 49 units of expected soft bin inventories. On average 43 units are used for testing which *does not utilize all of the available capacity*. There are on average 6 units and some unused capacity left which can be seen as safety stock to cover higher than expected demands. This reactive capacity is available due to preliminary yield information and results in a profit increase.

The tighter the capacity constraint the more important it becomes to test only those units for quality level 1 that have a high probability to pass this test. The same reasoning applies if test costs are increasing in test level. If $r_1 > r_2$, it increases profit to test only units with high success probability for quality level 1. Preliminary yield information can partially identify these units. Figure 5-4 shows that with preliminary yield information fewer units are tested with the more capacity consuming test process 1 yielding at the same time more units of quality 1 than without preliminary yield information. This demonstrates that preliminary yield information is an effective mean to make more efficient use of scarce test capacity.

With unlimited test capacity and equal test costs ($r_1 = r_2$) all units can be tested for quality level 1 without harming profits. Therefore preliminary yield information yields almost no profit increase for test cases with unlimited test capacity. The small profit increase for unlimited test capacity is due to the fact that test results are more predictable under preliminary yield information.

Next, we discuss the effect of increasing the planning horizon *T*. With increasing planning horizon T and limited capacity, preliminary yield information becomes more profitable. In the setting without preliminary yield information the limited capacity cannot be used as efficiently as with preliminary yield information resulting in more and longer backorders. Therefore the average number of backorders per period increases in *T*. Figure 5-5 displays this fact for test cases with low capacity and low yield rate. With preliminary yield information the average number of backorders per period is increasing at a much lower rate. The results are similar to the other test cases with limited capacity. For test cases with unlimited test capacity this effect is not observable. The results from this section show that signification profit increases can be achieved by preliminary yield information when test capacity is limited.

Figure 5-5 Average number of backorders per period with and without preliminary yield information



5.8. Conclusion

We analyzed a two-stage production process with co-production, random yields, and substitution as well as limited capacity at the second stage. We modeled the multi period problem as a nested dynamic program. To gain structural insights on the optimal policy a single period model and a static two-period model have been studied. We proved that the profit functions of both models are convex. Based on these properties we developed a heuristic that can be applied on a rolling horizon approach. Any direct solution approach of the dynamic program was considers as not tractable even for very small problem instances. We therefore applied a solution approach that relies on stochastic linear programming and can compute solution with arbitrary small optimality gap. This approach is used to evaluate the performance of the heuristic. We conduct numerical experiments that show that the heuristic performs well over a wide parameter setting and that preliminary yield information is of significant value when test capacity is limited.

Our research provides the algorithms that are necessary to plan the proposed two stage semiconductor production system. Companies can use the results of our research to determine whether or not to implement a pre-test after initial wafer production. Companies who decide to implement a pre-test after initial wafer production can use our solution approaches to compute close-to-optimal solutions with low computation times.

An extension of the current work is to study problems with more than two usable quality levels. For these models different heuristic solutions approaches would be needed. For example, one could consider a model with only one level downward substitution at the second stage. Another approach to deal with more than two quality levels would be to aggregate multiple quality levels for planning. Analysis of such approximate structures and comparisons with optimal solutions merit a separate study.

Another interesting extension is to include direct demand substitution with finished products. This would add a third stage to the dynamic program. Decision on whether or not to satisfy current demand for low quality products with products of higher quality could be modeled explicitly. The influences of this option on the production decisions in the previous two stages and the value of preliminary yield information in such a setting would be worthwhile to study.

Appendix 5.A. Proof of Theorem 5-1

We show that the function $\tilde{\pi}_1\,(X,\beta_{22},\beta_{21})$ is jointly concave in $X_{11} \geq 0$, $X_{12} \geq 0$, $X_{21} \geq 0$ and $X_{22} \geq 0$. Because the expectation of a jointly concave function is jointly concave, $\hat{\pi}_1(X,Y)$ is a combination of jointly concave functions and the desired result follows. Substituting $P_1, P_2,\ W_1$, and W_2 back into $\tilde{\pi}_1$ and using some basic transformations we get

$$\tilde{\pi}_1\,(X,\beta_{22},\beta_{21}) =$$

$$\int_0^1 f(\beta_{12}) \int_0^{(d_1 - X_{21}\beta_{21}\beta_{22})/X_{11}\beta_{12}} g(\beta_{11})[(p_1 + h_1 + b_1)(X_{11}\beta_{11}\beta_{12} + X_{21}\beta_{21}\beta_{22} - d_1)]\, d\beta_{11} d\beta_{21}$$

$$+ \int_0^1 g(\beta_{11}) \int_0^{(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})/(X_{11}(1-\beta_{11}) + X_{12})} f(\beta_{12})$$

$$[(p_2 + h_2 + b_2)(X_{11}\beta_{12}(1-\beta_{11}) + X_{12}\beta_{12} + X_{21}\beta_{22}(1-\beta_{21}) + X_{22}\beta_{22} - d_2)]\, d\beta_{21} d\beta_{11}$$

$$+h_1(d_1 - X_{11}E[\beta_{11}]E[\beta_{12}] - X_{21}E[\beta_{21}]E[\beta_{22}]) + h_2(d_2 - X_{11}E[\beta_{12}](1 - E[\beta_{11}]) -$$

$$X_{12}E[\beta_{12}] - X_{21}E[\beta_{22}](1 - E[\beta_{21}]) - X_{22}E[\beta_{22}]) + p_1 d_1 + p_2 d_2$$

The first order derivatives are

$$\frac{\boldsymbol{d\tilde{\pi}_1}}{\boldsymbol{dX_{11}}} = \int_0^1 f(\beta_{12}) \int_0^{(d_1 - X_{21}\beta_{21}\beta_{22})/X_{11}\beta_{12}} g(\beta_{11})[\beta_{11}\beta_{12}(p_1 + h_1 + b_1)]\, d\beta_{11} d\beta_{21}$$

$$+ \int_0^1 g(\beta_{11}) \int_0^{(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})/(X_{11}(1-\beta_{11}) + X_{12})} f(\beta_{12})$$

$$[\beta_{12}(1-\beta_{11})(p_2 + h_2 + b_2)]\, d\beta_{21} d\beta_{11}$$

$$-h_1 E[\beta_{11}]E[\beta_{12}] - h_2 E[\beta_{12}](1 - E[\beta_{11}])$$

$$\frac{\boldsymbol{d\tilde{\pi}_1}}{\boldsymbol{dX_{12}}} = \int_0^1 g(\beta_{11}) \int_0^{(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})/(X_{11}(1-\beta_{11}) + X_{12})} f(\beta_{12})$$

$$[\beta_{12}(p_2 + h_2 + b_2)]\, d\beta_{21} d\beta_{11}$$

$$-h_2 E[\beta_{12}]$$

$$\frac{\boldsymbol{d\tilde{\pi}_1}}{\boldsymbol{dX_{21}}} = \int_0^1 f(\beta_{12}) \int_0^{(d_1 - X_{21}\beta_{21}\beta_{22})/X_{11}\beta_{12}} g(\beta_{11})[\beta_{21}\beta_{22}(p_1 + h_1 + b_1)]\, d\beta_{11} d\beta_{21}$$

$$+ \int_0^1 g(\beta_{11}) \int_0^{(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})/(X_{11}(1-\beta_{11}) + X_{12})} f(\beta_{12})$$

$$[\beta_{22}(1 - \beta_{21})(p_2 + h_2 + b_2)]\, d\beta_{21} d\beta_{11}$$

$$-h_1\beta_{21}\beta_{22} - h_2\beta_{22}(1 - \beta_{21})$$

$$\frac{\boldsymbol{d\tilde{\pi}_1}}{\boldsymbol{dX_{22}}} = \int_0^1 g(\beta_{11}) \int_0^{(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})/(X_{11}(1-\beta_{11}) + X_{12})} f(\beta_{12})$$

$$[\beta_{22}(p_2 + h_2 + b_2)]\, d\beta_{21} d\beta_{11}$$

$$-h_2\beta_{22}$$

The second order derivatives are

$$a = \frac{d^2\tilde{\pi}_1}{dX_{11}^2} = -(p_1 + h_1 + b_1) \int_0^1 f(\beta_{12}) g\left(\frac{(d_1 - X_{21}\beta_{21}\beta_{22})}{X_{11}\beta_{12}}\right) \left[\frac{(d_1 - X_{21}\beta_{21}\beta_{22})^2}{X_{11}^3\beta_{12}}\right] d\beta_{21}$$

$$-(p_2 + h_2 + b_2) \int_0^1 (1 - \beta_{11})^2$$

$$g(\beta_{11}) f\left(\frac{(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})}{(X_{11}(1-\beta_{11}) + X_{12})}\right) \left[\frac{(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})^2}{(X_{11}(1-\beta_{11}) + X_{12})^3}\right] d\beta_{11}$$

$$b = \frac{d^2\tilde{\pi}_1}{dX_{11}X_{12}} = \frac{d^2\tilde{\pi}_1}{dX_{12}X_{11}} = -(p_2 + h_2 + b_2) \int_0^1 (1 - \beta_{11})$$

$$g(\beta_{11}) f\left(\frac{(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})}{(X_{11}(1-\beta_{11}) + X_{12})}\right) \left[\frac{(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})^2}{(X_{11}(1-\beta_{11}) + X_{12})^3}\right] d\beta_{11}$$

$$c = \frac{d^2\tilde{\pi}_1}{dX_{12}^2} = -(p_2 + h_2 + b_2) \int_0^1 g(\beta_{11}) f\left(\frac{(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})}{(X_{11}(1-\beta_{11}) + X_{12})}\right)$$

$$\left[\frac{(d_2 - X_{21}\beta_{22}(1-\beta_{21}) - X_{22}\beta_{22})^2}{(X_{11}(1-\beta_{11}) + X_{12})^3}\right] d\beta_{11}$$

$$d = \frac{d^2\tilde{\pi}_1}{dX_{11}X_{21}} = \frac{d^2\tilde{\pi}_1}{dX_{21}X_{11}} = -\beta_{21}\beta_{22}(p_1 + h_1 + b_1)$$

$$\int_0^1 f(\beta_{12})g\left(\frac{(d_1-X_{21}\beta_{21}\beta_{22})}{X_{11}\beta_{12}}\right)\left[\frac{d_1-X_{21}\beta_{21}\beta_{22}}{X_{11}^2\beta_{12}}\right]d\beta_{21}$$

$$-\beta_{22}(1-\beta_{21})(p_2+h_2+b_2)\int_0^1(1-\beta_{11})$$

$$g(\beta_{11})f\left(\frac{(d_2-X_{21}\beta_{22}(1-\beta_{21})-X_{22}\beta_{22})}{(X_{11}(1-\beta_{11})+X_{12})}\right)\left[\frac{d_2-X_{21}\beta_{22}(1-\beta_{21})-X_{22}\beta_{22}}{(X_{11}(1-\beta_{11})+X_{12})^2}\right]d\beta_{11}$$

$$e=\frac{d^2\tilde{\pi}_1}{dX_{11}X_{22}}=\frac{d^2\tilde{\pi}_1}{dX_{22}X_{11}}=-\beta_{22}(p_2+h_2+b_2)\int_0^1(1-\beta_{11})$$

$$g(\beta_{11})f\left(\frac{(d_2-X_{21}\beta_{22}(1-\beta_{21})-X_{22}\beta_{22})}{(X_{11}(1-\beta_{11})+X_{12})}\right)\left[\frac{d_2-X_{21}\beta_{22}(1-\beta_{21})-X_{22}\beta_{22}}{(X_{11}(1-\beta_{11})+X_{12})^2}\right]d\beta_{11}$$

$$f=\frac{d^2\tilde{\pi}_1}{dX_{12}X_{21}}=\frac{d^2\tilde{\pi}_1}{dX_{21}X_{12}}=-\beta_{22}(1-\beta_{21})(p_2+h_2+b_2)$$

$$\int_0^1 g(\beta_{11})f\left(\frac{d_2-X_{21}\beta_{22}(1-\beta_{21})-X_{22}\beta_{22}}{X_{11}(1-\beta_{11})+X_{12}}\right)\left[\frac{d_2-X_{21}\beta_{22}(1-\beta_{21})-X_{22}\beta_{22}}{(X_{11}(1-\beta_{11})+X_{12})^2}\right]d\beta_{11}$$

$$g=\frac{d^2\tilde{\pi}_1}{dX_{12}X_{22}}=\frac{d^2\tilde{\pi}_1}{dX_{22}X_{12}}=-\beta_{22}(p_2+h_2+b_2)$$

$$\int_0^1 g(\beta_{11})f\left(\frac{(d_2-X_{21}\beta_{22}(1-\beta_{21})-X_{22}\beta_{22})}{(X_{11}(1-\beta_{11})+X_{12})}\right)\left[\frac{d_2-X_{21}\beta_{22}(1-\beta_{21})-X_{22}\beta_{22}}{(X_{11}(1-\beta_{11})+X_{12})^2}\right]d\beta_{11}$$

$$h=\frac{d^2\tilde{\pi}_1}{dX_{21}X_{22}}=\frac{d^2\tilde{\pi}_1}{dX_{22}X_{21}}=-\beta_{22}^2(1-\beta_{21})(p_2+h_2+b_2)$$

$$\int_0^1 g(\beta_{11})f\left(\frac{(d_2-X_{21}\beta_{22}(1-\beta_{21})-X_{22}\beta_{22})}{(X_{11}(1-\beta_{11})+X_{12})}\right)\left[\frac{1}{(X_{11}(1-\beta_{11})+X_{12})}\right]d\beta_{11}$$

$$k=\frac{d^2\tilde{\pi}_1}{dX_{21}^2}=-\beta_{21}^2\beta_{22}^2(p_1+h_1+b_1)\int_0^1 f(\beta_{12})g\left(\frac{(d_1-X_{21}\beta_{21}\beta_{22})}{X_{11}\beta_{12}}\right)\left[\frac{1}{X_{11}\beta_{12}}\right]d\beta_{21}$$

$$-\beta_{22}^2(1-\beta_{21})^2(p_2+h_2+b_2)$$

$$\int_0^1 g(\beta_{11})f\left(\frac{d_2-X_{21}\beta_{22}(1-\beta_{21})-X_{22}\beta_{22}}{X_{11}(1-\beta_{11})+X_{12}}\right)\left[\frac{1}{(X_{11}(1-\beta_{11})+X_{12})}\right]d\beta_{11}$$

$$l=\frac{d^2\tilde{\pi}_1}{dX_{22}^2}=$$

$$-\beta_{22}^2(p_2+h_2+b_2)\int_0^1 g(\beta_{11})f\left(\frac{d_2-X_{21}\beta_{22}(1-\beta_{21})-X_{22}\beta_{22}}{X_{11}(1-\beta_{11})+X_{12}}\right)\left[\frac{1}{X_{11}(1-\beta_{11})+X_{12}}\right]d\beta_{11}$$

The Hessian matrix has the following structure.

$$H=\begin{pmatrix} a & b & d & e \\ b & c & f & g \\ d & f & k & h \\ e & g & h & l \end{pmatrix}$$

We show that the Hessian matrix is negative semidefinite by showing that $x^T Hx \le 0$ for all $x \in \mathbb{R}^4$. [*Comment:* $x^T = (x_1, x_2, x_3, x_4)$]

We get the following result $x^T Hx = ax_1^2 + cx_2^2 + kx_3^2 + lx_4^2 + 2bx_1x_2 + 2dx_1x_3 + 2ex_1x_4 + 2fx_2x_3 + 2gx_2x_4 + 2hx_3x_4$. Because all second order derivatives are negative (≤ 0), there exists always an ε_1 that is $\ge a, b$, and c, an $\varepsilon_2 \ge k, l$, and h and an $\varepsilon_3 \ge d, e, f$ and g but $\varepsilon_1, \varepsilon_2, \varepsilon_3 \le 0$. We can establish that $x^T Hx \le \varepsilon_1(x_1+x_2)^2 + \varepsilon_2(x_3+x_4)^2 + 2\varepsilon_3\big((x_1+x_2)(x_3+x_4)\big)$. Following the same reasoning there will always be an ε_4 with $0 \ge \varepsilon_4 \ge \varepsilon_1, \varepsilon_2, \varepsilon_3$ so that $x^T Hx \le \varepsilon_4(x_1+x_2+x_3+x_4)^2 \le 0$, which is true for all $x \in \mathbb{R}^4$. ■

Appendix 5.B. Profit function for two-period model

Let P_{it} be the test yield of quality level i in period t given as $P_{11} = X_{11}\beta_{11,1}\beta_{12,1} + X_{21}\beta_{21,1}\beta_{22,1}$, $P_{21} = X_{11}\beta_{12,1}(1-\beta_{11,1}) + X_{12}\beta_{12,1} + X_{21}\beta_{22,1}(1-\beta_{21,1}) + X_{22}\beta_{22,1}$ $P_{12} = Y_{11}\beta_{11,2}\beta_{12,2} + Y_{21}\beta_{21,2}\beta_{22,2}$, and $P_{22} = Y_{11}\beta_{12,2}(1-\beta_{11,2}) + Y_{12}\beta_{12,2} + Y_{21}\beta_{22,2}(1-\beta_{21,2}) + Y_{22}\beta_{22,2}$.

To model the profit function $\pi_2(X,Y)$ we have to distinguish 16 cases. They are given as:

For $P_{11} \ge d_1$ and $P_{21} \ge d_2$

$$\pi_2(X,Y) = -r_1(X_{11}+X_{21}+Y_{11}+Y_{21}) - r_2(X_{12}+X_{22}+Y_{12}+Y_{22})$$
$$+h_{S1}(2X_{11}+2X_{12}+Y_{11}+Y_{12}) + h_{S2}(2X_{21}+2X_{22}+Y_{21}+Y_{22}) + p_1d_1 + p_2d_2 - h_1(P_{11}-d_1)$$
$$-h_2(P_{21}-d_2) \tag{54}$$
$$+\begin{cases} p_1D_1 + p_2D_2 - h_1(P_{12}+P_{11}-d_1-D_1) - h_2(P_{22}+P_{21}-d_2-D_2) \\ p_1D_1 + p_2(P_{22}+P_{21}-d_2) - h_1(P_{12}+P_{11}-d_1-D_1) - b_2\big(D_2-(P_{22}+P_{21}-d_2)\big) \\ p_1(P_{12}+P_{21}-d_2) + p_2D_2 - b_1\big(D_1-(P_{12}+P_{21}-d_2)\big) - h_2(P_{22}+P_{21}-d_2-D_2) \\ p_1(P_{12}+P_{21}-d_2) + p_2(P_{22}+P_{21}-d_2) - b_1\big(D_1-(P_{12}+P_{21}-d_2)\big) - b_2\big(D_2-(P_{22}+P_{21}-d_2)\big) \end{cases} \tag{55}$$

$$\text{if} \quad \begin{array}{c} \left(d_1 + D_1 - Y_{21}\beta_{21,2}\beta_{22,2} - P_{11}\right)/Y_{11}\beta_{12,2} \leq \beta_{11,2} \text{ and} \\ \left(d_2 + D_2 - Y_{21}\beta_{22,2}\left(1 - \beta_{21,2}\right) - Y_{22}\beta_{22,2} - P_{21}\right)/\left(Y_{11}\left(1 - \beta_{11,2}\right) + Y_{12}\right) \leq \beta_{12,2} \end{array} \tag{56}$$

$$\text{if} \quad \begin{array}{c} \left(d_1 + D_1 - Y_{21}\beta_{21,2}\beta_{22,2} - P_{11}\right)/Y_{11}\beta_{12,2} \leq \beta_{11,2} \text{ and} \\ \left(d_2 + D_2 - Y_{21}\beta_{22,2}\left(1 - \beta_{21,2}\right) - Y_{22}\beta_{22,2} - P_{21}\right)/\left(Y_{11}\left(1 - \beta_{11,2}\right) + Y_{12}\right) > \beta_{12,2} \end{array} \tag{57}$$

$$\text{if} \quad \begin{array}{c} \left(d_1 + D_1 - Y_{21}\beta_{21,2}\beta_{22,2} - P_{11}\right)/Y_{11}\beta_{12,2} > \beta_{11,2} \text{ and} \\ \left(d_2 + D_2 - Y_{21}\beta_{22,2}\left(1 - \beta_{21,2}\right) - Y_{22}\beta_{22,2} - P_{21}\right)/\left(Y_{11}\left(1 - \beta_{11,2}\right) + Y_{12}\right) \leq \beta_{12,2} \end{array} \tag{58}$$

$$\text{if} \quad \begin{array}{c} \left(d_1 + D_1 - Y_{21}\beta_{21,2}\beta_{22,2} - P_{11}\right)/Y_{11}\beta_{12,2} > \beta_{11,2} \text{ and} \\ \left(d_2 + D_2 - Y_{21}\beta_{22,2}\left(1 - \beta_{21,2}\right) - Y_{22}\beta_{22,2} - P_{21}\right)/\left(Y_{11}\left(1 - \beta_{11,2}\right) + Y_{12}\right) > \beta_{12,2} \end{array}$$

For $P_{11} \geq d_1$ and $P_{21} < d_2$

$$\pi_2(X,Y) = -r_1(X_{11} + X_{21} + Y_{11} + Y_{21}) - r_2(X_{12} + X_{22} + Y_{12} + Y_{22}) + h_{S1}(2X_{11} + 2X_{12} + Y_{11} + Y_{12})$$

$$+h_{S2}(2X_{21} + 2X_{22} + Y_{21} + Y_{22}) + p_1 d_1 + p_2 P_{21} - h_1(P_{11} - d_1) - b_2(d_2 - P_{21})$$

$$+\begin{cases} p_1 D_1 + p_2(D_2 + d_2 - P_{21}) - h_1(P_{12} + P_{11} - d_1 - D_1) - h_2\left(P_{22} - (D_2 + d_2 - P_{21})\right) \text{ if } (\mathbf{55}) \\ p_1 D_1 + p_2 P_{22} - h_1(P_{12} + P_{11} - d_1 - D_1) - b_2(D_2 + d_2 - P_{21} - P_{22}) \text{ if } (\mathbf{56}) \\ p_1(P_{12} + P_{11} - d_1) + p_2(D_2 + d_2 - P_{21}) - b_1\left(D_1 - (P_{12} + P_{11} - d_1)\right) - h_2\left(P_{22} - (D_2 + d_2 - P_{21})\right) \text{ if } (\mathbf{57}) \\ p_1(P_{12} + P_{11} - d_1) + p_2 P_{22} - b_1\left(D_1 - (P_{12} + P_{11} - d_1)\right) - b_2(D_2 + d_2 - P_{21} - P_{22}) \text{ if } (\mathbf{58}) \end{cases} \tag{59}$$

For $P_{11} < d_1$ and $P_{21} \geq d_2$

$$\pi_2(X,Y) = -r_1(X_{11} + X_{21} + Y_{11} + Y_{21}) - r_2(X_{12} + X_{22} + Y_{12} + Y_{22}) + h_{S1}(2X_{11} + 2X_{12} + Y_{11} + Y_{12})$$

$$+h_{S2}(2X_{21} + 2X_{22} + Y_{21} + Y_{22}) + p_1 P_{11} + p_2 d_2 - b_1(d_1 - P_{11}) - h_2(P_{21} - d_2)$$

$$+\begin{cases} p_1(D_1 + d_1 - P_{11}) + p_2 D_2 - h_1\left(P_{12} - (D_1 + d_1 - P_{11})\right) - h_2(P_{22} + P_{21} - d_2 - D_2) \text{ if } (\mathbf{55}) \\ p_1(D_1 + d_1 - P_{11}) + p_2(P_{22} + P_{21} - d_2) - h_1\left(P_{12} - (D_1 + d_1 - P_{11})\right) - b_2\left(D_2 - (P_{22} + P_{21} - d_2)\right) \text{ if } (\mathbf{56}) \\ p_1 P_{12} + p_2 D_2 - b_1(D_1 + d_1 - P_{11} - P_{12}) - h_2(P_{22} + P_{21} - d_2 - D_2) \text{ if } (\mathbf{57}) \\ p_1 P_{12} + p_2(P_{22} + P_{21} - d_2) - b_1(D_1 + d_1 - P_{11} - P_{12}) - b_2\left(D_2 - (P_{22} + P_{21} - d_2)\right) \text{ if } (\mathbf{58}) \end{cases} \tag{60}$$

For $P_{11} < d_1$ and $P_{21} < d_2$

$$\pi_2(X,Y) = -r_1(X_{11} + X_{21} + Y_{11} + Y_{21}) - r_2(X_{12} + X_{22} + Y_{12} + Y_{22}) + h_{S1}(2X_{11} + 2X_{12} + Y_{11} + Y_{12})$$

$$+h_{S2}(2X_{21} + 2X_{22} + Y_{21} + Y_{22}) + p_1 P_{11} + p_2 P_{21} - b_1(d_1 - P_{11}) - b_2(d_2 - P_{21})$$

$$+\begin{cases} p_1(D_1 + d_1 - P_{11}) + p_2(D_2 + d_2 - P_{21}) - h_1\left(P_{12} - (D_1 + d_1 - P_{11})\right) - h_2\left(P_{22} - (D_2 + d_2 - P_{21})\right) \text{ if } (\mathbf{55}) \\ p_1(D_1 + d_1 - P_{11}) + p_2 P_{22} - h_1\left(P_{12} - (D_1 + d_1 - P_{11})\right) - b_2(D_2 + d_2 - P_{21} - P_{22}) \text{ if } (\mathbf{56}) \\ p_1 P_{12} + p_2(D_2 + d_2 - P_{21}) - b_1(D_1 + d_1 - P_{11} - P_{12}) - h_2\left(P_{22} - (D_2 + d_2 - P_{21})\right) \text{ if } (\mathbf{57}) \\ p_1 P_{12} + p_2 P_{22} - b_1(D_1 + d_1 - P_{11} - P_{12}) - b_2(D_2 + d_2 - P_{21} - P_{22}) \text{ if } (\mathbf{58}) \end{cases} \tag{61}$$

s. t.:

$$v_1(X_{11} + X_{21}) + v_2(X_{12} + X_{22}) \leq V \tag{62}$$

$$v_1(Y_{11} + Y_{21}) + v_2(Y_{12} + Y_{22}) \leq V \tag{63}$$

$$X_{11} + X_{12} + Y_{11} + Y_{12} \leq S_1 \tag{64}$$

$$X_{21} + X_{22} + Y_{21} + Y_{22} \leq S_2 \tag{65}$$

Non-negativity constraints

Appendix 5.C. First order derivatives for two-period model

We provide expressions for the first order derivatives of $\hat{\pi}_2(X, Y)$ as they are used to compute the optimal solution for the two period model. The calculations can be found in the online appendix.

We set $W_1 = \frac{d_1 - X_{21}\beta_{21,1}\beta_{22,1}}{X_{11}\beta_{12,1}}$, $W_2 = \frac{d_2 - X_{21}\beta_{22,1}(1-\beta_{21,1}) - X_{22}\beta_{22,1}}{X_{11}(1-\beta_{11,1}) + X_{12,1}}$, $W_3 = \frac{d_1 + D_1 - Y_{21}\beta_{21,2}\beta_{22,2} - X_{11}\beta_{11,1}\beta_{12,1} - X_{21}\beta_{21,1}\beta_{22,1}}{Y_{11}\beta_{12,2}}$ and

$W_4 = \frac{d_2 + D_2 - Y_{21}\beta_{22,2}(1-\beta_{21,2}) - Y_{22}\beta_{22,2} - X_{11}\beta_{12,1}(1-\beta_{11,1}) - X_{12}\beta_{12,1} - X_{21}\beta_{22,1}(1-\beta_{21,1}) - X_{22}\beta_{22,1}}{Y_{11}(1-\beta_{11,2}) + Y_{12}}$.

$$\frac{\boldsymbol{d\hat{\pi}_2}}{\boldsymbol{dX_{11}}} = -r_1 + 2h_{S1} + \int_{D_1=0}^{\infty} m(D_1) \int_{D_2=0}^{\infty} n(D_2) \int_{\beta_{21,2}=0}^{1} h(\beta_{22}) \int_{\beta_{22,1}=0}^{1} k(\beta_{21}) \int_{\beta_{21,2}=0}^{1} h(\beta_{21}) \int_{\beta_{22,2}=0}^{1} k(\beta_{22}) \left[\int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \int_{\beta_{11,1}=0}^{W_1} g(\beta_{11}) [\beta_{11,1}\beta_{12,1}(h_1 + b_1)] \, d\beta_{11} d\beta_{21} + \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{W_2} f(\beta_{12}) \left[\beta_{12,1}(1 - \beta_{11,1})(h_2 + b_2)\right] d\beta_{21} d\beta_{11} + \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \left[\beta_{11,1}\beta_{12,1}(p_1 + h_1 + b_1) \int_{\beta_{12,2}=0}^{1} f(\beta_{12}) G_{\beta_{11.2}}(W_3) \, d\beta_{12} + \beta_{12,1}(1 - \beta_{11,1})(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \, F_{\beta_{12.2}}(W_4) d\beta_{11}\right] d\beta_{12} d\beta_{11}\right] d\beta_{22} d\beta_{21} d\beta_{22} d\beta_{21} d_{D_1} d_{D_2} - 2E[\beta_{12,1}] \left(h_1 E[\beta_{11,1}] + h_2(1 - E[\beta_{11,1}])\right)$$

$$\frac{\boldsymbol{d\hat{\pi}_2}}{\boldsymbol{dX_{12}}} = -r_2 + 2h_{S1} + \int_{D_1=0}^{\infty} m(D_1) \int_{D_2=0}^{\infty} n(D_2) \int_{\beta_{21,2}=0}^{1} h(\beta_{22}) \int_{\beta_{22,1}=0}^{1} k(\beta_{21}) \int_{\beta_{21,2}=0}^{1} h(\beta_{21}) \int_{\beta_{22,2}=0}^{1} k(\beta_{22}) \left[\int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{(d_2 - X_{21}\beta_{22,1}(1-\beta_{21,1}) - X_{22}\beta_{22,1})/(X_{11}(1-\beta_{11,1}) + X_{12,1})} f(\beta_{12}) \left[\beta_{12,1}(h_2 + b_2)\right] d\beta_{21} d\beta_{11} + \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \left[\beta_{12,1}(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \, F_{\beta_{12.2}}(W_4) d\beta_{11}\right] d\beta_{12} d\beta_{11}\right] d\beta_{22} d\beta_{21} d\beta_{22} d\beta_{21} d_{D_1} d_{D_2} - 2h_2 E[\beta_{12,1}]$$

$$\frac{\boldsymbol{d\hat{\pi}_2}}{\boldsymbol{dX_{21}}} = -r_1 + 2h_{S2} + \int_{D_1=0}^{\infty} m(D_1) \int_{D_2=0}^{\infty} n(D_2) \int_{\beta_{21,2}=0}^{1} h(\beta_{22}) \int_{\beta_{22,1}=0}^{1} k(\beta_{21}) \int_{\beta_{21,2}=0}^{1} h(\beta_{21}) \int_{\beta_{22,2}=0}^{1} k(\beta_{22}) \left[\beta_{21,1}\beta_{22,1}(h_1 + b_1) \int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \, G_{\beta_{11.1}}(W_1) d\beta_{21} + \beta_{22,1}(1 - \beta_{21,1})(h_2 + b_2) \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) F_{\beta_{12.1}}(W_2) \, d\beta_{11} + \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \left[\beta_{21,1}\beta_{22,1}(p_1 + h_1 + b_1) \int_{\beta_{12,2}=0}^{1} f(\beta_{12}) \, G_{\beta_{11.2}}(W_3) d\beta_{12} + \beta_{22,1}(1 - \beta_{21,1})(p_2 + h_2 + \right.\right.$$

$b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, F_{\beta_{12.2}}(W_4)d\beta_{11}\Big]\, d\beta_{12}d\beta_{11}\Big]\, d\beta_{22}d\beta_{21}d\beta_{22}d\beta_{21}d_{D_1}d_{D_2} - 2E[\beta_{22,1}]\left(h_1E[\beta_{21,1}] + h_2(1 - E[\beta_{21,1}])\right)$

$\frac{\boldsymbol{d\hat{\pi}_2}}{\boldsymbol{dX_{22}}} = -r_2 + 2h_{S2} + \int_{D_1=0}^{\infty} m(D_1) \int_{D_2=0}^{\infty} n(D_2) \int_{\beta_{21,2}=0}^{1} h(\beta_{22}) \int_{\beta_{22,1}=0}^{1} k(\beta_{21}) \int_{\beta_{21,2}=0}^{1} h(\beta_{21}) \int_{\beta_{22,2}=0}^{1} k(\beta_{22})$ $\Big[\beta_{22,1}(h_2 + b_2)\int_{\beta_{11,1}=0}^{1} g(\beta_{11})F_{\beta_{12.1}}(W_2)\, d\beta_{11} + \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \Big[\beta_{22,1}(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, F_{\beta_{12.2}}(W_4)d\beta_{11}\Big]\, d\beta_{12}d\beta_{11}\Big]\, d\beta_{22}d\beta_{21}d\beta_{22}d\beta_{21}d_{D_1}d_{D_2} - 2h_2E[\beta_{22,1}]$

$\frac{d\hat{\pi}_2}{dY_{11}} = -r_1 + h_{S1} + \int_{D_1=0}^{\infty} m(D_1) \int_{D_2=0}^{\infty} n(D_2) \int_{\beta_{21,2}=0}^{1} h(\beta_{22}) \int_{\beta_{22,1}=0}^{1} k(\beta_{21}) \int_{\beta_{21,2}=0}^{1} h(\beta_{21}) \int_{\beta_{22,2}=0}^{1} k(\beta_{22})$ $\Big[\int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \Big[\int_{\beta_{12,2}=0}^{1} f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11})\, [\beta_{11,2}\beta_{12,2}(p_1 + h_1 + b_1)]d\beta_{11}d\beta_{12} + \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12})\, [\beta_{12,2}(1 - \beta_{11,2})(p_2 + h_2 + b_2)]d\beta_{12}d\beta_{11}\Big]\, d\beta_{12}d\beta_{11}\Big]\, d\beta_{22}d\beta_{21}d\beta_{22}d\beta_{21}d_{D_1}d_{D_2} - E[\beta_{12,2}]\left(h_1E[\beta_{11,2}] + h_2(1 - E[\beta_{11,2}])\right)$

$\frac{\boldsymbol{d\hat{\pi}_2}}{\boldsymbol{dY_{12}}} = -r_2 + h_{S1} + \int_{D_1=0}^{\infty} m(D_1) \int_{D_2=0}^{\infty} n(D_2) \int_{\beta_{21,2}=0}^{1} h(\beta_{22}) \int_{\beta_{22,1}=0}^{1} k(\beta_{21}) \int_{\beta_{21,2}=0}^{1} h(\beta_{21}) \int_{\beta_{22,2}=0}^{1} k(\beta_{22})$ $\Big[\int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \Big[\int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12})\, [\beta_{12,2}(p_2 + h_2 + b_2)]d\beta_{12}d\beta_{11}\Big]\, d\beta_{12}d\beta_{11}\Big]\, d\beta_{22}d\beta_{21}d\beta_{22}d\beta_{21}d_{D_1}d_{D_2} - h_2E[\beta_{12,2}]$

$\frac{\boldsymbol{d\hat{\pi}_2}}{\boldsymbol{dY_{21}}} = -r_1 + h_{S2} + \int_{D_1=0}^{\infty} m(D_1) \int_{D_2=0}^{\infty} n(D_2) \int_{\beta_{21,2}=0}^{1} h(\beta_{22}) \int_{\beta_{22,1}=0}^{1} k(\beta_{21}) \int_{\beta_{21,2}=0}^{1} h(\beta_{21}) \int_{\beta_{22,2}=0}^{1} k(\beta_{22})$ $\Big[\int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \Big[\beta_{21,2}\beta_{22,2}(p_1 + h_1 + b_1)\int_{\beta_{12,2}=0}^{1} f(\beta_{12})\, G_{\beta_{11.2}}(W_3)d\beta_{12} + \beta_{22,1}(1 - \beta_{21,1})(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, F_{\beta_{12.2}}(W_4)d\beta_{11}\Big]\, d\beta_{12}d\beta_{11}\Big]\, d\beta_{22}d\beta_{21}d\beta_{22}d\beta_{21}d_{D_1}d_{D_2} - E[\beta_{22,2}]\left(h_1E[\beta_{21,2}] + h_2(1 - E[\beta_{21,2}])\right)$

$\frac{\boldsymbol{d\hat{\pi}_2}}{\boldsymbol{dY_{22}}} = -r_2 + h_{S2} + \int_{D_1=0}^{\infty} m(D_1) \int_{D_2=0}^{\infty} n(D_2) \int_{\beta_{21,2}=0}^{1} h(\beta_{22}) \int_{\beta_{22,1}=0}^{1} k(\beta_{21}) \int_{\beta_{21,2}=0}^{1} h(\beta_{21}) \int_{\beta_{22,2}=0}^{1} k(\beta_{22})$ $\Big[\int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{1} f(\beta_{12})\beta_{22,2}(p_2 + h_2 + b_2)\, \Big[\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, F_{\beta_{12.2}}(W_4)d\beta_{12}d\beta_{11}\Big]\, d\beta_{12}d\beta_{11}\Big]\, d\beta_{22}d\beta_{21}d\beta_{22}d\beta_{21}d_{D_1}d_{D_2} - h_2E[\beta_{22,2}]$

Appendix 5.D. Proof of Theorem 5-2

We show that the function $\tilde{\pi}_2(X, Y, D_1, D_2, \beta_{21,1}, \beta_{22,1}, \beta_{21,2}, \beta_{22,2})$ is jointly concave in $X \geq 0$ and $Y \geq 0$. Because the expectation of a jointly concave function is jointly concave, $\hat{\pi}_2(X, Y)$ is a combination of jointly concave functions and the desired result follows.

We set

$W_1 = (d_1 - X_{21}\beta_{21,1}\beta_{22,1})/X_{11}\beta_{12,1}$, $W_2 = (d_2 - X_{21}\beta_{22,1}(1 - \beta_{21,1}) - X_{22}\beta_{22,1})/(X_{11}(1 - \beta_{11,1}) + X_{12})$

$W_3 = (d_1 + D_1 - Y_{21}\beta_{21,2}\beta_{22,2} - P_{11})/Y_{11}\beta_{12,2}$,

$W_4 = (d_2 + D_2 - Y_{21}\beta_{22,2}(1 - \beta_{21,2}) - Y_{22}\beta_{22,2} - P_{21})/(Y_{11}(1 - \beta_{11,2}) + Y_{12})$,

$P_{11} = X_{11}\beta_{11,1}\beta_{12,1} + X_{21}\beta_{21,1}\beta_{22,1}$, $P_{21} = X_{11}\beta_{12,1}(1 - \beta_{11,1}) + X_{12}\beta_{12,1} + X_{21}\beta_{22,1}(1 - \beta_{21,1}) + X_{22}\beta_{22,1}$

$P_{12} = Y_{11}\beta_{11,2}\beta_{12,2} + Y_{21}\beta_{21,2}\beta_{22,2}$, and $P_{22} = Y_{11}\beta_{12,2}(1 - \beta_{11,2}) + Y_{12}\beta_{12,2} + Y_{21}\beta_{22,2}(1 - \beta_{21,2}) + Y_{22}\beta_{22,2}$.

We get

$$
\begin{aligned}
&\tilde{\pi}_2(X, Y, D_1, D_2, \beta_{21,1}, \beta_{22,1}, \beta_{21,2}, \beta_{22,2}) = \\
&\int_{\beta_{12,1}=0}^{W_2} f(\beta_{12}) \int_{\beta_{11,1}=0}^{W_1} g(\beta_{11})[p_1P_{11} + p_2P_{21} - b_1(d_1 - P_{11}) - b_2(d_2 - P_{21})] \\
&+ \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11})[p_1P_{12} + p_2P_{22} - b_1(D_1 + d_1 - P_{11} - P_{12}) - b_2(D_2 + d_2 - P_{21} - P_{22})]\, d\beta_{11} \\
&+ \int_{W_3}^{\beta_{11,2}=1} g(\beta_{11})\big[p_1(D_1 + d_1 - P_{11}) + p_2P_{22} - h_1\big(P_{12} - (D_1 + d_1 - P_{11})\big) - b_2(D_2 + d_2 - P_{21} - P_{22})\big]\, d\beta_{11} d\beta_{12} \\
&+ \int_{W_4}^{\beta_{12,2}=1} f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11}) \begin{bmatrix} p_1P_{12} + p_2(D_2 + d_2 - P_{21}) - b_1(D_1 + d_1 - P_{11} - P_{12}) \\ -h_2\big(P_{22} - (D_2 + d_2 - P_{21})\big) \end{bmatrix} d\beta_{11} \\
&+ \int_{W_3}^{\beta_{11,2}=1} g(\beta_{11}) \begin{bmatrix} p_1(D_1 + d_1 - P_{11}) + p_2(D_2 + d_2 - P_{21}) \\ -h_1\big(P_{12} - (D_1 + d_1 - P_{11})\big) - h_2\big(P_{22} - (D_2 + d_2 - P_{21})\big) \end{bmatrix} d\beta_{11} d\beta_{12} d\beta_{11} d\beta_{12} \\
&+ \int_{\beta_{12,1}=0}^{W_2} f(\beta_{12}) \int_{W_1}^{\beta_{11,1}=1} g(\beta_{11})[p_1d_1 + p_2P_{21} - h_1(P_{11} - d_1) - b_2(d_2 - P_{21})] \\
&+ \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11}) \begin{bmatrix} p_1(P_{12} + P_{11} - d_1) + p_2P_{22} - b_1\big(D_1 - (P_{12} + P_{11} - d_1)\big) \\ -b_2(D_2 + d_2 - P_{21} - P_{22}) \end{bmatrix} d\beta_{11} \\
&+ \int_{W_3}^{\beta_{11,2}=1} g(\beta_{11})[p_1D_1 + p_2P_{22} - h_1(P_{12} + P_{11} - d_1 - D_1) - b_2(D_2 + d_2 - P_{21} - P_{22})]\, d\beta_{11} d\beta_{12} \\
&+ \int_{W_4}^{\beta_{12,2}=1} f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11}) \begin{bmatrix} p_1(P_{12} + P_{11} - d_1) + p_2(D_2 + d_2 - P_{21}) - b_1\big(D_1 - (P_{12} + P_{11} - d_1)\big) \\ -h_2\big(P_{22} - (D_2 + d_2 - P_{21})\big) \end{bmatrix} d\beta_{11} \\
&+ \int_{W_3}^{\beta_{11,2}=1} g(\beta_{11}) \begin{bmatrix} p_1D_1 + p_2(D_2 + d_2 - P_{21}) - h_1(P_{12} + P_{11} - d_1 - D_1) \\ -h_2\big(P_{22} - (D_2 + d_2 - P_{21})\big) \end{bmatrix} d\beta_{11} d\beta_{12} d\beta_{11} d\beta_{12} \\
&+ \int_{W_2}^{\beta_{12,1}=1} f(\beta_{12}) \int_{\beta_{11,1}=1}^{W_1} g(\beta_{11})[p_1P_{11} + p_2d_2 - b_1(d_1 - P_{11}) - h_2(P_{21} - d_2)] \\
&+ \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11}) \begin{bmatrix} p_1P_{12} + p_2(P_{22} + P_{21} - d_2) - b_1(D_1 + d_1 - P_{11} - P_{12}) \\ -b_2\big(D_2 - (P_{22} + P_{21} - d_2)\big) \end{bmatrix} d\beta_{11}
\end{aligned}
$$

$$+\int_{W_3}^{\beta_{11,2}=1} g(\beta_{11})\begin{bmatrix} p_1(D_1+d_1-P_{11})+p_2(P_{22}+P_{21}-d_2) \\ -h_1\big(P_{12}-(D_1+d_1-P_{11})\big)-b_2\big(D_2-(P_{22}+P_{21}-d_2)\big)\end{bmatrix} d\beta_{11}d\beta_{12}$$

$$+\int_{W_4}^{\beta_{12,2}=1} f(\beta_{12})\int_{\beta_{11,2}=0}^{W_3} g(\beta_{11})[p_1P_{12}+p_2D_2-b_1(D_1+d_1-P_{11}-P_{12})-h_2(P_{22}+P_{21}-d_2-D_2)]\,d\beta_{11}$$

$$+\int_{W_3}^{\beta_{11,2}=1} g(\beta_{11})\begin{bmatrix} p_1(D_1+d_1-P_{11})+p_2D_2-h_1\big(P_{12}-(D_1+d_1-P_{11})\big) \\ -h_2(P_{22}+P_{21}-d_2-D_2)\end{bmatrix} d\beta_{11}d\beta_{12}d\beta_{11}d\beta_{12}$$

$$+\int_{W_2}^{\beta_{12,1}=1} f(\beta_{12})\int_{W_1}^{\beta_{11,1}=1} g(\beta_{11})[p_1d_1+p_2d_2-h_1(P_{11}-d_1)-h_2(P_{21}-d_2)]$$

$$+\int_{\beta_{12,2}=0}^{W_4} f(\beta_{12})\int_{\beta_{11,2}=0}^{W_3} g(\beta_{11})\begin{bmatrix} p_1(P_{12}+P_{21}-d_2)+p_2(P_{22}+P_{21}-d_2) \\ -b_1\big(D_1-(P_{12}+P_{21}-d_2)\big)-b_2\big(D_2-(P_{22}+P_{21}-d_2)\big)\end{bmatrix} d\beta_{11}$$

$$+\int_{W_3}^{\beta_{11,2}=1} g(\beta_{11})[p_1D_1+p_2(P_{22}+P_{21}-d_2)-h_1(P_{12}+P_{11}-d_1-D_1)-b_2\big(D_2-(P_{22}+P_{21}-d_2)\big)]\,d\beta_{11}d\beta_{12}$$

$$+\int_{W_4}^{\beta_{12,2}=1} f(\beta_{12})\int_{\beta_{11,2}=0}^{W_3} g(\beta_{11})\begin{bmatrix} p_1(P_{12}+P_{21}-d_2)+p_2D_2-b_1\big(D_1-(P_{12}+P_{21}-d_2)\big) \\ -h_2(P_{22}+P_{21}-d_2-D_2)\end{bmatrix} d\beta_{11}$$

$$+\int_{W_3}^{\beta_{11,2}=1} g(\beta_{11})[p_1D_1+p_2D_2-h_1(P_{12}+P_{11}-d_1-D_1)-h_2(P_{22}+P_{21}-d_2-D_2)]\,d\beta_{11}d\beta_{12}d\beta_{11}d\beta_{12}$$

After some algebraic transformations we get

$$\tilde{\pi}_2\big(X,Y,D_1,D_2,\beta_{21,1},\beta_{22,1},\beta_{21,2},\beta_{22,2}\big)=$$

$$\int_{\beta_{12,1}=0}^{1} f(\beta_{12})\int_{\beta_{11,1}=0}^{W_1} g(\beta_{11})[(h_1+b_1)(P_{11}-d_1)]\,d\beta_{11}d\beta_{12}$$

$$+\int_{\beta_{11,1}=0}^{1} g(\beta_{11})\int_{\beta_{12,1}=0}^{W_2} f(\beta_{12})\,[(h_2+b_2)(P_{21}-d_2)]d\beta_{12}d\beta_{11}$$

$$+\int_{\beta_{11,1}=0}^{1} g(\beta_{11})\int_{\beta_{12,1}=0}^{1} f(\beta_{12})[h_1(d_1-P_{11})+h_2(d_2-P_{21})]$$

$$+\int_{\beta_{12,2}=0}^{1} f(\beta_{12})\int_{\beta_{11,2}=0}^{W_3} g(\beta_{11})\,[(p_1+h_1+b_1)(P_{12}+P_{11}-D_1-d_1)]d\beta_{11}d\beta_{12}$$

$$+\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\int_{\beta_{12,2}=0}^{W_4} f(\beta_{12})\,[(p_2+h_2+b_2)(P_{22}+P_{21}-D_2-d_2)]d\beta_{12}d\beta_{11}$$

$$+\int_{\beta_{12,2}=0}^{1} f(\beta_{12})\int_{\beta_{11,2}=0}^{1} g(\beta_{11})[h_1(D_1+d_1-P_{12}-P_{11})+h_2(D_2+d_2-P_{22}-P_{21})]\,d\beta_{11}d\beta_{12}d\beta_{12}d\beta_{11}$$

$$+p_1(D_1+d_1)+p_2(D_2+d_2)$$

$\tilde{\pi}_2$ is a sum of three functions ψ_1, ψ_2 and ψ_3. ψ_1 and ψ_2 depend only on X. Their structure is very similar to the first and second function of $\tilde{\pi}_1$ (see Appendix 5.A) for which we proved concavity. The proof of concavity for ψ_1 and ψ_2 is therefore omitted. Before we prove the concavity of ψ_3, we provide the first order derivatives for ψ_1 and ψ_2, because they are needed for computing the optimal solution.

Substituting W_1 and P_{11}back we get

$\psi_1 =$

$\int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \int_{\beta_{11,1}=0}^{(d_1-X_{21}\beta_{21,1}\beta_{22,1})/X_{11}\beta_{12,1}} g(\beta_{11})\big[(h_1+b_1)\big(X_{11}\beta_{11,1}\beta_{12,1}+X_{21}\beta_{21,1}\beta_{22,1}-d_1\big)\big]\, d\beta_{11} d\beta_{12}$

with the first order derivatives

$\frac{d\psi_1}{dX_{11}} = \int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \int_{\beta_{11,1}=0}^{(d_1-X_{21}\beta_{21,1}\beta_{22,1})/X_{11}\beta_{12,1}} g(\beta_{11})[\beta_{11,1}\beta_{12,1}(h_1+b_1)]\, d\beta_{11} d\beta_{21}$

$\frac{d\psi_1}{dX_{21}} = \int_{\beta_{12,1}=0}^{1} f(\beta_{12}) \int_{\beta_{11,1}=0}^{(d_1-X_{21}\beta_{21,1}\beta_{22,1})/X_{11}\beta_{12,1}} g(\beta_{11})[\beta_{21,1}\beta_{22,1}(h_1+b_1)]\, d\beta_{11} d\beta_{21}$

$\frac{d\psi_1}{dX_{12}} = \frac{d\psi_1}{dX_{22}} = \frac{d\psi_1}{dY_{11}} = \frac{d\psi_1}{dY_{12}} = \frac{d\psi_1}{dY_{21}} = \frac{d\psi_1}{dY_{22}} = 0$

Substituting W_2 and P_{21} back we get

$\psi_2 = \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{(d_2-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1})/(X_{11}(1-\beta_{11,1})+X_{12,1})} f(\beta_{12})$

$\big[(h_2+b_2)\big(X_{11}\beta_{12,1}\big(1-\beta_{11,1}\big)+X_{12}\beta_{12,1}+X_{21}\beta_{22,1}\big(1-\beta_{21,1}\big)+X_{22}\beta_{22,1}-d_2\big)\big] d\beta_{12} d\beta_{11}$

with the first order derivatives

$\frac{d\psi_2}{dX_{11}} = \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{(d_2-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1})/(X_{11}(1-\beta_{11,1})+X_{12,1})} f(\beta_{12}) \big[\beta_{12,1}\big(1-$

$\beta_{11,1}\big)(h_2+b_2)\big]\, d\beta_{21} d\beta_{11}$

$\frac{d\psi_2}{dX_{12}} = \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{(d_2-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1})/(X_{11}(1-\beta_{11,1})+X_{12,1})} f(\beta_{12}) \big[\beta_{12,1}(h_2+b_2)\big]\, d\beta_{21} d\beta_{11}$

$\frac{d\tilde{\pi}_2}{dX_{21}} = \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{(d_2-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1})/(X_{11}(1-\beta_{11,1})+X_{12,1})} f(\beta_{12}) \big[\beta_{22,1}\big(1-$

$\beta_{21,1}\big)(h_2+b_2)\big]\, d\beta_{21} d\beta_{11}$

$\frac{d\psi_2}{dX_{22}} = \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{(d_2-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1})/(X_{11}(1-\beta_{11,1})+X_{12,1})} f(\beta_{12}) \big[\beta_{22,1}(h_2+b_2)\big]\, d\beta_{21} d\beta_{11}$

$\frac{d\psi_2}{dY_{11}} = \frac{d\psi_2}{dY_{12}} = \frac{d\psi_2}{dY_{21}} = \frac{d\psi_2}{dY_{22}} = 0$

Next we prove the concavity of ψ_3. We show that the function $\widehat{\psi}_3$ is jointly concave in $X \geq 0$ and $Y \geq 0$. Because the expectation of a jointly concave function is jointly concave, the desired result follows. Substituting P_{11}, P_{21}, P_{12} and P_{22} back we get

$\psi_3 = \int_{\beta_{11,1}=0}^{1} g(\beta_{11}) \int_{\beta_{12,1}=0}^{1} f(\beta_{12})[\hat{\psi}_3] d\beta_{12} d\beta_{11}$ with

$$
\begin{aligned}
\hat{\psi}_3 = {} & h_1\big(d_1 - X_{11}\beta_{11,1}\beta_{12,1} - X_{21}\beta_{21,1}\beta_{22,1}\big) \\
& + h_2\big(d_2 - X_{11}\beta_{12,1}\big(1-\beta_{11,1}\big) - X_{12}\beta_{12,1} - X_{21}\beta_{22,1}\big(1-\beta_{21,1}\big) - X_{22}\beta_{22,1}\big) \\
+ \int_{\beta_{12,2}=0}^{1} & f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11}) \,\big[(p_1+h_1+b_1)\big(Y_{11}\beta_{11,2}\beta_{12,2} + Y_{21}\beta_{21,2}\beta_{22,2} + X_{11}\beta_{11,1}\beta_{12,1} + X_{21}\beta_{21,1}\beta_{22,1} \\
& - D_1 - d_1\big)\big] d\beta_{11} d\beta_{12}
\end{aligned}
$$

$$+\int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12}) \big[(p_2+h_2+b_2)\big(Y_{11}\beta_{12,2}(1-\beta_{11,2}) + Y_{12}\beta_{12,2} + Y_{21}\beta_{22,2}(1-\beta_{21,2})$$
$$+ Y_{22}\beta_{22,2} + X_{11}\beta_{12,1}(1-\beta_{11,1}) + X_{12}\beta_{12,1} + X_{21}\beta_{22,1}(1-\beta_{21,1}) + X_{22}\beta_{22,1} - D_2$$
$$- d_2\big)\big] d\beta_{12} d\beta_{11}$$
$$+\int_{\beta_{12,2}=0}^{1} f(\beta_{12}) \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \big[h_1\big(D_1 + d_1 - Y_{11}\beta_{11,2}\beta_{12,2} - Y_{21}\beta_{21,2}\beta_{22,2} - X_{11}\beta_{11,1}\beta_{12,1} - X_{21}\beta_{21,1}\beta_{22,1}\big)$$
$$+ h_2\big(D_2 + d_2 - Y_{11}\beta_{12,2}(1-\beta_{11,2}) - Y_{12}\beta_{12,2} - Y_{21}\beta_{22,2}(1-\beta_{21,2}) - Y_{22}\beta_{22,2}$$
$$- X_{11}\beta_{12,1}(1-\beta_{11,1}) - X_{12}\beta_{12,1} - X_{21}\beta_{22,1}(1-\beta_{21,1}) - X_{22}\beta_{22,1}\big)\big]\, d\beta_{11} d\beta_{12} d\beta_{12} d\beta_{11}$$

and the first order derivatives

$$\frac{d\hat{\psi}_3}{dX_{11}} = -2h_1\beta_{11,1}\beta_{12,1} - 2h_2\beta_{12,1}(1-\beta_{11,1})$$
$$+\beta_{11,1}\beta_{12,1}(p_1+h_1+b_1) \int_{\beta_{12,2}=0}^{1} f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11})\, d\beta_{11} d\beta_{12}$$
$$+\beta_{12,1}(1-\beta_{11,1})(p_2+h_2+b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12})\, d\beta_{12} d\beta_{11}$$
$$\frac{d\hat{\psi}_3}{dX_{12}} = -2h_2\beta_{12,1}$$
$$+\beta_{12,1}(p_2+h_2+b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12})\, d\beta_{12} d\beta_{11}$$
$$\frac{d\hat{\psi}_3}{dX_{21}} = -2h_1\beta_{21,1}\beta_{22,1} - 2h_2\beta_{22,1}(1-\beta_{21,1})$$
$$+\beta_{21,1}\beta_{22,1}(p_1+h_1+b_1) \int_{\beta_{12,2}=0}^{1} f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11})\, d\beta_{11} d\beta_{12}$$
$$+\beta_{22,1}(1-\beta_{21,1})(p_2+h_2+b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12})\, d\beta_{12} d\beta_{11}$$
$$\frac{d\hat{\psi}_3}{dX_{22}} = -2h_2\beta_{22,1}$$
$$+\beta_{22,1}(p_2+h_2+b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12})\, d\beta_{12} d\beta_{11}$$

$$\frac{d\hat{\psi}_3}{dY_{11}} = \int_{\beta_{12,2}=0}^{1} f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11}) \big[\beta_{11,2}\beta_{12,2}(p_1+h_1+b_1)\big] d\beta_{11} d\beta_{12}$$
$$+\int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12}) \big[\beta_{12,2}(1-\beta_{11,2})(p_2+h_2+b_2)\big] d\beta_{12} d\beta_{11}$$
$$-h_1 E[\beta_{11,2}] E[\beta_{12,2}] - h_2 E[\beta_{12,2}]\big(1 - E[\beta_{11,2}]\big)$$

$$\frac{d\hat{\psi}_3}{dY_{12}} = \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12}) \big[\beta_{12,2}(p_2+h_2+b_2)\big] d\beta_{12} d\beta_{11}$$
$$-h_2 E[\beta_{12,2}]$$

$$\frac{d\hat{\psi}_3}{dY_{21}} = \int_{\beta_{12,2}=0}^{1} f(\beta_{12}) \int_{\beta_{11,2}=0}^{W_3} g(\beta_{11}) \left[\beta_{21,2}\beta_{22,2}(p_1 + h_1 + b_1)\right] d\beta_{11} d\beta_{12}$$

$$+ \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12}) \left[\beta_{22,1}\left(1 - \beta_{21,1}\right)(p_2 + h_2 + b_2)\right] d\beta_{12} d\beta_{11}$$

$$-h_1\beta_{21,2}\beta_{22,2} - h_2\beta_{22,2}\left(1 - \beta_{21,2}\right)$$

$$\frac{d\hat{\psi}_3}{dY_{22}} = \int_{\beta_{11,2}=0}^{1} g(\beta_{11}) \int_{\beta_{12,2}=0}^{W_4} f(\beta_{12}) \left[\beta_{22,2}(p_2 + h_2 + b_2)\right] d\beta_{12} d\beta_{11}$$

$$-h_2\beta_{22,2}$$

and second order derivatives

$$a_{xx} = \frac{d^2\hat{\psi}_3}{dX_{11}^2}$$

$$-\beta_{11,1}^2\beta_{12,1}^2(p_1 + h_1 + b_1) \int_{\beta_{12,2}=0}^{1} f(\beta_{12})\, g(W_3) \left[\frac{1}{Y_{11}^2\beta_{12,2}^2}\right] d\beta_{12}$$

$$-\beta_{12,1}^2\left(1 - \beta_{11,1}\right)^2(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4) \left[\frac{1}{\left(Y_{11}\left(1-\beta_{11,2}\right)+Y_{12}\right)^2}\right] d\beta_{11}$$

$$b_{xx} = \frac{d^2\hat{\psi}_3}{dX_{11}X_{12}} = \frac{d^2\hat{\psi}_3}{dX_{12}X_{11}} = -\beta_{12,1}^2\left(1 - \beta_{11,1}\right)(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4) \left[\frac{1}{\left(Y_{11}\left(1-\beta_{11,2}\right)+Y_{12}\right)^2}\right] d\beta_{11}$$

$$c_{xx} = \frac{d^2\hat{\psi}_3}{dX_{11}X_{21}} = \frac{d^2\hat{\psi}_3}{dX_{21}X_{11}} = -\beta_{11,1}\beta_{12,1}\beta_{21,1}\beta_{22,1}(p_1 + h_1 + b_1) \int_{\beta_{12,2}=0}^{1} f(\beta_{12})\, g(W_3) \left[\frac{1}{Y_{11}^2\beta_{12,2}^2}\right] d\beta_{12}$$

$$-\beta_{12,1}\left(1 - \beta_{11,1}\right)\beta_{22,1}\left(1 - \beta_{21,1}\right)(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4) \left[\frac{1}{\left(Y_{11}\left(1-\beta_{11,2}\right)+Y_{12}\right)^2}\right] d\beta_{11}$$

$$d_{xx} = \frac{d^2\hat{\psi}_3}{dX_{11}X_{22}} = \frac{d^2\hat{\psi}_3}{dX_{22}X_{11}} - \beta_{12,1}\left(1 - \beta_{11,1}\right)\beta_{22,1}(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4) \left[\frac{1}{\left(Y_{11}\left(1-\beta_{11,2}\right)+Y_{12}\right)^2}\right] d\beta_{11}$$

$$e_{xx} = \frac{d^2\hat{\psi}_3}{dX_{12}^2} = -\beta_{12,1}^2(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4) \left[\frac{1}{\left(Y_{11}\left(1-\beta_{11,2}\right)+Y_{12}\right)^2}\right] d\beta_{11}$$

$$f_{xx} = \frac{d^2\hat{\psi}_3}{dX_{12}X_{21}} = \frac{d^2\hat{\psi}_3}{dX_{21}X_{12}} = -\beta_{12,1}\beta_{22,1}\left(1 - \beta_{21,1}\right)(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4) \left[\frac{1}{\left(Y_{11}\left(1-\beta_{11,2}\right)+Y_{12}\right)^2}\right] d\beta_{11}$$

$$g_{xx} = \frac{d^2\hat{\psi}_3}{dX_{12}X_{22}} = \frac{d^2\hat{\psi}_3}{dX_{22}X_{12}} = -\beta_{12,1}\beta_{22,1}(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4) \left[\frac{1}{\left(Y_{11}\left(1-\beta_{11,2}\right)+Y_{12}\right)^2}\right] d\beta_{11}$$

$$h_{xx} = \frac{d^2\hat{\psi}_3}{dX_{21}^2} = -\beta_{21,1}^2\beta_{22,1}^2(p_1 + h_1 + b_1) \int_{\beta_{12,2}=0}^{1} f(\beta_{12})\, g(W_3) \left[\frac{1}{Y_{11}^2\beta_{12,2}^2}\right] d\beta_{12}$$

$$-\beta_{22,1}^2\left(1 - \beta_{21,1}\right)^2(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4) \left[\frac{1}{\left(Y_{11}\left(1-\beta_{11,2}\right)+Y_{12}\right)^2}\right] d\beta_{11}$$

$$i_{xx} = \frac{d^2\hat{\psi}_3}{dX_{21}X_{22}} = \frac{d^2\hat{\psi}_3}{dX_{22}X_{21}} = -\beta_{22,1}^2\left(1 - \beta_{21,1}\right)(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4) \left[\frac{1}{\left(Y_{11}\left(1-\beta_{11,2}\right)+Y_{12}\right)^2}\right] d\beta_{11}$$

$$j_{xx} = \frac{d^2\hat{\psi}_3}{dX_{22}^2} = -\beta_{22,1}^2(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4) \left[\frac{1}{\left(Y_{11}\left(1-\beta_{11,2}\right)+Y_{12}\right)^2}\right] d\beta_{11}$$

$$a_{xy} = \frac{d^2\hat{\psi}_3}{dX_{11}Y_{11}} = \frac{d^2\hat{\psi}_3}{dY_{11}X_{11}} = -\beta_{11,1}\beta_{12,1}(p_1 + h_1 +$$

$$b_1) \int_{\beta_{12,2}=0}^{1} f(\beta_{12})\, g(W_3) \left[\frac{d_1+D_1-Y_{21}\beta_{21,2}\beta_{22,2}-X_{11}\beta_{11,1}\beta_{12,1}-X_{21}\beta_{21,1}\beta_{22,1}}{Y_{11}^2\beta_{12,2}}\right] d\beta_{12}$$

$$-\beta_{12,1}\left(1 - \beta_{11,1}\right)(p_2 + h_2 + b_2) \int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4) \left[\left(1 -\right.\right.$$

$$\left.\left.\beta_{11,2}\right) \frac{\left(d_2+D_2-Y_{21}\beta_{22,2}\left(1-\beta_{21,2}\right)-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}\left(1-\beta_{11,1}\right)-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}\left(1-\beta_{21,1}\right)-X_{22}\beta_{22,1}\right)}{\left(Y_{11}\left(1-\beta_{11,2}\right)+Y_{12}\right)^2}\right] d\beta_{11}$$

$$b_{xy} = \frac{d^2\hat{\psi}_3}{dX_{11}Y_{12}} = \frac{d^2\hat{\psi}_3}{dY_{12}X_{11}} = -\beta_{12,1}\big(1-\beta_{11,1}\big)(p_2 + h_2 +$$
$$b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\left[\frac{\big(d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1}\big)}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)^2}\right] d\beta_{11}$$

$$c_{xy} = \frac{d^2\hat{\psi}_3}{dX_{11}Y_{21}} = \frac{d^2\hat{\psi}_3}{dY_{21}X_{11}} = -\beta_{11,1}\beta_{12,1}\beta_{21,2}\beta_{22,2}(p_1 + h_1 + b_1)\int_{\beta_{12,2}=0}^{1} f(\beta_{12})\, g(W_3)\left[\frac{1}{Y_{11}\beta_{12,2}}\right] d\beta_{12}$$
$$-\beta_{12,1}\big(1-\beta_{11,1}\big)\beta_{22,2}\big(1-\beta_{21,2}\big)(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\left[\frac{1}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)}\right] d\beta_{11}$$

$$d_{xy} = \frac{d^2\hat{\psi}_3}{dX_{11}Y_{22}} = \frac{d^2\hat{\psi}_3}{dY_{22}X_{11}} = -\beta_{12,1}\big(1-\beta_{11,1}\big)\beta_{22,2}(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\left[\frac{1}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)}\right] d\beta_{11}$$

$$e_{xy} = \frac{d^2\hat{\psi}_3}{dX_{12}Y_{11}} = \frac{d^2\hat{\psi}_3}{dY_{11}X_{12}} = -\beta_{12,1}(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\Bigg[\big(1-$$
$$\beta_{11,2}\big)\frac{\big(d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1}\big)}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)^2}\Bigg] d\beta_{11}$$

$$f_{xy} = \frac{d^2\hat{\psi}_3}{dX_{12}Y_{12}} = \frac{d^2\hat{\psi}_3}{dY_{12}X_{12}} = -\beta_{12,1}(p_2 + h_2 +$$
$$b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\left[\frac{\big(d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1}\big)}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)^2}\right] d\beta_{11}$$

$$g_{xy} = \frac{d^2\hat{\psi}_3}{dX_{12}Y_{21}} = \frac{d^2\hat{\psi}_3}{dY_{21}X_{12}} = -\beta_{12,1}\beta_{22,2}\big(1-\beta_{21,2}\big)(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\left[\frac{1}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)}\right] d\beta_{11}$$

$$h_{xy} = \frac{d^2\hat{\psi}_3}{dX_{12}Y_{22}} = \frac{d^2\hat{\psi}_3}{dY_{22}X_{12}} = -\beta_{12,1}\beta_{22,2}(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\left[\frac{1}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)}\right] d\beta_{11}$$

$$i_{xy} = \frac{d^2\hat{\psi}_3}{dX_{21}Y_{11}} = \frac{d^2\hat{\psi}_3}{dY_{11}X_{21}} = -\beta_{21,1}\beta_{22,1}(p_1 + h_1 +$$
$$b_1)\int_{\beta_{12,2}=0}^{1} f(\beta_{12})\, g(W_3)\left[\frac{d_1+D_1-Y_{21}\beta_{21,2}\beta_{22,2}-X_{11}\beta_{11,1}\beta_{12,1}-X_{21}\beta_{21,1}\beta_{22,1}}{Y_{11}^2\beta_{12,2}}\right] d\beta_{12}$$
$$-\beta_{22,1}\big(1-\beta_{21,1}\big)(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\Bigg[\big(1-$$
$$\beta_{11,2}\big)\frac{\big(d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1}\big)}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)^2}\Bigg] d\beta_{11}$$

$$j_{xy} = \frac{d^2\hat{\psi}_3}{dX_{21}Y_{12}} = \frac{d^2\hat{\psi}_3}{dY_{12}X_{21}} = -\beta_{22,1}\big(1-\beta_{21,1}\big)(p_2 + h_2 +$$
$$b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\left[\frac{\big(d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1}\big)}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)^2}\right] d\beta_{11}$$

$$k_{xy} = \frac{d^2\hat{\psi}_3}{dX_{21}Y_{21}} = \frac{d^2\hat{\psi}_3}{dY_{21}X_{21}} = -\beta_{21,1}\beta_{22,1}\beta_{21,2}\beta_{22,2}(p_1 + h_1 + b_1)\int_{\beta_{12,2}=0}^{1} f(\beta_{12})\, g(W_3)\left[\frac{1}{Y_{11}\beta_{12,2}}\right] d\beta_{12}$$
$$-\beta_{22,1}\big(1-\beta_{21,1}\big)\beta_{22,2}\big(1-\beta_{21,2}\big)(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\left[\frac{1}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)}\right] d\beta_{11}$$

$$l_{xy} = \frac{d^2\hat{\psi}_3}{dX_{21}Y_{22}} = \frac{d^2\hat{\psi}_3}{dY_{12}X_{22}} = -\beta_{22,1}\big(1-\beta_{21,1}\big)\beta_{22,2}(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\left[\frac{1}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)}\right] d\beta_{11}$$

$$m_{xy} = \frac{d^2\hat{\psi}_3}{dX_{22}Y_{11}} = \frac{d^2\hat{\psi}_3}{dY_{11}X_{22}} = -\beta_{22,1}(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\Bigg[\big(1-$$
$$\beta_{11,2}\big)\frac{\big(d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1}\big)}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)^2}\Bigg] d\beta_{11}$$

$$n_{xy} = \frac{d^2\hat{\psi}_3}{dX_{22}Y_{12}} = \frac{d^2\hat{\psi}_3}{dY_{12}X_{22}} = -\beta_{22,1}(p_2 + h_2 +$$
$$b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\left[\frac{\big(d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1}\big)}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)^2}\right] d\beta_{11}$$

$$o_{xy} = \frac{d^2\hat{\psi}_3}{dX_{22}Y_{21}} = \frac{d^2\hat{\psi}_3}{dY_{21}X_{22}} = -\beta_{22,1}\beta_{22,2}\big(1-\beta_{21,2}\big)(p_2 + h_2 + b_2)\int_{\beta_{11,2}=0}^{1} g(\beta_{11})\, f(W_4)\left[\frac{1}{\big(Y_{11}(1-\beta_{11,2})+Y_{12}\big)}\right] d\beta_{11}$$

$$p_{xy}=\frac{d^2\hat{\psi}_3}{dX_{22}Y_{22}}=\frac{d^2\hat{\psi}_3}{dY_{22}X_{22}}=-\beta_{22,1}\beta_{22,2}(p_2+h_2+b_2)\int_{\beta_{11,2}=0}^{1}g(\beta_{11})\,f(W_4)\left[\frac{1}{(Y_{11}(1-\beta_{11,2})+Y_{12})}\right]d\beta_{11}$$

$$a_{yy}=\frac{d^2\tilde{\pi}_1}{dY_{11}^2}=-(p_1+h_1+b_1)\int_{\beta_{12,2}=0}^{1}f(\beta_{12})\,g(W_3)\left[\frac{(d_1+D_1-Y_{21}\beta_{21,2}\beta_{22,2}-X_{11}\beta_{11,1}\beta_{12,1}-X_{21}\beta_{21,1}\beta_{22,1})^2}{Y_{11}^3\beta_{12,2}}\right]d\beta_{12}$$
$$-(p_2+h_2+b_2)\int_{\beta_{11,2}=0}^{1}g(\beta_{11})\,f(W_4)\left[(1-\beta_{11,2})^2\frac{(d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1})^2}{(Y_{11}(1-\beta_{11,2})+Y_{12})^3}\right]d\beta_{11}$$

$$b_{yy}=\frac{d^2\hat{\psi}_3}{dY_{11}Y_{12}}=\frac{d^2\hat{\psi}_3}{dY_{12}Y_{11}}=-(p_2+h_2+b_2)\int_{\beta_{11,2}=0}^{1}g(\beta_{11})\,f(W_4)\left[(1-\beta_{11,2})\frac{(d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1})^2}{(Y_{11}(1-\beta_{11,2})+Y_{12})^3}\right]d\beta_{11}$$

$$c_{yy}=\frac{d^2\hat{\psi}_3}{dY_{11}Y_{21}}=\frac{d^2\hat{\psi}_3}{dY_{21}Y_{11}}=-\beta_{21,1}\beta_{22,1}(p_1+h_1+b_1)\int_{\beta_{12,2}=0}^{1}f(\beta_{12})\,g(W_3)\left[\frac{d_1+D_1-Y_{21}\beta_{21,2}\beta_{22,2}-X_{11}\beta_{11,1}\beta_{12,1}-X_{21}\beta_{21,1}\beta_{22,1}}{Y_{11}^2\beta_{12,2}}\right]d\beta_{12}$$
$$-\beta_{22,2}(1-\beta_{21,2})(p_2+h_2+b_2)\int_{\beta_{11,2}=0}^{1}g(\beta_{11})\,f(W_4)\left[(1-\beta_{11,2})\frac{d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1}}{(Y_{11}(1-\beta_{11,2})+Y_{12})^2}\right]d\beta_{11}$$

$$d_{yy}=\frac{d^2\hat{\psi}_3}{dY_{11}Y_{22}}=\frac{d^2\hat{\psi}_3}{dY_{22}Y_{11}}=-\beta_{22,2}(p_2+h_2+b_2)\int_{\beta_{11,2}=0}^{1}g(\beta_{11})\,f(W_4)\left[(1-\beta_{11,2})\frac{d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1}}{(Y_{11}(1-\beta_{11,2})+Y_{12})^2}\right]d\beta_{11}$$

$$e_{yy}=\frac{d^2\hat{\psi}_3}{dY_{12}^2}=-(p_2+h_2+b_2)\int_{\beta_{11,2}=0}^{1}g(\beta_{11})\,f(W_4)\left[\frac{(d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1})^2}{(Y_{11}(1-\beta_{11,2})+Y_{12})^3}\right]d\beta_{11}$$

$$f_{yy}=\frac{d^2\hat{\psi}_3}{dY_{12}Y_{21}}=\frac{d^2\hat{\psi}_3}{dY_{21}Y_{12}}=-\beta_{22,2}(1-\beta_{21,2})(p_2+h_2+b_2)\int_{\beta_{11,2}=0}^{1}g(\beta_{11})\,f(W_4)\left[\frac{d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1}}{(Y_{11}(1-\beta_{11,2})+Y_{12})^2}\right]d\beta_{11}$$

$$g_{yy}=\frac{d^2\hat{\psi}_3}{dY_{12}Y_{22}}=\frac{d^2\hat{\psi}_3}{dY_{22}Y_{12}}=-\beta_{22,2}(p_2+h_2+b_2)\int_{\beta_{11,2}=0}^{1}g(\beta_{11})\,f(W_4)\left[\frac{d_2+D_2-Y_{21}\beta_{22,2}(1-\beta_{21,2})-Y_{22}\beta_{22,2}-X_{11}\beta_{12,1}(1-\beta_{11,1})-X_{12}\beta_{12,1}-X_{21}\beta_{22,1}(1-\beta_{21,1})-X_{22}\beta_{22,1}}{(Y_{11}(1-\beta_{11,2})+Y_{12})^2}\right]d\beta_{11}$$

$$h_{yy}=\frac{d^2\hat{\psi}_3}{dY_{21}^2}=-\beta_{21,2}^2\beta_{22,2}^2(p_1+h_1+b_1)\int_{\beta_{12,2}=0}^{1}f(\beta_{12})\,g(W_3)\left[\frac{1}{Y_{11}\beta_{12,2}}\right]d\beta_{12}$$
$$-\beta_{22,1}^2(1-\beta_{21,1})^2(p_2+h_2+b_2)\int_{\beta_{11,2}=0}^{1}g(\beta_{11})\,f(W_4)\left[\frac{1}{(Y_{11}(1-\beta_{11,2})+Y_{12})}\right]d\beta_{11}$$

$$i_{yy}=\frac{d^2\hat{\psi}_3}{dY_{21}Y_{22}}=\frac{d^2\hat{\psi}_3}{dY_{22}Y_{21}}=-\beta_{22,1}^2(1-\beta_{21,1})(p_2+h_2+b_2)\int_{\beta_{11,2}=0}^{1}g(\beta_{11})\,f(W_4)\left[\frac{1}{(Y_{11}(1-\beta_{11,2})+Y_{12})}\right]d\beta_{11}$$

$$j_{yy}=\frac{d^2\hat{\psi}_3}{dY_{22}^2}=-\beta_{22,1}^2(p_2+h_2+b_2)\int_{\beta_{11,2}=0}^{1}g(\beta_{11})\,f(W_4)\left[\frac{1}{(Y_{11}(1-\beta_{11,2})+Y_{12})}\right]d\beta_{11}$$

The Hessian matrix has the following structure.

$$H = \begin{pmatrix} a_{xx} & b_{xx} & c_{xx} & d_{xx} & a_{xy} & b_{xy} & c_{xy} & d_{xy} \\ b_{xx} & e_{xx} & f_{xx} & g_{xx} & e_{xy} & f_{xy} & g_{xy} & h_{xy} \\ c_{xx} & f_{xx} & h_{xx} & i_{xx} & i_{xy} & j_{xy} & k_{xy} & l_{xy} \\ d_{xx} & g_{xx} & i_{xx} & j_{xx} & m_{xy} & n_{xy} & o_{xy} & p_{xy} \\ a_{xy} & e_{xy} & i_{xy} & m_{xy} & a_{yy} & b_{yy} & c_{yy} & d_{yy} \\ b_{xy} & f_{xy} & j_{xy} & n_{xy} & b_{yy} & e_{yy} & f_{yy} & g_{yy} \\ c_{xy} & g_{xy} & k_{xy} & o_{xy} & c_{yy} & f_{yy} & h_{yy} & i_{yy} \\ d_{xy} & h_{xy} & l_{xy} & p_{xy} & d_{yy} & g_{yy} & i_{yy} & j_{yy} \end{pmatrix}$$

We show that the Hessian matrix is negative semi definite by showing that $x^T Hx \leq 0$ for all $x \in \mathbb{R}^8$. [*Comment:* $x^T = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$].

Because all second order derivatives are negative (≤ 0), there exists always an $\varepsilon \leq 0$ that is larger than or equal to the each second order derivative. Following the same reasoning as for the proof of Theorem 5-1 we can establish that $x^T Hx \leq \varepsilon(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8)^2 \leq 0$, which is true for all $x \in \mathbb{R}^8$. ■

Chapter 6

Conclusion

This thesis extends the literature on inventory management under random yield by analyzing the value of supply chain visibility that is gained by real time yield information. Perfect, imperfect, as well as costly real time yield information are considered. Optimal and heuristic solution approaches are provided. The optimal solution approaches enable the exact quantification of benefits gained from increased supply chain visibility and enhance the understanding of the optimal policy. The heuristics are capable of solving large problems efficiently and facilitate the transfer of our research into real world applications.

In Section 6.1, we summarize the key results of this thesis. In Section 6.2, we critically review our modeling and solution approaches. In Section 6.3, we discuss promising areas for future research.

6.1. Summary of Key Results

In **Chapter 3** we consider a periodic review inventory system with random yield, random demand and positive lead time. The decision maker has access to real time yield information. This information is perfect and free of charge. To contribute to a better understanding of the value of this information, we develop a mathematical model of the inventory system and derive structural properties. We build on these properties to develop an optimal solution approach that can be used to solve small to medium sized problems. To solve large problems, we develop two heuristics. We conduct numerical experiments to test the performances of our approaches and to identify conditions under which real time yield information is particularly beneficial. Chapter 3 provides the approaches that are necessary to implement inventory control policies that utilize real time yield information. The results can also be used to quantify the cost savings that can be achieved by using real time yield information. These cost savings can then be compared against the required investments to decide if such an investment is profitable. The analysis is extended to consider a setting with fixed order cost.

In **Chapter 4** we consider basically the same inventory system as in Chapter 3. Order batches can be tracked to get access to real time information about the actual yield realizations. In difference to the setting in Chapter 3 tracking induces fixed costs per order and the decision maker can decide for each order whether or not to obtain yield information. We develop a mathematical model of the inventory system. We prove that the cost function is convex for a given tracking decision and that a solution for the infinite horizon problem exists. Based on these properties we apply an optimal solution approach for discrete state spaces. We conduct numerical experiments to quantify the benefits of a flexible tracking system versus systems that track all orders or do not track any order. We identify conditions under which real time yield information with flexible tracking is particularly beneficial and identify the key drivers for the tracking decision. Our research provides the approaches that are necessary to implement inventory control policies that utilize costly real time yield information on an order-by-order basis.

In **Chapter 5** we consider a two-stage production system which produces a hierarchy of multiple grades of outputs. In the first stage, a single type of input (wafer) is used to produce products (chips) of different quality levels with random yield rates. After first stage production is finished chips are pre-tested for their quality level. This fast and inexpensive pre-test can determine the final quality level of a chip with a certain probability. In the second stage, chips are tested for their final quality level. Test capacity at the second stage is limited and test processes reveal if the quality of a chip is of the tested quality or any lower quality level. Therefore all first and second stage processes have the characteristic features of co-production and random yields. Customer demands for chips of different quality levels are random. We develop a mathematical model to plan the input quantity for the first stage and the respective quantities at the second stage so as to maximize profit over a finite horizon. We use the optimal approach to solve small problems and develop a heuristic that can solve larger problems. We conduct numerical experiments to test the accuracy of the heuristic and to quantify the value of preliminary yield information gathered by the pre-test after first stage production is completed.

6.2. Critical Review of Modeling Approach

We had to make certain assumptions to keep our models tractable. In this section, we review the most critical assumptions. Our assumptions are in line with existing literature on inventory management under random yield but may not always correspond to settings faced in practice. We assume complete backordering for all our models. This might not always be possible in practice, where some or all unsatisfied demand might be lost or substituted. We also assume that demand and yield rates are independent and identically distributed random variables over time. In real world applications one might think of situations where both variables might be correlated between periods and may change over time. The primary goal of all our models is to provide guidance for the use and the quantification of the value of real time yield information. The models need to be rather generic to enable derivation of analytical results and to keep them applicability to a broad spectrum of scenarios. The proposed models should be useful in any further attempt to relax or change the underlying assumptions.

All our inventory models incorporate the proportional yield model. This model is commonly used in research on inventory management under random yield but is limited to some extent. The main assumption of the proportional yield model is that the yield of units in the same batch is perfectly correlated. This might not always be true in practice as there can be a significant individual yield risk for each unit. The other extreme is the binomial yield model where the yield of units is uncorrelated. Yield processes in practice might be a mix of these two models. However, research on new yield models would be a topic on its own and distract the focus from our work on real time yield information as well as hinder the comparability of our results with previous research.

6.3. Directions for Further Research

In the conclusion sections of the main chapters of this thesis, we point out possible further extensions of our models. Two main directions for future research can be applied to all our models.

A research stream that naturally evolves after a certain problem has been modeled and its structural properties have been derived is the development of efficient optimal and heuristic solution approaches. Especially the inventory problems, analyzed in Chapter 4 and Chapter 5, create a field where advanced heuristics are of great value. Also, more efficient algorithms to compute the optimal solution could be developed. Especially the field of approximate dynamic programming might be promising in this context. More efficient algorithms might support the implementation of technologies that enable the use of real time yield information in practice as well as the analysis of more complex problems.

From a theoretical point of view an opportunity for future research is to change some of our main assumptions. The impact of real time yield information on continuous review inventory systems with the same or other yield models could be analyzed. First results of our work in this direction indicate that results will be similar by trend. However the modeling and the development of efficient solutions approaches for other types of inventory systems might be worthwhile. This would facilitate the understanding of how real time yield information can be optimally used in different settings.

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