L^2-Cohomology of Coverings of q-convex Manifolds and Stein Spaces

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Abstract

We are interested in the L^2 cohomology groups of coverings of different geometric objects.
In the first part we are dealing with a non-compact manifold with boundary X that admits a free, holomorphic and properly discontinuous group action of a discrete group Γ such that the orbit space $\widetilde{X} = X/\Gamma$ is a compact q-convex manifold with boundary. Assume furthermore that there is a holomorphic Γ-invariant holomorphic line bundle E. We show that the Γ-dimension of the L^2-cohomology groups $H^{0,j}_{(2)}(X, E)$ is finite if $j > q$.
In the second part we are dealing with infinite coverings of a relatively compact pseudoconvex domain X in a normal Stein space with isolated singularities that are generated by a group action of a discrete group Γ. We assume that the group action is again free, holomorphic and properly discontinuous. We show that the space of L^2 holomorphic functions on X has infinite Γ-dimension.

Zusammenfassung

Wir untersuchen die L^2-Kohomologie Gruppen überlagerter geometrischer Objekte.
Im ersten Teil behandeln wir den Fall einer nicht-kompakten Mannigfaltigkeit mit Rand X, die eine freie, holomorphe und eigentlich diskontinuirliche Gruppenwirkung einer diskreten Gruppe Γ zulässt derart, dass der Quotient nach der Gruppenwirkung $\widetilde{X} = X/\Gamma$ eine kompakte q-konvexen Mannigfaltigkeit mit Rand ist. Ebenfalls setzen wir voraus, dass ein Γ-invariantes holomorphes Geradenbündel E auf X existiert. Wir zeigen, dass die L^2-Kohomologie Gruppen $H^{0,j}_{(2)}(X, E)$ endliche Γ-Dimension haben sofern die Ungleichung $j > q$ gilt.
Im zweiten Teil behandeln wir den Fall einer unendlichen Überlagerung eines relativ kompakten pseudokonvexen Gebietes X in einem normalen Stein Raum mit isolierten Singularitäten, die durch eine Gruppenwirkung einer diskreten Gruppe Γ erzeugt wird. Die Gruppenwirkung wird dabei als frei, holomorph und eigentlich diskontinuierlich vorausgesetzt. Wir zeigen, dass der Raum der L^2-holomorphen Funktionen auf X unendliche Γ-Dimension hat.

Contents

Introduction

The classical motivation to study cohomology groups comes from de Rham cohomology groups which arise as follows. Given a smooth compact manifold M, denote by $\Omega^r(M)$ the vector bundle of smooth differential r-forms and let $d\colon \Omega^r(M) \to \Omega^{r+1}(M)$ denote the exterior differential. Hence it is natural to ask: Given any closed r-form u (i.e. $du = 0$), does there exist a $(r-1)$-form v such that the equation

$$dv = u$$

is satisfied?

This leads automatically to the notion of cohomology groups which are defined as

$$H^r_{dR}(M) = \frac{\operatorname{Ker} d\colon \Omega^r(M) \to \Omega^{r+1}(M)}{\operatorname{Im} d\colon \Omega^{r-1}(M) \to \Omega^r(M)}.$$

This is always well defined since we have $d^2 = 0$ and thus $\operatorname{Im} d|_{\Omega^r(M)} \subset \operatorname{Ker} d|_{\Omega^r(M)}$. An important tool to compute these cohomology groups is the *Hodge isomorphism* which states that on a compact smooth manifold the cohomology groups $H^r_{dR}(M)$ are isomorphic to the kernel of the Laplace operator $\Delta = dd^* + d^*d$ restricted to the space of smooth r-forms. Here d^* denotes the adjoint operator of d with respect to the standard inner product.

Pierre Dolbeault transferred this concept to complex manifolds in 1953 and the resulting cohomology theory, as an analogue to the de Rham theory, is named after him *Dolbeault cohomology*. Here he used the natural splitting of the exterior differential $d = \partial + \bar{\partial}$ to define the $\bar{\partial}$-complex $\bar{\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. Now one can ask again:

Given $u \in \Omega^{p,q}(M)$ with $\bar{\partial}u = 0$, does there exist $v \in \Omega^{p,q-1}(M)$ such that

$$\bar{\partial}v = u?$$

This question, also known as the $\bar{\partial}$-equation, engaged several mathematicians in research for a long time and still today. This includes famous mathematicians like for example Lars Hörmander, John Joseph Kohn and Kunihiko Kodaira. There are several ways to determine the $\bar{\partial}$-cohomology which is defined in the same way as the de Rham cohomology. Dolbeault showed that there is an isomorphism

$$H^{p,q}_{DB}(M) \cong H^q(M, \Omega^p(M))$$

which relates the cohomology of the complex of differential forms on a complex manifold to sheaf cohomology in the sense of algebraic geometry ($\Omega^p(M)$ is the sheaf of holomorphic p-forms). There is also a complex analog of the Hodge isomorphism that states that $H^{p,q}_{DB}(M) \cong \mathscr{H}^{p,q}(M)$ with $\mathscr{H}^{p,q}(M)$ being the kernel of the associated complex laplacian $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ - but only in the case that M is compact.

One approach to study cohomology on non-compact manifolds is L^2-cohomology. This theory is based on the "classical" theory in the sense that we take $\Omega^{p,q}_0(M)$ to be the space of compactly supported (p,q)-forms and take its completion with respect to the metric on M. The result of this procedure is a Hilbert space, called the L^2 space of square-integrable (p,q)-forms and denoted by $L^{p,q}(X)$.
In order to perform analysis on these spaces, we need to extend the operators $\bar{\partial}, \bar{\partial}^*$ and $\square$ to the L^2 space and this causes several problems, for example we need boundary conditions for the operators mentioned before.

Suppose now that we have a holomorphic vector bundle E on M. All approaches mentioned before can be extended to vector-bundle-valued differential forms which are often used in physics, for example in the theory of quantization. There are several papers and books dealing with L^2 theory on certain classes of manifolds like for example [Hör65] and [Lüc02].

The paper which can be seen as the starting point of this thesis is [GHS98] written by Mikhail Gromov, Gennadi Henkin and Mikhail Shubin in 1998. They consider covering manifolds that are strictly pseudoconvex of complex manifolds and analyze the L^2-holomorphic functions on the covering. The coverings considered are infinite and generated by a discrete group Γ that acts on the strictly pseudoconvex manifold M such that the quotient M/Γ is a compact complex manifold with boundary. They show that the space of L^2-holomorphic functions is in some sense infinite-dimensional (one has to take the group action into account in order to get a well-defined dimension for infinite coverings).
In 2002 George Marinescu, Radu Todor and Ionut Chiose generalized their results to weakly pseudoconvex coverings in [MTC02]. Pseudoconvexity might be expressed in terms of eigenvalues of a certain hermitian $(1,1)$-form which is called the Levi form. If all eigenvalues are positive, the manifold is called strictly pseudoconvex, if they are non-negative, it is called weakly pseudoconvex.

The first part of this thesis deals with a more general situation, that means we require that the Levi form has only $n-q+1$ positive eigenvalues outside a compact set, which is also known as the notion of q-convexity. We prove the following Theorem.

Theorem 1. *Let M be a q-convex manifold with boundary bM that admits a free and holomorphic group action by a discrete group Γ such that the quotient $\overline{M}/\Gamma$ is compact and let E be a Γ-invariant holomorphic line bundle M. Then*

$$\dim_\Gamma H^{0,j}_{(2)}(M,E) < \infty \quad \textit{for } q > j.$$

In the second part of this thesis we are considering infinite coverings of strictly pseudoconvex domains X in a normal Stein space S with isolated singularities and determine the dimension of the space of L^2-holomorphic functions. To do this, we first blow up the singular locus of X in order to get a manifold with complete metric in a neighbourhood of the exceptional divisor which we push down on the regular locus of X. Then we prove the following Theorem.

Theorem 2. *Let $X \subset S$ be a relatively compact strictly pseudoconvex domain in a normal Stein space S and let $q\colon \widetilde{S} \to S$ be a Galois covering of S by a discrete group Γ. Set $\widetilde{X} = q^{-1}(X)$. Then*

$$\dim_\Gamma H^{0,0}_{(2)}(\widetilde{X}) = \infty.$$

1 Preliminaries

In this chapter we will start with a brief summary of notions and concepts from many areas of mathematics that will be used frequently in the sequel.

1.1 Vector bundles on complex manifolds

In this Section we will recall some foundational material of complex geometry and fix notations that are used in this thesis.

Let (X, J) be a n-dimensional complex manifold with complex structure J and let (E, h^E) be a holomorphic vector bundle of rank r on X with hermitian metric h^E. We denote by

$$\Omega^r(X, E) = \mathcal{C}^\infty(X, \Lambda^r(T^*X) \otimes E)$$

the vector bundle of smooth E-valued differential r-forms on X. Recall that a **hermitian metric** on E is an assignment of a hermitian inner product $\langle \cdot, \cdot \rangle_x$ to each fiber E_x of E such that for any open set $U \subset X$ and sections $f, g \in \Omega(U, E)$ the map

$$\langle f, g \rangle \colon U \to \mathbb{C}, \qquad \langle f, g \rangle(x) = \langle f(x), g(x) \rangle_x$$

is smooth.
On E we have a **connection** ∇ which is a first order linear differential

operator acting on $\Omega^r(X,E)$ that is given by

$$\begin{aligned}\nabla\colon \Omega^r(X,E) &\to \Omega^{r+1}(X,E),\\ \nabla(u\wedge v) &= du\wedge v+(-1)^p u\wedge \nabla v,\end{aligned}$$

for any $u\in\Omega^p(X)$ and $v\in\mathcal{C}^\infty(X,E)$, with du being the usual exterior derivative of u. The connection is called **hermitian** if it is compatible with the hermitian structure of E, i.e.

$$d\langle u,v\rangle_{h^E}=\langle\nabla u,v\rangle_{h^E}+\langle u,\nabla v\rangle_{h^E}.$$

On a complex manifold we have a canonical splitting of the complexified tangent bundle

$$TX\otimes_{\mathbb{R}}\mathbb{C}=T^{(1,0)}X\oplus T^{(0,1)}X$$

induced by the complex structure J, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenspaces corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ of J respectively. Let $T^{*(1,0)}X$ and $T^{*(0,1)}X$ be the corresponding dual bundles. This inherits a splitting

$$\Omega^r(X,E)=\bigoplus_{p+q=r}\Omega^{p,q}(X,E).$$

Sections of $\Omega^{p,q}(X,E)$ are called smooth E-valued (p,q)-forms on X. The connection ∇ splits also in a natural way into two first order linear differential operators

$$\nabla=\nabla^{1,0}+\nabla^{0,1},$$

which act on E-valued (p,q)-forms imitating the usual operators ∂ and $\bar\partial$. More precisely we have

$$\begin{aligned}\nabla^{0,1}\colon \Omega^{p,q}(X,E) &\to \Omega^{p,q+1}(X,E),\\ \nabla^{0,1}(u\wedge v) &= \bar\partial u\wedge v+(-1)^{\deg u}u\wedge\nabla^{0,1}v,\end{aligned}$$

for any $u\in\Omega^{p_1,q_1}(X)$ and $v\in\Omega^{p_2,q_2}(X,E)$ with $p_1+p_2=p$ and $q_1+q_2=q$. The definition for the $(1,0)$ part of the connection is similar. On a vector bundle we have in general a lot of different connections, but there is one which is special in a certain sense.

Theorem 1.1.1. *On any holomorphic vector bundle E on X there exists a unique hermitian connection ∇ that is compatible with the holomorphic structure, i.e. $(\nabla)^{0,1} = \bar{\partial}$. This connection is called* ***Chern connection*** *of E.*

A proof of this can be found for example in [Wel08, Theorem 2.1]. A fundamental property of a connection ∇ is that in contrast to the operators ∂ and $\bar{\partial}$, its square does not have to be zero. This allows us to study the operator ∇^2 which plays an important role in differential geometry.

Definition 1.1.2. The **curvature** $\Theta(E)$ of a connection ∇ on a vector bundle E is defined to be the composition

$$\Theta(E) = \nabla \circ \nabla \colon \Omega(X, E) \to \Omega^2(X, E).$$

Let $(e_1, \dots, e_r)$ be a smooth orthonormal frame of E over an open set $U \subset X$ with complex coordinates $(z_1, \dots, z_n)$. Since by assumption E is a holomorphic hermitian vector bundle, we can express the curvature in these coordinates as

$$\sqrt{-1}\Theta(E) = \sqrt{-1} \sum_{j,k=1}^{n} \sum_{\alpha,\beta=1}^{r} c_{jk\alpha\beta} dz_j \wedge d\bar{z}_k \otimes e_\alpha^* \otimes e_\beta. \tag{1.1}$$

The property of being a hermitian vector bundle implies that the matrix $(c_{jk\alpha\beta})$ is hermitian, i.e. $\bar{c}_{jk\alpha\beta} = c_{kj\beta\alpha}$.
Note, that in the case that E is a line bundle, $E^* \otimes E \cong \mathbb{C}$ and thus the formula above simplifies to

$$\sqrt{-1}\Theta(E) = \sqrt{-1} \sum_{j,k=1}^{n} c_{jk} dz_j \wedge d\bar{z}_k. \tag{1.2}$$

Definition 1.1.3. Let L be a holomorphic hermitian line bundle on X. We say that L is **positive** (respectively **negative**) if the hermitian matrix $(c_{jk}(z))$ of its curvature form (1.2) is positive (respectively negative) definite at every point $z \in X$.

Remark 1.1.4. Let L be a holomorphic line bundle on a complex manifold X and denote by L^* the corresponding dual line bundle. Note that L positive implies that the dual bundle L^* is negative.

We denote by $L^k = \underbrace{L \otimes \ldots \otimes L}_{k-\text{times}}$ the k-fold tensor product of L.

Example 1.1.5. Consider the compact Kähler manifold $X = \mathbb{CP}^n$ and let

$$\mathcal{O}(-1) = \left\{(\zeta, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid z \in \zeta\right\} \tag{1.3}$$

be the tautological line bundle on $\mathbb{CP}^n$. The dual bundle of $\mathcal{O}(-1)$ is $\mathcal{O}(-1)^* = \mathcal{O}(1)$ and the curvature of $\mathcal{O}(1)$ is positive since on compact Kähler manifolds we can identify a Kähler form ω on X to a hermitian metric on L and a Kähler form is by definition positive definite.

Definition 1.1.6. The **canonical bundle** K_X on X is defined as the top dimension exterior power of the $(1,0)$ part of the cotangent bundle on X, i.e.

$$K_X = \Lambda^n T^{*(1,0)} X.$$

Let K_X^* be the dual bundle of K_X. Since it is again a line bundle, we have $K_X^* \otimes K_X \cong \mathbb{C}$ and also we have an isometry

$$\Psi\colon \Lambda^{0,q}(T^*X) \otimes E \to \Lambda^{n,q}(T^*X) \otimes E \otimes K_X^*, \tag{1.4}$$

which will be useful later on.

1.2 Pseudoconvexity and the Levi Form

In this Section we are going to introduce the notion of pseudoconvexity in terms of plurisubharmonic functions and in terms of the Levi form.

We start with the definition of plurisubharmonicity.

Definition 1.2.1. Let $\varphi\colon \mathbb{C}^n \supset U \to [-\infty, \infty)$ be an upper semicontinuous function defined on an open set U in $\mathbb{C}^n$. We call the function φ

subharmonic if for any closed ball $\overline{B_r(z_0)} \subset U$ with radius r, center z_0 and any real function $h \in \mathcal{C}^0(\overline{B_r(z_0)}) \cap \mathcal{C}^2(B_r(z_0))$ that is harmonic in the interior satisfying $\varphi(z) \leq h(z)$ for all boundary points z, then $\varphi(z) \leq h(z)$ for all $z \in B_r(z_0)$. We call φ **plurisubharmonic**, if φ is upper semicontinuous and the restriction of φ to any complex line $L \subset \mathbb{C}^n$ is subharmonic on $U \cap L$. Moreover we call φ **strictly plurisubharmonic** if $\varphi \in L^1_{\text{loc}}(U)$ and for any point $z \in U$ there is a neighbourhood V of z and a positive constant $c > 0$ such that the function $\varphi(z) - c\,|z|^2$ is plurisubharmonic on V. Here

$$L^1_{\text{loc}}(U) = \left\{ f\colon U \to \mathbb{C} \text{ measurable} \,\middle|\, \int_{\mathbb{C}^n} f(z)\phi(z)d\lambda < \infty,\ \phi \in \mathbb{C}_0^\infty(U) \right\}$$

denotes the set of locally integrable functions on U and $d\lambda$ is the Lebesgue measure. We call φ additionally **exhaustion function** or **exhaustive** on U if all sublevel sets

$$U_c = \{z \in U \mid \varphi(z) < c\}$$

are relatively compact for any $c \in \mathbb{R}$.

Plurisubharmonic exhaustion functions play a quite important role in complex geometry because many geometric properties of spaces can be written in terms of them, as can be seen in the following definition.

Definition 1.2.2. Let X be a complex manifold of (complex) dimension n. Then X is called

(i) **weakly pseudoconvex**, if there exists a smooth plurisubharmonic exhaustion function and

(ii) **strictly pseudoconvex**, if there exists a smooth strictly plurisubharmonic exhaustion function.

In the case that we are dealing with an n-dimensional compact complex manifold X with boundary bX, $\overline{X} = X \cup bX$, such that $\overline{X}$ is contained in some open manifold M, it turns out that pseudoconvexity is actually a local property of the boundary. To see this, we introduce the Levi form which uses the concept of defining functions for manifolds.

Definition 1.2.3. Consider a function $\rho \in \mathcal{C}^2(M, \mathbb{R})$. We call ρ a **defining function** for X, if ρ has the following properties:

(i) $\rho(z) < 0$ for all $z \in X$,

(ii) $\rho(z) = 0$ for all $z \in bX$ and

(iii) $|d\rho| \neq 0$ on bX.

It turns out that we can actually substitute condition (iii) in Definition 1.2.3 without loss of generality by $|d\rho| = 1$ on bX since we can replace ρ by $\rho' = \rho/\,|d\rho|$ and a short calculation shows that ρ' is again a defining function for X.
Since X is by assumption a n-dimensional manifold, the boundary bX is of real dimension $2n-1$. We define the **holomorphic tangent space** $T_z^{\mathbb{C}} bX$ at a boundary point $z \in bX$ as the maximal complex subspace of $T_z bX$. Choosing local coordinates $\{z_k\}_{k=1}^n$ around a boundary point $z \in bX$, we can describe the holomorphic tangent space in terms of the defining function as follows:

$$T_z^{\mathbb{C}} bX = \left\{ w = (w_1, \dots, w_n) \in T_z X \;\middle|\; \sum_{k=1}^n \frac{\partial \rho}{\partial z_k}(z) w_k = 0 \right\}. \tag{1.5}$$

Definition 1.2.4. The **Levi form** of ρ at a point $z \in bX$ is the hermitian form defined by

$$L_\rho(z; w) = (\partial\bar{\partial}\rho)(z; w) = \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) w_j \bar{w}_k. \tag{1.6}$$

Since the Levi form depends on the defining function, it is a priori not clear which kind of (geometric) information is contained in L_ρ. Note also that L_ρ is a hermitian form, hence the eigenvalues of its coefficient matrix are real.

Lemma 1.2.5. *The number of positive and negative eigenvalues of the Levi form is independent of the choice of the defining function ρ.*

Proof. See [MM07, Lemma B.3.8]. □

Lemma 1.2.5 is extremely useful in practice because given a complex manifold X we can work with a defining function for X which is easy to handle in the sense that for example the eigenvalues of the matrix $\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right)_{j,k}$ are easy to compute. This observation is used in the following Lemma.

Lemma 1.2.6. *Let X be a complex manifold with boundary bX. Let $L_\rho(z)$ be the Levi form of ρ at a boundary point $z \in bX$. Then*

(i) X *is weakly pseudoconvex* $\iff$ $L_\rho(z;\cdot)$ *is positive semidefinite,*

(ii) X *is strictly pseudoconvex* $\iff$ $L_\rho(z;\cdot)$ *is positive definite.*

A proof of this can be found for example in [Dem12, Chapter 1, §7, Theorem 7.12].

1.3 Functional Analysis

In this Section we will recall some basic facts from functional analysis. Suppose throughout the Section that $\mathcal{H}$ is a complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$.

Definition 1.3.1. Let $T \in \mathcal{B}(\mathcal{H})$. The **weak operator topology** on $\mathcal{B}(\mathcal{H})$, abbreviated WOT, is the topology of the pointwise weak convergence, i.e. it is the topology induced by the seminorms

$$T \mapsto |\langle Tx, y\rangle|, \qquad x, y \in \mathcal{H}.$$

The WOT is the weakest topology among all topologies on $\mathcal{B}(\mathcal{H})$. It has several useful properties, for example for fixed $S \in \mathcal{B}(\mathcal{H})$, the maps $T \mapsto TS$ and $T \mapsto ST$ are continuous. Also the adjoint mapping $T \to T^*$ is continuous in the WOT. We need the WOT later on in Section 1.7 to define the von Neumann dimension.
In the following we recall some basic properties of linear (differential) operators.

Recall that the **adjoint** of a densely defined operator $T\colon \operatorname{Dom}(T) \to \mathcal{H}$ is defined by

$$\operatorname{Dom}(T^*) = \{u \in \mathcal{H} \mid \exists C > 0 : |\langle Tu, v\rangle| \le C\,\|v\| \ \forall\, v \in \operatorname{Dom}(T)\},$$

and for $u \in \operatorname{Dom}(T^*)$ we set T^*u to be the unique $w \in \mathcal{H}$ such that $\langle Tv, u\rangle = \langle v, w\rangle$ for any $v \in \operatorname{Dom}(T)$ by using the Riesz representation theorem.

Definition 1.3.2. Let $T\colon \operatorname{Dom}(T) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator where $\operatorname{Dom}(T)$ is a dense linear subspace in $\mathcal{H}$. We call T

(i) **closed** if $\operatorname{Graph}(T) = \{(u, Tu) \mid u \in \operatorname{Dom}(T)\}$ is closed,

(ii) **preclosed** if the closure $\overline{\text{Graph(T)}}$ is again the graph of a linear operator,

(iii) **self-adjoint** if $T = T^*$,

(iv) **positive** if $\langle Tu, u\rangle \ge 0$ for all $u \in \mathcal{H}$.

Another concept which we use is that of Friedrichs extensions. This is a particular way of extending positive operators to self-adjoint operators on a Hilbert space $\mathcal{H}$ which makes use of quadratic forms. We start with a brief description of the latter.

Definition 1.3.3. A **quadratic form** is a sesquilinear map

$$Q\colon \operatorname{Dom}(Q) \times \operatorname{Dom}(Q) \to \mathbb{C},$$

where $\operatorname{Dom}(Q)$ is a dense linear subspace of $\mathcal{H}$.

On $\operatorname{Dom}(Q)$ we introduce a norm by setting

$$\|u\|_Q = \left(Q(u,u) + \|u\|^2\right)^{\frac{1}{2}}.$$

We call the quadratic form Q

(i) **closed**, if $(\operatorname{Dom}(Q), \|.\|_Q)$ is complete, and

(ii) **positive**, if $Q(u,u) \geq 0$ for any $u \in \mathrm{Dom}(Q)$.

The connection to operators on Hilbert spaces is established by the following propositions.

Proposition 1.3.4. *To any positive and self-adjoint operator T we can associate a closed quadratic form Q_T such that*

$$\mathrm{Dom}(T) = \{u \in \mathrm{Dom}(Q_T) \mid \exists v \in \mathcal{H}\ Q_T(u,w) = \langle v, w\rangle\ \forall w \in \mathrm{Dom}(Q_T)\},$$
$$Tu = v \quad \forall u \in \mathrm{Dom}(T).$$

We call Q_T the quadratic form associated to T.

Proof. See [MM07, Proposition C.1.4]. □

The whole procedure works also vice versa.

Proposition 1.3.5. *Let Q be a closed positive quadratic form. Then there exists a positive self-adjoint operator T such that $Q_T = Q$.*

Proof. See [MM07, Proposition C.1.5]. □

Propositions 1.3.4 and 1.3.5 ensure that we can without loss of generality jump back and forth between the levels of positive operators and quadratic forms and work with what is more convenient in the present situation.

Let $T\colon \mathrm{Dom}(T) \subset \mathcal{H} \to \mathcal{H}$ be a positive operator. We set

$$\mathrm{Dom}(Q'_T) := \mathrm{Dom}(T), \quad Q'_T(u,v) = \langle Tu, v\rangle \quad \text{for } u,v \in \mathrm{Dom}(T).$$

There is one last technicality that is needed to define the Friedrichs extension.

Proposition 1.3.6. *Let T be a positive operator. Then Q'_T is closable, i.e. there exists a positive closed form $\widehat{Q}$ that extends Q_T.*

Proof. See [MM07, Proposition C.1.6]. □

We consider the smallest closed positive extension $\widehat{Q}_T$.

Definition 1.3.7. Let T be a positive operator on $\mathcal{H}$. The **Friedrichs extension** T_F of T is the self-adjoint operator with $Q_{T_F} = \widehat{Q}_T$.

Keeping this small summary in mind, we will focus now on differential operators on manifolds.

Definition 1.3.8. Let M be a differentiable manifold and suppose E and F are arbitrary vector bundles on M. Let $T\colon \Omega(X,E) \to \Omega(X,F)$ be a differential operator of degree k given in local coordinates $(x_1,\dots,x_n)$ by the formula

$$Tu(x) = \sum_{|\alpha|\le k} t_\alpha(x) D^\alpha u(x),$$

where $D^\alpha = (\partial/\partial x_1)^{\alpha_1}\dots(\partial/\partial x_n)^{\alpha_n}$, $u \in \Omega(X,E)$.
The map $\sigma_T\colon T^*M \to \mathrm{Hom}(E,F)$ defined by

$$\begin{aligned} T^*_xM \ni \xi &\mapsto \sigma_T(x,\xi) \in \mathrm{Hom}(E_x,F_x),\\ \sigma_T(x,\xi) &= \sum_{|\alpha|=k} t_\alpha(x)\xi^\alpha, \end{aligned}$$

is called the **principal symbol** of T. We call T **elliptic** , if σ_T is injective for every $x \in M$ and every $\xi \in T^*_xM \setminus \{0\}$. Moreover we call T **strongly elliptic** , if there exists $c > 0$ such that

$$|\sigma_T(x,\xi)| \ge c\,|\xi|^k$$

for all $(x,\xi) \in M \times T^*M$.

When we study the behavior of self-adjoint operators like for example the Laplacian, one thing we can start with is to analyze the spectrum. Let T be self-adjoint. A complex number λ lies in the **resolvent set** if the operator $(T - \lambda\,\mathrm{id})$ is a bounded operator of $\mathrm{Dom}(T)$ onto $\mathcal{H}$. The complement of the resolvent set in $\mathbb{C}$ is called **spectrum** of T and is denoted by $\sigma(T)$. The spectrum itself is again subdivided into two parts. The **discrete spectrum** $\sigma_d(T)$ is the subset of $\sigma(T)$ which is the set of all eigenvalues λ of T of finite multiplicity which are isolated in the sense

that for small $\varepsilon > 0$ the eigenvalue λ is the only one in $B_\varepsilon(\lambda)$.
The other part of the spectrum is called **essential spectrum** and is defined as the complement of discrete spectrum, i.e. $\sigma_{ess}(T) = \sigma(T) \setminus \sigma_d(T)$.
Determining the essential spectrum of an operator can a priori be complicated, but there is a nice statement about the non-existence of σ_{ess}.

Theorem 1.3.9. *Let T be a positive self-adjoint (unbounded) operator on a Hilbert space $\mathcal{H}$. Then the following conditions are equivalent:*

(i) *The resolvent operator $(T - \lambda\,\mathrm{id})^{-1}$ is compact.*

(ii) *The operator T has empty essential spectrum.*

(iii) *There exists a complete orthonormal set of eigenvectors $\{v_k\}_{k\in\mathbb{N}}$ of T with corresponding eigenvalues $\lambda_k \geq 0$ which converge to $+\infty$ as $k \to \infty$.*

Proof. See [Dav95, Corollary 4.2.3]. □

Definition 1.3.10. The **spectral family** or **spectral resolution** associated to T is the family $(E_\lambda)_{\lambda\in\mathbb{R}}$ with $E_\lambda = E((-\infty, \lambda])$, where $E\colon \mathrm{Bor}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ is the spectral measure of T and $\mathrm{Bor}(\mathbb{R})$ denotes the family of Borel sets on $\mathbb{R}$. If we want to stress the dependence on the operator T, we write $\{E_\lambda(T)\}$.

Remark 1.3.11. Note that any self-adjoint operator T has a unique spectral measure in the sense that for any $u \in \mathrm{Dom}\, T$ we have

$$Tu = \left(\int_{\mathbb{R}} x \, dE(x)\right) u.$$

For more details we refer to [Lüc02, §1.4].

We are also making use of Sobolev spaces at some point.

Definition 1.3.12. Let M be a compact differentiable manifold, k an integer and let E be an arbitrary hermitian vector bundle on M. The space $W^k(M, E)$ which consists of sections $s\colon M \to E$ whose derivates up to order k are in $L^2(M, E)$ is called **Sobolev space** .

The Sobolev spaces $W^k(M,E)$ have several benefits, for example they are all complete metric spaces with respect to the Sobolev norm

$$\|u\|_k^2 = \sum_{|\alpha|\leq k} \left\| \frac{\partial^\alpha}{\partial^\alpha x^\alpha} u \right\|_{L^2}^2, \quad u \in W^k(M,E).$$

Obviously we have $W^{k+1}(M,E) \subset W^k(M,E)$ which leads to the following observation.

Lemma 1.3.13 (Rellich Lemma)**.** *For every integer k, the embedding*

$$W^{k+1}(M,E) \hookrightarrow W^k(M,E)$$

is compact.

Proof. See [Tay11, Proposition 4.4]. □

1.4 Analytic Sets in $\mathbb{C}^n$

In this Section we will give a brief summary about analytic sets in $\mathbb{C}^n$ and we will focus on the structural background which comes from (classical) algebraic geometry. The goal of this will be the formulation of Hironakas theorem about resolutions of singularities of complex spaces which will be essential in Chapter 3. We start with some definitions.

Definition 1.4.1. Let $A \subset \mathbb{C}^n$. We call the set A **complex analytic** set if A is closed and every point $z_0 \in A$ has a neighbourhood $U \subset A$ and there exist finitely many holomorphic functions $f_1, \dots, f_n \in \mathcal{O}(U)$ such that

$$A \cap U = \{z \in U \mid f_1(z) = \ldots = f_n(z) = 0\}.$$

It is clear from the definition that there are in general many systems of functions $f_1, \dots, f_k$ which describe the same analytic set A. But among all these systems there exists one defined in a suitably small neighbourhood U of z_0 where the k is minimal. The number k is called the **analytic dimension** of A at z_0 and is denoted by $\dim_{z_0} A$. Obviously we have

$$\dim_{z_0} A = 0 \iff z_0 \text{ is an isolated point of } A.$$

Recall that a **germ** (A, z_0) of an analytic set A at a point $z_0 \in A$ is an equivalence class of elements of the power set $\mathfrak{P}(A)$, where

$$B \sim C \iff \exists\, V \text{ open neighbourhood of } z_0 : B \cap V = C \cap V$$

for any $B, C \in \mathfrak{P}(A)$.

Definition 1.4.2. A germ (A, z_0) of an analytic set A is called **reducible** if there are analytic subsets $A_1, A_2 \subset A$ with $(A_1, z_0) \cup (A_2, z_0) = (A, z_0)$ and $(A_j, z_0) \neq (A, z_0)$, $j \in \{1, 2\}$. Otherwise we call (A, z_0) **irreducible** .

Given an analytic set A, it is in practice hard to check from the definition whether A is irreducible or not, but fortunately there is an algebraic condition which is easier to verify.
We assume from now on that A is an analytic subset of a complex manifold M. We denote by $\mathcal{O}(M)_{z_0}$ the ring of holomorphic functions at $z_0 \in M$. Consider the set

$$\mathcal{I}_{A,z_0} = \{ f \in \mathcal{O}(M) \mid f|_A = 0 \}.$$

Then $\mathcal{I}_{A,z_0}$ is an ideal which is called the **defining ideal** for A.

Proposition 1.4.3. *Let (A, z_0) be a germ of an analytic set and let $\mathcal{I}_{A,z_0}$ be the defining ideal for A. Then (A, z_0) is irreducible if and only if $\mathcal{I}_{A,z_0}$ is a prime ideal in the ring $\mathcal{O}(M)_{z_0}$.*

The proof of this is direct and can be found for example in [Dem12, Chapter 2, Proposition 4.5].
The notion of irreducibility is quite powerful since it provides a classification of analytic sets.

Theorem 1.4.4. *Every germ (A, z_0) of an analytic set has a finite decomposition into irreducible germs (A_j, z_0) such that*

$$(A, z_0) = \bigcup_{j=1}^{n} (A_j, z_0),$$

with $(A_j, z_0) \subsetneq (A_k, z_0)$ for $j \neq k$ and the decomposition is unique apart from ordering.

Proof. See [Dem12, Chapter 2, Theorem 4.7]. □

1.5 Complex Spaces

In this Section we will give a short introduction to complex spaces which are the objects of study in Chapter 3. We use primarily the notation from [For11].

Let X be a complex manifold. We denote the set of **germs of holomorphic functions** at a point $x \in X$ as $\mathcal{O}_x$. The **germ** is denoted by $[f]_x \in \mathcal{O}_x$ and it is represented by a holomorphic function f in a neighbourhood U of x. By definition of the germ, two functions f and g represent the same germ if they agree on U.
The ring $\mathcal{O}_x$ is isomorphic to the ring $\mathcal{O}_{\mathbb{C}^n,0}$ via any coordinate map sending x to 0 and thus it is a Noetherian ring and a unique factorization domain. Moreover, $\mathcal{O}_x$ has no zero divisors so that we can form its **quotient field** $\mathcal{M}_x$. The elements of $\mathcal{M}_x$ are called **germs of meromorphic functions** on X and locally any meromorphic function $f \in \mathcal{M}_x$ at a point $x \in X$ has a representation $f = g/h$ with $f, g \in \mathcal{O}_x$ and f, g being relatively prime.
Set $\mathcal{O}_X = \cup_{x \in X} \mathcal{O}_x$. We equip the set $\mathcal{O}_X$ with the topology that is given by the basis of sets $\{[f]_x \mid x \in U\}$ with $f \colon U \to \mathbb{C}$ being holomorphic. This turns $\mathcal{O}_X$ into a sheaf, called the **sheaf of holomorphic functions** or **structure sheaf** on X.
Recall that a topological space X is called **countable at infinity** if

there exists a countable family of compact subsets K_j of X such that $X \subset \cup_j K_j$ and $K_j \subset \operatorname{int} K_{j+1}$.

Definition 1.5.1. A **reduced complex space** is a pair $(X, \mathcal{O}_X)$ consisting of a paracompact Hausdorff space X, countable at infinity, and a sheaf of continuous functions $\mathcal{O}_X$ such that for every point $x \in X$ there is a neighbourhood U and a homeomorphism $\phi\colon U \to A \subset \mathbb{C}^N$ onto an analytic subset $A \subset \mathbb{C}^N$ such that the comorphism $\phi^*\colon \mathcal{O}_A \to \mathcal{O}_{X|U}$ induces an isomorphism of sheaves of rings. The triple (U, ϕ, V) is called a **chart** on X and the map ϕ is called a **local embedding** of X.
We call X **pure dimensional** if we have

$$\dim_x X = \dim X \qquad \text{for all } x \in X.$$

Further on we say that X is pure dimensional at a point $x \in X$ if there exists an open neighbourhood U of x such that U is pure dimensional.

Remark 1.5.2. The term *reduced* in Definition 1.5.1 actually means that the nilradical $N_X = \cup_{x \in X} N_x$ (which is an ideal subsheaf of $\mathcal{O}_X$) is zero, where N_x is the ideal of all nilpotent germs in $\mathcal{O}_x$.

Roughly speaking X is obtained by gluing analytic subsets in $\mathbb{C}^N$ via biholomorphic maps. One main part in the analysis of complex spaces is the presence of singular points.

Definition 1.5.3. Let X be a complex space. A point $x \in X$ is called **regular** or **smooth**, if X has the structure of a complex manifold in a neighbourhood of x. A point which is not regular is called **singular**. The sets of regular and singular points respectively are denoted by X_{reg} and X_{sing}. We will refer to them as the regular locus and singular locus respectively.

Singular points of a complex space are of particular interest because their presence can be seen as a kind of generalization of the the concept of manifolds as any complex manifold X is a complex space with the property that $X_{\text{sing}} = \emptyset$. The singular locus itself provides some (topological) structure as can be seen in the following Theorem from [Fis76, § 2.15].

Theorem 1.5.4. *X_{sing} is an analytic subset of X.*

In order to analyze the singular locus of a complex space, we know by Theorem 1.5.4 that X_{sing} has a well-defined codimension in X. It turns out that among the singular points there are some which are in some sense "simpler".

Definition 1.5.5. A point $x \in X$ is called **normal** , if the ring $\mathcal{O}_x$ is integrally closed in its quotient field $\mathcal{M}_x$ and X is called normal if every point $x \in X$ is normal. All points which are not normal are called **non-normal**. The set of non-normal points is denoted by $X_{\text{n-n}}$.

By looking at the definition of regular points, we obtain that any regular point $x \in X$ is normal since we have $\mathcal{O}_x \cong \mathcal{O}_{\mathbb{C}^n,0}$ and thus $\mathcal{O}_x$ is a factorial ring and hence integrally closed. Therefore we also have $X_{\text{n-n}} \subset X_{\text{sing}}$. Moreover we have a result (cf. [Dem12, Chapter 2, Theorem 7.6]) that describes the connection between non-normal and singular points.

Theorem 1.5.6. *Let $x \in X$ be a normal point. Then* $\operatorname{codim}_x(X_{sing}) \geq 2$.

Consider a normal complex space X. Let us also mention a topological characterization of normal points.

Proposition 1.5.7. *If $x \in X$ is a normal point, then X is irreducible at x.*

Proof. See [GR84, Chapter 6, §4]. □

Hence every normal complex space is locally irreducible and also locally pure dimensional since a complex space is pure dimensional at all irreducible points $x \in X$ (cf. [GR84, Chapter 5, § 4]). The following Theorem from [AG06, Theorem 7.5] relates irreducibility to a topological obstruction of the regular locus.

Theorem 1.5.8. *Let X be a reduced complex space. Then X is irreducible if and only if X_{reg} is connected.*

We also have the following extension Theorem for holomorphic functions on normal complex spaces.

Theorem 1.5.9 (Riemann Extension Theorem)**.** *Let X be a normal complex space. Then every holomorphic function on the complex manifold $X_{reg} = X \setminus X_{sing}$ extends uniquely to a holomorphic function on X.*

The proof of Riemann's Extension Theorem involves the concept of so called weakly holomorphic functions which we will not introduce here and so we refer to [GR84, Chapter 7, § 4.2] for the proof.

Let X be a complex space. A complex-valued function f on X is called **smooth**, if for every point $x \in X$ there exists an isomorphic embedding of a neighbourhood U of x as an analytic subset A in a ball $B \subset \mathbb{C}^N$ such that f (considered as a function on A) is the restriction to A of a smooth function in B. We define plurisubharmonic functions on a complex space X similar to the definitions given in Section 1.2.

Definition 1.5.10. Let φ be an upper semicontinuous function on X. We call φ **plurisubharmonic** if the following holds: Any point $x \in X$ has a neighbourhood U which is realized isomorphically as an analytic subset V in a domain $G \subset \mathbb{C}^N$ by $\psi\colon U \to V$ such that $\varphi \circ \psi^{-1}$ is the restriction to V of a plurisubharmonic function in G. If $\varphi \circ \psi^{-1}$ is locally the restriction of a strictly plurisubharmonic function, φ is called **strictly plurisubharmonic**.

When we are talking about metrics on complex spaces, we also have to make some slight modifications of the Definition given in Section 1.1.

Recall that a Hermitian form on a complex manifold is a smooth positive $(1,1)$-form and can be identified to a Hermitian metric. Consider a covering $\{U_\alpha\}$ of X and local embeddings $\tau_\alpha\colon U_\alpha \to \mathbb{C}^{N_\alpha}$. A **Hermitian form** on a complex space X is a hermitian form ω on the regular locus X_{reg} which on every open set U_α is the pullback of a Hermitian form on the ambient space $\mathbb{C}^{N_\alpha}$, i.e. $\omega = \tau_\alpha^* \omega_\alpha$. A positive Hermitian $(1,1)$-form is called **Kähler**, if $d\omega = 0$.

We will also consider line bundles on a complex space X. Let L be a holomorphic line bundle on X and assume that $L|_{U_\alpha}$ is the inverse image by τ_α of the trivial line bundle $\mathbb{C}_\alpha$ on $\mathbb{C}^{N_\alpha}$. In order to introduce a metric on L we have to modify the definition given in Section 1.1.

Definition 1.5.11. Let L be a holomorphic line bundle on X. Consider Hermitian metrics $h_\alpha = e^{-2\chi_\alpha}$ on $\mathbb{C}_\alpha, \chi_\alpha \in L^1_{\text{loc}}(\mathbb{C}^{N_\alpha})$ and smooth outside $\tau(X_{\text{sing}})$ such that $\tau_\alpha^* h_\alpha = \tau_\beta^* h_\beta$ on $U_\alpha \cap U_\beta \cap X_{\text{reg}}$. The family $\{\tau_\alpha^* h_\alpha)$ is called a **Hermitian metric** on L.

We will also work with Stein spaces in Chapter 3.

Definition 1.5.12. A second countable complex space X is called **Stein**, if

(i) the global holomorphic functions separate points, i.e. for all $x \neq y$ in X there is a holomorphic function $f \in \mathcal{O}_X$ such that $f(x) \neq f(y)$,

(ii) X is holomorphically convex, i.e. for every compact set $K \subset X$ the convex hull

$$\widehat{K} = \left\{ x \in X \;\middle|\; |f(x)| \leq \max_{y \in K} |f(y)|\,,\ \forall f \in \mathcal{O}_X \right\}$$

is compact again and

(iii) every local ring $\mathcal{O}_x$ is generated by functions in $\mathcal{O}_X$.

Example 1.5.13. (i) $\mathbb{C}^n$ is Stein.

(ii) Any closed analytic subspace of a Stein space (in particular of $\mathbb{C}^n$) is a Stein space.

(iii) An open subset $U \subset X$ of a Stein space is Stein if and only if it is holomorphically convex.

(iv) $\mathbb{C}^n \setminus \{0\}$ is not Stein for $n \geq 2$.

Stein spaces provide a lot of structure as can be seen in the following Proposition.

Proposition 1.5.14. *Let X be a complex space such that the global holomorphic functions separate points. Then every compact analytic subset $A \subset X$ is finite.*

The proof of Proposition 1.5.14 is straightforward since we can use that a holomorphic function on an irreducible component of A must be constant and hence the irreducible component consists of one point by the open map theorem. Since the global holomorphic functions of Stein spaces always separate points, the assumptions of Proposition 1.5.14 are fulfilled in any case.

1.6 Coverings

In this Section we will introduce infinite coverings of manifolds and complex spaces.

Let X be a complex manifold of complex dimension n and suppose Γ is a discrete group which acts on X freely and properly discontinuously. Note that the assumptions on the group action ensure that the quotient X/Γ is a manifold since a proper discontinuous action implies that the quotient is Hausdorff and a free action that there are no fixed points. Denote by

$$\pi_\Gamma : X \to X/\Gamma =: \widetilde{X}$$

the canonical projection. We assume further on that X is paracompact so that Γ is countable.

Remark 1.6.1. Given a manifold $\widetilde{X}$ we can always consider its universal cover X. In this case the group Γ can be chosen to be the group of Deck transformations which is isomorphic to the fundamental group $\pi_1(\widetilde{X})$ and it acts on X via representations of $\pi_1(\widetilde{X})$.

We consider further a holomorphic vector bundle $\widetilde{E}$ on $\widetilde{X}$ and denote by $E = \pi_\Gamma^* \widetilde{E}$ its pullback bundle. To keep things straight, we will decorate objects living on the quotient with a tilde as we visualize in the following diagram:

$$\begin{tikzcd} E = \pi_\Gamma^* \widetilde{E} \arrow[r] \arrow[d] & \widetilde{E} \arrow[d] \\ X \arrow[r, "\pi_\Gamma"] & \widetilde{X} \end{tikzcd} .$$

If we have a closer look at the construction of the pullback bundle, we see that it is Γ-invariant because by definition we have

$$\pi_\Gamma^* E = \left\{ (x, e) \in X \times \widetilde{E} \,\middle|\, \pi_\Gamma(x) = \pi_{\widetilde{E}}(e) \right\}$$

and by setting

$$\gamma.(x, e) = (\gamma.x, e) \qquad \gamma \in \Gamma, (x, e) \in X \times \widetilde{E},$$

we ensure that the action of Γ lifts.

Definition 1.6.2. Let Γ be a discrete group acting on a manifold X. A **fundamental domain** for the Γ-action is an open set $U \subset X$ such that

(i) $X = \bigcup_{\gamma \in \Gamma} \gamma U$,

(ii) $U \cap \gamma U = \emptyset \qquad \forall \gamma \neq e$ and

(iii) the topological boundary $\overline{U} \setminus U$ has measure zero.

Later on we will define L^2-cohomology which depends on the choice of the metrics on both the manifold as well as on the vector bundle. Due to the presence of the Γ-action, invariant metrics are a key ingredient to define these spaces.

Remark 1.6.3. Given any Lie group G acting properly and differentiably on a paracompact differentiable manifold M, there exists a G-invariant Riemannian metric on M (cf. [Kos65, Chapter 1, §4, Theorem 2]).

A discrete group Γ can be seen as a 0-dimensional Lie group and a complex manifold is clearly differentiable, hence existence of Γ-invariant metrics is always guaranteed.

Let us assume for the moment that M is a compact manifold with boundary bM and set $\overline{M} = M \cup bM$.

Lemma 1.6.4. *Given any Γ-invariant metric on $\overline{M}$, any point $x \in \overline{M}$ and any positive constant $r \in \mathbb{R}^+$ the ball of the corresponding geodesic metric $\left\{ y \in \overline{M} \mid \operatorname{dist}(x, y) < r \right\}$ is relatively compact in $\overline{M}$.*

Proof. See [Kos65, Chapter 1, §4, Theorem 3]. □

Coverings which are generated by a discrete group Γ are infinite coverings and of fundamental importance in this thesis. When we are taking quotients by the action of the group, there are several things to keep in mind. One natural question is for example whether or not the structure of a manifold or a complex space is preserved. In the case of complex spaces, we have the following important result.

Theorem 1.6.5. *Let $\widetilde{X}$ be a reduced Stein space and let $\pi \colon X \to \widetilde{X}$ be a covering of $\widetilde{X}$ such that $X/\Gamma = \widetilde{X}$ and Γ is a discrete group that acts holomorphically, freely and properly discontinously on X. Then X is Stein.*

Proof. See [Ste56, Satz 2.1]. □

1.7 The Γ-dimension

We will give a brief introduction into the theory of von Neumann Dimensions in this Section, for details we refer for example to [Kol95, Chapter 6] or [Lüc02, Chapter 1, §1].

In general the von Neumann Dimension, which we will shortly call Γ-dimension, is a tool to measure dimensions of invariant spaces with respect to a group action of a discrete group Γ on a vector space.

Suppose throughout this Section that $\mathcal{H}$ is a complex Hilbert space and let Γ be a discrete group acting on $\mathcal{H}$. As a motivating example consider the Hilbert space $L^2\Gamma$ which consists of all complex valued L^2-functions on Γ. Here the L^2 structure is chosen with respect to the Dirac measure. The group Γ acts on this space unitarily by left regular representations of the form $\gamma \mapsto l_\gamma, \gamma \in \Gamma$ with l_γ being defined in the following way: If $x \in \Gamma$ and $f \in L^2\Gamma$, the group action is defined as

$$l_\gamma f(x) = f(\gamma^{-1}x).$$

Most of the times we will work on arbitrary (complex) Hilbert spaces. In this context we call the tensor product $L^2\Gamma \otimes \mathcal{H}$ a free Hilbert Γ-module with Γ-action given by

$$\gamma \mapsto L_\gamma := l_\gamma \otimes 1. \tag{1.7}$$

Since all the spaces considered are supposed to be Hilbert spaces, we will drop the word "Hilbert" and call $L^2\Gamma \otimes \mathcal{H}$ simply a **free Γ-module**. Subspaces of this which are invariant under the Γ-action (i.e. under the maps L_γ) and closed in the induced topology are called **Γ-modules**.

We consider the von Neumann algebra $\mathcal{N}(\mathcal{H})$ on $\mathcal{H}$, which is defined in the following way.

Definition 1.7.1. The **von Neumann Algebra** $\mathcal{N}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ is defined as the $*$-algebra of bounded linear operators which are closed in the weak operator topology (cf. Definition 1.3.1) containing the identity.

Remark 1.7.2. Recall that a $*$-algebra $\mathcal{A}$ is an algebra where the multiplication $*\colon \mathcal{A} \to \mathcal{A}$ is given by an antiautomorphic and involutive mapping, i.e. for any $A, B \in \mathcal{A}$ we have

$$(A+B)^* = A^* + B^*, \qquad (AB)^* = B^*A^*,$$
$$(A^*)^* = A \quad \text{and} \quad 1^* = 1.$$

By definition $\mathcal{N}(\mathcal{H})$ is a $*$-algebra and according to this the multiplication is given by an involutive mapping. The group Γ acts by assumption on the Hilbert space via unitary transformations which are continuous and thus bounded. Since the map that sends an unitary operator to its adjoint is involutive, we can form the von Neumann algebra $\mathcal{N}_\Gamma$ of bounded linear operators on $L^2\Gamma \otimes \mathcal{H}$ which additionally commute with the action of Γ. This algebra $\mathcal{N}_\Gamma$ is a tensor product in the sense of von Neumann algebras:

$$\mathcal{N}_\Gamma = \mathcal{R}_\Gamma \otimes \mathcal{B}(\mathcal{H}),$$

where $\mathcal{R}_\Gamma$ is the von Neumann algebra of bounded operators on $L^2\Gamma$ and $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on the Hilbert space $\mathcal{H}$. The algebra $\mathcal{R}_\Gamma$ is generated by right translations, i.e. by the maps

$$R_\gamma f(x) = f(x\gamma),$$

again with $f \in L^2\Gamma$ and $x \in \Gamma$ (cf. [Kol95, Chapter 6]). We define a trace on $\mathcal{R}_\Gamma$ in the following way: The L^2-structure on Γ is chosen with respect to the Dirac measure, that means if we consider the group Γ as the diagonal element in $\Gamma \times \Gamma$, we have an orthonormal basis $(\delta_\gamma)_{\gamma\in\Gamma}$ in $L^2\Gamma$ where δ_γ denotes the Dirac Delta function at γ. According to this we get $\delta_\gamma(x) = 1 \iff x = \gamma$. In this basis every operator $R_\gamma \in \mathcal{R}_\Gamma$ has the property that all of its diagonal elements are equal. We define a trace function for any $A \in \mathcal{R}_\Gamma$ by

$$\operatorname{tr}_\Gamma A = (A\delta_e, \delta_e).$$

If Tr denotes the usual trace on $\mathcal{B}(\mathcal{H})$, we get an induced trace on the algebra $\mathcal{N}_\Gamma$ setting

$$\operatorname{Tr}_\Gamma = \operatorname{tr}_\Gamma \otimes \operatorname{Tr}.$$

This leads to the definition of the Γ-dimension.

Definition 1.7.3. Let L be an arbitrary Γ-invariant subspace of $L^2\Gamma \otimes \mathcal{H}$. We define the **Γ-dimension** of L as

$$\dim_\Gamma L = \operatorname{Tr}_\Gamma P_L$$

where P_L is the orthogonal projection on L in $L^2\Gamma \otimes \mathcal{H}$.

Remark 1.7.4. The idea of introducing the Γ-dimension is a normalization of the usual complex dimension. In the case that Γ is a finite group and $\mathcal{H}$ a Hilbert module, we simply have $\dim_\Gamma \mathcal{H} = \frac{1}{|\Gamma|} \dim \mathcal{H}$.

Obviously $\dim_\Gamma \in [0, \infty]$. We introduce the notion of Γ-morphisms. Consider two given Γ-modules $L_1, L_2 \subset L^2\Gamma \otimes \mathcal{H}$. A bounded linear operator $T\colon L_1 \to L_2$ is called **Γ-morphism** if it commutes with the action of Γ. Let us recall some basic properties of the Γ-dimension as proved in [Kol95, Chapter 6].

Lemma 1.7.5. *Let $T\colon L_1 \to L_2$ be a Γ-morphism. Then the following holds:*

(i) $\dim_\Gamma L^2\Gamma = 1$.

(ii) *If T is injective, then* $\dim_\Gamma L_1 \leq \dim_\Gamma L_2$.

(iii) *If T has dense image, then* $\dim_\Gamma L_1 \geq \dim_\Gamma L_2$.

(iv) $\dim_\Gamma (\operatorname{Ker} T)^\perp = \dim_\Gamma [\operatorname{Im} T]$,

where the brackets in (iv) denote the closure as a vector space.

In Chapter 3 we will also need an unbounded analogue of Γ-morphisms which we define in the following way.

Definition 1.7.6. Let L_1, L_2 be Γ-modules and let $T\colon L_1 \to L_2$ be a closed and densely defined operator which commutes with the action of Γ. We call the operator T **Γ-Fredholm** if

(i) $\dim_\Gamma \operatorname{Ker} T < \infty$ and

(ii) there exists a closed Γ-invariant subspace $Q \subset L_2$ such that $Q \subset \operatorname{Im} T$ and $\operatorname{codim}_\Gamma Q < \infty$.

When we are dealing with Γ-modules we also need a slightly different notion of density which we define in the following way.

Definition 1.7.7. Let L be a Γ-module and let $Q \subset L$ be a Γ-invariant subspace. Then we call Q **Γ-dense** in L if for any $\delta > 0$ there exists a Γ-invariant subspace $Q_\delta \subset Q$ such that Q_δ is closed in L and $\operatorname{codim}_\Gamma Q_\delta < \delta$ in L.

We conclude this Section with a Lemma that connects the notions of Γ-Fredholm operators and Γ-density as introduced in Definitions 1.7.6 and 1.7.7, respectively.

Lemma 1.7.8. *Let $T\colon L_1 \to L_2$ be a Γ-Fredholm operator and let $L_3 \subset L_2$ be a closed Γ-invariant subspace such that $L_3 \subset \overline{\operatorname{Im} T}$. Then $L_3 \cap \operatorname{Im} T$ is Γ-dense in $L-3$.*

Proof. See [GHS98, Corollary 2.6]. □

1.8 L^2 Cohomology

The main goal of this thesis is to study L^2 cohomology spaces. In this Section we will describe how they are constructed.

Let X be a complex manifold with boundary bX and set as before $\overline{X} = X \cup bX$. Let E be a hermitian vector bundle on X and let $\Omega^{p,q}(X, E)$ be the vector bundle of smooth E-valued (p, q)-forms on X. We denote by $\Omega_0^{p,q}(\overline{X}, E)$ the space of smooth E-valued p, q-forms with compact support which are smooth up to the boundary (the zero as subscript will always mean compactly supported forms). In the following we set $p = 0$.

Definition 1.8.1. The completion of $\Omega_0^{0,q}(\overline{X}, E)$ with respect to the metrics on X and E is called the corresponding L^2 space and denoted by $L^{0,j}(X, E)$. By convention we write $L^2(X, E)$ for $L^{0,0}(X, E)$.

Actually $L^{0,q}(X, E)$ consists of $(0, q)$-forms with values in E which have measurable coefficients that satisfy

$$\int_X |u|^2_{h^E} \, dV_X < \infty, \tag{1.8}$$

where $|.|_{h^E}$ denotes the pointwise norm induced by the hermitian metric h^E on E.

In order to perform analysis on these L^2 spaces, we need to extend differential operators, for example the $\bar{\partial}$-operator, which is a priori only defined on forms with compact support. There are different possibilities to do this, but we will focus on one of them.

Let in general $D\colon \Omega_0^{0,\bullet}(X,E) \to L^{0,\bullet}(X,E)$ be a differential operator, which is in particular a preclosed operator. Let further D^* be the **formal adjoint** operator of D, i.e. the operator defined by

$$\langle Du, v\rangle = \langle u, D^* v\rangle \qquad \forall u, v \in \Omega_0^{0,\bullet}(X,E).$$

Definition 1.8.2. The **maximal extension** of D is defined by

$$\begin{aligned} \mathrm{Dom}(D_{\max}) &= \left\{ u \in L^{0,\bullet}(X,E) \,\middle|\, Du \in L^{0,\bullet}(X,E) \right\}, \\ D_{\max} u &= Du, \quad \text{for } u \in \mathrm{Dom}(D_{\max}), \end{aligned} \tag{1.9}$$

where Du is calculated in the sense of distributions.

This way of extending D has several benefits as can be seen in the next lemma.

Lemma 1.8.3. *The operator D_{max} is a densely defined and closed operator.*

The proof of Lemma 1.8.3 is straightforward and can be found in [Ada07, Theorem 2.5]. We will use these properties for the operators $\bar{\partial}^E$ and $\Box^E$, which are defined in the following way.

Definition 1.8.4. The extension of $\bar{\partial}$ to E-valued $(0,q)$-forms is defined as

$$\begin{aligned} \bar{\partial}^E \colon \Omega_0^{0,q}(X,E) &\to \Omega^{0,q+1}(X,E) \\ \bar{\partial}^E &= \bar{\partial} \otimes 1. \end{aligned}$$

We shall work with the maximal extension $\bar{\partial}^E_{\max}$ given in (1.9) in the sequel and denote for simplicity $\bar{\partial}^E = \bar{\partial}^E_{\max}$. Clearly $(\bar{\partial}^E)^2 = 0$, so we get a complex

$$\bar{\partial}^E \colon L^{0,\bullet}(X,E) \to L^{0,\bullet+1}(X,E).$$

The cohomology of this complex is the L^2 **cohomology** of X with values in E and is defined as

$$H^{0,q}_{(2)}(X,E) = \frac{\operatorname{Ker} \bar{\partial}^E \colon L^{0,q}(X,E) \to L^{0,q+1}(X,E)}{\operatorname{Im} \bar{\partial}^E \colon L^{0,q-1}(X,E) \to L^{0,q}(X,E)}.$$

The **reduced L^2 cohomology** is defined as

$$\overline{H}^{0,q}_{(2)}(X,E) = \frac{\operatorname{Ker} \bar{\partial}^E \colon L^{0,q}(X,e) \to L^{0,q+1}(X,E)}{[\operatorname{Im} \bar{\partial}^E \colon L^{0,q-1}(X,E) \to L^{0,q}(X,E)]}, \tag{1.10}$$

where the brackets in the denominator stand for the closure of vector spaces. Hence the reduced L^2 cohomology spaces are again Hilbert spaces. For several reasons we are interested in the dimension of these spaces, for example consider

$$H^{0,0}_{(2)}(X,E) = \operatorname{Ker} \bar{\partial}^E \colon L^2(X,E) \to L^{0,1}(X,E).$$

Since by definition $u \in H^{0,0}_{(2)}(X,E)$ if and only if $\bar{\partial}^E u = 0$, it is convenient to determine the dimension if you want to know how many L^2-holomorphic functions on X with values in E exist. In the case that X is a covering manifold as described in Section 1.6 we use the Γ-dimension from Section 1.7.
One important tool to compute the dimension of these cohomology spaces is the Laplace operator.

Definition 1.8.5. Let $\bar{\partial}^E$ be as in Definition 1.8.4 and let $\bar{\partial}^{E*}$ be its formal adjoint operator. The **Kodaira Laplacian** is defined as

$$\begin{aligned} &\square^E \colon \Omega^{0,q}_0(X,E) \to \Omega^{0,q}(X,E) \\ &\square^E = \bar{\partial}^E \bar{\partial}^{E*} + \bar{\partial}^{E*} \bar{\partial}^E. \end{aligned} \tag{1.11}$$

It is a densely defined and positive operator which makes it possible to introduce its Friedrichs extension (cf. Definition 1.3.7). For simplicity we denote the Friedrichs extension by the same symbol $\square^E$. We denote by $\mathscr{H}^{0,j}(X,E)$ the space of **harmonic** $(0,j)$-forms which is defined by

$$\mathscr{H}^{0,j}(X,E) \coloneqq \operatorname{Ker} \square^E \cap L^{0,j}(X,E).$$

One goal of using the Laplace operator is that we have an L^2 analogue of the Hodge decomposition theorem similar to the classical one. But before we can state this, we need an estimate which will play an important role in the following chapters.

Definition 1.8.6. Let K be a compact subset of X. We say that the **fundamental estimate** holds if there exists a constant $C \in \mathbb{R}^+$ such that the inequality

$$\|u\|_{L^2}^2 \leq C \left(\left\|\bar{\partial}^E u\right\|_{L^2}^2 + \left\|\bar{\partial}^{E*} u\right\|_{L^2}^2 + \int_K |u|^2 \, dV \right) \tag{1.12}$$

is satisfied for any $u \in \operatorname{Dom} \bar{\partial}^E \cap \operatorname{Dom} \bar{\partial}^{E*} \cap L^{0,j}(X,E)$.

The fundamental estimate is a very useful tool because it provides a weak Hodge-Kodaira decomposition.

Theorem 1.8.7 (Hodge Theory)**.** *Let X be a complex manifold with boundary. Then the following weak orthogonal decomposition holds:*

$$L^{0,j}(X,E) = [\operatorname{Im}(\bar{\partial}^E \bar{\partial}^{E*})] \oplus \mathscr{H}^{0,j}(X,E) \oplus [\operatorname{Im}(\bar{\partial}^{E*} \bar{\partial}^E)],$$

where the brackets denote the closure as a vector space. In particular we have an isomorphism of Γ-modules

$$\overline{H}_{(2)}^{0,j}(X,E) = \mathscr{H}^{0,j}(X,E).$$

Proof. See [GHS98, Proposition 1.4]. □

2 Coverings of q-convex Manifolds

In this chapter we will be concerned with coverings of q-complete manifolds. Let Γ be a discrete group acting freely and properly discontinuously on a complex manifold X and set $\widetilde{X} = X/\Gamma$. We consider a q-complete submanifold $\widetilde{M}$ of $\widetilde{X}$ and define by $\pi\colon M = \pi^{-1}\widetilde{M} \to \widetilde{M}$ the induced covering. Let $\widetilde{E}$ be a holomorphic line bundle on $\widetilde{M}$ and denote by $E = \pi^{-1}\widetilde{E}$ its pullback. In Section 2.1 we introduce the notion of q-convex and q-complete manfifolds. Afterwords we study spectra of differential operators in Section 2.2 on this class of manifolds and finally in Section 2.3 we study the L^2 cohomology groups $H^{0,j}_{(2)}(M, E)$.

2.1 q-Convexity

In this Section we will describe the general setup for this Chapter.

Let X be a complex manifold of complex dimension n, Γ be a discrete group acting holomorphically and properly discontinuously on X and let E be a holomorphic vector bundle on X. Set $\widetilde{X} = X/\Gamma$. Then X becomes naturally a covering space of $\widetilde{X}$. We fix Γ-invariant metrics on both X and E (cf. Remark 1.6.3).
Next let us consider a relatively compact open set $\widetilde{M} \subset \widetilde{X}$ with smooth boundary and let $M = \pi_\Gamma^{-1}\widetilde{M}$ be its preimage. Since M is a subset of X, the group Γ acts on it. Let $U \subset M$ be a fundamental domain for the

Γ-action (cf. Definition 1.6.2).
We define the Hilbert space of square integrable sections on M with values in E with respect to the Γ-invariant metrics on X and E chosen before and denote it as in Section 1.8 by $L^2(M,E)$. The corresponding L^2-space on U is defined in the same way and denoted by $L^2(U,E|_U)$. These spaces are indeed related, as the following result shows (cf. [MM07, §3.6.1]).

Lemma 2.1.1. *Under the assumptions of the situation above we have isomorphisms*

$$L^2(M,E) \cong L^2\Gamma \otimes L^2(U,E|_U) \cong L^2\Gamma \otimes L^2(\tilde{M},\tilde{E}).$$

This formula will play an important role in the analysis of differential operators acting on L^2-sections of the bundle E because due to this splitting we can keep apart the action of the group and the differential operators on them.

Let $\Omega_0(M,E)$ be the space of smooth sections of E with compact support. Let $\tilde{D}\colon \Omega^{\bullet,\bullet}(\tilde{X},\tilde{E}) \to \Omega^{\bullet,\bullet}(\tilde{X},\tilde{E})$ be a formally self-adjoint, strongly elliptic and positive first order differential operator acting on the sections of $\tilde{E}$ (cf. Definitions 1.3.2 and 1.3.8). Denote by D its pullback on X which is a Γ-invariant first order differential operator acting on $\Omega^{\bullet,\bullet}(X,E)$. Note that the Γ-invariance is a direct consequence of the second isomorphism of Theorem 2.1.1 since D acts only on the second factor.

Consider two extensions of the previously described operator D on the domains U and M introduced earlier: Let D_0 denote the Friedrichs extension (cf. Definition 1.3.7) of D to $L^2(U,E|_U)$ and for simplicity we call the Friedrichs extension to $L^2(M,E)$ again D. Note that the assumption that D is formally self-adjoint and positive ensure that these Friedrichs extensions exist (cf. Propositions 1.3.4, 1.3.5). Since any formally self-adjoint operator is closed, its extensions are closed, too. Both extensions of D coincide with the corresponding Dirichlet Laplacians on

U and M respectively because these extensions are constructed out of compactly supported sections which by definition vanish outside their domain and thus satisfy the Dirichlet boundary conditions on U and M respectively.
We will later analyze the spectra of the operators constructed above in detail, so let us sum up some properties which should be kept in mind. Let $\{E_\lambda(D)\}_{\lambda\in\mathbb{R}}$ as defined in 1.3.10 be the spectral family associated to D. We saw already that the operator D is Γ-invariant which means that it commutes with the maps L_γ defined in (1.7). Thus the spectral projections $E_\lambda(D)$ commute with L_γ, too, and are according to this also Γ-invariant. Note also that by the Rellich Lemma 1.3.13 the operator D_0 has compact resolvent (since the embedding of $W^1(U,E) \hookrightarrow L^2(U,E)$ is compact) and because of this has discrete spectrum.

From now on let M be a complex manifold of complex dimension n with smooth boundary bM. We denote by $\overline{M} = M \cup bM$ and we will assume that there exists a complex neighbourhood X of $\overline{M}$ such that every boundary point of M is an interior point of X. We will further on suppose that bM is given as the zero set of a Γ-invariant defining function for X. Let $T_x^{\mathbb{C}}bM$ be the holomorphic tangent space at $x \in bM$ as defined in (1.5).

Definition 2.1.2. The manifold X is called q**-convex** in the sense of Andreotti and Grauert which we will shortly call q-convex if there exists a smooth exhaustion function $\varphi\colon X \to [a,b)$, $a \in \mathbb{R}$, $b \in \mathbb{R}\cup\{+\infty\}$, such that the sublevel sets $X_c = \{x \in X \mid \varphi(x) < c\}$ are relatively compact in X and the complex Hessian $\partial\bar{\partial}\varphi$ has at least $n-q+1$ positive eigenvalues outside a compact set $K \subset X$. We call X q**-complete** if the choice $K = \emptyset$ is possible.

Remark 2.1.3. The notion of q-convexity in Definition 2.1.2 goes back to Andreotti and Grauert who introduced this in [AG62].

Definition 2.1.4. Let M be a domain in a complex manifold X with

$\mathcal{C}^2$-boundary. Then M is called **Levi q-convex** if the Levi form $L_\rho(x;w)$ has $n-q$ positive eigenvalues at every boundary point $x \in bM$.

The concepts of q-convexity and Levi q-convexity are related as the following Lemma from [HL88, Lemma 5.8] shows.

Lemma 2.1.5. *Let M be a complex manifold of complex dimension n with smooth boundary bM and let ρ be a defining function for M. If the Levi form of ρ*

$$L_\rho(x;w) = \sum_{k,l=1}^{n} \frac{\partial^2 \rho}{\partial z_k \partial \bar{z}_l}(x) w_k \overline{w}_l. \tag{2.1}$$

has $n-q$ positive eigenvalues at every boundary point $x \in bM$, then M is q-convex.

Proof. Consider the holomorphic tangent space $T_x^{\mathbb{C}}(bM)$. Since the Levi form has $n-q$ positive eigenvalues at $x \in bM$, we choose an $n-q$-dimensional subspace T of $T_x^{\mathbb{C}}(bM)$ such that $L_\rho(x,w)|_T$ is positive definite. Choose further a $n-q+1$-dimensional subspace T' of $T_x^{\mathbb{C}}(bM)$ such that
$T' \cap T_x^{\mathbb{C}}(bM) = T$ and set

$$K = \{t \in T' \mid \|t\| = 1, \quad L_\rho(x;t) < 0\},$$

where $\|.\|$ is some norm on $T_x^{\mathbb{C}}(bM)$. Then K is compact and $t(\rho) \neq 0$ for all $t \in K$. Then we know by the min-max theorem that we can find a constant $c_x > 0$ such that

$$\max_{t\in K} |L_\rho(x;t)| < c_x \min_{t\in K} |t(\rho)|^2$$

for a constant $C > c_x$. Hence

$$L_{e^{C\rho}}(x;t) = e^{C\rho}\left(CL_\rho(x;t) + C^2 |t(\rho)|^2\right),$$

and thus we get $L_{e^{C\rho}}(x;t) > 0$ if $0 \neq t \in T'$ and $C > c_x$. $\square$

Let X be complex manifold and Γ be a discrete group that acts on X freely and properly discontinuously and let as before E be a Γ-invariant holomorphic vector bundle on X. Then X/Γ is a normal covering and we fix Γ-invariant metrics on X and E. Let M be a smooth open set in X and suppose that

(i) M is q-convex,

(ii) $\overline{M} = M \cup bM$ is Γ-invariant and

(iii) M/Γ is compact.

Let $\Omega^{0,j}(M,E)$ be the space of smooth $(0,j)$-forms on M with values in E and denote by $\Omega_0^{0,j}(\overline{M},E)$ the space of compactly supported $(0,j)$-forms which are smooth up to the boundary. Let $L^{0,j}(M,E)$ be the L^2-space of $(0,j)$-forms with respect to the Γ-invariant metrics on M and E. We denote by $\bar{\partial}^E$ the maximal extension of the usual $\bar{\partial}$-operator as in Definition 1.8.4 and (1.9) respectively. It is a closed and densely defined operator with domain

$$\operatorname{Dom}\bar{\partial}^E = \left\{u \in L^{0,j}(M,E) \,\middle|\, \bar{\partial}^E u \in L^{0,j+1}(M,E)\right\},$$

where $\bar{\partial}^E u$ is defined in the sense of distributions. We consider the reduced L^2 cohomology spaces $H^{0,j}_{(2)}(M,E)$ as described in (1.10). Using Theorem 2.1.1 we obtain that $\bar{\partial}^E$ is a Γ-invariant operator since it acts only on the second factor. Thus the cohomology spaces are naturally Γ-modules. Let $\bar{\partial}^{E*}$ be the Hilbert space adjoint of $\bar{\partial}^E$.

Remark 2.1.6. Note that the cohomology spaces are naturally Γ-modules since the operators $\bar{\partial}^E$ and $\bar{\partial}^{E*}$ are Γ-invariant.

Later on we will use for technical reasons also the formal adjoint operator of $\bar{\partial}^E$ which is denoted by ϑ^E. Since we are working on manifolds with boundary, these two operators do not have to coincide since there might appear boundary values. The following formula is taken from [FK72, Proposition 1.3.1].

Lemma 2.1.7. *Let $u,v \in \Omega^{0,j}(\overline{M},E)$ and let $\sigma(\vartheta^E,\cdot)$ be the principal symbol of the formal adjoint operator ϑ^E. Then*

$$(\bar{\partial}^E u, v) = (u, \vartheta^E v) + \int_{bM} \langle \sigma(\vartheta^E, d\rho)u, v\rangle \, dV_{bM}.$$

As we will see, there is a relation between $\bar{\partial}^{E*}$ and ϑ^E. We introduce the boundary condition $\sigma(\vartheta^E, d\rho)u = 0$ to the effect that the boundary integral in the preceded lemma vanishes and thus the operators coincide. We set

$$B^{0,j}(M,E) := \left\{ u \in \Omega_0^{0,j}(\overline{M},E) \,\middle|\, \sigma(\vartheta^E, d\rho)u = 0 \text{ on } bM \right\}.$$

Integration by parts as in [FK72, Proposition 1.3.1 and 1.3.2] shows that

$$B^{0,j}(M,E) = \Omega_0^{0,j}(\overline{M},E) \cap \operatorname{Dom} \bar{\partial}^{E*}, \tag{2.2}$$

$$\bar{\partial}^{E*} = \vartheta^E \text{ on } B^{0,j}(M,E). \tag{2.3}$$

Moreover, (2.2) and (2.3) implie

$$\langle \bar{\partial}^E u, v\rangle = \langle u, \bar{\partial}^{E*} v\rangle \quad \text{for } u \in \Omega_0^{0,j}(\overline{M},E), v \in B^{0,j+1}(M,E). \tag{2.4}$$

Thus we can consider the operator $\square^E$ with domain

$$\operatorname{Dom}(\square^E) = \left\{ u \in B^{0,j}(M,E) \,\middle|\, \bar{\partial}^E u \in B^{0,j+1}(M,E) \right\}, \tag{2.5}$$

$$\square^E u = \bar{\partial}^E \bar{\partial}^{E*} u + \bar{\partial}^{E*} \bar{\partial}^E u, \quad \text{for } u \in \operatorname{Dom}(\square^E),$$

for all j which is by (2.4) a positive operator.

Definition 2.1.8. The boundary conditions of $\operatorname{Dom}(\square^E)$ in (2.5) are called **$\bar{\partial}$-Neumann boundary conditions** and are given by

$$\operatorname{Dom}(\square^E) = \left\{ u \in \Omega^{0,\bullet}(\overline{M},E) \,\middle|\, \sigma(\vartheta^E, d\rho)u = \sigma(\vartheta^E, d\rho)\bar{\partial}^E u = 0 \text{ on } bM \right\}.$$

2.2 Spectral Counting Functions

In this Section we will analyze the spectrum of differential operators via spectral distribution functions.

In Section 1.3 we introduced the Friedrichs extension of a formally self-adjoint, strongly elliptic and positive differential operator. This extension enables us to study spectral properties of the operators considered. Let $\{E_\lambda(D)\}_{\lambda\in\mathbb{R}}$ and $\{E_\lambda(D_0)\}_{\lambda\in\mathbb{R}}$ respectively be the spectral resolutions of the operators D and D_0 from Section 2.1.

Definition 2.2.1. The **spectral distribution function** associated to D_0 is the function $N(\lambda, D_0) = \dim \operatorname{Im} E_\lambda(D_0)$.

Since $N(\lambda, D_0)$ equals the number of eigenvalues $\leq \lambda$ (counted with multiplicity), we will refer to it as the "counting function". Similarily we denote by $N_\Gamma(\lambda, D) = \dim_\Gamma E_\lambda(D)$ the counting function associated to D. According to the remarks before the image of the spectral projection is a Γ-module and has thus a well-defined Γ-dimension. We want to compare the counting functions $N_\Gamma(\lambda, D)$ and $N(\lambda, D_0)$. The following lemma provides a formula for $N_\Gamma(\lambda, D)$.

Lemma 2.2.2 (Variational Principle)**.** *Let D be a positive self-adjoint Γ-invariant differential operator and let D act on a free Hilbert module of the form $L^2\Gamma \otimes \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space. If Q denotes the quadratic form associated to D, the counting function satisfies*

$$N_\Gamma(\lambda, D) = \sup \left\{ \dim_\Gamma L \,\middle|\, L \subset \operatorname{Dom}(Q),\ Q(u,u) \leq \lambda \left\| u \right\|^2 \ \forall u \in L \right\}, \tag{2.6}$$

whereas L runs over all Γ-modules that are contained in $\operatorname{Dom}(Q)$*.*

Proof. See [Shu96, Lemma 2.4]. □

We are looking for upper and lower bounds of $N_\Gamma(\lambda, D)$. Since the spectrum of D_0 is discrete, the spectral projections onto eigenspaces bounded by λ are of finite dimension. The first estimate relating $N_\Gamma(\lambda, D)$ to $N(\lambda, D_0)$ is what we intuitively expect.

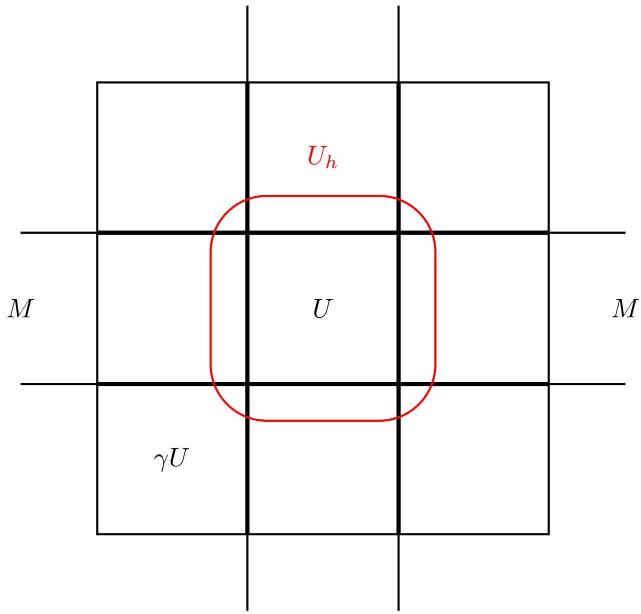
Lemma 2.2.3 (Estimate from below)**.** *Under the assumptions of Lemma 2.2.2 the following estimate holds for any $\lambda \in \mathbb{R}$:*

$$N_\Gamma(\lambda, D) \geq N(\lambda, D_0). \tag{2.7}$$

Proof. Let $\lambda_0 \leq \lambda_1 \leq \ldots$ be the spectrum of D_0. Next we consider the set of eigenfunctions $\{\widetilde{\psi}_j\}$ corresponding to the eigenvalues λ_j which form an orthonormal basis in $L^2(U, E)$. Then we can extend these eigenfunctions by 0 to the whole of M to get an orthonormal basis $\{L_\gamma \psi_j\}$ of $L^2(M, E)$

with L_γ as in (1.7). Finally we consider the Γ-module Ψ_λ spanned by the orthonormal functions $\{L_\gamma \psi_j, \lambda_j \leq \lambda\}$. Then $\dim_\Gamma \Psi_\lambda = N(\lambda, D_0)$, $\Psi_\lambda \subset \mathrm{Dom}(Q)$ and $Q(u,u) \leq \lambda \|u\|^2, u \in \mathrm{Dom}(Q)$ and the claim follows now from Lemma 2.2.2 as $\mathrm{Dom}(Q)$ is complete in the graph norm. □

In order to get an estimate from above, there are some things to be kept in mind. The strategy will be to enlarge the fundamental domain U of the Γ-action a bit and compare the counting functions of D on U and on the enlarged domain with Dirichlet boundary conditions. The latter means that we consider the Friedrich extension of D on the enlarged domain restricted to compactly supported forms. For more details see [TCM01, Chapter 1].
For $h \in \mathbb{R}^+$ we let $U_h = \{x \in M \mid \mathrm{dist}(x, U) < h\}$ be the enlarged domain where the distance is chosen with respect to the metric on M. Let h be fixed and let us visualize the situation.



The extension of D onto $\Omega_0(U_h, E)$ will be denoted by $D_0^{(h)}$ where the

h indicates the chosen parameter.

Lemma 2.2.4 (Estimate from above)**.** *Under the assumptions of Lemma 2.2.2 there exists a constant $C \in \mathbb{R}^+$ such that for all $\lambda \in \mathbb{R}$*

$$N_\Gamma(\lambda, D) \leq N(\lambda + C, D_0^{(h)}). \tag{2.8}$$

Proof. See [TCM01, Proposition 1.4]. □

Summing up both implications of Lemma 2.2.3 and Lemma 2.2.4 we get

$$N(\lambda, D_0) \leq N_\Gamma(\lambda, D) \leq N(\lambda + C, D_0^{(h)}). \tag{2.9}$$

So the counting function of the Γ-dimensions gets wedged in between two "usual" dimensions.

In order to determine the Γ-dimension of the L^2 cohomology spaces we will work with the Laplacian $\Box^E$ defined in (2.5).

Lemma 2.2.5. $\Box^E$ *is a strongly elliptic, positive and formally self-adjoint operator.*

Proof. First, the principal symbol of $\bar{\partial}^E$ is $\sigma_{\bar{\partial}^E}(x, \xi) = \xi^{0,1} \wedge \cdot$ with $\xi^{0,1}$ being the $(0,1)$-part of ξ. Similarly we have $\sigma_{\bar{\partial}^{E*}}(x, \xi) = \iota_{\xi^{0,1}}$, where $\iota.$ denotes the contraction operator. Hence we have $\sigma_{\Box^E}(x, \xi) = \left|\xi^{0,1}\right|^2$, so $\Box^E$ is elliptic. Positivity and formally self-adjointness follow automatically from the definition. □

Thus we are able to use the results described in the preceded lemmas in our context. Sometimes it will be useful to work rather with the associated quadratic form Q of $\Box^E$.

Lemma 2.2.6. *The quadratic form Q (as introduced in Section 1.3) associated to $\Box^E$ on $\mathrm{Dom}(Q) := \mathrm{Dom}(\bar{\partial}^E) \cap \mathrm{Dom}(\bar{\partial}^{E*})$ is given by*

$$Q(u, v) := (\bar{\partial}^E u, \bar{\partial}^E v) + (\bar{\partial}^{E*} u, \bar{\partial}^{E*} v) \qquad \textit{for all} \quad u, v \in \mathrm{Dom}(Q). \tag{2.10}$$

For a proof see e.g. [MTC02, Lemma 3.2]. There is a close connection between the spaces $B^{0,j}(M,E)$ and $\mathrm{Dom}(Q)$.

Lemma 2.2.7. *$B^{0,j}(M,E)$ is dense in $\mathrm{Dom}(Q)$ with respect to the norm*

$$\|u\|_Q = \left(\|u\|^2 + \|\bar\partial^E u\|^2 + \|\bar\partial^{E*}\|^2\right)^{\frac{1}{2}}.$$

Proof. We will give in the proof in two steps. First we reduce to the case of compactly supported forms and second we proof the assertion in the reduced case.
Let $\varepsilon > 0$. Using Lemma 1.6.4 we can construct $[0,1]$-valued cut-off functions $\eta_\varepsilon \in \mathcal{C}_0^\infty(\overline{M})$ such that $\operatorname{supp}\eta_\varepsilon \Subset \overline{M}$ exhaust $\overline{M}$ and $\sup|d\eta_\varepsilon| \to O(\varepsilon)$ as $\varepsilon \to 0$.
For any $u \in \mathrm{Dom}(\bar\partial^E) \cap \mathrm{Dom}(\bar\partial^{E*})$ we consider the form $\eta_\varepsilon u$. It follows that $\eta_\varepsilon \in \mathrm{Dom}(\bar\partial^E) \cap \mathrm{Dom}(\bar\partial^{E*})$ and moreover we have

$$\begin{aligned} \left\|\bar\partial^E(\eta_\varepsilon u) - \eta_\varepsilon \bar\partial^E u\right\|_{L^2} &= O(\varepsilon)\,\|u\|_{L^2}\,, \\ \left\|\bar\partial^{E*}(\eta_\varepsilon u) - \eta_\varepsilon \bar\partial^{E*} u\right\|_{L^2} &= O(\varepsilon)\,\|u\|_{L^2}\,. \end{aligned}$$

Hence we have for $\varepsilon \to 0$ convergence of $\eta_\varepsilon u \to u$, $\bar\partial^E(\eta_\varepsilon u) \to \bar\partial^E u$ and finally $\bar\partial^{E*}(\eta_\varepsilon u) \to \bar\partial^{E*} u$. Since the form $\eta_\varepsilon u$ has compact support, any form $u \in B^{0,j}(M,E)$ is the limit of a sequence $\{\eta_\varepsilon u\}$ with respect to the graph norm. So we can prove the assertion for forms with compact support. But in this case this is a consequence of the Friedrichs Theorem on the identity of weak and strong extensions of differential operators, see e.g. [Hör65, Proposition 1.2.4] for the proof. □

Later on we will use cut-off functions, i.e. smooth functions $\eta \in \mathcal{C}_0^\infty(X)$ that satisfy $\|\eta\| \le 1, \eta = 1$ on some relatively compact set $V \Subset U$ with $U \subset X$ being an open set in X.

Lemma 2.2.8. *Let $V \Subset U \subset X$ as described before and let $\eta \in \mathcal{C}_0^\infty(X)$ be a cut-off function. Let Q be the quadratic form from Lemma 2.2.6. Then*

$$Q(\eta u, \eta u) \le \frac{3}{2}Q(u,u) + 6\sup|d\eta|^2\,\|u\|^2$$

holds for any $u \in \mathrm{Dom}(Q)$.

Proof. This is a direct computation. □

2.3 Γ-dimension of coverings

We will estimate this Section the Γ-dimension for the reduced Γ-modules $\overline{H}^{0,j}_{(2)}(M,E)$. Our strategy here is to compute upper and lower bounds for the dimension of the cohomology spaces with the help of the spectral counting functions from Section 2.2 associated to the Laplacian $\square^E$ on the fundamental domain of the Γ-action.

Our starting point for the sequel is the Bochner-Kodaira-Nakano formula for manifolds with boundary. To state it, we need some further terminology. Let End $(\Omega^{\bullet,\bullet}(M,E))$ be the algebra of endomorphisms of smooth differential forms on M with values in E. We denote by $[\cdot,\cdot]$ the **graded commutator** (or graded Lie bracket) on it which is defined as

$$[A,B] = AB - (-1)^{\deg A \deg B} BA$$

for $A, B \in \operatorname{End}(\Omega^{\bullet,\bullet}(M,E))$. Here $\deg A$ denotes the degree of A with respect to the bigrading on $\Omega^{\bullet,\bullet}(M,E)$. Let further L be the **Lefschetz operator** associated to the real $(1,1)$-form ω of the metric on M and let Λ be the adjoint operator of L, i.e. the operators

$$\begin{aligned} L\colon \Omega^{\bullet,\bullet}(M,E) &\to \Omega^{\bullet+1,\bullet+1}(M,E), \qquad u \mapsto \omega\wedge u, \quad \text{and}\\ \Lambda\colon \Omega^{\bullet,\bullet}(M,E) &\to \Omega^{\bullet-1,\bullet-1}(M,E), \qquad u \mapsto \iota_\omega u, \end{aligned}$$

where ι denotes the contraction operator. The **torsion operator** of the metric ω on M is defined by $\mathcal{T} = [\Lambda, \partial\omega]$. There is one last thing we need to define in order to state the Bochner-Kodaira-Nakano formula. Let L_ρ be the Levi form as defined in (2.1). We set

$$\mathcal{L}_\rho(u,u) = [L_\rho, \Lambda]u \tag{2.11}$$

for any $u \in B^{0,j}(M,E)$.

Theorem 2.3.1 (Bochner-Kodaira-Nakano). *Let K_X be the canonical bundle on X and denote by $\nabla^{1,0}$ the $(1,0)$-part of the connection on $E \otimes K_X$. Then for every $u \in B^{0,j}(M,E)$ we have*

$$\begin{aligned} Q(u,u) &= \left\|\bar{\partial}^E u\right\|^2 + \left\|\bar{\partial}^{E*} u\right\|^2 \\ &= \left\|\nabla^{1,0*} u\right\|^2 + \langle i[\Theta(E \otimes K_X^*), \Lambda]u, u\rangle + \int_{bM} \mathcal{L}_\rho(u,u) dV_{bM} \\ &\quad + \langle \mathcal{T}^* u, \nabla^{1,0*} u\rangle - \langle [\bar{\partial}^E, \overline{\mathcal{T}}^*]u, u\rangle. \end{aligned} \tag{2.12}$$

This formula is studied intensively in [Paragraph 1.4.4][MM07]. We compute the spectrum of the Laplacian (2.5) and use the Bochner-Kodaira-Nakano formula to get an estimate for the quadratic form. We start with the latter and follow [MM07, Chapter 3.5]. Let M be q-convex.

Remark 2.3.2. We work rather with the twisted bundle $E \otimes K_X$ than with E itself because there are estimates of the eigenvalues of hermitian forms which are in some sense "preferable". The twist also does not change dimensions of the cohomology spaces since the complex of $(0,j)$-forms is canonically isomorphic to the one of (n,j)-forms via the isomorphism (1.4).

At first we give an estimate for $\mathcal{L}_\rho(u,u)$ in Theorem (2.3.1). Let $\gamma \in \Omega^{1,1}(X)$ be an arbitrary real $(1,1)$-form on X. By the Gram-Schmidt process we can find an ω-orthonormal basis $(\zeta_1, \dots, \zeta_n)$ of $T^{*(1,0)}X$ which diagonalizes ω and γ at the same time, hence we can write

$$\omega = i\sum_{j=1}^{n} \zeta_j \wedge \overline{\zeta}_j, \qquad \gamma = \sum_{j=1}^{n} \gamma_j \zeta_j \wedge \overline{\zeta}_j.$$

Lemma 2.3.3. *For any $u \in \Omega^{0,q}(X), u = \sum_{|J|=q} u_J \overline{\zeta_J}$ we have*

$$[\gamma, \Lambda]u = \sum_{|J|=q} \left(\sum_{k\in J} \gamma_k - \sum_{j=1}^{n} \gamma_j \right) |u_J|^2 .$$

Proof. See [Dem12, Chapter VI, §5.2] with $p = 0$. □

Since $\mathcal{L}_\rho$ is a hermitian $(1,1)$-form, it is real and so we can apply Lemma 2.3.3 with $\gamma = \mathcal{L}_\rho$.

Secondly we show that the torsion in (2.3.1) is bounded and therefore negligible.

Lemma 2.3.4. *Let X be a complex manifold of complex dimension n and let γ be a real smooth $(1,1)$-form having at least $n-q+1$ positive eigenvalues. For any $N \in \mathbb{N}$ there exists a metric ω on X such that*

(i) *at least $n-q+1$ eigenvalues of γ are bigger than N and*

(ii) *the negative eigenvalues of γ are bigger than $-\frac{1}{N}$.*

Proof. See [Ohs82, Lemma 4.3]. □

Lemma 2.3.5. *Let M be Levi q-convex and let (E, h^E) be a holomorphic line bundle on M. For any $C \in \mathbb{R}^+$ there is a metric ω on X such that for all $u \in \Omega_0^{0,j}(X, E)$ with support in a neighbourhood U of bM with $j \geq q$, we have*

$$\mathcal{L}_\rho(u, u) \geq C \, |u|^2 . \tag{2.13}$$

Proof. Since M is Levi q-convex, the Levi form has $n-q$ positive eigenvalues at every boundary point. Consider local coordinates $(U, z_1, \dots, z_n)$ around $x \in bM$ such that $L_\rho(x; w)$ is positive definite on the subbundle of $T_x X|_U$ generated by $\frac{\partial}{\partial z_q}, \dots, \frac{\partial}{\partial z_n}$. We define a metric on U depending on $\varepsilon \in \mathbb{R}^+$ by

$$\omega_\varepsilon = \sum_{k=1}^{n} \varepsilon_k dz_k \otimes d\bar{z}_k,$$

with $\varepsilon_k = \frac{1}{\varepsilon}$ if $k < q$ and $\varepsilon_k = \varepsilon$ else. Since the form $\partial\bar{\partial}\rho$ is a hermitian form, all eigenvalues are real. Denote them with respect to ω_ε by $\mu_1 \leq \ldots \leq \mu_n$. By choosing ε small enough we get the estimates $\mu_1 \geq -\frac{1}{c_0}$ and $\mu_q \geq c_0$ for some $c_0 \in \mathbb{R}^+$ by Lemma 2.3.4. Using Lemma 2.3.3 we get

$$\begin{aligned}
\mathcal{L}_\rho(u, u) &\geq (\mu_1 + \ldots + \mu_n) \, |u|^2 \\
&\geq \left((j - q + 1) c_0 - \frac{1}{c_0}(q-1) \right) |u|^2 .
\end{aligned} \tag{2.14}$$

Choosing c_0 sufficiently large implies (2.13) for $u \in \Omega_0^{0,j}(U, E)$. Patching the metrics constructed above with the help of a partition of unity we obtain the Lemma. $\square$

Lemma 2.3.6. *The the torsion operator* $\mathcal{T} \colon L^{0,j}(\overline{X_c}, E) \to L^{1,j}(\overline{X_c}, E)$ *given by* $\mathcal{T} = [\Lambda, \partial\omega]$ *is bounded.*

Proof. By definition the Torsion operator acts by a composition of interior multiplication with the metric ω and multiplication with $\partial\omega$. Since this means that we multiply continuous functions on a compact set this implies that $\mathcal{T}$ is bounded. $\square$

We also multiply the hermitian metric on the fibers of E with $\exp(-\chi(\rho))$, where χ a smooth **convex increasing weight function** χ on $\mathbb{R}$, i.e. a function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $\chi' > 0$ and $\chi'' \geq 0$, such that

$$h_\chi^E = \exp(-\chi(\rho))h^E \qquad \text{and} \qquad E_\chi = (E, h_\chi^E).$$

A direct computation yields

$$i\Theta(E_\chi) = i\Theta(E) + i\chi'(\rho)\partial\bar{\partial}\rho + i\chi''(\rho)\partial\rho \wedge \bar{\partial}\rho. \tag{2.15}$$

Then $i\chi''(\rho)\partial\rho \wedge \bar{\partial}\rho$ is positive semidefinite since ρ is real and $\chi'' \geq 0$ and by assumption we have that $i\chi'(\rho)\partial\bar{\partial}\rho$ has at least $n - q + 1$ positive eigenvalues outside X_c. Thus for $j \geq q$ we obtain similar to (2.14)

$$\langle [i\Theta(E_\chi \otimes K_X^*, \Lambda)]u, u\rangle \geq (\chi'(\rho)A - B)\,|u|^2 \tag{2.16}$$

with $A, B \in \mathbb{R}^+$. Choosing χ increasing sufficiently fast, the bracket becomes positive.

In order to get an estimate for the boundary term in (2.3.1), we need that bM is the sublevel set X_0. We obtain this by replacing the defining function ρ for M by $\rho_0 = (\rho - c)/\,|d\rho|$ near bM. This implies that the boundary term is $\mathcal{L}_{\rho_0}(u, u)$ with u being compactly supported in a neighbourhood of bM. Accordingly we conclude

$$\begin{aligned}
L_{\rho_0} &= \partial\bar{\partial}\rho_0 \\
&= \partial\bar{\partial}\left(\frac{\rho - c}{|d\rho|}\right) \\
&= \frac{1}{|d\rho|}\partial\bar{\partial}\rho + (\rho - c)\partial\bar{\partial}\frac{1}{|d\rho|} + \partial\frac{1}{|d\rho|} \wedge \bar{\partial}\rho + \partial\rho \wedge \bar{\partial}\frac{1}{|d\rho|}.
\end{aligned}$$

This computation in collaboration with Lemma 2.3.5 implies

$$\mathcal{L}_{\rho_0}(u, u) \geq \frac{C}{|d\rho|}|u|^2, \qquad C \in \mathbb{R}^+ \tag{2.17}$$

for any form $u \in B^{0,j}(M, E)$ provided $j \geq q$. Using (2.15), (2.16) and (2.17), we get that (2.12) simplifies to

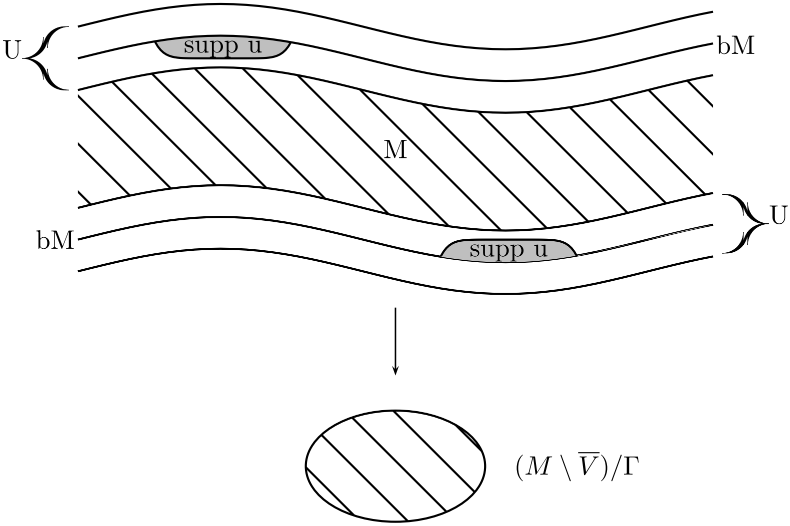
$$\begin{aligned}
Q(u, u) &\geq \left\|\nabla^{1,0*}u\right\|^2 + \langle[i\Theta(E \otimes K_X^*), \Lambda]u, u\rangle - D\|u\|^2 \\
&\geq \langle[i\Theta(E \otimes K_X^*), \Lambda]u, u\rangle - D\|u\|^2 \\
&\geq (\chi'(\rho)A - B - D)\|u\|^2 > 0,
\end{aligned}$$

where $D \in \mathbb{R}^+$ is the constant depending on the torsion and the norms are taken with respect to the metrics ω and h_χ^E. The last conclusion is obtained as in (2.16).

Theorem 2.3.7. *Let E be a Γ-invariant holomorphic line bundle on X with a Γ-invariant metric on the fibers and let $M \subset X$ be a smooth open Γ-invariant q-convex subset of X. Let $U, V \subset X$ be Γ-invariant neighbourhoods of bM such that $V \subset U$. Suppose there is a Γ-invariant cut-off function η which satisfies $\eta = 1$ on V and $\eta = 0$ outside U. There exists a constant $C \in \mathbb{R}^+$ such that for every $u \in \operatorname{Dom}(Q)$ provided $j > q$ we have*

$$\|u\|^2 \leq C\left(\left\|\bar{\partial}^E u\right\|^2 + \left\|\bar{\partial}^{E*}u\right\|^2 + \int_{M\setminus\overline{V}}|(1-\eta)u|^2\,dV\right).$$

Proof. First, Lemma 2.2.7 implies that the space $B^{0,j}(M, E)$ is dense in $\operatorname{Dom} Q$. Let $u \in B^{0,j}(M, E)$ with $\operatorname{supp} u \subset U$ and let us visualize the setup we are working in the following picture.



The neighbourhoods U, V as well as the bundle E are by assumption Γ-invariant and thus we can repeat the arguments from the proof of Lemma 2.3.5 as well as (2.17) before to get

$$\|u\|^2 \leq C_0 \left(\left\|\bar{\partial}^E u\right\|^2 + \left\|\bar{\partial}^{E*} u\right\|^2 \right) \tag{2.18}$$

for some constant $C_0 \in \mathbb{R}^+$ and $\operatorname{supp} u \subset U$. Taking η into account this yields

$$\left\|\bar{\partial}^E(\eta u)\right\|^2 + \left\|\bar{\partial}^{E*}(\eta u)\right\|^2 \leq \frac{3}{2} \left(\left\|\bar{\partial}^E u\right\|^2 + \left\|\bar{\partial}^{E*} u\right\|^2 \right) + 6 \sup |d\eta|^2 \|u\|^2 , \tag{2.19}$$

using the elementary estimate of cut-off functions (Lemma 2.2.8). By the choice of η its gradient $d\eta$ is bounded, we put $C_1 = 6 \sup |d\eta|^2$. Summing up we have by (2.18) and (2.19)

$$\begin{aligned}
\|u\|^2 &\leq C_0 \left(\left\|\bar{\partial}^E u\right\|^2 + \left\|\bar{\partial}^{E*} u\right\|^2 \right) \\
&\leq C_0 \left(\frac{3}{2} \left(\left\|\bar{\partial}^E u\right\|^2 + \left\|\bar{\partial}^{E*} u\right\|^2 \right) + \int_{M \setminus \overline{V}} |(1-\eta)u|^2 \, dV \right) .
\end{aligned}$$

□

Note that since we have Neumann boundary conditions on M and Dirichlet boundary conditions on $\overline{V}$, we can interpret the function $(1-\eta)$ as a kind of "transition" of these two conditions.

In Section 2.2 we introduced some properties of spectral spaces of coverings with respect to a formally self-adjoint, strongly elliptic and positive differential operator. Lemma 2.2.5 ensures that $\square^E$ satisfies these conditions and thus we can apply everything achieved before to the operator $\square^E$. We set

$$E^j(\lambda) := \operatorname{Im} E_\lambda(\square^E) \cap L^{0,j}(M,E)$$

and

$$E^j_D(\lambda) := \operatorname{Im} E_\lambda(\square^E_0|_{M\setminus\overline{V}}) \cap L^{0,j}(M\setminus\overline{V},E),$$

where the subscript 0 is chosen to indicate the Dirichlet boundary conditions on this domain.

Lemma 2.3.8. *Suppose the assumptions of Theorem 2.3.7 hold and let C_0 and C_1 be as in the proof of 2.3.7. For $j \geq q$ and $\lambda < \frac{1}{2C_0}$, the Γ-morphism given by*

$$E^j(\lambda) \to E^j_0(3C_0\lambda + 2C_0C_1) \qquad u \mapsto E_{3C_0\lambda+2C_0C_1}(\square^E_0|_{M\setminus\overline{V}})(1-\eta)u \tag{2.20}$$

is injective. In the sense of counting functions, we have

$$N_\Gamma(\lambda,\square^E) \leq N(3C_0\lambda + 2C_0C_1, \square^E_0|_{M\setminus\overline{V}}). \tag{2.21}$$

Proof. First we show that the map given by (2.20) is a Γ-morphism. Since η is Γ-invariant and the spectral projections are Γ-morphisms, the map considered is also a Γ-invariant morphism.
For injectivity, we take $u \in E^j(\lambda)$ with $\lambda < \frac{1}{2C_0}$. So by the properties of spectral spaces:

$$\left\|\bar\partial^E u\right\|^2 + \left\|\bar\partial^{E*}u\right\|^2 \leq \lambda\left\|u\right\|^2 \leq \frac{1}{2C_0}\left\|u\right\|^2.$$

Using theorem 2.3.7 we obtain

$$\|u\|^2 \leq 2C_0 \int_{M\setminus\overline{V}} |(1-\eta)u|^2 \, dV, \tag{2.22}$$

for $u \in E^j(\lambda)$ with $\lambda < \frac{1}{2C_0}$. We denote by $\|.\|_0$ the norm of the quadratic form associated to $\Box_0^E|_{M\setminus\overline{V}}$. Combining (2.19) and (2.22) yields

$$\begin{aligned}\left\|\bar\partial^E(1-\eta)u\right\|_0^2 + \left\|\bar\partial^{E*}(1-\eta)u\right\|_0^2 &\leq \frac{3}{2}\left(\left\|\bar\partial^E u\right\|^2 + \left\|\bar\partial^{E*}u\right\|^2\right) + C_1\|u\|^2 \\ &\leq (3C_0\lambda + 2C_0C_1)\int_{M\setminus\overline{V}} |(1-\eta)u|^2 \, dV.\end{aligned}$$

The previous estimate shows that $(1-\eta)u \in \operatorname{Im} E_{3C_0\lambda+2C_0C_1}(\Box_0^E|_{M\setminus\overline{V}})$, hence

$$E_{3C_0\lambda+2C_0C_1}(\Box_0^E|_{M\setminus\overline{V}}) = (1-\eta)u.$$

If now $E_{3C_0\lambda+2C_0C_1}(\Box_0^E|_{M\setminus\overline{V}}) = 0$, it follows $(1-\eta)u = 0$. $\Box$

Note that Lemma 2.3.8 implies, that the spectral spaces $E^j(\lambda)$ are of finite Γ-dimension for $\lambda < \frac{1}{2C_0}$ and $j > q$.

Theorem 2.3.9. *Let M be a complex manifold which is acted upon by a discrete group Γ freely and properly discontinuously such that M is a covering of a compact q-convex manifold $\widetilde{M}$ with smooth boundary $b\widetilde{M}$. Assume that E is a Γ-invariant holomorphic line bundle on M. Then we have for $j > q$*

$$\dim_\Gamma \overline{H}^{0,j}_{(2)}(M,E) < \infty.$$

Proof. At first we sum up some remarks. Since the operator $\bar\partial^E$ is Γ-invariant and commutes with $\Box^E$, it follows that $\bar\partial^E E^j(\lambda) \subset E^{j+1}(\lambda)$. The restriction of $\bar\partial^E$ onto the spectral spaces is by construction also bounded by λ, we denote it by $\bar\partial^E_\lambda$. Additionally, $\bar\partial^E$ is a Γ-morphism and by Lemma 1.7.5 it has the property $\dim_\Gamma \operatorname{Ker}(\bar\partial^E)^\perp = \dim_\Gamma \overline{\operatorname{Im}\bar\partial^E}$ and thus

$$\dim_\Gamma \operatorname{Ker}\bar\partial^E_\lambda + \dim_\Gamma \overline{\operatorname{Im}\bar\partial^E_\lambda} = \dim_\Gamma E^j(\lambda), \qquad j > q.$$

Especially, $\dim_\Gamma \operatorname{Ker} \bar{\partial}^E_\lambda$ is finite and because of this we have found a lower bound for the L^2-cohomology since

$$\dim_\Gamma H^{0,j}_{(2)}(M, E) \geq \dim_\Gamma \operatorname{Ker} \bar{\partial}^E_\lambda.$$

We remark that Hodge theory (1.8.7) implies that we have an isomorphism of Γ-modules

$$\operatorname{Ker} \square^E \cap L^{0,j}(M, E) \cong H^{0,j}_{(2)}(M, E).$$

Since

$$\operatorname{Ker} \square^E|_{E^j(\lambda)} \cap L^{0,j}(M, E) \subset \operatorname{Ker} \bar{\partial}^E|_{E^j(\lambda)} \cap L^{0,j}(M, E),$$

we have that $\dim_\Gamma \operatorname{Ker} \square^E \leq \dim_\Gamma E^j(\lambda)$ for any $\lambda \in \mathbb{R}$. Using Lemma 2.3.8, we obtain

$$N_\Gamma(\lambda, \square^E) \leq N(3C_0\lambda + 2C_0C_1, \square^E_0|_{M \setminus \overline{V}})$$

and the number on the right hand side is finite. $\square$

3 Stein Coverings

In this Chapter we will be concerned with coverings of normal Stein spaces with isolated singularities. Let S be a normal Stein space with isolated singularities and of (pure) dimension n. Consider a relatively compact strictly pseudoconvex domain $X \subset S$ with smooth boundary as in Section 1.2. In particular we assume that $X_{\mathrm{sing}} \cap bX = \emptyset$.
We consider further a Galois covering $\widetilde{S}$ of S, that is, $\widetilde{S}$ admits a free holomorphic and properly discontinuous action of a discrete group Γ such that the orbit space $\widetilde{S}/\Gamma = S$. We denote by $q : \widetilde{S} \to S$ the canonical projection and define the induced covering $\widetilde{X} := q^{-1}(X)$ of X. Our goal is to study the L^2 holomorphic functions on $\widetilde{X}$.

3.1 Stein spaces

All complex spaces considered in this Section are assumed to be countable at infinity and reduced. Let S be a complex space.

Let us sum up some properties of the singular locus S_{sing} that have been mentioned in Sections 1.4 and 1.5 respectively. By Theorem 1.5.4 S_{sing} is an analytic subset of S and hence a closed complex subspace of S. By assuming normality Theorem 1.5.6 implies that $\operatorname{codim} S_{\mathrm{sing}} \geq 2$ which means that we only have isolated singularities.

From now on we will additionally assume that S is Stein, i.e. S is holomorphically convex and the global holomorphic functions $f \in \mathcal{O}_S$ separate the points of S (cf. Definition 1.5.12).
The following Lemma by Naramsinhan (cf. [Nar61, Chapter 3]) establishes a connection between Stein spaces and plurisubharmonic functions as

introduced in Definition 1.5.10.

Lemma 3.1.1. *Let S be a Stein space. Then there exists a real analytic, strictly plurisubharmonic function $\varphi\colon S \to \mathbb{R}$ such that the sets $\{x \in S \mid \varphi(x) < c\}$ are relatively compact in S for any $c \in \mathbb{R}$.*

Let φ be the plurisubharmonic function from Lemma 3.1.1. We set $\omega = \sqrt{-1}\partial\bar{\partial}\varphi$ on the regular locus S_{reg}. Hence we conclude the following.

Corollary 3.1.2. *Any Stein space admits a Kähler form ω defined on the regular locus.*

This can be seen quite easily since $\omega = \sqrt{-1}\partial\bar{\partial}\varphi$ is a Kähler form if and only if the function φ is strictly plurisubharmonic.
Strictly plurisubharmonic functions are used to describe pseudoconvex domains in a complex space X in the same way as they are used in the context of manifolds.

Definition 3.1.3. Let S be a complex space. An open subset $X \subset\subset S$ of S is called **strictly pseudoconvex** if for any $x_0 \in bX$ there exists a neighbourhood U and a strictly plurisubharmonic function ρ on U such that $U \cap X = \{x \in U \mid \varphi(x) < 0\}$.

We also have the notion of Levi pseudoconvexity as defined in Definition 1.2.4.

Definition 3.1.4. Let X be a domain in a complex space S and let ρ be a smooth defining function for X and suppose $bX \cap X_{\mathrm{sing}} = \emptyset$. Then X is called **Levi pseudoconvex** if and only if the Levi form

$$L_\rho = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$$

is positive definite on the complex tangent space of bX.

3.2 Resolution of Singularities

In this Section we will give a short introduction to the notion of blow ups and state the famous desingularization Theorem by Hironaka. We start by introducing the blow up procedure.

Let X be a Stein space and denote as before by X_{sing} the singular locus of X. The singular points (and thus the non-normal points) can be resolved by blow ups. The general idea of blowing up points or blowing up along submanifolds is to cut out a point (or a submanifold respectively) and replace it with a projective space. Because of this we need the notion of projectivized bundles. We illustrate the method in the setup of manifolds because we have the following theorem by Bishop, Remmert and Narasimhan (see [Nar60, Theorem 6]).

Theorem 3.2.1. *Let X be a Stein space of dimension n and of finite type $m > n$ which means that we can locally realize X as an analytic set in $\mathbb{C}^m$. Then the set of proper regular embeddings of X in $\mathbb{C}^{n+m}$ is dense in the set of all holomorphic mappings from X to $\mathbb{C}^{m+n}$ endowed with the topology of uniform convergence.*

Remark 3.2.2. Note that any *reduced* complex space X is of finite type $m > n$ since being reduced implies that for any point $x \in X$ there is a local embedding of a neighbourhood U of x as an analytic subset of some $\mathbb{C}^N$.

According to Theorem 3.2.1 we can assume that our Stein space is embedded in some $\mathbb{C}^N$ for a (possibly very large) $N \in \mathbb{N}$. This allows us to introduce the notion of blow ups as mentioned before in the context of manifolds.

Definition 3.2.3. Let M be a complex manifold of complex dimension n and let Y be a closed submanifold with codimesion s in M.

(i) The **normal bundle** of Y in M is defined as the quotient bundle

$$NY = TM|_Y/TY$$

with fibers

$$(NY)_y = T_yM/T_yY.$$

(ii) The **projectivized bundle** associated to the normal bundle NY is the vector bundle

$$P(NY) \to Y, \qquad (P(NY))_y := P(NY_y).$$

This means that every fiber of the normal bundle is replaced by its projective space.

The **blow up** of M along Y is the complex manifold $\widehat{M}$ together with a holomorphic map $\pi\colon \widehat{M} \to M$ with the following properties:

(i) The set $E = \pi^{-1}(Y)$ is a smooth hypersurface in $\widehat{M}$ an there is an isomorphism of fiber bundles

$$\begin{tikzcd} P(NY) \arrow[rr, "\cong"] \arrow[dr, "\nu"'] & & E . \arrow[dl, "\pi'"] \\ & Y & \end{tikzcd}$$

(ii) The restriction $\pi\colon \widehat{M} \setminus E \to M \setminus Y$ is a biholomorphism.

Concretely, the manifold $\widehat{M}$ is given as

$$\widehat{M} = (M \setminus Y) \,\dot{\cup}\, E$$

with induced projection map

$$\pi = \mathrm{id}_{M\setminus Y} \cup \pi'.$$

Y is called the **center** of the blow up π and E is called the **exceptional variety** (or **exceptional divisor**). This means that we obtain $\widehat{X}$ by replacing any point $y \in Y$ by the projective space of the directions normal to Y.

In general there are several ways to resolve singularities besides blow ups (e.g. by normalization, see e.g. [Kol07] for further information), but the concept of blow ups has a big advantage as we see in the following Theorem.

Theorem 3.2.4 (Hironaka's resolution of singularities). *Let X be a complex space. There exists a locally finite sequence of blow ups*

$$\widehat{X} = X_r \xrightarrow{\tau_r} X_{r-1} \longrightarrow \ldots \longrightarrow X_1 \xrightarrow{\tau_1} X_0 = X$$

along smooth centers $C_j \subset X_j$, $\pi = \tau_r \circ \tau_{r-1} \circ \ldots \circ \tau_1$, such that

(i) $\widehat{X}$ *is smooth (i.e. a complex manifold),*

(ii) $\Sigma = \pi^{-1}(X_{sing})$ *is a divisor with normal crossings and*

(iii) $\pi \colon \widehat{X} \setminus \Sigma \to X \setminus X_{sing} = X_{reg}$ *is a biholomorphism.*

Moreover we have that for any local embedding $X|_U \hookrightarrow \mathbb{C}^N$ this sequence of blow-ups is induced by the embedded desingularization of $X|_U$.

The original proof of Hironaka he gave in [Hir64] was more than 200 pages long which indicates that the assertion of Theorem 3.2.4 is hardly non-trivial. However, Bierstone and Milman gave a shorter proof (which is still 96 pages long) in their paper [BM97] to which we refer for the proof.

In the sequel we will study properties of the exceptional divisor Σ which is why we recall some basic facts about divisors. We start with a formal definition.

Definition 3.2.5. A **divisor** on a complex manifold M is a locally finite linear combination

$$D = \sum_j c_j V_j,$$

with coefficients $c_j \in \mathbb{Z}$ and irreducible analytic hypersurfaces V_j of M. We denote the set of all divisors on M by $\mathrm{Div}(M)$. Moreover a divisor D is said to have **normal crossings** if $c_j = 1$ for all j and all V_j are distinct irreducible hypersurfaces which intersect transversally, i.e. for all $x \in \mathrm{supp}(D)$ there are local coordinates $(U, z_1, \ldots, z_n)$ around x such that

$$\mathrm{supp}(D) \cap U = \{z_1 z_2 \ldots z_r = 0\} \quad \text{for some } 1 \le r \le n.$$

We say that D has **simple normal crossings** if the hypersurfaces V_j are smooth.

The advantage of studying divisors is that we can associate a line bundle to each divisor as can be seen as follows. Let $\mathcal{O}_M^*$ be the sheaf of nonvanishing holomorphic functions on M and let $\mathcal{M}_M^*$ be the sheaf of nonvanishing meromorphic functions on M. We have a natural short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_M^* \longrightarrow \mathcal{M}_M^* \longrightarrow \mathcal{M}_M^*/\mathcal{O}_M^* \longrightarrow 0\,. \tag{3.1}$$

By applying the (left exact) global section functor to the sequence (3.1) we get an induced sequence in cohomology

$$0 \longrightarrow H^0(X, \mathcal{O}_M^*) \longrightarrow H^0(X, \mathcal{M}_M^*) \longrightarrow\quad .$$

$$\longrightarrow H^0(X, \mathcal{M}_M^*/\mathcal{O}_M^*) \xrightarrow{\delta} H^1(X, \mathcal{O}_M^*) \longrightarrow \ldots$$

The space $H^0(X, \mathcal{M}_M^*/\mathcal{O}_M^*)$ can actually be identified with the group of divisors $\mathrm{Div}(M)$ on M (cf. [Dem12, Chapter 5]) and by the discussion in [MM07, Chapter 2, §1] we have an isomorphism of groups from $\mathrm{Pic}(M) \to H^1(X, \mathcal{O}_M^*)$, where $\mathrm{Pic}(M)$ denotes the group of isomorphism classes of holomorphic line bundles on M. Note that the map δ is the usual connecting homomorphism that occurs in homological algebra. By now we only know that the association of a divisor to a holomorphic line bundle is natural, so let us turn to the explicit construction of the line bundle.

Let $D = \sum_j c_j V_j$ be a divisor on M and let $\{U_\alpha\}$ an open cover for M such that every V_j has a local defining function $g_{j\alpha} \in \mathcal{O}(U_\alpha)$. We set

$$g_\alpha = \prod_j g_{j\alpha}^{c_j} \in \mathcal{M}_M(U_\alpha), \quad h_{\alpha\beta} := \frac{g_\alpha}{g_\beta}. \tag{3.2}$$

Then $h_{\alpha\beta} \in \mathcal{O}_M(U_\alpha \cap U_\beta)$.

Definition 3.2.6. The holomorphic line bundle defined by the cocycle $(h_{\alpha\beta})_{\alpha\beta}$ in (3.2) is called the **line bundle associated to D** and is denoted by $\mathcal{O}_M(D)$. When it is clear from the context where bundle is defined, we write shortly $\mathcal{O}(D) = \mathcal{O}_M(D)$.

3.3 Poincaré metric

In this Section we introduce the generalized Poincaré metric which we will use in the sequel as the metric on the regular locus of our normal Stein space X.

We fix from now on a resolution of singularities $\pi : \widehat{X} \longrightarrow X$ of our normal space X as described in Section 3.2, and we denote by $\Sigma = \pi^{-1}(X_{\text{sing}})$ the exceptional divisor. By further applying Hironaka's theorem of resolution of singularities we can blow up $\widehat{X}$ and achieve that the divisor Σ has simple normal crossings (cf. [MM07, Theorem 2.1.13]).

Lemma 3.3.1. *There exists a metric h^{Σ} with positive curvature on the line bundle $\mathcal{O}(\Sigma)$ associated to the divisor in a neighbourhood of Σ.*

Proof. Consider the blow up of the singular points Σ . This yields a divisor in normal crossings. We choose a sufficiently small neighbourhood U_x around a singular point $x \in X_{\text{sing}}$ such that no other singular point is contained in U_x. Furthermore let $\iota\colon U_x \to \mathbb{C}^N$ be an embedding of U_x as an analytic subset of a ball $B_\varepsilon^N = \left\{ z \in \mathbb{C}^N \,\middle|\, \sum_{j=1}^N |z_j|^2 < \varepsilon^2 \right\} \subset \mathbb{C}^N$ in $\mathbb{C}^N$. By definition of the blow-up there exist holomorphic sections $s_1, \dots , s_k$ of $\mathcal{O}(\Sigma)$ that give a map

$$\phi\colon \pi^{-1}(U_x) \to \mathbb{CP}^k,$$
$$\phi(x) = [s_1(x), \dots , s_k(x)]$$

such that the map

$$\pi \times \phi\colon \pi^{-1}(U_x) \to B_\varepsilon^N \times \mathbb{CP}^k \tag{3.3}$$

is an embedding. Consider also the projection map

$$p\colon B_\varepsilon^N \times \mathbb{CP}^k \to \mathbb{CP}^k. \tag{3.4}$$

Composing the maps from (3.3) and (3.4), we get can express $\mathcal{O}(\Sigma)$ as

$$\mathcal{O}(\Sigma) = \phi^*\mathcal{O}(1) = (\pi \times \phi)^* p^*\mathcal{O}(1),$$

where $\mathcal{O}(1)$ is the hyperplane bundle on $\mathbb{CP}^k$ defined as the dual bundle of $\mathcal{O}(-1)$ in (1.3). Thus endowing $p^*\mathcal{O}(1)$ with a metric with positive curvature (e.g. the Fubini-Study metric), we get that the pullback by $\pi \times \phi$ remains positive. $\square$

In the sequel we will also consider line bundles on X as introduced in Section 1.5. Let L be a positive line bundle on X and denote by π^*L its pullback bundle to the manifold $\widehat{X}$. In general the bundle π^*L is only semi-positive, but there is a way to fix this problem (see also [MM07, Lemma 6.2.2]).

Lemma 3.3.2. *Let L be a positive holomorphic line bundle on X. Then there exists a positive holomorphic line bundle $(\widehat{L}, h^{\widehat{L}})$ on $\widehat{X}$ such that $\widehat{L}|_{\widehat{X}\setminus\Sigma} \cong \pi^*L^k$ for $k \in \mathbb{N}$.*

Proof. By induction. Let $\Sigma_0 = \tau_1^{-1}(X_{\text{sing}})$. By the definition of the blow-up and Lemma 3.3.1 there is a positive smooth Hermitian metric h_0 on the line bundle $\mathcal{O}_{X_1}(-\Sigma_0)$ whose curvature is positive along Σ_0, bounded on X_1 and equal to 0 outside a neighbourhood U_0 of Σ_0.
Consider the line bundle $L_1 := \tau_1^*(L^{k_0}) \otimes \mathcal{O}_{X_1}(-\Sigma_0), k_0 \in \mathbb{N}$ with metric $(h^L)^{\otimes k_0} \otimes h_0$ on X_1. The curvature of the line bundle L_1 is given by

$$\Theta(L_1) = k_0\tau_1^*\Theta(L) + \Theta(\mathcal{O}_{X_1}(-\Sigma_0))$$

and by taking k_0 sufficiently large we obtain that $\Theta(L_1)$ is positive. Note that taking the k_0-fold tensor power is necessary since $\tau_1^*\Theta(L)$ is positive outside a small neighbourhood V of X_{sing}, $V \subset U_0$ and $\Theta(\mathcal{O}_{X_1}(-\Sigma_0))$ might be negative on $U_0 \setminus V$.
Continuing inductively, the result is a positive line bundle $\widehat{L}$ on $\widehat{X}$ that satisfies $\widehat{L} = \pi^*L^k$ for $k \in \mathbb{N}$. $\square$

Corollary 3.3.3. *$\widehat{X}$ carries a Kähler metric $\widehat{\omega}$.*

Proof. By lemma 3.3.2, there exists a positive line bundle $\widehat{L}$ on $\widehat{X}$. Take $\widehat{\omega} = R^{\widehat{L}}$, then $\widehat{\omega}$ is clearly a Kähler form. $\square$

In the sequel we consider the case that L is globally trivial, i.e. $L = X \times \mathbb{C}$ equipped with the metric $h = e^{-\varphi}$ where φ is the smooth strictly plurisubharmonic function from Lemma 3.1.1. Then

$$\Theta(L) = \sqrt{-1}\partial\bar{\partial}\varphi > 0.$$

We recall the construction and properties of the generalized Poincaré metric on $\widehat{X} \setminus \Sigma \cong X_{\mathrm{reg}}$ (cf. [MM07, Lemma 6.2.1]).

Let $\Sigma = \cup_j \Sigma_j$ be the decomposition of Σ into irreducible components Σ_j, σ_j the defining holomorphic section of the line bundle $\mathcal{O}(\Sigma_j)$ and let $\|\cdot\|_j$ be the norm for a smooth Hermitian metric on the associated line bundle such that $\|\sigma_j\| < 1$. Let ω_0 be the hermitian $(1,1)$-form associated to any smooth hermitian metric on $T\widehat{X}$ that is invariant under the complex structure of $\widehat{X}$.

Definition 3.3.4. The **generalized Poincaré metric** ω_P on X_{reg} is defined by the Hermitian form

$$\omega_P = \omega_0 + \varepsilon\sqrt{-1}\sum_j \bar{\partial}\partial \log\left(\left(-\log\left(\|\sigma_j\|_j^2\right)\right)^2\right), \quad \varepsilon \in [0,1] \text{ fixed.} \tag{3.5}$$

The Poincareé metric plays a fundamental role since the L^2 cohomology spaces we defined in Section 1.8 depend on the choice of a metric in a natural way and the following Lemma from [MM07, Lemma 6.2.1] clarifies the importance of the Poincaré metric.

Lemma 3.3.5. *The metric ω_P is a complete Hermitian metric of finite volume with bounded Hermitian torsion $\mathcal{T}_P$.*

Proof. Let us start with a local description of the metric. Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. Consider $(\mathbb{D}^*)^l \times \mathbb{D}^{n-l} =: D$. We define on D a metric Ω_p by

$$\Omega_p = \frac{\sqrt{-1}}{2}\sum_{k=1}^{l} \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2\left(\log|z_k|^2\right)^2} + \frac{\sqrt{-1}}{2}\sum_{k=l+1}^{n} dz_k \wedge d\bar{z}_k.$$

For any $z_0 \in \Sigma$ there exists an open neighbourhood $U = U_{z_0}, U \cong \mathbb{D}^n$ in which we have

$$(\mathbb{D}^*)^l \times \mathbb{D}^{n-l} \cong \left(\widehat{X} \setminus \Sigma\right) \cap U = \{(z_1, \dots, z_n) \mid z_1 \neq 0, \dots, z_l \neq 0\}, \tag{3.6}$$

since Σ is a divisor with normal crossings. In order to get a representation for Ω_p, we analyze the summands appearing in the definition separately. We have

$$\begin{aligned} &\sqrt{-1}\bar{\partial}\partial \log\left((-\log \|\sigma_j\|_j^2)^2\right) = \\ =&2\sqrt{-1}\frac{\Theta(\mathcal{O}_{\widehat{X}}(\Sigma_j))}{\log \|\sigma_j\|_j^2} + 2\sqrt{-1}\frac{\partial \log \|\sigma_j\|_j^2 \wedge \bar{\partial} \log \|\sigma_j\|_j^2}{\log \|\sigma_j\|_j^2 \log \|\sigma_j\|_j^2}. \end{aligned}$$

The second summand is non-negative since $\bar{\partial} f \wedge \partial f \geq 0$ for any real function f. The first summand tends to zero when we are getting close to the divisor and can be handled as follows: σ_j is the defining section of Σ_j, i.e. Σ_j is given by the equation $\sigma_j = 0$. Thus $\log \|\sigma_j\|_j$ tends to minus infinity and thus the whole fraction to zero. This implies that in a neighbourhood of the divisor we have

$$\Omega_p = \omega_0 + 2\varepsilon\sqrt{-1}\sum_j \frac{\Theta(\mathcal{O}_{\widehat{X}}(\Sigma_j))}{\log \|\sigma_j\|_j^2}$$

and this is positive for small ε. Let us choose now an open neighbourhood U of the point z_0 such that

(i) Σ_j is defined by the equation $z_j = 0$ for $j \in \{1, \dots, k\}$ and

(ii) $\Sigma_j \cap U = \emptyset$ for $j > k$.

Then we can locally write $\|\sigma_j\|_j^2 = \varphi_j |z_j|^2$ with φ_j being a positive smooth function on U. Hence locally

$$\frac{\partial \log \|\sigma_j\|_j^2 \wedge \bar{\partial} \log \|\sigma_j\|_j^2}{\log \|\sigma_j\|_j^2 \log \|\sigma_j\|_j^2} = \frac{\partial \log(\varphi_j |z_j|^2) \wedge \bar{\partial} \log(\varphi_j |z_j|^2)}{|z_j|^2 \left(\log(\varphi_j |z_j|^2)\right)^2}$$
$$= \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2 \left(\log(\varphi_j |z_j|^2)\right)^2} + \frac{\nu_j}{|z_j|^2 \left(\log(\varphi_j |z_j|^2)\right)^2},$$

where $\nu_j \in \Omega^{1,1}(U)$ vanishes on set $\{z_j = 0\}$.
We will now show that the Poincare metric is complete. Note that according to the previous computations it is sufficient to prove completeness on the set $\{z \in \mathbb{C} \mid 0 < |z| < c\}$, where $c > 0$. Let $\gamma\colon [0, c] \to \mathbb{C}$ be a curve defined by $\gamma(t) = tz$. Then $\gamma'(t) = z$ and thus

$$\int_\gamma \frac{dz \wedge d\bar{z}}{|z|^2 \left(\log |z|^2\right)^2} = \int_0^c \frac{1}{t\,|\log t|} dt = \infty.$$

Hence ω_p is complete.
To compute the volume of ω_p, we switch to polar coordinates.

$$\operatorname{Vol} \omega_p = \int_0^{2\pi} \int_0^c \frac{r}{r^2 (\log r)^2} dr d\theta = -2\pi \left[\frac{1}{\log r}\right]_0^c$$

and this is a finite number.
It remains to show that the Hermitian torsion $\mathcal{T}_P$ is bounded. By definition we have $\mathcal{T}_P = [\iota(\omega_p), \partial\omega_P]$ and from (3.5) we infer that $\partial\omega_P = \partial\omega_0$, hence $\partial\omega_P$ extends smoothly over $\widehat{X}$ and thus we are done. $\square$

Let us sum up what we achieved so far. By Lemma 3.3.2 we know that the blow up $\widehat{X}$ is a Kähler manifold with Kähler form $\omega = \Theta(\widehat{L})$, so we can take $\omega_0 = \omega$ as our refenrence metric in (3.5). Hence ω_P is a complete metric in a neighbourhood of the singular locus X_{sing}.

3.4 Coverings of Stein Spaces

In this Section we are going to lay down the foundations that are needed to prove the main Theorem in Section 3.5.

Let us start with a visualization of the spaces we are considering. Our starting point was a normal Stein space S which we blow up in order to get a complex manfiold $\widehat{S}$ such that $\widehat{S} \setminus \Sigma \cong S_{\mathrm{reg}}$ and we have complete metrics on both spaces. Now we are considering a Galois covering $\widetilde{S}$ of the space S and a relatively compact pseudoconvex domain $X \subset S$ with induced metrics as can be read off from the following diagram:

$$\begin{tikzcd}
\widehat{L} \arrow[d] & L \arrow[d] & \widetilde{L} = q^* L \arrow[d] \\
\widehat{X} \arrow[r, "\pi"] & X & \widetilde{X} \arrow[l, "q"']
\end{tikzcd}$$

Let us consider the trivial line bundle $L' \cong \pi^* L^k$ over $X_{\mathrm{reg}} \cong \widehat{X} \setminus \Sigma$. We introduce on L a new metric $(h^{\widehat{L}})^{1/k}$ in order to get a positive metric on the whole of X.
Set

$$h^L_\varepsilon := (h^{\widehat{L}})^{1/k} \prod_i (-\log(\|\sigma_i\|_i^2))^\varepsilon\,, \quad 0 < \varepsilon \ll 1, \tag{3.7}$$

with $h^{\widehat{L}}$ and k as in Lemma 3.3.2. The curvature of this new metric is then

$$\Theta_\varepsilon(L') = \pi^*\Theta(L) + \Theta(\Sigma) + \varepsilon\sqrt{-1}\sum_j \bar{\partial}\partial \log\left(\left(-\log\|\sigma_j\|_j^2\right)^2\right), \tag{3.8}$$

where $\Theta(\Sigma)$ is the curvature of the metric from Lemma 3.3.1. Note that $\Theta_\varepsilon(L')$ is positive on the whole of X.
We introduce on $\widetilde{X}_{\mathrm{reg}}$ the pullback metrics

$$\widetilde{\omega}_P = q^*\omega_P\,, \quad \widetilde{h}^L_\varepsilon = q^* h^L_\varepsilon, \tag{3.9}$$

and we denote the L^2-space of square integrable (p,q)-forms with respect to $\widetilde{\omega}_P$ and $\widetilde{h}^L_\varepsilon$ on $\widetilde{X}_{\mathrm{reg}}$ by $L^{p,q}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P, \widetilde{h}^L_\varepsilon)$. We introduce now the L^2 cohomology groups $H^{p,q}_{(2)}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P, \widetilde{h}^L_\varepsilon)$ with respect to the metrics $\widetilde{\omega}_P$ from (3.9) and $\widetilde{h}^L_\varepsilon$ as in (3.7) as in Section 1.8.

Let $\Omega_0^{p,q}(\widetilde{X}_{\mathrm{reg}})$ denote space of compactly supported smooth (p,q)-forms on $\widetilde{X}_{\mathrm{reg}}$ and let $L^{p,q}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P, \widetilde{h}^{L_\varepsilon}) := L^{p,q}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$ denote the set of square-integrable (p,q)-forms with respect to the metrics constructed above, i.e. $L^{p,q}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$ is the closure of $\Omega_0^{p,q}(\widetilde{X}_{\mathrm{reg}})$ with respect to the norm induced by the inner product

$$(u,v) = \int_{\widetilde{X}_{\mathrm{reg}}} \tilde{h}^L_\varepsilon(u,v) dV_{\widetilde{\omega}_P}. \tag{3.10}$$

By convention we write $L^{0,0}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P) = L^2(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$. We denote the maximal extension of the $\bar{\partial}$-operator to $L^{p,q}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$ for simplicity again by $\bar{\partial}$. It is a closed, linear and densely defined operator with domain

$$\mathrm{Dom}\,\bar{\partial} = \left\{ u \in L^{p,q}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P) \,\middle|\, \bar{\partial}u \in L^{p,q+1}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P) \right\},$$

where the form $\bar{\partial}u$ is calculated in the sense of distributions (cf. Section 1.3). This induces a complex $\bar{\partial}\colon L^{\bullet,\bullet}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P) \to L^{\bullet,\bullet+1}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$ and its cohomology groups are defined as

$$H^{p,q}_{(2)}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P) := \frac{\mathrm{Ker}\,\bar{\partial} \cap L^{p,q}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)}{\mathrm{Im}\,\bar{\partial} \cap L^{p,q-1}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)}. \tag{3.11}$$

In particular we consider the space

$$H^{0,0}_{(2)}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P) = \left\{ u \in L^2(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P) \mid \bar{\partial}u = 0 \right\}. \tag{3.12}$$

Let $\bar{\partial}^*$ and ϑ be the Hilbert space and formal adjoint operators of the maximal extension of $\bar{\partial}$. Note that due to the presence of the boundary $b\widetilde{X}_{\mathrm{reg}}$ both operators do not coincide. Let $\sigma(\vartheta, d\xi)$ denote the principal symbol of the operator ϑ evaluated at the cotangent vector $d\xi$ which was introduced in Definition 1.3.8. Set

$$B^{p,q}(\widetilde{X}_{\mathrm{reg}}) = \left\{ u \in \Omega_0^{p,q}(\widetilde{X}_{\mathrm{reg}} \cup b\widetilde{X}_{\mathrm{reg}}) \mid \sigma(\vartheta, d\varphi)u = 0 \right\}. \tag{3.13}$$

Integration by parts as in [FK72, Propositions 1.3.1 and 1.3.2] yields

$$B^{p,q}(\widetilde{X}_{\text{reg}}) = \Omega_0^{p,q}(\widetilde{X}_{\text{reg}} \cup b\widetilde{X}_{\text{reg}}) \cap \operatorname{Dom} \bar{\partial}^* \quad \text{and} \quad \vartheta = \bar{\partial}^* \text{ on } B^{p,q}(\widetilde{X}_{\text{reg}}). \tag{3.14}$$

Consider from now on the case $p = 0$. We recall that since the connected components of $\widetilde{X}_{\text{reg}}$ are complex submanifolds of $\mathbb{C}^N$, we can use the fundamental estimate 1.12 and the L^2 Hodge Theorem 1.8.7 on $\widetilde{X}_{\text{reg}}$.

We will work with the Laplacian associated to $\bar{\partial}$.

Definition 3.4.1. Let $\bar{\partial} \colon \Omega_0^{0,q}(\widetilde{X}_{\text{reg}}) \to \Omega_0^{0,q}(\widetilde{X}_{\text{reg}})$ and let $\bar{\partial}^*$ be the Hilbert space adjoint of $\bar{\partial}$. Then we define the **Laplacian** $\square$ as

$$\begin{aligned} &\square \colon \Omega_0^{0,q}(\widetilde{X}_{\text{reg}}) \to \Omega_0^{0,q}(\widetilde{X}_{\text{reg}}), \\ &\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}. \end{aligned}$$

Note that $\square$ is a self-adjoint, elliptic differential operator. Since $\widetilde{X}_{\text{reg}}$ is not compact we define the **Gaffney extension** , which is a positive self-adjoint extension of $\square$ to $L^{\bullet,\bullet}(\widetilde{X}_{\text{reg}}, \widetilde{\omega}_P)$ (cf. [MM07, Proposition 3.1.2]) as

$$\begin{aligned} \operatorname{Dom} \square &= \left\{ u \in \operatorname{Dom} \bar{\partial} \cap \operatorname{Dom} \bar{\partial}^* \;\middle|\; \bar{\partial} u \in \operatorname{Dom} \bar{\partial}^*, \bar{\partial}^* u \in \operatorname{Dom} \bar{\partial} \right\}, \\ \square &= \bar{\partial}\bar{\partial}^* u + \bar{\partial}^*\bar{\partial} u \quad \text{for } u \in \operatorname{Dom} \square. \end{aligned}$$

The quadratic form associated to the Gaffney extension is the form Q defined by

$$\begin{aligned} \operatorname{Dom}(Q) &= \operatorname{Dom} \bar{\partial}|_{L^{0,q}(\widetilde{X}_{\text{reg}})} \cap \operatorname{Dom} \bar{\partial}^*|_{L^{0,q}(\widetilde{X}_{\text{reg}})}, \\ Q(u,v) &= \langle \bar{\partial} u, \bar{\partial} v \rangle + \langle \bar{\partial}^* u, \bar{\partial}^* v \rangle \quad \text{for } u, v \in \operatorname{Dom}(Q). \end{aligned}$$

Lemma 3.4.2. *The space $B^{0,q}(\widetilde{X}_{reg})$ is dense in $\operatorname{Dom} \bar{\partial} \cap \operatorname{Dom} \bar{\partial}^*$ with respect to the norm*

$$u \mapsto \left(\|u\|^2 + \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\| \right)^{1/2}.$$

Proof. The idea is to show that every form in $B^{0,q}(\widetilde{X}_{\mathrm{reg}})$ is a limit of a sequence of compactly supported forms because in this case the assertion follows from the Friedrichs identity lemma, see e.g. [MM07], Lemma 3.1.3.
Since the metric $\widetilde{\omega}_P$ is complete in the neighbourhood of the singular points, we can construct a sequence of smooth cut-off functions $\{\eta_j\}_j \subset \mathcal{C}_0^\infty(\widetilde{X}_{\mathrm{reg}} \cup b\widetilde{X}_{\mathrm{reg}})$ such that $0 \le \eta_j \le 1$, $\eta_{j+1} = 1$ on $\operatorname{supp}\eta_j$ and $|d\eta_j| \le 1$ and $\operatorname{supp}\eta_j$ exhaust $\widetilde{X}_{\mathrm{reg}} \cup b\widetilde{X}_{\mathrm{reg}}$ as $j \to \infty$. Clearly $u \in \operatorname{Dom}\bar{\partial} \cap \operatorname{Dom}\bar{\partial}^*$ implies $\eta_j u \in \operatorname{Dom}\bar{\partial} \cap \operatorname{Dom}\bar{\partial}^*$ and following Andreotti-Vesentini, $\{\eta_j u\} \to u$, $\bar{\partial}(\eta_j u) \to \bar{\partial}u$ and $\bar{\partial}^*(\eta_j u) \to \bar{\partial}^* u$ as $j \to \infty$. □

We will use the Bochner-Kodaira-Nakano formula with boundary term (see e.g. [MM07], §1.4).
Let $T\widetilde{X}_{\mathrm{reg}} = T^{1,0}\widetilde{X}_{\mathrm{reg}} \oplus T^{0,1}\widetilde{X}_{\mathrm{reg}}$ denote the splitting of the tangent bundle of $\widetilde{X}_{\mathrm{reg}}$ with respect to the complex structure of S and denote by $K_{\widetilde{X}_{\mathrm{reg}}}$ the canonical bundle on $\widetilde{X}_{\mathrm{reg}}$. We set $\check{L} = L' \otimes K^*_{\widetilde{X}_{\mathrm{reg}}}$. By (1.4) there is a natural isometry

$$\begin{aligned} \vee \colon \Lambda^{0,q}(T^*\widetilde{X}_{\mathrm{reg}}) \otimes L &\to \Lambda^{n,q}(T^*\widetilde{X}_{\mathrm{reg}}) \otimes \check{L}, \\ u \mapsto \check{u} &= (\xi^1 \wedge \ldots \wedge \xi^n \wedge u) \otimes (\xi_1 \wedge \ldots \wedge \xi_n), \end{aligned} \tag{3.15}$$

with $\{\xi_j\}_{j=1^n}$ being an orthonormal frame of $T^{1,0}\widetilde{X}_{\mathrm{reg}}$ and $\{\xi^j\}_{j=1}^n$ the corresponding dual frame.
Let $\operatorname{End}(\Omega^{\bullet,\bullet}(\widetilde{X}_{\mathrm{reg}}))$ denote the algebra of endomorphisms of smooth differential forms on $\widetilde{X}_{\mathrm{reg}}$ and let $[\cdot,\cdot]$ be the graded commutator on it. The Hermitian torsion $\mathcal{T}$ of the Poincare metric mentioned before is defined as $\mathcal{T} = [i(\widetilde{\omega}_P), \partial\widetilde{\omega}_P]$.
We assume further on that $|d\varphi| = 1$ on $b\widetilde{X}_{\mathrm{reg}}$ (this is possible since if necessary, we can replace φ by $\varphi/|d\varphi|$ near $b\widetilde{X}_{\mathrm{reg}}$ and thus have a normalized defining function).

Theorem 3.4.3. *For any $u \in B^{0,1}(\widetilde{X}_{reg})$ we have*

$$\frac{3}{2}\left(\left\|\bar{\partial}u\right\|^2+\left\|\bar{\partial}^*u\right\|^2\right)\geq\left(\Theta(\check{L})(\xi_j,\bar{\xi}_k)\bar{\xi}^k\wedge i_{\bar{\xi}_j}u,u\right)$$
$$+\int_{b\widetilde{X}_{reg}}L_\varphi(u,u)dV_{b\widetilde{X}_{reg}}$$
$$-\frac{1}{2}\left(\left\|\mathcal{T}^*\check{u}^2\right\|+\left\|\overline{\mathcal{T}}\check{u}\right\|^2+\left\|\overline{\mathcal{T}}^*\check{u}\right\|^2\right). \tag{3.16}$$

By assumption the boundary $b\widetilde{X}_{\text{reg}}$ is strictly pseudoconvex, so the boundary integral in (3.16) is non-negative.
We set $T=\frac{1}{2}\left(\mathcal{T}\mathcal{T}^*+\overline{\mathcal{T}}^*\overline{\mathcal{T}}+\overline{\mathcal{T}}\overline{\mathcal{T}}^*\right)$ and define a continuous function

$$\tau\colon\widetilde{X}_{\text{reg}}\to\mathbb{R},\quad\tau(x)=\sup\left\{\frac{\langle Tv,v\rangle}{\langle v,v\rangle}\;\middle|\;v\in\Lambda^{n,1}T_x^*\widetilde{X}_{\text{reg}}\setminus\{0\}\right\}. \tag{3.17}$$

Then (3.16) simplifies to

$$\frac{3}{2}\left(\left\|\bar{\partial}u\right\|^2+\left\|\bar{\partial}^*u\right\|^2\right)\geq\left(\Theta(\check{L})(\xi_j,\bar{\xi}_k)\bar{\xi}^k\wedge i_{\bar{\xi}_j}u,u\right)-\int_{\widetilde{X}_{\text{reg}}}\tau(x)\left|u\right|^2. \tag{3.18}$$

Lemma 3.4.4. *There are constants $A,B,C>0$ such that*

(i) $\sqrt{-1}\Theta(\check{L})>A\widetilde{\omega}_P$,

(ii) $\sqrt{-1}\Theta(K^*_{\widetilde{X}_{reg}})>-B\widetilde{\omega}_P$ *and*

(iii) $|\mathcal{T}_{\widetilde{\omega}_P}|<C$.

Proof. The first assertion follows for a small ε from (3.5) and (3.7) and the assumption that X is relatively compact. The second and the third assertion are a direct consequence of Lemma 3.3.5 since $\widetilde{\omega}_P=q^*\omega_P$ and hence $\widetilde{\omega}_P$ is complete in a neighbourhood of the singular locus. □

Combining Theorem 3.4.3 and Lemma 3.4.4 we obtain the following Theorem.

Theorem 3.4.5. *For any $u \in B^{0,1}(\widetilde{X}_{reg})$ we have*

$$\frac{3}{2}\left(\left\|\bar{\partial}u\right\|^2 + \left\|\bar{\partial}^* u\right\|^2\right) \geq (A - B)\left\|u\right\|^2. \tag{3.19}$$

Moreover Lemma 3.4.2 states that $B^{0,1}(\widetilde{X}_{\mathrm{reg}})$ is dense in $\operatorname{Dom}\bar{\partial} \cap \operatorname{Dom}\bar{\partial}^*$, so (3.19) holds for any $u \in \operatorname{Dom}\bar{\partial} \cap \operatorname{Dom}\bar{\partial}^* \cap L^{0,1}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$. Note that the right hand side of (3.19) is positive without loss of generality since by taking higher tensor powers of the curvature of $\widehat{L} \cong \pi^* L^k$ we ensure that $A > B$.
In particular, Theorem 1.8.7 tells us that

$$\operatorname{Ker}\Box|_{L^{0,1}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)} \cong H^{0,1}_{(2)}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P).$$

3.5 Proof of the Main Theorem

In this Section we will prove the main Theorem of this thesis which is as follows.

Theorem 3.5.1. *Let $X \subset S$ be a relatively compact strictly pseudoconvex domain in a normal Stein space S and let $q \colon \widetilde{S} \to S$ be a Galois covering of S by a discrete group Γ. Set $\widetilde{X} = q^{-1}(X)$. Then*

$$\dim_\Gamma H^{0,0}_{(2)}(\widetilde{X}) = \infty.$$

The proof of Theorem 3.5.1 is organized in different steps which are inspired by the paper of Gromov, Henkin and Shubin [GHS98].
Let us start with an important remark. Since S is a Stein space by assumption we know by Theorem 1.6.5 that $\widetilde{S}$ is Stein, too. Hence Riemann's Extension Theorem 1.5.9 holds also on $\widetilde{S}$ and thus we get

$$H^{0,0}_{(2)}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P) \subset H^{0,0}_{(2)}(\widetilde{X}, \widetilde{\omega}_P).$$

For technical reasons we will make use of Sobolev spaces with real exponents which are defined in the following way (see [GHS98], [Ati76] or [Shu92]).

Definition 3.5.2. Let M be a differentiable manifold with a free action of a discrete group Γ such that $\overline{M}/\Gamma$ is compact, E be a vector bundle on $\overline{M}$ with Γ-invariant metric and let $s \in R$. Let $\tilde{M}$ be a Γ-invariant complex neighbourhood of $\overline{M}$.The space $W^s(M, E)$ is defined to be the Hilbert space which consists of all restrictions to M finite linear combinations of all sections Tu where $u \in L^2(\tilde{M}, E)$ and T is a properly supported Γ-invariant pseudodifferential operator of order $-s$ on $\tilde{M}$.

Next we state a condition for the Γ-dimension which is sufficient to show finiteness of our L^2-cohomology modules.

Lemma 3.5.3. *Let $L \subset L^2(\widetilde{X}_{reg}, \widetilde{\omega}_P)$ be a closed Γ-invariant subspace such that $L \subset W^\varepsilon(\widetilde{X}_{reg}, \widetilde{\omega}_P)$ for some $\varepsilon > 0$ and suppose that there is a constant $C > 0$ such that for $u \in L$ we have*

$$\|u\|_\varepsilon \leq C \|u\|_{L^2} . \tag{3.20}$$

Then $\dim_\Gamma L < \infty$.

Proof. See [GHS98, Proposition 1.5]. □

We can now use Lemma 3.5.3 to state a condition for an operator to be Γ-Fredholm in the sense of Definition 1.7.6.

Lemma 3.5.4. *Let T be a self-adjoint operator on $L^2(\widetilde{X}_{reg}, \widetilde{\omega}_P)$ such that T commutes with the action of Γ and assume that $\mathrm{Dom}(T) \subset W^\varepsilon(\widetilde{X}_{reg}, \widetilde{\omega}_P)$ for some $\varepsilon > 0$ and moreover that there exists a constant $C > 0$ such that the inequality*

$$\|u\|_\varepsilon^2 \leq C \left(\|Tu\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \tag{3.21}$$

is satisfied for any $u \in \mathrm{Dom}(T)$. Then T is Γ-Fredholm.

Proof. We have to check two conditions: First that $\dim_\Gamma \mathrm{Ker}\, T < \infty$ and second that there exists a closed Γ-invariant subspace $Q \subset L^2(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$ such that $Q \subset \mathrm{Im}(T)$ and $\mathrm{codim}_\Gamma Q < \infty$.
To prove the first assertion we notice that (3.21) implies that (3.20)

is satisfied for any $u \in \operatorname{Ker}(T)$ and hence Lemma 3.5.3 implies that $\dim_\Gamma \operatorname{Ker}(T) < \infty$.
For the second assertion we put $Q = \operatorname{Im} E_\delta$, where E_δ is the spectral projection of T onto the interval $(-\delta, \delta)$. Then we get by Lemma 3.5.3 that $\dim_\Gamma E_\delta < \infty$. But we also know that

$$\operatorname{Im}(\mathrm{id} - E_\delta) = (\operatorname{Im} E_\delta)^\perp \subset \operatorname{Im} T,$$

and hence T is Γ-Fredholm. □

Lemma 3.5.5. *Let U be an arbitrary set and let $f\colon U \to \mathbb{C}$ be an unbounded function. Then for any $N \in \mathbb{N}$ the functions $f, f^2, \dots, f^N$ are linearly independent modulo bounded functions on U which means that if*

$$c_1 f + c_2 f^2 + \ldots + c_N f^N$$

is a bounded function on U, then $c_1 = c_2 = \ldots = c_N = 0$.

The proof is straightforward and left to the reader.

Theorem 3.5.6. *Let $\widetilde{X}$ be as in Theorem 3.5.1 and assume that $b\widetilde{X} \neq \emptyset$. Then every boundary point $x \in b\widetilde{X}$ is a local peak point for $H^{0,0}_{(2)}(\widetilde{X})$.*

Proof. By Lemma 3.1.1 we know that there exists a strictly plurisubharmonic function φ on $\widetilde{X}$ such that its Levi form L_φ is positive definite on $\widetilde{X}$. We know that we can rescale φ if necessary such that the boundary $b\widetilde{X} = \left\{ z \in \widetilde{X} \mid \varphi(z) = 0 \right\}$, hence we choose (the possibly rescaled) φ as defining function for $\widetilde{X}$. Since φ is smooth we can consider its Taylor expansion at a boundary point $x \in b\widetilde{X}$

$$\varphi(z) = \varphi(x) + 2 \operatorname{Re} f(z, x) + L_\varphi(z - x, \bar{z} - \bar{x}) + O(|z - x|^3), \quad (3.22)$$

where $f(z, x)$ is a complex quadratic polynomial and L_φ is the Levi form of φ at x. Morover (3.22) implies that $\operatorname{Re} f(z, x)$ must be negative if z is sufficiently close to x because the Levi form is by assumption positive definite, $\varphi(x) = 0$ and z is an interior point of $\widetilde{X}$. Hence we can choose a branch of $\log f(z,x)$ in a possibly very small neighbourhood U_x of x such that the function $h_x(z) = \log f(z,x)$ is holomorphic on $U_x \cap \widetilde{X}$. Consider the functions $h_x, h_x^2, \dots$. Then these functions clearly satisfy $h_x^N \in L^2\left(U_x \cap \left(\widetilde{X} \cup b\widetilde{X}\right)\right)$ and additionally have a peak point at x. Using Lemma 3.5.5 we obtain that the functions h_x^N are linearly independent modulo bounded functions on U_x. □

Remark 3.5.7. Note that it is essential in the proof of Theorem 3.5.6 that we assume that there is no singularity of $\widetilde{X}$ at the boundary.

In order to proof Theorem 3.5.1 we are now going to construct a subspace of $\Omega^{0,1}(\widetilde{X} \cup b\widetilde{X})$ on which we will solve the $\bar{\partial}$-equation in order to be able to apply the results on $H^{0,0}_{(2)}(\widetilde{X})$.

Let us consider the set of functions $\{h_x^N\}$ from the proof of Theorem 3.5.6. Since these functions are only defined on a small neighbourhood U_x of a boundary point $x \in b\widetilde{X}$, we use the group action of Γ to cover the whole boundary.
Let $\chi \in \mathcal{C}^\infty(U_x)$ be a smooth cutoff function such that $\chi = 1$ on a neighbourhood $V \subset U_x$ of x and we will also identify χ with its extension by 0 to $\widetilde{X} \cup b\widetilde{X}$. Let $\gamma \in \Gamma$ and denote the translation on $\widetilde{X}$ induced by the group element again for simplicity by γ. Keeping this in mind we can define the translation of χ by γ as

$$\gamma^*\chi(z) = \chi\left(\gamma^{-1}z\right), \tag{3.23}$$

where the function $\gamma^*\chi$ is supported in a neighbourhood of γx. This means basically that we move U_x by $\gamma \in \Gamma$ around the boundary. Let us consider now the functions χh_x^j for $j \in \{1, \dots, N\}$. Then we obtain a closed Γ-invariant subspace L in $L^2(\widetilde{X})$ that is spanned by these functions

$$L = \left\{ u \;\middle|\; u = \sum_{\gamma\in\Gamma}\sum_{j=1}^{N} a_{j,\gamma}\gamma^*(\chi h_x^j),\ \sum_{j,\gamma} \left|a_{j,\gamma}\right|^2 < \infty \right\}, \tag{3.24}$$

where the $a_{j,\gamma}$ are constants. The set L consists of functions u that are a linear combination of the functions χh_x^j that are moved around by $\gamma \in \Gamma$. Hence $\dim_\Gamma L = N$.
Consider next the forms

$$\bar{\partial}(\chi h_x^j), \quad j = 1, \ldots, N. \tag{3.25}$$

Since the $\bar{\partial}$-operator is linear and the functions h_x^j are holomorphic in a neighbourhood of the boundary, the functions $\bar{\partial}(\chi h_x^j)$ are linearly independent, too, as a consequence of Lemma 3.5.5. Note that due to the construction all these forms have compact support. Now we can play the same game again and consider the closed Γ-invariant subspace L' of $L^{0,1}(\widetilde{X})$ that is spanned by the forms (3.25), i.e.

$$L' = \left\{ u \;\middle|\; u = \sum_{\gamma\in\Gamma}\sum_{j=1}^{N} a_{j,\gamma}\bar{\partial}\left(\gamma^*(\chi h_x^j)\right), \; \sum_{j,\gamma} |a_{j,\gamma}|^2 < \infty \right\}. \tag{3.26}$$

By the same arguments presented before we obtain $\dim_\Gamma L' = N$.
Let us describe the elements of L' a bit more in detail. First, any $u \in L'$ has compact support and is smooth on $\widetilde{X} \cup b\widetilde{X}$. Obviously we have $L' \subset \operatorname{Im}\bar{\partial}$ and thus also $L' \subset \overline{\operatorname{Im}\Box}$ by the Hodge decomposition Theorem 1.8.7.

Lemma 3.5.8. *Suppose* (3.19) *holds. The domain* $\operatorname{Dom}(\Box)|_{L^{0,q}(\widetilde{X}_{reg},\widetilde{\omega}_P)}$ *is included into* $W^1(\widetilde{X}_{reg},\widetilde{\omega}_P)$ *for* $q > 0$ *and there is a constant* C *such that for any* $u \in \operatorname{Dom}(\Box)|_{L^{0,q}(\widetilde{X}_{reg},\widetilde{\omega}_P)}$ *we have*

$$\|u\|_1^2 \le C\left(\|\Box u\|_{L^2}^2 + \|u\|_{L^2}^2\right). \tag{3.27}$$

Proof. Let $\{U_j\}$ be a Γ-invariant covering of $\widetilde{X}$ and suppose that there exists another Γ-invariant open covering $\{V_j\}$ such that $\overline{U_j} \subset V_j$. Let us further choose partitions of unity $\{\psi_j\}$ and $\{\Psi_j\}$ subordinate to the chosen coverings such that $\Psi_j \equiv 1$ on $\operatorname{supp}\psi_j$. Note that by Lemma 3.4.2 the space $B^{0,q}(\widetilde{X}_{\mathrm{reg}})$ is dense in $\operatorname{Dom}\bar{\partial} \cap \operatorname{Dom}\bar{\partial}^*$ and hence we can start

with $u \in B^{0,q}(\widetilde{X}_{\mathrm{reg}})$.
Assume first that u is supported in a neighbourhood of the boundary. Then by (3.19) we can use Theorem 1.2 from [GHS98] to get for any $u \in \Omega^{p,q}(\widetilde{X}_{\mathrm{reg}} \cup b\widetilde{X}_{\mathrm{reg}}) \cap \mathrm{Dom}(\Box)$ the estimate

$$\|\psi_j u\|_1^2 \leq C_2 \left(\|\Psi_j \Box u\|_{L^2}^2 + \|\Psi_j u\|_{L^2}^2\right).$$

Since the coverings chosen are Γ-invariant and the quotient $\widetilde{X}/\Gamma = X$ is relatively compact, the constant can be chosen independent of j and hence the proof is completed in this case.
Let us assume now that u is supported in a neighbourhood of the singular locus $\widetilde{X}_{\mathrm{sing}}$. Since the Poincare metric $\widetilde{\omega}_P$ is complete in a neighbourhood of the singular locus by Lemma 3.3.5, the Andreotti-Vesentini density lemma [MM07, Lemma 3.3.1] implies that (3.27) holds for any $u \in \mathrm{Dom}(\Box)|_{L^{0,q}(\widetilde{X}_{\mathrm{reg}},\widetilde{\omega}_P)}$. Using a partition of unity we can patch forms u and v that have support near the boundary and near the singular locus respectively. Hence the proof is complete. $\Box$

Theorem 3.5.9. *The Laplacian $\Box$ is Γ-Fredholm on $\mathrm{Dom}(\Box)|_{L^{0,q}(\widetilde{X}_{reg},\widetilde{\omega}_P)}$ and $\mathrm{Im}\,\Box|_{L^{0,1}(\widetilde{X}_{reg},\widetilde{\omega}_P)} \cap L'$ is Γ-dense in L' defined in (3.26).*

Proof. Lemma 3.5.8 implies that the conditions of Lemma 3.5.4 are satisfied on $\mathrm{Dom}(\Box)|_{L^{0,q}(\widetilde{X}_{\mathrm{reg}},\widetilde{\omega}_P)}$ and hence $\Box$ is Γ-Fredholm.
Since $\Box$ is Γ-Fredholm, Lemma 1.7.7 implies the second assertion. $\Box$

Finally we are able to prove Theorem 3.5.1.

Proof of Theorem 3.5.1. The idea of the final step is to solve the $\bar{\partial}$-equation on a certain subspace of $L^2(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$ and to generalize it to an arbitrary choice of a metric.
Note that by Theorem 3.5.9 the Laplacian $\Box$ is Γ-Fredholm on $L^{0,1}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$ and $\mathrm{Im}(\Box)|_{L^{0,1}(\widetilde{X}_{\mathrm{reg}},\widetilde{\omega}_P)}$ is Γ-dense in the space L' defined in (3.26). By the definition of Γ-density we know that for any $\delta > 0$ there exists a closed Γ-invariant subspace $Y' \subset L'$ such that $Y' \subset \mathrm{Im}\,\Box$ and $\dim_\Gamma Y' > N - \delta$. Since $Y' \subset \mathrm{Im}\,\Box$ we know that for any $u \in Y'$

there exists a v with the property that $v \perp \operatorname{Ker} \Box$ such that $\Box v = u$ and moreover v is unique with this property.
Call the space of all such solutions Z. Then we have immediately

$$\dim_\Gamma Z = \dim_\Gamma Y' > N - \delta.$$

Consider now the $\bar\partial$-problem for the equation $\Box v = u$. Then

$$\bar\partial(\Box v) = \bar\partial\left((\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)v\right) = \bar\partial\bar\partial^*\bar\partial v = \bar\partial u = 0,$$

and according to that $\bar\partial v = 0$. This means that $\bar\partial\bar\partial^* v = u$ since by assumption $v \perp \operatorname{Ker} \Box$.
Let us now consider the space Y defined by

$$Y = \left\{ f \in L \,\middle|\, \bar\partial f = g \in Y' \right\}.$$

By Lemma 3.5.5 and the definition of L we know that the $\bar\partial$-operator is injective on L as we can see easily: Assume

$$\mu_1\bar\partial(\chi h_x) + \mu_2\bar\partial(\chi h_x^2) + \ldots + \mu_N\bar\partial(h_x^N) = 0,$$

then by linearity of $\bar\partial$ we get immediately that

$$\mu_1\chi h_x + \ldots + \mu_N\chi h_x^N = 0$$

is a holomorphic function with compact support and hence is identially 0 which implies $\mu_1 = \ldots = \mu_N = 0$.
By injectivity of the $\bar\partial$-operator on L we conclude

$$\dim_\Gamma Y = \dim_\Gamma Y' > N - \delta.$$

Let us analyze elements $f \in Y$ a bit more in detail. For $f \in Y$ we know that by the construction of Z we can find a unique solution $g \in Z$ such that

$$\Box g = u = \bar\partial f.$$

But this means that the function $h = f - \bar\partial^* g$ is in $L^2(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$. Since all these functions h form again a closed Γ-invariant subspace $H \subset L^2(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$ with $\dim_\Gamma H > N - \delta$ for any $N \in \mathbb{N}$, and

$$H^{0,0}_{(2)}(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P) \subset H^{0,0}_{(2)}(\widetilde{X}, \widetilde{\omega}_P),$$

we proved Theorem 3.5.1 for the Poincare metric $\widetilde{\omega}_P$.
In order to get the result in general let θ be a different hermitian metric on $\widetilde{X}_{\mathrm{reg}}$. If the metric θ is dominated by $\widetilde{\omega}_P$, i.e. $\theta \leq \widetilde{\omega}_P$, then we are done since for any $u \in L^2(\widetilde{X}_{\mathrm{reg}}, \widetilde{\omega}_P)$ we have

$$\int_{\widetilde{X}_{\mathrm{reg}}} |u|^2 \, dV_\theta = \int_{\widetilde{X}_{\mathrm{reg}}} |u|^2 \frac{\theta^n}{n!} \leq \int_{\widetilde{X}_{\mathrm{reg}}} |u|^2 \frac{\widetilde{\omega}_P^n}{n!} = \int_{\widetilde{X}_{\mathrm{reg}}} |u|^2 \, dV_{\widetilde{\omega}_P},$$

which immediately shows that $u \in L^2(\widetilde{X}_{\mathrm{reg}}, \theta)$. Hence we have to show that $\theta \leq \widetilde{\omega}_P$ for any given metric θ.
Let us choose coordinates as in (3.6). Then it is clear that $\widetilde{\omega}_P$ dominates the Euclidean metric in these coordinates in a neighbourhood of $\widetilde{X}_{\mathrm{sing}}$ and thus it dominates some positive multiple of any hermitian metric θ on $\widetilde{X}_{\mathrm{reg}}$. Taking $\theta = \omega_0$ in (3.5) this means that there exists some constant $C > 0$ such that $\widetilde{\omega}_P \geq C\theta$ on $\widetilde{X}_{\mathrm{reg}}$, hence we proved Theorem 3.5.1 in full generality. $\square$

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