Torsion structures, subobjects and unique filtrations in non-abelian categories

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Abstract

In this thesis we study torsion theory, subobjects and filtration properties in quasi-abelian, exact and right triangulated categories. All such categegories fit into the larger framework of extriangulated categories. Our work falls into three parts. In the first, we define torsion pairs for quasi-abelian categories and give several characterisations. We show that many of the torsion theoretic concepts translate from abelian categories to quasi-abelian categories. As an application, we generalise the recently defined algebraic Harder-Narasimhan filtrations to quasi-abelian categories.

Secondly, we investigate how the concepts of intersection and sums of subobjects carry to exact categories. We obtain a new characterisation of quasi-abelian categories in terms of admitting admissible intersections in the sense of [60]. There are also many alternative characterisations of abelian categories as those that additionally admit admissible sums and in terms of properties of admissible morphisms. We then define a generalised notion of intersection and sum which every exact category admits. Using these new notions, we define and study classes of exact categories that satisfy the Jordan-Hölder property for exact categories, namely the Diamond exact categories and Artin-Wedderburn exact categories. By explicitly describing all exact structures on $\mathcal{A} = \operatorname{rep} \Lambda$ for a Nakayama algebra Λ we characterise all Artin-Wedderburn exact structures on $\mathcal{A}$ and show that these are precisely the exact structures with the Jordan-Hölder property.

Thirdly, we study right triangulated categories; which can be thought of as triangulated categories whose shift functor is not an equivalence. We give intrinsic characterisations of when such categories have a natural extriangulated structure and are appearing as the (co-)aisle of a (co-)t-structure in an associated triangulated category.

Zusammenfassung

In diese Doktorarbeit untersuchen wir Torsiontheorie, Unterobjekte und Filtrierung-eigenschaften in quasi-abelschen, exakten und rechtstriangulierten Kategorien. Alle derartige Kategorien passen in den großeren Rahmen der sogenannten 'extriangulatierten' Kategorien. Unsere Arbeit ist in drei Teile aufgeteilt. In die Erste definieren wir Torsionpaare für quasi-abelsche Kategorien und präsentieren verschiedene Charakterisierungen. Wir zeigen, dass sich viele Konzepte der Torsiontheorie direkt aus abelschen Kategorien in quasi-abelsche Kategorien übertragen lassen. Als Anwendung generalisieren wir die kurzlich definierten algebraischen Harder-Narasimhan Filtrierungen auf quasi-abelsche Kategorien.

Zweitens untersuchen wir, wie sich die Konzepte der Schnittmengen und Summen von Unterobjekten in exakte Kategorien übersetzen lassen. Wir erhalten eine neue Charakterisierung von quasi-abelschen Kategorien bezüglich 'admissible' Schnittmengen im Sinne von [60] sowie alternative Charakterisierungen abelscher Kategorien als genau diese Kategorien, die zusätzlich admissible Summen unterstützen, sowie in Bezug auf 'admissible' Morphismen. Wir führen eine generalisierte Vorstellung von Schnittmenge und Summe des Unterobjekte ein, die in jeder exakten Kategorie funktionieren. Mit diesen definieren und untersuchen wir exakten Kategorien, die die Jordan-Hölder Eigenschaft besitzen, nämlich die 'Diamond' und Artin-Wedderburn exakten Kategorien. Indem wir ausdrücklich jede exakte Struktur auf $\mathcal{A} = \operatorname{rep}\Lambda$ für eine Nakayama Algebra Λ beschreiben, charakterisieren wir alle Artin-Wedderburn exakten Strukturen auf $\mathcal{A}$ und zeigen, dass diese genau die exact Strukturen sind, die die Jordan-Hölder Eigenschaft aufweisen.

Im dritten Teil untersuchen wir rechtstriangulierte Kategorien; diese kann man sich als triangulierte Kategorien vorstellen, deren die Shiftfunktor keine Äquivalenz ist. Wir geben intrinische Charakterisierungen, wann solche Kategorien ein natürliche extriangulatierte Struktur haben und als eine (Co-)Aisle von einer (Co-)-t-struktur in einer verknüpften triangulierten Kategorie vorkommen.

Declaration

'Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne die Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Werken dem Wortlaut oder dem Sinn nach entnommen wurden, sind als solche kenntlich gemacht. Ich versichere an Eides statt, dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen und eingebundenen Artikeln und Manuskripten - noch nicht veröffentlicht worden ist sowie, dass ich eine Veröffentlichung der Dissertation vor Abschluss der Promotion nicht ohne Genehmigung des Promotionsausschusses vornehmen werde. Die Bestimmungen dieser Ordnung sind mir bekannt. Darüber hinaus erkläre ich hiermit, dass ich die Ordnung zur Sicherung guter wissenschaftlicher Praxis und zum Umgang mit wissenschaftlichem Fehlverhalten der Universität zu Köln gelesen und sie bei der Durchführung der Dissertation zugrundeliegenden Arbeiten und der schriftlich verfassten Dissertation beachtet habe und verpflichte mich hiermit, die dort genannten Vorgaben bei allen wissenschaftlichen Tätigkeiten zu beachten und umzusetzen. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.'

Teilpublikationen:
Chapter III is based on the published paper [123] authored by myself.
Chapter IV is based on the published paper [32] written by Thomas Brüstle, Souheila Hassoun and myself. The contribution of my coauthors to this work should be treated as equal to my own. This paper was also used in an application for a doctoral degree to the Université de Sherbrooke, Canada submitted by Souheila Hassoun in 2021.
Chapter V is based on the pre-print [122] authored by myself.
The works mentioned above also contribute to the Introduction and Chapter II.

April 12, 2022 Aran Tattar

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Contents

Chapter I

Introduction

In representation theory, to study an associative (finite dimensional) algebra, A, (over a field k) one looks to understand the category of modules of A. It is well-known that a (hom-finite) abelian k-category, $\mathcal{A}$ with a projective generator, P, is equivalent to a module category of a k-algebra, namely of $\operatorname{End}_{\mathcal{A}} P$ which naturally inherits the structure of an associative (finite dimensional) k-algebra from $\mathcal{A}$.

Subcategories of these module categories can aid in the understanding of the whole category for example, in the study of torsion(free) classes, subcategories of filtered modules, and wide subcategories; which appear in the study of (τ)-tilting theory [2, 57], stratifying systems [49, 91] and stability conditions [79] and other areas respectively. Often, and in the case of the above examples, the property of a subcategory that is 'closed under extenstions' or 'extension closed' arises. The formalism of Quillen's exact categories [107] axiomatises the property of extension closed subcategories of abelian categories. These are additive categories equipped with an *exact structure*, which provides the minimal framework required for studying a class of extensions/ short exact sequences in a (bi)functorial way. Choosing a Quillen exact structure allows to define various cohomology theories for locally compact abelian groups, Banach spaces, or other categories studied in functional analysis [62].

Alternatively, one may seek to understand the derived category of an algebra, which is the natural setting for the study of dervied functors [125]. This has the structure of a triangulated category [106, 125]. Triangulated categories have become an important and powerful tool throughout representation theory [109, 75], homological algebra [121] and algebraic geometry [59].

Thus the study of triangulated, exact and abelian (which are a special class of exact categories) categories is of general importance in representation theory and other mathematical disciplines. The language of extriangulated categories allows one to study such structures and more simultaneously. Extriangulated categories were introduced by Nakaoka and Palu in [95] as a generalisation of exact categories and triangulated categories. The framework of extriangulated categories allows one to axiomatise properties of categories that have structural similarities to exact and/or triangulated categories but fall into neither class, for example extension closed subcategories of triangulated categories. The formalism of extriangulated categories then allows homological algebra to be applied to such categories which has been done successfully by many authors, for example [48, 98, 130].

In this thesis, we investigate how some important concepts in abelian categories carry over to (classes of) exact categories. Namely, torsion pairs, sums and intersections of subobjects and the Jordan-Hölder property. We also look at certain torsion pairs in triangulated categories and study the structures behind these. For the rest of the introduction, we discuss these concepts in more detail.

I.1 Torsion pairs

Torsion classes were introduced for abelian categories by Dickson [41] to generalise the notion of torsion and torsionfree groups. Since then they have been widely studied in various contexts including (τ-)tilting theory [2, 57], lattice theory [40] and, more recently, stability conditions [33, 124].

Bondal & Van den Bergh [25] and Rump [112] characterised the structure of torsion(free) classes in abelian categories: Each torsion(free) class in an abelian category is quasi-abelian and every quasi-abelian category, $\mathcal{Q}$, appears as the torsionfree class of a 'left associated' abelian category $\mathcal{L}_{\mathcal{Q}}$ and as a torsion class of an abelian category $\mathcal{R}_{\mathcal{Q}}$.

Quasi-abelian categories are a particular class of exact categories whose maximal exact structure ([114, 119]) coincides with the class of all short exact sequences in the category (see Definition II.1.19). As the name suggests, they are a weaker structure than abelian categories. Quasi-abelian categories appear naturally in cluster theory [118] and in the context of Bridgeland's stability conditions [29].

Based on a characterisation of torsion pairs in abelian categories [41], we define a torsion pair for an arbitrary additive category as follows.

Definition. (Definition III.1.1) Let $\mathcal{A}$ be an additive category. A *torsion pair in $\mathcal{A}$* is an ordered pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ satisfying the following.

(T1) $\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$.

(T2) For all A in $\mathcal{Q}$ there exists a short exact sequence

$$0 \longrightarrow {}_{\mathcal{T}}A \longrightarrow A \longrightarrow A_{\mathcal{F}} \longrightarrow 0$$

with ${}_{\mathcal{T}}A \in \mathcal{T}$ and $A_{\mathcal{F}} \in \mathcal{F}$.

In this case we call $\mathcal{T}$ a *torsion class* and $\mathcal{F}$ a *torsionfree class.*

We seek to define and study torsion classes in quasi-abelian categories by describing torsion classes of quasi-abelian categories in terms of the torsion(free) classes in the associated abelian category. We note that torsion pairs in pre-abelian and semi-abelian categories, which are weaker structures still than quasi-abelian categories, have been studied in [70]. In this more general context, torsion pairs no longer have the well-known characterisations that they have in the abelian set up. In [28] torsion theory in non-abelian, so-called homological categories has also been considered.

We call torsion pairs $(\mathcal{C}, \mathcal{D})$, $(\mathcal{C}', \mathcal{D}')$ in an additive category satisfying $\mathcal{C} \subseteq \mathcal{C}'$ *twin torsion pairs* and the intersection $\mathcal{C}' \cap \mathcal{D}$ their *heart.* In Theorems III.2.2 we show that, in abelian categories, the heart of twin torsion pairs is quasi-abelian, then in Theorem III.3.2 we establish a bijection between torsion pairs in the heart and certain torsion pairs in the ambient abelian category.

With this machinery in hand, our strategy for studying properties of torsion classes in a quasi-abelian category $\mathcal{Q}$ is to translate the problem to the associated abelian category $\mathcal{L}_{\mathcal{Q}}$ using the above bijection, utilise the properties of torsion in abelian categories, then translate back to $\mathcal{Q}$. We see that in general, torsion theoretic concepts of abelian categories carry well to quasi-abelian categories (Propositions III.4.13, III.4.14 and III.4.17). Furthermore, in Theorem III.4.9 we characterise when $\mathcal{L}_{\mathcal{Q}}$ is a small module category over a right noetherian (resp. right artinian ring).

In [124], Treffinger has shown that every chain of torsion classes satisfying mild finiteness conditions in an abelian category induces Harder-Narasimhan filtrations. Such filtrations were extensively studied in [108] and named after Harder and Narasimhan for their work [58]. Furthermore, Rudakov [110] showed that every stability function on an abelian category induces a Harder-Narasimhan filtration of each object. In [16] and [33] it was observed, for abelian categories, that each stability function induces a chain of torsion classes; which served as inspiration for [124]. As an application of our results, we show that chains of torsion classes in quasi-abelian categories also induce Harder-Narasimhan filtrations (Corollary III.5.9).

I.2 Intersections and sums

The concept of intersection and sum of subobjects, which is readily available for groups, modules or objects in an abelian category is very useful. For instance, in the definition of the standard modules in stratified algebras [43] and also in torsion theory: When working in an abelian category, the module ${}_{\mathcal{T}}A$ is isomorphic to $\sum_{\substack{\psi:T\to A\\ T\in\mathcal{T}}} \operatorname{Im}\psi$ [11]. In addition, many proofs of the Jordan-Hölder-Schreier theorem make use of these notions e.g. [17].

In a general categorical setup, the intersection is defined as pullback of two monomorphisms, if it exists. However, in order to define a sensible cohomology theory, one needs to restrict the notion of subobjects to *admissible subobjects*, which allow to form kernel-cokernel pairs. In the context of functional analysis, for instance, this leads to the study of closed subspaces, often giving rise to the structure of a quasi-abelian category. More generally, the setup is that of exact categories [107]. In this generality, one requires not only that the intersection of admissible subobjects exists, but it needs to be an admissible subobject itself.

Motivated by these ideas, we generalise the abelian notions of intersection and sum to exact categories. We do this in two ways. Firstly, by considering intersections as pullbacks and sums as pushouts of intersections - as is the case in the abelian setting, see [53, Section 5] and [104, Definition 2.6] - we recall in Definitions IV.2.1 and IV.2.2 the classes of *AI-categories* (Admissible Intersection) and *AIS-categories* (Admissible Intersection and Sum) from [60], which are exact categories that admit intersections (respectively intersections and sums) of subobjects in a similar way to the abelian setting.

It transpires that these notions of intersection and sums are quite restrictive: The AI-categories are necessarily quasi-abelian with the maximal exact structure $\mathcal{E}_{max}$ Proposition IV.2.5. The converse has also been proved by Hassoun, Shah & Wegner and thus we have a new characterisation of quasi-abelian categories. As for the (AIS)-categories, it turns out are precisely the abelian categories endowed with the maximal exact structure (Theorem IV.2.9). Along the way, we also show that the class of admissible morphisms is typically poorly behaved, unless we are working in an abelian category with maximal exact structure (Theorem IV.1.4).

As a conclusion of the above, these notions of sums and intersection for exact categories are not suitable for the general setting, see also Examples IV.3.2 and IV.3.3. This leads us to define, in Definition IV.3.4, a general notion of admissible intersection and sum that works for all exact categories. For two admissible subobjects (A, f) and (B, g) of X, their intersection, $\mathrm{Int}_X(A, B)$, is the set of all their maximal common proper admissible subobjects. Dually, their sum, $\mathrm{Sum}_X(A, B)$ is the set of all their minimal common proper admissible superobjects that are subobjects of X. As an application of these new definitions, we study the Jordan-Hölder property for exact categories which we discuss next.

I.3 The Jordan Hölder property

In a classical theorem in group theory, Camille Jordan stated in 1869 that any two composition series of the same finite group have the same number of quotients. Later, in 1889, Otto Hölder reinforced this result by proving the theorem known as the Jordan-Hölder-Schreier theorem, which states that any two composition series of a given group are equivalent, that is, they have the same length and the same factors, up to permutation and isomorphism. This theorem has been generalised to many other contexts, such as operator groups, modules over rings or general abelian categories.

Definition (Definition IV.3.1)**.** Let $(\mathcal{A}, \mathcal{E})$ be an exact category. A finite $\mathcal{E}$-composition series for an object X of $\mathcal{A}$ is a sequence

$$0 = X_0 \overset{i_0}{\rightarrowtail} X_1 \overset{i_1}{\rightarrowtail} \dots \overset{i_{n-2}}{\rightarrowtail} X_{n-1} \overset{i_{n-1}}{\rightarrowtail} X_n = X$$

where all i_l are *proper admissible monics* with $\mathcal{E}$-simple cokernel. We say an exact category $(\mathcal{A}, \mathcal{E})$ has the *($\mathcal{E}$-)Jordan-Hölder property* or is a *Jordan-Hölder exact category* if any two finite $\mathcal{E}$-composition series of X are equivalent, that is, they have the same length and the same composition factors, up to permutation and isomorphism.

This is an interesting problem since the Jordan-Hölder property does not hold in general for any exact category, see [31, Example 6.9], [46] and Examples IV.3.2 and IV.3.9 for counter-examples. This problem is also studied by Enomoto in [46], using the Grothendieck monoid which is a lesser-known invariant of exact categories defined by the same universal property as the Grothendieck group. He shows that the relative Jordan-Hölder property holds if and only if the Grothendieck monoid of the exact category is free. We use our concepts of intersections and sums of subobjects to define classes of exact categories. Firstly, the Diamond exact categories, which are exact categories satisfying the *Diamond axiom* (Definition IV.3.6). These categories generalise abelian categories (Remark IV.3.2), and satisfy the relative Jordan-Hölder property (Theorem IV.3.8). Thus the Diamond axiom provides a sufficient condition for the Jordan-Hölder property to hold.

We also define an analog of the Jacobson radical for exact categories, the *$\mathcal{E}$-Jacobson radical*, $\mathrm{rad}_{\mathcal{E}}(X)$, as the generalised intersection of all maximal $\mathcal{E}$-subobjects of X and also introduce the notion of $\mathcal{E}$*-semisimple* objects (see Definitions IV.4.1 and IV.4.3). We show some basic properties of the $\mathcal{E}$-Jacobson radical motivated by the properties of the classical Jacobson radical. We then use this to introduce the $\mathcal{E}$*-Artin-Wedderburn categories*, which are exact categories where an analog of the classical Artin-Wedderburn theorem holds (Definition IV.4.4). We give examples of such categories and prove in Lemma IV.4.6, that every additive category with the minimal exact structure $\mathcal{E}_{min}$ in the lattice $(Ex(\mathcal{A}), \subseteq)$; the split exact structure, is an $\mathcal{E}$-Artin-Wedderburn category. Then, by showing that the Diamond axiom is satisfied, we see that Krull-Schmidt $\mathcal{E}$-Artin-Wedderburn categories are Jordan-Hölder (Theorem IV.4.7).

We then give for any Nakayama algebra, Λ, an explicit description of all exact structures on $\operatorname{rep}\Lambda$ in Theorem IV.4.8 and use this to characterise all Artin-Wedderburn exact structures on $\operatorname{rep}\Lambda$ in Theorem IV.4.9. It turns out these they are exactly the Jordan-Hölder exact structures on $\operatorname{rep}\Lambda$ (Theorem IV.4.10).

Once satisfied, the $\mathcal{E}$-Jordan-Hölder property allows to define *the* $\mathcal{E}$*-Jordan-Hölder length function* (compare also [46, §4.1]):

Definition (Definition IV.5.1)**.** The $\mathcal{E}$-Jordan-Hölder length $l_{\mathcal{E}}(X)$ of an object X in $\mathcal{A}$ is the length of an $\mathcal{E}$-composition series of X. That is $l_{\mathcal{E}}(X) = n$ if and only if there exists an $\mathcal{E}$-composition series

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots \rightarrowtail X_{n-1} \rightarrowtail X_n = X.$$

This $\mathcal{E}$-Jordan-Hölder length function has good properties (Propositions IV.5.5 and IV.5.7) that improves the general length defined and studied on any exact category in [31, Definition 6.1, Theorem 6.6]. For instance, it is additive along admissible short exact sequences (Corollary IV.5.1).

I.4 Aisles and co-aisles

As we noted above, the structures behind torsion pairs in abelian categories are quasi-abelian categories. In the final chapter, we seek to find a triangulated analog of this characterisation. The torsion pairs we are interested in for triangulated categories are t-structures and co-t-structures.

Definition. Let $\mathcal{T}$ be a triangulated category with shift functor Σ. A pair of additive subcategories $(\mathcal{U}, \mathcal{V})$ is a *torsion pair* in $\mathcal{T}$ if

(a) $\operatorname{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$;

(b) For all $A \in \mathcal{T}$ there exists a triangle

$$U \to A \to V \to \Sigma U$$

such that $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Additionally, $(\mathcal{U}, \mathcal{V})$, is a *t-structure* [18] (resp. *co-t-structure* [26, 101]) if $\Sigma\mathcal{U} \subseteq \mathcal{U}$ (resp. $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$).

t-structures play a central role in stability conditions [29] and tilting theory [57] on triangulated categories. Co-t-structures are a more recent development and have become very important in silting theory [3, 67] which is intimately connected to τ-tilting theory [2]. We aim to characterise the aisles of the t-structures and the co-aisles of the co-t-structures. Both of these classes of subcategories are right triangulated categories so first we must understand this structure. To do this, we employ the use of extriangulated categories.

The data of a right triangulated category (or suspended category), first introduced in [76] consists of an additive category $\mathcal{R}$, an endofunctor $\Sigma : \mathcal{R} \to \mathcal{R}$ called 'the shift of $\mathcal{R}$' and a class of right triangles of the form $A \to B \to C \to \Sigma A$ subject to essentially the same axioms as the triangles of a triangulated category (see Definition V.1.1). Informally, a right triangulated category is a triangulated category whose shift functor is not necessarily an equivalence. Such categories (or their left-handed analogues) have been the subject of study in many articles [12, 21, 74, 83, 86, 87]. In [10], the class of 'right triangulated categories with right semi-equivalence' were introduced, these are the right triangulated categories whose shift functor is fully faithful and with image that is closed under extensions. Such right triangulated categories enjoy homological properties close to those of triangulated categories, we formalise this similarity by showing that a right triangulated category has the natural structure of an extriangulated category precisely when the shift functor is a right semi-equivalence (Corollary V.2.15). Moreover, we are able to characterise which extriangulated categories have a natural right triangulated structure (Theorem V.2.14).

To prove this, we use an extriangulated generalisation of the constructions of a right triangulated quotient category from a contravariantly finite subcategory (of an additive category) due to [22] and [10] (see Proposition V.2.8). We also use ideas from relative homological algebra to characterise which extriangulated structures give rise to right triangulated quotient categories (with right semi-equivalence) (see Proposition V.2.9).

The construction of these right triangulated structures can be thought of as a 'one-sided' analogue of the triangulated structure of the stable category of a Frobenius exact (or even extriangulated) category [56, 63]. See [20, 69] for similar constructions.

We may now turn our attention to aisles and co-aisles. To every right triangulated category $\mathcal{R}$ there is an associated triangulated category: the stabilisation $\mathcal{S}(\mathcal{R})$ (see Section V.1.1 for details and construction). In the case where $\mathcal{R}$ has a right semi-equivalence, $\mathcal{S}(\mathcal{R})$ can be thought of as the smallest triangulated category containing $\mathcal{R}$ as a subcategory. Furthermore, we show in Lemma V.3.3 that if $\mathcal{R}$ is a (co-)aisle in a triangulated category then it is also a (co-)aisle in $\mathcal{S}(\mathcal{R})$. Thus we look to characterise when $\mathcal{R}$ is a (co-)aisle in $\mathcal{S}(\mathcal{R})$.

We give an intrinsic characterisations of when a right triangulated category with right semi-equivalence, $\mathcal{R}$, appears as the co-aisle of a co-t-structure in terms of internal torsion pairs of $\mathcal{R}$ and homological properties (Theorem V.3.5). As a direct consequence, we obtain that silting subcategories of triangulated categories correspond precisely to bounded right triangulated categories with right semi-equivalence that have enough projectives (Corollary V.3.6). This adds to the interpretations of silting subcategories in a triangulated category, which are surveyed in [6].

For the case of t-structures, there are related works [4, 77, 84] that give various characterisations of aisles. We note that our approach differs in the sense that we look to give characterisations intrinsic to the right triangulated category, that is, the aisle, rather than properties of the aisle related to the ambient triangulated category which we do in (Theorem V.3.10).

In [115, Proposition 3.9], it was shown that t-structures in an algebraic triangulated category correspond bijectively to certain complete cotorsion pairs in the associated Frobenius exact category. Additionally, it was observed in [94, Proposition 2.6] that t-structures in a triangulated category are precisely cotorsion pairs satisfying a shift closure property. We add to this picture by showing that in the case of Frobenius extriangulated categories, aisles of t-structures in the triangulated stable category may be constructed as (shifts of) right triangulated quotients (Theorem V.4.1).

I.5 Outline

In Chapter II we recall and discuss the background concepts that provide the setting for our work. Some other chapters also contain preliminaries that are not relevant for the whole thesis. Chapter III is based on [123] and corresponds to the introduction Section I.1. Chapter IV is based on [32] and corresponds to the sections I.2 and I.3 of the introduction. Lastly, Chapter V goes with the introductory section I.4 and the material can mostly be found in [122]. A more detailed breakdown is given at the beginning of each chapter. We have chosen to order the contents in this way as it is more or less chronological.

I.6 Conventions

Throughout the document categories are assumed to be essentially small and additive (though we often restate this) and subcategories are assumed to be full and closed under isomorphisms.

Chapter II

Preliminaries

In this section we introduce the relevant definitions and concepts that will form the backdrop to our work. We begin by discussing additional structures on additive categories, namely extriangulations, exact structures and idempotent conditions. We then discuss some concepts from relative homological algebra. Most of this Chapter is unoriginal, with the exceptions being short Lemmas.

II.1 Additive categories with addtional structure

Additive categories are categories with finite biproducts that are enriched over the category of abelian groups, Ab. Such categories are ubiquitous throughout algebra. This is not surprising, since they are categories where one may 'add' objects or morphisms together (in finite amounts, at least,) and the categories of the more elementary algebraic structures, (groups, matrices, modules...) are all naturally additive categories. Many of the above examples actually have a richer structure, they are naturally k-categories for a field k, that is, they are enriched over the category of k-modules.

In the study of homological algebra, one often looks to study long exact sequences of functors/ modules / groups. To do this in a functorial way, extra data is required, for example an extriangulation or an exact structure, which we discuss next.

II.1.1 Extriangulated categories

In this section we put the background on extriangulated categories based on [95, Section 2] where such categories were introduced. Let $\mathcal{A}$ be an additive category and $\mathbb{E} : \mathcal{A}^{op} \times \mathcal{A} \to \mathsf{Ab}$ be an additive bifunctor.

Definition II.1.1. For any $A, C \in \mathcal{A}$, two pairs of composable morphisms in $\mathcal{A}$

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A \xrightarrow{x'} B' \xrightarrow{y'} C$$

are *equivalent* if there exists an isomorphism $b : B \to B'$ such that

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & \cong \downarrow b & & \| \\ A & \xrightarrow[x']{} & B' & \xrightarrow[y']{} & C \end{array}$$

commutes. We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$, and by $\mathcal{S}(C, A)$ we denote the class of all such equivalence classes.

Notation II.1.2. For $a : A \to A'$ write $a_* = \mathbb{E}(-, a) : \mathbb{E}(-, A) \to \mathbb{E}(-, A')$. Similarly, we write $c^* = \mathbb{E}(c, -) : \mathbb{E}(C', -) \to \mathbb{E}(C, -)$ for $c : C \to C'$.

Definition II.1.3. [95, Definition 2.6] Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$.. Then $\delta \oplus \delta$ is the unique element, η, of $\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A')$ satisfying

$$(p_A)_*(j_C)^*\eta = \delta, \quad (p_{A'})_*(j_C)^*\eta = 0, \quad (p_A)_*(j_{C'})^*\eta = 0, \text{ and } (p_{A'})_*(j_{C'})^*\eta = \delta'$$

where $C \xrightarrow{j_C} C \oplus C' \xleftarrow{j_{C'}} C'$ and $A \xleftarrow{p_A} A \oplus A' \xrightarrow{p_{A'}} A'$ are a co-product and product in $\mathcal{A}$ respectively.

Definition II.1.4 ([95, Definitions 2.9, 2.10])**.** An assignment $\mathfrak{s}_{C,A} : \mathbb{E}(C, A) \to \mathcal{S}(C, A)$ for all $C, A \in \mathcal{A}$ is an *additive realisation of* $\mathbb{E}$ if it satisfies the following axioms.

(S1) $\mathfrak{s}(0) = [A \to A \oplus C \to C]$.

(S2) $\mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta') = \mathfrak{s}(\delta \oplus \delta') \in \mathcal{S}(C \oplus C', A \oplus A')$ for $\delta \in \mathbb{E}(C, A)$, $\delta' \in \mathbb{E}(C', A')$.

(S3) For all $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $\mathfrak{s}(\delta) = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$ such that $a_*\delta = c^*\delta'$ for $a : A \to A'$ and $c : C \to C'$ (that is, $(a, c) : \delta \to \delta'$ is a *morphism of* $\mathbb{E}$*-extensions*). There is a commutative diagram in $\mathcal{A}$

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \exists b} & & \downarrow{\scriptstyle c} \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

and in this case we say that the triple (a, b, c) *realises the morphism of* $\mathbb{E}$*-extensions* $(a, c) : \delta \to \delta'$.

Notation II.1.5. If $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ we may also write

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

and call this an *extriangle*. In this situation we also call x an $\mathbb{E}$*-inflation* and y an $\mathbb{E}$*-deflation*. We may also say that $y : B \to C$ is the *cone of* x and $x : A \to B$ is the *co-cone of* y.

Definition II.1.6 ([95, Definition 2.12])**.** A triple $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ is an *extriangulated category* if it satisfies the following conditions. In this case we call the pair $(\mathbb{E}, \mathfrak{s})$ an *external triangulation of* $\mathcal{A}$ and $\mathfrak{s}$ an $\mathbb{E}$*-triangulation of* $\mathcal{A}$.

(ET1) $\mathbb{E} : \mathcal{A}^{op} \times \mathcal{A} \to \mathsf{Ab}$ is an additive bifunctor.

(ET2) $\mathfrak{s}$ is an additive realisation of $\mathbb{E}$.

(ET3) For all commutative diagrams with extriangles as rows

$$\begin{array}{cccccc} A & \longrightarrow & B & \longrightarrow & C & \overset{\delta}{\dashrightarrow} \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \exists c} & \\ A' & \longrightarrow & B' & \longrightarrow & C' & \overset{\delta'}{\dashrightarrow} \end{array}$$

there exists $c : C \to C'$ making the diagram commute and such that $(a, c) : \delta \to \delta'$ is a morphism of extriangles.

(ET3)$^{\mathrm{op}}$ Dual to (ET3).

(ET4) For any pair of extriangles of the form

$$A \xrightarrow{f} B \xrightarrow{f'} D \overset{\delta}{\dashrightarrow} \quad , \qquad B \xrightarrow{g} C \xrightarrow{g'} F \overset{\delta'}{\dashrightarrow}$$

there exists a commutative diagram

$$\begin{array}{cccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D & \overset{\delta}{\dashrightarrow} \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & \\ A & \xrightarrow{gf} & C & \xrightarrow{e} & E & \overset{\delta''}{\dashrightarrow} \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h'} & \\ & & F & = & F & \\ & & \vdots{\scriptstyle \delta'} & & \vdots{\scriptstyle (f')_*\delta'} & \end{array}$$

with columns and rows being extriangles such that $h^*\delta'' = \delta$ and $f_*\delta'' = (h')^*\delta'$ (that is, $(f, h') : \delta'' \to \delta'$ is a morphism of extriangles).

(ET4)$^{\mathrm{op}}$ Dual to (ET4).

For the rest of this section, let $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

Remark II.1.7 ([95, Remark 2.18]). Let $\mathcal{X} \subseteq \mathcal{A}$ be a subcategory of $\mathcal{A}$ that is closed under $\mathbb{E}$-extensions, that is, for all extriangles

$$A \longrightarrow B \longrightarrow C \dashrightarrow$$

the implication $A, C \in \mathcal{X} \Rightarrow B \in \mathcal{X}$ holds. Then $(\mathcal{X}, \mathbb{E}|_{\mathcal{X}}, \mathfrak{s}|_{\mathcal{X}})$ is an extriangulated category.

Lemma II.1.8 ([95, Corollary 3.12])**.** *Let* $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ *be an extriangle. Then there are long exact sequences*

$$\mathrm{Hom}_{\mathcal{A}}(-, A) \xrightarrow{x\circ-} \mathrm{Hom}_{\mathcal{A}}(-, B) \xrightarrow{y\circ-} \mathrm{Hom}_{\mathcal{A}}(-, C) \xrightarrow{\delta_\#} \mathbb{E}(-, A) \xrightarrow{\mathbb{E}(-,x)}$$
$$\xrightarrow{\mathbb{E}(-,x)} \mathbb{E}(-, B) \xrightarrow{\mathbb{E}(-,y)} \mathbb{E}(-, C),$$

$$\mathrm{Hom}_{\mathcal{A}}(C, -) \xrightarrow{-\circ y} \mathrm{Hom}_{\mathcal{A}}(-, B) \xrightarrow{-\circ x} \mathrm{Hom}_{\mathcal{A}}(-, A) \xrightarrow{\delta^\#} \mathbb{E}(C, -) \xrightarrow{\mathbb{E}(y,-)}$$
$$\xrightarrow{\mathbb{E}(y,-)} \mathbb{E}(B, -) \xrightarrow{\mathbb{E}(x,-)} \mathbb{E}(A, -)$$

of functors and natural transformation in the functor categories $[\mathcal{A}^{op}, \mathsf{Ab}]$ *and* $[\mathcal{A}, \mathsf{Ab}]$ *respectively. The maps* $\delta_\#$ *and* $\delta^\#$ *are given at* $X \in \mathcal{A}$ *by*

$$(\delta_\#)_X : \operatorname{Hom}_{\mathcal{A}}(X, C) \to \mathbb{E}(X, A), \qquad (\delta^\#)_X : \operatorname{Hom}_{\mathcal{A}}(A, X) \to \mathbb{E}(C, X)$$
$$f \mapsto f^*\delta \qquad\qquad\qquad\qquad g \mapsto g_*\delta.$$

Lemma II.1.9 ([88, Proposition 1.20])**.** *Let* $\delta \in \mathbb{E}(C, A)$ *be an extriangle and* $a \in \operatorname{Hom}_{\mathcal{A}}(A, A')$ *a morphism. Then there exists a triple of morphisms* $(a, b, 1)$ *realising the morphism of extriangles* $(a, 1) : \delta \to a_*\delta$

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C \arrow[r, dashed, "a_*\delta"] & {} \end{tikzcd} , \tag{II.1}$$

and such that the sequence $A \xrightarrow{\left[\begin{smallmatrix} -a \\ x \end{smallmatrix}\right]} A' \oplus B \xrightarrow{[\, y \; b \,]} C \overset{y'^*\delta}{\dashrightarrow}$ *is an extriangle.*

Definition II.1.10. A commutative square

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "f"'] & B \arrow[d, "f'"] \\ A' \arrow[r, "x'"'] & B' \end{tikzcd}$$

in $\mathcal{A}$ is a *weak pushout* if for all pairs of morphisms $g : A' \to C$, $y : B \to C$ such that $gf = yx$, there exists a (not necessarily unique) morphism $h : B' \to C$ such that $hf' = y$ and $hx' = g$:

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "f"'] & B \arrow[d, "f'"] \arrow[ddr, bend left, "y"] & \\ A' \arrow[r, "x'"'] \arrow[drr, bend right, "g"'] & B' \arrow[dr, dashed, "\exists h"] & \\ & & C. \end{tikzcd}$$

Lemma II.1.11. *The left hand square in Diagram II.1 is a weak pushout and weak pullback.*

Proof. By Lemma II.1.8, every $\mathbb{E}$-deflation of an extriangle is a weak cokernel of the $\mathbb{E}$-inflation; that is, every every morphism that post-composes with the $\mathbb{E}$-inflation to give the zero morphism factors, not necessarily uniquely, through the $\mathbb{E}$-deflation. Similarly, every $\mathbb{E}$-inflation of an extriangle is a weak kernel of the $\mathbb{E}$-deflation. The claim now follows from Lemma II.1.9, since the weak pushout property is equivalent to $[\, y \; b \,] : A' \oplus B \to C$ being a weak cokernel of $\left[\begin{smallmatrix} -a \\ x \end{smallmatrix}\right] : A \to A' \oplus B$. □

Lemma II.1.12 ([95, Corollary 3.5])**.** *Let*

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd}$$

be a morphism of extriangles. Then the following are equivalent

(i) *a factors through x;*

(ii) $a_*\delta = c^*\delta' = 0$*;*

(iii) *c factors through y'.*

Definition II.1.13. Let $\mathcal{A} = (\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. An object $P \in \mathcal{A}$ is $\mathbb{E}$-*projective* if $\mathbb{E}(P, C) = 0$ for all $C \in C$. By $\mathrm{Proj}_{\mathbb{E}}\mathcal{A}$ we denote the subcategory of $\mathbb{E}$-projective objects. We say that $\mathcal{A}$ has *enough* $\mathbb{E}$*-projectives* if for all $C \in \mathcal{A}$ there exists an extriangle

$$A \longrightarrow P \longrightarrow C \dashrightarrow$$

with $P \in \mathrm{Proj}_{\mathbb{E}}\mathcal{A}$. The notions of $\mathbb{E}$-injectives and having enough $\mathbb{E}$-injectives are defined dually. The subcategory of $\mathbb{E}$-injective objects is denoted by $\mathrm{Inj}_{\mathbb{E}}\mathcal{A}$. An object $Q \in \mathrm{Proj}_{\mathbb{E}}\mathcal{A}$ is an $\mathbb{E}$*-projective generator* if for all $A \in \mathcal{A}$ there exists an $\mathbb{E}$-deflation $Q^I \to X$ for some set $\mathcal{I}$. When the external triangulation is implicit, we drop the subscript $\mathbb{E}$.

Definition II.1.14. For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{A}$, by $\mathcal{X} * \mathcal{Y}$ we denote the subcategory of $\mathcal{A}$ consisting of objects $A \in \mathcal{A}$ for which there exists an extriangle

$$X \longrightarrow A \longrightarrow Y \dashrightarrow$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Remark II.1.15. It follows immediately from the definition that if $0 \in \mathcal{X}$ then $\mathcal{Y} \subseteq \mathcal{X} * \mathcal{Y}$ and dually, if $0 \in \mathcal{Y}$ then $\mathcal{X} \subseteq \mathcal{X} * \mathcal{Y}$.

Lemma II.1.16. *Let $\mathcal{A} = (\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Consider a commutative diagram of extriangles*

$$\begin{array}{cccccc}
A & \longrightarrow & B & \longrightarrow & C & \overset{\delta}{\dashrightarrow} \\
\downarrow{\scriptstyle i_A} & & \downarrow & & \downarrow{\scriptstyle j_C} & \\
A \oplus A' & \longrightarrow & B \oplus B' & \longrightarrow & C \oplus C' & \overset{\eta}{\dashrightarrow} \\
\downarrow{\scriptstyle p_{A'}} & & \downarrow & & \downarrow{\scriptstyle q_{C'}} & \\
A' & \longrightarrow & B' & \longrightarrow & C' & \overset{\delta'}{\dashrightarrow} \\
\downarrow{\scriptstyle \varepsilon_A} & & \downarrow{\scriptstyle \varepsilon_B} & & \downarrow{\scriptstyle \varepsilon_C} &
\end{array}$$

where $\varepsilon_A, \varepsilon_B, \varepsilon_C$ are all split extriangles and the upper (resp. lower) sets of vertical maps are morphisms of extriangles $(i_A, j_C) : \delta \to \eta$ (resp. $(p_{A'}, q_{C'}) : \eta \to \delta'$). Then $\eta = \delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$.

Proof. We verify that η satisfies the conditions of Definition II.1.3. Using the upper morphism of extriangles we have that

$$(p_A)_*(j_C)^*\eta = (p_A)_*(i_A)_*\delta = (1_A)_*\delta = \delta$$

where p_A is the retraction of the section i_A; and

$$(p_{A'})_*(j_C)^*\eta = (p_{A'})_*(i_A)_*\delta = 0_*\delta = 0.$$

The other properties are checked similarly using the lower morphism of extriangles. □

The motivating examples of extriangulated categories, as we have hinted, are exact categories and triangulated categories. In the next section, we look at exact categories, and specific classes thereof, in more detail. We do not treat triangulated categories on their own, but rather through the more general lens of right triangulated categories in Chapter V.

II.1.2 Exact categories

Let us first recall the full definition of a exact category, which were introduced by Quillen in [107].

Definition II.1.17. Let $\mathcal{A}$ be an additive category. A *kernel-cokernel* or *short exact sequence* in $\mathcal{A}$ is a pair of composable morphisms, (i, d), such that i is kernel of d and d is cokernel of i. Let $\mathcal{E}$ be a class of kernel-cokernel pairs that is closed under isomorphism in $\mathcal{A}$. The pair $(\mathcal{A}, \mathcal{E})$ is an *exact category* and $\mathcal{E}$ *is an exact structure on* $\mathcal{A}$ if the axioms (Ex0), (Ex1),(Ex1)^{op}, (Ex2) and (Ex2)^{op} hold.

- (Ex0) For all objects $A \in \operatorname{Ob}\mathcal{A}$, we have that $(1_A, 0), (0, 1_A) \in \mathcal{E}$.
- (Ex1) The class of kernels in $\mathcal{E}$ is closed under composition. That is, if i and i' are kernels in $\mathcal{E}$ (that is, $(i, \operatorname{Coker} i), (i', \operatorname{Coker} i') \in \mathcal{E}$) the the composition $i'i$, if it is well-defined, is also a kernel in $\mathcal{E}$.
- (Ex1)$^{\text{op}}$ The class of cokernels in $\mathcal{E}$ is closed under composition.
- (Ex2) The class of kernels in $\mathcal{E}$ is closed under pushout along arbitrary morphisms. That is, the pushout of a kernel $i : A \to B$ in $\mathcal{E}$ along an arbitrary morphism $f : A \to C$ exists

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & B \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow{\scriptstyle g} \\ C & \xrightarrow[\;j\;]{} & D, \end{array}$$

  and yields a kernel j in $\mathcal{E}$.
- (Ex2)$^{\text{op}}$ The class of cokernels in $\mathcal{E}$ is closed under pullback along arbitrary morphisms.

In this case we call the pairs (i, d) in $\mathcal{E}$ *conflations* or *admissible sequences*. The kernels i in $\mathcal{E}$ are referred to as *inflations* or *admissible monomorphisms* and are depicted as $\rightarrowtail$; the cokernels d in $\mathcal{E}$ are referred to as *deflations* or *admissible epimorphisms* and are depicted as $\twoheadrightarrow$.

Remark II.1.18. Let us explain the connection of exact categories to extriangulated categories following [95, Example 2.13, Corollary 3.18]. Let $(\mathcal{A}, \mathcal{E})$ be an exact category and set $\mathbb{E}(C, A)$ to be collection of isomorphism classes of admissible sequences starting at A and ending at C. Let us assume that $\mathbb{E}(C, A)$ is a set for all $A, C \in \mathcal{A}$. Then, it follows from the existence of pushouts and pullbacks that $\mathbb{E}(-, ?)$ is functorial in both arguments and the Baer sum provides $\mathbb{E}(C, A)$ with the structure of an abelian group. We also set $\mathfrak{s} : \mathbb{E}(C, A) \to \mathcal{S}(C, A)$ to be the identity for all $A, C \in \mathcal{A}$, then it follows that $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category.

Conversely, an extriangulated category, $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$, gives rise to an exact structure $\mathcal{E} = \bigcup_{A,C\in\mathcal{A}} \mathfrak{s}\mathbb{E}(C, A)$ on $\mathcal{A}$ if and only if each $\mathbb{E}$-inflation is a monomorphism and each $\mathbb{E}$-deflation is an epimorphism.

In this work, we often work with specific classes of additive categories.

Definition II.1.19. Let $\mathcal{A}$ be an additive category.

(a) [34, §5.4] $\mathcal{A}$ is *pre-abelian* if every morphism in $\mathcal{A}$ admits a kernel and a cokernel.

(b) [112, p168] $\mathcal{A}$ is *quasi-abelian* (or *almost abelian*) if it is pre-abelian and if the pullback (resp. pushout) of any cokernel (resp. kernel) in $\mathcal{A}$ along an arbitrary morphism is again a cokernel (resp. kernel)

In pre-abelian categories, pushout and pullbacks always exist, which will be a useful property in Chapter IV. It is well-known that the class of all split short exact sequences, $\mathcal{E}_{min}$, in an additive category $\mathcal{A}$ is an exact structure and is minimal (that is, $\mathcal{E}_{min}$ is contained in every other exact structure on $\mathcal{A}$). For pre-abelian categories, a maximal exact structure also always exists and can be described.

Proposition II.1.20. *Let $\mathcal{A}$ be a pre-abelian category and let $\mathcal{E}_{max}$ be the class of all short exact sequences in $\mathcal{A}$ satisfying the following:*

(i) *The pushout of any kernel in $\mathcal{E}_{max}$ along an arbitrary morphism is again a kernel in $\mathcal{A}$;*

(ii) *The pullback of any cokernel in $\mathcal{E}_{max}$ along an arbitrary morphism is again a cokernel in $\mathcal{A}$.*

Then $\mathcal{E}_{max}$ is an exact structure on $\mathcal{A}$ and contains every other exact structure on $\mathcal{A}$. Furthermore, $\mathcal{A}$ is quasi-abelian if and only if $\mathcal{E}_{max}$ coincides with the class, $\mathcal{E}_{all}$ of all short exact sequences in $\mathcal{A}$.

Proof. The statement for pre-abelian categories is [119, Theorem 3.3]. The second is a manipulation of the definition of a quasi-abelian category. □

Remark II.1.21. In the sequel, when we make reference to a quasi-abelian or abelian category, $\mathcal{A}$, without specifying an exact structure, it is implicit that we are working with the exact category $(\mathcal{A}, \mathcal{E}_{all})$.

Note that certain properties of the underlying additive category $\mathcal{A}$ determine which exact structures can exist on $\mathcal{A}$. See [31, Section 2] for a summary on the minimal and maximal exact structures on any additive category. Moreover, under some finiteness conditions, the exact structures on $\mathcal{A}$ are parametrized by subsets of Auslander-Reiten sequences. This phenomenon was observed in [31, Theorem 5.7], and is based on [47]:

Theorem II.1.22. *Let $\mathcal{A}$ be a skeletally small, Hom-finite, idempotent complete additive category which has finitely many indecomposable objects up to isomorphism. Then every exact structure $\mathcal{E}$ on $\mathcal{A}$ is uniquely determined by the set $\mathcal{B}$ of Auslander-Reiten sequences that are contained in $\mathcal{E}$. We write in this case $\mathcal{E} = \mathcal{E}(\mathcal{B})$.*

The poset of $\mathcal{E}$-subobjects

When working in an exact category $(\mathcal{A}, \mathcal{E})$, one looks to study subobjects relative to the exact structure $\mathcal{E}$. Here we recall the important definitions of classes of these '$\mathcal{E}$-subobjects'.

Definition II.1.1. Let A and B be objects of an exact category $(\mathcal{A}, \mathcal{E})$. If there is an admissible monic $i : A \rightarrowtail B$ we say the pair (A, i) is an *admissible subobject* or *$\mathcal{E}$-subobject of B*. Often we will refer to the pair (A, i) by the object A. If i is not an isomorphism and $A \ncong 0$ we say that (A, i) is a *proper* admissible subobject of B.

Definition II.1.2 ([31, Definition 3.3])**.** A non-zero object S in $(\mathcal{A}, \mathcal{E})$ is *$\mathcal{E}$-simple* if S admits no $\mathcal{E}$-subobjects except 0 and S, that is, whenever $A \rightarrowtail S$, then A is the zero object or isomorphic to S.

Remark II.1.23. Let A be an $\mathcal{E}$-subobject of B given by the monic $i : A \rightarrowtail B$. We denote by B/A the cokernel of i, thus we denote the corresponding admissible sequence as

$$A \overset{i}{\rightarrowtail} B \twoheadrightarrow B/A$$

Remark II.1.24. An admissible monic $i : A \rightarrowtail B$ is proper precisely when its cokernel is not an isomorphism and is non-zero.

Definition II.1.3 ([31, Section 6.1])**.** An object X of $(\mathcal{A}, \mathcal{E})$ is *$\mathcal{E}$-Noetherian* if any increasing sequence of $\mathcal{E}$-subobjects of X

$$X_1 \rightarrowtail X_2 \rightarrowtail \dots \rightarrowtail X_{n-1} \rightarrowtail X_n \rightarrowtail X_{n+1} \dots$$

becomes stationary. Dually, an object X of $(\mathcal{A}, \mathcal{E})$ is *$\mathcal{E}$-Artinian* if any descending sequence of $\mathcal{E}$-subobjects of X

$$\dots X_{n+1} \rightarrowtail X_n \rightarrowtail X_{n-1} \rightarrowtail \dots \rightarrowtail X_2 \rightarrowtail X_1$$

becomes stationary. An object X which is both $\mathcal{E}$-Noetherian and $\mathcal{E}$-Artinian is called *$\mathcal{E}$-finite*. The exact category $(\mathcal{A}, \mathcal{E})$ is called *$\mathcal{E}$-Artinian* (respectively *$\mathcal{E}$-Noetherian*, *$\mathcal{E}$-finite*) if every object is $\mathcal{E}$-Artinian (respectively $\mathcal{E}$-Noetherian, $\mathcal{E}$-finite).

Now let us recall a definition similar to [46, Definition 2.1]:

Definition II.1.4. Two $\mathcal{E}$-subobjects ($Y_i \overset{f_i}{\rightarrowtail} X$) for $i = 0, 1$ are *isomorphic $\mathcal{E}$-subobjects of* X if there exists an isomorphism $\phi \in \mathrm{Hom}_{\mathcal{A}}(Y_0, Y_1)$ such that $f_0 = f_1 \circ \phi$. We denote by $\mathcal{P}_X^{\mathcal{E}}$ the set of isomorphism classes of $\mathcal{E}$-subobjects of X. The relation

$$(Y, f) \leq (Z, g) \iff \exists \quad \begin{array}{ccc} Y & \overset{\exists h}{\rightarrowtail} & Z \\ & \underset{f}{\searrow} & \downarrow g \\ & & X \end{array}$$

turns $(\mathcal{P}_X^{\mathcal{E}}, \leq)$ into a poset. Sometimes, to avoid clutter, we will drop the superscript $\mathcal{E}$ and write $\mathcal{P}_X$. By $\mathcal{S}_X$ we denote the set of isomorphism classes of proper $\mathcal{E}$-subobjects of X, thus $\mathcal{P}_X^{\mathcal{E}} = \mathcal{S}_X \cup \{0\} \cup \{X\}$ and $\mathcal{S}_X$ inherits a poset structure from $\mathcal{P}_X$.

Remark II.1.25. An $\mathcal{E}$-subobject (Y, f) of X is a maximal element of $\mathcal{P}_X^{\mathcal{E}}$ if and only if Coker f is $\mathcal{E}$-simple. For a poset $(P, \leq)$, by $\mathrm{Max}(P)$ we denote the maximal elements of the poset. Thus $\mathrm{Max}(\mathcal{S}_X)$ is the class of *maximal $\mathcal{E}$-subobjects* of X.

II.1.3 Idempotent completness

Here we discuss the property of splitting idempotents in an additive category which is strongly related to the existence of direct summands and see how these concepts fit into the extriangulated setting. Until Lemma II.1.40, everything is well-known, we tend to include short proofs for completeness and convenience.

(Weak) idempotent completeness

Definition II.1.26. An additive category, $\mathcal{A}$, is *idempotent complete* if for every idempotent morphism $e : B \to B$ there exists a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{s} & B & & \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle r} & \searrow{\scriptstyle e} & \\ & & A & \xrightarrow{s} & B \end{array}$$

which we refer to as the *splitting of* e.

Remark II.1.27. In a splitting of an idempotent e as above, r is a retraction and s is a section.

Lemma II.1.28. *Let $\mathcal{A}$ be an additive category, then the following are equivalent.*

(i) *$\mathcal{A}$ is idempotent complete;*

(ii) *Every idempotent morphism in $\mathcal{A}$ admits a kernel;*

(iii) *Every idempotent morphism in $\mathcal{A}$ admits a cokernel.*

Proof. We prove the equivalence of (i) and (ii) whence the remaining equivalences will follow by duality. Firstly, let us show that (i) implies (ii). Let $e : B \to B$ be idempotent, then $(1 - e)$ is also idempotent and admits a splitting

$$\begin{array}{ccccc} A & \xrightarrow{s} & B & & \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle r} & \searrow{\scriptstyle 1-e} & \\ & & A & \xrightarrow{s} & B. \end{array}$$

We claim that s is a kernel of e. Indeed, we have that $(1-e)s = srs = s$ from which we deduce that $es = 0$. Now, let $w : W \to B$ be such that $ew = 0$; then we have that $w = (1 - e)w = srw$ and we are done.

For the converse, let e be idempotent and let $s : A \to B$ be kernel of the idempotent $(1-e)$. Then, as $(1-e)e = 0$ there exists a unique morphism $r : B \to A$ such that $e = sr$. Observe that $(1 - e)s = 0$ gives us that $es = s$. Therefore we have that $srs = s$ and since s is monic, we conclude that $rs = 1_A$ and we are done. □

Definition II.1.29. An additive category, $\mathcal{A}$, is *weakly idempotent complete* if every retraction in $\mathcal{A}$ admits a kernel.

As the terminology suggests, being weakly idempotent complete is a more general than being idempotent complete.

Lemma II.1.30. *Suppose an additive category $\mathcal{A}$ is idempotent complete, then $\mathcal{A}$ is weakly idempotent complete.*

Proof. Suppose that $\mathcal{A}$ is idempotent complete and let $r : B \to C$ be a retraction and $s : C \to B$ be such that $rs = 1_C$. Then, sr is idempotent. Let k be a kernel of sr then it is straightforward to verify that k is also a kernel of r. □

Lemma II.1.31. *Let $\mathcal{A}$ be weakly idempotent complete additive category and let $r : B \to C$ be a retraction in $\mathcal{A}$. Then $B \cong K \oplus C$ where $k : K \to B$ is a kernel of r.*

Proof. This is essentially [35, Remark 7.4]. We repeat the argument here for convenience.

Let $s : C \to B$ be such that $1_C = rs$. We have that $r(1_B - sr) = r - rsr = 0$. Thus there exists a unique morphism $t : B \to K$ such that $kt = 1_B - sr$. We make two observations: Firstly, that k is a section since $ktk = (1_B - sr)k = k - srk = k$ and k is monic. Secondly, that $ts = 0$ since $kts = (1_B - sr)s = 0$ and k is monic. From these two observations, it follows that the morphisms $\left[\begin{smallmatrix} t \\ r \end{smallmatrix}\right] : B \to K \oplus C$ and $\left[\begin{smallmatrix} k & s \end{smallmatrix}\right] : K \oplus C \to B$ are mutually inverse isomorphisms. □

In other words, a weakly idempotent category is one that contains all the direct summands of its objects.

Definition II.1.32. For a subcategory $\mathcal{X}$ of an additive category $\mathcal{A}$ by $\operatorname{add}(\mathcal{X})$ (resp. $\operatorname{Add}(\mathcal{X})$) we denote the subcategory consisting of all direct summands (that exist in $\mathcal{A}$) of finite (resp. infinite) direct sums of objects in $\mathcal{X}$ (that exist in $\mathcal{A}$). We also call the subcategory $\mathcal{X}$ *additive* if $\mathcal{X} = \operatorname{Add}(\mathcal{X})$.

Lemma II.1.33. *Let $\mathcal{A}$ be an additive category, then the following are equivalent*

(i) *$\mathcal{A}$ is weakly idempotent complete;*

(ii) *Every section in $\mathcal{A}$ admits a cokernel.*

Proof. The equivalence of (i) and (ii) is [35, Lemma 7.1] and we recall the argument. By duality, it is enough to show that (i) implies (ii). To this end, let $s : C \to B$ be a section and let $r : B \to C$ be such that $1_C = rs$. Then r is a retraction and by assumption admits a kernel $k : K \to B$. We claim that, in the notation of the proof of Lemma II.1.31, that $t : B \to K$ is a cokernel of s. We have already seen that $ts = 0$ and that t is a retraction, in particular an epimorphism. It remains to verify that t is a weak cokernel of s: let $w : B \to W$ be a morphism such that $ws = 0$, then $wkt = w(1_B - sr) = w - wsr = w$ and we are done. □

Remark II.1.34. Every additive category, $\mathcal{A}$, admits an *idempotent (resp. weak idempotent) completion*. That is, there exists an idempotent (resp. weakly idempotent) complete category $\mathcal{B}$ and a fully faithful additive functor $F : \mathcal{A} \to \mathcal{B}$ with the following universal property: Every additive functor $G : \mathcal{A} \to \mathcal{C}$ with $\mathcal{C}$ being idempotent (resp. weakly idempotent) complete factors uniquely through $\mathcal{B}$ via F. For constructions and more discussion see [35, §6,7], [73, 1.2], and [96, Remark 1.12]. We note that, without a smallness condition on $\mathcal{A}$, the weak idempotent completion of $\mathcal{A}$ may not be defined see [27, pg. 373ff].

Krull-Schmidt categories

Definition II.1.35. An additive category $\mathcal{A}$ is *Krull-Schmidt* if each object decomposes into a finite direct sum of indecomposable objects having local endomorphism rings and that this decomposition is unique up to isomorphism and permutation of summands.

First we recall the following.

Definition II.1.36 ([82, Proposition 4.1])**.** A ring R is *semi-perfect* if it satisfies the following equivalent conditions

(a) proj$R$ is Krull-Schmidt;

(b) Every simple R-module admits a projective cover;

(c) Every finitely generated R-module admits a projective cover;

(d) As an R-module, R admits a decomposition $R = P_1 \oplus \cdots \oplus P_r$ such that the P_i each have a local endomorphism ring.

The Krull-Schmidt property is related to that of idempotent completeness.

Proposition II.1.37 ([82, Corollary 4.4])**.** *An additive category is Krull-Schmidt if and only if it is idempotent complete and the endomorphism ring of every object is semi-perfect.*

Extriangulated categories with (WIC)

Definition II.1.38 ([95, Condition 5.8])**.** Let $\mathcal{A} = (\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Then $\mathcal{A}$ satisfies the condition *(WIC)* if the following hold

(a) For all $f \in \mathcal{A}(A, B)$ and $g \in \mathcal{A}(B, C)$, if gf is an $\mathbb{E}$-inflation then so is f.

(b) For all $f \in \mathcal{A}(A, B)$ and $g \in \mathcal{A}(B, C)$, if gf is an $\mathbb{E}$-deflation then so is g.

Example II.1.39. There are two large classes of examples.

(a) Every triangulated category (viewed naturally as an extriangulated category) satisfies the condition (WIC). Indeed, every morphism in a triangulated category is an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation.

(b) An exact category $(\mathcal{A}, \mathcal{E})$ (viewed naturally as an extriangulated category) satisfies the condition (WIC) if and only if $\mathcal{A}$ is weakly idempotent complete [35, Proposition 7.6].

In general the condition (WIC) is not equivalent to the ambient additive category being (weakly) idempotent complete. For instance, there exist triangulated categories whose underlying additive category is not idempotent complete [97]. However, we may characterise weak idempotent completeness through the lens of extriangulated categories.

Lemma II.1.40. *Let $\mathcal{A} = (\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Then the following statements equivalent*

(i) *$\mathcal{A}$ is weakly idempotent complete;*

(ii) *Every retraction in $\mathcal{A}$ is an $\mathbb{E}$-deflation;*

(iii) *Every section in $\mathcal{A}$ is an $\mathbb{E}$-inflation.*

Proof. The fact that (i) implies (ii) follows from Lemma II.1.31. Indeed, using the notation of the proof, we have an equivalence of composable morphisms

$$\begin{array}{ccccc} K & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & K\oplus C & \xrightarrow{[0\ 1]} & C \\ \| & & \downarrow{\scriptstyle [k\ s]}\ \cong & & \| \\ K & \xrightarrow[k]{} & B & \xrightarrow[r]{} & C, \end{array}$$

and thus both rows lie in the same equivalence class of $\mathcal{S}(C, A)$. Our claim follows since the upper sequence is always an extriangle [NP Remark 2.11].

Now we show that (ii) implies (i). Let $r : B \to C$ be a retraction and suppose that there is an extriangle $A \xrightarrow{x} B \xrightarrow{r} C \dashrightarrow$. Let $s : C \to B$ be such that $rs = 1_C$. Then we apply (ET3) to the commutative diagram

$$\begin{array}{cccccc} A & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & A\oplus C & \xrightarrow{[0\ 1]} & C & \dashrightarrow \\ \| & & \downarrow{\scriptstyle [x\ s]} & & \exists \downarrow{\scriptstyle c} & \\ A & \xrightarrow[x]{} & B & \xrightarrow[r]{} & C & \dashrightarrow \end{array}$$

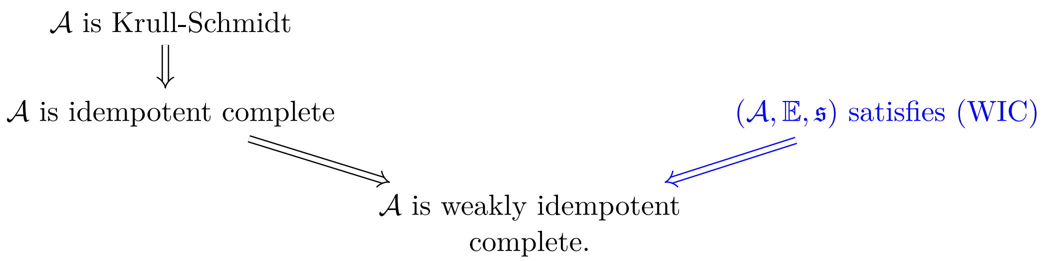
and by the commutativity of the right hand square we have that $[\,0\ c\,] = r[\,x\ s\,] = [\,0\ 1_C\,]$. Thus $c = 1_C$. It follows from [95, Corollary 3.6] that $[\,x\ s\,] : A \oplus C \to B$ is an isomorphism and we deduce that x is monic and therefore a kernel of r. The equivalence of (i) and (iii) follows from dual arguments. □

We have the following immediate consequence.

Corollary II.1.41. *Let $\mathcal{A} = (\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Suppose that $\mathcal{A}$ satisfies the condition (WIC), then $\mathcal{A}$ is weakly idempotent complete.*

Remark II.1.42. Let $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Then idempotent completion and weak idempotent completion of $\mathcal{A}$ admit a natural external triangulation [93, 126].

We summarise the relationships that we have discussed: For an additive category, $\mathcal{A}$, the following implications hold for an external triangulation $(\mathbb{E}, \mathfrak{s})$ of $\mathcal{A}$.



We finish this section with a technical lemma which is useful for detecting when an extriangle is decomposable.

Lemma II.1.43. *Let $\mathcal{A} = (\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Suppose that $\mathcal{A}$ is weakly idempotent complete, then any extriangle of the form*

$$A \xrightarrow{\left[\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix}\right]} B_0 \oplus B_1 \xrightarrow{[0\ b_1]} C \overset{\delta}{\dashrightarrow}$$

with $B_0 \ncong 0$ is decomposable. In particular, b_1 is an $\mathbb{E}$-deflation, $A \cong B_0 \oplus A'$ and $\delta \cong \delta_0 \oplus \delta_1$ where $\delta_0 = 0 \in \mathbb{E}(0, B_0)$ and $\delta_1 \in \mathbb{E}(C, A')$.

Proof. By (ET3)$^{\mathrm{op}}$ there is a morphism $x : B_0 \to A$ rendering the diagram

$$\begin{array}{ccccccc}
B_0 & \xrightarrow{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} & B_0 \oplus B_1 & \xrightarrow{[0\ 1]} & B_1 & \overset{\eta_0}{\dashrightarrow} & \\
{\scriptstyle \exists x}\downarrow & & \| & & \downarrow{\scriptstyle b_1} & & \\
A & \xrightarrow[\left[\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix}\right]]{} & B_0 \oplus B_1 & \xrightarrow[{[0\ b_1]}]{} & C & \overset{\delta}{\dashrightarrow} &
\end{array}$$

commutative and such that $(x, b_1) : \eta_0 \to \delta$ is a morphism of extriangles. Observe that commutativity of the left square tells us that $1_{B_0} = a_0 x$, thus x is a section. By assumption, we have a split extriangle

$$B_0 \xrightarrow{x} A \xrightarrow{y} A' \overset{\eta_1}{\dashrightarrow} .$$

Now we apply (ET4) to obtain

$$\begin{array}{ccccccc}
B_0 & \xrightarrow{x} & A & \xrightarrow{y} & A' & \overset{\eta_1}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle \left[\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \alpha} & & \\
B_0 & \xrightarrow{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} & B_0 \oplus B_1 & \xrightarrow{[0\ \phi]} & B_1 & \dashrightarrow & \\
 & & \downarrow{\scriptstyle [0\ b_1]} & & \downarrow{\scriptstyle \beta} & & \\
 & & C & = & C & & \\
 & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_1 = y_* \delta} & &
\end{array}$$

where $\phi : B_1 \to B_1$ is an automorphism. Then, by Lemma II.1.16, we deduce that $\delta = \delta_0 \oplus \delta_1$ where $\mathfrak{s}(\delta_0) = [B_0 \to B_0 \to 0]$. The fact that b_1 is an $\mathbb{E}$-deflation follows from using the automorphism ϕ to find a representative of $\mathfrak{s}(\delta_1)$ with b_1 as the second morphism. □

II.2 Approximations and relative homological algebra

We briefly recall some basic definitions from relative homological algebra introduced in [66] and state some useful properties.

By $\mathrm{Mor}_{\mathcal{A}}$, we denote the category of morphisms in $\mathcal{A}$.

Definition II.2.1. [13] $\mathcal{X} \subseteq \mathcal{A}$ be a full subcategory and let $A \in \mathcal{A}$. A *right $\mathcal{X}$-approximation of* A is a morphism $\alpha : X \to A$ with $X \in \mathcal{X}$ such that all morphisms

$X' \to A$ with $X' \in \mathcal{X}$ factor through α:

$$\begin{array}{ccc} X' & & \\ {\scriptstyle\exists}\downarrow & \searrow^{\forall} & \\ X & \xrightarrow{\alpha} & A \end{array}$$

Dually, we define a *left $\mathcal{X}$-approximation of A*. The subcategory $\mathcal{X}$ is called *contravariantly finite* (resp. *covariantly finite*) in $\mathcal{A}$ if every $A \in \mathcal{A}$ admits a right (resp. left) $\mathcal{X}$-approximation. $\mathcal{X}$ is called *functorially finite* if it is both contravariantly and covariantly finite in $\mathcal{A}$.

Definition II.2.2. Let $\mathcal{D} \subset \mathrm{Ob}(\mathcal{A})$ be a class of objects. A morphism $f : A \to B$ in $\mathcal{A}$ is *$\mathcal{D}$-monic* if all morphisms $A \to D$ with $D \in \mathcal{D}$ factor through f

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\forall}\downarrow & \swarrow_{\exists} & \\ \mathcal{D} \ni D. & & \end{array}$$

or, equivalently, $\mathcal{A}(f,-)|_{\mathcal{D}} : \mathcal{A}(B,-)|_{\mathcal{D}} \to \mathcal{A}(A,-)|_{\mathcal{D}}$ is an epimorphism. By $\mathrm{Mon}(\mathcal{D})$ we denote the class of all $\mathcal{D}$-monic morphisms in $\mathcal{A}$. We define the notion of a $\mathcal{D}$-epic morphism and the class $\mathrm{Epi}(\mathcal{D})$ dually.

Similarly, for a class of morphisms $\omega \subset \mathrm{Ob}(\mathrm{Mor}_{\mathcal{A}})$, an object $J \in \mathcal{A}$ is *ω-injective* if for every morphism $f : A \to B$ in ω, all morphisms $A \to J$ factor through f

$$\begin{array}{ccc} A & \xrightarrow{\forall f \in \omega} & B \\ {\scriptstyle\forall}\downarrow & \swarrow_{\exists} & \\ J. & & \end{array}$$

or, equivalently, $\mathcal{A}(f, J)$ is an epimorphism for all $f \in \omega$. By $\mathrm{Inj}(\omega)$ we denote the class of all ω-injective objects in $\mathcal{A}$. We define the notion of an ω-projective object and the class $\mathrm{Proj}(\omega)$ dually.

Example II.2.3. (a) $\mathcal{A}$-monics are precisely sections.

(b) If $\mathcal{A}$ has enough injectives, then $\mathrm{Inj}(\mathcal{A})$-monics are monomorphisms.

(c) A left $\mathcal{D}$-approximation of an object A is just a $\mathcal{D}$-monic morphism $A \to B$ such that $B \in \mathcal{D}$.

(d) In an extriangulated category, $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$, the $\mathbb{E}$-injective objects coincide with $\mathrm{Inj}(\{\mathbb{E}\text{-inflations}\})$.

We collect some useful properties.

Lemma II.2.4. *Let $\mathcal{D}$ be a class of objects in $\mathcal{A}$ and ω a class of morphisms. Then the following hold*

(i) $\mathrm{Mon}(\mathcal{D})$ *is closed under composition and direct summands in* $\mathrm{Mor}\,\mathcal{A}$*;*

(ii) $\mathrm{Mon}(\mathcal{D})$ *is left divisive, that is,* $gf \in \mathrm{Mon}(\mathcal{D})$ *implies that* $f \in \mathrm{Mon}(\mathcal{D})$*;*

(iii) $\mathrm{Mon}(\mathcal{D})$ *is closed under weak pushouts.*

Proof. (i) and (ii) are easily verified. Let us show (iii). Let

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A' & \xrightarrow[x']{} & B' \end{array}$$

be a weak pushout square in $\mathcal{A}$ with $x \in \mathrm{Mon}\mathcal{D}$. We must show that $x' \in \mathrm{Mon}\mathcal{D}$. To this end, let $g : A' \to D$ be a morphism with $D \in \mathcal{D}$. Then, as x is $\mathcal{D}$-monic, there exists a morphism $y : B \to D$ such that $yx = gf$. Now, by the weak pushout property there exists a morphism $h : B' \to C$ such that $hx' = g$ as required. $\square$

The interested reader may look at [90] and [113] for more properties of $\mathcal{D}$-monics and ω-injectives. For other applications of these notions, look, for instance in [23, 45, 7].

Chapter III

Torsion pairs and quasi-abelian categories

In this chapter we define and study torsion pairs in quasi-abelian categories. The chapter is organised as follows. In Section III.1 we define torsion pairs for additive and extriangulated categories, prove some basic properties of these objects and compare with the characterisations of torsion pairs in the abelian setting. In the next section (III.2), we prove that the heart of twin torsion pairs is quasi-abelian. This provides us with a way to generate examples of quasi-abelian categories that are not naturally arising as torsion(free) classes. Section III.3 is devoted to proving a bijection between torsion pairs in the heart of a twin torsion pairs and certain torsion pairs in the ambient abelian category. We furthermore show that, under mild assumptions, this bijection preserves the functorially finiteness of the torsion(free) classes. In the fourth section (III.4), we recall the construction of the left associated abelian category, $\mathcal{L}_{\mathcal{Q}}$, for a quasi-abelian category, $\mathcal{Q}$, due to Schneiders [116] and give criteria for when it is equivalent to a small module category. We then use the results of the previous sections to completely characterise torsion pairs for quasi-abelian categories. As an application of the newly developed theory, in the final section (III.5) we show the existence of Harder-Narasimhan filtrations for chains of torsion classes in a quasi-abelian category. Furthermore, we also explore topological properties of the set of chains of torsion classes in a quasi-abelian category.

III.1 Defining torsion pairs

We will consider two types of torsion pairs in our work, one that depends only on the ambient category and another that depends on an external triangulation.

Definition III.1.1. Let $\mathcal{A}$ be an additive category. A *torsion pair in* $\mathcal{A}$ is an ordered pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ satisfying the following.

(T1) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$.

(T2) For all A in $\mathcal{A}$ there exists a short exact sequence

$$0 \longrightarrow {}_{\mathcal{T}}A \xrightarrow{i_A} A \xrightarrow{p_A} A_{\mathcal{F}} \longrightarrow 0 \tag{III.1}$$

with ${}_{\mathcal{T}}A \in \mathcal{T}$ and $A_{\mathcal{F}} \in \mathcal{F}$.

In this case we call $\mathcal{T}$ a *torsion class*, $\mathcal{F}$ a *torsionfree class* and the short exact sequence in (T2) is called the *$(\mathcal{T},\mathcal{F})$-canonical short exact sequence of M*.

Definition III.1.2. Let $\mathcal{A} = (\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. A pair of additive subcategories, $(\mathcal{U}, \mathcal{V})$, of $\mathcal{A}$ is an $\mathbb{E}$-*torsion pair in* $\mathcal{A}$ if

($\mathbb{E}$T1) $\operatorname{Hom}_{\mathcal{A}}(\mathcal{U}, \mathcal{V}) = 0$;

($\mathbb{E}$T2) For all $C \in \mathcal{A}$ there exists an extriangle

$$U \xrightarrow{u} C \xrightarrow{v} V \dashrightarrow \tag{III.2}$$

with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. We call this sequence a $\mathbb{E}$-*torsion extriangle* of C.

Let us note that the term $\mathbb{E}$-torsion pair is non-standard, but we employ this term to make a distinction between our two definitions. We collect some properties of ($\mathbb{E}$-)torsion classes.

Lemma III.1.3. *Let $\mathcal{A}$ be an additive category and $(\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{A}$. Then*

(i) $\mathcal{T} = {}^{\perp}\mathcal{F} := \{A \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(A, \mathcal{F}) = 0\}$*;*

(i') $\mathcal{F} = \mathcal{T}^{\perp} := \{C \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, C) = 0\}$*;*

(ii) *$\mathcal{T}$ is closed under extensions, quotients, and coproducts;*

(ii') *$\mathcal{F}$ is closed under extensions, subobjects, and products;*

(iii) *For all $A \in \mathcal{A}$, the morphism i_A is a right $\mathcal{T}$-approximation of A;*

(iii') *For all $A \in \mathcal{A}$, the morphism p_A is a left $\mathcal{F}$-approximation of A.*

If $(\mathcal{U}, \mathcal{V})$ is an $\mathbb{E}$-torsion pair with respect to an external triangulation $(\mathbb{E}, \mathfrak{s})$ of $\mathcal{A}$ then the above properties (with an appropriate change of notation) also hold. In addition, $\mathcal{U}$ is also closed under $\mathbb{E}$-extensions and $\mathbb{E}$-quotients. Dually, $\mathcal{V}$ is also closed under $\mathbb{E}$-extensions and $\mathbb{E}$-subobjects.

Recall that a subcategory $\mathcal{X}$ of an additive category $\mathcal{A}$ is:

(a) *closed under extensions* if for all short exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ such that $A, C \in \mathcal{X}$, then $B \in \mathcal{X}$.

(b) *closed under quotients* if for all epimorphisms $B \to C$ in $\mathcal{A}$ such that $B \in \mathcal{X}$, then $C \in \mathcal{X}$. Being *closed under subobjects* is defined dually.

(c) *closed under coproducts* if for all families of objects $\{A_i \mid i \in I\}$ in $\mathcal{X}$ indexed by a set I such that the coproduct $\coprod_{i\in I} A_i$ exists in $\mathcal{A}$, then $\coprod_{i\in I} A_i \in \mathcal{X}$. Being *closed under products* is defined dually.

(d) *closed under* $\mathbb{E}$*-quotients* (with respect to a fixed external triangulation $(\mathbb{E}, \mathfrak{s})$ of $\mathcal{A}$) if for all $\mathbb{E}$-deflations, $B \to C$, if $B \in \mathcal{X}$ then $C \in \mathcal{X}$. Being *closed under* $\mathbb{E}$*-subobjects* is defined dually.

Proof of Lemma III.1.3. Let $(\mathcal{T}, \mathcal{F})$ be a torsion class in an additive category $\mathcal{A}$. We prove the statements for the torsion class $\mathcal{T}$, whence the statements for $\mathcal{F}$ follow by dualising the arguments.

(i): Clearly $\mathcal{T} \subseteq^{\perp} \mathcal{F}$. To see the converse, let $A \in A$ be such that $\operatorname{Hom}_{\mathcal{A}}(A, \mathcal{F}) = 0$. Then in the $(\mathcal{T}, \mathcal{F})$-canonical short exact sequence of A we have that $p_A = 0$ and thus $A \cong T \in \mathcal{T}$.

(ii): The fact that $\mathcal{T}$ is closed under extensions, follows from part (i) after by applying the left exact functor $\operatorname{Hom}_{\mathcal{A}}(-, F)$ for all $F \in \mathcal{F}$ to a short exact sequence $0 \to A \to B \to C \to 0$ with $A, C \in \mathcal{T}$. Next, let $e : B \to C$ be an epimorphism in $\mathcal{A}$ with $B \in \mathcal{T}$. Observe that the composition $p_C e : B \to C_{\mathcal{F}}$ is zero by (T1). Thus, as e is an epimorphism, p_C is zero and $C \cong {}_{\mathcal{T}}C \in \mathcal{T}$. Now we show that $\mathcal{T}$ is closed under coproducts. Let $\{A_i \mid i \in I\}$ be a family of objects in $\mathcal{T}$ indexed by a set I such that the coproduct $\coprod_{i \in I} A_i$ exists in $\mathcal{A}$ and let $\iota_i : A_i \to \coprod_{i \in I} A_i$ denote the canonical inclusions. Suppose there is a morphism $f : \coprod_{i \in I} A_i \to F$ for some $F \in \mathcal{F}$. Then, as all $A_i \in \mathcal{T}$ the composition $f\iota_i = 0$ for all $i \in I$. Thus f is the unique morphism such that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\iota_i} & \coprod_{i \in I} A_i \\ & \searrow^{0} & \downarrow f \\ & & F \end{array}$$

commutes for all $i \in I$. Since the zero morphism satisfies this property, we conclude that $f = 0$ and $\coprod_{i \in I} B_i \in \mathcal{T}$ by part (i).

(iii): Let $A \in \mathcal{A}$ and $t : T \to A$ be a morphism with $T \in \mathcal{T}$. Then, by (T1) the composition $p_A t = 0$ and therefore, by the universal property of kernels, there exists a (unique) $t' : T \to {}_{\mathcal{T}}A$ such that $t = i_A t'$.

Now we fix an external triangulation $(\mathbb{E}, \mathfrak{s})$ of $\mathcal{A}$ and let $(\mathcal{U}, \mathcal{V})$ be an $\mathbb{E}$-torsion pair in $\mathcal{A}$. The proof of everything apart from the closure of $\mathcal{U}$ under $\mathbb{E}$-extensions and $\mathbb{E}$-quotients is a straightforward generalisation of the above. The fact that $\mathcal{U}$ is closed under $\mathbb{E}$-extensions follows from the axiom (ET1) together with Lemma II.1.8. It remains to show the closure under $\mathbb{E}$-quotients, to this end let

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

be an extriangle with $B \in \mathcal{U}$. We apply $(\text{ET4})^{\text{op}}$ to this extriangle and an $\mathbb{E}$-torsion extriangle III.2 of C to obtain

$$\begin{array}{ccccccc} A & \longrightarrow & E & \longrightarrow & U & \dashrightarrow & \\ \| & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u} & & \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C & \dashrightarrow & \\ & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle v} & & \\ & & V & = & V. & & \\ & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \end{array}$$

Now since $B \in U$ we have that $g = 0$ and thus $f : E \xrightarrow{\cong} B$. From this, we deduce that u is also an isomorphism and so $C \cong U \in \mathcal{U}$ as required. The claims for $\mathcal{V}$ follow dually. $\square$

We make the following observation.

Remark III.1.4. For a quasi-abelian category equipped with its maximal exact structure, the notions of torsion pair and $\mathbb{E}$-torsion pair coincide. Thus in the sequel, for brevity, we only use the term torsion pair in this setting.

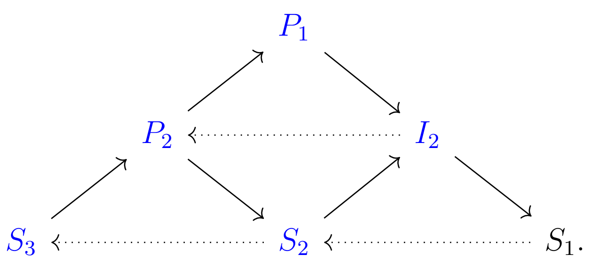
For the rest of this section, we will discuss only torsion pairs for additive categories. Then, for the remainder of this Chapter we will be working with torsion pairs in quasi-abelian categories, which are the same as $\mathbb{E}$-torsion pairs. In Section V.3 we will discuss $\mathbb{E}$-torsion pairs in (right) triangulated categories.

Remark III.1.5. In an abelian category, the conditions (i) & (i') of Lemma III.1.3 together are sufficient conditions for a pair for for $(\mathcal{T}, \mathcal{F})$ to be a torsion pair [41, Theorem 3.1]; we will show in Proposition III.4.14 that this is also the case for quasi-abelian categories.

Also in the abelian setting, the condition (ii) (resp. (ii')) of Lemma III.1.3 is sufficient for a subcategory to be a torsion (resp. torsionfree) class [41, Theorem 4.1]. However, in general, this is not the case in a pre-abelian (or even quasi-abelian) category. Let Q be the quiver

$$1 \longrightarrow 2 \longrightarrow 3$$

and consider the abelian category $\mathcal{A} = \operatorname{mod} KQ$ whose Auslander-Reiten quiver is given by



Consider the additive subcategory $\mathcal{C}$ generated by the indecomposables S_3, S_2, P_2, P_1 and I_2. which is quasi-abelian as it is a torsionfree class of $\mathcal{A}$. Now the subcategory $\mathcal{T} = \operatorname{add}\{S_2, P_2\}$ of $\mathcal{C}$ is closed under coproducts, extensions and quotients in $\mathcal{C}$ but it is not a torsion class in $\mathcal{C}$. Indeed, $\mathcal{T}^\perp = \operatorname{add}\{S_2 \oplus P_2\}$ but ${}^\perp(\mathcal{T}^\perp) = \operatorname{add}\{S_2 \oplus P_2 \oplus I_2 \oplus P_1\} \neq \mathcal{T}$ which contradicts Lemma III.1.3(i) thus $\mathcal{T}$ is not a torsion class in $\mathcal{C}$.

Torsion pairs in additive categories also admit a functorial description.

Proposition III.1.6. *Let $\mathcal{A}$ be an additive category. Then a full subcategory $\mathcal{T} \subseteq \mathcal{A}$ is a torsion class in $\mathcal{A}$ if and only if there exists a functor $\mathtt{t} : \mathcal{A} \to \mathcal{T}$ that is an idempotent and radical kernel subfunctor of the identity such that $\mathcal{T} = \{A \in \mathcal{A} \mid \mathtt{t}A \cong A\}$. Moreover, in this situation such a functor is a right adjoint to the canonical inclusion $\mathcal{T} \hookrightarrow \mathcal{A}$.*

Recall that a functor $F : \mathcal{A} \to \mathcal{A}$ is

(a) *idempotent* if $F(FA) \cong FA$ for all $A \in \mathcal{A}$.

(b) a *kernel subfunctor of the identity* if for all $A \in \mathcal{A}$ there exists a morphism $FA \to A$ which is a kernel and part of a kernel-cokernel pair

$$FM \hookrightarrow M \twoheadrightarrow M/FM$$

for all $M \in \mathcal{A}$ and furthermore that for all $f : A \to B$ in $\mathcal{A}$ the diagram

$$\begin{array}{ccc} FA & \longrightarrow & A \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FB & \longrightarrow & B \end{array}$$

commutes.

(c) *radical* if $F(\mathrm{Coker}(FA \to A)) \cong 0$ for all $A \in \mathcal{A}$.

Proof of Proposition III.1.6. ($\Rightarrow$) Let $\mathcal{F}$ be the torsionfree class associated to $\mathcal{T}$. We verify that the assignment $A \mapsto {}_{\mathcal{T}}A$ satisfies the conditions above. Firstly, let $f : A \to B$ in $\mathcal{A}$ then, by (T1), $p_B f i_A : {}_{\mathcal{T}}A \to B_{\mathcal{F}}$ is zero, hence by the universal property of kernels, there exists a unique ${}_{\mathcal{T}}f : {}_{\mathcal{T}}A \to {}_{\mathcal{T}}B$ such that $i_B({}_{\mathcal{T}}f) = f i_A$. It is clear that this defines a functor $\mathcal{A} \to \mathcal{T}$ which is, by construction, a subfunctor of the identity. To see that it is idempotent, consider the $(\mathcal{T}, \mathcal{F})$-canonical short exact sequence of ${}_{\mathcal{T}}A$ for any $A \in \mathcal{A}$.

$$0 \longrightarrow {}_{\mathcal{T}}({}_{\mathcal{T}}A) \xrightarrow{i_{\mathcal{T}A}} {}_{\mathcal{T}}A \xrightarrow{p_{\mathcal{T}A}} ({}_{\mathcal{T}}A)_{\mathcal{F}} \longrightarrow 0$$

and observe that $p_{\mathcal{T}A} = 0$ by (T1) therefore $i_{\mathcal{T}A}$ is an isomorphism and also $({}_{\mathcal{T}}A)_{\mathcal{F}} \cong 0$. The fact that ${}_{\mathcal{T}}(-)$ is radical follows from applying a dual argument to the $(\mathcal{T}, \mathcal{F})$-canonical short exact sequence of $A_{\mathcal{F}}$. It remains to check that $\mathcal{T} = \{A \in \mathcal{A} \mid {}_{\mathcal{T}}A \cong A\}$, but this follows from the fact that, for all $A \in \mathcal{T}$, $p_A = 0$ by (T1).

($\Leftarrow$) Let $\mathtt{t} : \mathcal{A} \to \mathcal{T}$ be a functor as in the statement and set $\mathcal{F} = \{A \in \mathcal{A} \mid \mathtt{t}A \cong 0\}$. Then as $\mathtt{t}$ is radical, $A/\mathtt{t}A \in \mathcal{F}$, for all $A \in \mathcal{A}$. Thus (T2) is satsified. To verify (T1), let $A \in \mathcal{T}, B \in \mathcal{F}$ and $f : A \to B$ be a morphism in $\mathcal{A}$, then there is a commutative diagram

$$\begin{array}{ccc} \mathtt{t}A & \xrightarrow{\cong} & A \\ \downarrow{\scriptstyle \mathtt{t}f} & & \downarrow{\scriptstyle f} \\ 0 \cong \mathtt{t}B & \longrightarrow & B \end{array}$$

from which we conclude $f = 0$ and (T1) is satisfied.

The fact that such a $\mathtt{t}$ is a right adjoint follows from Lemma III.1.3(iii). □

As a direct consequence, we justify some of our terminology.

Corollary III.1.7. *Let $\mathcal{A}$ be an additive category and $(\mathcal{T}, \mathcal{F})$ be a torsion pair on $\mathcal{A}$. Then for all $A \in \mathcal{A}$ the $(\mathcal{T}, \mathcal{F})$-canonical short exact sequence is unique up to isomorphism.*

We end with a motivating family of examples for studying torsion pairs in non-abelian settings.

Example III.1.8. Let Λ be a finite dimensional k-algebra for a field k. We fix an order on the indecomposable projective Λ-modules $P(1), P(2), \ldots, P(n)$ and define the modules

$$U(i) := \sum_{\substack{\phi: P(j) \to P(i) \\ j > i}} \mathrm{Im}\, \phi, \text{ and } \Delta(i) := P(i)/U(i).$$

We write $\Delta = \{\Delta(1), \ldots, \Delta(n)\}$ and by $\mathrm{Filt}(\Delta)$ we denote the smallest extension closed subcategory of $\operatorname{mod}\Lambda$ containing Δ (by convention $0 \in \mathrm{Filt}(\Delta)$). The algebra Λ is called *standardly stratified* [42] if $\operatorname{proj}\Lambda \in \mathrm{Filt}(\Delta)$ and *quasi-hereditary* [117] if in addition $\mathrm{End}_\Lambda \Delta(i)$ is a skew-field for all $1 \leq i \leq n$. The concept of a quasi-hereditary algebra, which was introduced as a tool to study highest weight categories that appeared in the representation theory of Lie algebras and algebraic groups [100, 39], pre-dates that of a standardly stratified algebra. Let us note that the definition of a quasi-hereditary algebra that we gave here, due to Dlab & Ringel [43], is different from Scott's original formulation (in which the class of quasi-hereditary algebras was defined recursively using *heredity ideals*) but is equivalent.

Proposition III.1.9. *Let Λ be a standardly stratified algebra. Then for all $1 \leq j \leq n$ the pair of subcategories $(\mathrm{Filt}(\Delta_{>j}), \mathrm{Filt}(\Delta_{\leq j}))$ is a torsion pair in $\mathrm{Filt}(\Delta)$; where we use $\Delta_{>j}$ to denote the set $\{\Delta_i \mid j < i \leq n\}$ and $\Delta_{\leq j}$ is used in a similar way.*

We export most of the proof to the following lemma.

Lemma III.1.10. *In the situation of Proposition III.1.9, for all $X \in \mathrm{Filt}(\Delta)$ there exists $s \geq 0$ and a short exact sequence in $\operatorname{mod}\Lambda$*

$$0 \to \Delta(n)^{\oplus s} \to X \to X' \to 0$$

with $X' \in F(\Delta_{<n})$.

We do not believe this to be an original result. However, since we have found no explicit proof of this in the literature, we include one.

Proof. Note that for a subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$ we have that $\mathrm{Filt}(\mathcal{X}) = \bigcup_{k\geq 1} \mathcal{X}^{*k}$ where $\mathcal{X}^{*k} = \mathcal{X} * \mathcal{X} * \cdots * \mathcal{X}$ (k-times). If $X \cong 0$ there is nothing to show. Let $0 \ncong X \in \mathrm{Filt}(\Delta)$, we proceed by induction on $\ell = \ell_X := \min\{k \in \mathbb{N} \mid X \in \Delta^{*k}\}$. For $\ell = 1$ we have that $X \in \Delta$ and the assertion is trivial. Suppose that $\ell > 1$, then by definition of $\Delta^{*\ell}$ there is an short exact sequence in $\operatorname{mod}\Lambda$

$$0 \longrightarrow Y \xrightarrow{\ f\ } X \longrightarrow \Delta(j) \longrightarrow 0$$

where $\ell_Y < \ell$ and $1 \leq j \leq n$. By the induction hypothesis, there is a short exact sequence

$$0 \longrightarrow \Delta(n)^{\oplus r} \xrightarrow{\ f'\ } Y \longrightarrow Y' \longrightarrow 0$$

for some $0 \leq r$ and $Y' \in \mathrm{Filt}(\Delta_{<n})$. From these sequences, we build the commutative diagram (this can be thought of as applying (ET4))

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta(n)^{\oplus r} & \xrightarrow{f'} & Y & \longrightarrow & Y' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow & & \\
0 & \longrightarrow & \Delta(n)^{\oplus r} & \xrightarrow{x} & X & \longrightarrow & E & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Delta(j) & = & \Delta(j) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

In the case that $j < n$, the right hand vertical sequence shows that $E \in \mathrm{Filt}(\Delta_{<n})$ and we have the second row sequence as required. If $j = n$. then by [43, Lemma 1.3] and [129, Lemma 3.3] the right hand vertical sequence splits. We form the pullback of the second row along a section $s : \Delta(n) \to E$ to obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta(n)^{\oplus r} & \longrightarrow & F & \longrightarrow & \Delta(n) & \longrightarrow & 0 \\ & & \| & & \downarrow y & & \downarrow s & & \\ 0 & \longrightarrow & \Delta(n)^{\oplus r} & \xrightarrow{x} & X & \longrightarrow & E & \longrightarrow & 0. \end{array}$$

It again follows from [43, Lemma 1.3] that the top row splits, that is $F \cong \Delta(n)^{\oplus r+1}$ and by the Snake lemma we have that y is a monomorphism with cokernel $Y' \in \mathrm{Filt}(\Delta_{<n})$ and we are done. □

Proof of Proposition III.1.9. The fact that (T1) holds follows from, for example, [129, Lemma 3.3]. (T2) follows from repeated application of Lemma III.1.10. □

III.2 The heart of twin torsion pairs

We begin by recalling the structure of torsion(free) classes in abelian categories:

Lemma III.2.1 ([25, Proposition B.3], [112, §4, Corollary])**.** *Every torsion class and torsionfree class of an abelian category has the structure of a quasi-abelian category.*

In this section we generalise the above result. Namely, we consider the intersection $\mathcal{C}' \cap \mathcal{D}$ where $(\mathcal{C}, \mathcal{D})$, $(\mathcal{C}', \mathcal{D}')$ are torsion pairs in $\mathcal{A}$ such that $\mathcal{C} \subseteq \mathcal{C}'$ or, equivalently, $\mathcal{D}' \subseteq \mathcal{D}$. We shall refer to such couples of torsion pairs as *twin torsion pairs* and the intersection $\mathcal{C}' \cap \mathcal{D}$ as their *heart*. We denote twin torsion pairs by $[(\mathcal{C}, \mathcal{D}),\ (\mathcal{C}', \mathcal{D}')]$. We will show that such hearts are quasi-abelian.

Theorem III.2.2. *Let $\mathcal{A}$ be an abelian category and let $[(\mathcal{C}, \mathcal{D}),\ (\mathcal{C}', \mathcal{D}')]$ be twin torsion pairs on $\mathcal{A}$. Then the heart, $\mathcal{C}' \cap \mathcal{D}$, is quasi-abelian.*

Remark III.2.3. We remark that, in general, distinct twin torsion pairs can have the same heart. Indeed, consider the quiver A_3 as in Remark III.1.5 and the twin torsion pairs

$$\Big[\Big(\mathrm{add}\{S_1\},\ \mathrm{add}\{S_2 \oplus S_3 \oplus P_2 \oplus I_2 \oplus P_1\}\Big),\quad \Big(0, \mathcal{A}\Big)\Big]$$

and

$$\Big[\Big(\mathrm{add}\{S_3\},\ \mathrm{add}\{S_1 \oplus S_2 \oplus \oplus I_2\}\Big),\quad \Big(\mathrm{add}\{S_1 \oplus S_3\},\ \mathrm{add}\{S_2 \oplus I_2\}\Big)\Big]$$

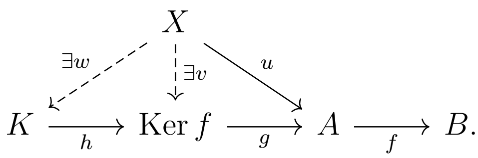
which both have heart $\mathrm{add}\{S_1\}$.

The first step of the proof of Proposition III.2.2 follows the argument in [112, Theorem 2] and does not require the assumption that the torsion pairs are twin.

Lemma III.2.4. *Let $\mathcal{A}$ be an abelian category and let $(\mathcal{C}, \mathcal{D})$, $(\mathcal{C}', \mathcal{D}')$ be torsion pairs in $\mathcal{A}$. Then $\mathcal{C}' \cap \mathcal{D}$ is pre-abelian.*

Proof. We check the existence of kernels in $\mathcal{C}' \cap \mathcal{D}$, whence existence of cokernels will follow by duality. Let $f : A \to B$ be a morphism in $\mathcal{C}' \cap \mathcal{D}$, let $g : \operatorname{Ker} f \to A$ be a kernel of f in $\mathcal{A}$ and let $h : {}_{\mathcal{C}'}(\operatorname{Ker} f) \to \operatorname{Ker} f$ be the right $\mathcal{C}'$-approximation of $\operatorname{Ker} f$. Set $K := {}_{\mathcal{C}'}(\operatorname{Ker} f)$. We claim that $hg : K \to A$ is a kernel of f in $\mathcal{C}' \cap \mathcal{D}$. Firstly, note that $K \in \mathcal{D}$. Indeed, $\mathcal{D}$ is closed under subobjects, and K is a subobject of $\operatorname{Ker} f$ which in turn is a subobject of A.

Now let $u : X \to A$ be a morphism in $\mathcal{C}' \cap \mathcal{D}$ such that $fu = 0$. Then by the universal property of kernels, there exists a unique morphism $v : X \to \operatorname{Ker} f$ such that $vg = u$. Since h is a right $\mathcal{C}'$-approximation of $\operatorname{Ker} f$ and $X \in \mathcal{C}'$ there exists a morphism $w : X \to K$ such that $wh = v$. Together, we have that $u = vg = whg$, thus u factors through hg:



It remains to show that this factorisation is unique. Let $w' : X \to K$ be such that $u = w'(hg)$. Observe that h and g are both monomorphisms and hence so is hg. Then $w(hg) = u = w'(hg)$ and we conclude that $w = w'$. $\square$

The previous result shows that kernels (resp. cokernels) in $\mathcal{C}' \cap \mathcal{D}$ are given by kernels in $\mathcal{C}'$ (resp. cokernels in $\mathcal{D}$).

Notation III.2.5. When it exists, we denote the kernel of a morphism f in a subcategory $\mathcal{C}$ of an ambient category $\mathcal{A}$ by $\operatorname{Ker}_{\mathcal{C}} f$.

We observe that the exact structure on $\mathcal{C}' \cap \mathcal{D}$ inherited from $\mathcal{A}$ and the exact structure arising from short exact sequences in $\mathcal{C}' \cap \mathcal{D}$ coincide.

Proposition III.2.6. *Let $\mathcal{A}$ be an abelian category and let $[(\mathcal{C}, \mathcal{D}), (\mathcal{C}', \mathcal{D}')]$ be twin torsion pairs in $\mathcal{A}$. Then a pair of composable morphisms (f, g) in $\mathcal{C}' \cap \mathcal{D}$ is a short exact sequence in $\mathcal{C}' \cap \mathcal{D}$ if and only if it is a short exact sequence in $\mathcal{A}$.*

Proof. Let $(f : A \to B, g : B \to C)$ be a short exact sequence in $\mathcal{C}' \cap \mathcal{D}$. Then it follows from Lemma III.2.4 that $A = \operatorname{Ker}_{\mathcal{C}'} g = {}_{\mathcal{C}'}(\operatorname{Ker} g)$ and $C = \operatorname{Coker}_{\mathcal{D}} f = (\operatorname{Coker} f)_{\mathcal{D}}$. Consider the commutative diagram with rows and columns that are exact in $\mathcal{A}$

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & {}_{\mathcal{C}}(\operatorname{Coker} f) & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & A & \xrightarrow{f} & B & \longrightarrow & \operatorname{Coker} f & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} g & \rightarrowtail & B & \xrightarrow{g} & C & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & (\operatorname{Ker} g)_{\mathcal{D}'} & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

By the Snake lemma we see that ${}_{\mathcal{C}}(\operatorname{Coker} f) \cong (\operatorname{Ker} g)_{\mathcal{D}'} \in \mathcal{C} \cap \mathcal{D}'$. But as $\mathcal{C} \subseteq \mathcal{C}'$ we have that $\mathcal{C} \cap \mathcal{D}' = 0$. Thus $A \cong \operatorname{Ker} g$, $C \cong \operatorname{Coker} f$ proving the assertion.

The reverse implication is trivial. □

Proof of Theorem III.2.2. This follows directly from Lemma III.2.4 and Proposition III.2.6 since the exact structure inherited from $\mathcal{A}$ consists of all short exact sequences in $\mathcal{C}' \cap \mathcal{D}$. □

Remark III.2.7. Not every quasi-abelian subcategory of an abelian category arises this way. For example, consider the linearly oriented quiver Q of type A_3 as in Remark III.1.5. Then the subcategory $\mathcal{X} = \operatorname{add}\{P_2 \oplus I_2\}$ of $\operatorname{mod} KQ$ is quasi-abelian. Indeed, the kernel and cokernel of the morphism $P_2 \to I_2$ are both the zero morphism and there are no non-trivial short exact sequences. Suppose that $\mathcal{X} = \mathcal{C}' \cap \mathcal{D}$ for some twin torsion pairs $[(\mathcal{C}, \mathcal{D}), (\mathcal{C}', \mathcal{D}')]$. Then $\operatorname{add}\{S_2\} \subset \operatorname{Fac}\mathcal{X} \subseteq \mathcal{C}'$ and $\operatorname{add}\{S_2\} \subset \operatorname{Sub}\mathcal{X} \subseteq \mathcal{D}$, but $\operatorname{add}\{S_2\} \not\subset \mathcal{X}$.

III.3 A bijection of torsion pairs

In this section, we develop a bijection between the torsion pairs of the heart of two twin torsion pairs and a class of torsion pairs of the ambient category. We remind the reader that when we work with a (quasi-)abelian category $\mathcal{A}$, we mean that we work with the exact category $(\mathcal{A}, \mathcal{E}_{all})$ and that, in this case, the terms 'short exact sequence', 'conflation', and '$\mathcal{E}$-sequence' are all interchangeable. We begin with a technical lemma.

Lemma III.3.1. *Let $\mathcal{A}$ be an abelian category and let $[(\mathcal{C}, \mathcal{D}), (\mathcal{C}', \mathcal{D}')]$ be twin torsion pairs in $\mathcal{A}$. Then for all $A \in \mathcal{A}$, we have the following isomorphisms*

(i) $({}_{\mathcal{C}'}A)_{\mathcal{D}} \cong {}_{\mathcal{C}'}(A_{\mathcal{D}}) =: {}_{\mathcal{C}'}A_{\mathcal{D}}$;

(ii) ${}_{\mathcal{C}}(A_{\mathcal{D}'}) \cong ({}_{\mathcal{C}}A)_{\mathcal{D}'} \cong 0$;

(iii) ${}_{\mathcal{C}}({}_{\mathcal{C}'}A) \cong {}_{\mathcal{C}}A \cong {}_{\mathcal{C}'}({}_{\mathcal{C}}A)$;

(iv) $(A_{\mathcal{D}})_{\mathcal{D}'} \cong A_{\mathcal{D}'} \cong (A_{\mathcal{D}'})_{\mathcal{D}}$.

Proof. Let $A \in \mathcal{A}$, using the $(\mathcal{C}, \mathcal{D})$-canonical short exact sequence of A and the $(\mathcal{C}', \mathcal{D}')$-canonical short exact sequence of $A_{\mathcal{D}}$, we build the following commutative diagram

$$\begin{array}{ccccc}
{}_{\mathcal{C}}A & \rightarrowtail & E & \twoheadrightarrow & {}_{\mathcal{C}'}(A_{\mathcal{D}}) \\
\| & & \downarrow f & \lrcorner & \downarrow \\
{}_{\mathcal{C}}A & \rightarrowtail & A & \twoheadrightarrow & A_{\mathcal{D}} \\
 & & & & \downarrow \!\!\! \downarrow \\
 & & & & (A_{\mathcal{D}})_{\mathcal{D}'}.
\end{array}$$

We make some observations. First note that as $\mathcal{C} \subseteq \mathcal{C}'$ and $\mathcal{C}'$ is closed under extensions, the upper short exact sequence shows that $E \in \mathcal{C}'$. Secondly, by using the Snake Lemma we see that f is a monomorphism and we have a short exact sequence

$$E \overset{f}{\rightarrowtail} A \twoheadrightarrow (A_{\mathcal{D}})_{\mathcal{D}'}$$

with first term in $\mathcal{C}'$ and last term in $\mathcal{D}'$. Hence, by uniqueness of torsion canonical short exact sequences, we have that $E \cong {}_{\mathcal{C}'}A$. Now the top row can be written as

$${}_{\mathcal{C}}A \rightarrowtail {}_{\mathcal{C}'}A \twoheadrightarrow {}_{\mathcal{C}'}(A_{\mathcal{D}})$$

which has first term in $\mathcal{C}$ and, as $\mathcal{D}$ is closed under submodules, last term in $\mathcal{D}$. Thus we conclude that $({}_{\mathcal{C}'}A)_{\mathcal{D}} \cong {}_{\mathcal{C}'}(A_{\mathcal{D}})$ and ${}_{\mathcal{C}}({}_{\mathcal{C}'}A) \cong {}_{\mathcal{C}}A$.

The fact that ${}_{\mathcal{C}}(A_{\mathcal{D}'}) \cong 0$ and $(A_{\mathcal{D}})_{\mathcal{D}'} \cong A_{\mathcal{D}'}$ follows from the commutative diagram with rows being short exact sequences

$$\begin{array}{ccccc}
{}_{\mathcal{C}}(A_{\mathcal{D}'}) & \rightarrowtail & A_{\mathcal{D}'} & \twoheadrightarrow & (A_{\mathcal{D}})_{\mathcal{D}'} \\
\downarrow & & \| & & \downarrow \\
0 & \rightarrowtail & A_{\mathcal{D}'} & \overset{1_{A_{\mathcal{D}'}}}{\twoheadrightarrow} & A_{\mathcal{D}'}
\end{array}$$

and the uniqueness of torsion short exact sequences. The remaining isomorphisms are proved similarly. □

The main result of this section is the following.

Theorem III.3.2. *Let $\mathcal{A}$ be an abelian category and let $[(\mathcal{C}, \mathcal{D}),\ (\mathcal{C}', \mathcal{D}')]$ be twin torsion pairs in $\mathcal{A}$. Then there is an inclusion preserving bijection:*

$$\begin{array}{rcl}
\{(\mathcal{X}, \mathcal{Y}) \text{ torsion pair in } \mathcal{A} \mid \mathcal{C} \subseteq \mathcal{X} \subseteq \mathcal{C}'\} & \longleftrightarrow & \{(\mathcal{T}, \mathcal{F}) \text{ torsion pair in } \mathcal{C}' \cap \mathcal{D}\} \\
(\mathcal{X}, \mathcal{Y}) & \longmapsto & (\mathcal{X} \cap \mathcal{D}, \mathcal{Y} \cap \mathcal{C}') \\
(\mathcal{C} * \mathcal{T}, \mathcal{F} * \mathcal{D}') & \longleftarrow\!\shortmid & (\mathcal{T}, \mathcal{F}).
\end{array}$$

Proof. We begin by showing the maps are well-defined. First, let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in $\mathcal{A}$ and suppose that $\mathcal{C} \subseteq \mathcal{X} \subseteq \mathcal{C}'$. Observe that $\mathcal{X} \cap \mathcal{D}$ and $\mathcal{Y} \cap \mathcal{C}'$ are subcategories of $\mathcal{C}' \cap \mathcal{D}$ and we have that $\mathrm{Hom}_{\mathcal{C}' \cap \mathcal{D}}(\mathcal{X} \cap \mathcal{D}, \mathcal{Y} \cap \mathcal{C}') = 0$ thus (T1) is satisfied. To verify (T2), let $A \in \mathcal{C}' \cap \mathcal{D}$ and consider the $(\mathcal{X}, \mathcal{Y})$-canonical short exact sequence of A

$$ {}_{\mathcal{X}}A \rightarrowtail A \twoheadrightarrow A_{\mathcal{Y}}. $$

Now as $\mathcal{D}$ is closed under subobjects, ${}_{\mathcal{X}}A \in \mathcal{D}$ and thus ${}_{\mathcal{X}}A \in \mathcal{X} \cap \mathcal{D}$. Similarly, as $\mathcal{C}'$ is closed under quotients we have that $A_{\mathcal{Y}} \in \mathcal{Y} \cap \mathcal{C}'$. Thus, $(\mathcal{X} \cap \mathcal{D}, \mathcal{Y} \cap \mathcal{C}')$ is a torsion pair in $\mathcal{C}' \cap \mathcal{D}$.

Conversely, let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{C}' \cap \mathcal{D}$. By definition, we have that $\mathcal{C} \subseteq \mathcal{C} * \mathcal{T}$ and as $\mathcal{C}'$ is closed under extensions and $\mathcal{T} \subseteq \mathcal{C}'$ we have that $\mathcal{C} * \mathcal{T} \subseteq \mathcal{C}'$. Now to show that $(\mathcal{C} * \mathcal{T}, \mathcal{F} * \mathcal{D}')$ satsifies (T1), let $f : A \to B$ be an arbitrary morphism with $A \in \mathcal{C} * \mathcal{T}$ and $B \in \mathcal{F} * \mathcal{D}'$. Consider the diagram

$$
\begin{array}{ccccc}
C & \rightarrowtail & A & \twoheadrightarrow & T \\
{\scriptstyle \exists f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \exists f''} \\
F & \rightarrowtail & B & \twoheadrightarrow & D'
\end{array}
$$

where the top (respectively bottom) row shows that A (respectively B) is an element of $\mathcal{C} * \mathcal{T}$ (respectively $\mathcal{F} * \mathcal{D}'$). That is, $C \in \mathcal{C}$, $T \in \mathcal{T}$, $F \in \mathcal{F}$ and $D' \in \mathcal{D}'$. Observe that as $\operatorname{Hom}_{\mathcal{A}}(\mathcal{C}, \mathcal{D}') = 0$, by the universal property of kernels (respectively, cokernels) there exists $f' : C \to F$ (resp. $f'' : T \to D'$) rendering the diagram commutative. But since $\mathcal{F} \subseteq \mathcal{D}$, we have that $\operatorname{Hom}_{\mathcal{A}}(\mathcal{C}, \mathcal{F}) = 0$, so $f' = 0$. Similarly, as $\mathcal{T} \subseteq \mathcal{C}'$, $f'' = 0$. By the Snake Lemma there is an exact sequence

$$ 0 \longrightarrow C \longrightarrow \operatorname{Ker} f \longrightarrow T \xrightarrow{\delta} F \longrightarrow \operatorname{Coker} f \longrightarrow D' \longrightarrow 0. $$

Then $\delta = 0$ as $\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = \operatorname{Hom}_{\mathcal{C}' \cap \mathcal{D}}(\mathcal{T}, \mathcal{F}) = 0$. We conclude that $\operatorname{Ker} f \cong A$, $\operatorname{Coker} f \cong B$ and $f = 0$.

To show (T2) let $A \in \mathcal{A}$. We begin by using the $(\mathcal{C}', \mathcal{D}')$-canonical short exact sequence of A and the $(\mathcal{C}, \mathcal{D})$-canonical short exact sequence of ${}_{\mathcal{C}'}A$ to form the pushout of short exact sequences

$$
\begin{array}{ccccc}
{}_{\mathcal{C}}A & & & & \\
\downarrow & & & & \\
{}_{\mathcal{C}'}A & \rightarrowtail & A & \twoheadrightarrow & A_{\mathcal{D}'} \\
\downarrow & \ulcorner & \downarrow & & \| \\
{}_{\mathcal{C}'}A_{\mathcal{D}} & \rightarrowtail & P & \twoheadrightarrow & A_{\mathcal{D}'}.
\end{array}
$$

Note that, by the Snake Lemma we have a short exact sequence

$$ {}_{\mathcal{C}}A \rightarrowtail A \twoheadrightarrow P. \tag{III.3} $$

Now, we use the lower short exact sequence of the above diagram and the $(\mathcal{T}, \mathcal{F})$-canonical short exact sequence of ${}_{\mathcal{C}'}A_{\mathcal{D}}$ to form the pushout of short exact sequences

$$
\begin{array}{ccccc}
{}_{\mathcal{T}}({}_{\mathcal{C}'}A_{\mathcal{D}}) & & & & \\
\downarrow & & & & \\
{}_{\mathcal{C}'}A_{\mathcal{D}} & \rightarrowtail & P & \twoheadrightarrow & A_{\mathcal{D}'} \\
\downarrow & \ulcorner & \downarrow & & \| \\
({}_{\mathcal{C}'}A_{\mathcal{D}})_{\mathcal{F}} & \rightarrowtail & Q & \twoheadrightarrow & A_{\mathcal{D}'}.
\end{array}
$$

Then the lower short exact sequence shows that $Q \in \mathcal{F} * \mathcal{D}'$ and by the Snake Lemma we have a short exact sequence

$$_{\mathcal{T}}({}_{\mathcal{C}'}A_{\mathcal{D}}) \rightarrowtail P \twoheadrightarrow Q.$$

Finally we use this short exact sequence and Sequence (III.3) to form the pullback of short exact sequences

$$\begin{array}{ccccc}
{}_{\mathcal{C}}A & \rightarrowtail & R & \twoheadrightarrow & {}_{\mathcal{T}}({}_{\mathcal{C}'}A_{\mathcal{D}}) \\
\Vert & & \downarrow & & \downarrow \\
{}_{\mathcal{C}}A & \rightarrowtail & A & \twoheadrightarrow & P \\
 & & & & \downarrow \\
 & & & & Q.
\end{array}$$

We observe that the upper short exact sequence shows that $R \in \mathcal{C} * \mathcal{T}$. Now by the Snake lemma, there is a short exact sequence

$$R \rightarrowtail A \twoheadrightarrow Q$$

which shows that (T2) is satisfied.

We show that the mappings are mutually inverse. Let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in $\mathcal{A}$ such that $\mathcal{C} \subseteq \mathcal{X} \subseteq \mathcal{C}'$. We claim that $\mathcal{X} = \mathcal{C} * (\mathcal{X} \cap \mathcal{D})$. Let $A \in \mathcal{X}$. Observe that, as $\mathcal{C}'$ is closed under quotients and $\mathcal{X} \subseteq \mathcal{C}'$, we have $A_{\mathcal{D}} \in \mathcal{C}' \cap \mathcal{D}$. Therefore we may build the pullback of short exact sequences using the $(\mathcal{C}, \mathcal{D})$-canonical short exact sequence of A in $\mathcal{A}$ and the $(\mathcal{X} \cap \mathcal{D}, \mathcal{Y} \cap \mathcal{C}')$-canonical short exact sequence of $A_{\mathcal{D}}$ in $\mathcal{C}' \cap \mathcal{D}$

$$\begin{array}{ccccc}
{}_{\mathcal{C}}A & \rightarrowtail & E & \twoheadrightarrow & {}_{\mathcal{X}\cap\mathcal{D}}(A_{\mathcal{D}}) \\
\Vert & & \downarrow f & & \downarrow \\
{}_{\mathcal{C}}A & \rightarrowtail & A & \twoheadrightarrow & A_{\mathcal{D}} \\
 & & & & \downarrow \\
 & & & & (A_{\mathcal{D}})_{\mathcal{Y}\cap\mathcal{C}'}
\end{array}$$

Observe that the upper short exact sequence shows that E is an element of $\mathcal{C} * (\mathcal{X} \cap \mathcal{D})$. By the Snake Lemma, we see that $\operatorname{Coker} f \cong (A_{\mathcal{D}})_{\mathcal{Y}\cap\mathcal{C}'}$. As $A \in \mathcal{X}$, $\operatorname{Hom}_{\mathcal{A}}(A, \mathcal{Y}) = 0$ and so $\operatorname{Coker} f = 0$. Thus $A \cong E \in \mathcal{C} * (\mathcal{X} \cap \mathcal{D})$. The reverse inclusion is clear since both $\mathcal{C}$ and $\mathcal{X} \cap \mathcal{D}$ are contained in $\mathcal{X}$ and $\mathcal{X}$ is closed under extensions. The fact that $\mathcal{Y} = (\mathcal{Y} \cap \mathcal{C}') * \mathcal{D}'$ follows by a dual argument.

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{C}' \cap \mathcal{D}$. We claim that $\mathcal{T} = (\mathcal{C} * \mathcal{T}) \cap \mathcal{D}$. Let $A \in (\mathcal{C} * \mathcal{T}) \cap \mathcal{D}$. As $A \in \mathcal{C} * \mathcal{T}$ there is a short exact sequence

$$C \overset{f}{\rightarrowtail} A \twoheadrightarrow T$$

with $C \in \mathcal{C}$ and $T \in \mathcal{T}$. Now, as $A \in \mathcal{D}$, $\operatorname{Hom}_{\mathcal{A}}(\mathcal{C}, A) = 0$ and, in particular, $f = 0$. Thus $A \cong T \in \mathcal{T}$. The reverse inclusion is clear since $\mathcal{T} \subseteq \mathcal{D}$ by assumption and $\mathcal{T} \subseteq \mathcal{C} * \mathcal{T}$ trivially. The fact that $\mathcal{F} = (\mathcal{F} * \mathcal{D}') \cap \mathcal{C}'$ holds follows by a dual argument. $\square$

The following Corollary is a direct consequence of the inclusion preserving property of the bijection in Theorem III.3.2. We note that this generalises Theorem 4.2 in [9], where the same result is shown to hold in the case that $\mathcal{C}' \cap \mathcal{D}$ is wide, that is, closed under kernels, cokernels and extensions.

Corollary III.3.3. *Let $\mathcal{A}$ be an abelian length category and let $[(\mathcal{C}, \mathcal{D}),\ (\mathcal{C}', \mathcal{D}')]$ be twin torsion pairs in $\mathcal{A}$. Then the set of all torsion classes in $\mathcal{C}' \cap \mathcal{D}$ is a complete lattice isomorphic to the lattice interval $[\mathcal{C}, \mathcal{C}']$ of the complete lattice of torsion classes in $\mathcal{A}$.*

Let us also note that Theorem III.3.2 has since been generalised in [1, Theorem 3.9] for extriangulated categories to the class of '$\mathfrak{s}$-torsion pairs' which are a special class of $\mathbb{E}$-torsion pairs. We discuss this more in Section V.4. For now, we look at consequences of the above Theorem. The following Lemma shows that the hearts of twin torsion pairs are preserved under the bijection of Theorem III.3.2.

Lemma III.3.4. *Let $\mathcal{A}$ be an abelian category, $[(\mathcal{C}, \mathcal{D}),\ (\mathcal{C}', \mathcal{D}')]$ twin torsion pairs in $\mathcal{A}$ and let $[(\mathcal{T}, \mathcal{F}),\ (\mathcal{T}', \mathcal{F}')]$ be twin torsion pairs in $\mathcal{C}' \cap \mathcal{D}$. Then*

$$\mathcal{T}' \cap \mathcal{F} = (\mathcal{C} * \mathcal{T}') \cap (\mathcal{F} * \mathcal{D}').$$

Proof. Let $A \in (\mathcal{C} * \mathcal{T}') \cap (\mathcal{F} * \mathcal{D}')$ and consider the commutative diagram

$$\begin{array}{ccccc} {}_{\mathcal{C}}A & \rightarrowtail & A & \twoheadrightarrow & A_{\mathcal{D}} \\ \downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle g} \\ {}_{\mathcal{C}'}A & \rightarrowtail & A & \twoheadrightarrow & A_{\mathcal{D}'} \end{array}$$

with top (resp. bottom) row being the $(\mathcal{C}, \mathcal{D})$-canonical (resp. $(\mathcal{C}', \mathcal{D}')$-canonical) short exact sequences of X. The existence of the vertical maps f and g follows from the fact that $\mathcal{C} \subseteq \mathcal{C}'$. Moreover, it follows from Lemma III.3.6 and its dual that ${}_{\mathcal{C}'}A \in \mathcal{F}$ and $A_{\mathcal{D}} \in \mathcal{T}$; in particular, ${}_{\mathcal{C}'}A, A_{\mathcal{D}} \in \mathcal{C}' \cap \mathcal{D}$. Thus, as $\operatorname{Hom}_{\mathcal{A}}(\mathcal{C}, \mathcal{D}) = 0$, $f = 0$ and we deduce that $A_{\mathcal{C}} \cong 0$ and $A \cong A_{\mathcal{D}} \in \mathcal{T}'$. Similarly, $A \cong {}_{\mathcal{C}'}A \in \mathcal{F}$ and we have $A \in \mathcal{T}' \cap \mathcal{F}$. The reverse inclusion is trivial. □

Example III.3.5. In [43] the class of *left-strongly quasi-hereditary* algebras were introduced. These are quasi-hereditary algebras such that the subcategory $\operatorname{Filt}(\Delta)$ is a torsionfree class itself in $\operatorname{mod}\Lambda$. Let Λ be a left strongly quasi-hereditary algebra. Then it follows from Proposition III.1.9 and Theorem III.3.2 that for all $j < i$ we have twin torsion pairs in $\operatorname{mod}\Lambda$

$$\Big[\big(\operatorname{Filt}(\Delta_{>j}) * {}^{\perp}\operatorname{Filt}(\Delta)\ ,\ \operatorname{Filt}(\Delta_{\leq j})\big), \big(\operatorname{Filt}(\Delta_{>i}) * {}^{\perp}\operatorname{Filt}(\Delta)\ ,\ \operatorname{Filt}(\Delta_{\leq i})\big)\Big]$$

With heart

$$\operatorname{Filt}(\Delta_{\leq i}) \cap \Big(\operatorname{Filt}(\Delta_{>j}) * {}^{\perp}\operatorname{Filt}(\Delta)\Big) = \operatorname{Filt}(\Delta_{(j,i]})$$

which, by Corollary III.4.18, is quasi-abelian.

The following lemma will be useful later.

Lemma III.3.6. *Let $\mathcal{A}$ be an abelian category, $[(\mathcal{C}, \mathcal{D}),\ (\mathcal{C}', \mathcal{D}')]$ be twin torsion pairs in $\mathcal{A}$ and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{C}' \cap \mathcal{D}$. Then for $A \in \mathcal{A}$, we have that $A \in \mathcal{C} * \mathcal{T}$ if and only if $A_{\mathcal{D}} \in \mathcal{T}$. In particular, any short exact sequence showing A as an element of $\mathcal{C} * \mathcal{T}$ is isomorphic to the $(\mathcal{C}, \mathcal{D})$-canonical short exact sequence of A.*

Proof. Suppose that $A \in \mathcal{C} * \mathcal{T}$. Then there is a short exact sequence $C \rightarrowtail A \twoheadrightarrow T$ with $C \in \mathcal{C}$ and $T \in \mathcal{T}$. Since $\mathcal{T} \subseteq \mathcal{D}$, by the uniqueness of the $(\mathcal{C}, \mathcal{D})$-canonical short exact sequence of A, we deduce that $A_{\mathcal{D}} \cong T \in \mathcal{T}$.

Conversely, suppose that $A_{\mathcal{D}} \in \mathcal{T}$. Then the $(\mathcal{C}, \mathcal{D})$-canonical short exact sequence of A exhibits A as an element of $\mathcal{C} * \mathcal{T}$. □

III.3.1 Functorial finiteness

In this section, we investigate how the bijection in Theorem III.3.2 reflects the functorially finite property of torsion(free) classes. Recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ is *functorially finite* if both $\mathcal{T}$ and $\mathcal{F}$ are functorially finite subcategories. In the following result, we see how functorially finite torsion pairs behave under the bijection of Theorem III.3.2. The proof of part (ii) was privately communicated by Gustavo Jasso who used a similar argument in his work on τ-tilting reduction [71, Theorem 3.13].

Proposition III.3.7. *Let $\mathcal{A}$ be an abelian category and let $[(\mathcal{C}, \mathcal{D}), (\mathcal{C}', \mathcal{D}')]$ be twin torsion pairs in $\mathcal{A}$.*

(i) *If $(\mathcal{X}, \mathcal{Y})$ is a functorially finite torsion pair in $\mathcal{A}$ such that $\mathcal{C} \subseteq \mathcal{X} \subseteq \mathcal{C}'$, then $(\mathcal{X} \cap \mathcal{D}, \mathcal{Y} \cap \mathcal{C}')$ is a functorially finite torsion pair in $\mathcal{C}' \cap \mathcal{D}$.*

(ii) *Suppose that $\mathcal{A}$ has enough projectives and injectives and that $(\mathcal{C}, \mathcal{D})$ and $(\mathcal{C}', \mathcal{D}')$ are functorially finite as torsion pairs in $\mathcal{A}$. Suppose $(\mathcal{T}, \mathcal{F})$ is a functorially finite torsion pair in $\mathcal{C}' \cap \mathcal{D}$, then $(\mathcal{C} * \mathcal{T}, \mathcal{F} * \mathcal{D}')$ is a functorially finite torsion pair in $\mathcal{A}$.*

For the proof we will need the following result.

Lemma III.3.8 ([68, Proposition 5.33])**.** *Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ and $\mathcal{Y}$ be full subcategories of $\mathcal{T}$. If $\mathcal{X}$ and $\mathcal{Y}$ are contravariantly finite in $\mathcal{T}$, then so is $\mathcal{X} * \mathcal{Y}$.*

Proof of Proposition III.3.7. **(i):** Let $(\mathcal{X}, \mathcal{Y})$ be a functorially finite torsion pair in $\mathcal{A}$. In light of Lemma III.1.3(iii)(iii') we only need to check that $\mathcal{X} \cap \mathcal{D}$ is covariantly finite in $\mathcal{C}' \cap \mathcal{D}$ and that $\mathcal{Y} \cap \mathcal{C}'$ is contravariantly finite in $\mathcal{C}' \cap \mathcal{D}$. We will show the first property, the second will follow by a dual argument. Let $A \in \mathcal{C}' \cap \mathcal{D}$ and let $\beta : A \to X$ be a left $\mathcal{X}$-approximation of A, which exists as $A \in \mathcal{A}$. Consider the canonical $(\mathcal{C}, \mathcal{D})$-short exact sequence of X

$$_{\mathcal{C}}X \xrightarrow{f} X \xrightarrow{g} X_{\mathcal{D}}.$$

Observe that, as $\mathcal{X}$ is closed under factor objects, $X_{\mathcal{D}} \in \mathcal{X}$ and therefore $X_{\mathcal{D}} \in \mathcal{X} \cap \mathcal{D}$. We claim that $g\beta : A \to X_{\mathcal{D}}$ is a left $\mathcal{X} \cap \mathcal{D}$-approximation of A in $\mathcal{C}' \cap \mathcal{D}$. Indeed, let $r : A \to X'$ be a morphism with $X' \in \mathcal{X} \cap \mathcal{D}$ then, as $X' \in \mathcal{X}$ and g is a left $\mathcal{X}$-approximation of A, there exists a morphism $\gamma : X \to X'$ such that $\gamma\beta = r$. Now, as $X' \in \mathcal{D}$, $(\gamma f : {}_{\mathcal{C}}X \to X') = 0$ and as $g = \operatorname{Coker} f$, there exists a morphism $\delta : X_{\mathcal{D}} \to X'$ such that $\delta g = \gamma$. Together we have $r = \gamma\beta = \delta(g\beta)$ and thus r factors through $g\beta$ as required.

(ii): Suppose that $(\mathcal{C}, \mathcal{D})$ and $(\mathcal{C}', \mathcal{D}')$ are functorially finite torsion pairs in $\mathcal{A}$ and let $(\mathcal{T}, \mathcal{F})$ be a functorially finite torsion pair in $\mathcal{C}' \cap \mathcal{D}$. We claim that

$(\mathcal{C} * \mathcal{T}, \mathcal{F} * \mathcal{D}')$ is a functorially finite torsion pair in $\mathcal{A}$. By Lemma III.1.3, we only need to show that $\mathcal{C} * \mathcal{T}$ (resp. $\mathcal{F} * \mathcal{D}'$) is covariantly (resp. contravariantly) finite in $\mathcal{A}$. Both facts follow from Lemma III.3.8 and its dual by using the equivalences $\mathsf{D}^{\mathrm{b}}(\mathcal{A}) \cong \mathsf{K}^{-}(\mathsf{proj}\mathcal{A})$ and $\mathsf{D}^{\mathrm{b}}(\mathcal{A}) \cong \mathsf{K}^{+}(\mathsf{inj}\mathcal{A})$ respectively which hold as $\mathcal{A}$ has enough projectives (resp. injectives) together with the observation that, in this case, $\mathcal{A}$ is a functorially finite subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. □

III.4 Torsion pairs in quasi-abelian categories

The aim of this section is to characterise torsion pairs in quasi-abelian categories. For a torsionfree class $\mathcal{F}$ of an abelian category $\mathcal{A}$ (which are quasi-abelian as we noted in Section 3) we have already done this in the previous sections: By taking the twin torsion pairs $[(\mathcal{T}, \mathcal{F}), (\mathcal{A}, 0)]$, Theorem III.3.2 tells us that torsionfree classes in $\mathcal{F}$ are precisely torsionfree classes of $\mathcal{A}$ that lie in $\mathcal{F}$ with corresponding torsion classes obtained by intersecting with $\mathcal{F}$. Using results of Bondal & van den Bergh and Rump, we may do this for all quasi-abelian categories.

Lemma III.4.1 ([25, Proposition B.3], [112, Theorem 2], [111, Theorem 1])**.** *Let $\mathcal{Q}$ be a quasi-abelian category. Then $\mathcal{Q}$ is a torsionfree class in an abelian category $\mathcal{L}_{\mathcal{Q}}$. Moreover, $\mathcal{Q}$ is a cotilting torsionfree class in $\mathcal{L}_{\mathcal{Q}}$ (that is, every object in $\mathcal{L}_{\mathcal{Q}}$ is a quotient of an object in $\mathcal{Q}$). This gives a correspondence between cotiltings in abelian categories and quasi-abelian categories.*

$\mathcal{L}_{\mathcal{Q}}$ is sometimes referred to as the *(left) associated abelian category of* $\mathcal{Q}$.

III.4.1 The category $\mathcal{L}_{\mathcal{Q}}$.

Following [111] and originally due to Schneiders [116, §1.2], we give a construction of $\mathcal{L}_{\mathcal{Q}}$. Then we investigate the conditions on $\mathcal{Q}$ such that $\mathcal{L}_{\mathcal{Q}}$ is a (small) module category.

Recall the homotopy category of $\mathcal{Q}$, $\mathsf{K}(\mathcal{Q})$, whose objects are chain complexes of objects of $\mathcal{Q}$ and morphisms are chain complex morphisms modulo homotopy, see [116, 1.2.1] for details. Let $\mathcal{X}$ be the subcategory of $\mathsf{K}(\mathcal{Q})$ consisting of complexes concentrated in degrees 0 and 1 with the non-trivial differential being an monomorphism. That is, complexes of the form

$$\dots \longrightarrow 0 \longrightarrow X^0 \xrightarrow{f} X^1 \longrightarrow 0 \longrightarrow \dots$$

that are exact in X^0. In practice, we identify the above complex with the monomorphism f.

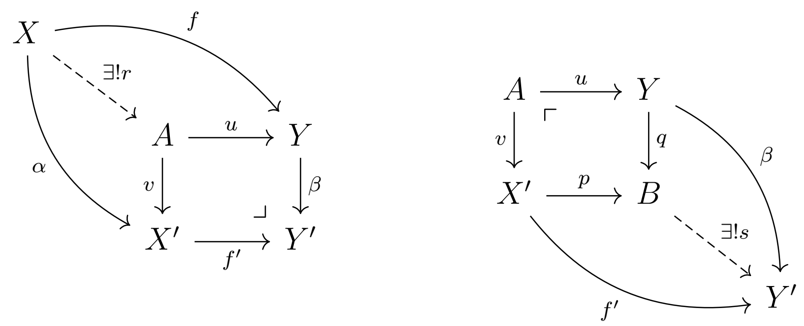
Remark III.4.2. We make some observations.

(a) A morphism, $(\alpha, \beta) : f \to f'$ in $\mathcal{X}$ is just a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ X' & \xrightarrow{f'} & Y' \end{array} \tag{III.4}$$

and is null homotopic if there exists $h : Y \to X'$ in $\mathcal{Q}$ such that $hf = \alpha$ and $f'h = \beta$. Observe that, as f is monic, if $f'h = \beta$ then $hf = \alpha$ is automatically satisfied.

(b) [111, Proposition 6] We may describe kernels and cokernels of a morphism as in (III.4) explicitly in $\mathcal{X}$. Consider the commutative diagrams in $\mathcal{Q}$



Then it is easily verified that the morphisms in $\mathcal{X}$

$$
\begin{array}{ccc}
X & \xrightarrow{r} & A \\
\| & & \downarrow{\scriptstyle u} \\
X & \xrightarrow[f]{} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{\scriptstyle p} & & \| \\
B & \xrightarrow[s]{} & Y'
\end{array}
$$

give a kernel and cokernel of (III.4) in $\mathcal{X}$ respectively. It follows that a morphism as in (III.4) is monic if and only if it is a pullback and it is regular (that is, both monic and epic) if and only if it is an exact square in $\mathcal{Q}$. Futhermore, we can naturally decompose any morphism as in (III.4) into a cokernel followed by a regular morphism followed by a kernel:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle r}\downarrow & & \| \\
A & \xrightarrow{u} & Y \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle q} \\
X' & \xrightarrow[p]{} & B \\
\| & & \downarrow{\scriptstyle s} \\
X' & \xrightarrow[f']{} & Y'.
\end{array}
$$

By [111, Proposition 1, Proposition 3], we may formally invert all regular morphisms to obtain the category $\mathcal{L}_{\mathcal{Q}}$ which is abelian by [51, 3.2].

Remark III.4.3. Schneiders originally observed $\mathcal{L}_{\mathcal{Q}}$ as the heart of a canonical t-structure on the category $\mathsf{K}(\mathcal{Q})$ as part of his work to study the derived category of a quasi-abelian category. We also note that $\mathcal{L}_{\mathcal{Q}}$ is a special case of [18, Exemple 1.3.22] where the authors made the corresponding construction in the more general setting of exact categories. Similar categories have also been considered in other contexts, for example [127].

Remark III.4.4. There is a canonical inclusion

$$\begin{array}{ccc} \mathcal{Q} & \hookrightarrow & \mathcal{L}_{\mathcal{Q}} \\ A & \mapsto & (0 \to A) \end{array}$$

which is full, faithful, additive, exact and preserves monomorphisms. We implicitly identify $\mathcal{Q}$ with its image in $\mathcal{L}_{\mathcal{Q}}$.

In the following results, for brevity we say 'projective' when we really mean '$\mathbb{E}$-projective where $(\mathbb{E}, \mathfrak{s})$ is the external triangulation corresponding to the exact structure $\mathcal{E}_{all}$' (Remark II.1.18).

Lemma III.4.5 ([111, Lemma 4])**.** *Let $\mathcal{Q}$ be a quasi-abelian category. Then* $\mathrm{Proj}\mathcal{Q} = \mathrm{Proj}\mathcal{L}_{\mathcal{Q}}$.

Corollary III.4.6. *Let $\mathcal{Q}$ be a quasi-abelian category. Then $\mathcal{L}_{\mathcal{Q}}$ has a (small) projective generator if and only if $\mathcal{Q}$ has a (small) projective generator.*

Proof. Suppose that $\mathcal{L}_{\mathcal{Q}} =: \mathcal{L}$ has a projective generator, then by Lemma III.4.5 we may assume that is is of the form $(0 \to P)$ with $P \in \mathrm{Proj}\mathcal{Q}$. Then, for all $A \in \mathcal{Q}$ there exists a morphism $p : P^I \to A$ in $\mathcal{Q}$ for some set I such that

$$\begin{array}{ccc} 0 & \longrightarrow & P^I \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ 0 & \longrightarrow & A \end{array} \tag{III.5}$$

is an epimorphism in $\mathcal{L}$. We claim that p is a cokernel in $\mathcal{Q}$. By computations as in Remark III.4.2(b), the cokernel of III.5 is

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \big\downarrow & & \big\| \\ \mathrm{Coim}_{\mathcal{Q}}\, p & \xrightarrow[\tilde{p}]{} & A \end{array}$$

where $\tilde{p} : \mathrm{Coim}_{\mathcal{Q}}\, p \to A$ is the canonical morphism. By assumption, this morphism is null-homotopic, so there exists a morphism $h : A \to \mathrm{Coim}_{\mathcal{Q}}\, p$ such that $\tilde{p}h = 1_A$. In particular, $\tilde{p}$ is a retraction. It is well-known (e.g. [112, §1]) that in a quasi-abelian category the morphism $\tilde{p}$ is always a monomorphism. Thus we deduce that $\mathrm{Coim}_{\mathcal{Q}}\, p \cong A$ and that p is a cokernel in Q.

Conversely, suppose that P is a projective generator of $\mathcal{Q}$. We will show that $(0 \to P)$ is a projective generator of $\mathcal{L}$. First note that $(0 \to P)^I \cong (0 \to P^I)$ for a set I. By Proposition III.4.5, $(0 \to P)$ is projective, it remains to show that for all $(f : A \to B) \in \mathcal{L}$ there exists an epimorphism $(0 \to P) \twoheadrightarrow (f : A \to B)$. To this end, let $p : P^I \twoheadrightarrow B$ be a cokernel in $\mathcal{Q}$. We will show that the composition

$$\begin{array}{ccc} 0 & \longrightarrow & P^I \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ 0 & \longrightarrow & B \\ \big\downarrow & & \big\| \\ A & \xrightarrow{f} & B \end{array}$$

is epic. Since the embedding $\mathcal{Q} \hookrightarrow \mathcal{L}$ is exact, the upper commutative square is an epimorphism. By computations as in III.4.2(b), the cokernel of the lower commutative square is

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow{\scriptstyle f} & & \big\| \\ B & = & B \end{array}$$

which is null-homotopic by the identity morphism $B \to B$. Thus the lower commutative square is an epimorphism. We conclude that the composition is an epimorphism. □

As a consequence, there is a 'Gabriel-Mitchell theorem for quasi-abelian categories', giving conditions on $\mathcal{Q}$ such that $\mathcal{L}_{\mathcal{Q}}$ is a module category.

Proposition III.4.7 ([111, Proposition 12])**.** *Let $\mathcal{Q}$ be a quasi-abelian category. Then $\mathcal{L}_{\mathcal{Q}} \cong \mathrm{Mod}\Lambda$ if and only if $\mathcal{Q}$ has a small basic projective generator P. Moreover, in this case $\Lambda \cong \mathrm{End}_{\mathcal{Q}}P$.*

We may also describe when $\mathcal{L}_{\mathcal{Q}}$ is a small module category over certain classes of rings. Note that in the next results we are working with subobjects and not $\mathcal{E}$-subobjects.

Lemma III.4.8. *Let $\mathcal{Q}$ be a quasi-abelian category. Then $\mathcal{L}_{\mathcal{Q}}$ is noetherian (resp. artinian) with respect to subobjects (that is, every ascending (resp. descending) chain of subobjects stabilises) if and only if $\mathcal{Q}$ is noetherian (resp. artinian) with respect to subobjects.*

Proof. Since $\mathcal{L}_{\mathcal{Q}}$ is abelian, we may identify subobjects with kernels, thus by Remark III.4.2(b) we may assume all subobjects of $(f : A \to B)$ are of the form

$$\begin{array}{ccc} A & \xrightarrow{r} & X \\ \big\| & & \big\downarrow{\scriptstyle u} \\ A & \xrightarrow{f} & B \end{array}$$

and note that r and u are monomorphisms. Hence, an ascending chain of f subobjects in $\mathcal{L}_{\mathcal{Q}}$ corresponds to an ascending chain of subobjects

$$A \subset X_1 \subset X_2 \subset \cdots \subset B$$

in $\mathcal{Q}$ and the claim follows. The dual argument holds for descending chains of subobjects. □

Theorem III.4.9. *Let $\mathcal{Q}$ be a quasi-abelian category. Then $\mathcal{L}_{\mathcal{Q}} \cong \mathrm{mod}\Lambda$ for a right noetherian (resp. artinian) ring if and only if $\mathcal{Q}$ is noetherian (resp. noetherian and artinian) with respect to subobjects and has a basic projective generator P. Moreover, in this case $\Lambda \cong \mathrm{End}_{\mathcal{Q}}(P)$.*

Proof. By, for example, [85, Theorem 1 and Corollary 1] an abelian category is equivalent to a small module category over a right noetherian (resp. artinian ring) if and only if it admits a projective generator and is noetherian (resp. noetherian and artinian) with respect to subobjects; whence the claim follows from Corollary III.4.6 and Lemma III.4.8. □

We also discuss the existence of an injective cogenerator in $\mathcal{Q}$.

Lemma III.4.10. *Let $\mathcal{Q}$ be a quasi-abelian category with an injective cogenerator I. Then the following implication holds:*

$$\forall X \in \mathcal{L}_{\mathcal{Q}},\ \operatorname{Ext}^1_{\mathcal{L}}(X,(0\to I)) = 0 = \operatorname{Hom}_{\mathcal{L}}(X,(0\to I)) = 0 \Rightarrow X \cong 0 \qquad \text{(III.6)}$$

Proof. Let $X = (f : A \to B) \in \mathcal{L} := \mathcal{L}_{\mathcal{Q}}$, then, by construction there is a short exact sequence in $\mathcal{L}$

$$(0\to A) \xrightarrow{(0,f)} (0\to B) \xrightarrow{(0,1_B)} X$$

By assumption there is a short exact sequence $0 \to A \to I \to A/I \to 0$ in $\mathcal{Q}$ which, after embedding into $\mathcal{L}$, remains a short exact sequence. We form the pushout of these short exact sequences to obtain

$$\begin{array}{ccccc}
(0\to A) & \xrightarrow{(0,f)} & (0\to B) & \xrightarrow{(0,1_B)} & X \\
\downarrow & & \downarrow & & \| \\
(0\to I) & \longrightarrow & E & \longrightarrow & X \\
\downarrow & & \downarrow & & \\
(0\to I/A) & = & (0\to I/A). & &
\end{array}$$

Since $\mathcal{Q}$ is closed under extensions, we have that $E \in \mathcal{Q}$. Now suppose that $\operatorname{Ext}^1_{\mathcal{L}}(X,(0\to I)) = 0$ then the second row splits which gives that $X \in \mathcal{Q}$. Thus the claim follows since I is a cogenerator of Q. □

Recall from [57] that an object T in an abelian category $\mathcal{A}$ is called *cotilting* if $\operatorname{Sub} T$ is a cotilting torsionfree class such that T is Ext-injective with respect to this torsionfree class and that the implication III.6 of Lemma III.4.10 holds (with appropriate change of notation). We note that by [57, Lemma 4.5] over an artin algebra Λ, cotilting modules and cotilting objects in $\operatorname{mod}\Lambda$ coincide. We have the following as an immediate consequence.

Corollary III.4.11. *Let $\mathcal{Q}$ be a quasi-abelian category with an injective cogenerator I. Then $(0 \to I)$ is a cotilting object in $\mathcal{L}_{\mathcal{Q}}$.*

Remark III.4.12. Dually, for a quasi-abelian category $\mathcal{Q}$, one may construct an abelian category $\mathcal{R}_{\mathcal{Q}}$ such that $\mathcal{Q}$ is a torsion class in $\mathcal{R}_{\mathcal{Q}}$. This gives another proof of Proposition III.2.2. Moreover, the categories $\mathcal{L}_{\mathcal{Q}}$ and $\mathcal{R}_{\mathcal{Q}}$ are derived equivalent and are related by tilting induced by $\mathcal{Q}$, see [25], [50], [111] and [116] for more details.

III.4.2 Properties of torsion(free) classes

We compare the properties of torsion and torsion free classes in quasi-abelian categories with the abelian setting.

Proposition III.4.13. *Every torsion and torsionfree class in a quasi-abelian category has the structure of a quasi-abelian category.*

Proof. By Theorem III.3.2 every torsionfree class, $\mathcal{F}$, of a quasi-abelian category $\mathcal{Q}$ is a torsion class of $\mathcal{L}_\mathcal{Q}$ that happens to lie in $\mathcal{Q}$ and is therefore quasi-abelian. Theorem III.3.2 also tells us that the associated torsion class in $\mathcal{Q}$ is $\mathcal{T} \cap \mathcal{Q}$ where $\mathcal{T} = {}^\perp\mathcal{F}$ in $\mathcal{L}_\mathcal{Q}$. But $\mathcal{T} \cap \mathcal{Q}$ is the intersection of a torsion and torsionfree class, and as $\mathcal{F} \subseteq \mathcal{Q}$, by Proposition III.2.2, it is quasi-abelian. □

We now prove that the converse of Lemma III.1.3(i)(i') holds in quasi-abelian categories giving a familiar characterisation of torsion classes.

Proposition III.4.14. *Let $\mathcal{Q}$ be a quasi-abelian category. Then a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ is a torsion pair on $\mathcal{Q}$ if and only if the following hold.*

(i) *For all $A \in \mathcal{Q}$, $\mathrm{Hom}_\mathcal{Q}(A, \mathcal{F}) = 0$ iff $A \in \mathcal{T}$;*

(ii) *For all $B \in \mathcal{Q}$, $\mathrm{Hom}_\mathcal{Q}(\mathcal{T}, B) = 0$ iff $B \in \mathcal{F}$.*

Proof. The fact that the conditions are necessary was proved in Lemma III.1.3. We now prove that they are sufficient. Let $\mathcal{T}, \mathcal{F}$ be full subcategories of $\mathcal{Q}$ such that

$$\begin{aligned} \mathcal{T} = & \ \{A \in \mathcal{Q} \mid \mathrm{Hom}_\mathcal{Q}(A, \mathcal{F}) = 0\} \\ \mathcal{F} = & \ \{B \in \mathcal{Q} \mid \mathrm{Hom}_\mathcal{Q}(\mathcal{T}, B) = 0\}. \end{aligned}$$

Observe that if $({}^\perp\mathcal{Q} * \mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{L} = \mathcal{L}_\mathcal{Q}$, then $(({}^\perp\mathcal{Q} * \mathcal{T}) \cap \mathcal{Q}, \mathcal{F}) = (\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{Q}$ which proves the statement. It remains to show that $({}^\perp\mathcal{Q} * \mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{L}$. Since $\mathcal{L}$ is abelian, it suffices to show that

$$\begin{aligned} {}^\perp\mathcal{Q} * \mathcal{T} = & \ \{A \in \mathcal{L} \mid \mathrm{Hom}_\mathcal{L}(A, \mathcal{F}) = 0\} \\ \mathcal{F} = & \ \{B \in \mathcal{L} \mid \mathrm{Hom}_\mathcal{L}({}^\perp\mathcal{Q} * \mathcal{T}, B) = 0\} \end{aligned}$$

Let $B \in \mathcal{L}$ be such that $\mathrm{Hom}_\mathcal{L}({}^\perp\mathcal{Q} * \mathcal{T}, B) = 0$. In particular, we have that $\mathrm{Hom}_\mathcal{L}({}^\perp\mathcal{Q}, B) = 0$ thus $B \in \mathcal{Q}$ and as $0 = \mathrm{Hom}_\mathcal{L}(\mathcal{T}, B) = \mathrm{Hom}_\mathcal{Q}(\mathcal{T}, B)$, $B \in \mathcal{F}$. Now let $A \in \mathcal{L}$ be such that $\mathrm{Hom}_\mathcal{L}(A, \mathcal{F}) = 0$ and consider the $({}^\perp\mathcal{Q}, \mathcal{Q})$-canonical short exact sequence of A

$$0 \longrightarrow {}_{{}^\perp\mathcal{Q}}A \longrightarrow A \xrightarrow{p} A_\mathcal{Q} \longrightarrow 0.$$

Observe that $\mathrm{Hom}_\mathcal{L}(A_\mathcal{Q}, \mathcal{F}) = \mathrm{Hom}_\mathcal{Q}(A_\mathcal{Q}, \mathcal{F}) = 0$, else by pre-composing with the epimorphism p we would obtain a non-zero morphism $A \to \mathcal{F}$. Thus $A_\mathcal{Q} \in \mathcal{T}$ and the sequence shows A is an element of ${}^\perp\mathcal{Q} * \mathcal{T}$. □

In the case of torsion pairs in small module categories over artin algebras, which are abelian, there is a well-known symmetry:

Proposition III.4.15 ([120, Theorem])**.** *Let $\mathcal{A} \cong \mathrm{mod}\,\Lambda$ with Λ an artin algebra be an abelian category and $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. Then $\mathcal{T}$ is functorially finite in $\mathcal{A}$ if and only if $\mathcal{F}$ is functorially finite in $\mathcal{A}$.*

We will see that this symmetry extends to the quasi-abelian setting. We note that $\mathcal{L}_\mathcal{Q} \cong \mathrm{mod}\Lambda$ for an artin algebra Λ is satisfied, for instance, the conditions of Theorem III.4.9 are met and $\mathrm{End}_\mathcal{Q}(P)$ is an artin algebra. In particular, if $\mathcal{Q}$ is a k-linear hom-finite quasi-abelian category for a field k, then $\mathrm{End}_\mathcal{Q}(P)$ is finite dimensional and hence artin. We first prove a lemma that does not depend on $\mathrm{End}_\mathcal{Q}(P)$.

Lemma III.4.16. *Let $\mathcal{Q}$ be a quasi-abelian category. Then $\mathcal{Q}$ is functorially finite in $\mathcal{L} = \mathcal{L}_{\mathcal{Q}}$.*

Proof. As we noted in Lemma III.4.1, by [25, Proposition B.3] $\mathcal{Q}$ is a cotilting torsionfree class of $\mathcal{L}_{\mathcal{Q}}$ and is therefore covariantly finite in $\mathcal{L}$. It remains to show that $\mathcal{Q}$ is contravariantly finite. Let $(f : A \to B) \in \mathcal{L}$, we claim that the morphism

$$\begin{array}{ccc} 0 & \longrightarrow & B \\ \downarrow & & \| \\ A & \xrightarrow{f} & B \end{array}$$

in $\mathcal{L}$ is a right $\mathcal{Q}$-approximation of f. Indeed, if $(\alpha, \beta) : (0 \to B') \to f$ is some morphism in $\mathcal{L}$ (note that necessarily $\alpha = 0$), then $(0, \beta) : (0 \to B') \to (0 \to B)$ gives the required factorisation. □

The author thanks Lidia Angeleri-Hügel for pointing out an inaccuracy in a previous version of the following result.

Proposition III.4.17. *Let $\mathcal{Q}$ be a quasi-abelian category such that $\mathcal{L}_{\mathcal{Q}} \cong mod\Lambda$ for an artin algebra Λ. and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{Q}$. Then $\mathcal{T}$ is functorially finite if and only if $\mathcal{F}$ is functorially finite.*

Proof. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in a quasi-abelian category $\mathcal{Q}$ and suppose that $\mathcal{F}$ is functorially finite in $\mathcal{Q}$. We begin by showing that $\mathcal{F}$ is functorially finite in $\mathcal{L} = \mathcal{L}_{\mathcal{Q}}$ (this was not done in Proposition III.3.7). As $\mathcal{F}$ is a torsionfree class in $\mathcal{L}$, it is covariantly finite in $\mathcal{L}$. We now show that every $X \in \mathcal{L}$ admits a right $\mathcal{F}$-approximation. Let $Q \to X$ be a right $\mathcal{Q}$-approximation of X, which exists by Lemma III.4.16, and let $F \to Q$ be a right $\mathcal{F}$-approximation of Q, which exists by assumption. Then it is easily verified that the composition $F \to X$ is a right $\mathcal{F}$-approximation of X. Thus $\mathcal{F}$ is a functorially finite torsion class in $\mathcal{A}$ and therefore so is its associated torsion free class ${}^{\perp}\mathcal{Q} * T$ in $\mathcal{L}$ by Theorem III.4.9 and Proposition III.4.15. It now follows from Proposition III.3.7(i) that $\mathcal{T}$ is functorially finite in $\mathcal{Q}$.

For the converse, we reverse the argument but use Proposition III.3.7(ii) to see that ${}^{\perp}\mathcal{Q} * \mathcal{T}$ is functorially finite in $\mathcal{L}$ which we may do since $\mathcal{L}$ has enough projectives by Corollary III.4.6 as $\mathcal{Q}$ has a projective generator by assumption. □

We close this section by noting some other properties of torsion classes the quasi-abelian setting are also inherited from the abelian case.

Lemma III.4.18. *The intersection of torsion classes in a quasi-abelian category is again a torsion class. Also, the heart of twin torsion pairs in a quasi-abelian category is quasi-abelian.*

III.5 Harder-Narasimhan filtrations

In this section we apply the results of the previous sections to show the existence of Harder-Narasimhan filtrations arising from chains of torsion classes in quasi-abelian categories. We recall the pseudometric on the space of chains of torsion classes defined by [124] building on [29] and then investigate topological properties of this space. To begin, we recall the necessary concepts following [124, §2] where it was shown that abelian categories admit Harder-Narasimhan filtrations.

Definition III.5.1. For an additive category, $\mathcal{A}$, consider the order reversing functions of posets, η, from the real interval $[0,1]$ to the set of all torsion classes of $\mathcal{A}$ such that $\eta(0) = \mathcal{A}$ and $\eta(1) = 0$. Equivalently, the data of such a map is a chain of torsion classes in $\mathcal{A}$

$$\eta: \quad 0 = \mathcal{T}_1 \subseteq \cdots \subset \mathcal{T}_r \subset \cdots \subseteq \mathcal{T}_0 = \mathcal{A}$$

with $r \in [0,1]$ satisfying $\mathcal{T}_r \subseteq \mathcal{T}_{r'}$ if and only if $r \geq r'$. We call such an η *quasi-Noetherian* (resp. *weakly-Artinian*) if for every interval $(a,b) \subset [0,1]$ there exists $s \in (a,b)$ such that ${}_{\mathcal{T}_r}A \hookrightarrow {}_{\mathcal{T}_s}A$ (resp. $s' \in (a,b)$ such that ${}_{\mathcal{T}_{s'}}A \hookrightarrow {}_{\mathcal{T}_r}A$) for all $r \in (a,b)$ and $A \in \mathcal{A}$. By $\mathfrak{T}(\mathcal{A})$ we denote the set of all η that are quasi-Noetherian and weakly-Artinian.

Notation III.5.2. Let $\eta = (\mathcal{T}_i)_{i \in [0,1]} \in \mathfrak{T}(\mathcal{A})$. For any $j \in [0,1]$, by $\mathcal{F}_j$ we denote the associated torsionfree class of $\mathcal{T}_j$ in $\mathcal{A}$.

Lemma III.5.3. *Let $\mathcal{Q}$ be a quasi-abelian category and $\eta = (\mathcal{T}_i)_{i \in [0,1]} \in \mathfrak{T}(\mathcal{Q})$. Then for every $r \in [0,1]$, the pairs of subcategories*

$$\Big(\bigcup_{s>r} \mathcal{T}_s, \bigcap_{s>r} \mathcal{F}_s\Big) \text{ and } \Big(\bigcap_{s<r} \mathcal{T}_s, \bigcup_{s<r} F_s\Big)$$

are torsion pairs in $\mathcal{Q}$. Moreover, for all $\mathcal{X} \in \operatorname{tors} \mathcal{Q}$, if $\mathcal{X} \subset \mathcal{T}_s$ for all $s < r$ then $\mathcal{X} \subset \bigcap_{s<r} \mathcal{T}_s$. Similarly, if $\mathcal{T}_s \subset \mathcal{X}$ for all $s > r$ then $\bigcup_{s>r} \mathcal{T}_s \subset \mathcal{X}$.

Proof. Let $r \in [0,1]$, we will show that the pair $(\mathcal{T}, \mathcal{F}) = (\bigcup_{s>r} \mathcal{T}_s, \bigcap_{s>r} \mathcal{F}_s)$ satisfies the hom-orthogonality conditions of Proposition III.4.14. Since, for all $s > r$, $\mathcal{F} \subset \mathcal{F}_s$ we have that $\operatorname{Hom}_{\mathcal{Q}}(\mathcal{T}_s, \mathcal{F}) = 0$ and hence $\operatorname{Hom}_{\mathcal{Q}}(\mathcal{T}, \mathcal{F}) = 0$.

Let $Y \in \mathcal{Q}$ be such that $\operatorname{Hom}_{\mathcal{Q}}(\mathcal{T}, Y) = 0$. Then for all $s > r$, $\operatorname{Hom}_{\mathcal{Q}}(\mathcal{T}_s, Y) = 0$ and so $Y \in \mathcal{F}_s$ for all $s > r$, thus $Y \in \mathcal{F}$.

Let $X \in \mathcal{Q}$ and suppose that $X \notin \mathcal{T}$. Then for all $s > r$, $X \notin \mathcal{T}_s$. Since $(\mathcal{T}_s, \mathcal{F}_s)$ is a torsion pair in $\mathcal{Q}$ it follows that $\operatorname{Hom}_{\mathcal{Q}}(X, \mathcal{F}_s) \neq 0$ for all $s > r$. We deduce that $\operatorname{Hom}_{\mathcal{Q}}(X, \mathcal{F}) \neq 0$ as the $\mathcal{F}_i$ are ordered by inclusion.

Thus we have shown that $\mathcal{T} = \{X \in \mathcal{Q} \mid \operatorname{Hom}_{\mathcal{Q}}(X, \mathcal{F}\} = 0$ and $\mathcal{F} = \{X \in \mathcal{Q} \mid \operatorname{Hom}_{\mathcal{Q}}(\mathcal{T}, X) = 0\}$, so by Proposition III.4.14, $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{Q}$. The fact that $(\bigcap_{s<r} \mathcal{T}_s, \bigcup_{s<r} F_s)$ is a torsion pair follows from a similar argument and the remaining claims are basic set theory. $\square$

Definition III.5.4. Let $\mathcal{A}$ be an additive category and $\eta = (\mathcal{T}_i)_{i \in [0,1]} \in \mathfrak{T}(\mathcal{A})$. For $r \in [0,1]$, we define the subcategories $\mathcal{P}_r^\eta$ as follows

$$\mathcal{P}_r^\eta = \begin{cases} \bigcap_{s>0} \mathcal{F}_s & \text{if } r = 0 \\ \left(\bigcap_{s<r} \mathcal{T}_s\right) \cap \left(\bigcap_{s>r} \mathcal{F}_s\right) & \text{if } r \in (0,1) \\ \bigcap_{s<1} \mathcal{T}_s & \text{if } r = 1. \end{cases}$$

Remark III.5.5. In a quasi-abelian category $\mathcal{Q}$, for every $\eta = (\mathcal{T}_i)_{i \in [0,1]} \in \mathfrak{T}(\mathcal{Q})$, each $\mathcal{P}_r^\eta$ is quasi-abelian. For $r = 0, 1$ this is obvious. For $r \in (0,1)$ observe that

$$\bigcup_{s>r} \mathcal{T}_s \subseteq \mathcal{T}_r \subseteq \bigcap_{s<r} \mathcal{T}_s$$

thus $\bigcup_{s>r}\mathcal{T}_s$ and $\bigcap_{s<r}\mathcal{T}_s$ define twin torsion pairs with heart $\mathcal{P}_r^\eta$ and is hence quasi-abelian by Theorem III.2.2. We also note that for all twin torsion pairs $[(\mathcal{C},\mathcal{D}),(\mathcal{C}',\mathcal{D}')]$ in $\mathcal{Q}$ their heart, $\mathcal{C}'\cap\mathcal{D}$, appears as $\mathcal{P}_r^\eta$ in the chain of torsion classes

$$\eta: \quad 0\subset\mathcal{C}\subset\mathcal{C}'\subset\mathcal{Q}$$

for some $r\in[0,1]$.

Set-up III.5.6. Let $\mathcal{A}$ be an abelian category and fix twin torsion pairs $[(\mathcal{C},\mathcal{D}),(\mathcal{C}',\mathcal{D}')]$ in $\mathcal{A}$ and set $\mathcal{Q}=\mathcal{C}'\cap\mathcal{D}$. By Theorem III.3.2, we may identify $\mathfrak{T}(\mathcal{Q})$ bijectively with a subset of $\mathfrak{T}(\mathcal{A})$ along the map

$$\begin{aligned}\phi_{\mathcal{C}}=\phi:\mathfrak{T}(\mathcal{Q})&\hookrightarrow\mathfrak{T}(\mathcal{A})\\ \eta=(\mathcal{T}_i)_{i\in[0,1]}&\mapsto\phi(\eta)=(\mathcal{X}_i)_{i\in[0,1]}\end{aligned}$$

where

$$\mathcal{X}_i=\begin{cases}\mathcal{A} & \text{if } i=0\\ \mathcal{C}*\mathcal{T}_i & \text{if } i\in(0,1)\\ 0 & \text{if } i=1.\end{cases}$$

We denote the image of $\phi_{\mathcal{C}}$ by $\mathfrak{T}_{\mathcal{C}}(\mathcal{Q})$. Thus $\mathfrak{T}_{\mathcal{C}}(\mathcal{Q})$ consists of all $\eta=(\mathcal{T}_i)_{i\in[0,1]}\in\mathfrak{T}(\mathcal{A})$ such that $\mathcal{C}\subseteq\mathcal{T}_i\subseteq\mathcal{C}'$ for all $i\in(0,1)$. We remark that, in light of Remark III.2.3, this map does indeed depend on $\mathcal{C}$ (since then $\mathcal{Q}$ determines $\mathcal{C}'$ by Theorem III.3.2).

We investigate the subcategories $\mathcal{P}_r^{\phi(\eta)}$.

Lemma III.5.7. *In the situation of Set-up III.5.6. Let $\eta=(\mathcal{T}_i)_{i\in[0,1]}\in\mathfrak{T}(\mathcal{Q})$. Then*

$$\mathcal{P}_r^{\phi(\eta)}=\begin{cases}\mathcal{P}_0^\eta*\mathcal{D}' & \text{if } r=0;\\ \mathcal{P}_r^\eta & \text{if } r\in(0,1);\\ \mathcal{C}*\mathcal{P}_1^\eta & \text{if } r=1.\end{cases}$$

Proof. Let $r\in(0,1)$, and observe

$$\begin{aligned}\mathcal{P}_r^{\phi(\eta)}&=\Big(\bigcap_{s\in(0,r)}(\mathcal{C}*\mathcal{T}_s)\Big)\cap\Big(\bigcup_{s\in(r,1)}(\mathcal{C}*\mathcal{T}_s)\Big)^{\perp}\\ &=\Big(\mathcal{C}*\bigcap_{s\in(0,r)}\mathcal{T}_s\Big)\cap\Big(\mathcal{C}*\bigcup_{s\in(r,1)}\mathcal{T}_s\Big)^{\perp}\\ &=\Big(\mathcal{C}*\bigcap_{s\in(0,r)}\mathcal{T}_s\Big)\cap\Big(\Big(\bigcup_{s\in(r,1)}\mathcal{T}_s\Big)^{\perp_{\mathcal{Q}}}*\mathcal{D}'\Big)\\ &=\Big(\bigcap_{s\in(0,r)}\mathcal{T}_s\Big)\cap\Big(\bigcup_{s\in(r,1)}\mathcal{T}_s\Big)^{\perp_{\mathcal{Q}}}=\mathcal{P}_r^\eta\end{aligned}$$

where the first and last equalities follow from Lemma III.5.3 and the definitions knowing that $\mathcal{T}_1=0$ and $\mathcal{T}_0=\mathcal{Q}$. The second equality is straightforward set theory, the third equality follows from Theorem III.3.2 and the fourth equality holds by Lemma III.3.4. The cases $r=0,1$ follow by similar arguments. □

We now show that every $\eta \in \mathfrak{T}(\mathcal{Q})$ induces a unique Harder-Narasimhan filtration of each object.

Theorem III.5.8. *In the situation of Set-up III.5.6. Let $\eta = (\mathcal{T}_i)_{i\in[0,1]} \in \mathfrak{T}(\mathcal{Q})$. Then for all $A \in \mathcal{Q}$ there exists a unique (up to isomorphism) Harder-Narasimhan filtration of A with respect to η in $\mathcal{Q}$. That is, a filtration*

$$0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = A$$

of A in $\mathcal{Q}$ such that

(HN1) $A_k/A_{k-1} \in \mathcal{P}^{\eta}_{r_k}$ *for all* $1 \leq k \leq n$;

(HN2) $r_k > r_{k'}$ *if and only if* $k < k'$.

Proof. Let $A \in \mathcal{Q}$ and $\eta = (\mathcal{T}_i)_{i\in[0,1]} \in \mathfrak{T}(\mathcal{Q})$. Let

$$0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$$

be the Harder-Narasimhan filtration of A with respect to $\phi(\eta)$ in $\mathcal{A}$, which exists by [124, 2.9], so that for all $1 \leq k, k' \leq n$, $A_k/A_{k-1} \in \mathcal{P}^{\phi(\eta)}_{r_k}$ and $r_k > r_{k'}$ precisely when $k < k'$. We claim that this is also the Harder-Narasimhan filtration of A with respect to η in $\mathcal{Q}$.

We first show that for all $1 \leq k \leq n$, $A_k/A_{k-1} \in \mathcal{P}^{\eta}_{r_k}$. When $r_k \in (0,1)$, this is trivially true by Lemma III.5.7. It remains to check for $r_k = 0, 1$. Observe that the only case where $r_k = 0$ (resp. $r_k = 1$) can occur is when $k = n$ (resp. $k = 1$). So suppose that $r_n = 0$, then $A_n/A_{n-1} = A/A_{n-1} \in \mathcal{P}^{\phi(\eta)}_0 = \mathcal{P}^{\eta}_0 * \mathcal{D}'$. As $A \in \mathcal{Q}$ is an element of $\mathcal{C}'$, so is the quotient A/A_{n-1}. Thus $A/A_{n-1} \in \mathcal{C}' \cap (\mathcal{P}^{\eta}_0 * \mathcal{D}') = \mathcal{P}^{\eta}_0$ by Theorem III.3.2. Similarly, we see that $A_1/A_0 = A_1 \in \mathcal{D} \cap \mathcal{P}^{\phi(\eta)}_1 = \mathcal{D} \cap (\mathcal{C} * \mathcal{P}^{\eta}_1) = \mathcal{P}^{\eta}_1$. By Lemma III.5.7, this implies that (HN1) holds. Note that (HN2) holds since it is inherited from the abelian case $\mathcal{A}$ as is the uniqueness of the filtration up to isomorphism.

It remains to show that $A_i \in \mathcal{Q}$ for all $1 \leq k \leq n$. We proceed by induction on k. For $k = 1$ we have shown that $A_1 = A_1/A_0 \in \mathcal{P}^{\eta}_{r_1} \subset \mathcal{Q}$ for some $r_1 \in [0,1]$. The $k > 1$ case follows by using the short exact sequences

$$0 \longrightarrow A_{k-1} \longrightarrow A_k \longrightarrow A_k/A_{k-1} \longrightarrow 0$$

in $\mathcal{A}$ as $A_k/A_{k-1} \in \mathcal{P}^{\eta}_{r_k} \subset \mathcal{Q}$ and since $\mathcal{Q}$ is closed under extensions. □

Corollary III.5.9. *Let $\mathcal{Q}$ be a quasi-abelian category and $\eta = (\mathcal{T}_i)_{i\in[0,1]} \in \mathfrak{T}(\mathcal{Q})$. Then for all $A \in \mathcal{Q}$ there exists a unique (up to isomorphism) Harder-Narasimhan filtration with respect to η in $\mathcal{Q}$.*

Proof. The result follows from Theorem III.5.8 as $\mathcal{Q}$ appears as $\mathcal{P}^{\eta}_r$ for some $r \in [0,1]$ in the chain of torsion classes

$$\eta : \quad 0 \subset {}^{\perp}\mathcal{Q} \subset \mathcal{L}_{\mathcal{Q}}$$

in $\mathfrak{T}(\mathcal{L}_{\mathcal{Q}})$. □

Example III.5.10. Let Λ be a standardly stratified algebra. Then, it follows from Proposition III.1.9 that we have a chain of torsion classes in $\mathrm{Filt}(\Delta)$:

$$\eta : \quad \{0\} \subset \mathrm{Filt}(\Delta_{>n-1}) \subset \mathrm{Filt}(\Delta_{>n-2}) \subset \cdots \subset \mathrm{Filt}(\Delta_{>1}) \subset \mathrm{Filt}(\Delta).$$

Then if Λ is left strongly quasi-hereditary (Example III.3.5) then we may use Corollary III.5.9 to see that η induces Harder-Narasimhan filtrations of all objects in $\mathrm{Filt}\,\Delta$. Alternatively, we could deduce this by repeatedly applying Proposition III.1.9 and we note that this second method works even without the left strongly quasi-hereditary assumption.

We recall that, by [29, 6.1] and [124, 7.1], for an abelian category $\mathcal{A}$, $\mathfrak{T}(\mathcal{A})$ is a topological space with pseudometric given by

$$d(\eta, \eta') = \inf\{\varepsilon \in [0,1] \mid \mathcal{P}_r^{\eta'} \subset \mathcal{P}_{[r-\varepsilon, r+\varepsilon]}^{\eta} \forall r \in [0,1]\} \tag{III.7}$$

for $\eta, \eta' \in \mathfrak{T}(\mathcal{A})$. Where

$$\mathcal{P}_{[a,b]}^{\eta} := \mathrm{Filt}\Big(\bigcup_{s\in[a,b]} \mathcal{P}_s^{\eta}\Big)$$

for $0 \leq a \leq b \leq 1$ and we set $\mathcal{P}_r^{\eta} = 0$ for all $r \notin [0,1]$.

Remark III.5.11. Note that, for $\eta, \eta' \in \mathfrak{T}(\mathcal{A})$, we have $d(\eta, \eta') = 0$ if and only if $\mathcal{P}_r^{\eta} = \mathcal{P}_r^{\eta'}$ for all $r \in [0,1]$.

As we remarked earlier, the embedding of $\mathfrak{T}(\mathcal{Q})$ (of Set-up III.5.6) in $\mathfrak{T}(\mathcal{A})$ depends on $\mathcal{C}$. So when $\mathcal{Q}$ occurs as the heart of many twin torsion pairs, $\mathfrak{T}(\mathcal{Q})$ can be embedded into $\mathfrak{T}(\mathcal{A})$ in as many ways. To finish, we see that the various embeddings of $\mathcal{Q}$ are at maximal distance apart in $\mathfrak{T}(\mathcal{A})$ and that each of these embeddings is closed.

Theorem III.5.12. *In the situation of Set-up III.5.6, $\mathfrak{T}_{\mathcal{C}}(\mathcal{Q})$ is a closed set of the topological space $\mathfrak{T}(\mathcal{A})$ and if $\mathcal{Q}$ also occurs as the heart of different twin torsion pairs $[(\mathcal{C}_1, \mathcal{D}_1), (\mathcal{C}_1', \mathcal{D}_1')]$, then*

$$d(\mathfrak{T}_{\mathcal{C}}(\mathcal{Q}), \mathfrak{T}_{\mathcal{C}_1}(\mathcal{Q})) = 1$$

where $\mathfrak{T}_{\mathcal{C}_1}(\mathcal{Q})$ is defined following Set-up III.5.6.

Proof. First, we show that $\mathfrak{T}_{\mathcal{C}}(\mathcal{Q})$ contains all of its accumulation points and is therefore a closed set of $\mathfrak{T}(\mathcal{A})$. To this end, let $x = (\mathcal{T}_i)_{i\in[0,1]}$ be an accumulation point of $\mathfrak{T}_{\mathcal{C}}(\mathcal{Q})$ so that there exists a sequence $(\eta_n)_{n\in\mathbb{N}}$ in $\mathfrak{T}_{\mathcal{C}}(\mathcal{Q})$ such that $d(x, \eta_n) < \frac{1}{n}$.

To this end, let $\eta = (\mathcal{T}_i)_{i\in[0,1]} \in \mathfrak{T}(\mathcal{A})$ such that there exists $\eta' \in \mathfrak{T}_{\mathcal{C}}(\mathcal{Q})$ with $d(\eta, \eta') = d(\eta', \eta) = 0$. We want to show that $x \in \mathfrak{T}_{\mathcal{C}}(\mathcal{Q})$ so we therefore must show that $\mathcal{C} \subseteq \mathcal{P}_1^x$ and $\mathcal{D}' \supseteq \mathcal{P}_0^x$. For the first of these claims, let s be the supremum of the set

$$\{i \in [0,1] \mid \mathcal{C} \subseteq \mathcal{P}_{[i,1]}^x\}.$$

Note that this set is non-empty since $\mathcal{A} = \mathcal{P}_{[0,1]}^x$ by Theorem III.5.8. We must consider the case when $s < 1$. Then for large enough n such that $\frac{n-1}{n} > s$ we have that $d(x, \eta_n) > \frac{1}{n}$ which is a contradiction. The second claim is shown dually. $\square$

Chapter IV

Intersections, sums and Jordan-Holder property for exact categories

The chapter is organised as follows. In the first section (IV.1) we give characterisations of abelian categories in terms of the behaviour of admissible morphisms and prove an exact analog of the fourth isomorphism theorem which will be a useful tool. In Section IV.2 we recall the (AI) and (AIS) categories from [60] which are the exact categories that admit admissible intersections and sums of $\mathcal{E}$-subobjects in a similar way to abelian categories. We then prove that the (AI) categories are precisely the quasi-abelian categories and the (AIS) categories are precisely the abelian categories. To end that section, we discuss how these results apply to the category of Banach spaces. In Section IV.3, we address the shortcomings of the (AI) and (AIS) categories by introducing a generalised intersection and sum which makes sense for arbitrary exact categories. Using this, we define the Diamond exact categories and show that these categories satisfy the $\mathcal{E}$-Jordan-Hölder property. In the next section (IV.4), we introduce an exact analog of the Jacobson radical and study exact categories that behave well with respect to the radical, which we call the $\mathcal{E}$-Artin-Wedderburn categories. We show that these Krull-Schmidt exact categories with this property are $\mathcal{E}$-Jordan Hölder and apply this theory to classify completely exact structures on the module category of a Nakayama algebra with the Jordan-Hölder property. We finish in Section IV.5, by discussing a length function that exists for an exact category with the $\mathcal{E}$-Jordan-Hölder property.

IV.1 General results

We show an $\mathcal{E}$-version of the fourth isomorphism theorem. We also give some results describing the behaviour of admissible morphisms, which yields a new characterisation of abelian categories in Theorem IV.1.4.

IV.1.1 Admissible morphisms and abelian categories

Definition IV.1.1 ([35, Definition 8.1])**.** A morphism $f : A \to B$ in an exact category is called *admissible* if it factors as $f = me$ where m is an admissible monic and e is an admissible epic. Admissible morphisms will sometimes be displayed as

$$A \overset{f}{-\!\circ\!\rightarrow} B$$

in diagrams, and the classes of admissible morphisms of $\mathcal{A}$ will be denoted as $\mathrm{Mor}^{ad}_{\mathcal{A}}$. Similarly, by $\mathrm{Hom}^{ad}_{\mathcal{A}}(A,B)$ (resp. $\mathrm{End}^{ad}_{\mathcal{A}}(A)$) we denote the set of admissible morphisms in $\mathcal{A}$ from A to B (resp. A). Note the abuse of notation here, one should really write $\mathrm{Mor}^{ad}(\mathcal{A},\mathcal{E})$ since the admissible property depends on the exact structure $\mathcal{E}$, however we avoid doing this for readability purposes.

In this subsection we show that the admissible morphisms in an exact category behave poorly, unless we work in an abelian category with the maximal exact structure. Let $(\mathcal{A},\mathcal{E})$ be an exact category. The following fact will be our main tool:

Lemma IV.1.1 ([51, Proposition 3.1])**.** *Suppose that every morphism in $\mathcal{A}$ is admissible, then $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{max} = \mathcal{E}_{all}$.*

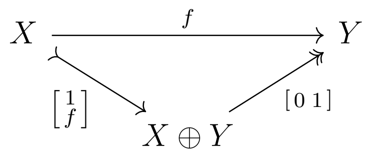
Lemma IV.1.2. *Suppose that the class of admissible morphisms in $\mathcal{A}$ is closed under composition. Then $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{max} = \mathcal{E}_{all}$.*

Proof. We show that every morphism can be written as the composition of a section followed by a retraction. Whence the claim will follow from Lemma IV.1.1 since sections and retractions are always admissible morphisms, since the split exact structure is the minimal exact structure on any additive category. To this end, let $f : X \to Y$ be an arbitrary morphism in $\mathcal{A}$ and consider the two split short exact sequences

$$X \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} X \oplus Y \xrightarrow{[0\ 1]} Y$$

$$X \xrightarrow{\left[\begin{smallmatrix}1\\f\end{smallmatrix}\right]} X \oplus Y \xrightarrow{[-f\ 1]} Y.$$

Then there is a commutative diagram



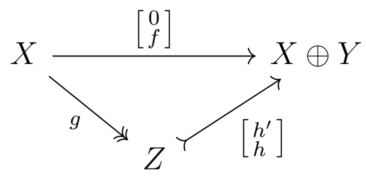
which proves the claim. □

Lemma IV.1.3. *Let $\mathcal{A}$ be a weakly idempotent complete additive category. Suppose that the class of admissible morphisms in $\mathcal{A}$ is closed under addition. Then $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{max} = \mathcal{E}_{all}$.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. Then

$$\left[\begin{smallmatrix}0\\f\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\f\end{smallmatrix}\right] + \left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right] : X \to X \oplus Y$$

is the sum of two sections and is hence admissible by assumption. Let



be a factorisation of $\left[\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right]$ into an admissible epic followed by an admissible monic. Observe that, as g is epic, $h' = 0$. By Lemma II.1.43, we have that if $\left[\begin{smallmatrix} 0 \\ h \end{smallmatrix}\right]$ is an admissible monic then so is h. Thus $f = hg$ and is therefore an admissible morphism. Now the claim follows from Lemma IV.1.1.

□

This shows that, in general, the set of admissible endomorphisms $\mathrm{End}^{ad}_{\mathcal{A}}(X)$ is not a subring of $\mathrm{End}_{\mathcal{A}}(X)$ under the usual addition and composition, also that $\mathrm{Hom}^{ad}(X, Y)$ is not a group under the usual addition. To finish, we summarise the results of this subsection.

Theorem IV.1.4. *Then the following conditions are equivalent:*

(i) $\mathcal{A}$ *is an abelian category and* $\mathcal{E} = \mathcal{E}_{all}$*;*

(ii) $\mathrm{Mor}(\mathcal{A}) = \mathrm{Mor}^{ad}(\mathcal{A})$*;*

(iii) $\mathrm{Mor}^{ad}(\mathcal{A})$ *is closed under composition;*

(iv) $\mathcal{A}$ *is weakly idempotent complete and* $\mathrm{Mor}^{ad}(\mathcal{A})$ *is closed under addition.*

Proof. The implications (ii) ⇒ (i), (iii) ⇒ (ii) and (iv) ⇒ (ii) follow from Lemmas IV.1.1, IV.1.2 and IV.1.3 respectively. Clearly, (ii) ⇒ (iii), (iv); thus it remains to verify that (i) ⇒ (ii). Indeed, let $f : A \to B$ be an arbitrary morphism in an abelian category, then the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle \mathrm{im}\, f} \searrow \qquad \nearrow {\scriptstyle \mathrm{coim}\, f} & \\ & \mathrm{Im}\, f \cong \mathrm{Coim}\, f & \end{array}$$

shows that f is an admissible morphism with respect to $\mathcal{E}_{all}$. □

IV.1.2 Isomorphism theorem

We give a generalisation of the fourth isomorphism theorem for modules to exact categories:

Proposition IV.1.5. (The fourth $\mathcal{E}$-isomorphism theorem) *Let* $(\mathcal{A}, \mathcal{E})$ *be an exact category and let*

$$X' \rightarrowtail X \twoheadrightarrow X/X'$$

be a short exact sequence in $\mathcal{E}$*. Then there is an isomorphism of posets*

$$\{A \in \mathcal{A} \mid X' \rightarrowtail A \rightarrowtail X\} \longleftrightarrow \{B \in \mathcal{A} \mid B \rightarrowtail X/X'\} = \mathcal{P}^{\mathcal{E}}_{X/X'}$$
$$A \longmapsto A/X'.$$

Proof. Let us begin by showing that the correspondence is bijective. First we note that the map $A \mapsto A/X'$ is well-defined by [35, Lemma 3.5]. Next, we define an inverse map ϕ. For $B \rightarrowtail X/X'$ define $\phi(B)$ to be the pullback

$$\begin{array}{ccccc} X' & \rightarrowtail & \phi(B) & \twoheadrightarrow & B \\ \| & & \downarrow{\scriptstyle \alpha} & \lrcorner & \downarrow \\ X' & \rightarrowtail & X & \twoheadrightarrow & X/X'. \end{array}$$

We observe that by [35, Proposition 2.15], α is an admissible monic and thus ϕ is a well-defined map. We now show that the maps are mutually inverse. For $X' \rightarrowtail A \rightarrowtail X$, we apply ϕ by taking the pull-back and by [35, Proposition 2.12, 2.13] we obtain the identity on the left of the diagarm and then the fact that $\phi(A/X') \cong M$ follows from applying the Five Lemma for exact categories [35, Corollary 3.2] to the diagram

$$\begin{array}{ccccc} X' & \rightarrowtail & \phi(A/X') & \twoheadrightarrow & A/X' \\ \| & & \downarrow & & \| \\ X' & \rightarrowtail & A & \twoheadrightarrow & A/X'. \end{array}$$

For $B \rightarrowtail X/X'$, there is a short exact sequence

$$X' \rightarrowtail \phi(B) \twoheadrightarrow B.$$

Thus, $\phi(B)/X' \cong B$ and we are done.

Now we show that this is an isomorphism of posets. First we show that if $X' \rightarrowtail A' \rightarrowtail A \rightarrowtail X$ then $A'/X' \rightarrowtail A/X'$. This follows from applying [35, Lemma 3.5] to the diagram

$$\begin{array}{ccccc} X' & \rightarrowtail & A' & \twoheadrightarrow & A'/X' \\ \| & & \downarrow & \lrcorner & \downarrow \\ X' & \rightarrowtail & A & \twoheadrightarrow & A/X'. \end{array}$$

Finally, we show the converse, that is if $A'/X' \rightarrowtail A/X' \rightarrowtail X/X'$ then $A' \rightarrowtail A$. From earlier in the proof, there is a commutative diagram

$$\begin{array}{ccc} A' & \rightarrowtail & A'/X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ A & \twoheadrightarrow & A/X' \\ \downarrow & \lrcorner & \downarrow \\ X & \twoheadrightarrow & X/X'. \end{array}$$

with the outer rectangle being a pullback. Thus, by the Pullback Lemma and [35, Proposition 2.15], α is an admissible monic. ☐

Remark IV.1.2. By the Fourth $\mathcal{E}$-isomorphism theorem (Proposition IV.1.5), an $\mathcal{E}$-subobject (Y, f) of an object X is $\mathcal{E}$-maximal if and only if for all commutative diagrams

$$\begin{array}{ccc} Y & & \\ {\scriptstyle g}\downarrow & \overset{f}{\searrow} & \\ Z & \underset{h}{\rightarrowtail} & X \end{array}$$

either g or h is an isomorphism.

IV.2 The AI and AIS exact categories

In abelian categories, the notions of intersection and sum of subobjects are given by pullbacks and pushouts respectively, see [53, Section 5] and [104, Definition 2.6]. In this paragraph, we investigate whether these concepts carry to exact categories. We recall the definitions of admissible intersection and sum that were first defined in [60], then show that these lead to characterisations of quasi-abelian and abelian categories respectively. To finish, we look at the category of Banach spaces as an example.

IV.2.1 Definitions and properties

The intersection, which exists and is well defined in a pre-abelian exact category, is not necessarily an *admissible* subobject. We recall the definition of exact categories satisfying the admissible intersection property and the admissible sum property from [60]. Note that, in a previous version of [60], the name quasi-n.i.c.e. was used in the sense that they are **n**ecessarily **i**ntersection **c**losed **e**xact categories, and which we will call **A.I** since they admit **A**dmissible **I**ntersections:

Definition IV.2.1 ([60, Definition 4.3, 4.6])**.** An exact category $(\mathcal{A}, \mathcal{E})$ is called an *AI-category* if $\mathcal{A}$ is pre-abelian additive category satisfying the following additional axiom:

(AI) The pullback A of two admissible monics $j : C \rightarrowtail D$ and $g : B \rightarrowtail D$ exists and yields two admissible monics i and f.

$$\begin{array}{ccc} A & \overset{i}{\rightarrowtail} & B \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ C & \underset{j}{\rightarrowtail} & D \end{array} \tag{IV.1}$$

The object A in the diagram above is called the *intersection* of the $\mathcal{E}$-subobjects (B, g) and (C, j) of D; we also use the notation $B \cap_D C$ for this intersection.

Let us now introduce a special sub-class of the AI exact categories, that we call **A.I.S** exact categories, since they admit **A**dmissible **I**ntersections and **S**ums:

Definition IV.2.2 ([60, Definition 4.5, 4.6])**.** An exact category $(\mathcal{A}, \mathcal{E})$ is called an *AIS-category* if it is an AI-category and moreover it satisfies the following additional axiom:

(AS) The morphism u in the diagram below, given by the universal property of the pushout E of i and f coming from the pullback diagram of the axiom (AI) above, is an admissible monic.

$$\begin{array}{ccccc} A & \overset{i}{\rightarrowtail} & B & & \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow{\scriptstyle l} & \searrow{\scriptstyle g} & \\ C & \underset{k}{\rightarrowtail} & E & & \\ & \searrow_{j} & & \overset{u}{\searrow} & \\ & & & & D \end{array} \tag{IV.2}$$

The object E in the above diagram is called the *sum* of the subobjects (B, g) and (C, j) of D; we also use the notation $B +_D C$ to denote the sum.

Let us note that these definitions generalise the abelian versions.

Lemma IV.2.3 ([60, Corollary 4.11])**.** *Let $\mathcal{A}$ be an abelian category. Then $(\mathcal{A}, \mathcal{E}_{all})$ is an AIS-category.*

Remark IV.2.1. One may consider the duals of the above definitions by taking admissible epics instead of monics. Since our focus is on $\mathcal{E}$-subobjects we only study the above and simply remark that the dual definitions lead to the duals of the results of the rest of Section IV, which hold without statement.

Remark IV.2.2. Equivalently, in the notation of the above definitions, we have

$$B \cap_D C = \mathrm{Ker}\left(B \oplus D \xrightarrow{[g\ -j]} D \right)$$

and

$$B +_D C = \mathrm{Coker}\left(B \cap_D C \xrightarrow{[i\ -f]^t} B \oplus C \right).$$

Lemma IV.2.4. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category and let $f : X \rightarrowtail Z$ and $g : Y \rightarrowtail Z$ be admissible monics. Suppose that $X \cap_Z Y$ exists and is the zero object, then $X +_Z Y \cong X \oplus Y$.*

Proof. By assumption, there is a pullback diagram in $\mathcal{A}$:

$$\begin{array}{ccc} 0 & \rightarrowtail & X \\ \downarrow & \lrcorner & \downarrow f \\ Y & \underset{g}{\rightarrowtail} & Z. \end{array}$$

By direct computation we have that

$$\begin{array}{ccc} 0 & \rightarrowtail & X \\ \downarrow & \ulcorner & \downarrow s \\ Y & \underset{t}{\rightarrowtail} & X \oplus Y \end{array}$$

is a pushout diagram for any pair of morphisms s and t satisfying the universal property of the coproduct. Thus, by definition, $X +_Z Y \cong X \oplus Y$. □

IV.2.2 AI-categories and quasi-abelian categories

It is not difficult to see that the split exact structure $\mathcal{E}_{min}$ does *not* satisfy axiom (AI) unless every sequence splits in $\mathcal{A}$. Compare also [78, Remark 2.4 and 5.3] which helps to show that the category of abelian groups equipped with $\mathcal{E}_{min}$ does *not* satisfy axiom (AI). In fact, an exact structure needs to contain *all* short exact sequences in order to satisfy the (AI) axiom:

Proposition IV.2.5. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. If $(A, \mathcal{E})$ is an AI-category, then $\mathcal{E} = \mathcal{E}_{all}$.*

Proof. Let us suppose that the exact structure $\mathcal{E}$ is strictly included in $\mathcal{E}_{all}$, then there exists a short exact sequence

$$\eta : \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

such that $\eta \notin \mathcal{E}$. Consider the two sections $\left[\begin{smallmatrix}1\\g\end{smallmatrix}\right] : B \to B \oplus C$ and $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] : B \to B \oplus C$. It is easy to verify that the pull-back of these two morphisms is:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle \left[\begin{smallmatrix}1\\g\end{smallmatrix}\right]} \\ B & \xrightarrow[\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]]{} & B \oplus C. \end{array}$$

Since f is not admissible in $\mathcal{E}$, the (AI) axiom is not satisfied and $(\mathcal{A}, \mathcal{E})$ is therefore not an AI-category. □

Thus, in light of Proposition II.1.20, every AI-category is a quasi-abelian category equipped with its maximal exact structure. It has been proved recently in [61, Theorem 6.1] that the converse also holds, and hence together a new characterisation of quasi-abelian categories is established:

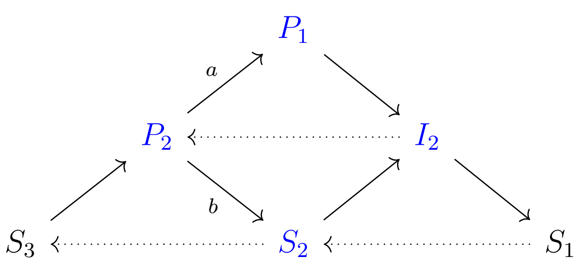
Theorem IV.2.6. (Brüstle, Hassoun, Shah, Tattar, Wegner) *An exact category $(\mathcal{A}, \mathcal{E})$ is an AI-category if and only if $\mathcal{A}$ is quasi-abelian and $\mathcal{E} = \mathcal{E}_{all}$.*

However, as we see in the next example, not every quasi-abelian category with its maximal exact structure is an AIS-category.

Example IV.2.7. As we did in Chapter III, consider the quiver

$$Q : \qquad 1 \longrightarrow 2 \longrightarrow 3$$

The Auslander-Reiten quiver of $\operatorname{rep} Q$ is as follows:



Let $\mathcal{A}$ be the full additive subcategory generated by the indecomposables P_2, P_1, S_2 and I_2. Then $\mathcal{A}$ is an intersection $\mathcal{F} \cap \mathcal{T}'$ where $\mathcal{F}$ is the torsion free class of the hereditary torsion pair $(\mathcal{T}, \mathcal{F}) = (\operatorname{add}(S_1), \operatorname{add}(S_3 \oplus P_2 \oplus P_1 \oplus S_2 \oplus I_2))$ and $\mathcal{T}'$ is the torsion class of the cohereditary torsion pair $(\mathcal{T}', \mathcal{F}') = (\operatorname{add}(P_2 \oplus P_1 \oplus S_2 \oplus I_2 \oplus S_1), \operatorname{add}(S_3))$ of $\operatorname{rep} Q$. By Theorem III.2.2 we conclude that $\mathcal{A}$ is an quasi-abelian category and the only non-split short exact sequence in $\mathcal{A}$ is the Auslander-Reiten sequence

$$0 \longrightarrow P_2 \longrightarrow P_1 \oplus S_2 \longrightarrow I_2 \longrightarrow 0$$

We verify that the axiom (AS) fails; to that end, we consider the following admissible monics in $\mathcal{A}$:

$$\begin{array}{ccc} & & P_2 \\ & & \downarrow{\scriptstyle \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]} \\ P_1 & \xrightarrow[\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]]{} & P_1 \oplus S_2 \end{array}$$

The pullback along these monics in the abelian category $\operatorname{rep} Q$ is given by the object S_3, but this is not available in $\mathcal{A}$. Being quasi-abelian and so pre-abelian, $\mathcal{A}$ admits a pullback which is a subobject of the abelian pullback, thus the zero object. Hence, we have in $\mathcal{A}$ that the intersection along the given monics is $P_1 \cap P_2 = 0$, and therefore, by Lemma IV.2.4, $P_1 + P_2 = P_1 \oplus P_2$. However, the direct sum $P_1 \oplus P_2$ is not an admissible subobject of $P_1 \oplus S_2$, thus the axiom (AS) fails.

IV.2.3 AIS-categories and abelian categories

In this subsection we prove that the categories satisfying both the (AI) and the (AS) axioms are exactly the abelian categories. First we need a Lemma.

Lemma IV.2.8. *Let $\mathcal{A}$ be a quasi-abelian category and $\mathcal{E} = \mathcal{E}_{all}$. Suppose that every monomorphism in $\mathcal{A}$ is a kernel, then $\mathcal{A}$ is abelian. Dually, if every epimorphism is a cokernel, then $\mathcal{A}$ is abelian.*

Proof. Let $f : X \to Y$ be an arbitrary morphism in $\mathcal{A}$ we will show that f is admissible, whence it follows that $\mathcal{A}$ is abelian by Lemma IV.1.1. Recall that there is a commutative diagram in $\mathcal{A}$

$$\begin{array}{ccc} \operatorname{Ker} f & & \operatorname{Coker} f \\ \downarrow & & \uparrow \\ X & \xrightarrow{f} & Y \\ {\scriptstyle c}\downarrow & & \uparrow{\scriptstyle i} \\ \operatorname{Coim} f & \xrightarrow{\bar{f}} & \operatorname{Im} f \end{array}$$

where $\bar{f}$ is both monic and epic (see [112, Section 1] for details) and the columns are $\mathcal{E}$-sequences since $\mathcal{A}$ is quasi-abelian and $\mathcal{E} = \mathcal{E}_{all}$. By assumption, the composition $i\bar{f}$ is a kernel and therefore an admissible monic since $\mathcal{A}$ is quasi-abelian. Thus the decomposition $f = (i\bar{f})c$ shows that f is admissible and we conclude that $\mathcal{A}$ is abelian by Lemma IV.1.1. The proof of the dual statement is similar. □

Now we can prove the main result.

Theorem IV.2.9. *An exact category $(\mathcal{A}, \mathcal{E})$ is an AIS-category if and only if $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{all}$.*

Proof. We noted earlier in Lemma IV.2.3 that every abelian category with its maximal exact structure is AIS. To prove the converse, let $(\mathcal{A}, \mathcal{E})$ be an exact AIS-category. By Theorem IV.2.6, $\mathcal{A}$ is quasi-abelian and $\mathcal{E} = \mathcal{E}_{all}$. Thus, by Lemma

IV.2.8, it is enough to show that every monomorphism $f : X \to Y$ in $\mathcal{A}$ is a kernel. To this end, consider the $\mathcal{E}$-subobjects given by two sections

$$\begin{bmatrix}1\\f\end{bmatrix} : X \rightarrowtail X \oplus Y$$
$$\begin{bmatrix}1\\0\end{bmatrix} : X \rightarrowtail X \oplus Y.$$

By computation, their intersection is the zero-object

$$\begin{array}{ccc} 0 & \rightarrowtail & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \begin{bmatrix}1\\f\end{bmatrix}} \\ X & \underset{\begin{bmatrix}1\\0\end{bmatrix}}{\rightarrowtail} & X \oplus Y. \end{array}$$

Thus, by Lemma IV.2.4, their sum is given by the direct sum $X \oplus X$

$$\begin{array}{ccccc} 0 & \rightarrowtail & X & & \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \begin{bmatrix}1\\1\end{bmatrix}} & \searrow{\scriptstyle \begin{bmatrix}1\\f\end{bmatrix}} & \\ X & \underset{\begin{bmatrix}1\\0\end{bmatrix}}{\rightarrowtail} & X \oplus X & \overset{u}{\rightarrowtail} & X \oplus Y \\ & \searrow{\scriptstyle \begin{bmatrix}1\\0\end{bmatrix}} & & & \nearrow \end{array}$$

where $u = \begin{bmatrix}1 & 0\\0 & f\end{bmatrix}$ is an admissible monic since $(\mathcal{A}, \mathcal{E})$ is (AIS). Now, by [35, Corollary 2.18], f is an admissible monic and we are done. □

Remark IV.2.3. Let $\mathcal{A}$ be an abelian category. Then for all objects $X \in \mathcal{A}$ the poset $\mathcal{P}_X^{\mathcal{E}_{all}}$ of $\mathcal{E}$-subobjects (which is also the poset of subobjects in the classical sense) is a lattice with join and meet operations given by the intersection and sum of $\mathcal{E}$-subobjects.

IV.2.4 Banach spaces: an example

Typical examples of such quasi-abelian but non-abelian categories arise in functional analysis (see [61] for many examples). We next look in detail at one important example, the category of Banach spaces and see how the theory we have developed fits there. We thank Theo Bühler for his helpful remarks about this category.

Definition IV.2.10. We denote by **Ban** the category of Banach spaces (over the field of real numbers). The objects of **Ban** are the complete normed $\mathbb{R}-$vector spaces, and morphisms are continuous linear maps.

The kernel of a morphism $f : X \to Y$ in **Ban** is the linear kernel $f^{-1}(0) \rightarrowtail X$, however the cokernel

$$Y \twoheadrightarrow Y/\overline{f(X)}$$

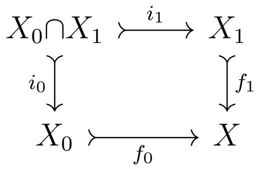
in **Ban** is in general different from the linear cokernel $Y \to Y/f(X)$. Thus $f : X \to Y$ is an admissible monic in **Ban** precisely when f is a monomorphism such that $f(X)$ is closed in Y. The Open Mapping Theorem for Banach spaces guarantees that an admissible monic $f : X \to Y$ is an isomorphism onto $f(X)$. In fact the class

$\mathcal{E} = \mathcal{E}_{all}$ of all kernel-cokernel pairs coincides with the class of short exact sequences of bounded linear maps, see [36, IV.2].

It is well-known that the category **Ban** is quasi-abelian with the maximal exact structure $\mathcal{E}_{all}$, but it is not abelian. We verify here the admissible intersection property and we reprove, using Theorem IV.2.6 that **Ban** is quasi-abelian:

Theorem IV.2.11. *The category* **Ban** *of Banach spaces, equipped with the maximal exact structure* $\mathcal{E} = \mathcal{E}_{all}$*, is an AI-category.*

Proof. Consider two $\mathcal{E}$-subobjects (X_0, f_0), (X_1, f_1) of an object X in **Ban**. Since the admissible monics f_i are isomorphisms onto their range ,$f_i(X_i)$, we can identify X_0 and X_1 with closed subspaces of X. The intersection of closed subspaces is closed, therefore we have the following diagram of closed embeddings (which are admissible monics):

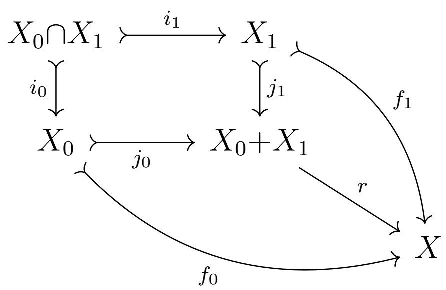


From [60], we know that the object $X_0 \cap X_1$ satisfies the pullback property from axiom (AI) in mod $\mathbb{R}$. Since the pullback can be written as kernel (Remark IV.2.2) and kernels in **Ban** are the kernels in mod $\mathbb{R}$, we conclude that the (AI)-axiom is satisfied: The pullback along admissible monics exists, and yields admissible monics. □

Remark IV.2.4. While **Ban** satisfies the admissible intersection property, it does not satisfy the admissible sum property and so it is not abelian by Theorem IV.2.9. Indeed, it is shown in [24, Chapter 3.1] that *both,* the intersection $X_0 \cap X_1$ and the sum $X_0 + X_1$ (as subvector spaces of X) admit norms turning them into Banach spaces, satisfying that

$$X_0 \cap X_1 \hookrightarrow X_i \hookrightarrow X_0 + X_1$$

are continuous embeddings for $i = 0, 1$. In fact, the whole interval between $X_0 \cap X_1$ and $X_0 + X_1$ is studied in [89], as *interpolations* between intersection and sum. We summarize the situation in the following diagram:



The sum $X_0 + X_1$ is the pushout in mod $\mathbb{R}$, hence satisfies the pushout property in **Ban** since the kernel-cokernel pairs of bounded maps in mod $\mathbb{R}$ are also exact in **Ban**. However, the inclusion map $r : X_0 + X_1 \to X$ (which is bounded, thus continuous) is not an admissible monic in general: The norm on $X_0 + X_1$ is given in [37, Chapter 3.1] by

$$\|x\|_{X_0+X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} \mid x_0 + x_1 = x\},$$

and with respect to this norm, the subspace $X_0 + X_1$ in X is not necessarily closed. Indeed, consider any morphism of Banach spaces $f : X \to Y$ such that the range of f is not closed in Y (see, for instance, [37, Chapter 1.2] for examples). Then, since f is continuous, the subspace

$$G(f) := \{(x, y) \in X \oplus Y \mid fx = y\} \subseteq X \oplus Y$$

which is called the *graph of f* is closed in $X \oplus Y$. However, the sum, Z of $G(f)$ (equipped with the canonical inclusion) and of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] : X \rightarrowtail X \oplus Y$ is not closed in $X \oplus Y$. Since, if it were then the intersection of Z and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] : Y \rightarrowtail X \oplus Y$ would be closed as a subspace of Y by Theorem IV.2.11 but, this intersection is $f(Y)$, which is not closed in Y by assumption.

Remark IV.2.5. We may also describe the $\mathcal{E}$-simple objects of **Ban**. There is only one (up to isomorphism) - $\mathbb{R}$ itself. This follows from the facts that every finite dimensional subspace of a Banach space is closed and that any two norms on a finite dimensional real vector space are equivalent [37, Chapter 1.2].

Furthermore, this tells us that the maximal $\mathcal{E}$-subobjects of a Banach space, X are its closed subspaces of codimension 1.

We also see that the poset $\mathcal{P}_X^{\mathcal{E}}$ of closed subspaces of X is a lattice: Meets in $\mathcal{P}_X^{\mathcal{E}}$ are given by intersections, which are closed. The join of two closed subspaces Y and Z is given by $\overline{Y + Z}$.

It also follows from this discussion that **Ban** satisfies the $\mathcal{E}$-Jordan-Hölder property since every $\mathcal{E}$-composition series of a Banach space X of dimension $n < \infty$ will be of length n and each composition factors is isomorphic to $\mathbb{R}$.

IV.3 The diamond exact categories

In this section we address the drawbacks of the intersection and sum in the previous section by introducing a general notion of intersection and sum that applies to exact categories. We then use this to introduce a class of exact categories - the diamond exact categories - and show that these satisfy the $\mathcal{E}$-Jordan-Hölder property as in Definition IV.3.1.

IV.3.1 Jordan-Hölder property

Definition IV.3.1. Let $(\mathcal{A}, \mathcal{E})$ be an exact category. A finite $\mathcal{E}$-composition series for an object X of $\mathcal{A}$ is a sequence

$$0 = X_0 \overset{i_0}{\rightarrowtail} X_1 \overset{i_1}{\rightarrowtail} \dots \overset{i_{n-2}}{\rightarrowtail} X_{n-1} \overset{i_{n-1}}{\rightarrowtail} X_n = X \qquad \text{(IV.3)}$$

where all i_l are *proper admissible monics* with $\mathcal{E}$-simple cokernel. We say an exact category $(\mathcal{A}, \mathcal{E})$ has the *($\mathcal{E}$-)Jordan-Hölder property* or is a *Jordan-Hölder exact category* if any two finite $\mathcal{E}$-composition series for an object X of $\mathcal{A}$

$$0 = X_0 \overset{i_0}{\rightarrowtail} X_1 \overset{i_1}{\rightarrowtail} \dots \overset{i_{m-2}}{\rightarrowtail} X_{m-1} \overset{i_{m-1}}{\rightarrowtail} X_m = X$$

and

$$0 = X'_0 \overset{i'_0}{\rightarrowtail} X'_1 \overset{i'_1}{\rightarrowtail} \dots \overset{i'_{n-2}}{\rightarrowtail} X'_{n-1} \overset{i'_{n-1}}{\rightarrowtail} X'_n = X$$

are equivalent, that is, they have the same length and the same composition factors, up to permutation and isomorphism.

Remark IV.3.1. As shown in [60, Theorem 6.2], one can use the same steps as in [17] and the $\mathcal{E}$-Schur lemma [60, Proposition 3.5] one may prove that every abelian category (equipped with maximal exact structure) is a Jordan-Hölder exact category using only the (AI) and (AIS) axioms.

IV.3.2 General intersection and sum

For an AIS-category $(\mathcal{A}, \mathcal{E})$, or equivalently, for an abelian category $\mathcal{A}$ with maximal exact structure $\mathcal{E}_{all}$, the intersection of two subobjects of X is defined as the pullback of their monomorphisms in X and their sum is defined as the pushout of this pullback, which is also admissible. In terms of the poset $\mathcal{P}_X^{\mathcal{E}}$ of $\mathcal{E}$-subobjects of X, this means that $\mathcal{P}_X^{\mathcal{E}}$ forms a lattice as we remarked before. However, in general the poset $\mathcal{P}_X^{\mathcal{E}}$ is not a lattice, even when the $\mathcal{E}$-Jordan-Hölder property holds for the exact category $(\mathcal{A}, \mathcal{E})$, as the following simple examples demonstrate.

Example IV.3.2. Let $\mathcal{A}$ be the category of all even dimensional k-vector spaces endowed with the split exact structure $\mathcal{E} = \mathcal{E}_{min}$. Then the $\mathcal{E}$-simple objects are precisely the two-dimensional vector spaces, and the Jordan-Hölder property is clearly satisfied. Consider the object $X = k^6$ with basis $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and the two elements of $\mathcal{P}_X^{\mathcal{E}}$ given by

$$V_1 = < v_1, v_2, v_3, v_4 > \quad \text{and} \quad V_2 = < v_2, v_3, v_4, v_5 > .$$

The intersection $V_1 \cap V_2$ in $\operatorname{mod} k$ is $V_3 = < v_2, v_3, v_4 >$. But since V_3 is not in $\mathcal{A}$, every two-dimensional subspace U of V_3 is a maximal lower bound for both V_1 and V_2, when we view (U, f) as an element in $\mathcal{P}_X^{\mathcal{E}}$ with its inclusion map f. Therefore $\mathcal{P}_X^{\mathcal{E}}$ is not a lattice, and the intersection of V_1 and V_2 is not unique in $(\mathcal{A}, \mathcal{E})$, in fact it is an infinite set formed by all embeddings (U, f) of maximal proper subspaces U of V_3.

Example IV.3.3. A similar phenomenon can be observed studying the additive category $\mathcal{A} = \operatorname{rep} A_2$ of representations of the quiver of type A_2, endowed with the minimal exact structure $\mathcal{E} = \mathcal{E}_{min}$. We denote the simple representations by S_1 and S_2, and the indecomposable projective-injective representation by P_1. Then there is a non-split indecomposable short exact sequence in $\mathcal{A}$

$$0 \longrightarrow S_2 \xrightarrow{f} P_1 \xrightarrow{g} S_1 \longrightarrow 0$$

which is not admissible in $\mathcal{E}_{min}$. Therefore $(\mathcal{A}, \mathcal{E}_{min})$ is not an AI-category by Proposition IV.2.5. Choosing the object $X = S_2 \oplus P_1 \oplus S_1$, we observe that there are many maximal $\mathcal{E}$-subobjects of X with quotient S_1 given by $(S_2 \oplus P_1, \alpha_\lambda)$ with $\lambda \in k$, where

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \lambda g \end{bmatrix} = \alpha_\lambda : S_2 \oplus P_1 \to X = S_2 \oplus P_1 \oplus S_1.$$

Each of these admit many maximal $\mathcal{E}$-subobjects with quotient P_1 given by $(S_2 \oplus P_1, \beta_\mu)$ with $\mu \in k$ where

$$\begin{bmatrix} 1 \\ \mu f \end{bmatrix} = \beta_\mu : S_2 \to S_2 \oplus P_1.$$

The preceding examples motivate the following definition, where we allow the (generalised) intersection and sum to be a set of objects:

Definition IV.3.4. Let (A_i, f_i), $i \in I$, be a collection of $\mathcal{E}$-subobjects of X indexed by a set I. We denote the set of all their common admissible subobjects with respect to X as

$$\mathrm{Sub}_X(\{(A_i, f_i) \mid i \in I\}) := \{(Y, h) \in P^{\mathcal{E}}_X \ \mid \ Y \in \mathcal{P}^{\mathcal{E}}_{A_i}; \forall i \in I\}$$

and define the $\mathcal{E}$-relative intersection of the (A_i, f_i) in $\mathcal{P}^{\mathcal{E}}_X$ as

$$\mathrm{Int}_X(\{(A_i, f_i) \mid i \in I\}) := \mathrm{Max}(\mathrm{Sub}_X(\{(A_i, f_i) \mid i \in I\}),$$

the set of maximal elements in $\mathrm{Sub}_X(\{(A_i, f_i) \mid i \in I\})$ (where we define the generalised intersection over the empty set to be $\{0\}$). Dually, we denote the set of all common superobjects of $A_i, i \in I$

$$\mathrm{Sup}_X(\{(A_i, f_i) \mid i \in I\}) := \{ \ (Y, h) \in \mathcal{P}^{\mathcal{E}}_X \ \mid \ (A_i, f_i) \in \mathcal{P}^{\mathcal{E}}_Y, \ \forall i \in I\}$$

and define the $\mathcal{E}$-relative sum of the $A_i, i \in I$ in $\mathcal{P}^{\mathcal{E}}_X$ as

$$\mathrm{Sum}_X(\{(A_i, f_i) \mid i \in I\}) := \mathrm{Min}(\mathrm{Sup}_X(\{(A_i, f_i) \mid i \in I\})),$$

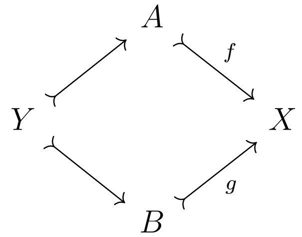
the set of minimal elements in $\mathrm{Sup}_X(\{(A_i, f_i) \mid i \in I\})$.

Example IV.3.5. In the setup of Example IV.3.2, the objects V_1 and V_2 have as $\mathcal{E}$-relative intersection in $\mathcal{P}^{\mathcal{E}}_X$ the Grassmannian $\mathrm{Int}_X(V_1, V_2) = Gr(2, 3)$ of all maximal proper subspaces of V_3. The set $\mathrm{Sum}_X(V_1, V_2)$ however consists only of the element X itself. In Example IV.3.3, any two of the objects $(S_2 \oplus P_1, \alpha_\lambda)$ have an infinite intersection containing all elements (S_2, β_μ) of $\mathcal{P}^{\mathcal{E}}_X$, and conversely, any two of the (S_2, β_μ) have an infinite sum containing all the objects $(S_2 \oplus P_1, \alpha_\lambda)$.

IV.3.3 The diamond categories are Jordan-Hölder exact categories

In this section we prove the $\mathcal{E}$-Jordan-Hölder property in a more general context than abelian categories, namely for exact categories that we call *the diamond exact categories*:

Definition IV.3.6. (**Diamond Axiom**) Let (A, f) and (B, g) be two distinct maximal $\mathcal{E}$-subobjects in $\mathcal{P}_X$, that is, their cokernels X/A and X/B are $\mathcal{E}$-simple. We say that (A, f) and (B, g) satisfy the *diamond axiom* if for every $Y \in \mathrm{Int}_X(A, B)$ we have that A/Y and B/Y are both $\mathcal{E}$-simple and the elements of the sets $\{X/A, A/Y\}$, $\{X/B, B/Y\}$ are equal up to permuation and isomorphism.



A *diamond* exact category $(\mathcal{A}, \mathcal{E})$ is an exact category that satisfies the diamond axiom for each pair of maximal subobjects A and B of a fixed object X.

Remark IV.3.2. When $\mathcal{A}$ is an abelian category equipped with its maximal exact structure, then for each object X we have that $\mathrm{Int}_X(A, B)$ and $\mathrm{Sum}_X(A, B)$ are given by the unique objects $A \cap_X B$ and $A +_X B$, respectively. We also have the second isomorphism theorem, which tells us that

$$B/A \cap_X B \cong A +_B B/A.$$

When A and B are maximal $\mathcal{E}$-subobjects of X, we have that $A +_X B = X$ and we deduce from the above that $(\mathcal{A}, \mathcal{E}_{max})$ satisfies the Diamond axiom. Moreover, in this case, we always have 'crosswise' isomorphisms

$$X/A \cong B/A \cap_X B \quad \text{and} \quad X/B \cong A/A \cap_X B.$$

In Example IV.3.3 we see that one can have the lengthwise isomorphisms

$$X/A \cong A/Y \quad \text{and} \quad X/B \cong B/Y$$

when the poset $\mathcal{P}_X^{\mathcal{E}}$ is not a lattice.

Lemma IV.3.7. *Assume that an object X in a diamond exact category $\mathcal{A}$ has a composition series of length n:*

$$0 = B_0 \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_n = X.$$

If (C, f) is a maximal element in $\mathcal{P}_X$, then there exists a composition series of X through C of length n:

$$0 = C_0 \rightarrowtail C_1 \rightarrowtail \dots \rightarrowtail C_{n-2} \rightarrowtail C \overset{f}{\rightarrowtail} X.$$

Proof. By induction on n. For $n = 1$, this is obvious because $C = 0$. Assume now $n \geq 2$. If $B_{n-1} = C$ as elements in $\mathcal{P}_X^{\mathcal{E}}$, we can use the given composition series of X. Otherwise, consider an element $Y \in \mathrm{Int}_X(B_{n-1}, C)$:

$$\begin{array}{ccccc} 0 = B_0 \rightarrowtail \dots \rightarrowtail & B_{n-1} & & & \\ \nearrow & & \searrow & & \\ Y & & & X. & \\ \searrow & & \nearrow & & \\ & C & & & \end{array}$$

By the diamond axiom, both quotients B_{n-1}/Y and C/Y are $\mathcal{E}$-simple since B_{n-1} and C are maximal elements in $\mathcal{P}_X^{\mathcal{E}}$. By assumption we have a composition series of length $n - 1$ of B_{n-1}:

$$0 = B_0 \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_{n-1} = X'.$$

Since Y is maximal in $\mathcal{P}_{B_{n-1}}^{\mathcal{E}}$, we may apply our induction hypothesis so that there exists a composition series of B_{n-1} through Y of length $n - 1$. We remove the final arrow $Y \rightarrowtail B_{n-1}$ in this series and append $Y \rightarrowtail C \rightarrowtail X$. This yields a composition series of X through C of length n:

$$0 = Y_0 \rightarrowtail \dots \rightarrowtail Y_{n-3} \rightarrowtail Y \rightarrowtail C \xrightarrow{f} X.$$

□

Theorem IV.3.8. *Every diamond exact category is a Jordan-Hölder exact category.*

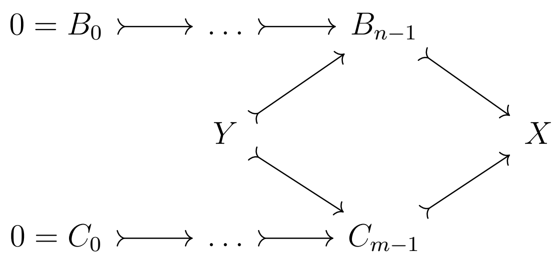
Proof. Following the strategy of the proof in [99, Chapter 4.5], assume we are given two composition series

$$0 = B_0 \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_n = X$$

and

$$0 = C_0 \rightarrowtail C_1 \rightarrowtail \dots \rightarrowtail C_m = X.$$

We proceed by induction on n. For $n = 1$, the object X is $\mathcal{E}$-simple and the statement clearly holds. Assume now $n \geq 2$. For any object $Y \in \mathrm{Int}_X(B_{n-1}, C_{m-1})$ we obtain the following diagram:



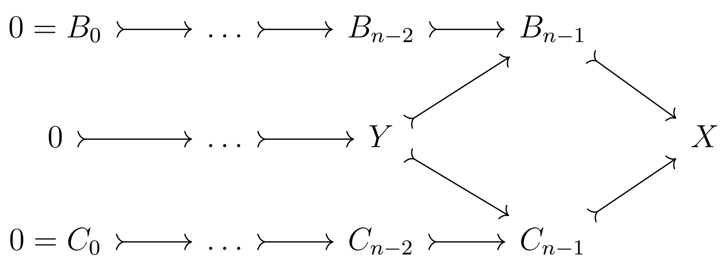
The diamond axiom applied to the maximal $\mathcal{E}$-subobjects B_{n-1}, C_{m-1} of X yields that Y is maximal in both B_{n-1} and C_{m-1}. Lemma IV.3.7 applied to the maximal element Y of B_{n-1} yields a composition series

$$0 = Y_0 \rightarrowtail \dots \rightarrowtail Y_{n-3} \rightarrowtail Y \rightarrowtail B_{n-1}$$

of length $n-1$. Moreover, Lemma IV.3.7 applied to the maximal element Y of C_{m-1} yields a composition series

$$0 = Y'_0 \rightarrowtail \dots \rightarrowtail Y'_{m-3} \rightarrowtail Y \rightarrowtail C_{m-1}$$

of length $m - 1$. This gives two composition series of the object Y of length $n - 2$ and $m - 2$, respectively. By induction hypothesis, we conclude that $n - 2 = m - 2$ (thus $n = m$), and that these two composition series of Y have the same composition factors, up to permutation and isomorphism. Consider now the following diagram:



By induction hypothesis, the two composition series

$$0 = B_0 \rightarrowtail \dots \rightarrowtail B_{n-2} \rightarrowtail B_{n-1}$$

and

$$0 = Y_0 \rightarrowtail \dots \rightarrowtail Y \rightarrowtail B_{n-1}$$

are equivalent. In the same way, the two composition series

$$0 = C_0 \rightarrowtail \ldots \rightarrowtail C_{n-2} \rightarrowtail C_{n-1}$$

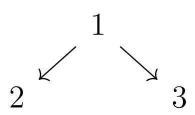
and

$$0 = Y_0' \rightarrowtail \ldots \rightarrowtail Y \rightarrowtail C_{n-1}$$

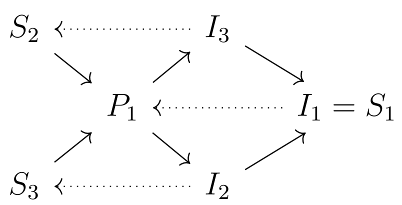
are equivalent. By the diamond axiom, the sets of quotients $\{X/B_{n-1}, B_{n-1}/Y\}$ and $\{X/C_{n-1}, C_{n-1}/Y\}$ are equal, up to isomorphism. Using this fact and comparing the four composition series of length $n-1$ above, we conclude that the two composition series given in the beginning have the same composition factors up to permutations and isomorphism. □

We provide in Section IV.4 examples of diamond categories that are not abelian categories with $\mathcal{E}_{all}$. However, not every exact category $(\mathcal{A}, \mathcal{E})$ is diamond, or Jordan-Hölder, even if $\mathcal{A}$ is abelian, as the following example demonstrates.

Example IV.3.9. Consider the category $\mathcal{A} = \operatorname{rep} Q$ of representations of the quiver



The Auslander-Reiten quiver of $\mathcal{A}$ is as follows:



By Theorem II.1.22, each exact structure on $\mathcal{A}$ is uniquely determined by the set of Auslander-Reiten sequences which it contains. Consider the exact structure $\mathcal{E}$ containing the Auslander-Reiten sequences

(AR1) $$0 \longrightarrow S_2 \longrightarrow P_1 \longrightarrow I_3 \longrightarrow 0$$

(AR2) $$0 \longrightarrow S_3 \longrightarrow P_1 \longrightarrow I_2 \longrightarrow 0$$

Then $(\mathcal{A}, \mathcal{E})$ is not Jordan-Hölder, and it is also not a diamond category. Indeed, we have that the simples S_2 and S_3 are maximal subobjects of P_1, but the quotient sets $\{S_2, P_1/S_2 = I_3\}$ and $\{S_3, P_1/S_3 = I_2\}$ are not isomorphic.

IV.4 $\mathcal{E}$-Artin-Wedderburn Categories

We use the notion of generalised intersection to define a version of the Jacobson radical relative to an exact structure $\mathcal{E}$. This allows us to show the Jordan-Hölder property for Krull-Schmidt categories under the assumption that this $\mathcal{E}$-radical behaves well with respect to direct sums of $\mathcal{E}$-simple objects, that is, the exact structure satisfies an exact analog of the Artin-Wedderburn theorem. We then classify all such exact structures in $\operatorname{rep}\Lambda$ where Λ is a Nakayama algebra and furthermore note that these are all Jordan-Hölder exact structures on $\operatorname{rep}\Lambda$.

Throughout this section, we assume all categories to be Krull-Schmidt. Recall from Definition II.1.35 that a Krull-Schmidt category is an additive category, $\mathcal{A}$, such that each object decomposes into a finite direct sum of indecomposable objects having local endomorphism rings and that this decomposition is unique up to isomorphism and permutation of summands. In particular, in this case $(\mathcal{A}, \mathcal{E}_{min})$ is a Jordan-Hölder category.

IV.4.1 $\mathcal{E}$-Jacobson Radical

Let $(\mathcal{A}, \mathcal{E})$ be an essentially small Krull-Schmidt exact category. We introduce a Jacobson radical for exact categories.

Definition IV.4.1. Let $X \in \mathcal{A}$, we define the *$\mathcal{E}$-Jacobson radical* to be the generalised intersection

$$\operatorname{rad}_{\mathcal{E}}(X) := \operatorname{Int}_X\{(Y, f) \in \mathcal{S}_X \mid (Y, f) \in \operatorname{Max}(\mathcal{S}_X)\}$$

and $\mathcal{S}_X$ is as defined previously in II.1.4. Note that, by Definition IV.3.4, $\operatorname{rad}_{\mathcal{E}} S = \{0\}$ for all $\mathcal{E}$-simple objects S.

Proposition IV.4.2. *Consider $X, Y \in \mathcal{A}$ and $r : R \rightarrowtail X$.*

(i) *For all $(R, r) \in \operatorname{rad}_{\mathcal{E}}(X)$, $\operatorname{rad}_{\mathcal{E}}(\operatorname{Coker}(r)) = \{0\}$.*

(ii) *For all $(Z, g) \in \mathcal{S}_X$, Z is an $\mathcal{E}$-subobject of some $(R, r) \in \operatorname{rad}_{\mathcal{E}}(X)$ if and only if $pg = 0$ for all $\mathcal{E}$-simple quotients $p : X \twoheadrightarrow S$ of X.*

Proof. **(i):** Let $(R, r) \in \operatorname{rad}_{\mathcal{E}}(X)$ and $(Q, q) \in \operatorname{rad}_{\mathcal{E}}(X/R)$ corresponding to $Q' \rightarrowtail X$ via the Fourth $\mathcal{E}$-isomorphism Theorem (Proposition IV.1.5). By same result and since $(R, r) \in \operatorname{rad}_{\mathcal{E}}(X)$ we have that the maximal $\mathcal{E}$-subobjects of X correspond exactly to maximal $\mathcal{E}$-subobjects of X/R. Hence, as Q is an $\mathcal{E}$-subobject of every $\mathcal{E}$-maximal subobject of X/R, we have that Q' is an $\mathcal{E}$-subobject of every maximal $\mathcal{E}$-subobject of X. Thus, by definition of the generalised intersection, since $R \rightarrowtail Q'$ we deduce that $R \cong Q'$ so $Q \cong Q'/R \cong 0$.

(ii): The claim follows from the observation that admissible epimorphisms $X \twoheadrightarrow S$ with S being $\mathcal{E}$-simple correspond exactly to maximal $\mathcal{E}$-subobjects of X. ☐

Definition IV.4.3. An object $X \in \mathcal{A}$ is called *$\mathcal{E}$-semisimple* if it can be written as a finite direct sum of $\mathcal{E}$-simple objects.

We study exact categories where the $\mathcal{E}$-semisimple objects have nice characterisations:

Definition IV.4.4. An exact structure $\mathcal{E}$ on $\mathcal{A}$ is called *Artin-Wedderburn* if for any object $X \in \mathcal{A}$ the following properties are equivalent:

(AW1) Every sequence in $\mathcal{E}$ of the form $A \rightarrowtail X \twoheadrightarrow X/A$ splits;

(AW2) X is $\mathcal{E}$-semisimple;

(AW3) $\operatorname{rad}_{\mathcal{E}}(X) = \{0\}$.

We say in this case that $(\mathcal{A}, \mathcal{E})$ is an *$\mathcal{E}$-Artin-Wedderburn category.*

Remark IV.4.1. (a) The implication (AW1) $\Rightarrow$ (AW2) always holds for Krull-Schmidt categories. Indeed, suppose X is not $\mathcal{E}$-semisimple. Then in the decomposition of X as a direct sum of indecomposables, $X \cong \bigoplus_{i=1}^{n} X_i$, there exists $1 \leq i \leq n$ such that X_i is not $\mathcal{E}$-simple. Thus there exists a non-split $\mathcal{E}$-inflation $f : Y \rightarrowtail X$ and observe that composing f with the canonical inclusion $X_i \rightarrowtail X$ results in a non-split $\mathcal{E}$-inflation $Y \rightarrowtail X$.

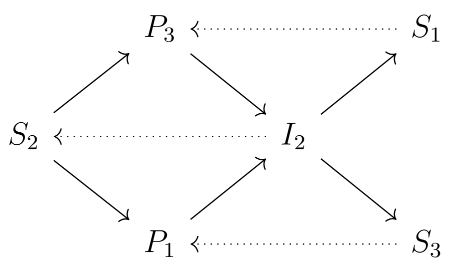
We note that without the Krull-Schmidt assumption on our categories, this implication in general does not hold, even in the abelian case. A class of counterexamples is given by the continuous spectral categories. These are Grothendieck categories where every short exact sequence splits but there are no simple objects as every object is decomposable, see [105, Example 2.9] for examples of such categories.

(b) The implication (AW2) $\Rightarrow$ (AW3) also always holds. Indeed, let S_i, $1 \leq i \leq n$ be $\mathcal{E}$-simple objects and $X = \bigoplus_{i=1}^{n} S_i$. Then observe that for all $1 \leq j \leq n$ that $\bigoplus_{i=1, i\neq j}^{n} S_i$ equipped with the canonical inclusion $f_j : \bigoplus_{i \neq j} S_i \rightarrowtail X$ is an $\mathcal{E}$-maximal subobject of X. Thus for every $(r : R \rightarrowtail X) \in \operatorname{rad}_{\mathcal{E}}(X)$, r factors through f_j for all $1 \leq j \leq n$ and we deduce that $r = 0$.

Example IV.4.5. Consider the category $\mathcal{A} = \operatorname{rep} Q$ of representations of the quiver

$$Q : \qquad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

We classify which exact structures $\mathcal{E}$ on $\mathcal{A}$ are Artin-Wedderburn, and when $(\mathcal{A}, \mathcal{E})$ is a diamond or Jordan-Hölder category. The Auslander-Reiten quiver of $\mathcal{A}$ is



and the Auslander Reiten sequences in $\mathcal{A}$ are

(1) $S_2 \to P_1 \oplus P_3 \to I_2$;

(2) $P_3 \to I_2 \to S_1$;

(3) $P_1 \to I_2 \to S_3$.

This example has been studied in [31, Example 4.2], and $\mathcal{A}$ admits precisely $2^3 = 8$ exact structures $\mathcal{E}$ corresponding to choosing some subset $\mathcal{B}$ of the three Auslander-Reiten sequences in $\mathcal{A}$, as discussed in Theorem II.1.22. We denote the different exact structures accordingly as $\mathcal{E}_{min}, \mathcal{E}(1), \mathcal{E}(2), \mathcal{E}(3), \mathcal{E}(1,2), \mathcal{E}(1,3), \mathcal{E}(2,3), \mathcal{E}_{max}$, indicating the Auslander-Reiten sequences that are included.

Consider first the exact structure $\mathcal{E}(1)$ generated by the Auslander-Reiten sequence (1). Then the only non-split indecomposable $\mathcal{E}(1)$-sequence is (1) thus

$P_1 \oplus P_3$ is $\mathcal{E}(1)$-semisimple and $(\mathcal{A}, \mathcal{E}(1))$ does not satisfy the implication (AW2) $\Rightarrow$ (AW1). The same object $P_1 \oplus P_3$ also shows that $(\mathcal{A}, \mathcal{E}(1))$ is not Jordan-Hölder (and hence not diamond) since there are non-equivalent $\mathcal{E}(1)-$composition series $0 \to S_2 \to P_1 \oplus P_3$ and $0 \to P_1 \to P_1 \oplus P_3$.

Now consider the exact structure $\mathcal{E}(2,3)$ on $\mathcal{A}$ generated by the sequences (2) and (3). As in Example IV.3.9 one can see that $(\mathcal{A}, \mathcal{E}(2,3))$ is not Jordan-Hölder nor diamond. Moreover, $\operatorname{rad}_{\mathcal{E}(2,3)}(I_2) = \{0\}$ but I_2 is not $\mathcal{E}(2,3)-$semisimple thus $(\mathcal{A}, \mathcal{E}(2,3))$ satisfies neither the implication (AW3) $\Rightarrow$ (AW1) nor (AW3)$\Rightarrow$ (AW2).

One may verify that all other exact structures $\mathcal{E}$ on $\mathcal{A}$ are Artin-Wedderburn, and also satisfy the diamond and Jordan-Hölder property, but only $(\mathcal{A}, \mathcal{E}_{max})$ is an AIS-category. We conclude that six of the eight exact structures are Jordan-Hölder, and in this example, the conditions being $\mathcal{E}$-Artin-Wedderburn, diamond and Jordan-Hölder are equivalent.

A further example of $\mathcal{E}$-Artin-Wedderburn categories is provided by the split exact structure:

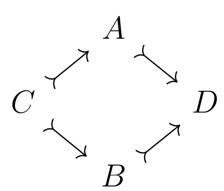
Lemma IV.4.6. *$\mathcal{A}$ is an $\mathcal{E}_{min}$-Artin-Wedderburn category.*

Proof. For the exact structure $\mathcal{E} = \mathcal{E}_{min}$, we have that the admissible monics are precisely the sections, and the $\mathcal{E}$-simple objects are the indecomposables. Every object in $\mathcal{A}$ is thus $\mathcal{E}$-semisimple, and we clearly have the equivalence of (AW1) and (AW2). Since every X is $\mathcal{E}$-semisimple, the implication (AW3) $\Longrightarrow$ (AW2) is always true. □

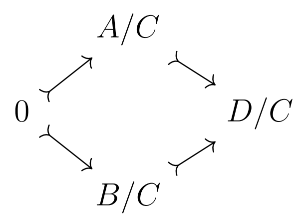
As we have noted, for Krull-Schmidt categories, $(\mathcal{A}, \mathcal{E}_{min})$ is a Jordan-Hölder category. The following result further studies the relationship between Krull-Schmidt categories and the Jordan-Hölder property.

Theorem IV.4.7. *Let $(\mathcal{A}, \mathcal{E})$ be an $\mathcal{E}$-Artin-Wedderburn category. Then $(\mathcal{A}, \mathcal{E})$ is a Jordan-Hölder exact category.*

Proof. We show that $(\mathcal{A}, \mathcal{E})$ satisfies the Diamond Axiom IV.3.6. For that purpose, let



be a commutative diagram in $(\mathcal{A}, \mathcal{E})$ with D/A and D/B being $\mathcal{E}$-simple and $C \in \operatorname{Int}_D(A, B)$. By the Fourth $\mathcal{E}$-Isomorphism Theorem (Proposition IV.1.5), there is a commutative diagram



with $(D/C)/(A/C) \cong D/A$ and $(D/C)/(B/C) \cong D/B$ both being $\mathcal{E}$-simple and $\mathrm{Int}_{D/C}(A/C, B/C) = \{0\}$. Thus, it is enough to consider diagrams of the form

$$\begin{array}{ccccc} & & X_0 & & \\ & \nearrow & & \overset{f_0}{\searrow} & \\ 0 & & & & Y \\ & \searrow & & \underset{f_1}{\nearrow} & \\ & & X_1 & & \end{array}$$

with Y/X_i being $\mathcal{E}$-simple for $i = 0, 1$ and $\mathrm{Int}_Y(X_0, X_1) = \{0\}$.

We must show that the X_i are $\mathcal{E}$-simple and that the sets $\{X_0, Y/X_0\}, \{X_1, Y/X_1\}$ are equal up to permuation and isomorphism of their elements.

If (X_0, f_1) and (X_1, f_1) are isomorphic as $\mathcal{E}$-subobjects of Y it follows that $X_0 \cong X_1$ is $\mathcal{E}$-simple since $\mathrm{Int}_Y(X_0, X_1) = \{0\}$. So we may assume that this is not the case. Observe that the (X_i, f_i) are both maximal $\mathcal{E}$-subobjects of Y. It follows that $\mathrm{rad}_{\mathcal{E}}(Y) \subset \mathrm{Int}_Y((X_0, f_0), (X_1, f_1)) = \{0\}$. Since $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$-Artin-Wedderburn, the short exact sequences $X_i \overset{f_i}{\rightarrowtail} Y \twoheadrightarrow Y/X_i$ both split and Y is $\mathcal{E}$-semisimple. Thus $X_0 \oplus Y/X_0 \cong Y \cong X_1 \oplus Y/X_1$ and $X_0 \cong \bigoplus_{j=0}^{n} S_j$ and $X_1 \cong \bigoplus_{j=0}^{m} T_j$ with the S_j and T_j being $\mathcal{E}$-simple. As $(\mathcal{A}, \mathcal{E})$ is Krull-Schmidt, $n = m$ and the sets $\{S_0, \dots, S_n, Y/X_0\}, \{T_0, \dots, T_n, Y/X_1\}$ consist of the same objects, up to permutation and isomorphism. Without loss of generality, we may suppose that $S_0 \cong Y/X_1$ and $T_0 \cong Y/X_0$. Now $\oplus_{j=1}^{n} S_j \rightarrowtail X_i$, but since $\mathrm{Int}_Y(X_0, X_1) = \{0\}$ we conclude that $n = 0$ and that the X_i are $\mathcal{E}$-simple and we are done. □

IV.4.2 Artin-Wedderburn exact structures for Nakayama algebras

We characterise all Artin-Wedderburn exact structures for any Nakayama algebra Λ. It turns out they are exactly the Jordan-Hölder exact categories for $\mathrm{mod}\,\Lambda$, the category of finitely generated left $\Lambda-$modules.

A finite-dimensional algebra Λ is called *Nakayama* if every indecomposable right and left projective Λ-module is uniserial. The representation theory of Nakayama algebras is well-known (see e.g. [11, Chapter V] or [14, Section VI.2]), we recall some details here:

The indecomposable $\Lambda-$modules are all uniserial, thus determined by the list of its composition factors from top to socle, which can be represented by a word w in the vertices of the quiver of Λ. Denote the module corresponding to a word w by $[w]$. Equivalently, indecomposable $\Lambda-$modules are parametrized by the non-zero paths in the quiver Q of Λ.

If we label the vertices of the path in Q corresponding to the indecomposable module $[w]$ as

$$c \to c+1 \to \cdots \to d-1 \to d$$

then we denote the module $[w]$ also by $[w] = [c, d]$. In this case, the Auslander-Reiten quiver of Λ contains a subquiver of the form described in Figure IV.1 where we label the Auslander-Reiten sequences $\eta_{[c,d-1]}$ in $\mathcal{A} = \mathrm{mod}\,\Lambda$ by the module $[c, d-1]$ where the sequence ends; the sequence starts in the Auslander-Reiten translate $\tau[c, d-1] = [c+1, d]$. For indecomposables $[w]$ and $[w']$, the space

$$\mathrm{Ext}^1_\Lambda([w], [w'])$$

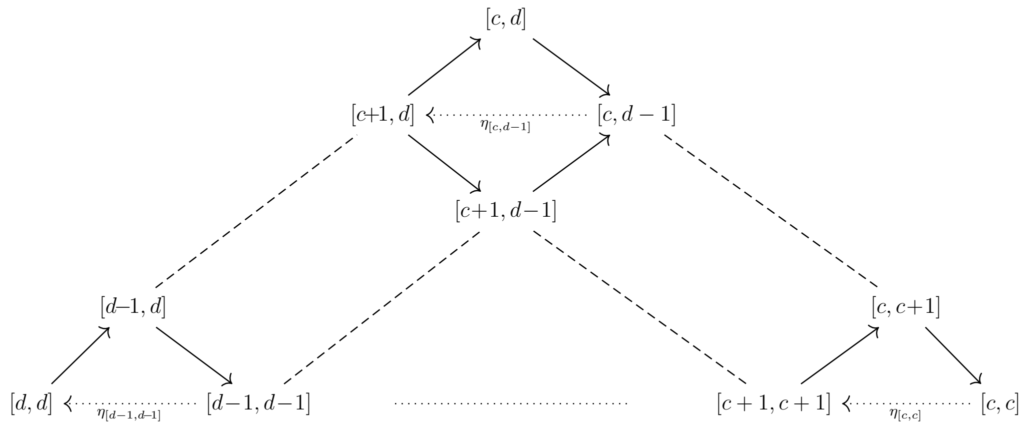


Figure IV.1: Part of the Auslander-Reiten quiver of $\operatorname{rep}\Lambda$ containing the module $[c,d]$ and all of its simple composition factors

is at most one-dimensional, and a basis can be given by the following non-split short exact sequences: If $[ww']$ is indecomposable, then a basis is given by

$$\eta_{w,w'} : 0 \longrightarrow [w] \longrightarrow [ww'] \longrightarrow [w'] \longrightarrow 0.$$

If $w = uv$ and $w' = vt$ such that $[uvt]$ is indecomposable, then a basis is given by

$$\eta_{w,w'} : 0 \longrightarrow [w] \longrightarrow [uvt] \oplus [v] \longrightarrow [w'] \longrightarrow 0.$$

We refer to $[ww']$ respectively $[uvt]$ as the *top module* in the extension $\eta_{w,w'}$. Thus for the Auslander-Reiten sequence $\eta_{[c,d-1]}$, the top module is $[c,d]$. The description of the indecomposables and the Auslander-Reiten sequences in $\mathcal{A}$ can be obtained from [38] for the more general case of string algebras, and the basis for the Ext^1-spaces is given in [30] for gentle algebras.

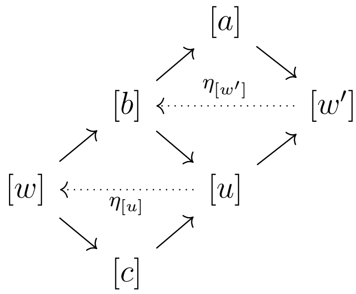
Our first step is to give a more precise description of an exact structure on $\mathcal{A}$ using the Auslander-Reiten sequences it contains.

Theorem IV.4.8. *Let $\mathcal{B}$ be a set of Auslander-Reiten sequences in $\mathcal{A}$ and $\mathcal{E} = \mathcal{E}(\mathcal{B})$ be the corresponding exact structure on $\mathcal{A}$. Then the short exact sequence $\eta_{w,w'} \in \mathcal{E}$ if and only if the Auslander-Reiten sequence $\eta_{[u]}$ belongs to $\mathcal{B}$ whenever there is a non-zero morphism from $[w]$ to $\tau[u]$ and from $[u]$ to $[w']$.*

Proof. Necessity follows directly from axioms (A2) and $(\mathrm{A2})^{op}$: One can easily verify that forming push-outs and pull-backs of the given exact sequence $\eta_{w,w'}$ along the morphisms from $[w]$ to $\tau[u]$ and from $[u]$ to $[w']$ yields the desired Auslander-Reiten sequences, which thus belong to $\mathcal{E}$.

Sufficiency follows from the fact that exact structures $\mathcal{E}$ on $\mathcal{A}$ correspond to *closed* subfunctors of the bifunctor $\operatorname{Ext}^1(-,-)$ on $\mathcal{A}$, see [44]. Auslander-Reiten theory implies that the socle of $\operatorname{Ext}^1(-,-)$ is given by the Auslander-Reiten sequences, and a closed subfunctor $\mathcal{E} = \mathcal{E}(\mathcal{B})$ is uniquely determined by its socle $\mathcal{B}$, see [15]. We show in [15] that $\mathcal{E} = \mathcal{E}(\mathcal{B})$ is the maximal subfunctor of $\operatorname{Ext}^1(-,-)$ whose socle is $\mathcal{B}$, therefore the sequence $\eta_{w,w'}$ (which induces the socle elements in $\mathcal{B}$ as we showed above when discussing necessity) must belong to $\mathcal{E} = \mathcal{E}(\mathcal{B})$. Here we indicate how to verify this directly from the axioms and leave the details to the reader:

Consider first the case where $[w] = \tau[u]$ and there is an arrow in the Auslander-Reiten quiver from $[u]$ to $[w']$:



By assumption, the Auslander-Reiten sequence $\eta_{[u]}$ belongs to $\mathcal{B}$ since there is an irreducible morphism from $[u]$ to $[w']$. Moreover, since the identity is a non-zero morphism, the Auslander-Reiten sequence $\eta_{[w']}$ also belongs to $\mathcal{B}$. We wish to apply axiom (A1) of an exact structure to this situation, however the monics from $\eta_{[u]}$ and $\eta_{[w']}$ cannot be composed directly, only when considering the direct sum of the split exact sequence $(1_{[c]}, 0)$ with $\eta_{[w']}$ this becomes possible. It turns out that the composition of the monic from $\eta_{[u]}$ with the monic of the short exact sequence $\eta_{[w']} \oplus (1_{[c]}, 0)$ yields the monic of the short exact sequence $\eta_{w,w'} \oplus (1_{[c]}, 0)$, which belongs to $\mathcal{E}$ by axiom (A1). Then [35, Corollary 2.18] shows that $\eta_{w,w'} \in \mathcal{E}$. To finish the proof, proceed by induction along paths from $[w]$ to $\tau[w']$ and from $\tau^{-1}[w]$ to $[w']$. □

Remark IV.4.2. As Λ is Nakayama, the poset of submodules of an indecomposable $[c, d]$ is totally ordered. In particular, for any exact structure $\mathcal{E}$ on $\mathcal{A}$, the poset of proper $\mathcal{E}$-subobjects $\mathcal{S}_{[c,d]}$ is also totally ordered. Hence all indecomposable non $\mathcal{E}$-simple objects have a unique maximal $\mathcal{E}$-subobject. Moreover, all ($\mathcal{E}$-)subobjects of $[c, d]$ are of the form $[x, d]$ for some $c \leq x \leq d$, whereas all quotients are of the form $[c, y]$ for some $c \leq y \leq d$, see Figure 1.

Now we may classify all Artin-Wedderburn exact structures on $\mathcal{A} = \operatorname{mod}\Lambda$ when Λ is Nakayama.

Theorem IV.4.9. *Let $\mathcal{B}$ be a set of Auslander-Reiten sequences in $\mathcal{A} = \operatorname{mod}\Lambda$ and $\mathcal{E} = \mathcal{E}(\mathcal{B})$ be the corresponding exact structure on $\mathcal{A}$. Then $\mathcal{E}$ is Artin-Wedderburn if and only if for all Auslander-Reiten sequences $\eta_{[w]} \in \mathcal{B}$ the top module of this sequence is not $\mathcal{E}$-simple.*

Proof. We use the notation from Figure IV.1. To simplify the presentation of the proof, we introduce phantom zero objects $[x, y] = 0$ whenever $x > y$. In this notation, all Auslander-Reiten sequences

$$\eta_{[c,d-1]} : [c+1, d] \rightarrowtail [c, d] \oplus [c+1, d-1] \twoheadrightarrow [c, d-1]$$

have two middle terms, with top module $[c, d]$, and where $[c+1, d-1]$ denotes the zero object when $c+1 > d-1$.

We first suppose that there exists an Auslander-Reiten sequence $\eta_{[c,d-1]}$ in $\mathcal{B}$ such that the top module $[c, d]$ is $\mathcal{E}$-simple, and we show that this implies $\mathcal{E}$ being not Artin-Wedderburn. Let $y \leq d-1$ be such that $[c, y]$ is $\mathcal{E}$-simple and $\eta_{[c,j]} \in \mathcal{B}$ for all

$j \in (y, d-1]$. Such a y always exists. Indeed, if $[c, d-1]$ is $\mathcal{E}$-simple then we take $y = d-1$. Else, let $[c, y]$ be an $\mathcal{E}$-simple factor module of $[c, d-1]$, then y satisfies the required conditions by Theorem IV.4.8.

Now, let $x \geq c$ be maximal such that there is an indecomposable non-split short exact sequence in $\mathcal{E}$ of the form

$$[x, d] \rightarrowtail [c, d] \oplus [x, y] \twoheadrightarrow [c, y]. \tag{IV.4}$$

Note that $[x, y] \ncong 0$ by the assumption that $[c, d]$ is $\mathcal{E}$-simple. If $[x, y]$ is $\mathcal{E}$-simple then this sequence shows that the implication (AW2) $\Rightarrow$(AW1) does not hold. Suppose that $[x, y]$ is not $\mathcal{E}$-simple and let $[w, y]$ be its unique maximal $\mathcal{E}$-subobject, note that $w > x$. Thus

$$\mathrm{rad}_{\mathcal{E}}\left([c, d] \oplus [x, y]\right) \subseteq \mathrm{Int}_{[c,d]\oplus[x,y]}\left([c, d] \oplus [w, y], [x, d]\right).$$

Observe that the $\mathcal{E}$-subobjects of $[x, d]$ are of the form $[i, d]$ with $i \in (x, d]$ and the only possible indecomposable $\mathcal{E}$-subobjects of $[c, d] \oplus [w, y]$ are of the form $[j, y]$ with $j \in (w, y]$ or $[w, d]$. We deduce that, if $\mathrm{rad}_{\mathcal{E}}([c, d] \oplus [x, y]) \neq \{0\}$ then $[w, d]$ is an $\mathcal{E}$-subobject of $[c, d] \oplus [w, y]$. But this is a contradiction to the maximality of x, thus the implication (AW3) $\Rightarrow$ (AW1) does not hold.

For the converse, consider a non-split $\mathcal{E}$-sequence. Since $\mathrm{Ext}^1(-, -)$ is an additive bifunctor, it suffices to consider short exact sequences with indecomposable end terms, which are for Nakayama algebras of the form

$$[c, d] \rightarrowtail [a, d] \oplus [b, c] \twoheadrightarrow [a, b]$$

where $[b, c]$ may denote the zero object. Note that the inequalities $a \leq c-1 \leq b \leq d-1$ must hold. By assumption and Theorem IV.4.8; $[a, d]$ is not $\mathcal{E}$-simple. Thus, since $\mathcal{A}$ is Krull-Schmidt, $[a, d] \oplus [b, c]$ is not $\mathcal{E}$-semisimple. This shows that the implication (AW2) $\Rightarrow$ (AW1) holds. It remains to show that $\mathrm{rad}_{\mathcal{E}}([a, d] \oplus [b, c]) \neq \{0\}$. Observe that

$$\mathcal{S}_{[a,d]\oplus[b,c]} = \left\{[c, d],\ [i, d] \oplus [c, b],\ [a, d] \oplus [j, b] \;\middle|\; [i, d] \in \mathcal{S}_{[a,d]},\ [j, b] \in \mathcal{S}_{[c,b]}\right\}$$

Let $[x, d] \rightarrowtail [a, d]$ be the unique maximal $\mathcal{E}$-subobject which exists by assumption that the top module $[a, d]$ is not $\mathcal{E}$-simple, and let $[y, b] \rightarrowtail [c, b]$ be the unique maximal $\mathcal{E}$-subobject of $[c, b]$ if it exists or the identity if not. Then

$$\mathrm{Max}(\mathcal{S}_{[a,d]\oplus[c,b]}) \subseteq \left\{[c, d], [x, d] \oplus [c, b],\ [a, d] \oplus [y, b]\right\}.$$

First suppose that $x \geq c$. Then, as $a < c$ by Theorem IV.4.8, $[x, d] \rightarrowtail [c, d]$ and

$$\mathrm{rad}_{\mathcal{E}}\left([a, d] \oplus [c, b]\right) \supseteq \mathrm{Int}_{[a,d]\oplus[c,b]}\left([c, d], [x, d] \oplus [c, b],\ [a, d] \oplus [y, b]\right) = \left\{[x, d]\right\}.$$

Which shows that $\mathrm{rad}_{\mathcal{E}}([a, d] \oplus [c, b]) \neq \{0\}$. Now suppose that $x < c$. Then $b > x$ as $c - 1 \leq b$. Now $\mathrm{rad}_{\mathcal{E}}([a, d] \oplus [c, b])$ contains

$$\mathrm{Int}_{[a,d]\oplus[c,b]}\left([c, d], [x, d] \oplus [c, b],\ [a, d] \oplus [y, b]\right) = \left\{[x, d] \oplus [y, b]\right\} \neq \{0\}$$

which again shows that $\mathrm{rad}_{\mathcal{E}}([a, d] \oplus [c, b]) \neq \{0\}$; and we are done. $\square$

Note that Enomoto studies in [46] the Jordan-Hölder property for torsion-free classes in the module category of a Nakayama algebra endowed with the maximal exact structure. We investigate now when $\mathcal{A} = \operatorname{mod} \Lambda$ with any exact structure $\mathcal{E}$ is Jordan-Hölder:

Theorem IV.4.10. *Let Λ be a Nakayama algebra, and denote $\mathcal{A} = \operatorname{mod} \Lambda$. Then an exact category $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$-Artin-Wedderburn precisely when it is Jordan-Hölder.*

Proof. The $\mathcal{E}$-Artin-Wedderburn categories are Jordan-Hölder by Theorem IV.4.7. Conversely, assume that $(\mathcal{A}, \mathcal{E}) = (\mathcal{A}, \mathcal{E}(\mathcal{B}))$ is Jordan-Hölder. By [46, Theorem 4.13], we know that the number s of $\mathcal{E}$-simple objects equals the number p of $\mathcal{E}$-projective indecomposable objects. Every non-$\mathcal{E}$-projective indecomposable admits an Auslander-Reiten sequence in $\mathcal{B}$, therefore

$$s = p = |\operatorname{ind}(\mathcal{A})| - |\mathcal{B}|$$

where $\operatorname{ind}(\mathcal{A})$ denotes the (isoclasses of) indecomposables in $\mathcal{A}$. We conclude

$$|\operatorname{ind}(\mathcal{A})| = |\mathcal{B}| + s,$$

and parametrise the set of indecomposables by the $\mathcal{E}$-simples together with the top module for every Auslander-Reiten sequence. Clearly this top module cannot be $\mathcal{E}$-simple in this case, thus by Theorem IV.4.9, $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$-Artin-Wedderburn. □

IV.5 The length function

In this section, we consider a Jordan-Hölder exact category $(\mathcal{A}, \mathcal{E})$ and we study the $\mathcal{E}$-Jordan-Hölder length function $l_{\mathcal{E}}$ that the $\mathcal{E}$-Jordan-Hölder theorem allows us to define over the set $\operatorname{Obj} \mathcal{A}$ of isomorphism classes of objects. Throughout, $(\mathcal{A}, \mathcal{E})$ denotes an $\mathcal{E}$-finite essentially small Jordan-Hölder exact category. To simplify notation, we will not distinguish here between the isomorphism class $[X]$ of an object X of $\mathcal{A}$ and the object X.

Definition IV.5.1. We define the $\mathcal{E}$-Jordan-Hölder length $l_{\mathcal{E}}(X)$ of an object X in $\mathcal{A}$ as the length of an $\mathcal{E}$-composition series of X. That is $l_{\mathcal{E}}(X) = n$ if and only if there exists an $\mathcal{E}$-composition series

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots \rightarrowtail X_{n-1} \rightarrowtail X_n = X.$$

We say in this case that X is $\mathcal{E}$-finite. If no such bound exists, we say that X is $\mathcal{E}$-infinite. Clearly, isomorphic objects have the same length, and therefore this definition gives rise to a length function $l_{\mathcal{E}} : Obj \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ defined on isomorphism classes.

Now we prove some corollaries of the $\mathcal{E}$-Jordan-Hölder theorem:

Corollary IV.5.1. *Let*

$$X \rightarrowtail Z \twoheadrightarrow Y$$

be an admissible short exact sequence of finite length objects. Then

$$l_{\mathcal{E}}(Z) = l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Y).$$

Proof. We know that X is a subobject of Z and that $Y \cong Z/X$. By the fourth $\mathcal{E}$-isomorphism theorem (Proposition IV.1.5) we may assume that there is an $\mathcal{E}$-composition series of Y of the form

$$0 = Z_0/X \rightarrowtail Z_1/X \rightarrowtail \dots \rightarrowtail Z_{l-1}/X \rightarrowtail Z_l/X \cong Y$$

with $Z_i \rightarrowtail Z$; we also take an $\mathcal{E}$-composition series

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots \rightarrowtail X_{n-1} \rightarrowtail X_n = X$$

of X. Since

$$(Z_{i+1}/X)/(Z_i/X) \cong (Z_{i+1}/Z_i)$$

by [35, Lemma 3.5], the following is a composition series of Z:

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots \rightarrowtail X_{n-1} \rightarrowtail X_n = X = Z_0 \overset{i}{\rightarrowtail}$$
$$\overset{i}{\rightarrowtail} Z_1 \rightarrowtail \dots \rightarrowtail Z_{l-1} \rightarrowtail Z_l = Z$$

Thus

$$l_{\mathcal{E}}(Z) = n + l = l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Y).$$

□

We show now that the function $l_{\mathcal{E}}$ is a *length function* in the sense of [81]:

Definition IV.5.2. A *measure for a poset* $\mathcal{S}$ is a morphism of posets $\mu : \mathcal{S} \to \mathcal{P}$ where $(\mathcal{P}, \leq)$ is a totally ordered set. A measure μ is called a *length function* when $\mathcal{P} = \mathbb{N}$ with the natural order.

Theorem IV.5.3. *The function $l_{\mathcal{E}}$ of an $\mathcal{E}$-finite Jordan-Hölder exact category $(\mathcal{A}, \mathcal{E})$ is a* length function *for the poset* $\operatorname{Obj} \mathcal{A}$.

Proof. The function $l_{\mathcal{E}} : \operatorname{Obj} \mathcal{A} \to \mathbb{N}$ is defined on the set $\operatorname{Obj} \mathcal{A}$, which is partially ordered by the $\mathcal{E}$-subset relation $X \subset_{\mathcal{E}} Y$, see [31, Proposition 6.11]. Moreover, consider X and Y in $\operatorname{Obj} \mathcal{A}$ with $X \subset_{\mathcal{E}} Y$. Then by Corollary IV.5.1 we have

$$l_{\mathcal{E}}(X) \leq l_{\mathcal{E}}(Y),$$

so $l_{\mathcal{E}}$ is a morphism of posets. □

As a consequence of the previous result, an $\mathcal{E}$-finite object is an object with $\mathcal{E}$-finite length.

Lemma IV.5.4. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Then every $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian object X of $(\mathcal{A}, \mathcal{E})$ admits an $\mathcal{E}$-composition series.*

Proof. Let X be an $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian object. Using the artinian hypothesis, one can construct a sequence of strict $\mathcal{E}$-subobjets with $\mathcal{E}$-simple quotients:

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots$$

Since X is noetherian too, this sequence became stationary and ends with X at some point. Finally, this sequence gives an $\mathcal{E}$-composition series in the sense of IV.3.1. □

The following result improves and uses [31, Lemma 6.5] and Lemma IV.5.4:

Theorem IV.5.5. *Let $(\mathcal{A}, \mathcal{E})$ be a Jordan-Hölder exact category. An object X of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian if and only if it has an $\mathcal{E}$-finite length.*

Proof. For an $\mathcal{E}$-finite object X of length $l_{\mathcal{E}}(X) = n \in \mathbb{N}$, the composition series is of length n. Thus any increasing or decreasing sequence of $\mathcal{E}$-subobjects of X must become stationary and X is $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian.
Conversely, let X be an $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian object, then X admits an $\mathcal{E}$-composition series by Lemma IV.5.4. Since $\mathcal{E}$ satisfies the Jordan-Hölder property, all composition series ending with X have the same finite length, so X is $\mathcal{E}$-finite. □

Remark IV.5.2. Note that a length function for exact categories in general was studied in [31, Section 6]. The notion there was defined as maximum over all lengths of an $\mathcal{E}$-composition series; in the case of an $\mathcal{E}$-Jordan-Hölder category all composition series of an object have the same length, so the definition we use here is compatible with the one from [31].

Definition IV.5.6. We denote by $(Ex(\mathcal{A}), \subseteq)$ the poset of exact structures $\mathcal{E}$ on $\mathcal{A}$, where the partial order is given by containment $\mathcal{E}' \subseteq \mathcal{E}$. This *containment* partial order is studied in [31, Section 4].

We conclude by noting that, similarly to [31, Lemma 8.1], the $\mathcal{E}$-Jordan Hölder length function can only decrease under reduction of exact structures:

Proposition IV.5.7. *Let $\mathcal{E}$ and $\mathcal{E}'$ are exact structures on $\mathcal{A}$ such that $\mathcal{E}' \subseteq \mathcal{E}$, then $l_{\mathcal{E}'}(X) \leq l_{\mathcal{E}}(X)$ for all objects X in $\mathcal{A}$.*

Proof. Let us consider an $\mathcal{E}'-$composition series of ending by X

$$0 = X_0 \overset{i_1}{\rightarrowtail} X_1 \overset{i_2}{\rightarrowtail} \dots \overset{i_{n-1}}{\rightarrowtail} X_{n-1} \overset{i_n}{\rightarrowtail} X_n = X.$$

where $l_{\mathcal{E}'}(X) = n$. Since $\mathcal{E}' \subseteq \mathcal{E}$, all these pairs (i_j, d_j) will also be in $\mathcal{E}$. So the $\mathcal{E}'$-composition series is also an $\mathcal{E}$-composition series and therefore by definition $l_{\mathcal{E}}(X) \geq n$. □

Chapter V

Right triangulated categories

The chapter is organised as follows. In Section V.1 we recall the definition of a right triangulated category and associated triangulated categories - the stabilisation and co-stabilisation. We also show some properties. In Section V.2 we show how one may use relative homological algebra to construct new extriangulated structures and characterise the projectives and injectives of these new structures. We then investigate how such extriangulated structures induce a right triangulated structure on a quotient category and use this to prove our characterisations of extriangulated categories as right triangulated categories and vice versa. We begin Section V.3 with a discussion of torsion pairs in a right triangulated category with right semi-equivalence. We then go on to prove the characterisations of right triangulated categories as (co-)aisles of (co-)t-structures and use these to describe related classes of (co-)t-structures. We end in Section V.4 by proving that in the case of Frobenius extriangulated categories, aisles of t-structures in the triangulated stable category may be constructed as (shifts of) right triangulated quotients using tools from Section V.2.

Throughout, dual results for left triangulated categories (with left semi-equivalence) hold but remain unstated.

V.1 Preliminaries

We begin by recalling in full the definition of a right triangulated (or suspended) category first introduced in [76].

Definition V.1.1. Let $\mathcal{R}$ be an additive category and $\Sigma : \mathcal{R} \to \mathcal{R}$ an endofunctor. A *right triangulation* of the pair $(\mathcal{R}, \Sigma)$ is a collection Δ of sequences of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

in $\mathcal{R}$ that satisfy the following axioms.

(R1) (a) Δ is closed under isomorphisms. That is, for every commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma f} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma(A') \end{array}$$

in $\mathcal{R}$ whose vertical arrows are isomorphisms, one row belongs to Δ if and only if the other row also belongs to Δ.

(b) For every $A \in \mathcal{R}$, the sequence

$$0 \longrightarrow A \xrightarrow{1_A} A \longrightarrow 0$$

belongs to Δ.

(c) Every morphism $x : A \to B$ in $\mathcal{R}$ can be embedded into a sequence

$$A \xrightarrow{x} B \longrightarrow C_x \longrightarrow \Sigma x$$

in Δ.

(R2) If the sequence

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{z} \Sigma A$$

is in Δ then so is the sequence

$$B \xrightarrow{y} C \xrightarrow{z} \Sigma A \xrightarrow{-\Sigma z} \Sigma B.$$

(R3) Every commutative diagram

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\
\downarrow{\scriptstyle f} & & \downarrow & & & & \downarrow{\scriptstyle \Sigma f} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma(A')
\end{array}$$

in $\mathcal{R}$ whose rows belong to Δ can be extended to a commutative diagram

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\
\downarrow{\scriptstyle f} & & \downarrow & & \vdots & & \downarrow{\scriptstyle \Sigma f} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma(A').
\end{array}$$

(R4) Let

$$A \xrightarrow{x} B \xrightarrow{x'} X \xrightarrow{x''} \Sigma A, \qquad B \xrightarrow{y} C \xrightarrow{y'} Y \xrightarrow{y''} \Sigma B,$$

and

$$A \xrightarrow{yx} C \xrightarrow{z'} Z \xrightarrow{z''} \Sigma A$$

be sequences in Δ. Then there is a commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{x} & B & \xrightarrow{x'} & X & \xrightarrow{x''} & \Sigma A \\
\| & & \downarrow{\scriptstyle y} & & \vdots{\scriptstyle \alpha} & & \| \\
A & \xrightarrow{yx} & B & \xrightarrow{z'} & Y & \xrightarrow{z''} & \Sigma A \\
\downarrow{\scriptstyle x} & & \| & & \vdots{\scriptstyle \beta} & & \downarrow{\scriptstyle \Sigma x} \\
B & \xrightarrow{y} & C & \xrightarrow{y'} & Z & \xrightarrow{y''} & \Sigma B \\
 & & & & \vdots{\scriptstyle \gamma} & \swarrow{\scriptstyle \Sigma(x')} & \\
 & & & & \Sigma X & &
\end{array}$$

in $\mathcal{R}$ such that the dotted column belongs to Δ.

If Δ is a right triangulation of $(\mathcal{R}, \Sigma)$, then the triple $(\mathcal{R}, \Sigma, \Delta)$ is called a *right triangulated category* and the sequences in Δ are called *right triangles.* Following [10], if the functor Σ is fully faithful and its image, $\Sigma\mathcal{R}$, is closed under extensions we call Σ a *right semi equivalence* and $\mathcal{R}$ a *right triangulated category with right semi equivalence.*

Definition V.1.2. Let $(\mathcal{R}, \Sigma, \Delta)$ and $(\mathcal{R}', \Sigma', \Delta')$ be right triangulated categories. An additive functor $F : \mathcal{R} \to \mathcal{R}'$ is a *right triangle functor* if

(a) There is a natural isomorphism $\zeta : F\Sigma \xrightarrow{\cong} \Sigma' F$;

(b) For all right triangles $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{z} \Sigma A$ in Δ, the sequence

$$FA \xrightarrow{Fx} FB \xrightarrow{Fy} FC \xrightarrow{\zeta_A \circ Fz} \Sigma' FA$$

is in Δ'.

Remark V.1.3. Let $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category such that $\Sigma\mathcal{R}$ is closed under extensions. Then the subcategory $\Sigma\mathcal{R}$ naturally inherits the structure of a right triangulated category. Clearly, if Σ is a right semi-equivalence then $\Sigma\mathcal{R}$ is also a right triangulated category with right semi-equivalence. We will see in Section V.4 examples where $\Sigma\mathcal{R}$ is a right triangulated category with right semi-equivalence but $\mathcal{R}$ is not.

For homological properties of right triangulated categories we point the reader to [10].

We finish this subsection by noting that there are no finite right triangulated categories with right semi-equivalence that are not triangulated. This is one of the many ways to see that any additively finite triangulated category admits no non-trivial (co-)t-structures. We refer to [5] for a discussion of such categories.

Lemma V.1.4. *Let $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category with right semi-equivalence. Suppose that $\mathcal{R}$ is idempotent complete and additively finite, then Σ is an autoequivalence. In particular, $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ is a triangulated category.*

Proof. Let $S = \{X_1, \dots, X_n\}$ be a set of isomorphism classes of indecomposable objects in $\mathcal{R}$. Since Σ is additive, fully faithful and $\mathcal{R}$ is idemopotent complete. Σ acts as a permutation on the set S and we deduce that $\Sigma^m = 1_{\mathcal{R}}$ for some $m \in \mathbb{N}$. $\square$

V.1.1 (Co-)stabilisation of a right triangulated category

We recall the definitions of two triangulated categories associated to a right triangulated category: the stabilisation and costabilisation.

The *stabilisation of a right triangulated category*, $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$, consists of a pair $(\mathcal{S}(\mathcal{R}), s)$ where $\mathcal{S}(\mathcal{R})$ is a triangulated category and $s : \mathcal{R} \to \mathcal{S}(\mathcal{R})$ is a right triangle functor satisfying a universal property: For all right triangle functors $F : \mathcal{R} \to \mathcal{T}$ with $\mathcal{T}$ being a triangulated category, there exists a unique triangle functor $F' : \mathcal{S}(\mathcal{R}) \to \mathcal{T}$ such that $F's = F$

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{s} & \mathcal{S}(\mathcal{R}) \\ {\scriptstyle \forall F}\downarrow & \swarrow {\scriptstyle \exists! F'} & \\ \mathcal{T}. & & \end{array}$$

Dually, the *co-stabilisation of* $\mathcal{R}$ consists of a pair $(\mathcal{C}(\mathcal{R}), c)$ where $\mathcal{C}(\mathcal{R})$ is a triangulated category and $c : \mathcal{C}(\mathcal{R}) \to \mathcal{R}$ is a right triangle functor satisfying a universal property: For all right triangle functors $G : \mathcal{T} \to \mathcal{R}$ with $\mathcal{T}$ being a triangulated category, there exists a unique triangle functor $G' : \mathcal{T} \to \mathcal{C}(\mathcal{R})$ such that $G = cG'$.

The stabilisation and co-stablisation of a right triangulated category always exist. Since we will use it explicitly, we recall the construction of the stablisation from [19, Section 3.1] and [64]; see also [52]. For more information on the co-stabilisation, which may be constructed as the category of spectra, we refer to [55, Section 4.5-4.7]; see also [19, 72].

Let $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category.

Definition V.1.5. We define the additive category $\mathcal{S}(\mathcal{R})$ as follows. The objects of $\mathcal{S}(\mathcal{R})$ are pairs (A, n) with $A \in \mathcal{R}$ and $n \in \mathbb{Z}$. The spaces of morphisms are given by

$$\operatorname{Hom}_{\mathcal{S}(\mathcal{R})}\big[(A,n),(B,m)\big] = \underset{\longrightarrow k\in J}{\operatorname{colim}}\big(\operatorname{Hom}_{\mathcal{R}}(\Sigma^{n-k}A, \Sigma^{m-k}B)\big)$$

where $J = \{k \in \mathbb{Z} \mid k \leq \operatorname{Min}\{n, m\}\}$. There is an autoequivalence of $\mathcal{S}(\mathcal{R})$ which is given on objects by $\Sigma(A, n) = (A, n+1)$ and induced on morphisms by the natural map

$$\operatorname{Hom}_{\mathcal{R}}(\Sigma^{n-k}A, \Sigma^{m-k}B) \to \operatorname{Hom}_{\mathcal{R}}(\Sigma^{n+1-k}A, \Sigma^{m+1-k}B)$$

for all $k \leq \operatorname{Min}\{m, n\}$. By abuse of notation, we denote this autoequivalence also by Σ.

The functor $s : \mathcal{R} \to \mathcal{S}(\mathcal{R})$ is given on objects by $s(A) = (A, 0)$ and for a morphism $f : A \to B$, $s(f : A \to B)$ is the zero-representative of $\operatorname{Hom}_{\mathcal{S}(\mathcal{R})}((A, 0), (B, 0))$.

Triangles in $\mathcal{S}(\mathcal{R})$ are given by sequences isomorphic to sequences of the form

$$(A,n) \xrightarrow{\ x\ } (B,m) \xrightarrow{\ y\ } (C,l) \xrightarrow{\ z\ } (A,n+1)$$

such that there exists $k \leq \operatorname{Min}\{n, m, l\}$ such that

$$\Sigma^{n-k}A \xrightarrow{(-1)^k\Sigma^{-k}x} \Sigma^{m-k}B \xrightarrow{(-1)^k\Sigma^{-k}y} \Sigma^{l-k}C \xrightarrow{(-1)^k\Sigma^{-k}z} \Sigma^{n+1-k}A \qquad \text{(V.1)}$$

is a right triangle in $\mathcal{R}$.

One may verify that $S\Sigma \cong \Sigma S$ and that S is a right triangle functor.

Remark V.1.6. We make some observations.

(a) If the endofunctor $\Sigma : \mathcal{R} \to \mathcal{R}$ is fully faithful, then the morphism spaces in $\mathcal{S}(\mathcal{R})$ are neater

$$\operatorname{Hom}_{\mathcal{S}(\mathcal{R})}\big((A,n),(B,m)\big) \cong \operatorname{Hom}_{\mathcal{R}}(\Sigma^{n-k}A, \Sigma^{m-k}B),\ \forall k \leq \operatorname{Min}\{n,m\}.$$

Further, in this case, $s : \mathcal{R} \to \mathcal{S}(\mathcal{R})$ is fully faithful and $\mathcal{S}(\mathcal{R})$ is the smallest triangulated category that contains $\mathcal{R}$ as a full right triangulated subcategory.

(b) If $\Sigma : \mathcal{R} \to \mathcal{R}$ is a right semi-equivalence then the 'there exists' preceding Equation (V.1) can be replaced by 'for all'.

(c) If $\mathcal{R}$ is not triangulated, then every indecomposable object in $\mathcal{S}(\mathcal{R})$ is isomorphic to an object of the form (A, n) with $A \in (\mathcal{R} \setminus \Sigma\mathcal{R}) \cup \{0\}$.

Many categorical properties of $\mathcal{S}(\mathcal{R})$ are inherited from $\mathcal{R}$, for instance if $\mathcal{R}$ has (co-)products then so does $\mathcal{S}(\mathcal{R})$. The following properties will be useful for our work.

Lemma V.1.7. *Suppose that for all $A, B \in \mathcal{R}$ we have that $\mathrm{Hom}_{\mathcal{R}}(A, \Sigma^i B) = 0$ for $i >> 0$. Then, for all $X, Y \in \mathcal{S}(\mathcal{R})$ we have that $\mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(X, \Sigma^i Y) = 0$ for all $i >> 0$.*

Proof. Let $X, Y \in \mathcal{S}(\mathcal{R})$ then there exists $A, B \in \mathcal{R}$ and $k > 0$ such that $\Sigma^k X \cong A$ and $\Sigma^k Y \cong B$. Therefore, for $i >> 0$, we have that

$$\mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(X, \Sigma^i Y) \cong \mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(A, \Sigma^i B) \cong \mathrm{Hom}_{\mathcal{R}}(A, \Sigma^i B) = 0.$$

□

We call a right triangulated category satisfying the condition of the above Lemma V.1.7 *bounded.*

Lemma V.1.8. *Let $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category with right semi-equivalence. Then $\mathcal{R}$ is extension closed as a subcategory of $\mathcal{S}(\mathcal{R})$.*

Proof. Let

$$(A, n) \xrightarrow{x} (B, m) \xrightarrow{y} (C, l) \xrightarrow{z} (A, n+1)$$

be a triangle in $\mathcal{S}(\mathcal{R})$ with n, m, l, minimal such that $\Sigma^{-n}A$, $\Sigma^{-m}B$, $\Sigma^{-l}C \in \mathcal{R}$. We must show that if $n, l = 0$ then $m = 0$. For that purpose, suppose that $m < 0$, then $\mathrm{Min}\{m, n, l\} = m$ and, by definition, there is a right triangle in $\mathcal{R}$

$$\Sigma^{n-m}A \xrightarrow{(-1)^m\Sigma^{-m}x} B \xrightarrow{(-1)^m\Sigma^{-m}y} \Sigma^{l-m}C \xrightarrow{(-1)^m\Sigma^{-m}z} \Sigma^{n+1-m}A.$$

As $n - m, l - m > 0$ and $\Sigma : \mathcal{R} \to \mathcal{R}$ is a right semi-equivalence, we deduce that $B \in \Sigma\mathcal{R}$ which is a contradiction to the minimality of m. □

Example V.1.9. Let $\mathcal{T}$ be a triangulated category and $\mathcal{R} \subset \mathcal{T}$ a subcategory that is closed under positive shifts and extensions. Then $\mathcal{R}$ is a right triangulated category with right semi-equivalence and $\mathcal{S}(\mathcal{R}) \cong \mathrm{cosusp}_{\mathcal{T}}(\mathcal{R})$, the smallest subcategory of $\mathcal{T}$ containing $\mathcal{R}$ that is closed under negative shifts and extensions.

For more examples of the stabilisation of right triangulated categories, see [19, Section 3.1] and [76].

V.2 Right triangulated categories as extriangulated categories

The aim of this section is show that right triangulated categories have a natural extriangulated structure precisely when the shift functor is a right semi-equivalence. We also describe which extriangulated categories have a right triangulated structure. To do this, we begin by using relative homological algebra to define new extriangulated structures from existing ones (Section V.2.1). We then show when these new extriangulations induce right triangulated structures on a quotient category (Section V.2.2). With these tools in hand we complete the above aims in Section V.2.3.

V.2.1 Extriangulated structures using relative homological algebra

In this section we show how one can use relative homological algebra to construct new extriangulated structures and characterise when these exact structures have enough injectives/ projectives. The existence of these extriangulated structures also follows from [65, Proposition 3.17] for $n = 1$ but we give an alternative proof using relative notions. We begin with an easy lemma.

Lemma V.2.1. *Let $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, $\mathcal{D}$ be a class of objects in $\mathcal{A}$ and δ be an extriangle. Then the following are equivalent*

(i) *There exists representative of $\mathfrak{s}(\delta)$, $A \xrightarrow{x} B \xrightarrow{y} C$, such that x is $\mathcal{D}$-monic;*

(ii) *The $\mathbb{E}$-inflation of every representative of $\mathfrak{s}(\delta)$ is $\mathcal{D}$-monic.*

Proof. (i)⇒(ii) follows from Lemma II.2.4(iii). (ii)⇒(i) is obvious. □

Proposition V.2.2. *Let $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{D}$ a class of objects in $\mathcal{A}$. Consider the classes of extriangles for $A, C \in \mathcal{A}$*

$$\mathbb{I}_{\mathcal{D}}(C, A) = \left\{ \ A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} \ \in \mathbb{E}(C, A) \mid x \in \mathrm{Mon}(\mathcal{D}) \right\},$$
$$\mathbb{P}_{\mathcal{D}}(C, A) = \left\{ \ A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} \ \in \mathbb{E}(C, A) \mid y \in \mathrm{Epi}(\mathcal{D}) \right\}, \ \textit{and}$$
$$\mathbb{D}_{\mathcal{D}}(C, A) = \mathbb{I}_{\mathcal{D}} \cap \mathbb{P}_{\mathcal{D}}.$$

Then $(\mathbb{I}_{\mathcal{D}}, \mathfrak{s}|_{\mathbb{I}_{\mathcal{D}}}), (\mathbb{P}_{\mathcal{D}}, \mathfrak{s}|_{\mathbb{P}_{\mathcal{D}}})$ and $(\mathbb{D}_{\mathcal{D}}, \mathfrak{s}|_{\mathbb{D}_{\mathcal{D}}})$ all define external triangulations of $\mathcal{A}$. Moreover,

$$\mathcal{D} \subseteq \mathrm{Inj}_{\mathbb{I}_{\mathcal{D}}}\mathcal{A} \subseteq \mathrm{Inj}_{\mathbb{D}_{\mathcal{D}}}\mathcal{A} \ \textit{and}$$
$$\mathcal{D} \subseteq \mathrm{Proj}_{\mathbb{P}_{\mathcal{D}}}\mathcal{A} \subseteq \mathrm{Proj}_{\mathbb{D}_{\mathcal{D}}}\mathcal{A}.$$

Additionally, suppose that $\mathcal{D} = Add(\mathcal{D})$ then the following hold.

(i) *If $\mathcal{D}$ is covariantly finite and left $\mathcal{D}$-approximations are $\mathbb{E}$-inflations then $\mathcal{D} = \mathrm{Inj}_{\mathbb{I}_{\mathcal{D}}}\mathcal{A}$ and $\mathcal{A}$ has enough $\mathbb{I}_{\mathcal{D}}$-injectives.*

(ii) *If $\mathcal{D}$ is contravariantly finite and right $\mathcal{D}$-approximations are $\mathbb{E}$-deflations then $\mathcal{D} = \mathrm{Proj}_{\mathbb{P}_{\mathcal{D}}}\mathcal{A}$ and $\mathcal{A}$ has enough $\mathbb{P}_{\mathcal{D}}$-projectives.*

(iii) *If $\mathcal{D}$ is functorially finite, left (resp. right) $\mathcal{D}$-approximations are $\mathbb{E}$-inflations (resp. $\mathbb{E}$-deflations) and cones (resp. co-cones) of left (resp. right) $\mathcal{D}$-approximations are $\mathcal{D}$-epic (resp. $\mathcal{D}$-monic) then $\mathcal{D} = \mathrm{Inj}_{\mathbb{D}_{\mathcal{D}}}\mathcal{A} = \mathrm{Proj}_{\mathbb{D}_{\mathcal{D}}}\mathcal{A}$ and $\mathcal{A}$ has enough $\mathbb{D}_{\mathcal{D}}$-injectives and $\mathbb{D}_{\mathcal{D}}$-projectives.*

Proof. We prove the statements for $\mathbb{I}_{\mathcal{D}}$ whence the remaining claims follow from dual and combined arguments. Let $\delta \in \mathbb{I}_{\mathcal{D}}(C', A)$, $a \in \mathrm{Hom}_{\mathcal{A}}(A, A')$ and $c \in \mathrm{Hom}_{\mathcal{A}}(C, C')$. By [65, Proposition 3.14] it is enough to show that $a_*\delta \in \mathbb{I}_{\mathcal{D}}(C', A')$, $c^*\delta \in \mathbb{I}_{\mathcal{D}}(C, A)$ and $\mathbb{I}_{\mathcal{D}}$-inflations are closed under composition. The first follows from Lemma II.1.11 and Lemma II.2.4(iii) and the third from Lemma II.2.4(i). To show the second claim consider the morphism of extriangles $(1, c) : c^*\delta \to \delta$

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{c^*\delta}{\dashrightarrow} & \\ \| & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{a_*\delta}{\dashrightarrow} & . \end{array}$$

We see that $x' = bx \in \mathrm{Mon}(\mathcal{D})$ and hence x is $\mathcal{D}$-monic by Lemma II.2.4(ii).

We verify the additional claims. By construction we have that the class of $\mathcal{D}$-monic morphisms is contained in the class of $\mathbb{I}_{\mathcal{D}}$-inflations, ω. Thus

$$\mathcal{D} \subseteq \mathrm{Inj}(\mathrm{Mon}(\mathcal{D})) \subseteq \mathrm{Inj}(\omega) = \mathrm{Inj}_{\mathbb{I}_{\mathcal{D}}}\mathcal{A}.$$

Now additionally suppose that $\mathcal{D} = \mathrm{Add}(\mathcal{D})$, $\mathcal{D}$ is covariantly finite and left $\mathcal{D}$-approximations are $\mathbb{E}$-inflations. Then for every object $I \in \mathrm{Inj}_{\mathbb{I}_{\mathcal{D}}}\mathcal{A}$ there is an $\mathbb{I}_{\mathcal{D}}$-inflation $i : I \to D$ with $D \in \mathcal{D}$. We deduce that I is a direct summand of $\mathcal{D}$ and we are done. □

Remark V.2.3. It follows from Lemma II.2.4(ii) that the external triangulation $(\mathbb{E}, \mathfrak{s})$ satisfies (WIC) then so do $(\mathbb{I}_{\mathcal{D}}, \mathfrak{s}|_{\mathbb{I}_{\mathcal{D}}})$, $(\mathbb{P}_{\mathcal{D}}, \mathfrak{s}|_{\mathbb{P}_{\mathcal{D}}})$ and $(\mathbb{D}_{\mathcal{D}}, \mathfrak{s}|_{\mathbb{D}_{\mathcal{D}}})$, in the notation of the above result.

Notation V.2.4. Let $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. For brevity, in the sequel we will say a subcategory $\mathcal{D}$ of $\mathcal{A}$ has property $(*)$ if $\mathcal{D} = \mathrm{Add}(\mathcal{D})$, $\mathcal{D}$ is covariantly finite and all left $\mathcal{D}$-approximations are $\mathbb{E}$-inflations.

We note that one may view the Frobenius property [63, 56] of external triangulations through the lens of relative homological algebra.

Corollary V.2.5. *An extriangulated category $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ is Frobenius if and only if there exists a functorially finite subcategory $\mathcal{P}$ such that*

(a) All left $\mathcal{P}$-approximations are $\mathbb{E}$-inflations and all $\mathbb{E}$-inflations are $\mathcal{P}$-monic;

(b) All right $\mathcal{P}$-approximations are $\mathbb{E}$-deflations and all $\mathbb{E}$-deflations are $\mathcal{P}$-epic.

In other words, $\mathcal{P} = \mathrm{Proj}_{\mathbb{D}_{\mathcal{P}}}\mathcal{A} = \mathrm{Inj}_{\mathbb{D}_{\mathcal{P}}}\mathcal{A}$ and $\mathbb{D}_{\mathcal{P}} = \mathbb{E}$.

V.2.2 Right triangulated stable categories

We show how extriangulations can induce right triangulated structures on a quotient category. The construction is reminiscent of the triangulated structure of the stable category of a Frobenius exact category [63, 56].

Set-up V.2.6. For the rest of this section, let $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{D} \subseteq \mathcal{A}$ be a subcategory satisfying property $(*)$. For each $A \in \mathcal{A}$ we make a choice of extriangle

$$A \xrightarrow{i_A} \mathcal{D}(A) \xrightarrow{p_A} \Sigma_{\mathcal{D}}A \overset{\delta_A}{\dashrightarrow} \tag{V.2}$$

where i_A is a left $\mathcal{D}$-approximation of A. We also define Σ_D on morphisms: Let $f : A \to B$. Then, as i_A is $\mathcal{D}$-monic, there exists $\mathcal{D}(f) : \mathcal{D}(A) \to \mathcal{D}(B)$ such that $\mathcal{D}(f)i_A = i_B f$ and thus, by (ET3) there exists a morphism $\Sigma_{\mathcal{D}}f : \Sigma_{\mathcal{D}}A \to \Sigma_{\mathcal{D}}B$ such that $(f, \Sigma_{\mathcal{D}}) : \delta_A \to \delta_B$ is a morphism of extriangles.

$$\begin{array}{ccccccc} A & \xrightarrow{i_A} & \mathcal{D}(A) & \xrightarrow{p_A} & \Sigma_{\mathcal{D}}A & \overset{\delta_A}{\dashrightarrow} & \\ \downarrow{\scriptstyle f} & & \exists \downarrow{\scriptstyle \mathcal{D}(f)} & & \exists \downarrow{\scriptstyle \Sigma_{\mathcal{D}}f} & & \\ B & \xrightarrow[i_B]{} & \mathcal{D}(B) & \xrightarrow[p_B]{} & \Sigma_{\mathcal{D}}B & \overset{\delta_B}{\dashrightarrow} & . \end{array} \tag{V.3}$$

By $\underline{\mathcal{A}}_{\mathcal{D}}$ we denote the stable category of $\mathcal{A}$ by the ideal of morphisms factoring through objects of $\mathcal{D}$.

The following lemma is similar to [95, Claim 6.1].

Lemma V.2.7. *The construction above of $\Sigma_{\mathcal{D}}$ defines an additive endofunctor $\underline{\mathcal{A}}_{\mathcal{D}} \to \underline{\mathcal{A}}_{\mathcal{D}}$. Further, any other choice of extriangles δ_A yields a naturally isomorphic endofunctor.*

Proof. We keep the notation of the above paragraph. First we show that $\Sigma_{\mathcal{D}}$ is well-defined on morphisms in $\underline{\mathcal{A}}_{\mathcal{D}}$, that is, the construction of $\Sigma_{\mathcal{D}} f$ is independent of the choices made of $\mathcal{D}(f)$ and in (ET3). Indeed, let $g : \Sigma_{\mathcal{D}} A \to \Sigma_{\mathcal{D}} B$ be any morphism such that $(f, g) : \delta_A \to \delta_B$ is a morphism of extriangles. Then $g^* \delta_B = f_* \delta_A = (\Sigma_{\mathcal{D}} f)^* \delta_B$ and therefore $(g - \Sigma_{\mathcal{D}} f)^* \delta_B = 0$. By Lemma II.1.12, $g - \Sigma_{\mathcal{D}} f$ factors through p_B and hence $\underline{g - \Sigma_{\mathcal{D}} f} = 0$ in $\underline{\mathcal{A}}_{\mathcal{D}}$.

Now suppose for all $A \in \mathcal{A}$ we have another choice of extriangle

$$A \xrightarrow{i'_A} \mathcal{D}'(A) \xrightarrow{p'_A} \Sigma'_{\mathcal{D}} A \overset{\delta'_A}{\dashrightarrow}$$

which then results in another endofunctor $\Sigma'_{\mathcal{D}} : \underline{\mathcal{A}}_{\mathcal{D}} \to \underline{\mathcal{A}}_{\mathcal{D}}$. Then, since i_A is $\mathcal{D}$-monic, there exists $s : \mathcal{D}(A) \to \mathcal{D}'(A)$ such that $s i_A = i'_A$. Then, by (ET3) there exists $t_A = t : \Sigma_{\mathcal{D}} A \to \Sigma'_{\mathcal{D}} A$ such that $(1, t) : \delta_A \to \delta'_A$ is a morphism of extriangles. Similarly, we obtain a morphism $t'_A = t' : \Sigma'_{\mathcal{D}} A \to \Sigma_{\mathcal{D}} A$ such that $(1, t') : \delta'_A \to \delta_A$ is a morphism of extriangles

$$\begin{array}{ccccccc}
A & \xrightarrow{i_A} & \mathcal{D}(A) & \xrightarrow{p_A} & \Sigma_{\mathcal{D}} A & \overset{\delta_A}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle \exists s} & & \downarrow{\scriptstyle \exists t} & & \\
A & \xrightarrow{i'_A} & \mathcal{D}'(A) & \xrightarrow{p'_A} & \Sigma'_{\mathcal{D}} A & \overset{\delta'_A}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle \exists} & & \downarrow{\scriptstyle \exists t'} & & \\
A & \xrightarrow{i_A} & \mathcal{D}(A) & \xrightarrow{p_A} & \Sigma_{\mathcal{D}} A & \overset{\delta_A}{\dashrightarrow} & .
\end{array}$$

We claim that $\{t_A\}_{A \in \underline{\mathcal{A}}_{\mathcal{D}}}$ is an isomorphism of functors $\Sigma_{\mathcal{D}} \to \Sigma'_{\mathcal{D}}$ with inverse given by $\{t'_A\}_{A \in \underline{\mathcal{A}}_{\mathcal{D}}}$. The fact that the t_A are isomorphisms in $\underline{\mathcal{A}}_{\mathcal{D}}$ follows from the observation that $(1, t't) : \delta_A \to \delta_A$ is a morphism of extriangles and hence, by using a similar argument to the above, we see that $\underline{t't} = \underline{1_{\Sigma_{\mathcal{D}} A}}$ in $\underline{\mathcal{A}}_{\mathcal{D}}$. Dually, $\underline{tt'} = \underline{1_{\Sigma'_{\mathcal{D}} A}}$.

It remains to verify that $\{t_A\}_{A \in \underline{\mathcal{A}}_{\mathcal{D}}}$ is a natural transformation. Let $f : A \to B$ we must show that $\underline{t_B \Sigma_{\mathcal{D}} f} = \underline{\Sigma'_{\mathcal{D}} f t_A}$. This follows from the observation that the pairs $(f, (t_B \Sigma_{\mathcal{D}} f))$ and $(f, (\Sigma'_{\mathcal{D}} f t_A))$ both define morphisms of extriangles $\delta_A \to \delta'_B$. $\square$

Proposition V.2.8. *The stable category $\underline{\mathcal{A}}_{\mathcal{D}}$ with the endofunctor $\Sigma_{\mathcal{D}}$ admits a right triangulation given by the collection of all sequences isomorphic to sequences of the form*

$$A \xrightarrow{\underline{f}} B \xrightarrow{\underline{g}} C \xrightarrow{\underline{h}} \Sigma x$$

that fit into a commutative diagram in $\mathcal{A}$

$$\begin{array}{ccccccc}
A & \xrightarrow{i_A} & \mathcal{D}(A) & \xrightarrow{p_A} & \Sigma_{\mathcal{D}} A & \overset{\delta_A}{\dashrightarrow} & \\
{\scriptstyle f}\downarrow & & \downarrow & & \| & & \\
B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma_{\mathcal{D}} A & \overset{f_* \delta_A}{\dashrightarrow} & .
\end{array}$$

Furthermore, $\Sigma_{\mathcal{D}} \underline{\mathcal{A}}_{\mathcal{D}}$ is always closed under extensions and $\Sigma_{\mathcal{D}}$ is fully faithful if and only if $\mathcal{D} \subset \mathrm{Proj}_{\mathbb{I}_{\mathcal{D}}} \mathcal{A}$.

Proof. The arguments of [10, Theorem 3.3] and [22, Theorem 3.1] may be recycled to the extriangulated setting. □

We can precisely describe when $\Sigma_{\mathcal{D}}$ is a right semi-equivalence.

Proposition V.2.9. *Suppose additionally that cones of left $\mathcal{D}$-approximations are $\mathcal{D}$-epic. Then the following hold.*

(i) $\mathbb{I}_{\mathcal{D}} = \mathbb{D}_{\mathcal{D}}$.

(ii) $\underline{\mathcal{A}}_{\mathcal{D}}$ *is a right triangulated category with right-semi equivalence.*

(iii) *For all $A, C \in \mathcal{A}$ there is a functorial isomorphism of abelian groups*

$$\begin{aligned} F = F_{C,A} : \operatorname{Hom}_{\underline{\mathcal{A}}_{\mathcal{D}}}(C, \Sigma_{\mathcal{D}} A) &\xrightarrow{\cong} \mathbb{D}_{\mathcal{D}}(C, A) \\ \underline{f} &\longmapsto f^* \delta_A. \end{aligned}$$

Proof. Claims (i) and (ii) follow directly from Propositions V.2.2 and V.2.8. It remains to show (iii):

F **is well-defined:** We must check that if $\underline{f} = \underline{g} \in \operatorname{Hom}_{\underline{\mathcal{A}}_{\mathcal{D}}}(C, \Sigma_{\mathcal{D}} A)$ then $f^*\delta_A = g^*\delta_A$. Indeed, in this case, $\underline{f-g} = 0$ and, since p_A is $\mathcal{D}$-epic by part (i), $f - g$ factors through p_A. Now, by Lemma II.1.12, $(f-g)^*\delta_A = 0$ and we are done. Note that this also shows the injectivity of F, since $f^*\delta_A = 0$ implies that f factors through p_A and thus $\underline{f} = 0$.

F **is bijective:** It remains to show that F is surjective. Let $\gamma \in \mathbb{D}_{\mathcal{D}}(C, A)$ be realised by $A \xrightarrow{x} B \xrightarrow{y} C$. Then, since x is $\mathcal{D}$-monic there exists $g : B \to \mathcal{D}(A)$ such that $gx = i_A$. By (ET3), there then exists $f : C \to \Sigma_{\mathcal{D}} A$ such that $(1, f) : \gamma \to \delta_A$ is a morphism of extriangles. In other words, $\gamma = f^*\delta_A =: F(f)$.

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{f^*\delta_A}{\dashrightarrow} & \\ \| & & \downarrow{\scriptstyle \exists g} & & \downarrow{\scriptstyle \exists f} & & \\ A & \xrightarrow[i_A]{} & \mathcal{D}(A) & \xrightarrow[p_A]{} & \Sigma_{\mathcal{D}} A & \overset{\delta_A}{\dashrightarrow} & . \end{array}$$

F **is a homomorphism of abelian groups:** This is straightforward:

$$F(f + f') = (f + f')^*\delta_A = f^*\delta_A + f'^*\delta_A = F(f) + F(f').$$

Functorality in the first argument: Let $\underline{c} \in \mathcal{A}(C, C')$. Then for all $\underline{f} \in \operatorname{Hom}_{\underline{\mathcal{A}}_{\mathcal{D}}}(C', \Sigma_{\mathcal{D}} A)$

$$\begin{aligned} F_{C,A} \operatorname{Hom}_{\underline{\mathcal{A}}_{\mathcal{D}}}(\underline{c}, \Sigma_{\mathcal{D}} A) : \underline{f} &\longmapsto \underline{fc} \quad \longmapsto (fc)^*\delta_A \\ \mathbb{D}_{\mathcal{D}}(c, A) F_{C',A} : \underline{f} &\longmapsto f^*\delta_A \longmapsto c^* f^* \delta_A = (fc)^*\delta_A \end{aligned}$$

which proves the claim.

Functorality in the second argument: Let $\underline{a} \in \operatorname{Hom}_{\underline{\mathcal{A}}_{\mathcal{D}}}(\Sigma_D A, \Sigma_{\mathcal{D}} A')$. Since $\Sigma_{\mathcal{D}}$ is full by part (b), there exists $\underline{\alpha} \in \operatorname{Hom}_{\underline{\mathcal{A}}_{\mathcal{D}}}(A, A')$ such that $\Sigma_{\mathcal{D}} \underline{\alpha} = \underline{a}$. Then for all $\underline{f} \in \operatorname{Hom}_{\underline{\mathcal{A}}_{\mathcal{D}}}(C, \Sigma_{\mathcal{D}} A)$

$$\begin{aligned} F_{C,A'} \operatorname{Hom}_{\underline{\mathcal{A}}_{\mathcal{D}}}(C, \underline{a}) : \underline{f} &\longmapsto \underline{af} \quad \longmapsto (af)^*\delta_A \\ \mathbb{D}_{\mathcal{D}}(C, \alpha) F_{C,A} : \underline{f} &\longmapsto f^*\delta_A \longmapsto \alpha_* f^* \delta_{A'} = (f)^* \alpha_* \delta_A = (f)^* a^* \delta_{A'} \end{aligned}$$

where the last equality in the second line follows from the definition of $\Sigma_{\mathcal{D}}$. □

In light of Corollary V.2.5, extriangulated categories $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ such that the injective stable category $\underline{\mathcal{A}}_{\operatorname{Inj}\mathbb{E}}$ is right triangulated with right semi-equivalence have a 'one-sided Frobenius' property: There are enough injectives and each injective object is projective. This imbalance of projectives and injectives and also Lemma V.1.4 indicate that we must look in extriangulated categories with infinitely many objects for examples of quotients that are right triangulated with right semi-equivalence.

Example V.2.10. (a) Let Q be the infinite quiver

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \dots$$

and consider the category $\mathcal{A} = \operatorname{mod} KQ/\operatorname{rad}^m$ for some $m > 1$. $\mathcal{A}$ is an abelian category and with its maximal exact structure it is an extriangulated category. Observe that $\mathcal{A}$ has enough injectives and that each injective object is projective. Indeed, $I_r = P_{r+m-1}$. But not every projective object is injective, for instance $P_1 = S_1$ is not injective. Thus, by Theorem V.2.8, the quotient category $\underline{\mathcal{A}}_{\operatorname{Inj}\mathcal{A}}$ is a right triangulated category with semi-equivalence with the shift given by the co-syzygy functor.

(b) Let $\mathcal{A} = (\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in $\mathcal{A}$ [95, Definition 4.1]. It is easily verified that $\mathcal{V}$ satisfies property $(*)$. Thus $\underline{\mathcal{A}}_{\mathcal{V}}$ is a right triangulated category. We will investigate examples of this flavour for the case of Frobenius extriangulated categories further in Section V.4. Let us note that the class of subcategories satisfying property $(*)$ is more general than the class of cotorsion pairs, since the subcategories giving cotorsion pairs must be closed under extensions.

(c) Let $\mathcal{T}$ be a compactly generated triangulated category. Recall that a subcategory $\mathcal{X}$ of $\mathcal{T}$ is *definable* if there is a class of morphisms ω between compact objects in $\mathcal{T}$ such that $\mathcal{X} = \operatorname{Inj}(\omega)$ [8, Section 4.1]; where it was also shown that every definable category admits left approximations. Thus, since in a triangulated category every morphism is an $\mathbb{E}$-inflation, it follows that $\underline{\mathcal{T}}_{\mathcal{X}}$ is a right triangulated category.

A future avenue of investigation could be to classify the examples in (b) and (c) that result in right triangulated categories with right semi-equivalence.

V.2.3 Right triangulated extriangulated categories

We characterise right triangulated categories as extriangulated categories. We begin with some terminology and a useful lemma.

Definition V.2.11. Let $(\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category. We say that the right triangulation Δ *induces an extriangulated structure on* $\mathcal{R}$ if there exists an external triangluation $(\mathbb{E}, \mathfrak{s})$ of $\mathcal{R}$ such that for all right triangles $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{z} \Sigma A$, there is an extriangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$.

Lemma V.2.12 ([95, Proposition 3.30])**.** *Let* $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated category and* $\mathcal{D} \subseteq \operatorname{Proj}_{\mathbb{E}}(\mathcal{A}) \cap \operatorname{Inj}_{\mathbb{E}}(\mathcal{A})$ *be a full, additive, replete subcategory. Then the stable category* $\underline{\mathcal{A}}_{\mathcal{D}}$ *inherits an external triangulation,* $(\underline{\mathbb{E}}, \underline{\mathfrak{s}})$*, given by*

(i) $\underline{\mathbb{E}}(C, A) = \mathbb{E}(C, A)$ *for all* $A, C \in \mathcal{A}$*;*

(ii) $\underline{\mathbb{E}}(\underline{c}, \underline{a}) = \mathbb{E}(c, a)$ *for all* $a \in \operatorname{Hom}_{\mathcal{A}}(A, A')$, $c \in \operatorname{Hom}_{\mathcal{A}}(C', C)$;

(iii) $\underline{\mathfrak{s}}(\delta) = [A \xrightarrow{\underline{x}} B \xrightarrow{\underline{y}} C]$ *where* $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ *for all extriangles* δ.

Example V.2.13. We give two important classes of examples.

(a) Let $\mathcal{T}$ be a triangulated category. Then the triangulation of $\mathcal{T}$ is a right triangulation which induces an extriangulated structure on $\mathcal{T}$. See [95, Section 3.3] This was a motivating example for the introduction of extriangulated categories.

(b) Let $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{D} \subset \mathcal{A}$ be a subcategory satisfying property $(*)$ and such that all cones of left $\mathcal{D}$-approximations are $\mathcal{D}$-epic. By Proposition V.2.9(i) we have that $\mathcal{D} \subseteq \operatorname{Proj}_{\mathbb{E}}(\mathcal{A}) \cap \operatorname{Inj}_{\mathbb{E}}(\mathcal{A})$ and it follows from part (iii) of the same result that the right triangulated structure on $\underline{\mathcal{A}}_{\mathcal{D}}$ of Proposition V.2.8 coincides with the extriangulated structure of Lemma V.2.12. In other words, the right triangulation of $\underline{\mathcal{A}}_{\mathcal{D}}$ (with right semi-equivalence) induces an extriangulation on $\underline{\mathcal{A}}_{\mathcal{D}}$. Let us note that this could also be deduced from a combination of Lemma V.1.8, Remark II.1.7 and the above Example.

We may now state and prove the main result of this section.

Theorem V.2.14. *Let* $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated category. Then the following are equivalent*

(i) *There exists a fully faithful additive endofunctor* $\Sigma : \mathcal{A} \to \mathcal{A}$ *such that* $\mathbb{E}(-, ?) \cong \operatorname{Hom}_{\mathcal{A}}(-, \Sigma ?)$ *and that the image* $\Sigma\mathcal{A}$ *is closed under* $\mathbb{E}$*-extensions;*

(ii) $\operatorname{Inj}_{\mathbb{E}}(\mathcal{A}) = \{0\}$ *and there are enough* $\mathbb{E}$*-injectives;*

(iii) *There is a right triangulation of* $\mathcal{A}$ *that induces the extriangulated structure* $(\mathbb{E}, \mathfrak{s})$.

Proof. **(i)⇒(ii):** We claim that, for each $A \in \mathcal{A}$, the identity morphism $1_{\Sigma A} \in \operatorname{Hom}_{\mathcal{A}}(\Sigma A, \Sigma A) \cong \mathbb{E}(A, \Sigma A)$ is realised by the sequence $[A \to 0 \to \Sigma A]$. Indeed, we may use a similar argument to that of [95, Lemma 3.21]: Let $\mathfrak{s}(1_{\Sigma A}) = [A \xrightarrow{x} E \xrightarrow{y} \Sigma A]$, then by Lemma II.1.8 there is a long exact sequence in $[\mathcal{A}^{\mathrm{op}}, \mathsf{Ab}]$

$$\operatorname{Hom}_{\mathcal{A}}(-, A) \xrightarrow{x \circ -} \operatorname{Hom}_{\mathcal{A}}(-, E) \xrightarrow{y \circ -} \operatorname{Hom}_{\mathcal{A}}(-, \Sigma A) \xrightarrow{(1_{\Sigma A})_{\#} = \mathrm{id}}$$
$$\xrightarrow{(1_{\Sigma A})_{\#} = \mathrm{id}} \operatorname{Hom}_{\mathcal{A}}(-, \Sigma A) \xrightarrow{\Sigma x \circ -} \operatorname{Hom}_{\mathcal{A}}(-, \Sigma E).$$

It follows that $y = 0 = \Sigma x$. Thus $x = 0$ since Σ is faithful. Now the exactness of $0 \to \mathcal{A}(-, E) \to 0$ implies that $E \cong 0$. Thus, if I is an $\mathbb{E}$-injective object then it is a direct summand of 0 and so $I = 0$.

(ii)⇒(i),(iii): In this case, the subcategory $\{0\}$ satisfies $\mathbb{E} = \mathbb{I}_{\{0\}} = \mathbb{D}_{\{0\}}$. Thus the stable category $\underline{\mathcal{A}}_{\{0\}} \cong \mathcal{A}$ has a right triangulated structure with right semi-equivalence by Propositions V.2.8 and V.2.9. The claims follow from Example V.2.13(b).

(iii)⇒(ii): By the axiom (R3)(c), for all $A \in \mathcal{A}$, the morphism $A \to 0$ is the first morphism in a right triangle. Thus, by assumption, $A \to 0$ is an $\mathbb{E}$-inflation and the claim follows. □

As a direct consequence, we see that, in general, right triangulated categories do not have a natural extriangulated structure.

Corollary V.2.15. *Let $(\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category. Then Δ induces an extriangulated structure on $\mathcal{R}$ if and only if Σ is a right semi-equivalence.*

Remark V.2.16. The concept of *negative (first) extensions* of an extriangulated category has been recently introduced and studied [1, 54]. For a triangulated category $\mathcal{T}$ one may take $\mathbb{E}^{-1}(-,?) = \mathrm{Hom}_{\mathcal{T}}(-, \Sigma^{-1}?)$ as a negative first extension. Since by Lemma V.1.8 a right triangulated category with right semi equivalence $\mathcal{R}$ is an extension closed subcategory of the triangulated category $\mathcal{S}(\mathcal{R})$ there is a natural first negative extension structure on $\mathcal{R}$ given by $\mathbb{E}^{-1}(C, A) := \mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(C, \Sigma^{-1}A) \cong \mathrm{Hom}_{\mathcal{R}}(\Sigma C, A)$ for all $A, C \in \mathcal{R}$.

V.3 Aisles and co-aisles

In this section, we show that the language of extriangulated categories allows us, under some assumptions, to give an intrinsic characterisation of which right triangulated categories with right semi-equivalence occur as (co)-aisles of (co-)t-structures in its stabilisation. Let $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category with right semi-equivalence.

V.3.1 $\mathbb{E}$-torsion pairs

We are interested in certain classes of $\mathbb{E}$-torsion pairs.

Definition V.3.1. If $\mathcal{A}$ is a triangulated category (viewed naturally as an extriangulated category) with shift functor Σ, an $\mathbb{E}$-torsion pair in $\mathcal{A}$, $(\mathcal{U}, \mathcal{V})$, is a *t-structure* [18] (resp. *co-t-structure* [26, 101]) if $\Sigma\mathcal{U} \subseteq \mathcal{U}$ (resp. $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$) with *heart* $\mathcal{H} = \mathcal{U} \cap \Sigma\mathcal{V}$ (resp. *co-heart* $\mathcal{M} = \mathcal{U} \cap \Sigma^{-1}\mathcal{V}$). We call the subcategory $\mathcal{U}$ the *aisle* of the (co-)t-structure and $\mathcal{V}$ the *co-aisle*. A (co-)t-structure is *bounded* if the equalities $\mathcal{A} = \bigcup_{n\in\mathbb{Z}} \Sigma^n\mathcal{U} = \bigcup_{m\in\mathbb{Z}} \Sigma^m\mathcal{V}$ hold.

The reader should be aware that the terminology and notation of torsion pairs and (co-)t-structures in a triangulated category varies, often by a shift, throughout the literature and that co-t-structures were introduced under the name 'weight structures' in [26].

Before we proceed, let us compare the above definition of a torsion pair in $\mathcal{R}$ with other notions appearing in the literature:

(a) An $\mathbb{E}$-torsion pair $(\mathcal{U}, \mathcal{V})$ in $\mathcal{R}$ is a *right torsion pair* [87] if Σ preserves $\mathcal{V}$-monics.

(b) When $\mathcal{R}$ is equipped with a negative first extension structure, $\mathbb{E}^{-1}$, (see Remark V.2.16) an $\mathbb{E}$-torsion pair $(\mathcal{U}, \mathcal{V})$ in $\mathcal{R}$ is an $\mathfrak{s}$*-torsion pair* [1] if $\mathbb{E}^{-1}(\mathcal{U}, \mathcal{V}) = 0$. In this case the extriangle III.2 is essentially unique and assigments $C \mapsto U$ and $C \mapsto V$ are functorial [1, Proposition 3.7].

The next lemma shows that right torsion pairs and $\mathfrak{s}$-torsion pairs in $\mathcal{R}$ coincide (when we equip $\mathcal{R}$ with the natural first negative extension structure $\mathbb{E}^{-1}(-.?) =$

$\mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(-, \Sigma^{-1}?))$ and that such $\mathbb{E}$-torsion pairs remind us of t-structures; which are precisely the $\mathfrak{s}$-torsion pairs in a triangulated category. For a class of objects $\mathcal{X}$ in $\mathcal{R}$, by $\Sigma^{-1}\mathcal{X}$ we denote the class of objects $\{X \in \mathcal{R} \mid \Sigma X \in \mathcal{X}\}$.

Lemma V.3.2. *Let $(\mathcal{U}, \mathcal{V})$ be an $\mathbb{E}$-torsion pair in $\mathcal{R}$. Then the following are equivalent*

(i) $\Sigma\mathcal{U} \subseteq \mathcal{U}$;

(ii) $\Sigma^{-1}\mathcal{V} \subseteq \mathcal{V}$;

(iii) $\mathrm{Hom}_{\mathcal{R}}(\Sigma\mathcal{U}, \mathcal{V}) = 0$;

(iv) Σ *preserves $\mathcal{V}$-monics.*

Proof. **(i)⇒(ii):** Suppose that $\Sigma\mathcal{U} \subseteq \mathcal{U}$ and let $Y \in \Sigma^{-1}\mathcal{V}$. Then there is a right triangle

$$U \xrightarrow{u} Y \longrightarrow V \longrightarrow$$

with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Consider the morphism $\Sigma u : \Sigma U \to \Sigma Y$. By assumption, $\Sigma U \in \mathcal{U}$ and $\Sigma Y \in \mathcal{V}$; thus, $\Sigma u = 0$. Since Σ is faithful, $u = 0$ and we deduce that $Y \cong V \in \mathcal{V}$.

(ii)⇒(iii): Follows from the fact that $\mathrm{Hom}_{\mathcal{R}}(\mathcal{U}, \mathcal{V}) = 0$ and Σ is fully faithful.

(iii)⇒(iv): Let $f : A \to B$ be a $\mathcal{V}$-monic morphism. There are right triangles

$$U \xrightarrow{u} A \xrightarrow{v} V \longrightarrow \Sigma U$$

and

$$U' \xrightarrow{u'} \Sigma A \xrightarrow{v'} V' \longrightarrow \Sigma U'$$

with $U, U' \in \mathcal{U}$ and $V, V' \in V$. Then, as f is $\mathcal{V}$-monic, there exists $g : B \to V$ such that $gf = v$. Since $\mathrm{Hom}_{\mathcal{R}}(\Sigma\mathcal{U}, V) = 0$ there exists $c : \Sigma V \to V'$ such that $c(\Sigma v) = v'$

$$\begin{array}{ccccccc}
\Sigma U & \xrightarrow{\Sigma u} & \Sigma A & \xrightarrow{\Sigma v} & \Sigma V & \longrightarrow & \Sigma^2 U \\
 & & \| & & \downarrow{\scriptstyle \exists c} & & \\
U' & \xrightarrow{u'} & \Sigma A & \xrightarrow{v'} & V' & \longrightarrow & \Sigma U'.
\end{array}$$

Thus

$$v' = c(\Sigma v) = c\Sigma(gf) = c(\Sigma g)(\Sigma f).$$

This finishes the proof since v' is a left $\mathcal{V}$-approximation of ΣA and so all morphisms from ΣA to $\mathcal{V}$ factor through v'.

(iv)⇒(i): Let $U \in \mathcal{U}$. Then, since $\mathrm{Hom}_{\mathcal{R}}(U, \mathcal{V}) = 0$ we have that $U \to 0$ is $\mathcal{V}$-monic. Thus, by assumption $\Sigma U \to 0$ is also $\mathcal{V}$-monic from which we deduce that $\Sigma U \in \mathcal{U}$.

□

V.3.2 Co-aisles of co-t-structures

The next lemma justifies why we will look to describe $\mathcal{R}$ as a (co-)aisle in $\mathcal{S}(\mathcal{R})$.

Lemma V.3.3. *Let $\mathcal{T}$ be a triangulated category, $(\mathcal{V}, \mathcal{W})$ (resp. $(\mathcal{U}, \mathcal{V})$) be a t-structure (resp. co-t-structure) in $\mathcal{T}$ and $\mathcal{S} := \operatorname{cosusp}_{\mathcal{T}}\mathcal{V} \cong \mathcal{S}(\mathcal{V})$. Then $(\mathcal{V}, \mathcal{W} \cap \mathcal{S})$ (resp. $(\mathcal{U} \cap \mathcal{S}, \mathcal{V})$) is a t-structure (resp. co-t-structure) in $\mathcal{S}$.*

Proof. We prove the t-structure case, whence statement for co-t-structures will follow dually. Let $(\mathcal{V}, \mathcal{W})$ be a t-structure in $\mathcal{T}$ and set $\mathcal{S} := \operatorname{cosusp}_{\mathcal{T}}\mathcal{V} \cong \mathcal{S}(\mathcal{V})$. Clearly, $\operatorname{Hom}_{\mathcal{S}}(\mathcal{V}, \mathcal{W} \cap \mathcal{S}) = 0$ and $\Sigma\mathcal{V} \subset \mathcal{V}$. Thus it remains to show that $\mathcal{S} = \mathcal{V} * (\mathcal{W} \cap \mathcal{S})$. Since $\mathcal{V}, (\mathcal{W} \cap \mathcal{S}) \subset \mathcal{S}$ and $\mathcal{S}$ is closed under extensions, $\mathcal{V} * (\mathcal{W} \cap \mathcal{S}) \subset \mathcal{S}$. To show the converse, let $A \in \mathcal{S}$ then there exists a triangle in $\mathcal{T}$

$$V \longrightarrow A \longrightarrow W \longrightarrow \Sigma V$$

with $V \in \mathcal{V}$ and $W \in \mathcal{W}$. Thus $W \in \mathcal{S} * \Sigma\mathcal{V} \subset \mathcal{S} * \mathcal{S} \subset \mathcal{S}$. □

We also note that the boundedness of right triangulated categories (Lemma V.1.7) relates to the boundedness of (co-)t-structures.

Lemma V.3.4. *Suppose that $\mathcal{R}$ is the co-aisle of a co-t-structure, $(\mathcal{U}, \mathcal{R})$ in $\mathcal{S}(\mathcal{R})$. If $\mathcal{R}$ is bounded then the co-t-structure $(\mathcal{U}, \mathcal{R})$ is bounded. A dual statement holds for a t-structure $(\mathcal{R}, \mathcal{V})$ in $\mathcal{S}(\mathcal{R})$.*

Proof. Let $(\mathcal{U}, \mathcal{R})$ be a co-t-structure in $\mathcal{S}(\mathcal{R})$. By construction of $\mathcal{S}(\mathcal{R})$, the equality $\mathcal{S}(\mathcal{R}) = \bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{R}$ holds. It remains to verify that $\mathcal{S}(\mathcal{R}) = \bigcup_{m \in \mathbb{Z}} \Sigma^m \mathcal{U}$. Observe that since $\mathcal{R}$ is bounded, in light of Lemma V.1.7, for all $X \in \mathcal{S}(\mathcal{R})$ we have that $\operatorname{Hom}_{\mathcal{S}(\mathcal{R})}(\mathcal{R}, \Sigma^i X) = 0$ for $i << 0$. Thus, by Lemma III.1.3, $\Sigma^i X \in \mathcal{V}$ for $i << 0$; whence the claim follows. □

Before we give our characterisations of $\mathcal{R}$ as the co-aisle of a co-t-structure, we make some comments. Note, by Proposition V.2.9, the $\mathbb{E}$-projectives of $\mathcal{R}$ are the precisely the objects $P \in \mathcal{R}$ satisfying $\operatorname{Hom}_{\mathcal{R}}(P, \Sigma -) = 0$. Recall from [3] that a subcategory $\mathcal{X} = \operatorname{add}\mathcal{X}$ of a triangulated category $\mathcal{T}$ is *silting* if $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, \Sigma^{>0}\mathcal{X}) = 0$ and $\mathcal{T} = \operatorname{thick}\mathcal{X}$, the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{X}$ that is closed under direct summands. We are now ready to give characterisations of $\mathcal{R}$ as the co-aisle of a co-t-structure in $\mathcal{S}(\mathcal{R})$ in terms of torsion pairs, $\mathbb{E}$-projectives of $\mathcal{R}$ and silting subcategories.

Theorem V.3.5. *The following are equivalent*

(i) *$\mathcal{R}$ is the co-aisle of a co-t-structure $(\mathcal{U}, \mathcal{R})$ in $\mathcal{S}(\mathcal{R})$;*

(ii) *$\mathcal{R}$ has enough $\mathbb{E}$-projectives;*

(iii) *There is a torsion pair $(\operatorname{Proj}_{\mathbb{E}}\mathcal{R}, \Sigma\mathcal{R})$ in $\mathcal{R}$.*

Moreover, if $\mathcal{R}$ is bounded then the above conditions are also equivalent to

(d) *$\operatorname{Proj}_{\mathbb{E}}\mathcal{R}$ is a silting subcategory of $\mathcal{S}(\mathcal{R})$.*

Proof. **(i)⇒(ii):** Suppose there is a co-t-structure $(\mathcal{U}, \mathcal{R})$ in $\mathcal{S}(\mathcal{R})$. We must show that for all $X \in \mathcal{R}$ there is a right triangle

$$R \to P \to X \to \Sigma R$$

in $\mathcal{R}$ with $P \in \operatorname{Proj}\mathcal{R}$. Since $(\mathcal{U}, \mathcal{R})$ is a co-t-structure in $\mathcal{S}(\mathcal{R})$ there is a triangle in $\mathcal{S}(\mathcal{R})$

$$U \to \Sigma^{-1}X \to R \to \Sigma U$$

with $U \in \mathcal{U}$, $R \in \mathcal{R}$. By rotating we obtain the triangle

$$R \to \Sigma U \to X \to \Sigma R.$$

We claim that $\Sigma U \in \operatorname{Proj}_{\mathbb{E}}\mathcal{R}$. The fact that $\Sigma U \in \mathcal{R}$ follows from this triangle since $\mathcal{R}$ is closed under extensions in $\mathcal{S}(\mathcal{R})$ by Lemma V.1.8. It remains to verify that ΣU is $\mathbb{E}$-projective in $\mathcal{R}$, that is $\operatorname{Hom}_{\mathcal{R}}(\Sigma U, \Sigma\mathcal{R}) = 0$:

$$0 = \operatorname{Hom}_{\mathcal{S}(\mathcal{R})}(U, \mathcal{R}) \cong \operatorname{Hom}_{\mathcal{S}(\mathcal{R})}(\Sigma U, \Sigma\mathcal{R}) \cong \operatorname{Hom}_{\mathcal{R}}(\Sigma U, \Sigma\mathcal{R})$$

where we have the first equality from the properties of co-t-structures.

(ii)⇒(iii): Let $X \in \mathcal{R}$. By assumption there is a right triangle $R \to P \to X \to \Sigma\mathcal{R}$ in $\mathcal{R}$ with $P \in \operatorname{Proj}_{\mathbb{E}}\mathcal{R}$. We rotate this to the right triangle

$$P \to X \to \Sigma R \to \Sigma P$$

whence the claim follows as $\operatorname{Hom}_{\mathcal{R}}(P, \Sigma\mathcal{R}) = 0$.

(iii)⇒(i): We use a similar argument to [92, Theorem 3.11]. Suppose that there is a torsion pair $(\operatorname{Proj}_{\mathbb{E}}\mathcal{R}, \Sigma\mathcal{R})$ in $\mathcal{R}$ and set $\mathcal{P} := \operatorname{Proj}_{\mathbb{E}}\mathcal{R}$ and $\mathcal{U} := \{Y \in \mathcal{S}(\mathcal{R}) \mid \operatorname{Hom}_{\mathcal{S}(\mathcal{R})}(Y, \mathcal{R}) = 0\}$. Let $0 \neq X \in \mathcal{S}(\mathcal{R})$, we must show that there is a triangle

$$U \to X \to R \to \Sigma U$$

in $\mathcal{S}(\mathcal{R})$ with $U \in \mathcal{U}$ and $R \in \mathcal{R}$. By the definition of $\mathcal{S}(\mathcal{R})$ there exists $A \in \mathcal{R}$ and $n \in \mathbb{Z}$ such that $X \cong \Sigma^n A$. If $n \geq 0$ then $X \in \mathcal{R}$ and we take the triangle $0 \to X \to X \to 0$. For $n < 0$ we proceed by induction. By assumption, there is a right triangle in $\mathcal{R}$

$$P \to A \to \Sigma R \to \Sigma P$$

with $P \in \mathcal{P}$ and $R \in \mathcal{R}$. This rotates to a triangle

$$\Sigma^n P \to \Sigma^n A \to \Sigma^{n+1} R \to \Sigma^{n+1} P$$

in $\mathcal{S}(\mathcal{R})$. By the induction hypothesis, there is a triangle in $\mathcal{S}(\mathcal{R})$

$$U \to \Sigma^{n+1} R \to V \to \Sigma U$$

with $U \in \mathcal{U}$ and $V \in \mathcal{R}$. We apply (ET4)$^{\mathrm{op}}$ to these triangles

$$\begin{array}{ccccccc}
\Sigma^n P & \longrightarrow & E & \longrightarrow & U & \longrightarrow & \Sigma^{n+1} P \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^n P & \longrightarrow & \Sigma^n A & \longrightarrow & \Sigma^{n+1} R & \longrightarrow & \Sigma^{n+1} P \\
 & & \downarrow & & \downarrow & & \\
 & & V & = & V & & \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma E & \longrightarrow & \Sigma U & &
\end{array} \tag{V.4}$$

and claim that the triangle $E \to \Sigma^n A \to V \to \Sigma E$ satisfies the required conditions. Since $V \in \mathcal{R}$ by construction, we only have to show that $E \in \mathcal{U}$. Observe that $\Sigma^n P$ is in $\mathcal{U}$:

$$\mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(\Sigma^n P, \mathcal{R}) \cong \mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(P, \Sigma^{-n}\mathcal{R}) \cong \mathrm{Hom}_{\mathcal{R}}(P, \Sigma^{-n}\mathcal{R}) = 0$$

since $\mathrm{Hom}_{\mathcal{R}}(P, \Sigma\mathcal{R}) = 0$ and $n < 0$. Thus the top row of the Diagram V.4 shows that $E \in \mathcal{U}$ since $\mathcal{U}$ is closed under extensions.

(iv)⇔(i): Since $\mathcal{R}$ is bounded, by Lemma V.1.7 the co-t-structure $(\mathcal{U}, \mathcal{V})$ is bounded in $\mathcal{S}(\mathcal{R})$. Observe that $\mathrm{Proj}_{\mathbb{E}}\mathcal{R}$ is the co-heart of this co-t-structure. The claim then follows from [92, Corollary 5.9] where it was shown that a subcategory of a triangulated category is silting precisely when it is the co-heart of a bounded co-t-structure. □

The characterisations of Theorem V.3.5 adds to the numerous interpretations of silting subcategories in triangulated categories which are surveyed in [6]. We also obtain the following as a direct consequence.

Corollary V.3.6. *There is a correspondence between silting subcategories of triangulated categories and bounded right triangulated categories with right semi-equivalence that have enough projectives.*

The above result is a triangulated analogue of Lemma III.4.1 due to Bondal & van den Bergh and Rump, which states that there is a correspondence between tiltings in abelian categories and quasi-abelian categories (this is actually the dual to what was stated before). When working with derived categories/ module categories of nice finite dimensional algebras these correspondences have a natural connection which we explain in the next paragraph.

V.3.3 Tilting and silting connection

Let us first describe the tilting–quasi-abelian correspondence in the abelian setting: Let $\mathcal{Q}$ be a quasi-abelian category and let $\mathcal{A}$ be the right associated abelian category of $\mathcal{Q}$ so that $\mathcal{Q}$ is a tilting torsion class in $\mathcal{A}$. Then $\mathrm{Proj}\,\mathcal{Q}$ is a tilting subcategory of $\mathcal{A}$, that is, every object is a subobject of an object in $\mathrm{Proj}\,\mathcal{Q}$.

There is a natural way to associate a right triangulated category (with right semi-equivalence) to a quasi-abelian category. For a quasi-abelian category $\mathcal{Q}$ as above, we define the right triangulated category $\mathcal{R}_{\mathcal{Q}} := \mathrm{susp}_{\mathsf{D}^b(\mathcal{A})}\mathcal{Q}$, that is, the smallest subcategory of $\mathsf{D}^b(\mathcal{A})$ that is closed under positive shifts and extensions that contains $\mathcal{Q}$.

Lemma V.3.7. *In the notation as above, the following hold.*

(i) $\mathcal{R}_{\mathcal{Q}} = \{X^\bullet \in \mathsf{D}^b(\mathcal{A}) \mid \mathsf{H}^0(X^\bullet) \in \mathcal{Q}, \mathsf{H}^i(X^\bullet) = 0 \, \forall \, i < 0\}$*;*

(ii) $\mathcal{S}(\mathcal{R}_{\mathcal{Q}}) = \mathsf{D}(\mathcal{A})$*;*

(iii) *If $\mathcal{A}$ has finite global dimension, $\mathcal{R}_{\mathcal{Q}}$ is bounded;*

(iv) *If $\mathcal{Q}$ has a projective generator then* $\mathrm{Proj}\,\mathcal{R}_{\mathcal{Q}} = \mathrm{Proj}\,\mathcal{Q}$ *and $\mathcal{R}_{\mathcal{Q}}$ has enough projectives.*

Proof. The left-to-right inclusion of part (i) is obvious. For the converse, note that it is enough to show that $\Sigma\mathcal{A} = \{\Sigma X \mid X \in \mathcal{A}\} \subseteq \mathcal{R}_{\mathcal{Q}}$. To this end, let $X \in \mathcal{A}$, then, as $\operatorname{Proj}\mathcal{Q}$ is a tilting subcategory of $\mathcal{A}$, there is a short exact sequence

$$0 \to X \to Q \to Q' \to 0$$

in $\mathcal{A}$ with $Q \in \mathcal{Q}$. Since $\mathcal{Q}$ is a torsion class in $\mathcal{A}$, it is closed under quotients (Lemma III.1.3) and thus $Q' \in \mathcal{Q}$. This short exact sequence gives rise to a triangle in $\mathsf{D}^b(\mathcal{A})$

$$X \to Q \to Q' \to \Sigma X, \tag{V.5}$$

which we rotate to the obtain the triangle

$$Q' \to \Sigma X \to \Sigma Q \to \Sigma Q'.$$

The claim now follows since $Q', \Sigma Q \in \mathcal{R}_{\mathcal{Q}}$ and $\mathcal{R}_{\mathcal{Q}}$ is closed under extensions. (ii) and (iii) follow from (i). For (iv), let P be a projective generator of $\mathcal{Q}$. We will show that every object A in $\mathcal{R}_{\mathcal{Q}}$ admits a right triangle

$$B \to P^I \to A \to \Sigma B$$

in $\mathcal{R}_{\mathcal{Q}}$ for a set I. Since, for $X \in \Sigma\mathcal{R}_{\mathcal{Q}}$ we may make take the triangle

$$\Sigma^{-1}X \to 0 \to X \to X$$

we only have to show the claim for objects in $\mathcal{R}_{\mathcal{Q}} \setminus \Sigma\mathcal{R}_{\mathcal{Q}} \subset \mathcal{Q} \sqcup \Sigma\mathcal{A}$. For every object $Q' \in Q$ there is a short exact sequence in $\mathcal{Q}$ (and hence also in $\mathcal{A}$) $K \rightarrowtail P^I \twoheadrightarrow Q'$ which induces a triangle in $\mathsf{D}^b(\mathcal{A})$

$$K \to P^I \to Q \to \Sigma K$$

which is a right triangle in $\mathcal{R}_{\mathcal{Q}}$ since all terms are in $\mathcal{R}_{\mathcal{Q}}$. For $X \in \mathcal{A}$ we apply $(\mathrm{ET4})^{\mathrm{op}}$ to the above triangle and an appropriate rotation of the triangle V.5 to obtain a right triangle in $\mathcal{R}_{\mathcal{Q}}$ of the form $E \to P^I \to \Sigma X \to \Sigma E$. □

Corollary V.3.8. *Let $\mathcal{Q}$ be a quasi-abelian category with an injective cogenerator I and a projective generator P and let $\mathcal{A}$ be the right associated abelian category of $\mathcal{Q}$. Suppose that, $\mathcal{Q}$ is artinian with respect to subobjects and* $\operatorname{End}_{\mathcal{Q}}(I)$ *is an artin algebra of finite global dimension. Then P is a tilting module in $\mathcal{A}$ and* $\operatorname{add} P$ *is a silting subcategory of* $\mathsf{D}^b(\mathcal{A})$.

Proof. The first claim follows from our work in Chapter III, in particular it is a combination of (the duals of) Theorem III.4.9 and Corollary III.4.11. The second claim follows from Theorem V.3.5 and Lemma V.3.7 □

Remark V.3.9. Let Λ be a finite dimensional algebra over a field k and set $\mathcal{A} = \operatorname{mod}\Lambda$. By [2, Theorems 2.7 & 3.2] and [67, Theorem 2.2] (see also [80]) there are bijections between the following

(a) Functorially finite torsion classes in $\mathcal{A}$;

(b) Isomorphism classes of τ-tilting modules in $\mathcal{A}$;

(c) Two-term silting complexes in $\mathsf{K}^b(\operatorname{proj}\mathcal{A})$;

(d) Intermediate co-t-structures of $\mathsf{K}^b(\operatorname{proj}\mathcal{A})$.

and that the first bijection restricts to a bijection between tilting torsion classes in $\mathcal{A}$ and isomorphism classes of tilting modules in $\mathcal{A}$. Along these bijections, a functorially finite torsion class $\mathcal{T}$ corresponds to the τ-tilting module M where M is a projective generator of $\mathcal{T}$ which in turn corresponds to the 2-term silting complex $(P^1 \to P^0) \in \mathsf{K}^b(\operatorname{proj}\mathcal{A})$ where $P^1 \to P^0 \to M \to 0$ is a projective presentation of M in $\mathcal{A}$. The co-t-structure corresponding to $(P^1 \to P^0)$ has co-aisle $\operatorname{susp}_{\mathsf{K}^b(\operatorname{proj}\mathcal{A})}(P^1 \to P^0)$.

When Λ is of finite global dimension we have that $\mathsf{K}^b(\operatorname{proj}\mathcal{A}) \cong \mathsf{D}^b(\mathcal{A})$. Thus, by Lemma V.3.7, following the correspondences above from (a) to (d), a tilting torsion class $\mathcal{T}$ is associated to (the co-t-structure with co-aisle) $\mathcal{R}_{\mathcal{T}}$ as we defined at the start of this subsection. Of course, without the finite global dimension assumption on $\mathcal{A}$, one could define $\mathcal{R}_{\mathcal{T}}$ as $\operatorname{susp}_{\mathsf{K}^b(\operatorname{proj}\mathcal{A})}(P^1 \to P^0)$ but this is somewhat less natural.

V.3.4 Aisles of t-structures

We now present our characterisations of aisles of t-structures.

Theorem V.3.10. *The following are equivalent*

(i) *$\mathcal{R}$ is the aisle of a t-structure $(\mathcal{R}, \mathcal{V})$ in $\mathcal{S}(\mathcal{R})$;*

(ii) *There is a torsion pair $(\Sigma\mathcal{R}, \mathcal{F})$ in $\mathcal{R}$;*

(iii) *There is an equivalence of triangulated categories $\phi : \mathcal{S}(\mathcal{R}) \to \mathcal{C}(\mathcal{R})$*

$$\begin{array}{ccc} & \mathcal{R} & \\ {}^{s}\swarrow & & \nwarrow^{c} \\ \mathcal{S}(\mathcal{R}) & \xrightarrow[\cong]{\phi} & \mathcal{C}(\mathcal{R}) \end{array}$$

such that $c\phi s = 1_{\mathcal{R}}$.

Proof. The equivalence of (i) and (ii) follows from dualising the arguments used in the proof of Theorem V.3.5.

(i)⇒(iii): We show that in this case, $\mathcal{S}(\mathcal{R})$ satisfies the universal property of the co-stabilsation of $\mathcal{R}$. Let $r : \mathcal{S}(\mathcal{R}) \to \mathcal{R}$ denote the functor induced by the t-structure $(\mathcal{R}, \mathcal{V})$. Let $\mathcal{T}$ be a triangulated category and $F : \mathcal{T} \to \mathcal{R}$ be a right triangle functor. Define the functor $F' : \mathcal{T} \to \mathcal{S}(\mathcal{R})$ by $F'X = (FX, 0)$. Clearly $rF' = F$ and F' is a triangle functor.

We verify that F' is unique with this property. Suppose that $F'' : \mathcal{T} \to \mathcal{R}$ is another triangle functor such that $rF'' = F$. Then for all $X \in \mathcal{T}$ there are triangles in $\mathcal{S}(\mathcal{R})$

$$rF'X \cong FX \to F'X \to V' \to \Sigma FX$$
$$rF''X \cong FX \to F''X \to V'' \to \Sigma FX$$

with $\mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(\mathcal{R}, V') = 0 = \mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(\mathcal{R}, V'')$. Observe that, for $n >> 0$, $\Sigma^n F'X$, $\Sigma^n F''X \in \mathcal{R}$. Thus, $\Sigma^n F'X \cong \Sigma^n FX \cong \Sigma^n F''X$. So, since F' and F'' are triangle functors, $F''\Sigma^n X \cong F'\Sigma^n X$ and we are done.

(iii)⇒(i): We will show that that $c\phi$ is right adjoint to the inclusion s whence we will be done by [77, Proposition 1.2]. It is enough to show that for all $X, Y \in \mathcal{R}$ and $m \in \mathbb{Z}$ that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{R}}(X, c\phi(Y, m)) \cong \mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(sX, (Y, m)).$$

If $m \geq 0$ then $(Y, m) = s(\Sigma^m Y)$ and thus, by assumption $c\phi(Y, m) = c\phi s(\Sigma^m Y) \cong \Sigma^m Y$. On the other hand

$$\mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(sX, (Y, m)) = \mathrm{Hom}_{\mathcal{S}(\mathcal{R})}((X, 0), (Y, m)) \cong \mathrm{Hom}_{\mathcal{R}}(X, \Sigma^m Y)$$

and we are done.

If $m < 0$, then $\mathrm{Hom}_{\mathcal{S}(\mathcal{R})}(sX, (Y, m)) \cong \mathrm{Hom}_{\mathcal{R}}(\Sigma^{-m} X, Y)$. On the other hand,

$$\mathrm{Hom}_{\mathcal{R}}(X, c\phi(Y, m)) \cong \mathrm{Hom}_{\mathcal{R}}(\Sigma^{-m} X, \Sigma^{-m} c\phi(Y, m))$$

and

$$\Sigma^{-m} c\phi(Y, m) \cong c\phi\Sigma^{-m}(Y, m) = c\phi(Y, 0) \cong c\phi sY \cong Y$$

and we are done since all isomorphisms used are natural. □

We end this section by noting that the language of the torsion pairs in $\mathcal{R}$ also allows us to describe related classes of (co-)t-structures.

Proposition V.3.11. $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ *be a right triangulated category with right semi-equivalence.*

(i) *Suppose that there is a co-t-structure* $(\mathcal{U}, \mathcal{R})$ *in* $\mathcal{S}(\mathcal{R})$. *Then there is a bijection*

$$\begin{aligned}&\{(\mathcal{X}, \mathcal{Y}) \text{ co-t-structure in } \mathcal{S}(\mathcal{R}) \mid \mathcal{Y} \subseteq \mathcal{R}\}\\ &\qquad\longleftrightarrow \{(\mathcal{A}, \mathcal{B}) \ \mathbb{E}\text{-torsion pair in } \mathcal{R} \mid \Sigma\mathcal{B} \subseteq \mathcal{B} \subseteq \Sigma\mathcal{R}\}\\ &(\mathcal{X}, \mathcal{Y}) \longmapsto (\mathcal{R} \cap \Sigma\mathcal{X}, \Sigma\mathcal{Y})\\ &(\Sigma^{-1}(\mathcal{U} * \mathcal{A}), \Sigma^{-1}\mathcal{B}) \longleftarrow\!\shortmid (\mathcal{A}, \mathcal{B})\end{aligned}$$

which preserves the inclusion of aisles and restricts to a bijection

$$\begin{aligned}&\{(\mathcal{X}, \mathcal{Y}) \text{ co-t-structure in } \mathcal{S}(\mathcal{R}) \mid \Sigma\mathcal{R} \subseteq \mathcal{Y} \subseteq \mathcal{R}\}\\ &\qquad\longleftrightarrow \{(\mathcal{A}, \mathcal{B}) \ \mathbb{E}\text{-torsion pair in } \mathcal{R} \mid \Sigma^2\mathcal{R} \subseteq \mathcal{B} \subseteq \Sigma\mathcal{R}\}.\end{aligned}$$

(ii) *Suppose that there is a t-structure* $(\mathcal{R}, \mathcal{V})$ *in* $\mathcal{S}(\mathcal{R})$. *Then there is a bijection*

$$\begin{aligned}&\{(\mathcal{X}, \mathcal{Y}) \text{ t-structure in } \mathcal{S}(\mathcal{R}) \mid \mathcal{X} \subseteq \mathcal{R}\}\\ &\qquad\longleftrightarrow \{(\mathcal{A}, \mathcal{B}) \ \mathbb{E}\text{-torsion pair in } \mathcal{R} \mid \Sigma\mathcal{A} \subseteq \mathcal{A} \subseteq \Sigma\mathcal{R}\}\\ &(\mathcal{X}, \mathcal{Y}) \longmapsto (\Sigma\mathcal{X}, \mathcal{R} \cap \Sigma\mathcal{Y})\\ &(\Sigma^{-1}\mathcal{A}, \Sigma^{-1}(\mathcal{V} * \mathcal{B})) \longleftarrow\!\shortmid (\mathcal{A}, \mathcal{B})\end{aligned}$$

which preserves the inclusion of aisles and restricts to a bijection

$$\begin{aligned}&\{(\mathcal{X}, \mathcal{Y}) \text{ t-structure in } \mathcal{S}(\mathcal{R}) \mid \Sigma\mathcal{R} \subseteq \mathcal{X} \subseteq \mathcal{R}\}\\ &\qquad\longleftrightarrow \{(\mathcal{A}, \mathcal{B}) \ \mathbb{E}\text{-torsion pair in } \mathcal{R} \mid \Sigma^2\mathcal{R} \subseteq \mathcal{A} \subseteq \Sigma\mathcal{R}\}.\end{aligned}$$

Proof. We do not write the proof of (i) since it is a straightforward generalisation of the proof of Theorem III.3.2. Part (ii) can be proved in a similar way. Alternatively, since all $\mathbb{E}$-torsion pairs in (ii) are $\mathfrak{s}$-torsion pairs, the bijections of (ii) are, after shifting, special cases of [1, Theorem 3.9]. □

The (co-)t-structures in the second correspondences in parts (i) and (ii) are called *intermediate* with respect to the (co-)t-structure of $\mathcal{R}$. Let us mention related work on this property. Intermediate t-structures also correspond to torsion pairs in the heart [57, 23] and have applications in, for example, stability conditions [128] and algebraic geometry [103]. In the case of co-t-structures, it has been shown that intermediate co-t-structures correspond to certain two term silting subcategories [67] and to cotorsion pairs in the 'extended coheart' [102].

V.4 Aisles through quotients

We fix some conventions. Let $\mathcal{A} = (\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be a Frobenius extriangulated category, with projectives-injectives $\mathcal{I}$ and $\underline{\mathcal{A}} = \underline{\mathcal{A}}_{\mathcal{I}}$ be the stable category, which has the natural structure of triangulated category [95, Corollary 7.4] with shift functor Σ. By $\mathcal{I}(X)$ we denote the object such that $X \to \mathcal{I}(X)$ is a minimal left $\mathcal{I}$-approximation.

In this section, we show the following.

Theorem V.4.1. *Let $(\mathcal{U}, \mathcal{V})$ be a t-structure in $\underline{\mathcal{A}}$ and set $\mathcal{D} = \Sigma^{-1}\mathcal{V}$. Then there is an equivalence of right triangulated categories $\Sigma_{\mathcal{D}}\underline{\mathcal{A}}_{\mathcal{D}} \cong \mathcal{U}$.*

Remark V.4.2. In [115, Proposition 3.9], where it was shown that t-structures in an algebraic triangulated category correspond bijectively to certain complete cotorsion pairs in the associated Frobenius exact category. Along that bijection a t-structure $(\mathcal{U}, \mathcal{V})$ in $\underline{\mathcal{A}}$ corresponds to the cotorsion pair $(\mathcal{U}, \Sigma^{-1}\mathcal{V})$ in $\mathcal{A}$. This bijection generalises immediately to the extriangulated setting. Further, when then applied to the case of a triangulated category $\mathcal{T}$ (which is Frobenius extriangulated category [95, Proposition 3.22]), this specialises to the observation of [94, Proposition 2.6] that t-structures in $\mathcal{T}$ are in bijection with cotorsion pairs $(\mathcal{U}, \mathcal{V})$ in $\mathcal{T}$ such that $\Sigma\mathcal{U} \subseteq \mathcal{U}$. Thus, these results and Theorem V.4.1 complement each other.

We note that a class of objects $\mathcal{V} \subset \mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(\underline{\mathcal{A}})$ defines subcategories of both $\mathcal{A}$ and $\underline{\mathcal{A}}$.

Lemma V.4.3. *Let $\mathcal{D} = \mathrm{Add}(\mathcal{D})$ be a covariantly finite additive subcategory of $\underline{\mathcal{A}}$. Then $\mathcal{D}$ is also a covariantly finite additive subcategory of $\mathcal{A}$. Moreover, left $\mathcal{D}$-approximations are $\mathbb{E}$-inflations in $\mathcal{A}$.*

Proof. For any $X \in \mathcal{A}$, let $\underline{\alpha} : X \to D$ and $\beta : X \to \mathcal{I}(X) = I$ be a left $\mathcal{D}$-approximation of X in $\underline{\mathcal{A}}$ and a left $\mathcal{I}$-approximation of X in $\mathcal{A}$ respectively. We claim that $\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right] : X \to D \oplus I$ is a left $\mathcal{D}$-approximation of X in $\mathcal{A}$.

Indeed, since $\mathcal{D}$ is additive, $0 \cong \mathcal{I} \subset \mathcal{D}$, so $D \oplus I \in \mathcal{D}$. Let $f : X \to W$ be a morphism in $\mathcal{A}$ with $W \in \mathcal{D}$. Then there exists $(\underline{g} : D \to W) \in \mathrm{Hom}_{\underline{\mathcal{A}}}(D, W)$ such that $\underline{f} = \underline{g\alpha}$. Now, $\underline{f - g\alpha} = 0$ so $f - g\alpha$ factors through $\mathcal{I}$ and so must factor through the left $\mathcal{I}$-approximation of X. Thus there exists $g' : I \to W$ such that

$f - g\alpha = g'\beta$. Rearranging we have

$$\begin{array}{ccc} X & \xrightarrow{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]} & D \oplus I \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle [g\ g']} & \\ W. & & \end{array}$$

The remaining claim follows from Lemma II.1.9. □

Let $(\mathcal{U}, \mathcal{V})$ be a t-structure on $\underline{\mathcal{A}}$, so that for all objects X there exists a unique (up to isomorphism) triangle in $\underline{\mathcal{A}}$

$$\mathcal{U}(X) \xrightarrow{u_X} X \xrightarrow{v_X} \mathcal{V}(X) \xrightarrow{w_X} \Sigma\mathcal{U}(X)$$

therefore there is a unique (up to isomorphism) triangle in $\underline{\mathcal{A}}$

$$\Sigma^{-1}\mathcal{U}(\Sigma X) \xrightarrow{u'_X} X \xrightarrow{v'_X} \Sigma^{-1}\mathcal{V}(\Sigma X) \xrightarrow{w'_X} \mathcal{U}(\Sigma X). \tag{V.6}$$

Note that this triangle is also the torsion extriangle of X with respect to the t-structure $(\Sigma^{-1}U, \Sigma^{-1}\mathcal{V})$. Set $\mathcal{D} := \Sigma^{-1}\mathcal{V} \subset \mathcal{A}$, which by Lemma V.4.3 satisfies property $(*)$ and thus the category $\underline{\mathcal{A}}_{\mathcal{D}}$ is a right triangulated category by Proposition V.2.8. We also set $\mathcal{V}'(X) := \Sigma^{-1}\mathcal{V}(\Sigma X)$ and $J(X) := \mathcal{V}'(X) \oplus \mathcal{I}(X)$.

Lemma V.4.4. *For all $X \in \mathcal{A}$, $\Sigma_{\mathcal{D}} X \cong \mathcal{U}(\Sigma X)$ in $\underline{\mathcal{A}}$.*

Proof. By the definition of the functor $\Sigma_{\mathcal{D}}$ and Lemmas II.1.9 and V.4.3, $\Sigma_{\mathcal{D}} X$ fits into a commutative diagram of extriangles in $\mathcal{A}$

$$\begin{array}{ccccccc} X & \longrightarrow & \mathcal{I}(X) & \longrightarrow & \Sigma X & \dashrightarrow & \\ {\scriptstyle v'_X}\downarrow & & \downarrow & & \| & & \\ \mathcal{V}'(X) & \longrightarrow & \Sigma_{\mathcal{D}} X & \longrightarrow & \Sigma X & \dashrightarrow & . \end{array}$$

where $\underline{v}'_X : X \to \mathcal{V}'(X)$ fits into the triangle (V.6). Thus $\Sigma_{\mathcal{D}} X = \mathrm{cone}_{\underline{\mathcal{A}}}(\underline{v}'_X) \cong \mathcal{U}(\Sigma X)$ in $\underline{\mathcal{A}}$. □

Consider the diagram

$$\begin{array}{ccccc} & & \mathcal{A} & \xrightarrow{\text{can}} & \underline{\mathcal{A}}_{\mathcal{D}} \\ & & {\scriptstyle \text{can}}\downarrow & \nearrow {\scriptstyle \exists! G} & \\ \mathcal{U} & \underset{\text{can}}{\overset{i}{\hookrightarrow}} & \underline{\mathcal{A}} & & \end{array}$$

where G is the unique additive functor rendering the diagram commutative which exists since $\mathcal{I} \subseteq \mathcal{D}$. The next result finishes the proof of Theorem V.4.1.

Proposition V.4.5. *The composition $\psi := Gi : \mathcal{U} \to \underline{\mathcal{A}}_{\mathcal{D}}$ is a full and faithful right triangle functor and induces an equivalence of right triangulated catgeories $U \cong \Sigma_{\mathcal{D}} \underline{\mathcal{A}}_{\mathcal{D}}$.*

Proof. ψ **is a right triangulated functor:** Note that for all $U \in \mathcal{U}$, $\mathcal{U}(U) \cong U$ and $\mathcal{V}'(U) \cong 0$ in $\underline{\mathcal{A}}$. Thus, by Lemma V.4.4, there is a natural isomorphism of functors $\Sigma|_{\mathcal{U}} \cong \Sigma_{\mathcal{D}}$ and $I(U) \cong J(U)$ so that (right) triangles in $\mathcal{U}$ are also right triangles in $\underline{C}_{\mathcal{D}}$.

ψ **is full:** Follows from the fact that i and G are both full.

ψ **is faithful:** Let $\underline{f} : X \to Y$ be a morphism in $\mathcal{U}$. Suppose that $\psi(\underline{f}) = 0$. Then $\underline{f}$ must factor through $\mathcal{D} = \Sigma^{-1}\mathcal{V}$ in $\underline{C}$. But since $\mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{U}, \mathcal{V}) = 0$ and $\Sigma^{-1}\mathcal{V} \subset \mathcal{V}$ this is only possible if $\underline{f} = 0$.

Im$\psi \cong \Sigma_{\mathcal{D}}\underline{\mathcal{A}}_{\mathcal{D}}$**:** Let $\Sigma_{\mathcal{D}}X \in \Sigma_{\mathcal{D}}\underline{C}_{\mathcal{D}}$, by Lemma V.4.4, $\Sigma_{\mathcal{D}}X \cong \mathcal{U}(\Sigma X)$ in $\mathcal{A}$ and hence also in $\underline{\mathcal{A}}_{\mathcal{D}}$. Thus $\psi(\mathcal{U}(\Sigma X)) \cong \Sigma_{\mathcal{D}}X$. Conversely, let $Y \in \mathcal{U}$. Then $Y \cong \Sigma_{\mathcal{D}}(\Sigma^{-1}Y) \in \underline{\mathcal{A}}_{\mathcal{D}}$ by Lemma V.4.4. □

Corollary V.4.6. $\Sigma_{\mathcal{D}}G\Sigma^{-1} : \underline{\mathcal{A}} \to \Sigma_{\mathcal{D}}\underline{\mathcal{A}}_{\mathcal{D}} \cong \mathcal{U}$ *is right adjoint to the inclusion* $\mathcal{U} \hookrightarrow \underline{\mathcal{A}}$. *In particular,* $\Sigma_{\mathcal{D}}G\Sigma^{-1} \cong \mathcal{U}(-)$.

Remark V.4.7. Theorem V.4.1 cannot be directly dualised to give co-aisles of co-t-structures since the aisle of a co-t-structure is contravariantly finite.

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