Generalized sections and representations of copolarity one and two

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Abstract

In the present thesis we analyze, based on the definition of polar representations, representations of connected compact groups, whose orbit space is isometric to the orbit space of a low dimensional group. Is the smallest of those reductions a one or two dimensional group, this will affect the dimension of the orbit space. In the case of a two dimensional minimal reduction we show that, up to one exceptional case and providing that the orbit space is not a product, the dimension of the orbit space is equal to four. We show that the existence of a one dimensional minimal reduction is equivalent to the existence of a generalized section with one dimensional part in regular orbits. For a singular Riemannian foliation on a simply connected manifold of constant curvature, we give necessary and sufficient conditions for the existence of a generalized section with one dimensional vertical part.

Kurzzusammenfassung

In der vorliegenden Arbeit untersuchen wir, angelehnt an die Definition polarer Darstellungen, Darstellungen kompakter zusammenhängender Gruppen, deren Bahnenraum isometrisch zum Bahnenraum einer niedrig dimensionalen Gruppe ist. Ist die kleinste dieser Reduktionen eine ein- oder zweidimensionale Gruppe, hat dies Einfluss auf die Dimension des Bahnenraums. Mit Ausnahme von Produktdartstellungen und einem exzeptionellen Fall zeigen wir, dass für Darstellungen mit einer zweidimensionale minimalen Reduktion die Dimension des Bahnenraumes gleich vier ist. Für den exzeptionellen Fall geben wir ein Beispiel. Wir zeigen, dass die Existenz einer eindimensionalen minimalen Reduktion equivalent zur Existenz von verallgemeinerten Schnitten mit eindimensionalem Anteil in regulären Bahnen ist. Für eine singuläre Riemannsche Blätterung eines einfach zusammenhängenden Raumes konstanter Krümmung, geben wir notwendige und hinreichende Bedingungen für die Existenz eines verallgemeinerten Schnittes mit eindimensionalem Blattanteil an.

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Contents

Introduction

An isometric action (G, M) of a compact Lie group G on a complete Riemannian manifold M decomposes the manifold M into equidistant submanifolds, the so called orbits. The relative positions of the orbits in M is encoded in the horizontal geometry, which is reflected in the orbit space M/G. In mathematics one often tries to reduce a given problem to a lower dimensional one, having the same attributes one is interested in. In our context we are interested in lowering the dimension of the group G such that the horizontal information is unchanged. Two isometric actions (G, M) and (G', M') are called quotient-equivalent if their orbit spaces are isometric. We call (G', M') a reduction of (G, M) if both actions belong to the same quotient-equivalence class and if the dimension of G' is strictly smaller than the dimension of G. We are interested in a minimal representative of a quotient-equivalence class, which we call reduced.

An extremal situation occurs if (G, M) is a polar action, i.e. if there exists a complete totally geodesic submanifold $\Sigma \subset M$, called a section, which intersects every orbit and is perpendicular to the orbits at intersection points. Then the intersections of Σ with the orbits are parameterized by a finite group W such that $M/G = \Sigma/W$ (cf. [**21**]). In fact, a polar action admits a minimal reduction to a finite group. A point in M is called regular if it is contained in an orbit of maximal dimension. Due to the homogeneity of an orbit, for a polar action there exists a unique section through every regular point. The existence of sections is equivalent to the integrability of the horizontal distribution (cf. [**15**]).

In [**12**] the authors generalize the notion of polarity. They call a complete, totally geodesic, embedded submanifold a generalized section of (G, M) if it intersects every orbit and regular orbits perpendicular. The intersections of the orbits with a generalized section Σ can be parametrized by a compact group W, and (W, Σ) is a reduction of (G, M). If (W, Σ) is minimal under all reductions, which can be realized by a generalized sections, the number $k = \dim W$ is called the copolarity of (G, M). Of course, a polar action has copolarity zero.

In the first chapter we give sufficient and necessary conditions for a non-polar action (G, M) on a simply connected space form M to have copolarity one.

Theorem I. *Let M be a simply connected space form and (G, M) a non-polar isometric action. Then (G, M) has copolarity one if and only if there exists a one dimensional vertical autoparallel distribution $\mathcal{D} \subset \mathcal{V}$, i.e. $\nabla^v_{\mathcal{D}} \mathcal{D} \subset \mathcal{D}$, over the set of regular points such that $\mathcal{D} \oplus \mathcal{H}$ is integrable.*

In general, a reduction (G', M') of (G, M) is not necessarily realized by a submanifold M' of M. The minimal dimension of G' with respect to all reductions is called the abstract copolarity of (G, M). Of course, the copolarity is an upper bound for the abstract copolarity. It is an interesting question under which assumptions both notions coincide, i.e. if a non-reduced representation admits an appropriate generalized section.

In the second chapter we restrict our attention to representations with low abstract copolarity and give some partial answers to the above question. That a representation of abstract copolarity zero is in fact polar, is an easy consequence of the O'Neill formulas for submersions. In [**11**] it is proven that both concepts also coincide for irreducible representations of connected compact groups with abstract copolarity up to three. We show this result for reducible representations of connected compact groups of abstract copolarity one. To underline connectedness, we write (H, W) for a representation of a connected compact group H on the vector space W. Following [**11**], the idea is to calculate the cohomogeneity of a representation of abstract copolarity one. It turns out that the interesting ones have cohomogeneity three, what was already known in the irreducible case (cf. [**11**]).

Theorem II. *Let (H, W) be a non-reduced representation of a connected compact group H of abstract copolarity one, which is not orbit equivalent to a product representation. Then (H, W) has cohomogeneity three.*

From the classification of Straume (cf. [**23**]) it follows that each non-polar representation of cohomogeneity three admits a generalized section, such that the copolarity is equal to one. The abstract copolarity of a product representation is the sum of the abstract copolarities of its factor representations. In fact, a product representation of abstract copolarity one has a polar factor and a factor of cohomogeneity three. In this case, the product of the generalized sections is a generalized section for the original representation, and therefore its copolarity is equal to one.

The representation $(SO(n) \times SO(2), \rho_n \otimes \rho_2 \oplus \rho_2^{k_1} \oplus \rho_2^{k_2} \ldots \oplus \rho_2^{k_l})$, for $n \geq 3$ and $k_i \in \mathbb{Z} - \{0\}$, has copolarity two. In fact, this is an example of a reducible representation of abstract copolarity two, where the cohomogeneity is not restricted.

Theorem III. *Let (H, W) be a non-reduced representation of a connected compact group H of abstract copolarity two, which is not orbit equivalent to a product representation. Then either*

- *(H, W) has cohomogeneity four, or*
- *each minimal reduction (G, V) of (H, W) has two connected components, i.e. $G/G_0 = \mathbb{Z}/2\mathbb{Z}$. Furthermore, there exists an irreducible subspace $W' \subset W$ such that H acts with cohomogeneity two on W', and $(H, (W')^{\perp})$ is orbit equivalent to a non-polar S^1-representation.*

That the cohomogeneity of an irreducible representation of abstract copolarity two is equal to four, is proven in [**11**], including a classification. There are three classes of reducible representations of abstract copolarity two and cohomogeneity four. We give examples for two of these classes. Our examples have copolarity two. In the case that the cohomogeneity is bigger than four, we show that the restriction (H, W') is orbit equivalent to the isotropy representation of a rank two symmetric space with $g = 4$.

Finally, in the third chapter we define generalized sections for singular Riemannian foliations. Roughly speaking, a singular Riemannian foliation $(M, \mathcal{F})$ of a complete Riemannian manifold M is a decomposition of M into locally equidistant submanifolds, called the leaves, not necessarily all of the same dimension. An isometric action is an example of a singular Riemannian foliation. Generalizing polarity, a singular Riemannian foliation with sections is a singular Riemannian foliation, such that for every regular point $p \in M$ there exists a totally geodesic submanifolds containing p, which intersects every leaf and is perpendicular to the leaves at intersection points. There are many results concerning singular Riemannian foliations with sections (cf. [**2**]). That the existence of sections is equivalent to the integrability of the horizontal distribution, even for singular Riemannian foliations, is proven in [**1**]. We prove some basic properties for a generalized section of a singular Riemannian foliation and generalize the existence result of the first chapter.

THEOREM IV. *Let M be a simply connected space form and $\mathcal{F}$ a singular Riemannian foliation without sections. Then $(M, \mathcal{F})$ admits a generalized section, which intersects the regular leaves in a one-dimensional submanifold, if and only if there exists a one-dimensional vertical autoparallel distribution $\mathcal{D}$, i.e. $\nabla^v_{\mathcal{D}}\mathcal{D} \subset \mathcal{D}$, over the set of regular points such that $\mathcal{D} \oplus \mathcal{H}$ is integrable.*

CHAPTER 1

Copolarity of isometric actions

1.1. Isometric actions

Let G be a compact Lie group and M a connected complete Riemannian manifold and denote by $\mathrm{I}(M)$ the Lie group of isometries of M. A Lie group homomorphism $\Phi : G \to \mathrm{I}(M)$ induces an isometric action of the group G on M by

$$G \times M \to M,\ (g,p) \mapsto gp := \Phi(g)(p).$$

If Φ is injective, the action is called *effective* and we identify G with its image in $\mathrm{I}(M)$. For the special case that $M = V$ is a finite dimensional Euclidean vector space and $\Phi(G) \subset O(V)$ is a subgroup of the orthogonal group, we call Φ an *(orthogonal) representation.* Two representations $\rho : G \to O(V)$ and $\rho' : G' \to O(V')$ are called *orbit equivalent* if there exists an isomorphism $\Phi : V \to V'$, such that for every $g \in G$ there exists $g' \in G'$ with

$$\Phi\left(\rho(g)v\right) = \rho'(g')\Phi(v).$$

In the following we write (G, M) for an isometric action of a compact group G on a connected complete Riemannian manifold M.

In this section we recall some basic facts of isometric actions and we will fix our notation. For a short introduction to Lie groups and homogeneous spaces we refer to [**5**] and [**27**]. A detailed discussion of the following statements can be found in [**21**].

The orbits of an isometric action decompose M into compact, embedded, equidistant submanifolds. We define an equivalence relation on these submanifolds by saying that two orbits belong to the same *orbit type* if their isotropy groups are conjugate in G. We call Gp a *principal orbit* if there exists a neighborhood U of p such G_p is conjugate to a subgroup of G_q for each $q \in U$. A point is called *principal* if it lies on a principal orbit. The connected component through p of the set of points whose orbits belong to the same equivalence class as Gp is called the *stratum* $\mathrm{Str}(p)$ *of* p. The strata are embedded submanifolds and the principal stratum M_{pr} is an open and dense subset of M.

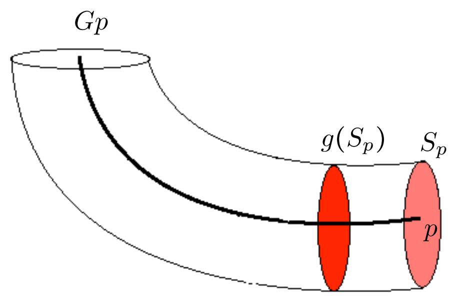
Since the group G acts by isometries on M, the isotropy representation $G_p \to O(T_pM), g \mapsto g_*$ leaves the decomposition $T_pM = T_pGp \oplus \nu_pGp$ of T_pM into the spaces tangent and normal to the orbit Gp invariant. The *slice representation* is the restriction of the isotropy representation to the normal space

$$G_p \to O(\nu_pGp), \quad g \mapsto g_*|_{\nu_pGp},$$

and for $v \in \nu_p Gp$ we have

$$\exp_p(g_* v) = g \exp_p(v).$$

The *geodesic slice* S_p *through* p is the image of a small neighborhood $U_r^\nu(0) \subset \nu_p Gp$ under the normal exponential map, such that this image is embedded (cf. [**5**]). Since the orbits are compact, embedded submanifolds, we can choose $r > 0$ so small that $G \cdot S_p$ is a distance tube around Gp.



Theorem 1.1.1 (Slice Theorem). *Let $S_p = \exp_p(U_r^\nu(0))$ be a geodesic slice of the isometric action (G, M). Then $G_q \subset G_p$ for each element $q \in S_p$ and G_p parameterizes the intersection of the orbits with S_p. In particular, $(G_p)_v = G_{\exp_p(v)}$ for $v \in U_r^\nu(0)$, i.e. the orbit types of the slice representation coincide locally with the orbit types of the original action.*

From the Slice Theorem we conclude that a point p is principal if and only if its slice representation is trivial. We call a point *exceptional* if its slice representation is finite. If p is neither principal nor exceptional we call p a *singular point*. We also say that an orbit is exceptional, resp. singular, if one and hence all of its points is exceptional, resp. singular. The codimension of a principal orbit is called the *cohomogeneity* of the action.

Corollary 1.1.2. *The cohomogeneity of (G, M) coincides with the cohomogeneity of the slice representation at any point $p \in M$.*

Proof. Let $v \in \nu_p Gp$ be a principal points of the slice representation, with $\|v\| < r$, such that $\exp_p(v) = p' \in S_p$. Then p' is a principal point of the G-action. The cohomogeneity of (G, M) equals

$$\begin{aligned}\dim M - \dim G + \dim G_{p'} &= (\dim G - \dim G_p) + \dim \nu_p Gp - \dim G + \dim G_{p'} \\ &= \dim \nu_p Gp - \dim G_p + \dim G_{p'}.\end{aligned}$$

Since $G_{p'} = (G_p)_v$ equals the isotropy group of v this is exactly the the cohomogeneity of the slice representation. $\square$

Let $M_0^{G_p}$ be the connected component of set of G_p-fixed points through p. Then following the Slice Theorem $M_0^{G_p} \cap S_p = \mathrm{Str}(p) \cap S_p$ and therefore the intersection of the tube $G \cdot S_p$ with the stratum $\mathrm{Str}(p)$ equals $G \cdot (M_0^{G_p} \cap S_p)$. Now for any G_p-fixed point p' in the tube $G \cdot S_p$ we can write $p' = gv$ for some $v \in S_p$. Then $G_p(gv) = gv$ is equivalent to $c_g(G_p) \subset G_v \subset G_p$. For dimension reasons we conclude that $G_v = G_p$

and $p' \in \mathrm{Str}(p)$. Furthermore, g is an element of the normalizer $N_G(G_p)$. We have just proven that locally $\mathrm{Str}(p)$ equals $G \cdot M_0^{G_p}$ and that intersections of $M_0^{G_p}$ with an orbit near Gp are parametrized by $N_G(G_p)$. Therefore, we get the following dimension formula, which we will use extensively in the next chapter.

$$\dim \mathrm{Str}(p) = \dim(G \cdot M_0^{G_p}) = \dim G + \dim M_0^{G_p} - \dim N_G(G_p). \tag{1}$$

For an isometric action (G, M) a geodesic in M is called a *horizontal geodesic* if it is everywhere perpendicular to the orbits it meets.

LEMMA 1.1.3. *Let (G, M) be an isometric action, then a geodesic is perpendicular to the orbits at all or non of its points.*

PROOF. Let $\gamma : [a, b] \to M$ be a geodesic and denote by $X_p^* = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{Exp}(tX)p$ the Killing field corresponding to $X \in \mathfrak{g}$. Then the skew-symmetry of ∇X^* implies that $\frac{\mathrm{d}}{\mathrm{d}t} < X^* \circ \gamma, \dot\gamma >=< \nabla_{\dot\gamma} X^*, \dot\gamma >= 0$. Since $X^* \circ \gamma$ is everywhere tangent to the orbits, the claim follows. □

Let M/G denote the set of orbits and let $\pi : M \to M/G$, $p \mapsto Gp$ be the corresponding projection. The set M/G equipped with the quotient topology is called the *orbit space* of the G-action. Since the orbits are equidistant, there is a natural metric on M/G induced by the distance of the orbits in M, such that M/G is a complete length metric space. Any minimal geodesic segment in M/G is the projections of a horizontal geodesics in M. The action of G on a stratum $\mathrm{Str}(p)$ has only one orbit type and the restriction $\pi : \mathrm{Str}(p) \to \mathrm{Str(p)}/G$ is a Riemannian submersion (cf. [**21**]). The image $\mathrm{Str(p)}/G$ is also called a *stratum* and the principal stratum M_{pr}/G is dense in M/G.

Assume that (G, M) and (G', M') are two effective isometric actions, such that their quotient spaces M/G and M'/G' are isometric. Then we say that (G, M) and (G', M') are *quotient-equivalent*. We call (G', M') a *reduction of* (G, M) if $\dim G' < \dim G$. If (G', M') is minimal in its quotient equivalent class, the number $k' = \dim G'$ is called the *abstract copolarity of* (G, M). In the next chapter we will analyze representations of abstract copolarity 1 and abstract copolarity 2.

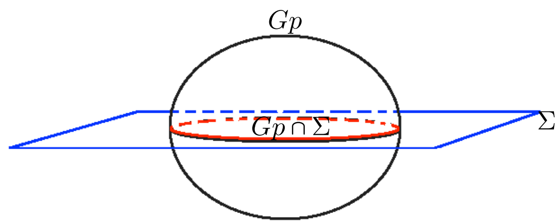
DEFINITION 1.1.4. Let (G, M) be an isometric action of a compact group G on a complete Riemannian manifold M. A complete, embedded submanifold Σ is called a *k-section* of the action (G, M) if it satisfies the following conditions

(C1) Σ is a totally geodesic submanifold of M,
(C2) Σ intersects every G-orbit,
(C3) for each principal point $p \in \Sigma$ the normal space $\nu_p Gp$ is a vector subspace of the tangent space $T_p\Sigma$ of codimension k,
(C4) if $gp \in \Sigma$ for $g \in G$ and some principal point $p \in \Sigma$, then $g\Sigma = \Sigma$.

Note that if there exists a k-section Σ through a principal point p, then the set $g\Sigma$ is a k-section through gp. The orbits are compact submanifolds, therefore for $p \in M$ there exists a minimal geodesic $\gamma : [0, 1] \to M$ from p to every orbit Gq and this geodesic is necessarily perpendicular to Gq by the first variation formula, hence $\dot\gamma(0) \in \nu_p Gp$ by the previous lemma and we get

LEMMA 1.1.5. *For every point p, the set $\exp_p(\nu_p Gp)$ intersects every orbit.*

Therefore, for a complete submanifold Σ condition (C2) follows from condition (C1) and (C3).



Example of a 1-section Σ

The manifold M itself is always a n-section, where n denotes the dimension of a principal orbit. If the dimension of Σ is not important we call Σ a *generalized section*. Let Σ_1, Σ_2 be a k_1, k_2-section respectively, and $p \in \Sigma_1 \cap \Sigma_2$ a principal point. The connected component of the intersection $\Sigma_1 \cap \Sigma_2$ is a k-section through p, with $k \leq k_i$ for $i = 1, 2$. Hence, among the generalized sections of (G, M) through p there exists a unique one of minimal dimension. Let Σ be the minimal generalized section through a principal point p. If Σ is a k-section, then the corresponding integer k is called the *copolarity of* (G, M). If M itself is the minimal generalized section we say that the action has *trivial copolarity*.

We will later see that the existence of a generalized section causes a reduction of (G, M). The next chapter deals with the question if the notions of abstract copolarity and copolarity are connected in the special case that $M = V$ is an Euclidean vector space. In this chapter we are interested in finding geometric conditions for the existence generalized sections.

Remark 1.1.6. If we drop the condition (C4) in the definition of a generalized section the corresponding submanifold Σ is called a *pre-section*. Note that for a pre-sections Σ, the set $g\Sigma$ is again a pre-section and also the intersection of two pre-sections is again a pre-section. Now a minimal pre-section fulfills (C4) and is actually a generalized section.

Polar actions. An isometric action (G, M) is called *polar* if it has copolarity 0. Then a 0-section is simply called a *section*. In the following we list some important properties of polar actions, which we want to generalize to actions of copolarity > 0 in the next subsection. Most of the following results can be found in [**21**] and [**5**]. We also refer to [**24**] for an introduction to polar actions and especially to [**9**] for polar representations.

Example (1). The standard action $(SO(2), \mathbb{R}^2)$ is polar. The sections are given by lines through the origin.

EXAMPLE (2). Let V be the vector space of real symmetric $(n \times n)$-matrices and the action $(O(n), V)$ given by conjugation. If V is equipped with the inner product $\langle X, Y \rangle = \operatorname{tr}(XY)$, then the action is isometric and polar. A section is given by the subspace of diagonal matrices. More generally, let $M = G/K$ be a symmetric space. Then the action (K, M) is polar and a section is given by a maximal flat, totally geodesic submanifold through eK. Furthermore, the isotropy representation $(K, T_{eK}M)$ is polar. Here a section is given by a maximal abelian subspace.

An important property of a section Σ is that the set $\Sigma_{pr} = \Sigma \cap M_{pr}$ of principal points in Σ is open and dense in it (cf. [**25**]).

LEMMA 1.1.7. *Let (G, M) be a polar action and let Σ be a section of this action. Then Σ intersects every orbit perpendicular.*

PROOF. The tangent space to an orbit Gp is spanned by Killing fields X^* induced by the G-action. Since Σ is complete, each two points $p, q \in \Sigma$ can be connected by a geodesic contained in Σ. The restriction of X^* to any such geodesic is a Jacobi field, which is tangent to the orbits. Since Σ is totally geodesic, this Jacobi field is everywhere perpendicular to Σ if and only if the initial condition X^*_p and $(\nabla X^*)_p$ are perpendicular to Σ. We can assume that p is principal, then $X^*_p \in \nu_p\Sigma$ and we only have to show $(\nabla_\mu X^*)_p \in \nu_p\Sigma$ for all $\mu \in T_p\Sigma$.
Let γ be the horizontal geodesic starting in p and choose $t \in [0, \epsilon)$, such that $\gamma(t) \in \Sigma_{pr}$ and $\dot\gamma(0) = \mu$. Then for a parallel vector field v tangent to Σ along $\gamma(t)$ we get

$$\begin{aligned} 0 &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle X^* \circ \gamma(t), v_t \rangle \\ &= \langle (\nabla_\mu X^*)_p, v_0 \rangle + < X^*_p, \left.\frac{\nabla}{\mathrm{d}t}\right|_{t=0} v_t > \\ &= \langle (\nabla_\mu X^*)_p, v_0 \rangle, \end{aligned}$$

and the claim follows. □

REMARK 1.1.8. Usually a section is by definition a complete immersed submanifold, which meets every orbit and intersects the orbits perpendicularly. Such a submanifold has to be totally geodesic. In this work we concentrate on the case that M is a simply connected space form, i.e. a simply connected complete Riemannian manifold of constant curvature. For simply connected space forms we will prove that a section has to be embedded, so that the above definition is not restrictive for our purpose.

The existence of sections is purely geometric and the orbits of (G, M) locally coincide with the orbits of its identity component (G_0, M), therefore

LEMMA 1.1.9. *An isometric action (G, M) is polar if and only if the restriction to the identity component (G_0, M) is polar.*

The tangent bundle over the principal stratum M_{pr} splits in a so called *vertical distribution* $\mathcal{V}$ tangent to the orbits and a *horizontal distribution* $\mathcal{H} = \mathcal{V}^\perp$, i.e.

$$TM_{pr} = \mathcal{V} \oplus \mathcal{H},$$

and $\mathcal{V} = \ker \mathrm{d}\pi$ for the Riemannian submersion $\pi : M_{pr} \to M_{pr}/G$. For a polar action the horizontal distribution $\mathcal{H}$ over the principal stratum M_{pr} is integrable.

The sections induce the corresponding integral manifolds. That the converse is also true is an important theorem which was first proven in [**15**]. We will recall the proof for simply connected space forms and show that a section is embedded.

THEOREM 1.1.10. *If M is a simply connected space form, the isometric action (G, M) is polar if and only if the horizontal distribution over the principal stratum is integrable.*

PROOF. The projection $\pi : M_{pr} \to M_{pr}/G$ is a Riemannian submersion and $\mathcal{H}$ is the corresponding horizontal distribution. Let $\tilde{\Sigma}$ be a leaf of the horizontal distribution through the point p, then $\tilde{\Sigma}$ is totally geodesic. Since M is a simply connected space form of curvature κ, there exists a complete, connected, totally geodesic submanifold Σ containing $\tilde{\Sigma}$, with $\dim \Sigma = \dim \tilde{\Sigma}$ (cf. [**5**]). Furthermore, Σ is isometric to a simply connected space form $M'(\kappa)$, which is canonically embedded in M. For principal points $p \in \Sigma$ the intersection with the orbit Gp is perpendicular, since $T_p\Sigma = \nu_p Gp$. Furthermore, Σ is totally geodesic and complete, and we conclude that $\exp_p(\nu_p Gp)$ is contained in Σ, i.e. Σ intersects every orbit. □

In the case of a polar action (G, M) the intersection of Σ with a G-orbit is parameterized by the group $W(\Sigma) = N(\Sigma)/Z(\Sigma)$, where $N(\Sigma) = \{h \in G \,|\, h(\Sigma) = \Sigma\}$ and $Z(\Sigma) = \{h \in G \,|\, hp = p \text{ for all } p \in \Sigma\}$ are called the *normalizer* and *centralizer* of Σ, respectively. The group $W(\Sigma)$ is a finite group called *Weyl group*. Since Σ intersects every orbit the spaces $\Sigma/W(\Sigma) = M/G$ are isometric (cf. [**21**]).

In the special case of a polar representation (G, V) we have some simple results, which we will often use in the sequel.

PROPOSITION 1.1.11. *A representation (G, V) of cohomogeneity 1 or 2 is polar.*

PROOF. Let (G, V) be of cohomogeneity 1, then every horizontal geodesic is a section. Every totally geodesic submanifold Σ' of $S(V)$ is the intersection of $S(V)$ with a vector subspace Σ and Σ' is a section of $(G, S(V))$ if and only if Σ is a section of (G, V). Therefore, a representation of cohomogeneity 2 induces a cohomogeneity 1, hence polar action on $S(V)$ and vice versa. □

For representations, we have the following interesting converse of Example (2).

THEOREM 1.1.12 ([**9**]). *A polar representation (G, V) is orbit equivalent to the isotropy representation of a symmetric space.*

A further interesting property of a polar action is that its slice representations inherits the property of being polar.

PROPOSITION 1.1.13. *Every slice representation of a polar action (G, M) is polar.*

PROOF. Let Σ be a section and $p \in \Sigma$. First note that Σ intersects every orbit of the induced G_p-action on the normal slice $S_p = \exp(U_r^\nu(0))$. Then $T_p\Sigma$ intersects the orbits of the slice representation in $U_r^\nu(0)$ and the linearity of the slice representation implies that $T_p\Sigma$ intersects every orbit. The codimension of the slice representation equals the codimension of (G, M), hence we are left to show that $T_p\Sigma$ intersects the orbits of the slice representation perpendicular. The tangent spaces to the orbits of $(G_p, \nu_p Gp)$ are given by the Lie algebra $\mathfrak{g}_p$ of G_p.

The elements of $\mathfrak{g}_p$ can be regarded as skew-symmetric endomorphism of $\nu_p Gp$ of the form $(\nabla X^*)_p$, where X^* is a Killing vector field on M induced by G. But X^* is always perpendicular to Σ and so are its starting conditions, i.e. $(\nabla_v X^*)_p$ is perpendicular to $T_p\Sigma$ for every $v \in T_p\Sigma$ □

Generalized sections. The notion copolarity of an isometric action was introduced as a generalization of polar actions (cf. [**12**]). In this section we generalize the statements concerning polar actions, given in the last subsection, to generalized sections. Note that all results are reformulations of the statements in [**12**] and can also be found in [**18**].

At first we give an example. The main result of [**12**] states that the irreducible representations of copolarity 1 are exactly the three exceptional irreducible representation of cohomogeneity 3, which are not polar.

Theorem 1.1.14. [**12**] *Let (G, V) be an irreducible representation of copolarity 1, then it is one of the following orthogonal representations ($n \geq 2$):*

(G,V)	Φ
$(SO(2) \times \mathrm{Spin}(9), \mathbb{R}^{32})$	$\rho_2 \otimes \Delta_9$
$(U(2) \times \mathrm{Sp}(n), \mathbb{R}^{8n})$	$[\mu_2 \otimes_{\mathbb{C}} \nu_n]_{\mathbb{R}}$
$(\mathrm{Sp}(1) \times \mathrm{Sp}(n), \mathbb{R}^{8n})$	$S^3\nu_1 \otimes_{\mathbb{H}} \nu_n$

We now present a general method to find generalized sections.

The reduction principal. Let (G, M) be an isometric action and let p be a principal point. Restrict the action to the principal isotropy group G_p and denote by M^{G_p} the set fixed points in M. Let M_c be the closure of $M^{G_p} \cap M_{pr}$ in M, then its connected components are generalized sections (cf. [**14**] where the set M_c is called the *core*). To see this note, that in [**14**] it is shown that the connected components Σ of M_c are those components of M^{G_p}, which contain principal points. In fact Σ is totally geodesic as a connected component of M^{G_p}. Since for a principal point $p' \in \Sigma$ the slice representation is trivial $\nu_{p'}Gp' \subset T_{p'}M^{G_p} = T_{p'}\Sigma$. Therefore, Σ intersects every orbit, and fulfills condition (C3) in each of its principal points. If $p, gp \in \Sigma$ for a principal point p and $g \in G$, then the element g normalizes G_p, hence $g\Sigma = \Sigma$, which is condition (C4). Note that each minimal generalized section has to be contained in M^{G_p}.

A principal point of the action (G, M) is also a principal point of the action (G_0, M), where G_0 denotes the connected component of G. Unfortunately, the converse is not true, since a principal point of (G_0, M) can be exceptional for the G-action. But we can state

Lemma 1.1.15. *Let Σ be a generalized section of (G_0, M), then it is also a generalized section of (G, M).*

In analogy to Proposition 1.1.13 our first goal is to show that a tangent space of a generalized section induces a generalized section of the slice representation. Therefore, we need the following

LEMMA 1.1.16. *The set Σ_{pr} of G-principal points is open and dense in Σ.*

PROOF. Since Σ is an embedded submanifold Σ_{pr} is open in Σ. The density follows from the fact that non-principal points are isolated along horizontal geodesics starting in a principal point (cf. [**17**]). □

THEOREM 1.1.17. *Let (G, M) be an isometric action and Σ a k-section. For each $p \in \Sigma$ the intersection $T_p\Sigma \cap \nu_p Gp$ is a k_1-section of the slice representation, with $k_1 \leq k$.*

PROOF. Let $V_p = T_p\Sigma \cap \nu_p Gp$ then it is clearly totally geodesic in $\nu_p Gp$. We first prove condition (C3), which is equivalent to prove that $V_p^\perp \subset T_v(G_p v)$, for G_p principal points v. Note that $V_p^\perp \subset T_v(G_p v)$ is invariant under rescaling v, since the G_p-action is linear. Therefore, let $v \in V_p^\perp$ be a principal point of the slice representation, such that $\exp_p(v) = p' \in S_p$ is contained in the geodesic slice through p. Then p' is a principal point for the G-action. Let J be the Jacobi field along $\exp_p(tv)$ with $J(0) = 0$ and $J'(0) = w \in V_p^\perp$. Then J is everywhere perpendicular to Σ and tangent to S_p, especially $\mathrm{d}(\exp)_v(w) = J(1) \in \nu_{p'}\Sigma \cap T_{p'}S_p \subset T_{p'}Gp'$, since Σ fulfills (C3). The intersections of an G-orbit with the slice S_p is parametrize by G_p, hence $J(1) \in T_{p'}(G_p p')$ which is the image of $T_v(G_p v)$ under the differential of the normal exponential map and therefore $w \in T_v(G_p v)$. We directely conclude that V_p intersects every orbit, which is condition (C2). To see condition (C4), let $v \in V_p$ be a principal vector of the slice representation and let $h \in G_p$ such that $hv \in V_p$. Then, after eventually rescaling v, we can assume that v, hv where mapped by the normal exponential map to the G-principal points $p', gp' \in S_p$. Note that $g \in G_p$ with $h = (g_*)_p$. Since Σ is totally geodesic the points $p', hp' \in \Sigma$ and with (C4) for Σ it follows that $g\Sigma = \Sigma$. Differentiation now implies $(g_*)_p(T_p\Sigma) = T_p\Sigma$ and since $(g_*)_p = h$ is an isometry $hV_p = V_p$. □

The next step will be to find a group $W(\Sigma)$ parameterizing the intersections of a principal orbit with a generalized section Σ, such that $(W(\Sigma), \Sigma)$ is a reduction of (G, M).

LEMMA 1.1.18. *Let (G, M) be an isometric action and Σ be a k-section. Let $p \in \Sigma$ and denote by $\mathcal{K}_p = \{g\Sigma \,|\, g \in G, p \in g\Sigma\}$ the set of k-sections through p which are G-translates of Σ. Then the isotropy group G_p acts transitively on $\mathcal{K}_p$.*

PROOF. If p is a G-principal point the statement is a reformulation of (C4). So let $p \in \Sigma$ be a non-principal point. Clearly G_p acts on $\mathcal{K}_p$. Let $\Sigma_1, \Sigma_2 \in \mathcal{K}_p$ and denote by S_p the normal slice at p, then $\Sigma_i \cap S_p$ intersects every G_p-orbit in S_p, for $i = 1, 2$. For a G-principal point $p' \in S_p$ we find $h_i \in G_p$ such that $h_i p' \in \Sigma_i$ and $p' \in h_1^{-1}\Sigma_1 \cap h_2^{-1}\Sigma_2$. But $p' \in S_p$ is a principal point, hence $h_2 h_1^{-1}\Sigma_1 = \Sigma_2$ and $h_2 h_1^{-1} \in G_p$. □

Let $N(\Sigma) = \{g \in G \,|\, g\Sigma = \Sigma\}$ be the *normalizer of* Σ and let $p, q \in \Sigma$ be in the same G-orbit. If they are principal, then $q = gp \in \Sigma \cap g\Sigma$ and (C4) in the definition of a k-section implies that $g\Sigma = \Sigma$, hence $g \in N(\Sigma)$. Suppose that $p, q \in \Sigma$ are not principal. Since $\Sigma, g\Sigma$ are both k-sections through q we deduce from the last lemma that there exists an element $h \in G_q$ such that $g\Sigma = h\Sigma$. Then $h^{-1}g \in N(\Sigma)$ and $gp = q = hq$ and therefore $q = h^{-1}gp$. We have just proven that

$$N(\Sigma)p = Gp \cap \Sigma, \text{ for all } p \in \Sigma,$$

i.e. $N(\Sigma)$ parameterizes the intersection of Σ with a G-orbit. We equip the quotient space $\Sigma/N(\Sigma)$ with the induced metric structure, then

THEOREM 1.1.19. [**18**] *The inclusion* $i : \Sigma \to M$ *induces an isometry* $I : \Sigma/N(\Sigma) \to M/G$.

PROOF. The distance between two points $Gp, Gp' \in M/G$ is given by the distance in M between the orbits Gp and Gp'. This distance is realized through a geodesic γ intersecting Gp and Gp' perpendicularly. For p, p' principal we may assume that both lie in Σ and therefore γ is a segment in Σ. Since in general $d(N(\Sigma)p, N(\Sigma)p') \geq d(Gp, Gp')$ the distance between $N(\Sigma)p$ and $N(\Sigma)p'$ is realized by γ. Therefore, I is an isometry for an open and dense subset and using that $\Sigma/N(\Sigma)$ and M/G are complete metric spaces and I is continuous, we see that I is an isometry. □

The *centralizer* $Z(\Sigma) = \{g \in G \,|\, gp = p, \text{ for } p \in \Sigma\}$ of Σ equals the kernel of the action $(N(\Sigma), \Sigma)$. The group $W(\Sigma) = N(\Sigma)/Z(\Sigma)$ is called the *generalized Weyl group of* (G, M) and we conclude

PROPOSITION 1.1.20 ([**18**]). *Let* Σ *be a generalized section and* W *the corresponding generalized Weyl group. Then* (W, Σ) *is effective and* W *parametrizes the intersection of* Σ *with the* G*-orbits, i.e.*

$$Wp = \Sigma \cap Gp,$$

for all $p \in \Sigma$. *Therefore,* Σ/W *and* M/G *are isometric.*

In other words an isometric action (G, M) of copolarity k admits a reduction (W, Σ) with $\dim W = k$. Note that, a priori, k can be larger than the abstract copolarity.

We will later need the following observation.

LEMMA 1.1.21. *For a* G*-principal point* $p \in \Sigma$ *the orbit* Wp *is totally geodesic as a submanifold of* Gp.

PROOF. Let ∇ be the Levi-Civita-connection of M. We denote with ∇' the induced connection on the submanifold Gp. For G-principal points we have that $\nu_p\Sigma \subset T_pGp$, then

$$T_pGp = T_p(Wp) \oplus \nu_p\Sigma.$$

Let X, Y be vector fields tangent to the orbit Wp and α' be the second fundamental form of Wp as a submanifold of Gp. Let $\nu(Wp)$ the normal bundle of the orbit Wp in M, then $\nu(Wp) \cap T(Gp) = \nu\Sigma$ and

$$\alpha'(X, Y) = (\nabla'_X Y)^{\nu(Wp) \cap T(Gp)} = (\nabla_X Y)^{\nu\Sigma} = \alpha^{\Sigma}(X, Y),$$

where α^{Σ} is the second fundamental form of Σ. Since Σ is a totally geodesic submanifold of M we conclude that

$$\alpha'(X, Y) = \alpha^{\Sigma}(X, Y) = 0.$$

main □

1.2. Copolarity one actions in simply connected space forms

In the following let M be a simply connected space form, i.e. a simply connected complete Riemannian manifold of constant sectional curvature. We have seen in Section 1.1 that an isometric action (G,M) is polar if and only if the horizontal distribution $\mathcal{H}$ over the principal stratum is integrable. In this section we present an equivalent description for isometric actions (G,M) of copolarity 1. Recall that (G,M) is of copolarity 1 if it is not-polar, i.e. $\mathcal{H}$ is not integrable, and there exists a 1-section Σ through one and hence any principal point. The intersection of Σ with a principal orbit Gp is a totally geodesic, 1-dimensional submanifold of Gp. Therefore, the existence of Σ induces a 1-dimensional subdistribution $\mathcal{D}$ of the vertical distribution $\mathcal{V}$ over the principal stratum, whose restriction to any orbit Gp is autoparallel with respect to the connection of Gp. We call such a distribution $\mathcal{D}$ *vertical autoparallel*, i.e. $\nabla^v_{\mathcal{D}}\mathcal{D} \subset \mathcal{D}$. Furthermore, $\mathcal{D} \oplus \mathcal{H}$ is integrable with totally geodesic leaves, the connected components of $\Sigma \cap M_{pr}$. The next theorem tells us that also the converse is true.

THEOREM I. *Let M be a simply connected space form and (G,M) a non-polar isometric action. Then (G,M) has copolarity 1 if and only if there exists a 1-dimensional vertical autoparallel distribution $\mathcal{D} \subset \mathcal{V}$ over the principal stratum M_{pr}, such that $\mathcal{D} \oplus \mathcal{H}$ is integrable.*

We just explained one direction. So assume that there exists a 1-dimensional, distribution $\mathcal{D} \subset \mathcal{V}$ over the principal stratum M_{pr}, such that $\mathcal{D} \oplus \mathcal{H}$ is integrable. Recall that the restriction $\pi : M_{pr} \to M_{pr}/G$ is a Riemannian submersion, $\ker \mathrm{d}\pi = \mathcal{V}$ and since (G,M) is not polar, the horizontal distribution $\mathcal{H}$ is not integrable. The O'Neill tensor associated to a Riemannian submersion measures the integrability of $\mathcal{H}$ and we will show that $\mathcal{D}$ equals the image of the O'Neill tensor. Therefore, we have to insert a short excursion to Riemannian submersions. The reader familiar with this subject can skip the following explanations and should just notice the last two results, which describe some special properties of the O'Neill tensor in simply connected space forms.

Riemannian submersions. Let M and B be Riemannian manifolds and let $\pi : M \to B$ be a Riemannian submersion. Then π induces a splitting of the tangent bundle $TM = \mathcal{V} \oplus \mathcal{H}$ into a vertical distribution $\mathcal{V} = \ker \mathrm{d}\pi$ tangent to the *fibers* $\pi^{-1}(b)$ and a horizontal distribution $\mathcal{H} = \mathcal{V}^{\perp}$, such that $\mathrm{d}\pi_p|_{\mathcal{H}_p} : \mathcal{H}_p \to T_bB$ is an isometry, for $\pi(p) = b$. We assume all vector fields to be smooth and denote by $\mathfrak{X}^v, \mathfrak{X}^h$ the vertical vector fields and horizontal vector fields, respectively. A horizontal vector field $X \in \mathfrak{X}^h$ is called *basic* if it is π-related to a vector field of B. Denote by $\mathcal{B}$ the set of basic vector fields, then for $X \in \mathcal{B}$ and $U \in \mathfrak{X}^v$ we get $\pi_*[X,U] = [\pi_*(X), \pi_*(U)] = [\pi_*(X), 0] = 0$, i.e. $[\mathcal{B}, \mathfrak{X}^v]^h = 0$ and therefore

$$[\mathcal{B}, \mathfrak{X}^v] \subset \mathfrak{X}^v.$$

The restriction of a basic vector field X to a fiber $\pi^{-1}(b)$ projects to a single vector $\bar{X}_b \in T_bB$. Therefore, $\langle X, Y\rangle$ is constant along along fibers for $X, Y \in \mathcal{B}$ and especially $\|X\|$ is constant along the fibers.

DEFINITION 1.2.1. The *O'Neill-tensor* of a Riemannian submersion is the tensor field $A : \mathcal{H} \times \mathcal{H} \to \mathcal{V}$ given by $A_XY = \nabla^v_XY = \frac{1}{2}[X,Y]^v$, where X, Y are basic vector fields.

The O'Neill tensor A measures the integrability of the horizontal distribution $\mathcal{H}$. In fact $\mathcal{H}$ is integrable if and only if A vanishes. Furthermore, it measures the difference of the horizontal sectional curvatures of M and the sectional curvatures of B. Let $X, Y \in \mathcal{B}$, then

$$K^B(\pi_*(X), \pi_*(Y)) = K^M(X, Y) + 3\|A_X Y\|^2,$$

where K^M, K^B denote the sectional curvatures of M and B, respectively (cf. [**13**]).

EXAMPLE. The quotient of the Hopf action (S^1, S^3) is $S^3/S^1 = S^2(r)$ a 2-sphere of radius $r < 1$. The sectional curvature of S^3 equals 1 and the sectional curvature of $S^2(r)$, for $r < 1$, is bigger that 1. By the O'Neill formula the O'Neill-tensor A of the Hopf action does not vanish. Therefore, the horizontal distribution is not integrable and the Hopf action is not polar.

If the manifold M is of constant curvature κ, the O'Neill tensor has strong properties. To prove the next lemma we need the following notation. For $X \in \mathcal{B}$ let $A_X^* : \mathcal{V} \to \mathcal{H}$ be the point wise adjoint map of $A_X : \mathcal{H} \to \mathcal{V}$. Then for $X, Y \in \mathcal{B}$ and $U \in \mathfrak{X}^v$ we get

$$< A_X^* U, Y >=< U, A_X Y >=< U, \nabla_X Y >=< -\nabla_X U, Y >,$$

and with $[X, U]^h = \nabla_X^h U - \nabla_U^h X = 0$ follows

$$-A_X^* U = \nabla_U^h X = \nabla_X^h U.$$

LEMMA 1.2.2. [**13**] *Let M be a Riemannian manifold with constant curvature κ and $\pi : M \to B$ a Riemannian submersion. Let $X, Y \in \mathcal{B}$ be basic vector fields, then $A_X^* A_X Y$ is a basic vector field, i.e. $\|A_X Y\|$ is constant along the fibers.*

PROOF. The curvature tensor of M is given by

$$R(X, Y, Z) = \kappa\left(\langle Y, Z\rangle X - \langle X, Z\rangle Y\right),$$

hence $R(X, Y, Z)$ is basic for $X, Y, Z \in \mathcal{B}$. In general one can show (cf. [**13**]) that

$$\pi_*(R(X, Y, Z)) = R^B(\bar{X}, \bar{Y}, \bar{Z}) \circ \pi + \pi_*(A_X^* A_Y Z - A_Y^* A_X Z + 2A_Z^* A_X Y),$$

where $\bar{X} = \pi_*(X), \bar{Y} = \pi_*(y)$ and $\bar{Z} = \pi_*(Z)$. From this equation we conclude that $A_X^* A_Y Z - A_Y^* A_X Z + 2A_Z^* A_X Y$ is basic, too. Set $Z = X$, then the skew-symmetry of the A-tensor implies that $A_X X = 0$ and we conclude that $-A_X^* A_X Y + 2A_X^* A_X Y = A_X^* A_X Y$ is basic. Therefore, $< A_X^* A_X Y, Y >=< A_X Y, A_X Y >$ is constant along the fibers. □

Let γ be a horizontal geodesic. The next proposition tells us that the rank of $A_{\dot{\gamma}}$ is constant along γ.

PROPOSITION 1.2.3. [**13**] *Let $\pi : M \to B$ be a Riemannian submersion and M of constant curvature. If X is the tangent field of a horizontal geodesic γ, then the kernel of A_X is horizontal parallel, hence the the rank of A_X is constant. Furthermore, if Y is horizontal parallel along γ, then*

$$\nabla_X^v A_X Y = 2S_X A_X Y.$$

Here S denotes the shape operator of the respective fiber.

We will need the last equation in our construction of generalized sections.

PROOF. For $X, Y, Z \in \mathcal{B}$ we have

$$\frac{1}{2}[X,[Y,Z]^v] = \frac{1}{2}[X,[Y,Z]^v]^v = \nabla^v_X A_Y Z - \nabla^v_{A_Y Z} X = \nabla^v_X A_Y Z + S_X A_Y Z.$$

Thus by the Jacobi identity

$$\begin{aligned} 0 &= ([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]])^v \\ &= [X,[Y,Z]^v]^v + [Y,[Z,X]^v]^v + [Z,[X,Y]^v]^v + \\ &\quad [X,[Y,Z]^h]^v + [Y,[Z,X]^h]^v + [Z,[X,Y]^h]^v \\ &= 2(\mathcal{C}\,\nabla^v_X A_Y Z + \mathcal{C}\,S_X A_Y Z) + 2(\mathcal{C}\,A_X[Y,Z]^h), \end{aligned}$$

here $\mathcal{C}$ denotes cyclic summation. Therefore,

$$\mathcal{C}\,\nabla^v_X A_Y Z + \mathcal{C}\,S_X A_Y Z = -\mathcal{C}\,A_X[Y,Z]^h. \tag{2}$$

We have seen that for constant curvature $R(X,Y,Z)$ is basic. Therefore,

$$\begin{aligned} 0 &= R^v(X,Y,Z) \\ &= \nabla^v_X \nabla_Y Z - \nabla^v_Y \nabla_X Z - \nabla^v_{[X,Y]} Z \\ &= \nabla^v_X A_Y Z - \nabla^v_Y A_X Z - A_{[X,Y]^h} Z + \nabla^v_X \nabla^h_Y Z - \nabla^v_Y \nabla^h_X Z - \nabla^v_{[X,Y]^v} Z \end{aligned}$$

and

$$\nabla^v_X A_Y Z - \nabla^v_Y A_X Z = \nabla^v_{[X,Y]^v} Z + A_{[X,Y]^h} Z - A_X \nabla^h_Y Z + A_Y \nabla^h_X Z. \tag{3}$$

Combining (2) and (3) we get

$$\begin{aligned} \nabla^v_Z A_X Y &= \mathcal{C}\,\nabla^v_X A_Y Z - \nabla^v_X A_Y Z + \nabla^v_Y A_X Z \\ &= -(\mathcal{C}\,S_X A_Y Z + \mathcal{C}\,A_X[Y,Z]^h) - \\ &\quad (-2S_Z A_X Y + A_{[X,Y]^h} Z - A_X \nabla^h_Y Z + A_Y \nabla^h_X Z) \\ &= S_Z A_X Y - S_X A_Y Z - S_Y A_Z X + A_X \nabla^h_Z Y - A_Y \nabla^h_Z X. \end{aligned}$$

For $Z = X = \dot\gamma$ and $Y \in \ker A_{\dot\gamma}$, we get $0 = A_{\dot\gamma} \nabla^h_{\dot\gamma} Y$ and therefore $\nabla^h_{\dot\gamma} Y \in \ker A_{\dot\gamma}$. Hence the kernel $\ker A_{\dot\gamma}$ is horizontal parallel and rank $A_{\dot\gamma}$ is constant.

Now let Y be horizontal parallel, i.e. $\nabla^h Y = 0$, and $Z = X = \dot\gamma$ be tangent to a horizontal geodesic γ, then from the above equation we immediately get

$$\nabla^v_X A_X Y = 2 S_X A_X Y.$$

$\square$

The proof of Theorem I. Going back to the assumptions of Theorem I let (G, M) be a non-polar isometric action. Then the horizontal distribution $\mathcal{H}$ of the Riemannian submersion $\pi : M_{pr} \to M_{pr}/G$ is not integrable, hence the O'Neill tensor $A : \mathcal{H} \times \mathcal{H} \to \mathcal{V}, (X,Y) \mapsto \frac{1}{2}[X,Y]^v$ does not vanish. Let $\mathcal{D}$ be a 1-dimensional, vertical autoparallel subdistribution of $\mathcal{V}$, such that $\mathcal{D} \oplus \mathcal{H}$ is integrable. Then for basic vector fields $X, Y \in \mathcal{B}$ the image of the O'Neill tensor $A_X Y = \frac{1}{2}[X,Y]^v \in \mathcal{D}$ is contained in $\mathcal{D}$. We will show that $\dim \mathcal{D} = 1$ implies that $\mathcal{D}_p = \operatorname{Im} A_p$ for almost all principal points $p \in M_{pr}$.

LEMMA 1.2.4. *Assume that $\mathcal{D} \oplus \mathcal{H}$ is integrable, with $\dim \mathcal{D} = 1$ and that $A_p \neq 0$ for the principal point p. Then the set of vectors $x \in \mathcal{H}_p$, such that the map $A_p(x, .) : \mathcal{H}_p \to \mathcal{V}_p$ has rank 1, are open and dense in $\mathcal{H}_p$.*

PROOF. Let $X, Y \in \mathcal{B}$ be basic vector fields of the Riemannian submersion $\pi : M_{pr} \to M_{pr}/G$. Then $A_X Y = \frac{1}{2}[X,Y]^v \in \mathcal{D}$, since $\mathcal{D} \oplus \mathcal{H}$ is integrable. Therefore, $A_p(x,.) : \mathcal{H}_p \to \mathcal{V}_p$ has maximal rank equal to $\dim \mathcal{D} = 1$. The set $x \in \mathcal{H}_p$ for which $A_p(x,.)$ has maximal rank is open. Now assume that it is not dense. Then there exists $y \in \mathcal{H}_p$ and a neighborhood $U \subset \mathcal{H}_p$ of y such that $A_p(z,.) = 0$ for all $z \in U$. Let $x \in \mathcal{H}_p$ with $A_p(x,.) \neq 0$. The map A_p is skew-symmetric, hence $A_p(x,z) = -A_p(z,x) = 0$ for z in an open subset U, i.e. $A_p(x,.)$ vanishes on an open subset. But A_p is also bilinear, hence $A_p(x,.) = 0$, contradicting the choice of x. □

Let p be a principal point and γ a horizontal geodesic starting in p. Then Proposition 1.2.3 implies that the rank of $A_{\dot\gamma}$ is constant as long as γ stays in the principal stratum. Using the last statements we can finally prove the following proposition, which provides us with the key argument to prove Theorem I.

PROPOSITION 1.2.5. *Let (G, M) be a non-polar isometric action on a space form M. Assume there exists a 1-dimensional vertical distribution $\mathcal{D}$ over the principal stratum, such that $\mathcal{D} \oplus \mathcal{H}$ is integrable. Then $\mathcal{D}_p = \operatorname{Im} A_p$ for almost every principal point p, and in those points it is invariant under the shape operator of the orbit Gp. This means $S_{X_p}(\mathcal{D}_p) \subset \mathcal{D}_p$ for all $X \in \mathcal{B}$ basic.*

PROOF. Since (G, M) is non-polar, the A-tensor does not vanish, and $\mathcal{D}$ contains the image $\operatorname{Im} A \subset \mathcal{D}$. Now for a principal point p with $A_p \neq 0$, the map $A_p(X_p,.)$ has rank 1 for almost all basic vector fields X, and for dimension reasons $\mathcal{D}_p = \operatorname{Im} A_p$. Since $\|A_X Y\|$ constant along Gp, for basic vector fields $X, Y \in \mathcal{B}$, we get $\mathcal{D}_{p'} = \operatorname{Im} A_{p'}$ for all $p' \in Gp$. The rank of A is also constant along horizontal geodesics, as long as they stay in the principal stratum. Therefore, the last lemma implies that set of principal points for which $D_p = \operatorname{Im} A_p$ is open and dense in M_{pr}. We now restrict our attention to those points. Let $X, Y, Z \in \mathcal{B}$ be basic vector fields, then the integrability of $\mathcal{D} \oplus \mathcal{H}$ implies

$$[A_X Y, Z] \in \mathcal{D}.$$

Remember that $A_X Y \in \mathcal{D}$ is always vertical and Z was chosen basic. Assume that $(A_X Y)_p \neq 0$, i.e. $A_X Y$ span $\mathcal{D}$ along Gp. Then for almost every $Z \in \mathcal{B}$ we find $Y' \in \mathcal{B}$, such that $(A_X Y)_p = (A_Z Y')_p$. Now Lemma 1.2.2 implies that $\|A_X Y\|$ and $\|A_Z Y'\|$ are constant along Gp, hence the initial conditions imply that $A_X Y = A_Z Y'$ along Gp. We can assume that Y, Y' are horizontal parallel, then Proposition 1.2.3 implies

$$\nabla^v_X A_X Y = 2 S_X A_X Y.$$

Therefore,

$$\begin{aligned} [A_X Y, Z] &= -S_Z(A_X Y) - \nabla^v_Z A_X Y \\ &= -S_Z(A_Z Y') - \nabla^v_Z A_Z Y' \\ &= -3 S_Z(A_Z Y') \in \mathcal{D}, \end{aligned}$$

i.e. $S_Z(\mathcal{D}) \subset \mathcal{D}$. The hole statement now follows from the density of the chosen Z and the continuity of all involved tensors. □

For the proof of Theorem I we need a well known application of the theorem on the reduction of the codimension due to J. Erbacher (cf. [**5**]). Let $\phi : \beta \to M$ be an isometric immersion of a k-dimensional connected Riemannian manifold β into a $(k+n)$-dimensional simply connected space form M. The first normal space $\mathcal{N}_p^1$ is the vector subspace of $\nu_p\beta$ spanned by the image of the second fundamental form α of β

$$\mathcal{N}_p^1 := \operatorname{span}\{\alpha(X,Y) \,|\, X,Y \in T_p\beta\} \subset \nu_p\beta.$$

The equality $\langle\alpha(X,Y),\xi\rangle = \langle S_\xi X, Y\rangle$ for all normal vector $\xi \in \nu_p\beta$ implies that its complement $(\mathcal{N}_p^1)^\perp \subset \nu_p\beta$ consists of those normal vectors ξ for which the shape operator S_ξ of β vanishes

$$(\mathcal{N}_p^1)^\perp = \{\xi \in \nu_p\beta \,|\, S_\xi \equiv 0\}.$$

THEOREM 1.2.6. [**5**] *If there exists a subdistribution $\mathcal{M} \subset (\mathcal{N}^1)^\perp$, such that $\mathcal{M}$ is invariant under normal parallel translation along $\phi(\beta)$ then there exists a complete, embedded totally geodesic submanifold $\Sigma_{\mathcal{M}}$, such that $\phi(\beta) \subset \Sigma_{\mathcal{M}}$. Furthermore, $\mathcal{M}$ is the normal bundle of $\Sigma_{\mathcal{M}}$ along $\phi(\beta)$.*

We can finally prove the existence of 1-sections in a simply connected space form M.

PROOF OF THEOREM I. Let M be simply connected space form and (G,M) an non-polar isometric action. Let $\mathcal{D}$ be a 1-dimensional vertical autoparallel subdistribution $\mathcal{D}$ over M_{pr}, such that $\mathcal{D} \oplus \mathcal{H}$ is integrable. We will now prove the existence of a pre-section with vertical part $\mathcal{D}$. Since (G,M) is non-polar, each pre-section is actually a minimal section.

Let p be a principal point, then our assumption implies that $\nabla^v_{\mathcal{D}}\mathcal{D} \subset \mathcal{D}$. Especially, the integral manifold β of $\mathcal{D}$ through p is a totally geodesic submanifold of the orbit Gp. Let $\mathcal{E}$ be the normal bundle of β in $T(Gp)$. The orbit Gp is embedded in M, hence we can regard β as a submanifold of M. For $p' \in \beta$ we have the following splitting

$$T_{p'}M = \mathcal{E}_{p'} \oplus \mathcal{D}_{p'} \oplus \nu_{p'}Gp.$$

We claim that $\mathcal{E} \subset (\mathcal{N}^1)^\perp$ is contained in the complement of the first normal space $\mathcal{N}^1$ of β regarded as a submanifold of M. Denote with ∇ the Levi-Civita connection of M and with ∇' the induced Levi-Civita connection of Gp. Let $U \in \mathcal{E}|_\beta$ and let S' be the shape operator of β as a submanifold of Gp and $\tilde{S}$ be the shape operator of β as a submanifold of M. Then $-S'_U(V) = (\nabla'_V U)^{\mathcal{D}} = (\nabla_V U)^{\mathcal{D}} = -\tilde{S}_U(V)$ for each $V \in T\beta$. Since $\beta \subset Gp$ is totally geodesic $S' = 0$ and therefore $\mathcal{E} \subset (\mathcal{N}^1)^\perp$. We have to show that $\mathcal{E}$ is parallel along β with respect to the normal connection $\nabla^\perp$ of M. We can assume that p is a principal point such that $\mathcal{D}_p = \operatorname{Im} A_p$. Then from Proposition 1.2.5, we know that $\mathcal{D}$ is invariant under the shape S of the orbit Gp. Denote by α the second fundamental form of the orbit Gp and let $\xi \in \nu(Gp)$ be a normal vector field. For $V \in \mathcal{D}$ and $U \in \mathcal{E}$ along β we get

$$0 =< S_\xi(U), V >=< \xi, \alpha(U,V) >,$$

i.e. $\alpha(\mathcal{D},\mathcal{E}) = 0$. Note that the above equations implies that $\mathcal{E}$ is also invariant under the shape operator S of the orbit Gp. The condition $\alpha(\mathcal{D},\mathcal{E}) = 0$ implies

$$\begin{aligned}
(\nabla^\perp_V U) &= (\nabla_V U)^{\nu\beta} \\
&= (\nabla_V U)^{\mathcal{E}} + (\nabla_V U)^{\nu(Gp)} \\
&= (\nabla_V U)^{\mathcal{E}} + \alpha(V,U) \\
&= (\nabla_V U)^{\mathcal{E}} \in \mathcal{E}.
\end{aligned}$$

Hence, $\mathcal{E}$ is invariant under the normal parallel translation along β and Theorem 1.2.6 implies that there exists a complete, embedded totally geodesic submanifold Σ of M, with $\beta \subset \Sigma$.

We will now prove that Σ is a pre-section. Theorem 1.2.6 implies that the distribution $\mathcal{E}$ coincides with the normal space of the totally geodesic submanifold Σ, i.e. $\mathcal{E}_p = \nu_p\Sigma \subset T_pGp$ for $p \in \beta$. Taking orthogonal complements yields $\nu_pGp \subset T_p\Sigma$ and condition (C3) is fulfilled for points in β. Since Σ is complete $\exp_p(\nu_pGp) \subset \Sigma$, hence it intersects every orbit and this is condition (C2). We are left to prove that $\nu_pGp \subset T_p\Sigma$ for principal points not contained in β. Therefore, let $p' \in \Sigma$ be a principal point not contained in β and let γ be the shortest geodesic in Σ between p' and β. Then γ intersects β perpendicular, i.e. $\dot\gamma(1) \in \nu_{\gamma(1)}\beta \cap T_{\gamma(1)}\Sigma = \nu_{\gamma(1)}Gp$, and γ is in fact a horizontal geodesic of the G-action. Denote $\gamma(1) = p$, and recall that $\mathcal{D}_p = \operatorname{Im} A_p$. For $U \in \mathcal{E}$ we get

$$0 = \langle A_XY, U\rangle = \langle Y, A_X^*U\rangle\,,$$

hence $\mathcal{E} = \bigcap_{X\in\mathcal{B}} \ker A_X^*$. For every $u \in \mathcal{E}_p \subset T_pGp$ there exists a Jacobifield J along γ, with $J(0) = u$, such that J is everywhere tangent to the G-orbits. Then J fulfills the starting condition $J'(0) = -A_{\dot\gamma}^*u - S_{\dot\gamma}u$ (cf. [**22**] or [**13**]). The distribution $\mathcal{D}, \mathcal{E}$ are invariant under the shape operator S of the G-orbit, hence $J'(0) = -S_{\dot\gamma}u \in \mathcal{E}_p = \nu_p\Sigma$. Now Σ is totally geodesic, and the starting conditions $J(0), J'(0)$ are perpendicular to Σ. Therefore, the Jacobifield J is everywhere perpendicular to Σ, in fact we get $\nu_{p'}\Sigma \subset T_{p'}Gp'$ and this is condition (C3). $\square$

CHAPTER 2

Representations of abstract copolarity one and two

2.1. Introduction

Let G be a compact Lie group and let V be a finite dimensional Euclidean vector space. A group homomorphism $\rho : G \to O(V)$ is called an *(orthogonal) representation*. It induces an isometric action of G on V through

$$\begin{array}{ccc} G \times V & \to & V \\ (g,p) & \mapsto & \rho(g)p = gp. \end{array}$$

An injective representation is called *faithful*. In the following we will only consider faithful representations, hence we identify G with its image in $O(V)$ and write (G,V) for the induced linear action, which we also call a representation. A representation (G,V) is called *irreducible* if there exist no G-invariant subspaces beside $\{0\}$ and V itself. Otherwise it is called *reducible*. We say that a representation (G,V) is *without fixed points*, if there exists no non-trivial fixed point in V.

For a representation (G,V) we denote its space of orbits by V/G. From Chapter 1, we know that V/G inherits a natural metric given by the distance of the G-orbits in V. Let (G',V') be another representation, such that the orbit spaces V'/G' and V/G are isometric. Then (G',V') and (G,V) are called *quotient-equivalent*. A representation (G',V') which is quotient-equivalent to (G,V) is called a *reduction of* (G,V) if $\dim G' < \dim G$. The representation (G,V) is called *reduced* if the dimension of G is minimal in its quotient equivalence class. Recall that two representations (G,V), (G',V') are called *orbit equivalent* if there exists an isometry $\Phi : V \to V'$, such that for every $g \in G$ there exists $g' \in G'$ with $\Phi(gp) = g'\Phi(p)$. We see that the representations (G,V), (G',V') are quotient-equivalent.

Example (Generalized sections). Let (G,V) be a representation and Σ a generalized section. The induced action of the generalized Weyl group $(W(\Sigma),\Sigma)$ is a reduction of the original representation, although it is not necessarily minimal.

For a minimal reduction (G',V') of (G,V), the dimension of the group G' is called the *abstract copolarity of* (G,V). From the above example follows that the copolarity is an upper bound for the abstract copolarity, hence it is an interesting question under which assumptions these two notations coincide. We restrict our attention to connected groups and denote them in the following by H.

At first we consider polar representations, which turn out to be exactly the representations of abstract copolarity 0 (cf. Proposition 2.4.1).

Therefore, a representation of copolarity 1 has abstract copolarity equal to 1. But does there always exist a 1-section for a representation of abstract copolarity 1? In the irreducible case, when the group is connected, the answer is positive as we explain now. In [**23**] E. Straume analyzes representations of cohomogeneity 3. He proves that the non-polar representation of cohomogeneity 3 admit 1-sections. In [**11**] it is proven that an irreducible representation of a connected group with abstract copolarity 1 has cohomogeneity 3. Hence, the result of Straume implies that it admits a 1-sections. Therefore, in the irreducible case, the concepts of copolarity 1 and abstract copolarity 1 coincide for connected groups.

That for irreducible representations of connected groups the notions of copolarity 2 and abstract copolarity 2 also coincide, is shown in [**11**]. First note that the above discussion implies that an irreducible representation of copolarity 2 has abstract copolarity 2. In [**11**] is proven that the irreducible representations of abstract copolarity 2 are exactly the following representations

H	Φ
$SO(3) \times G_2$	$\rho_3 \otimes_{\mathbb{R}} \phi$
$SU(3)$	$S^2\mu_3$
$SU(6)$	$\bigwedge^2 \mu_6$
$SU(3) \times SU(3)$	$\mu_3 \otimes_{\mathbb{C}} \mu_3$
E_6	ϕ

Here ϕ stands for the respective standard representation. All the above representations admit 2-sections, hence their copolarities are equal to 2. The listed representations have cohomogeneity equal to 4.

In general, for non-polar irreducible representations small minimal reductions restrict the cohomogeneity of the representation.

THEOREM 2.1.1. [**11**] *Let $\rho : H \to O(W)$ be an irreducible representation of a compact connected group H. Let $\tau : G \to O(V)$ be a minimal reduction of ρ. If* $\dim G \leq 6$*, then G_0 is a torus T^k. Moreover, if $k \geq 1$, then* $\dim V = 2k + 2$.

The main result of this chapter is a partial answer to the question if the concepts of copolarity and abstract copolarity up to two coincide for reducible representations. For a non-reduced representation (H, W) of a connected group, we show that the properties copolarity 1 and abstract copolarity 1 are equivalent (cf. Corollary 2.1.3). In the case that (H, W) has abstract copolarity 2, we are able to generalize the estimation of the cohomogeneity.

For a representation of abstract copolarity 1 or abstract copolarity 2, the identity component of every minimal reduction (G, V) equals $G = S^1$ or $G = T^2$, respectively, and we say that (G, V) has a toric identity component.

Before we state our main results we explain a simple method to construct a reducible representation of arbitrary abstract copolarity.

EXAMPLE. The product representation $(H_1 \times H_2, W_1 \times W_2)$ of a representation of abstract copolarity l and a representation of abstract copolarity m has abstract copolarity $l + m$. The quotient $W_1 \times W_2/(H_1 \times H_2) = W_1/H_1 \times W_2/H_2$ is a product.

Assume that the quotient of a representation (H, W) is isometric to the quotient of a product representation $(G_1 \times G_2, V_1 \times V_2)$, with $\dim V_i \neq 0$, for $i = 1, 2$. Then we say that the quotient W/H *splits*. The interesting representations are those with non-splitting quotient. We now state our main results.

THEOREM II. *Let (H, W) be a non-reduced representation of a connected compact group H of abstract copolarity* 1 *with non-splitting quotient W/H. Then (H, W) has cohomogeneity* 3.

By the classification of Straume (cf. [**23**]), the representations of Theorem II coincide with the non-polar representations of cohomogeneity 3 listed below. They admit 1-sections.

(H, W)	Φ
$(SO(n), \mathbb{R}^{2n}), n \geq 3$	$2\rho_n$
$(U(2), \mathbb{R}^7)$	$\rho_3 + [\mu_2]_{\mathbb{R}}$
$(\mathrm{Sp}(1) \times \mathrm{Sp}(2), \mathbb{R}^{13})$	$\rho_5 + \nu_1 \otimes_{\mathbb{H}} \nu_2$
$(\mathrm{Spin}(9), \mathbb{R}^{25})$	$\Delta_9 + \rho_9$
$(U(1) \times SU(n) \times U(1), \mathbb{R}^{4n}), n \geq 2$	$[\mu_1 \otimes_{\mathbb{C}} \mu_n + \mu_n \otimes_{\mathbb{C}} \mu_1]_R$
$(Sp(1) \times \mathrm{Sp}(n) \times \mathrm{Sp}(1), \mathbb{R}^{8n}) n \geq 2$	$\nu_1 \otimes_{\mathbb{H}} \nu_n + \nu_n \otimes_{\mathbb{H}} \nu_1$
$(SO(2) \times \mathrm{Spin}(9), \mathbb{R}^{32})$	$\rho_2 \otimes \Delta_9$
$(U(2) \times \mathrm{Sp}(n), \mathbb{R}^{8n}), n \geq 2$	$[\mu_2 \otimes_{\mathbb{C}} \nu_n]_{\mathbb{R}}$
$(\mathrm{Sp}(1) \times \mathrm{Sp}(n), \mathbb{R}^{8n}), n \geq 2$	$S^3\nu_1 \otimes_{\mathbb{H}} \nu_n$

If the quotient of a representation (H, W) splits, we show that actually (H, W) is itself orbit equivalent to a product representation $(H_1 \times H_2, W_1 \times W_2)$. As in the above example, the abstract copolarity of the factor representations sum up to one. Therefore, one factor is polar and the other has abstract copolarity 1. Choosing the abstract copolarity 1 factor (H_2, W_2) with non-splitting quotient, (H_2, W_2) is either reduced and $H_2 = S^1$, or Theorem II implies that (H_2, W_2) has cohomogeneity 3.

THEOREM 2.1.2. *Let (H, W) be a non-reduced representation of a connected compact group H of abstract copolarity* 1. *Assume that the quotient W/H splits. Then (H, W) is orbit equivalent to a product representation $(H_1 \times H_2, W_1 \times W_2)$, such that one factor (H_1, W_1) is polar and the other (H_2, W_2) is either one of the representations of Theorem II, or reduced.*

For a product representation a generalized section is the product of the generalized sections of the factors. Therefore, the representations of the last theorem admit 1-sections, too.

COROLLARY 2.1.3. *A representation* (H, W) *of a connected compact group has abstract copolarity* 1 *if and only if the copolarity is equal to* 1*.*

The representation of abstract copolarity 2 are a bit more complicated to describe. We prove

THEOREM III. *Let* (H, W) *be a non-reduced representation of a connected compact group* H *of abstract copolarity two, with non-splitting quotient* W/H*. Then either*

- (H, W) *has cohomogeneity* 4*, or*
- *each minimal reduction* (G, V) *of* (H, W) *has two connected components, i.e.* $G/G_0 = \mathbb{Z}/2\mathbb{Z}$*. Furthermore, there exist an irreducible subspace* $W' \subset W$ *such that* H *acts with cohomogeneity two on* W'*, and* $(H, (W')^{\perp})$ *is orbit equivalent to a non-polar* S^1*-representation.*

For a representation of abstract copolarity 2 with non-splitting quotient, there are three classes of reducible representations with cohomogeneity 4. We give examples for two of these classes. Our examples have copolarity two. In the case that the cohomogeneity is bigger than four, we show that the restriction (H, W') is orbit equivalent to the isotropy representation of a rank two symmetric space with $g = 4$.

A representation (H, W) of abstract copolarity 2 with splitting quotient is orbit equivalent to a product representation $(H_1 \times H_2, W_1 \times W_2)$, where the abstract copolarities of the factor representations sum up to two. Therefore, we have two possibilities

- both factors have abstract copolarity 1, or
- one factor is polar and the other has abstract copolarity 2.

In the second case we can assume that the abstract copolarity 2 factor (H_2, W_2) has non-splitting quotient. Then (H_2, W_2) is either reduced and $H_2 = T^2$, or it is one of the representations of Theorem III. We summarize

THEOREM 2.1.4. *Let* (H, W) *be a non-reduced representation of a connected compact group of abstract copolarity* 2*. Assume that the quotient* W/H *splits. Then* (H, W) *is orbit equivalent to a product representation* $(H_1 \times H_2, W_1 \times W_2)$ *and either*

- (H_i, W_i) *has abstract copolarity* 1*, or*
- (H_1, W_1) *is polar and* $H_2 = T^2$*, or*
- (H_1, W_1) *is polar and* (H_2, W_2) *is a representation of Theorem III.*

This chapter is constructed as follows. The next section explains the results of [**11**]. We have a closer look at the orbit space and its invariants. In Section 2.3 we recall some general facts from representation theory and fix our notation. In Section 2.4 the minimal reductions of representations of abstract copolarity up to two are considered. Finally, in Section 2.5 we prove Theorem II, and Section 2.6 is devoted to the proof of Theorem III.

2.2. The space of orbits

Stratification. Let (G, V) be a faithful representation. Then there is a natural stratification of V by orbit types (cf. Section 1.1). The strata are embedded submanifolds and the restriction of the quotient map $\pi : V \to V/G$ to any stratum is a Riemannian submersion onto its image, which is also called stratum. For a representation the isotopy groups are constant along lines through the origin. Therefore, the quotient V/G is the cone over the quotient of the induced action on the unit sphere $(G, S(V))$, and we often restrict our attention to the second action.

We will need the existence of certain strata in the quotient V/G. Set $X = V/G$ and let X_{pr} be the image of the principal stratum. Then X_{pr} is an open and dense set in V/G and we define the *quotient-dimension* $\dim V/G = \dim X_{pr}$, which equals the cohomogeneity of (G, V). For a stratum $\mathrm{Str}(\bar{p})$ in the quotient the *quotient-codimension* is

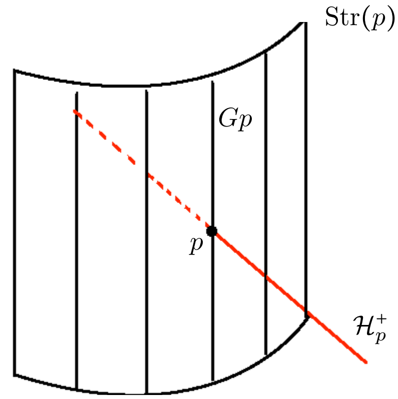
$$\operatorname{qcodim} \mathrm{Str}(\bar{p}) = \dim X_{pr} - \dim \mathrm{Str}(\bar{p}).$$

The closure of the strata of quotient-codimension 1 in X is called the *boundary of* X and will be denoted by ∂X. The existence of boundary will be an important tool in our discussion, since it indicates if (G, V) is reduced.

Let G_p be the isotropy group of p and denote by $\mathrm{Str}(p) \subset V$ the stratum through p. Following the discussion of Section 1.1 the stratum fulfills the following dimension formula

$$(1) \qquad \dim \mathrm{Str}(p) = \dim G \cdot V^{G_p} = \dim V^{G_p} + \dim G - \dim N_G(G_p),$$

where V^{G_p} denotes the set of G_p-fixed points. In the following this formula will be used frequently to estimate the dimension $\dim V^{G_p}$ of V^{G_p}. Denote by $\mathcal{H}_p^+$ the normal space to the stratum at the point p. This is exactly the non-trivial part of the slice representation $(G_p, \nu_p Gp)$.



Let ν^{G_p} denote the set of fixed point of G_p, then $\nu_p Gp = \nu^{G_p} \times \mathcal{H}_p^+$ and the quotient of the slice representation $\nu_p Gp/G_p = \nu^{G_p} \times \mathcal{H}_p^+/G_p$ splits. The cohomogeneity of the slice representation coincides with the cohomogeneity of (G, V) and since

$\dim \mathrm{Str}(\bar{p}) = \dim \nu^{G_p}$ we get

$$\begin{aligned} \operatorname{qcodim} \mathrm{Str}(\bar{p}) &= \dim V/G - \dim \mathrm{Str}(\bar{p}) \\ &= (\dim \nu^{G_p} + \dim \mathcal{H}_p^+/G_p) - \dim \nu^{G_p} \\ &= \dim \mathcal{H}_p^+/G_p. \end{aligned}$$

As an immediate consequence we denote

LEMMA 2.2.1. *Let $p \in V$ be a point, which projects to a point $\bar{p}$ contained in a stratum of quotient-codimension* 1*. Then the restriction of the slice representation to the subspace $\mathcal{H}_p^+$ is transitive on the unit sphere $S(\mathcal{H}_p^+)$.*

Invariant of orbit spaces. Some important properties of a representation can be read of the quotient, for example fixed points and invariant subspaces (cf. [**11**]). Therefore, quotient-equivalent representations have the same number of invariant subspaces and their set of fixed points even have the same dimension. Especially, a reduction of a reducible representation is itself reducible. In the following we will recall the results of [**11**] including their proofs for the sake of completeness.

PROPOSITION 2.2.2. *Let (G, V) and (G', V') be quotient-equivalent representations. Then (G, V) has non-trivial set of fixed points if and only if (G', V') has non-trivial set of fixed points. Actually, the dimensions of the set of fixed points coincide.*

PROOF. If the representation (G, V) has fixed points V^G the quotient $V/G = V^G \times (V^G)^\perp/G$ splits and V^G is an Euclidean factor. Let $p, \bar{p}$ be two points in the unit sphere $S(V)$. The distance in V between p and the orbit $G\bar{p}$ is less or equal to 2. Equality holds if $G\bar{p}$ is the antipodal point $-p$ and therefore $\bar{p}$ is a fix point of the G-action. Hence, for $p, \bar{p} \notin V^G$ a shortest line between Gp and $G\bar{p}$ does not cross the origin 0_V. Then its projection to V/G does not cross the cone point $0_{V/G}$ and $Gp, G\bar{p}$ are not contained in an Euclidean factor. Therefore, V^G is the maximal Euclidean factor in V/G. The isometry $I = V/G \to V'/G'$ maps Euclidean factors to Euclidean factors, and the statement follows. □

By changing an isometry along the maximal Euclidean factor we always find a cone point preserving isometry $I : V/G \to V'/G'$.

We will show that in fact all invariant subspaces can be recognized metrically in the quotient, i.e. are invariant under quotient equivalence. Recall that for an G-invariant subspace of V its orthogonal complement is again G-invariant and the same is true for their intersections with $S(V)$. Therefore, the images of these intersections have distance $\frac{\pi}{2}$ in $S(V)/G$. In fact, we will prove that the converse is also true. Therefore, we have to recall some elementary facts from spherical geometry and show some preparing statements. The first is

PROPOSITION 2.2.3. *A representation (G, V) has non-trivial fixed points if and only if the diameter of $S(V)/G$ is greater than $\frac{\pi}{2}$. If the diameter is greater than $\frac{\pi}{2}$, it equals π.*

A subset $C \subset S(V)$ is called *convex* if for two points $p, q \in C$ a shortest geodesic $\gamma_{p,q}$ between p and q is contained in C. For a bounded set $B \subset S(V)$, we denote by r the *radius* of B, which is the infimum of the positive numbers r' such that

$B \subset B_{r'}(p)$ for some $p \in S(V)$. A point $p \in S(V)$ with $B \subset \overline{B_r(p)}$ is called *a center of* B. In the following we denote the distance function on $S(V)$ by d_s and the induced distance function on the quotient $S(V)/G$ by d_q.

LEMMA 2.2.4. *Let* $(G, S(V))$ *be an isometric action of a group* G *and* $p \in S(V)$. *Then the set* $\{p' \in S(V) \,|\, d_s(p', Gp) \geq r \text{ with } r > \frac{\pi}{2}\}$ *is convex.*

PROOF. Let $x, q \in \{p' \in S(V) \,|\, d_s(p', Gp) \geq r\}$ and denote by $\gamma(t)$ a minimal geodesic connecting $x = \gamma(0)$ and $q = \gamma(1)$. For a fix time t there exists a point $g_t p \in Gp$ realizing the distance between the point $\gamma(t)$ and the orbit Gp. Taking an open ball of radius r around $g_t p$, the points x and q are contained in its complement, which is a closed ball $\bar{B}$ of radius $\pi - r < \frac{\pi}{2}$. Therefore, γ is unique and contained in $\bar{B}$. Especially, $d_s(\gamma(t), Gp) = d_s(\gamma(t), g_t p) \geq r$. □

LEMMA 2.2.5. *A bounded set* $B \subset S(V)$ *with radius* $r < \frac{\pi}{2}$ *has a unique center.*

PROOF. Choose a sequence $(p_n)_{n \in \mathbb{N}} \subset S(V)$ and corresponding $r_n > r$, such that $B \subset B_{r_n}(p_n)$ and $\lim r_n = r$. We will show that (p_n) is a Cauchy sequence. Since $S(V)$ is complete (p_n) will have a unique limit point $\bar{p}$, such that $B \subset \overline{B_r(\bar{p})}$, i.e. $\bar{p}$ is a center. The group of isometries is transitive on $S(V)$. We fix a point $x \in S(V)$, then for every point $q \in B$ we can find an isometry I_q sending q to x. Note that for every q the image $I_q(p_n)$ is contained in a ball of radius r_n around x. To estimate the distance between p_n, p_m, we will estimate the distance of the images $I_q(p_n), I_q(p_m)$. First look at a stripe $B_R(x) - B_{R'}(x)$ for $R' < R < \frac{\pi}{2}$. Let $\epsilon > 0$ and choose $R' < r < R < \frac{\pi}{2}$, such that each geodesic segment in $B_R(x) - B_{R'}(x)$ has length less than ϵ. To see that this is possible, assume that there exists an $\epsilon > 0$, such that for each pair (R, R') as above, we find a geodesic segment $\gamma_{R,R'}$ in $B_R(x) - B_{R'}(x)$, such that $L(\gamma_{R,R'}) > \epsilon$. For $R, R' \to r$ the geodesic segment $\gamma_{R,R'} \mapsto \gamma \subset S_r(x)$ converges to a geodesic segment γ contained in the distance sphere $S_r(x)$ with $L(\gamma) > 0$. This contradicts the fact that $S_r(x)$ does not contain geodesics since $r < \frac{\pi}{2}$. Now we can choose n, m large enough, such that $r_n, r_m < R$. Let p denote the midpoint of the geodesic segment joining p_n, p_m. Since $R < \frac{\pi}{2}$ the ball $B_R(x)$ is convex and the image $I_q(p)$ of the midpoint is contained in it for each $q \in B$. If $I_q(p) \in B_{R'}(x)$ for all $q \in B$, then $B \subset B_{R'}(x)$, contradicting $R' < r$. Hence, there exists a point $q \in B$ such that $I_q(p) \subset B_R(x) - B_{R'}(x)$ and we conclude $d_s(p_n, p_m) < 2\epsilon$.
To show that the center $\bar{p}$ is unique, let p' be another point with $B \subset \overline{B_r(p')}$. Then $\bar{p}, p' \in \bar{B}_r(q)$ for each $q \in B$. The midpoint p'' of the geodesic segment joining $\bar{p}$ and p' is contained in $B_r(q)$ and the distance between $q \in B$ and p'' is smaller than r, contradicting the choice of r. □

PROOF OF PROPOSITION 2.2.3. Assume the representation (G, V) has fixed points V^G. Then the diameter of $S(V)/G = \pi$. On the other hand, assume that the diameter of $S(V)/G$ is larger than $\frac{\pi}{2}$. Then for some point $p \in S(V)$ the set B of points, whose distance to Gp is greater or equal to $\frac{\pi}{2} + \epsilon$, is non-empty, compact, convex and G-invariant. From the proof of Lemma 2.2.4 we know that for each pair $x, q \in B$ the shortest geodesic between x and q in $S(V)$ is unique and contained in B. Therefore, B does not contain a great sphere. A convex and compact subset of the sphere $S(V)$ is either a great i-sphere ($i \geq 1$), an i-hemisphere H_i, or a proper subset of H_i. Since, B does not contain a great sphere $B \subset H_i - \partial H_i$. As a compact subset of $S(V)$ the set B is bounded and $B \subset H_i - \partial H_i$ implies that its radius $r < \frac{\pi}{2}$. From Lemma 2.2.5 we deduce that B has a unique center $\bar{p}$. For any $q \in B$, and $g \in G$ we have $d_s(q, g\bar{p}) = d_s(g^{-1}q, \bar{p}) < r$, since B is G-invariant. But the center is unique, hence $g\bar{p} = \bar{p}$ and $\bar{p}$ is a fixed point of the G-action. □

From now on assume that the representation (G, V) is free of fixed points. Then the above proposition implies that the diameter of the quotient $S(V)/G$ is at most $\frac{\pi}{2}$. We can finally prove

Proposition 2.2.6. *Let (G, V) and (G', V') be quotient-equivalent representations without fixed points. If $W \subset V$ is a G-invariant subspace, then there exists a G'-invariant subspace $W' \subset V'$, such that $W/G = W'/G'$.*

Proof. Consider the restricted action of G on the unit sphere $S(V) = S^n$. We claim, that a closed subset $Z \subset S(V)/G$ has the form $Z = S(W)/G$ for an G-invariant subspace W if and only if there is a subset $Z' \subset S(V)/G$, such that Z is the set of all points $z \in S(V)/G$ with $d_q(z, z') = \frac{\pi}{2}$ for all $z' \in Z'$. For an G-invariant subspace W it is clear that $S(W)/G$ and $S(W^\perp)/G$ have distance $\frac{\pi}{2}$ in the quotient. On the other hand, assume that Z is given in terms of Z' as above. The pre-image of Z under the projection $\pi : S(V) \to S(V)/G$ is a closed subset of $S(V)$ hence compact and of course G-invariant. Let $p \in \pi^{-1}(Z)$ and $p' \in \pi^{-1}(Z')$, then $d_s(p, Gp') = \frac{\pi}{2}$. The action has no fixed points hence the diameter $\operatorname{diam} S(V)/G = \frac{\pi}{2}$. Similar to Lemma 2.2.4 we argue that the set $\{p \in S(V) \,|\, d_s(p, Gp') = \frac{\pi}{2}\}$ is convex. Let p, q with $d_s(p, Gp') = d_s(q, Gp') = \frac{\pi}{2}$ and $\gamma(t)$ a shortest geodesic between p and q. Assume that $q \neq -p$, then γ is unique and for fixed t_0 we find a point $g_{t_0}p' \in Gp'$, such that $d_s(\gamma(t_0), g_{t_0}p') = d(\gamma(t_0), Gp') \leq \frac{\pi}{2}$, since the diameter equals $\frac{\pi}{2}$. Then p and q lie on the sphere with distance $\frac{\pi}{2}$ to the point $g_{t_0}p'$. Hence γ is also contained in this distance sphere, i.e. $d(\gamma(t_0), g_{t_0}p') = \frac{\pi}{2}$ and $\gamma(t_0) \in \{p \in S(V) \,|\, d_s(p, Gp') = \frac{\pi}{2}\}$. Assume that $q = -p$ are antipodal. Then Gp' is contained in the sphere $S^{n-1}(p)$ of distance $\frac{\pi}{2}$ to p, since $d(p, Gp') = d(-p, Gp') = \frac{\pi}{2}$. This is in fact true for every element in the orbit Gp, i.e.

$$Gp' \subset \bigcap_{g \in G} S^{n-1}(gp) \subset S^{n-2}(p),$$

since $Gp \neq \{p\} \cup \{-p\}$. Hence, p and $-p$ are contained in the complementary sphere S^2 and there exists a minimizing geodesic γ connecting p and $-p$ with $\gamma(t) \subset S^2$, i.e. $d_s(\gamma(t), Gp') = \frac{\pi}{2}$. Therefore, the set $\{p \in S(V) \,|\, d_s(p, Gp') = \frac{\pi}{2}\}$ is convex. As the intersection of convex sets the set $\pi^{-1}(Z)$ is convex, too, and we noticed before, that it is compact and G-invariant. A convex and compact subset of S^n is either a great i-sphere $(i \geq 1)$, an i-hemisphere H_i, or a proper subset of H_i. If G leaves an hemisphere $H_i = \overline{(S^i - S^{(i-1)})_0}$ invariant, it has to leave S^i and $S^{(i-1)}$ invariant, too. In fact, the image $G \subset O(V^i)$ maps $GS^{(i-1)} = S^{(i-1)}$ and therefore G also leaves the complementary sphere $S^0 \cap H^i$ invariant, which is a point. Since G has no fixed points, we conclude that $\pi^{-1}(Z)$ is a great sphere and $Z = S(W)/G$ for some G-invariant subspace W. Finally, there exists an isometry $I : V/G \to V'/G'$ and the statement follows. □

Note that we have just shown that for a representation of abstract copolarity k, each reduction to an invariant subspace has abstract copolarity $\leq k$. Another immediate consequence is

COROLLARY 2.2.7. *A reduction of a reducible representation is itself reducible.*

We will now show that, up to orbit-equivalence, it can be read of the quotient if (G, V) is a product representation.

PROPOSITION 2.2.8. *Assume that the quotient V/G is isometric to the quotient of a product representation $(G_1' \times G_2', V_1' \times V_2')$, where $\dim V_i' \neq 0$ for $i = 1, 2$. Then (G, V) is orbit equivalent to a product representation $(G_1 \times G_2, V_1 \times V_2)$, with $\dim V_i \neq 0$, and $V = V_1 \times V_2$.*

PROOF. If (G, V) has fixed points V_0 then $V/G = V_0^\perp/G \times V_0$ and (G, V) is orbit equivalent to $(G \times \{e\}, V_0^\perp \times V_0)$. Therefore, assume that (G, V) is without fixed points and $V/G = V_1'/G_1' \times V_2'/G_2'$. Set $G' = G_1' \times G_2'$, then V_1', V_2' are complementary G'-invariant subspaces and the last proposition implies that there is a G-invariant subspaces $W \subset V$, such that $W/G = V_1'/G'$ and $W^\perp/G = V_2'/G'$. Denote by G_1 the image of G in $O(W)$ and by G_2 the image of G in $O(W^\perp)$. We have to show that the orbits of (G, V) and $(G_1 \times G_2, W \times W^\perp)$ coincide. For $p = (p_1, p_2) \in W \times W^\perp$ we clearly have $G(p_1, p_2) \subset G_1 p_1 \times G_2 p_2$. On the other hand the quotient $V/G = W/G_1 \times W^\perp/G_2$ is a direct product. Hence, for $G(p_1, p_2)$ there exists points $p \in W$ and $q \in W^\perp$, such that $G(p_1, p_2) = G_1 p \times G_2 q$. Then

$$G_1 \times G_2(p, q) = G(p_1, p_2) \subset G_1 \times G_2(p_1, p_2).$$

Two orbits of an isometric action, with non-empty intersections coincide and we conclude $G(p_1, p_2) = G_1 \times G_2(p, q) = G_1 \times G_2(p_1, p_2)$. □

The dimension of $G_1 \times G_2$ is in general much bigger than the dimension of G. For example consider the representation $(\mathrm{Spin}(8), \rho_8 \oplus \Delta_8^+)$, where ρ_8 is the induced standard $SO(8)$-representation and Δ_8 is a half-spin-representation. A principal orbit of $\rho_8 \oplus \Delta_8^+$ equals $S^7 \times S^7$ (cf. [**4**]), which is also a principal orbit of $(SO(8) \times SO(8), 2\rho_8)$. Since their orbits coincide on the dense set of principal points, the representations $(\mathrm{Spin}(8), \rho_8 \oplus \Delta_8^+)$ and $(SO(8) \times SO(8), 2\rho_8)$ are orbit equivalent.

For a non-reduced representation (G, V) let (G', V') be a minimal reduction, i.e. $\dim G' < \dim G$. We now show that we can also estimate the dimension of the representation spaces, in fact $\dim V' < \dim V$. To see this, assume that the representation (G, V) has trivial principal isotropy group G_p, since otherwise it can be replaced by $(N/Z, V^{G_p})$. The representation (G', V') is a minimal reduction and has therefore trivial principal isotropy groups, too. The definition implies that $\dim V/G = \dim V'/G'$, then

$$\dim V' = \dim G' + \dim V'/G' < \dim G + \dim V/G = \dim V.$$

LEMMA 2.2.9. *For a product representation $(G_1 \times G_2, V_1 \times V_2)$ the abstract copolarity of $(G_1 \times G_2, V_1 \times V_2)$ is the sum of the abstract copolarities of (G_i, V_i), $i = 1, 2$.*

PROOF. The quotient of the product representation splits in

$$V_1 \times V_2/G_1 \times G_2 = V_1/G_1 \times V_2/G_2.$$

The last proposition implies that the minimal reduction (G', V') is orbit equivalent to an product representation $(G_1' \times G_2', V_1' \times V_2')$, such that $V_1' \times V_2' = V'$. Then the factor representations are minimal reductions of (G_i, V_i), $i = 1, 2$, respectively, and $\dim G_1' + \dim G_2' \geq \dim G'$. Assume that $\dim G_1' + \dim G_2' > \dim G'$, then (G', V') is a minimal reduction of $(G_1' \times G_2', V_1' \times V_2')$ and $\dim V' < \dim V_1' + \dim V_2'$, which is a contradiction. □

In general, conjugation with an element $g \in G$ leaves the identity component G_0 invariant, i.e. $c_g : G_0 \to G_0$ is an group isomorphism. Then

$$d(G_0(gp), G_0(gp')) = d(gG_0g^{-1}gp, gG_0g^{-1}(gp')) = d(gG_0p, gG_0p') = d(G_0p, G_0p'),$$

hence the group G/G_0 acts by isometries on V/G_0 through

$$\begin{array}{ccc} G/G_0 \times V/G_0 & \to & V/G_0 \\ (gG_0, G_0p) & \mapsto & G_0(gp). \end{array}$$

The quotient of this action is $(V/G_0)/(G/G_0) = V/G$. We can finally state

PROPOSITION 2.2.10. [**11**] *Let $\rho : G \to V$ be a faithful representation and let G_0 be the identity component of G. If V/G_0 has empty boundary then the representation ρ is reduced.*

PROOF. Assume V/G_0 has empty boundary but the representation (G, V) is not reduced. Since V/G_0 is the cone over $S(V)/G_0$, this quotient does not have boundary, too. The finite group of connected components G/G_0 acts by isometries on $S(V)/G_0$ and the orbit space $(S(V)/G_0)/(G/G_0) = S(V)/G$.
We denote by $Y = S(V)/G_0$ the orbit space of the G_0-action and the G_0-principal stratum with Y_{pr}. Note that the set of G-principal points V_{pr} in V is contained in the set of G_0-principal points V_{pr_0}. The orbit space $S(V)/G_0$ has no strata of quotient-codimension 1, therefore we can find an infinite geodesic $\bar{\gamma} \subset S(V)/G_0$, which is contained in the principal stratum Y_{pr}. Note that $\bar{\gamma}$ is a geodesic of the manifold Y_{pr} in the usual sense. We assume that $\bar{\gamma}(0)$ projects to an G-principal point in $S(V)/G$ and regard a part of $\bar{\gamma}$, which has length π.
Let γ be a horizontal lift of $\bar{\gamma}$. As a submanifold of the unit sphere, the $G_0\gamma(0)$-index of a horizontal geodesic of length π is exactly the dimension of the principal orbit $G_0\gamma(0)$ and coincides with the index of $\bar{\gamma}$. Since the dimension of a G_0-principal orbit equals the dimension of an G-principal orbit we get

$$\dim S(V) = \dim S(V)/G + \dim G_0\gamma(0) = \dim S(V)/G + \mathrm{ind}(\bar{\gamma}).$$

The action of G/G_0 on $S(V)/G_0$ leaves the principal stratum invariant and the geodesic $\bar{\gamma}$ projects to a geodesic γ' in $Y_{pr}/(G/G_0)$. In [**17**] is proven that for an isometric action (G, M) the Gp-index of a horizontal geodesic γ starting in Gp equals

$$\mathrm{ind}(\gamma) = \mathrm{ind}(\bar{\gamma}) + \sum(\dim G - \dim G\gamma(t)),$$

where $\bar{\gamma} \subset M/G$ is the projection of γ, its index $\mathrm{ind}(\bar{\gamma})$ is defined in the appendix. In our special case, the finite group G/G_0 acts on Y_{pr}, hence $\mathrm{ind}(\bar{\gamma}) = \mathrm{ind}(\gamma')$.

Let (G', V') be a minimal reduction of (G, V), then $V'/G' = V/G$ and $\dim V' < \dim V$. Denote by $G'p' \subset S(V')$ the principal G'-orbit, which projects to $\gamma'(0)$ and let μ' be a horizontal lift of γ' starting in $G'p'$, then

$$\mathrm{ind}(\mu') = \mathrm{ind}(\gamma') + \sum(\dim G' - \dim G'\mu(t)).$$

The horizontal geodesic μ' can intersect singular orbit of the G'-action. In those points $\dim G' - \dim G'\mu(t) > 0$.

$$\begin{aligned} \dim S(V') &= \dim S(V')/G' + \dim G'p' \\ &= \dim S(V)/G + \operatorname{ind}(\mu') \\ &\geq \dim S(V)/G + \operatorname{ind}(\gamma') \\ &= \dim S(V)/G + \operatorname{ind}(\bar{\gamma}) = \dim S(V), \end{aligned}$$

contradicting $\dim V' < \dim V$. □

We will know applying the last proposition to our general setting. Therefore, let (H, W) be a non-reduced representation of a connected group and (G, V) a minimal reduction. Since (H, W) is not reduced, the last proposition tells us that $W/H = V/G$ has boundary. But the quotient V/G_0 may have no boundary, hence the group of connected components G/G_0 has to create it. We have seen that the group of connected components G/G_0 act by isometries on V/G_0. An isometry $I : V/G_0 \to V/G_0$ is called a *reflection* if its restriction to the principal part $(V/G_0)_{pr}$ is a reflection, i.e. it fixes a totally geodesic subset of codimension 1.

Proposition 2.2.11. *Let (H, W) be a representation of a connected group H and (G, V) a representation, such that the quotients W/H and V/G are isometric. Then G/G_0 acts on V/G_0 as a reflection group.*

Since G/G_0 is generated by reflections, each generator creates boundary in V/G. Unfortunately, the proof of the above proposition needs the theory of Riemannian orbifolds. We will give a basic introduction to orbifolds and a detailed proof of the above proposition in the appendix.

2.3. Some general facts about representations

In this section we will present some general facts about representations and will fix our notation. For an introduction to representations we refer to [**7**].

Let V be a finite dimensional real vector space, G a compact group and let $\rho : G \to O(V)$ be a faithful representation, i.e. an injective group homomorphism. Two representations $\rho_1 : G \to O(V)$ and $\rho_2 : G \to O(V')$ are called *equivalent* if there exists an G-equivariant isometry $\Phi : V \to V'$, i.e. $\Phi(\rho_1(g)v) = \rho_2(g)\Phi(v)$ for all $v \in V$.

A G-invariant vector subspace $V_i \subset V$ is called *irreducible* if the induced representation $\rho_i = \rho|_{V_i} : G \to O(V_i)$ is an irreducible representation, i.e. V_i has no G-invariant subspaces beside $\{0\}$ and V_i itself. Since G is compact, the vector space V decomposes into irreducible vector subspaces $V = V_1 \oplus \ldots \oplus V_n$ and we say that the representation is *completely reducible*. We define $\rho_1 + \ldots + \rho_n : G \times \ldots \times G \to V_1 \oplus \ldots \oplus V_n$ by

$$\rho_1 + \ldots + \rho_n(g_1, \ldots, g_n)(v_1, \ldots, v_n) = (\rho_1(g_1)v_1, \ldots, \rho_n(g_n)v_n)$$

and let $\rho_1 \oplus \ldots \oplus \rho_n = \rho_1 + \ldots + \rho_n|_{\Delta G}$ denote the restriction of $\rho_1 + \ldots + \rho_n : G \times \ldots \times G \to V_1 \oplus \ldots \oplus V_n$ to the diagonal ΔG, then $\rho = \rho_1 \oplus \ldots \oplus \rho_n$. Unfortunately, this decomposition is not unique.

If V is a complex vector spaces, a homomorphism $\rho : G \to U(V)$ from G into the unitary group is called a *complex representation*. Note that the unitary group $U(n)$ is a subgroup of $O(2n)$, hence any complex representation induces a real representation.

Fix an irreducible complex representation $\phi : G \to U(W)$ and let $\rho : G \to U(V)$ be a complex representation. Denote by $\mathrm{Hom}^{\mathbb{C}}_G(W, V)$ the set of all G-equivariant complex linear maps $\varphi : W \to V$. The image of the map

$$\begin{array}{rcl} \mathrm{Hom}^{\mathbb{C}}_G(W, V) \otimes W & \to & V \\ (\varphi \otimes w) & \mapsto & \varphi(w) \end{array}$$

is called an *isotypical component* V_ϕ of V. The induced representation on each irreducible subspace of V_ϕ is complex equivalent to $\phi : G \to U(W)$. The Lemma of Schur implies that the decomposition of V into isotypical components is unique.

It is known, that a real irreducible representation of a torus T^n is either trivial or the induced representation of an irreducible complex representation $\phi : T^n \to U(1)$, called a *weight* (cf. [**7**]). Therefore, we identify complex and real irreducible representations of a torus and call them weights. Let $\rho : T^n \to O(V)$ be a representation and denote by V_0 the set of fixed points. For a weight $\phi : T^n \to S^1$ we call $V^i = V_\phi \oplus V_{\bar{\phi}}$ the *(real) isotypical component of* ϕ. The decomposition $V = V^1 \oplus \ldots \oplus V^m \oplus V_0$ into isotypical components is unique.

For a representation of abstract copolarity 1, or abstract copolarity 2 the identity component of a minimal reduction is either a circle or a two torus. Therefore, we will now have a closer look on their representations.

Circle representation. The weights of the circle group S^1 equal

$$\phi : S^1 \to U(1),\ \lambda \mapsto \lambda^k,\ \text{for } k \in \mathbb{Z} - \{0\}.$$

The induced action on $\mathbb{C}$ is given by

$$S^1 \times \mathbb{C} \to \mathbb{C}\ ,\ (\lambda, z) \mapsto \lambda^k z.$$

Example. Let $\rho = \rho_1 \oplus \rho_2 \oplus \rho_3 : S^1 \to SO(V)$ be a representation, such that the irreducible representations ρ_1, ρ_2, ρ_3 are equivalent to the weights $\phi_1(\lambda) = \lambda, \phi_2(\lambda) = \bar{\lambda} = \bar{\phi}_1(\lambda)$ and $\phi_3(\lambda) = \lambda^3$, respectively. Then ρ is equivalent to

$$\begin{array}{rcl} S^1 \times \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} & \to & \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \\ (\lambda, (z_1, z_2, z_3)) & \mapsto & \left(\lambda z_1, \bar{\lambda} z_2, \lambda^3 z_3\right). \end{array}$$

Since complex conjugation is $\mathbb{R}$-linear and $\overline{\phi(\lambda)z} = \overline{\lambda^k z} = \bar{\lambda}^k \bar{z} = \bar{\phi}(\lambda)\bar{z}$, the weights ϕ and $\bar{\phi}$ are equivalent as real representations. Therefore, given a real isotypical component $V_\phi \oplus V_{\bar{\phi}}$ of a representation $\rho : S^1 \to SO(V)$, each restricting to an irreducible subspace is equivalent to the weight ϕ and we will write $V^\rho_\phi = V_\phi \oplus V_{\bar{\phi}}$. In the above example the isotypical components of V are $V^\rho_{\phi_1} \simeq \mathbb{C} \oplus \mathbb{C}$ and $V^\rho_{\phi_3} \simeq \mathbb{C}$.

Representation of the torus T^2. Let $\rho : T^2 \to SO(V)$ be a faithful representation of the 2-torus, such that the induced action (T^2, V) is without fixed points. Since T^2 is compact each representation is completely reducible.

A weight $\phi : T^2 \to S^1$ induces a linear functional $\mathrm{d}\phi = \mathfrak{t}^2 \to \mathbb{R}$ called an *infinitesimal weight*. We identify $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the Lie algebra $\mathfrak{t}^2 = \mathbb{R}^2$ with the Euclidean plane. Then the exponential map $\mathrm{Exp} : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ can be identified with the projection and the following diagram commutes

$$\begin{tikzcd} \mathbb{Z}^2 \arrow[r, hook, "i"] \arrow[d, "\mathrm{d}\phi|_{\mathbb{Z}^2}"] & \mathbb{R}^2 \arrow[r, "\mathrm{Exp}"] \arrow[d, "\mathrm{d}\phi"] & T^2 \arrow[d, "\phi"] \\ \mathbb{Z} \arrow[r, hook, "i"] & \mathbb{R} \arrow[r, "e^{2\pi i}"] & S^1 \end{tikzcd}$$

An element $x \in \mathbb{R}^2$ is mapped to $\mathrm{d}\phi(x) = \langle \alpha, x \rangle$ for some $\alpha \in \mathbb{Z}^2$. Therefore, each irreducible representation of T^2 is given by $\phi([x]) = e^{2\pi i \langle \alpha, x \rangle}$ for some $\alpha \in \mathbb{Z}^2$. If a weight $\phi' : T^2 \to S^1$ equals $\phi'([x]) = \phi([x])^m$ for all $[x] \in T^2$ we will write $\phi' = \phi^m$.

Example. We identify $\mathbb{R}^2/\mathbb{Z}^2 \to S^1 \times S^1, ([x_1, x_2]) \mapsto (e^{2\pi i x_1}, e^{2\pi i x_2})$ and let $\phi_1 : T^2 \to S^1, (\lambda, \mu) \mapsto \lambda\mu$ and $\phi_2 : T^2 \to S^1, (\lambda, \mu) \mapsto \lambda$ be weights. The representation $\phi = \phi_1 \oplus \phi_1^3 \oplus \phi_2 : T^2 \to SO(\mathbb{C}^3)$ induces the following T^2-action

$$\begin{array}{rcl} T^2 \times \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} & \to & \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \\ ((\lambda, \mu), (z_1, z_2, z_3)) & \mapsto & (\lambda\mu z_1, \lambda^3\mu^3 z_2, \lambda z_3). \end{array}$$

In this example the isotypical components are the irreducible subspaces $\mathbb{C}$.

The weights $\phi : T^2 \to S^1$ are non-trivial homomorphisms between Lie groups. Their kernels are 1-dimensional Lie subgroups of T^2 and $(\ker\phi)_0 \simeq S^1$, where $(\ker\phi)_0$ denotes the identity component of $\ker\phi$. The Lie algebra $\mathfrak{s} = \ker \mathrm{d}\phi$ of $(\ker\phi)_0$ is a line through the origin in $\mathfrak{t}^2$ of rational slope. For a representation $\rho : T^2 \to SO(V)$, we will say that a line $\mathfrak{s} \subset \mathfrak{t}^2$ is induced by ρ if there exists an isotypical component $V^\rho_\phi \neq \{0\}$, such that $\ker \mathrm{d}\phi = \mathfrak{s}$. Let $V = V^1 \oplus \ldots \oplus V^m$ be the decomposition of V into isotypical components and denote by $\rho^i : T^2 \to SO(V^i)$ the restriction to V^i. Since we assume that ρ is faithful, i.e. $\bigcap_{i=1}^m \ker \rho^i = \{e\}$, there are at least to lines $\mathfrak{s}_1, \mathfrak{s}_2$ in $\mathfrak{t}^2$ induced by ρ. The decomposition of V into isotypical components is unique, hence the lines in $\mathfrak{t}^2$ induced by ρ are unique, too. Unfortunately, we cannot read of the number of isotypical components from the number of lines in $\mathfrak{t}^2$, since two weights ϕ and ϕ', which are powers of each other, i.e. $\phi' = \phi^m$, induce the same line in $\mathfrak{t}^2$.

LEMMA 2.3.1. *For every line $\mathfrak{s} \subset \mathfrak{t}^2$ with rational slope there exists a weight $\phi : T^2 \to S^1$, such that $\mathrm{d}\phi(\mathfrak{s}) = 0$. Furthermore, we can choose ϕ normed in the sense that for each weight $\phi' : T^2 \to S^1$ with $\ker \mathrm{d}\phi' = \mathfrak{s}$ there exists $m \in \mathbb{Z}$ such that $\phi' = \phi^m$.*

PROOF. For each line $\mathfrak{s} \subset \mathfrak{t}^2$ with rational slope, there exists minimal positiv integers $\alpha_1, \alpha_2 \in \mathbb{N}$, such that $\mathfrak{s} = \{x \in \mathfrak{t}^2 \mid \langle \alpha, x \rangle = 0\}$ with $\alpha = (\alpha_1, \alpha_2)$. Then $\phi([x]) = e^{2\pi i \langle \alpha, x \rangle}$. Let $\phi' : T^2 \to S^1$ be a weight with $\ker \mathrm{d}\phi' = \mathfrak{s}$. Then $\phi'([x]) = e^{2\pi i \langle \alpha', x \rangle}$ for some $\alpha' \in \mathbb{Z}^2$. Therefore, $\langle \alpha', x \rangle = 0 = \langle \alpha, x \rangle$ and $\alpha' = m\alpha$ for some $m \in \mathbb{Q}$. Since α was chosen minimal $m \in \mathbb{Z}$ and

$$\phi'([x]) = e^{2\pi i \langle \alpha', x \rangle} = e^{2\pi i \langle m\alpha, x \rangle} = \phi([x])^m.$$

$\square$

In the next section we are interested in the quotient space V/T^2 of a 2-torus representation, especially if it splits or not. Let $\rho : T^2 \to SO(V)$ be a representation and assume that there are two invariant subspaces $\tilde{V}^1, \tilde{V}^2 \subset V$, such that ρ can be written as a sum $\rho = \psi_1 + \psi_2$ of two, not necessarily irreducible, S^1-representations $\psi_i : S^1 \to SO(\tilde{V}^i)$ for $i = 1, 2$. Then the quotient $V/T^2 = \tilde{V}^1/S^1 \times \tilde{V}^2/S^1$ splits.

PROPOSITION 2.3.2. *Let $\rho : T^2 \to SO(V)$ be a faithful representation without fixed points and assume that there are exact two lines $\mathfrak{s}_1, \mathfrak{s}_2$ in $\mathfrak{t}^2$ induced by ρ. Then the quotient V/T^2 splits.*

PROOF. Assume that there are exactly two lines $\mathfrak{s}_1, \mathfrak{s}_2$ in $\mathfrak{t}^2$ induced by ρ. Then each isotypical component $V_\phi^\rho \subset V$ either belongs to $\mathfrak{s}_1$ or $\mathfrak{s}_2$. Let ϕ be the minimal weight in the sense of the last lemma, such that $\ker \mathrm{d}\phi = \mathfrak{s}_1$ and let ϕ' be the minimal weight, such that $\ker \mathrm{d}\phi' = \mathfrak{s}_2$. We define

$$\tilde{V}^1 = \bigoplus_{m \in \mathbb{Z}} V_{\phi^m}^\rho \text{ and } \tilde{V}^2 = \bigoplus_{n \in \mathbb{Z}} V_{(\phi')^n}^\rho.$$

Then $\tilde{V}^i$ contains all isotypical components of V, which correspond to $\mathfrak{s}_i$ for $i = 1, 2$, respectively. In fact, $\tilde{V}^1$ and $\tilde{V}^2$ are T^2-invariant subspaces, such that $V = \tilde{V}^1 \oplus \tilde{V}^2$. Let $\tilde{\rho}_i : T^2 \to SO(\tilde{V}^i)$ denote the restriction of ρ to $\tilde{V}^i$, then $\tilde{\rho}_i$ is not faithful, i.e. $(\ker \tilde{\rho}_1)_0 = (\ker \phi)_0 = \mathrm{Exp}(\mathfrak{s}_1) =: S_a^1$ and $(\ker \tilde{\rho}_2)_0 = (\ker \phi')_0 = \mathrm{Exp}(\mathfrak{s}_2) =: S_b^1$. The lines $\mathfrak{s}_1, \mathfrak{s}_2$ span $\mathfrak{t}^2$ and since the exponential map is surjective, each element $t \in T^2$ can be written as a product $t = t_b \cdot t_a$ with $t_a, t_b \in S_a^1, S_b^1$, respectively. Then $\rho(t) = \rho(t_b t_a) = \rho(t_b)\rho(t_a)$ and for $(v_1, v_2) \in \tilde{V}^1 \oplus \tilde{V}^2$ we get

$$\rho(t_b)\rho(t_a)(v_1, v_2) = (\tilde{\rho}_1(t_b)v_1, \tilde{\rho}_2(t_a)v_2).$$

Therefore, $\rho = \tilde{\rho}_1 + \tilde{\rho}_2 : S_b^1 \times S_a^1 \to SO(\tilde{V}^1 \oplus \tilde{V}^2)$ and the quotient $V/T^2 = \tilde{V}^1/S_b^1 \times \tilde{V}^2/S_a^1$ splits. $\square$

A Lie group automorphism $\omega : T^2 \to T^2$ induces a linear map $\mathrm{d}\omega : \mathfrak{t}^2 \to \mathfrak{t}^2$ preserving $\mathbb{Z}^2$. Then $\mathrm{d}\omega \in GL(2,\mathbb{Z})$ and each automorphism $\omega : T^2 \to T^2$ is given by

$$\begin{array}{rccc} \omega : & S^1 \times S^1 & \to & S^1 \times S^1 \\ & (\lambda,\mu) & \mapsto & (\lambda^a \mu^b, \lambda^c \mu^d), \end{array}$$

with $a,b,c,d \in \mathbb{Z}$ and $ad - bc \neq 0$.

LEMMA 2.3.3. *If* $\mathrm{d}\omega : \mathfrak{t}^2 \to \mathfrak{t}^2$ *is an involution, then* $\mathrm{d}\omega$ *is diagonalizable with eigenvalues* ± 1.

PROOF. Let $v \in \mathfrak{t}^2$, then $\mathrm{d}\omega(v \pm \mathrm{d}\omega(v)) = \mathrm{d}\omega(v) \pm v = \pm(v \pm \mathrm{d}\omega(v))$ and $v = \frac{1}{2}(v + \mathrm{d}\omega(v)) + \frac{1}{2}(v - \mathrm{d}\omega(v))$ can be written as the sum of eigenvectors. □

2.4. Minimal reductions

Let us start by proving the equivalence between polar representations and representations of abstract copolarity 0, mentioned in the introduction.

PROPOSITION 2.4.1. *Let* (G,V) *be a representation of abstract copolarity* 0*, then* (G,V) *is polar.*

PROOF. Let (Γ, V') be a minimal reduction of (G,V). Then Γ is a finite group and $V/G = V'/\Gamma = X$. The representation (G,V) is polar if and only if the horizontal distribution over the principal stratum V_{pr} is integrable. The foliation induced by the G-action over the principal stratum is given by the Riemannian submersion $V_{pr} \to X_{pr}$, where X_{pr} is the principal stratum of the quotient $V/G = X$. Since X_{pr} is isometric to the principal stratum of V'/Γ, their curvatures coincide and are equal to zero. Therefore, the horizontal sectional curvatures of the total space V_{pr} and the corresponding sectional curvatures of the base space X_{pr} of the submersion coincide. By the O'Neill formula the sectional curvatures of the base space of a submersion are bigger or equal to the horizontal sectional curvatures of the total space, and equality holds if and only if the O'Neill tensor vanishes. This is equivalent to the integrability of the horizontal distribution, hence (G,V) is polar. □

A polar representation has a reduction to a finite group, hence its abstract copolarity equals 0. In the contexts of abstract copolarity we are, of course, interested in minimal reductions. In this section we will discuss the minimal reductions of a representation of abstract copolarity 1, or abstract copolarity 2, respectively. For the following results we refer to [**11**]. Since the set of fixed points of a principal isotropy group gives rise to a reduction, a minimal reduction has trivial principal isotropy groups. This has some important consequences.

LEMMA 2.4.2. *Let* (G,V) *be a representation such that* G *acts with trivial principal isotropy groups. Then the action of* G/G_0 *on* V/G_0 *is effective.*

PROOF. The action of G/G_0 on V/G_0 is effective, if we can show that an element $[\omega] \in G/G_0$ which acts trivially on V/G_0 is represented by an element $\omega \in G_0$. So take $[\omega] \in G/G_0$ and assume $[\omega]\bar{p} = \bar{p}$ for all $\bar{p} \in V/G_0$. Let $p \in V$ be a G-principal point which projects to $\bar{p}$ and assume $\omega \in G$ represents $[\omega]$. Then $[\omega]\bar{p} = \bar{p}$ is equivalent to $\omega(G_0 p) = G_0 p$. Since ω normalizes G_0 this means $\omega p \in G_0 p$, hence

$\omega p = g_0 p$ for some $g_0 \in G_0$. Then $g_0^{-1}\omega \in G_p = \{e\}$, since p was chosen G-principal, and $\omega \in G_0$. □

Another application is the next lemma, which follows directly from the Slice Theorem.

LEMMA 2.4.3. *Assume that the principal isotropy group is trivial. Then each slice representation is faithful and has itself trivial principal isotropy groups.*

PROOF. Let q be a point in V and assume that $g_q \in G_q$ acts trivially on the normal space $\nu_q Gq$. Let v be a principal vector of the slice representation, then $\exp_q(v) = p$ is a G-principal and the Slice Theorem implies $(G_q)_v = G_p = \{e\}$, hence $g_q = e$. Therefore, the slice representation is faithful. The second statement follows directly from the above equality of the isotropy groups. □

For a non-reduced representation of a connected group (H, W) the quotient W/H has boundary. In fact, the quotient of each minimal reduction (G, V) has boundary.

PROPOSITION 2.4.4. *Let (S^1, V) be a faithful representation with trivial principal isotropy group. Then the quotient V/S^1 has boundary if and only if (S^1, V) is polar.*

PROOF. Assume first that (S^1, V) is polar. Then (S^1, V) is not reduced and V/S^1 has boundary (cf. Proposition 2.2.10). On the other hand assume that V/S^1 has boundary. We can ignore the S^1-fixed points, since they do not influence the polarity of (S^1, V). Let $p \in V$ be a point which projects to a stratum of quotient-codimension 1 in V/S^1. Denote by S^1_p the isotropy group at p, then S^1_p acts transitive on the unit sphere in $\mathcal{H}_p^+$, the normal space to the stratum of p. As we have seen the slice representation has trivial principal isotropy groups and we conclude that $S^1_p \simeq S(\mathcal{H}_p^+) = S^a$. For dimension reasons we get $a = 0, 1$. First assume that $S^1_p = S^0 = \mathbb{Z}/2\mathbb{Z}$. Then $S(\mathcal{H}_p^+) = S^0$ and the stratum $\mathrm{Str}(p)$ has codimension 1. Let $\omega \in S^1$ be the generator of the isotropy group S^1_p, i.e. $\langle\omega\rangle = S^1_p$. Since S^1 is abelian, the generator ω commutes with the whole group S^1. The dimension formula (1) (cf. page 7)

$$\begin{aligned} \dim \mathrm{Str}(p) &= \dim S^1 + \dim V^{S^1_p} - \dim N_{S^1}(S^1_p) \\ &= \dim S^1 + \dim V^{\omega} - \dim Z_{S^1}(\omega) \end{aligned}$$

implies that $\dim V^\omega = \dim V - 1$ and therefore ω is a reflection. This contradicts the fact that $\omega \in S^1 \subset SO(V)$. Now assume that $S^1_p = S^1$. Since we excluded fixed points $p = 0$ and $\mathcal{H}_p^+ = V$. The transitivity of S^1_p on the unit sphere in $\mathcal{H}_p^+$ implies $\dim V = 2$ and the representation (S^1, V) is polar. □

PROPOSITION 2.4.5. *Let (T^2, V) be a faithful representation with trivial principal isotropy group. If the quotient V/T^2 has boundary, then (T^2, V) admits a generalized section.*

PROOF. Recall that the set of T^2-fixed points is the maximal Euclidean subset of V/T^2 and splits of in the quotient. Therefore, it neither influence the existence of boundary, nor the existence of reductions and we will ignore it. Assume that V/T^2

has boundary and let $p \in V$ be a point which projects to a stratum of quotient-codimension 1. With the same argumentation as in the proof of the last proposition the triviality of the principal isotropy groups implies that $T_p^2 = S(\mathcal{H}_p^+) = S^a$. The unit sphere is a group for $a = 0, 1, 3$ and since $\dim T_p^2$ is abelian we conclude $a = 0, 1$. The case $T_p^2 = S^0$ can be excluded by the same arguments as before. Therefore, $T_p^2 = S^1$, the stratum $\operatorname{Str}(p)$ has codimension 2 and with formula (1), using the commutativity of T^2, we get

$$\dim V^{T_p^2} = \dim V - 2.$$

Furthermore, the commutativity of T^2 implies that the set of fixed points $V^{T_p^2}$ of the isotropy group T_p^2 is T^2-invariant. We have a decomposition

$$V = V^{T_p^2} \oplus \bar{V}$$

into T^2-invariant subspaces, where $\bar{V}$ denotes the 2-dimensional orthogonal complement of $V^{T_p^2}$. We write $\rho = \tilde{\rho} \oplus \bar{\rho}$ for the representation (T^2, V), respecting the above decomposition. Following the considerations of Section 2.3, for each irreducible representation $\tilde{\rho}_i$ of $\tilde{\rho}$ the identity component $(\ker \tilde{\rho}_i)_0 = T_p^2$. Therefore, $\tilde{\rho}$ induces only one line $\mathfrak{s}_1$ in $\mathfrak{t}^2$. Since the representation is faithful $(\ker \bar{\rho})_0 \neq T_p^2$ and $\bar{\rho}$ induces a line $\mathfrak{s}_2 \neq \mathfrak{s}_1$. Hence, there are exact two lines in $\mathfrak{t}^2$ induced by ρ and Proposition 2.3.2 implies that V/T^2 splits into $V/T^2 = V^{T_p^2}/S^1 \times \bar{V}/S^1$. But $(S^1, \bar{V})$ is polar, hence (T^2, V) is polar, too, or has 1-sections. □

In the following, assume that (G, V) is a minimal reduction of a non-reduced representation (H, W), where H is connected, then V/G has boundary. Of course, (G, V) does not admit generalized sections. Therefore, the last propositions imply that for $\dim G = k \in \{1, 2\}$ the group G cannot be connected and that the induced representation (T^k, V) of its toric identity component is reduced. The group of connected components G/T^k is a non-trivial reflection group acting on V/T^k, creating the boundary of V/G. To any reflection $[\omega] \in G/T^k$ there exists a regular point $x \in V/T^k$, which is fixed under $[\omega]$ and projects to a stratum of quotient-codimension 1 in V/G. Let $p \in V$ be a lift of x, then the isotropy group G_p acts transitive on the unit sphere in the space $\mathcal{H}_p^+$ normal to the stratum $\operatorname{Str}(p)$ through p. Since p is a T^k-principal point, the slice representation of T^k at p is trivial. Therefore, a principal orbit of $(G_p, \nu_p(Gp))$ is finite and $G_p = S(\mathcal{H}_p^+) = S^0$. Let $\omega' = \omega g_0$ denote the lift of $[\omega]$ fixing p, then $G_p = \langle \omega' \rangle$ and ω' is unique. Furthermore, $\dim \mathcal{H}_p^+ = 1$ and the discussion in Chapter 1 implies that the dimension of the stratum through p equals $\dim \operatorname{Str}(p) = \dim G \cdot V^\omega = \dim V^\omega + \dim G - \dim N_G(\omega)$, where $N_G(\omega)$ is the normalizer of ω in G. For a single element ω its normalizer and centralizer coincide, $N_G(\omega) = Z_G(\omega)$, and we get a new dimension formula

$$\dim V^\omega + \dim G - \dim Z_G(\omega) = \dim V - 1. \tag{4}$$

Corollary 2.4.6. *Let (H, W) be non-reduced representation of a connected group H with abstract copolarity 1 or abstract copolarity 2. For each minimal reduction (G, V) the induced representation (T^k, V) of the toric identity component is reduced. Furthermore, G/T^k is a non-trivial reflection group. To each reflection $[\omega] \in G/T^k$, there exist a lift ω, which fulfills the dimension formula (4).*

The image of the representation

$$\begin{array}{rcl}\Phi : G/T^k & \to & \mathrm{Aut}(\mathfrak{t}^k)\\ [g] & \mapsto & \mathrm{Ad}_g\end{array}$$

is a finite group generated by involutions. Each reflection $[\omega]$ in G/T^k has a lift ω which is an involution and fulfills the above dimension formula. Therefore, $[w] \in Z_G(T^k)/T^k$ if and only if ω is a reflection, i.e. $\operatorname{codim} V^\omega = 1$. We will show that Φ is not trivial, i.e. $G \neq Z_G(T^k)$. For $k = 1$ the image $\Phi(G/S^1) = \mathbb{Z}/2\mathbb{Z}$. For $k = 2$, the group $\Phi(G/T^2)$ interchanges the lines in $\mathfrak{t}^2$, which are induced by (T^2, V). In fact, it acts on the set of weights corresponding to (T^2, V). We show that $\Phi(G/T^2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

To prove Theorem II and Theorem III, we have to understand the correspondence between the action of a reflection $[\omega]$ on $\mathfrak{t}^2$ and the codimension of the set of fixed points V^ω of a lift ω as above. The details are discussed in the next to sections. We here summarize some elementary statements needed in both proofs.

Let $\rho : T^k \to SO(V)$, $k = 1, 2$, be the representation induced by restricting a minimal reduction $\rho : G \to O(V)$ to its identity component. We decompose V into T^k-isotypical components $V^i = V^\rho_\phi$ and denote the set of fixed points by V_0. Then $V = V^1 \oplus \ldots \oplus V^m \oplus V_0$ and we write $V_* = V^1 \oplus \ldots \oplus V^m$ for the non trivial part. Let $c_g : T^k \to T^k$ denote the conjugation with an element $g \in G$. As before, all representations are faithful and we identify G with its image in $O(V)$ and write g instead of $\rho(g)$.

Lemma 2.4.7. *An involution $g \in G$ permutes the T^k-isotypical components. In fact $g(V^\rho_\phi) = V^\rho_{\phi\circ c_g}$ and g leaves V_0 invariant.*

Proof. First let $v_0 \in V_0$, then for $t \in T^2$ we get

$$\rho(t)gv_0 = g^2\rho(t)gv_0 = g\rho(c_g(t))v_0 = gv_0.$$

Therefore, $gv_0 \in V_0$ and for dimension reason $g(V_0) = V_0$. Let $\phi : T^k \to S^1$ be a weight and identify $\mathbb{R}^2 = \mathbb{C}$. Denote by $\mathrm{Hom}_{T^k}(\mathbb{C}_\phi, V_\rho)$ the set of T^k-equivariant (real) homomorphisms $\varphi : \mathbb{C} \to V$, i.e. for $\varphi \in \mathrm{Hom}_{T^k}(\mathbb{C}_\phi, V_\rho)$

$$\varphi(\phi(t)z) = \rho(t)\varphi(z), \text{ for all } z \in \mathbb{C}.$$

Then the isotypical component V^ρ_ϕ in V corresponding to ϕ is the image of

$$\begin{array}{rcl}\mathrm{Hom}_{T^k}(\mathbb{C}_\phi, V_\rho) \otimes \mathbb{C} & \to & V,\\ (\varphi \otimes z) & \mapsto & \varphi(z).\end{array}$$

For $\varphi \in \mathrm{Hom}_{T^k}(\mathbb{C}_\phi, V_\rho)$ and $z \in \mathbb{C}$ we get that

$$\begin{array}{rcl}(g \circ \varphi)(\phi \circ c_g(t)z) & = & g(\varphi(\phi(c_g t)z))\\ & = & g(\rho(c_g t)\varphi(z))\\ & = & g^2\rho(t)g\varphi(z)\\ & = & \rho(t)(g \circ \varphi)(z),\end{array}$$

i.e. $g \circ \varphi \in \mathrm{Hom}_{T^k}(\mathbb{C}_{\phi\circ c_g}, V_\rho)$ and since g in an involution the following map is an isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{T^k}(\mathbb{C}_\phi, V_\rho) & \to & \mathrm{Hom}_{T^k}(\mathbb{C}_{\phi\circ c_g}, V_\rho) \\ \varphi & \mapsto & g \circ \varphi, \end{array}$$

Therefore, $g \circ \varphi(z) \in V^\rho_{\phi\circ c_g}$ and $g \circ \psi(z) \in V^\rho_\phi$ and in fact $g(V^\rho_\phi) = V^\rho_{\phi\circ c_g}$. □

From the last lemma we immediately get that an involution $g \in G$ leaves the decomposition $V = V_* \oplus V_0$ invariant. In general g maps T^k-invariant subspaces to T^k-invariant subspaces of the same dimension. To see this let W be a T^k-invariant subspace of V, then for $w \in W$ we get

$$\rho(t)g(w) = g^2\rho(t)g(w) = g\left(\rho(c_g(t))w\right) \in g(W) \text{ for all } t \in T^k.$$

Therefore, $g(W)$ is T^k-invariant and, since g is one to one, of the same dimension as W. Assume that W is an irreducible subspace of V, then its image $g(W)$ is irreducible, too. To see this, assume that $W' \subset g(W)$ is a T^k-invariant subspace. Since g is an involution $g(W')$ is an T^k-invariant subspace of W, hence $g(W')$ equals $\{0\}$, or W. Then W' equals $\{0\}$, or $g(W)$. Note that this implies $W \cap g(W) = \{0\}$ or $W \cap g(W) = W$.

LEMMA 2.4.8. [**11**] *Let (G, V) be a minimal reduction with $\dim G = k \in \{1, 2\}$ and let W and $\bar{W}$ be T^k-invariant subspaces with $W \cap \bar{W} = \{0\}$. Let $\omega \in G$ denote the appropriate involutive lift of a reflection $[\omega] \in G/T^2$, which fulfills the dimension formula (4). Assume that $\omega(W) = \bar{W}$, then the action of T^k on $W, \bar{W}$ is of cohomogeneity 1, respectively.*

PROOF. Since $G_p = \{\omega\}$ for some $p \in V^\omega$, the stratum through p has codimension 1. From the discussions in Chapter 1 we know that the set $G \cdot V^\omega$ has codimension 1. Since $G \cdot V^\omega$ locally coincide with the set $T^k \cdot V^\omega$, their dimensions are the same. Now $(W \oplus \bar{W}) \oplus (W \oplus \bar{W})^\perp$ is a T^k-invariant decomposition of V, which is also invariant under ω. Therefore, $V^\omega = (V^\omega \cap (W \oplus \bar{W})) \oplus (V^\omega \cap (W \oplus \bar{W})^\perp)$ and the codimension of $T^k \cdot (V^\omega \cap (W \oplus \bar{W}))$ in $W \oplus \bar{W}$ is less or equal 1. The intersection $V^\omega \cap (W \oplus \bar{W}) = F$ equals the set $\{w + \omega(w) \,|\, w \in W\}$. We will show that $T^k \cdot F \neq W \oplus \bar{W}$. Therefore, let $w, w' \in W$ with $w \neq 0$ and $t \in T^k$, then $w = t(w' + \omega(w')) \in T^k \cdot F$ implies that $w - tw' = t\omega(w') \in W \cap \bar{W}$. Therefore, $\omega(w') = 0$, hence $w' = 0$ and this implies $w = 0$, contradicting our assumption. We conclude that $w \notin T^k \cdot F$ and $T^k \cdot F$ has codimension 1 in $W \oplus \bar{W}$.
Let Δ be the subset $v + u \in W \oplus \bar{W}$ with $|v| = |u|$, then $T^k \cdot F$ is contained in Δ. Now, up to the origin, both sets are locally submanifolds of $W \oplus \bar{W}$. Since $\mathrm{codim}\,(T^k \cdot F) = 1$ and $\Delta \neq W \oplus \bar{W}$ they have the same dimension and $T^k \cdot F$ contains a neighborhood of Δ. Let v be a unit vector in W and $u \in U$, where U is a neighborhood in the unit sphere $S(\bar{W})$. Then $v + U$ is a neighborhood in Δ and we can choose U, such that $v + u \in T^k \cdot F$ for each element $u \in U$. This implies that there exists elements $w' \in W$ and $t \in T^k$, such that $v + u = t(w' + \omega(w'))$. Then $v + u = tw' + t\omega(w')$ and in particular we get $v = tw', u = t\omega(w')$. Then $u = t\omega t^{-1}v$, thus $\omega u = (\omega t \omega)t^{-1}v = c_\omega(t)t^{-1}v$. Since ω is an isometry $\omega(U)$ is an open set in the sphere $S(W)$. Furthermore, $c_\omega(t) \in T^k$, hence the orbit $T^k v$ contains an open set of $S(W)$. If $\dim W \geq 2$, then T^k is transitive on the sphere. For $\dim W = 1$ the statement is clear. □

2.5. Representation of abstract copolarity one

Let (H, W) be a representation of a connected compact group H of abstract copolarity 1. We mentioned in the introduction to this chapter, that one can construct new representations of abstract copolarity 1 by multiplying a polar representation (H', W') to (H, W). The quotient of such a product representation equals $W \times W'/H \times H' = W/H \times W'/H'$, and we say that the quotient splits. In this section we are interested in representation of abstract copolarity 1 with non-splitting quotient.

Let (G, V) be a minimal reduction of (H, W), i.e. $\dim G = 1$. We proved in the previous sections that (G, V) has trivial principal isotropy groups and that the quotient space $W/H = V/G$ has boundary. Furthermore, we have shown that G/S^1 is a non-trivial reflection group. The representation $\rho : G \to O(V)$ is assumed to be faithful and we identify G with its image in $O(V)$. The discussion in Section 2.4 implies that each reflections $[\omega] \in G/S^1$ has a lift ω, such that its set of fixed points V^ω fulfills the dimension formula

$$\dim V^\omega + \dim G - \dim Z_G(\omega) = \dim V - 1.$$

Since $Z_G(\omega)$ is a subgroup of the 1-dimensional group G, $\dim Z_G(\omega) \le 1$. Therefore, the dimension formula implies that the codimension of V^ω is either 1 or 2, i.e. ω acts as reflection or pseudo-reflection on V. Furthermore, conjugation with $\omega \in G$ induces a group isomorphism $c_\omega : S^1 \to S^1$, hence $c_\omega(\lambda) = \lambda^{\pm 1}$. On the one hand $\dim Z_G(\omega)$ determines the isomorphism c_ω, since $Z_G(\omega)$ is its kernel, and on the other hand it describes if the image $\omega \in O(V)$ is a reflection or pseudo-reflection.

The restriction of ρ to the identity component S^1 induces a representation $\rho : S^1 \to SO(V)$. We decompose $V = V^1 \oplus \ldots \oplus V^m \oplus V_0$ into S^1-isotypical components with respect to ρ. Here V_0 is the set of S^1-fixed points and the non-trivial part is denoted by $V_* = V^1 \oplus \ldots \oplus V^m$. Recall that the irreducible S^1-representations are given by $\phi : S^1 \to S^1, \lambda \mapsto \lambda^k$, for some $k \in \mathbb{Z} - \{0\}$. The isotypical component V^i corresponding to a weight ϕ will be denoted by $V^i = V^\rho_\phi$. Then for any irreducible subspace $W' \subset V^\rho_\phi$ the induced S^1-representation is equivalent to ϕ. We have seen that the involution $\omega \in O(V)$ permutes the isotypical components of $\rho : S^1 \to SO(V)$ and leaves the decomposition $V_* \oplus V_0$ invariant, hence we will write $\omega = (\omega_*, \omega_0)$.

First we estimate the dimension of the non-trivial part V_*. The isotypical components in V_* admit a decomposition into S^1-irreducible subspaces, which all have dimension 2. Therefore, the dimension of V_* is even. The dimension $\dim V_* \ge 4$, since otherwise the representation (S^1, V), which has trivial principal isotropy groups, has codimension 1. Then (S^1, V) would be polar, contradicting the minimality of (G, V).

The condition $\operatorname{codim} V^\omega = 1$ is equivalent to $\dim Z_G(\omega) = 1$. A 1-dimensional subgroup of the 1-dimensional group G contains the identity component, hence $S^1 \subset Z_G(\omega)$ and $c_\omega = \mathrm{id}$. Since $\rho : G \to O(V)$ is an injective homomorphism, the image $\omega \in O(V)$ commutes with the induced S^1-action on V.

PROPOSITION 2.5.1. *If ω is a reflection, then $\omega = (\mathrm{id}_*, r_0)$ acts trivially on V_* and is a reflection on V_0.*

PROOF. The set of fixed points $V_*^{\omega_*}$ and $V_0^{\omega_0}$ of the restrictions ω_* and ω_0 fulfill
$$\operatorname{codim} V_*^{\omega_*} + \operatorname{codim} V_0^{\omega_0} = \operatorname{codim} V^{\omega} = 1.$$
Assume that ω_* is not trivial, then ω_* is a reflection. Recall from the last section that for an irreducible subspace $W \subset V_*$
$$W \cap \omega(W) = \{0\} \text{ or } W \cap \omega(W) = W.$$
In the first case W does not contain ω-fixed points and $\operatorname{codim} V_*^{\omega_*} \geq \dim W = 2$. Therefore, if ω_* is a reflection it leaves all irreducible subspaces invariant. Then there exists an irreducible subspace $W \subset V_*$, such that $\omega_* = r$ is a reflection on it and trivial on $W^{\perp}$. The assumption implies that r has to commute with the induced representation (S^1, W). Since (S^1, W) is irreducible it is equivalent to a S^1-representation given by rotations in $\mathbb{R}^2$. But these rotations do not commute with a reflection r in $\mathbb{R}^2$. Therefore, ω_* is trivial and we conclude that $\omega = (\mathrm{id}_*, r_0)$. □

Assume that ω is a pseudo-reflection, i.e. $\operatorname{codim} V^{\omega} = 2$. The dimension formula implies that $\dim Z_G(\omega) = 0$ and therefore $c_\omega(\lambda) = \bar{\lambda}$. Now Lemma 2.4.7 tells us that ω interchanges the isotypical components V_ϕ^ρ and $V_{\phi \circ c_\omega}^\rho$. For $\lambda \in S^1$ we get
$$\phi \circ c_\omega(\lambda) = \phi(\bar{\lambda}) = \overline{\phi(\lambda)} = \bar{\phi}(\lambda).$$
We have seen before that the weights ϕ and $\bar{\phi}$ are equivalent as real representations. In particular, they belong to the same S^1-isotypical component of V, hence ω leaves all isotypical components invariant.

LEMMA 2.5.2. *If ω is a pseudo-reflection it cannot act trivially on any non-trivial S^1-invariant subspaces $W \subset V_*$.*

PROOF. Let ω be a pseudo-reflection, then $c_\omega(\lambda) = \bar{\lambda}$. Assume that ω is trivial on the S^1-invariant subspace $W \subset V_*$. Denote the induced representation on W by ρ_W. Then for $\lambda \in S^1$ and $w \in W$ we get
$$\rho_W(c_\omega(\lambda))w = \rho(c_\omega(\lambda))w = \omega\rho(\lambda)\omega w = \rho_W(\lambda)w,$$
and $c_\omega(\lambda)\bar{\lambda} = \bar{\lambda}^2 \in \ker \rho_W$ for all $\lambda \in S^1$. Hence, $\dim \ker \rho_W = 1$ and therefore (S^1, W) is trivial. Now $V_* \cap V_0 = \{0\}$ implies $W = \{0\}$. □

PROPOSITION 2.5.3. *If ω is a pseudo-reflection. Then $\omega = (\omega_*, \mathrm{id}_0)$ is trivial on V_0 and $\dim V_* = 4$.*

PROOF. The set of fixed points $V_*^{\omega_*}$ and $V_0^{\omega_0}$ of the restrictions ω_*, ω_0 fulfill
$$\operatorname{codim} V_*^{\omega_*} + \operatorname{codim} V_0^{\omega} = 2.$$
The last lemma implies that ω_* cannot be trivial, i.e. $\operatorname{codim} V_*^{\omega_*} \geq 1$. Assume that ω_* is a reflection on V_*. The considerations of the last proposition imply that ω_* leaves all irreducible subspaces $W \subset V_*$ invariant. Then there exists an irreducible subspace $W \subset V_*$, such that ω_* is a reflection on it and trivial on $W^{\perp} \subset V_*$. The last lemma implies that $W^{\perp} = \{0\}$ and $V_* = W$ has dimension 2, contradicting $\dim V_* \geq 4$. Therefore, $\omega = (\omega_*, \mathrm{id}_0)$ and ω_* is a pseudo-reflection. Since ω leaves all isotypical components invariant, denote by $\omega_i = \omega_*|_{V^i}$ the restriction to an

isotypical component V^i. Let $V^{i\omega_i}$ be the set of fixed point of these restriction, then
$$\sum \operatorname{codim} V^{i\omega_i} = 2.$$
The last lemma implies that ω_i cannot be trivial and we conclude that either there is only one isotypical component $V_* = V^1$ and ω_* is a pseudo-reflection on it, or there exist two isotypical components $V_* = V^1 \oplus V^2$ and ω_* is a reflection on both. In the second case, the reflections ω_1 and ω_2 are not trivial on invariant subspaces, too, and we conclude as above that $\dim V^1 = \dim V^2 = 2$, hence $\dim V_* = 4$. If there is only one isotypical component, ω_* can either interchange two irreducible subspaces W and $\omega_*(W)$ in V_* and is trivial on $(W \oplus \omega_*(W))^\perp \subset V_*$, or it leaves all irreducible subspaces invariant. In the first case, since ω_* cannot be trivial on invariant subspaces $(W \oplus \omega_*(W))^\perp = \{0\}$ and $\dim V_* = \dim(W \oplus \omega(W)) = 4$. So assume that ω_* leaves all irreducible subspaces invariant. Then either there exists two irreducible subspaces W_1 and W_2, such that ω_* is a reflection on each and trivial on $(W_1 \oplus W_2)^\perp \subset V_*$, or ω_* acts without fixed points on an irreducible subspace $W \subset V_*$. In the first case $(W_1 \oplus W_2)^\perp = \{0\}$ and $\dim V_* = 4$. In the second case ω_* is trivial on $W^\perp \subset V_*$ and we conclude $\dim V_* = \dim W = 2$, contradicting $\dim V_* \geq 4$. □

We can finally prove

THEOREM II. *Let (H, W) be a non-reduced representation of a connected compact group H of abstract copolarity 1 with non-splitting quotient W/H. Then (H, W) has cohomogeneity 3.*

PROOF. Let (H, W) be a non-reduced representation of a connected compact group H of abstract copolarity 1. Let (G, V) be a minimal reduction, i.e. $\dim G = 1$. Then the quotient $W/H = V/G$ has boundary and we assume that it does not split. Since (G, V) is minimal, Proposition 2.4.4 implies that G/S^1 is not trivial. The group of connected components G/S^1 is generated by reflections. Each reflection $[\omega] \in G/S^1$ has an involutive lift ω and we know that V^ω fulfills
$$\dim V^\omega + \dim G - \dim Z_G(\omega) = \dim V - 1.$$
Decompose $V = V_* \oplus V_0$ into the set of points fixed by the induced S^1-action and its orthogonal complement V_*. Then we have shown that ω leave the decomposition $V_* \oplus V_0$ invariant, and write $\omega = (\omega_*, \omega_0)$. The results of Proposition 2.5.1 and Proposition 2.5.3 imply that either ω is a reflection and $\omega = (\mathrm{id}_*, r_0)$, or ω is pseudo-reflection and $\omega = (\omega_*, \mathrm{id}_0)$. In any case, $[\omega]$ leaves the splitting $V/S^1 \times V_0$ invariant and therefore the quotient V/G splits. Then $V_0 = \{0\}$ and no generator ω is a reflection. Finally, Proposition 2.5.3 implies that $\dim V = \dim V_* = 4$ and $\dim W/H = \dim V/G = 3$. □

2.6. Representation of abstract copolarity two

In this sections we analyze the representations of abstract copolarity 2, which are not orbit equivalent to a product representation, i.e. whose quotient does not split.

In the following, let (H, W) be a non-reduced representation of a connected compact group H of abstract copolarity 2 and (G, V) a minimal reduction, i.e. $\dim G = 2$. All representations are assumed to be faithful and we identity G with its image in $O(V)$. We know that the quotient space $W/H = V/G$ has boundary, while V/T^2 does not have boundary. Note that the absence of boundary in V/T^2 implies that (T^2, V) is reduced. This also follows from the fact that a k-section of (T^2, V) is a k-section for (G, V), and the last section, where we proved that the abstract copolarity and the copolarity coincide for values $k \leq 1$.

The finite group of connected components G/T^2 is a non-trivial group generated by reflections of the quotient V/T^2. Each reflection $[\omega]$ in G/T^2 has a lift $\omega \in G$ which is an involution and fulfills the dimension formula

$$\dim V^\omega + \dim G - \dim Z_G(\omega) = \dim V - 1.$$

The image of the representation

$$\begin{array}{rcl} \Phi : G/T^2 & \to & \mathrm{Aut}(\mathfrak{t}^2) = GL(2, \mathbb{Z}) \\ [g] & \mapsto & \mathrm{Ad}_g \end{array}$$

is a finite group generated by involutions and we will show that for reducible reductions (G, V) the image $\Phi(G/T^2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

We have seen that an involution $g \in G$ interchanges the isotypical components of (T^2, V). Therefore, the above representation G/T^2 interchanges the lines in $\mathfrak{t}^2$ induced by the weights of (T^2, V). For a reflection $[\omega] \in G/T^2$, the dimension formula implies that an appropriate lift ω even has restricted set of fixed points. Since reflections generate G/T^2, it is enough to understand the action of those lifts on the set of weights of (T^2, V). In the following ω will denote an involutive lift of a reflection $[\omega] \in G/T^2$, which fulfills the dimension formula. As a subgroup of the 2-dimensional group G, $\dim Z_G(\omega) \leq 2$, i.e. $\dim V^\omega \in \{1, 2, 3\}$, and we will distinguish between these three cases.

The strategy for the proof of Theorem III is similar to the proof of Theorem II. We analyze the interaction of a single element ω with the representation (T^2, V) before we draw the conclusion for the whole group G.

Let $\rho : T^2 \to SO(V)$ be the representation induced by restricting (G, V) to the identity component and decompose $V = V^1 \oplus \ldots \oplus V^m \oplus V_0$ into the T^2-isotypical components. Here V_0 is the set of T^2-fixed points and the non-trivial part is denoted by $V_* = V^1 \oplus \ldots \oplus V^m$. Recall that an isotypical component V^i correspond to a weight $\phi : T^2 \to S^1$ in the sense that every restriction of ρ to an irreducible subspace $W \subset V^i$ is equivalent to ϕ. We sometimes write $V^i = V^\rho_\phi$, to make this correspondence clear. Note that the decomposition of V into isotypical components is unique and all V^i are pairwise orthogonal.

Example. The representation

$$\begin{array}{rcl} T^2 \times \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} & \to & \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \\ ((\lambda,\mu),(z_1,z_2,z_3)) & \mapsto & (\lambda^k z_1, \lambda^k z_2, \lambda\mu z_3) \end{array}$$

has two isotypical components. The weights are $\phi_1 : T^2 \to S^1, (\lambda,\mu) \mapsto \lambda^k$ and $\phi_2 : T^2 \to S^1, (\lambda,\mu) \mapsto \lambda\mu$.

Since every T^2-representation is completely reducible, each isotypical component V^i has a decomposition into irreducible subspaces. This decomposition is no longer unique as can be seen in the above example. The isotypical component $V^\rho_{\phi_1} = \mathbb{C} \oplus \mathbb{C}$ corresponding to ϕ_1 equals the first to components. The diagonal $D = \{(z,z,0) \in \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\}$ is also a T^2-invariant subspace and the induced representation is equivalent to ϕ_1. In the same way $D' = \{(z,-z,0) \in \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\}$ is an invariant subspace and the restricted representation is equivalent to ϕ_1. Hence, $V^\rho_{\phi_1} = D \oplus D^\perp$ is another decomposition of $V^\rho_{\phi_1}$ into irreducible subspaces.

We have seen that an involution $g \in G$ permutes the isotypical components of $\rho : T^2 \to SO(V)$ and leaves the decomposition $V_* \oplus V_0$ invariant, hence we will write $g = (g_*, g_0)$. Furthermore, Lemma 2.4.7 tells us which isotypical components are interchanged by g, i.e. V^ρ_ϕ is mapped by ω to $V^\rho_{\phi\circ c_g}$.

Example. Assume $c_g : T^2 \to T^2, (\lambda,\mu) \mapsto (\bar\lambda,\bar\mu)$ and assume $\rho : T^2 \to SO(V)$ is equivalent to the representation of the last example. We define the weight

$$\bar\phi(\lambda,\mu) = \overline{\phi(\lambda,\mu)}, \text{ for all } (\lambda,\mu) \in T^2,$$

then $\phi \circ c_g = \bar\phi$ and $\omega(V^\rho_{\phi_i}) = V^\rho_{\bar\phi_i}$, for $i = 1,2$. Since ϕ and $\bar\phi$ are equivalent as real representations, $V^\rho_\phi = V^\rho_{\bar\phi}$ and g leaves all isotypical components of ρ invariant.

Lemma 2.6.1. *The dimension of V_* is bigger or equal to* 6.

Proof. Assume that $\dim V_* < 6$. Since V_* has no T^2-fixed points, it admits a decomposition into irreducible subspaces, which all have dimension 2. Therefore, the dimension of V_* is even. The dimension cannot be equal to 2, because the representation (T^2, V_*) is faithful. Furthermore, (T^2, V_*) has trivial principal isotropy groups. If $\dim V_* = 4$, the representation (T^2, V_*) has cohomogeneity 2 and is in fact polar. Then (T^2, V) and consequently (G, V) are polar, too, contradicting the fact that (G, V) is minimal in its quotient equivalence class. □

We now discuss the behavior of an appropriate lift ω depending on the codimension of its set of fixed points V^ω.

The case $\operatorname{codim} V^\omega = 1$: Assume that $\operatorname{codim} V^\omega = 1$, then the involution ω is a reflection. The dimension formula implies that $\dim Z_G(\omega) = 2$. Therefore, $T^2 \subset Z_G(\omega)$ is a subgroup of the centralizer and ω commutes with T^2, i.e. $c_\omega = \mathrm{id}$. Since we identify G with its image $\rho(G)$, $\omega \in O(V)$ commutes with the induced T^2-action on V. The elements of T^2 and ω can be simultaneously diagonalized, hence ω leaves all isotypical components invariant. We write $\omega = (\omega_*, \omega_0)$, then $V^\omega = V_*^{\omega_*} \oplus V_0^{\omega_0}$, where $V_*^{\omega_*}$ and $V_0^{\omega_0}$ denotes the set of fixed points of the restrictions ω_*, ω_0. Therefore,

$$\operatorname{codim} V_*^{\omega_*} + \operatorname{codim} V_0^{\omega_0} = 1.$$

If ω_* is a reflection, it has to leave all irreducible subspaces W in V_* invariant, since otherwise $\operatorname{codim} V_*^{\omega_*} \geq \dim W = 2$ (cf. the proof of Proposition 2.5.1). Hence, there exists an irreducible subspace $W \subset V_*$, such that ω_* is a reflection on W. Our assumption implies that ω_* commutes with the induced T^2-action on W. But the representation (T^2, W) is equivalent to an irreducible S^1-representation, which is given by rotations in $\mathbb{R}^2$, hence the reflection $\omega_*|_W$ cannot commute with the representation (T^2, W) and ω_* is trivial. We summarize

PROPOSITION 2.6.2. *Assume that ω is a reflection. Then $\omega = (\mathrm{id}_*, r_0)$ is trivial on V_* and acts as a reflection on V_0.*

The case $\operatorname{codim} V^\omega = 3$: The case $\operatorname{codim} V^\omega = 3$ is equivalent to $\dim Z_G(\omega) = 0$, hence the set of fixed points of the involution $c_\omega : T^2 \to T^2$ is finite. Differentiation implies that $\mathrm{Ad}_\omega : \mathfrak{t}^2 \to \mathfrak{t}^2$ is an involution with no non-trivial fixed points. From Lemma 2.3.3 we know that Ad_ω is diagonalizable with eigenvalues ± 1, hence $\mathrm{Ad}_\omega = -\mathrm{Id}$ and $c_\omega : T^2 \to T^2, t \mapsto t^{-1}$.

LEMMA 2.6.3. *If ω is an involution with $\operatorname{codim} V^\omega = 3$, then ω is not trivial on any non trivial T^2-invariant subspace $W \subset V_*$.*

PROOF. Assume that ω is trivial on the T^2-invariant subspace $W \subset V_*$. Denote the restriction of $\rho : T^2 \to SO(V)$ to the subspace W by $\rho_W : T^2 \to SO(W)$, then

$$\rho_W(t^{-1}) = \rho_W(c_\omega(t)) = \omega \rho_W(t) \omega = \rho_W(t).$$

This implies $t^2 \in \ker \rho_W$ for all $t \in T^2$, hence $W \subset V_0$. Since $V_* \cap V_0 = \{0\}$ we get $W = \{0\}$. $\square$

Let $\phi : T^2 \to S^1$ be a weight. For $c_\omega(t) = t^{-1}$ the weight $\phi \circ c_\omega : T^2 \to S^1$ equals

$$\phi \circ c_\omega(t) = \phi(t)^{-1} = \bar{\phi}(t)$$

and ϕ, $\phi \circ c_\omega$ are equivalent as real representations. Therefore, ω leaves all isotypical components invariant.

PROPOSITION 2.6.4. *Assume that ω has $\operatorname{codim} V^\omega = 3$. Then $\omega = (\omega_*, \mathrm{id}_0)$ is trivial on V_0 and $\dim V_* = 6$.*

PROOF. The codimension of the set of fixed points of the restrictions ω_* and ω_0 sum up to the codimension of V^ω

$$\operatorname{codim} V_*^{\omega_*} + \operatorname{codim} V_0^{\omega_0} = 3.$$

The last lemma implies that ω_* cannot be trivial, hence $\operatorname{codim} V_*^{\omega_*} \in \{1, 2, 3\}$. Furthermore, ω_* leaves all isotypical components invariant and we get

$$\sum \operatorname{codim} V^{i\,\omega_i} = \operatorname{codim} V_*^{\omega_*},$$

where $V^{i\,\omega_i}$ is the set of fixed points of the restriction $\omega_i = \omega_*|_{V^i}$ to an isotypical component V^i. The last lemma also implies that ω_* is not trivial on an isotypical component $V^i \subset V_*$, hence there are at most three isotypical components in V_*. Assume that $\operatorname{codim} V_*^{\omega_*} = 1$, then there is only one isotypical component, contradicting the fact that (T^2, V_*) is faithful. Assume that $\operatorname{codim} V_*^{\omega_*} = 2$. Since there are at least two isotypical components we conclude that there are exact two V^1, V^2 and $\omega_* = (\omega_1, \omega_2)$ is reflection on both. As in the first case (**$\operatorname{codim} V^\omega = 1$**) we conclude that the ω_i leaves all irreducible subspaces $W \subset V^i$ invariant. Hence, ω_i is a reflection on one irreducible subspace W and trivial on $W^\perp \subset V^i$. Since ω_i cannot

be trivial on invariant subspaces, $W^\perp = \{0\}$ and therefore $\dim V^1 = \dim V^2 = 2$. Then $\dim V_* = 4$, contradicting $\dim V_* \geq 6$.
Therefore, $\operatorname{codim} V_*^{\omega_*} = 3$ and $\omega = (\omega_*, \mathrm{id}_0)$. Recall that there are either two or three isotypical components, i.e. $V_* = V^1 \oplus V^2$ or $V_* = V^1 \oplus V^2 \oplus V^3$. In the case of three isotypical components ω_* is a reflection on each and the same argumentation as above implies that $\dim V^1 = \dim V^2 = \dim V^3 = 2$ and therefore $\dim V_* = 6$. If there are two isotypical components we write $\omega_* = (\omega_1, \omega_2)$ and assume without loss of generality that $\operatorname{codim} {V^1}^{\omega_1} = 1$ and $\operatorname{codim} {V^2}^{\omega_2} = 2$. The fact that ω_1 is not trivial on invariant subspaces implies that $\dim V^1 = 2$. If ω_2 moves an irreducible subspace $W \subset V^2$, i.e. $W \cap \omega_2(W) = \{0\}$, the discussion in Section 2.4 implies that $\omega(W)$ is a T^2-irreducible subspace and $W \oplus \omega(W) \subset V^2$ is T^2-invariant. Then $\operatorname{codim} {V^2}^{\omega_2} = 2$ implies that ω_2 has to be trivial on $(W \oplus \omega(W))^\perp \subset V^2$. We conclude $(W \oplus \omega(W))^\perp = \{0\}$ and $\dim V^2 = \dim(W \oplus \omega(W)) = 4$. If ω_2 leaves all irreducible subspaces invariant, there exists an irreducible subspace W, such that ω_2 is not trivial on it. If ω_2 has no fixed points in W beside $\{0\}$, i.e. $\operatorname{codim} W^{\omega_2} = 2$, ω_2 is trivial on $W^\perp \subset V^2$, hence $V^2 = W$ and $\dim V_* = \dim V^1 + \dim V^2 = 4$, contradicting $\dim V_* \geq 6$. Therefore, ω_2 is a reflection on W and also a reflection on the T^2-invariant subspace $W^\perp \subset V^2$. The same argumentation as above implies that $\dim W^\perp = 2$ and $\dim V^2 = 4$. In any case $\dim V_* = 6$. □

We summarize that either there exist two isotypical components in V_*, i.e. $V = V^1 \oplus V^2 \oplus V_0$, with $\dim V^1 = 2$ and $\dim V^2 = 4$ and $\omega = (r_1, \omega_2, \mathrm{id}_0)$ is a reflection on V^1 and a pseudo-reflection on V^2, or there exists three isotypical components in V_*, i.e. $V = V^1 \oplus V^2 \oplus V^3 \oplus V_0$ with $\dim V^i = 2$, such that $\omega = (r_1, r_2, r_3, \mathrm{id}_0)$ is a reflection on each V^i.

A T^2-isotypical component $V^i = V^\rho_\phi$ of V is characterized by a weight $\phi : T^2 \to S^1$. A weight ϕ induce a line $\mathfrak{s} = \ker \mathrm{d}\phi$ in $\mathfrak{t}^2$. Note that the weight ϕ^n induces the same line in $\mathfrak{t}^2$ as ϕ. We say that a line $\mathfrak{s} \subset \mathfrak{t}^2$ is induced by $\rho : T^2 \to SO(V)$, if $\mathfrak{s} = \ker \mathrm{d}\phi$ and $V^\rho_\phi \neq \{0\}$.

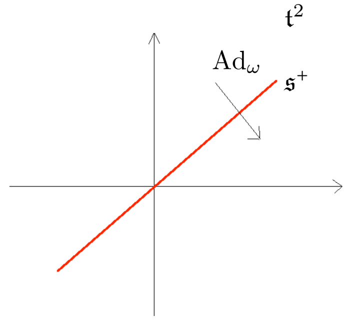
Proposition 2.6.5. *Assume that* $\operatorname{codim} V^\omega = 3$, *then* (T^2, V_*) *induces three lines in* $\mathfrak{t}^2$. *In fact,* $V = V^1 \oplus V^2 \oplus V^3 \oplus V_0$ *and* $\omega = (r_1, r_2, r_3, \mathrm{id}_0)$ *is a reflection on each* V^i.

Proof. From the last proposition we know that either $\omega = (r_1, \omega_2, \mathrm{id}_0)$, or $\omega = (r_1, r_2, r_3, \mathrm{id}_0)$ with respect to the respective decomposition of V. Denote by $\mathfrak{s}_i$ the respective lines in $\mathfrak{t}^2$ induced by the isotypical components of (T^2, V_*). Assume that there are only two lines in $\mathfrak{t}^2$ induced by (T^2, V_*), then the quotient V_*/T^2 splits. Let $V^{\mathfrak{s}_i} = \bigoplus_{n \in \mathbb{Z}} V^\rho_{\phi_i^n}$ denote the sum of all isotypical components in V_* associated to $\mathfrak{s}_i = \ker \mathrm{d}\phi_i$, then without loss of generality we can assume that $\dim V^{\mathfrak{s}_1} = 2$, $\dim V^{\mathfrak{s}_2} = 4$ and the quotient

$$V_*/T^2 = V^{\mathfrak{s}_1}/S^1 \times V^{\mathfrak{s}_2}/S^1.$$

Since $\dim V^{\mathfrak{s}_1} = 2$, the action $(S^1, V^{\mathfrak{s}_1})$ is polar and (T^2, V) is not reduced, contradicting the minimality of (G, V). □

The case $\operatorname{codim} V^\omega = 2$: The codimension of V^ω is equal to 2 if and only if the dimension $\dim Z_G(\omega) = 1$. In fact the kernel $Z_G(\omega)$ of $c_\omega : T^2 \to T^2$ is 1-dimensional and its lie algebra $\mathfrak{s}^+ \subset \mathfrak{t}^2$ is a line, which equals the set of fixed points of Ad_ω. Recall that Ad_ω is diagonalizable with eigenvalues ± 1 and denote with $\mathfrak{s}^- \subset \mathfrak{t}^2$ the -1-eigenspace of Ad_ω.



Let $\phi_+ : T^2 \to S^1$ be the normed weight with $\mathfrak{s}^+ = \ker \mathrm{d}\phi_+$ and ϕ_- the normed weight corresponding to $\mathfrak{s}^-$. Define

$$V^+ = \bigoplus_{m\in\mathbb{Z}} V^\rho_{\phi_+^m} \text{ and } V^- = \bigoplus_{n\in\mathbb{Z}} V^\rho_{\phi_-^n},$$

then V^+, V^- contain all isotypical components of ρ, which correspond to $\mathfrak{s}^+$ and $\mathfrak{s}^-$, respectively. Finally denote by $\bar{V}$ the orthogonal complement of $V^+ \oplus V^-$ in V_* Then

$$V = V^+ \oplus V^- \oplus \bar{V} \oplus V_0,$$

is a T^2-invariant decomposition, which is also invariant under ω and we write $\omega = (\omega_+, \omega_-, \bar{\omega}, \omega_0)$. We get the following splitting

$$\begin{aligned} 2 &= \operatorname{codim} V^\omega \\ &= \operatorname{codim} V^{+\omega_+} + \operatorname{codim} V^{-\omega_-} + \operatorname{codim} \bar{V}^{\bar{\omega}} + \operatorname{codim} V_0^{\omega_0}, \end{aligned}$$

and analyze the restrictions $\omega_+, \omega_-, \bar{\omega}, \omega_0$. Assume, that $V^i \neq \{0\}$ is an isotypical components in $\bar{V}$. Let $\mathfrak{s}_i$ denote the corresponding line in $\mathfrak{t}^2$, then Ad_ω does not leave $\mathfrak{s}_i$ invariant. Hence, V^i is mapped by $\bar{\omega}$ to another isotypical component $V^j \subset \bar{V}$ and $V^i \cap \omega(V^i) = \{0\}$. Comparing with Lemma 2.4.8 the T^2-action has cohomogeneity 1 on V^i and V^j, respectively. A T^2-orbit in V^i is actually 1-dimensional, hence $\dim V^i = \dim V^j = 2$. Since $\bar{\omega}$ moves V^i to V^j the codimension $\operatorname{codim} \bar{V}^{\bar{\omega}} \geq \dim V^i = 2$.

Lemma 2.6.6. *The restrictions ω_+, ω_- leave isotypical components invariant.*

Proof. We will show the statement for ω_+, the proof for ω_- acting on V^- is analogous. Assume without loss of generality that $V^\rho_{\phi_+} \neq \{0\}$ and that ω_+ maps $V^\rho_{\phi_+}$ to $V^\rho_{\phi_+^k}$, i.e.

$$\phi_+ \circ c_\omega = \phi_+^k.$$

Differentiation implies that

$$\mathrm{d}\phi_+ \circ \mathrm{Ad}_\omega(x) = \mathrm{d}\phi_+^k(x) = k\mathrm{d}\phi_+(x) \text{ for all } x \in \mathfrak{t}^2.$$

Therefore, $\mathrm{d}\phi_+ (\mathrm{Ad}_\omega(x) - kx) = 0$ and in fact $\mathrm{Ad}_\omega(x) - kx \in \mathfrak{s}^+$ for all $x \in \mathfrak{t}^2$. Then

$$\mathrm{Ad}_\omega (\mathrm{Ad}_\omega(x) - kx) = x - k\mathrm{Ad}_\omega(x) = \mathrm{Ad}_\omega(x) - kx$$

implies $(1+k)x = (1+k)\mathrm{Ad}_\omega(x)$ for all $x \in \mathfrak{t}^2$. For $k \neq -1$ we get $x = \mathrm{Ad}_\omega(x)$, contradicting the fact that $\mathrm{Ad}_\omega \neq \mathrm{Id}$. Therefore, $k = -1$ and since $V^\rho_{\phi_+} = V^\rho_{\bar{\phi}_+}$, ω_+ leaves all isotypical component in V^+ invariant. $\square$

We have a closer cool at the action of ω_-. Denote the induced T^2-representation on V^- by $\rho_- : T^2 \to SO(V^-)$. Since $\mathfrak{s}^+$ and $\mathfrak{s}^- = \ker \mathrm{d}\rho_-$ span $\mathfrak{t}^2$ and the exponential map is surjective, we can write every $t \in T^2$ as $t = \mathrm{Exp}(x_- + x_+)$, where $x_- \in \mathfrak{s}^-$ and $x_+ \in \mathfrak{s}^+$. Using the fact that T^2 is abelian

$$\begin{aligned}
\omega_-\rho_-(t)\omega_- &= \rho_-(c_\omega(t)) \\
&= \rho_-\left(\mathrm{Exp}\left(\mathrm{Ad}_\omega(x_- + x_+)\right)\right) \\
&= \rho_-\left(\mathrm{Exp}(-x_-)\cdot \mathrm{Exp}(x_+)\right) \\
&= \rho_-\left(\mathrm{Exp}(-x_-)\right)\cdot \rho_-\left(\mathrm{Exp}(x_+)\right) \\
&= \rho_-\left(\mathrm{Exp}(x_+)\right) \\
&= \rho_-(\left(\mathrm{Exp}(x_-)\right)\cdot \rho_-\left(\mathrm{Exp}(x_+)\right) \\
&= \rho_-(t),
\end{aligned}$$

i.e. ω_- commutes with the induced T^2-action on V^-. Note that (T^2, V^-) is equivalent to a S^1-representation, since ρ_- is not faithful.

The next lemma implies that ω_+ cannot act trivial on non-trivial T^2-invariant subspaces in V^+.

LEMMA 2.6.7. *If the involution ω acts as the identity on a non-trivial T^2-invariant subspace $W \subset V_*$, then $W \subset V^-$.*

PROOF. Assume $W \subset V_*$ is a T^2-invariant subspace and that ω is trivial on W. Denote by $\rho_W : T^2 \to SO(W)$ the induced representation on W, then

$$\rho_W(t) = \omega\rho_W(t)\omega = \rho_W(c_\omega(t)) \text{ for all } t \in T^2.$$

Hence, $c_\omega(t)t^{-1} \in \ker \rho_W$ for all $t \in T^2$ and differentiation implies that $\mathrm{Ad}_\omega(x) - x \in \ker \mathrm{d}\rho_W$ for all $x \in \mathfrak{t}^2$. Now $\mathrm{Ad}_\omega(\mathrm{Ad}_\omega(x) - x) = -(\mathrm{Ad}_\omega(x) - x)$ for all $x \in \mathfrak{t}^2$ and therefore, $\mathfrak{s}^- \subset \ker \mathrm{d}\rho_W$. If $\dim \ker \rho_W = 2$, then $\ker \mathrm{d}\rho_W = \mathfrak{t}^2$ and $W \subset V_* \cap V_0$, i.e. $W = \{0\}$. Hence, $\dim \ker \mathrm{d}\rho_W = 1$ and $\mathfrak{s}^- = \ker \mathrm{d}\rho_W$. In fact, $W \subset V^-$. □

In the following we will distinguish the cases that $\mathfrak{s}^+$ is induced by the representation $\rho : T^2 \to V$, or not.

PROPOSITION 2.6.8. *Let ω be a pseudo-reflection.*

(a) *Assume that the set $\mathfrak{s}^+$ of Ad_ω fixed points coincide with a line induced by (T^2, V). Then $\mathfrak{s}^-$ is also induced by (T^2, V). Furthermore, these are the only lines in $\mathfrak{t}^2$ induced by the representation (T^2, V), i.e. $V = V^+ \oplus V^- \oplus V_0$, and the quotient of (T^2, V_*) splits. Then $\omega = (\omega_+, \mathrm{id}_-, \mathrm{id}_0)$, where ω_+ is pseudo-reflection and $\dim V^+ = 4$ and $\dim V^- \geq 4$.*

(b) *Assume that the set $\mathfrak{s}^+$ of Ad_ω fixed points is not induced by the representation (T^2, V). Then $V = V^- \oplus V^1 \oplus V^2 \oplus V_0$ and $\omega = (\mathrm{id}_-, \bar{\omega}, \mathrm{id}_0)$, where $\bar{\omega}$ interchanges the two isotypical components $V^1, V^2 \subset \bar{V}$. Especially, V^1, V^2 are 2-dimensional.*

PROOF. Case (a): Assume that $\mathfrak{s}^+$ is induced by (T^2, V). Then $V^+ \neq \{0\}$ is invariant under ω. Since (T^2, V) is faithful, at least one further component V^- or $\bar{V}$ is not trivial. The last lemma implies that ω_+ cannot act trivially on V^+, or any T^2-invariant subspace contained in V^+. Then the splitting of the codimension implies that $\operatorname{codim} V^{+\omega_+} \in \{1, 2\}$. Since ω_+ is not trivial on invariant subspaces, we can use the same argumentation as in the case $\mathbf{codim}\, \boldsymbol{V^{\omega} = 3}$ to estimate the dimension of V^+. Assume, ω_+ is a reflection, then $\dim V^+ = 2$. Since $\operatorname{codim} V^{\omega} = 2$, ω is a reflection on $V^- \oplus \bar{V} \oplus V_0$. The restriction $\bar{\omega}$ interchanges isotypical components of $\bar{V}$, hence it cannot be a reflection nor is $\bar{\omega}$ trivial on $\bar{V}$, hence $\bar{V} = \{0\}$. Then $V^- \neq \{0\}$, i.e. $V = V^+ \oplus V^- \oplus V_0$, and $\mathfrak{s}^+, \mathfrak{s}^-$ are the only lines in $\mathfrak{t}^2$ induced by (T^2, V). Therefore, the quotient

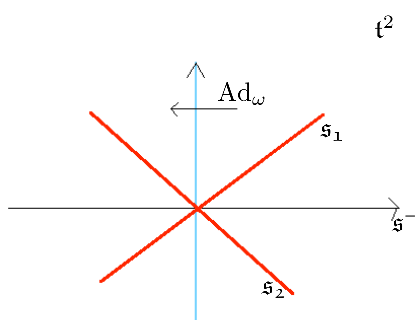
$$V/T^2 = V^+/S^1 \times V^-/S^1 \times V_0$$

splits and $\dim V^+ = 2$ implies that (S^1, V^+) is polar. Hence, V/T^2 is not reduced, contradicting the minimality of (G, V). Assume that ω_+ is a pseudo-reflection. We have seen that ω_+ leaves all isotypical components invariant and can use the splitting of the codimensions to see that

$$2 = \operatorname{codim} V^{+\omega_+} = \sum_{k \in \mathbb{Z}} \operatorname{codim} (V^{\rho}_{\phi^k_+})^{\omega_+}.$$

Since ω_+ is not trivial on isotypical components contained in V^+, there are either one or two isotypical components in V^+. In the second case $V^+ = V^1 \oplus V^2$ and ω_+ is a reflection on both. Using that ω_+ is not trivial on invariant subspaces, we conclude that $\dim V^1 = \dim V^2 = 2$. If there exists only one isotypical component, ω_+ is a pseudo-reflection on it. Then either ω_+ interchanges two irreducible subspaces, then $\dim V^+ = 4$ since ω_+ is not trivial on invariant subspaces, or it leaves all invariant. If ω_+ leaves all irreducible subspaces invariant, there exists either one W and ω_+ is a pseudo-reflection on it, or there exists two irreducible subspaces W_1, W_2, such that ω_+ is a reflection on both. In any case ω_+ is trivial on $W^\perp, (W_1 \oplus W_2)^\perp \subset V^+$, respectively and $W^\perp = \{0\}$, or $(W_1 \oplus W_2)^\perp = \{0\}$. Therefore, $\dim V^+ = 2$ or $\dim V^+ = 4$. In any case, if ω_+ is a pseudo-reflection then $\omega = (\omega_+, \mathrm{id}_-, \mathrm{id}_0)$ and $\dim V^+ \in \{2, 4\}$. The splitting of the codimension implies that ω is trivial on the orthogonal complement of V^+ and the previous lemma implies $\bar{V} = \{0\}$. Since (T^2, V) is faithful, $V^- \neq \{0\}$ and $V = V^+ \oplus V^- \oplus V_0$. Hence, there are exactly two lines in $\mathfrak{t}^2$ induced by (T^2, V). Again the quotient $V/T^2 = V^+/S^1 \times V^-/S^1 \times V_0$ splits, and since V/T^2 is reduced, $\dim V^+ = 4$ and $\dim V^- \geq 4$.

Case(b): Assume that $\mathfrak{s}^+$ is not induced by the representation (T^2, V). Since the representation is faithful, $\bar{V} \neq \{0\}$ and there exists lines $\mathfrak{s}_1, \mathfrak{s}_2$, not invariant under Ad_ω, such that $\mathrm{Ad}_\omega(\mathfrak{s}_1) = \mathfrak{s}_2$.



Now $\bar{\omega}$ interchanges two isotypical components $V^1, V^2 \subset \bar{V}$ corresponding to $\mathfrak{s}_1, \mathfrak{s}_2$, respectively. From Lemma 2.4.8 we know that T^2 acts with cohomogeneity 1 on V^1 and V^2, hence $\dim V^1 = \dim V^2 = 2$. The splitting of the codimension of V^ω implies that $\bar{\omega}$ is trivial on $(V^1 \oplus V^2)^\perp \subset \bar{V}$, hence there are no more isotypical components contained in $\bar{V}$. Since $\dim V_* \geq 6$, $V^- \neq \{0\}$ and $V = V^- \oplus V^1 \oplus V^2 \oplus V_0$. Then $\omega = (\mathrm{id}_-, \bar{\omega}, \mathrm{id}_0)$, where $\bar{\omega}$ interchanges V^1 and V^2. □

Coming so far we summarize the possible shapes of ω in the next table. Let $[\omega] \in G/T^2$ be a reflection and let ω be an appropriate lift. We say that $[\omega]$ is of type I, II, III, if the respective codimension of V^ω is equal to $\operatorname{codim} V^\omega = 1$, $\operatorname{codim} V^\omega = 2$ or $\operatorname{codim} V^\omega = 3$. Then ω has the following shape.

Table I

Type	V	ω	$\#\mathfrak{s}_\mathfrak{i}$
I	$V_* \oplus V_0$	(id_*, r_0)	arb
II	$V^+ \oplus V^- \oplus V_0$, $\dim V^+ = 4$, $\dim V^- > 2$	$(\omega_+, \mathrm{id}_-, \mathrm{id}_0)$, ω_+ pseudo-reflection	2
II	$V^- \oplus V^1 \oplus V^2 \oplus V_0$, $\dim V^- \geq 2$, $\dim V^i = 2$	$(\mathrm{id}_-, \bar{\omega}, \mathrm{id}_0)$ $\bar{\omega}$ interchanges V^1, V^2	3
III	$V^1 \oplus V^2 \oplus V^3 \oplus V_0$, $\dim V^i = 2$	$(r_1, r_2, r_3, \mathrm{id}_0)$	3

With the help of the above table we can finally prove

Theorem III. *Let (H, W) be a non-reduced representation of a connected compact group H of abstract copolarity two, with non-splitting quotient W/H. Then either*

- *(H, W) has cohomogeneity 4, or*
- *each minimal reduction (G, V) of (H, W) has two connected components, i.e. $G/G_0 = \mathbb{Z}/2\mathbb{Z}$. Furthermore, there exist an irreducible subspace $W' \subset W$ such that H acts with cohomogeneity two on W', and $(H, (W')^\perp)$ is orbit equivalent to a non-polar S^1-representation.*

Proof. Let (H, W) be a non-reduced representation of a connected compact group H of abstract copolarity 2 and (G, V) a minimal reduction, i.e. $\dim G = 2$. The quotient space $W/H = V/G$ has boundary, while the quotient V/T^2 of the identity component of G does not have boundary. Furthermore, the finite group of connected components G/T^2 is a non-trivial group generated by reflections. Each reflection $[\omega]$ in G/T^2 has a lift $\omega \in G$ which is an involution and fulfills the dimension formula

$$\dim V^\omega + \dim G - \dim Z_G(\omega) = \dim V - 1.$$

We decompose $V = V^1 \oplus \ldots V^m \oplus V_0$ into isotypical components with respect to the representation (T^2, V). Then each ω acts on this decomposition by permuting the isotypical components and fixing a subspace V^ω of codimension $\operatorname{codim} V^\omega \in \{1, 2, 3\}$. The weights corresponding to the isotypical components induce lines in $\mathfrak{t}^2$ and Ad_ω permutes these lines. The previous propositions imply that, depending on

$\operatorname{codim} V^\omega$, the possible shapes of ω are listed in Table I.

Let $V_* = V^1 \oplus \ldots \oplus V^m$ be the non- trivial part of the representation (T^2, V). Then, comparing with Table I, an involution ω is either trivial on V_* or trivial on V_0, i.e. $\omega = (\mathrm{id}_*, \omega_0)$ or $\omega = (\omega_*, \mathrm{id}_0)$. In any case, their projections $[\omega] \in G/T^2$ leave the splitting

$$V/T^2 = V_*/T^2 \times V_0$$

invariant. Since the reflections $[\omega]$ generate G/T^2, the action of G/T^2 leaves the above splitting invariant and the quotient V/G splits. Hence, $V_0 = \{0\}$ and no involution ω is of type I. From Table I we conclude that (T^2, V) induces at most three lines in $\mathfrak{t}^2$. In fact, we show that (T^2, V) induces exactly three lines, i.e. V/T^2 does not split.

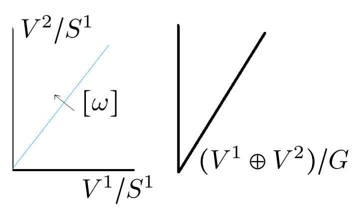
Assume that (T^2, V) induces 2 lines $\mathfrak{s}_1, \mathfrak{s}_2 \subset \mathfrak{t}^2$. In this case, comparing with Table I, each reflection $[\omega] \in G/T^2$ is of type II. For an appropriate lift ω, the lines $\mathfrak{s}_i$ are the ± 1-eigenspaces of Ad_ω. Since (T^2, V) induces exactly two lines, the quotient V/T^2 splits, i.e. $V/T^2 = V^{\mathfrak{s}_1}/S^1 \oplus V^{\mathfrak{s}_2}/S^1$, where $V^{\mathfrak{s}_i}$ is the sum of all isotypical components corresponding to $\mathfrak{s}_i$. Each involution ω leaves the subspaces $V^{\mathfrak{s}_i}$ invariant and is trivial on one of them, i.e. $\omega = (\omega_{\mathfrak{s}_1}, \mathrm{id}_{\mathfrak{s}_2})$ or $\omega = (\mathrm{id}_{\mathfrak{s}_1}, \omega_{\mathfrak{s}_2})$. In particular, a reflection $[\omega] \in G/T^2$ preserves the splitting

$$V/T^2 = V^{\mathfrak{s}_1}/S^1 \times V^{\mathfrak{s}_2}/S^1$$

and therefore V/G splits.

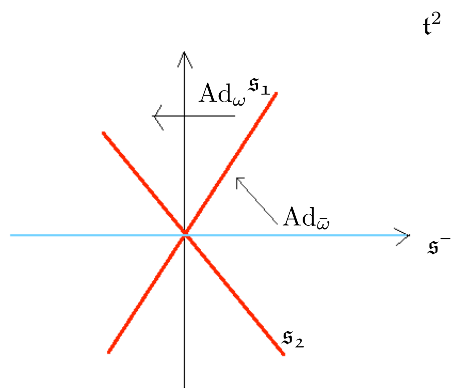
We conclude that (T^2, V) induces three lines in $\mathfrak{t}^2$. If there exists at least one reflection of type III, the above table implies that $\dim V = 6$. Then the cohomogeneity of (G, V) is equal to 4, since G acts with trivial principal isotropy groups. Hence, the cohomogeneity of (H, W) is equal to 4.

We are left with the case that G/T^2 is generated only by involutions of type II. Let $[\omega] \in G/T^2$ be a generator. If $[\omega]$ is the single generator of G/T^2, i.e. $G/T^2 = \mathbb{Z}/2\mathbb{Z}$, then $V = V^- \oplus V^1 \oplus V^2$ and $\omega = (\mathrm{id}_-, \bar{\omega}, \mathrm{id}_0)$, where $\bar{\omega}$ interchanges the isotypical components V^1 and V^2. Furthermore, $\dim V^1 = \dim V^2 = 2$. If $\dim V^- = 2$ then $\dim V = 6$ and the cohomogeneity of (G, V) is equal to 4. As above we conclude that (H, W) has cohomogeneity 4. For $\dim V^- > 2$, we have to show that we are in the second case of Theorem III. First note that V^- and $V^1 \oplus V^2$ are G-invariant subspaces. The sub representation $(G, V^1 \oplus V^2)$ is irreducible and has cohomogeneity 2. In fact, $(G, V^1 \oplus V^2)$ is polar. Since $\mathfrak{s}_1 \neq \mathfrak{s}_2$, the quotient of the identity component $(V^1 \oplus V^2)/T^2 = V^1/S^1 \times V^2/S^1$ splits and $[\omega] \in G/T^2$ interchanges the two half lines V^1/S^1 and V^2/S^1. Therefore, $(V^1 \oplus V^2)/G = \left(V^1/S^1 \times V^2/S^1\right)/[\omega]$ is isometric to a segment of angle $\frac{\pi}{4}$ in the Euclidean plane.



There exists an irreducible subspace $W' \subset W$, such that $W'/H = (V^1 \oplus V^2)/G$. In fact, (H, W') has cohomogeneity 2 and is therefore polar. Then (H, W') is orbit equivalent to the isotropy representation of a rank 2 symmetric space and the quotient implies that $g = 4$. Since ω is trivial on $V^- - = \bigoplus V^{\rho}_{\phi^n_-}$, the restriction $(G, V^-) = (S^1, V^-)$. Then $\dim V^- > 2$ implies that (S^1, V^-) is not polar, and actually reduced. In fact, it is the minimal reduction of the representation $(H, (W')^\perp)$. We can apply Proposition 2.4.4 and conclude that V^-/S^1, and therefore $(W')^\perp/H$, does not have boundary. Then the restriction $(H, (W')^\perp)$ is itself reduced, i.e. $\mathrm{pr}_{(W')^\perp}(H) \subset SO((W')^\perp)$ is 1-dimensional.

Now assume that G/T^2 is generated by at least two reflections $[\omega], [\bar{\omega}]$ of type II. We show that (G, V), and in fact (H, W), is irreducible. Let $V = V^- \oplus V^1 \oplus V^2$ be the decomposition of V with respect to the involution ω, i.e. V^- correspond to the -1-eigenspace of Ad_ω. Denote by $\mathfrak{s}^-, \mathfrak{s}_1, \mathfrak{s}_2$ the induced lines in $\mathfrak{t}^2$. We claim that for $[\omega] \neq [\bar{\omega}]$ the product $\omega\bar{\omega} \notin Z_G(T^2)$ and prove this claim later. Since $\bar{\omega}$ is compatible with the decomposition $V = V^- \oplus V^1 \oplus V^2$, the differential $\mathrm{Ad}_{\bar{\omega}}$ permutes the lines $\mathfrak{s}^-, \mathfrak{s}_1, \mathfrak{s}_2$. The involution $\bar{\omega}$ fulfills the same conditions as ω, hence the $+1$-eigenspace of $\mathrm{Ad}_{\bar{\omega}}$ is not induced by (T^2, V). The claim implies that $\mathrm{Ad}_\omega \neq \mathrm{Ad}_{\bar{\omega}}$, hence the line $\mathfrak{s}^-$ is not the -1-eigenspace of $\mathrm{Ad}_{\bar{\omega}}$. We can assume without loss of generality that $\mathrm{Ad}_{\bar{\omega}}(\mathfrak{s}^-) = \mathfrak{s}^1$, then $\bar{\omega}$ interchanges V^- and V^1.



We can use Lemma 2.4.8, then $\dim V^- = 2$ and therefore $\dim V = 6$. We conclude that (G, V) and hence (W, H) has cohomogeneity 4. Note that in this case, V^-, V^1, V^2 are 2-dimensional isotypical components, pairwise interchanged by G, i.e (G, V) is irreducible.

Claim: Let $[\omega], [\bar{\omega}| \in G/T^2$ be reflections of type II. If $\mathrm{Ad}_\omega = \mathrm{Ad}_{\bar{\omega}}$, then $[\omega] = [\bar{\omega}]$. Assume that $[\omega], [\bar{\omega}]$ are reflections of type II, such that $\mathrm{Ad}_\omega = \mathrm{Ad}_{\bar{\omega}}$, then $\bar{\omega}\omega \in Z_G(T^2)$. Therefore, $\omega\bar{\omega}$ and the elements of T^2 have a common decomposition of V into eigenspaces, i.e $\bar{\omega}\omega$ leaves the isotypical components invariant. Let $V = V^- \oplus V^1 \oplus V^2$ be the decomposition of V with respect to the involution ω, where ω interchanges V^1 and V^2. Then $\bar{\omega}(\omega(V^1)) = \bar{\omega}(V^2) = V^1$, and $\bar{\omega}$ interchanges V^1, V^2. Then $\operatorname{codim} V^{\bar{\omega}} = 2$ implies that $\bar{\omega}$ is trivial on V^-. We identify $V^- = \mathbb{C}^n, V^1 = \mathbb{C}$ and $V^2 = \mathbb{C}$. There exists isometries $p, \bar{p} : \mathbb{R}^2 \to \mathbb{R}^2$ and with respect to the decomposition $V = V^- \oplus V^1 \oplus V^2$ we write

$$\omega(v_-, v_1, v_2) = (v_-, p^{-1}(v_2), p(v_1)) \quad \text{and} \quad \bar{\omega}(v_-, v_1, v_2) = (v_-, \bar{p}^{-1}(v_2), \bar{p}(v_1)).$$

Let $T^2 = S^1 \times S^1$, and assume that the factor spheres are tangent to $\mathfrak{s}^-$ and $\mathfrak{s}^+$, respectively, where $\mathfrak{s}^\pm$ denotes the ± 1-eigenspaces of Ad_ω. Let x_-, x_+ be unit vectors spanning $\mathfrak{s}^-, \mathfrak{s}^+$, then $\mathfrak{s}_1 = kx_- + lx_+$, for $k, l \in \mathbb{N} - \{0\}$. Since Ad_ω interchanges $\mathfrak{s}_1, \mathfrak{s}_2$, we get $\mathfrak{s}_2 = -kx_- + lx_+$ and the weights $\phi_i : T^2 \to S^1$ corresponding to V^i equal

$$\phi_1(\lambda,\mu) = \lambda^k \mu^l \quad \text{and} \quad \phi_2(\lambda,\mu) = \bar{\lambda}^k \mu^l.$$

Furthermore, let $\rho_- : T^2 \to SO(V^-)$, denote the restriction of (T^2, V) to V^-, then $(\ker \rho_-)_0 = S^1 \times \{e\}$. With respect to the chosen coordinates $(\lambda, \mu) \in S^1 \times S^1$ the involution $c_\omega : T^2 \to T^2$ is given by $c_\omega(\lambda,\mu) = (\bar{\lambda}, \mu)$. Then we get

$$\begin{aligned} \omega(\lambda,\mu)\omega(v_-, v_1, v_2) &= \omega(\lambda,\mu)(v_-, p^{-1}(v_2), p(v_1)) \\ &= \omega(\rho_-(\lambda,\mu)v_-, \lambda^k\mu^l p^{-1}(v_2), \bar{\lambda}^k\mu^l p(v_1)) \\ &= (\rho_-(\lambda,\mu)v_-, p^{-1}\bar{\lambda}^k\mu^l p(v_1), p\lambda^k\mu^l p^{-1}(v_2)). \end{aligned}$$

Furthermore,

$$(\bar{\lambda},\mu)(v_-, v_1, v_2) = (\rho_-(\bar{\lambda},\mu)v_-, \bar{\lambda}^k\mu^l v_1, \lambda^k\mu^l v_2),$$

and the above equations implies that the isometry $p : \mathbb{R}^2 \to \mathbb{R}^2$ commutes with the rotations $\bar{\lambda}^k\mu^l, \lambda^k\mu^l$, hence it muss be a rotation itself and we can write $p = \lambda_1 \in S^1$. The same argumentation implies that $\bar{p} = \lambda_2 \in S^1$ and we get

$$\begin{aligned} \omega\bar{\omega}(v_-, v_1, v_2) &= \omega(v_-, \bar{\lambda}_2 v_2, \lambda_2 v_1) \\ &= (v_-, \bar{\lambda}_1\lambda_2 v_1, \lambda_1\bar{\lambda}_2 v_2) \\ &= \rho(e, (\bar{\lambda}_1\lambda_2)^{1/l})(v_-, v_1, v_2). \end{aligned}$$

Since $\rho : G \to SO(V)$ is injective, we get $\omega\bar{\omega} \in T^2$ and in fact $[\omega] = [\bar{\omega}]$. □

In the following, let (H, W) be as in the last theorem and (G, V) a minimal reduction. We have seen in the proof of the last theorem, that V/T^2 does not split and that G/T^2 is generated by reflections of type II and type III. A reflection $[\omega]$ is of type II, if and only if $\det \mathrm{Ad}_\omega = -1$. A reflection $[\omega]$ is of type III, if and only if $\mathrm{Ad}_\omega = -\mathrm{Id}$. Therefore, the representation

$$\begin{aligned} \Phi : G/T^2 &\to \mathrm{Aut}(\mathfrak{t}^2) = GL(2,\mathbb{Z}) \\ [g] &\mapsto \mathrm{Ad}_g \end{aligned}$$

is not trivial. The image $\Phi(G/T^2)$ is a finite group generated by involutions, which interchanges the three lines induced by (T^2, V). For a reducible representation, there exists at most one reflection of type II, hence $\Phi(G/T^2) = G/Z_G(T^2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

If G/T^2 is generated only by reflections of type III, then clearly $G/Z_G(T^2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and (G, V) is reducible, with three irreducible subspaces $V = V_1 \oplus V_2 \oplus V_3$. The restrictions (G, V_i) are of cohomogeneity 1, for $i = 1, 2, 3$. Furthermore, $(G, V_i \oplus V_j)$, for $i \neq j$, is reducible and has cohomogeneity 2. Then (H, W) is the sum $\rho_1 \oplus \rho_2 \oplus \rho_3$ of three irreducible representations, each of cohomogeneity 1. Furthermore, each pair $\rho_i \oplus \rho_j$, for $i \neq j$, is reducible, has cohomogeneity 2 and is in fact orbit equivalent to a reducible polar representation.

For G/T^2 only generated by reflections of type II, the claim implies that Φ is injective and if (G, V) is reducible, $G/T^2 = \mathbb{Z}/2\mathbb{Z}$. Then for (G, V) the representation space V has two invariant subspaces $V = V_1 \oplus V^2$. The restriction (G, V_1) is irreducible and has cohomogeneity 2. Furthermore, the quotient V_1/G is isometric to a segment of angle $\frac{\pi}{4}$ in the Euclidean plane. For $\dim V^2 > 2$, the restriction (G, V^2) is orbit equivalent to a non-polar S^1-representation. For $\dim V^2 = 2$, the restriction (G, V^2) has cohomogeneity 1. Then (H, W) is the sum of two representations $\rho_1 \oplus \rho_2$, where ρ_1 is orbit equivalent to a rank two symmetric space with $g = 4$. If (H, W) has cohomogeneity 4, ρ_2 has cohomogeneity 1, otherwise ρ_2 is orbit equivalent to a non-polar S^1 representation.

Finally, for mixed types, $G/Z_G(T^2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and (G, V) has two irreducible subspaces $V = V_1 \oplus V_2$, where (G, V_1) has cohomogeneity 2 and (G, V_1) has cohomogeneity 1. Then (H, W) is the sum $\rho_1 \oplus \rho_2$ of two irreducible representations, where ρ_1 is orbit equivalent to a rank two symmetric space and ρ_2 is of cohomogeneity 1.

Corollary 2.6.9. *Let (H, W) be a non-reduced reducible representation of a connected compact group H of abstract copolarity 2, with non-splitting quotient W/H and let (G, V) be a minimal reduction. Then the quotient V/T^2 of identity component (T^2, V) has no boundary and does not split. Furthermore, $G/Z_G(T^2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

We know give examples of reducible representation of abstract copolarity 2, whose minimal reduction fulfills $G/Z_G(T^2) = \mathbb{Z}/2\mathbb{Z}$. Note that all the following examples have copolarity 2.

Example. The representation $(\mathrm{Spin}(8), \rho_8 \oplus \Delta_8^+ \oplus \Delta_8^-)$ has cohomogeneity 4. It admits a 2-section, whose induced representation is a minimal reduction.

We follow the calculations in [**10**] and identify $\mathbb{R}^8 = \mathbb{R}\langle 1, i, j, k, e, ie, je, ke\rangle$ with the octonion. Then the principal isotropy group $H_p \simeq SU(3)$ and its set of fixed points in $\mathbb{R}^8 \oplus \mathbb{R}^8 \oplus \mathbb{R}^8$ is given by $V^{H_p} = \mathbb{R}\langle 1, i\rangle \oplus \mathbb{R}\langle 1, i\rangle \oplus \mathbb{R}\langle 1, i\rangle \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. The identity component of the normalizer $N(H_p)$ equals $N_0 = T^2$. It acts on V^{H_p} by

$$\begin{array}{rcl} S^1 \times S^1 \times (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}) & \to & \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \\ (\lambda, \mu), (v_1, v_2, v_3) & \mapsto & (\lambda v_1, \mu v_2, \lambda\mu v_3). \end{array}$$

Therefore, V^{H_p} is a 2-section. The induced representation (N, V^{H_p}) is a reduction of $(\mathrm{Spin}(8), \rho_8 \oplus \Delta_8^+ \oplus \Delta_8^-)$, with $\dim N = 2$. Since the quotient of the above T^2-action does not split, the same is true for the quotient of $(\mathrm{Spin}(8), \rho_8 \oplus \Delta_8^+ \oplus \Delta_8^-)$. In [**10**] it is proven that the representation $(\mathrm{Spin}(8), \rho_8 \oplus \Delta_8^+ \oplus \Delta_8^-)$ is not polar. Furthermore, the cohomogeneity of $(\mathrm{Spin}(8), \rho_8 \oplus \Delta_8^+ \oplus \Delta_8^-)$ is equal to

$$c(\rho) = 3 \cdot 8 - \dim \mathrm{Spin}(8) + \dim H_p = 24 - 28 + 8 = 4.$$

Since a non-splitting representation of abstract copolarity 1, has cohomogeneity 3, (N, V^H) is a minimal reduction. Each representation $(\mathrm{Spin}(8), \rho_8)$, $(\mathrm{Spin}(8), \Delta_8^+)$, $(\mathrm{Spin}(8), \Delta_8^-)$ is transitive on the unit sphere S^7. Furthermore, from [**4**] we know that for each sub representation $(\mathrm{Spin}(8), \rho_8 \oplus \Delta_8^\pm), (\mathrm{Spin}(8), \Delta_8^+ \oplus \Delta_8^-)$ are polar. It principal orbits are $S^7 \times S^7$ and the sub representations have cohomogeneity 2.

Example. The representation $(SO(2) \times SO(n), \rho_2 \otimes \rho_n \oplus \rho_2^k)$ $(n \geq 3,\ k \in \mathbb{Z} - \{0\})$ has cohomogeneity 4. It admits a 2-section, whose induced representation is a minimal reduction.

The principal isotropy group equals $H_p = \mathrm{Id} \times (\mathrm{Id}, SO(n-2)) \subset SO(2) \times SO(n)$. Unfortunately, for $n = 3$ it is trivial. So assume $n > 3$, then the set of fixed points equals $V^{H_p} = \mathbb{R}^2 \otimes (\mathbb{R}\langle 1, 0, \ldots, 0\rangle \oplus \mathbb{R}\langle 0, 1, \ldots, 0\rangle) \oplus \mathbb{R}^2$ and the normalizer $N(H_p) = SO(2) \times (O(2), \det(O(2)) \cdot \mathrm{Id}_{n-2}) \simeq SO(2) \times O(2)$. The representation (N, V^{H_p}) is orbit equivalent to $(SO(2) \times O(2), \rho_2 \otimes \rho_2 \oplus \rho_2^k)$, where only the first $SO(2)$ factor acts on the last entry. We identify $\mathbb{R}^2 \otimes \mathbb{R}^2$ with the set of real 2×2 matrices $\mathrm{M}(2 \times 2, \mathbb{R}) \simeq \mathbb{R}^4$. The representation $(SO(2) \times SO(2), \rho_2 \otimes \rho_2)$ is given by

$$\begin{array}{rcl} SO(2) \times SO(2) \times \mathrm{M}(2 \times 2, \mathbb{R}) & \to & \mathrm{M}(2 \times 2, \mathbb{R}) \\ (A, B, X) & \mapsto & AXB^{-1}. \end{array}$$

An invariant subspace is the set $\mathbb{M}$ of symmetric matrices with trace 0. With respect to the natural metric let $\mathbb{M}^{\perp}$ be the orthogonal complement. An easy calculation shows that $(SO(2) \times SO(2), \mathbb{M} \oplus \mathbb{M}^{\perp})$ is equivalent to

$$\begin{array}{rcl} S^1 \times S^1 \times (\mathbb{C} \oplus \mathbb{C}) & \to & \mathbb{C} \oplus \mathbb{C} \\ (\lambda, \mu), (v_1, v_2) & \mapsto & (\lambda\mu v_1, \lambda\bar{\mu} v_2). \end{array}$$

Therefore, (N_0, V^{H_p}) is given by

$$\begin{array}{rcl} S^1 \times S^1 \times (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}) & \to & \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \\ (\lambda, \mu), (v_1, v_2, v_3) & \mapsto & (\lambda\mu v_1, \lambda\bar{\mu} v_2, \lambda^k v_3). \end{array}$$

We see that (N_0, V^{H_p}) has cohomogeneity 4 and is not polar. The quotient V^{H_p}/N_0 does not split, hence the same is true for the quotient of the original representation. Since a non-splitting representation of abstract copolarity 1, has cohomogeneity 3, (N, V^{H_p}) is a minimal reduction.

For $n = 3$ the representation $(SO(2) \times SO(3), \rho_2 \otimes \rho_3 \oplus \rho_2^k)$ is orbit equivalent to $(SO(2) \times O(3), \rho_2 \otimes \rho_3 \oplus \rho_2^k)$. The principal isotropy group of $\rho_2 \otimes \rho_3$ is given by $(A, A) \times (A, A, B)$, where $A, B = \pm 1$. Intersection with the trivial principal isotropy of the second summand yields $H_p = (\mathrm{Id}_2) \times (\mathrm{Id}_2, \pm 1)$. Its normalizer is $N(H_p) = SO(2) \times (O(2) \times \pm 1)$. The set of fixed points is $V^{H_p} = \mathbb{R}^2 \otimes (\mathbb{R}\langle 1, 0, 0\rangle \oplus \mathbb{R}\langle 0, 1, 0\rangle) \oplus \mathbb{R}^2$ and (N, V^{H_p}) is orbit equivalent to $(SO(2) \times O(2), \rho_2 \otimes \rho_2 \oplus \rho_2^k)$ as in the former cases. Again V^{H_p} is a 2-section and (N, V^{H_p}) a minimal reduction.

Example. The representation $(SO(2) \times SO(n), \rho_2 \otimes \rho_n \oplus \rho_2^{k_1} \oplus \rho_2^{k_2} \ldots \oplus \rho_2^{k_l})$ $(n \geq 3,\ k_i \in \mathbb{Z} - \{0\})$ admits a 2-section, whose induced representation is a minimal reduction.

Proof. The principal isotropy H_p equals the principal isotropy group of the last example. Let $\Sigma_1 = V^{H_p}$ be the 2-section of the representation $(SO(2) \times SO(n), \rho_2 \otimes \rho_n \oplus \rho_2^{k_1}$ as in the last example, then $\Sigma_1 \oplus \mathbb{R}^2_{k_2} \oplus \ldots \oplus \mathbb{R}^2_{k_l}$ is a 2-section for $(SO(2) \times SO(n), \rho_2 \otimes \rho_n \oplus \rho_2^{k_1} \oplus \rho_2^{k_2} \ldots \oplus \rho_2^{k_l})$. Recall that for a representation of abstract copolarity k, each reduction to an invariant subspace has abstract copolarity $\leq k$. Since, $(SO(2) \times SO(n), \rho_2 \otimes \rho_n \oplus \rho_2^{k_1})$ has abstract copolarity 2, $(SO(2) \times SO(n), \rho_2 \otimes \rho_n \oplus \rho_2^{k_1} \oplus \rho_2^{k_2} \ldots \oplus \rho_2^{k_l})$ has abstract copolarity 2. □

CHAPTER 3

Copolarity of Singular Riemannian foliations

The orbits of an isometric action (G, M) decompose the manifold M into equidistant submanifolds, not necessarily all of the same dimension. Roughly speaking, if we drop the condition that the submanifolds are homogeneous, we call such a decomposition a singular Riemannian foliation. In this chapter we define generalized sections for singular Riemannian foliations and generalize our results from Chapter 1. In Subsection 3.2 we will prove the existence of 1-sections for a singular Riemannian foliation of a simply connected space form.

3.1. Singular Riemannian foliations

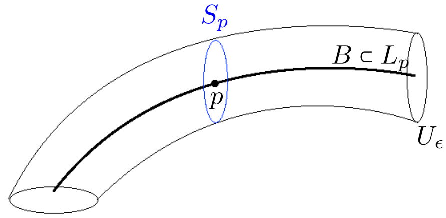
Let M be a complete Riemannian manifold. A decomposition $\mathcal{F}$ of M into connected immersed submanifolds (called *leaves*) is called a *transnormal system* if a geodesic perpendicular to one leaf is perpendicular to all leaves it meets. Denote by $\mathfrak{X}(\mathcal{F})$ the module of smooth vector fields, which are everywhere tangent to the leaves of $\mathcal{F}$. A transnormal system $\mathcal{F}$ is called a *singular Riemannian foliation* if $\mathfrak{X}(\mathcal{F})$ is transitive on each leaf. We denote a singular Riemannian foliation by $(M, \mathcal{F})$. A singular Riemannian foliation is called a *regular Riemannian foliation* if all leaves have the same dimension. For an introduction to regular and singular Riemannian foliations we refer to the book of P. Molino [**19**].

Example. The standard model of a singular Riemannian foliation is the orbit decomposition of an isometric action (G, M) of a connected group G. The geodesics are everywhere perpendicular to the leaves and the module $\mathcal{K}$ of Killing fields, induced by the G-action is transitive on each leaf.

In this work we always assume the leaves of a singular Riemannian foliation to be closed.

Decomposition of M. In the following we will define strata and an analogue of the Slice Theorem for singular Riemannian foliation. All proofs can be found in [**19**].

Let $(M, \mathcal{F})$ be a singular Riemannian foliation, $p \in M$ and let B be a relative compact connected open neighborhood of p in the leaf L_p through p. Let $\nu_r B$ be the set of r-discs in the normal bundle νB. Then there exists an $\epsilon > 0$, such that $\exp : \nu_\epsilon B \to U_\epsilon$ is a diffeomorphism onto a tubular neighborhood U_ϵ of B.



The connected components of $L_p \cap U_\epsilon$, for $L_p \in \mathcal{F}$, define a singular Riemannian foliation $(U_\epsilon, \mathcal{F}|_{U_\epsilon})$. Its leaves are called *plaques.* The plaque P_p through p is entirely contained in the cylinder C_δ^B of radius δ around B, where $\delta = d(p, B)$. Therefore, the distance between the neighboring leaves is locally constant. The orthogonal projection $\pi : U_\epsilon \to B$, restricted to a plaque is a surjective submersion. The set $S_p = \pi^{-1}(p)$ is called *geodesic slice at* p and the neighborhood U_ϵ is called a *distinguished open neighborhood of* p.

We can now formulate a first important local result, due to P. Molino [**19**]. Let U_ϵ be a distinguished open neighborhood of p. The homothetic transformation $h_\lambda : \nu B \to \nu B, \xi \mapsto \lambda\xi$, for $\lambda > 0$, induces via the diffeomorphism $\exp : \nu_\epsilon B \to U_\epsilon$, a differentiable map $h_\lambda : U_\epsilon \to U_\epsilon$. Then for λ small enough, the cylinder $C_\delta^B \subset U_\epsilon$ of radius δ is mapped to the cylinder $C_{\lambda\delta} B \subset U_\epsilon$. Moreover, h_λ leaves the induced foliation $(U_\epsilon, \mathcal{F}|_{U_\epsilon})$ invariant.

Lemma 3.1.1 (Homothetic-Transformation-Lemma). [**19**] *The homothetic transformation $h_\lambda : U_\epsilon \to U_\epsilon$ sends plaques to plaques.*

Let M_d denote the union of all leaves of dimension d in M. A connected component of M_d is called a *stratum.* If n is the maximal dimension among all leaf dimensions, the corresponding stratum M_n is called the *regular stratum* (M_{reg}), otherwise we call a stratum *singular.* A point is called *regular* if it lies on a leaf of maximal dimension, otherwise we call it *singular.* Note that a regular point of an isometric action is either principal or exceptional.

Proposition 3.1.2. [**19**] *The strata are embedded submanifolds of M. Furthermore, a horizontal geodesic in a distinguished open neighborhood of p which is tangent to the stratum through p stays in this stratum until it leaves the neighborhood.*

As an immediate consequence Molino proves that the regular stratum M_{reg} is an open connected and dense subset of M. The restriction of the singular Riemannian foliation $(M, \mathcal{F})$ to a stratum is a regular Riemannian foliation $(M_d, \mathcal{F}|_{M_d})$. Following [**19**] closed leaves of a regular foliation are embedded and we summarize

Lemma 3.1.3. *Let $(M, \mathcal{F})$ be a singular Riemannian foliation with closed leaves. Then the leaves are embedded.*

Since each leaf is an embedded submanifold there exists a global ϵ-tube (cf. [**20**]). The distance of the leaves is globally constant and as in the case of isometric actions this defines a metric on the space of leaves $M/\mathcal{F}$.

In [**17**] it is proven that the plaques are even invariant under negative homothetic transformation.

LEMMA 3.1.4. [**17**] *Let γ be a horizontal geodesic in M and let $d(\gamma)$ be the maximal dimension of $L_{\gamma(t)}$. Then for all but discrete many t, the leaf $L_{\gamma(t)}$ has dimension $d(\gamma)$.*

In particular, singular points are isolated along horizontal geodesics starting in the regular stratum M_{reg}.

Infinitesimal foliation. In the case of an isometric action (G, M), the induced foliation on a geodesic slice is equivalent to the foliation given by the orbits of the slice representation. We now want to introduce a similar concept for a singular Riemannian foliation $(M, \mathcal{F})$.

Let $p \in M$ and L_p be the leaf through p with $\dim L_p = d$. Then there exists linear independent vector fields $X^1, \dots, X^d \in \mathfrak{X}(\mathcal{F})$ spanning the tangent space T_pL_p. In the following let ϕ^i denote the corresponding local flows. Since the X^i are everywhere tangent to the leaves, their flows leave $\mathcal{F}$ invariant. Let $X \in T_pM$, then the decomposition $T_pM = T_pL_p \oplus \nu_pL_p$ induces a unique decomposition of $X = \sum_{i=1}^d t_i X_p^i + v$, with $v \in \nu_pL_p$.

LEMMA 3.1.5. *There exists a neighborhood U of 0 in T_pM, such that*

$$\begin{aligned} \varphi : \quad U \quad &\to \varphi(U) \\ (X, v) \quad &\mapsto \phi_{t_1}^1 \circ \dots \circ \phi_{t_d}^d \circ \exp_p(v) \end{aligned}$$

is a diffeomorphism.

PROOF. The map

$$\begin{aligned} \varphi : \quad T_pL + \nu_pL \quad &\to M \\ (X, v) \quad &\mapsto \phi_{t_1}^1 \circ \dots \circ \phi_{t_d}^d \circ \exp_p(v) \end{aligned}$$

is differentiable. We will show that the differential in 0 equals $\mathrm{d}\varphi_0 = Id_{T_pM}$. We identify $T_0(T_pM)$ with T_pM as usual. Then for X^i as above

$$\mathrm{d}\varphi_0(X_p^i, 0) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \phi_t^i(p) = X_p^i.$$

Choose $v \in \nu_pL$, then

$$\mathrm{d}\varphi_0(0, v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp_p(tv) = v.$$

The statement follows now from the linearity of $\mathrm{d}\varphi_0$. □

The map of the previous lemma allows us to define a singular Riemannian foliation of T_pM, the so called *infinitesimal foliation*.

DEFINITION 3.1.6. Let $\mathcal{F}$ be a singular Riemannian foliation of a complete Riemannian manifold (M, g) and $p \in M$. There exists a neighborhood V of p in M and a diffeomorphism $\Phi : V \to \Phi(V) \subset T_pM$, such that $\Phi(p) = 0$ and $\mathrm{d}\Phi_p = \mathrm{Id}_{T_pM}$. Furthermore, there exists a singular Riemannian foliation $\hat{\mathcal{F}}$ of (T_pM, g_p), which coincide with $(\Phi)_*\mathcal{F}$ on $\Phi(V)$ and which is invariant under homothetic transformations $h_\lambda : T_pM \to T_pM$, where $h_\lambda(v) = \lambda v$. The foliation $\hat{\mathcal{F}}$ is called the *infinitesimal foliation*.

We can think of the infinitesimal foliation as a blow up. Set $\Phi = \varphi^{-1}$, where φ is the map of the previous lemma and identify V and $\Phi(V)$. Then we define a metric g_λ on $V^\lambda = h_\lambda(V)$ through $g^\lambda = \lambda^2(h_\lambda)_* g'$, where $g' = (\Phi_*)g$ is the push forward. Then $\bigcup_{\lambda>0} V^\lambda = T_pM$ and it follows that

LEMMA 3.1.7. *On compact sets the metric $g^\lambda = \lambda^2(h_\lambda)_* g'$ tends smoothly to the flat metric g_p on T_pM. Furthermore, the restriction of $\hat{\mathcal{F}}$ to V^λ is a singular Riemannian foliation with respect to g^λ.*

From the construction of φ we deduce that a leaf $\hat{L}$ of the infinitesimal foliation splits in $\hat{L} = T_pL_p \times \hat{L}_2$, such that the foliation $(\nu_pL_p, \hat{\mathcal{F}}_2)$ is equivalent to the induced foliation on the geodesic slice S_p given by the intersections of the leaves.

3.2. Generalized sections for singular Riemannian foliations

In this section we will define generalized sections for singular Riemannian foliations and generalize the results of Chapter 1. As in the homogenous case we will require that a generalized section is an embedded submanifold, although at least sections, i.e. 0-sections, of a singular Riemannian foliation are in general assumed to be immersed. We restrict our attention to simply connected space forms, then our definition is no restriction.

DEFINITION 3.2.1. Let M be a complete Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliations of M with closed leaves. A submanifold $\Sigma \subseteq M$ is called a k-section of $\mathcal{F}$, if the following conditions hold

- (C1) Σ is a connected, complete, embedded and totally geodesic submanifold of M,
- (C2) Σ intersects every leaf of $\mathcal{F}$,
- (C3) for each regular leaf $L \in \mathcal{F}$ and each point $p \in \Sigma \cap L$ the normal space $\nu_pL \subset T_p\Sigma$ is a vector subspace of codimension k.

If the number k is not important we will speak of a *generalized section.* Note that at first glance the above definition is a bit stronger that the definition of a pre-section in Chapter 1, since also exceptional points are required to fulfill (C3). But condition (C3) is equivalent to $\dim(T_p\Sigma \cap \mathcal{V}_p) = k$ for all regular points, where $\mathcal{V}_p = T_pL_p$ denotes the vertical distribution. In the homogeneous case the set Σ_{pr} of principal points is dense in $\Sigma_{reg} = \Sigma \cap M_{reg}$. Since $T_p\Sigma$ and $\mathcal{V}_p$ are smooth distributions over Σ_{reg}, continuity arguments imply that (C3) is also fulfilled in exceptional points.

Let p be regular point and consider the intersection of a k_1-section Σ_1 through p with a k_2-section Σ_2 through p. Then its connected component $(\Sigma_1 \cap \Sigma_2)_\circ$ through p is again a k-section with $k \leq \min\{k_1, k_2\}$. Through every regular point there exists a unique generalized section of minimal dimension, which we call a *minimal generalized section.* Unfortunately, in the inhomogeneous case, the dimension of the minimal sections may depend on the given regular point, i.e. there could be minimal generalized sections of different dimension through different regular points.

DEFINITION 3.2.2. We say that a singular Riemannian foliation $\mathcal{F}$ of a Riemannian manifold M has *copolarity* k, if through every regular point there exists a minimal generalized section Σ with $\dim \Sigma = k + \operatorname{codim} \mathcal{F}$.

Note that we assume the leaves of $\mathcal{F}$ to be closed, then the set $\exp_p(\nu_p L)$ intersects every other leaf. Therefore, for a generalized section Σ, condition (C3) and (C1) already imply condition (C2).

In the following we list some properties of generalized sections generalizing the results of Chapter 1. We will show that the induced foliation on a generalized section Σ is again a singular Riemannian foliation, and that $T_p\Sigma$ is a generalized section of the infinitesimal foliation at every point $p \in \Sigma$.

Let $p \in \Sigma$ and denote by $\beta_p = (L_p \cap \Sigma)_0$ the connected component of the intersection containing p. Then β_p are embedded submanifolds of Σ and we will show that (Σ, β) is a singular Riemannian foliation. Our first goal is to show that every geodesic in Σ which is perpendicular to a leaf β_p is in fact a $\mathcal{F}$-horizontal geodesic, i.e. it intersect every leaf $L \in \mathcal{F}$ perpendicular. Then it is of course perpendicular to every leaf of β. Note that this assertion is clear for $\mathcal{F}$-regular points $p \in \Sigma$, since

$$T_p\Sigma = T_p\beta_p \oplus \nu_p L_p$$

in those points. In the following let $\nu_p^\Sigma \beta_p$ be the normal space of $T_p\beta_p$ in $T_p\Sigma$. The next lemma implies that the set Σ_{reg} of $\mathcal{F}$-regular points is dense in Σ.

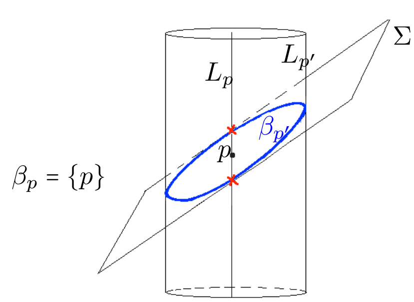
Lemma 3.2.3. *For every $p \in \Sigma$ there exists $v \in \nu_p^\Sigma \beta_p$, such that $\exp_p(v)$ is a $\mathcal{F}$-regular point. Furthermore, $\exp_p(v)$ can be chosen arbitrary close to p.*

Proof. Let $p \in \Sigma$ and let $q \in \Sigma$ be a $\mathcal{F}$-regular point. Let $\gamma(t)$ be a minimal geodesic in Σ connecting p and β_q. Then $\gamma(t)$ intersects the submanifold β_q perpendicular in the $\mathcal{F}$-regular point $\gamma(1)$. Since, q is an $\mathcal{F}$-regular point

$$\nu_q^\Sigma \beta_q = \nu_q L_q$$

and γ is in fact a horizontal geodesic of $\mathcal{F}$. Therefore, γ also intersects L_p perpendicular. Now $\gamma(t) = \exp_p(tv)$ for some $v \in T_p\Sigma \cap \nu_p L_p \subset \nu_p^\Sigma \beta_p$. Since singular points are isolated along horizontal geodesics starting in a regular leaf, there exists an $\epsilon > 0$ such that $\gamma(t) = \exp_p(tv)$ are regular points for $0 < t < \epsilon$. $\square$

Let S_p^Σ denote the geodesic slice at p of β_p in Σ, i.e. $S_p^\Sigma = \exp_p(U_\epsilon(0) \cap \nu_p^\Sigma \beta_p)$, where $U_\epsilon(0)$ is the ϵ-ball in T_pM. Denote by d_Σ, d_M the distance in Σ, M respectively. The leaves $L_p \in \mathcal{F}$ are equidistant. We will show that the distance in Σ of two neighboring leaves β_p, β_q equals the distance in M of the corresponding leaves L_p, L_q. That this is in general not the case for arbitrary totally geodesic submanifolds is illustrated in the next picture.



LEMMA 3.2.4. *Let $q \in S_p^\Sigma$ be an $\mathcal{F}$-regular point, then $d_\Sigma(\bar{q}, \beta_p)$ is constant for $\bar{q} \in \beta_q$.*

PROOF. Let η be the geodesic in Σ minimizing the distance between p and β_q and let $\tilde{q} = \eta(1)$. Then η intersects β_q perpendicular, i.e. $\dot{\eta}(1) \in \nu_{\tilde{q}}^\Sigma \beta_q = \nu_{\tilde{q}} L_q$, since $\tilde{q}$ is $\mathcal{F}$-regular. We see that η is in fact a horizontal geodesic of $\mathcal{F}$ and $\tilde{q} = \exp_p(v)$ for $v \in \nu_p L_p$. Since $\tilde{q}$ minimizes the distance between p and β_q, we conclude that $\tilde{q} \in S_p^\Sigma$ is contained in the geodesic slice. After eventually decreasing S_p^Σ we can assume that $\tilde{q}$ is contained in a distinguished neighborhood of L_p. Let $\pi : U_\epsilon(L_p) \to L_p$ be the orthogonal projection, then $\pi(\tilde{q}) = p$. Remember that $\pi|_{L_q} : L_q \to L_p$ is a submersion. We want to show that $\beta_q \subset \pi^{-1}(\beta_p)$, then $d_M(\bar{q}, \beta_p)$ is constant for $\bar{q} \in \beta_q$. Let $\bar{q} \in \Sigma_{reg} \cap U_\epsilon(L_p)$ be a point in a distinguished open neighborhood of L_p. Let $\gamma_{\bar{q}, L_p}$ be the minimizing geodesic between $\bar{q}$ and L_p in M. Then $\pi(\bar{q}) = \gamma_{\bar{q}, L_p}(1)$. But $\gamma_{\bar{q}, L_p}$ is perpendicular to L_p, hence to $L_{\bar{q}}$ and again we conclude $\dot{\gamma}_{\bar{q}, L_p}(0) \in \nu_{\bar{q}} L_{\bar{q}} \subset T_{\bar{q}}\Sigma$. Therefore, $\gamma_{\bar{q}, L_p}$ is completely contained in Σ, especially $\pi(\Sigma_{reg}) \subset \Sigma$. Since $\pi|_{L_q} : L_q \to L_p$ is continuos, connected components where mapped to connected components and $\pi(\beta_q) = \beta_p$, since $\pi(\tilde{q}) = p$. Now β_p and β_q have constant distance in M equal to $d_M(L_q, L_p)$. For $\bar{q} \in \beta_q$

$$d_\Sigma(\bar{q}, \beta_p) \geq d_M(\bar{q}, \beta_p) = L(\gamma_{\bar{q}, L_p}).$$

Since $\gamma_{\bar{q}, L_p}$ is a geodesic in Σ equality holds. Therefore, $d_\Sigma(\bar{q}, \beta_p) = d_M(\bar{q}, \beta_p) = d_M(L_q, L_p)$ is constant for all $\bar{q} \in \beta_q$. □

With the above lemma one easily proves

LEMMA 3.2.5. *Let Σ be a k-section of the singular Riemannian foliation $(M, \mathcal{F})$. Let $p \in \Sigma$ and L_p the leaf through p, then*

$$T_p\Sigma = T_p\Sigma \cap T_p L_p \oplus T_p\Sigma \cap \nu_p L_p.$$

In particular, $\nu_p^\Sigma \beta_p = T_p\Sigma \cap \nu_p L_p$.

PROOF. For every $p \in \Sigma$ we have $T_p\beta_p = T_p\Sigma \cap T_p L_p$. Therefore, it is left to prove that $\nu_p^\Sigma \beta_p = T_p\Sigma \cap \nu_p L_p$. Obviously, $T_p\Sigma \cap \nu_p L_p \subset \nu_p^\Sigma \beta_p$. For every $\mathcal{F}$-regular point $q \in S_p^\Sigma$, with $q = \exp_p(v)$ for $v \in \nu_p^\Sigma \beta_p$, the geodesic $\exp_p(tv)$ meets the distance cylinder $C_r^\Sigma(\beta_p)$ of β_p with respect to Σ perpendicular. From the last lemma we know that $\beta_q \subset C_r^\Sigma(\beta_p)$ is contained in a distance cylinder and therefore $\exp_p(tv)$ is perpendicular to β_q. Since q is $\mathcal{F}$-regular, the geodesic $\exp_p(tv)$ is in fact $\mathcal{F}$-horizontal and $v \in T_p\Sigma \cap \nu_p L_p$. Since the set of $\mathcal{F}$-regular points is open in Σ, the two vector subspaces $\nu_p^\Sigma \beta_p$ and $T_p\Sigma \cap \nu_p L_p$ coincide in an open set, hence they are equal. □

We have implicit proven that a geodesic γ in Σ, which is perpendicular to β_p is in fact perpendicular to L_p, i.e. an $\mathcal{F}$-horizontal geodesic. We can finally state

PROPOSITION 3.2.6. *Let Σ be a k-section of $(M, \mathcal{F})$. The connected components of the intersections $\Sigma \cap L$ induce a singular Riemannian foliation of Σ. If Σ is minimal, the induced foliation (Σ, β) has trivial copolarity.*

PROOF. Denote by β the foliation given by the connected components of the intersections of Σ with $\mathcal{F}$. First we have to show that $\mathfrak{X}(\beta)$, i.e. the smooth vector field on Σ which are everywhere tangent to the submanifolds β acts transitive on

β. We know that $\mathfrak{X}(\mathcal{F})$ is transitive on $\mathcal{F}$. Let $X \in \mathfrak{X}(\mathcal{F})$ and let $p \in \Sigma$. Denote the restriction to Σ by $\tilde{X} = X|_\Sigma$ and let $\mathrm{pr}^\Sigma : TM|_\Sigma \to T\Sigma, (p,v) \mapsto (p, \mathrm{pr}^{T_p\Sigma}(v))$ be the projection onto $T\Sigma$. Then $X' = \mathrm{pr}^\Sigma \circ \tilde{X}$ is smooth and the splitting condition of Lemma 3.2.5 implies that $X' \in \mathfrak{X}(\beta)$. Hence, the vector fields $\mathfrak{X}(\beta)$ are transitive on β. We are left to prove that (Σ, β) is a transnormal system. Let $p \in \Sigma$ and let γ be a geodesic in Σ starting perpendicular to β_p. Since $\nu_p^\Sigma \beta_p = T_p\Sigma \cap \nu_p L_p$ this geodesic is in fact an $\mathcal{F}$-horizontal geodesic, hence γ intersects the leaves of β perpendicular. Now assume the induced foliation (Σ, β) admits a k'-section, with $k' < k = \dim \beta$. Then this would also be k'-section for $(M, \mathcal{F})$, contradicting the minimality. □

Our next purpose is to show that for every point $p \in \Sigma$ the tangent space $T_p\Sigma$ is a generalized section for the infinitesimal foliation.

LEMMA 3.2.7. *Let $(M, \mathcal{F})$ be a singular Riemannian foliation of copolarity k. Then each infinitesimal foliation has copolarity less or equal to k.*

PROOF. Let $p \in M$ and let Σ be the minimal section through p. Then Σ is a k-section and $\dim T_p\Sigma = k + \operatorname{codim} \hat{\mathcal{F}}$. We will show that $T_p\Sigma$ is generalized section of $(T_pM, \hat{\mathcal{F}})$. Let $(X, v) \in T_p\Sigma$ be a regular point of $\hat{\mathcal{F}}$. Since $T_p\Sigma = T_p\Sigma \cap T_pL_p \oplus T_p\Sigma \cap \nu_pL_p$ we conclude that $(X,0), (0,v) \in T_p\Sigma$. The leaf $\hat{L}_{(X,v)} = T_pL_p \times \hat{L}_2$ splits, hence $\nu_{(X,v)}\hat{L}_{(X,v)} = (X,v) + W$, where W is the normal space of $\hat{L}_2$ in ν_pL_p. Then the condition (C3) of a generalized section is equivalent to $W \subset T_p\Sigma$. We can also assume that $\|v\| < \epsilon$, since $\hat{\mathcal{F}}$ and $T_p\Sigma$ are invariant under homothetic transformations. Then $\exp_p(\hat{L}_2)$ is contained in a $\mathcal{F}$-regular leaf. The condition $W \subset T_p\Sigma$ is equivalent to $\nu_p\Sigma \subset W^\perp$. Now $\nu_p\Sigma = \nu_p\Sigma \cap T_pL_p \oplus \nu_p\Sigma \cap \nu_pL_p$ and we have to show that $\nu_p\Sigma \cap \nu_pL_p$ is tangent to $\hat{L}_2$ at v. Therefore, let $w \in \nu_p\Sigma \cap \nu_pL_p$ and let J be the Jacobi field along the geodesic $\exp_p(tv)$ with starting conditions $J(0) = 0$ and $J'(0) = w$. Note that the geodesic $\exp_p(tv)$ is entirely contained in Σ and $q = \exp_p(v)$ is $\mathcal{F}$-regular. Then

$$J(t) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp_p(t(v + sw))$$

is everywhere perpendicular to Σ and tangent to the slice S_p, i.e. $J(1) \in \nu_q\Sigma \cap T_qS_p \subset T_qL_q \cap T_qS_p$. Since $J(1) = (\mathrm{d}\exp_p)_v(w)$, we conclude that

$$w \in (\mathrm{d}\exp_p)_v^{-1}(T_qL_q \cap T_qS_p) = T_v\hat{L}_2.$$

For an arbitrary regular point (Y, v') of the infinitesimal foliation, the point $q' = \exp_p(v')$ is an $\mathcal{F}$-regular point, hence there exists a k-section Σ' through it. Then $T_p\Sigma'$ is a k-section through (Y, v') and the copolarity of the infinitesimal foliation is less or equal to k. □

Existence of small generalized sections in simply connected space forms. We will finally prove, in analogy to Theorem 2.6, the existence of 1-sections for singular Riemannian foliation of simply connected space forms. The tangent spaces to the leaves form a smooth distribution $\mathcal{V}$ over the regular stratum M_{reg}. We denote the orthogonal distribution by $\mathcal{V}^\perp = \mathcal{H}$ and call it the *horizontal distribution.* In the following a 0-section is simply called a section and the next important theorem, due to M. Alexandrino, states that even for singular Riemannian foliations the existence of sections is equivalent to the integrability of the horizontal distribution.

THEOREM 3.2.8. [**1**] *Let $(M, \mathcal{F})$ be a a singular Riemannian foliation of a simply connected space form. Then $(M, \mathcal{F})$ admits sections if and only if the horizontal distribution $\mathcal{H}$ over the regular stratum is integrable.*

In the following, let M be a simply connected space form and $\mathcal{F}$ a singular Riemannian foliation without sections. If the singular Riemannian foliation has copolarity 1, there exists a 1-section through every regular point, i.e. there exists a 1-dimensional vertical distribution, such that $\mathcal{D} \oplus \mathcal{H}$ is integrable. Since a generalized section Σ is totally geodesic $\nabla^v_{\mathcal{D}}\mathcal{D} \subset \mathcal{D}$, i.e. $\mathcal{D}$ is vertical autoparallel.

Assume there exists a 1-dimensional vertical autoparallel distribution $\mathcal{D}$, i.e. $\nabla^v_{\mathcal{D}}\mathcal{D} \subset \mathcal{D}$, over the regular stratum M_{reg}, such that $\mathcal{D} \oplus \mathcal{H}$ is integrable. To prove the existence of a generalized section, we need an analogue of the O'Neill tensor for the regular Riemannian foliation $(M_{reg}, \mathcal{F}_{reg})$. Every regular Riemannian foliation is locally given by a Riemannian submersion (cf. [**13**]), hence locally the O'Neill tensor A exists. Like in the homogeneous case, the non-existence of sections implies that the horizontal distribution over the regular stratum is not integrable. Integrability is a local property, hence each tensor A does not vanish. We can argue as in the homogenous case (cf. Proposition 1.2.5) and show that $\mathcal{D}_p = \operatorname{Im} A_p$ for almost every $p \in M_{reg}$. Now the proof of Theorem I can be applied to prove the existence of an embedded submanifold Σ, tangent to $\mathcal{D} \oplus \mathcal{H}$, which meets every leaf. We have to verify condition (C3). Let β denote the connected component of $\Sigma \cap L_p$ through p, then (C3) is satisfied for β. Let $p' \in \Sigma$ be another regular point and let γ be a shortest geodesic in Σ connecting p' and β. Then γ is in fact a horizontal geodesic of $(M, \mathcal{F})$, entirely contained in Σ. In [**17**] is proven that the set of Jacobi fields W^γ, which are everywhere tangent to the leaves of $(M, \mathcal{F})$ along a horizontal geodesic γ, span every tangent space $T_{\gamma(t)}L_{\gamma(t)}$. In regular points a vector field $J \in W^\gamma$ has starting conditions

$$J(0) \in T_{\gamma(0)}L_{\gamma(0)} \text{ and } J'(0) = -A^*_{\dot\gamma}(J(0)) - S_{\dot\gamma}(J(0)),$$

where A^* is the point wise conjugate of the O'Neill tensor A. Since $D_p = \operatorname{Im} A_p$, the vertical orthogonal complement $\mathcal{E}$ of $\mathcal{D}$ is given by

$$\mathcal{E}_p = \bigcap_{x \in \mathcal{H}_p} \ker A^*_x.$$

Now $\mathcal{E}_p = \nu_p\Sigma$ equals the normal space of Σ. For each $J \in W^\gamma$, with $J(0) \in \mathcal{E}_p = \nu_p\Sigma$ follows $J'(0) = -S_{\dot\gamma}(J(0)) \in \mathcal{E}_p$ and J stays perpendicular to Σ along γ. This proves (C3) and Σ is a 1-section for $(M, \mathcal{F})$.

THEOREM IV. *Let M be a simply connected space form and $\mathcal{F}$ a singular Riemannian foliation without sections. Then $(M, \mathcal{F})$ has copolarity* 1 *if and only if there exists a* 1*-dimensional vertical autoparallel distribution $\mathcal{D}$, i.e. $\nabla^v_{\mathcal{D}}\mathcal{D} \subset \mathcal{D}$, over the regular stratum M_{reg}, such that $\mathcal{D} \oplus \mathcal{H}$ is integrable.*

Orbifolds

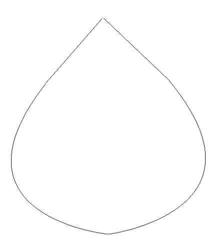
In this appendix we give a short introduction to Riemannian orbifolds and add the missing proof of Proposition 2.2.11. The following statements can be found in [**6**].

A *Riemannian orbifold* of dimension n is a topological Hausdorff space Q with an *Riemannian orbifold structure*, given by the following data:

(1) An open cover $(U_i)_{i \in I}$ of Q by a set I, closed under finite intersections.
(2) For each $i \in I$ there is a finite subgroup Γ_i of the isometry group of a simply connected n-dimensional Riemannian manifold X_i, and a continuous map $q_i : X_i \to U_i$, called a *chart*, such that q_i induces a homeomorphism from X_i/Γ_i onto U_i.
(3) For all $x_i \in X_i$ and $x_j \in X_j$ with $q_i(x_i) = q_j(x_j)$, there is an isometry h from an open connected neighborhood W of x_i to a neighborhood of x_j such that $q_j \circ h = q_i|_W$. Such a map h is called a *change of chart*.

The family $\mathcal{A} = (X_i, q_i)_{i \in I}$ is called an *atlas* for the orbifold structure on Q. Two atlases $\mathcal{A} = (X_i, q_i)_{i \in I}$ and $\mathcal{A}' = (X_j, q_j)_{j \in J}$ define the same orbifold structure on Q if $(X_k, q_k)_{k \in I \cup J}$ satisfies the compatibility condition (3). The action of Γ_i on X_i is assumed to have trivial principal isotropy groups. Then, for $i = j$ a change of chart h is the restriction of an element of Γ_i.

Let Q be a Riemannian orbifold and define a metric on Q, by requiring that each chart $q_i : X_i \to U_i$ induces an isometry between X/Γ_i and U_i. In the following, we will always think of Q as a metric space equipped with the metric coming from the underlying Riemannian orbifold structure.



Teardrop

Example. A Riemannian manifold is a Riemannian orbifold, locally isometric to $B_r(x)/\{e\}$, where $B_r(x)$ denote ball of radius r around $x \in M$. A Riemannian manifold with boundary is also a Riemannian orbifold with charts $B_r(x)/\mathbb{Z}_2$, where

$B_r(x)$ is the doubling of the ball around x in boundary points and $\mathbb{Z}_2$ acts by a reflection. The teardrop is a Riemannian orbifold. The singular point has a neighborhood isometric to $B_r(x)/\mathbb{Z}_n$, where $B_r(x)$ denotes a ball in the two sphere.

Let (X, q) be a chart with $x \in q(X) = U$. For each $p \in q^{-1}(x)$ the isotropy group $(\Gamma)_p$ is independent of the chart U_i, up to an isomorphism of groups. We call it the *local group at* x and denote it by Γ_x. A *stratum* $Q_{(G)}$ *of type* G in an orbifold, is a connected component of the set of points $x \in Q$ with local group isomorphic to G.

Proposition A.0.9. *Let Q be a Riemannian orbifold. Then each stratum is a manifold.*

Proof. The orbifold Q is locally isometric to X/Γ, where Γ is a finite group of isometries of the Riemannian manifold X. For $G < \Gamma$, the stratum $Q_{(G)}$ is therefore locally isometric to the stratum of G in X/Γ, which is a manifold (cf. Chapter 1). □

Note that the *regular stratum* $Q_{(e)}$ is open and dense in Q. The other strata are called *singular*. A point is called *regular* if is contained in a regular stratum, otherwise we call appoint *singular*.

Covering orbifolds. A *covering orbifold* of a Riemannian orbifold Q is a Riemannian orbifold Q' with a surjective projection $p : Q' \to Q$, such that each point $x \in Q$ has a neighborhood $U = X/\Gamma$ for which each connected component U'_i of $p^{-1}(U)$ is isometric to X/Γ_i, where $\Gamma_i \leq \Gamma$ is a subgroup. The isometries must respect the projection.

Lemma A.0.10. *Let M be a complete Riemannian manifold and let Γ be a discrete subgroup of isometries of M. Then M/Γ is a Riemannian orbifold and the natural projection is an orbifold covering.*

Proof. Each point $x \in M$ has a ball neighborhood $B_r(x)$, where the action is given by the action of the isotropy group Γ_x. Since the action is proper, the isotropy groups are compact, hence finite. The open sets $B_r(x)/\Gamma_x$ cover M/Γ and the projections $q_x : B_r(x) \to B_r(x)/\Gamma_x$ are the restriction of the natural projection. Let $\bar{p} \in B_r(x)/\Gamma_x \cap B_\delta(y)/\Gamma_y$ and choose points $p_1 \in B_r(x)$ and $p_2 \in B_\delta(y)$, which projects to $\bar{p}$. Then $p_1 \in \Gamma p_2$ and there exists an element $g \in \Gamma$ such that $g(p_1) = p_2$. Let W be a neighborhood of p_1 in $B_r(x)$, such that $g(W) \subset B_\delta(y)$, then for $p \in W$ we have $\Gamma(gp) = \Gamma p$, i.e. $q_x(p) = q_y \circ g(p)$ and the restriction of g to W is a change of charts. The last statement follows immediately, since the manifold M has the orbifold charts $U = B_r(x)/\{e\}$. □

A Riemannian orbifold is called *good* if it is the orbit space of a discrete isometric group action on a Riemannian manifold, as in the previous lemma. For example the orbit space of a polar isometric action is a good Riemannian orbifold. For a good Riemannian orbifold M/Γ, an *orbifold geodesic* $\bar{\gamma}$ is the projection of a geodesic in $\gamma \in M$. We set $\mathrm{ind}(\bar{\gamma}) = \mathrm{ind}(\gamma)$.

Let Q be a Riemannian orbifold and $\mathcal{A} = (X_i, q_i)_{i \in I}$ a maximal atlas. A homeomorphism $f : Q \to Q$ is called an *orbifold-isometry* if for each point $x \in Q$ and each pair (X_i, Γ_i, q_i) and (X_j, Γ_j, q_j) of charts of $\mathcal{A}$, such that $x \in U_i = X_i/\Gamma_i$ and

$f(U_i) = U_j = X_j/\Gamma_j$, there exists a lift $f_{ij} : X_i \to X_j$ that is an isometry of Riemannian manifolds. An orbifold-isometry $f : Q \to Q$ respects the metric of Q. Therefore, the restriction of an orbifold-isometry to the regular stratum $Q_{(e)}$ is an isometry of the Riemannian manifold $Q_{(e)}$. Let $p : Q' \to Q$ be an orbifold covering of Riemannian orbifolds. Then a *deck transformation* $f : Q' \to Q'$ is an orbifold-isometry, such that $p \circ f = p$. The set of all deck transformations form a group Γ and $Q'/\Gamma = Q$.

Lemma A.0.11. *Let Q be a Riemannian orbifold and Γ a discrete group of isometries of Q, then Q/Γ is a Riemannian orbifold.*

Proof. Since the elements of Γ are orbifold-isometries, the action is proper (cf. [**3**]), hence the isotropy groups Γ_x are finite. There exist a neighborhood U of x invariant under the action of Γ_x, such that Γ_x parameterizes the orbits of Γ in this neighborhood and U/Γ_x is a neighborhood in Q/Γ. Without loss of generality we assume this neighborhood to be a chart domain $U = X_i/\Gamma_i$. Then each $f \in \Gamma_x$ is an orbifold-isometry, i.e. $f : X_i/\Gamma_i \to X_i/\Gamma_i$ is an isometry, which comes from an isometry $f_{ii} : X_i \to X_i$ per definition. The set of all lifts of Γ_x is a group $\tilde{\Gamma}_x$ of isometries of X_i, such that Γ_i is normal in $\tilde{\Gamma}_x$ and $\tilde{\Gamma}_x/\Gamma_i \simeq \Gamma_x$. In fact, $\tilde{\Gamma}_x$ is finite and $X_i/\tilde{\Gamma}_x = U/\Gamma_x$ is a chart domain for Q/Γ. Since $\Gamma_i \leq \tilde{\Gamma}_x$, the projection $Q \to Q/\Gamma$ is an orbifold covering. □

A *universal covering orbifold* is an orbifold covering $\tilde{p} : \tilde{Q} \to Q$, where $\tilde{Q}$ is a connected Riemannian orbifold such that for every orbifold covering $p' : Q' \to Q$, where Q' is a connected Riemannian orbifold, there exists an orbifold covering $p : \tilde{Q} \to Q'$ such that the following diagram commutes

$$\begin{tikzcd} \tilde{Q} \arrow[rr, "p"] \arrow[dr, "\tilde{p}"'] & & Q' \arrow[dl, "p'"] \\ & Q & \end{tikzcd}$$

The group of deck transformation of the universal orbifold is called the *orbifold fundamental group* and we denote it by $\pi_1^{orb}(Q)$.

Thursten proved in [**26**] the existents of a universal covering orbifold, a detailed proof can be found in [**8**].

Proposition A.0.12 (Thurston). *Let Q be a connected Riemannian orbifold. Then there exists a universal covering orbifold $\tilde{Q}$ unique up to covering isomorphism. Moreover, the orbifold fundamental group of Q acts transitive and free on the preimage of a regular point $x \in Q$.*

A *stratum of codimension* 1 in a Riemannian orbifold Q is a stratum of type $\Gamma_q = \mathbb{Z}/2\mathbb{Z}$. The closure of the union of all strata of codimension 1 is called the *boundary* of Q and will be denoted by ∂Q.

Proposition A.0.13. *Let Q be a Riemannian orbifold with $\pi_1^{orb}(Q) = 1$. Then Q has no strata of codimension* 1.

Proof. Assume there exists a codimension 1 strata $Q_{(\mathbb{Z}/2\mathbb{Z})}$. Let Q' be the Riemannian orbifold given by doubling Q, i.e. take two copies of Q and glue them along the boundary. For every boundary point $q \in Q$ replace the local chart $U_q =$

$X/(\mathbb{Z}/2\mathbb{Z})$ by X. Then $p : Q' \to Q$ is an orbifold covering, which induces a double cover of the regular part, contradicting $\pi_1^{orb}(Q) = 1$. □

Reflections in orbifolds. An isometry of the Riemannian orbifold Q is called a *reflection* if its restriction to the regular stratum $Q_{(e)}$ fixes a submanifold of codimension 1. Let Q be a connected Riemannian orbifold and Γ a discrete group of isometries. We denote by Γ_{refl} the subgroup of Γ that is generated by reflections of Q which are contained in Γ. Since the conjugate of a reflection is a reflection, Γ_{refl} is a normal subgroup. The following statements can be found in [**11**].

LEMMA A.0.14. *Let Q be a connected Riemannian orbifold and let Γ be a discrete group of isometries of Q. Let Q' be the orbifold $Q' = Q/\Gamma$ and $Q'_* = Q' - \partial Q'$. If the orbifold fundamental group of Q'_* is trivial, then Γ is generated by reflections.*

PROOF. The quotient group $\bar{\Gamma} = \Gamma/\Gamma_{\text{refl}}$ acts by isometries on $\bar{Q} = Q/\Gamma_{\text{refl}}$ with quotient Q'. We want to show that $\bar{Q} \to Q'$ induces an orbifold covering of $\bar{Q}_* \to Q'_*$. For this, assume that an element $\omega \in \bar{\Gamma}$ acts as a reflection on $\bar{Q}$. Then there exists a point $\bar{p} \in \bar{Q}_{(e)}$, which is only fixed by ω and therefore projects to a point p' contained in a stratum of codimension 1 in Q'. A lift $p \in Q$ of p' is contained in the regular stratum $Q_{(e)}$ and is therefore fixed by a reflection in Γ not contained in Γ_{refl}, which contradicts the definition of Γ_{refl}. Hence, no element of $\bar{\Gamma}$ acts as a reflection on $\bar{Q}$ and the projection $\bar{Q} \to Q'$ has the property that the preimage of a boundary point in Q' is a boundary point in $\bar{Q}$. Therefore, the preimage of Q'_* is exactly $\bar{Q}_* = \bar{Q} - \partial\bar{Q}$, i.e. a connected orbifold. Then $Q'_* = \bar{Q}_*/\bar{\Gamma}$ and $\bar{Q}_*$ is an orbifold covering. But $\pi_1^{\text{orb}}(Q'_*) = 1$, therefore $\bar{\Gamma}$ acts trivial on the dense set $\bar{Q}_*$ and hence on all of $\bar{Q}$, this implies $\Gamma = \Gamma_{\text{refl}}$. □

Orbifold points in quotients. For a representation (G, V) let $X = V/G$ denote the space of orbits and $\pi : V \to V/G$ the corresponding projection. A point $\bar{p} \in X$ is called an *orbifold point* if it has a neighborhood isometric to a Riemannian orbifold. We denote the set of all orbifold points in X by X_{orb}. In [**17**] is proven that $\bar{p} \in X_{orb}$ if and only if the slice representation of each lift $p \in \pi^{-1}(\bar{p})$ is polar. Therefore, X_{orb} contains the principal stratum and the strata of quotient codimension 1 and 2.

REMARK A.0.15. The notion of boundary in V/G and X_{orb} coincide. Let $\bar{p}$ be a point on a codimension 1 strata of V/G, then the slice representation $(G_p, \mathcal{H}_p^+)$ is transitive on the unit sphere and therefore admits a 1-dimensional section $\mathbb{R}$. Hence, $\bar{p}$ has a neighborhood $U_{\bar{p}}$ homeomorphic to $\nu^{G_p} \times \mathcal{H}_p^+/G_p = \mathbb{R}^k \times \mathbb{R}/(\mathbb{Z}/2\mathbb{Z})$, i.e. $\Gamma_{\bar{p}} = \mathbb{Z}/2\mathbb{Z}$, and $\bar{p}$ lies on a codimension 1 strata in the Riemannian orbifold X_{orb}. Note that after changing the metric on the section $\nu^{G_p} \times \Sigma_p$, the neighborhood U_p is even isometric to the Riemannian orbifold $\nu^{G_p} \times \Sigma_p/\Gamma_{\bar{p}}$.

PROPOSITION A.0.16. [**16**] *Let (H, W) be a representation of a connected compact group H. Then the set B of non-singular orbits in W/H is a Riemannian orbifold with $\pi_1^{\text{orb}}(B) = 1$.*

The next theorem characterizes the singular points in X_{orb}.

THEOREM A.0.17. *Let (H, W) be a representation of a connected compact group H. Let X_{orb} be the set of orbifold points in W/H and set $X_* = X_{orb} - \partial X_{orb}$. Then X_* is exactly the set of non-singular H-orbits. Moreover, X_* has trivial orbifold fundamental group.*

Proof. The second statement follows from the last proposition. Let q be a singular point which projects to an orbifold point $\bar{q} \in X_{orb}$. A neighborhood $U_{\bar{q}}$ is isometric to S_q/H_q, where S_q is the normal slice at q. Since $\bar{q} \in X_{orb}$, the slice representation is polar. Let Σ_q be a section of the slice representation and denote by Σ_q^ϵ a neighborhood of 0 in Σ_q. Then after changing the metric on Σ_q^ϵ (cf. [**17**]) we can assume that $U_{\bar{q}}$ is isometric to Σ_q^ϵ/W, where $W \subset H_q$ denotes the Weyl group of Σ_q. Now W is a Coxeter group, i.e. generated by reflections, and Σ_q^ϵ/W has boundary. Since the boundary is closed, $\bar{q} \in \partial X_{orb}$.
On the other hand let $\bar{q} \in \partial X_{orb}$ and assume q is not singular. Then $\bar{q} \in B$, where $B = W_0/H$ as in the last proposition. From the previous proposition we know that $\pi_1^{orb}(B) = 1$, and therefore it does not contain strata of codimension 1. Since $B \subset X_{orb}$ is open, there is a neighborhood $U_{\bar{q}}$ in X_{orb} which does not contain points of strata of codimension 1. But this contradict $\bar{q} \in \partial X_{orb}$, which is per definition the closure of the of the codimension 1 strata. $\square$

Let (H, W) be a representation of a connected compact group H and (G, V) a minimal reduction. Let G_0 denote the connected component of G. Then we have seen in Chapter 2.4 that the finite group G/G_0 acts isometrically on the quotient V/G_0.

Corollary A.0.18. *The group G/G_0 is generated by reflection on V/G_0.*

Proof. Let $X = V/G_0$ be the quotient of the connected group G_0 and $X' = V/G = W/H$. Since H is connected, the last theorem implies that the connected orbifold $X'_* = X'_{orb} - \partial X'_{orb}$ has trivial orbifold fundamental group. Now $X_{orb} \to X'_{orb} = X_{orb}/(G/G_0)$ is an orbifold covering and Lemma A.0.16 implies that G/G_0 is generated by reflections. $\square$

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Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Herrn Professor Doktor Gudlaugur Thorbergsson betreut worden.

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