Asymptotic Analysis of Mixed Mock Modular Forms and Related q-products

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Abstract

This thesis contains results of three research projects which study asymptotics for the Fourier coefficients of mixed mock modular forms and twisted q-products arising in combinatorics. To begin, we compute an asymptotic distribution for generalizations of unimodal sequences called odd-balanced unimodal sequences which were defined by Kim, Lim, and Lovejoy in 2016. We find the interesting result that the odd-balanced unimodal sequences with certain restrictions on their rank, are asymptotically related to the overpartition function. This is in contrast to strongly unimodal sequences which are asymptotically related to the partition function. In the second part of this thesis, we compute asymptotic estimates for the Fourier coefficients of two mock theta functions originating from Bailey pairs derived by Lovejoy and Osburn in 2012. We encounter cancellation in our estimates for one of the functions, which requires a careful study of secondary asymptotic terms. We deal with this by using higher order asymptotic expansions for the Jacobi theta functions. In our final result, we find asymptotic estimates for the complex Fourier coefficients of the product $(\zeta q; q)_\infty^{-1}$, with ζ a root of unity. This result has interesting applications in analysis and combinatorics. For large n, we are able to predict sign changes of arbitrary linear combinations of the function $p(a, b; n)$ for fixed b, where $p(a, b; n)$ counts the number of partitions of n where the number of parts is congruent to a modulo b. We see that simple differences of the type $p(a_1, b; n) - p(a_2, b; n)$ have sign change patterns that oscillate like a cosine.

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Contents

Chapter 1

Statement of main results and motivation

1.1 Statement of results

The work in this thesis is built around stating and proving the following results. Our first result deals with the growth of certain types of unimodal sequences called odd-balanced unimodal sequences, which first appeared in [50]. A *unimodal sequence of size n* is a sequence of positive integers $\{a_j\}_{j=1}^{m}$, such that $\sum_{j=1}^{m} a_j = n$ and

$$a_1 \leq a_2 \leq ... \leq a_k \leq a_c \geq a_{k+1} \geq ... \geq a_{m-1} \geq a_m. \qquad (1.1.1)$$

The number a_c (which is allowed to repeat) is referred to as a *peak*. In particular, unimodal sequences are partitions of n, and if we replace $\leq$ and $\geq$ with strict inequalities in Eq. (1.1.1), we arrive at the *strongly* unimodal sequences, which then have a unique peak a_c. In both the general and strict case, we can define a rank statistic, usually denoted by m, which is defined as

$$m{:=}\text{number of parts after the peak(s)}-\text{number of parts before the peak(s)}.$$

Example 1.1. We write down the unimodal sequences of size 4 and their corresponding ranks m:

$$\begin{aligned} (4)\ \text{ with }\ m=0, &\quad (3,1)\ \text{ with }\ m=1, \\ (1,3)\ \text{ with }\ m=-1, &\quad (2,2)\ \text{ with }\ m=0, \end{aligned}$$

$$(1,2,1) \text{ with } m=0, \quad (2,1,1) \text{ with } m=2,$$
$$(1,1,1,1) \text{ with } m=0, \quad (1,1,2) \text{ with } m=-2.$$

An *odd-balanced unimodal sequence of size* n is a unimodal sequence where the peak is even and unique, odd parts can repeat but the odd parts must be the same on each side of the peak, and the even parts satisfy strict inequalities. In Ch. 3, we will prove the following distribution result for odd-balanced unimodal sequences of size $2n+2$ with rank congruent to a modulo c denoted by $v(a,c;n)$.

Result A. Let $c>1$ be odd. Then as $n \to \infty$,

$$v(a,c;n) \sim \frac{1}{16cn^{\frac{3}{4}}} e^{\pi\sqrt{n}} \sim \frac{n^{\frac{1}{4}}}{2c}\overline{p}(n).$$

The function $\overline{p}(n)$ is the overpartition function, which we will define at the beginning of Ch. 3.

In Ch. 4, we will prove the following for the coefficients of two Bailey-type mock theta functions defined in Def. 4.2. The Fourier coefficients for these functions are denoted by $a(n)$ and $b(n)$.

Result B. The following estimates hold as $n \to \infty$:

$$a(n) \sim (-1)^n \frac{\sqrt{6}}{12\sqrt{n}} e^{\pi\sqrt{\frac{n}{12}}}, \qquad b(n) \sim \left(\frac{1}{2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{12}\right)}+1\right)\frac{e^{\pi\sqrt{\frac{n}{6}}}}{\sqrt{24n}}.$$

The final result of this thesis is discussed in Ch. 5 and was proven with Walter Bridges and Johann Franke, both members of the University of Cologne at the time of writing. We find and prove asymptotic formulas for the complex coefficients of the twisted q-product $(\zeta q;q)_\infty^{-1}$, where ζ is a root of unity. With this result, we are able to accurately model the sign changes in generic linear combinations of $p(a,b;n)$ when n is large. The function $p(a,b;n)$ denotes the number of partitions of n with number of parts congruent to a modulo b. Let $Q_n(\zeta_b)$ denote the n^{th} Fourier coefficient of the product above with ζ_b a b^{th} root of unity, let Li_2 denote the dilogarithm function, and let $\theta_{13} \approx 2.06672$ and $\theta_{23} \approx 2.36170$ be numbers implicitly defined later in Ch. 5.

Result C. In the limit $n \to \infty$, we have the following results:
(1) If $2\pi\frac{a}{b} \in (0, \theta_{13})$, then

$$Q_n(\zeta_b^a) \sim \frac{\sqrt{1-\zeta_b^a}\,\mathrm{Li}_2(\zeta_b^a)^{\frac{1}{4}}}{2\sqrt{\pi}n^{\frac{3}{4}}} \exp\left(2\sqrt{\mathrm{Li}_2\left(\zeta_b^a\right)}\sqrt{n}\right).$$

(2) If $2\pi\frac{a}{b} \in (\theta_{23}, \pi)$, then

$$Q_n(\zeta_b^a) \sim \frac{(-1)^n\sqrt{1-\zeta_b^a}\,\mathrm{Li}_2\left(\zeta_b^{2a}\right)^{\frac{1}{4}}}{2\sqrt{2\pi}n^{\frac{3}{4}}} \exp\left(\sqrt{\mathrm{Li}_2\left(\zeta_b^{2a}\right)}\sqrt{n}\right).$$

(3) If $2\pi\frac{a}{b} \in (\theta_{13}, \theta_{23}) \setminus \left\{\frac{2\pi}{3}\right\}$, then

$$Q_n(\zeta_b^a) \sim (\zeta_3^{-n}\omega_{1,3}(\zeta_b^a) + \zeta_3^{-2n}\omega_{2,3}(\zeta_b^a)) \frac{\mathrm{Li}_2\left(\zeta_b^{3a}\right)^{\frac{1}{4}}}{2\sqrt{3\pi}n^{\frac{3}{4}}} \exp\left(\frac{2}{3}\sqrt{\mathrm{Li}_2\left(\zeta_b^{3a}\right)}\sqrt{n}\right).$$

(4) We have

$$Q_n(\zeta_3) \sim \frac{\zeta_3^{-2n}\left(1-\zeta_3^2\right)^{\frac{1}{6}}(1-\zeta_3)^{\frac{1}{2}}\Gamma\left(\frac{1}{3}\right)}{2(6\pi n)^{\frac{2}{3}}} \exp\left(\frac{2\pi}{3}\sqrt{\frac{n}{6}}\right).$$

The $\omega_{j,m}(z)$ are products which are defined later in Eq. (5.2.1). For now it is sufficient to note that they are non-zero and independent of n.

1.2 Motivation

This thesis aims to understand the Fourier coefficients of two classes of functions defined on the upper half-plane: the mixed mock modular forms (MMMFs) and the twisted q-products. Generically, modular forms are holomorphic functions on the upper-half-plane which transform under a specific group action with respect to elements of a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. MMMFs are certain generalizations of modular forms, and are not required to be holomorphic. Modular forms and MMMFs are of particular interest in number theory due to many of them being closely related to fundamental arithmetic functions. The most ubiquitous example is that of the partition function, $p(n)$, whose generating function has been studied since the time of Euler and was most famously studied by Ramanujan and Hardy [45] and Rademacher [63] in the early 20th century. The function $p(n)$ counts the number of partitions of n, i.e, the number of ways to write n as a sum of positive integers, ordered in increasing order. For example, 3 can be partitioned as

$$3,\ 1+2,\ 1+1+1.$$

That is, $p(3) = 3$. Each one of the sums is called a *partition* of n, and each summand within a partition is called a *part*.

Notation 1.2. Sometimes we denote a partition of n with the notation $\lambda \vdash n$.

Let $(a;q)_\infty := \prod_{j\geq 0}(1-aq^j)$ with $q := q(\tau) := e^{2\pi i\tau}$ and $\tau \in \mathbb{H}$. The $p(n)$ are generated by

$$P(q) := \sum_{n\geq 0} p(n)q^n = \prod_{j\geq 1}\left(1-q^j\right)^{-1} = (q;q)_\infty^{-1} =: (q)_\infty^{-1}. \quad (1.2.1)$$

Note with this definition of the generating function, $p(0) := 1$. The function $P(q)$ is equal to $\frac{1}{\eta(\tau)}$ up to a factor of q, where

$$\eta(\tau) := q^{\frac{1}{24}}(q)_\infty$$

is *Dedekind's η-function*: a premier example of a modular form. The generating function in Eq. (1.2.1) is one of the nice examples that is both a modular form and a q-product. In contrast, some slight changes to the combinatorics make a big difference in the analytic properties of the generating function. If we now consider $p(a,b;n)$, the number of partitions of n with number of parts congruent to a modulo b, the generating function takes the shape

$$P_{a,b}(q) := \sum_{j\geq 0} \frac{q^{bj+a}}{(q)_{bj+a}}, \quad (1.2.2)$$

where $(w;q)_k := \prod_{m=0}^{k-1}(1-wq^m)$, $(w)_k := (w;q)_k$, and $(q)_0 := 1$. The generating function in Eq. (1.2.2) is neither a modular form nor a product, except for the case $b=2$ where one has (see Ch. 5 for a proof)

$$P_{a,2}(q) = \frac{1}{2}\left(\frac{1}{(q)_\infty} + (-1)^a \frac{(q)_\infty}{(q^2;q^2)_\infty}\right).$$

However, the $p(a,b;n)$ are closely related to the so-called twisted q-products, which we will study in detail in Ch. 5.

Many of the generating functions encountered in this thesis do not fit in the category of modular forms or twisted q-products. Much of the time, these functions can be related to MMMFs. The close ties that MMMFs have to combinatorics will become more apparent in Ch. 3 and Ch. 4. To illustrate this point for now, we provide an example of a generating function that is a MMMF. Recall the unimodal sequences described in the previous section. The generating function for the unimodal sequences is given by [20, 28]

$$U(q) := \sum_{n\geq 1} u(n)q^n := \frac{1}{(q;q)_\infty^2}\sum_{n\geq 1}(-1)^{n+1}q^{\frac{n(n+1)}{2}}. \tag{1.2.3}$$

Note with this definition for the generating function, $u(0) := 0$. It is important to note that there are asymptotically more unimodal sequences than partitions, which was proven by Auluck in 1951 [10]. Similarly, the strongly unimodal sequencs are generated by [30]

$$U^*(q) := \sum_{n\geq 1} u^*(n)q^n := \sum_{n\geq 0}(-q)_n^2 q^{n+1}. \tag{1.2.4}$$

Note once more that $u^*(0) := 0$. There are in general roughly one fourth the same amount of partitions as strongly unimodal sequences when n is large, which was proven by Rhoades in 2014 [64]. Much more related to this thesis is the fact that Rhoades also showed (see Thm. 1.3 therein) that $U^*(q)$ is of the form

$$U^*(q) = \text{mock theta function} + \text{modular form} \times \text{mock theta function},$$

which we will shortly see is a typical case of a MMMF. In light of the unimodal sequences, we study a related counting function coming from [50] that is also a MMMF. Let

$$V(q) := \sum_{n\geq 0} v(n)q^n := \sum_{n\geq 0} \frac{(-q)_n^2 q^n}{(q;q^2)_{n+1}}. \tag{1.2.5}$$

Notice that $v(0) := 1$. The function V is the generating function for the number of aptly named odd-balanced unimodal sequences, which are as previously discussed, types of unimodal sequences. More specifically, the $v(n)$ count the number of unimodal sequences of size $2n+2$ such that the peak is unique and even, the odd parts can repeat, but must be identical on each side of the peak, and the rest of the parts satisfy strict inequalites. A few examples of odd-balanced sequences of size 12 are $(1,1,2,4,2,1,1), (1,3,4,3,1), (12)$, and $(1,8,2,1)$. In Ch. 3, we will study the asymptotic distribution of the $v(n)$ by exploiting the fact that V is a MMMF.

Notation 1.3. It is customary to write generating functions like that for $p(n)$ as depending on the variable q, e.g., $P(q)$, whereas functions whose transformation properties are being studied are usually written as depending directly on τ, for example, $\eta(\tau)$. We will use the τ dependence when doing asymptotic analysis or studying transformation formulas.

Chapter 2

Basic definitions

We introduce some of the basic notions associated with the theory of modular forms and combinatorics needed to understand this thesis. We assume throughout that $\tau =: u + iv$, unless stated otherwise.

2.1 Congruence subgroups and cusps

Throughout this work, we will deal with the notion of a cusp for congruence subgroups. We let $\mathrm{SL}_2(\mathbb{Z})$ be the special linear group with integer entries. We then define a congruence subgroup of level N to be the following (see any book on modular forms like [9, 20, 32, 35, 57]).

Definition 2.1. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a normal subgroup of finite index. We say Γ is a *congruence subgroup of level* N if there exists a positive integer N such that the group

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon a \equiv d \equiv 1 \pmod N \text{ and } c \equiv b \equiv 0 \pmod N \right\}$$

is contained in Γ.

Remark 2.2. We often refer to groups of the type in Def. 2.1 as merely "congruence subgroups" without reference to the level N.

Remark 2.3. Many theorems on modular forms are stated in terms of the group

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon c \equiv 0 \pmod N \right\}.$$

Note that $\Gamma(N) \subset \Gamma_0(N)$, which shows $\Gamma_0(N)$ is a congruence subgroup of level N.

Let $\tau \in \mathbb{H}$, let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and define $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. The *Möbius transformation* is the standard group action on $\mathbb{H}$ which appears in the theory of modular forms and it is defined by

$$A\tau := \frac{a\tau + b}{c\tau + d}.$$

We can associate a connected set in the complex plane to the group Γ modulo the equivalence classes defined by the Möbius transformation. That is, two elements $\tau_1, \tau_2 \in \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ are Γ-*equivalent* if there is an $A \in \Gamma$ such that $A\tau_1 = \tau_2$. A fundamental domain for Γ is defined as follows.

Definition 2.4 (Def. 4.3.1, [32] or pg. 20, [57])**.** Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A *fundamental domain for* Γ, denoted by $\mathscr{F}(\Gamma) \subset \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ is defined by the following properties:
(1) For every $\tau \in \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$, there is an $A \in \Gamma$ such that $A\tau \in \mathscr{F}(\Gamma)$. Furthermore, two distinct points that are Γ-equivalent must be in the boundary of $\mathscr{F}(\Gamma)$.
(2) The set $\mathscr{F}(\Gamma)$ is topologically connected with respect to the Euclidean metric.
(3) The set of all Γ-orbits of $\mathscr{F}(\Gamma)$ span $\mathbb{H}$. That is

$$\overline{\mathbb{H}} = \bigcup_{A \in \Gamma} A\mathscr{F}(\Gamma),$$

where $\overline{\mathbb{H}}$ is the closure of $\mathbb{H}$.

The boundary of $\mathscr{F}(\Gamma)$ is essential for the study of modular forms, and leads to the notion of a cusp of a group Γ.

Definition 2.5. A *cusp* for the congruence subgroup Γ is a point, $c \in \mathscr{F}(\Gamma) \cap (\mathbb{Q} \cup \{i\infty\})$. The number of cusps for Γ is the finite number of equivalence classes in $\mathscr{F}(\Gamma)$ whose orbits intersect $\mathbb{Q} \cup \{i\infty\}$.

Remark 2.6. The notion of a cusp is sometimes defined in terms of Riemann surfaces. However, the definition above can be shown to be equivalent to that of the geometric version usually stated. The fact that there are only finitely many inequivalent cusps is also related to this fact (see [32] or Lem. 2.4.1, [35]).

2.1.1 Modular forms, harmonic Maass forms, and mixed mock modular forms

We define the notions of the various modular and mock modular objects that we will encounter in this work. We begin with the classical definitions of a modular form, cusp form, and weakly holomorphic modular form.

Definition 2.7. A *modular form of weight* $k \in \frac{1}{2}\mathbb{Z}$ for a congruence subgroup Γ with multiplier system or nebentypus $\chi : \Gamma \to \mathbb{C}$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which is also holomorphic at each cusp κ of Γ, and which satisfies the transformation law for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$:

$$f(A\tau) = \chi(A)(c\tau + d)^k f(\tau). \tag{2.1.1}$$

If the function f decays to 0 at each cusp, it is called a *cusp form* and if it is allowed to grow at any cusp, it is called a *weakly holomorphic modular form.*

Remark 2.8. The multiplier system χ is not explicitly defined and is really only relevant for us in the case of the η-function, which we list as an example below. Multiplier systems allow for one to incorporate more functions into the definition of a modular form by allowing multiplicative factors in the transformation formula Eq. (2.1.1). When k is a half integer, it is necessary to include more elaborate factors which we will see in the definition for harmonic Maass forms below. Why this is so is due to Shimura's work in the 1970's [65].

We now state the definition of a harmonic Maass form, which is a generalization of a modular form.[1]

Definition 2.9 (see pg. 54 of [29], Def. 15.3.2 of [32], or Def. 4.2 of [20])**.** Let $f : \mathbb{H} \to \mathbb{C}$ be a smooth function and[2] let $k \in \frac{1}{2}\mathbb{Z} \setminus \{1\}$. We say that f is a *harmonic Maass form of weight* k for $\Gamma_0(N)$ if the following properties hold:
(1) If $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ then $4|N$. Notice that d is odd in this case.
(2) Let $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. The function f satisfies the transformation law

$$f(A\tau) = \begin{cases} (c\tau + d)^k f(\tau) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{-2k} (c\tau + d)^k f(\tau) & \text{otherwise,} \end{cases}$$

where $\left(\frac{\bullet}{\bullet}\right)$ is the Kronecker symbol and

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

(3) The function f is annihilated by the *weight* k *hyperbolic Laplacian*, defined by

$$\Delta_k := -4v^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \overline{\tau}} + 2ikv \frac{\partial}{\partial \overline{\tau}}.$$

(4) For each cusp κ of $\Gamma_0(N)$, the function is bounded by a linear exponential involving v. That is, if $M > 0$ is fixed and $|u - \kappa| < Mv$, then as $v \to 0$

$$f(\tau) \ll e^{\alpha v}, \qquad \alpha \in \mathbb{R}.$$

Remark 2.10. It is worth noting that any holomorphic function on $\mathbb{H}$ that satisfies the transformation law

$$f(A\tau) = (c\tau + d)^k f(\tau)$$

[1] The following functions are also sometimes referred to as "weak Maass forms".
[2] We will not encounter any weight 1 harmonic Maass forms in this thesis.

with k an integer and $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ for some congruence subgroup Γ automatically satisfies the differential equation in Def. 2.9 by noting that the real and imaginary parts of holomorphic functions are harmonic, and then applying the Cauchy–Riemann equations.

Our first example of a modular form is the previously defined Dedekind η-function. Let $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and define (see Thm. 5.8.1, [32])

$$\nu(A) := \begin{cases} \left(\frac{d}{|c|}\right) e^{\frac{\pi i}{12}\left((a+d-3)c-bd(c^2-1)\right)} & \text{if } c \text{ is odd,} \\ \rho(c,d)\left(\frac{c}{|d|}\right) e^{\frac{\pi i}{12}\left((a-2d)c-bd(c^2-1)+3d-3\right)} & \text{if } c \text{ is even,} \end{cases} \tag{2.1.2}$$

and

$$\rho(c,d) := \begin{cases} -1 & \text{if } c \leq 0,\ d < 0, \\ 1 & \text{else.} \end{cases}$$

We then have the following.

Theorem 2.11 (see [9, 32, 35]). *The Dedekind η-function satisfies*

$$\eta(A\tau) = \nu(A)(c\tau+d)^{\frac{1}{2}}\eta(\tau).$$

Modular forms of the types in Def. 2.7 for Γ have Fourier expansions at a cusp κ of the form [32, 35, 57]

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^{\frac{n}{d(\kappa)}},$$

where $a(n)$ are complex numbers, $d(\kappa) \in \mathbb{N}$ is the *width*[3] of κ and m is a fixed positive integer. One can identify cusp forms easily by

[3]The width of a cusp is related to the fact that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ may not always be in Γ. There always is an N such that $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$ is in Γ since it is a congruence subgroup. Therefore, for each κ, $d(\kappa)$ is essentially the number of left cosets of certain conjugacy classes related to κ, which are in turn related to the level N. When $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, $d(\kappa) = 1$ for all κ. See pgs. 55 and 56 in [35] for an exact definition.

their Fourier expansions in that $a(0) = 0$. How the coefficients $a(n)$ behave in the limit $n \to \infty$ differs greatly depending on whether f is holomorphic or weakly holomorphic. If f is holomorphic, a simple bound is that the $a(n)$ must be bounded by a polynomial in n, otherwise the series would not converge absolutely as one approaches the cusp $i\infty$. The story changes drastically for f which are weakly holomorphic, in that the $a(n)$ can grow exponentially, as in the case of the partition function generating function $P(q)$, which is a weakly holomorphic modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $-\frac{1}{2}$ up to a power of q.

From the modular objects, we can now build towards defining the MMMFs and related functions. It is important to note that harmonic Maass forms are a rather modern discovery in mathematics, however, their power in describing the century-old-problem of the mock theta functions indicates their important place in analytic number theory. Mock theta functions and more generally, mock modular forms, go back to the work of Ramanujan, and more specifically to his last letter to Hardy in 1920 where he gave examples of mock theta functions which we now know to be associated with harmonic Maass forms (see [6] and Ch. 9, [20] for detailed accounts of this letter). The classical definition of a mock theta function that Ramanujan was working with is the following. Let $M(q)$ be a complex-valued function of q which has essential singularities at infinitely many roots of unity. We say $M(q)$ is a *mock theta function* (in the classical sense) if for every root of unity ξ, there is a weakly holomorphic modular form $f_\xi(q)$ and a rational number a_ξ such that as $q \to \xi$ inside the unit disk,

$$M(q) - q^{a_\xi} f_\xi(q) = O(1).$$

It is also required that no single f exists that satisfies the above equation for every root of unity. Most of the mock theta functions we will encounter in this thesis are of the classical type. An example of a such a classical mock theta function comes from Ramanujan's lost notebook (see 8.1.1, [8]),

$$\phi(q) := \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(q;q^2)_{n+1}}.$$

The function ϕ is normally referred to as a *tenth order* mock theta function, which will make an appearance again in Ch. 4.

A more modern interpretation of mock theta functions stems from the decomposition of harmonic Maass forms into their holomorphic and principle parts. That is, if H is a harmonic Maass form, then near each cusp we can write

$$H(\tau) = H^{+}(\tau) + H^{-}(\tau). \tag{2.1.3}$$

To simplify the discussion, assume that $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \in \Gamma$. Otherwise the forthcoming equations have to incorporate the width $d(\kappa)$ as previously discussed. The portion $H^{+}(\tau)$ (resp. $H^{-}(\tau)$) is referred to as the *holomorphic part of* H (resp. *principle part*) and takes the shape near κ (see pg. 64, [20] or [29])[4]

$$\begin{aligned} H^{+}(\tau) &= \sum_{n\geq m_{+}} a^{+}(n)q^{n}, \\ H^{-}(\tau) &= a^{-}(0)v^{1-k} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{m_{-}} a^{-}(n)\Gamma\left(1-k, -4\pi n v\right) q^{n}, \end{aligned}$$

where m_{+} and m_{-} are integers and

$$\Gamma(x,z) := \int_{z}^{\infty} t^{x-1}e^{-t}dt, \qquad \mathrm{Re}(x) > 0,\ z \in \mathbb{C}\setminus\mathbb{R}_{<0}.$$

is the *incomplete gamma function.* If $z = 0$, we have the classical Γ-function which has poles at $x \in \mathbb{Z}_{\leq 0}$ which arise after extending values

[4] The reader may be worried about negative bounds in the incomplete Γ-function. For our work, $m_{-} < 0$ and this will not be a problem. If it were the case that $m_{-} \geq 0$, one would have to define values of the incomplete Γ-function by analytic continuation (see Sect. 4.2 of [20]).

of the integral via analytic continuation. We define the standard branch cut for $\Gamma(x, z)$ to lie on the negative real axis. Thus, the path of integration is dependent on the lower integration bound z. If z is in the lower left quadrant, the path of integration must remain in the lower-half-plane until reaching the imaginary axis. The analogous holds for z in the upper left quadrant. Following Sect. 5 of [20], we define a *mock modular form* of weight k to be the holomorphic part of a harmonic Maass form, where $H^-(\tau)$ is not always 0. The modern version of mock theta functions are essentially mock modular forms that get mapped under a special operator called the shadow operator to single variable theta functions of weight $k = \frac{1}{2}$ or $\frac{3}{2}$. This interpretation is due to Zwegers [73]. Therein, he discovered that the classical mock theta functions can be embedded in the larger theory of harmonic Maass forms as described above. As a result, Zwegers was able to write down transformation laws for mock theta functions, which showed that they transform like modular forms up to a non-holomorphic piece. A consequence of these transformation laws is that one can accurately study the asymptotic growth of the Fourier coefficients of mock theta functions: a fact that we will make use of often in this thesis.

If we denote the space of harmonic Maass forms of weight k for the congruence subgroup $\Gamma = \Gamma_0(N)$ by $\mathscr{H}_k(\Gamma)$, there is a canonical map $\xi_k : \mathscr{H}_{2-k}(\Gamma) \to S_k(\Gamma)$ called the *shadow operator* [20, 59]. If $f \in \mathscr{H}_{2-k}(\Gamma)$ and the image $\xi_k(f)$ is a single variable theta function, then f^+ is said to be a *mock theta function* of weight k. One can then generalize the notion of a holomorphic part for functions that are not quite harmonic Maass forms, but something close. This leads to the idea of a mixed harmonic Maass form.

Definition 2.12 (see Ch. 13 of [20], or [56])**.** A *mixed harmonic Maass form of total weight* k for a congruence subgroup $\Gamma = \Gamma_0(N)$ is a sum of the form

$$T(\tau) := \sum_{j=1}^{N} W_j(\tau) H_j(\tau),$$

where the $W_j \in M^!_{w_j}(\Gamma)$ and the $H_j \in \mathscr{H}_{h_j}(\Gamma)$ for weights w_j and h_j, such that for each j, we have $w_j + h_j = k$.

In a similar matter to harmonic Maass forms, we define the holomorphic part of T by

$$T^+(\tau) := \sum_{j=1}^{N} W_j(\tau) H_j^+(\tau).$$

A *mixed mock modular form* (MMMF) is a function that is equal to $T^+(\tau)$ for some mixed harmonic Maass form T. Throughout, we also refer to objects that are powers of q times a MMMF as MMMFs. The MMMFs we will encounter in this thesis are generally defined in terms of q-series and thus will have an absolutely convergent series of the form[5]

$$T^+(\tau) = \sum_{n \geq 0} a(n) q^n,$$

when q is inside the punctured unit disk. In the same spirit of weakly holomorphic modular forms, the coefficients of this series are allowed to grow exponentially as τ approaches points in the set $\mathbb{Q} \cup \{i\infty\}$.

2.1.2 Mellin transform and gamma functions

A useful idea in number theory involves relating power series to integrals. One of the most prominent tools which does this involves the Mellin transform. Let g be a complex valued function defined on the interval $(0, \infty)$. Assuming the integral below exists, we define the *Mellin transform* of g by (see Def. 16.1, [66])

$$M(g)(s) := \int_0^\infty t^{s-1} g(t) dt.$$

[5] We still refer to this series as a Fourier series, even though $T^+(\tau)$ may not be periodic under the shift $\tau \mapsto \tau + 1$.

The most famous Mellin transform is undoubtedly the Γ-function, which is the Mellin transform of the exponential function:

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt, \qquad \mathrm{Re}(s) > 0,$$

where additional values are defined by analytic continuation when $s \notin \mathbb{Z}_{\leq 0}$. Given a Mellin transform, the question arises if one can recover the original function. In some cases, the answer is yes, and this is the subject of the famous Mellin inversion theorem (see Prop. 3.1.22, [32] or Def. 16.1, [66]).

Theorem 2.13 (Mellin Inversion Theorem)**.** *If g is continuous on $(0,\infty)$, $g(t) \ll t^b$ for all $b > 0$ as $t \to \infty$, and $g(t) \ll t^a$ for some $a \in \mathbb{R}$ as $t \to 0$, then*

$$g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s}M(g)(s)ds,$$

where $c > a$.

Remark 2.14. The notation

$$\int_{(c)} := \int_{c-i\infty}^{c+i\infty}$$

is used extensively in the literature and we will also adopt it here. We take the usual definition of the principle argument of a complex number $-\pi \leq \mathrm{Arg}(z) < \pi$.

The applicability of Thm. 2.13 to q-series is related to the rapid decay of the Γ-function for arguments of large modulus. This behavior is sometimes referred to as *Stirling's approximation.* We recommend Ch. 6 of [66] or the work of Paris [60] for discussions involving asymptotic expansions of Γ-functions.

Lemma 2.15. *For a fixed distance away from the branch cut at $\mathrm{Arg}(s) = \pi$, we have as $|s| \to \infty$*

$$\Gamma(s) = \sqrt{\frac{2\pi}{s}}\left(\frac{s}{e}\right)^s\left(1 + O\left(\frac{1}{s}\right)\right). \tag{2.1.4}$$

Remark 2.16. Note that when s is away from the branch cut, the main exponential term in Eq. (2.1.4) is

$$e^{\mathrm{Re}(s)(\log|s|-1)}e^{-\mathrm{Im}(s)\operatorname{Arg}(s)},$$

which implies exponential decay as $|s|\to\infty$ in the left half-plane, or if $|s|\to\infty$ on vertical strips.

We now show an example of how Thm. 2.13 and Lem. 2.1.4 can be used to write a q-series in terms of an inverse Mellin transfom, which we then use to find an asymptotic expansion for the series. This is also now a good time to introduce some notation which we will use throughout this thesis.

Notation 2.17. As usual, let $\tau = u + iv$ for $u \in \mathbb{R}$ and $v > 0$. By "$\tau \to c$ within a fixed angular region" or "$\tau \to c$ within a cone" we mean that there is an $M > 0$ such that $|u - c| < Mv$ as $v \to 0^+$.

Example 2.18. We define the Lambert series[6]

$$\mathscr{L}(q) := \sum_{n\geq 1} \frac{q^n}{1+q^n}.$$

Our goal is to find an alternative formula for $\mathscr{L}(q)$ as well as the asymptotic behavior as $q \to 1$ from inside the unit circle. We first relate $\mathscr{L}(q)$ to an inverse Mellin transform of the Γ-function. Let $\tau =: \frac{iz}{2\pi}$, which implies that $\mathrm{Re}(z) > 0$. Expanding the denominator in a geometric series, we have

$$\begin{aligned}\mathscr{L}(q) &= \sum_{n\geq 1} \frac{q^n}{1+q^n} = -\sum_{n,r\geq 1}(-1)^r q^{nr} = -\sum_{n,r\geq 1}(-1)^r e^{-znr} \\ &= -\frac{1}{2\pi i}\int_{(c)} \sum_{r,n\geq 1}(-1)^r (nrz)^{-s}\,\Gamma(s)ds\end{aligned}$$

[6]This series appears prominently in the study of so-called moments of partitions, for example in [58]. Series of this form were also extensively studied by Ramanujan [7].

$$= -\frac{1}{2\pi i}\int_{(c)} z^{-s}\zeta(s)\mathrm{Li}_s(-1)\Gamma(s)ds,$$

where $c > 1$ is chosen to avoid the poles of the integrand and

$$\mathrm{Li}_s(w) := \sum_{n\geq 1}\frac{w^n}{n^s}, \qquad s, w \in \mathbb{C},\ |w| < 1,$$

is the *polylogarithm*[7]. Let N be a positive integer and $\mathscr{C}$ denote the rectangular path with vertices at $(-N-\frac{1}{2}, \pm iR)$ and $(c, \pm iR)$ where $R \sim \frac{1}{\mathrm{Re}(z)}$. Traversing the arc $\mathscr{C}$ counter-clockwise, we denote $\mathscr{C} := \mathscr{C}_r \cup \mathscr{C}_t \cup \mathscr{C}_L \cup \mathscr{C}_b$, where $\mathscr{C}_r, \mathscr{C}_t, \mathscr{C}_L, \mathscr{C}_b$ are the rightmost, top, leftmost, and bottom arcs of the rectangle respectively. For fixed N the integrals on the arcs $\mathscr{C}_t$ and $\mathscr{C}_b$ are negligible due to the rapid decay of the Γ-function discussed in Thm. 2.1.4. In the same vein, the rapid decay of the Γ-function also gives as $z \to 0$

$$-\frac{1}{2\pi i}\int_{\left(-N-\frac{1}{2}\right)} z^{-s}\zeta(s)\mathrm{Li}_s(-1)\Gamma(s)ds = O\left(z^{\frac{1}{2}+N}\right).$$

The integrand has poles at odd negative integers and at $s \in \{0, 1\}$. Additionally around $s = 0$ we have[8]

$$\Gamma(s) = \frac{1}{s} - \gamma + O(s) \quad \text{and} \quad \zeta(0) = -\frac{1}{2} = \mathrm{Li}_0(-1),$$

where γ is the Euler–Mascheroni constant. When $s = 1$,

$$\Gamma(1) = 1,\ \zeta(s) = \frac{1}{s-1} + \gamma + O(s-1),$$
$$\text{and} \quad \mathrm{Li}_1(-1) = -\log(2).$$

[7]The series for the polylogarithm is not always convergent when $z \geq 1$. For example, when $s = 1$ and $z = 1$, the series is divergent. As with the ζ-function, one can define values for the polylogarithm via analytic continuation.

[8]One can use the integral representations of the forthcoming functions and refer to books such as [43, 66], or use MAPLE's "laurent()" command.

Thus,

$$\begin{aligned}
&\mathrm{Res}_{s=1}\left(z^{-s}\zeta(s)\mathrm{Li}_s(-1)\Gamma(s)\right) = -\log(2)z^{-1},\\
&\mathrm{Res}_{s=0}\left(z^{-s}\zeta(s)\mathrm{Li}_s(-1)\Gamma(s)\right) = \frac{1}{4}.
\end{aligned}$$

Therefore, if we choose $N = 1$ and switch variables back to τ, as $q \to 1$ inside the unit circle

$$\mathscr{L}(q) = \frac{i}{2\pi}\log(2)\tau^{-1} - \frac{1}{4} + O\left(\tau^{\frac{3}{2}}\right). \tag{2.1.5}$$

2.1.3 Jacobi theta functions and Appell sums

We now collect some information on the building blocks of modular and mock modular objects. We start by recalling the definitions of the normalized Appell sum and the Jacobi theta function. The normalized Appell sum is defined by [73]

$$\mu(z_1, z_2; \tau) := \frac{\zeta_1^{\frac{1}{2}}}{\vartheta(z_2;\tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n^2+n}{2}}\zeta_2^n}{1-q^n\zeta_1}, \tag{2.1.6}$$

where $\zeta_j := e^{2\pi i z_j}$, $z_1, z_2 \in \mathbb{C}$ with $z_1, z_2 \notin \mathbb{Z} + \tau\mathbb{Z}$, and ϑ is the *Jacobi theta function* (or ϑ-function for short) given by

$$\vartheta(z_2;\tau) := \sum_{m\in\frac{1}{2}+\mathbb{Z}} (-1)^m q^{\frac{m^2}{2}}\zeta_2^m. \tag{2.1.7}$$

Furthermore, we have the *Jacobi product representation* (see [2] for a proof)

$$\vartheta(z;\tau) = -iq^{\frac{1}{8}}\zeta^{-\frac{1}{2}}(\zeta;q)_\infty(q;q)_\infty\left(\zeta^{-1}q;q\right)_\infty, \tag{2.1.8}$$

where $\zeta := e^{2\pi i z}$. The following identities involving ϑ and μ will be used frequently throughout.

Proposition 2.19 (see Ch. 1, [73] and Eq. 2.1, [20])**.** *The normalized Appell sum and ϑ-function satisfy*
(1) $\mu(z_1, z_2; \tau+1) = e^{-\frac{\pi i}{4}}\mu(z_1, z_2; \tau)$,
(2) $\mu(z_1+1, z_2; \tau) = \mu(z_1, z_2+1; \tau) = -\mu(z_1, z_2; \tau)$,
(3) We have

$$\mu\left(\frac{z_1}{\tau}, \frac{z_2}{\tau}; -\frac{1}{\tau}\right) = -\sqrt{-i\tau}e^{-\pi i\frac{(z_1-z_2)^2}{\tau}}\mu(z_1, z_2; \tau) + \frac{\sqrt{-i\tau}}{2i}e^{-\pi i\frac{(z_1-z_2)^2}{\tau}}h(z_1-z_2; \tau),$$

where $h(z;\tau)$ is the Mordell integral given by

$$h(z;\tau) := \int_{\mathbb{R}} \frac{e^{\pi i\tau x^2}e^{-2\pi zx}}{\cosh(\pi x)}dx,$$

(4) $h\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) = \sqrt{-i\tau}e^{-\frac{\pi i z^2}{\tau}}h(z;\tau)$,
(5) $\vartheta(z+\tau;\tau) = -e^{-\pi i\tau - 2\pi i z}\vartheta(z;\tau)$,
(6) $\vartheta(z;\tau+1) = e^{\frac{\pi i}{4}}\vartheta(z;\tau)$,
(7) $\vartheta(z+1;\tau) = -\vartheta(z;\tau)$,
(8) $\vartheta\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) = -i\sqrt{-i\tau}e^{\frac{\pi i z^2}{\tau}}\vartheta(z;\tau)$.
(9) *We have the relation*

$$\left[\frac{\partial\vartheta(z;\tau)}{\partial z}\right]_{z=0} = -2\pi\eta(\tau)^3.$$

It is sometimes the case in the combinatorics literature that different normalizations for the Jacobi theta functions and Appell sums are used, normally denoted by j and m, respectively. One can go between the m-j notation used in the relevant works [46, 52, 53] and the μ-ϑ via the formulas

$$\vartheta(z_2;\tau) = -iq^{\frac{1}{8}}\zeta_2^{-\frac{1}{2}}j(\zeta_2, q),$$
$$m(\zeta_1; q; \zeta_2) = iq^{\frac{1}{8}}\zeta_1^{-\frac{1}{2}}\mu(z_1+z_2, z_2; \tau).$$

Additionally, the last few items of Prop. 2.19 come from the fact that ϑ is a Jacobi form of weight and index $\frac{1}{2}$. Jacobi forms are like modular forms in that they transform nicely under the action of a group, with the added caveat that the transformation must account now for the variable z. A nice account of the theory of Jacobi forms can be found in [37]. We state now the Jacobi transformations for ϑ in a concise form.

Proposition 2.20 (see Ch. 2, [20]). *Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\lambda, k \in \mathbb{Z}$. Then*

$$\vartheta\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = \chi(A)(c\tau+d)^{\frac{1}{2}} e^{\frac{\pi i c z^2}{c\tau+d}} \vartheta\left(z;\tau\right), \tag{2.1.9}$$

$$\vartheta\left(z+\lambda\tau+k;\tau\right) = \beta(\lambda) e^{-\pi i\left(\lambda^2\tau+2\lambda z\right)} \vartheta\left(z;\tau\right), \tag{2.1.10}$$

where χ and β are multipliers.[9]

As Zwegers showed in his thesis [73], the building blocks of mock modular forms are the Appell sums defined in Eq. (2.1.6), and he later showed in [74] that the so-called higher level Appell sums are generically mixed mock modular forms (see Lem. 2 therein). For $z_1, z_2 \in \mathbb{C} \setminus \{\mathbb{Z} + \tau\mathbb{Z}\}$ the *higher level Appell sum of level ℓ* is defined by

$$A_\ell(z_1, z_2; \tau) := \zeta_1^{\frac{\ell}{2}} \sum_{n\in\mathbb{Z}} \frac{(-1)^{\ell n} \zeta_2^n q^{\frac{\ell n(n+1)}{2}}}{1-\zeta_1 q^n}, \qquad \ell \in \mathbb{N}. \tag{2.1.11}$$

The function A_1 is the non-normalized version of Eq. (2.1.6), whose transformation properties were studied at length in [73]. In particular,

$$-\frac{1}{\tau} e^{\frac{\pi i\left(z_1^2 - 2z_1 z_2\right)}{\tau}} A_1\left(\frac{z_1}{\tau}, \frac{z_2}{\tau}; -\frac{1}{\tau}\right) + A_1(z_1, z_2; \tau) \tag{2.1.12}$$

$$= \frac{1}{2i} h(z_1 - z_2; \tau) \vartheta(z_2; \tau). \tag{2.1.13}$$

[9] We will not use the explicit values of the multipliers in this thesis.

As can be seen with Eq. (2.1.12), that knowing how the Mordell integral behaves for τ near the real line is essential for understanding the growth of many mixed mock modular forms near the real line. In light of this, we first prove a simple lemma related to the positivity of the Mordell integral for certain z and τ of small modulus.

Lemma 2.21. *Let $-\frac{1}{2} < z < \frac{1}{2}$. Then as $\tau \to 0$ within a fixed angular region,*

$$0 < h(z;\tau) \ll 1.$$

We also have the specific value

$$h(0;0) = 1.$$

Proof. By definition,

$$|h(z;\tau)| \leq \int_{-\infty}^{\infty} \left| \frac{e^{\pi i \tau x^2 - 2\pi z x}}{\cosh(\pi x)} \right| dx \leq \int_{-\infty}^{\infty} \frac{e^{-2\pi z x}}{\cosh(\pi x)} dx,$$

where the last integrand is integrable for z in the specified range. Therefore, by dominated convergence

$$\lim_{\tau \to 0} h(z;\tau) = h(z;0) = \int_{-\infty}^{\infty} \frac{e^{-2\pi z x}}{\cosh(\pi x)} dx > 0.$$

When $z = 0$, we have (see pg. 116, [43])

$$h(0;0) = \int_{-\infty}^{\infty} \frac{1}{\cosh(\pi x)} dx = 1,$$

which completes the proof. $\square$

In a similar vein, we have the following, slightly different estimate for the Mordell integral.

Lemma 2.22. *Let $0 \leq \alpha < \frac{1}{2}$. Then as $\tau \to 0$ in a fixed angular region,*

$$h(\alpha\tau;\tau) \ll 1.$$

Proof. The proof follows from the transformation law for h given in Prop. 2.19.4:

$$\begin{aligned}h(\alpha\tau;\tau) &= \frac{q^{\frac{\alpha^2}{2}}}{\sqrt{-i\tau}}\, h\left(\alpha; -\frac{1}{\tau}\right) = \frac{q^{\frac{\alpha^2}{2}}}{\sqrt{-i\tau}} \int_{-\infty}^{\infty} \frac{e^{-\frac{\pi i w^2}{\tau}} e^{-2\pi\alpha w}}{\cosh(\pi w)} dw \\ &= \frac{q^{\frac{\alpha^2}{2}}}{\sqrt{-i\tau}} \left(\int_0^{\infty} \frac{e^{-\frac{\pi i w^2}{\tau}} e^{-2\pi\alpha w}}{\cosh(\pi w)} + \int_0^{\infty} \frac{e^{-\frac{\pi i w^2}{\tau}} e^{2\pi\alpha w}}{\cosh(\pi w)} \right) dw,\end{aligned}$$

which implies that

$$|h(\alpha\tau;\tau)| \leq \left| \frac{q^{\frac{\alpha^2}{2}}}{\sqrt{-i\tau}} \right| \left| \int_0^{\infty} e^{-\frac{\pi i w^2}{\tau}} \left(\frac{e^{(2\alpha-1)\pi w}}{1+e^{-2\pi w}} + \frac{e^{-(2\alpha+1)\pi w}}{1+e^{-2\pi w}} \right) dw \right|. \tag{2.1.14}$$

Since $0 \leq \alpha < \frac{1}{2}$, the term in the parentheses is bounded above by a constant. Therefore,

$$|h(\alpha\tau;\tau)| \ll \left| \frac{q^{\frac{\alpha^2}{2}}}{\sqrt{-i\tau}} \right| \int_0^{\infty} \left| e^{-\frac{\pi i w^2}{\tau}} \right| dw \ll \frac{\sqrt{\tau}}{\sqrt{\tau}} = 1,$$

where we used the fact that $v > 0$ and that $\int_{\mathbb{R}} e^{-\frac{v w^2}{|\tau|^2}} dw = \sqrt{\frac{\pi}{v}} |\tau|$. This leads to the claimed estimate as $\tau \to 0$. $\square$

2.2 The Hardy–Ramanujan circle method

Much of the concepts covered in this section are addressed in the classic texts [5] and [9]. Throughout, we assume that the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ to avoid having to incorporate the width of a cusp in the forthcoming discussion. Given a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, we are interested in the asymptotic behavior of the coefficients a_n for large n of the expansion

$$f(q) = \sum_{n \geq 0} a_n q^n. \tag{2.2.1}$$

When f is a weakly holomorphic modular form, for example, the a_n grow rapidly since as one approaches $\mathbb{Q}$ within a fixed angular region, the function f is not necessarily bounded. This includes weakly holomorphic modular forms, mock modular forms, and MMMFs, to name a few. Using Cauchy's Integral Formula, we can formally write the coefficients in Eq. (2.2.1) as

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(q)dq}{q^{n+1}} = \frac{r^{-n}}{2\pi}\int_0^{2\pi} f\left(re^{i\theta}\right)e^{-in\theta}d\theta, \tag{2.2.2}$$

where $C := re^{i\theta}$ is a circular path enclosing the origin, oriented counter-clockwise. This is the set up for the famous circle method first studied in 1918 by Hardy and Ramanujan [45]. They proved that as $n \to \infty$

$$p(n) \sim \frac{1}{4\sqrt{3}n}e^{\pi\sqrt{\frac{2n}{3}}}. \tag{2.2.3}$$

The main idea for the Hardy–Ramanujan circle method is two-fold. Firstly, one must precisely study the asymptotic behavior of the function f near points on the unit circle. Secondly, one must divide the arc C in such a way as to capture the growth found in the first part, while allowing one to bound precisely the smaller terms coming from the other roots of unity. The "best choice" for a curve C stems from the theory of Farey fractions and the saddle point method. Once the curve is chosen, certain parts of the curve contribute more to the overall estimate for the a_n than others. This leads to the notion of the major and minor arcs. There are many detailed accounts of the circle method, however the application of the method usually requires much work to apply to the function in question (as we will see in Ch. 5). As an introduction, we follow the standard method adapted for the partition function, which is covered in great detail in Ch. 5 of [5] and Ch. 5 of [9]. The partition function $p(n)$ is one of the special cases where the circle method yields an exact formula. Rademacher adjusted the methodology developed by Hardy and Ramanujan to yield this exact formula for $p(n)$ [63]. When we generically refer to the "circle method", we refer to the Hardy–Ramanujan version where we are not attempting to find exact formulas for a_n, rather an asymptotic series.

We start with the first part in the circle method, i.e. the asymptotic estimations for the function f near roots of unity. There are two distinct situations to contend with: when f has a modular-type transformation and when f does not.

2.2.1 Modular transformations: asymptotic estimates

As an example, we find the behavior at a generic cusp $\frac{h}{k}$ for the partition generating function.

Example 2.23. We compute the growth of the partition generating function near a generic cusp. We have two cases: (1) cusps on the real line not equal to 0 and (2) the cases 0 and ∞. We start with Case (1). Let $\frac{h}{k}$ be a cusp such that $\gcd(h,k)=1$. Then there exist $\alpha,\beta\in\mathbb{Z}$ such that $A := \left(\begin{smallmatrix} h & \beta \\ k & \alpha \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Define $w := \tau - \frac{h}{k}$ and $\sigma := -\frac{h}{wk^2} - \frac{\alpha}{k}$. Then it follows that in a fixed angular region

$$\lim_{\tau\to\frac{h}{k}} A\sigma = \lim_{w\to 0}\frac{h}{k} + \alpha w = \frac{h}{k}.$$

Therefore,

$$\begin{aligned}\eta(A\sigma) &= \sqrt{-\frac{h}{k\left(\tau-\frac{h}{k}\right)}}\nu(A)e^{\frac{\pi i\sigma}{12}}\prod_{j\geq 1}\left(1-e^{2\pi i j\sigma}\right)\\ &= \sqrt{-\frac{h}{k\left(\tau-\frac{h}{k}\right)}}\nu(A)e^{-\frac{\pi i\alpha}{12k}}e^{-\frac{h\pi i}{12\left(\tau-\frac{h}{k}\right)h^2}}\left(1+O\left(e^{-\frac{2h\pi i}{\left(\tau-\frac{h}{k}\right)h^2}}\right)\right).\end{aligned}$$

Thus, as $\tau\to\frac{h}{k}$ in a fixed angular region the partition generating function satisfies

$$P(q) = e^{\frac{\pi i h}{12k}} \eta^{-1}(\tau)$$
$$= \sqrt{-\frac{k\left(\tau - \frac{h}{k}\right)}{h}} \nu(A)^{-1} e^{\frac{\pi i (h+\alpha)}{12k}} e^{\frac{h\pi i}{12\left(\tau - \frac{h}{k}\right)h^2}} \left(1 + O\left(e^{-\frac{2h\pi i}{\left(\tau - \frac{h}{h}\right)k^2}}\right)\right). \tag{2.2.4}$$

We now investigate Case (2). When $h = 0$, we let $S := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We have

$$\eta(S\tau) = \sqrt{-i\tau}\eta(\tau).$$

This implies that as $q \to 1$

$$\frac{1}{(q)_\infty} = \sqrt{-i\tau} e^{\frac{\pi i}{12\tau}} \left(1 + O\left(e^{-\frac{2\pi i}{\tau}}\right)\right). \tag{2.2.5}$$

The function $P(q)$ is clearly bounded as $q \to 0$, or rather as $\tau \to i\infty$. That is,

$$\lim_{q \to 0} P(q) = 1.$$

Using the idea illustrated in Ex. 2.23, we have the following concise statements for the ϑ- and η-functions.

Lemma 2.24. *Let $\alpha \in [0, 1)$, let $q := e^{2\pi i \tau}$, $q_0 := e^{-\frac{2\pi i}{\tau}}$, and $r > 1$ be a rational number. As $\tau \to 0$ within a fixed angular region,*

$$\vartheta(\alpha\tau; \tau) = -\frac{2i \sin(\pi\alpha) q^{-\frac{\alpha^2}{2}} q_0^{\frac{1}{8}}}{\sqrt{-i\tau}} \left(1 + O(q_0)\right), \tag{2.2.6}$$

$$\vartheta\left(\frac{1}{r} + \alpha\tau; \tau\right) = -\frac{q^{-\frac{\alpha^2}{2}} e^{\pi i \alpha \left(1 - \frac{2}{r}\right)}}{\sqrt{-i\tau}} q_0^{\frac{1}{2r^2} - \frac{1}{2r} + \frac{1}{8}} \left(1 + O\left(q_0^{\frac{1}{r}}\right)\right), \tag{2.2.7}$$

$$\eta(\tau) = \frac{q_0^{\frac{1}{24}}}{\sqrt{-i\tau}} \left(1 + O(q_0)\right). \tag{2.2.8}$$

Proof. We begin with Eq. (2.2.6). Using the Jacobi product formula and Prop. 2.19, we have

$$\vartheta(\alpha\tau;\tau) = \frac{iq^{-\frac{\alpha^2}{2}}}{\sqrt{-i\tau}}\vartheta\left(\alpha;-\frac{1}{\tau}\right)$$
$$= \frac{e^{-\pi i\alpha}q^{-\frac{\alpha^2}{2}}q_0^{\frac{1}{8}}}{\sqrt{-i\tau}}\left(e^{2\pi i\alpha};q_0\right)_\infty\left(q_0e^{-2\pi i\alpha};q_0\right)_\infty\left(q_0;q_0\right)_\infty.$$

Thus as $\tau \to 0$ in a fixed angular region,

$$\vartheta(\alpha\tau;\tau) = \frac{e^{-\pi i\alpha}q^{-\frac{\alpha^2}{2}}q_0^{\frac{1}{8}}}{\sqrt{-i\tau}}\left(1-e^{2\pi i\alpha}+O(q_0)\right)\left(1+O(q_0)\right)\left(1+O(q_0)\right)$$
$$= \frac{e^{-\pi i\alpha}q^{-\frac{\alpha^2}{2}}q_0^{\frac{1}{8}}}{\sqrt{-i\tau}}\left(1-e^{2\pi i\alpha}+O(q_0)\right)$$
$$= \frac{e^{-\pi i\alpha}q^{-\frac{\alpha^2}{2}}\left(1-e^{2\pi i\alpha}\right)q_0^{\frac{1}{8}}}{\sqrt{-i\tau}}\left(1+O(q_0)\right)$$
$$= \frac{-2i\sin(\pi\alpha)q^{-\frac{\alpha^2}{2}}q_0^{\frac{1}{8}}}{\sqrt{-i\tau}}\left(1+O(q_0)\right),$$

where the second to last step follows from the fact that $1-e^{2\pi i\alpha}$ is $O(1)$. Similarly for Eq. (2.2.7),

$$\vartheta\left(\frac{1}{r}+\alpha\tau;\tau\right) = \frac{ie^{-\frac{\pi i\left(\frac{1}{r^2}+\frac{2\alpha\tau}{r}+\alpha^2\tau^2\right)}{\tau}}}{\sqrt{-i\tau}}\vartheta\left(\frac{1}{r\tau}+\alpha;-\frac{1}{\tau}\right)$$
$$= \frac{ie^{-\frac{2\pi i\alpha}{r}}q^{-\frac{\alpha^2}{2}}q_0^{\frac{1}{2r^2}}}{\sqrt{-i\tau}}\left(-iq_0^{\frac{1}{8}}e^{-\pi i\left(\alpha+\frac{1}{r\tau}\right)}\right)$$
$$\times\left(e^{2\pi i\left(\alpha+\frac{1}{r\tau}\right)};q_0\right)_\infty\left(q_0e^{-2\pi i\left(\alpha+\frac{1}{r\tau}\right)};q_0\right)_\infty(q_0;q_0)_\infty$$
$$= -\frac{q^{-\frac{\alpha^2}{2}}e^{\pi i\alpha\left(1-\frac{2}{r}\right)}}{\sqrt{-i\tau}}q_0^{\frac{1}{2r^2}-\frac{1}{2r}+\frac{1}{8}}\left(1+O\left(q_0^{\frac{1}{r}}\right)\right).$$

Finally, the estimate for the η-function follows directly from Ex. 2.23. $\square$

2.2.2 Non-modular objects: asymptotic estimates

When a modular transformation is not available, we have to turn to other means to get the asymptotic behavior of f near the cusps. Let $B_n(x)$ denote the n^{th} Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1} =: \sum_{n\geq 0} \frac{B_n(x)t^n}{n!}.$$

We now state a modern version of the famous Euler–Maclaurin summation formula which is useful in dealing with non-modular functions defined by infinite sums.

Theorem 2.25 (asymptotic Euler–Maclaurin summation Thm. 1.2, [22])**.** *Suppose that $0 \leq \theta < \frac{\pi}{2}$ and suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic in a domain containing $\{re^{i\alpha} : |\alpha| \leq \theta\}$ with derivatives of sufficient decay; i.e., $f^{(m)}(z) \ll z^{-1-\varepsilon}$ as $z \to \infty$ for some $\varepsilon > 0$. Then for $a \in [0,1]$, we have*

$$\sum_{\ell\geq 0} f(w(\ell+a)) = \tfrac{1}{w}\textstyle\int_0^\infty f(y)dy - \sum_{n=0}^{N-1} \frac{f^{(n)}(0)B_{n+1}(a)}{(n+1)!} w^n + O_N\left(w^N\right),$$

uniformly as $w \to 0$ in $\operatorname{Arg}(w) \leq \theta$.

We illustrate the utility of Thm. 2.25 through an example involving functions from [49]. We also apply our knowledge from Ex. 2.23.

Example 2.26. The following function appears in [49]:

$$\sum_{n\geq 0} \mathbf{B}(n)q^n = \frac{1}{(q)_\infty^2}\left(1 - 2\sum_{n\geq 1} q^{\frac{n(3n-1)}{2}}(1-q^n)\right). \tag{2.2.9}$$

The combinatorics and generating function were studied extensively in [49], but the asymptotics for the $\mathbf{B}(n)$ were not stated.[10] However,

[10] The $\mathbf{B}(n)$ count a certain difference of bipartitions. Bipartitions are the number of tuples (λ_m, λ_k), such that λ_m and λ_k are partitions of m and k respectively, and $k+m=n$. See Sect. 3.1 of [49] for more information on the generating function, and how one exactly arrives at the formula in Eq. (2.2.9).

the study is nearly identical to that of Eq. 1.14 therein. We fill in the details not provided in [49]. Let

$$F(q) := \sum_{n\geq 1} q^{\frac{n(3n-1)}{2}}(1-q^n).$$

Throughout, we take the principle branch of the squareroot. The authors of [49] decompose $F\left(e^{-z}\right)$ (with $z := -2\pi i\tau$) as

$$F\left(e^{-z}\right) = e^{-\frac{z}{24}} \sum_{n\geq 0} \left(f\left(\left(n+\frac{5}{6}\right)\sqrt{z}\right) - f\left(\left(n+\frac{7}{6}\right)\sqrt{z}\right)\right),$$

where we defined $f(z) := e^{-\frac{3}{2}z^2}$. Let

$$z_1 := \left(n+\frac{5}{6}\right)\sqrt{z} \text{ and } z_2 := \left(n+\frac{7}{6}\right)\sqrt{z}.$$

Thm. 2.25 says that

$$\begin{aligned} F\left(e^{-z}\right) &= \sum_{n=0}^{N} \frac{B_{n+1}\left(\frac{7}{6}\right) - B_{n+1}\left(\frac{5}{6}\right)}{(n+1)!} f^{(n)}(0) z^{\frac{n}{2}} + O\left(z^{\frac{N+1}{2}}\right) \\ &:= \sum_{n=0}^{N} \alpha(n) z^{\frac{n}{2}} + O\left(z^{\frac{N+1}{2}}\right), \end{aligned}$$

where for $k > 0$

$$f^{(k)}(0) = \begin{cases} (-1)^{\frac{k}{2}} \frac{n!}{\left(\frac{n}{2}\right)!} 3^k \cdot 6^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{else.} \end{cases}$$

From Eq. (2.2.5), we have that near $q = 1$,

$$\frac{1}{(q)_\infty^2} = -i\tau e^{\frac{\pi i}{6\tau}} \left(1 + O\left(e^{-\frac{2\pi i}{\tau}}\right)\right) = \frac{z}{2\pi} e^{\frac{\pi^2}{3z}} \left(1 + O\left(e^{-\frac{1}{z}}\right)\right).$$

This implies overall that as $z \to 0$ within a fixed angular region in the right-half-plane,

$$\sum_{n\geq 0} \mathbf{B}(n) q^n = \frac{z}{6\pi} e^{\frac{\pi^2}{3z}} + O\left(z^2 e^{\frac{1}{12}z}\right). \tag{2.2.10}$$

Remark 2.27. It is worth noting, that with Eq. (2.2.10), one can apply the upcoming Thm. 2.29 to find

$$\mathbf{B}(n) \sim \frac{1}{3^{\frac{3}{4}} \cdot 12 \cdot n^{\frac{5}{4}}} e^{2\pi\sqrt{\frac{n}{3}}}.$$

Notation 2.28. The notation $\sim$ is understood to mean that two functions are asymptotically the same. This means $f(x) \sim h(x)$ if and only if

$$f(x) = h(x)(1 + o(1)), \qquad x \to \infty.$$

2.2.3 Choosing the correct curve

Much of this section is based on Ch. 5 of [5], Ch. 5 of [9], and Ch. 8 of [38]. We advise the interested reader to consult these texts for a detailed analysis of the circle method, its history, and its applications in number theory. The real magic of the circle method arises in the freedom to choose a curve C in Eq. (2.2.2). Let N be a fixed positive integer and let $0 \leq \frac{h}{k} \leq 1$ be any fraction in reduced form such that $k \leq N$. Then the *Farey sequence of order* N is defined by the set

$$\mathscr{F}_N := \left\{ 0 \leq \frac{h}{k} \leq 1 : \ \frac{h}{k} \in \mathbb{Q} \ \text{ and } \ k \leq N \right\},$$

with ordering from smallest to largest. For example, the Farey sequence of order 3 is given by

$$\mathscr{F}_3 = \left\{ 0, \frac{1}{3}, \ \frac{1}{2}, \ \frac{2}{3}, \ 1 \right\}.$$

Regardless of how the radius of C is chosen, the Farey sequences cut the circle into distinct regions. That is to say, we dissect the path of integration into multiple intervals built around $\mathscr{F}_N$ by first rewriting $re^{i\theta} := e^{t_n + 2\pi i(-\frac{h}{k} + \theta)}$, where t_n is chosen via the saddle point method and $\frac{h}{k} \in \mathscr{F}_N$. This yields

$$a_n = \sum_{\frac{h}{k} \in \mathscr{F}_N} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} f\left(e^{-t_n+2\pi i\left(-\frac{h}{k}+\theta\right)}\right) e^{nt_n-2\pi in\left(-\frac{h}{k}+\theta\right)} d\theta. \tag{2.2.11}$$

The values $\theta'_{h,k}$ and $\theta''_{h,k}$ are defined by shifting the so-called mediant of two consecutive Farey fractions. Precisely stated, if $\frac{h_1}{k_1} < \frac{h}{k}$ are consecutive Farey fractions then the *mediant* is the value ([5], [9], and Sect. 4, [61])

$$\frac{h_1 + h}{k_1 + k}.$$

A well-known property of Farey sequences is that adjacent fractions satisfy the relation $kh_1 - k_1 h = 1$. Let $\frac{h_1}{k_1}, \frac{h}{k}$, and $\frac{h_3}{k_3}$ be a triple of consecutive Farey fractions. Thus, we define our integration bounds in Eq. (2.2.11) using this fact and shifting the mediants by $\frac{h}{k}$, and addressing the fractions 0 and 1 as special cases [61]:

$$\theta'_{h,k} := \frac{1}{k(k+k_1)}, \quad \theta''_{h,k} := \frac{1}{k(k+k_3)}, \quad \theta'_{0,k} = \theta''_{0,k} := 0,$$
$$\theta'_{1,1} = \theta''_{1,1} := \frac{1}{N+1}.$$

As Andrews points out in [5], the intervals of integration have different lengths, with the largest intervals given to cusps with smaller denominators. The intervals where $\frac{h}{k}$ contribute to the main term of the asymptotic behavior of the a_n are called *major arcs* and the others which amount to error terms are referred to as the *minor arcs*. In principle, there are infinitely many minor arcs in the classical circle method.

The saddle point method applied to the integral in Eq. (2.2.2) in the context of the Hardy–Ramanujan circle method tells us how to choose the radius of the circle C. Specifically, we want to find the point where

$$\frac{d}{dq}\left(f(q)q^{-n}\right) = 0. \tag{2.2.12}$$

After reparameterizing C as above it is possible for many generating functions to write the integrand as $e^{S(\theta)}$ where $S(\theta)$ has a convergent Taylor series near $\theta = 0$. Thus, the condition in Eq. (2.2.12) amounts to finding where $S'(\theta) = 0$. If t_s is such that $S'(t_s) = 0$, then

$$S(\theta) = S(t_s) + \frac{S^{''}(t_s)}{2}(\theta - t_s)^2 + O\left(\theta^3\right).$$

What the saddle point essentially does, is it allows one to then approximate the portion of the integral in Eq. (2.2.2) near t_s by a Gaussian integral. One can show that any f with positive q-series coefficients, guarantees that $S(\theta)$ has a unique saddle point (see pg. 549, [38]). An excellent account of the saddle point method for Cauchy-type integrals is given in Ch. 8 of [38], including uniqueness proofs and a multitude of examples. In order to find the saddle point of the integrand above, one must have an idea of which cusps lead to the largest growth. This is ideally sorted out by either having a modular transformation, or some other additional information on the function as discussed in the previous section. In the case of the partition function, the largest growth (see Ex. 2.23) is near $q = 1$. It is well known that for sufficiently large n the saddle point of the integrand involving the partition generating function yields the arc radius (see Ch.8, [38])

$$r := 1 - \frac{\pi}{\sqrt{6n}}.$$

2.3 The Wright circle method

The classical circle method of Hardy and Ramanujan demands a careful study of the integrand in Eq. (2.2.2) at each cusp $\frac{h}{k}$. In general, there are infinitely many minor arcs and at least one major arc to contend with. However, if one wants an asymptotic formula

$$a_n \sim g(n)$$

for some function $g(n)$, it sometimes suffices to have only one major arc and one minor arc. That is to say, the integration path C is bisected into only two parts. This is the case when the generating function has extreme growth near a single pole (for example $q = 1$). A prototype fitting into this scheme is a function of the type

$$f(q) = \frac{1}{h(q)} M(q),$$

where h is a weakly holomorphic modular form, and M is a function of polynomial growth, typically a mock theta function or false theta function[11] (see [19, 20, 49] for some examples). If $M(q)$ is a mock theta function, then $f(q)$ is generically a MMMF. This version of the circle method is called the Wright circle method which stems from Wright's work on stacks in the late 60's and early 70's [69, 70]. The Wright circle method is usually the method of choice when studying the coefficients of MMMFs. We will apply Wright's version of the circle method in Ch. 4.

2.4 Tauberian theorems

If we are only interested in the main asymptotic term for the a_n, i.e., $a_n \sim g(n)$, we can sometimes avoid dealing with the technicalities of bounding the minor arcs in the circle method if we can show that the generating function satisfies some basic properties. This is the subject of the so-called Ingham's Tauberian Theorem.

[11] False theta functions are like regular theta functions, with an extra sign function added in. Until recently, false theta functions had no known transformations like the mock theta functions. However, this was settled by Bringmann and Nazaroglu in [28].

Theorem 2.29 (see Thm. 1.1 of [23]). *Let $c(n)$ denote the coefficients of a power series $C(q) := \sum_{n=0}^{\infty} c(n)q^n$ with radius of convergence equal to 1. Define $z := x + iy \in \mathbb{C}$ with $x > 0$. If the $c(n)$ are non-negative, are weakly increasing, and we have as $t \to 0^+$ along the real axis that*

$$C\left(e^{-t}\right) \sim \lambda t^{\alpha} e^{\frac{A}{t}},$$

and if for each $M > 0$ such that $|y| \leq M|x|$,

$$C\left(e^{-z}\right) \ll |z|^{\alpha} e^{\frac{A}{|z|}} \tag{2.4.1}$$

with $A > 0$ holds, then as $n \to \infty$

$$c(n) \sim \frac{\lambda A^{\frac{\alpha}{2}+\frac{1}{4}}}{2\sqrt{\pi} n^{\frac{\alpha}{2}+\frac{3}{4}}} e^{2\sqrt{An}}.$$

2.5 Outline of this thesis

The rest of this thesis is organized as follows. In Ch. 3, we show an asymptotic equidistribution result for odd-balanced unimodal sequences using standard transformation formulas for the generating function and a Tauberian theorem. In Ch. 4, we apply a modified version of the Wright circle method to study functions coming from Bailey pairs to find asymptotic formulas for their Fourier coefficients. Finally, in Ch. 5, we apply the classical circle method to study the complex coefficients of a twisted q-product, which allows us to predict sign changes in differences of certain partition counts. We finish this thesis by offering some concluding remarks and some open problems.

Chapter 3

Asymptotic distribution of odd-balanced unimodal sequences with rank congruent to a modulo c

This chapter is an adapted version of the work [41] published by the author of this thesis in *Research in Number Theory*. Throughout this chapter, we will continue to use the convention $\tau := u + iv$, $w := e^{2\pi iz}$, and $q_0 := e^{-\frac{2\pi i}{\tau}}$. It is important to note that positive powers of q_0 indicate decay as $\tau \to 0$, and negative powers of q_0 indicate growth as $\tau \to 0$.

3.1 Recent studies and the setup

Our goal is to study the asymptotic properties of the odd-balanced unimodal sequences defined by the generating function in Eq. (1.2.5). We defined the unimodal sequence generating function $U(q)$ in Eq. (1.2.3) and the strongly unimodal sequence generating function $U^*(q)$ in Eq. (1.2.4). Many authors have studied these generating functions for their analytic properties [21, 24, 26, 30]. One of the profound features of the generating function $U^*(q)$ is its relation to analytic functions on the real line. The authors of [30] showed that scaled versions of the function $U^*(q)$ can be defined as functions on $\mathbb{Q}$, and these functions are quantum modular forms in the sense of Zagier [72]. The *quantum modular forms* are those functions whose obstruction to having a modular type transformation on $\mathbb{Q}$ of the type in Def. 2.9 can be characterized by a function that has an analytic extension to an open subset of the real line. More interesting for this chapter is the relation between the strongly unimodal sequneces with rank m, denoted by $u^*(m,n)$, and the classical partition function $p(n)$. Rhoades proved in 2014 an asymptotic formula for the number of strongly unimodal sequences $u^*(n)$ [64], and the authors of [25] extended this result for fixed rank m. The main result in [25] reads

$$u^*(m,n) \sim \frac{p(n)}{4}. \tag{3.1.1}$$

In light of the special analytic properties of $U^*(w;q)$, Kim, Lim, and Lovejoy defined odd-balanced unimodal sequences of rank m by interpreting the coefficients of the generating function in Eq. (1.2.5). The authors of [50] showed that $V(1;q^{-1})$ exhibits quantum modular properties analogous to the generating function for strongly unimodal sequences.

Let $v(m,n)$ denote the number of odd-balanced unimodal sequences of rank m and let $v(a,c;n)$ denote the same count but with the relaxed condition that the rank is congruent to $a \pmod{c}$. Let $\zeta_c^a := e^{2\pi i \frac{a}{c}}$. Using orthogonality of roots of unity, we have that the generating function for the $v(a,c;n)$, $V(a,c;q)$, can be written formally as

$$V(a,c;q) := \frac{V(1;q)}{c} + \frac{1}{c}\sum_{j=1}^{c-1} \zeta_c^{-aj} V(\zeta_c^j;q), \tag{3.1.2}$$

with

$$V(w;q) := \sum_{n\geq 0} \frac{(-wq;q)_n(-w^{-1}q;q)_n q^n}{(q;q^2)_{n+1}}.$$

For a proof of Eq. (3.1.2), we refer the reader to Prop. 5.7, which follows identically. It is not hard to see that the $v(a, c; n)$ are weakly increasing in n. To see this, notice that $v(a, c; n+1)$ counts sequences of size $2(n+1)+2 = 2n+4$. Therefore, we can take each of the sequences of size $2n+2$ and add a 1 to each side of the peak without changing the rank. That is,

$$v(a, c; n) \leq v(a, c; n+1). \tag{3.1.3}$$

Our main theorem involves the $v(a, c; n)$ and describes how they are distributed with respect to the overpartition function $\overline{p}(n)$. Overpartitions are generalizations of partitions, in which an overpartition of n is a partition of n, where the first occurrence of each part may or may not be over-lined. For example, the overpartitions of 3 are

$$(1)\,,\ \left(\overline{1}\right)\ ,(1,1,2)\,,\ \left(1,1,\overline{2}\right),\ \left(\overline{1},1,2\right),\ \left(\overline{1},1,\overline{2}\right),\ (3)\,,\ \text{and}\ \left(\overline{3}\right).$$

That is $\overline{p}(3) = 8$. For an account of overpartitions and the combinatorics that accompany them, we refer the reader to [33]. Due to the modularity of the generating function for the $\overline{p}(n)$, an analogous asymptotic result to the $p(n)$ in Eq. (2.2.3) follows again from the techniques of Hardy and Ramanujan [45]:

$$\overline{p}(n) \sim \frac{1}{8n} e^{\pi\sqrt{n}}, \qquad n \to \infty.$$

In an attempt to find an analogy to Eq. (3.1.1) for $v(m, n)$, we find and prove the following for $v(a, c; n)$.

Theorem 3.1. *Let $c > 1$ be odd. Then as $n \to \infty$,*

$$v(a, c; n) \sim \frac{1}{16cn^{\frac{3}{4}}} e^{\pi\sqrt{n}} \sim \frac{n^{\frac{1}{4}}}{2c}\overline{p}(n).$$

Remark 3.2. There are two things worth noting here:
(1) The exclusion of even c is related to the fact that the generating function $V(w; q)$ is not a mixed mock modular form at $w = -1$.

Since $V(-1;q)$ will occur at $j = \frac{c}{2}$ in the sum in Eq. (3.1.2) when c is even, we require c to be odd.
(2) The result above can be interpreted as an equidistribution result with respect to the modulus c. That is,

$$v(a,c;n) \sim \frac{v(n)}{c}.$$

Results of this type were explored by Ciolan for the overpartition function [31] and by Males for the partition function [54].

An immediate consequence of Thm. 3.1 involves the following nice bound in terms of adjacent overpartition counts.

Corollary 3.3. *Let $c > 1$ odd. Then there exists an $N \in \mathbb{N}$, such that for all $n > N$,*

$$v(a,c;2n) \leq v(a,c;n-1)v(a,c;n+1) < \sqrt{n} \cdot \overline{p}(n-1)\overline{p}(n+1).$$

The rest of this chapter is organized as follows. In Sect. 3.2, we calculate the growth of the generating function $V(w;q)$ for w a fixed root of unity. In Sect. 3.3, we prove Thm. 3.1 and Cor. 3.3.

3.2 Analytic properties of the generating function $V(w;q)$

Our proof relies on the fact that $V(w;q)$ can be written as a mixed mock modular form for fixed $z \neq \frac{1}{2}$. Kim, Lim, and Lovejoy in [50] used the theory of indefinite theta series and a corresponding result of Mortenson and Osburn [46] to write[1]

$$\left(1 + w^{-1}\right) qV(w;q) = -T_1(w;q) + T(w;q) - wT_2(w;q), \tag{3.2.1}$$

[1] The authors of [46] showed that one can essentially go from Hecke double sums, which are closely related to indefinite theta functions of the type studied by Zwegers [73], to sums of Appell functions and modular forms of the type defined later in Def. 4.3. Eq. (3.2.1) comes from an application of Thm. 0.3 of [46].

where

$$\begin{aligned} T_1(w;q) &:= -iq^{\frac{1}{8}}w^{-\frac{1}{2}}\mu\left(z+\frac{1}{2},\frac{1}{2};\tau\right), \\ T(w;q) &:= -q^{-\frac{1}{8}}w^{-\frac{1}{2}}\frac{\vartheta(\frac{1}{2}+z;\tau)}{\vartheta(\tau;2\tau)}\mu\left(2z+\frac{1}{2},\frac{1}{2};2\tau\right), \\ T_2(w;q) &:= iq^{\frac{11}{8}}w^{-\frac{1}{2}}\frac{\vartheta(4\tau;12\tau)^3}{\vartheta(2\tau;6\tau)^3}\frac{\vartheta(z;\tau)\vartheta(2z+\tau;2\tau)}{\vartheta(4z;4\tau)}. \end{aligned}$$

With this decomposition, we can begin our study of the growth of the generating function near $\tau=0$.

3.2.1 Odd-balanced with $z=0$

In this section, we capture the growth of $V(1;q)$ near $\tau=0$, or equivalently, the growth of the coefficients $v(n)$. To do this properly, we take the limit as $z\to 0$ first. We will find that the term $T(w;q)$ provides the main estimate. We find and prove the following.

Lemma 3.4. *As $n\to\infty$,*

$$v(n)\sim\frac{1}{16n^{\frac{3}{4}}}e^{\pi\sqrt{n}}\sim\frac{n^{\frac{1}{4}}}{2}\overline{p}(n).$$

Proof. Let $M>0$ and let $|u|\le Mv$. That is, we investigate the limit inside a fixed angular region. The proof amounts to calculating estimates near $\tau=0$ for $T_1(1;q)$, $T(1;q)$, and $T_2(1;q)$ where we have to explicitly calculate

$$T_j(1;q):=\lim_{w\to 1}T_j(w;q).$$

We begin with calculating the main term for $T(1;q)$, which will turn out to give us the main contribution:

$$T(1;q)=-q^{-\frac{1}{8}}\frac{\vartheta\left(\frac{1}{2};\tau\right)}{\vartheta(\tau;2\tau)}\mu\left(\frac{1}{2},\frac{1}{2};2\tau\right).$$

For the ϑ-functions we turn to Lem. 2.24, which gives

$$\frac{\vartheta\left(\frac{1}{2};\tau\right)}{\vartheta(\tau;2\tau)} = \frac{-\frac{1}{\sqrt{-i\tau}}\left(1+O\left(q_0^{\frac{1}{2}}\right)\right)}{-2i\frac{1}{\sqrt{-2i\tau}}q_0^{\frac{1}{16}}\left(1+O\left(q_0^{\frac{1}{2}}\right)\right)} \tag{3.2.2}$$

$$= -i\frac{\sqrt{2}}{2}q_0^{-\frac{1}{16}}\left(1+O\left(q_0^{\frac{1}{2}}\right)\right). \tag{3.2.3}$$

On the other hand, we write $\mu\left(\frac{1}{2},\frac{1}{2};2\tau\right) = \frac{A_1\left(\frac{1}{2},\frac{1}{2};2\tau\right)}{\vartheta\left(\frac{1}{2};2\tau\right)}$ and use the transformation formula in Eq. (2.1.12) to obtain

$$\mu\left(\frac{1}{2},\frac{1}{2};2\tau\right) = \frac{A_1\left(\frac{1}{2},\frac{1}{2};2\tau\right)}{\vartheta\left(\frac{1}{2};2\tau\right)}$$

$$=\frac{1}{2\tau\vartheta\left(\frac{1}{2};2\tau\right)}e^{\frac{\pi i\left(-\frac{1}{4}\right)}{2\tau}}A_1\left(\frac{1}{4\tau},\frac{1}{4\tau};-\frac{1}{2\tau}\right)+\frac{1}{2i}h(0;2\tau).$$

Lemmas 2.24 and 2.21 tell us that as $\tau\to 0$

$$\vartheta\left(\frac{1}{2};2\tau\right) \ll |\tau|^{-\frac{1}{2}}, \tag{3.2.4}$$

$$\lim_{\tau\to 0} h(0;2\tau) = 1. \tag{3.2.5}$$

Writing $A_1\left(\frac{1}{4\tau},\frac{1}{4\tau};-\frac{1}{2\tau}\right)$ as a unilateral sum by swapping $n<0$ for $-n$, we have

$$A_1\left(\frac{1}{4\tau},\frac{1}{4\tau};-\frac{1}{2\tau}\right) = q_0^{-\frac{1}{8}}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q_0^{\frac{n^2}{4}}}{1-q_0^{\frac{n}{2}-\frac{1}{4}}}$$

$$=q_0^{-\frac{1}{8}}\left(\frac{1}{1-q_0^{-\frac{1}{4}}}+\sum_{n>0}(-1)^n q_0^{\frac{n^2}{4}}\left(\frac{1}{1-q_0^{\frac{n}{2}-\frac{1}{4}}}+\frac{1}{1-q_0^{-\frac{n}{2}-\frac{1}{4}}}\right)\right)$$

$$\ll q_0^{\frac{1}{8}}.$$

Combining this with Eqs. (3.2.4) and (3.2.5) gives

$$\mu\left(\frac{1}{2},\frac{1}{2};2\tau\right) = \frac{1}{2i}+O\left(|\tau|^{-\frac{1}{2}}q_0^{\frac{3}{16}}\right). \tag{3.2.6}$$

Combining Eqs. (3.2.2) and (3.2.6), and then multiplying by -1 gives the estimate for $T(1;q)$ as $\tau \to 0$,

$$T(1;q) \sim \frac{\sqrt{2}}{4} q_0^{-\frac{1}{16}}. \tag{3.2.7}$$

We now show that the estimates coming from $T_1(1;q)$ and $T_2(1;q)$ are negligible. For $T_1(1;q)$, we can use Eq. (3.2.6) with $\tau = \frac{\tau}{2}$ to obtain

$$T_1(1;q) \ll 1,$$

which shows $T_1(1;q)$ is negligible when compared with $T(1;q)$. We now turn to $T_2(1;q)$, which requires formally taking the limit and using the famous formula from Prop. 2.19,

$$\left[\frac{\partial \vartheta(z;\tau)}{\partial z}\right]_{z=0} = -2\pi\eta(\tau)^3.$$

Doing so, we find

$$\begin{aligned}
\lim_{z\to 0} T_2(w;q) &= iq^{\frac{11}{8}} \frac{\vartheta(4\tau;12\tau)^3\vartheta(\tau;2\tau)}{\vartheta(2\tau;6\tau)^3} \lim_{z\to 0} \frac{\vartheta(z;\tau)}{\vartheta(4z;4\tau)} \\
&= iq^{\frac{11}{8}} \frac{\vartheta(4\tau;12\tau)^3\vartheta(\tau;2\tau)}{\vartheta(2\tau;6\tau)^3} \lim_{z\to 0} \frac{\frac{d}{dz}\vartheta(z;\tau)}{4\frac{d}{dz}\vartheta(z;4\tau)} \\
&= \frac{iq^{\frac{11}{8}}}{4} \frac{\vartheta(4\tau;12\tau)^3}{\vartheta(2\tau;6\tau)^3} \frac{\eta(\tau)^3\vartheta(\tau;2\tau)}{\eta(4\tau)^3}.
\end{aligned}$$

Using Lem. 2.24, we find

$$T_2(1;q) \ll \frac{q_0^{\frac{1}{8}}}{\sqrt{|\tau|}}.$$

We now apply Thm. 2.29 to $\frac{1}{2}T(1;e^{-t})$ with $\lambda = \frac{\sqrt{2}}{8}$, $\alpha = 0$, and $A = \frac{\pi^2}{4}$ along with the fact that $\overline{p}(n) \sim \frac{1}{8n}e^{\pi\sqrt{n}}$. $\square$

3.2.2 Odd-balanced for fixed roots of unity

We now consider the growth of $V(w;q)$ where w is a generic root of unity, not equal to $\pm i$ or -1.[2] We break the study into four cases by splitting the interval $(0,1)$ into four pieces of length $\frac{1}{4}$.

Case: $0 < z < \frac{1}{4}$

With this restriction, we are able to prove the following.

Lemma 3.5. *Let* $0 < z < \frac{1}{4}$. *Then as* $n \to \infty$

$$V(w;q) \sim \frac{\sqrt{2}}{4} \frac{w^{-\frac{1}{2}}}{1+w^{-1}} q_0^{\frac{z^2}{2}-\frac{1}{16}} h(2z;2\tau).$$

Proof. We address the Appell sum first:

$$\begin{aligned}\mu\left(2z+\frac{1}{2},\frac{1}{2};2\tau\right) &= \frac{1}{\vartheta\left(\frac{1}{2};2\tau\right)} A\left(2z+\frac{1}{2},\frac{1}{2};2\tau\right)\\ &= \frac{h(2z;2\tau)}{2i} + \frac{e^{\frac{\pi i}{2\tau}\left(4z^2+\frac{1}{4}+2z\right)}}{2\tau\vartheta\left(\frac{1}{2};2\tau\right)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q_0^{\frac{n^2}{4}}}{1-q_0^{\frac{n}{2}-z-\frac{1}{4}}}.\end{aligned} \tag{3.2.8}$$

We showed in Lem. 2.21 that the Mordell integral is of $O(1)$ in the regime $\tau \to 0$. This leaves the sum in the parentheses to deal with. Splitting up the sum into positive and negative n, we find

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q_0^{\frac{n^2}{4}}}{1-q_0^{\frac{n}{2}-z-\frac{1}{4}}} &= \frac{1}{1-q_0^{-z-\frac{1}{4}}} + \sum_{n>0} (-1)^n q_0^{\frac{n^2}{4}} \left(\frac{1}{1-q_0^{\frac{n}{2}-z-\frac{1}{4}}} + \frac{1}{1-q_0^{-\frac{n}{2}-z-\frac{1}{4}}}\right)\\ &= -q_0^{\frac{1}{4}} + O\left(q_0^{\frac{1}{4}+z}\right).\end{aligned}$$

[2] We exclude the cases $\pm i$ to simplify the proof, which is allowed since $\zeta_c^j \neq \pm i$ for all j in Eq. (3.1.2) when c is odd.

Plugging this back into Eq. (3.2.8), we find

$$\begin{aligned}\mu\left(2z+\frac{1}{2},\frac{1}{2};2\tau\right) &= \frac{h(2z;2\tau)}{2i} - \frac{q_0^{\frac{3}{16}-z^2-\frac{z}{2}}}{2\tau\vartheta\left(\frac{1}{2};2\tau\right)}\left(1+O\left(q_0^z\right)\right) \\ &= \frac{h(2z;2\tau)}{2i} + \frac{\sqrt{2}}{2\tau}\sqrt{-i\tau}\,q_0^{\frac{3}{16}-z^2-\frac{z}{2}}\left(1+O(q_0^z)\right).\end{aligned} \tag{3.2.9}$$

Substituting in $z=\frac{z}{2}, \tau=\frac{\tau}{2}$, we also find

$$\begin{aligned}T_1(w;q) &= -iw^{-\frac{1}{2}}\mu\left(z+\frac{1}{2},\frac{1}{2};\tau\right) \\ &= -w^{-\frac{1}{2}}\frac{h(z;\tau)}{2} \\ &+ \frac{w^{-\frac{1}{2}}}{2\tau}\sqrt{-i\tau}\,q_0^{\frac{6}{16}-\frac{z^2}{2}-\frac{z}{2}}\left(1+O\left(q_0^{\frac{z}{2}}\right)\right) \\ &= O(1),\end{aligned} \tag{3.2.10}$$

where the last line follows since $\frac{6}{16}-\frac{z^2}{2}-\frac{z}{2}$ is positive on the interval $(0,\frac{1}{4})$. With regard to the ϑ-quotients, we can directly apply Lem. 2.24. We find that for $z<\frac{1}{2}$,

$$\frac{\vartheta(\frac{1}{2}+z;\tau)}{\vartheta(\tau;2\tau)} = \frac{-\frac{q_0^{\frac{z^2}{2}}}{\sqrt{-i\tau}}\left(1+O\left(q_0^{z+\frac{1}{2}}\right)\right)}{-2i\frac{q_0^{\frac{1}{16}}}{\sqrt{-2i\tau}}\left(1+O\left(q_0^{\frac{1}{2}}\right)\right)} \tag{3.2.11}$$

$$= \frac{\sqrt{2}}{2i}q_0^{\frac{z^2}{2}-\frac{1}{16}}\left(1+O\left(q_0^{\frac{1}{2}}\right)\right). \tag{3.2.12}$$

For $z < \frac{1}{2}$, we have

$$\begin{aligned}
&\frac{\vartheta(4\tau;12\tau)^3\vartheta(z;\tau)\vartheta(2z+\tau;2\tau)}{\vartheta(2\tau;6\tau)^3} \\
&= \left(\frac{q_0^{-\frac{1}{12\cdot 8}}}{\sqrt{2}}\left(1+O\left(q_0^{\frac{1}{12}}\right)\right)\right)^3 \frac{iw^{-1}}{\sqrt{2}(-i\tau)} \\
&\quad \times q_0^{\frac{3}{2}z^2-z+\frac{3}{16}}\left(1+O(q_0^z)\right) \\
&= -\frac{w^{-1}}{4\tau}q_0^{\frac{5}{32}+\frac{3}{2}z^2-z}\left(1+O\left(q_0^{\min\left(z,\frac{1}{12}\right)}\right)\right).
\end{aligned} \tag{3.2.13}$$

Finally for $z < \frac{1}{4}$, we have

$$\vartheta(4z;4\tau) = -\frac{q_0^{\frac{16z^2}{8}-\frac{4z}{8}+\frac{1}{32}}}{2\sqrt{-i\tau}}\left(1+O\left(q_0^z\right)\right). \tag{3.2.14}$$

Combining Eqs. (3.2.9) and (3.2.11) gives

$$\begin{aligned}
T(w;q) &= \frac{\sqrt{2}}{4}w^{-\frac{1}{2}}q_0^{\frac{z^2}{2}-\frac{1}{16}}h(2z;2\tau)\left(1+O\left(q_0^{\frac{1}{2}}\right)\right) \\
&\quad + \frac{w^{-\frac{1}{2}}}{2}\sqrt{-i\tau}q_0^{\frac{1}{8}-\frac{z^2}{2}-\frac{z}{2}}\left(1+O(q_0^z)\right).
\end{aligned} \tag{3.2.15}$$

Combining Eq. (3.2.13) and the inverse of Eq. (3.2.14) gives,

$$T_2(w;q) = \frac{w^{-\frac{3}{2}}}{2\sqrt{-i\tau}}q_0^{\frac{1}{8}-\frac{z^2}{2}-\frac{z}{2}}\left(1+O\left(q_0^{\min\left(z,\frac{1}{12}\right)}\right)\right). \tag{3.2.16}$$

We now study the following polynomials on the interval $\left(0,\frac{1}{4}\right)$, which correspond to the exponents of q_0 in Eqs. (3.2.15) and (3.2.16):

$$f(z) := \frac{z^2}{2} - \frac{1}{16} \text{ and } f_2(z) := \frac{1}{8} - \frac{z^2}{2} - \frac{z}{2}.$$

The values $f(z)$ are the smallest on the interval $0 < z < \frac{1}{4}$, since

$$-\frac{1}{16} < f(z) < -\frac{1}{32} \quad \text{and} \quad f_2(z) > -\frac{1}{32}.$$

Therefore T gives the primary estimate and T_1 and T_2 are error terms. $\square$

Case: $\frac{1}{4} < z < \frac{1}{2}$

We see in this interval that T_2 will provide the main term. More precisely, we prove the following.

Lemma 3.6. *Let* $\frac{1}{4} < z < \frac{1}{2}$. *Then as* $n \to \infty$

$$V(w;q) \sim -\frac{1}{2\sqrt{-i\tau}} \frac{w^{-\frac{1}{2}}}{(1+w^{-1})} q_0^{-\frac{1}{8}-\frac{z^2}{2}+\frac{z}{2}} \left(1+O\left(q_0^{\min\left(z-\frac{1}{4},\frac{1}{12}\right)}\right)\right).$$

Proof. We only need to modify the steps taken to obtain Eqs. (3.2.9) and (3.2.14). Let $r := z - \frac{1}{4}$. Starting with the Appell sum, we have

$$\begin{aligned}
\mu\left(2z+\frac{1}{2},\frac{1}{2};2\tau\right) &= \mu\left(2\left(\frac{1}{4}+r\right)+\frac{1}{2},\frac{1}{2};2\tau\right) \\
&= -\mu\left(2r,\frac{1}{2};2\tau\right) \\
&= -\frac{A\left(2r,\frac{1}{2};2\tau\right)}{\vartheta\left(\frac{1}{2};2\tau\right)} \\
&= \frac{h\left(2r-\frac{1}{2};2\tau\right)}{2i} + \frac{e^{\frac{\pi i}{2\tau}(4r^2-2r)}}{2\tau\vartheta\left(\frac{1}{2};2\tau\right)} A\left(\frac{r}{\tau},\frac{1}{4\tau};-\frac{1}{2\tau}\right) \\
&= -\frac{h\left(2r-\frac{1}{2};2\tau\right)}{2i} - \frac{e^{\frac{2\pi i}{\tau}r^2}}{2\tau\vartheta\left(\frac{1}{2};2\tau\right)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q_0^{\frac{n^2}{4}}}{1-q_0^{-r+\frac{n}{2}}}.
\end{aligned}$$

We now split the sum as before into positive and negative n and expand the denominators in terms of geometric series, noting that $\frac{1}{1-q_0^{-a}} = -q_0^a\left(1+O(q_0^a)\right)$ for $a > 0$. Doing so, we find

$$\begin{aligned}
\mu\left(2z+\frac{1}{2},\frac{1}{2};2\tau\right) &= -\frac{h\left(2r-\frac{1}{2};2\tau\right)}{2i} - \frac{e^{2\frac{\pi i}{\tau}r^2}}{2\tau\vartheta\left(\frac{1}{2};2\tau\right)}\left(-q_0^r+O\left(q_0^{2r}\right)\right) \\
&= -\frac{h\left(2r-\frac{1}{2};2\tau\right)}{2i} + \frac{q_0^{r-r^2}}{2\tau\vartheta\left(\frac{1}{2};2\tau\right)}(1+O(q_0^r)).
\end{aligned}$$

Lem. 2.24 tells us that $\vartheta\left(\frac{1}{2}; 2\tau\right) \ll \frac{1}{\sqrt{|\tau|}}$. Combining this with the fact that $r - r^2 > 0$ since $0 < r < \frac{1}{4}$, we have by Prop. 2.24

$$\frac{q_0^{r-r^2}}{2\tau\vartheta\left(\frac{1}{2}; 2\tau\right)}\left(1 + O(q_0^r)\right) \ll \frac{q_0^{r-r^2}}{\sqrt{\tau}} \ll q_0^{\varepsilon}$$

for some $\varepsilon > 0$. By Lem. 2.21 we have that $0 \neq h\left(2r - \frac{1}{2}; 2\tau\right) \ll 1$. Therefore,

$$\mu\left(2z + \frac{1}{2}, \frac{1}{2}; 2\tau\right) = -\frac{h\left(2r - \frac{1}{2}; 2\tau\right)}{2i}\left(1 + O\left(\frac{q_0^{r-r^2}}{\sqrt{|\tau|}}\right)\right). \tag{3.2.17}$$

Combining Eq. (3.2.17) with Eq. (3.2.11) gives

$$T(w; q) = -w^{-\frac{1}{2}}\frac{\sqrt{2}}{4}h\left(2r - \tfrac{1}{2}; 2\tau\right)q_0^{\frac{z^2}{2} - \frac{1}{16}}\left(1 + O\left(\frac{q_0^{r-r^2}}{\sqrt{|\tau|}}\right) + O\left(q_0^{\frac{1}{2}}\right)\right). \tag{3.2.18}$$

Now we turn to the analog of Eq. (3.2.14). We have

$$\vartheta\left(4z; 4\tau\right) = \vartheta\left(1 + 4r; 4\tau\right) = -\vartheta\left(4r; 4\tau\right) \tag{3.2.19}$$

$$= \frac{q_0^{2r^2 - \frac{r}{2} + \frac{1}{32}}}{2\sqrt{-i\tau}}\left(1 + O\left(q_0^r\right)\right). \tag{3.2.20}$$

Combining Eqs. (3.2.13) and (3.2.19) and changing variables back to z gives

$$T_2(w; q) = -\frac{w^{-\frac{1}{2}}}{2\sqrt{-i\tau}}q_0^{-\frac{1}{8} - \frac{z^2}{2} + \frac{z}{2}}\left(1 + O\left(q_0^{\min\left(z - \frac{1}{4}, \frac{1}{12}\right)}\right)\right).$$

We now recycle the same estimate from Eq. (3.2.10), and thus we only need to consider the polynomials

$$f(z) := \frac{z^2}{2} - \frac{1}{16} \quad \text{and} \quad f_2(z) := -\frac{1}{8} - \frac{z^2}{2} + \frac{z}{2}.$$

We see that for $\frac{1}{4} < z < \frac{1}{2}$ that $f_2(z)$ is the smallest and dividing by $1 + w^{-1}$ proves the claim. $\square$

Case: $\frac{1}{2} < z < \frac{3}{4}$

Since the Appell sum $\mu\left(2z+\tau, \frac{1}{2}; 2\tau\right) \mapsto -\mu\left(2z+\tau, \frac{1}{2}; 2\tau\right)$ under the shift $z \mapsto z + \frac{1}{2}$, we can recycle many of the estimates from the previous case to find the following.

Lemma 3.7. *Let $\frac{1}{2} < z < \frac{3}{4}$. As $n \to \infty$*

$$V(w;q) \sim -\frac{w^{-\frac{1}{2}}}{1+w^{-1}} \frac{1}{\sqrt{-i\tau}} q_0^{-\frac{z^2}{2}+\frac{z}{2}-\frac{1}{8}}.$$

Proof. We let $z := \frac{1}{2} + r$. Using identical arguments as in Lemmas 3.5 and 3.6, we have

$$\begin{aligned} T_1(w;q) &= -iq^{\frac{1}{8}} w^{-\frac{1}{2}} \mu\left(r+1, \frac{1}{2}; \tau\right) = iq^{\frac{1}{8}} w^{-\frac{1}{2}} \mu\left(r, \frac{1}{2}; \tau\right) \\ &= iq^{\frac{1}{8}} w^{-\frac{1}{2}} \left(\frac{h\left(r-\frac{1}{2};\tau\right)}{2i} + \frac{e^{\frac{\pi i}{\tau}\left(r^2-r\right)}}{\tau\vartheta\left(\frac{1}{2};\tau\right)} A\left(\frac{r}{\tau}, \frac{1}{2\tau}; -\frac{1}{\tau}\right)\right) \\ &= \frac{1}{2} w^{-\frac{1}{2}} h\left(r-\frac{1}{2};\tau\right)\left(1+O\left(\frac{q_0^{r-\frac{r^2}{2}}}{\sqrt{|\tau|}}\right)\right) = O(1). \end{aligned}$$

We now look at T:

$$\begin{aligned} T(w,q) &= -w^{-\frac{1}{2}} \frac{\vartheta(1+r;\tau)}{\vartheta(\tau;2\tau)} \mu\left(2r+1+\frac{1}{2}, \frac{1}{2}; 2\tau\right) \\ &= -w^{-\frac{1}{2}} \frac{\vartheta(r;\tau)}{\vartheta(\tau;2\tau)} \mu\left(2r+\frac{1}{2}, \frac{1}{2}; 2\tau\right) \\ &= -w^{-\frac{1}{2}} \frac{\vartheta(r;\tau)}{\vartheta(\tau;2\tau)} \frac{h(2r;2\tau)}{2i} \left(1+O\left(\sqrt{|\tau|} q_0^{\frac{3}{16}-r^2+\frac{r}{2}}\right)\right) \\ &= -w^{-\frac{1}{2}} \frac{\sqrt{2}}{4} q_0^{\frac{r^2}{2}-\frac{r}{2}+\frac{1}{16}} h(2r;2\tau)\left(1+O\left(q_0^{\min\left(r, \frac{3}{16}-r^2-\frac{r}{2}\right)}\right)\right), \end{aligned}$$

where we used Lem. 2.24 in the last step. We finally need T_2:

$$T_2(w;q) = iw^{-\frac{1}{2}} \frac{\vartheta(4\tau;12\tau)^3}{\vartheta(2\tau;6\tau)^3} \frac{\vartheta\left(r+\frac{1}{2};\tau\right)\vartheta\left(2r+1+\tau;2\tau\right)}{\vartheta(4r+2;4\tau)}$$

$$= -iw^{-\frac{1}{2}} \frac{\vartheta(4\tau;12\tau)^3}{\vartheta(2\tau;6\tau)^3} \frac{\vartheta\left(r+\frac{1}{2};\tau\right)\vartheta\left(2r+\tau;2\tau\right)}{\vartheta(4r;4\tau)}.$$

We can recycle all of the estimates for these functions from Lem. 3.5, with the exception of the function $\vartheta\left(r+\frac{1}{2};\tau\right)$, which we can obtain from Lem. 2.24. This gives

$$\begin{aligned} T_2(w;q) =& -iw^{-\frac{1}{2}} \left(\frac{q_0^{-\frac{1}{12\cdot 8}}}{\sqrt{2}} \left(1+O\left(q_0^{\frac{1}{12}}\right)\right)\right)^3 \left(-\frac{q_0^{\frac{r^2}{2}}}{\sqrt{-i\tau}} \left(1+O\left(q_0^{r+\frac{1}{2}}\right)\right)\right) \\ &\times \left(-2\sqrt{-i\tau}q_0^{-2r^2+\frac{r}{2}-\frac{1}{32}} \left(1+O\left(q_0^r\right)\right)\right) \vartheta(2r+\tau;2\tau) \\ =& -\frac{w^{-\frac{1}{2}}e^{-2\pi i r}}{2\sqrt{-i\tau}} q_0^{-\frac{r^2}{2}} \left(1+O\left(q_0^{\min\left(\frac{1}{12},r\right)}\right)\right). \end{aligned}$$

We now compare the polynomials for $0<r<\frac{1}{4}$,

$$f(r) := \frac{r^2}{2} - \frac{r}{2} + \frac{1}{16} \quad \text{and} \quad f_2(r) := -\frac{r^2}{2}.$$

The values $f_2(r)$ are smaller on this interval, which shows that our main estimate comes from T_2, and the contributions from T_1 and T are error terms. Changing variables back to z yields the result. $\square$

Case: $\frac{3}{4} < z < 1$.

As in the previous case, we can recycle some estimates to prove the following.

Lemma 3.8. *Let $\frac{3}{4}<z<1$. Then as $n\to\infty$,*

$$V(w;q) = -\frac{w^{-\frac{1}{2}}}{1+w^{-1}} \frac{\sqrt{2}}{4} q_0^{\frac{z^2}{2}-z+\frac{7}{16}} h(2z-2;2\tau)\left(1+O\left(|\tau|^{-\frac{1}{2}} q_0^{\left(z-\frac{3}{4}\right)-\left(z-\frac{3}{4}\right)^2}\right)\right).$$

Proof. Let $r_0 := 1-z$. We look first at T_1 with this change of variables:

$$T_1(w;q) = -iw^{-\frac{1}{2}} \mu\left(1-r_0+\tfrac{1}{2},\tfrac{1}{2};\tau\right) = iw^{-\frac{1}{2}} \mu\left(-r_0+\tfrac{1}{2},\tfrac{1}{2};\tau\right)$$

$$
\begin{aligned}
&= iw^{-\frac{1}{2}} \frac{A_1\left(-r_0+\frac{1}{2}, \frac{1}{2}; \tau\right)}{\vartheta\left(\frac{1}{2}; \tau\right)} \\
&= \frac{h(-r_0;\tau)}{2i} + \frac{e^{\frac{\pi i}{\tau}\left(r_0^2-r_0\right)}}{2\tau\vartheta\left(\frac{1}{2};\tau\right)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q_0^{\frac{n^2}{2}}}{1-q_0^{n+\frac{r_0}{2}-\frac{1}{4}}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
T_1(w;q) = &\frac{h(-r_0;\tau)}{2i} \\
&+\frac{e^{\frac{\pi i}{\tau}\left(r_0^2-r_0\right)}}{2\tau\vartheta\left(\frac{1}{2};\tau\right)}\left(\frac{1}{1-q_0^{\frac{r_0}{2}-\frac{1}{4}}}+\sum_{n>0}(-1)^n q_0^{\frac{n^2}{2}}\left(\frac{1}{1-q_0^{n+\frac{r_0}{2}-\frac{1}{4}}}+\frac{1}{1-q_0^{-n+\frac{r_0}{2}-\frac{1}{4}}}\right)\right) \\
&= \frac{h(-r_0;\tau)}{2i} + O\left(q_0^{\frac{1}{4}-\frac{r_0^2}{2}}\right) = O(1).
\end{aligned}
$$

Next we look closer at T. The ϑ-functions are standard, and come directly from from Lem. 2.24. The Appell sum comes from Eq. (3.2.17), which we see by making the substitution $z = r + \frac{3}{4}$, so that $0 < r < \frac{1}{4}$. This gives

$$
\begin{aligned}
T(w;q) &= -w^{-\frac{1}{2}} \frac{\vartheta\left(\frac{1}{2}+1-r_0;\tau\right)}{\vartheta(\tau;2\tau)} \mu\left(2r+2, \frac{1}{2}; 2\tau\right) \\
&= w^{-\frac{1}{2}} \frac{\vartheta\left(\frac{1}{2}-r_0;\tau\right)}{\vartheta(\tau;2\tau)} \mu\left(2r, \frac{1}{2}; 2\tau\right) \\
&= w^{-\frac{1}{2}} \left(-i\frac{\sqrt{2}}{2} q_0^{\frac{z^2}{2}-z+\frac{7}{16}}\left(1+O\left(q_0^{z-\frac{1}{2}}\right)\right)\right) \\
&\quad \times \left(\frac{h(2z-2;2\tau)}{2i}\left(1+O\left(|\tau|^{-\frac{1}{2}} q_0^{\frac{3}{4}\left(z-\frac{3}{4}\right)-\left(z-\frac{3}{4}\right)^2}\right)\right)\right) \\
&= -w^{-\frac{1}{2}} \frac{\sqrt{2}}{4} q_0^{\frac{z^2}{2}-z+\frac{7}{16}} h(2z-2;2\tau)\left(1+O\left(|\tau|^{-\frac{1}{2}} q_0^{\frac{3}{4}\left(z-\frac{3}{4}\right)-\left(z-\frac{3}{4}\right)^2}\right)\right).
\end{aligned}
$$

We finally study T_2 using the variable changes $r_1 := z - \frac{1}{2}$ and the previous change $r_0 := 1 - z$. We apply Lem. 2.24 which gives

$$\begin{aligned} T_2(w;q) &= iw^{-\frac{1}{2}} \frac{\vartheta(4\tau;12\tau)^3}{\vartheta(2\tau;6\tau)^3} \frac{\vartheta(z;\tau)\,\vartheta(2r_1+1+\tau;2\tau)}{\vartheta(4r_1+2;4\tau)} \\ &= -iw^{-\frac{1}{2}} \frac{\vartheta(4\tau;12\tau)^3}{\vartheta(2\tau;6\tau)^3} \frac{\vartheta(z;\tau)\,\vartheta(2r_1+\tau;2\tau)}{\vartheta(4r_1;4\tau)} \\ &= \frac{iw}{2\sqrt{-i\tau}} q_0^{-\frac{z^2}{2}+\frac{3z}{2}-\frac{7}{8}} \left(1 + O\left(q_0^{1-z}\right)\right). \end{aligned}$$

As before, we consider the polynomials

$$f(z) := \frac{z^2}{2} - z + \frac{7}{16} \ \text{ and } \ f_2(z) := -\frac{z^2}{2} + \frac{3z}{2} - \frac{7}{8}.$$

The function f is smaller on the interval $\frac{3}{4} < z < 1$, which tells us that T gives us the main estimate and T_1 and T_2 amount to error terms, which completes the proof. $\square$

3.3 Proofs of Thm. 3.1 and Cor. 3.3

The proof of Thm. 3.1 amounts to showing that Lem. 3.4 provides the dominant term in Eq. (3.1.2), since the $v(a,c;n)$ are weakly increasing in n.

Proof of Thm. 3.1: We can prove the result by recalling in the proof of Lem. 3.4, that the main term came from $T(1;q)$, and in particular, Eq. (3.2.7). This amounts to showing that there exists $\beta > 0$ such that each of the estimates in Lemmas 3.5-3.8 is bounded by $q_0^{\beta-\frac{1}{16}}$. This is not difficult to see by considering the polynomials of the main terms in each of these lemmas on their corresponding intervals for z:

$$\tfrac{z^2}{2} - \tfrac{1}{16} \ \left(\text{for } 0<z<\tfrac{1}{4}\right), \qquad -\tfrac{z^2}{2} + \tfrac{z}{2} - \tfrac{1}{8} \ \left(\text{for } \tfrac{1}{4}<z<\tfrac{1}{2} \text{ and } \tfrac{1}{2}<z<\tfrac{3}{4}\right),$$
$$\tfrac{z^2}{2} - z + \tfrac{7}{16} \ \left(\text{for } \tfrac{3}{4}<z<1\right).$$

All of these polynomials are strictly bounded below by the critical value $-\frac{1}{16}$, which completes the proof. $\square$

We now prove Cor. 3.3 which is a simple application of our main theorem.

Proof of Cor. 3.3: Thm. 3.1 and Lem. 3.4 imply that for sufficiently large n

$$\begin{aligned}\frac{v(a,c;n+1)v(a,c;n-1)}{v(a,c,2n)} &= \frac{v(n-1)v(n+1)}{c\cdot v(2n)} \\ &= \frac{(2n)^{\frac{3}{4}}e^{\pi\left(\sqrt{n+1}+\sqrt{n-1}-\sqrt{2n}\right)}}{16c(n^2-1)^{\frac{3}{4}}} \geq 1,\end{aligned}$$

where the last step follows from the fact that for positive integers $\sqrt{a+b} < \sqrt{a}+\sqrt{b}$. $\square$

Chapter 4

Asymptotics for Bailey-type mock theta functions

This is a corrected version of the work published by the author in the *Ramanujan Journal* under the same name [42]. We add some extra commentary in Sections 4.5.2-4.5.4. We hope that much of the new commentary provides the reader with a larger framework to hopefully explore more functions of this type. Additionally, much more precise estimates for the minor arcs are discovered which may allow one to find exact formulas in the future for the coefficients studied here. This will be stated as an open problem at the end of this thesis.

4.1 What are Bailey-type mock theta functions?

Our goal in this chapter is to study the asymptotic behavior of some examples of mock theta functions which are derived from Bailey pairs. Since the work of Zwegers in 2002 [73], it is known that normalized Appell sums are prototypes for mock theta functions. Based on the transformations that Appell sums exhibit under the modular group $\mathrm{SL}_2(\mathbb{Z})$ described in Prop. 2.19, the addition of a weakly holomorphic modular form will preserve the mock modularity of the function in question. That is, a mock theta function plus a weakly holomorphic modular form is still a mock theta function. However, many mock theta functions only differ by a modular form and still their Fourier coefficients are quite different. Thus, asymptotic formulas for the coefficients of mock theta functions are important, especially when a combinatorial interpretation is available. As was shown by Lovejoy and others in works such as [52, 53], there is a combinatorial machinery that yields large families of mock theta functions, many of them only differing by the addition of a weakly holomorphic modular form. This is based on the theory of Bailey pairs and the Bailey chain, which we briefly discuss now. Let

$$\alpha_n(q) =: \alpha_n \quad \text{and} \quad \beta_n(q) =: \beta_n$$

be two sequences of q-series. The tuple (α_n, β_n) is referred to as a *Bailey pair with respect to* $a \in \mathbb{C}$ (assuming a causes no poles in what follows) if

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}. \tag{4.1.1}$$

The fact that Bailey pairs and mock theta functions are related is not immediately obvious, and it was not until Andrews showed in the 1980's that Eq. (4.1.1) can be iterated to obtain an infinite family of Bailey pairs that a true connection was found [3, 4]. This is the content of Bailey's lemma [3, 4, 11, 12]. Bailey's lemma leads to families of sums, known as higher level Appell sums of the kind introduced in Eq. (2.1.11), which are not necessarily mock theta functions, but MMMFs [20, 53]. Occasionally, certain pairs lead to normal mock theta functions via Bailey's lemma, and we call the resulting functions *Bailey-type mock theta functions.*

The study of Bailey-type mock theta functions became more interesting with a key result by Hickerson and Mortenson [46], which gave an explicit decomposition of indefinite theta functions in terms of Appell sums and ϑ-functions. This result was used by authors in works such as [44, 52, 53] to write families of Bailey-type mock theta functions in terms of classical mock theta functions. For example, Lovejoy and Osburn in [53] derived a Bailey-type mock theta function, $R_{1,4}(q)$, and used the decomposition of [46] to find the formula[1]

$$R_{1,4}(q) = -\phi\left(q^4\right) + M_1(q), \tag{4.1.2}$$

where ϕ is the 10th order classical mock theta function given by

$$\phi(q) := \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q;q^2)_{n+1}}$$

and $M_1(q)$ is a weakly holomorphic modular form. Understanding how the coefficients of certain Bailey-type mock theta functions grow is an interesting question, which was proposed by Lovejoy and Osburn in [53], and which we will begin to answer in this chapter. To the best of our knowledge, no works have investigated the growth of Bailey-type mock theta functions in depth. Doing so here for two example functions, we hope to lay the groundwork for future and more advanced studies of the asymptotic properties of Bailey-type mock theta functions. Let $a(n)$ denote the coefficients of $R_{3,3}(q)$ and $b(n)$ the coefficients of $R_{1,3}(q)$, which are two Bailey-type mock theta functions defined in Def. 4.2. We will show the following, which was stated as Result B at the beginning of this thesis.

Theorem 4.1. *The following estimates hold as $n \to \infty$:*

$$a(n) \sim (-1)^n \frac{\sqrt{6}}{12\sqrt{n}} e^{\pi\sqrt{\frac{n}{12}}}, \qquad b(n) \sim \left(\frac{1}{2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{12}\right)} + 1\right) \frac{e^{\pi\sqrt{\frac{n}{6}}}}{\sqrt{24n}}.$$

The following table shows the ratio between the estimated values in Thm. 4.1 and the actual values.

[1] The authors of [53] used the notation $R_1^{(4)}(q)$. However, we do not use this notation in order to avoid confusion with the derivative.

n	$a(n)$	$b(n)$
100	0.96315	0.98067
500	0.98249	0.99081
1000	0.98740	0.99343

To obtain asymptotic estimates like the ones we give in our main Thm. 4.1, it is often useful to use the Wright circle method [71] which allows one to consider only a finite number of major and minor arcs. In most cases one can get away with just studying one minor arc in detail, but sometimes more are needed, as will be the case here. Wright's technique has been used by several authors in recent years [19, 27, 36, 40] to deal with combinatorial generating functions. In our case, we will need to overcome some difficulties specific to the structure of Bailey-pairs and the related work of Hickerson and Mortenson [46]. As we will see, cancellation will occur when trying to compute our main asymptotic terms on the major arc. How far back in the asymptotic expansions one needs to go to extract the main term boils down to a careful analysis of some trigonometric identities. We begin our chapter by reviewing the important work of Lovejoy and Osburn [53].

4.2 The work of Lovejoy and Osburn

For $k \geq 3$ Lovejoy and Osburn showed that the following family of functions are mock theta functions.

Definition 4.2 (see Thm 1.2, [53])**.** Let $k \geq 3$ and $n_1, ..., n_k$ be integers such that $1 \leq n_1 \leq ... \leq n_k$. Define

$$B_k(n_k, n_{k-1}, ..., n_1; q) := (-1)^{n_1} (-q)_{n_{k-1}} q^{\binom{n_{k-1}+1}{2}} \times \frac{\prod_{j=2}^{k-1} q^{2^{j-2} n_{k-j}} \left(-q^{2^{j-2}}; q^{2^{j-2}}\right)_{2n_{k-j}}}{\prod_{j=1}^{k} \left(q^{2^{j-1}}; q^{2^{j-1}}\right)_{n_{k-j+1}-n_{k-j}}},$$

with $n_0 := 0$. Then,

$$R_{1,k}(q) := \sum_{0\leq n_1\leq n_{k-1}\leq \ldots\leq n_k} q^{\binom{n_k+1}{2}} B_k(n_k, ..., n_1; q),$$

$$R_{3,k}(q) := \textstyle\sum_{0\leq n_1\leq n_{k-1}\leq\cdots\leq n_k} \frac{(-1)^{n_k} q^{n_k^2+2n_k}(q;q^2)_{n_k}}{(-q^2;q^2)_{n_k}} B_k\left(n_k,...,n_1;q^2\right).$$

The authors of [53] showed that

$$R_{1,3}(q) = \nu(-q), \tag{4.2.1}$$

where

$$\nu(q) := \sum_{n\geq 0} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}$$

is a classical third order mock theta function.

Our second definition comes from Hickerson and Mortenson's seminal work on decompositions of mock theta functions [46], and uses the standard combinatorial notation for the Jacobi triple product

$$j(w,q) := (w)_\infty \left(qw^{-1}\right)_\infty (q)_\infty,$$

where w is a non-zero complex number. Hickerson and Mortenson's results involving Def. 4.3 incorporate a larger class of elliptic variables than the modular ϑ-functions, which is why the j notation is used here, which also avoids having to state results in terms of logarithms. However, when w is an integral or half integral power of q, we will always write j in terms of a ϑ-function, as discussed in Sect. 4.2, via the transformations

$$\vartheta(a\tau; b\tau) = -iq^{\frac{b}{8}} q^{-\frac{a}{2}} j\left(q^a, q^b\right),$$
$$\vartheta\left(a\tau + \frac{1}{2}; b\tau\right) = -q^{\frac{b}{8}} q^{-\frac{a}{2}} j\left(-q^a, q^b\right).$$

The following auxiliary function from [46] appears frequently as the weakly holomorphic modular form portion of the mock theta functions produced by the Bailey chain.

Definition 4.3 (see Sect. 2, [53] and Thm. 1.3, [46]). Let w and z be complex numbers so that they do not cause poles in the quotients that follow. Then for positive integers n and ℓ, we define the function $\theta_{n,\ell}(x,y,q)$ by

$$\theta_{n,\ell}(w,z,q) := \frac{j^3\left(q^{\ell^2(2n+\ell)}, q^{3\ell^2(2n+\ell)}\right)}{j\left(-1, q^{n\ell(2n+\ell)}\right)}$$
$$\times \sum_{r^*=0}^{\ell-1}\sum_{s^*=0}^{\ell-1} q^{n\binom{r-\frac{(n-1)}{2}}{2}+(n+\ell)\left(r-\frac{(n-1)}{2}\right)\left(s+\frac{(n+1)}{2}+n\binom{s+\frac{(n+1)}{2}}{2}\right)}$$
$$\times (-w)^{r-\frac{n-1}{2}}(-z)^{s+\frac{(n+1)}{2}}$$
$$\times \frac{j\left(-q^{\ell n(s-r)}\frac{w^n}{z^n}, q^{n\ell^2}\right) j\left(q^{\ell(2n+\ell)(r+s)+\ell(n+\ell)}w^\ell z^\ell, q^{\ell^2(2n+\ell)}\right)}{j\left(q^{\ell r(2n+\ell)+\frac{\ell(n+\ell)}{2}}\frac{(-z)^{n+\ell}}{(-w)^n}, q^{\ell^2(2n+\ell)}\right) j\left(q^{\ell s(2n+\ell)+\frac{\ell(n+\ell)}{2}}\frac{(-z)^{n+\ell}}{(-w)^n}, q^{\ell^2(2n+\ell)}\right)},$$

where $\binom{b}{c}$ is the standard binomial coefficient and

$$r := r^* + \left\{\frac{n-1}{2}\right\} \quad \text{and} \quad s := s^* + \left\{\frac{n-1}{2}\right\},$$

with $\{a\}$ denoting the fractional part of the number a.

From the above definition, Lovejoy and Osburn showed the following.

Theorem 4.4 (see pg. 13, [53]). *For $k \geq 3$ the function $R_{3,k}(q)$ is a mock theta function and satisfies the formula*

$$R_{3,k}(q) = 2iq^{-2^{k-3}}\mu\left(2^{k-2}\tau + \frac{1}{2}, \frac{1}{2}; \left(2^{2k-2}+2^k\right)\tau\right)$$
$$- 2q^{\frac{1}{8}}\frac{\theta_{1,4}\left(q^{2^{k-2}+1}, -q^{2^{k-2}+1}, q\right)}{\vartheta\left(\frac{1}{2};\tau\right)}.$$

We now take a closer look at the case $k=3$ from the above result.

Example 4.5 (The function $R_{3,3}$)**.** The Fourier expansion for $R_{3,3}(q)$ takes the shape

$$\begin{aligned}R_{3,3}(q) = & 1-q^3+q^4-q^5+q^6+q^8-q^9+q^{10}-2q^{11}+2q^{12}-2q^{13}\\ & +q^{14}-2q^{15}+2q^{16}-q^{17}+3q^{18}-3q^{19}+3q^{20}-4q^{21}\\ & +3q^{22}-2q^{23}+4q^{24}-4q^{25}+4q^{26}-6q^{27}+5q^{28}-6q^{29}\\ & +6q^{30}-5q^{31}+6q^{32}-6q^{33}+7q^{34}-9q^{35}+9q^{36}-9q^{37}\\ & +9q^{38}-9q^{39}+11q^{40}-10q^{41}+12q^{42}-14q^{43}+13q^{44}\\ & -16q^{45}+15q^{46}-14q^{47}+17q^{48}-16q^{49}+O\left(q^{50}\right).\end{aligned}$$

One can see the alternating sign changes and our main theorem shows that this behavior holds in the limit $n \to \infty$. We can explicitly get the function $R_{3,3}(q)$ in a form that is suitable for applying the circle method:

$$\begin{aligned}R_{3,3}(q) = & \, 2iq^{-1}\mu\left(2\tau+\frac{1}{2},\frac{1}{2};24\tau\right)+\frac{2iq^{\frac{417}{8}}}{\vartheta\left(\frac{1}{2};\tau\right)}\frac{\vartheta^3(96\tau;288\tau)}{\vartheta\left(\frac{1}{2};24\tau\right)}\\ & \times\left(\sum_{r,s=0}^{3}\left((-1)^r q^{Q(r,s)}\cdot\frac{\vartheta(4(s-r)\tau;16\tau)\ \vartheta(\{24(r+s)+44\}\tau;96\tau)}{\vartheta\left(\frac{1}{2}+\{24r+22\}\tau;96\tau\right)\ \vartheta\left(\frac{1}{2}+\{24s+22\}\tau;96\tau\right)}\right)\right),\end{aligned}$$

where we defined

$$Q(r,s) := \frac{r(r-1)}{2}+\frac{s(s+1)}{2}+5s+6r+5rs$$

above. This gives

$$R_{3,3}(q) =: 2iq^{-1}\mu\left(2\tau+\frac{1}{2},\frac{1}{2};24\tau\right)+T(\tau). \tag{4.2.2}$$

The study of $T(\tau)$ will be the main focus for the rest of the chapter.

Lovejoy and Osburn also proved a formula for a classical mock theta function via the Bailey chain.

Example 4.6 (The function $R_{1,3}$)**.** The authors of [53] found the formula

$$R_{1,3}(q) = \nu(-q),$$

which has the Fourier expansion

$$\begin{aligned}R_{1,3}(q) = & 1+q+2q^2+2q^3+2q^4+3q^5+4q^6+4q^7+5q^8+6q^9+6q^{10}+8q^{11}\\ & +10q^{12}+10q^{13}+12q^{14}+14q^{15}+15q^{16}+18q^{17}+20q^{18}22q^{19}+26q^{20}\\ & +29q^{21}+32q^{22}+36q^{23}+40q^{24}+44q^{25}+50q^{26}+56q^{27}+60q^{28}+68q^{29}\\ & +76q^{30}+82q^{31}+92q^{32}+101q^{33}+110q^{34}+122q^{35}+134q^{36}+146q^{37}\\ & +160q^{38}+176q^{39}+191q^{40}+210q^{41}+230q^{42}+248q^{43}+272q^{44}\\ & +296q^{45}+320q^{46}+350q^{47}+380q^{48}+410q^{49}+O\left(q^{50}\right).\end{aligned}$$

It is well-known that the following decomposition holds with $\tau \mapsto \tau+\frac{1}{2}$ (see A.2, [20]):

$$\begin{aligned}R_{1,3}(q) = & -2iq^{-\frac{1}{2}}\mu(5\tau, 3\tau; 12\tau)\\ & + e^{-\frac{\pi i}{12}}q^{-\frac{1}{3}}\frac{\eta\left(\tau+\frac{1}{2}\right)\eta\left(3\tau+\frac{1}{2}\right)\eta(12\tau)}{\eta(2\tau)\eta(6\tau)}.\end{aligned}$$

4.3 Higher order estimates

In this chapter, the main terms in the estimates of Prop. 2.24 alone will not be sufficient in proving the growth of the $a(n)$. We will find that many terms cancel in the main estimate for $T(\tau)$ near $\tau = \frac{1}{2}$. The transformation laws of the ϑ-functions allow one to keep an arbitrary amount of secondary terms, which we use to prove the following.

Lemma 4.7. *Let α be as in Lem. 2.24. Then as $\tau \to 0$ within a fixed angular region,*

$$\vartheta(\alpha\tau;\tau) = -2i\frac{\sin(\pi\alpha)q_0^{\frac{1}{8}}}{\sqrt{-i\tau}}\left(1-a_1q_0+a_3q_0^3+O\left(q_0^4\right)\right), \tag{4.3.1}$$

$$\vartheta\left(\frac{1}{2}+\alpha\tau;\tau\right)=-\tfrac{1}{\sqrt{-i\tau}}\left(1-2\cos(2\pi\alpha)q_0^{\frac{1}{2}}+2\cos(4\pi\alpha)q_0^2+O\left(q_0^4\right)\right), \quad (4.3.2)$$

where

$$a_1:=1+2\cos(2\pi\alpha) \quad \text{and} \quad a_3:=1+2\cos(2\pi\alpha)+2\cos(4\pi\alpha).$$

Proof. Let $w:=e^{2\pi i\alpha}$. The proof of Eq. (4.3.1) follows directly by applying the technique in the proof of Lem. 2.24 and observing that

$$\begin{aligned}&(w;q_0)_\infty(q_0w^{-1};q_0)_\infty(q_0;q_0)_\infty\\&=(1-w)\left(1-\left(1+w+w^{-1}\right)q_0+\left(1+w+w^{-1}+w^2+w^{-2}\right)q_0^3+O\left(q_0^4\right)\right).\end{aligned}$$

On the other hand, for Eq. (4.3.2), we consider the associated Jacobi product

$$\begin{aligned}&\left(wq_0^{-\frac{1}{2}};q_0\right)_\infty\left(w^{-1}q_0^{\frac{3}{2}};q_0\right)_\infty(q_0;q_0)_\infty\\&=-wq_0^{-\frac{1}{2}}+\left(1+w^2\right)-\left(w^{-1}+w^3\right)q_0^{\frac{3}{2}}+O\left(q_0^3\right).\end{aligned}$$

Writing the ϑ-functions as Jacobi products yields the result. $\square$

4.4 The $a(n)$

The function $T(\tau)$ defined in Ex. 4.5 can be simplified greatly.

Proposition 4.8. We have

$$\begin{aligned}T(\tau)=&\frac{4iq^{\frac{417}{8}}}{\vartheta\left(\frac{1}{2};\tau\right)}\frac{\vartheta^3(96\tau;288\tau)}{\vartheta\left(\frac{1}{2};24\tau\right)}\\&\times\left(q^6\frac{\vartheta(4\tau;16\tau)\vartheta(68\tau;96\tau)}{\vartheta\left(\frac{1}{2}+46\tau;96\tau\right)\vartheta\left(\frac{1}{2}+22\tau;96\tau\right)}-q^{-47}\frac{\vartheta(12\tau;16\tau)\vartheta(20\tau;96\tau)}{\vartheta\left(\frac{1}{2}+94\tau;96\tau\right)\vartheta\left(\frac{1}{2}+22\tau;96\tau\right)}\right.\\&\left.+q^{-39}\frac{\vartheta(4\tau;16\tau)\vartheta(20\tau;96\tau)}{\vartheta\left(\frac{1}{2}+46\tau;96\tau\right)\vartheta\left(\frac{1}{2}+70\tau;96\tau\right)}-q^{-52}\frac{\vartheta(4\tau;16\tau)\vartheta(68\tau;96\tau)}{\vartheta\left(\frac{1}{2}+94\tau;96\tau\right)\vartheta\left(\frac{1}{2}+70\tau;96\tau\right)}\right).\end{aligned}$$

Proof. Let

$$\begin{aligned}\overline{S}(\tau) &:= \sum_{r,s=0}^{3} (-1)^r q^{Q(r,s)} \cdot \frac{\vartheta(4(s-r)\tau;16\tau)\ \vartheta\left((24(r+s)+44)\tau;96\tau\right)}{\vartheta\left(\frac{1}{2}+(24r+22)\tau;96\tau\right)\ \vartheta\left(\frac{1}{2}+(24s+22)\tau;96\tau\right)} \\ &=: \sum_{r,s=0}^{3} \rho(r,s;\tau).\end{aligned}$$

It is clear that $\rho(r,r;\tau) = 0$ since $\vartheta(0;16\tau) = 0$. Furthermore, $Q(r,s) = Q(s,r)$. Using the fact that $\vartheta(-4(s-r)\tau;16\tau) = -\vartheta(4(s-r)\tau;16\tau)$, we deduce that

$$\rho(s,r;\tau) = -(-1)^{r+s}\rho(r,s;\tau).$$

This tells us that we can write

$$\overline{S}(q) = 2\left(\rho(1,0;\tau) + \rho(2,1;\tau) + \rho(3,0;\tau) + \rho(3,2;\tau)\right).$$

We can then apply Item (5) of Prop. 2.19 to $\rho(2,1;\tau)$, $\rho(3,0;\tau)$, and $\rho(3,2;\tau)$ to complete the proof. □

Using the simplification in Prop. 4.8, we will investigate the asymptotic growth of the coefficients of $R_{3,3}(q)$ in the next section.

4.4.1 The pole at $\tau = \frac{1}{2}$

It will turn out that the growth of $R_{3,3}$ is dominated by the cusp $\frac{1}{2}$. We first compute the growth near $\frac{1}{2}$ in this section, and we will later show that this growth dominates in Sect. 4.5. As was claimed in Sect. 4.3, we require higher order asymptotic expansions to accurately determine the growth of the $a(n)$. We break the study near $\tau = \frac{1}{2}$ into two parts: $T(\tau)$ and the Appell function, where Lem. 4.7 will prove useful for the study of $T(\tau)$. The first result involves the function $\vartheta(\frac{1}{2};\tau)$, appearing in the denominator of $T(\tau)$. Using the specific values of the η-multiplier defined in Eq. (2.1.2), we can prove the following.

Lemma 4.9. *Define $w := \tau - \frac{1}{2}$. As $w \to 0$ in a fixed angular region we have*

$$\vartheta\left(\frac{1}{2};\tau\right) = 2\frac{e^{-\frac{\pi i}{8}} e^{-\frac{\pi i}{16\left(\tau-\frac{1}{2}\right)}}}{\sqrt{-2\left(\tau-\frac{1}{2}\right)}}\left(1+O\left(e^{-\frac{\pi i}{2\left(\tau-\frac{1}{2}\right)}}\right)\right).$$

Proof. Let $z := -\frac{1}{4w} - \frac{1}{2}$ and define the matrices $A := \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. We apply the well-known formula $\vartheta\left(\frac{1}{2};\tau\right) = 2\frac{\eta^2(2\tau)}{\eta(\tau)}$,

$$\begin{aligned}
\vartheta\left(\frac{1}{2};Az\right) &= 2\frac{\eta^2(2Az)}{\eta(Az)} = 2\frac{\eta^2(B(2z))}{\eta(Az)} \\
&= 2\frac{\nu(B)^2(2z+1)^{\frac{1}{2}}\eta^2(2z)}{\nu(A)\eta(z)} \\
&= 2\frac{\nu(B)^2(2z+1)^{\frac{1}{2}} e^{\frac{8\pi i}{24}z}\left(e^{4\pi i z};e^{4\pi i z}\right)_\infty^2}{\nu(A)e^{\frac{2\pi i z}{24}}\left(e^{2\pi i z};e^{2\pi i z}\right)_\infty} \\
&= 2\frac{\nu(B)^2(2z+1)^{\frac{1}{2}} e^{\frac{\pi i}{4}z}\left(e^{4\pi i z};e^{4\pi i z}\right)_\infty^2}{\nu(A)\left(e^{2\pi i z};e^{2\pi i z}\right)_\infty} \\
&= 2\frac{e^{-\frac{\pi i}{8}} e^{-\frac{\pi i}{16w}}}{\sqrt{-2w}}\left(1+O\left(e^{-\frac{\pi i}{2w}}\right)\right),
\end{aligned}$$

where $\nu(A) = \nu(B)^2 = e^{-\frac{\pi i}{6}}$, which proves the claim. $\square$

We are now able to prove the growth of $T(\tau)$ near the pole at $\tau = \frac{1}{2}$.

Theorem 4.10. *Let $Q_0 := e^{-\frac{2\pi i}{\tau-\frac{1}{2}}}$ and write $\tau = u + iv$ as before. Define $v := \frac{1}{\sqrt{192n}}$ and let $M > 0$ such that $\left|u-\frac{1}{2}\right| < Mv$. Then as $n \to \infty$*

$$T(\tau) = \frac{\sqrt{3}}{6\sqrt{\tau-\frac{1}{2}}} e^{\frac{\pi i}{4}} Q_0^{-\frac{1}{192}}\left(1+O\left(e^{-\pi\sqrt{\frac{n}{12}}}\right)\right).$$

Proof. We focus our attention on

$$\begin{aligned}\overline{S}(\tau) := q^{6}&\frac{\vartheta(4\tau;16\tau)\vartheta(68\tau;96\tau)}{\vartheta\left(\frac{1}{2}+46\tau;96\tau\right)\vartheta\left(\frac{1}{2}+22\tau;96\tau\right)}\\ &- q^{-47}\frac{\vartheta(12\tau;16\tau)\vartheta(20\tau;96\tau)}{\vartheta\left(\frac{1}{2}+94\tau;96\tau\right)\vartheta\left(\frac{1}{2}+22\tau;96\tau\right)}\\ &+ q^{-39}\frac{\vartheta(4\tau;16\tau)\vartheta(20\tau;96\tau)}{\vartheta\left(\frac{1}{2}+46\tau;96\tau\right)\vartheta\left(\frac{1}{2}+70\tau;96\tau\right)}\\ &- q^{-52}\frac{\vartheta(4\tau;16\tau)\vartheta(68\tau;96\tau)}{\vartheta\left(\frac{1}{2}+94\tau;96\tau\right)\vartheta\left(\frac{1}{2}+70\tau;96\tau\right)}.\end{aligned}$$

We refer to the first, second, third, and fourth terms as $S_1(\tau)$, $S_2(\tau)$, $S_3(\tau)$, and $S_4(\tau)$ respectively. That is, $\overline{S}(\tau) = S_1(\tau)+S_2(\tau)+S_3(\tau)+S_4(\tau)$. Notice that

$$\begin{aligned}S_1\left(\tau+\frac{1}{2}\right)+S_4\left(\tau+\frac{1}{2}\right) &= S_1(\tau)+S_4(\tau),\\ S_2\left(\tau+\frac{1}{2}\right)+S_3\left(\tau+\frac{1}{2}\right) &= -S_2(\tau)-S_3(\tau).\end{aligned}$$

Thus, we can capture the behavior near the cusp $\frac{1}{2}$ by investigating the behavior near 0. We can apply Lem. 4.7 to the $S_j(\tau)$, and we find that as $\tau \to 0$

$$\begin{aligned}S_1(\tau)+S_4(\tau) = &-4\sqrt{6}\sin\left(\tfrac{\pi}{4}\right)\sin\left(\tfrac{17\pi}{24}\right)q_0^{\frac{7}{8\cdot 96}}\Bigg((a_1+a_2-c_1-c_2)q_0^{\frac{1}{192}}\\ &-\left(1+2\cos\left(\tfrac{17\pi}{12}\right)\right)(a_1+a_2-c_1-c_2)q_0^{\frac{3}{2\cdot 96}}+O\left(q_0^{\frac{5}{192}}\right)\Bigg),\end{aligned}$$

where,

$$a_1 := 2\cos\left(\frac{23\pi}{24}\right), \quad a_2 := 2\cos\left(\frac{11\pi}{24}\right),$$

$$c_1 := 2\cos\left(\frac{47\pi}{24}\right), \quad c_2 := 2\cos\left(\frac{35\pi}{24}\right).$$

Similarly as $\tau \to 0$,

$$S_2(\tau) + S_3(\tau) = -4\sqrt{6}\sin\left(\tfrac{\pi}{4}\right)\sin\left(\tfrac{5\pi}{24}\right)q_0^{\frac{7}{8\cdot 96}}\Bigg((h_1+h_2-\ell_1-\ell_2)q_0^{\frac{1}{192}} \\ -\left(1+2\cos\left(\tfrac{5\pi}{12}\right)\right)(h_1+h_2-\ell_1-\ell_2)q_0^{\frac{3}{2\cdot 96}} + O\left(q_0^{\frac{5}{192}}\right)\Bigg),$$

where

$$h_1 := 2\cos\left(\frac{23\pi}{24}\right), \quad h_2 := 2\cos\left(\frac{35\pi}{24}\right),$$
$$\ell_1 := 2\cos\left(\frac{47\pi}{24}\right), \quad \ell_2 := 2\cos\left(\frac{11\pi}{24}\right).$$

We then observe that

$$\sin\left(\tfrac{\pi}{4}\right)\sin\left(\tfrac{17\pi}{24}\right)\Big(a_1+a_2-c_1-c_2\Big) - \sin\left(\tfrac{\pi}{4}\right)\sin\left(\tfrac{5\pi}{24}\right)\Big(h_1+h_2-\ell_1-\ell_2\Big) = 0,$$

and

$$\sin\left(\frac{17\pi}{24}\right)\left(1+2\cos\left(\frac{17\pi}{12}\right)\right)\Big(a_1+a_2-c_1-c_2\Big) \\ -\sin\left(\frac{5\pi}{24}\right)\left(1+2\cos\left(\frac{5\pi}{12}\right)\right)\Big(h_1+h_2-\ell_1-\ell_2\Big) = 2\sqrt{2}.$$

Thus,

$$\begin{aligned}\lim_{q\to -1}\overline{S}(q) &= \lim_{q\to 1}\left(S_1(q) - S_2(q) - S_3(q) + S_4(q)\right) \\ &= \left(4\sqrt{6}\sin\left(\tfrac{\pi}{4}\right)\right)\left(2\sqrt{2}\right)Q_0^{\frac{19}{8\cdot 96}}\left(1+O\left(Q_0^{\frac{1}{96}}\right)\right) \\ &= 8\sqrt{6}Q_0^{\frac{19}{8\cdot 96}}\left(1+O\left(Q_0^{\frac{1}{96}}\right)\right).\end{aligned} \tag{4.4.1}$$

We now turn our attention to the outside ϑ-quotient on $T(\tau)$. Sending $\tau \mapsto \tau + \frac{1}{2}$ and using the appropriate ϑ transformations in Prop. 2.19, we have

$$\frac{4iq^{\frac{417}{8}}}{\vartheta\left(\frac{1}{2};\tau\right)}\frac{\vartheta^3(96\tau;288\tau)}{\vartheta\left(\frac{1}{2};24\tau\right)} \mapsto \quad -\frac{4iq^{\frac{417}{8}}e^{\frac{417\pi i}{8}}}{\vartheta\left(\frac{1}{2};\tau+\frac{1}{2}\right)}\frac{\vartheta^3(96\tau;288\tau)}{\vartheta\left(\frac{1}{2};24\tau\right)}.$$

Applying Lem. 4.9 to $\vartheta\left(\frac{1}{2};\tau\right)$ and Lem. 2.24 to the other functions leads to the near $\frac{1}{2}$ estimate

$$\begin{aligned}
&-\frac{4iq^{\frac{417}{8}}e^{\frac{417\pi i}{8}}}{\vartheta\left(\frac{1}{2};\tau+\frac{1}{2}\right)}\frac{\vartheta^3(96\tau;288\tau)}{\vartheta\left(\frac{1}{2};24\tau\right)} \\
&\sim -4i\frac{\left(-2i\sin\left(\frac{\pi}{3}\right)\right)^3 Q_0^{\frac{1}{8\cdot 96}}}{\left(-288i\left(\tau-\frac{1}{2}\right)\right)^{\frac{3}{2}}}\,\frac{\sqrt{-2\left(\tau-\frac{1}{2}\right)}\,Q_0^{-\frac{1}{32}}e^{\frac{\pi i}{4}}}{2}\left(-\sqrt{-24i\left(\tau-\frac{1}{2}\right)}\right) \qquad (4.4.2)\\
&= \frac{\sqrt{2}e^{\frac{\pi i}{4}}}{96\sqrt{\tau-\frac{1}{2}}}Q_0^{\frac{1}{8\cdot 96}-\frac{1}{32}}.
\end{aligned}$$

Combining Eqs. (4.4.1) and (4.4.2), we obtain the full estimate for $T(\tau)$ near $\tau = \frac{1}{2}$:

$$\begin{aligned}
T(\tau) &= \frac{\sqrt{3}}{6\sqrt{\tau-\frac{1}{2}}}e^{\frac{\pi i}{4}}Q_0^{\frac{20}{8\cdot 96}-\frac{24}{8\cdot 96}}\left(1+O\left(Q_0^{\frac{1}{96}}\right)\right)\\
&= \frac{\sqrt{3}}{6\sqrt{\tau-\frac{1}{2}}}e^{\frac{\pi i}{4}}Q_0^{-\frac{1}{192}}\left(1+O\left(Q_0^{\frac{1}{96}}\right)\right),
\end{aligned}$$

which proves the claim. $\square$

Since $R_{3,3}(q) = T(\tau) + 2iq^{-1}\mu\left(2\tau+\frac{1}{2},\frac{1}{2};24\tau\right)$, we now want to show that the Appell sum is dominated by the growth coming from the term $T(\tau)$. In order to simplify our calculations, we introduce some notation.

Notation 4.11. We use $\doteq$ to mean "equal up to a factor of $O(1)$" and $f(x) \dot{\sim} g(x)$ to mean "$f(x) = Mg(x)(1+o(1))$ for some $M = O(1)$".

Since our Appell sum is invariant under the transformation $\tau \mapsto \tau + \frac{1}{2}$, it suffices to look at the behavior near $\tau = 0$. Using the transformation law of Prop. 2.19.3 gives

$$\mu\left(2\tau + \frac{1}{2}, \frac{1}{2}; 24\tau\right) = -\frac{q^{\frac{1}{12}}}{\sqrt{-24i\tau}}\mu\left(\frac{1}{12} + \frac{1}{48\tau}, \frac{1}{48\tau}; -\frac{1}{24\tau}\right) + \frac{h(2\tau;24\tau)}{2i}. \tag{4.4.3}$$

Looking solely at the remaining Appell sum gives

$$\begin{aligned}
\mu\left(\frac{1}{12} + \frac{1}{48\tau}, \frac{1}{48\tau}; -\frac{1}{24\tau}\right) &= \frac{e^{\frac{\pi i}{12}} e^{\frac{\pi i}{48\tau}}}{\vartheta\left(\frac{1}{48\tau}; -\frac{1}{24\tau}\right)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q_0^{\frac{n^2+n}{48}} q_0^{-\frac{n}{48}}}{1 - e^{\frac{\pi i}{6}} q_0^{-\frac{1}{48}} q_0^{\frac{n}{24}}} \\
\doteq \frac{q_0^{-\frac{1}{96}}}{q_0^{\frac{1}{8\cdot 24}} q_0^{\frac{1}{96}} \left(1 - q_0^{-\frac{1}{48}} + O\left(q_0^{\frac{1}{48}}\right)\right)} &\left(-\frac{e^{-\frac{\pi i}{6}} q_0^{\frac{1}{48}}}{1 - e^{-\frac{\pi i}{6}} q_0^{\frac{1}{48}}} + O\left(q_0^{\frac{1}{24}}\right)\right) \\
\doteq \frac{q_0^{-\frac{1}{96}} q_0^{\frac{1}{24}}}{q_0^{\frac{1}{8\cdot 24}} q_0^{-\frac{1}{96}} \left(1 + O\left(q_0^{\frac{1}{48}}\right)\right)} &\left(1 + O\left(q_0^{\frac{1}{48}}\right)\right) \,\dot{\sim}\, q_0^{\frac{7}{192}}.
\end{aligned}$$

Plugging this back into Eq. (4.4.3) and using the estimate in Lem. 2.22, we find as $\tau \to 0$

$$\mu\left(2\tau + \frac{1}{2}, \frac{1}{2}; 24\tau\right) \ll \frac{1}{\sqrt{\tau}}. \tag{4.4.4}$$

Thus, we have the following.

Remark 4.12. The growth of $R_{3,3}$ near $\tau = \frac{1}{2}$ is determined by the estimate in Thm. 4.10.

4.5 Growth on the minor arcs

We now want to show that the growth at the other cusps is negligible when compared with the growth given in Thm. 4.10. Thus, our target is to beat the bound

$$\frac{Q_0^{-\frac{1}{96}}}{\sqrt{\tau - \frac{1}{2}}} \ll n^{\frac{1}{4}} e^{\pi\sqrt{\frac{n}{12}}},$$

exponentially where we chose the parameterization $v = \frac{1}{\sqrt{192n}}$. By beating this bound, we can incorporate the estimates away from $\frac{1}{2}$ into an error term and ignore them in our final estimate for the $a(n)$. Explicitly, we will prove the following.

Theorem 4.13. *Let $M > 0$ and let $v := \frac{1}{\sqrt{192n}}$. For $Mv < \left|u - \frac{1}{2}\right|$, there exists a $\beta > 0$ such that*

$$R_{3,3}(q) \ll e^{\pi\sqrt{\frac{n}{12}}(1-\beta)}$$

holds uniformly as $n \to \infty$.

4.5.1 A simple trick involving the logarithm

There is a standard trick which is based on the fundamental principle that taking logarithms reduces the study of products to studying sums. In our case, the Jacobi product formula representation in Eq. (2.1.8) indicates that by taking logarithms of ϑ-functions, we may be able to control the growth of $R_{3,3}(q)$ by manipulating q-series. This is a standard trick when using the Wright circle method that usually gives strict enough bounds to bound the minor arcs outright without any additional work (see [19] and [36] for examples involving partitions and overpartitions). While this is not the complete story for us, the following still gives us a way to explicitly determine which cusps could be potential problems in $T(\tau)$.

Lemma 4.14. *Let $|u| > Mv$, and let a and b be positive integers with $a < b$. Furthermore, let $v := \frac{1}{\delta\sqrt{n}}$ with $\delta > 0$ and for fixed*

$M > 0$ define the term $\varepsilon := -\frac{1}{\sqrt{1+M^2}} + 1 > 0$. Then as $n \to \infty$

$$\frac{1}{\vartheta(a\tau; b\tau)}, \frac{1}{\vartheta\left(\frac{1}{2} + a\tau; b\tau\right)} \ll \frac{1}{n^{\frac{1}{4}}} e^{\frac{3\delta\sqrt{n}}{2\pi b}\left(\frac{\pi^2}{6} - \varepsilon\right)}, \tag{4.5.1}$$

and

$$\frac{1}{\vartheta\left(\frac{1}{2}; b\tau\right)} \ll \frac{1}{n^{\frac{3}{4}}} e^{\frac{\delta\sqrt{n}}{\pi b}\left(\frac{\pi^2}{6} - \varepsilon\right)}. \tag{4.5.2}$$

Remark 4.15. Notice that the right hand side of Eq. (4.5.1) does not depend on a. Furthermore, the bounds above also hold for the functions $\vartheta(a\tau; b\tau)$, $\vartheta\left(\frac{1}{2}; b\tau\right)$ and $\vartheta\left(\frac{1}{2} + a\tau; b\tau\right)$ which can by seen by replacing $\operatorname{Log}(\bullet)$ with $-\operatorname{Log}(\bullet)$ in the proof below. We will use this fact later on in Sect. 4.5.2.

Proof of Lemma 4.14. The proof uses the same ideas as in [19, 36, 40] to prove their bounds away from the dominant pole. We recall the Taylor expansion for $\operatorname{Log}(1 - z) = -\sum_{n\geq 1} \frac{z^n}{n}$. This implies

$$\operatorname{Log}\left(\frac{1}{(q^a; q^b)_\infty (q^{b-a}; q^b)_\infty (q^b; q^b)_\infty}\right) = \sum_{n\geq 1} \frac{q^{an} + q^{(b-a)n} + q^{bn}}{n(1 - q^{bn})}. \tag{4.5.3}$$

The trick now, as described by many works such as [19, 36, 40], is to extract the first term in the sum and add an extra term, which will be the first term in the expansion for

$$\operatorname{Log}\left(\frac{1}{\left(|q|^a; |q|^b\right)_\infty \left(|q|^{b-a}; |q|^b\right)_\infty \left(|q|^b; |q|^b\right)_\infty}\right).$$

Explicitly, we have

$$\sum_{n\geq 1} \frac{q^{an} + q^{(b-a)n} + q^{bn}}{n\left(1 - q^{bn}\right)} = \sum_{n\geq 2} \frac{q^{an} + q^{(b-a)n} + q^{bn}}{n\left(1 - q^{bn}\right)} + \frac{q^a + q^{b-a} + q^b}{\left(1 - q^b\right)} + \left(|q|^a + |q|^{b-a} + |q|^b\right)\left(\frac{1}{1 - |q|^b} - \frac{1}{1 - |q|^b}\right).$$

Taking the absolute value of this equation and using the fact that $1-|q|^b \leq |1-q^b|$, we have the upper bound

$$\begin{aligned}
\left|\mathrm{Log}\left(\frac{1}{(q^a;q^b)_\infty (q^{b-a};q^b)_\infty (q^b;q^b)_\infty}\right)\right| \leq & \Bigg| \sum_{n\geq 2} \frac{q^{an}+q^{(b-a)n}+q^{bn}}{n(1-q^{bn})} + \frac{q^a+q^{b-a}+q^b}{1-q^b} \\
& + \left(|q|^a+|q|^{b-a}+|q|^b\right)\left(\frac{1}{1-|q|^b}-\frac{1}{1-|q|^b}\right)\Bigg| \\
\leq & \sum_{n\geq 2} \frac{|q|^{an}+|q|^{(b-a)n}+|q|^{bn}}{n(1-|q|^{bn})} + \frac{|q|^a+|q|^{b-a}+|q|^b}{|1-q^b|} \\
& + \left(|q|^a+|q|^{b-a}+|q|^b\right)\left(\frac{1}{1-|q|^b}-\frac{1}{1-|q|^b}\right) \\
= & \sum_{n\geq 1} \frac{|q|^{an}+|q|^{(b-a)n}+|q|^{bn}}{n(1-|q|^{bn})} + \left(|q|^a+|q|^{b-a}+|q|^b\right)\left(\frac{1}{|1-q^b|}-\frac{1}{1-|q|^b}\right) \\
= & \mathrm{Log}\left(\frac{1}{(|q|^a;|q|^b)_\infty (|q|^{b-a};|q|^b)_\infty (|q|^b;|q|^b)_\infty}\right) \\
& + \left(|q|^a+|q|^{b-a}+|q|^b\right)\left(\frac{1}{|1-q^b|}-\frac{1}{1-|q|^b}\right).
\end{aligned}$$

The Log term can be estimated by the asymptotic formulas derived for the ϑ-functions in Lem. 2.24. Namely, as $n \to \infty$

$$\begin{aligned}
\mathrm{Log}\left(\frac{1}{(|q|^a;|q|^b)_\infty (|q|^{b-a};|q|^b)_\infty (|q|^b;|q|^b)_\infty}\right) \ll & \mathrm{Log}\left(\frac{\sqrt{\frac{b}{\delta\sqrt{n}}}}{2\sin(\pi\frac{a}{b})}\right) + \frac{\delta\pi\sqrt{n}}{4b} \\
\ll & C_{a,b} + \mathrm{Log}\left(n^{-\frac{1}{4}}\right) + \frac{\delta\pi\sqrt{n}}{4b},
\end{aligned} \tag{4.5.4}$$

where $C_{a,b}$ is a constant. Now we bound the fractions. Recall that we are away from the root of unity $q=1$ by the amount $|u| > Mv$. Following the procedure on pg. 10 of [19], we can bound $\frac{1}{|1-q^b|}$ by using the fact that the cosine is even and decreasing near 0 for positive argument. Namely, as $n \to \infty$

$$|1-q^b|^2 = 1-2\cos(2\pi bu)e^{-2\pi vb}+e^{-4\pi bv} \gg 4\pi^2 b^2 v^2 (1+M^2). \tag{4.5.5}$$

For the other fraction, we have

$$\frac{1}{1-|q|^b} \sim \frac{1}{2\pi bv}. \tag{4.5.6}$$

Combining Eqs. (4.5.4), (4.5.5), and (4.5.6), we have

$$\begin{aligned}&\operatorname{Log}\left(\frac{1}{(q^a;q^b)_\infty(q^{b-a};q^b)_\infty(q^b;q^b)_\infty}\right)\\ &\quad \ll C_{a,b}+\log\left(n^{-\frac{1}{4}}\right)+\frac{\delta\pi\sqrt{n}}{4b}+\frac{3}{2\pi by\sqrt{1+M^2}}-\frac{3}{2\pi by},\end{aligned}$$

which proves the first part of Eq. (4.5.1).

The second part of Eq. (4.5.1) follows by noticing that

$$\begin{aligned}&\left|\operatorname{Log}\left(\frac{1}{(q^b;q^b)_\infty(-q^{b-a};q^b)_\infty(-q^a;q^b)_\infty}\right)\right|=\left|\sum_{n\geq 1}\frac{(-1)^n(q^a+q^{b-a})+q^b}{n(1-q^{bn})}\right|\\ &\ll \operatorname{Log}\left(\frac{1}{(|q|^a;|q|^b)_\infty(|q|^{b-a};|q|^b)_\infty(|q|^b;|q|^b)_\infty}\right)\\ &\qquad\qquad +\left(|q|^a+|q|^{b-a}+|q|^b\right)\left(\frac{1}{|1-q^b|}-\frac{1}{1-|q|^b}\right),\end{aligned}$$

as before.

For Eq. (4.5.2), we have that

$$\vartheta\left(\frac{1}{2},\tau\right)=-q^{\frac{1}{8}}(-1;q)_\infty\left(q^2;q^2\right)_\infty=-2q^{\frac{1}{8}}(-q;q)_\infty\left(q^2;q^2\right)_\infty. \tag{4.5.7}$$

Using this along with the fact that the function $n\mapsto|q|^n$ is decreasing gives

$$\begin{aligned}\left|\operatorname{Log}\left(\frac{1}{\vartheta\left(\frac{1}{2},\tau\right)}\right)\right| &= \left|-\operatorname{Log}\left(-\frac{q^{-\frac{1}{8}}}{2}\right)+\sum_{\substack{n\geq 1\\ j\geq 0}}\frac{(-1)^n q^{n(1+j)}}{n}+\sum_{\substack{n\geq 1\\ j\geq 0}}\frac{q^{2n(1+j)}}{n}\right|\\ &\leq B+\sum_{n\geq 1}\frac{2|q|^n}{n(1-|q|^n)}\ll\log\left(P^2\left(|q|\right)\right),\end{aligned}$$

where B is a constant and $P(q)=\frac{q^{\frac{1}{24}}}{\eta(\tau)}$. Using Lem. 4.6 of [36] we have

$$P^2\left(|q|^b\right)\ll\frac{1}{n^{\frac{3}{4}}}\,e^{\frac{\delta\sqrt{n}}{\pi b}\left(\frac{\pi^2}{6}+\frac{1}{\sqrt{1+M^2}}-1\right)}.$$

This completes the proof of Eq. (4.5.2). $\square$

We now look at the non-normalized Appell sum which forms the other piece of the function $R_{3,3}(q)$ in Eq. (4.2.2). Specifically, we want to investigate

$$\begin{aligned} A_1\left(2\tau+\frac{1}{2},\frac{1}{2};24\tau\right) &= \vartheta\left(\frac{1}{2};24\tau\right)\mu\left(2\tau+\frac{1}{2},\frac{1}{2};24\tau\right) \\ &= iq\sum_{n\in\mathbb{Z}}\frac{q^{12(n^2+n)}}{1+q^{24n+2}}. \end{aligned} \tag{4.5.8}$$

The following result shows that we can bound the above sum by a classical *single* variable theta function, $\Theta(\tau)$. A similar result was also mentioned by the authors of [19], but was not carried out explicitly.

Proposition 4.16. *Let* $\Theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$. *Then,*

$$\left|A_1\left(2\tau+\frac{1}{2},\frac{1}{2};24\tau\right)\right| \leq \frac{\Theta(iv)}{1-|q|^2}.$$

Proof. Splitting the sum in Eq. (4.5.8) into negative and positive index, and then recombining, we find

$$A_1\left(2\tau+\frac{1}{2},\frac{1}{2};24\tau\right) = i\tfrac{q}{1+q^2}+iq\textstyle\sum_{n\geq 1} q^{12(n^2+n)}\left(\tfrac{1}{1+q^{24n+2}}+\tfrac{q^{-2}}{1+q^{24n-2}}\right).$$

Since $|q|<1$, we have that $1-|q|\leq|1+q|$. Combined with the fact that $|q|^m$ is a decreasing function in m, we have that,

$$\begin{aligned} \left|A_1\left(2\tau+\frac{1}{2},\frac{1}{2};24\tau\right)\right| &\leq \frac{1}{1-|q|^2}+\sum_{n\geq 1}|q|^{12(n^2+n)}\left(\frac{2}{1+|q|^{24n+2}}\right) \\ &\leq \frac{1}{1-|q|^2}\left(1+2\sum_{n\geq 1}|q|^{n^2}\right) \\ &= \frac{1}{1-|q|^2}\Theta(iv), \end{aligned}$$

which proves the claim. $\square$

Remark 4.17. Recall that $\Theta(\tau)$ is a holomorphic modular form for the group $\Gamma_0(4)$. It is well-known (or one can compute using SAGE) that $\Gamma_0(4)$ has three inequivalent cusps represented by 0, $\frac{1}{2}$, and ∞.

Recall that in Eq. (4.4.4) we computed the estimate near 0 and $\frac{1}{2}$, which gives

$$\mu\left(2\tau+\frac{1}{2},\frac{1}{2};24\tau\right) \ll \frac{1}{\sqrt{|\tau|}}.$$

Due to Prop. 4.16 and the corresponding remark, we can see we only need to check the growth of $\Theta(\tau)$ near ∞. Since $\Theta(\tau)$ is modular, its growth at ∞ is at most $O(1)$. Since under the transformation $\tau \mapsto \tau + \frac{1}{2}$ the Jacobi theta remains unchanged up to a constant, that is $\vartheta\left(\frac{1}{2};24\tau\right) \mapsto -\vartheta\left(\frac{1}{2};24\tau\right)$, we can use Lemmas 4.14 and 4.16 to obtain the following.

Theorem 4.18. *Let $M>0$ such that $0 < vM < \left|u-\frac{h}{k}\right|$ where $\frac{h}{k}$ is a cusp. Then as $n\to\infty$ there is a $\beta>0$,*

$$\mu\left(2\tau+\frac{1}{2},\frac{1}{2};24\tau\right) \ll \frac{1}{\left|\tau-\frac{h}{k}\right|^{\beta}}.$$

Proof. We know that $\beta = \frac{1}{2}$ near $\tau \in \left\{0, \frac{1}{2}\right\}$. For the other cusps, we refer to Lem. 4.16 and compute near any cusp $\frac{h}{k}$,

$$\frac{1}{1-|q|^2} \dot{\sim} \frac{1}{\left(\tau-\frac{h}{k}\right)^2},$$

which proves the claim. $\square$

The estimates in Lem. 4.14 alone are not sufficient to bound $T(\tau)$ away from the dominant pole since they do not provide accurate information about the decay of the ϑ-functions near generic cusps $\frac{h}{k}$. However, we can use Lem. 4.14 in combination with a generalization of Lem. 4.9 to rule out contributions from cusps not equal to $\frac{1}{2}$. Before doing so, we note that $T(\tau)$ decays rapidly near 0.

Lemma 4.19. *As $\tau \to 0$ in a fixed angular region,*

$$T(\tau) \ll |\tau|^{-\frac{1}{2}} q_0^{\frac{1}{96}}.$$

Proof. From Lem. 2.24 we have near $\tau = 0$,

$$\vartheta(a_1\tau; b_1\tau) \ll |\tau|^{-\frac{1}{2}} q_0^{\frac{1}{8b_1}} \quad \text{and} \quad \vartheta\left(\frac{1}{2} + a_1\tau; b_1\tau\right) \ll |\tau|^{-\frac{1}{2}}.$$

The proof for the near 0 estimate follows by plugging in the estimates to T, which is a quotient of ϑ-functions. □

To deal with the other cusps, we recall the fact that $\vartheta(z;\tau)$ is a Jacobi form of weight and index $\frac{1}{2}$. We use this fact in the next section to write down asymptotic formulas for the ϑ-functions near any point on the unit circle.

4.5.2 Asymptotic formulas for Jacobi theta functions near generic cusps

We write down the growth of the Jacobi theta functions up to a constant near cusps $\frac{h}{k}$.

Theorem 4.20. *Suppose there is an $M > 0$ such that $\left|u - \frac{h}{k}\right| < Mv$. Let a_1 and b_1 be non-zero natural numbers with $a_1 < b_1$. Then as $v \to 0$,*

$$\vartheta(a_1\tau; b_1\tau) \dot{\sim} v^{-\frac{1}{2}} e^{-\frac{\pi i}{b_1 C^2 v}\left(\left\{\frac{C\gamma}{\kappa}\right\}^2 + \frac{1}{4} - \left\{\frac{C\gamma}{\kappa}\right\}\right)},$$

where

$$\gamma := \frac{a_1 h \pmod{k}}{\gcd(k, a_1 h \pmod{k})}, \qquad \Omega := \frac{b_1 h \pmod{k}}{\gcd(k, b_1 h \pmod{k})},$$

$$\kappa := \frac{k}{\gcd(k, a_1 h \pmod{k})}, \qquad C := \frac{k}{\gcd(k, b_1 h \pmod{k})}.$$

Remark 4.21. Notice that if $h = 0$, then $\gamma = 0$, $\kappa = 1$, and $C = 1$. As a result, we recover the same exponential term $q_0^{\frac{1}{8b_1}}$ as in Prop. 2.24.

Similarly, we have to address ϑ-functions with a shifted elliptic variable.

Theorem 4.22. *Suppose there is an $M > 0$ such that $\left|u - \frac{h}{k}\right| < Mv$. Let a_1 and b_1 be non-zero natural numbers with $a_1 < b_1$. Then as $v \to 0$,*

$$\vartheta\left(\frac{1}{2} + a_1\tau; b_1\tau\right) \mathrel{\dot\sim} v^{-\frac{1}{2}} e^{-\frac{\pi i}{b_1 C^2 w}\left(\left\{\frac{C\phi}{\omega}\right\}^2 + \frac{1}{4} - \left\{\frac{C\phi}{\omega}\right\}\right)},$$

where

$$\phi := \frac{k + 2a_1 h \pmod{2k}}{\gcd(2k, k + 2a_1 h \pmod{2k})} \qquad \text{and} \qquad \omega := \frac{2k}{\gcd(2k, k + 2a_1 h \pmod{2k})}.$$

Remark 4.23. Note that if $h = 0$, then $\phi = 1$, $\omega = 2$, and $C = 1$. Therefore, the main exponential term cancels and we are only left with the polynomial growth which we previously saw in Prop. 2.24.

Proof of Thm. 4.20. Let

$$A := \begin{pmatrix} \Omega & b \\ C & d \end{pmatrix},$$

where b, d are chosen such that $\det(A) = 1$ which is allowed since Ω and C are relatively prime. Let $w := \tau + \frac{h}{k}$. Define

$$\sigma := -\frac{1}{b_1 C^2 w} - \frac{d}{C}.$$

Notice that

$$\lim_{w \to 0} A(\sigma) = \lim_{w \to 0} \frac{\Omega}{C} + b_1 w = \frac{\Omega}{C}.$$

Applying Prop. 2.20 gives

$$\vartheta(a_1\tau; b_1\tau) \doteq \vartheta\left(a_1 w + \frac{\gamma}{\kappa}; b_1 w + \frac{\Omega}{C}\right).$$

We want to study this function as $w \to 0$ in a fixed angular region. We can write

$$\vartheta\left(a_1 w + \frac{\gamma}{\kappa}; b_1 w + \frac{\Omega}{C}\right) = \vartheta\left(\frac{z}{C\sigma + d}; A(\sigma)\right), \tag{4.5.9}$$

where

$$z := (C\sigma + d)\left(a_1 w + \frac{\gamma}{\kappa}\right) = -\frac{\gamma}{\kappa C b_1 w} + O(1).$$

Using the first Jacobi transform in Prop. 2.20 yields

$$\begin{aligned}
\vartheta\left(\frac{z}{C\sigma + d}; A(\sigma)\right) &\doteq v^{-\frac{1}{2}} e^{\frac{\pi i C z^2}{C\sigma + d}} \vartheta(z;\sigma) \\
&= v^{-\frac{1}{2}} e^{\frac{\pi i C z^2}{C\sigma + d}} \vartheta\left(a_1 C\sigma w + \frac{C\gamma}{\kappa}\sigma + a_1 d w + \frac{d\gamma}{\kappa}; \sigma\right) \\
&\doteq v^{-\frac{1}{2}} e^{\frac{\pi i C z^2}{C\sigma + d}} e^{-\pi i\left(\left\lfloor \frac{C\gamma}{\kappa}\right\rfloor^2 \sigma + 2\left\lfloor \frac{C\gamma}{\kappa}\right\rfloor\left(\Lambda + \left\{\frac{C\gamma}{\kappa}\right\}\sigma\right)\right)} \vartheta\left(\left\{\tfrac{d\gamma}{\kappa}\right\} + \left\{\tfrac{C\gamma}{\kappa}\right\}\sigma + \Lambda; \sigma\right),
\end{aligned}$$

where in the last step we used the elliptic property (the second Jacobi transformation) and we defined $\Lambda := a_1 w\,(C\sigma + d)$. Notice that $\Lambda = O(1)$. We now apply the Jacobi triple product to find that

$$\begin{aligned}
\vartheta\left(\left\{\tfrac{d\gamma}{\kappa}\right\} + \left\{\tfrac{C\gamma}{\kappa}\right\}\sigma + \Lambda; \sigma\right) &\doteq e^{\frac{\pi i}{4}\sigma} e^{-\pi i\left(\left\{\frac{d\gamma}{\kappa}\right\} + \left\{\frac{C\gamma}{\kappa}\right\}\sigma + \Lambda\right)} \left(e^{2\pi i\sigma}; e^{2\pi i\sigma}\right)_\infty \\
&\quad \times \left(e^{2\pi i\left(\left\{\frac{d\gamma}{\kappa}\right\} + \left\{\frac{C\gamma}{\kappa}\right\}\sigma + \Lambda\right)}; e^{2\pi i\sigma}\right)_\infty \\
&\quad \times \left(e^{2\pi i\sigma} e^{-2\pi i\left(\left\{\frac{d\gamma}{\kappa}\right\} + \left\{\frac{C\gamma}{\kappa}\right\}\sigma + \Lambda\right)}; e^{2\pi i\sigma}\right)_\infty \\
&\dot{\sim} e^{\frac{\pi i}{4}\sigma} e^{-\pi i\left(\left\{\frac{d\gamma}{\kappa}\right\} + \left\{\frac{C\gamma}{\kappa}\right\}\sigma + \Lambda\right)} \\
&\dot{\sim} e^{-\frac{\pi i}{b_1 C^2 w}\left(\frac{1}{4} - \left\{\frac{C\gamma}{\kappa}\right\}\right)}.
\end{aligned}$$

Furthermore, noting that $\sigma w = O(1)$,

$$\begin{aligned}
&e^{\frac{\pi i C z^2}{C\sigma + d}} e^{-\pi i\left(\left\lfloor \frac{C\gamma}{\kappa}\right\rfloor^2 \sigma + 2\left\lfloor \frac{C\gamma}{\kappa}\right\rfloor\left(\Lambda + \left\{\frac{C\gamma}{\kappa}\right\}\sigma\right)\right)} \\
&\quad \dot{\sim} e^{-\frac{\pi i \gamma^2}{\kappa^2 b_1 w}} e^{\frac{\pi i}{b_1 C^2 w}\left(\left\lfloor \frac{C\gamma}{\kappa}\right\rfloor^2 + 2\left\lfloor \frac{C\gamma}{\kappa}\right\rfloor\left\{\frac{C\gamma}{\kappa}\right\}\right)}
\end{aligned}$$

$$= e^{-\frac{\pi i \gamma^2}{\kappa^2 b_1 w}} e^{\frac{\pi i}{b_1 C^2 w}\left(\left(\frac{H\gamma}{\kappa}\right)^2 - \left\{\frac{H\gamma}{\kappa}\right\}^2\right)}$$
$$= e^{-\frac{\pi i}{b_1 C^2 w}\left\{\frac{H\gamma}{\kappa}\right\}^2}.$$

Combining the last two equations gives the desired result. □

Proof of Thm. 4.22. We begin again by writing $w := \tau + \frac{h}{k}$. Then, by Prop. 2.24

$$\vartheta\left(\frac{1}{2} + a_1\tau; b_1\tau\right) \doteq \vartheta\left(\frac{\phi}{\omega} + a_1\tau; \frac{\Omega}{C} + b_1\tau\right).$$

The proof is identical to the proof of Thm. 4.20 with $\gamma \mapsto \phi$ and $\kappa \mapsto \omega$. □

We finally take care of ϑ-functions that have a constant elliptic variable equal to $\frac{1}{2}$. This will then be all of the necessary transformation formulas needed to study the growth of $T(\tau)$ near any cusp.

Lemma 4.24. *Suppose there exists an $M > 0$ such that $\left|u - \frac{h}{k}\right| < Mv$. Then we have as $v \to 0$*

$$\vartheta\left(\frac{1}{2}; \tau\right) \dot{\sim} \begin{cases} v^{-\frac{1}{2}} e^{-\frac{\pi}{4k^2 v}} & \text{if } k \text{ is even,} \\ v^{-\frac{1}{2}} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let $A := \left(\begin{smallmatrix} h & b \\ k & d \end{smallmatrix}\right)$ with b, d so that $A \in \mathrm{SL}_2(\mathbb{Z})$, which exist since $(h,k) = 1$. Define $z := \frac{k\sigma + d}{2}$ with $\sigma := -\frac{1}{k^2 w} - \frac{d}{k}$ for $w \in \mathbb{H}$. Notice that

$$A(\sigma) = \frac{h}{k} + w(hd - kb) = \frac{h}{k} + w.$$

Thus, $\lim_{w \to 0} A(\sigma) = \dfrac{h}{k}$. Regardless of whether k is even or odd, using Eq. (2.1.9) yields

$$\vartheta\left(\frac{1}{2}; A(\sigma)\right) \doteq (k\sigma + d)^{\frac{1}{2}} e^{\pi i \frac{k^2\sigma + kd}{4}} \vartheta\left(\frac{k\sigma + d}{2}; \sigma\right). \tag{4.5.10}$$

If k is even, d must be odd since $\frac{h}{k}$ is in reduced form. Therefore, Eq. (2.1.10) implies that Eq. (4.5.10) reduces to

$$\vartheta\left(\frac{1}{2}; A(\sigma)\right) \doteq \left(-\frac{1}{kw}\right)^{\frac{1}{2}} \vartheta\left(\frac{1}{2}; \sigma\right),$$

which upon using the Jacobi triple product and taking the limit $w \to 0$ gives

$$\vartheta\left(\frac{1}{2}; A(\sigma)\right) \dot{\sim} \left(-\frac{1}{kw}\right)^{\frac{1}{2}} e^{-\frac{\pi i}{4k^2 w}}.$$

Substituting in $w = \tau - \frac{h}{k}$ and using that $\mathrm{Im}(w) = \mathrm{Im}(\tau) = v$ proves the first claim.

The second case when k is odd has two separate situations to contend with, depending on whether d is odd or even. Assume first that d is odd. Then $\frac{k}{2}$ and $\frac{d}{2}$ are both half integers. Therefore using Prop. 2.20 as before, we have

$$\begin{aligned}\vartheta\left(\frac{1}{2}; A(\sigma)\right) &\doteq (k\sigma+d)^{\frac{1}{2}} e^{\pi i \frac{k^2\sigma}{4}} \vartheta\left(\left(\left\lfloor\frac{k}{2}\right\rfloor+\frac{1}{2}\right)\sigma+\left\lfloor\frac{d}{2}\right\rfloor+\frac{1}{2};\sigma\right) \\ &\doteq (k\sigma+d)^{\frac{1}{2}} e^{-\pi i\sigma\left(\left\lfloor\frac{k}{2}\right\rfloor^2+\left\lfloor\frac{k}{2}\right\rfloor-\frac{k^2}{4}\right)} \vartheta\left(\frac{\sigma+1}{2};\sigma\right).\end{aligned} \tag{4.5.11}$$

Using the Jacobi product in Eq. (2.1.8) again and taking the limit as $w \to 0$, we have

$$\vartheta\left(\tfrac{1}{2};A(\sigma)\right) \dot{\sim} (-kw)^{-\frac{1}{2}} e^{\frac{\pi i}{k^2 w}\left(-\frac{k^2}{4}+\left\lfloor\frac{k}{2}\right\rfloor^2+\left\lfloor\frac{k}{2}\right\rfloor+\frac{1}{4}\right)} = (-2w)^{-\frac{1}{2}}, \tag{4.5.12}$$

where the last step follows since k odd implies $\lfloor\frac{k}{2}\rfloor = \frac{k-1}{2}$. If d is even, the only thing that changes in Eq. (4.5.11) is that $\vartheta\left(\frac{\sigma+1}{2};\sigma\right)$ becomes $\vartheta\left(\frac{\sigma}{2};\sigma\right)$, which both yield the same estimate up to a constant factor as $w \to 0$ by examining the triple product representations. $\square$

With the previous lemma, we have the following useful corollary.

Corollary 4.25. *Suppose there exists an $M > 0$ such that $\left|u - \frac{h}{k}\right| < Mv$. Then we have the lower bound as $v \to 0$*

$$\vartheta\left(\frac{1}{2}; 24\tau\right) \gg v^{-\frac{1}{2}} e^{-\frac{\pi}{96c^2 v}},$$

where $\mathbf{c} := \frac{k}{g}$ and $g := \gcd(24h \pmod{k}, k)$.

Proof. If $k \mid 24$, Prop. 2.24 implies that $\vartheta\left(\frac{1}{2}; 24\tau\right)$ is asymptotic to a polynomial in τ, and thus the claim follows. If $k \nmid 24$, defining $\tau := w + \frac{h}{k}$ and applying the transformations in Lem. 2.1.9 gives that

$$\vartheta\left(\frac{1}{2}; 24\tau\right) \doteq \vartheta\left(\frac{1}{2}; 24w + \frac{\gamma}{\mathbf{c}}\right),$$

where $\gamma := \frac{24h \pmod{k}}{g}$. The machinery of the the proof of the previous lemma with $h := \gamma$, $k = \mathbf{c}$, and $w := 24w$ now applies. If $\mathbf{c}$ is odd, we are done since the cancellation in the exponent in Eq. (4.5.12) only depends on the fact that c is odd. If $\mathbf{c}$ is even, we find that as $w \to 0$ in a cone

$$\vartheta\left(\frac{1}{2}; 24\tau\right) \dot{\sim} (-24\mathbf{c}w)^{-\frac{1}{2}} e^{-\frac{\pi i}{96\mathbf{c}^2 w}} = (-24\mathbf{c}w)^{-\frac{1}{2}} e^{-\frac{\pi i\left(\left(u+\frac{h}{k}\right)-iv\right)}{96\mathbf{c}^2\left(\left(u+\frac{h}{k}\right)^2+v^2\right)}}$$
$$\gg v^{-\frac{1}{2}} e^{-\frac{\pi i}{96\mathbf{c}^2 v}},$$

where the last step follows from the fact that $\mathrm{Im}(w) = \mathrm{Im}(\tau) = v$. $\square$

4.5.3 A helpful function

Let $\alpha := (a_1, a_2, a_3, a_4) \in \mathbb{N}^4$ and let us define the set of reduced fractions

$$\mathscr{X}(j) := \left\{\frac{h}{k} : h, k \in \mathbb{N} \text{ with } (h,k) = 1 \text{ and } k \text{ even with } k \le j\right\}.$$

For example,

$$\mathscr{X}(4) = \mathscr{X}(5) = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}.$$

Define the functions $P_j : \mathbb{N}^4 \times \mathbb{Q} \to \mathbb{Q}$ by

$$P_j\left(\alpha, \frac{h}{k}\right) := \frac{\{\beta_j\}^2 + \frac{1}{4} - \{\beta_j\}}{H_j^2},$$

where

$$H_j := \begin{cases} \frac{k}{\gcd(k,\ 288h \pmod{k})} & \text{if } j = 0, \\ \frac{k}{\gcd(k,\ 16h \pmod{k})} & \text{if } j = 1, \\ \frac{k}{\gcd(k,\ 96h \pmod{k})} & \text{if } j \in \{2,3,4\}, \end{cases}$$

$$\beta_j := \begin{cases} \frac{96h \pmod{k}}{\gcd(k,\ 288h \pmod{k})} & \text{if } j = 0, \\ \frac{a_1 h \pmod{k}}{\gcd(k,\ 16h \pmod{k})} & \text{if } j = 1, \\ \frac{a_2 h \pmod{k}}{\gcd(k,\ 96h \pmod{k})} & \text{if } j = 2, \\ \frac{k+2a_i h \pmod{2k}}{2\gcd(k,\ 96h \pmod{k})} & \text{if } j \in \{3,4\}. \end{cases}$$

We finally define the function $F : \mathbb{N}^4 \times \mathbb{Q} \to \mathbb{Q}$ by

$$\begin{aligned} F\left(\alpha, \frac{h}{k}\right) := \frac{24}{k^2} &+ \frac{\gcd(k,\ 24h \pmod{k})^2}{k^2} \\ &- P_0\left(\alpha, \frac{h}{k}\right) - 6P_1\left(\alpha, \frac{h}{k}\right) - P_2\left(\alpha, \frac{h}{k}\right) \\ &+ P_3\left(\alpha, \frac{h}{k}\right) + P_4\left(\alpha, \frac{h}{k}\right). \end{aligned}$$

Notice that $P\left(\alpha, 1 - \frac{h}{k}\right) = P\left(\alpha, \frac{h}{k}\right)$ when $\frac{h}{k} \leq \frac{1}{2}$ which implies the same holds for $F\left(\alpha, \frac{h}{k}\right)$. We are particularly interested in the values $F\left(\alpha, \frac{h}{k}\right)$, where $\frac{h}{k} \in \mathscr{X}(24)$. We will show in the next section that $F\left(\alpha, \frac{h}{k}\right)$ is a natural bound for the exponent for the growth of $R_{3,3}(q)$ on the minor arcs. Thus, it is important to have a tight bound for this function. We have the following lemma.

Lemma 4.26. *If* $\alpha = (4, 20, 46, 70),\ (4, 68, 46, 22),\ (4, 68, 94, 70),$ *or* $(12, 20, 94, 22)$, *then for all* $\frac{h}{k} \in \mathscr{X}(24)$,

$$F\left(\alpha, \frac{h}{k}\right) < 2.$$

Remark 4.27. These choices for α and $\mathscr{X}(\bullet)$ are relevant for the proof in Sect. 4.5.4.

Proof of Lem. 4.26. Since the set $\mathscr{X}(24)$ is finite, the proof amounts to checking all of the possible cases for h and k. There are 62 cases to check for each of the vectors α. Since the evaluation of the function F is modular arithmetic, this can be checked with a computer with a simple for-loop procedure, for example in MAPLE. We find that the largest value across these α is given by $\frac{13}{9}$. $\square$

4.5.4 Proof of Thm. 4.13

As stated at the beginning of this section, we claimed that there are only a finite number of cusps we need to check. The following proposition gives us a rough bound for this number, but it more importantly tells us that all of the cusps that could cause a large pole have k even. As a reminder, we have already checked explicitly in Lem. 4.19 that $T(\tau)$ does not grow near 0, which means we do not have to investigate this case in the proof below.

Proposition 4.28. *Let $v := \frac{1}{\delta\sqrt{n}}$ with $\delta := \sqrt{192}$. With this choice, the only cusps that could lead to $T(\tau)$ having larger growth than that at $\frac{1}{2}$ are cusps $\frac{h}{k}$ with $k \leq 24$ even.*

Proof. We use the notation from the proof of Thm. 4.10, where we saw that we could write $T(\tau)$ as

$$T(\tau) \doteq \frac{\vartheta^3(96\tau; 288\tau)}{\vartheta\left(\frac{1}{2}; \tau\right)\vartheta\left(\frac{1}{2}; 24\tau\right)}\overline{S}(\tau).$$

Referring back to Rem. 4.15 we can use Lem. 4.14 to bound the combination

$$D(\tau) := \vartheta^3(96\tau; 288\tau)\overline{S}(\tau).$$

Recalling that each one of the four terms in $\overline{S}(\tau)$ is of the form

$$\frac{\vartheta(\bullet;16\tau)\vartheta(\bullet;96\tau)}{\vartheta\left(\frac{1}{2}+\bullet;96\tau\right)\vartheta\left(\frac{1}{2}+\bullet;96\tau\right)},$$

where the " $\bullet$ " are all even multiples of τ. This implies that each term in $\overline{S}(\tau)$ is invariant up to a constant under the shift $\tau\mapsto\frac{1}{2}+\tau$. This means we can apply Lem. 4.14, recalling that the bounds only depend on the factor in the second slot therein. This implies that

$$\begin{aligned}D(\tau) &\ll n^{-\frac{3}{2}}e^{\sqrt{192}\sqrt{n}\left(\frac{9}{2\cdot 288\pi}+\frac{3}{2\cdot 16\pi}+\frac{9}{2\cdot 96\pi}\right)\left(\frac{\pi^2}{6}-\varepsilon\right)}\\ &= n^{-\frac{3}{2}}e^{\sqrt{192}\sqrt{n}\left(\frac{5}{32\pi}\right)\left(\frac{\pi^2}{6}-\varepsilon\right)}\\ &= n^{-\frac{3}{2}}e^{\frac{5\sqrt{192}\pi}{192}\sqrt{n}\left(1-\frac{6}{\pi^2}\varepsilon\right)} = n^{-\frac{3}{2}}e^{\frac{5\pi}{4}\sqrt{\frac{n}{12}}\left(1-\frac{6}{\pi^2}\varepsilon\right)},\end{aligned}$$

where $\varepsilon=1-\frac{1}{\sqrt{1+M^2}}$. We define

$$G(\tau):=\frac{1}{\vartheta\left(\frac{1}{2};\tau\right)\vartheta\left(\frac{1}{2};24\tau\right)}.$$

If k is odd, by Lem. 4.24 and Cor. 4.25, we have that

$$G(\tau)\ll ve^{\frac{\pi}{2\mathbf{c}^2}\sqrt{\frac{n}{12}}}.$$

Using Lem. 4.14 on D gives

$$G(\tau)D(\tau)\ll v^{-\frac{1}{2}}e^{\pi\sqrt{\frac{n}{12}}\left(\frac{1}{2\mathbf{c}^2}+\frac{5}{4}-\frac{15\varepsilon}{2\pi^2}\right)}.$$

We want that

$$\frac{1}{2\mathbf{c}^2}+\frac{5}{4}-\frac{15\varepsilon}{2\pi^2}<1.$$

If we chose $\varepsilon=0.99$, then (for example) all $\mathbf{c}>0.998$ satisfy the above inequality. That is, the inequality is always true since $\mathbf{c}\geq 1$.

Thus, we only need to consider even k. Applying Lem. 4.24 in the even case and Prop. 4.25 gives

$$G(\tau) \ll v e^{\pi\sqrt{192n}\left(\frac{1}{96\mathbf{c}^2}+\frac{1}{4k^2}\right)} = v e^{\pi\sqrt{192n}\left(\frac{k^2+24\mathbf{c}^2}{96(\mathbf{c}k)^2}\right)} \ll v e^{\pi\sqrt{\frac{n}{12}}\left(\frac{12\cdot 25}{k^2}\right)},$$

where in the last line we used that $k \leq 24\mathbf{c}$. Combining this with the previous estimate for D, we find

$$G(\tau)D(\tau) \ll e^{\pi\sqrt{\frac{n}{12}}\left(\frac{5}{4}-\frac{15}{2\pi^2}\varepsilon+\frac{12\cdot 25}{k^2}\right)}.$$

We want that

$$\frac{5}{4} - \frac{15}{2\pi^2}\varepsilon + \frac{12\cdot 25}{k^2} < 1,$$

which is satisfied for

$$k > 5\sqrt{\frac{24}{\frac{15}{\pi^2}\varepsilon - \frac{1}{2}}}.$$

Choosing again $\varepsilon = 0.99$, we find that (for example) all $k > 24.5$ satisfy the inequality. Thus, only poles with $k \leq 24$ and k even can cause growth as large as the major arc case $k = 2$. $\square$

We are now in a position to prove our main result for the minor arcs.

Proof of Thm. 4.13. Based on the form $T(\tau)$ given in Eq. (4.8), we need to investigate functions of the form

$$D(\alpha;\tau) := \frac{\vartheta^3(96\tau;288\tau)\,\vartheta(a_1\tau;16\tau)\,\vartheta(a_2\tau;96\tau)}{\vartheta\left(\frac{1}{2}+a_3\tau;96\tau\right)\,\vartheta\left(\frac{1}{2}+a_4\tau;96\tau\right)},$$

for even integers a_i. According to Prop. 4.28 we need to check the growth of $T(\tau)$ near cusps $\frac{h}{k} \in \mathscr{X}(24)$. We know from Lem. 4.24 and Cor. 4.25 with $v = \frac{1}{\sqrt{192n}}$ that

$$G(\tau) = \frac{1}{\vartheta\left(\frac{1}{2};\tau\right)\,\vartheta\left(\frac{1}{2};24\tau\right)} \ll v e^{\frac{\pi}{48}\sqrt{192n}\left(\frac{12}{k^2}+\frac{1}{2\mathbf{c}^2}\right)}, \tag{4.5.13}$$

where we recall that

$$\mathbf{c} = \frac{k}{\gcd(k,\ 24h \pmod{k})}.$$

On the other hand, directly applying Lemmas 4.20 and 4.22 gives

$$D(\alpha;\tau) \ll v^{-\frac{3}{2}} e^{\frac{\pi}{2}\sqrt{\frac{n}{12}}\left(-P_0\left(\alpha,\frac{h}{k}\right)-6P_1\left(\alpha,\frac{h}{k}\right)-P_2\left(\alpha,\frac{h}{k}\right)+P_3\left(\alpha,\frac{h}{k}\right)+P_4\left(\alpha,\frac{h}{k}\right)\right)}.$$

Therefore,

$$G(\tau)D(\alpha;\tau) \ll v^{-\frac{1}{2}} e^{\frac{\pi}{2}\sqrt{\frac{n}{12}}F\left(\alpha;\frac{h}{k}\right)}.$$

Applying Lem. 4.26 proves the claim. □

4.5.5 Integration and proof of Thm. 4.1

We follow the approach of [19] by approximating our integral with Bessel functions. We take the standard counter-clockwise path around the origin $\gamma := \{e^{-2\pi i x} : x \in (-\frac{1}{2}, \frac{1}{2}]\}$. By Cauchy's theorem, we have

$$a(n) = \int_\gamma R_{3,3}(q) q^{-n} dx = I_1 + I_2,$$

where

$$I_1 := \int_{\left|u-\frac{1}{2}\right|<Mv} R_{3,3}\left(e^{2\pi i\tau}\right) e^{-2\pi i n\tau} du,$$
$$I_2 := \int_{\left|u-\frac{1}{2}\right|>Mv} R_{3,3}\left(e^{2\pi i\tau}\right) e^{-2\pi i n\tau} du.$$

Due to the bound in Thm. 4.13, I_1 will be our main term. Applying Thm. 4.10, we have that

$$\begin{aligned} I_1 &= \int_{\left|u-\frac{1}{2}\right|<Mv} T(\tau) e^{-2\pi i n\tau} du \\ &= \frac{e^{\frac{\pi i}{4}}\sqrt{3}}{6} \int_{\left|u-\frac{1}{2}\right|<Mv} \frac{Q_0^{-\frac{1}{192}}}{\sqrt{\tau-\frac{1}{2}}} \left(1 + O\left(e^{-\pi\sqrt{\frac{n}{12}}}\right)\right) e^{-2\pi i\tau n} du \\ &= \frac{e^{\frac{\pi i}{4}}\sqrt{3}}{6} \int_{\left|u-\frac{1}{2}\right|<Mv} \frac{e^{\overline{\frac{\pi i}{96\left(\tau-\frac{1}{2}\right)}}}}{\sqrt{\tau-\frac{1}{2}}} e^{-2\pi i\tau n} du + E_1, \end{aligned}$$

where $E_1 \ll n^{\frac{1}{4}}$. Dealing with the remaining integral, we use the substitution $w := \tau - \frac{1}{2}$, and then $w = i\kappa v$ with $v = \frac{1}{\sqrt{192n}}$ to obtain,

$$\frac{e^{\frac{\pi i}{4}}\sqrt{3}}{6}\int_{\left|u-\frac{1}{2}\right|<Mv}\frac{e^{\frac{\pi i}{96\left(\tau-\frac{1}{2}\right)}}}{\sqrt{\tau-\frac{1}{2}}}e^{-2\pi i\tau n}du = -\frac{i(-1)^n\sqrt{3}\sqrt{v}}{6}\mathbb{I}, \quad (4.5.14)$$

where we define

$$\mathbb{I} := \int_{1-iM}^{1+iM}\frac{e^{\psi\left(\kappa+\frac{1}{\kappa}\right)}}{\sqrt{\kappa}}d\kappa$$

and $\psi := \frac{\sqrt{3n}\pi}{12}$. Lem. 7 of [19] gives the asymptotic expansion of $\mathbb{I}$ for such a ψ:

$$\mathbb{I} = i\frac{\sqrt{12}}{3^{\frac{1}{4}}n^{\frac{1}{4}}}e^{\pi\sqrt{\frac{n}{12}}} + O\left(\frac{e^{\pi\sqrt{\frac{n}{12}}}}{n^{\frac{3}{4}}}\right).$$

Subbing back into Eq. (4.5.14) and sending $n \to \infty$ gives

$$I_1 \sim a(n) \sim (-1)^n\frac{\sqrt{6}}{12\sqrt{n}}e^{\pi\sqrt{\frac{n}{12}}}.$$

This completes the proof of the asymptotic formula for $a(n)$. We will now investigate the $b(n)$.

4.6 The $b(n)$

We now turn our attention to $R_{1,3}(q) =: \sum_{n\geq 0} b(n)q^n$. To begin, we note that the $b(n)$ form a weakly increasing sequence, which is apparent from the definition of $\nu(q)$.

Lemma 4.29. *Let $b(n)$ denote the n^{th} Fourier coefficient of the function $\nu(-q)$. Then the sequence $\{b(n)\}_{n=0}^{\infty}$ is weakly increasing and $b(n) \geq 0$.*

We will show the bound in Thm. 2.29 for $R_{1,3}$ as $\tau \to 0$ with $\tau \in \mathbb{H}$, which is sufficient to show the bound for general z since we can define an even extension of $R_{1,3}$ into the lower half plane to get a function on all of $\mathbb{C}$.

4.6.1 Growth near $\tau = 0$

We focus on the Appell sum $\mu\left(\frac{5}{12}, \frac{1}{4}; -\frac{1}{12\tau}\right)$,

$$\mu\left(\frac{5}{12}, \frac{1}{4}; -\frac{1}{12\tau}\right) = \frac{e^{\frac{5\pi i}{12}}}{\vartheta\left(\frac{1}{4}; -\frac{1}{12\tau}\right)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q_0^{\frac{n(n+1)}{24}} e^{\frac{\pi i n}{2}}}{1-e^{\frac{5\pi i}{6}} q_0^{\frac{n}{12}}}$$
$$= \frac{e^{\frac{5\pi i}{12}}}{\vartheta\left(\frac{1}{4}; -\frac{1}{12\tau}\right)} \left(\frac{1}{1-e^{\frac{5\pi i}{6}}} + \sum\nolimits_{n>0} \left(\frac{(-1)^n q_0^{\frac{n(n+1)}{24}} e^{\frac{\pi i n}{2}}}{1-e^{\frac{5\pi i}{6}} q_0^{\frac{n}{12}}} + \frac{(-1)^n q_0^{\frac{n(n-1)}{24}} e^{-\frac{\pi i n}{2}}}{1-e^{\frac{5\pi i}{6}} q_0^{\frac{-n}{12}}}\right)\right). \tag{4.6.1}$$

The last line follows by splitting the sum into $n < 0$ and $n > 0$, and then swapping $n \mapsto -n$ in the sum over $n < 0$. We then have,

$$\sum_{n>0} \left(\frac{(-1)^n q_0^{\frac{n(n+1)}{24}} e^{\frac{\pi i n}{2}}}{1-e^{\frac{5\pi i}{6}} q_0^{\frac{n}{12}}} + \frac{(-1)^n q_0^{\frac{n(n-1)}{24}} e^{-\frac{n\pi i}{2}}}{1-e^{\frac{5\pi i}{6}} q_0^{-\frac{n}{12}}}\right) = O\left(q_0^{\frac{1}{12}}\right).$$

We substitute this back into Eq. (4.6.1) to obtain

$$\mu\left(\frac{5}{12}, \frac{1}{4}; -\frac{1}{12\tau}\right) = \frac{-1}{2i\sin\left(\frac{5\pi}{12}\right)\vartheta\left(\frac{1}{4}; -\frac{1}{12\tau}\right)}\left(1 + O\left(q_0^{\frac{1}{12}}\right)\right). \tag{4.6.2}$$

We can use the triple product formula in Eq. 2.1.8 to deal with the ϑ-function to find as $\tau \to 0$ in a fixed angular region that

$$\vartheta\left(\frac{1}{4}; -\frac{1}{12\tau}\right) = -ie^{-\frac{\pi i}{4}} q_0^{\frac{1}{96}} \left(q_0^{\frac{1}{12}}; q_0^{\frac{1}{12}}\right)_\infty \left(i; q_0^{\frac{1}{12}}\right)_\infty \left(-iq_0^{\frac{1}{12}}; q_0^{\frac{1}{12}}\right)_\infty$$
$$\sim -2\sin\left(\frac{\pi}{4}\right) q_0^{\frac{1}{96}}. \tag{4.6.3}$$

Substituting Eqs. (4.6.2) and (4.6.3) into (4.6.1) and using Lem. 2.22, we have as $\tau \to 0$ in a fixed angular region that

$$\begin{aligned}\mu(5\tau, 3\tau; 12\tau) &= -q^2 \frac{\mu\left(\frac{5}{12}, \frac{1}{4}; -\frac{1}{12\tau}\right)}{\sqrt{-12i\tau}} + \frac{h(2\tau; 12\tau)}{2i} \\ &\sim -\frac{q_0^{-\frac{1}{96}}}{4i\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{12}\right)\sqrt{-12i\tau}}.\end{aligned}$$

Therefore, we can state the following.

Theorem 4.30. *As $\tau \to 0$ within a fixed angular region, we have the estimate*

$$\mu(5\tau, 3\tau; 12\tau) \sim -\frac{e^{\frac{\pi i}{48\tau}}}{4i\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{12}\right)\sqrt{-12i\tau}}.$$

We now show that the η-product

$$\frac{\eta\left(\tau + \frac{1}{2}\right)\eta\left(3\tau + \frac{1}{2}\right)\eta(12\tau)}{\eta(2\tau)\eta(6\tau)},$$

has similar growth near $\tau = 0$. To get the behavior near 0 of the η-function involving the $\frac{1}{2}$ shift, we can proceed as we did in the proof of Lem. 4.9. Define the transformation $A := \left(\begin{smallmatrix}1 & 0\\ 2 & 1\end{smallmatrix}\right)$, with $w := -\frac{1}{4\tau} - \frac{1}{2}$ and then send $\tau \to 0$. We have that

$$\eta(A\tau) = \nu(A)(2\tau+1)^{\frac{1}{2}}\eta(\tau) = e^{-\frac{\pi i}{6}}(2\tau+1)^{\frac{1}{2}}\eta(\tau). \tag{4.6.4}$$

Lem. 2.24 and Eq. (4.6.4) say that near 0,

$$\begin{aligned}\eta(Aw) &= e^{-\frac{\pi i}{6}}(2w+1)^{\frac{1}{2}}\eta(w) = e^{-\frac{\pi i}{6}}(2w+1)^{\frac{1}{2}}e^{\frac{\pi i w}{12}}\left(e^{2\pi i w}; e^{2\pi i w}\right)_\infty \\ &\sim e^{-\frac{5\pi i}{24}}\left(-\frac{1}{2\tau}\right)^{\frac{1}{2}}e^{-\frac{\pi i}{48\tau}} = \frac{ie^{-\frac{5\pi i}{24}}}{\sqrt{2\tau}}q_0^{\frac{1}{96}}.\end{aligned} \tag{4.6.5}$$

Therefore as $\tau \to 0$ using Eq. (4.6.5),

$$\eta\left(\tau+\frac{1}{2}\right) \sim \eta\left(Aw\right) \sim \frac{ie^{-\frac{5\pi i}{24}}}{\sqrt{2\tau}} q_0^{\frac{1}{96}}, \tag{4.6.6}$$

$$\eta\left(3\tau+\frac{1}{2}\right) \sim \eta\left(A(3w)\right) \sim \frac{ie^{-\frac{5\pi i}{24}}}{\sqrt{6\tau}} q_0^{\frac{1}{288}}. \tag{4.6.7}$$

The other η-products satisfy the estimates near zero directly from Lem. 2.24 using the substitutions $\tau \mapsto 2\tau$, $\tau \mapsto 6\tau$, and $\tau \mapsto 12\tau$ respectively, we have as $\tau \to 0$ in a fixed angular region:

$$\frac{\eta\left(\tau+\frac{1}{2}\right)\eta\left(3\tau+\frac{1}{2}\right)\eta(12\tau)}{\eta(2\tau)\eta(6\tau)} \sim i\frac{e^{-\frac{5\pi i}{12}}}{\sqrt{-12i\tau}} q_0^{-\frac{1}{96}}. \tag{4.6.8}$$

4.6.2 Proof of the estimate for the $b(n)$

We now prove the main theorem for the $b(n)$.

Theorem 4.31. *Let $b(n)$ denote the coefficients of $\nu(-q)$. Then as $n \to \infty$,*

$$b(n) \sim \left(\frac{1}{2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{12}\right)}+1\right)\frac{e^{\pi\sqrt{\frac{n}{6}}}}{\sqrt{24n}}.$$

Proof. Combining Thm. 4.30 and Eq. (4.6.8), we have as $\tau \to 0$ that

$$\begin{aligned}\nu(-q) = R_{1,3}(q) &\sim 2i\frac{q_0^{-\frac{1}{96}}}{4i\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{12}\right)\sqrt{-12i\tau}} + \frac{1}{\sqrt{-12i\tau}} q_0^{\frac{1}{96}} \\ &= \frac{e^{\frac{\pi i}{48\tau}}}{\sqrt{-12i\tau}}\left(\frac{1}{2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{12}\right)}+1\right),\end{aligned} \tag{4.6.9}$$

where $q^{-\frac{1}{2}} \to 1$ in the limit $\tau \to 0$. Making the substitution $\tau := \frac{it}{2\pi}$ with $t > 0$, we have that as $t \to 0^+$ that

$$R_{1,3}\left(e^{-t}\right) = \left(\frac{1}{2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{12}\right)} + 1\right)\sqrt{\frac{\pi}{6}}\frac{e^{\frac{\pi^2}{24t}}}{\sqrt{t}}.$$

The bound for the complex variable, z, in Thm. 2.29 is trivially satisfied by combining the estimates in Thm. 4.30 and Eq. (4.6.8).

Define $A := \frac{\pi^2}{24}$ and $\lambda := \left(\frac{1}{2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{12}\right)} + 1\right)\sqrt{\frac{\pi}{6}}$. By Thm. 2.29 with $\alpha = \frac{1}{2}$, we have that

$$\begin{aligned} b(n) &\sim \frac{\lambda}{2\sqrt{\pi}n^{\frac{1}{2}}}e^{2\sqrt{An}} = \left(\frac{1}{2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{12}\right)} + 1\right)\sqrt{\frac{\pi}{6}}\frac{1}{2\sqrt{\pi}\sqrt{n}}e^{\sqrt{\frac{n}{6}}} \\ &\sim \left(\frac{1}{2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{12}\right)} + 1\right)\frac{e^{\pi\sqrt{\frac{n}{6}}}}{\sqrt{24n}}, \end{aligned}$$

which shows the claim. $\square$

Chapter 5

Twisted eta products and sign changes in partitions

This chapter is a modified version of a preprint written by Walter Bridges, Johann Franke, and the author of this thesis. The preprint can be found on the Arxiv [17] and is currently under review for publication. Much more detail is included here, which was unable to be included in the submitted version due to issues of length. We hope that this provides insight and clarity to the reader, especially in the proofs of supplementary lemmas. As a result, the chapter in itself is written to be self-contained. However, the reader is encouraged to reference the groundwork laid by Wright [68], Boyer and Goh [15], and Parry [61]. We also point out that we include our original proof of Lem. 5.24, which unbeknownst to the authors, was proven (as well as much more, including convexity) a few days after our work appeared in Nov. 2021 by Boyer and Parry in [16] using much more robust methodology developed by Lewis in the 1980's [51]. In the published version, we do not include this proof, since it was not our main objective.

5.1 Twisted eta products: from Wright to Parry

In this this chapter, we will study the *twisted eta product* defined by $(\zeta q;q)_\infty^{-1}$, with ζ a root of unity. Our original motivation stems from wanting to answer the following question: For ordinary partitions whose number of parts are congruent to a modulo b, is there a bias among residue classes for large n, and if so how are the residue classes distributed[1]? By "bias" we mean bias past the main exponential term, or rather how does the difference $p(a_1,b;n)-p(a_2,b;n)$ behave for $a_1 \neq a_2$? Here and throughout, $p(a,b;n)$ denotes the number of partitions with number of parts congruent to a modulo b. This distinction is important to note since it is well-known in analogy to our Ch. 3 of this thesis, that that the numbers $p(a,b;n)$ are asymptotically equidistributed; i.e.,

$$p(a,b;n) \sim \frac{p(n)}{b}, \qquad n \to \infty. \tag{5.1.1}$$

The proof of Eq. (5.1.1) again begins by writing the generating function for $p(a,b;n)$ in terms of non-modular eta-products twisted by roots of unity modulo b:

$$\sum_{n\geq 0} p(a,b;n)q^n = \frac{1}{b}\left(\frac{q^{\frac{1}{24}}}{\eta(\tau)} + \sum_{1\leq j\leq b-1} \frac{\zeta_b^{-ja}}{\left(\zeta_b^j q;q\right)_\infty}\right), \tag{5.1.2}$$

where $\zeta_b := e^{\frac{2\pi i}{b}}$. Since the first term does not depend on a and dominates the other summands, Eq. (5.1.1) can be seen as a corollary of Eq. (5.1.2). Similar results have also been proven for related statistics. For example, Males [55] showed that the Dyson rank function $N(a,b;n)$ (the number of partitions with Dyson rank congruent to a

[1] A similar sounding, but entirely different problem is to study the total number of parts that are all congruent to a modulo b. This problem was studied in detail by Beckwith–Mertens [13, 14] for ordinary partitions and recently by Craig [34] for distinct parts partitions.

modulo b) is asymptotically equidistributed; the rank of a partition is defined to be the largest part minus the number of parts. Males' proof exploited the mock modularity of the generating function for $N(a, b; n)$. In contrast, the twisted eta-products in Eq. (5.1.2) are not mock modular forms or MMMFs.

Moreover, if we now consider differences, $p(a_1, b; n) - p(a_2, b; n)$, the main terms in Eq. (5.1.2) cancel and the behavior must be determined by secondary terms. In the following example we look at the differences of two modulo 5 partition functions.

Example 5.1. Consider the case $a_1 = 1$, $a_2 = 4$, and $b = 5$. Let

$$\sum_{n\geq 0} \left(p(1,5,n) - p(4,5,n)\right) q^n := P_{1,4,5}(q).$$

Then,

$$\begin{aligned} P_{1,4,5}(q) = q &+ q^2 + q^3 - q^7 - 2q^8 - 3q^9 - 4q^{10} - 4q^{11} - 5q^{12} \\ &- 6q^{13} - 7q^{14} - 7q^{15} - 7q^{16} - \cdots + 2q^{22} + \cdots \\ &+ 109q^{40} + \cdots + 11q^{48} - 24q^{49} - \cdots - 1998q^{75} \\ &- \cdots - 266q^{85} + 163q^{86} \cdots + 40511q^{120} + \cdots \\ &+ 3701q^{133} - 3587q^{134} - \cdots . \end{aligned}$$

Notice that the sign change pattern is more elusive than $(-1)^n$ which we saw with the $a(n)$ in Ch. 4. We plot the above coefficients with log-scaling below in Fig. 5.1.

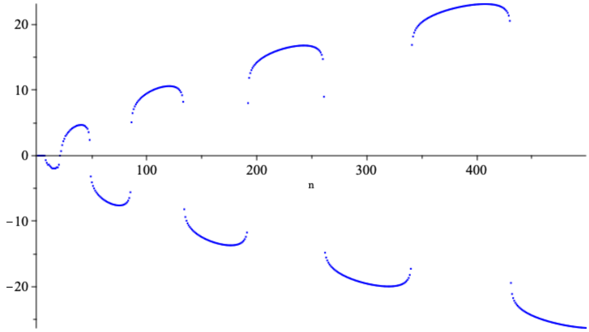

Figure 5.1: For $b = 5$, $a_1 = 1$, and $a_2 = 4$, we plot the difference $p(1, 5; n) - p(4, 5; n)$ with log-scaling. The abrupt vertical changes in the graph indicate the location of the sign changes in the sequence.

Our first result predicts this oscillation as follows. As usual, we define the *dilogarithm* for $|z| \leq 1$ as the $s = 2$ case of the previously defined polylogarithm:

$$\mathrm{Li}_2(z) := \sum_{n \geq 1} \frac{z^n}{n^2}.$$

Theorem 5.2. *Let $b \geq 5$ an integer. Then for any two residue classes $a_1 \neq a_2$ modulo b, we have as $n \to \infty$*

$$\frac{p(a_1, b; n) - p(a_2, b; n)}{Bn^{-\frac{3}{4}} \exp\left(2\lambda_1 \sqrt{n}\right)} = \cos\left(\beta + 2\lambda_2 \sqrt{n}\right) + o(1),$$

where $\lambda_1 + i\lambda_2 := \sqrt{\mathrm{Li}_2(\zeta_b)}$, $B > 0$, and $0 \leq \beta < 2\pi$ are implicitly defined by

$$Be^{i\beta} := \frac{1}{b}\left(\zeta_b^{-a_1} - \zeta_b^{-a_2}\right)\sqrt{\frac{(1 - \zeta_b)(\lambda_1 + i\lambda_2)}{\pi}}.$$

A more general version of Thm. 5.2 holds for values $b \geq 2$ where the special cases $b \in \{2, 3, 4\}$ have to be treated differently. This is taken care of in Thm. 5.8. Fig. 5.2 shows that the prediction above is surprisingly accurate even for small n.

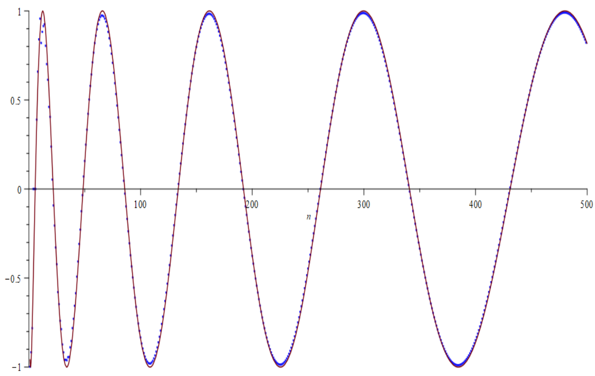

Figure 5.2: The plot shows the sign changes of $p(1,5;n) - p(4,5;n)$. The blue dots depict the function $\frac{p(1,5;n)-p(4,5;n)}{Bn^{-\frac{3}{4}}e^{2\lambda_1\sqrt{n}}}$ and the red line is the asymptotic prediction $\cos\left(\beta + 2\lambda_2\sqrt{n}\right)$. The approximate values of the constants are $B \approx 0.23268$, $\beta \approx 1.4758$, $\lambda_1 \approx 0.72984$, and $\lambda_2 \approx 0.68327$.

The proof of Thm. 5.2 makes use of Eq. (5.1.2) and a detailed study of the coefficients

$$\sum_{n\geq 0} Q_n(\zeta)q^n := (\zeta q; q)_\infty^{-1}. \tag{5.1.3}$$

Theorem 5.3. *Let $b \geq 5$. Then we have as $n \to \infty$*

$$Q_n\left(\zeta_b\right) \sim \frac{\sqrt{1-\zeta_b}\,\sqrt[4]{\mathrm{Li}_2\left(\zeta_b\right)}}{2\sqrt{\pi}n^{\frac{3}{4}}} \exp\left(2\sqrt{\mathrm{Li}_2\left(\zeta_b\right)}\sqrt{n}\right).$$

Again, a general version with the sporadic cases $b \in \{2,3,4\}$ is stated in Thm. 5.5. In fact, the asymptotic behavior of $Q_n(\zeta)$ is "not uniform" in ζ. One can check this for instance by comparing the result of Thm. 5.3 with formula Eq. (2.2.3) (which corresponds to the case $\zeta = 1$). That is to say,

$$\lim_{\zeta_b \to 1} Q_n(\zeta_b) = 0 \neq \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Historically, the coefficients in Eq. (5.1.3) were studied by Wright [68] in the case that $\zeta \in \mathbb{R}$. The problem of when ζ is inside the punctured unit disk $0 < |\zeta| < 1$ was settled more recently by Parry in 2017 [61]. The setup for Parry's work began in 2007 with Boyer and Goh's numerical experiments [15]. In [15] authors showed that the open unit disk can be divided into three regions such that the growth of the coefficients of $(\zeta q; q)_\infty^{-1}$ is uniform on compact subsets of each region. These regions come from studying the saddle point of the integrand for the coefficients in Cauchy's theorem, which in our case, is complex: a contrast to what one usually finds for cases like the partition function. The work of Boyer and Goh [15] can be summarized as follows. Define

$$U_k(\theta) := \mathrm{Re}\left(\frac{\sqrt{\mathrm{Li}_2(e^{ik\theta})}}{k}\right), \qquad \text{for } 0 \leq \theta \leq \pi, \tag{5.1.4}$$

and

$$0 < \theta_{13} < \frac{2\pi}{3} < \theta_{23} < \pi,$$

where each θ_{jk} is an implicit solution to $U_j(\theta) = U_k(\theta)$. Here,

$$\theta_{13} = 2.06672\ldots \quad \text{and} \quad \theta_{23} = 2.36170\ldots.$$

Theorem 5.4 (see discussion prior to Thm. 2, [15])**.** *For $0 \leq \theta \leq \pi$, we have*

$$\max_{k \geq 1} U_k(\theta) = \begin{cases} U_1(\theta) & \text{if } \theta \in [0, \theta_{13}), \\ U_2(\theta) & \text{if } \theta \in [\theta_{23}, \pi], \\ U_3(\theta) & \text{if } \theta \in [\theta_{13}, \theta_{23}). \end{cases}$$

As we will see, these regions determine our major arcs. This also was the case for Parry's study of $(\zeta q; q)_\infty^{-1}$ when $|\zeta| < 1$ [61]. We also shed light on the fact that the functions $(\zeta q; q)_\infty^{-1}$ also have been studied in recent work of Bringmann, Craig, Males, and Ono [18] in the context of the distribution of homology of Hilbert schemes and t-hook lengths.

Parry's application of the circle method when ζ is inside the unit disk gives a template for our proof, provided one can bound the minor arcs sufficiently. As we will see, this is a significant challenge in the root of unity case. For example, one technical issue we have to overcome when ζ is a root of unity is that the series representation of the polylogarithm $\mathrm{Li}_s(\zeta)$ does not converge for $\mathrm{Re}(s) < 0$. We will deal with the minor arcs in Sect. 5.5.

Our results can be extended further to general linear combinations of $p(a, b; n)$, with b fixed. Each combination $\sum_{0 \le a < b} c_a p(a, b; n)$ corresponds to a polynomial

$$P(x) := \sum_{0 \le j < b} c_a x^a,$$

via the mapping $p(a, b; n) \mapsto x^a$. Perhaps the most combinatorially interesting cases are $c_a \in \{0, \pm 1\}$; i.e., differences of partition numbers. This means that for any two nonempty disjoint subsets $S_1, S_2 \subset \{0, 1, \ldots, b-2, b-1\}$ of integers, we consider the differences

$$\sum_{a \in S_1} p(a, b; n) - \sum_{a \in S_2} p(a, b; n).$$

The asymptotic behavior of these differences is described in Thm. 5.10. When choosing the coefficients c_a properly we can reduce the growth of the difference terms by canceling main terms. By doing so, we see that actually all formulas of Thm. 5.3 are required (as well as the sporadic cases described in Thm. 5.5). In contrast, one can deduce formula Eq. (5.1.1) by only elementary means without a thorough analysis of the coefficients $Q_n(\zeta)$.

The chapter is organized as follows. In Sect. 5.2, we state our main results and applications, as well as work through some examples. In Sect. 5.3, we collect some tools and prove some key lemmas for the behavior of functions appearing in the study of the major and minor arcs. This includes a careful study of the dilogarithm function $\mathrm{Li}_2(z)$ for values $|z| = 1$. A proof of the main theorem, regarding the asymptotic behavior of the coefficients of $(\zeta q; q)_\infty^{-1}$ is given in Sect. 5.4 on the assumption of the needed bounds for the minor arcs. The minor arcs, whose study is the main focus of this chapter, are dealt with in Sect. 5.5.

5.2 Main results and applications

We collect all of our main results in this section including the asymptotic formulas for $Q_n(\zeta)$ and the applications to partitions described in the previous section. We begin with the results for $Q_n(\zeta)$.

5.2.1 Asymptotic formulas for $Q_n(\zeta)$

In this section, we record the general asymptotic formulas for $Q_n(\zeta)$ where ζ is any root of unity. This is our Result C stated in the introduction of this thesis. Note that $\overline{Q_n(\zeta)} = Q_n(\overline{\zeta})$, so it suffices to find asymptotic formulas for ζ in the upper-half plane. Following [61], we define

$$\omega_{h,k}(z) := \prod_{j=1}^{k} \left(1 - z\zeta_k^{-jh}\right)^{\frac{j}{k} - \frac{1}{2}}. \tag{5.2.1}$$

We have the following asymptotic formulas.

Theorem 5.5. (1) *If* $2\pi \frac{a}{b} \in (0, \theta_{13})$, *then*

$$Q_n\left(\zeta_b^a\right) \sim \frac{\sqrt{1-\zeta_b^a}\,\mathrm{Li}_2\left(\zeta_b^a\right)^{\frac{1}{4}}}{2\sqrt{\pi} n^{\frac{3}{4}}} \exp\left(2\sqrt{\mathrm{Li}_2\left(\zeta_b^a\right)}\sqrt{n}\right).$$

(2) *If* $2\pi\frac{a}{b} \in (\theta_{23}, \pi)$, *then*

$$Q_n\left(\zeta_b^a\right) \sim \frac{(-1)^n\sqrt{1-\zeta_b^a}\mathrm{Li}_2\left(\zeta_b^{2a}\right)^{\frac{1}{4}}}{2\sqrt{2\pi}n^{\frac{3}{4}}} \exp\left(\sqrt{\mathrm{Li}_2\left(\zeta_b^{2a}\right)}\sqrt{n}\right).$$

(3) *If* $2\pi\frac{a}{b} \in (\theta_{13}, \theta_{23}) \setminus \left\{\frac{2\pi}{3}\right\}$, *then*

$$Q_n\left(\zeta_b^a\right) \sim \left(\zeta_3^{-n}\omega_{1,3}(\zeta_b^a)+\zeta_3^{-2n}\omega_{2,3}(\zeta_b^a)\right)\frac{\mathrm{Li}_2(\zeta_b^{3a})^{\frac{1}{4}}}{2\sqrt{3\pi}n^{\frac{3}{4}}} \exp\left(\frac{2}{3}\sqrt{\mathrm{Li}_2(\zeta_b^{3a})}\sqrt{n}\right).$$

(4) *We have*

$$Q_n\left(\zeta_3\right) \sim \frac{\zeta_3^{-2n}(1-\zeta_3^2)^{\frac{1}{6}}(1-\zeta_3)^{\frac{1}{2}}\Gamma\left(\frac{1}{3}\right)}{2(6\pi n)^{\frac{2}{3}}} \exp\left(\frac{2\pi}{3}\sqrt{\frac{n}{6}}\right).$$

Remark 5.6. Recall that we have the asymptotic formula Eq. (2.2.3) for $Q_n(1) = p(n)$, whereas for $Q_n(-1)$, standard combinatorial methods give

$$\begin{aligned}\sum_{n\geq 0} Q_n(-1)q^n &= \frac{1}{(-q;q)_\infty} = \frac{(q;q)_\infty}{(q^2;q^2)_\infty} = \left(q;q^2\right)_\infty \\ &= \sum_{n\geq 0}(-1)^n p_{\mathcal{DO}}(n)q^n,\end{aligned}$$

where $p_{\mathcal{DO}}(n)$ counts the number of partitions of n into distinct odd parts. An asymptotic formula for $p_{\mathcal{DO}}(n)$ can be worked out using standard techniques. For example, Thm. 2.29 along with the modularity of the Dedekind η-function yields

$$Q_n(-1) = (-1)^n p_{\mathcal{DO}}(n) \sim \frac{(-1)^n}{2(24)^{\frac{1}{4}}n^{\frac{3}{4}}} \exp\left(\pi\sqrt{\frac{n}{6}}\right).$$

Note the lack of uniformity in the asymptotic formulas for ζ near ± 1; in particular, the asymptotic formulas for $Q_n(1)$ and $Q_n(-1)$ cannot be obtained by taking $\frac{a}{b} \to \pm 1$ in Cases (1) and (2) of Thm. 5.5.

5.2.2 Applications to differences of partition functions

In this section, we apply Thm. 5.5 to functions involving combinations of $p(a,b;n)$. We point out that we use a result in this section which we do not prove until later in Lem. 5.24. We do this in order to precisely state our main results early on without too much technical clutter. The reader is encouraged to visit the statement of that lemma, but is not required to in order to understand the upcoming discussion. The following elementary proposition relates the numbers $p(a,b;n)$ to the coefficients $Q_n(\zeta_b^j)$.

Proposition 5.7. *We have*

$$p(a,b;n) = \frac{1}{b} \sum_{0 \leq j \leq b-1} \zeta_b^{-ja} Q_n\left(\zeta_b^j\right).$$

Proof. Note that the number of partitions $\lambda \vdash n$ with largest part $\ell(\lambda)$ equals the number of partitions with number of parts equal to $\ell(\lambda)$. One can see this by conjugation of the Ferrers diagram which is covered in Ch. 1 of [5]. Using orthogonality of roots of unity, we can write the indicator functions of congruence classes as

$$1_{x \equiv a \pmod b} = \frac{1}{b} \sum_{j=0}^{b-1} \zeta_b^{-ja} \zeta_b^{jx},$$

which implies by standard combinatorial techniques (see [5], Ch. 1) that $Q_n(z) = \sum_{\lambda \vdash n} z^{\ell(\lambda)}$. Hence,

$$\begin{aligned} p(a,b;n) &= \sum_{\lambda \vdash n} \frac{1}{b} \sum_{0 \leq j \leq b-1} \zeta_b^{-ja} \zeta_b^{j\ell(\lambda)} = \frac{1}{b} \sum_{0 \leq j \leq b-1} \zeta_b^{-ja} \sum_{\lambda \vdash n} \left(\zeta_b^j\right)^{\ell(\lambda)} \\ &= \frac{1}{b} \sum_{0 \leq j \leq b-1} \zeta_b^{-ja} Q_n\left(\zeta_b^j\right), \end{aligned}$$

which completes the proof. $\square$

The asymptotics for differences of $p(a,b;n)$ can therefore be computed by using the asymptotic formulas found for $Q_n(\zeta)$ in Thm. 5.5. In particular, the equidistribution of the largest part in congruence classes follows immediately: $\frac{p(a,b;n)}{p(n)} \sim \frac{1}{b}$. Considering simple differences, $p(a_1,b;n) - p(a_2,b;n)$, it follows from Prop. 5.7 that when rewriting each $p(a,b;n)$ in terms of Q_n the two summands $Q_n(1) = p(n)$ cancel. With a few exceptions, the dominant terms are always $Q_n(\zeta_b)$ and $Q_n(\zeta_b^{b-1})$.

Theorem 5.8. *Let $0 \leq a_1 < a_2 \leq b-1$. We have the following asymptotic formulas for various cases of the modulus b:*
(1) *For $b = 2$, we have $p(0,2,n) - p(1,2,n) = Q_n(-1)$.*
(2) *For $b = 3$, we have*

$$\frac{p(a_1,3,n) - p(a_2,3,n)}{A_{a_1,a_2} n^{-\frac{2}{3}} \exp\left(\frac{2\pi}{3}\sqrt{\frac{n}{6}}\right)} = \cos\left(\alpha_{a_1,a_2} - \frac{4\pi n}{3}\right) + o(1),$$

where $A_{a_1,a_2} \geq 0$ and $\alpha_{a_1,a_2} \in [0,2\pi)$ are defined by

$$A_{a_1,a_2} e^{i\alpha_{a_1,a_2}} = (\zeta_3^{-a_1} - \zeta_3^{-a_2})(1-\zeta_3^2)^{\frac{1}{6}}(1-\zeta_3)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{3}\right)}{3(6\pi)^{\frac{2}{3}}}.$$

(3) *For $b = 4$, and a_1, a_2 of opposite parity, we have*

$$p(a_1,4,n) - p(a_2,4,n) \sim \frac{(-1)^{a_1} - (-1)^{a_2}}{4} Q_n(-1).$$

(4) *For $b \geq 5$, or for $b = 4$ and a_1, a_2 of the same parity, we have*

$$\frac{p(a_1,b;n) - p(a_2,b;n)}{B_{a_1,a_2,b} n^{-\frac{3}{4}} \exp(2\lambda_1\sqrt{n})} = \cos\left(\beta_{a_1,a_2,b} + 2\lambda_2\sqrt{n}\right) + o(1),$$

where $\lambda_1 + i\lambda_2 = \sqrt{\mathrm{Li}_2(\zeta_b)}$ and $B_{a_1,a_2,b} \geq 0$ and $\beta_{a_1,a_2,b} \in [0,2\pi)$ are defined by

$$B_{a_1,a_2,b} e^{i\beta_{a_1,a_2,b}} = \frac{\left(\zeta_b^{-a_1} - \zeta_b^{-a_2}\right)}{b} \sqrt{\frac{(1-\zeta_b)(\lambda_1 + i\lambda_2)}{\pi}}.$$

Proof. When $b=2$, Thm. 5.8 follows from Prop. 5.7. Let $b \geq 5$. By Prop. 5.7, it follows that

$$p(a_1,b;n)-p(a_2,b;n) = \frac{2}{b}\mathrm{Re}\left(\zeta_b^{-a}Q_n(\zeta_b)\right)+\frac{1}{b}\sum_{1\leq j\leq b-2}\zeta_b^{-ja}Q_n\left(\zeta_b^j\right).$$

Upon dividing both sides by $B_{a_1,a_2,b}n^{-\frac{3}{4}}\exp(2\lambda_1\sqrt{n})$, Lem. 5.24 implies that the sum on the left is $O(e^{-c\sqrt{n}})$, for some $c>0$, and it then follows from Thm. 5.5 that

$$\frac{p(a_1,b;n)-p(a_2,b;n)}{B_{a_1,a_2,b}n^{-\frac{3}{4}}\exp\left(2\lambda_1\sqrt{n}\right)} = \mathrm{Re}\left(e^{i\left(\beta_{a_1,a_2,b}+2\lambda_2\sqrt{n}\right)}\right)+o(1)+o\left(e^{-c\sqrt{n}}\right), \tag{5.2.2}$$

which gives the claim of Thm. 5.8.

The other cases are proved similarly by noting that Lem. 5.24 and Thm. 5.5 imply that the $Q_n(\zeta_b)$ and $Q_n(\zeta_b^{-1})$ terms always dominate $Q_n(\zeta_b^j)$ for $j \neq \pm 1$, except in the case that $b=4$, where $Q_n(-1)$ dominates when a_1, a_2 have opposite parity. But $Q_n(-1)$ vanishes from the sum in Prop. 5.7 when $b=4$ and a_1, a_2 have the same parity, which leads to the third case in Thm. 5.8. □

More generally, if $P_{\mathbf{v}}(x) := \sum_{0\leq a\leq b-1} v_a x^a \in \mathbb{R}[x]$ with $\mathbf{v} := (v_0, ..., v_{b-1}) \in \mathbb{R}^b$, then Thm. 5.5 implies asymptotic formulas for any weighted count $\sum_{a=0}^{b-1} v_a p(a,b;n)$. To state our general theorem, we let

$$L(e^{i\theta}) := \begin{cases} \sqrt{\mathrm{Li}_2(e^{i\theta})} & \text{if } 0 \leq \theta < \theta_{13}, \\ \frac{\sqrt{\mathrm{Li}_2(e^{3i\theta})}}{3} & \text{if } \theta_{13} < \theta < \theta_{23}, \\ \frac{\sqrt{\mathrm{Li}_2(e^{2i\theta})}}{2} & \text{if } \theta_{23} < \theta \leq \pi, \end{cases}$$

which is depicted in Fig. 5.3. Let $\mathcal{Z}(P_{\mathbf{v}})$ be the roots of $P_{\mathbf{v}}$, and let $\lambda_1 + i\lambda_2 = L(\zeta_b^{a_0})$ for $a_0 \leq \frac{b}{2}$, be defined via the equation

$$\mathrm{Re}\left(L(\zeta_b^{a_0})\right) = \max_{\substack{0\leq a\leq \frac{b}{2} \\ \zeta_b^a \notin \mathcal{Z}(P_{\mathbf{v}})}} \mathrm{Re}\left(L(\zeta_b^a)\right).$$

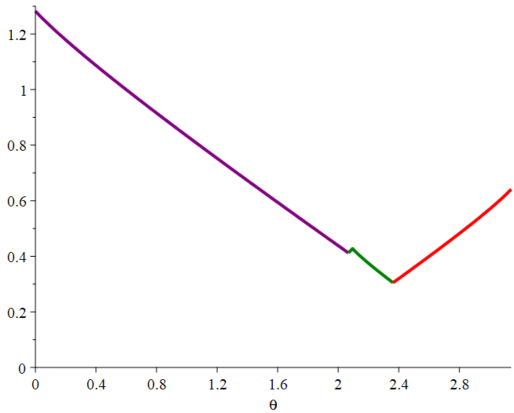

Figure 5.3: $\mathrm{Re}(L(e^{i\theta}))$ for $0 \le \theta \le \pi$. Each one of the colors from left to right corresponds to the functions defined in Eq. (5.1.4): $U_1(\theta)$, $U_3(\theta)$, and $U_2(\theta)$.

Theorem 5.9. *With notation as above, we have the following asymptotic formulas.*
(1) *If* $a_0 = 0$, *then* $\sum_{0\le a\le b-1} v_a p(a, b; n) \sim \frac{P_{\mathbf{v}}(1)}{b} p(n)$.
(2) *If* $0 < 2\pi\frac{a_0}{b} < \theta_{13}$, *then*

$$\frac{\sum_{0\le a\le b-1} v_a p(a, b; n)}{A_{a_0,b,\mathbf{v}} n^{-\frac{3}{4}} \exp(2\lambda_1\sqrt{n})} = \cos\left(\alpha_{a_0,b,\mathbf{v}} + 2\lambda_2\sqrt{n}\right) + o(1),$$

where $A_{a_0,b,\mathbf{v}} \ge 0$ *and* $\alpha_{a_0,b,\mathbf{v}} \in [0, 2\pi)$ *are defined by*

$$A_{a_0,b,\mathbf{v}} \cdot e^{i\alpha_{a_0,b,\mathbf{v}}} = \frac{P_{\mathbf{v}}(\zeta_b^{-a_0})}{b}\sqrt{\frac{(\lambda_1 + i\lambda_2)(1 - \zeta_b^{a_0})}{\pi}}.$$

(3) *If* $\theta_{13} < 2\pi\frac{a_0}{b} < \theta_{23}$ *and* $a_0 \ne \frac{b}{3}$, *then*

$$\frac{\sum_{0\le a\le b-1} v_a p(a, b; n)}{A_{a_0,b,\mathbf{v}} n^{-\frac{3}{4}} \exp(2\lambda_1\sqrt{n})} = B_{a_0,b} \cos\left(\alpha_{a_0,b,\mathbf{v}} + \beta_{a_0,b} - \frac{2\pi n}{3} + 2\lambda_2\sqrt{n}\right)$$

$$+C_{a_0,b}\cos\left(\alpha_{a_0,b,\mathbf{v}}+\gamma_{a_0,b}-\tfrac{4\pi n}{3}+2\lambda_2\sqrt{n}\right)+o(1),$$

where $B_{a_0,b}, C_{a_0,b} \geq 0$ *and* $\beta_{a_0,b}, \gamma_{a_0,b} \in [0,2\pi)$ *are defined by*

$$B_{a_0,b}e^{i\beta_{a_0,b}} = \omega_{1,3}(\zeta_b^{a_0}), \text{ and } C_{a_0,b}e^{i\gamma_{a_0,b}} = \omega_{2,3}(\zeta_b^{a_0}).$$

(4) *If* $a_0 = \frac{b}{3}$, *then*

$$\frac{b\sum_{0\leq a\leq b-1} v_a p(a,b;n)}{D_{\mathbf{v}} n^{-\frac{2}{3}} \exp\left(\frac{2\pi}{3}\sqrt{\frac{n}{6}}\right)} = \cos\left(\delta_{\mathbf{v}} - \frac{4\pi n}{3}\right) + o(1),$$

where $D_{\mathbf{v}} \geq 0$ *and* $\delta_{\mathbf{v}} \in [0,2\pi)$ *are defined by*

$$D_{\mathbf{v}} \cdot e^{i\delta_{\mathbf{v}}} = \frac{(1-\zeta_3^2)^{\frac{1}{6}}(1-\zeta_3)^{\frac{1}{2}}\Gamma\left(\frac{1}{3}\right)}{2(6\pi)^{\frac{2}{3}}} P_{\mathbf{v}}\left(\zeta_3^{-1}\right).$$

(5) *If* $\theta_{23} < 2\pi\frac{a_0}{b} < \pi$, *then*

$$\frac{\sum_{0\leq a\leq b-1} c_v p(a,b;n)}{A_{a_0,b,\mathbf{v}} n^{-\frac{3}{4}} \exp\left(2\lambda_1\sqrt{n}\right)} = \cos\left(\alpha_{a_0,b,\mathbf{v}} + \pi n + 2\lambda_2\sqrt{n}\right) + o(1).$$

(6) *If* $a_0 = \frac{b}{2}$, *then* $\sum_{0\leq a\leq b-1} v_a p(a,b;n) \sim \frac{P_{\mathbf{v}}(-1)}{b} Q_n(-1)$.

Proof. For Cases (1) and (6), the asymptotic formula for $p(n)$ and Prop. 5.7 directly implies

$$\sum_{0\leq a\leq b-1} v_a p(a,b;n) \sim \tfrac{1}{b}\left(v_0 Q_n(1)+v_1 Q_n(1)+\ldots+v_{b-1}Q_n(1)\right)$$
$$= \frac{P_{\mathbf{v}}(1)p(n)}{b},$$
$$\sum_{0\leq a\leq b-1} v_a p(a,b;n) \sim \tfrac{1}{b}\left(v_0 Q_n(-1)+v_1 Q_n(-1)+\ldots+v_{b-1}Q_n(-1)\right)$$
$$= \frac{P_{\mathbf{v}}(-1)Q_n(-1)}{b},$$

for Cases (1) and (6) respectively.

For Cases (2), (3), and (5), the asymptotic main term (using that $\overline{Q_n(\zeta)} = Q_n(\overline{\zeta})$) is

$$\begin{aligned} p(a,b;n) &\sim \frac{1}{b}\left(\zeta_b^{-a_o a} Q_n(\zeta_b^{a_0}) + \overline{\zeta_b^{-a_o a} Q_n(\zeta_b^{a_0})}\right) \\ &= \frac{2}{b}\mathrm{Re}\left(\zeta_b^{-a_o a} Q_n(\zeta_b^{a_0})\right). \end{aligned}$$

The proof then follows by applying Thm. 5.5 and dividing by the appropriate normalizing factors in analogy to Eq. (5.2.2).

For Case (4), there is only one main term in the sum for $p(a,b;n)$ coming from the third root of unity:

$$\zeta^{-\frac{a}{3}} Q_n\left(\zeta^{\frac{1}{3}}\right).$$

Applying Thm. 5.5 and the analogy to Eq. 5.2.2 again proves the claim. $\square$

One application of the above theorem generalizes Thm. 5.8 to differences of partitions with largest part modulo b in one of two disjoint sets of residue classes

$$S_1, S_2 \subset \{0, \ldots, b-1\}.$$

That is, we consider

$$P_{S_1,S_2}(x) := \sum_{a\in S_1} x^a - \sum_{a\in S_2} x^a,$$

and prove a more explicit version of Thm. 5.9 in this case. Since $P_{S_1,S_2}(x)$ is a polynomial of degree at most $b-1$ with integer coefficients, there must exist some $d \mid b$ such that $\zeta_d \notin \mathcal{Z}(P_{S_1,S_2})$, otherwise P_{S_1,S_2} would be divisible by $\prod_{d|b}\Phi_d(x) = x^b - 1$, where Φ_d is the d-th cyclotomic polynomial, a contradiction. Let $|S|$ denote the cardinality of a set S. We then have the following.

Theorem 5.10. *Let $b \geq 2$ and let $S_1, S_2 \subset \{0, \ldots, b-1\}$ be disjoint subsets of integers. If $|S_1| > |S_2|$, then*

$$\sum_{a\in S_1} p(a,b;n) - \sum_{a\in S_2} p(a,b;n) \sim \frac{(|S_1|-|S_2|)p(n)}{b}.$$

Otherwise, if $|S_1| = |S_2|$, then we have the following cases. Let d_0 be the largest integer such that $d_0 \mid b$ and $\zeta_{d_0} \notin \mathcal{Z}(P_{S_1,S_2})$.
(1) *If $\big(d_0 \geq 5\big)$ or $\big(d_0 = 4$ and $-1 \in \mathcal{Z}(P_{S_1,S_2})\big)$, then*

$$\frac{\sum_{a\in S_1} p(a,b;n) - \sum_{a\in S_2} p(a,b;n)}{A_{d_0,S_1,S_2} n^{-\frac{3}{4}} \exp(2\lambda_1\sqrt{n})} = \cos\left(\alpha_{d_0,S_1,S_2} + 2\lambda_2\sqrt{n}\right) + o(1),$$

where $\lambda_1 + i\lambda_2 = \sqrt{\mathrm{Li}_2(\zeta_{d_0})}$ and $A_{d_0,S_1,S_2} \geq 0$ and $\alpha_{d_0,S_1,S_2} \in [0, 2\pi)$ are defined by

$$A_{d_0,S_1,S_2} \cdot e^{i\alpha_{d_0,S_1,S_2}} = \frac{P_{S_1,S_2}\left(\zeta_{d_0}^{-1}\right)}{b}\sqrt{\frac{(\lambda_1+i\lambda_2)(1-\zeta_{d_0})}{\pi}}.$$

(2) *If $\big(d_0 = 4$ or 3 with b even $\big)$ and $-1 \notin \mathcal{Z}(P_{S_1,S_2})$, we have the following allowable*
sets:

- *If $d_0 = 4$ and $(k_1 + k_2) \not\equiv 0, 2 \pmod 4$,*

$$S_1 \times S_2 = \{a\colon a \leq b-1,\ a \equiv k_1 \pmod 4\} \times \{a\colon a \leq b-1,\ a \equiv k_2 \pmod 4\}.$$

- *If $d_0 = 3$ and b is even and $k_1 \not\equiv k_2 \pmod 2$,*

$$S_1 \times S_2 = \{a\colon a \leq b-1,\ a \equiv k_1 \pmod 2\} \times \{a\colon a \leq b-1,\ a \equiv k_2 \pmod 2\}.$$

The asymptotic formulas are then given by

$$\sum_{a\in S_1} p(a,b;n) - \sum_{a\in S_2} p(a,b;n) \sim \frac{N_{S_1,S_2}}{b} Q_n(-1),$$

where

$$N_{S_1,S_2} := \begin{cases} (-1)^{k_1} b & \text{if } d_0 = 3 \text{ and } b \text{ is even,} \\ (-1)^{k_1} \frac{b}{2} & \text{if } d_0 = 4. \end{cases}$$

(3) *If $d_0 = 3$ and b is even and $-1 \in \mathcal{Z}(P_{S_1,S_2})$, or $d_0 = 3$ and b is odd, we have the following sets S_1 and S_2:*

- *If b is odd, S_1 and S_2 must contain distinct residue classes modulo 3.*
- *If b is even,*

$$S_1 \times S_2 = \{a : a \leq b-1,\ a \equiv k_1 \text{ or } k_2 \pmod 6\} \\ \times \{a : a \leq b-2,\ a \equiv k_3 \text{ or } k_4 \pmod 6\},$$

where

$$(k_1, k_2, k_3, k_4) = (0,3,2,5),\ (2,5,0,3),\ (1,4,2,5), \\ (2,5,1,4),\ (1,3,1,4),\ \text{or } (1,4,1,3).$$

The asymptotic formula is then given by

$$\sum_{a \in S_1} p(a,b;n) - \sum_{a \in S_2} p(a,b;n) \sim \frac{2}{b} \mathrm{Re}\left(Q_n(\zeta_3) P_{S_1,S_2}\left(\zeta_3^{-1}\right)\right).$$

(4) *If $d_0 = 2$, then for some a_1, a_2 of opposite parity we have*

$$(S_1, S_2) = \{a \equiv a_1 \pmod 2\} \times \{a \equiv a_2 \pmod 2\}$$

and

$$\sum_{a \in S_1} p(a,b;n) - \sum_{a \in S_2} p(a,b;n) \sim (-1)^{a_1} Q_n(-1).$$

Proof. The asymptotic analysis is similar to the proof of Thm. 5.8, since Lem. 5.24 and Thm. 5.5 imply that the sequence $Q_n(\zeta_d)$, ranked in asymptotic order from least to greatest is

$$Q_n(\zeta_3),\ Q_n(i),\ Q_n(-1),\ Q_n(\zeta_5),\ Q_n(\zeta_6),\dots,Q_n(1)=p(n).$$

Proof of Case (1):

Since -1 is a root, the ranking above implies that $Q_n(i)$ is the main term when $d_0=4$, and when $d_0\geq 5$, the main term is $Q_n(\zeta_{d_0})$. To get the oscillatory term, we apply the same technique as in Thm. 5.9 and Eq. (5.2.2).

Proof of Case (2):

Assume first that $d_0=4$. In this case, we know by assumption that b is divisible by 4, and by the ranking above, the asymptotic main term must be $Q_n(-1)$. We now have to sort out the possible sets S_1 and S_2. By the definition of d_0, all divisors of b greater than d_0 must be roots of the polynomial $P_{S_1,S_2}(x)$. As a consequence, we have that

$$\frac{x^b-1}{\Phi_4(x)\Phi_2(x)}\Bigg|P_{S_1,S_2}(x).$$

By long division, we find that

$$\frac{x^b-1}{\Phi_4(x)\Phi_2(x)}=x^{b-3}-x^{b-4}+x^{b-7}-x^{b-8}+...+x-1,$$

where the remainder is always -1 since $b\equiv 0\pmod 4$. Notice the polynomial above can be grouped into pairs of two, where the next pairing is a distance (in degree) 3 away. The dividing polynomial is degree $b-3$, however, P_{S_1,S_2} can have degree as large as $b-1$. Thus, we multiply by a generic quadratic polynomial ax^2+Mx+c with integer coefficients. Since the pairings in the dividing polynomial are separated by degree 3, we need to only consider one pair to catch the pattern, so we can assume without loss of generality that $b=4$. We then have

$$\widehat{P}(x):=(ax^2+Mx+c)(x-1)=ax^3+(M-a)x^2+(c-M)x-c.$$

Since $P_{S_1,S_2}(x)$ can only have coefficients that are 0 or ± 1, so must $\widehat{P}(x)$. This means $|a| \leq 1$, $|c| \leq 1$, and

$$|M - a| \leq 1 \text{ and } |c - M| \leq 1.$$

This also means $|M| \leq 2$. Additionally, ζ_4 and -1 are not roots by assumption, and this holds for $\widehat{P}(x)$ as well. We can reduce our problem to finding the possible $\widehat{P}(x)$ using a computer. We find that

$$\widehat{P}(x) \in \{-x^3+1, -x^3+x^2, -x^2+x, -x+1, x-1, x^2-x, x^3-x^2, x^3-1\}.$$

This then tells us the possible sets S_1 and S_2 are the tuples of congruence classes modulo 4:

$$S_1 \times S_2 = (0,3),\ (3,0),\ (2,3),\ (3,2),\ (1,2),\ (2,1),\ (0,1),\ (1,0).$$

However, this can be rewritten as

$$S_1 \times S_2 = \{a\colon a \leq b-1,\ a \equiv k_1 \pmod 4\} \times \{a\colon a \leq b-1,\ a \equiv k_2 \pmod 4\},$$

where $(k_1 + k_2) \pmod 4 \neq 0, 2$. With these choices of S_1 and S_2, we can get the factor out front by evaluating the polynomial $P_{S_1,S_2}(x)$ at -1. That is

$$P_{S_1,S_2}(-1) = \sum_{a \in S_1} (-1)^a - \sum_{a \in S_2} (-1)^a = (-1)^{\tilde{a}} \left(|S_1| + |S_2|\right),$$

where $\tilde{a}$ is any element in S_1. Furthermore, since 4 divides b, we know that for each residue class modulo 4, there are exactly $\frac{b}{4}$ integers a that are congruent to $c \pmod 4$ such that $a \leq b-1$. Therefore,

$$|S_1| = |S_2| = \frac{b}{4}.$$

Thus,

$$P_{S_1,S_2}(-1) = (-1)^{\tilde{a}} \frac{b}{2},$$

which gives the claimed for-factor in the theorem statement.

Assume now that $d_0 = 3$ and b is even. We consider the polynomial

$$\frac{x^b - 1}{\Phi_3(x)\Phi_2(x)},$$

which divides $P_{S_1,S_2}(x)$ when $b \equiv 0 \pmod 6$. Proceeding by long division gives the polynomial

$$\frac{x^b - 1}{\Phi_3(x)\Phi_2(x)} = \left(x^{b-3} - 2x^{b-4} + 2x^{b-5} - x^{b-6}\right) + \left(x^{b-9} - 2x^{b-10} + 2x^{b-11} - x^{b-12}\right) + \cdots + \left(x^3 - 2x^2 + 2x - 1\right).$$

The coefficient sequence $\{1, -2, 2, -1, ...\}$ has period 4 and the powers in brackets are a distance of at least 3 (in degree) away from the next group. This means we can consider just one group of 4, or equivalently, assume $b = 6$. Doing so, we again multiply by a generic quadratic and define

$$\begin{aligned}\widehat{P}(x) &:= (ax^2 + Mx + c)\left(x^3 - 2x^2 + 2x - 1\right) \\ &= ax^5 + (M-2a)x^4 + (2a-2M+c)x^3 + (2M-a-2c)x^2 + (2c-M)x - c.\end{aligned}$$

Since we only want integer coefficients in the range -1 to 1, we have the inequalities

$$|a|,\ |c| \le 1 \quad \text{and} \quad |M| \le 3,$$

and

$$|2a - 2M + c| \le 1,\ |2M - a - 2c| \le 1,\ |2c - M| \le 1.$$

Additionally, ζ_3 and -1 cannot be roots of $\widehat{P}(x)$. Therefore, there are only two possibilities based on these restrictions:

$$\widehat{P}(x) \in \{-x^5 + x^4 - x^3 + x^2 - x + 1, x^5 - x^4 + x^3 - x^2 + x - 1\}.$$

Therefore, the possibilities for the sets S_1 and S_2 are given by

$$S_1 \times S_2 = \{a : a \le b - 1,\ a \equiv 0 \text{ or } 2 \text{ or } 4 \pmod 6\}$$

$$\times \{a : a \leq b-1,\ a \equiv 1 \text{ or } 3 \text{ or } 5 \pmod 6\}$$

or

$$S_1 \times S_2 = \{a : a \leq b-1,\ a \equiv 1 \text{ or } 3 \text{ or } 5 \pmod 6\}$$
$$\times \{a : a \leq b-1,\ a \equiv 0 \text{ or } 2 \text{ or } 4 \pmod 6\}.$$

That is, S_1 and S_2 have all odds/evens or all evens/odds. Since 6 divides b there are exactly $\frac{b}{6}$ integers $a \equiv c \pmod 6$ such that $a \leq b-1$. Therefore, we need to evaluate $P_{S_1,S_2}(x)$ at -1 based on the two possibilities for S_1 and S_2 above. This gives

$$P_{S_1,S_2}(-1) = (-1)^{\tilde{a}} b,$$

where $\tilde{a}$ is any element in S_1. This gives the second case in the for-factor of the theorem.

Proof of Case (3):

If b is odd and $d_0 = 3$ there is no way that -1 can be a b^{th} root of unity, and since we are concerned with b^{th} roots of unity which are also roots of $P_{S_1,S_2}(x)$, we see that $Q_n(\zeta_3)$ and $Q_n(\zeta_3^{-1})$ must build our main asymptotic term. On the other hand, we have $d_0 = 3$ even and $-1 \in \mathcal{Z}(P_{S_1,S_2})$, and the asymptotic behavior of $\sum_{a\in S_1} p(a,b;n) - \sum_{a\in S_2} p(a,b;n)$ is determined again by $Q_n(\zeta_3)$ and $Q_n(\zeta_3^{-1})$ since all other $Q_n(\zeta_b^a)$ vanish in $P_{S_1,S_2}(\zeta_b^{-j})Q_n(\zeta_b^j)$. Furthermore, in both the even and odd b case we must have

$$\left.\frac{x^b-1}{\Phi_3(x)}\right| P_{S_1,S_2}(x),$$

and the degree of $P_{S_1,S_2}(x)$ is at most $b-1$. Since

$$\frac{x^b-1}{\Phi_3(x)} = x^{b-2} - x^{b-3} + x^{b-5} - x^{b-6} + \cdots + x^4 - x^3 + x - 1,$$

we see that we need to multiply by a linear polynomial $Ax + B$ to get a polynomial of degree $b-1$. We use the same trick as before to get the pattern. If b is odd, we can assume that $b=3$ to find the sets S_1 and S_2, multiplying by the generic linear polynomial gives

$$\widehat{P}(x) = Ax^2 + (B-A)x - B,$$

where $|A|, |B| \leq 1$, and $|B-A| \leq 1$ This gives the possibilities for $\widehat{P}(x)$

$$\widehat{P}(x) \in \{\pm(x^2-1),\ \pm(x^2-x),\ \pm(x-1)\}.$$

Therefore, S_1 and S_2 must contain distinct residue classes modulo 3.

In the b even case, we can assume that $b=6$. Therefore, we multiply $x^4 - x^3 + x - 1$ by a generic linear term with the same restrictions as before, but now -1 can be a root of the polynomial. This gives that

$$\begin{aligned}\widehat{P}(x) \in \{&(-x-1)\left(x^4-x^3+x-1\right), -x\left(x^4-x^3+x-1\right), -x^4+x^3-x+1,\\ &x^4-x^3+x-1, x\left(x^4-x^3+x-1\right), (x+1)\left(x^4-x^3+x-1\right)\}.\end{aligned}$$

The possible sets S_1 and S_2 are thus given by

$$\begin{aligned}S_1 \times S_2 = \{&a : a \leq b-1,\ a \equiv k_1 \text{ or } k_2 \pmod 6\}\\ &\times \{a : a \leq b-2,\ a \equiv k_3 \text{ or } k_4 \pmod 6\},\end{aligned}$$

where

$$\begin{aligned}(k_1, k_2, k_3, k_4) = &(0,3,2,5),\ (2,5,0,3),\ (1,4,2,5),\ (2,5,1,4),\\ &(1,3,1,4),\ \text{or } (1,4,1,3).\end{aligned}$$

Finally, note that, given S_1 and S_2 defined in terms of a_1 and a_2 modulo b, we have

$$\begin{aligned}&P_{S_1,S_2}\left(\zeta_3^{-1}\right)\\ &= \sum_{\substack{0\leq a\leq b\\ a\equiv a_1 \pmod 3}} \zeta_3^{-a} - \sum_{\substack{0\leq a\leq b\\ a\equiv a_2 \pmod 3}} \zeta_3^{-a} = \frac{b}{3}(\zeta_3^{-a_1} - \zeta_3^{-a_2}).\end{aligned}$$

This is taken into account in the definition of the constant B_{S_1,S_2}.
Proof of Case (4):

When $d_0 = 2$, in addition to the fact that b is even, we know that the asymptotic main term is $Q_n(-1)$. That is,

$$p(a,b;n) \sim \frac{(-1)^a}{b} Q_n(-1).$$

To sort out the sets S_1 and S_2, we investigate the polynomial

$$\frac{x^b - 1}{\Phi_2(x)} = x^{b-1} - x^{b-2} + x^{b-3} - \cdots + x - 1,$$

which divides $P_{S_1,S_2}(x)$. However, we deduce from this that $P_{S_1,S_2}(x)$ is this polynomial up to a constant (ignoring the case where $P_{S_1,S_2}(x)$ is always 0). That is,

$$P_{S_1,S_2}(x) = \nu \left(x^{b-1} - x^{b-2} + x^{b-3} - \cdots + x - 1 \right),$$

where $\nu = \pm 1$. Therefore, the possibilities for the sets are

$$S_1 \times S_2 = \{a\colon a \le b-1,\ a \equiv 0 \pmod 2\} \times \{a\colon a \le b-2,\ a \equiv 1 \pmod 2\}$$

or

$$\{a\colon a \le b-2,\ a \equiv 1 \pmod 2\} \times \{a\colon a \le b-1,\ a \equiv 0 \pmod 2\}.$$

We have that in both cases $|S_1| = |S_2| = \frac{b}{2}$. Therefore, the for-factor in the theorem comes from evaluating $P_{S_1,S_2}(x)$ at $x = -1$:

$$P_{S_1,S_2}(-1) = (-1)^{\tilde{a}} b,$$

where $\tilde{a} \in S_1$. This completes the proof of the theorem. □

5.2.3 Examples

In this section, we provide some examples of the theorems discussed in the previous section.

Example 5.11 (Simple differences)**.** Let $b = 6$, and consider the difference $p(1,6;n) - p(5,6;n)$. Then according to Thm. 5.8, we obtain

$$\frac{p(1,6;n) - p(5,6;n)}{Bn^{-\frac{3}{4}} \exp\left(2\lambda_1\sqrt{n}\right)} = \cos\left(\beta + 2\lambda_2\sqrt{n}\right) + o(1), \tag{5.2.3}$$

where $\lambda_1 + i\lambda_2 = \sqrt{\mathrm{Li}_2(\zeta_6)}$, and $B > 0$ and $\beta \in [0, 2\pi)$ are given implicitly by

$$\begin{aligned} Be^{i\beta} &= \frac{\zeta_6^{-1} - \zeta_6^{-5}}{6}\sqrt{\frac{(1-\zeta_6)(\lambda_1 + i\lambda_2)}{\pi}} \\ &= \frac{i}{\sqrt{12\pi}}\sqrt{\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\sqrt{\mathrm{Li}_2(\zeta_6)}}. \end{aligned}$$

Note that this implies (choosing B to be the absolute value of the right hand side of the equation above)

$$\begin{aligned} \lambda_1 &= 0.81408\ldots, & \lambda_2 &= 0.62336\ldots, \\ B &= 0.23268\ldots, & \beta &= 1.37394\ldots. \end{aligned} \tag{5.2.4}$$

Considering the first 900 coefficients numerically yields

$$M := \{7, 26, 59, 104, 162, 233, 316, 412, 521, 642, 776, ...\}$$

which are the highest indices until a change of signs in $p(1,6;n) - p(5,6;n)$. We can compare this exact result to the prediction of formula Eq. (5.2.3). By considering the roots of the cosine, we find that it changes signs approximately at

$$M' := \{7, 27, 59, 104, 162, 233, 316, 412, 521, 642, 777, ...\}.$$

Note that in the first eleven cases, only case two with 27 and case eleven with 777 give slightly wrong predictions.

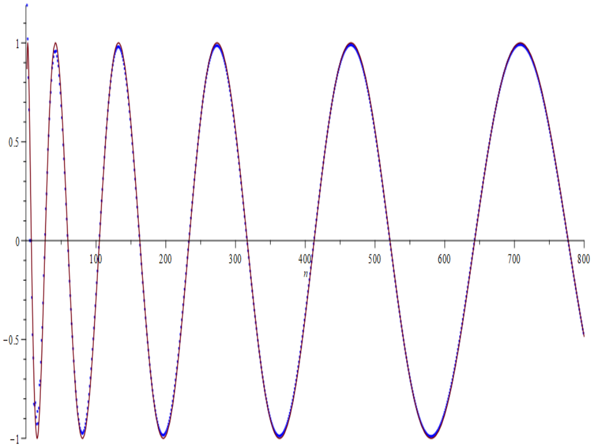

Figure 5.4: The plot shows the sign changes of $p(1,6;n) - p(5,6;n)$ (in blue dots) and the estimated sign changes (the red line).

The next example refers to higher differences of partition functions which we discussed in Thm. 5.10.

Example 5.12 (Differences with sets)**.** Again, we consider the case $b = 6$. In the spirit of Thm. 5.10 we want to consider multi-termed differences. Let $S_1 := \{1,3,5\}$ and $S_2 := \{0,2,4\}$. This implies that $P_{S_1,S_2}(x) = x^5 - x^4 + x^3 - x^2 + x - 1$ and as a result, $d_0 = 2$. This is exactly Case (4) of Thm. 5.10 which gives the asymptotic formula

$$\sum_{0 \le a \le 5} (-1)^{a+1} p(a,6;n) \sim -Q_n(-1) = \frac{(-1)^{n+1}}{2(24)^{\frac{1}{4}} n^{\frac{3}{4}}} \exp\left(\pi\sqrt{\frac{n}{6}}\right).$$

Also note that $\frac{\pi}{\sqrt{6}} = 1.2825\ldots$ is much smaller then the exponent $2\lambda_1 = 1.6281\ldots$ in Eq. (5.2.3), which one would get from considering just a simple difference of $p(a_1,b;n) - p(a_2,b;n)$ for residue classes a_1 and a_2 modulo b.

We now progress toward proving our main theorems.

5.3 Tools from analysis

We will need several analytical results in this work, which we collect in this section. Much of the items here are discussed in works such as [20, 22, 47, 62, 66]. The reader can skip this section and refer back to it as needed as we work through the proofs of our main theorems.

5.3.1 Asymptotic methods and integration formulas

A first tool is the well known Laplace's method for studying limits of definite integrals with oscillation, which we will use for evaluating Cauchy-type integrals.

Theorem 5.13 (Laplace's method, see Sect. 1.1.5 of [62])**.** *Let* $A, B : [a, b] \to \mathbb{C}$ *be continuous functions. Suppose* $x \neq x_0 \in [a, b]$ *such that* $\mathrm{Re}(B(x)) < \mathrm{Re}(B(x_0))$, *and that*

$$\lim_{x \to x_0} \frac{B(x) - B(x_0)}{(x - x_0)^2} = -z \in \mathbb{C},$$

with $\mathrm{Re}(z) > 0$. *Then as* $t \to \infty$

$$\int_a^b A(x) e^{tB(x)} dx = e^{tB(x_0)} \left(A(x_0) \sqrt{\frac{\pi}{tz}} + o\left(\frac{1}{\sqrt{t}}\right)\right).$$

We are ultimately interested in how the coefficients of a series $S(q) := \sum_{n \geq 0} a(n) q^n$ grow as $n \to \infty$. The Euler–Maclaurin summation formulas, one of which we saw in Thm. 2.25, can be applied in many cases to link the growth of $a(n)$ to the growth of $S(q)$ as q approaches the unit circle. We state here the classical Euler-Maclaurin summation formula for definite sums.

Theorem 5.14 (Classical Euler–Maclaurin summation, see pg. 66 of [47])**.** *Let* $\{x\} := x - \lfloor x \rfloor$ *denote the fractional part of* x. *For*

$N \in \mathbb{N}$ and $f : [1, \infty) \to \mathbb{C}$ a continuously differentiable function, we have

$$\sum_{1 \leq n \leq N} f(n) = \int_1^N f(x)dx + \tfrac{1}{2}(f(N)+f(1)) + \int_1^N f'(x)\big(\{x\} - \tfrac{1}{2}\big)dx.$$

To identify a constant term in our asymptotic formula, we cite the following integral calculation of Bringmann, Craig, Males and Ono.

Lemma 5.15 (Lem. 2.3, [18])**.** *For $a \in \mathbb{R}^+$,*

$$\int_0^\infty \left(\frac{e^{-ax}}{x(1-e^{-x})} - \frac{1}{x^2} - \left(\frac{1}{2} - a\right) \frac{e^{-ax}}{x} \right) dx$$
$$= \log\left(\Gamma\left(a\right)\right) + \left(\frac{1}{2} - a\right) \log\left(a\right) - \frac{1}{2}\log(2\pi).$$

Finally, we will use Abel partial summation extensively when bounding the twisted eta-products on the minor arcs.

Proposition 5.16 (Abel partial summation, see pg. 3 of [67])**.** *Let $N \in \mathbb{N}_0$ and $M \in \mathbb{N}$. For sequences $\{a_n\}_{n \geq N}$ and $\{b_n\}_{n \geq N}$ of complex numbers, we define $A_n := \sum_{N < m \leq n} a_m$. Then,*

$$\sum_{N < n \leq N+M} a_n b_n = A_{N+M} b_{N+M} + \sum_{N < n < N+M} A_n (b_n - b_{n+1}).$$

5.3.2 Results on holomorphic functions

The following bound for differences of holomorphic functions will be used in conjunction with Abel partial summation during the course of the circle method.

Lemma 5.17. *Let $f : U \to \mathbb{C}$ be a holomorphic function and let $\overline{B_r(c)} \subset U$ be a compact disc. Then, for all $a, b \in B_r(c)$ with $a \neq b$, we have*

$$|f(b) - f(a)| \leq \sup_{|z-c|=r} |f'(z)||b-a|.$$

Proof. Since the set $B_r(c)$ is convex, the straight line connecting a and b lies completely in $B_r(c)$. By the maximum modulus principle and the fundamental theorem of calculus (see Ch. 2 of [39] or any book on complex analysis), we have (where γ is the straight line between a and b)

$$|f(b) - f(a)| = \left| \int_{\gamma^{-1}(a)}^{\gamma^{-1}(b)} f^{'}(\gamma(x))dx \right| \leq |b-a| \sup_{|z-c|=r} |f'(z)|,$$

where in the final step we used the maximum modulus principle. $\square$

We apply this lemma to the function

$$\phi_a(w) := \frac{e^{-aw}}{1-e^{-w}} - \frac{1}{w}, \tag{5.3.1}$$

which will appear frequently in Sect. 4.5.4. Note that ϕ_a is holomorphic at 0 and in the cone $|\operatorname{Arg}(w)| \leq \frac{\pi}{2} - \delta$, for any $\delta > 0$.

Lemma 5.18. *Let x be a complex number with positive imaginary part and let $|x| \leq 1$. Then there is a constant $c > 0$ independent from a and x, such that for all $m \leq \frac{1}{|x|}$ we have*

$$|\phi_a(xm) - \phi_a(x(m+1))| \leq c|x|.$$

Proof. We first note that the function $\phi_a(z)$ is holomorphic in $B_3(0)$. The functions $\phi_a'(z)$ are uniformly bounded on $\overline{B_{\frac{5}{2}}(0)} \subset B_3(0)$ which implies that

$$\begin{aligned} &\max_{|z|\leq\frac{5}{2}} |\phi_a'(z)| \\ &= \max_{|z|=\frac{5}{2}} |\phi_a'(z)| \leq \sup_{|z|=\frac{5}{2}} \left| \frac{e^{-az-z}}{(1-e^{-z})^2} \right| + \sup_{|z|=\frac{5}{2}} \left| \frac{ae^{-az}}{1-e^{-z}} \right| \\ &\quad + \sup_{|z|=\frac{5}{2}} \left| \frac{1}{z^2} \right| \\ &\ll 1. \end{aligned}$$

On the other hand, by Lem. 5.17 applied to $f = \phi_a$, $U = B_3(0)$, and $\overline{B_{\frac{5}{2}}(0)} \subset U$, we find that since $|xm| \leq 1$ and $|x(m+1)| \leq |mx|+|x| \leq 2$

$$|\phi_a(xm + x) - \phi_a(mx)| \leq \max_{|z| \leq \frac{5}{2}} |\phi_a'(z)||xm + x - xm| = c|x|,$$

where c does not depend on $0 < a \leq 1$. $\square$

We also require the following lemma for large values of m, whose proof is a straightforward calculation.

Lemma 5.19. *Let x be a complex number with positive imaginary part. Then for all $m > \frac{1}{|x|}$ we have*

$$\begin{aligned}&|\phi_a(xm) - \phi_a(x(m+1))| \\ &\ll \frac{1}{|x|m(m+1)} + |x|e^{-m\mathrm{Re}(x)} + a|x|e^{-am\mathrm{Re}(x)}.\end{aligned}$$

Proof. We estimate the different parts separately. First we have

$$\left|\frac{1}{xm} - \frac{1}{x(m+1)}\right| \ll \frac{1}{|x|m(m+1)}.$$

Additionally,

$$\begin{aligned}&\left|\frac{e^{-amx}}{1 - e^{-mx}} - \frac{e^{-a(m+1)x}}{1 - e^{-(m+1)x}}\right| \\ &\leq e^{-am\mathrm{Re}(x)}\left|\frac{1}{1 - e^{-mx}} - \frac{1}{1 - e^{-(m+1)x}} - \frac{e^{-ax} - 1}{1 - e^{-(m+1)x}}\right| \\ &\leq e^{-am\mathrm{Re}(x)}\left(\left|\frac{1}{1 - e^{-mx}} - \frac{1}{1 - e^{-(m+1)x}}\right| + \left|\frac{e^{-ax} - 1}{1 - e^{-(m+1)x}}\right|\right) \\ &= e^{-am\mathrm{Re}(x)}\left(\left|e^{-mx} - e^{-(m+1)x} + O(e^{-2xm})\right| + \left|\frac{-ax + O(a^2x^2)}{1 - e^{-(m+1)x}}\right|\right).\end{aligned}$$

Since $m > \frac{1}{|x|}$,

$$\frac{1}{1 - e^{-xm}} \ll 1,$$

which gives

$$\left|\frac{e^{-amx}}{1-e^{-mx}}-\frac{e^{-a(m+1)x}}{1-e^{-(m+1)x}}\right| \ll e^{-am\mathrm{Re}(x)}\left(|x|e^{-m\mathrm{Re}(x)}+a|x|\right)$$
$$\ll |x|e^{-m\mathrm{Re}(x)}+a|x|e^{-a|x|m},$$

which proves the claim. □

We also need the following elementary maximum; for a proof see Lem. 19 of [61].

Lemma 5.20. *For $a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $b \in \mathbb{R}$, we have*

$$\mathrm{Re}\left(\frac{e^{ia}}{1+ib}\right) \leq \cos^2\left(\frac{a}{2}\right),$$

with equality if and only if b satisfies $\mathrm{Arg}(1+ib) = \frac{a}{2}$.

5.3.3 Bounds for trigonometric series and the polylogarithm

The following is well-known.

Proposition 5.21 (see pg. 1005, [1])**.** *We have for all* $0 \leq \theta < 2\pi$

$$\mathrm{Li}_2\left(e^{i\theta}\right) = \frac{\theta^2}{4} - \frac{\pi\theta}{2} + \frac{\pi^2}{6} - i\int_0^\theta \log\left|2\sin\left(\frac{t}{2}\right)\right| dt.$$

We need to consider the derivative of the function $\theta \mapsto \mathrm{Li}_2(e^{2\pi i\theta})$ and its partial sums, which requires an understanding of oscillating sums of the type

$$G_M(\theta) := \sum_{1\leq m\leq M} \frac{e^{2\pi i\theta m}}{m}. \tag{5.3.2}$$

The following uniform bound holds for the function in Eq. (5.3.2).

Lemma 5.22. *We have uniformly for* $0<\theta<1$*,*

$$G_M(\theta) \ll \log\left(\frac{1}{\theta}\right) + \log\left(\frac{1}{1-\theta}\right), \qquad M \to \infty.$$

Proof. Note that we have

$$G_M(\theta) = \sum_{1\le m\le M} \frac{\cos(2\pi\theta m)}{m} + i \sum_{1\le m\le M} \frac{\sin(2\pi\theta m)}{m},$$

and since it is well known that $\sum_{m=1}^{M} \frac{\sin(2\pi\theta m)}{m}$ is uniformly bounded in $\mathbb{R}$, we are left with only the cosine sum. Let $0<\theta\le\frac{1}{2}$ and $M\in\mathbb{N}$. Consider the meromorphic function

$$h_\theta(z) := \frac{\cos(2\pi\theta z)(\cot(\pi z)-i)}{2iz},$$

together with the rectangle R_M with vertices $\frac{1}{2}-iM$, $\frac{1}{2}+\frac{i}{\theta}$, $\frac{i}{\theta}+M+\frac{1}{2}$, and $-iM+M+\frac{1}{2}$. Notice that in punctured disks with radius $r<1$ centered at a positive integer k, we have

$$h_\theta(z) = \frac{\cos(2\pi\theta k)\left(\frac{1}{\pi(z-k)}+O(1)\right)}{2ik}.$$

With the Residue Theorem we find

$$\oint_{\partial R_M} h_\theta(z)dz = 2\pi i \sum_{m=1}^{M} \mathrm{Res}_{z=m} h_\theta(z) = \sum_{1\le m\le M} \frac{\cos(2\pi\theta m)}{m},$$

where the contour is taken in the counter-clockwise direction. By the invariance of the function under the transformation $\theta \mapsto 1-\theta$ it suffices to consider values $0<\theta\le\frac{1}{2}$. A straightforward calculation shows that the bottom side integrals are bounded uniformly in $0<\theta\le\frac{1}{2}$, since

$$\cot(\pi z) - i = 2i\frac{e^{-\pi iz}}{e^{i\pi z}-e^{-\pi iz}} \ll e^{-2\pi|\mathrm{Im}(z)|},$$

and

$$\cos(2\pi\theta z) \ll e^{2\pi\theta|\mathrm{Im}(z)|}.$$

Similarly, also using that $\cot(\pi z)$ is uniformly bounded on the vertical lines $\mathrm{Re}(z) \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, one finds

$$\begin{aligned}\int_{M+\frac{1}{2}-Mi}^{M+\frac{1}{2}+\frac{i}{\theta}} h_\theta(z)dz = O(1) + \int_{M+\frac{1}{2}}^{M+\frac{1}{2}+\frac{i}{\theta}} h_\theta(z)dz \ll \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{i}{\theta}} \left|\frac{dz}{z+M}\right| \\ \ll \log\left(\frac{1}{\theta}\right).\end{aligned} \tag{5.3.3}$$

Again using the decay of the integrand as $\mathrm{Im}(z) \to -\infty$ and splitting the integral, we have

$$\int_{\frac{1}{2}+\frac{\theta}{i}}^{\frac{1}{2}-Mi} h_\theta(z)dz \ll O(1) + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{\theta}{i}} \frac{dz}{z} \ll \log\left(\frac{1}{\theta}\right).$$

For the top portion of the rectangle, we make the substitution $z \mapsto \frac{z}{\theta}$:

$$\begin{aligned}\int_{M+\frac{1}{2}+\frac{i}{\theta}}^{\frac{1}{2}+\frac{i}{\theta}} h_\theta(z)dz = \int_{\theta\left(M+\frac{1}{2}\right)+i}^{\frac{\theta}{2}+i} \frac{\cos(2\pi z)(\cot\left(\frac{\pi z}{\theta}\right)+i)}{z}dz \\ - 2i\int_{\theta\left(M+\frac{1}{2}\right)+i}^{\frac{\theta}{2}+i} \frac{\cos(2\pi z)}{z}dz.\end{aligned} \tag{5.3.4}$$

We can split up the integral bounds by including a point at $i\infty$:

$$\int_{\frac{\theta}{2}+i}^{\theta(M+\frac{1}{2})+i} = \int_{\frac{\theta}{2}+i}^{i\infty} + \int_{\theta(M+\frac{1}{2})+i\infty}^{\theta(M+\frac{1}{2})+i}.$$

We then apply the Residue Theorem (with pole at ∞) to get the claimed bound for the second integral. Furthermore, the bound

$$\cos(2\pi z)\left(\cot\left(\frac{\pi z}{\theta}\right)+i\right) = O\left(e^{-2\pi\mathrm{Im}(z)}\right),$$

as $\mathrm{Im}(z) \to \infty$ (since $0 < \theta \leq \frac{1}{2}$), yields the uniform bound of the first integral in Eq. (5.3.4). The integral on the right hand side of Eq. (5.3.4) can be bounded by using that $\int_\alpha^\infty \frac{\cos(x)}{x} dx \ll 1$ for all $\alpha \geq 1$. $\square$

We also need to understand the behavior of the function $\theta \mapsto \mathrm{Re}(\sqrt{\mathrm{Li}_2(e^{2\pi i\theta})})$. More precisely, we would like to show that the function is monotonically decreasing, which is needed in the proof of our application to sign changes in partition differences. In order to do this, we first need a lemma which allows us to ignore the troublesome square-root. Throughout, we take the principle branch of the square-root.

Lemma 5.23. *Let I be an interval and let $f : I \to \mathbb{H}$ be a continuous function which satisfies the following*:
(1) *The induced function $|f| : I \to \mathbb{R}$ with $x \mapsto |f(x)|$ is decreasing.*
(2) *The induced function $\psi : I \to (0, \pi)$ with $x \mapsto \mathrm{Arg}(f(x)) := \theta(x)$ is increasing.*
Then the function $g : I \to \mathbb{R}$ with $g(x) := \mathrm{Re}(\sqrt{f(x)})$ is decreasing, where we consider the principle part of the square root.

Proof. Since f takes values in the upper half plane, we can write

$$f(x) = |f(x)|e^{i\theta(x)},$$

which implies that

$$\mathrm{Re}\sqrt{f(x)} = \sqrt{|f(x)|} \cos\left(\frac{\theta(x)}{2}\right).$$

The claim follows by noticing that $\cos\left(\frac{\theta(x)}{2}\right)$ is decreasing if $\theta(x)$ is increasing on the interval $(0, \pi)$. $\square$

We use the above tools to prove the following lemma. We define

$$H_n(s) := \sum_{1 \leq k \leq n} \frac{1}{k^s}.$$

As was stated at the beginning of this chapter, an alternative proof to the next lemma is available in the appendix of [16].

Lemma 5.24. *The function* $\theta \mapsto \mathrm{Re}\sqrt{\mathrm{Li}_2(e^{2\pi i\theta})}$ *is decreasing on the interval* $\left(0, \frac{1}{3}\right)$.

Proof. Our goal is to apply Lem. 5.23. We start by showing that $\theta \mapsto |\mathrm{Li}_2(e(\theta))|$ is decreasing on the interval $(0, \frac{1}{3})$. We have

$$\begin{aligned}|\mathrm{Li}_2(e(\theta))|^2 &= \mathrm{Im}(\mathrm{Li}_2(e(\theta)))^2 + \mathrm{Re}(\mathrm{Li}_2(e(\theta)))^2 \\ &= \sum_{n,m>0} \frac{\sin(2\pi\theta n)\sin(2\pi\theta m) + \cos(2\pi\theta n)\cos(2\pi\theta m)}{m^2n^2} \\ &= \sum_{n,m>0} \frac{\cos(2\pi\theta(n-m))}{m^2n^2} = \frac{\pi^4}{90} + 2\sum_{n\geq 1} A_n \cos(2\pi\theta n),\end{aligned}$$

where $A_n := \sum_{m\geq 1} \frac{1}{m^2(m+n)^2}$. Using the partial fraction decomposition

$$\frac{1}{x^2(x+n)^2} = \frac{2}{n^3(n+x)} - \frac{2}{n^3x} + \frac{1}{n^2x^2} + \frac{1}{n^2(n+x)^2}$$

we find

$$A_n = \frac{\pi^2}{3n^2} - \frac{H_n(2)}{n^2} - \frac{2H_n}{n^3},$$

where $H_n := H_n(1)$ is the n^{th} *harmonic number*. Since

$$A_n = \frac{\pi^2}{3n^2} + O\left(\frac{\log(n)}{n^3}\right),$$

we can take the derivative term by term to obtain

$$\frac{d}{d\theta}|\mathrm{Li}_2(e(\theta))|^2 = -2\pi\sum_{n\geq 1} nA_n \sin(2\pi\theta n).$$

We now show that the sum without the scaling factor -2π is positive in the interval $(0, \frac{1}{3})$. Using Prop. 5.16, we obtain

$$\sum_{n\geq 1} nA_n \sin(2\pi\theta n) = \sum_{n\geq 1}\left(\sum_{n\geq k\geq 1} \sin(2\pi\theta k)\right)(nA_n - (n+1)A_{n+1})$$
$$= \sum_{n\geq 1} \frac{\sin(\pi\theta n)\sin(\pi\theta(n+1))}{\sin(\pi\theta)}(nA_n - (n+1)A_{n+1}).$$

Since $(A_1 - 2A_2)\sin(2\pi\theta)$ is positive for $0 < \theta < \frac{1}{3}$, we are left with investigating the sum $\sum_{n\geq 2}$. Using Prop. 5.16 again, we find

$$\sum_{n\geq 2} \frac{\sin(\pi\theta n)\sin(\pi\theta(n+1))}{\sin(\pi\theta)}(nA_n - (n+1)A_{n+1})$$
$$= \frac{1}{\sin(\pi\theta)}\sum_{n\geq 2}\left(\sum_{n\geq k\geq 2} \sin(\pi\theta k)\sin(\pi\theta(k+1))\right)$$
$$\times (nA_n - 2(n+1)A_{n+1} + A_{n+2}).$$

By the trigonometric identity[2]

$$\sin(\pi\theta k)\sin(\pi\theta(k+1)) + \sin(\pi\theta(k+1))\sin(\pi\theta(k+2))$$
$$= 2\sin^2(\pi\theta(k+1))\cos(\pi\theta)$$

we see that for odd n

$$\sum_{n\geq k\geq 2} \sin(\pi\theta k)\sin(\pi\theta(k+1)) \geq 0$$

on $(0, \frac{1}{3})$, and for even n

$$\sum_{n\geq k\geq 2} \sin(\pi\theta k)\sin(\pi\theta(k+1))$$

[2] We would like to thank Caner Nazaroglu for pointing out this identity.

$$= \sin(2\pi\theta)\sin(3\pi\theta) + \sum_{n\geq k\geq 3} \sin(\pi\theta k)\sin(\pi\theta(k+1)) \geq 0$$

on $(0, \frac{1}{3})$. On the other hand, we see that

$$\begin{aligned}
&nA_n - 2(n+1)A_{n+1} + (n+2)A_{n+2} = \tfrac{\pi^2}{3}\left(\tfrac{1}{n} - \tfrac{2}{n+1} + \tfrac{1}{n+2}\right) \\
&- \tfrac{H_n(2)}{n} + \tfrac{2H_{n+1}(2)}{n+1} - \tfrac{H_{n+2}(2)}{n+2} - 2\left(\tfrac{H_n}{n^2} - \tfrac{2H_{n+1}}{(n+1)^2} + \tfrac{H_{n+2}}{(n+2)^2}\right) \\
&= \left(\tfrac{\pi^2}{3} - H_n(2)\right)\left(\tfrac{1}{n} - \tfrac{2}{n+1} + \tfrac{1}{n+2}\right) + \tfrac{2}{(n+1)^3} - \tfrac{1}{(n+2)(n+1)^2} - \tfrac{1}{(n+2)^3} \\
&\qquad -2\left(\tfrac{H_n}{n^2} - \tfrac{2H_n}{(n+1)^2} + \tfrac{H_n}{(n+2)^2} - \tfrac{2}{(n+1)^3} + \tfrac{1}{(n+1)(n+2)^2} + \tfrac{1}{(n+2)^3}\right) \\
&= \left(\tfrac{\pi^2}{3} - H_n(2)\right)\left(\tfrac{1}{n} - \tfrac{2}{n+1} + \tfrac{1}{n+2}\right) - 2\left(\tfrac{H_n}{n^2} - \tfrac{2H_n}{(n+1)^2} + \tfrac{H_n}{(n+2)^2}\right) \\
&\qquad + \tfrac{6}{(n+1)^3} - \tfrac{1}{(n+2)(n+1)^2} - \tfrac{2}{(n+1)(n+2)^2} - \tfrac{3}{(n+2)^3} \\
&\geq \left(\tfrac{\pi^2}{3} - H_n(2)\right)\left(\tfrac{1}{n} - \tfrac{2}{n+1} + \tfrac{1}{n+2}\right) - 2\left(\tfrac{H_n}{n^2} - \tfrac{2H_n}{(n+1)^2} + \tfrac{H_n}{(n+2)^2}\right) \\
&\geq \tfrac{9}{4n^3} - \tfrac{12}{n^{\frac{7}{2}}},
\end{aligned}$$

where the last step holds for $n \geq 10$, and we used the fact that $H_n < \sqrt{n}$ for $n \geq 7$. Furthermore, $\frac{9}{4n^3} - \frac{12}{n^{\frac{7}{2}}} > 0$ for all $n \geq 29$. One can then check explicitly for the cases $n = 2, ..., 28$ that

$$nA_n - 2(n+1)A_{n+1} + (n+2)A_{n+2} > 0.$$

Therefore, we have shown that $\frac{d}{d\theta}|\mathrm{Li}_2(e(\theta)))|^2 < 0$ on the interval $(0, \frac{1}{3})$. The cases $n = 2, .., 28$ are plotted in Fig. 5.5.

We now show that the function $\theta \mapsto \arg(\mathrm{Li}_2(e(\theta))$ is increasing on the interval $(0, \frac{1}{3})$. We denote $x := x(\theta) := \mathrm{Re}(\mathrm{Li}_2(e(\theta)))$ and $y := y(\theta) := \mathrm{Im}(\mathrm{Li}_2(e(\theta)))$. We note that on the specified interval, $x > 0$ only holds when $0 < \theta < \frac{1}{\sqrt{6}}(3 - \sqrt{3})$. Let $r_0 := \frac{1}{\sqrt{6}}(3 - \sqrt{3})$. Thus, using the principle branch, we have from $\theta \in (0, \frac{1}{3})$,

$$\mathrm{Arg}(\mathrm{Li}_2(e(\theta)) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } 0 < \theta < r_0, \\ \frac{\pi}{2} & \text{if } \theta = r_0, \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } \theta > r_0. \end{cases}$$

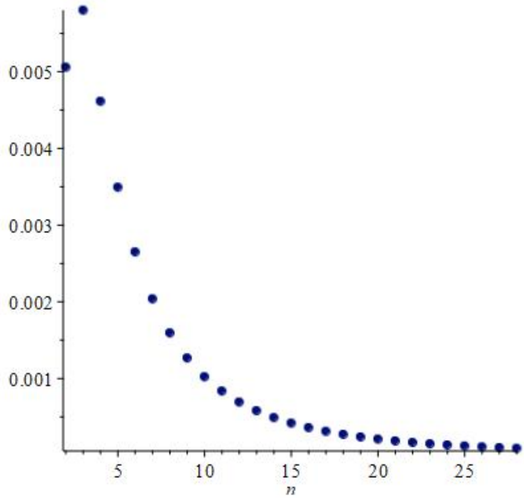

Figure 5.5: The function $nA_n - 2(n+1)A_{n+1} + (n+2)A_{n+2}$ is plotted above.

This function is differentiable everywhere on its domain, except at the point r_0. Away from the point, we compute

$$
\begin{aligned}
&\tfrac{d}{d\theta}\operatorname{Arg}(\mathrm{Li}_2(e(\theta)))=\tfrac{1}{1+\frac{y^2}{x^2}}\left(\tfrac{y'}{x}-\tfrac{yx'}{x^2}\right)=\tfrac{1}{|\mathrm{Li}_2(e(\theta))|^2}\left(y'x-yx'\right)\\
&=\tfrac{2\pi}{|\mathrm{Li}_2(e(\theta))|^2}\left(\textstyle\sum_{m,n\geq 1}\tfrac{\cos(2\pi\theta n)\cos(2\pi\theta m)}{nm^2}+\sum_{m,n\geq 1}\tfrac{\sin(2\pi\theta m)\sin(2\pi\theta n)}{mn^2}\right)\\
&=\tfrac{2\pi}{|\mathrm{Li}_2(e(\theta))|^2}\left(\textstyle\sum_{m,n\geq 1}\tfrac{\cos(2\pi\theta m)\cos(2\pi\theta n)}{mn^2}+\sum_{m,n\geq 1}\tfrac{\sin(2\pi\theta m)\sin(2\pi\theta n)}{mn^2}\right)\\
&=\tfrac{2\pi}{|\mathrm{Li}_2(e(\theta))|^2}\left(\textstyle\sum_{m,n\geq 1}\tfrac{\cos(2\pi\theta m)\cos(2\pi\theta n)+\sin(2\pi\theta n)\sin(2\pi\theta m)}{mn^2}\right)\\
&=\tfrac{2\pi}{|\mathrm{Li}_2(e(\theta))|^2}\left(\zeta(3)+\textstyle\sum_{n,m\geq 1}\left(\tfrac{1}{m(m+n)^2}+\tfrac{1}{n(n+m)^2}\right)\cos(2\pi\theta(n-m))\right)\\
&=\tfrac{2\pi}{|\mathrm{Li}_2(e(\theta))|^2}\left(\zeta(3)+\textstyle\sum_{n\geq 1}\tfrac{H_n(2)}{n}\cos(2\pi\theta n)\right).
\end{aligned}
$$

Note that in the fourth step we used

$$\sum_{N\geq n,m\geq 1} f(m,n)=\sum_{N\geq n,m\geq 1} f(n,m).$$

Since the corresponding products of partial sums $y_N'x_N-y_Nx_N'$ converge to $y'x-yx'$ in the observed interval for θ, and for sufficiently large N satisfying $n\geq N$ the coefficients $\frac{H_n(2)}{n}$ remain unchanged, the final expression converges to $\frac{d}{d\theta}\arg(\mathrm{Li}_2(e(\theta))$. Note that the term $\frac{2\pi}{|\mathrm{Li}_2(e(\theta))|^2}$ is clearly positive. With this in mind, we have to show that

$$\sum_{n\geq 1}\frac{H_n(2)}{n}\cos(2\pi\theta n)>-\zeta(3)$$

for all $0<\theta<\frac{1}{3}$. We note that

$$\lim_{N\to\infty} G_N(\theta)=-\log|2\sin(\pi\theta)|. \tag{5.3.5}$$

Let $\theta_0:=\arcsin\left(\frac{1}{2\pi}\right)=\frac{1}{6}$, which corresponds to the zero of the function in Eq. (5.3.5) in the interval $(0,\frac{1}{3})$. Using Prop. 5.16, we find

$$\begin{aligned}\sum_{n\geq 1}\frac{H_n(2)}{n}\cos(2\pi\theta n) &= \lim_{N\to\infty} G_N(\theta)H_N(2)\\ &\quad+\sum_{n\geq 1}\left(\sum_{n\geq k\geq 1}\frac{\cos(2\pi\theta k)}{k}\right)(H_n(2)-H_{n+1}(2))\\ &= -\frac{\pi^2}{6}\log(2|\sin(\pi\theta)|)\\ &\quad-\sum_{n\geq 1}\left(\cos(2\pi\theta)+\sum_{n\geq k\geq 2}\frac{\cos(2\pi\theta k)}{k}\right)\frac{1}{(n+1)^2}\\ &= -\frac{\pi^2}{6}\log(2|\sin(\pi\theta)|)-\left(\frac{\pi^2}{6}-1\right)\cos(2\pi\theta)\\ &\quad-\sum_{n\geq 1}\left(\sum_{n\geq k\geq 2}\frac{\cos(2\pi\theta k)}{k}\right)\frac{1}{(n+1)^2}.\end{aligned}$$

Since $\left|\sum_{n\geq k\geq 2}\frac{\cos(2\pi\theta k)}{k}\right| < H_n - 1$ in the range $0<\theta<\frac{1}{3}$, we find

$$\begin{aligned}&\sum_{n\geq 1}\frac{H_n(2)}{n}\cos(2\pi\theta n)\\ &\geq -\tfrac{\pi^2}{6}\log(2|\sin(\pi\theta)|)-\left(\tfrac{\pi^2}{6}-1\right)\cos(2\pi\theta)-\textstyle\sum_{n\geq 1}\frac{H_n-1}{(n+1)^2}\\ &= -\tfrac{\pi^2}{6}\log(2|\sin(\pi\theta)|)-\left(\tfrac{\pi^2}{6}-1\right)\cos(2\pi\theta)+\tfrac{\pi^2}{6}-1-\zeta(3)\\ &= \underbrace{-\tfrac{\pi^2}{6}\log(2|\sin(\pi\theta)|)+\left(\tfrac{\pi^2}{6}-1\right)(1-\cos(2\pi\theta))}_{:=\mathscr{L}(\theta)}-\zeta(3).\end{aligned}$$

When $\theta\in(0,\theta_0)$, $\mathscr{L}(\theta)>0$, and we are done since $\mathscr{L}(\theta)-\zeta(3)>-\zeta(3)$. We now assume $\theta\in(\theta_0,\frac{1}{3})$. We need to minimize the function $\mathscr{L}(\theta)$ on this interval. Since $\sin(\pi\theta)$ is positive on this interval, we can drop the absolute values and take derivatives to find the critical points. Doing so, the critical point(s), θ_c, satisfy the equation

$$\cot(\pi\theta) = 2\frac{\zeta(2)-1}{\zeta(2)}\sin(2\pi\theta) = 4\frac{\zeta(2)-1}{\zeta(2)}\sin(\pi\theta)\cos(\pi\theta).$$

Noting that $\cos(\pi\theta) \neq 0$ in the interval $(\theta_0, \frac{1}{3})$, we have

$$\sin(\pi\theta) = \pm\frac{1}{2}\sqrt{\frac{\zeta(2)}{\zeta(2)-1}}.$$

Thus, on the interval $(\theta_0, \frac{1}{3})$, we find the lone critical point

$$\theta_c = \frac{1}{\pi}\arcsin\left(\frac{1}{2}\sqrt{\frac{\zeta(2)}{\zeta(2)-1}}\right) \approx 0.29438. \tag{5.3.6}$$

One can easily check that this is a local minimum by the second derivative test:

$$\mathscr{L}''(\theta) = \frac{1}{6}\pi^4 + \frac{1}{6}\frac{\pi^4\left(\cos\left(\pi\,\theta\right)\right)^2}{\left(\sin\left(\pi\,\theta\right)\right)^2} + 4\left(\frac{1}{6}\pi^2 - 1\right)\pi^2\cos\left(2\,\pi\,\theta\right),$$

which implies $\mathscr{L}''(\theta_c) \approx 18.45339$. Substituting θ_c into the original function, we find that

$$\mathscr{L}(\theta_c) \approx 0.0524 > 0.$$

Therefore, $\mathscr{L}(\theta) - \zeta(3) > -\zeta(3)$, which completes the proof. □

5.4 Proof of Thm. 5.5

We prove Thm. 5.5 under the assumption that the minor arc bounds in Sect. 5.5 are true. Our proof is essentially the same as Parry's analysis of the major arcs in [61]. Since the asymptotic formulas for $Q_n(1)$ and $Q_n(-1)$ are well-known, we assume throughout that $1 \leq a < \frac{b}{2}$ with $\gcd(a,b) = 1$ and $b \geq 3$. The setup follows the standard Hardy–Ramanujan circle method discussed in Sect. 2.2, and we encourage the reader to review this section before proceeding. Let $N = \lfloor \delta\sqrt{n} \rfloor$, for some $\delta > 0$ to be chosen independently of n and small enough during the course of the proof. The number N will be the order of our Farey sequence. We write

$$t_\theta := t_n - 2\pi i\theta, \tag{5.4.1}$$

where t_n is determined below by the saddle point method and $k_0 \in \{1, 2, 3\}$ according to whether we are in Case (1), (2) or (3). By Cauchy's integral formula, we have

$$\begin{aligned} Q_n(z) &= \int_0^1 \left(\zeta_b^a e^{-t_n+2\pi i\theta}; e^{-t_n+2\pi i\theta}\right)_\infty^{-1} e^{nt_n-2\pi in\theta} d\theta \\ &= \sum_{\frac{h}{k} \in \mathcal{F}_N} \zeta_k^{-hn} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} \exp\left(\frac{\mathrm{Li}_2\left(\zeta_b^{ak}\right)}{k^2 t_\theta} + nt_\theta + E_{h,k}(\zeta_b^a, t_\theta)\right) d\theta, \end{aligned}$$

where

$$E_{h,k}(z,t) := \operatorname{Log}\left(\left(z^a\zeta_k^h e^{-t}; \zeta_k^h e^{-t}\right)_\infty^{-1}\right) - \frac{\mathrm{Li}_2\left(z^k\right)}{k^2 t}.$$

We will show that the integral(s) where $k = k_0$ dominate and constitute our major arcs. The trick of adding and subtracting off the dilog piece will allow us to later bound the functions $E_{h,k}(z,t)$. Under the assumption that we have sufficient bounds, we can calculate the saddle point t_n of the integrand.

5.4.1 The saddle point of $(\zeta q; q)_\infty^{-1}$

Recall the reasoning of the saddle point method discussed in Sect. 2.2 of this thesis. If our integrand is given by $S(t,\theta)$, we are after the point t_n where $\frac{d}{dt}S(t,\theta)|_{t=t_n} = 0$. In our case, or equivalently in the limit $n \to \infty$, we are after the point where we can approximate the integrand by a Gaussian in the variable θ. In our case, we want to approximate the function

$$S(t,\theta) := \frac{\mathrm{Li}_2\left(\zeta_b^{ak}\right)}{k^2 t_\theta} + n t_\theta + E_{h,k}(\zeta_b^a, t_\theta).$$

As we will show in Lem. 5.25 and Sect. 5.5, we can ignore the piece $E_{h,k}(\zeta_b^a, t_\theta)$. Assuming $k_0 = k$ constitutes a major arc, we have

$$\begin{aligned}\frac{\mathrm{Li}_2\left(\zeta_b^{ak_0}\right)}{k_0^2 t_\theta} + n t_\theta &= \frac{\mathrm{Li}_2\left(\zeta_b^{ak_0}\right)}{k_0^2 t_n\left(1 - \frac{2\pi i\theta}{t_n}\right)} + n t_\theta \\ &= \frac{\mathrm{Li}_2\left(\zeta_b^{ak_0}\right)}{k_0^2 t_n}\left(1 + \frac{2\pi i\theta}{t_n} - \frac{4\pi^2\theta^2}{t_n^2} + O\left(\theta^3\right)\right) \\ &\quad + n(t_n - 2\pi i\theta).\end{aligned}$$

We want to cancel the linear term in θ, which gives us the approximate saddle point

$$t_n = \frac{\sqrt{\mathrm{Li}_2\left(\zeta_b^{ak_0}\right)}}{k_0\sqrt{n}}.$$

5.4.2 Back to the proof

Bounding the $E_{h,k}$ on the major and minor arcs is the main difficulty for the rest of this chapter. We start with the growth of $E_{h,k}$ up to $o(1)$ on each of the possible major arcs.

Lemma 5.25. *Let $\omega_{h,k}(z)$ be the product defined in Eq.* (5.2.1)*. We have the following*: (1) *For $2\pi\frac{a}{b} \in (0,\theta_{13})$ and $-\theta'_{0,1} \le \theta \le \theta''_{0,1}$, we have*

$$E_{0,1}(\zeta_b^a, t_\theta) = \mathrm{Log}\left(\omega_{0,1}\left(\zeta_b^a\right)\right) + o(1).$$

(2) *For $2\pi\frac{a}{b} \in (\theta_{23},\pi)$ and $-\theta'_{1,2} \le \theta \le \theta''_{1,2}$, we have*

$$E_{1,2}(\zeta_b^a, t_\theta) = \mathrm{Log}\left(\omega_{1,2}\left(\zeta_b^a\right)\right) + o(1).$$

(3) *For* $2\pi\frac{a}{b} \in (\theta_{13}, \theta_{23}) \setminus \left\{\frac{2\pi}{3}\right\}$ *and* $-\theta'_{1,3} \leq \theta \leq \theta''_{1,3}$, *we have*

$$E_{1,3}(\zeta_b^a, t_\theta) = \operatorname{Log} \omega_{1,3}(\zeta_b^a) + o(1),$$

and for $-\theta'_{2,3} \leq \theta \leq \theta''_{2,3}$, *we have*

$$E_{2,3}(\zeta_b^a, t_\theta) = \operatorname{Log} \omega_{2,3}(\zeta_b^a) + o(1).$$

(4) *For* $\frac{a}{b} = \frac{1}{3}$ *and* $-\theta'_{1,3} \leq \theta \leq \theta''_{1,3}$, *we have*

$$\begin{aligned} E_{1,3}(\zeta_3, t_\theta) = &\operatorname{Log}\left(t_\theta^{\frac{1}{6}}\right) + \operatorname{Log}\left(\left(1-\zeta_3^2\right)^{-\frac{1}{6}} \left(1-\zeta_3\right)^{\frac{1}{2}}\right) \\ &+ \log\left(\Gamma\left(\frac{2}{3}\right)\right) + \frac{1}{6}\log(3) - \frac{1}{2}\log(2\pi) + o(1), \end{aligned}$$

and for $-\theta'_{2,3} \leq \theta \leq \theta''_{2,3}$, *we have*

$$\begin{aligned} E_{2,3}(\zeta_3, t_\theta) = &\operatorname{Log}\left(t_\theta^{-\frac{1}{6}}\right) + \operatorname{Log}\left(\left(1-\zeta_3^2\right)^{\frac{1}{6}} \left(1-\zeta_3\right)^{\frac{1}{2}}\right) + \log\Gamma\left(\frac{1}{3}\right) \\ &+ \frac{1}{6}\log\left(\frac{1}{3}\right) - \frac{1}{2}\log(2\pi) + o(1). \end{aligned}$$

The next two lemmas give uniform bounds for $E_{h,k}$ on the minor arcs. The proofs are quite intricate and are provided in the next section. We assume throughout that $0 < \varepsilon < \frac{3}{10}$.

Lemma 5.26. *Uniformly for* $k \leq n^\varepsilon$ *and* $-\theta'_{h,k} \leq \theta \leq \theta''_{h,k}$, *we have*

$$E_{h,k}(\zeta_b^a, t_\theta) = O\left(n^{3\varepsilon - \frac{1}{2}}\right) + O\left(n^{5\varepsilon - 1}\right) + O\left(n^\varepsilon\right).$$

The next lemma treats the case of large denominators.

Lemma 5.27. *Uniformly for* $k \geq n^\varepsilon$ *and* $-\theta'_{h,k} \leq \theta \leq \theta''_{h,k}$, *we have*

$$E_{h,k}(\zeta_b^a, t_\theta) = O\left(N\right).$$

With Lemmas 5.25-5.27 in hand, the proof of Thm. 5.5 follows [61] closely.

Proof of Thm. 5.5. Cases (1), (2) and (3). Assume $\frac{a}{b} \neq \frac{1}{3}$. We write

$$\lambda_1 + i\lambda_2 := \frac{\sqrt{\mathrm{Li}_2(\zeta_b^{ak_0})}}{k_0}.$$

It follows from Lemmas 5.26 and 5.27 that $E_{h,k}(\zeta_b^a, t_\theta) = \delta O(\sqrt{n})$ uniformly. Using this and Lem. 5.25, we have

$$\begin{aligned}
&e^{-2\lambda_1\sqrt{n}} Q_n(z) \\
&= \sum_{\substack{h \\ \frac{h}{k_0} \in \mathcal{F}_N}} \zeta_{k_0}^{-hn} \omega_{h,k_0}(\zeta_b^a) \\
&\qquad \times \int_{-\theta'_{h,k_0}}^{\theta''_{h,k_0}} \exp\left(-2\lambda_1\sqrt{n} + \frac{(\lambda_1 + i\lambda_2)^2}{t_\theta} + nt_\theta + o(1)\right) d\theta \\
&\quad + \sum_{\substack{\frac{h}{k} \in \mathcal{F}_N \\ k \neq k_0}} \zeta_k^{-hn} \\
&\qquad \times \int_{-\theta'_{h,k}}^{\theta''_{h,k}} \exp\left(-2\lambda_1\sqrt{n} + \frac{\mathrm{Li}_2\left(\zeta_b^{ak}\right)}{k^2 t_\theta} + nt_\theta + \delta O\left(\sqrt{n}\right)\right) d\theta.
\end{aligned}$$

Recalling Eq. (5.4.1), we rewrite the first term in the above equation as

$$\begin{aligned}
&\sum_{\substack{1 \leq h < k_0 \\ \gcd(h,k)=1}} \zeta_{k_0}^{-hn} \omega_{h,k_0}(z) \exp\left(i2\lambda_2\sqrt{n}\right) \\
&\quad \times \int_{-\theta'_{h,k_0}}^{\theta''_{h,k_0}} \exp\left(\sqrt{n}\left(\frac{\lambda_1 + i\lambda_2}{1 - \frac{2\pi i\theta\sqrt{n}}{\lambda_1 + i\lambda_2}} - (\lambda_1 + i\lambda_2)\right) - 2\pi i n\theta\right) d\theta.
\end{aligned}$$

We can estimate the integration endpoints as $\frac{1}{\sqrt{n}} \ll \theta'_{h,k_0}, \theta''_{h,k_0} \ll \frac{1}{\sqrt{n}}$ and setting $\theta \mapsto \theta n^{-\frac{1}{2}}$, the above integral is asymptotic to

$$\frac{1}{n^{\frac{1}{2}}} \int_{-c}^{c} \exp\left(\sqrt{n}\left(\frac{\lambda_1+i\lambda_2}{1-\frac{2\pi i\theta}{\lambda_1+i\lambda_2}} - (\lambda_1+i\lambda_2) - 2\pi i\theta\right)\right) d\theta =: \frac{1}{n^{\frac{1}{2}}} \int_{-c}^{c} e^{\sqrt{n}B(\theta)} d\theta, \tag{5.4.2}$$

for some $c > 0$. We claim that we can apply Lem. 5.13 with $x_0 \mapsto 0$, $A(x) \mapsto 1$ and B as above. Here, $B(0) = 0$ and expanding the geometric series gives

$$B(\theta) = -\frac{4\pi^2}{\lambda_1 + i\lambda_2}\theta^2 + o\left(\theta^2\right), \qquad \theta \to 0,$$

with $\operatorname{Re}\left(\frac{4\pi^2}{\lambda_1+i\lambda_2}\right) > 0$. Finally, we claim that $\operatorname{Re}(B(\theta)) \leq 0$ with equality if and only if $\theta = 0$. To this end,

$$\begin{aligned} \operatorname{Re}(B(\theta)) &= \operatorname{Re}\left(\frac{\lambda_1 + i\lambda_2}{1 - \frac{2\pi i\theta}{\lambda_1+i\lambda_2}}\right) - \lambda_1 \\ &= \frac{|\lambda_1 + i\lambda_2|^2}{\lambda_1}\operatorname{Re}\left(\frac{e^{i\psi}}{1 + i\left(\frac{\lambda_2}{\lambda_1} - \frac{2\pi\theta}{\lambda_1}\right)}\right) - \lambda_1, \end{aligned}$$

where $\lambda_1 + i\lambda_2 = |\lambda_1 + i\lambda_2|\, e^{i\frac{\psi}{2}}$ and $\psi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Applying Lem. 5.20 gives

$$\operatorname{Re}(B(\theta)) \leq \frac{|\lambda_1 + i\lambda_2|^2}{\lambda_1}\cos^2\left(\frac{\psi}{2}\right) - \lambda_1 = \frac{\lambda_1^2}{\lambda_1} - \lambda_1 = 0,$$

with equality if and only if

$$\operatorname{Arg}\left(1 + i\left(\frac{\lambda_2}{\lambda_1} - \frac{2\pi\theta}{\lambda_1}\right)\right) = \frac{\psi}{2} = \operatorname{Arg}\left(\lambda_1 + i\lambda_2\right).$$

That is, the above equation holds if and only if $\theta = 0$, as claimed. Now by Lem. 5.13, we conclude that Eq. (5.4.2) is asymptotic to

$$\frac{\sqrt{\lambda_1 + i\lambda_2}}{2\sqrt{\pi}n^{\frac{3}{4}}},$$

and overall

$$\sum_{\substack{1\leq h<k_0 \\ \gcd(h,k)=1}} \zeta_{k_0}^{-hn}\omega_{h,k_0}(z)$$

$$\times \int_{-\theta'_{h,k_0}}^{\theta''_{h,k_0}} \exp\left(-2\lambda_1\sqrt{n}+\frac{(\lambda_1+i\lambda_2)^2}{\frac{\lambda_1+i\lambda_2}{\sqrt{n}}-2\pi i\theta}+\sqrt{n}(\lambda_1+i\lambda_2)-2\pi in\theta\right)d\theta$$
$$\sim \frac{\sqrt{\lambda_1+i\lambda_2}}{2\sqrt{\pi}n^{\frac{3}{4}}}e^{2i\lambda_2\sqrt{n}} \sum_{\substack{1\le h<k_0\\(h,k_0)=1}} \zeta_{k_0}^{-hn}\omega_{h,k_0}(z).$$

When the $e^{2\lambda_1\sqrt{n}}$ is brought back to the right-hand side, this is the right-hand side of Thm. 5.5.

For $k \neq k_0$, we follow Parry's Lem. 19 of [61] and write

$$\begin{aligned}
&\mathrm{Re}\left(\frac{\mathrm{Li}_2\left(z^k\right)}{k^2t_\theta}+nt_\theta\right)\\
&= \lambda_1\sqrt{n}\left(\frac{\left|\mathrm{Li}_2\left(z^k\right)\right|}{k^2\lambda_1^2}\mathrm{Re}\left(\frac{e^{i\psi_k}}{1+i\left(\frac{\lambda_2}{\lambda_1}-\frac{2\pi\theta\sqrt{n}}{\lambda_1}\right)}\right)+1\right),
\end{aligned}$$

where ψ_k satisfies $\mathrm{Li}_2(z^k) = |\mathrm{Li}_2(z^k)|e^{i\psi_k}$. Arguing as for the major arcs, the expression $\mathrm{Re}(\cdot)$ is at most $\cos^2\left(\frac{\psi_k}{2}\right)$. Now let

$$\Delta := \inf_{k\neq k_0}\left(1-\left(\mathrm{Re}\frac{\sqrt{\mathrm{Li}_2(z^k)}}{k\lambda_1}\right)^2\right) > 0.$$

Then

$$\begin{aligned}
\mathrm{Re}\left(\frac{\mathrm{Li}_2(z^k)}{k^2t_\theta}+nt_\theta\right) &\le \lambda_1\sqrt{n}\left(\frac{\left|\mathrm{Li}_2\left(z^k\right)\right|}{k^2\lambda_1^2}\cos^2\left(\frac{\psi_k}{2}\right)+1\right)\\
&\le \lambda_1\sqrt{n}(2-\Delta).
\end{aligned}$$

Thus,

$$\left|\sum_{\substack{\frac{h}{k}\in\mathcal{F}_N\\k\neq k_0}} \zeta_k^{-hn}\int_{-\theta'_{h,k}}^{\theta''_{h,k}} \exp\left(-2\lambda_1\sqrt{n}+\frac{\mathrm{Li}_2(z^k)}{k^2t_\theta}+nt_\theta+\delta O(\sqrt{n})\right)d\theta\right|$$

$$\leq \exp\left(-\lambda_1\Delta\sqrt{n}+\delta O\left(\sqrt{n}\right)\right).$$

We can choose δ small enough so that the constant in the exponential is negative, and the minor arcs are exponentially smaller than the major arc(s). This completes the proof of Cases (1), (2) and (3).

For Case (4), we have $\lambda_1+i\lambda_2=\lambda_1=\frac{\sqrt{\mathrm{Li}_2(1)}}{3}=\frac{\pi}{3\sqrt{6}}$. Exactly as in Case (3) the minor arcs are those with $k\neq 3$, and these are shown to be exponentially smaller than for $k_0=3$. Thus, by Lem. 5.25 part (4), we have

$$\begin{aligned}
&e^{-2\lambda_1\sqrt{n}}Q_n(\zeta_3)\\
&=\zeta_3^{-n}C_{1,3}\int_{-\theta'_{1,3}}^{\theta''_{1,3}} t_\theta^{\frac{1}{6}}\exp\left(-2\lambda_1\sqrt{n}+\frac{\lambda_1^2}{t_\theta}+nt_\theta+o(1)\right)d\theta\\
&\quad+\zeta_3^{-2n}C_{2,3}\int_{-\theta'_{2,3}}^{\theta''_{2,3}} t_\theta^{-\frac{1}{6}}\exp\left(-2\lambda_1\sqrt{n}+\frac{\lambda_1^2}{t_\theta}+nt_\theta+o(1)\right)d\theta,
\end{aligned}\tag{5.4.3}$$

where

$$C_{1,3}:=\left(1-\zeta_3^2\right)^{-\frac{1}{6}}\left(1-\zeta_3\right)^{\frac{1}{2}}\Gamma\left(\frac{2}{3}\right)\frac{3^{\frac{1}{6}}}{\sqrt{2\pi}},$$
$$C_{2,3}:=\left(1-\zeta_3^2\right)^{\frac{1}{6}}\left(1-\zeta_3\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{3}\right)\frac{1}{3^{\frac{1}{6}}\sqrt{2\pi}},$$

and

$$t_\theta=\frac{\pi}{3\sqrt{6n}}\left(1-6i\sqrt{6n}\theta\right).\tag{5.4.4}$$

Overall, we have

$$\begin{aligned}
&e^{-2\lambda_1\sqrt{n}}Q_n(\zeta_3)\\
&=\zeta_3^{-n}C_{1,3}\frac{(\pi)^{\frac{1}{6}}}{3^{\frac{1}{6}}(6n)^{\frac{1}{12}}}\\
&\quad\times\int_{-\theta'_{1,3}}^{\theta''_{1,3}}\left(1-6i\sqrt{6n}\theta\right)^{\frac{1}{6}}\exp\left(-2\lambda_1\sqrt{n}+\frac{\lambda_1^2}{t_\theta}+nt_\theta+o(1)\right)d\theta
\end{aligned}\tag{5.4.5}$$

$$+\zeta_3^{-2n} C_{2,3} \frac{3^{\frac{1}{6}}(6n)^{\frac{1}{12}}}{(\pi)^{\frac{1}{6}}}$$
$$\times \int_{-\theta'_{2,3}}^{\theta''_{2,3}} \left(1-6i\sqrt{6n}\theta\right)^{-\frac{1}{6}} \exp\left(-2\lambda_1\sqrt{n}+\frac{\lambda_1^2}{t_\theta}+nt_\theta+o(1)\right) d\theta.$$

Setting $\theta \mapsto \theta n^{-\frac{1}{2}}$ and arguing as before, both integrals are asymptotic to

$$\frac{\sqrt{\lambda_1}}{2\sqrt{\pi}n^{\frac{3}{4}}} = \frac{1}{2^{\frac{5}{4}}3^{\frac{3}{4}}n^{\frac{3}{4}}}.$$

Hence, the second term in Eq. (5.4.5) dominates, and gives the claimed asymptotic formula. □

5.5 Proofs of Lemmas 5.25, 5.26, and 5.27

We prove Lem. 5.26 first then make use of these ideas in the proof of Lem. 5.25. We finish the section by proving Lem. 5.27.

5.5.1 Proof of Lem. 5.26

We rewrite $E_{h,k}$ as a sum of two functions: a function to which we can apply Euler–Maclaurin summation in the form of Thm. 2.25, and another to which we apply the tools in Prop. 5.16 and Lem. 5.22.

Lemma 5.28. *For $t \in \mathbb{C}$ with $\mathrm{Re}(t) > 0$ and $z = \zeta_b^a$, we have*

$$E_{h,k}(z,t) = \sum_{\substack{1\le m\le bk \\ 1\le j\le k}} \zeta_b^{ma}\zeta_k^{jmh} kt \sum_{\ell\ge 0} g_{j,k}\left(t(bk^2\ell+km)\right)$$
$$+ \mathrm{Log}\left(\prod_{j=1}^{k}\left(1-\zeta_b^a\zeta_k^{-jh}e^{-jt}\right)^{-\frac{1}{2}+\frac{j}{k}}\right), \qquad (5.5.1)$$

where

$$g_{j,k}(w) := \frac{e^{-\frac{j}{k}w}}{w(1-e^{-w})} - \frac{1}{w^2} - \left(\frac{1}{2} - \frac{j}{k}\right)\frac{e^{-\frac{j}{k}w}}{w}.$$

Proof. In $E_{h,k}$, we expand the logarithm using its Taylor series as

$$\begin{aligned}
&\operatorname{Log}\left(\zeta_b^a \zeta_k^h e^{-t}; \zeta_k^h e^{-t}\right)_\infty^{-1} = \sum_{\substack{\nu\geq 1\\ \ell\geq 1}} \frac{\zeta_b^{\ell a}\zeta_k^{\ell\nu h} e^{-\ell\nu t}}{\ell}\\
&= \sum_{\substack{1\leq j\leq k\\ 1\leq m\leq bk}} \sum_{\substack{\nu\geq 0\\ \ell\geq 0}} \frac{\zeta_{bk}^{m(ka+bjh)} e^{-(bk\ell+m)(\nu k+j)t}}{bk\ell+m}\\
&= \sum_{\substack{1\leq j\leq k\\ 1\leq m\leq bk}} \zeta_b^{ma}\zeta_k^{mjh} \sum_{\ell\geq 0} \frac{e^{-jt(bk\ell+m)}}{(bk\ell+m)(1-e^{-t(bk^2\ell+km)})}.
\end{aligned}$$

This corresponds to the left term in $g_{j,k}$. For the middle term in $g_{j,k}$, we compute (using $\gcd(h,k)=1$)

$$\begin{aligned}
&\sum_{\substack{1\leq j\leq k\\ 1\leq m\leq bk}} \zeta_b^{ma}\zeta_k^{mjh} kt \sum_{\ell\geq 0} \frac{1}{t^2(bk^2\ell+km)^2}\\
&= \frac{k^2}{t} \sum_{\substack{1\leq m\leq bk\\ m\equiv 0 \pmod{k}}} \zeta_b^{ma} \frac{1}{(bk^2\ell+mk)^2}\\
&= \frac{1}{tk^2}\operatorname{Li}_2\left(\zeta_b^{ka}\right).
\end{aligned}$$

Finally, it is simple to show that the logarithm of the product in Eq. (5.5.1) cancels with the sum of the right term in $g_{j,k}$, simply by expanding the logarithm into its Taylor series. $\square$

We estimate the first term in Eq. (5.5.1) using Thm. 2.25. To do so, we need a technical definition. It is elementary to show that in the sum in Eq. (5.5.1), there is at most one $j_0 = j_0(a,b,h,k)$ for which $\zeta_b^{ma}\zeta_k^{j_0 mh} = 1$; i.e., such that $ak + bj_0h \equiv 0 \pmod{bk}$. Moreover, since $\gcd(h,k) = 1$, the map $j \mapsto jh$ is a bijection, and so the existence of j_0 depends only on k and not on h. Define

$$\mathcal{S}_{a,b} := \{k \in \mathbb{N} : \exists\, j_0 \in [1,k] \text{ with } ak + j_0bk \equiv 0 \pmod{bk}\}.$$

Lemma 5.29. *Let j_0 be as above. For $k \leq n^{\varepsilon}$ and $-\theta'_{h,k} \leq \theta \leq \theta''_{h,k}$, we have*

$$\sum_{\substack{1\leq m\leq bk\\1\leq j\leq k}} \zeta_b^{ma}\zeta_k^{jmh} k t_\theta \sum_{\ell\geq 0} g_{j,k}\left(t_\theta(bk^2\ell + km)\right)$$
$$= \left(\log\left(\Gamma\left(\frac{j_0}{k}\right)\right) + \left(\frac{1}{2} - \frac{j_0}{k}\right)\log\left(\frac{j_0}{k}\right) - \frac{1}{2}\log(2\pi)\right) 1_{k\in\mathcal{S}_{a,b}}$$
$$+ O\left(\frac{k^3}{\sqrt{n}}\right) + O\left(\frac{k^5}{n}\right).$$

Proof. Note that the function $g_{j,k}(w)$ is holomorphic at 0 and in any cone $|\operatorname{Arg}(w)| \leq \frac{\pi}{2} - \eta$. Also, $\theta'_{h,k}, \theta''_{h,k} \leq \frac{1}{2N} = O\left(\frac{1}{\sqrt{n}}\right)$ implies that t_θ lies in such a fixed cone. Thus, we can apply Thm. 2.25 to $g_{j,d}(z)$ with $w \mapsto t_\theta bk^2$, $a \mapsto \frac{m}{bk}$, and $N \mapsto 0$,

$$\sum_{\ell\geq 0} g_{j,k}\left(t_\theta bk^2\left(\ell + \frac{m}{bk}\right)\right) = \frac{1}{t_\theta bk^2}\int_0^\infty g_{j,k}(w)dw$$
$$- \left(\frac{1}{12} - \frac{j^2}{2k^2}\right)\left(\frac{1}{2} - \frac{m}{bk}\right) + O\left(\frac{k^2}{\sqrt{n}}\right).$$

When summing the O-term, we get

$$\sum_{\substack{1\leq m\leq bk\\1\leq j\leq k}} k t_\theta O\left(\frac{k^2}{\sqrt{n}}\right) = O\left(\frac{k^5}{n}\right),$$

where we used the fact that t_θ lies in a cone. Summing first over m

gives

$$\sum_{1\le m\le bk} \zeta_b^{ma}\zeta_k^{jmh}\left(\frac{1}{12}-\frac{j^2}{2k^2}\right)\left(\frac{1}{2}-\frac{m}{bk}\right) = O\left(k\right).$$

Hence,

$$\begin{aligned}
&\sum_{\substack{1\le m\le bk\\ 1\le j\le k}} \zeta_b^{ma}\zeta_k^{jmh} k t_\theta \sum_{\ell\ge 0} g_{j,k}\left(t_\theta(bk^2\ell+km)\right)\\
&=\left(\int_0^\infty g_{j_0,k}(w)dw\right)1_{k\in\mathcal{S}_{a,b}} + O\left(\frac{k^3}{\sqrt{n}}\right) + O\left(\frac{k^2}{\sqrt{n}}\right) + O\left(\frac{k^5}{n}\right)\\
&=\left(\log\Gamma\left(\frac{j_0}{k}\right)+\left(\frac{1}{2}-\frac{j_0}{k}\right)\log\left(\frac{j_0}{k}\right)-\frac{1}{2}\log(2\pi)\right)1_{k\in\mathcal{S}_{a,b}}\\
&\quad + O\left(\frac{k^3}{\sqrt{n}}\right) + O\left(\frac{k^5}{n}\right),
\end{aligned}$$

where the last step follows by by Lem. 5.15. If $\varepsilon < \frac{3}{10}$, then the terms $O\left(\frac{k^3}{\sqrt{n}}\right)$ and $O\left(\frac{k^5}{n}\right)$ are smaller than $\sqrt{n}$ as needed since $N = O(\delta\sqrt{n})$ where δ is chosen small and independently of n. $\square$

It remains to estimate the product term in Eq. (5.5.3).

Lemma 5.30. *Uniformly for $k \le n^\varepsilon$ and $-\theta_{h,k} \le \theta \le \theta''_{h,k}$, we have*

$$\operatorname{Log}\left(\prod_{j=1}^{k}\left(1-\zeta_b^a\zeta_k^{-jh}e^{-jt_\theta}\right)^{-\frac{1}{2}+\frac{j}{k}}\right) = O\left(n^\varepsilon\right).$$

Proof. Recall the sums G_m defined in Lem. 5.22. Using the Taylor expansion for the logarithm followed by Prop. 5.16, we write

$$\left|\operatorname{Log}\left(\prod_{j=1}^{k}\left(1-\zeta_b^a\zeta_k^{-jh}e^{-jt_\theta}\right)^{-\frac{1}{2}+\frac{j}{k}}\right)\right|$$
$$=\left|\sum_{j=1}^{k}\left(\frac{1}{2}-\frac{j}{k}\right)\sum_{m\geq 1}\frac{\zeta_{bk}^{m(ak+bjh)}}{m}e^{-jmt_\theta}\right| \tag{5.5.2}$$
$$=\left|\sum_{j=1}^{k}\left(\frac{1}{2}-\frac{j}{k}\right)(1-e^{-jt_\theta})\sum_{m\geq 1}G_m\left(\frac{ak+bhj}{bk}\right)e^{-mjt_\theta}\right|.$$

It is elementary to show that at most one of $\{ak+bjh\}_{1\leq j\leq k}$ is divisible by bk. Suppose that this happens at j_0 (if it never happens, then the argument is similar). Then $G_m\left(\frac{ak+bhj_0}{bk}\right)=H_m$, the m^{th} harmonic number. Applying the formula

$$\sum_{m\geq 1}H_m x^m=-\frac{\log(1-x)}{1-x},$$

leads to the bound

$$\begin{aligned}\text{Eq. (5.5.2)} &\ll \left|1-e^{-j_0t_\theta}\right|\sum_{m\geq 1}H_m e^{-mj_0\operatorname{Re}(t_\theta)}\\ &\quad+\sum_{\substack{1\leq j\leq k\\ j\neq j_0}}\left|1-e^{-jt_\theta}\right|\max_{m\geq 1}\left|G_m\left(\frac{a}{b}+\frac{hj}{k}\right)\right|\sum_{m\geq 1}e^{-mj\operatorname{Re}(t_\theta)}\\ &\ll\left|\operatorname{Log}\left(1-e^{-j_0t_\theta}\right)\right|\frac{\left|1-e^{-j_0t_\theta}\right|}{1-e^{-j_0\operatorname{Re}(t_\theta)}}\\ &\quad+\sum_{\substack{1\leq j\leq k\\ j\neq j_0}}\frac{\left|1-e^{-jt_\theta}\right|}{1-e^{-j\operatorname{Re}(t_\theta)}}\max_{m\geq 1}\left|G_m\left(\frac{a}{b}+\frac{hj}{k}\right)\right|.\end{aligned}$$

The fact that t_θ lies in a cone $|\operatorname{Arg}(t_\theta)| \le \frac{\pi}{2}-\eta$ with $j \le k \le n^{\varepsilon} < \sqrt{n}$ gives

$$\frac{\left|1-e^{-jt_\theta}\right|}{1-e^{-j\operatorname{Re}(t_\theta)}} = O\left(\frac{|t_\theta|}{\operatorname{Re}(t_\theta)}\right) = O(1).$$

Recalling the definition of t_θ in Eq. (5.4.4), we see that $t_\theta \,\dot{\sim}\, \sqrt{n}$. This implies the estimate

$$\begin{aligned}&\left|\operatorname{Log}\left(\prod_{j=1}^{k}\left(1-\zeta_b^a\zeta_k^{-jh}e^{-jt_\theta}\right)^{-\frac{1}{2}+\frac{j}{k}}\right)\right|\\ &\qquad = O(\log(n)) + O\left(\sum_{\substack{1\le j\le k\\ j\ne j_0}}\max_{m\ge 1}\left|G_m\left(\frac{a}{b}+\frac{hj}{k}\right)\right|\right).\end{aligned}$$

Finally, $\gcd(h,k)=1$ implies that the map $j \mapsto hj$ is a bijection modulo k, so we obtain, with $G_m(x) = G_m(x+1)$,

$$\begin{aligned}\sum_{\substack{1\le j\le k\\ j\ne j_0}}\max_{m\ge 1}\left|G_m\left(\frac{a}{b}+\frac{hj}{k}\right)\right| &= \max_{m\ge 1}\left|G_m\left(\frac{\alpha_{a,b,k}}{bk}\right)\right|\\ &\quad + \max_{m\ge 1}\left|G_m\left(\frac{\alpha_{a,b,k}}{bk}+1-\frac{1}{k}\right)\right|\\ &\quad + \sum_{\substack{1\le j\le k-2\\ j\ne j_0}}\max_{m\ge 1}\left|G_m\left(\frac{\alpha_{a,b,k}}{bk}+\frac{j}{k}\right)\right|,\end{aligned}$$

for some $\alpha_{a,b,k}$ independent of j. By Lem. 5.22, we have

$$\begin{aligned}&\sum_{\substack{1\le j\le k-2\\ j\ne j_0}}\max_{m\ge 1}\left|G_m\left(\frac{\alpha_{a,b,k}}{bk}+\frac{j}{k}\right)\right|\\ &\ll \sum_{\substack{1\le j\le k-2\\ j\ne j_0}}\left(\log\left(\frac{k}{j}\right)+\log\left(\frac{k}{k-j}\right)\right)\end{aligned}$$

$$= O(k) = O(n^{\varepsilon}),$$

by Stirling's formula. Since clearly $\left|G_m\left(\frac{\alpha_{a,b,k}}{bk}\right)\right| = O(\log(k)) = O(\log(n))$ and likewise

$$\max_{m\geq 1}\left|G_m\left(\frac{\alpha_{a,b,k}}{bk}+1-\frac{1}{k}\right)\right| = O(\log(k)),$$

the lemma follows. $\square$

Lem. 5.26 now follows from Lemmas 5.28, 5.29 and 5.30.

5.5.2 Proof of Lem. 5.25

To prove 5.25, we need an elementary fact about the sets $\mathcal{S}_{a,b}$.

Lemma 5.31. *Let* $1 \leq a < \frac{b}{2}$ *with* $\gcd(a,b) = 1$ *and* $b \geq 3$*. Then* $1 \in \mathcal{S}_{a,b}$ *and* $2 \in \mathcal{S}_{a,b}$*, and* $3 \in \mathcal{S}_{a,b}$ *if and only if* $(a,b) = (1,3)$*.*

Proof. We have that $2 \in \mathcal{S}_{a,b}$ if and only if $2b$ divides $2a+b$ or $2a+2b$. Clearly, $2b \nmid (2a+2b)$, and since $2a+b < 3b < 2\cdot(2b)$,

$$2b \mid (2a+b) \Leftrightarrow 2a+b = 2b \Leftrightarrow 2a = b \Leftrightarrow (a,b) = (1,2).$$

The other proofs are analogous. $\square$

Proof of Lem. 5.25. Cases (1), (2) and (3) are simple consequences of Lemmas 5.28, 5.29, and 5.31. For Case (4), we suppose $\zeta = \zeta_3$. Then one finds $j_0(1,3,1,3) = 2$ and Lemmas 5.28 and 5.29 imply

$$\begin{aligned}
&E_{1,3}(\zeta_3, t_\theta)\\
&= \operatorname{Log}\left(\prod_{j=1}^{3}\left(1-\zeta_3\zeta_3^j e^{-jt_\theta}\right)^{-\frac{1}{2}+\frac{j}{3}}\right)\\
&\qquad + \log\left(\Gamma\left(\frac{2}{3}\right)\right) - \frac{1}{6}\log\left(\frac{2}{3}\right) - \frac{1}{2}\log(2\pi) + o(1)
\end{aligned}$$

$$= \mathrm{Log}\left(t_\theta^{\frac{1}{6}}\right) + \mathrm{Log}\left(\frac{(1-\zeta_3)^{\frac{1}{2}}}{(1-\zeta_3^2)^{\frac{1}{6}}}\right) + \log\left(\Gamma\left(\frac{1}{3}\right)\right)$$
$$+ \frac{1}{6}\log(3) - \frac{1}{2}\log(2\pi) + o(1),$$

as claimed, whereas $j_0(1,3,2,3) = 1$, and so

$$E_{2,3}(\zeta_b^a, t_\theta) = \mathrm{Log}\prod_{j=1}^{3}\left(1 - \zeta_3\zeta_3^{2j}e^{-jt_\theta}\right)^{-\frac{1}{2}+\frac{j}{3}}$$
$$+ \log\Gamma\left(\frac{1}{3}\right) + \frac{1}{6}\log\left(\frac{1}{3}\right) - \frac{1}{2}\log(2\pi) + o(1)$$
$$= \mathrm{Log}\left(t_\theta^{-\frac{1}{6}}\right) + \mathrm{Log}\left(\left(1-\zeta_3^2\right)^{\frac{1}{6}}(1-\zeta_3)^{\frac{1}{2}}\right)$$
$$+ \log\Gamma\left(\frac{1}{3}\right) + \frac{1}{6}\log\left(\frac{1}{3}\right) - \frac{1}{2}\log(2\pi) + o(1),$$

as claimed. □

5.5.3 Proof of Lem. 5.27

In preparation for the proof of Lem. 5.27, we rewrite $E_{h,k}$ as in Lem. 5.28, this time using only the first two terms of $g_{j,k}$. The proof is analogous.

Lemma 5.32. *For $t \in \mathbb{C}$ with $\mathrm{Re}(t) > 0$ and $z = \zeta_b^a$, we have*

$$E_{h,k}(z,t) = \sum_{\substack{1\le m\le bk \\ 1\le j\le k}} \zeta_b^{ma}\zeta_k^{jmh} kt \sum_{\ell\ge 0} \widetilde{g}_{j,k}\left(t\left(bk^2\ell + km\right)\right), \tag{5.5.3}$$

where

$$\widetilde{g}_{j,k}(w) := \frac{e^{-\frac{j}{k}w}}{w(1-e^{-w})} - \frac{1}{w^2}.$$

We need to estimate the sum in Eq. (5.5.3) separately for $\ell \geq 1$ and $\ell = 0$. For $\ell \geq 1$, we can first compute the sum over j as

$$\sum_{1\leq j\leq k} \zeta_k^{jmh}\widetilde{g}_{j,k}(z) = \frac{\zeta_k^{mh}e^{-\frac{z}{k}}}{z(1-\zeta_k^{mh}e^{-\frac{z}{k}})} - \frac{k}{z^2}\cdot 1_{k|m}.$$

Thus, writing $m = \nu k$ with $1 \leq \nu \leq b$ when $k \mid m$, we have

$$\begin{aligned}
&\sum_{\substack{1\leq m\leq bk\\ 1\leq j\leq k}} \zeta_b^{ma}\zeta_k^{jmh} kt \sum_{\ell\geq 1} \widetilde{g}_{j,k}\left(t(bk^2\ell + km)\right)\\
&= t\sum_{\nu=1}^{b} \zeta_b^{\nu k a}\sum_{\ell\geq 1} f_1\left(bkt\left(\ell + \frac{\nu}{b}\right)\right)\\
&\quad + t\sum_{\substack{1\leq m\leq bk\\ k\nmid m}} \zeta_b^{ma}\sum_{\ell\geq 1} f_2\left(bkt\left(\ell+\frac{m}{bk}\right)\right)\\
&=: S_1 + S_2,
\end{aligned}$$

where

$$f_1(z) := \frac{e^{-z}}{z(1-e^{-z})} - \frac{1}{z^2} \quad\text{and}\quad f_2(z) := \frac{\zeta_k^{mh}e^{-z}}{z(1-\zeta_k^{mh}e^{-z})}.$$

Lemma 5.33. *For $k \geq n^{\varepsilon}$, $-\theta'_{h,k} \leq \theta \leq \theta''_{h,k}$ and $t \mapsto t_\theta$, we have $|S_1| = O(\log(n))$.*

Proof. We use Thm. 5.14 (with $N \to \infty$) to write

$$\begin{aligned}
&\sum_{\ell\geq 1} f_1\left(bkt_\theta\left(\ell+\frac{\nu}{b}\right)\right)\\
&= \frac{f_1\left(bkt_\theta\left(1+\frac{\nu}{b}\right)\right)}{2} + \int_1^\infty f_1\left(bkt_\theta\left(x+\frac{\nu}{b}\right)\right)dx\\
&\quad + bkt_\theta\int_1^\infty f_1'\left(bkt_\theta\left(x+\frac{\nu}{b}\right)\right)\left(\{x\}-\frac{1}{2}\right)dx
\end{aligned}$$

$$= \frac{f_1\left(kt_\theta\left(b+\nu\right)\right)}{2} + \frac{1}{bkt_\theta}\int_{kt_\theta(b+\nu)}^{\infty} f_1\left(z\right)dz$$
$$+ \int_{kt_\theta(b+\nu)}^{\infty} f_1'\left(z\right)\left(\left\{\frac{z-\frac{\nu}{b}}{bkt_\theta}\right\} - \frac{1}{2}\right)dz.$$

Here, $f_1(z) \ll \frac{1}{z}$ as $z \to 0$, thus

$$f_1\left(kt_\theta(b+\nu)\right) = O\left(\frac{1}{k|t_\theta|}\right).$$

Furthermore $\int_{t\cdot\frac{c}{|t|}}^{t\infty} f_1(z)dz = O(1)$ with $|\operatorname{Arg}(t)| \leq \frac{\pi}{2} - \eta$, holds uniformly for any $\eta, c > 0$. As noted before, t_θ lies in such a cone, and as such, implies that

$$\frac{1}{bkt_\theta}\int_{kt_\theta(b+\nu)}^{\infty} f_1\left(z\right)dz = O\left(\frac{1}{k|t_\theta|}\int_{kt_\theta(b+\nu)}^{t\frac{2b}{|t|}} \frac{1}{z}dz\right)$$
$$= O\left(\frac{|\operatorname{Log}(dt_\theta)|}{k|t_\theta|}\right).$$

Similarly, one has

$$f_1'(z) = -\frac{e^{-z}}{z(1-e^{-z})} - \frac{e^{-z}}{z^2(1-e^{-z})} - \frac{e^{-2z}}{z(1-e^{-z})^2} + \frac{2}{z^3}.$$

Therefore,

$$\int_{t\cdot\frac{c}{|t|}}^{t\infty} f_1'(z)dz = O(1)$$

with $|\operatorname{Arg}(t)| \leq \frac{\pi}{2} - \eta$, which holds for any $\eta, c > 0$. Additionally, since $f_1'(z) \ll \frac{1}{z^2}$ as $z \to 0$,

$$\int_{kt_\theta(b+\nu)}^{\infty} f_1'\left(z\right)\left(\left\{\frac{z-\frac{\nu}{b}}{bkt_\theta}\right\} - \frac{1}{2}\right)dz = O\left(\int_{kt_\theta(b+\nu)}^{t_\theta\frac{2b}{|t_\theta|}} \frac{1}{z^2}dz\right)$$
$$= O\left(\frac{1}{k|t_\theta|}\right).$$

The above bounds are all clearly uniform in $1 \leq \nu \leq b$, thus overall

$$|S_1| = \sum_{\substack{1 \leq m \leq bk \\ k \nmid m}} |t_\theta| O\left(\frac{|\operatorname{Log} kt_\theta|}{k|t_\theta|}\right) = O\left(|\operatorname{Log}(kt_\theta)|\right) = O(\log n),$$

as claimed. □

Lemma 5.34. *For $k \geq n^\varepsilon$, $-\theta'_{h,k} \leq \theta \leq \theta''_{h,k}$ and $t \mapsto t_\theta$, we have $|S_2| = O\left(n^{\frac{1}{2}-\varepsilon}\right)$.*

Proof. For $\mathrm{Re}(z) > 0$, we see immediately that

$$|f_2(z)| \leq \frac{e^{-\mathrm{Re}(z)}}{\mathrm{Re}(z)(1 - e^{-\mathrm{Re}(z)})} =: \tilde{f}_2(\mathrm{Re}(z)).$$

Thus, since $\tilde{f}_2$ is decreasing, we have

$$\begin{aligned} |S_2| &\leq |t_\theta| bk \sum_{\ell \geq 1} \tilde{f}_2\left(bk\mathrm{Re}(t_\theta)\ell\right) \\ &\leq \frac{|t_\theta|}{\mathrm{Re}(t_\theta)} \left(\int_{bk\mathrm{Re}(t_\theta)}^{\infty} \tilde{f}_2(x)dx + bk\mathrm{Re}(t_\theta)\tilde{f}_2(bk\mathrm{Re}(t_\theta))\right), \end{aligned}$$

by integral comparison with the Riemann sum. We can bound the right term as

$$bk\mathrm{Re}(t_\theta)\tilde{f}_2(bk\mathrm{Re}(t_\theta)) = \frac{1}{e^{bk\mathrm{Re}(t_\theta)} - 1} \leq \frac{1}{bk\mathrm{Re}(t_\theta)}.$$

Furthermore, as $\eta \to 0^+$, we have

$$\begin{aligned} \int_\eta^\infty \tilde{f}_2(x)dx &= O(1) + \int_\eta^1 \frac{e^{-x}}{x(1-e^{-x})}dx \leq O(1) + \int_\eta^1 \frac{1}{x^2}dx \\ &= O\left(\frac{1}{\eta}\right). \end{aligned}$$

Hence,

$$|S_2| = O\left(\frac{1}{k\mathrm{Re}(t_\theta)}\right) = O\left(\frac{\sqrt{n}}{k}\right) = O\left(n^{\frac{1}{2}-\varepsilon}\right),$$

as claimed. $\square$

It remains to estimate the double sum Eq. (5.5.3) for the term $\ell = 0$; i.e.,

$$\sum_{\substack{1\le m\le bk\\ 1\le j\le k}} \zeta_b^{ma}\zeta_k^{jmh}\frac{\phi_{j/k}(tkm)}{m}, \tag{5.5.4}$$

where ϕ_a was defined in Eq. (5.3.1). The following lemma, when combined with Lemmas 5.32–5.34, completes the proof of Lem. 5.27, and thus the proof of Thm. 5.5.

Lemma 5.35. *For $n^\varepsilon \le k \le N$ and $-\theta'_{h,k} \le \theta \le \theta''_{h,k}$, we have*

$$\sum_{\substack{1\le m\le bk\\ 1\le j\le k}} \zeta_b^{ma}\zeta_k^{jmh}\frac{\phi_{j/k}(t_\theta km)}{m} = O(k).$$

Proof. Let $x := kt_\theta$ and $a := \frac{j}{k}$. Note that $0 < a \le 1$, $\mathrm{Re}(x) > 0$, and $\frac{|x|}{\mathrm{Re}(x)} \ll 1$ uniformly in k. We use Abel partial summation as in Prop. 5.16 and split the sum into two parts:

$$\sum_{\substack{1\le m\le bk\\ 1\le j\le k}} = \sum_{j=1}^{k}\left(\sum_{0<m\le\min\left\{bk,\frac{1}{|x|}\right\}} + \sum_{\min\left\{bk,\frac{1}{|x|}\right\}<m\le bk}\right).$$

In the case $|x| > 1$, the first sum is empty, so we can assume $|x| \le 1$. We first find with Prop. 5.16 that

$$\sum_{m\le\min\left\{bk,\frac{1}{|x|}\right\}} \zeta_{bk}^{m(ak+hjb)}\frac{1}{m}\phi_a(xm)$$

$$
\begin{aligned}
&= G_{\min\left\{bk,\frac{1}{|x|}\right\}}\left(\frac{a}{b}+\frac{hj}{k}\right)\phi_a\left(x\min\left\{bk,\left\lfloor\frac{1}{|x|}\right\rfloor\right\}\right)\\
&\quad+\sum_{m\leq\min\left\{bk,\frac{1}{|x|}\right\}-1} G_m\left(\frac{a}{b}+\frac{hj}{k}\right)\left(\phi_a(mx)-\phi_a((m+1)x)\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left|\sum_{j=1}^{k}\sum_{m\leq\min\left\{bk,\frac{1}{|x|}\right\}}\zeta_b^{ma}\zeta_k^{mhj}\cdot\frac{1}{m}\phi_a(mx)\right|\\
&\leq\sum_{j=1}^{k}\left|G_{\min\left\{bk,\frac{1}{|x|}\right\}}\left(\frac{a}{b}+\frac{hj}{k}\right)\right|\left|\phi_a\left(x\min\left\{bk,\left\lfloor\frac{1}{|x|}\right\rfloor\right\}\right)\right|\\
&\quad+\left|\sum_{j=1}^{k}\sum_{0<m\leq\min\left\{bk,\frac{1}{|x|}\right\}}G_m\left(\frac{a}{b}+\frac{hj}{k}\right)\left(\phi_a(mx)-\phi_a((m+1)x)\right)\right|\\
&\ll\sum_{j=1}^{k}\left|G_{\min\left\{bk,\frac{1}{|x|}\right\}}\left(\frac{a}{b}+\frac{hj}{k}\right)\right|\\
&\quad+\sum_{j=1}^{k}\max_{m=1,\ldots,\min\left\{bk,\frac{1}{|x|}\right\}}\left|G_m\left(\frac{a}{b}+\frac{hj}{k}\right)\right|\sum_{0<m\leq\frac{1}{|x|}}|x|\\
&\ll\sum_{j=1}^{k}\log\left(\frac{k}{j}\right)=O(k),
\end{aligned}
$$

where we used Lem. 5.18, the bounds for G_m found in the proof of Lem. 5.30 and Stirling's formula. That is,

$$\log(n!)=n\log(n)+O(n)$$

for large n. Similarly, we find with Lem. 5.19 (without loss of generality we assume $\frac{1}{|x|} < bk$)

$$\begin{aligned}
&\left|\sum_{j=1}^{k} \sum_{\frac{1}{|x|}<m\leq bk} \zeta_b^{ma}\zeta_k^{mhj} \cdot \frac{1}{m}\phi_a(mx)\right| \\
&\ll \sum_{j=1}^{k} \left|G_{bk}\left(\frac{a}{b}+\frac{hj}{k}\right)\right| |\phi_a\left(xbk\right)| \\
&\quad + \left|\sum_{j=1}^{k} \sum_{\frac{1}{|x|}<m\leq bk} G_m\left(\frac{a}{b}+\frac{hj}{k}\right)\left(\phi_a(mx)-\phi_a((m+1)x)\right)\right| \\
&\ll O(k) + \sum_{j=1}^{k} \max_{m=\frac{1}{|x|},\ldots,bk} \left|G_m\left(\frac{a}{b}+\frac{hj}{k}\right)\right| \\
&\quad \times \sum_{\frac{1}{|x|}<m\leq bk} \left(\frac{1}{|x|m(m+1)} + |x|e^{-m\mathrm{Re}(x)} + a|x|e^{-am\mathrm{Re}(x)}\right).
\end{aligned}$$

Note that $\phi_a(xbk) \ll 1$ holds uniformly (as $1 \ll |xbk|$ and x is part of a fixed cone $|\operatorname{Arg}(x)| \leq \frac{\pi}{2} - \eta$) as well as

$$\begin{aligned}
&\sum_{\frac{1}{|x|}<m\leq bk} \frac{1}{|x|m(m+1)} \leq \sum_{\frac{1}{|x|}<m<\infty} \frac{1}{|x|m(m+1)} \ll 1, \\
&\sum_{\frac{1}{|x|}<m\leq bk} |x|e^{-m\mathrm{Re}(x)} \leq \frac{|x|}{1-e^{-\mathrm{Re}(x)}} \ll 1, \\
&\qquad\qquad \text{which holds as} \quad \frac{|x|}{\mathrm{Re}(x)} \ll 1, |x| \ll 1,
\end{aligned}$$

and likewise

$$\sum_{\frac{1}{|x|}<m\leq bk} a|x|e^{-am\mathrm{Re}(x)} \ll 1.$$

As a result, again arguing as in the proof of Lem. 5.30,

$$\begin{aligned}
&\left|\sum_{j=1}^{k}\sum_{\frac{1}{|x|}<m\le bk}\zeta_b^{ma}\zeta_k^{mhj}\cdot\frac{1}{m}\phi_a(mx)\right| \\
&\ll O(k)+\sum_{j=1}^{k}\max_{m=\frac{1}{|x|},\ldots,bk}\left|G_m\left(\frac{a}{b}+\frac{hj}{k}\right)\right| \\
&= O(k)+O(k) \\
&= O(k),
\end{aligned}$$

which completes the proof. □

Chapter 6

Concluding remarks and open problems

We summarize the results proven in this thesis and discuss some open problems related to these results.

6.1 Ch. 3 discussion

In Ch. 3, we proved an asymptotic formula for the number of odd-balanced unimodal sequences of size $2n+2$ with rank congruent to a modulo c where $c>1$ is odd. The formal statement was introduced in Result A of the introduction of this thesis. We found that the number of such sequences is asymptotically related to the number of overpartitions up to a power of n. It would be interesting to prove Result A for all c, and it appears numerically that such a result holds. We state the following conjecture.

Conjecture 1. The statment in Thm. 3.1 holds for all $c>1$.

To prove this conjecture, it is equivalent to show the following.

Lemma 6.1. *The previous conjecture is equivalent to showing that $V(-1;q)$ is exponentially smaller than $q_0^{-\frac{1}{16}}$ as $q \to 1$ from inside the unit circle.*

Proof. As was mentioned at the end of Ch. 3, the authors of [50] showed (see just after Eq. 1.14, therein) that

$$V(\pm i; q) = q^{-1} A(q),$$

where $A(q)$ is the mock theta function defined by (see Appendix, [20])

$$A(q) := \sum_{n \geq 0} \frac{q^{n+1}(-q^2; q^2)_n}{(q; q^2)_{n+1}} = -i\mu(3\tau; 2\tau; 4\tau).$$

By a simple application of the transformation formula for μ in Prop. 2.19, one finds as $\tau \to 0$,

$$A(q) \ll q_0^{-\frac{1}{32}},$$

which is exponentially smaller than the main term $q_0^{-\frac{1}{16}}$ coming from $V(1; q)$. Since every other term in Eq. (3.1.2) satisfies this bound except possibly $V(-1; q)$, which only occurs when c is even, we are done. □

It is in fact believable that the bound is even better since the main subject of [50] dealt with showing that the formal dual[1] $V(-1; q^{-1})$ is a partial theta function, and thus by Thm. 5.14, is bounded by a polynomial in τ. This is also indicated in Fig. 6.1 where the Fourier coefficients of $V(1; q)$ may grow like $\log(n)$. This begs the following question.

[1]By formal, the representation of $V(-1; q^{-1})$ as a partial theta function, which are theta functions where the sum is over a partial lattice, comes from using a q-series transformation and taking the average of the even and odd partial sums to define the new function [50]. This however has the consequence that one cannot simply swap $q^{-1} \mapsto q$ to get an asymptotic expansion for $V(-1; q)$ since the series is divergent.

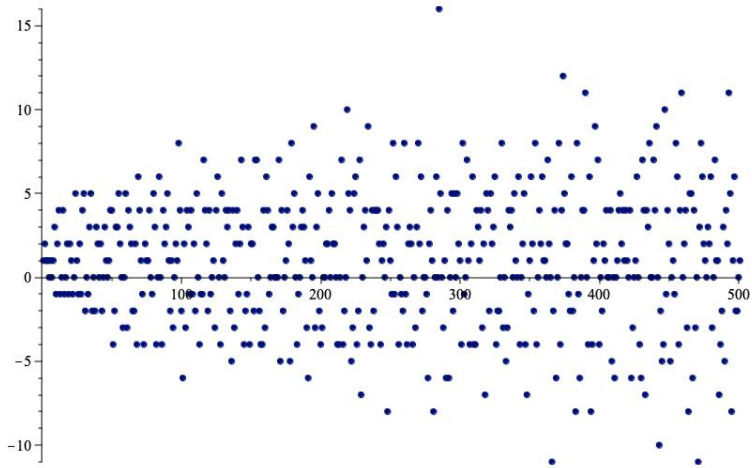

Figure 6.1: The first 500 Fourier coefficients of $V(-1;q)$.

Question 1. *Is the q-series*

$$V(-1;q) = \sum_{n\geq 0} \frac{(q)_n^2}{(q;q^2)_{n+1}} q^n$$

expressible in terms of a generalized Lambert series of the type discussed in the introduction, or expressible in terms of mock theta functions? Is the above series bounded by a polynomial in τ near $\tau = 0$?

Finally, we ask the following in connection with the results proven in Ch. 5.

Question 2. *Is there secondary-asymptotic bias among residue classes modulo c for the odd-balanced unimodal sequences? That is, if $a_1 \neq a_2$ are residue classes modulo c, does the difference $v(a_1, c; n) - v(a_2, c; n)$ have a predictable oscillation pattern?*

6.2 Ch. 4 discussion

In Ch. 4, we proved asymptotic formulas for the coefficients of the functions $R_{3,k}(q)$ and $R_{1,k}(q)$, when $k = 3$. This was stated as Result B at the beginning of this thesis. We expect that generalizing to $k > 3$ in both cases should be doable by brute-force methods given that many of the features of the $\theta_{1,\ell}$ that we encountered with $\ell = 4$ in this work generalize for $\ell > 4$. Much of this can be seen in Hickerson and Mortenson's original work [46]. This includes the symmetry of the indefinite quadratic form $Q(r, s)$ that appears in the exponent of q in the sum defining $\theta_{1,\ell}$, which will allow for simpler expressions for the $\theta_{1,\ell}$ like we found in this work. Albeit possible to do without, it would be nice to find more elegant methods for dealing with the asymptotics for these families of Bailey mock theta functions. Based on numerical checks of the Fourier coefficients, we expect that the $R_{1,k}(q)$ have weakly increasing coefficients for $k > 3$.

Conjecture 2. The $R_{1,k}(q)$ have weakly increasing coefficients for all $k \geq 3$.

Proving this by purely combinatorial means seems difficult, but possible using the many representations of $R_{1,k}$ given by Lovejoy and Osburn in [53]. One such way may involve appealing to some generalized q-binomial theorems and formulas for Gauss sums, like those posed in [48]. Such an idea seems reasonable since the B_k in the definition of $R_{k,1}$ can be expressed as weighted sums of Gaussian polynomials. For example,

$$\begin{aligned} B_4(n_4, n_3, n_2, n_1; q) &= (-1)^{n_1} q^{\frac{n_3(n_3+1)}{2}+n_2+2n_1} \\ &\quad \times \frac{(-q)_{n_3}(-q)_{2n_2}(-q^2;q^2)_{2n_1}}{(q)_{n_4-n_3}(q^2;q^2)_{n_3-n_2}(q^4;q^4)_{n_2-n_1}(q^8;q^8)_{n_1}} \\ &= (-1)^{n_1} q^{\frac{n_3(n_3+1)}{2}+n_2+2n_1} \end{aligned}$$

$$\times \begin{bmatrix} n_4 \\ n_3 \end{bmatrix}_q \begin{bmatrix} n_3 \\ n_2 \end{bmatrix}_{q^2} \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_{q^4} \frac{(-q;q^2)_{n_2}(-q^2;q^4)_{n_1}}{(q)_{n_4}},$$

where $\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{(q)_m}{(q)_{m-n}(q)_n}$ is the Gaussian q-binomial coefficient. Also of interest is trying to see if there are any patterns in the cancellation of the main terms for the asymptotic expansion of $R_{3,k}(q)$ on the major arc(s). We saw when $k = 3$, that the actual main term was "hiding" behind three other terms, which cancel when one computes the asymptotic expansion. However, for larger k there does not appear to be any obvious pattern of how many terms one needs to keep in order to get the main term in an asymptotic expansion for $R_{3,k}(q)$ near the dominant pole(s). Assuming that the piece $\theta_{1,\ell}$ constitutes the dominant piece, we ask the following.

Question 3. *For w and z integral or half-integral powers of q, does $\theta_{1,\ell}(w, z, q)$ have a generic asymptotic expansion near $q = \pm 1$, or any other dominant pole? If not, what values of ℓ, w, and z allow for such an expansion and in general, how many terms are needed in analogy to Lem. 4.7 to compute the main asymptotic term?*

6.3 Ch. 5 discussion

In Ch. 5, we proved our Result C as stated at the beginning of this thesis. We showed asymptotic formulas for the complex coefficients of the twisted q-product defined by

$$(\zeta q; q)^{-1} =: \sum_{n \geq 0} Q_n(\zeta) q^n,$$

with ζ a root of unity. This extended results proven by Wright [68] and Parry [61] for the cases ζ real, and $|\zeta| < 1$, respectively. By computing asymptotics for the $Q_n(\zeta)$, we were able to determine the sign-change behavior in the sequence $p(a_1, b; n) - p(a_2, b; n)$ where $p(a, b; n)$ is the number of partitions of n with number of parts congruent to a modulo b. We were able to extend these results to general combinations of $p(a, b; n)$ for fixed b. As was already alluded to in Question 2, it would be nice to know if there are other counting functions that exhibit oscillatory behavior when considering differences among congruence classes of certain statistics (like rank, or number of parts, for example). The difficulty with counting functions like that for the unimodal sequences and their generalizations like the odd-balanced unimodal sequences, is that their generating functions are not products, nor do they decompose nicely into sums of products. As we saw with the odd-balanced unimodal sequences in Ch. 3, this is not a problem in achieving an asymptotic formula where only the main term is needed. To find oscillatory behavior, one needs secondary terms which demands more information from the generating function. The reader may be asking "Can't one still determine secondary terms even if the generating function is not a product?". The answer is yes (for example, see an exact formula for the number of unimodal sequences in [28]). However, this brings us to the second difficulty. Bounding the minor arcs was, and is usually always the main difficulty in proving asymptotic formulas for Fourier coefficients of generating functions. In our Ch. 5, we had the luxury of being able turn products into sums by taking logarithms, computing bounds, and then exponentiating back. With non-product generating functions, such a method is not always advantageous. Thus, we pose the following question which generalizes Question 2.

Question 4. *Do the unimodal sequences, or any of their generalizations, have oscillation patterns of the type found in Ch. 5? For example, if $u(a, b; n)$ denotes the number of unimodal sequences whose peak is congruent to a modulo b, how does the sequence $u(a_1, b; n) - u(a_2, b; n)$ behave as $n \to \infty$ with $a_1 \neq a_2$?*

Chapter 7

Declarations

The publications which were the basis of Ch. 3 and Ch. 4, *Asymptotic distribution of odd-balanced unimodal sequences with rank congruent to a modulo c* and *Asymptotics of Bailey-type mock theta functions*, respectively are solely the work of the author of this thesis. The work *Asymptotics for the twisted eta-product and applications to sign changes in partitions*, which was the basis for Ch. 5 of this thesis, was a joint publication with Walter Bridges and Johann Franke. The author's contribution to this publication was 33.3 percent of the overall workload, and he was granted permission by the aforementioned authors to use this publication in this thesis.

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Notation

$\mathbb{N}$	the natural numbers
$\mathbb{Z}$	the integers
$\mathbb{Q}$	the rational numbers
$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
$\mathbb{H}$	the upper-half complex plane
τ	generic element in $\mathbb{H}$
u	real part of τ
v	imaginary part of τ
$\gcd(a, b)$	the greatest common divisor of a and b
$\lfloor x \rfloor$	integral part of x
$\{x\}$	fractional part of x
O	$f(x) = O(g(x))$ if $\exists C > 0$ such that $\lvert f(x)\rvert \leq Cg(x)$ as $x \to \infty$
$\ll$	$f(x) \ll g(x)$ if $f(x) = O(g(x))$
o	$f(x) = o(g(x))$ if $\forall C > 0\ \exists x_0$ such that $\lvert f(x)\rvert \leq Cg(x)$ for $x \geq x_0$
$\sim$	$f(x) \sim g(x)$ if $f(x) = g(x)(1 + o(1))$ as $x \to \infty$
$\dot\sim$	$f(x) \mathrel{\dot\sim} g(x)$ if $\exists M = O(1)$ such that $f(x) = Mg(x)(1 + o(1))$ as $x \to \infty$

$\doteq$	$f(x) \doteq g(x)$ if $\exists M = O(1)$ such that $f(x) = Mg(x)$
$\exp(z)$	placeholder for e^z when z takes up a lot of space
$\int_{(c)}$	placeholder for the integral $\int_{c-i\infty}^{c+i\infty}$
$\mathrm{Arg}(z)$	the principle argument of a complex number
$\mathrm{Log}(z)$	natural logarithm defined via the principle branch

Erklärung

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Teilpublikationen:

1. T. Garnowski. “Asymptotic distribution of odd-balanced unimodal sequences with rank congruent to a modulo c”. *Research in Number Theory* 8 (2022). doi: https: //doi.org/10.1007/s40993-021-00298-2.

2. T. Garnowski. “Asymptotics of Bailey-type mock theta functions”. *Ramanujan Journal* (2021). doi: https://doi.org/10.1007/s11139-021-00511-x.

3. W. Bridges, J. Franke, and T. Garnowski. “Asymptotics for the twisted eta- product and applications to sign changes in partitions”. *Preprint* (2021). doi: arXiv:2111.04183[math.NT].

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