Twist maps in low regularity and non-periodic settings

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Abstract

Symplectic twist maps already appeared in the works of Poincaré. They emerge naturally as discretizations of certain low-dimensional Hamiltonian systems and offer a nice handle for studying their dynamics. The associated theory had experienced a tremendous boost from the discoveries made by Kolmogorow, Arnold and Moser in the 1960's, and by Aubry and Mather in the 1980's. In this thesis, we will substantiate the usefulness of studying twist maps also in non-periodic and low regularity settings, that is in situations where the classical results are not applicable.

In particular, we will concentrate on perturbative methods for near-integrable systems. One typical feature of such systems is the existence of "approximate first integrals", so called adiabatic invariants, and their presence usually has strong consequences for the dynamics. Here, we derive growth rates for a large class of non-periodic twist maps depending on the regularity assumptions. As an application, the Fermi-Ulam ping-pong is considered, where the possible growth in velocity is linked to the number of bounded derivatives of the forcing function.

Moreover, in systems with adiabatic invariants and almost periodic time-dependence the underlying compact structure enables one to use a generalization of Poincaré's recurrence theorem. By harnessing this fact, we prove that in such systems the set of initial condition leading to escaping orbits typically has measure zero. This is again demonstrated using the ping-pong model. Other applications are found in the Littlewood boundedness problem, where we consider a periodically forced piecewise linear oscillator together with its discontinuous limit case, and also a super-linear oscillator with an almost periodic forcing term. These systems are given by differential equations and thus the mentioned results imply also

the Poisson stability of almost every solution. Even in the periodic case, these insights represent valuable contributions due to the low regularity assumptions necessary to obtain them.

Contents

Chapter 1

Introduction

Consider an interval $I \subset \mathbb{R}$ and a map $f : \mathbb{R} \times I \to \mathbb{R}^2$, $(\theta, r) \mapsto (\theta_1, r_1)$, which is a diffeomorphism with respect to its image and isotopic to the identity. Moreover suppose that f satisfies:

(i) *Twist condition*: We have $\frac{\partial \theta_1}{\partial r} \neq 0$ in $\mathbb{R} \times I$.

(ii) *Symplecticity*: There is a C^1 map $\eta(\theta, r)$ such that

$$d\eta = r_1 d\theta_1 - r d\theta.$$

(iii) *Periodicity condition*: We have

$$\theta_1(\theta + 2\pi, r) = \theta_1(\theta, r) + 2\pi, \qquad r_1(\theta + 2\pi, r) = r_1(\theta, r),$$

and also the primitive function $\eta(\theta, r)$ is 2π-periodic in θ.

After identifying the circle $\mathbb{S}^1$ with the quotient space $\mathbb{R}/2\pi\mathbb{Z}$, the function f can be seen as the lift of a map $\bar{f} : (\bar{\theta}, r) \mapsto (\bar{\theta}_1, r_1)$ defined on a cylinder. Then, $\bar{f}$ is called *exact symplectic twist map* (of the cylinder).

These maps arise naturally as Poincaré maps of certain low-dimensional Hamiltonian systems. In fact, the converse is also true. Under suitable assumptions, any exact symplectic twist map can be obtained as the time-2π map of non-autonomous Hamiltonian equations

$$\dot{q} = \frac{\partial H}{\partial p}(q, p, t), \qquad \dot{p} = -\frac{\partial H}{\partial q}(q, p, t),$$

where $(q, p) \in \mathbb{R}^2$ and $H(q, p, t + 2\pi) = H(q, p, t) = H(q + 2\pi, p, t)$ satisfying the Legendre condition $\partial_p^2 H > 0$ [Mos86]. Instead of studying the continuous equations, one can now consider its Poincaré map and the orbits that arise from its iteration. Often this comes with some advantages, for example in numerical calculations or when dealing with non-smooth Hamiltonians. In the last sixty years, the field has received a lot of attention and numerous applications (see e.g. [AP90, Gol01, MHO09] for the general theory), mainly due to the introduction of two powerful tools: the theories of Kolmogorow-Arnold-Moser on the one hand and Aubry-Mather on the other [Arn63, Mos62a, Mat82]. To illustrate these results, we briefly discuss their consequences for the completely integrable twist map

$$\theta_1 = \theta + \varphi(r), \qquad r_1 = r, \tag{1.1}$$

where $\varphi \in C^1(I)$ is such that $\varphi' \neq 0$. If $\mathfrak{h}_0$ denotes a primitive of $r\varphi'(r)$, then $d\mathfrak{h}_0 = r_1\, d\theta_1 - r\, d\theta$, i.e. the map is exact symplectic. Any embedded circle $\mathbb{S}^1 \times \{\tilde{r}\}$ is preserved by this function, and rotated by an angle increasing/decreasing with $\tilde{r}$. If the so-called rotation number $\omega = \varphi(\tilde{r})$ is commensurable with 2π, that is $\frac{\omega}{2\pi} \in \mathbb{Q}$, then all orbits on the corresponding circle are periodic. Otherwise, they are quasi-periodic with frequencies 2π and ω. In [Gol01], Golé aptly describes completely integrable maps as the paradise lost of mathematicians, physicists and astronomers. The dynamics are completely understood and the invariance of embedded circles prevents any point from escaping in the vertical direction. In applications, such systems mostly appear only as strongly idealized models. Fortunately, the mentioned theories imply that many of these nice features persist under small perturbations in the space of exact symplectic twist maps. On a non-rigorous level, the situation after such a perturbation is as follows. Let $\mathcal{I} \subset \mathbb{R}$ be a compact interval and a proper subset of $\varphi(I)$. For any $\omega \in \mathcal{I}$, there is a closed invariant set $M = M_\omega$, such that any orbit in M has rotation number ω. If ω is commensurable with 2π, the corresponding set contains periodic orbits and possibly heteroclinic orbits joining them. For ω not commensurable with 2π there are two possibilities: If ω satisfies a Diophantine condition and the perturbation is small enough, then M remains a closed invariant curve and orbits on this curve are quasi-periodic. Note, that these invariant loops act as barriers, since any orbit starting in a region between two such curves has to remain in this region. The other possibility is that the circle breaks down, in which case M is an invariant Cantor set with possibly homoclinic orbits in the gaps.

However, both Aubry-Mather-Theory and the different versions of the KAM-Theorem rely critically on the (generalized) periodicity of the involved functions with respect to the angle θ. Moreover, the latter also requires a considerable degree of smoothness. In this thesis, we will demonstrate that studying symplectic twist maps can still be very fruitful if these two conditions are relaxed. We consider functions f of the type depicted above, but without imposing the periodicity condition (iii). These maps are called *symplectic twist maps of the plane* and their analysis is non-standard in the literature. In a series of publications, Kunze and Ortega investigated this class of functions [KO08, KO10, KO11, KO12, KO13, KO20, KO21]. As applications, they considered the Fermi-Ulam ping-pong model and the Littlewood boundedness problem. Here, we shall continue their survey and examine the same two fields of application.

The thesis is in large parts based on the three papers [Sch19, Sch22, OS22] published by the author, the last one being a joint work with Rafael Ortega. It is structured as follows. First, we continue the introduction by stating the main abstract findings and also briefly discussing some of the key ideas employed to obtain these results. Then, we close it with two small sections containing descriptions of the Fermi-Ulam ping-pong and the Littlewood boundedness problem, including the most relevant references and also a presentation of the main non-abstract results. In the second chapter, some properties of measure-preserving transformations and a generalization of Poincaré's Recurrence Theorem for infinite measure spaces are discussed. This theorem is applied to a class of exact symplectic twist maps of the cylinder with low regularity in Chapter 3. In addition, the consequences for some piecewise linear oscillators are stated. Chapter 4 deals with almost periodic twist maps. As applications, the Fermi-Ulam ping-pong and a superlinear oscillator are considered. In Chapter 5, the completely non-periodic case is treated and growth rates depending on the regularity are established for a class of symplectic twist maps, which is again demonstrated using the ping-pong model. Finally, Chapter 6 contains some conclusions. Following the appendices, there is also a notation index listing some of the frequently used symbols with no claim to completeness.

1.1 Non-periodic twist maps and adiabatic invariants

As already indicated above, symplectic maps can be seen as the discrete analog to Hamiltonian motions. Twist maps are associated to those Hamiltonians for which the velocity is monotone in the canonical momentum. In general, these maps do not have to satisfy the periodicity assumption (iii), e.g. when studying a non-autonomous Hamiltonian $H(q, p, t)$ with general time dependence. Let $f : (\theta, r) \mapsto (\theta_1, r_1)$ be any symplectic twist map. Due to the twist condition (i), the map $(\theta, r) \mapsto (\theta, \theta_1)$ is a diffeomorphism with respect to its image. We write $\mathcal{R} : (\theta, \theta_1) \mapsto (\theta, r)$ for its inverse. If η denotes the primitive function from (ii), then $S(\theta, \theta_1) = -(\eta \circ \mathcal{R})(\theta, \theta_1)$ is a *generating function* since we have

$$\frac{\partial S}{\partial \theta}(\theta, \theta_1) = r, \qquad \frac{\partial S}{\partial \theta_1}(\theta, \theta_1) = -r_1.$$

So the canonical transformation f can be described by a single scalar function S. If f induces a map $\bar{f}$ on the cylinder, it can be easily shown that we have $\eta(\theta + 2\pi, r) = \eta(\theta, r)$ on $\mathbb{R} \times I$ if and only if

$$S(\theta + 2\pi, \theta_1 + 2\pi) = S(\theta, \theta_1)$$

holds for all suitable (θ, θ_1). Another important property of this particular generating function, depending on the old and new position variable, is that it yields a variational principle. There is a one-to-one correspondence between complete orbits $(\theta_n, r_n)_{n\in\mathbb{Z}}$, where $(\theta_n, r_n) = f(\theta_{n-1}, r_{n-1})$, and sequences $(\theta_n)_{n\in\mathbb{Z}}$ satisfying

$$\partial_2 S(\theta_{n-1}, \theta_n) + \partial_1 S(\theta_n, \theta_{n+1}) = 0, \quad n \in \mathbb{Z}.$$

The latter equation is sometimes called the discrete Euler-Lagrange equation and sequences $(\theta_n)_{n\in\mathbb{Z}}$ corresponding to orbits of f are critical points of a suitable action functional. In the case when $\bar{f}$ is an exact symplectic twist map of the cylinder, this observation is the starting point for

Aubry-Mather theory (see e.g. [Gol01]). However, even if condition (iii) is not satisfied, this approach yields many interesting insights. For example, under suitable assumptions on S, there are infinitely many orbits $(\theta_n, r_n)_{n\in\mathbb{Z}}$ so that $\sup_{n\in\mathbb{Z}} |r_n| < \infty$. Such orbits are simply called *bounded*. The variational character of twist maps is however no subject of this thesis. For a discussion of its implications in the non-periodic case we refer the reader to [KO08, KO10, KO11, KO12]. Instead, we will concentrate on pertubative methods for near-integrable systems.

In non-periodic settings, invariant curve theorems are not applicable, and thus, proving boundedness of all orbits seems generally out of reach. In fact, there are many examples of non-periodic twist maps leading to unbounded motions. One can even find such maps in the class C^∞ with bounded derivatives up to any preassigned order leading to *escaping orbits*, i.e. orbits with $\lim_{n\to\infty} |r_n| = \infty$ (see e.g. [KO11] or Section 5.5, where the same construction is depicted). Nevertheless, the regularity assumptions have an impact also for non-periodic maps. In some cases, where unbounded motions exist, one can at least determine upper bounds for their growth.

To this end, consider an autonomous Hamiltonian $H(q, p, \lambda)$ with one degree of freedom and depending on a fixed parameter λ. Moreover, assume that its level sets $L_h = \{(q, p) : H(q, p, \lambda) = h\}$ define compact closed curves encircling the origin. In this case, there is a canonical change of variables $(q, p) \mapsto (\phi, I)$, transforming the system into so called action-angle coordinates. Geometrically, the action I corresponding to a point $(q, p) \in L_h$ is defined to be the area bounded by the associated phase curve divided by 2π and ϕ is some conjugate angular variable. From an analytic perspective, these coordinates are chosen because then the equations of motion have the form

$$\dot{\phi} = \omega(I, \lambda), \qquad \dot{I} = 0, \tag{1.2}$$

i.e. its Hamiltonian $H_0 = H_0(I, \lambda)$ does not depend on the angle and I is a first integral. Now, suppose that the parameter is not fixed but a function $\lambda = \lambda(\varepsilon t)$. In this case, the transformation depicted above is time dependent and yields a Hamiltonian of the form $H_0(I, \lambda(\varepsilon t)) + \varepsilon H_1(\phi, I, \varepsilon t)$, where H_1 is 2π-periodic in ϕ. If the frequency $\omega(I, \lambda)$ does not vanish, an averaging procedure can be applied to show that the action I remains an "approximate first integral", that is $|I(t) - I(0)| = \mathcal{O}(\varepsilon)$ for times $0 \leq t \leq \varepsilon^{-1}$. Such a quantity is called *adiabatic invariant.* The time span over which I is nearly preserved can be improved by imposing stronger regularity assumptions on λ and H (up to exponential time spans in the analytic case [Nei84]). Note, that there are many subtleties involved, including the very definition of adiabatic invariants. For a more thorough discussion, we refer the reader to [Arn89] and also [Hen93]. If λ is fixed and $\partial_I^2 H_0 = \partial_I \omega \neq 0$, then clearly any time-$T$ map of system (1.2) is a completely integrable twist map of the form (1.1). Therefore, the reasoning described above also suggests the existence of "adiabatic invariants" for suitable perturbations of this map. This observation is one of the key ideas used throughout the thesis. However, we will obtain the perturbations differently. Moreover, we shall omit giving a precise definition of adiabatic invariants in the context of maps and instead always specify the exact properties of these quantities when needed. In Chapter 5 for example, we consider non-periodic symplectic twist maps $f : (\theta, r) \mapsto (\theta_1, r_1)$ of the form

$$\theta_1 = \theta + \frac{1}{r^\alpha}(\gamma + F_1(\theta, r)), \quad r_1 = r + r^{1-\alpha} F_2(\theta, r), \tag{1.3}$$

where $\alpha \in (0,1)$, $\gamma \in \mathbb{R} \setminus \{0\}$ and $F_i = \mathcal{O}(r^{-\alpha})$ uniformly in θ for $i = 1, 2$. These maps can indeed be seen as perturbations of the completely integrable twist map (1.1) with $\varphi(r) = \gamma r^{-\alpha}$ and $\mathfrak{h}_0(r) = -\frac{\alpha\gamma}{1-\alpha} r^{1-\alpha}$, since applying a rescaling $\rho = \delta r$ yields

$$\theta_1 = \theta + \delta^\alpha \gamma \rho^{-\alpha} + \mathcal{O}(\delta^{2\alpha}), \qquad \rho_1 = \rho + \mathcal{O}(\delta^{2\alpha}),$$

uniformly in θ as $\delta \to 0$. Thus, if one assumes periodicity of the relevant functions, the argument above leads to the existence of invariant curves, showing that all orbits are bounded. Without any periodicity assumptions one can still find adiabatic invariants. Note that f is exact symplectic since all closed 1-forms are exact in the plane. However, the exactness does not have the same strong implications as in the case of the cylinder. Therefore, Kunze and Ortega introduced the notion of *E-symplectic* families of maps in [KO21], where they studied holomorphic functions of the type (1.3). In Section 5.3, we adapt this notion to the non-analytic case. By using the rescaling $\xi = \varepsilon^{1/\alpha} r$, the map f can be brought into the form

$$\mathcal{P}_\varepsilon : x_1 = x + \varepsilon l(x, \varepsilon), \quad x_1 = (\theta_1, \xi_1), \quad x = (\theta, \xi).$$

Such a family $\{\mathcal{P}_\varepsilon\}$ is called *E-symplectic of class* C^{k+1} if the primitive function $\zeta(\theta, \xi, \varepsilon)$ satisfying

$$d\zeta(\cdot, \varepsilon) = \xi_1 \, d\theta_1 - \xi \, d\theta$$

is close to some function $\varepsilon\mathfrak{m}(x)$ with $\mathfrak{m} \in C_b^{k+1}$ in a suitable sense and both $l(x, \varepsilon)$ and $\zeta(x, \varepsilon)$ have sufficiently many bounded derivatives. Here, C_b^k denotes the space of bounded functions with bounded derivatives up to order k. For small $\varepsilon > 0$, any function in this class of near-identity symplectomorphisms can be realized as the Poincaré map of a 1-periodic Hamiltonian system. After k steps of an averaging procedure, one obtains the system in normal form

$$\dot{y} = \varepsilon J \nabla \mathcal{N}(y, \varepsilon) + \varepsilon J \nabla \mathcal{R}(y, t, \varepsilon),$$

where J denotes the standard symplectic matrix and the remainder satisfies $\mathcal{R}(y, t, \varepsilon) = \mathcal{O}(\varepsilon^k)$ in C^2. The function $E(x) = \mathcal{N}(x, 0)$ is an k-th order adiabatic invariant for $\mathcal{P}_\varepsilon$ in the sense that

$$|E(x_n) - E(x_0)| = \mathcal{O}(\varepsilon), \quad 0 \le n \le \min\{N, \varepsilon^{-k}\},$$

for a piece of orbit $(x_n)_{0 \le n \le N} = (\mathcal{P}_\varepsilon^n(x_0))_{0 \le n \le N}$ not leaving some domain G. This can be translated into growth rates for the original map. In the case of twist maps given by (1.3), we need to assume that F_1, F_2 and the symplectic primitive function lie in a suitable class. To this end, consider the space $\mathcal{F}^k(s)$ of functions $F(\tau, v)$ such that $F \in C^k(\mathbb{R} \times [v_*, \infty))$ for some $v_* > 0$ and

$$\sup_{(\tau, v) \in \mathbb{R} \times [v_*, \infty)} v^{s + \nu_2} |\partial^{\boldsymbol{\nu}} F(\tau, v)| < \infty$$

for every multi-index $\boldsymbol{\nu} = (\nu_1, \nu_2)$ with $|\boldsymbol{\nu}| \le k$. This class was first introduced in [Ort99] and it is very useful for describing expansions of twist maps. The depicted argument then leads to the following result.

Theorem 1.1 (Theorem 5.25)**.** *Given $k \in \mathbb{N}$, $r_* > 0$, $\alpha \in (0,1)$ and $\gamma \in \mathbb{R} \setminus \{0\}$, consider a map $f : \mathbb{R} \times [r_*, \infty) \to \mathbb{R} \times [0, \infty)$, $(\theta, r) \mapsto (\theta_1, r_1)$ of the form* (1.3) *with $F_1, F_2 \in \mathcal{F}^{k+2}(\alpha)$. Moreover, suppose there is a function $\mathfrak{h} \in C^{k+2}(\mathbb{R} \times [r_*, \infty))$ that satisfies $d\mathfrak{h} = r_1\, d\theta_1 - r\, d\theta$ with $(\mathfrak{h} - \mathfrak{h}_0) \in \mathcal{F}^{k+2}(2\alpha - 1)$, where $\mathfrak{h}_0(\theta, r) = -\left(\frac{\alpha\gamma}{1-\alpha}\right) r^{1-\alpha}$. Then, there is a constant $C > 0$ such that if $(\theta_n, r_n)_{n \in \mathbb{N}_0}$ denotes a complete forward orbit of f, there is $n_0 \in \mathbb{N}$ so that*

$$r_n \leq C n^{1/(k+1)\alpha}, \qquad n \geq n_0.$$

Remark 1.2. (a) $\mathbb{N}$ denotes the positive integers, whereas $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

(b) If the functions F_1, F_2 and $\mathfrak{h}$ are analytic, we have $r_n = \mathcal{O}((\log n)^{1/\alpha})$ for any real complete forward orbit $(\theta_n, r_n)_{n \in \mathbb{N}_0}$ [KO21].

Instead of abandoning condition (iii) altogether, one can also consider twist maps with an angle that is periodic in a generalized sense. A function $u \in C(\mathbb{R})$ is called *(Bohr) almost periodic*, if for all $\varepsilon > 0$ there exists $L = L(\varepsilon)$ so that any interval of length L contains a number T such that

$$|u(t+T) - u(t)| < \varepsilon, \qquad \forall t \in \mathbb{R}.$$

There are several equivalent ways to describe almost periodicity. The following will be particularly useful in our analysis. Let Ω be a commutative topological group, which is metrizable, compact and connected. Moreover, assume there is a continuous homomorphism $\psi : \mathbb{R} \to \Omega$ with dense image, inducing a flow $\psi_\omega(t) = \omega + \psi(t)$ on Ω. A function $u(t)$ is almost periodic if and only if there is such a pair (Ω, ψ) and a function $U \in C(\Omega)$ so that

$$u(t) = U(\psi(t)).$$

If $u \in C^1(\mathbb{R})$, the latter formula also implies $u'(t) = \partial_\psi U(\psi(t))$, where $\partial_\psi U(\omega) = \lim_{h \to 0} \frac{1}{h}(U(\omega + \psi(h)) - U(\omega))$ denotes the derivative along the flow. For example, taking $\Omega \cong \mathbb{S}^1$ leads to periodic functions. Another important subclass is given by the *quasi-periodic* functions. There Ω is the N-Torus $\mathbb{T}^N$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We denote its classes by $\bar{\Theta} = \Theta + \mathbb{Z}^N$. The image of the homomorphism $\psi(t) = \overline{\nu t}$ winds densely around $\mathbb{T}^N$, whenever the frequency vector $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{R}^N$ is rationally independent. Finally, note that also *limit periodic* functions are included, which are obtained as the uniform limit of continuous periodic functions.

Now, we consider a family $\{g_\omega\}_{\omega\in\Omega}$ of symplectomorphisms of the form

$$g_\omega(t,r) = (t + F(\psi_\omega(t), r), r + G(\psi_\omega(t), r)), \tag{1.4}$$

where $F, G \in C(\Omega \times (0,\infty))$. If these maps have twist and $\Omega \cong \mathbb{S}^1$, there are invariant curve theorems available. This is also true when $g_{\bar{\Theta}}(t,r)$ is quasi-periodic in t, provided that the frequencies satisfy a suitable Diophantine condition (see [HLL18] and the references therein). The situation is however fundamentally different if these Diophantine conditions are neglected (due to the problem of small denominators [Arn63]) or if e.g. there is an infinite number of frequencies. Moreover, even in the periodic case, there are minimal regularity assumptions for the applicability of these theorems. The necessary conditions have been decreased successively from the original C^{333} by Moser to C^5 by Rüssmann and finally to $C^{3,\beta}$ with $\beta > 0$ [Mos62a, Rü70, Her86]. On the other hand, there are again examples of periodic twist maps with unbounded orbits. The first one was given by Takens in [Tak71], where he constructed such a map as a perturbation of the completely integrable map (1.1) in the class of C^1 exact symplectic twist maps of the cylinder. Later, this was improved to $C^{2,1-\varepsilon}$ by Herman [Her83]. We will consider maps of the form (1.4) that are in general only assumed to be of class C^1. Despite the absence of invariant curve theorems, the special structure of these maps reveals valuable information about their dynamics. Note that F and G are uniquely determined by (1.4). So we may consider the function

$$g(\omega, r) = (\omega + \psi(F(\omega, r)), r + G(\omega, r)). \tag{1.5}$$

It is related to $\{g_\omega\}$ via the identity

$$g \circ (\psi_\omega \times \mathrm{id}) = (\psi_\omega \times \mathrm{id}) \circ g_\omega.$$

One of our main abstract results deals with these kind of functions.

Theorem 1.3 (Theorem 4.9)**.** *Let $g : \mathcal{D} \subset \Omega \times (0,\infty) \to \Omega \times (0,\infty)$ be a map of the form (1.5) that is continuous, injective and measure-preserving.*

Moreover, suppose there is a function $W \in C^1_\psi(\Omega \times (0,\infty))$ *satisfying*

$$0 < \beta \le \partial_r W(\omega, r) \le \delta \ \textit{ for } \ \omega \in \Omega, \ \ r \in (0,\infty),$$

with some constants $\beta, \delta > 0$, *and furthermore*

$$W(g(\omega, r)) \le W(\omega, r) + k(r) \ \textit{ for } \ (\omega, r) \in \mathcal{D},$$

where $k : (0,\infty) \to \mathbb{R}$ *is decreasing, bounded and* $\lim_{r\to\infty} k(r) = 0$.
Then, for almost all $\omega \in \Omega$, *the set*

$$E_\omega = \{(t_0, r_0) : \lim_{n\to\infty} r_n = \infty\}$$

of initial condition leading to escaping orbits of the map g_ω *has Lebesgue measure zero.*

Remark 1.4. (a) Here, $C^1_\psi(\Omega \times (0,\infty))$ denotes the space of continuous functions $U(\omega, r)$ with derivatives $\partial_\psi U$ and $\partial_r U$ in $C(\Omega \times (0,\infty))$. The spaces $C^k_\psi(\Omega)$ used down below are defined accordingly (see Section 4.1).

(b) This result should be compared to Theorem 3.1 in [KO20], where it is proven in the quasi-periodic case.

The reasoning behind this theorem is as follows. Since every map g_ω is symplectic and injective, also g is continuous, injective and moreover preserves a measure $\mu_\Omega \otimes \lambda$, where λ denotes the Lebesgue measure on the real line and μ_Ω is a Borel probability measure called the *Haar measure* of Ω. At first glance, g has a form suitable for the application of the Poincaré Recurrence Theorem. However, the underlying space may have infinite measure. This difficulty can be overcome by using the following generalization due to Maharam [Mah64]. Given a map $T : X \to X$ preserving a measure μ, assume there is a measurable set $\mathcal{M}$ with $\mu(\mathcal{M}) < \infty$, such that almost every orbit of T has to enter $\mathcal{M}$ in the future. Then T is recurrent. In the case under consideration, such a set can be constructed by using the function W. This function can be seen as a generalized adiabatic invariant, since any growth will be slow for large energies, i.e. where the system is close to integrable. Here, the trick is to consider the restriction of g to the set $\mathcal{U}$ of initial condition leading to unbounded orbits. In that case, a viable choice for the space engendering set is given by

$$\mathcal{M} = \bigcup_{j\in\mathbb{N}} \mathcal{M}_j, \;\; \mathcal{M}_j = \{(\omega, r) \in \Omega \times (0,\infty) : |W(\omega, r) - W_j| \leq 2^{-j}\},$$

where $(W_j)_{j\in\mathbb{N}}$ is a sequence of positive numbers growing to infinity sufficiently fast. Note, that this method is not specific to twist maps. However, the twist condition is very useful for finding a suitable adiabatic invariant W.

Next, we introduce two fields of application for these abstract results.

1.2 Fermi-Ulam ping-pong

The Fermi-Ulam ping-pong is a model describing how charged particles bounce off magnetic mirrors, and thus, gain energy. They undergo the so called Fermi acceleration and one central question is whether the particles velocities can get close to the speed of light that way. The model was introduced by Fermi [Fer49] in order to explain the origin of high energy cosmic radiation. A common one-dimensional mathematical formulation of this problem is as follows. The point particle bounces completely elastically between two vertical plates of infinite mass, one fixed at $x = 0$ and one moving in time as $x = p(t)$ for some forcing function $p = p(t) > 0$. The particle alternately hits the walls and experiences no external force in between the collisions. The motion can be described by the function $f : (t_0, v_0) \mapsto (t_1, v_1)$, mapping the time $t_0 \in \mathbb{R}$ of an impact at the left plate $x = 0$ and the corresponding velocity $v_0 > 0$ right after the collision to (t_1, v_1), representing the subsequent impact at $x = 0$. Since one is interested in the long term behavior, we study the forward iterates $(t_n, v_n) = f^n(t_0, v_0)$ for $n \in \mathbb{N}$. One can show, that the map $\mathcal{P} : (t_0, E_0) \mapsto (t_1, E_1)$, obtained from f by a change of variables $E = \frac{1}{2}v^2$, is a symplectic twist map. The most studied case is that of a periodic forcing $p(t)$. Ulam conjectured an increase in energy with time on the average [Ula61]. Based on some numerical simulations, he however realized that rather large fluctuations and no clear gain in energy seemed to be the typical behavior. Two decades later, the development of KAM theory allowed to prove that the conjecture is indeed false. If the forcing p is sufficiently smooth, all orbits stay bounded in the phase space, since the existence of invariant curves prevents the orbits from escaping [Pus83, LL91]. The proofs are based on Moser's twist thoerem [Mos62b], which relies on a higher regularity. And indeed, Zharnitsky showed the existence of escaping orbits if only continuity is imposed on p [Zha98]. In the non-periodic case, one can even find C^∞-forcings with this behavior [KO11]. More recently, Dolgopyat and De Simoi developed a new approach. They considered the periodic case and studied some maps which are basically approximations of the successor map f. This way they could prove several results regarding the Lebesgue measure of the escaping set

$$E = \{(t_0, v_0) : \lim_{n\to\infty} v_n = \infty\},$$

consisting of initial data, which lead to infinitely fast particles [Dol08b, Dol08a, DS12, Sim13]. Finally, Zharnitsky investigated the case of a quasi-periodic forcing function with frequencies satisfying a Diophantine condition. Again, using an invariant curve theorem, he was able to show that the velocity of every particle is uniformly bounded in time [Zha00]. Since no such theorem is available if the Diophantine condition is dropped, a different approach is necessary in this case. In [KO20], Kunze and Ortega proved an analog of Theorem 1.3 for quasi-periodic maps. Its application to the ping-pong map shows that typically the escaping set E has measure zero. They also raised the question whether this result can be generalized to the almost periodic case. By applying Theorem 1.3 with the adiabatic invariant $W(\omega, E) = P(\omega)^2 E$, one obtains the following result, giving an affirmative answer.

Theorem 1.5 (Theorem 4.11). *Assume $0 < a < b$ and $P \in C^2_\psi(\Omega)$ are such that*

$$a \le P(\omega) \le b, \qquad \forall \omega \in \Omega.$$

Denote by E_ω the escaping set for the ping-pong map with the almost periodic forcing function $p_\omega(t) = P(\omega + \psi(t))$. Then, for almost all $\omega \in \Omega$, the set $E_\omega \subset \mathbb{R}^2$ has Lebesgue measure zero.

Even in the completely non-periodic case one can obtain some interesting insights. Recently, Kunze and Ortega studied holomorphic twist maps of the form (1.3). After a suitable change of variables $(t, E) \mapsto (\tau, W)$, the ping-pong map can be written in that form. This way, they showed that the velocity v_n after the n-th impact satisfies $v_n = \mathcal{O}(\log n)$ if the forcing function is holomorphic [KO21]. This rigorously verifies an older result by Neishtadt [Nei84]. In the same way, an application of Theorem 1.1 yields the following.

Theorem 1.6 (Theorem 5.26)**.** *Given $k \geq 3$, let $p \in C_b^{k+1}(\mathbb{R})$ be so that $0 < a \leq p(t) \leq b$ for $t \in \mathbb{R}$. There are constants $\tilde{C}, \tilde{E} > 0$ such that if $(t_n, E_n)_{n\in\mathbb{N}_0}$ denotes any complete forward orbit of the ping-pong map $\mathcal{P} \in C^k(\mathbb{R} \times [\tilde{E}, \infty))$, then*

$$E_n \leq \tilde{C} n^{2/(k-1)}, \qquad n \geq n_0,$$

for some $n_0 \in \mathbb{N}$.

Moreover, we show how to construct a smooth forcing function $p(t)$ leading to escaping orbits. This example was first introduced in [KO11], where it is shown that bounded and unbounded motions coexist. We prove that if $\|p\|_{C^{k+1}(\mathbb{R})} \leq M$ for a prefixed parameter M, then there are a constant $C > 0$ and a complete forward orbit $(t_n, v_n)_{n\in\mathbb{N}_0}$ such that

$$v_n \geq C n^{1/(k+1)}, \qquad n \in \mathbb{N}.$$

Note that this example fails to show the optimality of the rates stated in Theorem 1.6. However, the influence of the regularity on possible growth is clearly demonstrated.

1.3 Littlewood boundedness problem

The dynamics of the Duffing-type equation

$$\ddot{x} + G'(x) = p(t) \tag{1.6}$$

have been studied extensively due to its relevance as a model for the motion of a classical particle in a one-dimensional potential field $G(x)$ affected by an external time-dependent force $p(t)$. In the 1960's, Littlewood [Lit66b] asked whether solutions of (1.6) stay bounded in the $(x, \dot{x})$-phase space if either

$$\begin{aligned} &(i)\ G'(x)/x \to +\infty \ \text{ as } \ x \to \pm\infty \\ \text{or} \ \ &(ii)\ \ \operatorname{sign}(x) \cdot G'(x) \to +\infty \ \text{ and } \ G'(x)/x \to 0 \ \text{ as } \ x \to \pm\infty. \end{aligned}$$

Despite it's harmless appearance, this question turned out to be a quite delicate matter. Whether resonance phenomena occur, does not only depend on the growth of G, but also on the properties of p. The most investigated case is that of a time-periodic forcing p. The first affirmative contribution in that regard is due to Morris [Mor76], who showed the boundedness of all solutions to

$$\ddot{x} + 2x^3 = p(t),$$

where p is continuous and periodic. Later, Dieckerhoff and Zehnder [DZ87] were able to show the same for

$$\ddot{x} + x^{2n+1} + \sum_{j=0}^{2n} p_j(t) x^j = 0,$$

where $n \in \mathbb{N}$ and $p_j \in C^\infty$ are 1-periodic. In the following years, this result was improved by several authors (see [Bin89], [LL91], [Lev91],[Nor92],[LZ95] and the references therein). If however the periodicity condition is dropped, Littlewood [Lit66b] himself showed that for any odd potential G satisfying the super-/sublinearity condition, there is a bounded forcing p leading to at least one unbounded trajectory. Later, Ortega [Ort05] was able to prove in a more general context that for any given C^2-potential one can find an arbitrarily small $p \in C^\infty$ such that most initial conditions (in the sense of a residual set) correspond to unbounded solutions of (1.6). Even in the time-periodic case, Littlewood [Lit66a] constructed $G \in C^\infty$ and a periodic p such that there is at least one unbounded solution. (Actually both [Lit66b] and [Lit66a] contain a computational mistake; see [Lev92, Lon91] for corrections.) Let us also mention [Zha97], where Zharnitsky improved the latter result for the superlinear case such that the periodic p can be chosen continuously. These counterexamples show that besides periodicity and regularity assumptions on p an additional hypothesis on G is needed if one hopes for boundedness of all solutions. Indeed, all positive results mentioned above suppose the monotone growth of $G'(x)/x$. This condition guarantees the monotonicity of the corresponding Poincaré map, and thus, enables the authors to use KAM theory.

We also want to point out that the Fermi-Ulam ping pong can be obtained as a limiting case, where the time-dependent potential becomes infinitely steep at the boundaries [LL91].

In the last twenty years a wealth of works on the Littlewood boundedness problem has been published, including the sublinear, semilinear and other cases. Since those are far too many to be presented here, we focus on the two particular cases studied in this thesis. First, consider the superlinear oscillator equation

$$\ddot{x} + |x|^{\alpha-1}x = p(t),$$

where $\alpha \geq 3$. In [LZ95], Levi and Zehnder were able to show that for a quasi-periodic forcing p all solutions are bounded, if the frequencies of p satisfy a Diophantine condition. Here, we shall investigate the case where p is only assumed to be almost periodic. Denote by $x(t) = x(t; \tilde{x}, \tilde{v}, \tilde{t})$ the solution to this equation satisfying the initial condition $x(\tilde{t}) = \tilde{x}$ and $\dot{x}(\tilde{t}) = \tilde{v}$. We consider solutions $x(t; 0, v_0, t_0)$ with $v_0 < 0$ and the map

$$\psi : (v_0, t_0) \mapsto (v_1, t_1),$$

where t_1 is the first zero to the right of t_0 such that corresponding velocity $v_1 = \dot{x}(t_1; 0, v_0, t_0)$ is negative. This Poincaré map will be well-defined for $|v_0|$ sufficiently large, since then the corresponding solution oscillates quickly.

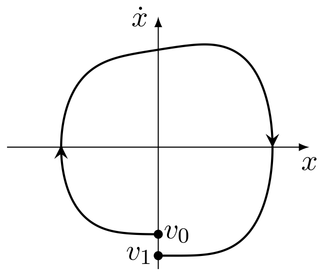

Figure 1.1: For large energies the trajectory spins clockwise around the origin

After applying several coordinate transformations ψ has a form suitable for the application of Theorem 1.3. Thus, the escaping set

$$E = \{(v_0, t_0) : \lim_{n\to\infty} v_n = -\infty\}$$

has Lebesgue measure zero. It follows that also the set of initial condition $(\tilde{x}, \tilde{v}, \tilde{t}) \in \mathbb{R}^3$ leading to solutions such that $\lim_{t\to\infty}(|x(t)| + |\dot{x}(t)|) = \infty$ has measure zero. In fact, we will even show that almost all solutions are *Poisson stable.* There are various ways to define this notion. We will use the following (cf. [CL20] and the references therein). A solution $x(t)$ is called Poisson stable, if there is a sequence $(t_n)_{n\in\mathbb{Z}}$ with $t_n \to \pm\infty$ as $n \to \pm\infty$ such that

$$|x(t + t_n) - x(t)| + |\dot{x}(t + t_n) - \dot{x}(t)| \to 0, \quad \text{as } n \to \pm\infty,$$

uniformly on every bounded interval in $\mathbb{R}$.

Theorem 1.7 (Theorem 4.15)**.** *Given $P \in C^4_\psi(\Omega)$, consider the family $\{p_\omega\}_{\omega\in\Omega}$ of almost periodic forcing functions defined by*

$$p_\omega(t) = P(\omega + \psi(t)), \quad t \in \mathbb{R}.$$

Let $x_\omega(t; \tilde{x}, \tilde{v}, \tilde{t})$ denote the solution of (4.22) *with forcing function $p(t) = p_\omega(t)$ satisfying the initial condition $x_\omega(\tilde{t}) = \tilde{x}$ and $\dot{x}_\omega(\tilde{t}) = \tilde{v}$. Then, for almost all $(\tilde{x}, \tilde{v}, \tilde{t}, \omega) \in \mathbb{R}^3 \times \Omega$, the solution $x_\omega(t; \tilde{x}, \tilde{v}, \tilde{t})$ is Poisson stable.*

Remark 1.8. For $\alpha > 3$ the same holds, if only $P \in C^2_\psi(\Omega)$ is imposed. Just Lemma 4.18 requires higher regularity and the latter is needed only for $\alpha = 3$.

The other case we will consider deals with a quadratic potential $G(x)$. More precisely, we study oscillators of the form

$$\ddot{x} + n^2 x + h(x) = p(t), \tag{1.7}$$

where $n \in \mathbb{N}$ and $h, p \in C(\mathbb{R})$ are bounded, with p also 2π-periodic. The n-th Fourier coefficient of p is given by

$$\hat{p}_n = \frac{1}{2\pi}\int_0^{2\pi} p(t)e^{-int}\,dt.$$

In the linear case $h = 0$, it is a well established fact that solutions of (1.7) are 2π-periodic (and hence bounded) if $\hat{p}_n = 0$, and otherwise unbounded and non-recurrent due to resonance phenomena. In [LL69], Lazer and Leach studied the case when h has two distinct finite limits $h(\pm\infty) = \lim_{x\to\pm\infty} h(x)$ at infinity and all values of h lie between those limits. They were able to show that (1.7) has a 2π-periodic solution if and only if

$$\pi|\hat{p}_n| < |h(+\infty) - h(-\infty)|. \tag{1.8}$$

Later, it was proven in [AO96] that the negation of this inequality implies that all solutions $x(t)$ satisfy

$$\lim_{t\to\pm\infty} \left[x(t)^2 + \dot{x}(t)^2\right] = \infty. \tag{1.9}$$

See also [Sei90] for a previous related work. Results with respect to boundedness were obtained in [Ort99]. There, it was shown that the same condition (1.8) leads to the boundedness of all solutions in the special case where $h = h_L$ with $L > 0$ is the piecewise linear function given by

$$h_L(x) = \begin{cases} L & \text{if } x \geq 1, \\ Lx & \text{if } |x| < 1, \\ -L & \text{if } x \leq -1, \end{cases}$$

provided that $p \in C^5(\mathbb{R})$ is 2π-periodic (see also [KKY97] for a related result with a discontinuous h). Moreover, this led to the insight that almost every solution $x(t)$ is Poisson stable. In the same year, Liu obtained a similar result for general $h \in C^6(\mathbb{R})$ such that $\lim_{|x|\to\infty} x^k h^{(k)}(x) = 0$ for $1 \leq k \leq 6$, if $p \in C^7(\mathbb{R})$ is 2π-periodic [Liu99]. Recent findings for more general non-linearities can be found in [PWW16, WWX19]. All latter results were obtained by using variants of Moser's small twist theorem. However, the application of any such invariant curve theorem requires a considerable degree of smoothness of either $h(x)$ or $p(t)$. It is an interesting question if any of the nice features of solutions survive if only mild regularity assumptions are made. Here, we investigate this question for the piecewise linear equation

$$\ddot{x} + n^2 x + h_1(x) = p(t). \tag{1.10}$$

By rescaling $\ddot{x} + n^2 x + h_L(x) = p(t)$, one obtains the same equation with $\tilde{p} = \frac{p}{L}$ and a function $\tilde{h}_L$ given by $\tilde{h}_L(x) = \mathrm{sign}(x)$ for $|x| \geq \frac{1}{L}$ and $\tilde{h}_L(x) = Lx$ for $|x| < \frac{1}{L}$. Since the slope L has basically no effect on the dynamics we have normalized the equation by setting $L = 1$. Besides giving a good starting point for more general non-linearities, such piecewise linear oscillators are also known in the engineering literature. E.g. (1.10) can be considered as a model for an oscillator with stops (see [Har85] and also [Ort99] for the derivation of (1.10)). Our main result is the following.

Theorem 1.9 (Theorem 3.4). *Suppose $p \in C(\mathbb{R})$ is 2π-periodic and satisfies the Lazer-Leach condition $\pi|\hat{p}_n| < 2$. If $x(t) = x(t; \tilde{x}, \tilde{v}, \tilde{t})$ denotes the solution of* (1.10) *with initial condition $x(\tilde{t}) = \tilde{x}$ and $\dot{x}(\tilde{t}) = \tilde{v}$, then $x(t; \tilde{x}, \tilde{v}, \tilde{t})$ is Poisson stable for almost every $(\tilde{x}, \tilde{v}, \tilde{t}) \in \mathbb{R}^3$.*

This theorem is an improvement of Corollary 2.1 in [Ort99].

Remark 1.10. (*a*) The result is also true for the discontinuous equation

$$\ddot{x} + n^2 x + \mathrm{sign}(x) = p(t), \tag{1.11}$$

if ∂N is countable, where $N = \{t \in \mathbb{R} : |p(t)| = 1\}$. This is basically the case considered in [KKY97] and it can be seen as a limit case of (1.10). Note that one first has to define a proper notion of solutions to (1.11) (see Definition 3.5 down below). We also refer the reader to [LZ20] for a discussion of chaos in second-order equations with signum non-linearities.
(*b*) The Lazer-Leach condition is not only sufficient but also necessary for recurrence, since all solutions satisfy (1.9) if $\pi|\hat{p}_n| \geq 2$ [AO96].
(*c*) Theorems 1.5, 1.7 and 1.9 state that the set of initial conditions leading to escaping orbits has measure zero. It should be noted, that the author knows of no example exhibiting unbounded orbits, provided the assumptions of the respective theorem are satisfied.

The proof is similar to the one of Theorem 1.7. A Poincaré map $P : (\tau, v) \mapsto (\tau_1, v_1)$ is constructed, which sends the initial condition (τ, v) of $x(t; 0, v, \tau)$ to (τ_1, v_1), representing a subsequent zero and its corresponding velocity. This map has an expansion of the form

$$\begin{cases} \tau_1 = \tau + 2\pi - \frac{L(\tau)}{v} + R(\tau, v), \\ v_1 = v + L'(\tau) + S(\tau, v), \end{cases}$$

where $L \in C^2$ and $R, S \in C^1$ are 2π-periodic in τ. Moreover, the Lazer-Leach condition guarantees that L is positive. Therefore, P can be viewed as the lift of a twist map $\bar{P}$ defined on the cylinder $\mathbb{S}^1 \times [0, \infty)$. Theorem 3.1 below states that any such map is recurrent if R and S satisfy certain bounds. This result should be compared to Theorem 1.3, only that the construction of the associated adiabatic invariant is already included in the proof. This again yields the Poisson stability of almost every solution.

Chapter 2

Recurrence

Let $(X, \mathcal{A}, \mu)$ be a measure space and introduce the following useful notation. For $A, B \in \mathcal{A}$ we write

$$A \subset B \mod \mu,$$

if $A \subset B \cup \mathcal{N}$, where $\mathcal{N}$ is a set of measure zero. Now, consider a map $T : X \to X$ which is bi-measurable, that is

$$T^{-1}(A), T(A) \in \mathcal{A} \qquad \text{for all } A \in \mathcal{A}.$$

Such a map T is said to be *measure-preserving*, if

$$\mu(T(A)) = \mu(A) \qquad \text{for all } A \in \mathcal{A}.$$

As a consequence, such a measure-preserving transformation satisfies

$$\mu(T^{-1}(A)) \leq \mu(A) \qquad \text{for all } A \in \mathcal{A},$$

with equality if $A \subset T(X) \mod \mu$.

Remark 2.1. In the literature, there is no unique way of defining the two properties above. In particular, T is often called measure-preserving, if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$. However, the definition in this work was chosen since it seems to be the most natural in the application to mechanical problems and suchlike.

Since T maps X into itself, the iterates $T^n = T^{n-1} \circ T$, where $T^0 = \mathrm{id}$, are well-defined for all $n \in \mathbb{N}$. We call the map T *recurrent*, if for every $A \in \mathcal{A}$ for almost all $x \in A$ there is $n \in \mathbb{N}$ such that $T^n(x) \in A$, that is

$$A \subset \bigcup_{n=1}^{\infty} T^{-n}(A) \mod \mu,$$

where $T^{-n}(A)$ denotes the pre-image under T^n. In other words, the set of points in A not returning to A has measure zero. Since T is measure-preserving, also any (iterated) pre-image of this set has measure zero. Hence T is even infinitely recurrent, i.e. for almost all $x \in A$ there is an increasing sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $T^{n_k}(x) \in A$ for all $k \in \mathbb{N}$.
In the case of a finite measure-space, the famous Poincaré recurrence theorem characterizes the relation between measure-preserving and recurrent maps. We will use it in the following form.

Lemma 2.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) < \infty$ and suppose $T : X \to X$ is measure-preserving. Then T is recurrent.*

Unfortunately, the situation is less clear if the space has infinite measure. However, the statement of the recurrence theorem stays valid if there exists a set $\mathcal{M}$ of finite measure which acts as some kind of bottleneck. This is described in the following generalization of Lemma 2.2 due to Maharam [Mah64], which also recently got some attention in the context of twist maps by Dolgopyat [Dol].

Lemma 2.3 (Maharam's Recurrence Theorem)**.** *Consider a measure space $(X, \mathcal{A}, \mu)$ and suppose $T : X \to X$ is measure-preserving. If there exists a set $\mathcal{M} \in \mathcal{A}$ with $\mu(\mathcal{M}) < \infty$, such that*

$$X \subset \bigcup_{n=1}^{\infty} T^{-n}(\mathcal{M}) \mod \mu,$$

then T is recurrent.

Proof. The "time of first return" $r(x) = \min\{k \in \mathbb{N} : T^k(x) \in \mathcal{M}\}$ is well-defined for almost all $x \in X$ by assumption. In particular, it can be shown that there is a set Γ of measure zero such that the induced map

$S : \mathcal{M} \setminus \Gamma \to \mathcal{M}$ given by $S(x) = T^{r(x)}(x)$ is well-defined and satisfies $S(\mathcal{M} \setminus \Gamma) \subset \mathcal{M} \setminus \Gamma$. Moreover, S is measure-preserving and hence, one can apply the Poincaré recurrence theorem to see that S is also recurrent. Now, let $A \in \mathcal{A}$ be a measurable set in X and for $k \in \mathbb{N}$ consider the sets

$$A_k = \{x \in A : r(x) = k\}.$$

Moreover, define $B_k = T^k(A_k) \subset \mathcal{M}$. Since S is recurrent, we have

$$B_k \subset \bigcup_{n=1}^{\infty} S^{-n}(B_k) \subset \bigcup_{n=1}^{\infty} T^{-n}(B_k) \mod \mu.$$

From this it follows

$$T^{-k}(B_k) \subset \bigcup_{n=1}^{\infty} T^{-(n+k)}(B_k) \mod \mu.$$

Since $A_k \subset T^{-k}(B_k)$ and $\mu(A_k) = \mu(B_k)$ we know that $T^{-k}(B_k) = A_k$ up to a set of measure zero. This in turn implies

$$A_k \subset \bigcup_{n=1}^{\infty} T^{-n}(A_k) \mod \mu.$$

Finally, taking the union over all $k \in \mathbb{N}$ shows that almost every point in A returns to A. $\square$

There are two drawbacks to Lemma 2.3. On the one hand, such a set $\mathcal{M}$ does not exist for every recurrent measure-preserving transformation, as already a trivial example like the identity shows. On the other hand, even when it does exist, it can be hard to find. A key idea of Chapters 3 and 4 will be that certain symplectic twist maps offer a class of measure-preserving transformations for which the construction of $\mathcal{M}$ can be done explicitly.

Remark 2.4. If X is σ-finite, the following observation can be made. A measure-preserving map $T : X \to X$ is recurrent if and only if there is a covering $\{X_j\}_{j\in\mathbb{N}}$ of X and a collection of sets $\{\mathcal{M}_j\}_{j\in\mathbb{N}}$ with $\mu(\mathcal{M}_j) < \infty$ such that for all $j \in \mathbb{N}$ we have $T(X_j) \subset X_j$ and $X_j \subset \cup_{n=1}^{\infty} T^{-n}(\mathcal{M}_j)$ mod μ.

Note, that there are several more generalizations to the Poincaré recurrence theorem and depending on the situation one might choose the appropriate version. For example, if $\{X_j\}_{j\in\mathbb{N}}$ is a covering of X with $\mu(X_j) < \infty$ and for every fixed $j \in \mathbb{N}$ the measure-preserving map T satisfies

$$\lim_{n\to\infty} \frac{1}{n}\mu\left(\bigcup_{k=1}^{n} T^k(X_j)\right) = 0,$$

then T is also recurrent (see [KS19]). For a more thorough discussion of maps preserving an infinite measure we refer the reader to [Aar97] and [Dol].

In Chapters 3 and 4 we will consider functions

$$f : \mathcal{D} \subset \Omega \times (0,\infty) \to \Omega \times (0,\infty),$$

where $\mathcal{D}$ is an open set and Ω is a compact commutative topological group. The space $\Omega \times (0,\infty)$ is equipped with the product measure $\mu_\Omega \otimes \tilde{\lambda}$, where $\tilde{\lambda}$ denotes an absolutely continuous measure on the real line and μ_Ω is a Borel probability measure called the Haar measure of Ω (see Section 4.1.3 for details). The map f is assumed to be a homeomorphism with respect to its image and measure-preserving, that is

$$(\mu_\Omega \otimes \tilde{\lambda})(f(\mathcal{B})) = (\mu_\Omega \otimes \tilde{\lambda})(\mathcal{B})$$

for all Borel sets $\mathcal{B} \subset \mathcal{D}$. In general, we can not assume that $f(\mathcal{D}) \subset \mathcal{D}$. Therefore, we have to carefully construct a suitable domain on which the forward iterations are well-defined. We initialize $\mathcal{D}_1 = \mathcal{D}$ and set

$$\mathcal{D}_{n+1} = f^{-1}(\mathcal{D}_n), \qquad \text{for } n \in \mathbb{N}.$$

This way f^n is well-defined on $\mathcal{D}_n$. Inductively it can be shown that $\mathcal{D}_{n+1} = \{(\omega, r) \in \mathcal{D} : f(\omega, r), \ldots, f^n(\omega, r) \in \mathcal{D}\}$ and thus $\mathcal{D}_{n+1} \subset \mathcal{D}_n \subset \mathcal{D}$ for all $n \in \mathbb{N}$. Initial conditions in the set

$$\mathcal{D}_\infty = \bigcap_{n=1}^{\infty} \mathcal{D}_n \subset \Omega \times (0,\infty)$$

correspond to complete forward orbits, i.e. if $(\omega_0, r_0) \in \mathcal{D}_\infty$, then

$$(\omega_n, r_n) = f^n(\omega_0, r_0)$$

is defined for all $n \in \mathbb{N}$. It could however happen that $\mathcal{D}_\infty = \emptyset$ or even $\mathcal{D}_n = \emptyset$ for some $n \geq 2$. The set of initial data leading to unbounded orbits is denoted by

$$\mathcal{U} = \{(\omega_0, r_0) \in \mathcal{D}_\infty : \limsup_{n\to\infty} r_n = \infty\}.$$

Complete orbits such that $\lim_{n\to\infty} r_n = \infty$ will be called *escaping orbits*. The corresponding set of initial data is

$$\mathcal{E} = \{(\omega_0, r_0) \in \mathcal{D}_\infty : \lim_{n\to\infty} r_n = \infty\}.$$

Note, that both $\mathcal{U}$ and $\mathcal{E}$ are Borel measurable. The *escaping set* $\mathcal{E}$ can be viewed as the transient part of the domain. Indeed, f is obviously non-recurrent on $\mathcal{E}$ and its complement $\mathcal{D}_\infty \setminus \mathcal{E}$ on the other hand can be covered by the measurable sets

$$\mathcal{B}_m = \{(\omega_0, r_0) \in \mathcal{D}_\infty : \liminf_{n\to\infty} r_n \leq m\}, \quad m \in \mathbb{N}.$$

Since every orbit starting in $\mathcal{B}_m$ eventually has to enter the set $\Omega \times [0, m+1]$, Lemma 2.3 can be applied to the restricted map $f : \mathcal{B}_m \to \mathcal{B}_m$. It follows easily, that f is recurrent on $\mathcal{D}_\infty \setminus \mathcal{E}$. Therefore, proving the recurrence of $f : \mathcal{D}_\infty \to \mathcal{D}_\infty$ is equivalent to showing $(\mu_\Omega \otimes \tilde{\lambda})(\mathcal{E}) = 0$. In particular, it is sufficient to find a set $\mathcal{M}$ of finite measure such that every escaping orbit enters $\mathcal{M}$ in the future, that is

$$\mathcal{E} \subset \bigcup_{n=1}^{\infty} f^{-n}(\mathcal{M}) \quad \mathrm{mod}\ (\mu_\Omega \otimes \tilde{\lambda}).$$

Chapter 3

The periodic case

In this chapter, we consider periodic twist maps of low regularity. First, the setup of Chapter 2 is employed to prove the recurrence for a class of these maps. Then, this is applied to deduce the Poisson stability of almost every solution to a piecewise linear oscillator and its discontinuous limit case in Sections 3.2 and 3.3, respectively. The content of this chapter stems mostly from the paper [OS22], which is a joint work of Rafael Ortega and the author.

3.1 Twist maps of the cylinder

We identify the circle $\mathbb{S}^1$ with the quotient space $\mathbb{R}/2\pi\mathbb{Z}$. With a small abuse of notation, $C^n(\mathbb{S}^1)$ denotes the space of n-times continuously differentiable functions $F : \mathbb{R} \to \mathbb{R}$ that are 2π-periodic. Sometimes, we will not differentiate between a map $\bar{F} : \mathbb{S}^1 \to \mathbb{S}^1$ and its lift satisfying

$$F(\theta + 2\pi) = F(\theta) \mod 2\pi.$$

Given $v_* > 0$, consider the cylinder $\mathbb{M}_{v_*} = \mathbb{S}^1 \times [v_*, \infty)$ equipped with the absolutely continuous measure $\mu = v\, d\bar{\theta} \otimes dv$. We write $M_{v_*} = \mathbb{R} \times [v_*, \infty)$ for its universal cover. In this section we will study twist maps of $\mathbb{M}_{v_*}$. These are functions

$$\bar{f} : \mathbb{M}_{v_*} \to \mathbb{S}^1 \times [0, \infty), \quad (\bar{\theta}, v) \mapsto (\bar{\theta}_1, v_1),$$

such that there is a corresponding lift $f : M_{v_*} \to \mathbb{R} \times [0, \infty)$ satisfying

$$\theta_1(\theta + 2\pi, r) = \theta_1(\theta, r) + 2\pi, \qquad r_1(\theta + 2\pi, r) = r_1(\theta, r),$$

which is a diffeomorphism with respect to its image and has twist, i.e.

$$\frac{\partial \theta_1}{\partial r} \neq 0, \quad \text{in } M_{v_*}.$$

Furthermore, suppose there is a function $\eta \in C^1(\mathbb{M}_{v_*})$ such that

$$d\eta = v_1^2 d\bar{\theta}_1 - v^2 d\bar{\theta}. \tag{3.1}$$

Then, we will say that $\bar{f}$ is *exact symplectic on* $\mathbb{M}_{v_*}$. Note, that this is not consistent with the (classical) definition given in the introduction. The map does not preserve the two form $d\bar{\theta} \wedge dv$, but it is symplectic in the sense that

$$v_1 d\bar{\theta}_1 \wedge dv_1 = v d\bar{\theta} \wedge dv. \tag{3.2}$$

However, within the scope of Chapter 3 we use the definition above, simply because it is more practical in the applications considered. After all, there are only minor qualitative differences, since changing the second coordinate to $E = \frac{v^2}{2}$ yields a map symplectic in the classical sense. Condition (3.1) also implies that for any embedded circle $C_v = \mathbb{S}^1 \times \{v\} \subset \mathbb{M}_{v_*}$ we have

$$0 = \int_{C_v} d\eta = \int_{\mathbb{S}^1} \left(v_1^2 \frac{\partial \bar{\theta}_1}{\partial \bar{\theta}} - v^2 \right) d\bar{\theta}.$$

In the introduction we already mentioned the spaces $\mathcal{F}^k(m)$ of functions $F(\tau, v)$ such that

$$\sup_{(\tau,v) \in M_{v_*}} v^{m+\nu_2} |\partial^{\boldsymbol{\nu}} F(\tau, v)| < \infty$$

for $|\boldsymbol{\nu}| \leq k$ and some $v_* > 0$. Let us also introduce the class $\mathcal{F}_{\mathrm{u}}(m)$ of continuous functions $F : M_{v_*} \to \mathbb{R}$, such that $v^m F(\cdot, v)$ converges uniformly as $v \to \infty$. We write $\mathcal{F}^k_{\mathrm{u}}(m)$ for the intersection $\mathcal{F}^k(m) \cap \mathcal{F}_{\mathrm{u}}(m)$. Some properties of these spaces can be found in Appendix A.

As depicted in the last chapter, we write $\mathcal{D}_\infty$ for the set of initial condition $(\bar{\theta}_0, v_0)$, such that the complete forward orbit $(\bar{\theta}_n, v_n) := \bar{f}^n(\bar{\theta}_0, v_0)$, $n \in \mathbb{N}$, is well-defined. The escaping set is given by

$$\mathcal{E} = \{(\bar{\theta}_0, v_0) \in \mathcal{D}_\infty : \lim_{n \to \infty} v_n = \infty\}.$$

Theorem 3.1. *Consider a map $\bar{f} : \mathbb{M}_{v_*} \to \mathbb{S}^1 \times [0, \infty)$ with a lift f given by*

$$\begin{cases} \theta_1 = \theta + 2\pi - \frac{L(\theta)}{v} + R(\theta, v), \\ v_1 = v + L'(\theta) + S(\theta, v), \end{cases}$$

where $L \in C^2(\mathbb{S}^1)$, $L > 0$, $R, S \in C^1(\mathbb{M}_{v_*})$, $R \in \mathcal{F}_u(2)$, $S \in \mathcal{F}_u(1)$ and

$$\sup_{(\theta,v)\in M_{v_*}} v^{\nu_2}|\partial^{\boldsymbol{\nu}} R(\theta, v)| < \infty, \qquad \lim_{v\to\infty} v^{\nu_2}\partial^{\boldsymbol{\nu}} R(\theta, v) = 0,$$

for every $\theta \in \mathbb{R}$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)$ with $|\boldsymbol{\nu}| = 1$. Moreover, assume that $\bar{f}$ is one-to-one and exact symplectic in the sense that there is a function $\eta \in C^1(\mathbb{M}_{v_*})$ with $d\eta = v_1^2 d\bar{\theta}_1 - v^2 d\bar{\theta}$. Then $\mu(\mathcal{E}) = 0$, i.e. $\bar{f}$ is recurrent.

Remark 3.2. (a) Under the stronger assumptions $L \in C^6(\mathbb{S}^1)$ as well as $R, S \in C^5(\mathbb{M}_{v_*})$, $R \in \mathcal{F}^5(2)$ and $S \in \mathcal{F}^5(1)$, KAM-theory is applicable and shows the boundedness of all orbits. See [Ort99] for a suitable invariant curve theorem and it's application to a map of the type under consideration.

(b) For forcing functions $p \in C^2(\mathbb{S}^1)$, the Fermi-Ulam ping-pong map $(t_0, v_0) \mapsto (t_1, v_1)$ from the introduction is a symplectic twist map in the sense above and has an expansion of the form

$$\begin{cases} t_1 = t_0 + \frac{2p(t_0)}{v_0} + \tilde{R}(t_0, v_0), \\ v_1 = v_0 - 2\dot{p}(t_0) + \tilde{S}(t_0, v_0), \end{cases}$$

with $\tilde{R} \in \mathcal{F}_{\mathrm{u}}^1(2)$ and $\tilde{S} \in C^1 \cap \mathcal{F}_{\mathrm{u}}(1)$. Thus it would be feasible for an application. However, Theorem 4.11 down below (dealing with almost periodic forcing terms) already includes this case.

In the proof, we will need the following auxiliary lemma, which is basically a variant of Lemma 4.1 in [KO20].

Lemma 3.3. *Consider a map $\bar{f} : D \to \mathbb{S}^1 \times [0, \infty)$, $(\bar{\theta}, v) \mapsto (\bar{\theta}_1, v_1)$, where $D \subset \mathbb{S}^1 \times [0, \infty)$. Let $\rho(\bar{\theta}, v) = v + \beta(\bar{\theta})$ with $\beta \in C(\mathbb{S}^1)$. Moreover, suppose there is $v_* > 0$ such that for all $(\bar{\theta}, v) \in D \cap \mathbb{M}_{v_*}$ we have*

$$|\rho(\bar{f}(\bar{\theta}, v)) - \rho(\bar{\theta}, v)| \le \frac{\delta(v)}{v}, \tag{3.3}$$

where $\delta : [v_*, \infty) \to [0, \infty)$ *is a decreasing function with* $\lim_{v\to\infty} \delta(v) = 0$. *Then, there is a set* $\mathcal{M} \subset \mathbb{S}^1 \times [0, \infty)$ *with* $\mu(\mathcal{M}) < \infty$ *such that every unbounded orbit of* $\bar{f}$ *enters* $\mathcal{M}$.

Proof. Let $(\rho_j)_{j\in\mathbb{N}} \subset [2v_*, \infty)$ be an increasing sequence with $\lim_{j\to\infty} \rho_j = \infty$ such that

$$\rho_1 > \frac{1}{2\rho_1} + \|\beta\|_\infty \qquad \text{and} \qquad \delta\left(\tfrac{\rho_j}{2}\right) < 2^{-(j+1)}$$

for all $j \in \mathbb{N}$. Now, define

$$\mathcal{M} = \bigcup_{j\in\mathbb{N}} \mathcal{M}_j, \qquad \mathcal{M}_j = \rho^{-1}\left(\left(\rho_j - \frac{1}{2^j\rho_j}, \rho_j + \frac{1}{2^j\rho_j}\right)\right).$$

Then, $\mathcal{M}_j \subset \mathbb{S}^1 \times [0, \infty)$ and moreover we have

$$\begin{aligned}
\mu(\mathcal{M}_j) &= \int_0^{2\pi} \int_{\rho_j - \frac{1}{2^j\rho_j} - \beta(\bar{\theta})}^{\rho_j + \frac{1}{2^j\rho_j} - \beta(\bar{\theta})} v\, dv\, d\bar{\theta} \\
&= \int_0^{2\pi} 2^{-j}\left(1 - \frac{\beta(\bar{\theta})}{\rho_j}\right) d\bar{\theta} \\
&\leq 2^{-j+1}\pi\left(1 + \frac{\|\beta\|_\infty}{2v_*}\right).
\end{aligned}$$

In particular, this implies $\mu(\mathcal{M}) < \infty$.

Fix some $(\bar{\theta}_0, v_0) \in D$ such that the corresponding complete forward orbit $(\bar{\theta}_n, v_n)$ is unbounded. Moreover, select $j_0 \in \mathbb{N}$ such that

$$\rho_{j_0} > 2\|\delta\|_\infty v_*^{-1} + 2\|\beta\|_\infty. \tag{3.4}$$

Since $\limsup_{n\to\infty} \rho(\bar{\theta}_n, v_n) = \infty$, there is $N \in \mathbb{N}$ so that $\rho(\bar{\theta}_N, v_N) > \rho_{j_0}$. Thus, $\rho(\bar{\theta}_N, v_N)$ lies in the interval $]\rho_{j_0}, \rho_{j_1}]$ for some $j_1 > j_0$. Since the orbit is unbounded, there must be a first index $K > N$ such that $\rho(\bar{\theta}_K, v_K) \notin]\rho_{j_0}, \rho_{j_1}]$. However, this can not happen without the orbit entering either $\mathcal{M}_{j_0}$ or $\mathcal{M}_{j_1}$. First, consider the case where we have

$\rho(\bar\theta_K, v_K) > \rho_{j_1} \geq \rho(\bar\theta_{K-1}, v_{K-1})$. Then, using (3.3) and (3.4) yields

$$\begin{aligned} v_{K-1} &= \rho(\bar\theta_{K-1}, v_{K-1}) - \beta(\bar\theta_{K-1}) \\ &\geq \rho(\bar\theta_K, v_K) - \frac{\delta(v_{K-1})}{v_{K-1}} - \beta(\bar\theta_{K-1}) \\ &> \rho_{j_1} - \|\delta\|_\infty v_*^{-1} - \|\beta\|_\infty \\ &> \tfrac{\rho_{j_1}}{2}. \end{aligned}$$

From this, it follows

$$|\rho(\bar\theta_K, v_K) - \rho_{j_1}| \leq |\rho(\bar\theta_K, v_K) - \rho(\bar\theta_{K-1}, v_{K-1})| \leq \frac{\delta(v_{K-1})}{v_{K-1}} < \frac{2\delta\left(\frac{\rho_{j_1}}{2}\right)}{\rho_{j_1}} < \frac{1}{2^{j_1}\rho_{j_1}}.$$

Thus $(\bar\theta_K, v_K) \in \mathcal{M}_{j_1}$. In the other case, $\rho(\bar\theta_{K-1}, v_{K-1}) > \rho_{j_0} \geq \rho(\bar\theta_K, v_K)$, we have

$$v_{K-1} > \rho_{j_0} - \beta(\bar\theta_{K-1}) > \tfrac{\rho_{j_0}}{2}.$$

Then, $(\bar\theta_K, v_K) \in \mathcal{M}_{j_0}$ follows analogously. □

Now, we are in position to prove the main result of this section.

Proof of Theorem 3.1. As the first step we perform the change of variables $\Phi : (\theta, v) \mapsto (\tau, r)$ defined by

$$\tau(\theta) = \gamma \int_0^\theta \frac{1}{L(s)^2}\, ds, \qquad r(\theta, v) = \gamma^{-\frac{1}{2}} L(\theta) v,$$

where

$$\gamma = 2\pi \left(\int_0^{2\pi} \frac{1}{L(s)^2}\, ds \right)^{-1} > 0.$$

The constant γ is chosen such that

$$\tau(\theta + 2\pi) = \tau(\theta) + 2\pi.$$

Since $\tau'(\theta) = \frac{\gamma}{L(\theta)^2} > 0$, the map $\tau \in C^3(\mathbb{S}^1, \mathbb{S}^1)$ is a diffeomorphism. Hence, also Φ is a diffeomorphism with regard to its image. The Taylor expansion of τ implies

$$\tau(\theta_1) = \tau(\theta_1 - 2\pi) = \tau(\theta) - \frac{\gamma}{L(\theta) v} + R_1(\theta, v),$$

where

$$\begin{aligned} R_1(\theta,v) &= \tfrac{\gamma R(\theta,v)}{L(\theta)^2} + \int_\theta^{\theta_1-2\pi} (\theta_1-2\pi-s)\tau''(s)\,ds \\ &= \tfrac{\gamma R(\theta,v)}{L(\theta)^2} + (\theta_1-2\pi-\theta)^2 \int_0^1 (1-\lambda)\tau''((1-\lambda)\theta+\lambda(\theta_1-2\pi))\,d\lambda \\ &= \tfrac{\gamma R(\theta,v)}{L(\theta)^2} + \left(R(\theta,v)-\tfrac{L(\theta)}{v}\right)^2 \int_0^1 (1-\lambda)\tau''\left(\theta+\lambda\left(R(\theta,v)-\tfrac{L(\theta)}{v}\right)\right)d\lambda. \end{aligned}$$

Then $R_1 \in \mathcal{F}_{\mathrm{u}}(2)$, since $R \in \mathcal{F}_{\mathrm{u}}(2)$ and

$$\lim_{v\to\infty} v^2(\theta_1-2\pi-\theta)^2 \int_0^1 (1-\lambda)\tau''((1-\lambda)\theta+\lambda(\theta_1-2\pi))\,d\lambda = L(\theta)^2\frac{\tau''(\theta)}{2}$$

holds uniformly in θ. Moreover, a direct calculation shows that also the derivatives have the same asymptotics, i.e. for $\boldsymbol{\nu}=(\nu_1,\nu_2)$ with $|\boldsymbol{\nu}|=1$ we have

$$\sup_{(\theta,v)\in M_{v*}} v^{\nu_2}|\partial^{\boldsymbol{\nu}} R_1(\theta,v)| < \infty, \qquad \lim_{v\to\infty} v^{\nu_2}|\partial^{\boldsymbol{\nu}} R_1(\theta,v)| = 0.$$

Similarly, the Taylor expansion of L yields

$$L(\theta_1) = L(\theta_1-2\pi) = L(\theta) + L'(\theta)\left(R(\theta,v)-\tfrac{L(\theta)}{v}\right) + I(\theta,v)$$

with $I \in \mathcal{F}_{\mathrm{u}}(2)$. Altogether, this yields

$$\begin{aligned} L(\theta_1)v_1 &= \left(L(\theta) - \tfrac{L(\theta)L'(\theta)}{v} + L'(\theta)R(\theta,v) + I(\theta,v)\right)\left(v + L'(\theta) + S(\theta,v)\right) \\ &= L(\theta)v + S_1(\theta,v), \end{aligned}$$

where $S_1 \in \mathcal{F}_{\mathrm{u}}(1)$. On $\Phi(M_{v_*})$ we define

$$R_2(\tau,r) = R_1\left(\Phi^{-1}(\tau,r)\right) \quad \text{and} \quad S_2(\tau,r) = \gamma^{-\frac{1}{2}} S_1\left(\Phi^{-1}(\tau,r)\right).$$

Then, the lift $g = \Phi\circ f\circ\Phi^{-1}$ of the transformed twist map $\bar{g}$ is given by

$$\begin{cases} \tau_1 = \tau + 2\pi - \frac{\sqrt{\gamma}}{r} + R_2(\tau,r), \\ r_1 = r + S_2(\tau,r), \end{cases}$$

with $R_2 \in \mathcal{F}_{\mathrm{u}}(2)$ and $S_2 \in \mathcal{F}_{\mathrm{u}}(1)$. Writing $(\theta(\tau), v(\tau, r)) = \Phi^{-1}(\tau, r)$, we get

$$\begin{aligned}\frac{\partial R_2}{\partial \tau} &= \frac{\partial R_1}{\partial \theta}(\theta, v)\theta' + \frac{\partial R_1}{\partial v}(\theta, v)\frac{\partial v}{\partial \tau} \\ &= \frac{\partial R_1}{\partial \theta}(\theta, v)\frac{L^2(\theta)}{\gamma} - \frac{\partial R_1}{\partial v}(\theta, v)\frac{vL(\theta)L'(\theta)}{\gamma}.\end{aligned}$$

In particular, it follows

$$\sup_{(\tau,r)\in\Phi(M_{v_*})}\left|\frac{\partial R_2}{\partial \tau}(\tau, r)\right| < \infty, \quad \text{and} \quad \lim_{r\to\infty}\frac{\partial R_2}{\partial \tau}(\tau, r) = 0,$$

for any $\tau \in \mathbb{R}$. Also, note that $\bar{g}$ is again one-to-one and

$$r_1^2 d\bar{\tau}_1 - r^2 d\bar{\tau} = d\hat{\eta}$$

holds for $\hat{\eta} = \eta \circ \Phi^{-1}$. Therefore, the new map is an exact symplectic twist map as well. Hence, Lemma 3.3 can be applied to $\bar{g}$ if a suitable adiabatic invariant $\rho(\bar{\tau}, r)$ can be found. In order to construct ρ, let

$$\alpha(\tau) = \lim_{r\to\infty} rS_2(\tau, r)$$

be the uniformly continuous and 2π-periodic limit. Due to the fact that $\bar{g}$ is exact symplectic, we know by (3.1) that

$$\int_0^{2\pi}\left(r_1^2\frac{\partial \tau_1}{\partial \tau} - r^2\right)d\tau = 0$$

holds for any fixed $r > r_* = \gamma^{-\frac{1}{2}}\max_{\theta\in\mathbb{R}} L(\theta)v_*$. Furthermore, we have

$$r_1(\tau, r)^2\frac{\partial \tau_1}{\partial \tau}(\tau, r) - r^2 = r^2\frac{\partial R_2}{\partial \tau}(\tau, r) + 2rS_2(\tau, r) + U(\tau, r),$$

where $U = S_2^2 + \frac{\partial R_2}{\partial \tau}(2rS_2 + S_2^2)$. In particular, U is bounded on M_{r_*} and $U(\tau, r) \to 0$, as $r \to \infty$. Since R_2 is 2π-periodic in τ, we get

$$\int_0^{2\pi}(2rS_2(\tau, r) + U(\tau, r))\,d\tau = 0.$$

Sending $r \to \infty$ yields $\int_0^{2\pi} \alpha(\tau)\, d\tau = 0$ by the dominated convergence theorem. Now, the sought adiabatic invariant $\rho(\tau, r)$ can be defined as $\rho(\tau, r) = r + \beta(\tau)$, where

$$\beta(\tau) = \gamma^{-\frac{1}{2}} \int_0^{\tau} \alpha(s)\, ds.$$

Note that $\beta \in C^1(\mathbb{S}^1)$, because α is purely periodic. With a similar argument as before, it follows

$$\begin{aligned}
\beta(\tau_1) &= \beta(\tau) + \gamma^{-\frac{1}{2}} \int_{\tau}^{\tau_1 - 2\pi} \alpha(s)\, ds \\
&= \beta(\tau) + \gamma^{-\frac{1}{2}} (\tau_1 - 2\pi - \tau) \int_0^1 \alpha((1-\lambda)\tau + \lambda(\tau_1 - 2\pi))\, d\lambda \\
&= \beta(\tau) + \left(\frac{R_2(\tau, r)}{\sqrt{\gamma}} - \frac{1}{r} \right) \int_0^1 \alpha((1-\lambda)\tau + \lambda(\tau_1 - 2\pi))\, d\lambda \\
&= \beta(\tau) - \frac{1}{r} \int_0^1 \alpha((1-\lambda)\tau + \lambda(\tau_1 - 2\pi))\, d\lambda + S_3(\tau, r),
\end{aligned}$$

with $S_3 \in \mathcal{F}_{\mathrm{u}}(2)$. From this, we obtain

$$\begin{aligned}
\rho(\tau_1, r_1) &= r + S_2(\tau, r) + \beta(\tau) - \frac{1}{r} \int_0^1 \alpha((1-\lambda)\tau + \lambda(\tau_1 - 2\pi))\, d\lambda + S_3(\tau, r) \\
&= \rho(\tau, r) + \frac{1}{r} \left(r S_2(\tau, r) - \int_0^1 \alpha((1-\lambda)\tau + \lambda(\tau_1 - 2\pi))\, d\lambda + r S_3(\tau, r) \right) \\
&= \rho(\tau, r) + \frac{S_4(\tau, r)}{r},
\end{aligned}$$

where $S_4(\tau, r) \to 0$ uniformly as $r \to \infty$. Thus, one can find a decreasing function $\delta : [r_*, \infty) \to \mathbb{R}$ with $|S_4(\tau, r)| \le \delta(r)$ on M_{r_*}, such that $\delta(r) \to 0$, as $r \to \infty$. Therefore, we have shown that all conditions of Lemma 3.3 are satisfied. The application yields a set $\mathcal{M}$ with $\mu(\mathcal{M}) < \infty$ such that every unbounded orbit of $\bar{g}$ enters $\mathcal{M}$. But since $\limsup_{n\to\infty} r_n = \infty$ holds if and only if $\limsup_{n\to\infty} v_n = \infty$, this means that every unbounded orbit of $\bar{f}$ enters $\mathcal{M}' = \Phi^{-1}(\mathcal{M})$. In particular, this implies

$$\mathcal{E} \subset \bigcup_{n=1}^{\infty} \bar{f}^{-n}(\mathcal{M}') \mod \mu.$$

Finally, due to the fact that $\mu\left(\Phi^{-1}(\mathcal{M})\right) = \mu(\mathcal{M}) < \infty$, we can apply Lemma 2.3 to deduce that the restricted map $\bar{f} : \mathcal{E} \to \mathcal{E}$ is recurrent and thus $\mu(\mathcal{E}) = 0$. □

3.2 A piecewise linear oscillator

As an application, we study the equation

$$\ddot{x} + n^2 x + h_1(x) = p(t), \tag{3.5}$$

where $p \in C(\mathbb{S}^1)$ and h_1 is the piecewise linear function given by

$$h_1(x) = \begin{cases} 1 & \text{if } x \geq 1, \\ x & \text{if } |x| < 1, \\ -1 & \text{if } x \leq -1. \end{cases}$$

In this section we prove the following result.

Theorem 3.4. *Let $p \in C(\mathbb{S}^1)$ satisfy the Lazer-Leach condition $\pi|\hat{p}_n| < 2$. If $x(t) = x(t; \tilde{x}, \tilde{v}, \tilde{t})$ denotes the solution of (3.5) with initial condition $x(\tilde{t}) = \tilde{x}$ and $\dot{x}(\tilde{t}) = \tilde{v}$, then $x(t; \tilde{x}, \tilde{v}, \tilde{t})$ is Poisson stable for almost every $(\tilde{x}, \tilde{v}, \tilde{t}) \in \mathbb{R}^3$.*

Proof. As indicated in the introduction, we start by constructing a twist map suitable for the application of Theorem 3.1, such that its orbits correspond to large amplitude solutions. In a second step, we then show that the recurrence of this twist map implies Poisson stability of almost every solution. In the context of 2π-periodic system, a more suitable definition of this property is as follows. There is a sequence of integers $\{\sigma_n\}_{n\in\mathbb{Z}}$ with $\sigma_n \to \pm\infty$ as $n \to \pm\infty$ such that

$$|x(t + 2\pi\sigma_n) - x(t)| + |\dot{x}(t + 2\pi\sigma_n) - \dot{x}(t)| \to 0 \quad \text{as } |n| \to \infty,$$

uniformly with respect to $t \in [0, 2\pi]$.

To this end, suppose $x(t)$ is a solution of (3.5) such that there are $\tau \in \mathbb{R}$ and $v > 0$ with $x(\tau) = 0$ and $\dot{x}(\tau) = v$. Then, x is also a solution of the integral equation

$$x(t) = v\frac{\sin n(t-\tau)}{n} + \int_\tau^t [p(s) - h_1(x(s))]\frac{\sin n(t-s)}{n}\, ds, \tag{3.6}$$

and the derivative is given by

$$\dot{x}(t) = v \cos n(t-\tau) + \int_\tau^t [p(s) - h_1(x(s))] \cos n(t-s)\, ds. \tag{3.7}$$

Given any time span $T > 0$, it follows from these formulas that $x(t)/v$ is arbitrarily close to $(\sin n(t-\tau))/n$ in $C^2([\tau, \tau+T])$ for large values of v. In particular, one can find $v_* > 0$ with the following property. If $v > v_*$, then $x(t)$ has $2n$ consecutive non-degenerate zeros

$$\tau = \tau_0 < \tau_1 < \ldots < \tau_{2n} = \tau'$$

and crosses the line $x = (-1)^i$ twice in each interval (τ_i, τ_{i+1}). We denote these crossings by $\tau_i^* < {}^*\tau_{i+1}$ and write

$$v_i = \dot{x}(\tau_i), \quad v_i^* = \dot{x}(\tau_i^*), \quad {}^*v_i = \dot{x}({}^*\tau_i),$$

for the corresponding velocities. For $i = 0, \ldots, 2n-1$, each of the three maps

$$(\tau_i, v_i) \to (\tau_i^*, v_i^*) \to ({}^*\tau_{i+1}, {}^*v_{i+1}) \to (\tau_{i+1}, v_{i+1})$$

can be described in terms of a forced linear oscillator. The arguments in Proposition 2.2 and Proposition 2.3 of [Ort96] show that these maps are of class C^1 and exact symplectic in the sense of (3.1). Since the induced function

$$\bar{P} : \mathbb{M}_{v_*} \to \mathbb{S}^1 \times [0, \infty), \qquad \bar{P}(\bar{\tau}, v) = (\bar{\tau}', v') = (\bar{\tau}_{2n}, v_{2n}),$$

is decomposable into $6n$ such maps, also $\bar{P} \in C^1(\mathbb{M}_{v_*})$ is exact symplectic. The map $\bar{P}$ is one-to-one due to the unique solvability of the corresponding initial value problem. Following the computations in Section 7 of [Ort99], it can be seen that for $p \in C(\mathbb{S}^1)$ the associated lift $P : M_{v_*} \to \mathbb{R} \times [0, \infty)$ has the form

$$\begin{cases} \tau' = \tau + 2\pi - (1/nv) L_1(\tau) + R_1(\tau, v), \\ v' = v + L_2(\tau) + R_2(\tau, v), \end{cases} \tag{3.8}$$

where

$$L_1(\tau) = 2\pi \Im(e^{in\tau} \hat{p}_n) + 4, \qquad L_2(\tau) = 2\pi \Re(e^{in\tau} \hat{p}_n),$$

and $R, S \in C^1(\mathbb{M}_{v_*})$, $R_1 \in \mathcal{F}^1(2)$, $R_2 \in \mathcal{F}^0(1)$. Throughout the computations in [Ort99], one can in fact replace the space $\mathcal{F}^k(r)$ by $\mathcal{F}^k_{\mathrm{u}}(r)$ with some obvious adjustments. This leads to the conclusion that $R_1 \in \mathcal{F}^1_{\mathrm{u}}(2)$ and $R_2 \in \mathcal{F}_{\mathrm{u}}(1)$. The Poincaré map of the discontinuous oscillator discussed in the next section has an expansion of the same form. This is shown in full detail in Appendix B. Finally, note that for $L_1 \in C^2(\mathbb{S}^1)$ we have $L_1' = nL_2$ and also $L_1 > 0$ is guaranteed by the Lazer-Leach condition $\pi|\hat{p}_n| < 2$. In total, $\bar{P}$ satisfies all assumptions of Theorem 3.1 and therefore the escaping set

$$\mathcal{E}_P = \{(\bar{\tau}, v) \in \mathbb{M}_{v_*} : (\bar{\tau}_j', v_j') = \bar{P}^j(\bar{\tau}, v) \in \mathbb{M}_{v_*} \forall j \in \mathbb{N} \text{ and } \lim_{j\to\infty} v_j' = \infty\}$$

has measure zero.

Going back to the question of Poisson stability, it is sufficient to consider initial time $\tilde{t} = 0$. We denote by $x(t) = x(t; \tilde{x}, \tilde{v})$ the solution of (3.5) satisfying the initial condition $x(0) = \tilde{x}$ and $\dot{x}(0) = \tilde{v}$. Thus, the time-2π map of (3.5) is given by

$$\Pi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (\tilde{x}, \tilde{v}) \mapsto (x(2\pi; \tilde{x}, \tilde{v}), \dot{x}(2\pi; \tilde{x}, \tilde{v})).$$

It can be shown that Π preserves the two-dimensional Lebesgue measure λ^2. We now prove that it is also recurrent. To this end, consider the solution $x(t) = x(t; \tilde{x}, \tilde{v})$ for some $(\tilde{x}, \tilde{v}) \in \mathbb{R}^2$. The corresponding solution of the unperturbed linear system $\ddot{z} + n^2 z = 0$ satisfying the same initial condition $z(0) = \tilde{x}$, $\dot{z}(0) = \tilde{v}$ is given by $z(t) = \hat{r}\frac{\sin n(t-\hat{\tau})}{n}$ for some $\hat{\tau} \in \mathbb{R}$ and $\hat{r} = \sqrt{n^2\tilde{x}^2 + \tilde{v}^2}$. Furthermore, $x(t)$ also solves the integral equation

$$x(t) = \hat{r}\frac{\sin n(t-\hat{\tau})}{n} + \int_0^t [p(s) - h_1(x(s))]\frac{\sin n(t-s)}{n}\, ds. \tag{3.9}$$

Again, $x(t)/\hat{r}$ is close to $(\sin n(t-\hat{\tau}))/n$ in $C^2([0, 4\pi])$ for large values of $\hat{r}$. Let $r(t) = \sqrt{n^2 x(t)^2 + \dot{x}(t)^2}$. Then, one can infer from (3.9) that there is a constant $C_p > 0$ (depending on $\|p\|_\infty$) such that

$$|r(t) - r(0)| \le C_p, \qquad \text{for } t \in [0, 4\pi].$$

Thus, if $\hat{r} = r(0) > v_* + C_p$, then $r(t) > v_*$ holds for all $t \in [0, 4\pi]$. In particular, there is a unique first $\tau \ge 0$ such that $x(\tau) = 0$ and

$v = \dot{x}(\tau) > v_*$. Let S be the induced map

$$S : \mathbb{R}^2 \setminus E \to M_{v_*}, \qquad (\tilde{x}, \tilde{v}) \mapsto (\tau, v),$$

where

$$E = \{(a, b) \in \mathbb{R}^2 : \sqrt{n^2a^2 + b^2} \leq v_* + C_p\}.$$

S is a diffeomorphism with respect to its image and the inverse map can be obtained by plugging $t = 0$ into (3.6) and (3.7). For a given solution $x(t) = x(t; \tilde{x}, \tilde{v})$ define $r_n = r(2\pi j)$ for $j \in \mathbb{N}$. Then, the escaping set E_Π of the map Π is given by

$$\mathcal{E}_\Pi = \{(\tilde{x}, \tilde{v}) \in \mathbb{R}^2 : \lim_{j\to\infty} r_j = \infty\}.$$

It can be shown by the same argument as in Section 3.1 that the restricted map $\Pi : \mathbb{R}^2 \setminus \mathcal{E}_\Pi \to \mathbb{R}^2 \setminus \mathcal{E}_\Pi$ is recurrent. Thus it remains to show that $\lambda^2(\mathcal{E}_\Pi) = 0$. Suppose $(\tilde{x}, \tilde{v}) \in \mathcal{E}_\Pi$. Then, there is $m \in \mathbb{N}$ such that $\Pi^j(\tilde{x}, \tilde{v}) \in \mathbb{R}^2 \setminus E$ for $j \geq m$. Thus, clearly we have $\iota\left(S\left(\Pi^m(\tilde{x}, \tilde{v})\right)\right) \in \mathcal{E}_P$, where $\iota : M_{v_*} \to \mathbb{M}_{v_*}$ denotes the covering map $\iota(\tau, v) = (\bar{\tau}, v)$. This leads to the inclusion

$$\mathcal{E}_\Pi \subset \bigcup_{m=0}^{\infty} \Pi^{-m}\left(S^{-1}(\iota^{-1}(\mathcal{E}_P))\right),$$

which in turn implies $\lambda^2(\mathcal{E}_\Pi) = 0$. In summary, we have shown that Π is recurrent. Due to the symmetry of the problem, the same is true for the inverse map Π^{-1}. Now, the Poisson stability of almost every solution $x(t; \tilde{x}, \tilde{v})$ follows from the fact that the corresponding flow is Lipschitz-continuous on $\mathbb{R}^2$.

□

3.3 An oscillator with jump discontinuity

Consider the piecewise linear oscillator

$$\ddot{x} + n^2 x + \operatorname{sign}(x) = p(t), \tag{3.10}$$

where $p \in C(\mathbb{S}^1)$. Let $N = \{t \in \mathbb{R} : |p(t)| = 1\}$ and suppose the set ∂N of its boundary points is countable. The goal of this section is to verify part (a) of Remark 1.10, that is to show that almost every solution of (3.10) is Poisson stable. But first we have to give the following

Definition 3.5. We say a function $x \in C^1(I)$ with $I = (\alpha, \beta) \subset \mathbb{R}$ is a *solution of* (3.10) if it satisfies the following conditions:

(i) $\dot{x}(t) \neq 0$ if $t \in Z$, where $Z = \{t \in \mathbb{R} : x(t) = 0\}$,

(ii) $x \in C^2(I \setminus Z)$ and x satisfies (3.10) on $I \setminus Z$.

Moreover, we say a solution is *global* if $I = \mathbb{R}$.

Between two consecutive zeros, any such solution must coincide with the solution of the corresponding linear problem. Thus given $(\tau, v) \in \mathbb{R}^2$, let $y_\pm(t) = y_\pm(t; 0, v, \tau)$ be the unique solution of

$$\ddot{y} + n^2 y \pm 1 = p(t), \quad y(\tau) = 0, \quad \dot{y}(\tau) = v.$$

The functions $(t, \tau, v) \mapsto y_\pm(t; 0, v, \tau), \dot{y}_\pm(t; 0, v, \tau)$ are both in $C^1(\mathbb{R}^3)$. Moreover, note that $y_\pm(t)$ also solves the integral equation

$$y(t) = \frac{v}{n} \sin(n(t-\tau)) + \int_\tau^t (p(s) \mp 1) \frac{\sin(n(t-s))}{n} \, ds. \tag{3.11}$$

In the following, we discuss properties of the solution $y_+(t)$. Its counterpart $y_-(t)$ can be dealt with completely analogously. It can be shown that all solutions of the linear equations are either oscillatory or of constant sign [Ort96]. In particular, if $v \neq 0$ there is a unique time $\hat{\tau} > \tau$ and a corresponding velocity $\hat{v}$ such that

$$y_+(\hat{\tau}) = 0, \quad y_+(t) \neq 0 \quad \forall t \in (\tau, \hat{\tau}), \quad \dot{y}_+(\hat{\tau}) = \hat{v}. \tag{3.12}$$

Therefore, we can define the map

$$S_+ : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \times (\mathbb{R}_- \cup \{0\}), \qquad S_+(\tau, v) = (\hat{\tau}, \hat{v}),$$

where $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$. This mapping is well-defined, one-to-one and satisfies

$$S_+(\tau + 2\pi, v) = S_+(\tau, v) + (2\pi, 0) \qquad \forall \tau \in \mathbb{R}.$$

Let $\Sigma_+ = \{(\tau, v) \in \mathbb{R} \times \mathbb{R}_+ : \hat{v} = 0\}$. The map S_+ can have discontinuities on Σ_+. On the open set $(\mathbb{R} \times \mathbb{R}_+) \setminus \Sigma_+$ however, the implicit function theorem can be applied to the equation $y_+(\hat{\tau}; 0, v, \tau) = 0$. This way one obtains a function $\hat{\tau} = \hat{\tau}(\tau, v)$ in $C^1\left((\mathbb{R} \times \mathbb{R}_+) \setminus \Sigma_+\right)$. Also $\hat{v}$ given by $\hat{v}(\tau, v) = \dot{y}_+(\hat{\tau}; 0, v, \tau)$ is in that class. Since the same argument can be applied to the inverse, this shows that S_+ restricted to $(\mathbb{R} \times \mathbb{R}_+) \setminus \Sigma_+$ is a diffeomorphism with respect to its image. Moreover, S_+ is symplectic in the sense of (3.1) on this domain (see Proposition 2.2 in [Ort96]). Next we will show that Σ_+ has measure zero. To this end, define

$$\mathcal{N}_+ = \{\hat{\tau} \in \mathbb{R} : (\hat{\tau}, 0) \in S_+(\Sigma_+)\}.$$

Given $\hat{\tau}_* \in \mathcal{N}_+$, let $(\tau_*, v_*) = S_+^{-1}(\hat{\tau}_*, 0)$. The equation $y_+(\tau; 0, 0, \hat{\tau}) = 0$ can be solved implicitly for τ at $\hat{\tau} = \hat{\tau}_*$. This yields an open interval $I_{\hat{\tau}_*}$ containing $\hat{\tau}_*$ and a function $\tau = \tau_{\hat{\tau}_*}$ of class $C^1(I_{\hat{\tau}_*})$ such that $\tau(\hat{\tau}_*) = \tau_*$ and

$$y_+(\tau(\hat{\tau}); 0, 0, \hat{\tau}) = 0, \qquad \text{for } \hat{\tau} \in I_{\hat{\tau}_*}.$$

Hence, the map $T = T_{\hat{\tau}_*}$ defined by

$$T : I_{\hat{\tau}_*} \to \mathbb{R}^2, \quad \hat{\tau} \mapsto (\tau(\hat{\tau}), \dot{y}_+(\tau(\hat{\tau}); 0, 0, \hat{\tau})),$$

is also C^1 and $\lambda^2(T(I_{\hat{\tau}_*})) = 0$. We also have $T_{\hat{\tau}_*}(\hat{\tau}_*) = S_+^{-1}(\hat{\tau}_*, 0)$ and therefore $\Sigma_+ = S_+^{-1}(\mathcal{N}_+ \times \{0\}) \subset \bigcup_{\hat{\tau}_* \in \mathcal{N}_+} T_{\hat{\tau}_*}(I_{\hat{\tau}_*})$. If one can extract a countable sub-covering, then clearly $\lambda^2(\Sigma_+) = 0$ follows.

First suppose $\hat{\tau}_* \in \mathcal{N}_+ \cap N_+$, where $N_+ = \{t \in \mathbb{R} : p(t) = 1\}$. If $\hat{\tau}_*$ would be in the interior of N_+, then $y_+(t) = 0$ needs to hold in a neighborhood of $\hat{\tau}_*$. But this contradicts the minimality condition in (3.12). Thus $\hat{\tau}_* \in \mathcal{N}_+ \cap \partial N_+$. By assumption, this set is countable and hence $\lambda^2(S_+^{-1}((\mathcal{N}_+ \cap N_+) \times \{0\})) = 0$.

Now, assume $\hat{\tau}_* \in \mathcal{N}_+ \setminus N_+$, that is $p(\hat{\tau}_*) \neq 1$. Then $y_+(t; 0, 0, \hat{\tau}_*)$ has a strict local extremum in $\hat{\tau}_*$. Due to the continuous dependence on initial condition, one can in fact find $\varepsilon > \delta > 0$ such that $\hat{\tau} \in (\hat{\tau}_* - \delta, \hat{\tau}_* + \delta)$ implies $y_+(t; 0, 0, \hat{\tau}) \neq 0$ for $t \in [\hat{\tau}_* - \varepsilon, \hat{\tau}_* + \varepsilon] \setminus \{\hat{\tau}\}$. Moreover, since $v_* > 0$, one can find a neighborhood U of (τ_*, v_*) such that $y_+(t; 0, v, \tau) \neq 0$ if $t \in (\tau, \hat{\tau}_* - \varepsilon)$ for all $(\tau, v) \in U$. By decreasing $\delta > 0$ if necessary, one can assume that $(\hat{\tau}_* - \delta, \hat{\tau}_* + \delta) \subset I_{\hat{\tau}_*}$ and $T((\hat{\tau}_* - \delta, \hat{\tau}_* + \delta)) \subset U$. Then $T_{\hat{\tau}_*}(\cdot) = S_+^{-1}(\cdot, 0)$ on $(\hat{\tau}_* - \delta, \hat{\tau}_* + \delta)$. In particular, it follows that $\mathcal{N}_+ \setminus N_+$ is open and that $S_+^{-1}(\cdot, 0) \in C^1(\mathcal{N}_+ \setminus N_+, \mathbb{R} \times \mathbb{R}_+)$. Thus $\lambda^2(S_+^{-1}((\mathcal{N}_+ \setminus N_+) \times \{0\})) = 0$.

In summary, we have shown that Σ_+ has measure zero. Using $y_-(t)$ instead of $y_+(t)$ in (3.12) one can define the successor map

$$S_- : \mathbb{R} \times \mathbb{R}_- \to \mathbb{R} \times (\mathbb{R}_+ \cup \{0\}), \qquad S_-(\tau, v) = (\hat{\tau}, \hat{v}),$$

and the set $\Sigma_- = S_-^{-1}(\mathbb{R} \times \{0\})$. Again, S_- restricted to $(\mathbb{R} \times \mathbb{R}_-) \setminus \Sigma_-$ is a symplectic diffeomorphism with respect to its image and Σ_- has measure zero. Now, define $\Sigma_\pm^1 = \Sigma_\pm$ and $\Sigma_\pm^r = S_\pm^{-1}(\Sigma_\mp^{r-1})$ for $r \geq 2$. Then $\Sigma_\pm^r$ consists of those points $(\tau_0, v_0) \in \mathbb{R} \times \mathbb{R}_\pm$ such that the corresponding orbit (τ_j, v_j) satisfies $v_j \neq 0$ for $j = 0, \ldots, r-1$ and $v_r = 0$. Finally, define

$$\Sigma = \bigcup_{r \in \mathbb{N}} (\Sigma_+^r \cup \Sigma_-^r) \cup (\mathbb{R} \times \{0\}),$$

then Σ has measure zero and every $(\tau_0, v_0) \in \mathbb{R}^2 \setminus \Sigma$ leads to a complete forward orbit $(\tau_j, v_j)_{j \in \mathbb{N}_0}$ that never touches the line $v = 0$. In particular, the map $P = (S_- \circ S_+)^n$ given by

$$P : (\mathbb{R} \times \mathbb{R}_+) \setminus \Sigma \to \mathbb{R} \times [0, \infty), \qquad P(\tau_0, v_0) = (\tau_{2n}, v_{2n}), \tag{3.13}$$

is well-defined. In Lemma B.2 of the appendix, we show that P has an expansion of the form (3.8) and moreover satisfies all conditions necessary for the application of Theorem 3.1. Hence, the corresponding twist map $\bar{P}$ is recurrent. Since $\bar{P}$ is recurrent for almost all $(\bar{\tau}, v) \in \mathbb{S}^1 \times \mathbb{R}_+$, we have $\lim_{j \to \infty} \tau_j = \infty$ for almost all orbits (τ_j, v_j) starting in $\mathbb{R}^2 \setminus \Sigma$. This leads to the following observation.

Lemma 3.6. *For almost every $(\tilde{x}, \tilde{v}) \in \mathbb{R}^2$ there exists a global solution $x(t) = x(t; \tilde{x}, \tilde{v})$ of* (3.10) *with initial condition $x(0) = \tilde{x}$, $\dot{x}(0) = \tilde{v}$.*

Proof. Let $\Omega_r \subset (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$ be the set of initial condition leading to solutions $x(t) = x(t; \tilde{x}, \tilde{v})$ such that $x(t) \neq 0$ for $t > 0$. For $(\tilde{x}, \tilde{v}) \in \Omega_r$ we have $x \in C^2([0, \infty))$, since then $x(t)$ solves the linear problem. Similar to $S_\pm$ we define

$$\tilde{S}_\pm : (\mathbb{R}_\pm \times \mathbb{R}) \setminus \Omega_r \to \mathbb{R} \times (\mathbb{R}_\mp \cup \{0\}), \qquad \tilde{S}_\pm(\tilde{x}, \tilde{v}) = (\hat{\tau}, \hat{v}),$$

where again $\hat{\tau} > 0$ denotes the first zero of $x(t; \tilde{x}, \tilde{v})$ to the right and $\hat{v}$ is the corresponding velocity. Let $\tilde{\Sigma}_\pm = \{(\tilde{x}, \tilde{v}) \in \mathbb{R}_\pm \times \mathbb{R} : \hat{v} = 0\}$. These sets have measure zero since we have $\tilde{\Sigma}_\pm \subset \gamma_\pm(\mathbb{R})$ for the C^1-maps $\gamma_\pm(\hat{\tau}) = (y_\pm(0;0,0,\hat{\tau}), \dot{y}_\pm(0;0,0,\hat{\tau}))$. Moreover, $\tilde{S}_\pm \in C^1\big((\mathbb{R}_\pm \times \mathbb{R}) \setminus (\Omega_r \cup \tilde{\Sigma}_\pm)\big)$ are diffeomorphisms with respect to their images and thus also the sets $\tilde{S}_\pm^{-1}(\Sigma)$ have measure zero. Therefore, $x \in C^1([0,\infty))$ is a solution in the sense of Definition (3.5) for almost all $(\tilde{x}, \tilde{v})$, since almost every $(\hat{\tau}_0, \hat{v}_0) \in \mathbb{R}^2 \setminus \Sigma$ leads to a complete forward orbit $(\hat{\tau}_j, \hat{v}_j)_{j \in \mathbb{N}_0}$ such that $\hat{\tau}_j \to \infty$, $x(\hat{\tau}_j) = 0$ and $\dot{x}(\hat{\tau}_j) = \hat{v}_j \neq 0$. Now, the assertion follows by repeating the whole argument for the set Ω_l of initial condition producing solutions such that $x(t) \neq 0$ for $t < 0$. $\square$

We have shown that there is a set Γ of measure zero such that all initial condition in $\mathbb{R}^2 \setminus \Gamma$ lead to global solutions of (3.10). In particular, the time-2π map $\Pi : \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}^2 \setminus \Gamma$ is well-defined. We will demonstrate that this map is also measure-preserving. To this end, we keep the notation introduced in the proof of Lemma 3.6. Given $(\tilde{x}, \tilde{v}) \in (\mathbb{R}_\pm \times \mathbb{R}) \setminus (\Omega_r \cup \Gamma)$ let $\tilde{S}_\pm(\tilde{x}, \tilde{v}) = (\hat{\tau}_0, \hat{v}_0)$, then there is an infinite series of non-degenerate consecutive zeros $(\hat{\tau}_j)_{j \in \mathbb{N}_0}$ of $x(t; \tilde{x}, \tilde{v})$. Moreover, let $\hat{\tau}_0 = \infty$ if $(\tilde{x}, \tilde{v}) \in \Omega_r$. We define the sets

$$A_j^\pm = \big\{(\tilde{x}, \tilde{v}) \in (\mathbb{R}_\pm \times \mathbb{R}) \setminus \Gamma : j = \min\{i \in \mathbb{N}_0 : \hat{\tau}_i \geq 2\pi\}\big\},$$

where the index j counts the number of zeros in the interval $[0, 2\pi]$. Clearly, $\mathbb{R}^2 \setminus \Gamma = \bigcup_{j \in \mathbb{N}_0} (A_j^+ \cup A_j^-)$. Moreover, the sets Ω_r and $A_j^\pm$ are measurable. For Ω_r this follows from the fact that $(\mathbb{R}_\pm \times \mathbb{R}) \setminus (\Omega_r \cup \tilde{\Sigma}_\pm)$ is open, so that Ω_r differs from a Borel set only by a set of measure zero. In the case of $A_j^\pm$ with $j \in \mathbb{N}$ consider the maps $g_j(\tilde{x}, \tilde{v}) = (\hat{\tau}_j, \hat{v}_j)$. These maps are well-defined and continuous almost everywhere on $\mathbb{R}^2 \setminus \Omega_r$ and hence measurable. Thus also

$$A_j^\pm = (\mathbb{R}_\pm \times \mathbb{R}) \cap g_j^{-1}([2\pi, \infty) \times \mathbb{R}) \setminus g_{j-1}^{-1}([2\pi, \infty) \times \mathbb{R})$$

is measurable. The argument for $j = 0$ is similar. On $A_0^\pm$ the map Π is just the time-2π map of a linear oscillator and thus it preserves the 2-dimensional Lebesgue measure λ^2. For $A_j^\pm$ with $j \in \mathbb{N}$ we again consider the maps g_j. Without loss of generality let $(\tilde{x}, \tilde{v}) \in A_j^+$, where $j = 2k$ with $k \in \mathbb{N}$. Then

$$(\hat{\tau}_j, \hat{v}_j) = (S_+ \circ S_-)^k(\tilde{S}_+(\tilde{x}, \tilde{v})).$$

$\tilde{S}_+$ restricted to $(\mathbb{R}_+ \times \mathbb{R}) \setminus (\Omega_r \cup \tilde{\Sigma}_+)$ is a diffeomorphism with respect to its image and the inverse is given by $\tilde{S}_+^{-1}(\hat{\tau}, \hat{v}) = (y_+(0; 0, \hat{v}, \hat{\tau}), \dot{y}_+(0; 0, \hat{v}, \hat{\tau}))$. Considering formula (3.11), one easily derives $\det D\tilde{S}_+^{-1}(\hat{\tau}, \hat{v}) = -\hat{v}$ for the Jacobian determinant. This implies that we have $\lambda^2(B) = \mu(\tilde{S}_+(B))$ for any measurable set $B \subset (\mathbb{R}_+ \times \mathbb{R}) \setminus (\Omega_r \cup \tilde{\Sigma}_+)$, where $\mu = v\, d\tau \otimes dv$. Furthermore, the maps $S_\pm$ are exact symplectic in the sense of (3.1) on the relevant domain and therefore preserve the measure μ. It follows that the sets

$$B_j^+ = \{(\tilde{x}, \tilde{v}) \in A_j^+ : \hat{\tau}_j = 2\pi\}$$

have measure zero. Finally, note that for $(\tilde{x}, \tilde{v}) \in A_j^+ \setminus B_j^+$ we have

$$\Pi(\tilde{x}, \tilde{v}) = \tilde{S}_+^{-1}(\hat{\tau}_j - 2\pi, \hat{v}_j).$$

In view of the argument above, the latter identity shows that Π preserves the 2-dimensional Lebesgue measure also on $A_j^\pm$ with $j \geq 1$ and hence on all of $\mathbb{R}^2 \setminus \Gamma$. Analogously to the continuous case, it now follows that Π is recurrent and that almost every solution $x(t; \tilde{x}, \tilde{v})$ is Poisson stable.

Chapter 4

The almost periodic case

Now, we will consider twist maps of the form

$$(t_1, r_1) = (t + f(t, r), r + g(t, r)), \tag{4.1}$$

where f and g are almost periodic in t. A definition of the latter notion and some associated properties are presented in the first section. Then, in the following section consequences for symplectic maps of the form (4.1) are discussed and a general framework is developed. This is applied to the Fermi-Ulam ping-pong in Section 4.3 and to a superlinear oscillator in Section 4.4. This chapter is based on the two publications [Sch22] and [Sch19] by the author, including improvements to the latter one.

4.1 Almost periodic functions and their representation

4.1.1 Compact topological groups and minimal flows

Let Ω be a commutative topological group, which is metrizable, compact and connected. We will consider the group operation to be additive. Moreover, suppose there is a continuous homomorphism $\psi : \mathbb{R} \to \Omega$, such that the image $\psi(\mathbb{R})$ is dense in Ω. This function ψ induces a canonical flow on Ω, namely

$$\Omega \times \mathbb{R} \to \Omega, \quad \omega \cdot t = \omega + \psi(t).$$

This flow is minimal, since

$$\overline{\omega \cdot \mathbb{R}} = \overline{\omega + \psi(\mathbb{R})} = \omega + \overline{\psi(\mathbb{R})} = \Omega$$

holds for every $\omega \in \Omega$. Let us also note that in general ψ can be nontrivial and periodic, but this happens if and only if $\Omega \cong \mathbb{S}^1$ [OT06].

Now consider the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and a continuous homomorphism $\varphi : \Omega \to \mathbb{S}^1$. Such functions φ are called *characters* and together with the point wise product they form a group, the so called dual group Ω^*. Its trivial element is the constant map with value 1. It is a well-known fact that nontrivial characters exist, whenever Ω is nontrivial [Pon66]. Also non-compact groups admit a dual group. Crucial to us will be the fact that

$$\mathbb{R}^* = \{t \mapsto e^{i\alpha t} : \alpha \in \mathbb{R}\}.$$

Now, for a nontrivial character $\varphi \in \Omega^*$ we define

$$\Sigma = \ker \varphi = \{\omega \in \Omega : \varphi(\omega) = 1\}.$$

Then Σ is a compact subgroup of Ω. If in addition $\Omega \not\cong \mathbb{S}^1$, it can be shown that Σ is perfect [OT06]. This subgroup will act as a global cross section to the flow on Ω. Concerning this, note that since $\varphi \circ \psi$ describes a nontrivial character of $\mathbb{R}$, there is a unique $\alpha \neq 0$ such that

$$\varphi(\psi(t)) = e^{i\alpha t}$$

for all $t \in \mathbb{R}$. Therefore, the minimal period of this function,

$$S = \frac{2\pi}{|\alpha|},$$

can be seen as a returning time on Σ in the following sense. If we denote by $\tau(\omega)$ the unique number in $[0, S)$ such that $\varphi(\omega) = e^{i\alpha\tau(\omega)}$, then one has

$$\varphi(\omega \cdot t) = \varphi(\omega + \psi(t)) = \varphi(\omega)\varphi(\psi(t)) = e^{i\alpha\tau(\omega)}e^{i\alpha t}$$

and thus

$$\omega \cdot t \in \Sigma \Leftrightarrow t \in -\tau(\omega) + S\mathbb{Z}.$$

Also τ as defined above is a function $\tau : \Omega \to [0, S)$ that is continuous where $\tau(\omega) \neq 0$, i.e. on $\Omega \setminus \Sigma$. From this we can derive that the restricted flow

$$\Phi : \Sigma \times [0, S) \to \Omega, \quad \Phi(\sigma, t) = \sigma \cdot t,$$

is a continuous bijection. Like $\tau(\omega)$, its inverse

$$\Phi^{-1}(\omega) = (\omega \cdot (-\tau(\omega)), \tau(\omega))$$

is continuous only on $\Omega \setminus \Sigma$. Therefore, Φ describes a homeomorphism from $\Sigma \times (0, S)$ to $\Omega \setminus \Sigma$.

Before giving some examples, we introduce the notion of morphism $(\Omega_1, \psi_1) \to (\Omega_2, \psi_2)$ between two such tuples. Those are continuous group homomorphisms $\Psi : \Omega_1 \to \Omega_2$ such that $\Psi \circ \psi_1 = \psi_2$.

Lemma 4.1. *There exists a morphism $(\Omega_1, \psi_1) \to (\Omega_2, \psi_2)$ if and only if $\psi_2(t_n) \to 0$ holds for every sequence $(t_n)_{n\in\mathbb{N}}$ such that $\psi_1(t_n) \to 0$.*

Proof. The condition is obviously necessary, since any such morphism is continuous. To show its sufficiency, define Ψ on $\psi_1(\mathbb{R})$ by setting $\Psi(\psi_1(t)) = \psi_2(t)$. By this relation, Ψ is well-defined even if ψ_1 is periodic, since for any two sequences $(t_n), (s_n)$ in $\mathbb{R}$ we have

$$\psi_1(t_n) - \psi_1(s_n) \to 0 \quad \text{implies} \quad \psi_2(t_n) - \psi_2(s_n) \to 0.$$

Due to this relation, Ψ can be extended to a continuous map on Ω_1, which is also a group homomorphism. □

Example 4.2. One important example for such a group Ω is the N-Torus $\mathbb{T}^N$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We will denote classes in $\mathbb{T}^N$ by $\bar{\theta} = \theta + \mathbb{Z}^N$. Then, the image of the homomorphism

$$\psi(t) = (\overline{\nu_1 t}, \dots, \overline{\nu_N t})$$

winds densely around the N-torus $\mathbb{T}^N$, whenever the frequency vector $\nu = (\nu_1, \dots, \nu_N) \in \mathbb{R}^N$ is nonresonant, i.e. rationally independent. It is easy to verify that the dual group of $\mathbb{T}^N$ is given by

$$(\mathbb{T}^N)^* = \{(\bar{\theta}_1, \dots, \bar{\theta}_N) \mapsto e^{2\pi i(k_1\theta_1 + \dots + k_N\theta_N)} : k \in \mathbb{Z}^N\}.$$

Therefore, one possible choice for the cross section would be

$$\Sigma = \{(\bar\theta_1, \ldots, \bar\theta_N) \in \mathbb{T}^N : e^{2\pi i\theta_1} = 1\} = \{0\} \times \mathbb{T}^{N-1},$$

so $\varphi(\bar\theta_1, \ldots, \bar\theta_N) = e^{2\pi i\theta_1}$. In this case, consecutive intersections of the flow and Σ would be separated by an interval of the length $1/\nu_1$.

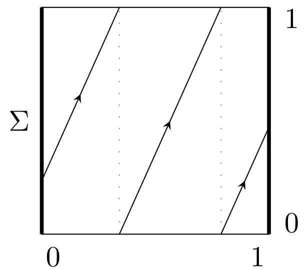

Figure 4.1: On the 2-torus $\mathbb{T}^2$, intersections of $\Sigma = \{0\} \times \mathbb{T}$ and the orbit of $\psi(t)$ are separated by time intervals of length $S = 1/\nu_1$.

Example 4.3. Lets consider another important topological group. Let $\Omega = \mathcal{S}_{\boldsymbol{p}}$ be the $\boldsymbol{p}$-adic solenoid, where $\boldsymbol{p} = (p_i)_{i\in\mathbb{N}}$ is a sequence of prime numbers. $\mathcal{S}_{\boldsymbol{p}}$ is defined as the projective limit of the inverse limit system

$$\mathcal{S}_{\boldsymbol{p}} : \mathbb{S}^1 \xleftarrow{z^{p_1}} \mathbb{S}^1 \xleftarrow{z^{p_2}} \mathbb{S}^1 \xleftarrow{z^{p_3}} \cdots,$$

where $\mathbb{S}^1 \xleftarrow{z^{p_i}} \mathbb{S}^1$ denotes the mapping $z \mapsto z^{p_i}$ of the circle $\mathbb{S}^1$ into itself. A point $z \in \mathcal{S}_{\boldsymbol{p}}$ has the form $z = (z_0, z_1, z_2, \ldots)$, where $z_{k-1} = z_k^{p_k}$ for $k \in \mathbb{N}$. Moreover, if we take the coordinatewise multiplication as the action, $\mathcal{S}_{\boldsymbol{p}}$ becomes a compact abelian group with neutral element $(1, 1, \ldots)$ [HR79, Theorem 10.13]. It can be endowed with the metric

$$d(z, w) = \sum_{k=0}^{\infty} \frac{d_{\mathbb{S}^1}(z_k, w_k)}{q_k},$$

where $d_{\mathbb{S}^1}$ denotes the canonical metric on $\mathbb{S}^1$ and $q_0 = 1$, $q_k = p_1 \cdots p_k$ for $k \in \mathbb{N}$. For $\lambda > 0$, the map

$$\psi(t) = (e^{2\pi i\lambda t/q_0}, e^{2\pi i\lambda t/q_1}, e^{2\pi i\lambda t/q_2}, \ldots)$$

provides a minimal flow on $\mathcal{S}_{\boldsymbol{p}}$. A cross section with return time $S = 1/\lambda$ is then given by

$$\Sigma = \{z \in \mathcal{S}_{\boldsymbol{p}} : z_0 = 1\}.$$

Geometrically, $\mathcal{S}_{\boldsymbol{p}}$ can be described as the intersections of a sequence of solid tori $T_1 \supset T_2 \supset \ldots$ in $\mathbb{R}^3$, where T_{k+1} is wrapped p_k times longitudinally inside T_k without self intersecting. See [BM95] for a nice description of the construction in the case of the dyadic solenoid $\mathcal{S}_{\boldsymbol{2}}$.

4.1.2 Almost periodic functions

The notion of almost periodic functions was introduced by H. Bohr as a generalization of strictly periodic functions [Boh25]. A function $u \in C(\mathbb{R})$ is called *(Bohr) almost periodic*, if for any $\varepsilon > 0$ there is a relatively dense set of ε-almost-periods of this function. By this we mean, that for any $\varepsilon > 0$ there exists $L = L(\varepsilon)$ such that any interval of length L contains at least one number T such that

$$|u(t+T) - u(t)| < \varepsilon \;\; \forall t \in \mathbb{R}.$$

Later, Bochner [Boc27] gave an alternative but equivalent definition of this property: For a continuous function u, denote by $u_\tau(t)$ the translated function $u(t+\tau)$. Then u is (Bohr) almost periodic if and only if every sequence $(u_{\tau_n})_{n\in\mathbb{N}}$ of translations of u has a subsequence that converges uniformly.

There are several other characterizations of almost periodicity, as well as generalizations due to Stepanov [Ste26], Weyl [Wey27] and Besicovitch [Bes26]. In this work we will only consider the notion depicted above and therefore call the corresponding functions just *almost periodic* (*a.p.*). We will however introduce one more way to describe a.p. functions using the framework of the previous section:

Consider (Ω, ψ) as above and a function $U \in C(\Omega)$. Then, the function defined by

$$u(t) = U(\psi(t)) \tag{4.2}$$

is almost periodic. This can be verified easily with the alternative definition due to Bochner. Since $U \in C(\Omega)$, any sequence $(u_{\tau_n})_{n\in\mathbb{N}}$ will be uniformly bounded and equicontinuous. Hence the Arzelà–Ascoli theorem guarantees the existence of a uniformly convergent subsequence. We will call any function obtainable in this manner *representable over* (Ω, ψ). Since the image of ψ is assumed to be dense, it is clear that the function $U \in C(\Omega)$ is uniquely determined by this relation. As an example take $\Omega \cong \mathbb{S}^1$, then ψ is periodic. Thus (4.2) gives rise to periodic functions. Conversely it is true, that any almost periodic function can be constructed this way. For this purpose we introduce the notion of hull. The *hull* $\mathcal{H}_u$ of a function u is defined by

$$\mathcal{H}_u = \overline{\{u_\tau : \tau \in \mathbb{R}\}},$$

where the closure is taken with respect to uniform convergence on the whole real line. Therefore if u is a.p. then $\mathcal{H}_u$ is a compact and connected metric space. If one uses the continuous extension of the rule

$$u_\tau * u_s = u_{\tau+s} \quad \forall \tau, s \in \mathbb{R}$$

onto all of $\mathcal{H}_u$ as the group operation, then the hull becomes a commutative topological group with neutral element u. This and some other properties of the hull are verified in Appendix C. If we further define the flow

$$\psi_u(\tau) = u_\tau, \tag{4.3}$$

then the pair $(\mathcal{H}_u, \psi_u)$ matches perfectly the setup of the previous section. Now, the representation formula (4.2) holds for $U \in C(\mathcal{H}_u)$ defined by

$$U(w) = w(0) \quad \forall w \in \mathcal{H}_u. \tag{4.4}$$

This function is sometimes called the ‘extension by continuity’ of the almost periodic function $u(t)$ to its hull $\mathcal{H}_u$. This construction is standard in the theory of a.p. functions and we refer the reader to [NS60] for a more detailed discussion.
Using Lemma 4.1, one easily shows the following (cf. [OT06], Lemma 12).

Lemma 4.4. *An a.p. function $u(t)$ is representable over (Ω, ψ) if and only if there exists a morphism $(\Omega, \psi) \to (\mathcal{H}_u, \psi_u)$.*

For a function $U : \Omega \to \mathbb{R}$ we introduce the derivative along the flow by

$$\partial_\psi U(\omega) = \lim_{t \to 0} \frac{U(\omega + \psi(t)) - U(\omega)}{t}.$$

Let $C^1_\psi(\Omega)$ be the space of continuous functions $U : \Omega \to \mathbb{R}$ such that $\partial_\psi U$ exists for all $\omega \in \Omega$ and $\partial_\psi U \in C(\Omega)$. The spaces $C^k_\psi(\Omega)$ for $k \geq 2$ are defined accordingly. Let us also introduce the norm

$$\|U\|_{C^k_\psi(\Omega)} = \sup_{1\leq m\leq k} \|\partial^m_\psi U\|_\infty.$$

For later reference, we also introduce the following. Let $C^1_\psi(\Omega \times \mathbb{R}^d)$ denote the space of functions $V : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $V(\omega, x_1, \ldots, x_d)$ such that the derivatives $\partial_\psi V$ and $\partial_{x_i} V$, $1 \leq i \leq d$, exist on $\Omega \times \mathbb{R}^d$ and $\partial_\psi V, \partial_{x_i} V \in C(\Omega \times \mathbb{R}^d)$. The spaces $C^k_\psi(\Omega \times \mathbb{R}^d)$ for $k \geq 2$ can now be defined recursively.

Consider $U \in C(\Omega)$ and assume the a.p. function $u(t) = U(\psi(t))$ is continuously differentiable. Then $\partial_\psi U$ exists on $\psi(\mathbb{R})$ and we have

$$u'(t) = \partial_\psi U\,(\psi(t)) \quad \text{for all } t \in \mathbb{R}.$$

Lemma 4.5. *Let $U \in C(\Omega)$ and $u \in C(\mathbb{R})$ be such that $u(t) = U(\psi(t))$. Then we have $u \in C^1(\mathbb{R})$ and $u'(t)$ is a.p. if and only if $U \in C^1_\psi(\Omega)$.*

One part of the equivalence is trivial. The proof of the other part can be found in [OT06, Lemma 13]. Furthermore, given $U \in C^k_\psi(\Omega)$ and $u(t) = U(\psi(t))$ we have

$$\|u\|_{C^k(\mathbb{R})} = \|U\|_{C^k_\psi(\Omega)},$$

since $\omega \cdot \mathbb{R}$ lies dense in Ω. Finally, we note that the derivative $u'(t)$ of an almost periodic function is itself a.p. if and only if it is uniformly continuous. This, and many other interesting properties of a.p. functions are demonstrated in [Bes26].

Example 4.6. Let us continue Example 4.2, where $\Omega = \mathbb{T}^N$. For $U \in C(\mathbb{T}^N)$ consider the function

$$u(t) = U(\psi(t)) = U(\overline{\nu_1 t}, \ldots, \overline{\nu_N t}).$$

Such functions are called *quasi-periodic*. In this case, ∂_ψ is just the derivative in the direction of $\nu \in \mathbb{R}^N$. So if U is in the space $C^1(\mathbb{T}^N)$ of functions in $C^1(\mathbb{R}^N)$, which are 1-periodic in each argument, then

$$\partial_\psi U = \sum_{i=1}^N \nu_i\, \partial_{\theta_i} U.$$

Note however, that in general $C^1(\mathbb{T}^N)$ is a proper subspace of $C^1_\psi(\mathbb{T}^N)$.

Example 4.7. The so called limit periodic functions are another important subclass of the a.p. functions. Here, a map $f : \mathbb{R} \to \mathbb{R}$ is called *limit periodic* if it is the uniform limit of continuous periodic functions. Now, in continuation of Example 4.3, let $\Omega = \mathcal{S}_{\boldsymbol{p}}$ and consider $U \in C(\mathcal{S}_{\boldsymbol{p}})$. Then, the function $u(t)$ defined by

$$u(t) = U(\psi(t)) = U(e^{2\pi i\lambda t/q_0}, e^{2\pi i\lambda t/q_1}, e^{2\pi i\lambda t/q_2}, \dots)$$

is limit periodic, since it is the uniform limit of a sequence of q_k-periodic functions u_k given by

$$u_k(t) = U(e^{2\pi i\lambda t/q_0}, \dots, e^{2\pi i\lambda t/q_k}, 1, 1, \dots).$$

Vice versa it is true, that for suitable $\boldsymbol{p}$ and $\lambda > 0$ any limit periodic function $v(t)$ can be obtained in this manner. To see this, first note that v can be expanded in a uniformly convergent series of continuous 1-periodic functions,

$$v(t) = \sum_{k=0}^{\infty} v_k(t/T_k),$$

with $T_0 > 0$ and T_k such that $T_k/T_{k-1} \in \mathbb{N}$ for all $k \in \mathbb{N}$ (cf. [Chu89]). W.l.o.g. we can assume that $p_k := \frac{T_k}{T_{k-1}}$ is a prime number for all k. Moreover, one can show that the hull $\mathcal{H}_v$ of $v(t)$ consists of those functions $w_\phi(t)$ which can be written in the form

$$w_\phi(t) = \sum_{k=0}^{\infty} v_k((t + \phi_k)/T_k),$$

where ϕ_k is an angle defined modulo T_k such that $\phi_{k-1} \equiv \phi_k \mod T_{k-1}$ for all $k \in \mathbb{N}$ (see [MS89] for a more detailed discussion in the case of a specific example). We can then define $z_k = e^{2\pi i\phi_k/T_k}$ to obtain a series $(z_k)_{k\in\mathbb{N}_0}$ in $\mathbb{S}^1$ so that

$$z_k^{p_k} = e^{2\pi i\phi_k/T_{k-1}} = e^{2\pi i\phi_{k-1}/T_{k-1}} = z_{k-1}.$$

So we have $z \in \mathcal{S}_{\boldsymbol{p}}$, where $\boldsymbol{p} = (p_k)_{k\in\mathbb{N}}$. We write $\eta : \mathcal{S}_{\boldsymbol{p}} \to \mathcal{H}_v$ for the resulting continuous map $(z_k) = (e^{2\pi i\phi_k/T_k}) \mapsto w_\phi(t)$. The translation flow restricted to $\mathcal{H}_v$ as described in (4.3) then corresponds to ψ as above with $\lambda = 1/T_0$, that is $\psi(\tau) = (e^{2\pi i\tau/T_0}, e^{2\pi i\tau/T_1}, \ldots)$. Indeed, defining $V = U \circ \eta$, where U is the extension by continuity as in (4.4), yields

$$V(\psi(\tau)) = U(w_\tau) = w_\tau(0) = \sum_{k=0}^{\infty} v_k(\tau/T_k) = v(\tau).$$

4.1.3 Haar measure and decomposition along the flow

It is a well-known fact, that for every compact commutative topological group Ω there is a unique Borel probability measure μ_Ω, which is invariant under the group operation, i.e. $\mu_\Omega(\mathcal{D} + \omega) = \mu_\Omega(\mathcal{D})$ holds for every Borel set $\mathcal{D} \subset \Omega$ and every $\omega \in \Omega$. This measure is called the *Haar measure* of Ω. (This follows from the existence of the invariant *Haar integral* of Ω and the Riesz representation theorem. Proofs can be found in [Pon66] and [HR79], respectively.) For Example if $\Omega = \mathbb{S}^1$ we have

$$\mu_{\mathbb{S}^1}(\mathcal{B}) = \frac{1}{2\pi}\lambda\{t \in [0, 2\pi) : e^{it} \in \mathcal{B}\},$$

where λ is the Lebesgue measure on $\mathbb{R}$. Let ψ, Σ and Φ be as in Section 4.1.1. Then Φ defines a decomposition $\Omega \cong \Sigma \times [0, S)$ along the flow. Since Σ is a subgroup, it has a Haar measure μ_Σ itself. Also the interval $[0, S)$ naturally inherits the probability measure

$$\mu_{[0,S)}(I) = \frac{1}{S}\lambda(I).$$

As shown in [CT13], the restricted flow $\Phi : \Sigma \times [0, S) \to \Omega, \Phi(\sigma, t) = \sigma \cdot t$ also allows for a decomposition of the Haar measure μ_Ω along the flow.

Lemma 4.8. *The map Φ is an isomorphism of measure spaces, i.e.*

$$\mu_\Omega(\mathcal{B}) = \frac{1}{S}(\mu_\Sigma \otimes \lambda)(\Phi^{-1}(\mathcal{B}))$$

holds for every Borel set $\mathcal{B} \subset \Omega$.

Before we prove this lemma, let us begin with some preliminaries. Consider the function $\chi : \Sigma \times [0, \infty) \to \Sigma \times [0, S)$ defined by

$$\chi(\sigma, t) = \Phi^{-1}(\sigma \cdot t) = \Phi^{-1}(\sigma + \psi(t)). \tag{4.5}$$

Since Φ is just the restricted flow, we have $\chi = \mathrm{id}$ on $\Sigma \times [0, S)$. This yields

$$\begin{aligned}\chi(\sigma, t) = \Phi^{-1}(\sigma + \psi(t)) &= \Phi^{-1}\left(\sigma + \psi\left(\left\lfloor \frac{t}{S} \right\rfloor S\right) + \psi\left(t - \left\lfloor \frac{s}{S} \right\rfloor S\right)\right) \\ &= \left(\sigma + \psi\left(\left\lfloor \frac{t}{S} \right\rfloor S\right), t - \left\lfloor \frac{t}{S} \right\rfloor S\right)\end{aligned}$$

for every $(\sigma, t) \in \Sigma \times \mathbb{R}$, where $\lfloor\cdot\rfloor$ indicates the floor function. This representation shows that χ is measure-preserving on any strip $\Sigma \times [t, t+S)$ of width S, since μ_Σ and λ are invariant under translations in Σ and $\mathbb{R}$, respectively. Moreover, the equality

$$\chi(\Phi^{-1}(\omega) + \Phi^{-1}(\tilde{\omega})) = \Phi^{-1}(\omega + \tilde{\omega}) \ \ \forall \omega, \tilde{\omega} \in \Omega \tag{4.6}$$

follows directly from the definition of χ.

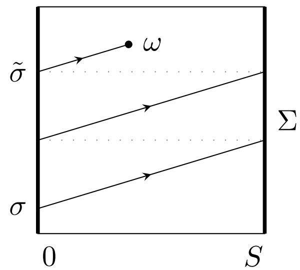

Figure 4.2: Let $\chi(\sigma, t) = (\tilde{\sigma}, s)$. The map χ 'divides out' every complete period of $\varphi \circ \psi$, that is $s = t \mod S$, while preserving the relation $\tilde{\sigma} \cdot s = \omega = \sigma \cdot t$.

Proof of Lemma 4.8. First we show that Φ^{-1} is Borel measurable. To prove this, it suffices to show that the image $\Phi(A \times I)$ of every open rectangle $A \times I \subset \Sigma \times [0, S)$ is a Borel set. If $0 \notin I$ this image is open in $\Omega \setminus \Sigma$, since Φ^{-1} is continuous. But if $0 \in I$, again $\Phi(A \times (I \setminus \{0\}))$ is open and $\Phi(A \times \{0\}) = A$ is it as well.
Now, consider the measure μ_Φ on Ω defined by

$$\mu_\Phi(\mathcal{B}) = \frac{1}{S}(\mu_\Sigma \otimes \lambda)(\Phi^{-1}(\mathcal{B})).$$

Since $\mu_\Phi(\Omega) = 1$, this is a Borel probability measure. We will show that μ_Φ is also invariant under addition in the group. For this purpose, let $\mathcal{B} \subset \Omega$ be a Borel set and let $\omega_0 \in \Omega$. Then, by (4.6) we have

$$\mu_\Phi(\mathcal{B} + \omega_0) = \frac{1}{S}(\mu_\Sigma \otimes \lambda)(\Phi^{-1}(\mathcal{B} + \omega_0))$$
$$= \frac{1}{S}(\mu_\Sigma \otimes \lambda)\left(\chi(\Phi^{-1}(\mathcal{B}) + \Phi^{-1}(\omega_0))\right).$$

Denoting $\Phi^{-1}(\omega_0) = (\sigma_0, s_0)$, we get $\Phi^{-1}(\mathcal{B}) + \Phi^{-1}(\omega_0) \subset \Sigma \times [s_0, s_0 + S)$. So it is contained in a strip of width S and therefore

$$\frac{1}{S}(\mu_\Sigma \otimes \lambda)\left(\chi(\Phi^{-1}(\mathcal{B}) + (\sigma_0, s_0))\right) = \frac{1}{S}(\mu_\Sigma \otimes \lambda)\left(\Phi^{-1}(\mathcal{B}) + (\sigma_0, s_0)\right)$$

But the product measure $\mu_\Sigma \otimes \lambda$ is invariant under translations in $\Sigma \times \mathbb{R}$. Thus, in total we have

$$\mu_\Phi(\mathcal{B} + \omega_0) = \frac{1}{S}(\mu_\Sigma \otimes \lambda)\left(\Phi^{-1}(\mathcal{B})\right) = \mu_\Phi(\mathcal{B}).$$

Therefore, μ_Φ is a Borel probability measure on Ω which is invariant under group action. Since the Haar measure is unique, it follows $\mu_\Omega = \mu_\Phi$. $\square$

4.2 A theorem about escaping sets

4.2.1 Almost periodic successor maps

From now on we will consider functions

$$f : \mathcal{D} \subset \Omega \times (0, \infty) \to \Omega \times (0, \infty),$$

where $\mathcal{D}$ is an open set. We will call such a function *measure-preserving embedding*, if f is continuous, injective and furthermore

$$(\mu_\Omega \otimes \lambda)(f(\mathcal{B})) = (\mu_\Omega \otimes \lambda)(\mathcal{B})$$

holds for all Borel sets $\mathcal{B} \subset \mathcal{D}$, where λ denotes the Lebesgue measure of $\mathbb{R}$. This matches the setup depicted at the end of Section 2. Thus, let $\mathcal{D}_\infty$

be the set of initial data $(\omega_0, r_0) \in \Omega \times (0,\infty)$ leading complete forward orbits $(\omega_n, r_n) = f^n(\omega_0, r_0)$, $n \in \mathbb{N}_0$. We again write

$$\mathcal{U} = \{(\omega_0, r_0) \in \mathcal{D}_\infty : \limsup_{n\to\infty} r_n = \infty\}$$

and

$$\mathcal{E} = \{(\omega_0, r_0) \in \mathcal{D}_\infty : \lim_{n\to\infty} r_n = \infty\}$$

for the set of initial condition leading to unbounded and escaping orbits, respectively. In particular, we are interested in measure-preserving embeddings of the form

$$f(\omega, r) = (\omega + \psi(F(\omega, r)), r + G(\omega, r)), \tag{4.7}$$

where $F, G : \mathcal{D} \to \mathbb{R}$ are continuous. For $\omega \in \Omega$ we introduce the notation $\psi_\omega(t) = \omega + \psi(t) = \omega \cdot t$ and define

$$D_\omega = (\psi_\omega \times \mathrm{id})^{-1}(\mathcal{D}) \subset \mathbb{R} \times (0,\infty).$$

On this open set, consider the map $f_\omega : D_\omega \subset \mathbb{R} \times (0,\infty) \to \mathbb{R} \times (0,\infty)$ given by

$$f_\omega(t, r) = (t + F(\psi_\omega(t), r), r + G(\psi_\omega(t), r)). \tag{4.8}$$

Then f_ω is continuous and meets the identity

$$f \circ (\psi_\omega \times \mathrm{id}) = (\psi_\omega \times \mathrm{id}) \circ f_\omega \text{ on } D_\omega,$$

i.e. the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f} & f(\mathcal{D}) \subset \Omega \times (0,\infty) \\ {\scriptstyle \psi_\omega \times \mathrm{id}}\uparrow & & {\scriptstyle \psi_\omega \times \mathrm{id}}\uparrow \\ D_\omega & \xrightarrow{f_\omega} & f_\omega(D_\omega) \subset \mathbb{R} \times (0,\infty) \end{array}$$

Therefore f_ω is injective as well. Again we define the sets $D_{\omega,1} = D_\omega$ and $D_{\omega,n+1} = f_\omega^{-1}(D_{\omega,n})$ to construct the domain

$$D_{\omega,\infty} = \bigcap_{n=1}^{\infty} D_{\omega,n} \subset \mathbb{R} \times (0,\infty),$$

where the forward iterates $(t_n, r_n) = f_\omega^n(t_0, r_0)$ are defined for all $n \in \mathbb{N}$. Analogously, unbounded orbits are generated by initial conditions in the set

$$U_\omega = \{(t_0, r_0) \in D_{\omega,\infty} : \limsup_{n\to\infty} r_n = \infty\}$$

and escaping orbits originate in

$$E_\omega = \{(t_0, r_0) \in D_{\omega,\infty} : \lim_{n\to\infty} r_n = \infty\}.$$

These sets can also be obtained through the relations

$$D_{\omega,\infty} = (\psi_\omega \times \mathrm{id})^{-1}(\mathcal{D}_\infty), \;\; U_\omega = (\psi_\omega \times \mathrm{id})^{-1}(\mathcal{U}), \;\; E_\omega = (\psi_\omega \times \mathrm{id})^{-1}(\mathcal{E}).$$

If there is a suitable adiabatic invariant $W(\omega, r)$ which is approximately constant for r sufficiently big, then one can show that f is recurrent. As a consequence, most escaping sets E_ω have measure zero.

Theorem 4.9. *Let $f : \mathcal{D} \subset \Omega \times (0, \infty) \to \Omega \times (0, \infty)$ be a measure-preserving embedding of the form (4.7) and suppose that there is a function $W = W(\omega, r)$ satisfying $W \in C^1_\psi(\Omega \times (0, \infty))$,*

$$0 < \beta \le \partial_r W(\omega, r) \le \delta \;\; \text{for} \;\; \omega \in \Omega, \;\; r \in (0, \infty), \tag{4.9}$$

with some constants $\beta, \delta > 0$, and furthermore

$$W(f(\omega, r)) \le W(\omega, r) + k(r) \;\; \text{for} \;\; (\omega, r) \in \mathcal{D}, \tag{4.10}$$

where $k : (0, \infty) \to \mathbb{R}$ is a decreasing and bounded function such that $\lim_{r\to\infty} k(r) = 0$. Then, for almost all $\omega \in \Omega$, the set $E_\omega \subset \mathbb{R} \times (0, \infty)$ has Lebesgue measure zero.

4.2.2 Proof of Theorem 4.9

Similar to the proof of Theorem 3.1, the main idea in the proof of Theorem 4.9 is to apply Maharam's Recurrence Theorem to the restricted map $f\big|_{\mathcal{U}}$. The necessary space engendering set $\mathcal{M}$ of finite measure can be constructed by means of the present adiabatic invariant $W(\omega, r)$. This is expressed in the following lemma.

Lemma 4.10. *Let $f : \mathcal{D} \subset \Omega \times (0,\infty) \to \Omega \times (0,\infty)$ be a measure-preserving embedding and suppose that there is a function $W = W(\omega, r)$ satisfying $W \in C^1_\psi(\Omega \times (0,\infty))$, (4.9) and (4.10). Then, there is a set $\mathcal{M} \subset \Omega \times (0,\infty)$ of finite measure such that every unbounded orbit of f enters $\mathcal{M}$, i.e. we have*

$$\mathcal{U} \subset \bigcup_{n=1}^{\infty} f^{-n}(\mathcal{M})$$

up to a set of measure zero.

Proof. Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a sequence of positive reals with $\sum_{j=1}^{\infty} \varepsilon_j < \infty$. By assumption, we can find a corresponding sequence $(W_j)_{j\in\mathbb{N}}$ with the properties $W_j > 0$, $\lim_{j\to\infty} W_j = \infty$ and $\lim_{j\to\infty} \varepsilon_j^{-1} k(\frac{1}{4\gamma} W_j) = 0$. The sought set can then be defined by

$$\mathcal{M} = \bigcup_{j\in\mathbb{N}} \mathcal{M}_j, \quad \mathcal{M}_j = \{(\omega, r) \in \Omega \times (0,\infty) : |W(\omega, r) - W_j| \le \varepsilon_j\}.$$

We start by showing that $\mathcal{M}$ has finite measure. By Fubini's theorem,

$$(\mu_\Omega \otimes \lambda)(\mathcal{M}_j) = \int_\Omega \lambda(\mathcal{M}_{j,\omega})\, d\mu_\Omega(\omega)$$

holds for the sections $\mathcal{M}_{j,\omega} = \{r \in (0,\infty) : (\omega, r) \in \mathcal{M}_j\}$. Now, consider the diffeomorphism $w_\omega : r \mapsto W(\omega, r)$. Its inverse w_ω^{-1} is Lipschitz continuous with constant β^{-1}, due to (4.9). But then, the fact that $\mathcal{M}_{j,\omega} = w_\omega^{-1}((W_j - \varepsilon_j, W_j + \varepsilon_j))$ implies $\lambda(\mathcal{M}_{j,\omega}) \le 2\beta^{-1}\varepsilon_j$. Thus in total we have

$$(\mu_\Omega \otimes \lambda)(\mathcal{M}) \le \sum_{j=1}^{\infty} (\mu_\Omega \otimes \lambda)(\mathcal{M}_j) \le \sum_{j=1}^{\infty} \frac{2\varepsilon_j}{\beta} < \infty.$$

Next we prove that every unbounded orbit has to go through $\mathcal{M}$. To this end, let $(\omega_0, r_0) \in \mathcal{U}$ be fixed and denote by (ω_n, r_n) the forward orbit under f. We will start with some preliminaries. Using (4.9) and the mean value theorem, we can find $\hat{r}$ such that

$$\frac{\beta}{2} \le \frac{W(\omega, r)}{r} \le 2\delta \quad \forall (\omega, r) \in \Omega \times (\hat{r}, \infty). \tag{4.11}$$

Furthermore, by definition of the sequence W_j we can find an index $j_0 \geq 2$ such that

$$W_{j_0} > \max\{W(\omega_1, r_1), \|k\|_\infty + \max_{\omega\in\Omega} W(\omega, \hat{r}), 2\|k\|_\infty\} \text{ and } k\left(\frac{1}{4\delta}W_{j_0}\right) \leq \varepsilon_{j_0}.$$

Moreover, we have $\limsup_{n\to\infty} W(\omega_n, r_n) = \infty$. Indeed, the fact that (ω_0, r_0) lies in $\mathcal{U}$ and (4.9) imply

$$W(\omega_n, r_n) \geq \beta(r_n - r_1) + W(\omega_n, r_1)$$

for some n sufficiently large. But then $\limsup_{n\to\infty} W(\omega_n, r_n) = \infty$ follows from the compactness of Ω. Now, since $W(\omega_1, r_1) < W_{j_0}$ we can select the first index $K \geq 2$ such that $W(\omega_K, r_K) > W_{j_0}$. So in particular this means $W(\omega_{K-1}, r_{K-1}) \leq W_{j_0}$. Since the estimate (4.10) yields $W(\omega_K, r_K) \leq W(\omega_{K-1}, r_{K-1}) + k(r_{K-1})$, we can derive the following inequality:

$$\begin{aligned} W(\omega_{K-1}, r_{K-1}) &\geq W(\omega_K, r_K) - \|k\|_\infty > W_{j_0} - \|k\|_\infty \\ &\geq \max_{\omega\in\Omega} W(\omega, \hat{r}) \geq W(\omega_{K-1}, \hat{r}) \end{aligned}$$

Then, the monotonicity of $w_{\omega_{K-1}}$ implies $r_{K-1} > \hat{r}$. Hence we can combine (4.11) with the previous estimate to obtain

$$r_{K-1} \geq \frac{1}{2\delta} W(\omega_{K-1}, r_{K-1}) \geq \frac{1}{2\delta}(W_{j_0} - \|k\|_\infty) \geq \frac{1}{4\delta} W_{j_0}.$$

Finally, since $k(r)$ is decreasing, $W(\omega_K, r_K) > W_{j_0} \geq W(\omega_{K-1}, r_{K-1})$ yields

$$\begin{aligned} |W(\omega_K, r_K) - W_{j_0}| &\leq W(\omega_K, r_K) - W(\omega_{K-1}, r_{K-1}) \\ &\leq k(r_{K-1}) \leq k\left(\frac{1}{4\delta}W_{j_0}\right) \leq \varepsilon_{j_0}, \end{aligned}$$

which implies $(\omega_K, r_K) \in \mathcal{M}_{j_0}$. □

Now, we are ready to prove the theorem.

Proof of Theorem 4.9. Consider the set

$$\mathcal{U} = \{(\omega_0, r_0) \in \mathcal{D}_\infty : \limsup_{n\to\infty} r_n = \infty\}.$$

We can assume that $\mathcal{U} \neq \emptyset$, since otherwise the assertion follows immediately.

Step 1: Almost all unbounded orbits are recurrent. We will prove the existence of a set $\mathcal{Z} \subset \mathcal{U}$ of measure zero such that if $(\omega_0, r_0) \in \mathcal{U} \setminus \mathcal{Z}$, then

$$\liminf_{n \to \infty} r_n < \infty.$$

In particular, we would have $\mathcal{E} \subset \mathcal{Z}$. To show this, we consider the restriction $T = f\big|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$. This map is well-defined, injective and, like f, measure-preserving. We will distinguish three cases:

(i) $(\mu_\Omega \otimes \lambda)(\mathcal{U}) = 0$,

(ii) $0 < (\mu_\Omega \otimes \lambda)(\mathcal{U}) < \infty$, and

(iii) $(\mu_\Omega \otimes \lambda)(\mathcal{U}) = \infty$.

In the first case $\mathcal{Z} = \mathcal{U}$ is a valid choice. In case *(ii)* we can apply the Poincaré recurrence theorem (Lemma 2.2), whereas in case *(iii)* the analogue by Maharam (Lemma 2.3) is applicable due to Lemma 4.10. Now, let us cover $\Omega \times \mathbb{R}$ by the sets $\mathfrak{B}_j = \Omega \times (j-1, j+1)$ for $j \in \mathbb{N}$. Then, for $\mathcal{B}_j = \mathfrak{B}_j \cap \mathcal{U}$ one can use the recurrence property to find sets $\mathcal{Z}_j \subset \mathcal{B}_j$ of measure zero such that every orbit $(\omega_n, r_n)_{n \in \mathbb{N}}$ starting in $\mathcal{B}_j \setminus \mathcal{Z}_j$ returns to $\mathcal{B}_j$ infinitely often. But this implies $\liminf_{n \to \infty} r_n \leq r_0 + 2 < \infty$. Therefore, the set $\mathcal{Z} = \bigcup_{j \in \mathbb{N}} \mathcal{Z}_j \subset \mathcal{U}$ has all the desired properties.

Step 2: We will show the existence of a subgroup $\Sigma \subset \Omega$ such that E_σ has Lebesgue measure zero for almost all $\sigma \in \Sigma$. Since $\mathcal{E} \subset \mathcal{Z}$ by construction, the inclusion

$$E_\omega = (\psi_\omega \otimes \mathrm{id})^{-1}(\mathcal{E}) \subset (\psi_\omega \otimes \mathrm{id})^{-1}(\mathcal{Z})$$

holds for all $\omega \in \Omega$. To $j \in \mathbb{Z}$ we can consider the restricted flow

$$\Phi_j : \Sigma \times [jS, (j+1)S) \to \Omega, \quad \Phi_j(\sigma, t) = \sigma \cdot t = \psi_\sigma(t).$$

It is easy to verify that just like $\Phi = \Phi_0$ of Lemma 4.8 those functions are isomorphisms of measure spaces. In other words, Φ_j is bijective up to a set of measure zero, both Φ_j and Φ_j^{-1} are measurable, and for every Borel set $\mathcal{B} \subset \Omega$ we have

$$\mu_\Omega(\mathcal{B}) = \frac{1}{S}(\mu_\Sigma \otimes \lambda)(\Phi_j^{-1}(\mathcal{B})). \tag{4.12}$$

This clearly implies

$$(\mu_\Omega \otimes \lambda)(B) = \frac{1}{S}(\mu_\Sigma \otimes \lambda^2)(\Phi_j^{-1} \times \mathrm{id})(B) \tag{4.13}$$

for every Borel set $B \subset \Omega \times (0, \infty)$. Let

$$C_j = \{(\sigma, t, r) \in \Sigma \times [jS, (j+1)S) \times (0, \infty) : (\Phi_j(\sigma, t), r) \in \mathcal{Z}\} = (\Phi_j^{-1} \times \mathrm{id})(\mathcal{Z}).$$

Since $\mathcal{Z}$ has measure zero, (4.13) yields $(\mu_\Sigma \otimes \lambda^2)(C_j) = 0$. Next we consider the cross sections

$$C_{j,\sigma} = \{(t, r) \in [jS, (j+1)S) \times (0, \infty) : (\sigma, t, r) \in C_j\}.$$

Then, $\lambda^2(C_{j,\sigma}) = 0$ for μ_Σ-almost all $\sigma \in \Sigma$ follows from Fubini's theorem. So for every $j \in \mathbb{Z}$ there is a set $M_j \subset \Sigma$ with $\mu_\Sigma(M_j) = 0$ such that $\lambda^2(C_{j,\sigma}) = 0$ for all $\sigma \in \Sigma \setminus M_j$. Thus $M = \bigcup_{j \in \mathbb{Z}} M_j$ has measure zero as well and

$$\lambda^2\Big(\bigcup_{j \in \mathbb{Z}} C_{j,\sigma}\Big) = 0$$

for all $\sigma \in \Sigma \setminus M$. But we have

$$\bigcup_{j \in \mathbb{Z}} C_{j,\sigma} = \{(t, r) \in \mathbb{R} \times (0, \infty) : (\psi_\sigma(t), r) \in \mathcal{Z}\} = (\psi_\sigma \times \mathrm{id})^{-1}(\mathcal{Z}),$$

and recalling that $E_\sigma \subset (\psi_\sigma \times \mathrm{id})^{-1}(\mathcal{Z})$, we therefore conclude $\lambda^2(E_\sigma) = 0$ for all $\sigma \in \Sigma \setminus M$.

Step 3: Concluding from Σ to Ω. If we denote by $T_s(t, r) = (t + s, r)$ the translation in time, then clearly

$$f_{\omega \cdot s} = T_{-s} \circ f_\omega \circ T_s \ \text{ on } \ D_{\omega \cdot s}$$

holds for all $\omega \in \Omega$ and $s \in \mathbb{R}$. But this implies $T_s(E_{\omega \cdot s}) = E_\omega$, since the identity above stays valid under iterations. In particular we have

$$\lambda^2(E_{\omega \cdot s}) = \lambda^2(E_\omega), \quad \forall \omega \in \Omega, s \in \mathbb{R}.$$

Again, we consider the restricted flow $\Phi : \Sigma \times [0, S) \to \Omega$, $\Phi(\omega, t) = \omega \cdot t$. Using $M \subset \Sigma$ of Step 2 we define $Z_* = \Phi(M \times [0, S)) \subset \Omega$. Then, (4.12) and $\mu_\Sigma(M) = 0$ imply that also Z_* has measure zero. Now let $\omega \in \Omega \setminus Z_*$ be fixed and let $(\sigma, \tau) = \Phi^{-1}(\omega)$. Then $\sigma \in \Sigma \setminus M$ and $\sigma \cdot \tau = \omega$. Therefore, Step 2 implies

$$\lambda^2(E_\omega) = \lambda^2(E_{\sigma\cdot\tau}) = \lambda^2(E_\sigma) = 0,$$

which proves the assertion. □

4.3 Application to the ping-pong

We start with a rigorous description of the ping-pong map. To this end, let $p \in C_b^2(\mathbb{R})$ be a forcing function such that

$$0 < a \leq p(t) \leq b \quad \forall t \in \mathbb{R}, \quad \|p\|_{C^2} = \max\{\|p\|_\infty, \|\dot{p}\|_\infty, \|\ddot{p}\|_\infty\} < \infty. \tag{4.14}$$

Now, consider the map

$$(t_0, v_0) \mapsto (t_1, v_1),$$

which sends a time t_0 of impact to the left plate $x = 0$ and the corresponding velocity $v_0 > 0$ immediately after the impact to their successors t_1 and v_1 describing the subsequent impact to $x = 0$. If we further denote by $\tilde{t} \in (t_0, t_1)$ the time of the particle's impact to the moving plate, then we can determine $\tilde{t} = \tilde{t}(t_0, v_0)$ implicitly through the equation

$$(\tilde{t} - t_0)v_0 = p(\tilde{t}), \tag{4.15}$$

since this relation describes the distance that the particle has to travel before hitting the moving plate. With that we derive a formula for the successor map:

$$t_1 = \tilde{t} + \frac{p(\tilde{t})}{v_1}, \quad v_1 = v_0 - 2\dot{p}(\tilde{t}) \tag{4.16}$$

To ensure that this map is well-defined, we will assume that

$$v_0 > v_* := 2\max\{\sup_{t\in\mathbb{R}} \dot{p}(t), 0\}. \tag{4.17}$$

This condition guarantees that v_1 is positive and also implies that there is a unique solution $\tilde{t} = \tilde{t}(t_0, v_0) \in C^1(\mathbb{R} \times (v_*, \infty))$ to (4.15). Thus we can take $\mathbb{R} \times (v_*, \infty)$ as the domain of the ping-pong map (5.5). Now, we are finally ready to state the main theorem.

Theorem 4.11. *Assume* $0 < a < b$ *and* $P \in C^2_\psi(\Omega)$ *are such that*

$$a \leq P(\omega) \leq b \ \ \forall \omega \in \Omega. \tag{4.18}$$

Consider the family $\{p_\omega\}_{\omega \in \Omega}$ *of almost periodic forcing functions defined by*

$$p_\omega(t) = P(\omega + \psi(t)), \ \ t \in \mathbb{R}. \tag{4.19}$$

Let $v_* = 2\max\{\max_{\varpi \in \Omega} \partial_\psi P(\varpi), 0\}$ *and denote by*

$$E_\omega = \{(t_0, v_0) \in \mathbb{R} \times (v_*, \infty) : (t_n, v_n)_{n \in \mathbb{N}} \text{ exists and } \lim_{n \to \infty} v_n = \infty\}$$

the escaping set for the ping-pong map with forcing function $p(t) = p_\omega(t)$*. Then, for almost all* $\omega \in \Omega$*, the set* $E_\omega \subset \mathbb{R}^2$ *has Lebesgue measure zero.*

Remark 4.12. The notation $v_* = 2\max\{\max_{\varpi \in \Omega} \partial_\psi P(\varpi), 0\}$ is consistent with (5.5), since for every $\omega \in \Omega$ the set $\omega \cdot \mathbb{R}$ lies dense in Ω and thus

$$\sup_{t \in \mathbb{R}} \dot{p}_\omega(t) = \sup_{t \in \mathbb{R}} \partial_\psi P(\omega + \psi(t)) = \max_{\varpi \in \Omega} \partial_\psi P(\varpi).$$

We will give some further preliminaries before starting the actual proof. First we note, that the ping-pong map $(t_0, v_0) \mapsto (t_1, v_1)$ is not symplectic. To remedy this defect, we reformulate the model in terms of time t and energy $E = \frac{1}{2}v^2$. In these new coordinates the map becomes $\mathcal{P} : (t_0, E_0) \mapsto (t_1, E_1)$ with

$$t_1 = \tilde{t} + \frac{p(\tilde{t})}{\sqrt{2E_1}}, \qquad E_1 = E_0 - 2\sqrt{2E_0}\dot{p}(\tilde{t}) + 2\dot{p}(\tilde{t})^2 = (\sqrt{E_0} - \sqrt{2}\dot{p}(\tilde{t}))^2,$$

where $\tilde{t} = \tilde{t}(t_0, E_0)$ is determined implicitly by the relation $\tilde{t} = t_0 + \frac{p(\tilde{t})}{\sqrt{2E_0}}$. This map is defined for $(t_0, E_0) \in \mathbb{R} \times (\frac{1}{2}v_*^2, \infty)$. Since it has a generating function [KO10, Lemma 3.7], the map $\mathcal{P}$ is indeed symplectic.

Now, we demonstrate that $W(t_0, E_0) = p(t_0)^2 E_0$ acts as an adiabatic invariant for the ping-pong map. For this purpose we cite the following lemma [KO20, Lemma 5.1]:

Lemma 4.13. *There is a constant* $C > 0$*, depending only upon* $\|p\|_{C^2}$ *and* $a, b > 0$ *from* (4.14)*, such that*

$$|p(t_1)^2 E_1 - p(t_0)^2 E_0| \leq C\Delta(t_0, E_0) \ \ \forall (t_0, E_0) \in \mathbb{R} \times (v_*^2/2, \infty),$$

where $(t_1, E_1) = \mathcal{P}(t_0, E_0)$ *denotes the ping-pong map for* $p(t)$*, and* $\Delta(t_0, E_0) = E_0^{-1/2} + \sup\{|\ddot{p}(t) - \ddot{p}(s)| : t, s \in [t_0 - C, t_0 + C], |t - s| \leq C E_0^{-1/2}\}$.

Proof. This is a consequence of the fact that $t_1 - t_0 \leq \tilde{C} E_0^{-1/2}$ for a suitable constant $\tilde{C} > 0$ together with formula (5.55) down below. □

So far we have depicted the case of a general forcing function p. Now, we will replace $p(t)$ by $p_\omega(t)$ from (4.19) and study the resulting ping-pong map. First, we note that due to $P \in C^2_\psi(\Omega)$ we have $p_\omega \in C^2_b(\mathbb{R})$. Also $0 < a \leq p_\omega(t) \leq b$ holds for all $\omega \in \Omega$ by assumption. Furthermore, we have $\|p_\omega\|_{C^2(\mathbb{R})} = \|P\|_{C^2_\psi(\Omega)}$ for all $\omega \in \Omega$. Therefore all considerations above apply with uniform constants. As depicted in Remark 4.12, also the threshold $v_* = 2 \max\{\max_{\varpi \in \Omega} \partial_\psi P(\varpi), 0\}$ is uniform in ω. Finally, since $\ddot{p}_\omega(t) = \partial^2_\psi P(\omega + \psi(t))$, the function $\Delta(t_0, E_0)$ can be uniformly bounded by

$$\Delta(E_0) = E_0^{-1/2} + \sup\{|\partial^2_\psi P(\varpi) - \partial^2_\psi P(\varpi')| : \varpi, \varpi' \in \Omega, \|\varpi - \varpi'\| \leq C E_0^{-1/2}\}.$$

Hence, from Lemma 4.13 we obtain

Lemma 4.14. *There is a constant* $C > 0$*, uniform in* $\omega \in \Omega$*, such that*

$$|p(t_1)^2 E_1 - p(t_0)^2 E_0| \leq C \Delta(E_0) \;\; \forall (t_0, E_0) \in \mathbb{R} \times (v_*^2/2, \infty),$$

where $(t_0, E_0) \mapsto (t_1, E_1)$ *denotes* $\mathcal{P}$ *for the forcing function* $p_\omega(t)$.

Consider the equation

$$\tau = \frac{1}{\sqrt{2E_0}} P(\omega_0 + \psi(\tau)). \tag{4.20}$$

Since $P \in C^1_\psi(\Omega)$ and $1 - (2E_0)^{-1/2} \partial_\psi P(\omega_0 + \psi(\tau)) \geq \frac{1}{2} > 0$ for $E_0 > \frac{1}{2} v_*^2$, (4.20) can be solved implicitly for $\tau = \tau(\omega_0, E_0) \in C(\Omega \times (v_*^2/2, \infty))$ (cf. [BGdS08] for a suitable implicit function theorem). For $\omega \in \Omega$ and $t_0 \in \mathbb{R}$ one can consider (4.20) with $\omega_0 = \omega + \psi(t_0)$. Then, $P \in C^1_\psi(\Omega)$ and the classical implicit function theorem yield $\tau \in C^1_\psi(\Omega \times (v_*^2/2, \infty))$. Moreover, comparing this to the definition of $\tilde{t}$, we observe the following relation:

$$\tilde{t}(t_0, E_0) = t_0 + \tau(\omega + \psi(t_0), E_0). \tag{4.21}$$

Now, we will give the proof of the main theorem, in which we will link the ping-pong map corresponding to $p_\omega(t)$ to the setup of Section 4.2.

Proof of Theorem 4.11. Let $\mathcal{D} = \Omega \times (E^*, \infty)$, where $E^* = \max\{\frac{1}{2}v_*^2, E_{**}\}$ and E_{**} will be determined below. Consider the function

$$f : \mathcal{D} \subset \Omega \times (0, \infty) \to \Omega \times (0, \infty), \qquad f(\omega_0, E_0) = (\omega_1, E_1)$$

given by

$$\omega_1 = \omega_0 + \psi(F(\omega_0, E_0)), \qquad E_1 = E_0 + G(\omega_0, E_0),$$

where

$$F(\omega_0, E_0) = \left(\frac{1}{\sqrt{2E_0}} + \frac{1}{\sqrt{2E_1}}\right) P(\omega_0 + \psi(\tau)),$$
$$G(\omega_0, E_0) = -2\sqrt{2E_0}\partial_\psi P(\omega_0 + \psi(\tau)) + 2\partial_\psi P(\omega_0 + \psi(\tau))^2,$$

for $\tau = \tau(\omega_0, E_0)$. Then f has special form (4.7) and therefore we can study the family $\{f_\omega\}_{\omega\in\Omega}$ of planar maps defined by (4.8). But plugging (4.21) into the definition of $\mathcal{P}$ shows, that f_ω is just the ping-pong map $\mathcal{P}$ in the case of the forcing $p_\omega(t)$. Independently of ω, these maps are defined on $D_\omega = (\psi_\omega \times \mathrm{id})^{-1}(\mathcal{D}) = \mathbb{R} \times (E^*, \infty)$.

We show that f is injective on $\Omega \times (E_{**}, \infty)$, if E_{**} is sufficiently large. Therefore suppose $f(\omega_0, E_0) = (\omega_1, E_1) = f(\tilde{\omega}_0, \tilde{E}_0)$. Since then $\omega_0 + \psi(F(\omega_0, E_0)) = \tilde{\omega}_0 + \psi(F(\tilde{\omega}_0, \tilde{E}_0))$, there is $\omega \in \Omega$ and $t_0, \tilde{t}_0 \in \mathbb{R}$ such that $\omega_0 = \omega + \psi(t_0)$ and $\tilde{\omega}_0 = \omega + \psi(\tilde{t}_0)$. Implicit differentiation yields $\partial_{t_0}\tau(\omega + \psi(t_0), E_0) = \mathcal{O}(E_0^{-1/2})$ and $\partial_{E_0}\tau(\omega + \psi(t_0), E_0) = \mathcal{O}(E_0^{-3/2})$. Moreover, $E_1 = \mathcal{O}(E_0)$ implies

$$D_{f_\omega}(t_0, E_0) = \begin{pmatrix} 1 + \mathcal{O}(E_0^{-1/2}) & \mathcal{O}(E_0^{-3/2}) \\ \mathcal{O}(E_0^{1/2}) & 1 + \mathcal{O}(E_0^{-1/2}) \end{pmatrix}$$

for the Jacobian matrix of f_ω. Throughout this paragraph C will denote positive constants depending on E_{**} and $\|P\|_{C^2_\psi(\Omega)}$, which will not be further specified. Without loss of generality we may assume $E_0 \le \tilde{E}_0$. Then, applying the mean value theorem yields

$$|t_0 - \tilde{t}_0| \le CE_0^{-1/2}|t_0 - \tilde{t}_0| + CE_0^{-3/2}|E_0 - \tilde{E}_0|$$

and $|E_0 - \tilde{E}_0| \le C\tilde{E}_0^{1/2}|t_0 - \tilde{t}_0| + CE_0^{-1/2}|E_0 - \tilde{E}_0|$, provided E_{**} is sufficiently big. Thus, for large E_{**} we get $|t_0 - \tilde{t}_0| \le CE_0^{-3/2}|E_0 - \tilde{E}_0|$

and $|E_0 - \tilde{E}_0| \le C\tilde{E}_0^{1/2}|t_0 - \tilde{t}_0|$. Now, combining these inequalities gives us $|t_0 - \tilde{t}_0| \le CE_0^{-3/2}\tilde{E}_0^{1/2}|t_0 - \tilde{t}_0|$. But since $E_1 = \mathcal{O}(E_0)$ and also $\tilde{E}_0 = \mathcal{O}(E_1)$, we can conclude $|t_0 - \tilde{t}_0| \le CE_0^{-1}|t_0 - \tilde{t}_0|$. In turn, this implies $t_0 = \tilde{t}_0$ and $E_0 = \tilde{E}_0$ for E_{**} sufficiently large, which proves the injectivity of f_ω and f.

Next we want to show that f is also measure-preserving. To this end, consider the maps $g : \Sigma \times [0, S) \times (E^*, \infty) \to \Sigma \times [0, \infty) \times (0, \infty)$ defined by

$$g(\sigma, s, E) = (\sigma, f_\sigma(s, E))$$

and $\chi : \Sigma \times [0, \infty) \to \Sigma \times [0, S)$, $\chi(\sigma, t) = \Phi^{-1}(\sigma \cdot t)$ from (4.5). Then, the identity

$$f = (\Phi \times \mathrm{id}) \circ (\chi \times \mathrm{id}) \circ g \circ (\Phi^{-1} \times \mathrm{id})$$

holds on $\mathcal{D}$. This can be illustrated as follows:

$$\begin{array}{ccccc}
(\omega_0, E_0) & \xmapsto{\;f\;} & (\omega_1, E_1) & & \\
\Phi^{-1}\times\mathrm{id}\downarrow & & & \nwarrow \Phi\times\mathrm{id} & \\
(\sigma_0, s_0, E_0) & \xmapsto{\;g\;} & (\sigma_0, s_1, E_1) & \xmapsto{\;\chi\times\mathrm{id}\;} & (\sigma_1, s_1', E_1)
\end{array}$$

Recalling Lemma 4.8 and the fact that f_ω has a generating function, it suffices to show that $\chi \times \mathrm{id}$ preserves the measure of any Borel set $\mathcal{B} \subset g\big((\Phi^{-1} \times \mathrm{id})(\mathcal{D})\big)$. Therefore, consider the sets

$$\mathcal{B}_k = \mathcal{B} \cap (\Sigma \times [(k-1)S, kS) \times (0, \infty)), \quad k \in \mathbb{N}.$$

Then we have

$$(\mu_\Sigma \otimes \lambda^2)\,((\chi \times \mathrm{id})(\mathcal{B}_k)) = (\mu_\Sigma \otimes \lambda^2)\,(\mathcal{B}_k),$$

as depicted in Section 4.1.3. Moreover, the injectivity of f implies the injectivity of $\chi \times \mathrm{id}$ on $\mathcal{B}$ and thus the sets $(\chi \times \mathrm{id})(\mathcal{B}_k)$ are mutually disjoint. Since $\mathcal{B} = \cup_{k\in\mathbb{N}}\mathcal{B}_k$, this yields $(\mu_\Sigma \otimes \lambda^2)\,((\chi \times \mathrm{id})(\mathcal{B})) = (\mu_\Sigma \otimes \lambda^2)\,(\mathcal{B})$.

Finally, we need to find a function $W \in C^1_\psi(\Omega \times (0, \infty))$ such that (4.9) and (4.10) are verified. For this define

$$W(\omega_0, E_0) = P(\omega_0)^2 E_0.$$

Conditions (4.9) clearly holds if we take $\beta = a^2$ and $\delta = b^2$ with a, b from (4.18). Moreover, the definition of f yields

$$\begin{aligned} W(f(\omega_0, E_0)) - W(\omega_0, E_0) &= P(\omega_1)^2 E_1 - P(\omega_0)^2 E_0 \\ &= P(\omega_0 + \psi(F(\omega_0, E_0)))^2 E_1 - P(\omega_0)^2 E_0 \\ &= p_{\omega_0}(F(\omega_0, E_0))^2 E_1 - p_{\omega_0}(0)^2 E_0. \end{aligned}$$

Now let $t_0 = 0$ and $(t_1, E_1) = f_{\omega_0}(t_0, E_0)$. Then $t_1 = F(\omega_0, E_0)$ and thus Lemma 4.14 yields

$$W(f(\omega_0, E_0)) - W(\omega_0, E_0) = p_{\omega_0}(t_1)^2 E_1 - p_{\omega_0}(t_0)^2 E_0 \leq C\Delta(E_0),$$

where $C > 0$ is uniform in ω_0. But then taking $k(E_0) = C\Delta(E_0)$ proves (4.10), since $\lim_{r\to\infty} \Delta(r) = 0$ follows from $\partial_\psi^2 P \in C(\Omega)$.

Now we have validated all conditions of Theorem 4.9 for the map f. Applying it yields $\lambda^2(\hat{E}_\omega) = 0$ for almost all $\omega \in \Omega$, where

$$\hat{E}_\omega = \{(t_0, E_0) \in \hat{D}_{\omega,\infty} : \lim_{n\to\infty} E_n = \infty\}$$

and $\hat{D}_{\omega,\infty}$ is defined as in Section 4.2.1. This can be translated back to the original coordinates $(t, v) = (t, \sqrt{2E})$. We write g_ω for the ping-pong map $(t_0, v_0) \mapsto (t_1, v_1)$ from (5.5) for the forcing $p(t) = p_\omega(t)$ and let

$$\tilde{D}_\omega = \mathbb{R} \times (\sqrt{2E^*}, \infty), \quad \tilde{D}_{\omega,1} = \tilde{D}_\omega, \quad \tilde{D}_{\omega,n+1} = g_\omega^{-1}(\tilde{D}_{\omega,n}), \quad \tilde{D}_{\omega,\infty} = \bigcap_{n=1}^{\infty} \tilde{D}_{\omega,n}.$$

Then $\lambda^2(\tilde{E}_\omega) = 0$ for almost all $\omega \in \Omega$, where

$$\tilde{E}_\omega = \{(t_0, v_0) \in \tilde{D}_{\omega,\infty} : \lim_{n\to\infty} v_n = \infty\}.$$

Now, consider the escaping set E_ω from the theorem and take $(t_0, v_0) \in E_\omega$. Since $\lim_{n\to\infty} v_n = \infty$, there is $n_0 \in \mathbb{N}$ such that $v_n > \sqrt{2E^*}$ for all $n \geq n_0$. But this just means $(t_n, v_n) \in \tilde{E}_\omega$ for $n \geq n_0$. In particular, this implies $E_\omega \subset \bigcup_{n\in\mathbb{N}} g_\omega^{-n}(\tilde{E}_\omega)$. Considering that g_ω is area-preserving, this proves the assertion: $\lambda^2(E_\omega) = 0$ for almost all $\omega \in \Omega$. □

4.4 A superlinear oscillator

For $\alpha \geq 3$, consider the second order differential equation

$$\ddot{x} + |x|^{\alpha-1}x = p(t), \tag{4.22}$$

where $p \in C_b^4(\mathbb{R})$. Here, $C_b^k(\mathbb{R})$ denotes the space of bounded functions with continuous and bounded derivatives up to order k.

Recall that a solution to (4.22) is called Poisson stable, if there is a sequence $(t_n)_{n\in\mathbb{Z}}$ with $t_n \to \pm\infty$ as $n \to \pm\infty$ such that

$$|x(t+t_n) - x(t)| + |\dot{x}(t+t_n) - \dot{x}(t)| \to 0, \quad \text{as } n \to \pm\infty,$$

uniformly on every bounded interval in $\mathbb{R}$. With this definition in mind we can state the main result of this section.

Theorem 4.15. *Given $P \in C_\psi^4(\Omega)$, consider the family $\{p_\omega\}_{\omega\in\Omega}$ of almost periodic forcing functions defined by*

$$p_\omega(t) = P(\omega + \psi(t)), \quad t \in \mathbb{R}.$$

Let $x_\omega(t;\tilde{x},\tilde{v})$ denote the solution of (4.22) with $p(t) = p_\omega(t)$ satisfying the initial condition $x_\omega(0) = \tilde{x}$ and $\dot{x}_\omega(0) = \tilde{v}$. Then, for almost all $(\tilde{x},\tilde{v},\omega) \in \mathbb{R}^2 \times \Omega$, the solutions $x_\omega(t;\tilde{x},\tilde{v})$ is Poisson stable.

The remainder of this chapter is dedicated to proving this theorem.

4.4.1 Transformation to suitable coordinates

We start by considering the case of a general (not necessarily almost periodic) forcing function $p \in C_b^4(\mathbb{R})$. Note that solutions $x(t)$ to (4.22) are unique and exist for all times. To see this, we define

$$E(t) = \frac{1}{2}\dot{x}(t)^2 + \frac{1}{\alpha+1}|x(t)|^{\alpha+1}.$$

Then $|\dot{E}| = |p(t)\dot{x}| \leq |p(t)|\sqrt{2E}$, and therefore

$$\sqrt{E(t)} \leq \sqrt{E(t_0)} + \frac{1}{\sqrt{2}}\left|\int_{t_0}^{t}|p(s)|ds\right|.$$

So E is bounded on finite intervals and thus x can be continued on $\mathbb{R}$.

Action-angle coordinates

First we want to reformulate (4.22) in terms of the action-angle coordinates of the unperturbed system, that is

$$\ddot{x} + |x|^{\alpha-1}x = 0. \tag{4.23}$$

The orbits of (4.23) are closed curves, defined by $\frac{1}{2}y^2 + \frac{1}{\alpha+1}|x|^{\alpha+1} = \text{const.}$ and correspond to periodic solutions. For $\lambda > 0$ let x_λ denote the solution of (4.23) having the initial values

$$x_\lambda(0) = \lambda, \quad \dot{x}_\lambda(0) = 0.$$

Using the homogeneity of the problem, we get $x_\lambda(t) = \lambda x_1(\lambda^{\frac{\alpha-1}{2}}t)$. In particular x_λ has a decreasing minimal period $T(\lambda) = \lambda^{\frac{1-\alpha}{2}}T(1)$. Thus we can find the unique number $\Lambda > 0$, such that $T(\Lambda) = 2\pi$. We will use the notation

$$c(t) = x_\Lambda(t), \quad s(t) = \dot{c}(t),$$

since in a lot of ways these functions behave like the trigonometric functions cos and sin: c is even, s is odd, and both are anti-periodic with period π. Hence they have zero mean value, i.e.

$$\int_0^{2\pi} c(t)\,dt = \int_0^{2\pi} s(t)\,dt = 0.$$

In this case however, $(c(t), s(t))$ spins clockwise around the origin of the $(x, \dot{x})$-plane. Furthermore c and s meet the identity

$$\frac{1}{2}s(t)^2 + \frac{1}{\alpha+1}|c(t)|^{\alpha+1} = \frac{1}{\alpha+1}\Lambda^{\alpha+1} \quad \forall t \in \mathbb{R}. \tag{4.24}$$

We define a change of variables $\eta : \mathbb{S}^1 \times (0,\infty) \to \mathbb{R}^2 \setminus \{0\},\ (\bar{\vartheta}, r) \mapsto (x, v)$ by

$$x = \gamma r^{\frac{2}{\alpha+3}} c(\bar{\vartheta}), \quad v = \gamma^{\frac{\alpha+1}{2}} r^{\frac{\alpha+1}{\alpha+3}} s(\bar{\vartheta}),$$

where $\gamma > 0$ is determined by

$$\gamma^{\frac{\alpha+3}{2}} \frac{2}{\alpha+3}\Lambda^{\alpha+1} = 1.$$

This choice of γ makes η a symplectic diffeomorphism, as can be shown by an easy calculation. Moreover, (4.24) implies the identity

$$\frac{1}{2}v^2 + \frac{1}{\alpha+1}|x|^{\alpha+1} = \kappa_1 r^{\frac{2(\alpha+1)}{\alpha+3}},$$

where $\kappa_1 = \frac{1}{\alpha+1}(\gamma\Lambda)^{\alpha+1}$. Adding a new component for the time, we define the transformation map

$$\mathcal{R} : \mathbb{R}^2 \setminus \{0\} \times \mathbb{R} \to \mathbb{S}^1 \times (0,\infty) \times \mathbb{R}, \;\; \mathcal{R}(x,v;t) = (\eta^{-1}(x,v);t).$$

Going back to the perturbed system, the old Hamiltonian

$$\mathcal{K}(x,\dot{x};t) = \frac{1}{2}\dot{x}^2 + \frac{1}{\alpha+1}|x|^{\alpha+1} - p(t)x$$

expressed in the new coordinates is

$$\mathcal{H}(\bar{\vartheta},r;t) = \kappa_1 r^{\frac{2(\alpha+1)}{\alpha+3}} - \gamma r^{\frac{2}{\alpha+3}} p(t) c(\bar{\vartheta}).$$

For simplicity's sake let us denote the lift of $\mathcal{H}$ onto $\mathbb{R} \times (0,\infty) \times \mathbb{R}$ by the same letter $\mathcal{H}$. The associated differential equations then become

$$\begin{cases} \dot{\vartheta} &= \partial_r \mathcal{H} = \frac{2(\alpha+1)}{\alpha+3}\kappa_1 r^{\frac{\alpha-1}{\alpha+3}} - \frac{2}{\alpha+3}\gamma r^{-\frac{\alpha+1}{\alpha+3}} p(t) c(\vartheta) \\ \dot{r} &= -\partial_\vartheta \mathcal{H} = \gamma r^{\frac{2}{\alpha+3}} p(t) s(\vartheta) \end{cases}. \tag{4.25}$$

It should be noted that solutions to (4.25) only exist on intervals $J \subset \mathbb{R}$, where $r(t) > 0$. Therefore, we can only make assertions about solutions of the original problem (4.22) defined on intervals, where $(x,\dot{x}) \neq 0$, when working with these action-angle coordinates.

Time-energy coordinates

In order to construct a measure preserving embedding one could take the Poincaré map of Hamiltonian system (4.25). However, to fit the setting of subsection 4.2.1, this map would need to have the time (and thus the almost periodic dependence of the system) as the first variable. Therefore, we will follow [Lev91] and take the time t as the new "position"-coordinate, the energy $\mathcal{H}$ as the new "momentum" and the angle ϑ as the new independent variable.

Since the first term in (4.25) is dominant for $r \to \infty$, one can find r_* such that

$$\partial_r \mathcal{H}(\vartheta, r; t) \geq 1 \text{ for all } r \geq r_*. \tag{4.26}$$

Remark 4.16. The value of r_* depends upon $\alpha, \gamma, \kappa_1, \|c\|_{C_b}$ and $\|p\|_{C_b^4}$, where again $\gamma, \kappa_1, \|c\|_{C_b}$ are uniquely determined by the choice of α. We will call quantities depending only upon α and $\|p\|_{C_b^4}$ constants. Let us also point out, that r_* can be chosen "increasingly in $\|p\|_{C_b^4}$". By this we mean, that if $r_* = r_*(\alpha, \|p\|_{C_b^4})$ is the threshold corresponding to some $p \in C_b^4(\mathbb{R})$, then (4.26) also holds for any forcing $\tilde{p} \in C_b^4$ with $\|\tilde{p}\|_{C_b^4} \leq \|p\|_{C_b^4}$. Indeed, all thresholds we will construct have this property.

Now consider a solution (ϑ, r) of (4.25) defined on an interval J, where $r(t) > r_*$ for all $t \in J$. Then, the function $t \mapsto \vartheta(t)$ is invertible, since

$$\dot{\vartheta}(t) = \partial_r \mathcal{H}(\vartheta(t), r(t); t) \geq 1.$$

Adopting the notation of [KO13], we will write $\tau = \vartheta(t)$ and denote the inverse by ϕ, i.e. $\phi(\tau) = t$. Since $\vartheta(t)$ is at least of class C^2, the same holds for the inverse function ϕ defined on $\vartheta(J)$. Let us now define

$$I(\tau) = \mathcal{H}(\tau, r(\phi(\tau)); \phi(\tau)) \text{ for } \tau \in \vartheta(J).$$

This function will be the new momentum. It is a well-known fact that the resulting system is again Hamiltonian. To find the corresponding Hamiltonian, we can solve the equation

$$\mathcal{H}(\vartheta, h; t) = I$$

implicitly for $h(t, I; \vartheta)$. Because of (4.26), this equation admits a solution, which is well-defined on the open set

$$G = \{(t, I; \vartheta) \in \mathbb{R}^3 : I > \mathcal{H}(\vartheta, r_*; t)\}.$$

Indeed, by implicit differentiation it can be verified that

$$\phi' = \partial_I h, \quad I' = -\partial_\phi h,$$

where the prime $'$ indicates differentiation with respect to τ. Using the new coordinates, we have to solve

$$\kappa_1 h^{\frac{2(\alpha+1)}{\alpha+3}} - \gamma h^{\frac{2}{\alpha+3}} p(\phi)c(\tau) = I \tag{4.27}$$

or equivalently

$$h = I^{\frac{\alpha+3}{2(\alpha+1)}} \kappa_1^{-\frac{\alpha+3}{2(\alpha+1)}} (1 - \kappa_1^{-1}\gamma h^{\frac{-2\alpha}{\alpha+3}} p(\phi)c(\tau))^{-\frac{\alpha+3}{2(\alpha+1)}}. \tag{4.28}$$

Since $p \in C^4$ and $c \in C^3$, also h will be of class C^3. Moreover, we can find $I_* > 0$ (depending upon $\alpha, \gamma, \kappa_1, \|c\|_{C_b}, \|p\|_{C_b}$ and r_*) such that

$$\{(\phi, I; \tau) \in \mathbb{R}^3 : I \geq I_*\} \subset G.$$

Furthermore, we can choose I_* so large that the solution h of (4.27) satisfies

$$\alpha_0 I^{\frac{\alpha+3}{2(\alpha+1)}} \leq h \leq \beta_0 I^{\frac{\alpha+3}{2(\alpha+1)}} \quad \text{for} \quad I \geq I_*$$

for some constants $\alpha_0, \beta_0 > 0$. Let

$$\kappa_0 = \kappa_1^{-\frac{\alpha+3}{2(\alpha+1)}} = \left(\frac{2(\alpha+1)}{\alpha+3}\gamma^{\frac{1-\alpha}{2}}\right)^{\frac{\alpha+3}{2(\alpha+1)}}.$$

To approximate the solution $h(\phi, I; \tau)$ of (4.28), one can use the Taylor polynomial of degree one for $(1-z)^{-\frac{\alpha+3}{2(\alpha+1)}}$ and then plug in the highest order approximation $\kappa_0 I^{\frac{\alpha+3}{2(\alpha+1)}}$ for the remaining h on the right-hand side. Therefore we define the remainder function $R \in C^3(G)$ through the relation

$$h(\phi, I; \tau) = \kappa_0 I^{\frac{\alpha+3}{2(\alpha+1)}} + \frac{(\alpha+3)}{2(\alpha+1)}\gamma\kappa_0^{\frac{\alpha+5}{\alpha+3}} p(\phi)c(\tau) I^{\frac{3-\alpha}{2(\alpha+1)}} + R(\phi, I; \tau). \tag{4.29}$$

The corresponding system is described by

$$\begin{cases} \phi' &= \partial_I h = \kappa_0 \frac{\alpha+3}{2(\alpha+1)} I^{\frac{1-\alpha}{2(\alpha+1)}} + \frac{9-\alpha^2}{4(\alpha+1)^2}\gamma\kappa_0^{\frac{\alpha+5}{\alpha+3}} p(\phi)c(\tau) I^{\frac{1-3\alpha}{2(\alpha+1)}} + \partial_I R, \\ I' &= -\partial_\phi h = -\frac{\alpha+3}{2(\alpha+1)}\gamma\kappa_0^{\frac{\alpha+5}{\alpha+3}} \dot{p}(\phi)c(\tau) I^{\frac{3-\alpha}{2(\alpha+1)}} - \partial_\phi R. \end{cases} \tag{4.30}$$

The change of variables $(\vartheta, r; t) \mapsto (\phi, I; \tau)$ can be realized via the transformation map $\mathcal{S} : \mathbb{R} \times [r_*, \infty) \times \mathbb{R} \to \mathbb{R} \times (0, \infty) \times \mathbb{R}$ defined by

$$\mathcal{S}(\vartheta, r; t) = (t, \mathcal{H}(\vartheta, r; t); \vartheta).$$

So $\mathcal{S}$ maps a solution $(\vartheta(t), r(t))$ of (4.25) with the initial condition $(\vartheta(t_0), r(t_0)) = (\vartheta_0, r_0)$ onto a solution $(\phi(\tau), I(\tau))$ of (4.30) with initial condition $(\phi(\vartheta_0), I(\vartheta_0)) = (t_0, \mathcal{H}(\vartheta_0, r_0; t_0))$.

The following lemma by Kunze and Ortega [KO13, Lemma 7.1] shows that R is small in a suitable sense:

Lemma 4.17. *There are constants $C_0 > 0$ and $I_{C_0} \geq I_* > 0$ (depending upon $\|p\|_{C_b^2(\mathbb{R})}$) such that*

$$|R| + |\partial_\phi R| + I|\partial_I R| + |\partial^2_{\phi\phi} R| + I|\partial^2_{\phi I} R| + I^2|\partial^2_{II} R| \leq C_0 I^{\frac{3(1-\alpha)}{2(\alpha+1)}} \quad (4.31)$$

holds for all $\phi, \tau \in \mathbb{R}$ and $I \geq I_{C_0}$.

In terms of the function space $\mathcal{F}^k(r)$ from Appendix A, this could be expressed as $R(\cdot, \cdot; \tau) \in \mathcal{F}^2\left(\frac{3(\alpha-1)}{2(\alpha+1)}\right)$ for all $\tau \in \mathbb{R}$. Now we could use these coordinates and a corresponding Poincaré map for Theorem 4.9. Unfortunately, the energy $I(\tau)$ fails to be an adiabatic invariant in the sense of (4.10), since for $\alpha = 3$ we do not have $I' \to 0$ as $I \to \infty$. Therefore we have to do one further transformation. For $\alpha > 3$ this last step would not be necessary.

A last transformation

In [KO13, Theorem 6.7] Ortega and Kunze constructed a change of coordinates, which reduces the power of the momentum variable in the second term of (4.29) while preserving the special structure of the Hamiltonian. Since in their paper they had to use this transformation several times consecutively, the associated theorem is somewhat general and too complicated for our purpose here. Thus we will cite it only in the needed form. For $\mu > 0$ we set

$$\Sigma_\mu = \mathbb{R} \times [\mu, \infty) \times \mathbb{R}. \quad (4.32)$$

Lemma 4.18. *Consider the Hamiltonian h from (4.29), i.e.*

$$h(\phi, I; \tau) = \kappa_0 I^{\frac{\alpha+3}{2(\alpha+1)}} + f(\phi)c(\tau)I^{\frac{3-\alpha}{2(\alpha+1)}} + R(\phi, I; \tau),$$

where $f(\phi) = \frac{(\alpha+3)}{2(\alpha+1)}\gamma\kappa_0^{\frac{\alpha+5}{\alpha+3}}p(\phi)$, *and* I_{C_0} *from Lemma 4.17. Then there exists* $I_{**} > I_{C_0}$, $\mathcal{I}_* > 0$ *and a* C^1*-diffeomorphism*

$$\mathcal{T} : \Sigma_{I_{**}} \to \mathcal{T}(\Sigma_{I_{**}}) \subset \Sigma_{\mathcal{I}_*}, \ \ (\phi, I; \tau) \mapsto (\varphi, \mathcal{I}; \tau),$$

which transforms the system (4.30) into $\varphi' = \partial_{\mathcal{I}} h_1, \mathcal{I}' = -\partial_\varphi h_1$, *where*

$$h_1(\varphi, \mathcal{I}; \tau) = \kappa_0 \mathcal{I}^{\frac{\alpha+3}{2(\alpha+1)}} + f_1(\varphi)c_1(\tau)\mathcal{I}^{b_\alpha} + R_1(\varphi, \mathcal{I}; \tau).$$

The new functions appearing in h_1 *satisfy*

(a) $f_1(\varphi) = -\frac{\alpha+3}{2(\alpha+1)}\kappa_0 \dot{f}(\varphi) = -\left(\frac{\alpha+3}{2(\alpha+1)}\right)^2 \gamma\kappa_0^{\frac{2\alpha+8}{\alpha+3}}\dot{p}(\varphi)$,

(b) $c_1 \in C^4(\mathbb{R}), c_1'(\tau) = c(\tau), \int_0^{2\pi} c_1(\tau)\, d\tau = 0$,

(c) $b_\alpha = -\frac{3\alpha^2-2\alpha-9}{2(\alpha+3)(\alpha+1)} < \frac{3-\alpha}{2(\alpha+1)} \leq 0$, *and*

(d) $R_1 \in C^3(\Sigma_{\mathcal{I}_*})$ *satisfies (4.31) for all* $\mathcal{I} \geq \mathcal{I}_*$ *and with some constant* $\tilde{C}_0 > 0$.

The quantities $I_{**}, \mathcal{I}_*$ *and* $\tilde{C}_0$ *can be estimated in terms of* $\alpha, \kappa_0, \|f\|_{C_b^4(\mathbb{R})}$, $\|c\|_{C_b(\mathbb{R})}$, *and* C_0 *from Lemma 4.17. Furthermore, the change of variables* $\mathcal{T}$ *has the following properties:*

(i) $\mathcal{T}(\cdot, \cdot; \tau)$ *is symplectic for all* $\tau \in \mathbb{R}$, *i.e.* $d\varphi \wedge d\mathcal{I} = d\phi \wedge dI$,

(ii) $\mathcal{T}(\phi, I; \tau + 2\pi) = \mathcal{T}(\phi, I; \tau) + (0, 0; 2\pi)$, *and*

(iii) $I/2 \leq \mathcal{I}(\phi, I; \tau) \leq 2I$ *for all* $(\phi, I; \tau)$.

Even if we omit the proof here, let us note that the change of variables can be realized via the generating function

$$\Psi(\phi, \mathcal{I}; \tau) = -\mathcal{I}^{\frac{3-\alpha}{2(\alpha+1)}} f(\phi)c_1(\tau),$$

where c_1 is uniquely determined by the conditions in (b). Therefore $\mathcal{T}$ is implicitly defined by the equations

$$I = \mathcal{I} + \partial_\phi \Psi, \quad \varphi = \phi + \partial_{\mathcal{I}} \Psi$$

and one can determine h_1 through the relation

$$h_1(\varphi, \mathcal{I}; \tau) = h(\phi, I; \tau) + \partial_\tau \Psi(\phi, \mathcal{I}; \tau).$$

4.4.2 The successor map

Consider the new Hamiltonian from Theorem 4.18, that is

$$h_1(\varphi, \mathcal{I}; \tau) = \kappa_0 \mathcal{I}^{\frac{\alpha+3}{2(\alpha+1)}} + f_1(\varphi) c_1(\tau) \mathcal{I}^{b_\alpha} + R_1(\varphi, \mathcal{I}; \tau),$$

which is well-defined on the set $\mathcal{T}(\Sigma_{I_{**}})$ and 2π-periodic in the time variable τ. The corresponding equations of motion are

$$\begin{cases} \varphi' &= \partial_{\mathcal{I}} h_1 = \kappa_0 \frac{\alpha+3}{2(\alpha+1)} \mathcal{I}^{\frac{1-\alpha}{2(\alpha+1)}} + b_\alpha f_1(\varphi) c_1(\tau) \mathcal{I}^{b_\alpha - 1} + \partial_{\mathcal{I}} R_1, \\ \mathcal{I}' &= -\partial_\varphi h_1 = -\dot{f}_1(\varphi) c_1(\tau) \mathcal{I}^{b_\alpha} - \partial_\varphi R_1, \end{cases} \tag{4.33}$$

where $\dot{f}_1(\varphi) = -\left(\frac{\alpha+3}{2(\alpha+1)}\right)^2 \kappa_0^{\frac{2\alpha+8}{\alpha+3}} \gamma \ddot{p}(\varphi)$.

Now, suppose $(\varphi_0, \mathcal{I}_0; \tau_0) \in \mathcal{T}(\Sigma_{I_{**}})$ and denote the solution of (4.33) with initial data $\varphi(\tau_0) = \varphi_0$, $\mathcal{I}(\tau_0) = \mathcal{I}_0$ by

$$(\varphi(\tau; \varphi_0, \mathcal{I}_0, \tau_0), \mathcal{I}(\tau; \varphi_0, \mathcal{I}_0, \tau_0)).$$

We want to construct a subset $\Sigma_{\mathcal{I}_{**}} = \mathbb{R} \times [\mathcal{I}_{**}, \infty) \times \mathbb{R} \subset \mathcal{T}(\Sigma_{I_{**}})$ such that $(\varphi, \mathcal{I})$ is defined on the whole interval $[\tau_0, \tau_0 + 2\pi]$ whenever $(\varphi_0, \mathcal{I}_0, \tau_0) \in \Sigma_{\mathcal{I}_{**}}$. Similar to [KO13, Lemma 4.1], we state:

Lemma 4.19. *There exists a constant $\mathcal{I}_{**} > \mathcal{I}_*$ (depending only upon $\alpha, \|f\|_{C_b^4(\mathbb{R})}, \|c\|_{C_b(\mathbb{R})}$ and $\tilde{C}_0$ from Theorem 4.18) such that $\Sigma_{\mathcal{I}_{**}} \subset \mathcal{T}(\Sigma_{I_{**}})$ and for any $(\varphi_0, \mathcal{I}_0, \tau_0) \in \Sigma_{\mathcal{I}_{**}}$ the solution $(\varphi, \mathcal{I})$ of (4.33) with initial data*

$$\varphi(\tau_0) = \varphi_0, \quad \mathcal{I}(\tau_0) = \mathcal{I}_0$$

exists on $[\tau_0, \tau_0 + 2\pi]$, where it satisfies

$$\frac{\mathcal{I}_0}{4} \le \mathcal{I}(\tau) \le 4\mathcal{I}_0 \quad \text{for } \tau \in [\tau_0, \tau_0 + 2\pi].$$

Proof. Suppose $\mathcal{I}_0 \geq \mathcal{I}_{**} \geq 4\mathcal{I}_*$, then (iii) from Theorem 4.18 yields $\Sigma_{\mathcal{I}_{**}} \subset \mathcal{T}(\Sigma_{I_{**}})$. Now, let $T > 0$ be maximal such that $\mathcal{I}_0/4 \leq \mathcal{I}(\tau) \leq 4\mathcal{I}_0$ holds for all $\tau \in [\tau_0, \tau_0 + T)$. On this interval we have

$$(\mathcal{I}^{1-b_\alpha})' = (1-b_\alpha)\mathcal{I}^{-b_\alpha}\mathcal{I}' = (1-b_\alpha)\mathcal{I}^{-b_\alpha}(-\dot{f}_1(\varphi)c_1(\tau)\mathcal{I}^{b_\alpha} - \partial_\varphi R_1)$$

and thus

$$\begin{aligned}|(\mathcal{I}^{1-b_\alpha})'| &\leq |(1-b_\alpha)| \left(\|\dot{f}_1\|_{C_b} \|c_1\|_{C_b} + |\partial_\varphi R_1|\mathcal{I}^{-b_\alpha}\right) \\ &\leq |(1-b_\alpha)| \left(\|\dot{f}_1\|_{C_b} \|c_1\|_{C_b} + \tilde{C}_0\right) = \hat{C},\end{aligned}$$

with $\tilde{C}_0 > 0$ from (d) of Theorem 4.18, since $b_\alpha = -\frac{3\alpha^2-2\alpha-9}{2(\alpha+3)(\alpha+1)} > \frac{3(1-\alpha)}{2(\alpha+1)}$. Now assume $T \leq 2\pi$, then for $\mathcal{I}_{**}$ sufficiently large we conclude

$$\left(\frac{\mathcal{I}_0}{2}\right)^{1-b_\alpha} \leq \mathcal{I}_0^{1-b_\alpha} - 2\pi\hat{C} \leq \mathcal{I}(\tau)^{1-b_\alpha} \leq \mathcal{I}_0^{1-b_\alpha} + 2\pi\hat{C} \leq (2\mathcal{I}_0)^{1-b_\alpha}$$

on the whole interval $[\tau_0, \tau_0 + T)$. This contradicts the definition of T and thus completes the proof. □

We can therefore consider the Poincaré map $\Phi : \mathbb{R} \times [\mathcal{I}_{**}, \infty) \to \mathbb{R}^2$ corresponding to the periodic system (4.33), defined by

$$\Phi(\varphi_0, \mathcal{I}_0) = (\varphi(5\pi/2; \varphi_0, \mathcal{I}_0, \pi/2), \mathcal{I}(5\pi/2; \varphi_0, \mathcal{I}_0, \pi/2)). \tag{4.34}$$

The choice $\tau_0 = \frac{\pi}{2}$ is basically due to computational advantages, since $c(\vartheta) = 0$ if and only if $\vartheta = \pi/2 + m\pi$ with $m \in \mathbb{Z}$. Moreover, values of $\tau = \vartheta$ in $\pi/2 + 2\pi\mathbb{Z}$ correspond exactly to those zeros of the solution $x(t)$, where $\dot{x} < 0$. We write

$$\Phi(\varphi_0, \mathcal{I}_0) = (\varphi_1, \mathcal{I}_1).$$

Now that we have defined a suitable successor map, we can prove that $\mathcal{I}$ is an adiabatic invariant in the sense of equation (4.10):

Lemma 4.20. *There is a constant $C > 0$ (depending only upon α, $\|f\|_{C_b^4(\mathbb{R})}$, $\|c\|_{C_b(\mathbb{R})}$ and $\tilde{C}_0$) such that*

$$|\mathcal{I}_1 - \mathcal{I}_0| \leq C\mathcal{I}_0^{b_\alpha}$$

holds for all $(\varphi_0, \mathcal{I}_0) \in \mathbb{R} \times [\mathcal{I}_{**}, \infty)$.

Proof. With a similar reasoning like in the proof of Lemma 4.19 we get

$$|\mathcal{I}'(\tau)| = |-\dot{f}_1(\varphi)c_1(\tau)\mathcal{I}^{b_\alpha}(\tau) - \partial_\varphi R_1| \leq \|\dot{f}_1\|_{C_b}\ \|c_1\|_{C_b}\mathcal{I}(\tau)^{b_\alpha} + \tilde{C}_0\mathcal{I}(\tau)^{\frac{3(1-\alpha)}{2(\alpha+1)}}$$
$$\leq \left(\|\dot{f}_1\|_{C_b}\ \|c_1\|_{C_b} + \tilde{C}_0\right)\mathcal{I}(\tau)^{b_\alpha} \leq \left(\|\dot{f}_1\|_{C_b}\ \|c_1\|_{C_b} + \tilde{C}_0\right)4^{-b_\alpha}\mathcal{I}_0^{b_\alpha}.$$

Now integrating over $[\pi/2, 5\pi/2]$ gives us

$$|\mathcal{I}_1 - \mathcal{I}_0| \leq 2\pi\left(\|\dot{f}_1\|_{C_b}\ \|c_1\|_{C_b} + \tilde{C}_0\right)4^{-b_\alpha}\mathcal{I}_0^{b_\alpha}.$$

□

4.4.3 Almost periodicity

So far all our considerations have dealt with the case of a general forcing function $p \in C_b^4(\mathbb{R})$. Now, let (Ω, ψ) be as in Section 4.1.1 and consider a map $P \in C_\psi^4(\Omega)$. We will replace $p(t)$ by

$$p_\omega(t) = P(\omega + \psi(t)), \quad \omega \in \Omega,$$

and show that the almost periodicity is inherited by the Hamiltonian system (4.33). We have $p_\omega \in C_b^4(\mathbb{R})$ and $\|p_\omega\|_{C_b^4(\mathbb{R})} = \|P\|_{C_\psi^4(\Omega)}$. Therefore all results of the previous sections are applicable. Considering Remark 4.16, this also implies that we can find new constants r_*, I_* etc. depending only upon α and $\|P\|_{C_\psi^4(\Omega)}$ such that corresponding estimates hold uniformly in $\omega \in \Omega$.

Since $\mathcal{R}$ and $\mathcal{S}$ basically leave the time variable t unchanged, it is straightforward to prove that the transformation to action-angle coordinates as well as the change to the time-energy coordinates (ϕ, I) preserve the almost periodic structure. Using the notation $\psi_\omega(t) = \omega + \psi(t)$, we find functions

$$H, \boldsymbol{R} : \Omega \times [I_*, \infty) \times \mathbb{R} \to \mathbb{R} \times [I_*, \infty) \times \mathbb{R}$$

in the class $C_\psi^3(\Omega \times [I_*, \infty) \times \mathbb{R})$ depending on P such that

$$h(\phi, I; \tau) = H(\psi_\omega(\phi), I; \tau), \quad R(\phi, I; \tau) = \boldsymbol{R}(\psi_\omega(\phi), I; \tau).$$

holds for every $\omega \in \Omega$.

Remark 4.21. Let us note, that the functions h, R etc. now depend on the choice of ω. Thus it would be more precise to write h_ω, R_ω and so on, but for reasons of clarity we will omit the index throughout this section. The functions $H, \boldsymbol{R}$ etc. on the other hand are uniquely determined by P.

However it requires a bit more work, to see that also the transformation $\mathcal{T}$ defined in Theorem 4.18 retains the almost periodic properties. We recall that this change of variables is defined by

$$I = \mathcal{I} + \partial_\phi \Psi, \quad \varphi = \phi + \partial_{\mathcal{I}} \Psi,$$

where $\Psi(\phi, \mathcal{I}; \tau) = -\mathcal{I}^{\frac{3-\alpha}{2(\alpha+1)}} f(\phi) c_1(\tau)$. In particular, we have $\varphi(\phi, \mathcal{I}; \tau) = \phi + \eta(\mathcal{I}, \tau) f(\phi)$ with

$$\eta(\mathcal{I}, \tau) = -\frac{3-\alpha}{2(\alpha+1)} \mathcal{I}^{\frac{1-3\alpha}{2(\alpha+1)}} c_1(\tau) \quad \text{and} \quad f(\phi) = \frac{(\alpha+3)}{2(\alpha+1)} \gamma \kappa_0^{\frac{\alpha+5}{\alpha+3}} P(\psi_\omega(\phi)).$$

Lemma 4.22. *Let* $F \in C^4_\psi(\Omega)$ *and* $f(\phi) = F(\psi_\omega(\phi))$ *be given. Moreover, suppose* $\eta \in C^4((0, \infty) \times \mathbb{R})$ *satisfies* $\lim_{I \to \infty} \eta(\mathcal{I}, \tau) = 0$ *uniformly and* $\tilde{\mathcal{I}} > 0$ *is such that* $|\eta(\mathcal{I}, \tau)| \|F\|_{C^4_\psi(\Omega)} < 1/4$ *for* $\mathcal{I} \geq \tilde{\mathcal{I}}$. *Finally, let*

$$\varphi(\phi, \mathcal{I}; \tau) = \phi + \eta(\mathcal{I}, \tau) f(\phi).$$

Then, the map $\phi \mapsto \varphi(\phi, \mathcal{I}; \tau)$ *is invertible for* $\mathcal{I} \geq \tilde{\mathcal{I}}$ *and its inverse can be written in the form*

$$\phi(\varphi, \mathcal{I}; \tau) = \varphi + q(\varphi, \mathcal{I}; \tau),$$

where $q(\varphi, \mathcal{I}; \tau) = Q(\psi_\omega(\varphi), \mathcal{I}; \tau)$ *with* $Q \in C^4_\psi(\Omega \times [\tilde{\mathcal{I}}, \infty) \times \mathbb{R})$.

Proof. First, consider the function $Z(\phi, \varphi, \mathcal{I}, \tau) = \phi + \eta(\mathcal{I}, \tau) f(\phi) - \varphi$. For $\mathcal{I} \geq \tilde{\mathcal{I}}$, we have $\partial_\phi Z \geq \frac{1}{2}$ and since $Z \in C^4(\mathbb{R}^2 \times [\tilde{\mathcal{I}}, \infty) \times \mathbb{R})$ the equation $Z = 0$ defines a unique solution $\phi(\varphi, \mathcal{I}; \tau)$ of class C^4. Thus also the function

$$q(\varphi, \mathcal{I}; \tau) = \phi(\varphi, \mathcal{I}; \tau) - \varphi$$

is of that class. At this point we fix $(\mathcal{I}, \tau) \in [\tilde{\mathcal{I}}, \infty) \times \mathbb{R}$ and for clarity we do not write the dependence explicitly. Thus, the reasoning above yields

$$q(\varphi) = \phi(\varphi) - \varphi = -\eta f(\phi(\varphi)) = -\eta f(\varphi + q(\varphi)) \tag{4.35}$$

and vice versa

$$q(\phi + \eta f(\phi)) = -\eta f(\phi). \tag{4.36}$$

Now, let (ϕ_n) be any sequence of reals. In order to show that q is almost periodic, we need to find a subsequence (ϕ_{n_k}) so that $q(\varphi + \phi_{n_k})$ converges uniformly. Since f is a.p., there is a subsequence (say the whole sequence) and a function $f_\infty \in C^4(\mathbb{R})$ such that $\lim_{n\to\infty} f(\phi+\phi_n) = f_\infty(\phi)$ uniformly. By Lemma C.2, there is $\tilde{\omega} \in \Omega$ such that $f_\infty(\phi) = F(\tilde{\omega} + \psi(\phi))$. Now, let $q_\infty(\varphi, \mathcal{I}; \tau)$ be the implicit solution of the equation $q + \eta f_\infty(\varphi + q) = 0$, which exists due to the assumption $|\eta| \|F\|_{C^4_\psi(\Omega)} < 1/4$. That is

$$q_\infty(\varphi) = -\eta f_\infty(\varphi + q_\infty(\varphi)), \qquad \varphi \in \mathbb{R}.$$

In fact, we also have

$$q_\infty(z + \eta f_\infty(z)) = -\eta f_\infty(z), \qquad z \in \mathbb{R}. \tag{4.37}$$

This can be seen as follows. The function q_∞ is of class C^4 and

$$\dot{q}_\infty(\varphi) = -\frac{\eta \dot{f}_\infty(\varphi + q_\infty(\varphi))}{1 + \eta \dot{f}_\infty(\varphi + q_\infty(\varphi))}.$$

So $\|q_\infty\|_{C^1} < 1/3$ and thus the function $z(\varphi) = \varphi + q_\infty(\varphi)$ is invertible. Its inverse $\varphi(z)$ satisfies

$$\varphi(z) = z - q_\infty(\varphi(z)) = z + \eta f_\infty(\varphi(z) + q_\infty(\varphi(z))) = z + \eta f_\infty(z).$$

Plugging this into the definition of q_∞ yields (4.37). Now, we show that $\lim_{n\to\infty} q(\varphi + \phi_n) = q_\infty(\varphi)$ holds uniformly for the same (sub-)sequence (ϕ_n). First, note that again $w \mapsto w + \eta f_\infty(w)$ is homeomorphism with respect to $\mathbb{R}$. Thus

$$\sup_{\varphi\in\mathbb{R}} |q(\varphi + \phi_n) - q_\infty(\varphi)| = \sup_{w\in\mathbb{R}} |q(w + \eta f_\infty(w) + \phi_n) - q_\infty(w + \eta f_\infty(w))|.$$

The right hand side can be split into two parts converging to 0. Since q is uniformly continuous we have

$$|q(w + \eta f_\infty(w) + \phi_n) - q(w + \eta f(w + \phi_n) + \phi_n)| \to 0$$

and moreover, using (4.36) and (4.37) we get

$$|q(w+\phi_n+\eta f(w+\phi_n))-q_\infty(w+\eta f_\infty(w))| = |\eta f(w+\phi_n)-\eta f_\infty(w)| \to 0,$$

uniformly. Thus in total, we have proven that q is almost periodic. Moreover, differentiating (4.35) yields

$$\dot{q}(\varphi) = -\frac{\eta \dot{f}(\varphi + q(\varphi))}{1+\eta\dot{f}(\varphi+q_\infty(\varphi))}.$$

So $\dot{q}(\varphi)$ is a continuous combination of a.p. functions and hence almost periodic. The fact that the higher derivatives $q^{(j)}$ are a.p. follows analogously. Finally, it remains to show that q is representable over (Ω, ψ). We need to find a morphism $(\Omega,\psi) \to (\mathcal{H}_q, \psi_q)$, which is equivalent to showing that $\psi(t_n) \to 0$ implies $\psi_q(t_n) \to 0$. But $\psi(t_n) \to 0$ yields

$$\lim_{n\to\infty} f(\phi + t_n) = \lim_{n\to\infty} F(\psi(\phi)+\psi(t_n)) = F(\psi(\phi)) = f(\phi)$$

and thus $\lim_{n\to\infty} q(\phi + t_n) = q(\phi)$ just follows from the argument above, in the special case $f_\infty = f$ and thus $q_\infty = q$.

□

Without loss of generality we can assume that $\tilde{\mathcal{I}} = \mathcal{I}_{**}$ satisfies the assumption of Lemma 4.22, since $\|f\|_{C^4(\Omega)} \leq \gamma\kappa_0^2\|P\|_{C^4_\psi(\Omega)}$. The new Hamiltonian is given by

$$h_1(\varphi,\mathcal{I};\tau) = h(\phi, I;\tau) + \partial_\tau\Psi(\phi,\mathcal{I};\tau),$$

where $\partial_\tau\Psi(\phi,\mathcal{I};\tau) = -\mathcal{I}^{\frac{3-\alpha}{2(\alpha+1)}} f(\phi)c(\tau)$, and because of Lemma 4.22 we have

$$h_1(\varphi,\mathcal{I};\tau) = h(\varphi + q(\varphi,\mathcal{I};\tau), I;\tau) - \mathcal{I}^{\frac{3-\alpha}{2(\alpha+1)}} f(\varphi + q(\varphi,\mathcal{I};\tau))c(\tau).$$

Moreover, I can be expressed as

$$I = \mathcal{I} - \mathcal{I}^{\frac{3-\alpha}{2(\alpha+1)}}\dot{f}(\phi)c_1(\tau) = \mathcal{I} - \mathcal{I}^{\frac{3-\alpha}{2(\alpha+1)}}\dot{f}(\varphi+q(\varphi,\mathcal{I};\tau))c_1(\tau).$$

Hereby motivated, we define the C^3-maps $\boldsymbol{I}, H_1 : \Omega \times [\mathcal{I}_{**},\infty) \times \mathbb{R} \to \mathbb{R}$ by

$$\boldsymbol{I}(\omega,\mathcal{I};\tau) = \mathcal{I} - \mathcal{I}^{\frac{3-\alpha}{2(\alpha+1)}}\partial_\psi F(\psi_\omega(Q(\omega,\mathcal{I};\tau)))c_1(\tau)$$

and further

$$H_1(\omega,\mathcal{I};\tau) = H(\psi_\omega(Q(\omega,\mathcal{I};\tau)),\boldsymbol{I}(\omega,\mathcal{I};\tau);\tau) - \mathcal{I}^{\frac{3-\alpha}{2(\alpha+1)}}F(\psi_\omega(Q(\omega,\mathcal{I};\tau)))c(\tau).$$

This way, the relation $h_1(\varphi,\mathcal{I};\tau) = H_1(\psi_\omega(\varphi),\mathcal{I};\tau)$ holds for all $(\varphi,\mathcal{I};\tau)$ in $\mathbb{R}\times[\mathcal{I}_{**},\infty)\times\mathbb{R}$.

4.4.4 Proof of the main result

Given $P \in C^4_\psi(\Omega)$, let $p_\omega(t) = P(\omega+\psi(t))$ be the induced almost periodic forcing function. We write $x_\omega(t;\tilde{x},\tilde{v},\tilde{t})$ for the solution of

$$\ddot{x} + |x|^{\alpha-1}x = p_\omega(t), \tag{4.38}$$

satisfying the initial condition $x_\omega(\tilde{t}) = \tilde{x}$ and $\dot{x}_\omega(\tilde{t}) = \tilde{v}$. Generally, it is sufficient to consider only $\tilde{t} = 0$. We show that $x_\omega(t;\tilde{x},\tilde{v},0)$ is Poisson stable for almost all $(\tilde{x},\tilde{v},\omega) \in \mathbb{R}^2\times\Omega$.

The proof follows a similar approach as in Section 3.2. First we are going to construct a measure-preserving embedding suitable for Theorem 4.9, which corresponds to the Poincaré map Φ of system (4.33). It follows that for almost all $\omega\in\Omega$ the corresponding escaping set E_ω has Lebesgue measure zero. In the second part we transform this map back to the original coordinates and deduce that also for this Poincaré map there are almost no escaping orbits. Finally, the last part contains the conclusion that almost every solution is Poisson stable.

Escaping orbits of the transformed system

For $\omega\in\Omega$, denote by $(\varphi_\omega(\tau;\varphi_0,\mathcal{I}_0,\tau_0),\mathcal{I}_\omega(\tau;\varphi_0,\mathcal{I}_0,\tau_0))$ the solution to system (4.33) with initial data $\varphi(\tau_0)=\varphi_0$, $\mathcal{I}(\tau_0)=\mathcal{I}_0$ and forcing function $p(t) = p_\omega(t)$. Furthermore, we will write $(\varphi(\tau;\omega_0,\mathcal{I}_0,\pi/2),\mathcal{I}(\tau;\omega_0,\mathcal{I}_0,\pi/2))$ for the solution of

$$\varphi' = \partial_{\mathcal{I}}H_1(\omega_0+\psi(\varphi),\mathcal{I};\tau),\;\; \mathcal{I}' = -\partial_\psi H_1(\omega_0+\psi(\varphi),\mathcal{I};\tau), \tag{4.39}$$

with H_1 defined as in the last section and satisfying the initial condition $\varphi(\pi/2)=0$, $\mathcal{I}(\pi/2)=\mathcal{I}_0$. If we have $\omega_0 = \psi_\omega(\varphi_0)$ these solutions meet the identity

$$\begin{pmatrix}\varphi_\omega(\tau;\varphi_0,\mathcal{I}_0,\pi/2)\\ \mathcal{I}_\omega(\tau;\varphi_0,\mathcal{I}_0,\pi/2)\end{pmatrix} = \begin{pmatrix}\varphi_0+\varphi(\tau;\omega_0,\mathcal{I}_0,\pi/2)\\ \mathcal{I}(\tau;\omega_0,\mathcal{I}_0,\pi/2)\end{pmatrix}. \tag{4.40}$$

Set $\mathcal{I}^* = \max\{4\mathcal{I}_{**}, (2\kappa_1)^{\frac{\alpha+3}{2}}\}$ and $\mathcal{D} = \Omega \times (\mathcal{I}^*, \infty)$. Moreover, let the functions $F, G : \mathcal{D} \to \mathbb{R}$ be defined by

$$F(\omega_0, \mathcal{I}_0) = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \varphi'(\tau; \omega_0, \mathcal{I}_0, \pi/2)\, d\tau$$

and

$$G(\omega_0, \mathcal{I}_0) = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \mathcal{I}'(\tau; \omega_0, \mathcal{I}_0, \pi/2)\, d\tau,$$

respectively, and consider $g : \mathcal{D} \to \Omega \times (0, \infty)$ given by

$$g(\omega_0, \mathcal{I}_0) = (\omega_0 + \psi(F(\omega_0, \mathcal{I}_0)), \mathcal{I}_0 + G(\omega_0, \mathcal{I}_0)).$$

Then F and G are continuous, since the solution of (4.39) depends continuously upon its initial condition and the parameter ω_0. Therefore g has special form (4.7). The corresponding family of maps of the plane $\{g_\omega\}_{\omega\in\Omega}$ as in (4.8) is

$$g_\omega : D_\omega \subset \mathbb{R} \times (0, \infty) \to \mathbb{R} \times (0, \infty),$$
$$g_\omega(\varphi_0, \mathcal{I}_0) = (\varphi_0 + F(\psi_\omega(\varphi_0), \mathcal{I}_0), \mathcal{I}_0 + G(\psi_\omega(\varphi_0), \mathcal{I}_0)),$$

where $D_\omega = (\psi_\omega \times \mathrm{id})^{-1}(\mathcal{D}) = \mathbb{R} \times (\mathcal{I}^*, \infty)$. Because of (4.40) this map coincides with the successor map Φ from (4.34) for the forcing function p_ω.

The injectivity of g is a consequence of the unique resolvability of the initial value problem

$$\varphi' = \partial_{\mathcal{I}} H_1(\omega_1 + \psi(\varphi), \mathcal{I}; \tau), \;\; \mathcal{I}' = -\partial_\omega H_1(\omega_1 + \psi(\varphi), \mathcal{I}; \tau),$$

with $\varphi(5\pi/2) = 0$ and $\mathcal{I}(5\pi/2) = \mathcal{I}_1$, where $(\omega_1, \mathcal{I}_1) = g(\omega_0, \mathcal{I}_0)$.

The proof that g is measure-preserving is analogue to the corresponding proof for the ping-pong map, since g_ω corresponds to a Hamiltonian flow and thus Liouville's theorem yields $\det J_{g_\omega}(\varphi_0, \mathcal{I}_0) = 1$.

Hence we have shown that g is a measure-preserving embedding. Now, we have to find suitable functions W, k as described in Theorem 4.9. Since C from Lemma 4.20 depends only upon $\|f\|_{C_b^4(\mathbb{R})}$, this constant is uniform in $\omega \in \Omega$. Therefore, if we take $W(\omega_0, \mathcal{I}_0) = \mathcal{I}_0$, Lemma 4.20 implies

$$W(g(\omega_0, \mathcal{I}_0)) - W(\omega_0, \mathcal{I}_0) = \mathcal{I}_1 - \mathcal{I}_0 \le k(\mathcal{I}_0),$$

where $k(\mathcal{I}_0) = C\mathcal{I}_0^{b_\alpha}$ with C as mentioned above and $b_\alpha < 0$ from Theorem 4.18. That way W and k meet all demanded criteria and thus the measure-preserving embedding g satisfies all conditions of Theorem 4.9. This gives us $\lambda^2(E_\omega) = 0$ for almost all $\omega \in \Omega$ for the escaping set

$$E_\omega = \{(\varphi_0, \mathcal{I}_0) \in D_{\omega,\infty} : \lim_{n\to\infty} \mathcal{I}_n = \infty\},$$

where $D_{\omega,\infty}$ is the set of initial conditions leading to complete forward orbits of g_ω as described in Section 4.2.1.

Undoing the transformations

Recall the transformations of Section 4.4.1:

$$(x, v; t) \overset{\mathcal{R}}{\to} (\bar{\vartheta}, r; t) \hookrightarrow (\vartheta, r; t) \overset{\mathcal{S}}{\to} (\phi, I; \tau) \overset{\mathcal{T}}{\to} (\varphi, \mathcal{I}; \tau)$$

Let $r^* > 0$ be such that

$$\kappa_1 r^{\frac{2(\alpha+1)}{\alpha+3}} - \gamma r^{\frac{2}{\alpha+3}} \|P\|_{C^4_\psi(\Omega)} \|c\|_\infty \geq 2\mathcal{I}^*, \qquad \forall r \geq r^*.$$

Using the notation introduced in (4.32), it follows

$$\mathcal{S}(\Sigma_{r^*}) \subset \Sigma_{2\mathcal{I}^*} \qquad \text{and} \qquad \mathcal{T}(\Sigma_{2\mathcal{I}^*}) \subset \Sigma_{\mathcal{I}^*}.$$

Moreover, define $G_{r^*} = \mathcal{R}^{-1}(\mathbb{S}^1 \times [r^*, \infty) \times \mathbb{R})$. Given $(x_0, v_0, t_0) \in G_{r^*}$, let $(\bar{\vartheta}_0, r_0, t_0) = \mathcal{R}(x_0, v_0, t_0)$ and $(\varphi_0, \mathcal{I}_0, \tau_0) = \mathcal{T}(\mathcal{S}(\vartheta_0, r_0, t_0))$. Furthermore, fix some $\omega \in \Omega$. Note however, that all considerations below hold uniformly in ω. Due to Lemma 4.19, we know that the corresponding solution $(\varphi_\omega(\tau; \varphi_0, \mathcal{I}_0, \tau_0), \mathcal{I}_\omega(\tau; \varphi_0, \mathcal{I}_0, \tau_0))$ of (4.33) satisfies $\mathcal{I}_\omega(\tau) \geq \mathcal{I}_{**}$ for all $\tau \in [\tau_0, \tau_0 + 2\pi]$. Also, we have

$$\mathcal{T}^{-1}(\Sigma_{\mathcal{I}_{**}}) \subset \Sigma_{I_*} \quad \text{and} \quad \mathcal{S}^{-1}(\Sigma_{I_*}) \subset \Sigma_{r_*}.$$

Hence, the solution $(\varphi_\omega, \mathcal{I}_\omega)(\tau)$ can be transformed back to the original coordinates for all $\tau \in [\tau_0, \tau_0 + 2\pi]$. Illustrative, this means that corresponding solution $(x_\omega, \dot{x}_\omega)(t)$ does a full turn around the origin. In particular, it enables us to consider the following Poincaré map. Let

$v_* \leq -\sqrt{2\kappa_1} r^{*\frac{\alpha+1}{\alpha+3}}$ and define the function ζ_ω that maps the initial values $(v_0, t_0) \in (-\infty, v_*) \times \mathbb{R}$ to (v_1, t_1), where $v_1 = \dot{x}_\omega(t_1; 0, v_0, t_0)$ and

$$t_1 = \inf\{s \in (t_0, \infty) : x_\omega(s; 0, v_0, t_0) = 0,\ \dot{x}_\omega(s; 0, v_0, t_0) < 0\}.$$

This map is well-defined by the argument above. In fact, it coincides with the Poincaré map g_ω of the last step if it is transformed back to the original coordinates. Since $x = 0$ and $v < 0$ corresponds to $\bar{\vartheta} = \pi/2$ and therefore $\tau \in \{\pi/2 + 2\pi\mathbb{Z}\}$, we consider restrictions of the transformation maps onto some 2-dimensional subspaces, namely

$$\begin{aligned}
&\mathcal{R}_0 : (-\infty, 0) \times \mathbb{R} \to (0, \infty) \times \mathbb{R}, \quad \mathcal{R}_0(v, t) = ((\pi_2, \pi_3) \circ \mathcal{R})(0, v; t),\\
&\mathcal{S}_0 : [r_*, \infty) \times \mathbb{R} \to \mathbb{R} \times (0, \infty), \quad \mathcal{S}_0(r, t) = ((\pi_1, \pi_2) \circ \mathcal{S})(\pi/2, r; t),\\
&\mathcal{T}_0 : \mathbb{R} \times [I_{**}, \infty) \to \mathbb{R} \times (0, \infty), \quad \mathcal{T}_0(\phi, I) = ((\pi_1, \pi_2) \circ \mathcal{T})(\phi, I; \pi/2),
\end{aligned}$$

where $\pi_j : \mathbb{R}^3 \to \mathbb{R}$ denotes the projection on to the j-th component. Then, ζ_ω can be written in the following way:

$$\zeta_\omega = (\mathcal{R}_0^{-1} \circ \mathcal{S}_0^{-1} \circ \mathcal{T}_0^{-1}) \circ g_\omega \circ (\mathcal{T}_0 \circ \mathcal{S}_0 \circ \mathcal{R}_0)$$

If $(v_n, t_n)_{n\in\mathbb{N}} = (\zeta_\omega^n(v_0, t_0))_{n\in\mathbb{N}}$ denotes a generic complete forward orbit, the escaping set $\mathcal{E}_\omega$ consists of those initial values $(v_0, t_0) \in (-\infty, v_*) \times \mathbb{R}$ such that $\lim_{n\to\infty} v_n = -\infty$. Clearly, we have

$$\mathcal{E}_\omega \subset (\mathcal{R}_0^{-1} \circ \mathcal{S}_0^{-1} \circ \mathcal{T}_0^{-1})(E_\omega).$$

Since $\mathcal{T}_0$, $\mathcal{S}_0$ and $\mathcal{R}_0$ are diffeomorphisms with respect to their images, the latter inclusion implies $\lambda^2(\mathcal{E}_\omega) = 0$.

Finally, note that by the same argument as above and by (4.26) we can also implicitly define a function $T_\omega : G_{r^*} \to [0, 2\pi)$ such that

$$T_\omega(\tilde{x}, \tilde{v}) = \inf\{s \in [0, \infty) : x_\omega(s; \tilde{x}, \tilde{v}, 0) = 0,\ \dot{x}_\omega(s; \tilde{x}, \tilde{v}, 0) < 0\}.$$

By restricting initial values to a set

$$G_{r^{**}} = \mathcal{R}^{-1}((\mathbb{S}^1 \setminus \{\pi/2\}) \times (r^{**}, \infty) \times \mathbb{R})$$

where $r^{**} < r^* < 0$ is a suitable constant, we can assure that the map

$$V_\omega : G_{r^{**}} \to (-\infty, v^*) \times (0, 2\pi), \quad V_\omega(\tilde{x}, \tilde{v}) = (\dot{x}_\omega(T_\omega(\tilde{x}, \tilde{v}); \tilde{x}, \tilde{v}, 0), T_\omega(\tilde{x}, \tilde{v}))$$

is well defined and in fact a diffeomorphism with respect to its image. Hence, also the set $\tilde{E}_\omega = V_\omega^{-1}(\mathcal{E}_\omega)$ of initial values $(\tilde{x}, \tilde{v}) \in G_{r^{**}}$ leading to escaping orbits has measure zero.

Poisson stability

Denote by $\Pi_\omega : \mathbb{R}^2 \to \mathbb{R}^2$ the time-2π map of (4.38), that is

$$\Pi_\omega(x, v) = (x_\omega(2\pi; x, v, 0), \dot{x}_\omega(2\pi; x, v, 0)).$$

Moreover, define $\tilde{\Pi} : \mathbb{R}^2 \times \Omega \to \mathbb{R}^2 \times \Omega$ by

$$\tilde{\Pi}(x, v, \omega) = (\Pi_\omega(x, v), \omega + \psi(2\pi)).$$

Then $\tilde{\Pi}$ preserves the measure $\lambda^2 \otimes \mu_\Omega$. Let $(x_n, v_n, \omega_n)_{n\in\mathbb{N}} = \tilde{\Pi}^n(x_0, v_0, \omega_0)$ denote a generic orbit. By construction we have

$$(x_n, v_n) = (x_{\omega_0}(2\pi n; x_0, v_0, 0), \dot{x}_{\omega_0}(2\pi n; x_0, v_0, 0)).$$

Consider the set

$$\mathcal{N} = \Big\{(x_0, v_0, \omega_0) : \lim_{n\to\infty} (|x_n| + |v_n|) = \infty\Big\}.$$

The reasoning depicted at the end of Chapter 2 leads to the conclusion that $\tilde{\Pi}$ is recurrent on $(\mathbb{R}^2 \times \Omega) \setminus \mathcal{N}$. Moreover, for every $(x_0, v_0, \omega_0) \in \mathcal{N}$ there is $m \in \mathbb{N}$ such that $(x_m, v_m) \in E_{\omega_m}$. So $\mathcal{N} \subset \bigcup_{m\in\mathbb{N}} \tilde{\Pi}^{-m}(\mathcal{Z})$, where

$$\mathcal{Z} = \Big\{(x, v, \omega) \in \mathbb{R}^2 \times \Omega : (x, v) \in \tilde{E}_\omega\Big\}.$$

Thus $\mathcal{N}$ has measure zero, since

$$(\lambda^2 \otimes \mu_\Omega)(\mathcal{Z}) = \int_\Omega \lambda^2(\tilde{E}_\omega)\, d\mu_\Omega(\omega) = 0.$$

Let $(x_0, v_0, \omega_0) = (x, v, \omega)$ be a point in $\mathbb{R}^2 \times \Omega$ and consider the corresponding orbit $(x_n, v_n, \omega_n)_{n\in\mathbb{N}}$. Then, for almost every such (x, v, ω) there is a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of natural numbers with $\sigma_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} (x_{\sigma_n}, v_{\sigma_n}, \omega_{\sigma_n}) = (x, v, \omega).$$

In particular, we have $p_{\omega_{\sigma_n}}(t) = P(\omega_{\sigma_n} + \psi(t)) \to p_\omega(t)$ uniformly. Thus, the right hand side of

$$\ddot{x} + |x|^{\alpha-1}x = p_{\omega_{\sigma_n}}(t),$$

converges uniformly as $n \to \infty$. Write $x_{\omega_{\sigma n}}(t) = x_{\omega_{\sigma n}}(t; x_{\sigma_n}, v_{\sigma_n}, 0)$ for the corresponding solution. Considering the fact that also for the initial condition we have $\lim_{n\to\infty}(x_{\sigma_n}, v_{\sigma_n}) = (x, v)$, it follows that

$$\begin{pmatrix} x_{\omega_{\sigma n}}(t) \\ \dot{x}_{\omega_{\sigma n}}(t) \end{pmatrix} \to \begin{pmatrix} x_\omega(t) \\ \dot{x}_\omega(t) \end{pmatrix}, \qquad \text{as } n \to \infty,$$

uniformly on compact intervals in $\mathbb{R}$ (cf. [Har82], Chapter II Theorem 3.2). But because we have $x_{\omega_{\sigma n}}(t) = x_\omega(t + 2\pi\sigma_n)$, this yields

$$|x_\omega(t + 2\pi\sigma_n) - x_\omega(t)| + |\dot{x}_\omega(t + 2\pi\sigma_n) - \dot{x}_\omega(t)| \to 0, \quad \text{as } n \to \infty,$$

uniformly on bounded intervals in $\mathbb{R}$. Since the inverse map $\tilde{\Pi}^{-1}$ is recurrent as well, the whole argument can be repeated to conclude that almost every solution $x_\omega(t)$ is Poisson stable. □

Chapter 5

The non-periodic case

In this chapter, we will study certain twist maps of the plane without any periodicity assumption. More precisely, we consider symplectic maps of the form

$$\theta_1 = \theta + \frac{1}{r^\alpha}(\gamma + F_1(\theta, r)), \quad r_1 = r + r^{1-\alpha} F_2(\theta, r),$$

with $\alpha \in (0,1)$, $\gamma \in \mathbb{R} \setminus \{0\}$ and $F_1, F_2 \in \mathcal{F}^{k+2}(\alpha)$. We denote by $\mathfrak{h}$ the primitive function satisfying $d\mathfrak{h} = r_1\, d\theta_1 - r\, d\theta$. The main result of this chapter, Theorem 5.25 down below, states that under suitable assumptions on $\mathfrak{h}$ all complete forward orbits are subjected to the growth condition $r_n = \mathcal{O}(n^{1/(k+1)\alpha})$. The chapter is essentially based on [KO21]. In this paper, Kunze and Ortega studied maps of the form above with F_1, F_2 and $\mathfrak{h}$ holomorphic. They were able to prove growth rates $r_n = \mathcal{O}((\log n)^{1/\alpha})$ for all forward complete real orbits.

The proof to Theorem 5.25 can be found in Section 5.4. To motivate the preliminary sections, we briefly sketch its strategy. On one hand, the fact that the involved functions are non-holomorphic simplifies some computations in contrast to [KO21]. On the other hand, this also implies that the Cauchy integral formula is not available. In particular, the norm $\|g\|_\infty$ of a function $g \in C^k$ yields no control over the norm $\|g\|_{C^k}$. As a consequence, we will be dealing with higher derivatives of implicitly defined functions. For this reason, a version of Faà di Bruno's formula is introduced in Section 5.1.

The first step of the proof will be to rescale the vertical coordinate. With $\xi = \varepsilon^{1/\alpha} r$ the map becomes

$$\psi_\varepsilon(\theta, \xi) = (\theta, \xi) + \varepsilon l(\theta, \xi, \varepsilon),$$

where $l = (l_1, l_2)$ is given by

$$l_1(\theta, \xi, \varepsilon) = \frac{1}{\xi^\alpha}\left(\gamma + F_1\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right)\right), \quad l_2(\theta, \xi, \varepsilon) = \xi^{1-\alpha} F_2\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right).$$

For $\varepsilon > 0$ sufficiently small, ψ_ε is well-defined on the ρ-neighborhood G_ρ of $G = \mathbb{R} \times (1, 2)$, where $\rho > 0$ is a small constant. The form of ψ_ε is reminiscent of the near identity symplectic maps

$$\mathcal{P}_\varepsilon : x_1 = x + \varepsilon l(x, \varepsilon), \quad x_1 = (q_1, p_1), \quad x = (q, p),$$

studied by Neishtadt in [Nei84]. In this context, Kunze and Ortega introduce the notion of E-symplectic families of maps. In Section 5.3, we adapt this notion to the non-analytic case and say that $\{\mathcal{P}_\varepsilon\}$ is an E-symplectic family of class C^{k+1} if the following conditions hold. There is a function $\zeta(q, p, \varepsilon)$ such that

$$p_1\, dq_1 - p\, dq = d\zeta(\cdot, \varepsilon).$$

Both $l(x, \varepsilon)$ and $\zeta(x, \varepsilon)$ lie in a suitable class $\mathcal{M}^{k+1}_{\rho,\sigma}$ of functions $(k+1)$-times continuously differentiable in x, differentiable in ε, but with possible discontinuities in $\varepsilon = 0$. Moreover, there is $\mathfrak{m} \in C^{k+1}_b(G_\rho)$ such that

$$\zeta(x, \varepsilon) = \varepsilon \mathfrak{m}(x) + \mathcal{O}(\varepsilon^2), \quad \partial_\varepsilon \zeta(x, \varepsilon) = \mathfrak{m}(x) + \mathcal{O}(\varepsilon),$$

uniformly in $x \in G_\rho$ as $\varepsilon \to 0$. Under these assumptions, it is possible to construct an adiabatic invariant for $\mathcal{P}_\varepsilon$. Indeed, any map in this family can be viewed as the Poincaré map of a 1-periodic Hamiltonian system

$$\dot{x} = \varepsilon J \nabla H(x, t, \varepsilon),$$

where $H(\cdot, t, \varepsilon)$ is of class C^{k+2}. Using a method of Hamiltonian averaging discussed in Section 5.2, the system can be brought into normal form

$$\dot{y} = \varepsilon J \nabla \mathcal{N}(y, \varepsilon) + \varepsilon J \nabla \mathcal{R}(y, t, \varepsilon),$$

where the remainder satisfies $\mathcal{R}(y,t,\varepsilon) = \mathcal{O}(\varepsilon^k)$. In Theorem 5.21, we conclude that $E(x) = \mathcal{N}(x,0)$ is an adiabatic invariant for $\mathcal{P}_\varepsilon$. More precisely, there are $\hat{\sigma}, \hat{C} > 0$ such that if $\varepsilon \in [0,\hat{\sigma}]$ and

$$(x_n)_{0\leq n\leq N} = (\mathcal{P}_\varepsilon^n(x_0))_{0\leq n\leq N}$$

denotes a forward orbit piece of $\mathcal{P}_\varepsilon$ with $x_n \in G$ for $0 \leq n \leq N$, then

$$|E(x_n) - E(x_0)| \leq \hat{C}\varepsilon, \quad 0 \leq n \leq \min\{N, \lfloor \varepsilon^{-k} \rfloor\},$$

where $\lfloor w \rfloor$ denotes the integer part of w. The content of Section 5.2 and 5.3 should be compared to [KO]. In our case, the family $\{\psi_\varepsilon\}$ is E-symplectic of class C^{k+1} and $E(\theta,\xi) = \frac{\gamma}{1-\alpha}\xi^{1-\alpha}$ can be chosen as the corresponding adiabatic invariant. After one additional change of variables $s_n \sim r_n^{1-\alpha}$, this leads to an estimate of the form

$$|s_n - s_m| \leq C s_m^\beta \qquad \text{for } m \leq n \leq m + \lfloor s_m^{k(1-\beta)} \rfloor,$$

where $\beta = \frac{1-2\alpha}{1-\alpha}$. In Section 5.4 we present a Lemma by Kunze and Ortega applicable in such situations. It is related to the notion of lower and upper solutions of the difference equation $x_{n+1} = x_n + C x_n^\beta$ and allows to conclude the growth rates asserted in Theorem 5.25 in a rigorous way.

Finally, Section 5.5 contains the application of this theorem to the Fermi-Ulam ping-pong. Given a forcing function $p \in C_b^{k+1}(\mathbb{R})$ with $k \geq 3$, we show that the ping-pong map can be transformed into a suitable form and deduce that there is a constant $\tilde{C} > 0$ so that any complete forward orbit $(t_n, v_n)_{n\in\mathbb{N}_0}$ must satisfy

$$v_n \leq \tilde{C} n^{1/(k-1)}, \qquad n \geq n_0,$$

for some $n_0 \in \mathbb{N}$. At this point, we also want to mention the paper [GN06], in which the same adiabatic invariant of the ping-pong map is derived by a Hamiltonian averaging procedure as well. At the end of this section, we show how to construct a smooth forcing function $p(t)$ leading to at least one escaping orbit. This example stems from [KO11]. We prove that for $\|p\|_{C^{k+1}(\mathbb{R})} \leq M$, with a prefixed parameter M, there are a constant $C > 0$ and a complete forward orbit $(t_n, v_n)_{n\in\mathbb{N}_0}$ such that

$$v_n \geq C n^{1/(k+1)}, \qquad n \in \mathbb{N}.$$

5.1 Derivatives of composite functions

We start by clarifying some notation. As the norm in $\mathbb{R}^d$ and $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ we take $|x| = \max_{1\le i\le d} |x_i|$ and $|A| = \max_{1\le i\le d_1, 1\le j\le d_2} |a_{ij}|$, respectively. Let $G \subset \mathbb{R}^d$ be an open set and $k \in \mathbb{N}_0$. We use the following definition of the C^k-norm. For $u \in C^k(G)$ we set

$$\|u\|_{C^k(G)} = \max_{|\boldsymbol{\nu}|\le k} \sup_{x\in G} |\partial^{\boldsymbol{\nu}} u(x)|.$$

Note, that for multi-indices $\boldsymbol{\nu} = (\nu_1, \dots, \nu_d) \in \mathbb{N}_0^d$ the term $|\boldsymbol{\nu}|$ sill denotes its length $|\nu_1| + \ldots + |\nu_d|$. For vector and matrix valued functions u the maximum in the definition is also taken over all components.

Lemma 5.1. *Let $f, g \in C^k(G)$ and consider $h(x) = f(x)g(x)$. Then*

$$\|h\|_{C^k(G)} \le 2^k \|f\|_{C^k(G)} \|g\|_{C^k(G)}.$$

Proof. Let $\boldsymbol{\nu} \in \mathbb{N}_0^d$ be a multi-index of length $|\boldsymbol{\nu}| \le k$. For $x \in G$ the Leibniz product formula yields

$$\partial^{\boldsymbol{\nu}} h(x) = \sum_{\boldsymbol{\mu}\le\boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\boldsymbol{\mu}} \partial^{\boldsymbol{\mu}} f(x) \partial^{\boldsymbol{\nu}-\boldsymbol{\mu}} g(x).$$

Moreover if we denote by **1** and **2** the vectors in $\mathbb{R}^d$ with every component equal 1 and 2 respectively, then the multi-binomial theorem implies

$$\sum_{\boldsymbol{\mu}\le\boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\boldsymbol{\mu}} = \sum_{\boldsymbol{\mu}\le\boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\boldsymbol{\mu}} \mathbf{1}^{\boldsymbol{\mu}} \mathbf{1}^{\boldsymbol{\nu}-\boldsymbol{\mu}} = \mathbf{2}^{\boldsymbol{\nu}} = 2^{|\boldsymbol{\nu}|}.$$

Thus the assertion follows. □

Since we will be dealing with many composite functions, we introduce a multivariate version of the Faà di Bruno formula [CS96]. Write $\boldsymbol{\mu} \prec \boldsymbol{\nu}$, if one of the following conditions hold:

1. $|\boldsymbol{\mu}| < |\boldsymbol{\nu}|$,

2. $|\boldsymbol{\mu}| = |\boldsymbol{\nu}|$ and $\mu_1 < \nu_1$, or

3. $|\boldsymbol{\mu}| = |\boldsymbol{\nu}|$, $\mu_1 = \nu_1, \ldots, \mu_r = \nu_r$ and $\mu_{r+1} < \nu_{r+1}$ for some $1 \le r < d$.

Lemma 5.2. *Let $\boldsymbol{\nu} \in \mathbb{N}_0^d$ be a multi-index with length $|\boldsymbol{\nu}| = k$ and consider $h = f \circ g$, where $g \in C^k(G, \mathbb{R}^m)$ and $f \in C^k(g(G))$. Then, for $x \in G$ we have*

$$\partial^{\boldsymbol{\nu}} h(x) = \sum_{1 \le |\boldsymbol{\mu}| \le k} \partial^{\boldsymbol{\mu}} f(g(x)) \sum_{p(\boldsymbol{\nu},\boldsymbol{\mu})} \boldsymbol{\nu}! \prod_{j=1}^{k} \frac{(\partial^{\boldsymbol{\ell}_j} g(x))^{\boldsymbol{r}_j}}{(\boldsymbol{r}_j!)(\boldsymbol{\ell}_j!)^{|\boldsymbol{r}_j|}}, \tag{5.1}$$

where $\boldsymbol{\mu} \in \mathbb{N}_0^m$ and

$$\begin{aligned} p(\boldsymbol{\nu},\boldsymbol{\mu}) = \\ &\Big\{(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k, \boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_k) : \boldsymbol{r}_i \in \mathbb{N}_0^m, \boldsymbol{\ell}_i \in \mathbb{N}_0^d \textit{ for } 1 \le i \le k, \textit{ there is } 0 \le s < k \\ &\quad \textit{so that } \boldsymbol{r}_i = \boldsymbol{0} \textit{ and } \boldsymbol{\ell}_i = \boldsymbol{0} \textit{ for } 1 \le i \le s; \quad |\boldsymbol{r}_i| > 0 \textit{ for } s+1 \le i \le k; \\ &\quad \textit{and moreover } \boldsymbol{0} \prec \boldsymbol{\ell}_{s+1} \prec \cdots \prec \boldsymbol{\ell}_n \textit{ are such that } \sum_{i=1}^{k} \boldsymbol{r}_i = \boldsymbol{\mu}, \ \sum_{i=1}^{k} |\boldsymbol{r}_i| \boldsymbol{\ell}_i = \boldsymbol{\nu} \Big\}. \end{aligned}$$

For a better comprehension of this formula, one may consider the special case $m = 1$. Then $\boldsymbol{\mu} = r$ is just a natural number and every element of $p(\boldsymbol{\nu}, r)$ corresponds to a partition of the multi-index $\boldsymbol{\nu}$ into smaller multi-indices $\boldsymbol{\ell}_i$. There are $k - s$ distinct parts $\boldsymbol{\ell}_i$, each with multiplicity r_i and the whole partition contains r non-zero multi-indices (counted with multiplicity).

In order to describe the occurring terms more thoroughly, we also mention the Stirling numbers of second kind defined by

$$S_{k,l} = \frac{1}{l!} \sum_{j=1}^{l} (-1)^{l-j} \binom{l}{j} j^k,$$

and the Bell numbers $B_k = \sum_{l=0}^{k} S_{k,l}$. Then, the number of terms occurring in Lemma 5.2 is given by the following equality (cf. [CS96], Lemma 2.7).

Lemma 5.3. *Let $\boldsymbol{\nu}$ be a multi-index of length $|\boldsymbol{\nu}| = k$. Then, for $1 \le l \le k$ we have the identity*

$$\sum_{\substack{|\boldsymbol{\mu}|=l \\ \boldsymbol{\mu} \in \mathbb{N}_0^m}} \sum_{p(\boldsymbol{\nu},\boldsymbol{\mu})} \boldsymbol{\nu}! \prod_{j=1}^{k} \frac{1}{(\boldsymbol{r}_j!)(\boldsymbol{\ell}_j!)^{|\boldsymbol{r}_j|}} = m^l S_{k,l}.$$

In particular, this yields the following estimate for composite functions.

Corollary 5.4. *Under the same assumptions of Lemma 5.2 we have*

$$\|h\|_{C^k(G)} \leq m^k B_k \|f\|_{C^k(g(G))} \max\{\|Dg\|_{C^{k-1}(G)}, \|Dg\|^k_{C^{k-1}(G)}\},$$

where Dg denotes the Jacobi matrix of g.

5.2 Hamiltonian normal forms

In this section, we establish a theorem about normal forms of periodic Hamiltonian systems and also give the fully detailed proof. The reader familiar with Hamiltonian averaging may only read Definition 5.5 and 5.7, as well as Theorem 5.13 below. The interested reader on the other hand is referred to [AKN85] for a comprehensive discussion of such topics and to [KO] for the full proof in the analytic case. For the sake of simplicity we will stick to the case relevant for our application. Thus, let $I \subset \mathbb{R}$ be an open bounded interval and consider the strip $G = \mathbb{R} \times I$. Given $\rho > 0$, we write $G_\rho = \mathbb{R} \times (\inf I - \rho, \sup I + \rho)$ for the ρ-neighborhood of G.

Definition 5.5. For $k \in \mathbb{N}$ and $\rho, \sigma > 0$, let $\mathcal{H}^k_{\rho,\sigma}$ be the class of all functions $H : G_\rho \times \mathbb{R} \times [0, \sigma] \to \mathbb{R}$, $H = H(x, t, \varepsilon)$ so that

(i) H is 1-periodic in t,

(ii) for every $t \in \mathbb{R}$, $\varepsilon \in [0, \sigma]$ we have $H(\cdot, t, \varepsilon) \in C^k(G_\rho)$ and

$$\partial_x^{\boldsymbol{\nu}} H \in C(G_r \times \mathbb{R} \times [0, \sigma]) \qquad \text{for } 0 \leq |\boldsymbol{\nu}| \leq k,$$

(iii) $\|H\|_{k,\rho,\sigma} < \infty$, where $\|H\|_{k,\rho,\sigma} = \sup_{\varepsilon \in [0,\sigma]} \sup_{t \in \mathbb{R}} \|H(\cdot, t, \varepsilon)\|_{C^k(G_\rho)}$.

Moreover, let $\tilde{\mathcal{H}}^k_{\rho,\sigma}$ be the subclass of functions $H \in \mathcal{H}^k_{\rho,\sigma}$ such that

$$\int_0^1 H(x, t, \varepsilon)\, dt = 0 \tag{5.2}$$

for all $x \in G_\rho$ and $\varepsilon \in [0, \sigma]$.

For functions $f = (f_1, \ldots, f_d)$ with values in $\mathbb{R}^d$ we set

$$\|f\|_{k,\rho,\sigma} = \max_{1 \le i \le d} \|f_i\|_{k,\rho,\sigma}.$$

Remark 5.6. Clearly the estimates both in Lemma 5.1 and Corollary 5.4 are also true for the norm $\|\cdot\|_{k,\rho,\sigma}$.

5.2.1 A near-identity transformation

On this class of Hamiltonian systems, we will consider a suitable type of transformations.

Definition 5.7. Let $0 < r < \rho$ and $\sigma > 0$. A map $\Psi : G_r \times \mathbb{R} \times [0, \sigma] \to G_\rho$, $\Psi(X, t, \varepsilon)$ will be called an *admissible change of variables of class* C^k, if it satisfies

(i) Ψ is 1-periodic in t and $\Psi(\cdot, 0, \varepsilon) = \mathrm{id}_X$ for $\varepsilon \in [0, \sigma]$,

(ii) for every $t \in \mathbb{R}$, $\varepsilon \in [0, \sigma]$ we have $\Psi(\cdot, t, \varepsilon) \in C^k(G_r)$ and

$$\partial_x^{\boldsymbol{\nu}} \Psi \in C(G_r \times \mathbb{R} \times [0, \sigma]) \qquad \text{for } 0 \le |\boldsymbol{\nu}| \le k,$$

(iii) for every $t \in \mathbb{R}$, $\varepsilon \in [0, \sigma]$ the map $\Psi(\cdot, t, \varepsilon)$ is an exact symplectic diffeomorphism with respect to its image.

The following canonical transformation is of this type.

Lemma 5.8. *Fix $k \in \mathbb{N}$, $0 < r < \rho$, $\sigma > 0$ and set $\sigma_1 = \min\{\frac{1}{2}, \rho - r, \sigma\}$. Then, for every $h \in \tilde{\mathcal{H}}^{k+1}_{\rho,\sigma}$ with $\|h\|_{k+1,\rho,\sigma} \le 1$, the equations*

$$q = Q + \varepsilon \int_0^t \frac{\partial h}{\partial P}(q, P, s, \varepsilon)\, ds, \quad p = P - \varepsilon \int_0^t \frac{\partial h}{\partial q}(q, P, s, \varepsilon)\, ds, \tag{5.3}$$

implicitly define an admissible change of variables of class C^k denoted by

$$\Psi : G_r \times \mathbb{R} \times [0, \sigma_1] \to G_\rho, \qquad (Q, P, t, \varepsilon) \mapsto (q, p).$$

Moreover, there is an increasing sequence of positive constants $(C_m)_{m \in \mathbb{N}}$ depending only on $m \in \mathbb{N}$ such that

$$\|\Psi(\cdot, t, \varepsilon) - id\,\|_{C^m(G_r)} \le C_m \varepsilon \|h\|_{k+1,\rho,\sigma} \qquad \text{for } 0 \le m \le k, \tag{5.4}$$

holds for all $t \in \mathbb{R}$ and $\varepsilon \in [0, \sigma_1]$.

Proof. First, we show that Ψ is well-defined. To this end, consider the function

$$Z(q,Q,P,t,\varepsilon) = q - Q - \varepsilon \int_0^t \frac{\partial h}{\partial P}(q,P,s,\varepsilon)\,ds,$$

where $q \in \mathbb{R}$, $(Q,P) \in G_r$, $t \in [0,1]$ and $\varepsilon \in [0,\sigma_1]$. We have

$$\frac{\partial Z}{\partial q} \geq 1 - \varepsilon \left| \int_0^t \frac{\partial^2 h}{\partial q \partial P}(q,P,s,\varepsilon)\,ds \right| \geq 1 - \varepsilon \|h\|_{k+1,\rho,\sigma} \geq 1 - \sigma_1 > 0.$$

Therefore, the equation $Z = 0$ has a unique solution $q = q(Q,P,t,\varepsilon)$ if (Q,P,t,ε) is fixed. A suitable implicit function theorem shows that locally q is continuous and due to the uniqueness we get $q \in C(G_r \times [0,1] \times [0,\sigma_1])$ [BGdS08]. And since Z is of class C^k in (q,Q,P), also the implicit solution $q(\cdot,t,\varepsilon)$ is in that class. Then, p is given by

$$p(Q,P,t,\varepsilon) = P - \varepsilon \int_0^t \frac{\partial h}{\partial q}(q(Q,P,t,\varepsilon),P,s,\varepsilon)\,ds.$$

Note that Z is 1-periodic in t due to (5.2). Thus, the definition of Ψ is completed by extending it periodically to $t \in \mathbb{R}$. In the following we also write $X = (Q,P)$ and $x = (q,p)$.
Next, we verify (5.4). For $m = 0$, it is clearly true with $C_0 = 1$. This also proves that $\Psi(G_r \times [0,1] \times [0,\sigma_1]) \subset G_\rho$. Differentiating the first equation in (5.3) with respect to Q yields

$$\frac{\partial q}{\partial Q}\Big(1 - \varepsilon \int_0^t \frac{\partial^2 h}{\partial q \partial P}(q,P,s,\varepsilon)\,ds\Big) = 1. \tag{5.5}$$

and consequently

$$\left| \frac{\partial q}{\partial Q} - 1 \right| \leq \frac{\varepsilon \|h\|_{k+1,\rho,\sigma}}{1 - \varepsilon \|h\|_{k+1,\rho,\sigma}} \leq 2\varepsilon \|h\|_{k+1,\rho,\sigma}.$$

In the same way, differentiation with respect to P yields

$$\frac{\partial q}{\partial P}\Big(1 - \varepsilon \int_0^t \frac{\partial^2 h}{\partial q \partial P}(q,P,s,\varepsilon)\,ds\Big) = \varepsilon \int_0^t \frac{\partial^2 h}{\partial P^2}(q,P,s,\varepsilon)\,ds. \tag{5.6}$$

And therefore

$$\left|\frac{\partial q}{\partial P}\right| \leq \frac{\varepsilon\|h\|_{k+1,\rho,\sigma}}{1-\varepsilon\|h\|_{k+1,\rho,\sigma}} \leq 2\varepsilon\|h\|_{k+1,\rho,\sigma}.$$

The bounds on $\left|\frac{\partial p}{\partial Q}\right|$ and $\left|\frac{\partial p}{\partial P}-1\right|$ can be obtained in an analogous way. Thus, we have verified (5.4) for $m=1$ and $C_1=2$. Now, assume that (5.4) holds for $j=0,1,\ldots,m$ with $m<k$. Let $\tilde{q}(Q,P,t,\varepsilon)=(q(Q,P,t,\varepsilon),P)$. Then for $\boldsymbol{\nu}\in\mathbb{N}_0^2$ with $|\boldsymbol{\nu}|=m+1$ Lemma 5.2 yields

$$\partial_X^{\boldsymbol{\nu}} q = \varepsilon\int_0^t \sum_{1\leq|\boldsymbol{\mu}|\leq m+1} \partial^{\boldsymbol{\mu}'} h(\tilde{q},s,\varepsilon) \sum_{p(\boldsymbol{\nu},\boldsymbol{\mu})} \boldsymbol{\nu}! \prod_{j=1}^{m+1} \frac{(\partial_X^{\boldsymbol{\ell}_j}\tilde{q})^{\boldsymbol{r}_j}}{(\boldsymbol{r}_j!)(\boldsymbol{\ell}_j!)^{|\boldsymbol{r}_j|}}\,ds, \tag{5.7}$$

where $\boldsymbol{\mu}'=\boldsymbol{\mu}+(0,1)$. Note, that on the right hand side there is exactly one term $\varepsilon\int_0^t \frac{\partial^2 h}{\partial q\partial P}(\tilde{q},s,\varepsilon)\partial_X^{\boldsymbol{\nu}} q\,dt$, all other terms contain only derivatives of q up to order m. Thus the inductive hypothesis and Lemma 5.3 lead to the estimate

$$|\partial_X^{\boldsymbol{\nu}} q| \leq \frac{\varepsilon\|h\|_{k+1,\rho,\sigma}}{1-\sigma_1}(1+C_m\varepsilon\|h\|_{k+1,\rho,\sigma})^{m+1}2^{m+1}B_{m+1} \leq C_{m+1}\varepsilon\|h\|_{k+1,\rho,\sigma},$$

for a suitable constant $C_{m+1}>0$. Again, the estimates for p can be obtained in a similar way. Thus (5.4) is verified. Moreover, (5.5), (5.6), (5.7) and their counterparts for p show that the derivatives $\partial_X^{\boldsymbol{\nu}}\Psi$ are also continuous in t and ε. It remains to show that also condition (iii) of Definition 5.7 is satisfied. Let $X,X'\in G_r$ and write $x=\Psi(X,t,\varepsilon)$, $x'=\Psi(X',t,\varepsilon)$ respectively. Then, by the convexity of G_ρ and the mean value theorem we have

$$\begin{aligned}|X-\tilde{X}| &\leq |x-x'|+\varepsilon\|h\|_{k+1,\rho,\sigma}(|q-q'|+|P-P'|)\\ &\leq |x-x'|+\tfrac{1}{2}(|x-x'|+|X-X'|)\end{aligned}$$

Hence Ψ is one-to-one. Finally, note that equations (5.3) can be derived from the generating function

$$S(q,P,t,\varepsilon)=qP-\varepsilon\int_0^t h(q,P,s,\varepsilon)\,ds. \tag{5.8}$$

Therefore, $\det D\Psi=1$ and Ψ is an exact symplectic diffeomorphism with respect to its image. □

Note, that in fact the slightly stronger estimate

$$\|\Psi(\cdot,t,\varepsilon)-\mathrm{id}\,\|_{C^m(G_r)} \leq C_m\varepsilon \sup_{t\in\mathbb{R}}\|h(\cdot,t,\varepsilon)\|_{C^{k+1}(G_\rho)}, \tag{5.9}$$

with $\varepsilon\in[0,\sigma]$, follows by the same argument.

Corollary 5.9. *Under the assumptions of Lemma 5.8, let $0<\hat{r}<r<\rho$ and denote by $\Psi : G_r\times\mathbb{R}\times[0,\sigma_1]\to G_\rho$ the map induced by (5.3). Let $\sigma_2=\min\{\frac{r-\hat{r}}{2},\sigma_1\}$. If $\varepsilon\in[0,\sigma_2]$, then $\Psi(G_r,t,\varepsilon)\supset G_{\hat{r}}$ for all $t\in\mathbb{R}$.*

In the proof we use the following Lemma (cf. [Shu87], Proposition I.3).

Lemma 5.10. *Let X,Y be Banach spaces and suppose that $U\subset Y$ is open. If $\Psi : U\to\Psi(U)\subset X$ is a homeomorphism, Ψ^{-1} is Lipschitz continuous with constant Lip$(\Psi^{-1})\leq\lambda$, and $\overline{B_r(y)}\subset U$, then*

$$\Psi(\overline{B_r(y)})\supset\overline{B_{r/\lambda}(\Psi(y))}.$$

Proof of Corollary 5.9. First fix some $t\in\mathbb{R}$ and $\varepsilon\in[0,\sigma_2]$. For simplicity we write $\Psi(y)=\Psi(y,t,\varepsilon)$. Using the von Neumann series and (5.4) yields

$$\|D\Psi^{-1}\|_{C^0(G_r)}=\left\|\sum_{j=0}^{\infty}(I-D\Psi(\cdot,t,\varepsilon))^j\right\|_{C^0(G_r)}\leq\sum_{j=0}^{\infty}(C_0\varepsilon)^j,$$

Since we know from the proof of Lemma 5.8 that $C_0=1$ is a viable choice, $\sigma_1\leq\frac{1}{2}$ shows that Ψ^{-1} is Lipschitz continuous with constant $\lambda=2$. Now let $y\in G_{\hat{r}}$, then $\|y-\Psi(y)\|\leq\varepsilon$ implies $y\in\overline{B_\varepsilon(\Psi(y))}$. Thus Lemma 5.10 yields

$$y\in\overline{B_\varepsilon(\Psi(y))}\subset\overline{B_{(r-\hat{r})/2}(\Psi(y))}\subset\Psi(\overline{B_{r-\hat{r}}(y)})\subset\Psi(G_r).$$

□

5.2.2 Hamiltonian averaging

Given $H\in\mathcal{H}^k_{\rho,\sigma}$ we denote the averaged part of H by

$$\bar{H}(x,\varepsilon):=\int_0^1 H(x,t,\varepsilon)\,dt,$$

and the purely periodic part by

$$\tilde{H}(x,t,\varepsilon) = H(x,t,\varepsilon) - \bar{H}(x,\varepsilon).$$

Then, $\bar{H} \in \mathcal{H}^k_{\rho,\sigma}$, $\tilde{H} \in \tilde{\mathcal{H}}^k_{\rho,\sigma}$ and

$$\|\bar{H}\|_{k,\rho,\sigma} \leq \|H\|_{k,\rho,\sigma}, \qquad \|\tilde{H}\|_{k,\rho,\sigma} \leq 2\|H\|_{k,\rho,\sigma}.$$

Moreover, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ denotes the symplectic matrix.

Lemma 5.11 (Basic transformation)**.** *For $\sigma > 0$, $k \in \mathbb{N}$ and $0 < r < \rho$, let $H \in \mathcal{H}^{k+1}_{\rho,\sigma}$ be such that $\|H\|_{k+1,\rho,\sigma} \leq \frac{1}{2}$. Set $\sigma_1 = \min\{\frac{1}{2}, \rho - r, \sigma\}$ and denote by Ψ the admissible change of variables of class C^k given by* (5.3) *with $h = \tilde{H}$, that is*

$$(Q,P,t,\varepsilon) = (X,t,\varepsilon) \mapsto x = (q,p) = \Psi(X,t,\varepsilon),$$

where $(X,t,\varepsilon) \in G_r \times \mathbb{R} \times [0,\sigma_1]$. Then, for every $\varepsilon \in [0,\sigma_1]$ the 1*-periodic Hamiltonian system*

$$\dot{x} = \varepsilon J \nabla_x H(x,t,\varepsilon)$$

is transformed into

$$\dot{X} = \varepsilon J \nabla_X K(X,t,\varepsilon),$$

where

$$K(X,t,\varepsilon) = H(\Psi(X,t,\varepsilon),t,\varepsilon) - \tilde{H}(q(X,t,\varepsilon),P,t,\varepsilon)$$

Moreover, $K \in \mathcal{H}^k_{r,\sigma_1}$ and for $t \in \mathbb{R}$, $\varepsilon \in [0,\sigma_1]$ we have

$$\|K(\cdot,t,\varepsilon) - \bar{H}(\cdot,\varepsilon)\|_{C^k(G_r)} \leq \kappa_k \|\tilde{H}\|_{k+1,\rho,\sigma}\|H\|_{k+1,\rho,\sigma}\varepsilon, \tag{5.10}$$

where $\kappa_k = 3 \cdot 2^{2k} B_k (2 + C_k)^{k+1}$ with B_k from Corollary 5.4 and C_k from Lemma 5.8.

Proof. Since Ψ is induced by a generating function of the form $S(q,P,t)$, the new Hamiltonian K is given by

$$K(Q,P,t) = H(q,p,t) + \frac{\partial S}{\partial t}(q(Q,P,t),Q,t).$$

Considering (5.8) and the additional ε this leads to

$$K(Q,P,t,\varepsilon) = H(\Psi(Q,P,t,\varepsilon),t,\varepsilon) - \tilde{H}(q(Q,P,t,\varepsilon),P,t,\varepsilon)$$

K is well-defined on $G_r \times [0,1] \times [0,\sigma_1]$ and clearly 1-periodic in t. Also $K(\cdot,t,\varepsilon) \in C^k(G_r)$ is satisfied for all $t \in \mathbb{R}$, $\varepsilon \in [0,\sigma_1]$ and furthermore $\partial_X^{\boldsymbol{\nu}} K \in C(G_r \times [0,1] \times [0,\sigma_1])$ for $|\boldsymbol{\nu}| \leq k$ follows from the continuity of Ψ and its derivatives. It remains to show the boundedness of $\|K\|_{k,r,\sigma_1}$ and to prove (5.10). To this end, fix $t \in \mathbb{R}$ and $\varepsilon \in [0,\sigma_1]$. For clarity's sake we omit the dependence of x with respect to (X,t,ε) and simply write $x = (q,p)$. Then, spitting up H yields

$$K(X,t,\varepsilon) - \bar{H}(X,\varepsilon) = \bar{H}(x,\varepsilon) + \tilde{H}(x,t,\varepsilon) - \tilde{H}(q,P,t,\varepsilon) - \bar{H}(X,\varepsilon).$$

We have $\bar{H}(x,\varepsilon) - \bar{H}(X,\varepsilon) = \int_0^1 \nabla\bar{H}(\lambda x + (1-\lambda)X,\varepsilon)(x-X)\,d\lambda$. Therefore, Lemma 5.1, Corollary 5.4 and (5.4) imply

$$\begin{aligned}
&\|\bar{H}(\Psi(\cdot,t,\varepsilon),\varepsilon) - \bar{H}(\cdot,\varepsilon)\|_{C^k(G_r)} \\
&\quad \leq \sup_{\lambda\in[0,1]} 2^k \|\nabla\bar{H}(\cdot,\varepsilon) \circ ((1-\lambda)\mathrm{id} + \lambda\Psi(\cdot,t,\varepsilon))\|_{C^k(G_r)} \|\Psi(\cdot,t,\varepsilon) - \mathrm{id}\|_{C^k(G_r)} \\
&\quad \leq 2^{2k} B_k \|\bar{H}(\cdot,t,\varepsilon)\|_{C^{k+1}(G_r)} (2+C_k)^k \|\tilde{H}\|_{k+1,\rho,\sigma} C_k \varepsilon \\
&\quad \leq \frac{\kappa_k}{3} \|\bar{H}\|_{k+1,\rho,\sigma} \|\tilde{H}\|_{k+1,\rho,\sigma} \varepsilon.
\end{aligned}$$

The second term is dealt with in the same fashion so that

$$\|\tilde{H}(\Psi(\cdot,t,\varepsilon),t,\varepsilon) - \tilde{H}(q,P,t,\varepsilon)\|_{C^k(G_r)} = \frac{2\kappa_k}{3} \|\bar{H}\|_{k+1,\rho,\sigma} \|\tilde{H}\|_{k+1,\rho,\sigma} \varepsilon.$$

Thus in total

$$\|K(\cdot,t,\varepsilon) - \bar{H}(\cdot,\varepsilon)\|_{C^k(G_r)} \leq \kappa_k \|\tilde{H}\|_{k+1,\rho,\sigma} \|H\|_{k+1,\rho,\sigma} \varepsilon.$$

Together with $\|\bar{H}\|_{k,r,\sigma_1} \leq \|H\|_{k,r,\sigma_1}$ this also yields $\|K\|_{k,r,\sigma_1} < \infty$. □

Corollary 5.12. *Under the assumptions of Lemma 5.11, estimate* (5.10) *also implies the following inequalities*

$$\|K(\cdot,t,\varepsilon)\|_{C^k(G_r)} \leq (\kappa_k \varepsilon + 1) \|H\|_{k+1,\rho,\sigma}, \tag{5.11}$$

$$\|\bar{K}(\cdot,\varepsilon) - \bar{H}(\cdot,\varepsilon)\|_{C^k(G_r)} \leq \kappa_k \|H\|_{k+1,\rho,\sigma} \varepsilon, \tag{5.12}$$

$$\|\tilde{K}(\cdot,t,\varepsilon)\|_{C^k(G_r)} \leq \kappa_k \sup_{t\in\mathbb{R}} \|\tilde{H}(\cdot,t,\varepsilon)\|_{C^{k+1}(G_\rho)} \varepsilon, \tag{5.13}$$

for $t \in \mathbb{R}$ *and* $\varepsilon \in [0,\sigma_1]$.

Proof. The first inequality follows immediately with $\|\tilde{H}\|_{k+1,\rho,\sigma} \leq 1$. In view of (5.9), also (5.10) can be improved to

$$\|K(\cdot,t,\varepsilon) - \bar{H}(\cdot,\varepsilon)\|_{C^k(G_r)} \leq \kappa_k \sup_{t\in\mathbb{R}} \|\tilde{H}(\cdot,t,\varepsilon)\|_{C^{k+1}(G_\rho)} \|H\|_{k+1,\rho,\sigma}\varepsilon,$$

for all $t \in \mathbb{R}$ and $\varepsilon \in [0,\sigma_1]$. Thus, we have

$$\begin{aligned}\|\bar{K}(\cdot,\varepsilon) - \bar{H}(\cdot,\varepsilon)\|_{C^k(G_r)} &= \max_{|\boldsymbol{\nu}|\leq k} \sup_{X\in G_r} \left| \int_0^1 [\partial_X^{\boldsymbol{\nu}} K(X,t,\varepsilon) - \partial_X^{\boldsymbol{\nu}} \bar{H}(X,\varepsilon)]\, dt \right| \\ &\leq \max_{|\boldsymbol{\nu}|\leq k} \sup_{\substack{X\in G_r\\ t\in\mathbb{R}}} \left| [\partial_X^{\boldsymbol{\nu}} K(X,t,\varepsilon) - \partial_X^{\boldsymbol{\nu}} \bar{H}(X,\varepsilon)] \right| \\ &\leq \kappa_k \sup_{t\in\mathbb{R}} \|\tilde{H}(\cdot,t,\varepsilon)\|_{C^{k+1}(G_\rho)} \|H\|_{k+1,\rho,\sigma}\varepsilon,\end{aligned}$$

implying (5.12). Finally, (5.13) follows from $\|H\|_{k+1,\rho,\sigma} \leq \frac{1}{2}$ together with the identity $\tilde{K} = (K - \bar{H}) + (\bar{H} - \bar{K})$. □

By iterating Lemma 5.11, we obtain the following.

Theorem 5.13 (Hamiltonian normal forms). *Let $k \in \mathbb{N}$, $\sigma > 0$ and fix $0 < \hat{r} < r < \rho$. For $H \in \mathcal{H}^{k+2}_{\rho,\sigma}$ consider the 1-periodic Hamiltonian system*

$$\dot{x} = \varepsilon J \nabla_x H(x,t,\varepsilon). \tag{5.14}$$

There are $\sigma_* \in (0,\sigma]$ and a constant $C_* > 0$ (depending only upon k and $\|H\|_{k+2,\rho,\sigma}$) with the following properties. There is an admissible change of variables $\Psi : G_r \times \mathbb{R} \times [0,\sigma_*] \to G_\rho$, $x = \Psi(X,t,\varepsilon)$, of class C^2 such that

$$\Psi\,(G_r,t,\varepsilon) \supset G_{\hat{r}} \tag{5.15}$$

for all $t \in \mathbb{R}$, $\varepsilon \in [0,\sigma_*]$, and (5.14) is transformed into

$$\dot{X} = \varepsilon(J\nabla_X \mathcal{N}(X,\varepsilon) + J\nabla_X \mathcal{R}(X,t,\varepsilon)),$$

where $\mathcal{N} \in \mathcal{H}^2_{r,\sigma_*}$, $\mathcal{R} \in \tilde{\mathcal{H}}^2_{r,\sigma_*}$. Moreover, for every $t \in \mathbb{R}$, $\varepsilon \in [0,\sigma_*]$ we have

$$\|\mathcal{N}\|_{2,r,\sigma_*} \leq 2\|H\|_{k+2,\rho,\sigma},$$

$$\|\mathcal{R}(\cdot,t,\varepsilon)\|_{C^2(G_r)} \leq C_*\varepsilon^k \tag{5.16}$$

$$\|\mathcal{N}(\cdot,\varepsilon) - \bar{H}(\cdot,\varepsilon)\|_{C^2(G_r)} \leq C_*\varepsilon. \tag{5.17}$$

Proof. **Step 1:** First, define $H_0 = H$ and assume that $\|H_0\|_{k+2,\rho,\sigma} \leq \frac{1}{4}$. For $0 \leq i \leq k$ let $r_i = \rho - \frac{i}{k}(\rho - r)$, i.e. $0 < r = r_k < r_{k-1} < \ldots < r_0 = \rho$. We are going to apply Lemma 5.11 k-times consecutively and concatenate the according transformations. For $1 \leq i \leq k$ (and starting at $i = 1$), the i-th step will be to apply Lemma 5.11 to H_{i-1} with

$$\sigma_i = \sigma_* = \min\left\{\sigma, \frac{1}{2}, \frac{\rho - r}{k}, \frac{2^{1/k} - 1}{4\kappa_{k+1}}, \frac{1}{2^{2k} k C_{k+1}}\right\}, \tag{5.18}$$

which yields an admissible change of variables of class C^{k+2-i}, that is

$$\Psi_i : G_{r_i} \times \mathbb{R} \times [0, \sigma_*] \to G_{r_{i-1}}, \quad x_i \mapsto x_{i-1} = \Psi_i(x_i, t, \varepsilon),$$

so that the Hamiltonian equations $\dot{x}_{i-1} = \varepsilon J \nabla H_{i-1}(x_{i-1}, t, \varepsilon)$ are transformed into

$$\dot{x}_i = \varepsilon J \nabla H_i(x_i, t, \varepsilon).$$

Since $H_i \in \mathcal{H}^{k+2-i}_{r_i,\sigma_*}$, the subsequent change of coordinates is well-defined if $\|H_i\|_{k+2-i,r_i,\sigma_*} \leq \frac{1}{2}$. This is guaranteed by the estimate

$$\|H_i\|_{k+2-i,r_i,\sigma_*} \leq 2^{i/k} \|H_0\|_{k+2,\rho,\sigma}, \tag{5.19}$$

which we will now prove inductively for $1 \leq i \leq k$. Using (5.11) and (5.18) yields

$$\|H_1(\cdot, t, \varepsilon)\|_{C^{k+1}(G_{r_1})} \leq (\kappa_{k+1}\varepsilon + 1)\|H_0\|_{k+2,\rho,\sigma} \leq 2^{1/k} \|H_0\|_{k+2,\rho,\sigma}$$

for $t \in \mathbb{R}$ and $\varepsilon \in [0, \sigma_*]$. Now, assume $i \in \{1, \ldots, k-1\}$ changes of coordinates have been done and $H_i \in \mathcal{H}^{k+2-i}_{r_i,\sigma_*}$ satisfies (5.19). In particular, the assumption $\|H_0\|_{k+2,\rho,\sigma} \leq \frac{1}{4}$ and (5.19) show that $\|H_i\|_{k+2-i,r_i,\sigma_*} \leq \frac{1}{2}$. Therefore Lemma 5.11 can be applied. Again, using (5.11), the inductive property and (5.18), we get

$$\begin{aligned}
\|H_{i+1}(\cdot, t, \varepsilon)\|_{C^{k+1-i}(G_{r_{i+1}})} &\leq (\kappa_{k+1-i}\varepsilon + 1)\|H_i\|_{k+2-i,r_i,\sigma_*} \\
&\leq (\kappa_{k+1}\varepsilon + 1) 2^{i/k} \|H_0\|_{k+2,\rho,\sigma} \\
&\leq 2^{(i+1)/k} \|H_0\|_{k+2,\rho,\sigma}.
\end{aligned}$$

Thus we have shown that Lemma 5.11 can be applied k-times consecutively so that the bound (5.19) holds for $1 \leq i \leq k$. Hence, the transformation

$$\Psi : G_r \times \mathbb{R} \times [0, \sigma_*] \to G_\rho,$$
$$\Psi(\cdot, t, \varepsilon) = \Psi_1(\cdot, t, \varepsilon) \circ \Psi_2(\cdot, t, \varepsilon) \circ \cdots \circ \Psi_k(\cdot, t, \varepsilon),$$

is well-defined and an admissible change of variables of class C^2. Also the new Hamiltonian H_k is in $\mathcal{H}^2_{r,\sigma_*}$ and we define $\mathcal{N} = \bar{H}_k$, $\mathcal{R} = \tilde{H}_k$. Due to (5.4) and (5.19) we have

$$\|\Psi_i(\cdot, t, \varepsilon) - \mathrm{id}\|_{C^{k+2-i}(G_{r_i})} \leq \|\tilde{H}_{i-1}\|_{k+3-i, r_{i-1}, \sigma_*} C_{k+2-i}\varepsilon \leq C_{k+1}\varepsilon \tag{5.20}$$

for $1 \leq i \leq k$, $t \in \mathbb{R}$ and $\varepsilon \in [0, \sigma_*]$. In particular, (5.18) thus implies $\|\Psi_i(\cdot, t, \varepsilon)\|_{C^{k+2-i}(G_{r_i})} \leq 2$. In order to prove (5.15) we introduce the following notation: For a fixed $t \in \mathbb{R}$ and $\varepsilon \in [0, \sigma_*]$ set $\chi_k = \Psi_k(\cdot, t, \varepsilon)$ and

$$\chi_i = \Psi_i(\cdot, t, \varepsilon) \circ \chi_{i+1}, \quad 1 \leq i \leq k-1.$$

In particular, it follows $\chi_1 = \Psi(\cdot, t, \varepsilon) = \Psi_1(\cdot, t, \varepsilon) \circ \ldots \circ \Psi_k(\cdot, t, \varepsilon)$. Moreover, $\|\chi_k\|_{C^1(G_r)} \leq 2$ and for $1 \leq i \leq k-1$ applying Corollary 5.4 yields

$$\|\chi_i\|_{C^1(G_r)} \leq 2\|\Psi_i(\cdot, t, \varepsilon)\|_{C^1(G_r)}\|\chi_{i+1}\|_{C^1(G_r)} \leq 4\|\chi_{i+1}\|_{C^1(G_r)}.$$

Hence $\|\chi_i\|_{C^1(G_r)} \leq 2^{2(k-i)+1}$. By using Corolary 5.4 and (5.20), we obtain

$$\|\chi_i - \chi_{i+1}\|_{C^1(G_r)} = \|(\Psi_i(\cdot, t, \varepsilon) - \mathrm{id}) \circ \chi_{i+1}\|_{C^1(G_r)} \leq 2^{2(k-i)} C_{k+1}\varepsilon.$$

In total, we have

$$\begin{aligned}\|\Psi(\cdot, t, \varepsilon) - \mathrm{id}\|_{C^1(G_r)} &\leq \sum_{i=1}^{k-1} \|\chi_i - \chi_{i+1}\|_{C^1(G_r)} + \|\chi_k - \mathrm{id}\|_{C^1(G_r)} \\ &\leq \sum_{i=1}^{k} 2^{2(k-i)} C_{k+1}\varepsilon.\end{aligned}$$

In particular, $\|D\Psi(\cdot, t, \varepsilon) - I\|_{C^0(G_r)} \leq \frac{1}{2}$ by (5.18) and thus (5.15) follows from the same argument as Corollary 5.9. Finally, define

$$C_* = 2\kappa_{k+1}^k \|H_0\|_{k+2, \rho, \sigma}.$$

In order to check the estimates, we fix $t \in \mathbb{R}$ and $\varepsilon \in [0, \sigma_*]$. We start by verifying the bound (5.16). By (5.13) we have

$$\|\tilde{H}_i(\cdot, t, \varepsilon)\|_{C^{k+2-i}(G_{r_i})} \leq \kappa_{k+2-i} \sup_{t \in \mathbb{R}} \|\tilde{H}_{i-1}(\cdot, t, \varepsilon)\|_{C^{k+3-i}(G_{r_{i-1}})} \varepsilon$$

for $1 \leq i \leq k$. Thus inductively it follows

$$\|\tilde{H}_k(\cdot, t, \varepsilon)\|_{C^2(G_r)} \leq \kappa_{k+1}^k \sup_{t \in \mathbb{R}} \|\tilde{H}_0(\cdot, t, \varepsilon)\|_{C^{k+2}(G_{r_0})} \varepsilon^k \leq C_* \varepsilon^k.$$

The bound $\|\bar{H}_k\|_{2,r,\sigma_*} \leq \|H_k\|_{2,r,\sigma_*} \leq 2\|H_0\|_{k+2,\rho,\sigma}$ follows directly from (5.19). Therefore it remains only to prove (5.17). Due to (5.12) and (5.19) we have

$$\|\bar{H}_i(\cdot, \varepsilon) - \bar{H}_{i-1}(\cdot, \varepsilon)\|_{C^{k+2-i}(G_{r_i})} \leq 2\kappa_{k+1} \|H_0\|_{k+2,\rho,\sigma} \varepsilon$$

for $1 \leq i \leq k$. Since $k \leq \kappa_{k+1}^{k-1}$, adding up yields

$$\|\bar{H}_k(\cdot, t, \varepsilon) - \bar{H}_0(\cdot, \varepsilon)\|_{C^2(G_r)} \leq C_* \varepsilon.$$

Step 2: Now, consider the case $A := 4\|H_0\|_{k+2,\rho,\sigma} > 1$. Rewrite (5.14) as $\dot{x} = \hat{\varepsilon} J \nabla \hat{H}_0(x, t, \hat{\varepsilon})$, where $\hat{\varepsilon} = A\varepsilon$ and $\hat{H}_0(x, t, \hat{\varepsilon}) = A^{-1} H_0(x, t, A^{-1}\hat{\varepsilon})$. Then $\|\hat{H}_0\|_{k+2,\rho,\hat{\sigma}} = A^{-1}\|H_0\|_{k+2,\rho,\sigma}$, where $\hat{\sigma} = A\sigma$. In particular, the rescaled Hamiltonian satisfies $\|\hat{H}_0\|_{k+2,\rho,\hat{\sigma}} \leq \frac{1}{4}$. Therefore, the first step can be applied to obtain an admissible change of variables

$$\hat{\Psi} : G_r \times \mathbb{R} \times [0, \hat{\sigma}_*] \to G_\rho, \quad x = \hat{\Psi}(X, t, \hat{\varepsilon}),$$

with its image contained in $G_{\hat{r}}$ and a new Hamiltonian $\hat{H}_k \in \mathcal{H}^2_{r,\hat{\sigma}_*}$ satisfying the required bounds with a constant $\hat{C}_*$. Then, defining

$$\Psi(X, t, \varepsilon) = \hat{\Psi}(X, t, A\varepsilon) \quad \text{and} \quad H_k(x, t, \varepsilon) = A\hat{H}_k(x, t, A\varepsilon)$$

for $\varepsilon \in [0, \sigma_*]$ with $\sigma_* = A^{-1}\hat{\sigma}_*$ proves the assertion. □

5.3 An adiabatic invariant for E-symplectic maps

Again, let $G = \mathbb{R} \times I$ and write G_ρ for its ρ-neighborhood. We study symplectic maps of the form

$$\mathcal{P}_\varepsilon : G \to \mathbb{R}^2, \quad x_1 = x + \varepsilon l(x, \varepsilon).$$

One can find a function $E(x)$ satisfying $J\nabla E(x) = l(x,0)$, which acts as an adiabatic invariant. In fact, for small ε the iteration $x_{n+1} = \mathcal{P}_\varepsilon(x_n)$ can be viewed as a numerical integration step for the autonomous Hamiltonian system $\dot{x} = J\nabla E(x)$. This was already observed in [Nei84], where it is shown that for holomorphic maps one has

$$|E(\mathcal{P}_\varepsilon^n(x)) - E(x)| \le C\varepsilon, \quad 0 \le n \le N_\varepsilon,$$

where N_ε is of the order $e^{D/\varepsilon}$ and $C, D > 0$ are suitable constants. However, since the domain G is unbounded, some additional assumptions are required. This is discussed in detail by Kunze and Ortega in [KO]. They introduce the notion of E-symplectic families of maps. Here, we shall follow their arguments with the according adaptation to the non-analytic case.

Definition 5.14. Let $k \in \mathbb{N}$, $\rho > 0$ and $\sigma > 0$. The class $\mathcal{M}_{\rho,\sigma}^k$ consists of those functions $l : G_\rho \times [0,\sigma] \to \mathbb{R}^d$, $l = l(x,\varepsilon)$, such that

(i) l is in $C^2(G_\rho \times (0,\sigma], \mathbb{R}^d)$,

(ii) for every $\varepsilon \in [0,\sigma]$ we have $l(\cdot,\varepsilon) \in C^k(G_\rho, \mathbb{R}^d)$ and

$$\partial_x^{\boldsymbol{\nu}} l \in C(G_\rho \times [0,\sigma], \mathbb{R}^d) \qquad \text{for } 0 \le |\boldsymbol{\nu}| \le k,$$

(iii) for every $\varepsilon \in (0,\sigma]$ we have $\partial_\varepsilon l(\cdot,\varepsilon) \in C^k(G_\rho, \mathbb{R}^d)$ and

$$\partial_x^{\boldsymbol{\nu}} \partial_\varepsilon l \in C(G_\rho \times (0,\sigma], \mathbb{R}^d) \qquad \text{for } 0 \le |\boldsymbol{\nu}| \le k,$$

(iv) one has $\|l\|_{k,\rho,\sigma}^* < \infty$, where

$$\|l\|_{k,\rho,\sigma}^* = \|l\|_{k,\rho,\sigma} + \sup_{\varepsilon\in(0,\sigma]} \|\partial_\varepsilon l(\cdot,\varepsilon)\|_{C^k(G_\rho)}.$$

Remark 5.15. (a) Here $\|\cdot\|_{k,\rho,\sigma}$ denotes the same norm as in Definition 5.5, that is

$$\|l\|_{k,\rho,\sigma} = \sup_{\varepsilon\in[0,\sigma]} \|l(\cdot,\varepsilon)\|_{C^k(G_\rho)} < \infty,$$

since the components of $l \in \mathcal{M}_{\rho,\sigma}^k$ can be regarded as functions in $\mathcal{H}_{\rho,\sigma}^k$ constant in $t \in \mathbb{R}$.

(b) Functions in $\mathcal{M}^k_{\rho,\sigma}$ have values in $\mathbb{R}^d$ for some $d \in \mathbb{N}$. Here, we will only encounter $d \in \{1, 2\}$ and the respective dimension will be clear from the context.

Definition 5.16. Suppose $l \in \mathcal{M}^k_{\rho,\sigma}$ and for $\varepsilon \in [0, \sigma]$ consider the family of maps $\mathcal{P}_\varepsilon : G_\rho \to \mathbb{R}^2$ given by

$$\mathcal{P}_\varepsilon(x) = X = x + \varepsilon l(x, \varepsilon), \quad X = (Q, P), \ x = (q, p).$$

We say $\{\mathcal{P}_\varepsilon\}$ is an *E-symplectic family of class* C^k, if there is a function $\zeta \in \mathcal{M}^k_{\rho,\sigma}$ such that

$$P \ dQ - p \ dq = d\zeta(\cdot, \varepsilon) \tag{5.21}$$

for all $\varepsilon \in [0, \sigma]$ and if moreover there is a function $\mathfrak{m} \in C^k_b(G_\rho)$ and a constant $C_\mathfrak{m} > 0$ such that

$$\|\zeta(\cdot, \varepsilon) - \varepsilon\mathfrak{m}\|_{C^1(G_\rho)} \leq C_\mathfrak{m}\varepsilon^2 \tag{5.22}$$

and

$$\|\partial_\varepsilon\zeta(\cdot, \varepsilon) - \mathfrak{m}\|_{C^0(G_\rho)} \leq C_\mathfrak{m}\varepsilon \tag{5.23}$$

for all $\varepsilon \in [0, \sigma]$.

Remark 5.17. (a) Equation (5.22) implies $\zeta(q, p, 0) = 0$ and thus dividing by ε shows that $\varepsilon^{-1}\zeta(\cdot, \varepsilon) \to \mathfrak{m}$ in $C^1(G_\rho)$ as $\varepsilon \to 0$. Moreover, for $\varepsilon > 0$, equation (5.21) can be reformulated as

$$\frac{\partial\zeta}{\partial q} = \varepsilon l_2 + \varepsilon p\frac{\partial l_1}{\partial q} + \varepsilon^2 l_2\frac{\partial l_1}{\partial q}, \quad \frac{\partial\zeta}{\partial p} = \varepsilon p\frac{\partial l_1}{\partial p} + \varepsilon^2 l_2\frac{\partial l_1}{\partial p},$$

where l_1, l_2 denote the components of l. Since $\varepsilon^{-1}\zeta(\cdot, \varepsilon) \to \mathfrak{m}$ in $C^1(G_\rho)$, this yields

$$\frac{\partial\mathfrak{m}}{\partial q}(q, p) = l_2(q, p, 0) + p\frac{\partial l_1}{\partial q}(q, p, 0), \quad \frac{\partial\mathfrak{m}}{\partial p}(q, p) = p\frac{\partial l_1}{\partial p}(q, p, 0),$$

or expressed differently

$$\nabla\mathfrak{m}(x) = p\nabla l_1(x, 0) + \begin{pmatrix} l_2(x, 0) \\ 0 \end{pmatrix}. \tag{5.24}$$

This relation will be used later. In fact, it shows that $J\nabla E = l(\cdot, 0)$ for the autonomous Hamiltonian

$$E(q, p) = l_1(q, p, 0)p - \mathfrak{m}(q, p).$$

(b) Condition (5.22) does not imply (5.23), as can be seen in the example

$$\zeta(x,\varepsilon) = \varepsilon\mathfrak{m}(x) + \varepsilon^2 \sin\left(\frac{1}{\varepsilon}\right).$$

First, we observe that certain symplectic isotopies can be realized as the solution map to a suitable periodic Hamiltonian system.

Lemma 5.18. *Consider a continuous map $\Phi : G \times [0,1] \to \mathbb{R}^2$ such that $\Phi(\cdot,t)$ is a diffeomorphism onto its image for every $t \in [0,1]$. We will write $\Psi(\cdot,t)$ for the inverse and further denote*

$$\Phi(x,t) = (Q,P) = X \quad \text{and} \quad \Psi(X,t) = (q,p) = x.$$

With a small abuse of notation we also write $Q(x,t)$, $P(x,t)$ and $q(X,t)$, $p(X,t)$ for the components of Φ and Ψ, respectively. Moreover, assume that there is a continuously differentiable function $\eta(x,t)$ such that

$$P(\cdot,t)\,dQ(\cdot,t) - p\,dq = d\eta(\cdot,t). \tag{5.25}$$

Finally, suppose $\Phi,\eta \in C^1(G\times[0,1])$ and that the cross-derivatives

$$\frac{\partial^2 Q}{\partial t\partial x} = \frac{\partial^2 Q}{\partial x\partial t}, \quad \frac{\partial^2 P}{\partial t\partial x} = \frac{\partial^2 P}{\partial x\partial t}, \quad \frac{\partial^2 \eta}{\partial t\partial x} = \frac{\partial^2 \eta}{\partial x\partial t} \tag{5.26}$$

exist, coincide and are continuous functions of (x,t). Then

$$\frac{\partial\Phi}{\partial t}(\Psi(X,t),t) = J\nabla h_{aux}(X,t). \tag{5.27}$$

where

$$h_{aux}(X,t) = \frac{\partial Q}{\partial t}(\Psi(X,t,\varepsilon),t)P - \frac{\partial\eta}{\partial t}(\Psi(X,t),t), \tag{5.28}$$

is defined on $\{(X,t) : t\in[0,1], X\in\Phi(G,t)\}$.

Proof. First note, that (5.25) can be stated as

$$\frac{\partial\eta}{\partial q} = P\frac{\partial Q}{\partial q} - p, \quad \frac{\partial\eta}{\partial p} = P\frac{\partial Q}{\partial p}.$$

Differentiation with respect to t yields

$$\frac{\partial^2\eta}{\partial t\partial q} = \left(\frac{\partial P}{\partial t}\right)\frac{\partial Q}{\partial q} + P\frac{\partial^2 Q}{\partial t\partial q}, \quad \frac{\partial^2\eta}{\partial t\partial p} = \left(\frac{\partial P}{\partial t}\right)\frac{\partial Q}{\partial p} + P\frac{\partial^2 Q}{\partial t\partial p}.$$

On the other hand, differentiating (5.28) with respect to P results in

$$\tfrac{\partial h_{\text{aux}}}{\partial P} = \left[\left(\tfrac{\partial^2 Q}{\partial t \partial q}\right)\left(\tfrac{\partial q}{\partial P}\right) + \left(\tfrac{\partial^2 Q}{\partial t \partial p}\right)\left(\tfrac{\partial p}{\partial P}\right)\right] P + \tfrac{\partial Q}{\partial t} - \left(\tfrac{\partial^2 \eta}{\partial t \partial q}\right)\left(\tfrac{\partial q}{\partial P}\right) - \left(\tfrac{\partial^2 \eta}{\partial t \partial p}\right)\left(\tfrac{\partial p}{\partial P}\right).$$

Together this leads to

$$\frac{\partial h_{\text{aux}}}{\partial P} = \frac{\partial Q}{\partial t} - \left(\frac{\partial P}{\partial t}\right)\left[\left(\frac{\partial Q}{\partial q}\right)\left(\frac{\partial q}{\partial P}\right) + \left(\frac{\partial Q}{\partial p}\right)\left(\frac{\partial p}{\partial P}\right)\right].$$

The second term vanishes, as can be seen by differentiating the identity $Q(q(Q,P,t), p(Q,P,t), t) = Q$ with respect to P. Similarly, we get

$$\frac{\partial h_{\text{aux}}}{\partial Q} = -\left(\frac{\partial P}{\partial t}\right)\left[\left(\frac{\partial Q}{\partial q}\right)\left(\frac{\partial q}{\partial Q}\right) + \left(\frac{\partial Q}{\partial p}\right)\left(\frac{\partial p}{\partial Q}\right)\right].$$

Thus, $\frac{d}{dQ} Q(q(Q,P,t), p(Q,P,t), t) = 1$ proves (5.27). □

Remark 5.19. In the notation of Lemma 5.18, let $X(t) = \Phi(x,t)$ for $x \in G$. Then $X(t)$ is a solution of the Hamiltonian system

$$\dot{X}(t) = J\nabla h_{\text{aux}}(X(t), t).$$

In the context of E-symplectic families this leads to the following result.

Lemma 5.20. *Let $G = \mathbb{R} \times I$, where $I \subset \mathbb{R}$ is an open bounded interval. For $l \in \mathcal{M}^k_{\rho,\sigma}$ and $\varepsilon \in [0,\sigma]$ assume that the family $\mathcal{P}_\varepsilon : G_\rho \to \mathbb{R}^2$ given by*

$$\mathcal{P}_\varepsilon(x) = x + \varepsilon l(x,\varepsilon)$$

is E-symplectic of class C^k. Given $0 < r < \hat{r} < \rho$, there is $\hat{\sigma} \in (0,\sigma]$ and a Hamiltonian $H_{aux} \in \mathcal{H}^{k+1}_{\hat{r},\hat{\sigma}}$ such that for $\varepsilon \in [0,\hat{\sigma}]$ the map $\mathcal{P}_\varepsilon$ restricted to G_r coincides with the time-1 map for the 1-periodic system

$$\dot{X} = \varepsilon J\nabla H_{aux}(X,t,\varepsilon),$$

and for $X \in G_r$ we have

$$J\nabla \bar{H}_{aux}(X,0) = l(X,0). \tag{5.29}$$

Moreover, there is a constant $C_{aux} > 0$ depending only upon k, ρ, $\hat{r}$, r, σ, $\|l\|^*_{k,\rho,\sigma}$, $\|\zeta\|^*_{k,\rho,\sigma}$, $C_I = \max\{\rho + |p| : p \in I\}$ and $C_{\mathfrak{m}}$ (where $\zeta \in \mathcal{M}^k_{\rho,\sigma}$ and $C_{\mathfrak{m}}$ are from Definition 5.16) such that $\|H_{aux}\|_{k+1,\hat{r},\hat{\sigma}} \le C_{aux}$ and

$$|H_{aux}(X,t,\varepsilon) - H_{aux}(X,t,0)| \le C_{aux}\varepsilon \tag{5.30}$$

uniformly in $X \in G_{\hat{r}}$, $t \in [0,1]$ and $\varepsilon \in [0,\hat{\sigma}]$.

Proof. Let $\chi : [0,1] \to [0,1]$ be a strictly increasing C^∞-function such that $\chi(0) = 0$, $\chi(1) = 1$ and $\dot\chi(0) = \dot\chi(1) = 0$. Consider the map

$$\Phi : G_\rho \times [0,1] \times [0,\hat\sigma] \to \mathbb{R}^2, \quad \Phi(x,t,\varepsilon) = x + \varepsilon\chi(t) l(x, \varepsilon\chi(t)),$$

where

$$\hat\sigma = \min\left\{\sigma, 1, \frac{1}{4\|l\|_{k,\rho,\sigma}}, \frac{\rho - \hat r}{2\|l\|_{k,\rho,\sigma}}, \frac{\hat r - r}{\|l\|_{k,\rho,\sigma}}, \frac{1}{2^{3k+1} B_k \|l\|_{k,\rho,\sigma}}\right\}. \tag{5.31}$$

We will show that Φ is suitable for the application of Lemma 5.18 if ε is fixed. First, note that $\Phi(x,t,\varepsilon) = \mathcal{P}_{\varepsilon\chi(t)}(x)$ and therefore it is exact symplectic in the sense of (5.25) with potential

$$\eta(x,t,\varepsilon) = \zeta(x, \varepsilon\chi(t)).$$

Obviously, Φ is continuous. Now fix $t \in [0,1]$ and $\varepsilon \in [0,\hat\sigma]$, then

$$(D_x\Phi)(x,t,\varepsilon) = I + \varepsilon\chi(t)(D_x l)(x, \varepsilon\chi(t)). \tag{5.32}$$

Thus $|(D_x\Phi)(x,t,\varepsilon) - I| \le \frac{1}{4}$ by (5.31), which shows that $\Phi(\cdot,t,\varepsilon)$ is a local C^k-diffeomorphism with respect to its image. Since G is convex, for $x_1, x_2 \in G_\rho$ we have

$$\begin{aligned}
&|\Phi(x_1,t,\varepsilon) - \Phi(x_2,t,\varepsilon)| \\
&= \left| x_1 - x_2 + \varepsilon\chi(t)\Big(\int_0^1 (D_x l)(\lambda x_1 + (1-\lambda)x_2, \varepsilon\chi(t))\, d\lambda\Big)(x_1 - x_2)\right| \\
&\ge |x_1 - x_2| - \tfrac{1}{2}|x_1 - x_2| = \tfrac{1}{2}|x_1 - x_2|.
\end{aligned}$$

Hence $\Phi(\cdot,t,\varepsilon)$ is also one-to-one and therefore it is a C^k-diffeomorphism with respect to its image. It also follows that the inverse $\Psi(\cdot,t,\varepsilon)$ is Lipschitz continuous with constant 2. Hence, (5.31) and the same argument as in Corollary 5.9 show that $\Phi(G_\rho,t,\varepsilon) \supset G_{\hat r}$. Moreover, since we have $\Phi \in C(G_\rho \times [0,1] \times [0,\hat\sigma], \mathbb{R}^2)$, the inverse Ψ is continuous with respect to all three variables as well.

We now prove that $\Phi(\cdot,\cdot,\varepsilon)$ is C^1 in $G_\rho \times [0,1]$ for all $\varepsilon \in [0,\hat\sigma]$ and that (5.26) is satisfied. For $\varepsilon = 0$ we have $\Phi = \mathrm{id}_x$, so one only has to consider the case $\varepsilon > 0$. Clearly, $\Phi(\cdot,\cdot,\varepsilon)$ is C^2 in $G_\rho \times (0,1]$ as $l \in \mathcal{M}^k_{\rho,\sigma}$.

Since $\Phi(\cdot,0,\varepsilon)=\mathrm{id}_x$, formula (5.32) together with $\|l\|_{k,\rho,\sigma}<\infty$ imply $D_x\Phi(\cdot,\cdot,\varepsilon)\in C(G_\rho\times[0,1],\mathbb{R}^4)$. The derivative with respect to t on the other hand is given by

$$\partial_t\Phi(x,t,\varepsilon)=\varepsilon\dot\chi(t)[l(x,\varepsilon\chi(t))+\varepsilon\chi(t)\partial_\varepsilon l(x,\varepsilon\chi(t))].$$

if $t>0$ and in the case $t=0$ it is

$$\partial_t\Phi(x,0,\varepsilon)=\lim_{t\downarrow 0}\tfrac{1}{t}(\Phi(x,t,\varepsilon)-\Phi(x,0,\varepsilon))=\lim_{t\downarrow 0}\tfrac{\varepsilon\chi(t)}{t}l(x,\varepsilon\chi(t))=0, \quad (5.33)$$

due to $\chi(0)=\dot\chi(0)=0$ and $\|l\|^*_{k,\rho,\sigma}<\infty$. The latter also implies the continuity of $\partial_t\Phi$ in $t=0$. Regarding the cross derivatives, we have $\partial_t\Phi(x,0,\varepsilon)=0$ so that $D_x\partial_t\Phi(x,0,\varepsilon)=0$. The fact that also $\partial_t D_x\Phi(x,0,\varepsilon)=0$ follows from an equation similar to (5.33). For $t>0$ on the other hand, we have

$$(\partial_t D_x\Phi)(x,t,\varepsilon)=\varepsilon\dot\chi(t)(D_x l)(x,\varepsilon\chi(t))+\varepsilon^2\chi(t)\dot\chi(t)(D_x\partial_\varepsilon l)(x,\varepsilon\chi(t)).$$

Thus, the continuity of the cross derivatives follows from $\|l\|^*_{k,\rho,\sigma}<\infty$. Since also $\zeta\in\mathcal{M}^k_{\rho,\sigma}$, the same argument can be done for η verifying $\eta(\cdot,\cdot,\varepsilon)\in C^1(G_\rho\times[0,1])$ and (5.26). In summary, we have shown that Lemma 5.18 is applicable. Considering the additional ε, let $h_{\mathrm{aux}}(X,t,\varepsilon)$ be the function given by (5.28), which is defined on

$$\mathcal{D}=\{(X,t,\varepsilon):t\in[0,1],\varepsilon\in[0,\hat\sigma],X\in\Phi(G_\rho,t,\varepsilon)\}.$$

In particular, it is well-defined for all $X\in G_{\hat r}$. We have $h_{\mathrm{aux}}(X,t,\varepsilon)=0$ if either $\varepsilon=0$ or $t\in\{0,1\}$. Otherwise,

$$\begin{aligned}h_{\mathrm{aux}}(X,t,\varepsilon)=&\varepsilon\dot\chi(t)\Big[l_1(\Psi(X,t,\varepsilon),\varepsilon\chi(t))+\varepsilon\chi(t)\partial_\varepsilon l_1(\Psi(X,t,\varepsilon),\varepsilon\chi(t))\Big]P\\&-\dot\chi(t)\partial_\varepsilon\zeta(\Psi(X,t,\varepsilon),\varepsilon\chi(t))\end{aligned}$$

For $X\in G_{\hat r}$ and $t\in[0,1]$ we define

$$H_{\mathrm{aux}}(X,t,\varepsilon)=\begin{cases}\varepsilon^{-1}h_{\mathrm{aux}}(X,t,\varepsilon), & \varepsilon\in(0,\hat\sigma]\\ \dot\chi(t)[l_1(X,0)P-\mathfrak{m}(X)], & \varepsilon=0,\end{cases} \quad (5.34)$$

where $\mathfrak{m}=\zeta(\cdot,0)$ is the function from Definition 5.16. Note also, that $\bar H_{\mathrm{aux}}(X,0)=E(X)$ with E defined in Remark 5.17. This verifies (5.29).

We will now show that H_{aux} has all the other required properties as well. First, we prove that $H_{\mathrm{aux}} \in \mathcal{H}^{k+1}_{\hat{r},\hat{\sigma}}$. To this end, observe that H_{aux} is continuous, since $l \in \mathcal{M}^k_{\rho,\sigma}$ and due to (5.23). Moreover, we have $H_{\mathrm{aux}}(X,0,\varepsilon) = H_{\mathrm{aux}}(X,1,\varepsilon) = 0$ so that one can extend H_{aux} continuously and 1-periodically to $t \in \mathbb{R}$. For all $\varepsilon \in [0,\hat{\sigma}]$ and $t \in \mathbb{R}$ we have

$$\|H_{\mathrm{aux}}(\cdot,t,\varepsilon)\|_\infty \leq \|\,\dot{\chi}\|_\infty \Big(\|l\|^*_{k,\rho,\sigma} C_I + \|\zeta\|^*_{k,\rho,\sigma} + C_{\mathfrak{m}} \Big),$$

since $\|\mathfrak{m}\|_\infty \leq \|\partial_\varepsilon \zeta(\cdot,\varepsilon)\|_\infty + \|\mathfrak{m} - \partial_\varepsilon \zeta(\cdot,\varepsilon)\|_\infty$ and $|P| \leq C_I$. Next, we will check the conditions regarding the derivatives. For the gradient one obtains

$$J\nabla H_{\mathrm{aux}}(X,t,\varepsilon) = \begin{cases} \varepsilon^{-1} J \nabla h_{\mathrm{aux}}(X,t,\varepsilon), & \varepsilon \in (0,\hat{\sigma}] \\ \dot{\chi}(t) J [P \nabla l_1(X,0) + (0, l_1(X,0)) - \nabla \mathfrak{m}(X)], & \varepsilon = 0. \end{cases}$$

But by definition, we have $J\nabla h_{\mathrm{aux}}(X,t,\varepsilon) = \frac{\partial \Phi}{\partial t}(\Psi(X,t,\varepsilon),t,\varepsilon)$ and this yields

$$\begin{aligned} &\varepsilon^{-1} J \nabla h_{\mathrm{aux}}(X,t,\varepsilon) \\ &\quad = \dot{\chi}(t)[l(\Psi(X,t,\varepsilon),\varepsilon\chi(t)) + \varepsilon\chi(t)\partial_\varepsilon l(\Psi(X,t,\varepsilon),\varepsilon\chi(t))], \end{aligned} \tag{5.35}$$

whereas for $\varepsilon = 0$ equation (5.24) leads to

$$J\nabla H_{\mathrm{aux}}(X,t,0) = \dot{\chi}(t) J \begin{pmatrix} -l_2(X,0) \\ l_1(X,0) \end{pmatrix} = \dot{\chi}(t) l(X,0).$$

Hence, ∇H_{aux} is continuous. In order to show the continuity of the higher derivatives in $\varepsilon = 0$ we differentiate (5.35). To this end, fix some $t \in [0,1]$, $\varepsilon \in [0,\hat{\sigma}]$ and a multi-index $\boldsymbol{\nu} \in \mathbb{N}_0^2$ with $1 \leq |\boldsymbol{\nu}| = n \leq k$. The Faá di Bruno formula (5.1) yields

$$\begin{aligned} &\partial_X^{\boldsymbol{\nu}} [l_i(\Psi(X,t,\varepsilon),\varepsilon\chi(t))] \\ &\quad = \sum_{1 \leq |\boldsymbol{\mu}| \leq n} \partial_x^{\boldsymbol{\mu}} l_i(\Psi(X,t,\varepsilon),\varepsilon\chi(t)) \sum_{p(\boldsymbol{\nu},\boldsymbol{\mu})} \boldsymbol{\nu}! \prod_{j=1}^n \frac{(\partial_X^{\boldsymbol{\ell}_j} \Psi(X,t,\varepsilon))^{\boldsymbol{k}_j}}{(\boldsymbol{k}_j!)(\boldsymbol{\ell}_j!)^{|\boldsymbol{k}_j|}}, \end{aligned}$$

for $i = 1,2$. We will show inductively, that

$$\|(D_X\Psi)(\cdot,t,\varepsilon)\|_{C^j(G_{\hat{r}})} \leq 2, \qquad \text{for } 0 \leq j \leq k-1. \tag{5.36}$$

By the Neumann series, the Jacobian of Φ satisfies

$$(D_x\Phi)^{-1}(x,t,\varepsilon) = \sum_{m=0}^{\infty}(I-(D_x\Phi)(x,t,\varepsilon))^m,$$

for all $x \in G_\rho$ and thus

$$(D_X\Psi)(X,t,\varepsilon) - I = \sum_{m=1}^{\infty}(I-(D_x\Phi)(\Psi(X,t,\varepsilon),t,\varepsilon))^m$$

for $X \in G_{\hat{r}}$. Consequently, the estimate $\|I-(D_x\Phi)(\cdot,t,\varepsilon)\|_{C^0(G_\rho)} \leq \frac{1}{2}$ verifies (5.36) for $j=0$. Now, for $1 \leq j \leq k-1$ we have

$$\begin{aligned}
&\|I-(D_x\Phi)(\Psi(\cdot,t,\varepsilon),t,\varepsilon)\|_{C^j(G_{\hat{r}})} \\
&\leq 2^j B_j\|I-(D_x\Phi)(\cdot,t,\varepsilon)\|_{C^j} \max\{\|(D_X\Psi)(\cdot,t,\varepsilon)\|_{C^{j-1}}, \|(D_X\Psi)(\cdot,t,\varepsilon)\|^j_{C^{j-1}}\} \\
&\leq 2^{2j} B_j\varepsilon\|l\|_{k,\rho,\sigma},
\end{aligned}$$

due to Corollary 5.4, (5.32) and by the inductive hypothesis. Together with Lemma 5.1 and the definition of $\hat{\sigma}$ this leads to the conclusion

$$\begin{aligned}
\|(D_X\Psi)(\cdot,t,\varepsilon) - I\|_{C^j(G_{\hat{r}})} &\leq \sum_{m=1}^{\infty} 2^{j(m-1)}\left(2^{2j}B_j\varepsilon\|l\|_{k,\rho,\sigma}\right)^m \\
&\leq 2^{2j+1}B_j\varepsilon\|l\|_{k,\rho,\sigma} \leq 2.
\end{aligned}$$

On the one hand this proves (5.36) and on the other hand it implies that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(\partial_X^{\boldsymbol{\ell}_j}\Psi(X,t,\varepsilon)\right)^{\boldsymbol{k}_j} \\
&\qquad = \begin{cases} 1, & (\boldsymbol{k}_j,\boldsymbol{\ell}_j) \in \left\{\big((m,0),(1,0)\big),((0,m),(0,1)) : 0 \leq m \leq k\right\} \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

holds uniformly. Therefore only one partition has to be considered in the limit $\varepsilon \to 0$, namely the one for which $(\boldsymbol{k}_{n-1},\boldsymbol{\ell}_{n-1}) = ((0,\nu_2),(0,1))$ and $(\boldsymbol{k}_n,\boldsymbol{\ell}_n) = ((\nu_1,0),(1,0))$. Hence

$$\lim_{\varepsilon\to 0}\partial_X^{\boldsymbol{\nu}}[l_i(\Psi(X,t,\varepsilon),\varepsilon\chi(t))] = \lim_{\varepsilon\to 0}\partial_x^{\boldsymbol{\nu}}l_i(\Psi(X,t,\varepsilon),\varepsilon\chi(t)) = \partial_x^{\boldsymbol{\nu}}l_i(X,0).$$

The second part in the derivative of (5.35) vanishes for $\varepsilon \to 0$, since $l \in \mathcal{M}^k_{\rho,\sigma}$ and (5.36) imply that $\partial^\nu_X[\partial_\varepsilon l(\Psi(X,t,\varepsilon),\varepsilon\chi(t))]$ is bounded. Thus we have shown the continuity of ∇H_{aux} and its derivatives. Also the bound

$$\|\nabla H_{\text{aux}}\|_{k,\hat{r},\hat{\sigma}} \leq 2^{2k} B_k \|\dot{\chi}\|_\infty \|l\|^*_{k,\rho,\sigma}$$

follows from (5.35) by applying Corollary 5.4 and (5.36). In summary, we have proven that $H_{\text{aux}} \in \mathcal{H}^{k+1}_{\hat{r},\hat{\sigma}}$.
According to Remark 5.19 the function $X(t) = \Phi(x,t,\varepsilon)$ is a solution to the differential equation

$$\dot{X} = \varepsilon J \nabla H_{\text{aux}}(X,t,\varepsilon),$$

for the initial value $X(0) = x \in G_r$. Since $\hat{\sigma}\|l\|_{k,\rho,\sigma} \leq \hat{r} - r$, we have $\Phi(G_r,t,\varepsilon) \subset G_{\hat{r}}$ for $t \in [0,1]$, $\varepsilon \in [0,\hat{\sigma}]$ and thus also $X(t) \in G_{\hat{r}}$ for $t \in [0,1]$. Hence we have shown, that $x \mapsto \Phi(x,1,\varepsilon) = x + \varepsilon l(x,\varepsilon) = \mathcal{P}_\varepsilon(x)$ coincides with the time-1 map for this Hamiltonian system on G_r.
It remains to prove (5.30), i.e. the existence of a constant $C_{\text{aux}} > 0$ so that

$$|H_{\text{aux}}(X,t,\varepsilon) - H_{\text{aux}}(X,t,0)| \leq C_{\text{aux}}\varepsilon$$

holds uniformly in $X = (Q,P) \in G_{\hat{r}}$, $t \in [0,1]$ and $\varepsilon \in [0,\hat{\sigma}]$. Again, for $t = 0$ or $\varepsilon = 0$ both terms are zero. Thus suppose $t, \varepsilon > 0$. Using (5.34) and the explicit formula for h_{aux} we obtain

$$\begin{aligned}
&|H_{\text{aux}}(X,t,\varepsilon) - H_{\text{aux}}(X,t,0)| \\
&= |\,\dot{\chi}(t)| \Big| [l_1(x,\varepsilon\chi(t)) + \varepsilon\chi(t)\partial_\varepsilon l_1(x,\varepsilon\chi(t))]P - \partial_\varepsilon\zeta(x,\varepsilon\chi(t)) - [l_1(X,0)P - \mathfrak{m}(X)] \Big| \\
&\leq \|\,\dot{\chi}\|_\infty \Big(T_1(X,t,\varepsilon)|P| + T_2(X,t,\varepsilon) + \varepsilon\chi(t)\,|\partial_\varepsilon l_1(x,\varepsilon\chi(t))|\,|P| \Big),
\end{aligned}$$

where again $x = \Psi(X,t,\varepsilon)$ and T_1, T_2 are given by

$$T_1(X,t,\varepsilon) = |l_1(x,\varepsilon\chi(t)) - l_1(X,0)|, \quad T_2(X,t,\varepsilon) = |\mathfrak{m}(X) - \partial_\varepsilon\zeta(x,\varepsilon\chi(t))|\,.$$

Now, we study the latter terms more thoroughly. First, observe that due to the definition of Φ we have

$$|x - X| = \varepsilon\chi(t)\|l(x,\varepsilon\chi(t))\| \leq \varepsilon\|l\|_{k,\rho,\sigma}.$$

Moreover, $X \in G_{\hat{r}}$ implies $x \in G_\rho$ and hence $\lambda x + (1-\lambda)X \in G_\rho$ for $\lambda \in [0,1]$ by convexity. Thus

$$\begin{aligned} |l_1(x, \varepsilon\chi(t)) - l_1(X, \varepsilon\chi(t))| &= \left| \int_0^1 \nabla l_1(\lambda x + (1-\lambda)X, \varepsilon\chi(t))(x - X)\, d\lambda \right| \\ &\leq \varepsilon \|l\|^2_{k,\rho,\sigma}. \end{aligned}$$

Therefore, we can find a bound for T_1. Namely, the mean value theorem yields

$$\begin{aligned} T_1(X,t,\varepsilon) &= |l_1(x, \varepsilon\chi(t)) - l_1(X, 0)| \\ &\leq |l_1(x, \varepsilon\chi(t)) - l_1(X, \varepsilon\chi(t))| + |l_1(X, \varepsilon\chi(t)) - l_1(X,0)| \\ &\leq \varepsilon\|l\|^2_{k,\rho,\sigma} + \varepsilon\|l\|^*_{k,\rho,\sigma}. \end{aligned}$$

The bound on T_2 can be obtained in a similar way. We have

$$\begin{aligned} T_2(X,t,\varepsilon) &\leq |\mathfrak{m}(X) - \partial_\varepsilon\zeta(X, \varepsilon\chi(t))| + |\partial_\varepsilon\zeta(X,\varepsilon\chi(t)) - \partial_\varepsilon\zeta(x, \varepsilon\chi(t))| \\ &\leq C_{\mathfrak{m}}\varepsilon + \varepsilon\|\zeta\|^*_{k,\rho,\sigma}\|l\|_{k,\rho,\sigma}, \end{aligned}$$

where $C_{\mathfrak{m}} > 0$ is the constant from (5.23). Therefore, in total we get

$$\begin{aligned} &|H_{\text{aux}}(X,t,\varepsilon) - H_{\text{aux}}(X,t,0)| \\ &\quad \leq \varepsilon\|\dot{\chi}\|_\infty \big[\big(\|l\|^2_{k,\rho,\sigma} + \|l\|^*_{k,\rho,\sigma}\big) C_I + C_{\mathfrak{m}} + \|\zeta\|^*_{k,\rho,\sigma}\|l\|_{k,\rho,\sigma} + \|l\|^*_{k,\rho,\sigma} C_I \big]. \end{aligned}$$

Thus we have verified (5.30). □

Before stating the main result of this section, we make the following simple observation, which will be used several times in the proof. Given $H \in \mathcal{H}^k_{\rho,\sigma}$ with $k \in \mathbb{N}$ and $0 < r < \rho$, we call $\Phi(x,t,\varepsilon)$ the *solution map* to the corresponding Hamiltonian system $\dot{X} = \varepsilon J \nabla H(X,t,\varepsilon)$ if $X(t) = \Phi(x,t,\varepsilon)$ is the solution satisfying $X(0) = x \in G_\rho$. Clearly, $\Phi(x,t,\varepsilon)$ can be written in the form

$$\Phi(x,t,\varepsilon) = x + \varepsilon \int_0^t J\nabla H(\Phi(x,s,\varepsilon), s, \varepsilon)\, ds.$$

Since ∇H is bounded, one can find $\sigma_* \in (0, \sigma]$ such that

$$\Phi(G_r, t, \varepsilon) \subset G_\rho$$

for all $t \in [0,1]$ and $\varepsilon \in [0, \sigma_*]$.

Theorem 5.21. *Let $G = \mathbb{R} \times I$ for an open bounded interval $I \subset \mathbb{R}$, $k \in \mathbb{N}$, $\sigma > 0$ and $\rho > 0$. Suppose $l \in \mathcal{M}^{k+1}_{\rho,\sigma}$ and for $\varepsilon \in [0, \sigma]$ consider the family of maps $\mathcal{P}_\varepsilon : G_\rho \to \mathbb{R}^2$ given by*

$$\mathcal{P}_\varepsilon : x_1 = x + \varepsilon l(x, \varepsilon).$$

Let $\{\mathcal{P}_\varepsilon\}$ be an E-symplectic family of class C^{k+1}. Then there exist $\hat{\sigma} \in (0, \sigma)$, $\hat{C} > 0$ (depending only upon k, ρ, σ, $\|l\|^*_{k+1,\rho,\sigma}$, $\|\zeta\|^*_{k+1,\rho,\sigma}$, $C_{\mathfrak{m}}$ from (5.23) and $C_I = \max\{\rho + |p| : p \in I\}$) and a function $E(x)$ satisfying $J\nabla E(x) = l(x, 0)$ with the following property. If $\varepsilon \in [0, \hat{\sigma}]$ and

$$(x_n)_{0 \le n \le N} = (\mathcal{P}^n_\varepsilon(x_0))_{0 \le n \le N}$$

denotes a forward orbit piece of $\mathcal{P}_\varepsilon$ such that $x_n \in G$ for $0 \le n \le N$, then

$$|E(x_n) - E(x_0)| \le \hat{C}\varepsilon, \quad 0 \le n \le \min\{N, \lfloor \varepsilon^{-k} \rfloor\}.$$

Proof. Let $0 < r_3 < r_2 < r_1 < r_0 = \rho$. First, applying Lemma 5.20 with $r = r_2$ and $\hat{r} = r_1$ yields $\sigma_1 \in (0, \sigma]$ and $H_{\text{aux}} \in \mathcal{H}^{k+2}_{r_1,\sigma_1}$ such that if $\varepsilon \in [0, \sigma_1]$, then $\mathcal{P}_\varepsilon$ restricted to G_{r_2} coincides with the time-1 map for the 1-periodic Hamiltonian system

$$\dot{x} = \varepsilon J\nabla H_{\text{aux}}(x, t, \varepsilon). \tag{5.37}$$

Moreover, there is a constant $C_{\text{aux}} > 0$ such that $\|H_{\text{aux}}\|_{k+2,r_1,\sigma_1} \le C_{\text{aux}}$ and

$$|H_{\text{aux}}(x, t, \varepsilon) - H_{\text{aux}}(x, t, 0)| \le C_{\text{aux}}\varepsilon \tag{5.38}$$

holds for all $x \in G_{r_1}$, $t \in \mathbb{R}$ and $\varepsilon \in [0, \sigma_1]$.

Now, since $0 < r_3 < r_2 < r_1$ we can apply Theorem 5.13 to the Hamiltonian $H_{\text{aux}} \in \mathcal{H}^{k+2}_{r_1,\sigma_1}$. This gives us $\sigma_2 \in (0, \sigma_1]$ and $C_* > 0$ such that the following holds. For $t \in \mathbb{R}$ and $\varepsilon \in [0, \sigma_2]$ there exists an admissible change of variables $\Gamma : G_{r_2} \times \mathbb{R} \times [0, \sigma_2] \to G_{r_1}$, $x = \Gamma(y, t, \varepsilon)$, of class C^2 such that

$$\Gamma(G_{r_2}, t, \varepsilon) \supset G_{r_3}$$

for all $t \in \mathbb{R}$, $\varepsilon \in [0, \sigma_2]$, and so that (5.37) is transformed into

$$\dot{y} = \varepsilon(J\nabla\mathcal{N}(y, \varepsilon) + J\nabla\mathcal{R}(y, t, \varepsilon)), \tag{5.39}$$

where $\mathcal{N} \in \mathcal{H}^2_{r_2,\sigma_2}$, $\mathcal{R} \in \tilde{\mathcal{H}}^2_{r_2,\sigma_2}$. Moreover, for every $t \in \mathbb{R}$, $\varepsilon \in [0, \sigma_2]$ we have

$$\|\mathcal{R}(\cdot, t, \varepsilon)\|_{C^2(G_{r_2})} \leq C_* \varepsilon^k \tag{5.40}$$

$$\|\mathcal{N}\|_{2,r_2,\sigma_2} \leq 2\|H_{\text{aux}}\|_{k+2,r_1,\sigma_1},$$

$$\|\mathcal{N}(\cdot, \varepsilon) - \bar{H}_{\text{aux}}(\cdot, \varepsilon)\|_{C^2(G_{r_2})} \leq C_* \varepsilon. \tag{5.41}$$

Denote by $\Phi(x, t, \varepsilon)$ and $\phi(y, t, \varepsilon)$ the solution maps to (5.37) and (5.39), respectively. Furthermore, let $\sigma_3 \in (0, \sigma_2]$ be such that $\Phi(x, t, \varepsilon) \in G_{r_3}$ for $x \in G$, $t \in [0, 1]$ and $\varepsilon \in [0, \sigma_3]$. Then, these maps satisfy

$$\phi(y, t, \varepsilon) = \Gamma^{-1}(\Phi(\Gamma(y, 0, \varepsilon), t, \varepsilon), t, \varepsilon) = \Gamma^{-1}(\Phi(y, t, \varepsilon), t, \varepsilon)$$

for $y \in G$, $t \in [0, 1]$ and $\varepsilon \in [0, \sigma_3]$. Since $\Gamma(\cdot, 1, \varepsilon) = \Gamma(\cdot, 0, \varepsilon) = \text{id}$, this also implies

$$\phi(y, 1, \varepsilon) = \Gamma^{-1}(\Phi(y, 1, \varepsilon), 1, \varepsilon) = \Phi(y, 1, \varepsilon) = \mathcal{P}_\varepsilon(y).$$

Thus, (5.39) is constructed in such a way that $\mathcal{P}_\varepsilon$ restricted to G coincides with the time-1-map of this new system. Finally, consider the autonomous system

$$\dot{y} = \varepsilon J \nabla \mathcal{N}(y, \varepsilon), \tag{5.42}$$

together with its solution map $\hat{\phi}(y, t, \varepsilon)$ and let $\hat{\sigma} \in (0, \sigma_3]$ be such that $\hat{\phi}(G \times [0, 1] \times [0, \hat{\sigma}]) \subset G_{r_3}$. Then, using (5.40) we conclude

$$\begin{aligned}
&|\phi(y, t, \varepsilon) - \hat{\phi}(y, t, \varepsilon)| \\
&= \varepsilon \left| \int_0^t J[\nabla \mathcal{N}(\phi(y, s, \varepsilon), \varepsilon) - \nabla \mathcal{N}(\hat{\phi}(y, s, \varepsilon), \varepsilon)]\, ds + \int_0^t \nabla \mathcal{R}(\phi(y, s, \varepsilon), s, \varepsilon)\, ds \right| \\
&\leq \varepsilon \|\mathcal{N}\|_{2,r_3,\hat{\sigma}} \int_0^t |\phi(y, t, \varepsilon) - \hat{\phi}(y, t, \varepsilon)|\, ds + C_* \varepsilon^{k+1}
\end{aligned}$$

Hence Gronwall's inequality yields

$$|\phi(y, t, \varepsilon) - \hat{\phi}(y, t, \varepsilon)| \leq C_* \varepsilon^{k+1} e^{\varepsilon \|\mathcal{N}\|_{2,r_3,\hat{\sigma}} t} \leq C_* e^{2C_{\text{aux}} t} \varepsilon^{k+1}.$$

Let $C = C_* e^{2C_{\text{aux}}}$ and define $\hat{\mathcal{P}}_\varepsilon(x) = \hat{\phi}(x, 1, \varepsilon)$, then the latter estimate implies

$$|\mathcal{P}_\varepsilon(x) - \hat{\mathcal{P}}_\varepsilon(x)| \leq C \varepsilon^{k+1} \tag{5.43}$$

for $x \in G$ and $\varepsilon \in [0, \hat{\sigma}]$. Moreover, since (5.42) is autonomous we have

$$\mathcal{N}(\hat{\mathcal{P}}_\varepsilon(x), \varepsilon) = \mathcal{N}(x, \varepsilon).$$

By (5.41), we have $|\mathcal{N}(x,\varepsilon) - \bar{H}_{\text{aux}}(x,\varepsilon)| \leq C_* \varepsilon$ and (5.38) implies

$$|\bar{H}_{\text{aux}}(X,\varepsilon) - \bar{H}_{\text{aux}}(X,0)| \leq C_{\text{aux}}\varepsilon.$$

Together this yields

$$|\mathcal{N}(x,\varepsilon) - \bar{H}_{\text{aux}}(x,0)| \leq (C_* + C_{\text{aux}})\varepsilon.$$

Let $(x_n)_{0\leq n\leq N} = (\mathcal{P}_\varepsilon^n(x_0))_{0\leq n\leq N}$ be such that $x_n \in G$ for $0 \leq n \leq N$. If we define $E(x) = \mathcal{N}(x, 0)$, then the latter estimate and (5.29) show that

$$J\nabla E(x) = J\nabla\mathcal{N}(x,0) = J\nabla \bar{H}_{\text{aux}}(x,0) = l(x,0).$$

Furthermore, it follows

$$\begin{aligned}
&|E(x_n) - E(x_0)| \\
&\leq |\mathcal{N}(x_n,0) - \mathcal{N}(x_n,\varepsilon)| + |\mathcal{N}(x_n,\varepsilon) - \mathcal{N}(x_0,\varepsilon)| + |\mathcal{N}(x_0,\varepsilon) - \mathcal{N}(x_0,0)| \\
&\leq 2(C_* + C_{\text{aux}})\varepsilon + |\mathcal{N}(x_n,\varepsilon) - \mathcal{N}(x_0,\varepsilon)|.
\end{aligned}$$

Finally, (5.43) yields

$$\begin{aligned}
|\mathcal{N}(x_n,\varepsilon) - \mathcal{N}(x_0,\varepsilon)| &\leq \sum_{j=0}^{n-1} |\mathcal{N}(\mathcal{P}_\varepsilon(x_j),\varepsilon) - \mathcal{N}(x_j,\varepsilon)| \\
&= \sum_{j=0}^{n-1} |\mathcal{N}(\mathcal{P}_\varepsilon(x_j),\varepsilon) - \mathcal{N}(\hat{\mathcal{P}}_\varepsilon(x_j),\varepsilon)| \\
&\leq n\|\mathcal{N}\|_{2,r_3,\hat{\sigma}} C\varepsilon^{k+1}.
\end{aligned}$$

Therefore, one can choose $\hat{C} = 2(C_* + C_{\text{aux}}) + 2C_{\text{aux}}C$ to verify the assertion. □

5.4 Application to twist maps

In this section, we will prove our main result of the chapter. Preliminary to the proof, we briefly discuss some properties of difference equations with the form

$$x_{n+1} = g(x_n), \tag{5.44}$$

where $g : \mathcal{I} \mapsto \mathbb{R}$ is an increasing function defined on $\mathcal{I} = (d, \infty)$. More specifically, we discuss the notion of upper and lower solutions. A sequence $(\gamma_n)_{n\in\mathbb{N}_0} \subset \mathcal{I}$ is called a *lower solution* of (5.44), if $\gamma_{n+1} \leq g(\gamma_n)$ for $n \in \mathbb{N}_0$. An *upper solution* has to satisfy the reversed inequality.

Lemma 5.22. *Suppose $(\gamma_n)_{n\in\mathbb{N}_0}$ is a lower solution and $(\Gamma_n)_{n\in\mathbb{N}_0}$ is an upper solution of* (5.44) *such that $\gamma_0 \leq \Gamma_0$, then $\gamma_n \leq \Gamma_n$ holds for all $n \in \mathbb{N}_0$.*

By imposing some additional properties one gets the following result (cf. [KO21], Lemma 3.4 and Remark 3.5).

Lemma 5.23. *Let $h : \mathbb{N}_0 \to \mathbb{N}_0$ be a function such that $h(n) \geq n + 1$ for all $n \in \mathbb{N}_0$. Moreover, suppose*

(a) $(\gamma_n)_{n\in\mathbb{N}_0} \subset \mathcal{I}$ is a sequence such that

$$\gamma_m \leq g(\gamma_n), \qquad n \leq m \leq h(n),$$

(b) $(\Gamma_n)_{n\in\mathbb{N}_0} \subset \mathcal{I}$ is an upper solution of (5.44),

(c) $\gamma_0 \leq \Gamma_0$,

(d) (Γ_n) is increasing and $\limsup_{n\to\infty} \gamma_n = \infty$.

Then, there is a non-decreasing function $\sigma : \mathbb{N}_0 \to \mathbb{N}$ such that

$$\gamma_{\sigma(m)} > \Gamma_m \quad \text{and} \quad \gamma_n \leq \Gamma_m, \quad n \in \{0, \ldots, \sigma(m) - 1\}.$$

In addition,

$$\sigma(n + 1) > h(\sigma(n) - 1), \quad n \in \mathbb{N}_0.$$

Remark 5.24. Condition (a) implies that (γ_n) is a lower solution. The statement of Lemma (5.23) remains true under the following relaxation. Instead of (a), one can also impose *(a*)* $(\gamma_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}$ *is a sequence such that there is a number* $d^* > d$ *with* $\Gamma_0 \geq d^*$ *so that*

$$\gamma_{n+1} \geq d^* \quad \Longrightarrow \quad d \leq \gamma_m \leq g(\gamma_n), \quad n \leq m \leq h(n).$$

Now we are in position to prove the main result. For the sake of clarity, we restate it.

Theorem 5.25. *For* $k \in \mathbb{N}$*,* $r_* > 0$*,* $\alpha \in (0,1)$ *and* $\gamma \in \mathbb{R} \setminus \{0\}$*, consider a twist map* $f : M_{r_*} \to \mathbb{R} \times [0,\infty)$ *with* $(\theta, r) \mapsto (\theta_1, r_1)$ *given by*

$$\theta_1 = \theta + \frac{1}{r^\alpha}(\gamma + F_1(\theta, r)), \quad r_1 = r + r^{1-\alpha} F_2(\theta, r),$$

where $F_1, F_2 \in \mathcal{F}^{k+2}(\alpha)$*. Moreover, suppose there is* $\mathfrak{h} \in C^{k+2}(M_{r_*})$ *such that* $d\mathfrak{h} = r_1\, d\theta_1 - r\, d\theta$ *and*

$$\mathfrak{h}(\theta, r) = \mathfrak{h}_0(\theta, r) + \mathcal{R}(\theta, r),$$

where $\mathfrak{h}_0(\theta, r) = -\left(\frac{\alpha\gamma}{1-\alpha}\right) r^{1-\alpha}$ *and* $\mathcal{R} \in \mathcal{F}^{k+2}(2\alpha - 1)$*. Then, there is a constant* $C > 0$ *such that if* $(\theta_n, r_n)_{n\in\mathbb{N}_0}$ *denotes a complete forward orbit of* f*, then there is* $n_0 \in \mathbb{N}$ *so that*

$$r_n \leq C n^{1/(k+1)\alpha}, \qquad n \geq n_0.$$

Proof. The proof will be presented in three parts. First, a rescaling brings the map f into a form suitable for the application of Theorem 5.21. In the second step, the Theorem is applied and the resulting estimate is translated into some growth condition for the original map. Finally, the last part contains a further change of variables allowing for the usage of Lemma 5.23. Then, the sought growth estimate can be concluded in a rigorous way. The corresponding constant $C > 0$ from the assertion will be obtained from a sequence of constants $C_1, \ldots, C_9 > 0$, all depending only upon the parameters.

<u>*Step 1:*</u> We rescale the map by using the transformation $\xi = \varepsilon^{1/\alpha} r$. The twist map f then becomes

$$\psi_\varepsilon(\theta, \xi) = (\theta, \xi) + \varepsilon l(\theta, \xi, \varepsilon),$$

where $l = (l_1, l_2)$ is given by

$$l_1(\theta,\xi,\varepsilon) = \frac{1}{\xi^\alpha}\left(\gamma + F_1\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right)\right), \quad l_2(\theta,\xi,\varepsilon) = \xi^{1-\alpha} F_2\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right)$$

for $\varepsilon \in (0, \sigma]$, and

$$l_1(\theta,\xi,0) = \frac{\gamma}{\xi^\alpha}, \quad l_2(\theta,\xi,0) = 0.$$

Define $\sigma = (2r_*)^{-\alpha}$ and let $\varepsilon \in (0, \sigma]$. Then, the map ψ_ε is well-defined for $\xi > \varepsilon^{1/\alpha} r_* \geq \frac{1}{2}$. In particular, ψ_ε is defined on $G = \mathbb{R} \times I$, where $I = (1, 2)$. In order to apply Theorem 5.21, we will show that $l \in \mathcal{M}^{k+1}_{\rho,\sigma}$ for $\rho > 0$ small enough and that $\{\psi_\varepsilon\}$ is an E-symplectic family of class C^{k+1}. We have seen that a function $g \in C^m(M_{r_*})$ lies in $\mathcal{F}^m(\lambda)$ iff

$$\partial^{\boldsymbol{\nu}} g(\theta, r) = \mathcal{O}(r^{-\lambda-\nu_2}), \quad |\boldsymbol{\nu}| \leq m,$$

holds uniformly in $\theta \in \mathbb{R}$. After applying the rescaling it follows

$$\frac{\partial^{|\boldsymbol{\nu}|}}{\partial\theta^{\nu_1}\partial\xi^{\nu_2}}\left[g\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right)\right] = \frac{1}{\varepsilon^{\nu_2/\alpha}} \frac{\partial^{|\boldsymbol{\nu}|} g}{\partial\theta^{\nu_1}\partial r^{\nu_2}}\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right) = \mathcal{O}(\varepsilon^{\lambda/\alpha}), \quad (5.45)$$

uniformly in $x = (\theta, \xi) \in G_\rho$. Clearly, $l \in C^2(G_\rho \times (0, \sigma], \mathbb{R}^2)$ and $l(\cdot, \varepsilon) \in C^{k+1}(G_\rho, \mathbb{R}^2)$ for $\varepsilon \in [0, \sigma]$ are satisfied. Considering (5.45), the fact that $F_1, F_2 \in \mathcal{F}^{k+2}(\alpha)$ implies

$$\partial_x^{\boldsymbol{\nu}}\left[F_i\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right)\right] = \mathcal{O}(\varepsilon), \quad |\boldsymbol{\nu}| \leq k+1,$$

for $i = 1, 2$ and uniformly in $(\theta, \xi) \in G_\rho$. Since also $(\theta, \xi) \mapsto \xi^s$ is in $C_b^\infty(G_\rho)$ for every $s \in \mathbb{R}$, it follows $\partial_x^{\boldsymbol{\nu}} l \in C(G_\rho \times [0, \sigma], \mathbb{R}^2)$ for $|\boldsymbol{\nu}| \leq k+1$ and

$$\sup_{\varepsilon \in [0,\sigma]} \|l(\cdot, \varepsilon)\|_{C^{k+1}(G_\rho)} < \infty.$$

Furthermore, for $\varepsilon \in (0, \sigma]$ it is

$$\partial_\varepsilon l_1(\theta,\xi,\varepsilon) = -\frac{1}{\alpha}\frac{\xi^{1-\alpha}}{\varepsilon^{1+1/\alpha}} \partial_r F_1\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right),$$

$$\partial_\varepsilon l_2(\theta,\xi,\varepsilon) = -\frac{1}{\alpha}\frac{\xi^{2-\alpha}}{\varepsilon^{1+1/\alpha}} \partial_r F_2\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right).$$

Thus, clearly $\partial_x^{\boldsymbol{\nu}}\partial_\varepsilon l \in C(G_\rho \times (0,\sigma], \mathbb{R}^2)$ for $|\boldsymbol{\nu}| \le k+1$. Since we have $\partial_r F_i \in \mathcal{F}^{k+1}(\alpha+1)$, equation (5.45) implies

$$\partial_x^{\boldsymbol{\nu}} \left[\partial_r F_i \left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right)\right] = \mathcal{O}(\varepsilon^{1+1/\alpha}),$$

which in turn yields $\sup_{\varepsilon\in(0,\sigma]} \|\partial_\varepsilon l(\cdot,\varepsilon)\|_{C^{k+1}(G_\rho)} < \infty$. So far we have shown that $l \in \mathcal{M}_{\rho,\sigma}^{k+1}$. Next, we prove that the family $\{\psi_\varepsilon\}$ is E-symplectic. Define $\zeta(\theta,\xi,0) = 0$ and $\zeta(\theta,\xi,\varepsilon) = \varepsilon^{1/\alpha}\mathfrak{h}\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right)$ for $\varepsilon > 0$. Then, ζ is a potential for ψ_ε, that is $\xi_1\, d\theta_1 - \xi\, d\theta = d\zeta(\cdot,\varepsilon)$. Moreover, we have

$$\zeta(\theta,\xi,\varepsilon) = \varepsilon\mathfrak{h}_0(\theta,\xi) + \varepsilon^{1/\alpha}\mathcal{R}\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right),$$

where $\mathcal{R} \in \mathcal{F}^{k+2}(2\alpha-1)$. Thus by (5.45), it follows

$$\partial_x^{\boldsymbol{\nu}}\zeta(\theta,\xi,\varepsilon) = \varepsilon\partial^{\boldsymbol{\nu}}\mathfrak{h}_0(\theta,\xi) + \mathcal{O}(\varepsilon^2)$$

for $|\boldsymbol{\nu}| \le k+1$ uniformly in $x = (\theta,\xi) \in G_\rho$. This equation shows that $\partial_x^{\boldsymbol{\nu}}\zeta \in C(G_\rho \times [0,\sigma])$. Similarly

$$\partial_\varepsilon\zeta(\theta,\xi,\varepsilon) = \mathfrak{h}_0(\theta,\xi) + \frac{1}{\alpha}\varepsilon^{\frac{1-\alpha}{\alpha}}\mathcal{R}\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right) - \frac{1}{\alpha}\frac{\xi}{\varepsilon}\partial_r\mathcal{R}\left(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\right)$$

and so $\partial_x^{\boldsymbol{\nu}}\partial_\varepsilon\zeta(\theta,\xi,\varepsilon) = \partial^{\boldsymbol{\nu}}\mathfrak{h}_0(\theta,\xi) + \mathcal{O}(\varepsilon)$ by (5.45). Together, these estimates verify $\|\zeta\|_{k+1,\rho,\sigma}^* < \infty$ and show that the conditions (5.22) and (5.23) are satisfied.

Step 2: Application of Theorem 5.21. Since

$$l(x,0) = \begin{pmatrix} \gamma\xi^{-\alpha} \\ 0 \end{pmatrix},$$

the function $E(\theta,\xi) = E(\xi) = \frac{\gamma}{1-\alpha}\xi^{1-\alpha}$ satisfies $J\nabla E(x) = l(x,0)$. If

$$(\theta_n,\xi_n) = \psi_\varepsilon^n(\theta_0,\xi_0), \qquad 0 \le n \le N,$$

is such that $\xi_n \in I$ for $0 \le n \le N$, then according to Theorem 5.21 there are $\hat\sigma \in (0,\sigma]$ and $\hat C > 0$ (depending only upon k, ρ, σ, $\|l\|_{k+1,\rho,\sigma}^*$, $\|\zeta\|_{k+1,\rho,\sigma}^*$, $C_I = \max\{\rho + |p| : p \in I\}$ and $C_{\mathfrak{m}}$ from (5.23)) such that

$$|E(\xi_n) - E(\xi_0)| \le \hat C\varepsilon, \qquad 0 \le n \le \min\{N, \lfloor\varepsilon^{-k}\rfloor\}. \tag{5.46}$$

Fix two numbers $1 < a < 3/2 < b < 2$. By decreasing $\hat{\sigma}$ if necessary, we can assume that

$$\hat{\sigma} \leq \frac{\gamma}{(1-\alpha)\hat{C}} \min\left\{ \left(\frac{3}{2}\right)^{1-\alpha} - a^{1-\alpha}, b^{1-\alpha} - \left(\frac{3}{2}\right)^{1-\alpha} \right\} \tag{5.47}$$

and

$$\hat{\sigma} \leq \min\left\{ \frac{a-1}{\|l\|_{k+1,\rho,\sigma}}, \frac{2-b}{\|l\|_{k+1,\rho,\sigma}} \right\} \tag{5.48}$$

Now let $(\theta_n, r_n)_{n\in\mathbb{N}_0}$ be a complete forward orbit. Assume there is $m \in \mathbb{N}_0$ such that $(2r_m/3)^{-\alpha} < \hat{\sigma}$. If we apply the rescaling introduced in the first step with $\varepsilon = (2r_m/3)^{-\alpha}$, then by construction we have $\xi_m = 3/2$. Thus

$$N_{\max} = \max\{M \geq m : 1 < \xi_n < 2 \text{ for } m \leq n \leq M\}$$

is well-defined. From (5.46) we know that

$$\left| \xi_n^{1-\alpha} - \left(\frac{3}{2}\right)^{1-\alpha} \right| \leq \frac{1-\alpha}{\gamma}\hat{C}\varepsilon, \qquad \text{for } m \leq n \leq \min\{N_{\max}, m + \lfloor \varepsilon^{-k} \rfloor\}.$$

For these n, (5.47) implies

$$a^{1-\alpha} < \left(\frac{3}{2}\right)^{1-\alpha} - \frac{1-\alpha}{\gamma}\hat{C}\varepsilon < \xi_n^{1-\alpha} \leq \left(\frac{3}{2}\right)^{1-\alpha} + \frac{1-\alpha}{\gamma}\hat{C}\varepsilon < b^{1-\alpha}.$$

In fact one can deduce that $N_{\max} \geq m + \lfloor \varepsilon^{-k} \rfloor$, since $\xi_n \in (a, b)$, (5.48) and $\xi_{n+1} = \xi_n + \varepsilon l_2(\theta_n, \xi_n, \varepsilon)$ yield $\xi_{n+1} \in (1, 2)$. In the original coordinates, this means

$$|r_n^{1-\alpha} - r_m^{1-\alpha}| \leq C_1 r_m^{1-2\alpha} \qquad \text{for } m \leq n \leq m + \lfloor (2r_m/3)^{k\alpha} \rfloor, \tag{5.49}$$

where $C_1 > 0$ depends on α, γ and $\hat{C}$.

<u>*Step 3:*</u> A difference equation. We write $s_n = \left(\frac{2r_n}{3}\right)^{1-\alpha}$ for all $n \in \mathbb{N}_0$. To complete the proof, one thus has to show that

$$\limsup_{n\to\infty} \frac{s_n}{n^{\frac{1-\alpha}{(k+1)\alpha}}} \leq C_* \tag{5.50}$$

for a suitable constant $C_* > 0$. This will be achieved by applying Lemma 5.23 and Remark 5.24 with $\gamma_n = s_n$. In terms of s_n, the estimate (5.49) yields that if $s_m > \hat{\sigma}^{\frac{\alpha-1}{\alpha}}$, then

$$|s_n - s_m| \leq C_2 s_m^\beta \qquad \text{for } m \leq n \leq m + \lfloor s_m^{k(1-\beta)} \rfloor,$$

where $\beta = \frac{1-2\alpha}{1-\alpha}$ and $C_2 = \left(\frac{2}{3}\right)^\alpha C_1$. The corresponding difference equation is thus induced by the function $g(x) = x + C_2 x^\beta$, which is increasing on $\mathcal{I} = (d, \infty)$, where

$$d = \begin{cases} 0, & \text{if } \beta \geq 0, \\ (C_2|\beta|)^{\frac{1}{1-\beta}}, & \text{if } \beta < 0. \end{cases}$$

Furthermore, let

$$h(n) = n + \max\{1, \lfloor s_n^{k(1-\beta)} \rfloor\}.$$

In order to find d^* such that condition (a^*) from the Remark 5.24 is satisfied, we will first establish an estimate of the type $s_n \geq C s_{n+1} - 1$ with a suitable constant $C > 0$. We have

$$r_{n+1} = r_n + r_n^{1-\alpha} F_2(\theta_n, r_n).$$

With $C_3 = \sup_{(\theta,r)\in M_{r_*}} |F_2(\theta, r)|$, this implies

$$\begin{aligned} s_{n+1} &= \left(\tfrac{2}{3}\right)^{1-\alpha} \left(r_n + r_n^{1-\alpha} F_2(\theta_n, r_n)\right)^{1-\alpha} \leq \left(\tfrac{2}{3}\right)^{1-\alpha} \left(r_n^{1-\alpha} + r_n^{(1-\alpha)^2} C_3^{1-\alpha}\right) \\ &= s_n + C_4 s_n^{1-\alpha} \leq (C_4 + 1)(s_n + 1), \end{aligned}$$

where $C_4 = \left(\frac{2}{3}\right)^{\alpha(1-\alpha)} C_3^{1-\alpha}$. In particular, it follows

$$s_n \geq (C_4 + 1)^{-1} s_{n+1} - 1. \tag{5.51}$$

Thus define

$$d_* = \max\{\hat{\sigma}^{\frac{\alpha-1}{\alpha}} + 1, 2d, (2C_2)^{\frac{1}{1-\beta}}\} \tag{5.52}$$

and let $d^* = (C_4+1)(d_*+1)$. Then $s_{m+1} \geq d^*$ implies that $s_m \geq d_* > \hat{\sigma}^{\frac{\alpha-1}{\alpha}}$ and hence

$$s_m - C_2 s_m^\beta \leq s_n \leq g(s_m), \qquad \text{for } m \leq n \leq m + h(m),$$

where we also used the fact that $s_m^{k(1-\beta)} \geq 1$. For the lower bound, (5.52) also yields

$$s_n \geq s_m(1 - C_2 s_m^{\beta-1}) \geq \frac{1}{2} s_m \geq d.$$

Finally, note that we can assume $\limsup_{n\to\infty} s_n = \infty$, since otherwise (5.50) is trivial. In summary, (s_n) has all the required properties and it only remains to find a suitable upper solution $(\Gamma_n)_{n\in\mathbb{N}_0}$. To this end, define $\Gamma_n = (A + Bn)^{\frac{1}{1-\beta}}$, where $B = 2C_2(1-\beta)$ and $A^{\frac{1}{1-\beta}} \geq \max\{s_0, d^*\}$ sufficiently big. The sequence is obviously increasing and $\Gamma_0 \geq \max\{s_0, d^*\}$. The mean value theorem yields

$$\Gamma_{n+1} - \Gamma_n = \frac{B}{1-\beta}(A + B\zeta_n)^{\frac{\beta}{1-\beta}}$$

for some $\zeta_n \in (n, n+1)$. In the case that $\beta \in [0, 1)$, it follows

$$\Gamma_{n+1} \geq \Gamma_n + \frac{B}{1-\beta}\Gamma_n^\beta = \Gamma_n + 2C_2\Gamma_n^\beta \geq g(\Gamma_n),$$

If $\beta < 0$ on the other hand, we have

$$\Gamma_{n+1} \geq \Gamma_n + \frac{B}{1-\beta}\Gamma_{n+1}^\beta.$$

Since

$$\frac{\Gamma_{n+1}^\beta}{\Gamma_n^\beta} = \left(1 + \frac{B}{A+Bn}\right)^{\frac{\beta}{1-\beta}} \geq \left(1 + \frac{B}{A}\right)^{\frac{\beta}{1-\beta}},$$

the latter estimate yields

$$\Gamma_{n+1} \geq \Gamma_n + 2C_2\left(1 + \frac{2C_2(1-\beta)}{A}\right)^{\frac{\beta}{1-\beta}}\Gamma_n^\beta \geq g(\Gamma_n),$$

provided that A is chosen sufficiently big (depending on C_2 and β). Hence, Lemma 5.23 yields a non-decreasing function $\sigma : \mathbb{N}_0 \to \mathbb{N}$ such that $s_{\sigma(m)} > \Gamma_m$, $s_n \leq \Gamma_m$ for $n \in \{0, \ldots, \sigma(m) - 1\}$ and moreover

$$\sigma(m+1) > \sigma(m) - 1 + \lfloor s_{\sigma(m)-1}^{k(1-\beta)} \rfloor, \qquad m \in \mathbb{N}_0.$$

The latter estimate, (5.51), $s_{\sigma(m)} > \Gamma_m$ and $\Gamma_m \geq d^* \geq 2(C_4+1)$ show that

$$\begin{aligned}\sigma(m+1)-\sigma(m) &> s_{\sigma(m)-1}^{k(1-\beta)} - 2 \\ &\geq ((C_4+1)^{-1}s_{\sigma(m)} - 1)^{k(1-\beta)} - 2 \\ &> ((C_4+1)^{-1}\Gamma_m - 1)^{k(1-\beta)} - 2 \\ &\geq (\tfrac{1}{2}(C_4+1)^{-1}\Gamma_m)^{k(1-\beta)} - 2 \\ &= C_5(A+Bm)^k - 2 \\ &\geq C_6 m^k - 2,\end{aligned}$$

with suitable constants $C_5, C_6 > 0$ (depending only upon C_2, C_4, k and β). Thus

$$\sigma(m) \geq \sigma(0) + \sum_{j=0}^{m-1}(\sigma(j+1)-\sigma(j)) \geq 1 + C_6\sum_{j=0}^{m-1} j^k - 2m$$

Since

$$\sum_{j=0}^{m-1} j^k \geq \int_0^{m-1} z^k\, dz = \frac{(m-1)^{k+1}}{k+1},$$

it follows

$$\sigma(m) \geq C_7(m-1)^{k+1}$$

for an appropriate constant $C_7 > 0$ and m sufficiently large. Now define

$$\psi(m) = \min\{n \in \mathbb{N}_0 : m < \sigma(n)\}.$$

If $\psi(m) \geq 1$, this yields $m \geq \sigma(\psi(m)-1)$. Thus for m large we have

$$m \geq \sigma(\psi(m)-1) \geq C_7(\psi(m)-2)^{k+1},$$

that is

$$\psi(m) \leq C_8 m^{\frac{1}{k+1}},$$

where $C_8 = C_7^{-\frac{1}{k+1}} + 2$. Moreover, the definition of ψ also implies that $m \leq \sigma(\psi(m)) - 1$ and hence $s_m \leq \Gamma_{\psi(m)}$. It follows

$$s_m \leq (A + B\psi(m))^{\frac{1}{1-\beta}} \leq (A + BC_8 m^{\frac{1}{k+1}})^{\frac{1}{1-\beta}} \leq C_9 m^{1/(k+1)(1-\beta)}$$

for a suitable constant $C_9 > 0$. In the original variables, this yields

$$r_m \leq C m^{1/(k+1)\alpha},$$

for a constant $C > 0$ and $m \geq m_0$ with m_0 sufficiently big. Note that m_0 is depending on the initial condition (θ_0, r_0), whereas C is independent of the specific orbit. □

5.5 Growth rates for the Fermi-Ulam ping-pong

Fix $k \in \mathbb{N}$ and let $p \in C_b^{k+1}(\mathbb{R})$ be a forcing function with $0 < a \leq p(t) \leq b$ for $t \in \mathbb{R}$. We consider the ping-pong map

$$(t_0, v_0) \mapsto (t_1, v_1),$$

introduced in Section 4.3. That is

$$t_1 = \tilde{t} + \frac{p(\tilde{t})}{v_1}, \qquad v_1 = v_0 - 2\dot{p}(\tilde{t}),$$

where $\tilde{t} = \tilde{t}(t_0, v_0)$ is defined implicitly through the equation

$$(\tilde{t} - t_0)v_0 = p(\tilde{t}).$$

To ensure that this map is well-defined, we will assume that

$$v_0 > v_* := 2\max\{\sup_{t\in\mathbb{R}} \dot{p}(t), 0\}.$$

The ping-pong map becomes symplectic if we change to time-energy coordinates with $E = \frac{1}{2}v^2$. It then has the form $\mathcal{P} : (t_0, E_0) \mapsto (t_1, E_1)$ given by

$$t_1 = \tilde{t} + \frac{p(\tilde{t})}{\sqrt{2E_1}}, \quad E_1 = (\sqrt{E_0} - \sqrt{2}\dot{p}(\tilde{t}))^2, \tag{5.53}$$

where $\tilde{t} = \tilde{t}(t_0, E_0)$ is defined by the relation

$$\tilde{t} = t_0 + \frac{p(\tilde{t})}{\sqrt{2E_0}}.$$

Note, that $\mathcal{P}$ should not be confused with the map $\mathcal{P}_\varepsilon$ from the previous sections. Clearly, we have $\mathcal{P} \in C^k(M_{E_*})$ for $E_* = \frac{1}{2}v_*^2$. By applying Theorem 5.25, one can derive the following upper bound on the growth in velocity.

Theorem 5.26. *Given $k \geq 3$, let $p \in C_b^{k+1}(\mathbb{R})$ be so that $0 < a \leq p(t) \leq b$ for $t \in \mathbb{R}$. There are constants $\tilde{C}, \tilde{E} > 0$ such that if $(t_n, E_n)_{n \in \mathbb{N}_0}$ denotes any complete forward orbit of the ping-pong map $\mathcal{P} \in C^k(M_{\tilde{E}})$ defined in* (5.53)*, then*

$$E_n \leq \tilde{C} n^{2/(k-1)}, \qquad n \geq n_0,$$

for some $n_0 \in \mathbb{N}$, or in terms of velocity $v_n \leq \sqrt{2\tilde{C}} n^{1/(k-1)}$.

The remainder of this section is divided into three parts. As a preparation for the proof, we will first determine some estimates for the ping-pong map and its derivatives. Afterwards, we introduce a change of variables that will be used to bring P into a form suitable for the application of Theorem 5.25. This enables us to state the proof of Theorem 5.26. Finally, the last part contains the construction of an admissible forcing function $p \in C_b^{k+1}(\mathbb{R})$, for arbitrary $k \in \mathbb{N}$, having a complete forward orbit $(t_n, v_n)_{n \in \mathbb{N}_0}$ such that

$$v_n \geq v_0 + C_* n^{1/(k+1)}, \qquad n \geq n_*,$$

where $C_* > 0$ and $n_* \in \mathbb{N}$ are appropriate constants. Note that this example fails to show the optimality of the upper bound on growth given above.

5.5.1 Expansions of the ping-pong map

Now, we establish some estimates for $\mathcal{P}$. Also note, that in this section the $\mathcal{O}$-notation is always understood to be uniform in $t_0 \in \mathbb{R}$, even when not explicitly stated. We start by showing that $(\tilde{t} - t_0) \in \mathcal{F}^k(1/2)$.

Lemma 5.27. *Let $f \in C_b^k(\mathbb{R})$. We have*

$$\partial^{\boldsymbol{\nu}}[\tilde{t} - t_0] = \mathcal{O}\big(E_0^{-1/2-\nu_2}\big)$$

and

$$\partial^{\boldsymbol{\nu}} f(\tilde{t}) = \begin{cases} \mathcal{O}\big(E_0^{-1/2-\nu_2}\big), & \nu_2 > 0, \\ \mathcal{O}(1), & \nu_2 = 0, \end{cases} \tag{5.54}$$

uniformly in t_0 and for all $\boldsymbol{\nu} \in \mathbb{N}_0^2$ with $|\boldsymbol{\nu}| \leq k$.

Proof. We use induction over $m = |\boldsymbol{\nu}|$. By definition

$$\tilde{t} - t_0 = \frac{p(\tilde{t})}{\sqrt{2E_0}}.$$

It follows easily that

$$\partial_{t_0}\tilde{t} = \left(1 - \frac{\dot{p}(\tilde{t})}{\sqrt{2E_0}}\right)^{-1} = \mathcal{O}(1).$$

Hence the assertion holds for $m = 0$ and $\boldsymbol{\nu} = (1,0)$. Now, assume that the hypothesis is true for $j = 0, \ldots, m \leq k-1$ and fix $\boldsymbol{\nu} \neq (1,0)$ with $|\boldsymbol{\nu}| = m+1$. Using $f(t) = p(t)$, we deduce

$$\partial^{\boldsymbol{\nu}}[\tilde{t} - t_0] = \sum_{\boldsymbol{\mu}\leq\boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\boldsymbol{\mu}} \partial^{\boldsymbol{\mu}} p(\tilde{t}) \partial^{\boldsymbol{\nu}-\boldsymbol{\mu}}\big(\tfrac{1}{\sqrt{2E_0}}\big) = \frac{\dot{p}(\tilde{t})}{\sqrt{2E_0}} \partial^{\boldsymbol{\nu}}\tilde{t} + \mathcal{O}\big(E_0^{-1/2-\nu_2}\big)$$

and thus $\partial^{\boldsymbol{\nu}}\tilde{t} = \mathcal{O}\big(E_0^{-1/2-\nu_2}\big)$ uniformly in t_0. Moreover, applying the Faà di Bruno formula (5.1) to $f(\tilde{t}(t_0, E_0))$ yields

$$\partial^{\boldsymbol{\nu}} f(\tilde{t}) = \dot{f}(\tilde{t})\partial^{\boldsymbol{\nu}}\tilde{t} + \sum_{r=2}^{m+1} f^{(r)}(\tilde{t}) \sum_{p(\boldsymbol{\nu},r)} \boldsymbol{\nu}! \prod_{j=1}^{m+1} \frac{(\partial^{\boldsymbol{\ell}_j}\tilde{t})^{r_j}}{(r_j!)(\boldsymbol{\ell}_j!)}.$$

In the case $\nu_2 = 0$, this clearly implies $\partial^{\boldsymbol{\nu}} f(\tilde{t}) = \mathcal{O}(1)$. This can not be improved, since $(0, \ldots, 0, r, \mathbf{0}, \ldots, \mathbf{0}, (1,0)) \in p(\boldsymbol{\nu}, r)$. If $\nu_2 > 0$ on the other hand, the inductive hypothesis yields

$$\sum_{p(\boldsymbol{\nu},r)} \boldsymbol{\nu}! \prod_{j=1}^{m+1} \frac{(\partial^{\boldsymbol{\ell}_j}\tilde{t})^{r_j}}{(r_j!)(\boldsymbol{\ell}_j!)} = \sum_{p(\boldsymbol{\nu},r)} \prod_{\substack{j=1\\ \boldsymbol{\ell}_j\neq(1,0)}}^{m+1} \mathcal{O}\big(E_0^{-1/2-\ell_{j,2}}\big)^{r_j} = \mathcal{O}\big(E_0^{-1/2-\nu_2}\big),$$

where $\boldsymbol{\ell}_j = (\ell_{j,1}, \ell_{j,2})$, because $\sum_{\boldsymbol{\ell}_j\neq(1,0)} r_j \geq 1$ and $\sum_{\boldsymbol{\ell}_j\neq(1,0)} r_j l_{j,2} = \nu_2$. Thus, we have shown that the assertion also holds for $|\boldsymbol{\nu}| = m+1$. □

As a consequence we get

Lemma 5.28. *For any $\boldsymbol{\nu} = (\nu_1, \nu_2) \in \mathbb{N}_0^2$ with $|\boldsymbol{\nu}| \leq k$ we have*

$$\partial^{\boldsymbol{\nu}}[t_1 - t_0] = \mathcal{O}\big(E_0^{-1/2-\nu_2}\big) \quad \textit{and} \quad \partial^{\boldsymbol{\nu}}[E_1 - E_0] = \mathcal{O}\big(E_0^{1/2-\nu_2}\big).$$

Furthermore,

$$\partial^{\boldsymbol{\nu}}\big[\sqrt{E_1} - \sqrt{E_0}\big] = \begin{cases} \mathcal{O}\big(E_0^{-\nu_2-1/2}\big), & \nu_2 > 0 \\ \mathcal{O}(1), & \nu_2 = 0. \end{cases}$$

Proof. First, recall that $E_1 = E_0 - 2\sqrt{2E_0}\dot{p}(\tilde{t}) + 2\dot{p}(\tilde{t})^2$. Thus, Lemma 5.27 implies

$$\begin{aligned}
\partial^{\boldsymbol{\nu}}[E_1 - E_0] &= 2\sum_{\boldsymbol{\mu}\leq\boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\boldsymbol{\mu}} \partial^{\boldsymbol{\mu}}\dot{p}(\tilde{t})\, \partial^{\boldsymbol{\nu}-\boldsymbol{\mu}}[\dot{p}(\tilde{t}) - \sqrt{2E_0}] \\
&= \sum_{\substack{\boldsymbol{\mu}\leq\boldsymbol{\nu} \\ \mu_2\neq 0}} \mathcal{O}\big(E_0^{-1/2-\mu_2}\big)\mathcal{O}\big(E_0^{1/2-(\nu_2-\mu_2)}\big) + \mathcal{O}\big(E_0^{1/2-\nu_2}\big) \\
&= \mathcal{O}\big(E_0^{1/2-\nu_2}\big)
\end{aligned}$$

for any $\boldsymbol{\nu} \in \mathbb{N}_0^2$ with $|\boldsymbol{\nu}| \leq k$. For $t_1 - t_0$ on the other hand we have

$$t_1 - t_0 = \left(\frac{1}{\sqrt{2E_0}} + \frac{1}{\sqrt{2E_1}}\right) p(\tilde{t}).$$

Here, the estimate for $\partial^{\boldsymbol{\nu}} E_1$ yields

$$\begin{aligned}
\partial^{\boldsymbol{\nu}}\left[\frac{1}{\sqrt{2E_1}}\right] &= \sum_{r=1}^{|\boldsymbol{\nu}|} (-1)^r \frac{(2r)!}{2^{2r+1/2} r! E_1^{r+1/2}} \sum_{p(\boldsymbol{\nu},r)} \boldsymbol{\nu}! \prod_{j=1}^{|\boldsymbol{\nu}|} \frac{(\partial^{\boldsymbol{\ell}_j} E_1)^{r_j}}{(r_j!)(\boldsymbol{\ell}_j!)} \\
&= \sum_{r=1}^{|\boldsymbol{\nu}|} \mathcal{O}\big(E_0^{-1/2-r}\big) \sum_{p(\boldsymbol{\nu},r)} \prod_{\substack{j=1 \\ \boldsymbol{\ell}_j\neq(0,1)}}^{|\boldsymbol{\nu}|} \mathcal{O}\big(E_0^{1/2-\ell_{j,2}}\big)^{r_j} \\
&= \sum_{r=1}^{|\boldsymbol{\nu}|} \mathcal{O}\big(E_0^{-1/2-r}\big) \sum_{p(\boldsymbol{\nu},r)} \mathcal{O}\big(E_0^{(r-r_J)/2-\nu_2+r_J}\big) \\
&= \mathcal{O}\big(E_0^{-1/2-\nu_2}\big)
\end{aligned}$$

where $r_J \in \mathbb{N}_0$ denotes the multiplicity of $\boldsymbol{\ell}_J = (0,1)$ in $p(\boldsymbol{\nu}, r)$ and thus $\sum_{\boldsymbol{\ell}_j \neq (0,1)} r_j(\frac{1}{2} - \ell_{j,2}) = \frac{r - r_J}{2} - \nu_2 + r_J \leq r - \nu_2$. So together with Lemma 5.27 we obtain

$$\partial^{\boldsymbol{\nu}}[t_1 - t_0] = \mathcal{O}\big(E_0^{-1/2-\nu_2}\big)$$

for all $\boldsymbol{\nu} \in \mathbb{N}_0^2$ with $|\boldsymbol{\nu}| \leq k$. Finally, Lemma 5.27 also leads to

$$\partial^{\boldsymbol{\nu}}\Big[\sqrt{E_1} - \sqrt{E_0}\Big] = -\sqrt{2}\partial^{\boldsymbol{\nu}}[\dot{p}(\tilde{t})] = \begin{cases} \mathcal{O}\big(E_0^{-1/2-\nu_2}\big), & \nu_2 > 0 \\ \mathcal{O}(1), & \nu_2 = 0. \end{cases}$$

□

Remark 5.29. There is an analogue to (5.54) for convex-combinations of $t_1, \tilde{t}$ and t_0. Let $t_\lambda = \lambda_1 t_1 + \lambda_2 \tilde{t} + \lambda_3 t_0$, where $\lambda_i \in [0,1]$ are such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ Given $f \in C_b^m(\mathbb{R})$ with $0 \leq m \leq k$, we have

$$D^{\boldsymbol{\nu}}\big[f(t_\lambda)\big] = \begin{cases} \mathcal{O}\big(E_0^{-\nu_2-1/2}\big), & \nu_2 > 0 \\ \mathcal{O}(1), & \nu_2 = 0, \end{cases}$$

for all $\boldsymbol{\nu} = (\nu_1, \nu_2) \in \mathbb{N}_0^2$ with $|\boldsymbol{\nu}| \leq m$. In particular, $f(t_\lambda) \in \mathcal{F}^m(0)$. Moreover, if $m \geq 1$ one can write

$$f(\tilde{t}) - f(t_0) = (\tilde{t} - t_0) \int_0^1 \dot{f}(\lambda\tilde{t} + (1-\lambda)t_0)\, d\lambda.$$

Thus, applying Lemma A.2 yields $(f(\tilde{t}) - f(t_0)) \in \mathcal{F}^{m-1}(1/2)$. In fact, if t_β denotes another convex-combination, then $(f(t_\lambda) - f(t_\beta)) \in \mathcal{F}^{m-1}(1/2)$ follows by the same argument.

5.5.2 Growth rates

Consider the change of variables $\Gamma : M_{E_*} \to \mathbb{R} \times (0, \infty)$ defined by

$$(t, E) \mapsto (\tau, W), \quad \tau(t) = \int_0^t \frac{ds}{p(s)^2}, \quad W(t, E) = p(t)^2 E.$$

Γ is a C^{k+1}-diffeomorphism between M_{E_*} and $\Gamma(M_{E_*})$. Moreover, note that $a \leq p(t) \leq b$. Therefore, defining $W_* = b^2 E_*$ and $E_{**} = \frac{W_*}{a^2}$ yields

$$M_{W_*} \subset \Gamma(M_{E_*}) \quad \text{and} \quad M_{E_{**}} \subset \Gamma^{-1}(M_{W_*}).$$

Lemma 5.30. *Given* $0 \le m \le k+1$ *and a function* $f \in C^m(M_{E_*})$*, denote by* $\hat{f} = (f \circ \Gamma^{-1}) \in C^m(\Gamma(M_{E_*}))$ *the same function in the new coordinates. Then,* $f \in \mathcal{F}^m(r)$ *if and only if* $\hat{f} \in \mathcal{F}^m(r)$*.*

Proof. We perform a proof by induction over m. Since $a \le p(t) \le b$, we have $f = \mathcal{O}(E_0^{-r})$ if and only if $\hat{f} = \mathcal{O}(W_0^{-r})$ uniformly in t_0 and τ_0, respectively. Now assume the hypothesis is true for some $m \in \{0, \dots, k\}$ and also let $\hat{f} \in \mathcal{F}^{m+1}(r)$. For $f = \hat{f} \circ \Gamma$ and $E_0 \ge E_{**}$ it follows

$$\partial_{t_0} f(t_0, E_0) = p(t_0)^{-2} \partial_{\tau_0} \hat{f}(\Gamma(t_0, E_0)) + 2p(t_0)\dot{p}(t_0) E_0 \partial_{W_0} \hat{f}(\Gamma(t_0, E_0))$$

and

$$\partial_{E_0} f(t_0, E_0) = p(t_0)^2 \partial_{W_0} \hat{f}(\Gamma(t_0, E_0)).$$

Moreover, $\partial_{\tau_0} \hat{f} \in \mathcal{F}^m(r)$, $\partial_{W_0} \hat{f} \in \mathcal{F}^m(r+1)$ and thus the inductive hypothesis implies that $\partial_{\tau_0} \hat{f} \circ \Gamma \in \mathcal{F}^m(r)$ and $\partial_{W_0} \hat{f} \circ \Gamma \in \mathcal{F}^m(r+1)$. Thus, $f \in \mathcal{F}^{m+1}(r)$ follows from Lemma A.2. The other direction can be shown analogously. □

Now we are ready to give the proof.

Proof of Theorem 5.26. Denote by $\hat{\mathcal{P}} = \Gamma \circ \mathcal{P} \circ \Gamma^{-1}$ the ping-pong map in the new coordinates. We will apply Theorem 5.25 to $\hat{\mathcal{P}} : M_{W_*} \to \mathbb{R} \times [0, \infty)$, $(\tau_0, W_0) \mapsto (\tau_1, W_1)$ with $\gamma = \sqrt{2}$ and $\alpha = \frac{1}{2}$. This map has the required form if we take $F_1 = F$ and $F_2 = G$, where

$$F(\tau_0, W_0) = \sqrt{W_0}(\tau_1 - \tau_0) - \sqrt{2}, \quad G(\tau_0, W_0) = \frac{W_1 - W_0}{\sqrt{W_0}}.$$

We must confirm that $F, G \in \mathcal{F}^k(1/2)$. Since $\hat{\mathcal{P}} \in C^k(M_{W_*})$, we clearly also have $F, G \in C^k(M_{W_*})$. Moreover,

$$\begin{aligned} \tau(t_1) - \tau(t_0) &= (t_1 - t_0) \int_0^1 \frac{1}{p(\lambda t_1 + (1-\lambda) t_0)^2} \, d\lambda \\ &= \left(\frac{1}{\sqrt{2E_0}} + \frac{1}{\sqrt{2E_1}} \right) p(\tilde{t}) \int_0^1 \frac{1}{p(\lambda t_1 + (1-\lambda) t_0)^2} \, d\lambda. \end{aligned}$$

Plugging this into F yields

$$\begin{aligned} F(\Gamma(t_0,E_0)) &= \frac{1}{\sqrt{2}}\left(1+\frac{\sqrt{E_0}}{\sqrt{E_1}}\right)\int_0^1 \frac{p(t_0)p(\tilde{t})}{p(\lambda t_1+(1-\lambda)t_0)^2}\,d\lambda - \sqrt{2} \\ &= \frac{1}{\sqrt{2}}\left(2+R_1\right)\left(1+R_2\right)-\sqrt{2} \\ &= \frac{1}{\sqrt{2}}R_1+\frac{2}{\sqrt{2}}R_2+\frac{1}{\sqrt{2}}R_1R_2, \end{aligned}$$

where the two auxiliary functions $R_1, R_2 \in C^k(\Gamma^{-1}(M_{W_*}))$ are defined by

$$R_1(t_0,E_0) = -\frac{\sqrt{E_1}-\sqrt{E_0}}{\sqrt{E_1}}$$

and

$$R_2(t_0,E_0) = \int_0^1 \frac{p(t_0)p(\tilde{t})}{p(\lambda t_1+(1-\lambda)t_0)^2}\,d\lambda - 1.$$

Lemma 5.28 yields $R_1 \in \mathcal{F}^k(1/2)$. Regarding R_2 we have

$$\frac{p(t_0)}{p(\lambda t_1+(1-\lambda)t_0)} = 1+\frac{p(t_0)-p(\lambda t_1+(1-\lambda)t_0)}{p(\lambda t_1+(1-\lambda)t_0)} = 1+R_3,$$

where $R_3 \in \mathcal{F}^k(1/2)$ by Remark 5.29. A similar argument shows that

$$\left(\frac{p(\tilde{t})}{p(\lambda t_1+(1-\lambda)t_0)}-1\right) \in \mathcal{F}^k(1/2),$$

and thus it follows $R_2 \in \mathcal{F}^k(1/2)$. So $F \in \mathcal{F}^k(1/2)$ is verified. For $G(\tau_0,W_0) = \frac{W_1-W_0}{\sqrt{W_0}}$ we will use a representation developed in [KO11, Lemma 3.11]. Denoting $\varphi(t)=p(t)^2$, we have

$$W_1-W_0 = \frac{1}{2}\varphi(\tilde{t})\int_0^1 (1-\lambda)[\ddot{\varphi}((1-\lambda)\tilde{t}+\lambda t_0)-\ddot{\varphi}((1-\lambda)\tilde{t}+\lambda t_1)]\,d\lambda. \quad (5.55)$$

Applying Remark 5.29 yields

$$\partial^{\boldsymbol{\nu}}[W_1-W_0] = \begin{cases} \mathcal{O}\big(E_0^{-1/2-\nu_2}\big), & \nu_2>0 \\ \mathcal{O}(1), & \nu_2=0, \end{cases}$$

for $\boldsymbol{\nu} = (\nu_1, \nu_2)$ with $0 \leq |\boldsymbol{\nu}| \leq k$. Hence, in total $G \in \mathcal{F}^k(1/2)$. It remains to prove that $\hat{\mathcal{P}}$ is E-symplectic with a suitable potential $\mathfrak{h} \in C^k(M_{W_*})$ satisfying

$$\mathfrak{h}(\tau_0, W_0) = -\sqrt{2W_0} + \mathcal{R}(\tau_0, W_0), \tag{5.56}$$

with a remainder $\mathcal{R} \in \mathcal{F}^k(0)$. According to [KO11, Lemma 3.8], the function $G(t_0, t_1)$ defined on a suitable domain and given by

$$G(t_0, t_1) = \frac{1}{2} p(\tilde{t})^2 \left(\frac{1}{\tilde{t} - t_0} + \frac{1}{t_1 - \tilde{t}} \right),$$

is a generating function for the ping-pong map $\mathcal{P}$ on M_{E_*}, if $E_* > 0$ is sufficiently big. Thus $g(\tau_0, \tau_1) = G(t(\tau_0), t(\tau_1))$ is a generating function for $\hat{\mathcal{P}}$, where $t(\tau)$ denotes inverse transformation of $\tau(t)$. That is $W_0 = \frac{\partial g}{\partial \tau_0}$ and $-W_1 = \frac{\partial g}{\partial \tau_1}$. Hence, the function

$$\mathfrak{h}(\tau_0, W_0) = -g(\tau_0, \tau_1(\tau_0, W_0)),$$

is indeed a potential for $\hat{P}$, since $\frac{\partial \mathfrak{h}}{\partial \tau_0} = -\frac{\partial g}{\partial \tau_0} - \frac{\partial g}{\partial \tau_1} \frac{\partial \tau_1}{\partial \tau_0} = W_1 \frac{\partial \tau_1}{\partial \tau_0} - W_0$ as well as $\frac{\partial \mathfrak{h}}{\partial W_0} = -\frac{\partial g}{\partial \tau_1} \frac{\partial \tau_1}{\partial W_0} = W_1 \frac{\partial \tau_1}{\partial W_0}$. Using the definitions of t_1 and $\tilde{t}$, together with the fact that $\sqrt{E_1} = \sqrt{E_0} - \sqrt{2}\dot{p}(\tilde{t})$, we obtain

$$\begin{aligned} G(t_0, t_1) &= \frac{1}{2} p(\tilde{t}) \left(\sqrt{2E_0} + \sqrt{2E_1} \right) \\ &= p(\tilde{t}) \sqrt{2E_0} - p(\tilde{t}) \dot{p}(\tilde{t}) \\ &= p(t_0) \sqrt{2E_0} + \tilde{R}(t_0, E_0), \end{aligned}$$

where

$$\tilde{R}(t_0, E_0) = (p(\tilde{t}) - p(t_0)) \sqrt{2E_0} - p(\tilde{t}) \dot{p}(\tilde{t}).$$

Since $(p(\tilde{t}) - p(t_0)) \in \mathcal{F}^k(1/2)$ by Remark 5.29, we have $\tilde{R} \in \mathcal{F}^k(0)$. Thus, $\mathcal{R} = -\tilde{R} \circ \Gamma^{-1}$ defines a function in $\mathcal{F}^k(0)$ satisfying (5.56). In summary, we have shown that all conditions of Theorem 5.25 are satisfied. This yields a constant $C > 0$ such that if $(\tau_n, W_n)_{n \in \mathbb{N}_0}$ denotes any complete forward orbit, then there is $n_0 \in \mathbb{N}$ so that

$$W_n \leq C n^{2/(k-1)}, \qquad n \geq n_0.$$

Choosing $\tilde{C} = a^{-2} C$ and $\tilde{E} = E_{**}$ proves the assertion.

5.5.3 An example with escaping orbits

Let $k \in \mathbb{N}$ with $k \geq 2$ and numbers $0 < a < b \leq M$ be given. We will construct $p \in C_b^k(\mathbb{R})$ with $a \leq p(t) \leq b$ for $t \in \mathbb{R}$ and $\|p\|_{C^k(\mathbb{R})} \leq M$ such that there is a corresponding orbit $(t_n, v_n)_{n\in\mathbb{N}_0}$ satisfying $v_n \geq v_0 + C_* n^{1/k}$ for $n \geq n_*$ with some $n_* \in \mathbb{N}$ and a constant $C_* > 0$. The example was introduced in [KO11] to prove the existence of unbounded orbits. Here, we will additionally determine a lower bound for the growth in v_n.

This forcing function $p(t)$, which oscillates between a and b, will be obtained by combining two basic motions: A "blocking motion", in which p moves monotonically from a to b (i.e. $\dot{p}(t) \geq 0$) so that $\dot{p}(t) = 0$ at all instances $t = \tilde{t}_n$ of collisions with the moving wall and an "accelerating motion", in which it moves monotonically from b to a such that $\dot{p}(t) < 0$ at all instances $t = \tilde{t}_n$. We start by constructing these isolated parts. Afterwards we combine them and show that the resulting function has the desired properties. To this end, consider a smooth function $w \in C^\infty([0,1])$ such that $w(0) = 0$, $w(1) = 1$, $\dot{w}(t) > 0$ for $t \in (0,1)$ and $w^{(j)}(0) = w^{(j)}(1) = 0$ for $j \in \mathbb{N}$. For example one could take

$$w(t) = \left(1 + \exp\left(\frac{1}{t} - \frac{1}{1-t}\right)\right)^{-1}.$$

The blocking motion $p_+(t)$

Define $\mu = \mu(v_0)$ by

$$\mu = \frac{Ma^k}{\|w\|_{C^k([0,1])} v_0^k}.$$

Now, fix $(t_0, v_0) \in \mathbb{R} \times (0, \infty)$ with $v_0 > a$ so large that $\mu < b - a$. Denote by $N_+ \geq 2$ the integer such that

$$(N_+ - 1)\mu < b - a \leq N_+ \mu.$$

Moreover, define the sequence $(\tau_n)_{n\in\mathbb{Z}}$ by setting $\tau_0 = t_0$ and

$$\tau_{n+1} - \tau_n = \begin{cases} \frac{2a}{v_0}, & n \leq -1 \\ 2\left(\frac{a+(n+1)\mu}{v_0}\right), & 0 \leq n \leq N_+ - 1 \\ \frac{2b}{v_0}, & n \geq N_+ \end{cases}.$$

Now, consider the function $p_+ : \mathbb{R} \to [a,b]$ given by

$$p_+(t) = \begin{cases} a, & t \le \tau_0 \\ a + n\mu + \mu w(\frac{v_0}{a}(t-\tau_n)), & n = 0,\dots,N_+ - 2;\ t \in [\tau_n, \tau_n + \frac{a}{v_0}] \\ a + (n+1)\mu, & n = 0,\dots,N_+ - 2;\ t \in [\tau_n + \frac{a}{v_0}, \tau_{n+1}] \\ a + (N_+ - 1)\mu + \hat{\mu} w(\frac{v_0}{a}(t - \tau_{N_+ - 1})), & t \in [\tau_{N_+-1}, \tau_{N_+-1} + \frac{a}{v_0}] \\ b, & t \ge \tau_{N_+-1} + \frac{a}{v_0} \end{cases},$$

where $\hat{\mu} \in (0, \mu]$ is given by $\hat{\mu} = b - a - (N_+ - 1)\mu$. Denote by $(t_n, v_n)_{n\in\mathbb{Z}}$ the complete orbit of the ping-pong map $\mathcal{P}$ using $p = p_+$ and the given initial condition (t_0, v_0). It is straight forward to check that

$$t_n = \tau_n \text{ and } v_n = v_0 \text{ for all } n \in \mathbb{Z}.$$

Moreover, we have $p_+ \in C^\infty(\mathbb{R})$ and $\|p_+\|_{C^k(\mathbb{R})} \le M$ by construction. Finally, note that the definition of N_+ implies the bound

$$C_1 v_0^k \le N_+ \le (C_1 + 1) v_0^k, \tag{5.57}$$

where

$$C_1 = \frac{(b-a)\|w\|_{C^k([0,1])}}{M a^k}.$$

The accelerating motion $p_-(t)$

Let

$$\sigma = \sup_{1\le m\le k} \left(\frac{\|w^{(m)}\|_\infty (b-a)}{M} \right)^{\frac{1}{m}}.$$

Given $(t_0, v_0) \in \mathbb{R} \times (0, \infty)$, consider the map $p_- : \mathbb{R} \to [a,b]$ defined by

$$p_-(t) = \begin{cases} b, & t \le t_0 \\ b - (b-a) w(\sigma^{-1}(t - t_0)), & t \in [t_0, t_0 + \sigma] \\ a, & t \ge t_0 + \sigma \end{cases}.$$

Then, again $p_- \in C^\infty(\mathbb{R})$ and $\|p_-\|_{C^k(\mathbb{R})} \le M$ by construction. Denote by $(t_n, v_n)_{n\in\mathbb{N}_0}$ the complete forward orbit of the ping-pong map $\mathcal{P}$ using

$p = p_-$ and the given initial condition. Note, that $\dot{p}_-(t) \leq 0$ for all $t \in \mathbb{R}$. Hence $v_{n+1} \geq v_n$ for all $n \in \mathbb{N}_0$. We have

$$\frac{2a}{v_{n+1}} \leq a\Big(\frac{1}{v_n} + \frac{1}{v_{n+1}}\Big) \leq t_{n+1} - t_n \leq b\Big(\frac{1}{v_n} + \frac{1}{v_{n+1}}\Big) \leq \frac{2b}{v_n}. \tag{5.58}$$

Let $N_- \in \mathbb{N}$ be the first integer such that $t_{N_-} \geq t_0 + \sigma$. We will determine constants $C_2, C_3 > 0$ depending only on the parameters such that

$$C_2 v_0 \leq N_- \leq C_3 v_0. \tag{5.59}$$

By (5.58), we get

$$\sigma \leq \sum_{n=0}^{N_- -1} (t_{n+1} - t_n) \leq N_- \frac{2b}{v_0}$$

and thus $N_- \geq C_2 v_0$ with $C_2 = \frac{\sigma}{2b}$. Since $v_{n+1} - v_n = 2|\dot{p}_-(\tilde{t}_n)|$, where $\tilde{t}_n \in (t_n, t_{n+1})$ denotes the time of collision with the moving wall, we have $v_n - v_0 = \sum_{j=0}^{n-1} 2|\dot{p}_-(\tilde{t}_j)| \leq 2Mn$. Using (5.58), it therefore follows

$$t_n - t_0 \geq \sum_{j=0}^{n-1} \frac{a}{v_j} \geq \sum_{j=0}^{n-1} \frac{a}{2Mj + v_0}.$$

Moreover, it is

$$\sum_{j=0}^{n-1} \frac{1}{j + \frac{v_0}{2M}} \geq \int_0^n \frac{1}{x + \frac{v_0}{2M}}\, dx = \log\Big(\frac{2Mn}{v_0} + 1\Big).$$

Hence $t_n - t_0 \geq \frac{a}{2M} \log\Big(\frac{2Mn}{v_0} + 1\Big)$, which verifies (5.59) with

$$C_3 = \frac{1}{2M} \exp\Big(\Big(\frac{2M\sigma}{a}\Big) - 1\Big).$$

Now, one can establish bounds on the growth $v_{N_-} - v_0$. Since we have $v_{N_-} \leq v_0 + 2MN_-$, estimate (5.59) yields

$$v_{N_-} \leq (1 + 2MC_3)v_0. \tag{5.60}$$

For the lower bound, note that $N_- \to \infty$ and $\sup_{n\in\mathbb{N}}(t_{n+1} - t_n) \to 0$ as $v_0 \to \infty$. Hence, we can choose v_0 so large that

$$\left| \int_{t_0}^{t_{N_-}} \dot{p}(t)\,dt - \sum_{n=0}^{N_- -1} \dot{p}(\tilde{t}_n)(t_{n+1} - t_n) \right| < \frac{b-a}{2}.$$

In particular, it follows

$$-\sum_{n=0}^{N_- -1} \dot{p}(\tilde{t}_n)(t_{n+1} - t_n) > -\int_{t_0}^{t_{N_-}} \dot{p}(t)\,dt - \frac{b-a}{2} = \frac{b-a}{2}.$$

But then this yields

$$v_{N_-} - v_0 = -2\sum_{n=0}^{N_- -1} \dot{p}(\tilde{t}_n) \geq -\frac{v_0}{b} \sum_{n=0}^{N_- -1} \dot{p}(\tilde{t}_n)(t_{n+1} - t_n) \geq \frac{v_0(b-a)}{2b}. \quad (5.61)$$

The combined function $p(t)$

Now, we glue p_- and p_+ together as follows. Given $t_0 = 0$ and v_0 large enough, we start by setting $p(t) = p_-(t)$ for $t \in [0, t_{N_-}]$, where $N_- = N_-(v_0)$ is as described above. In particular, this means $p(t_0) = b$, $p(t_{N_-}) = a$ and $v_n < v_{n+1}$ for $0 \leq n \leq N_- - 1$. Next, using (t_{N_-}, v_{N_-}) as the initial condition for p_+ yields some $N_+ = N_+(v_{N_-})$ as defined above. Let $N = N_- + N_+$ and set $p(t) = p_+(t)$ for $t \in [t_{N_-}, t_N]$. Thus $p(t_N) = b$ and $v_n = v_{N_-}$ for $N_- \leq n \leq N$. We want to find a lower bound for $(v_N - v_0)/N^{1/k}$. The numerator can be estimated using (5.61). For the denominator, applying (5.59), (5.57) and then (5.60) yields

$$N = N_- + N_+(v_{N_-}) \leq C_3 v_0 + (C_1+1)v_{N_-}^k \leq \left(C_3 + (C_1+1)(1+2MC_3)^k\right) v_0^k.$$

In summary, one obtains

$$v_N - v_0 \geq C_4 N^{\frac{1}{k}},$$

with a constant $C_4 > 0$ depending only on the parameters. Until this point, the forcing $p(t)$ is only defined on the interval $[0, t_N]$. However, since $p(t_N) = b$ and $v_N > v_0$ the procedure above can be repeated infinitely many times to obtain a forcing $p \in C^\infty([0,\infty)) \cap C_b^k([0,\infty))$ oscillating between a and b. Moreover, this yields sequences of natural numbers $N_-^{(j)}, N_+^{(j)}$ and $N^{(j)}$ with the following properties. For all $j \in \mathbb{N}$ we have

$$\begin{aligned}
&N_-^{(1)} = N_-, \quad N_+^{(1)} = N_+, \quad N^{(0)} = 0,\\
&N^{(j-1)} + N_-^{(j)} + N_+^{(j)} = N^{(j)},\\
&p(t_{N^{(j)}}) = b,\\
&v_n < v_{n+1} \qquad \text{for } N^{(j-1)} \le n \le N^{(j-1)} + N_-^{(j)} - 1,\\
&v_n = v_{N^{(j-1)}+N_-^{(j)}} \quad \text{for } N^{(j-1)} + N_-^{(j)} \le n \le N^{(j)}.
\end{aligned}$$

Since the constants C_4 depends only upon the parameters, the estimate

$$v_{N^{(j)}} - v_{N^{(j-1)}} \ge C_4(N_+^{(j)} + N_-^{(j)})^{\frac{1}{k}}$$

stays valid for all $j \in \mathbb{N}$. In particular, through summation it follows

$$v_{N^{(j)}} \ge v_0 + C_4(N^{(j)})^{\frac{1}{k}}.$$

Now, consider $n \ge N^{(1)}$ arbitrary but fixed and let $j \in \mathbb{N}$ be such that $N^{(j)} \le n < N^{(j+1)}$. If $n \ge N^{(j)} + N_-^{(j+1)}$, then

$$v_n - v_0 = v_{N^{(j+1)}} - v_0 \ge C_4(N^{(j+1)})^{\frac{1}{k}} \ge C_4 n^{\frac{1}{k}}.$$

If $N^{(j)} \le n < N^{(j)} + N_-^{(j+1)}$ on the other hand, it follows

$$v_n - v_0 \ge v_{N^{(j)}} - v_0 \ge C_4(N^{(j)})^{\frac{1}{k}}.$$

Moreover, $n^{\frac{1}{k}} \le (N^{(j)})^{1/k} + (N_-^{(j+1)})^{1/k}$ and by (5.59), we have

$$N_-^{(j+1)} \le C_3 v_{N^{(j)}} \le C_3(v_0 + 2MN^{(j)}) \le C_3(v_0 + 2M)N^{(j)}.$$

In total, we obtain

$$v_n - v_0 \ge C_* n^{1/k}, \qquad n \ge n_*,$$

where $n_* = N^{(1)}$ and

$$C_* = \frac{C_4}{1 + (C_3(v_0 + 2M))^{1/k}}.$$

□

Chapter 6

Conclusion

The development of the theories of Kolmogorow-Arnold-Moser and Aubry-Mather, respectively, have been major breakthroughs in the theory of dynamical systems, which led to a wealth of publications in the last decades. In this work, we successfully demonstrated the utility of studying twist maps even in cases where the classical theory is not available.

First, we used Maharam‘s Recurrence Theorem to prove the recurrence of a class of periodic twist maps under low regularity assumptions. This was shown to imply the Poisson stability of almost every solution to the piecewise linear oscillator

$$\ddot{x} + n^2 x + \tilde{h}_L(x) = p(t), \qquad \tilde{h}_L(x) = \begin{cases} \text{sign}(x) & \text{if } |x| \geq \frac{1}{L}, \\ Lx & \text{if } |x| < \frac{1}{L}, \end{cases}$$

with $p \in C(\mathbb{S}^1)$ and its discontinuous limit case.

Using a similar approach, the improbability of escaping orbits was proven for some near-integrable systems having adiabatic invariants and almost periodic time dependence. This includes the Fermi-Ulam ping-pong with forcing functions $p \in C_b^2$ and the super-linear oscillator

$$\ddot{x} + |x|^{\alpha-1} x = p(t),$$

with $p \in C_b^4$ and $\alpha \geq 3$. For the latter, one has again Poisson stability.

Note, that by Theorem 4.9, the set of initial condition leading to escaping orbits has measure zero. However, the author knows of no admissible example exhibiting even unbounded motions. Hence, it may be an interesting and also promising task to either construct such a counter-example or to show that there are indeed no escaping orbits, thereby overcoming a possible shortcoming of the measure-theoretical approach.

Finally, we determined growth rates for a large family of twist maps without imposing any periodicity condition. This class again covered the ping-pong model and we were able to show that the velocity satisfies $v_n = \mathcal{O}(n^{1/k})$ if $p \in C_b^{k+2}$ with $k \geq 2$.

Since there is a gap between the maximal growth proven for twist maps of the form (1.3) and the actual growth rates realized in the ping-pong example, it remains an open question whether the upper limit is optimal.

Appendices

A The space $\mathcal{F}^k(r)$

Given $k \in \mathbb{N}_0$ and $r \in \mathbb{R}$, we define $\mathcal{F}^k(r)$ to be the space of functions $F(\tau, v)$ such that $F \in C^k(M_{v_*})$ for some $v_* > 0$ and

$$\sup_{(\tau,v)\in M_{v_*}} v^{r+\nu_2} |\partial^{\boldsymbol{\nu}} F(\tau, v)| < \infty$$

for every multi-index $\boldsymbol{\nu} = (\nu_1, \nu_2)$ with $|\boldsymbol{\nu}| \leq k$. Moreover, we define $\mathcal{F}_{\mathrm{u}}(r)$ to be the space of functions $F \in \mathcal{F}^0(r)$ such that $v^r F(\cdot, v)$ converges uniformly as $v \to \infty$. We also write $\mathcal{F}_{\mathrm{u}}^k(r) = \mathcal{F}^k(r) \cap \mathcal{F}_{\mathrm{u}}(r)$.

In this section, we state some properties that are true for $\mathcal{F}^k$ and $\mathcal{F}_{\mathrm{u}}^k$ with any $k \in \mathbb{N} \cup \{0\}$. The proofs regarding $\mathcal{F}^k$ can be found in Section 5 of [Ort99]. For this reason, we only include proofs for the spaces $\mathcal{F}_{\mathrm{u}}$.

Remark A.1. In contrast to [Ort99], functions in $\mathcal{F}^k(r)$ are in general not periodic. If periodicity is assumed, the listed operations retain this property.

First, we state some algebraic features.

Lemma A.2. *(i) If $r_1 < r_2$, then $\mathcal{F}_u^k(r_2) \subset \mathcal{F}_u^k(r_1)$.*

(ii) If $r_1 \leq r_2$, $F_1 \in \mathcal{F}_u^k(r_1)$ and $F_2 \in \mathcal{F}_u^k(r_2)$, then $F_1 + F_2 \in \mathcal{F}_u^k(r_1)$.

(iii) If $F_1 \in \mathcal{F}_u^k(r_1)$ and $F_2 \in \mathcal{F}_u^k(r_2)$, then $F_1 \cdot F_2 \in \mathcal{F}_u^k(r_1 + r_2)$.

The remaining lemmata deal with different composite functions.

Lemma A.3. *Consider a function* $\varphi \in C^\infty([-\delta, \delta])$ *for some* $\delta > 0$ *satisfying*

$$\varphi(0) = 0, \qquad \text{or} \qquad \varphi(0) = \varphi'(0) = 0.$$

If $r > 0$ *and* $F \in \mathcal{F}_u^k(r)$*, then* $\varphi \circ F \in \mathcal{F}_u^k(r)$ *or* $\varphi \circ F \in \mathcal{F}_u^k(2r)$*, respectively.*

Proof. By taylor $\varphi(x) = \varphi'(0)x + \varphi''(0)x^2 + R(x)$ with $R \in C^\infty(\mathbb{R})$ and $R = o(x^2)$. Hence $\lim_{v\to\infty} v^r \varphi(F(\tau, v)) = \varphi'(0) \lim_{v\to\infty} v^r F(\tau, v)$. If also $\varphi'(0) = 0$, then

$$v^{2r}\varphi(F(\tau, v)) = \varphi''(0)v^{2r}F(\tau, v)^2 + v^{2r}F(\tau, v)^2 \frac{R(F(\tau, v)^2)}{F(\tau, v)^2}$$

converges uniformly as $v \to \infty$. □

Lemma A.4. *If* $G \in \mathcal{F}_u^k(0)$ *and* $F \in \mathcal{F}_u^k(1)$*, then*

$$\frac{1}{v + G(\tau, v) + F(\tau, v)} = \frac{1}{v} - \frac{G(\tau, v)}{v^2} + \tilde{F}(\tau, v),$$

with $\tilde{F} \in \mathcal{F}_u^k(3)$*.*

Proof. Consider the smooth functions

$$\varphi(\xi) = \frac{1}{1+\xi} - 1, \quad \psi(\xi) = \frac{1}{1+\xi} - 1 + \xi.$$

Then, we have

$$\frac{1}{v + G + F} = \frac{1}{v} - \frac{G}{v^2} + \frac{1}{v}\psi\left(\frac{G}{v}\right) + \frac{1}{v+G}\varphi\left(\frac{F}{v+G}\right)$$

Since $\varphi(0) = 0$ and $\psi(0) = \psi'(0) = 0$, Lemma A.3 proves the assertion. □

Lemma A.5. *Let* $R, S \in \mathcal{F}_u^k(0)$ *and* $F \in \mathcal{F}_u^k(r)$ *be given. If* F *is uniformly continuous, the map* $\tilde{F}$ *defined by*

$$\tilde{F}(\tau, v) = F(\tau + R(\tau, v), v + S(\tau, v)),$$

satisfies $\tilde{F} \in \mathcal{F}_u^k(r)$*.*

Proof. Denote by $\alpha(\tau) = \lim_{v\to\infty} v^r F(\tau, v)$ and $\beta(\tau) = \lim_{v\to\infty} R(\tau, v)$ the uniform limits. We show that $\tilde{F}(\tau, v) \to \alpha(\tau + \beta(\tau))$ uniformly. To this end, let $\varepsilon > 0$. Since $S(\tau, v)$ is bounded, we have

$$\alpha(\tau) = \lim_{v\to\infty} (v + S)^r F(\tau, v + S) = \lim_{v\to\infty} v^r F(\tau, v + S),$$

uniformly and independently of the arguments of S. Here, we used the expansion $(v + S)^r = v^r + rSv^{r-1} + o(v^{r-1})$. In particular, one can find $\tilde{v} > 0$ such that for $(\tau, v) \in M_{\tilde{v}}$ the estimate

$$|v^r F(\tau, v + S(t, w)) - \alpha(\tau)| < \tfrac{\varepsilon}{2}$$

holds for all $(t, w) \in M_{\tilde{v}}$. Moreover, $\alpha(\tau)$ is uniformly continuous. Therefore, we can find $\delta > 0$ such that $|\tau - \tau'| < \delta$ yields $|\alpha(\tau) - \alpha(\tau')| < \frac{\varepsilon}{2}$. Now, choose $\hat{v} \geq \tilde{v}$ so that $|R(\tau, v) - \beta(\tau)| < \delta$ for $v \geq \hat{v}$. Then, it follows $|v^r \tilde{F}(\tau, v) - \alpha(\tau + \beta(\tau))| < \varepsilon$ for all $(\tau, v) \in M_{\hat{v}}$.

□

Lemma A.6. *Given $f \in C_b^{k+1}(\mathbb{R})$ such that f' is uniformly continuous and $F \in \mathcal{F}_u^k(r)$ with $r \geq 0$, define*

$$G(\tau, v) = f(\tau + F(\tau, v)) - f(\tau).$$

Then $G \in \mathcal{F}_u^k(r)$.

Proof. Write $\alpha(\tau) = \lim_{v\to\infty} v^r F(\tau, v)$ for the uniform limit. First, consider the case $r = 0$. Since f is uniformly continuous, the limit

$$\lim_{v\to\infty} G(\tau, v) = f(\tau + \alpha(\tau)) - f(\tau)$$

follows directly from the definition of G. If $r > 0$ on the other hand, the identity

$$G(\tau, v) = F(\tau, v) \int_0^1 f'(\tau + \lambda F(\tau, v))\, d\lambda$$

implies the uniform limit $\lim_{v\to\infty} v^r G(\tau, v) = \alpha(\tau) f'(\tau)$. □

B Expansions for a piecewise linear oscillator

Here, we will verify the expansion of the map P introduced in (3.13). We start by determining the expansions in the linear case. To this end, let $y(t)$ be the solution of

$$\ddot{y} + n^2 y = f(t), \quad x(\tau) = 0, \quad \dot{y}(\tau) = v,$$

where $f \in C(\mathbb{S}^1)$. Moreover, consider the functions

$$F(\tau, t) = \int_\tau^t f(s) \sin(n(t-s))\, ds, \qquad G(\tau, t) = \int_\tau^t f(s) \cos(n(t-s))\, ds.$$

We have $F, G \in C_b^1(\mathbb{R}^2)$ and

$$\begin{aligned} &\partial_\tau F(\tau, t) = -f(\tau) \sin(n(t-\tau)), && \partial_\tau G(\tau, t) = -f(\tau) \cos(n(t-\tau)), \\ &\partial_t F(\tau, t) = nG(\tau, t), && \partial_t G(\tau, t) = f(t) - nF(\tau, t). \end{aligned}$$

Then, $y(t)$ has the form

$$y(t) = \frac{v}{n} \sin(n(t-\tau)) + \frac{1}{n} F(t, \tau). \tag{B.1}$$

As shown in Section 3.3, the successor maps

$$S_\pm : \mathbb{R} \times \mathbb{R}_\pm \to \mathbb{R} \times (\mathbb{R}_\mp \cup \{0\}), \qquad S_\pm(\tau, v) = (\hat{\tau}, \hat{v}),$$

sending (τ, v) to the coordinates of the subsequent zero, are well-defined and of class C^1 on $(\mathbb{R} \times \mathbb{R}_\pm) \setminus \Sigma_\pm$, where $\Sigma_\pm = \{(\tau, v) \in \mathbb{R} \times \mathbb{R}_\pm : \hat{v} = 0\}$. Also, note that

$$\Sigma_\pm \cap M_{v_*} = \emptyset$$

for $v_* > 0$ sufficiently large. This leads to the following.

Lemma B.1. *For v_* sufficiently large S_+ and S_- have expansions of the form*

$$\begin{cases} \hat{\tau} = \tau + \frac{\pi}{n} + \frac{1}{nv} F(\tau, \tau + \frac{\pi}{n}) + \hat{R}(\tau, |v|), \\ \hat{v} = -v + G(\tau, \tau + \frac{\pi}{n}) + \hat{S}(\tau, |v|), \end{cases}$$

where $\hat{R}, \hat{S} \in C^1(\mathbb{M}_{v_*})$, $\hat{R} \in \mathcal{F}_u^1(2)$ and $\hat{S} \in \mathcal{F}_u(1) \cap \mathcal{F}^1(0)$.

Proof. It follows from (B.1) that for v sufficiently large S_+ has the form

$$\begin{cases} \hat{\tau} = \tau + \frac{\pi}{n} + \frac{1}{n} \arcsin\left(\frac{F(\tau,\hat{\tau})}{v}\right), \\ \hat{v} = G(\tau,\hat{\tau}) - \sqrt{v^2 - F(\tau,\hat{\tau})^2}, \end{cases}$$

where arcsin maps $[-1,1]$ onto $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Since $F \in C_b^1(\mathbb{R}^2)$, it follows $\lim_{v\to\infty} \hat{\tau} = \tau + \frac{\pi}{n}$ and thus also $\lim_{v\to\infty} F(\tau,\hat{\tau}) = F(\tau, \tau + \frac{\pi}{n})$ uniformly in τ. Moreover, a direct calculation shows that

$$\partial_\tau \hat{\tau} = 1 + \mathcal{O}(v^{-1}), \quad \partial_v \hat{\tau} = \mathcal{O}(v^{-2}).$$

In summary, we have $F(\tau,\hat{\tau}) \in \mathcal{F}_{\mathrm{u}}^1(0)$. Since $\varphi(x) = \arcsin(x) - x$ satisfies $\varphi(0) = \varphi'(0) = 0$, we may apply Lemma A.3 to infer that

$$\hat{\tau} = \tau + \tfrac{\pi}{n} + \tfrac{1}{v} F(\tau,\hat{\tau}) + R_1(\tau, v),$$

where $R_1 \in \mathcal{F}_{\mathrm{u}}^1(2)$. The fact that $\partial_t F = nG \in C_b^1(\mathbb{R}^2)$ yields

$$F(\tau,\hat{\tau}) - F(\tau, \tau + \tfrac{\pi}{n}) = (\hat{\tau} - \tau - \tfrac{\pi}{n}) \int_0^1 nG(\tau, \tau + \tfrac{\pi}{n} + \lambda(\hat{\tau} - \tau - \tfrac{\pi}{n}))\, d\lambda.$$

Consequently, we have

$$F(\tau,\hat{\tau}) = F(\tau, \tau + \tfrac{\pi}{n}) + R_2(\tau, v)$$

with $R_2 \in \mathcal{F}_{\mathrm{u}}^1(1)$. Thus, we have proven the claimed expansion for $\hat{\tau}$. Note that in contrast to F, the function $G(\tau, t)$ is not of class C^2 in t. However, we still obtain $\left(G(\tau,\hat{\tau}) - G(\tau, \tau + \frac{\pi}{n})\right) \in \mathcal{F}_{\mathrm{u}}(1) \cap \mathcal{F}^1(0)$. Hence, one can derive the formula for $\hat{v}$ by applying Lemma A.3 with $\tilde{\varphi}(x) = \sqrt{1 - x^2} - 1$. The expansion of S_- follows by considering $z(t) = -y(t)$. □

Now, for $p \in C(\mathbb{S}^1)$ consider the piecewise linear oscillator

$$\ddot{x} + n^2 x + \mathrm{sign}(x) = p(t),$$

and the map P defined in (3.13).

Lemma B.2. *For $v > 0$ sufficiently big, $P(\tau, v) = (\tau', v')$ has an expansion of the form*

$$\begin{cases} \tau' = \tau + 2\pi - \frac{L_1(\tau)}{nv} + R'(\tau, v), \\ v' = v + L_2(\tau) + S'(\tau, v), \end{cases} \tag{B.2}$$

where

$$L_1(\tau) = 2\pi\Im(e^{in\tau}\hat{p}_n) + 4, \qquad L_2(\tau) = 2\pi\Re(e^{in\tau}\hat{p}_n),$$

and $R', S' \in C^1(\mathbb{M}_{v_*})$, $R' \in \mathcal{F}^1_u(2)$, $S' \in \mathcal{F}_u(1)$.

Proof. Define

$$\varphi(\tau) = \int_\tau^{\tau+\frac{\pi}{n}} p(s)\sin(n(s-\tau))\,ds, \qquad \psi(\tau) = \int_\tau^{\tau+\frac{\pi}{n}} p(s)\cos(n(s-\tau))\,ds.$$

For $t \in [\tau, \tau_1]$, the corresponding equation is $\ddot{x} + n^2x = p(t) - 1$. Hence, one could apply Lemma B.1 with $f = p - 1$ to obtain the expansions of τ_1 and v_1. Since

$$F(\tau, \tau + \tfrac{\pi}{n}) = \int_\tau^{\tau+\frac{\pi}{n}} (p(s) - 1)\sin(n(\tau + \tfrac{\pi}{n} - s))\,ds = \varphi(\tau) - \tfrac{2}{n}$$

and similarly $G(\tau, \tau + \frac{\pi}{n}) = -\psi(\tau)$, we get the expansion

$$\begin{cases} \tau_1 = \tau + \frac{\pi}{n} + \frac{1}{nv}\left(\varphi(\tau) - \frac{2}{n}\right) + R_1(\tau, v), \\ v_1 = -v - \psi(\tau) + S_1(\tau, v), \end{cases}$$

with $R_1 \in \mathcal{F}^1_{\mathrm{u}}(2)$ and $S_1 \in \mathcal{F}_{\mathrm{u}}(1) \cap \mathcal{F}^1(0)$. In the same way, one can derive the expansions

$$\begin{cases} \tau_{j+1} = \tau_j + \frac{\pi}{n} + \frac{1}{nv_j}\left(\varphi(\tau_j) + (-1)^{j+1}\frac{2}{n}\right) + R_{j+1}(\tau_j, |v_j|), \\ v_{j+1} = -v_j - \psi(\tau_j) + S_{j+1}(\tau_j, |v_j|), \end{cases}$$

for $j = 1, \ldots, 2n - 1$, where $R_{j+1} \in \mathcal{F}^1_{\mathrm{u}}(2)$ and $S_{j+1} \in \mathcal{F}_{\mathrm{u}}(1) \cap \mathcal{F}^1(0)$. In particular since $S_{j+1} \in \mathcal{F}^1(0)$, Lemma A.4 yields $\frac{1}{v_j} = -\frac{1}{v_{j-1}} + \ldots$ with a remainder in $\mathcal{F}^1_{\mathrm{u}}(2)$. By repeatedly using this fact as well as Lemmata A.6

and A.5, we obtain

$$\begin{cases} \tau_{2n} = \tau + 2\pi + \dfrac{1}{nv_0}\left[\displaystyle\sum_{j=0}^{2n-1}(-1)^j\varphi(\tau_0 + \tfrac{j\pi}{n}) - 4\right] + R'(\tau, v), \\ v_{2n} = v_0 + \displaystyle\sum_{j=0}^{2n-1}(-1)^j\psi(\tau_0 + \tfrac{j\pi}{n}) + S'(\tau, v), \end{cases}$$

where $R' \in \mathcal{F}_{\mathrm{u}}^1(2)$, $S' \in \mathcal{F}_{\mathrm{u}}(1) \cap \mathcal{F}^1(0)$. Thus, the assertion follows from

$$2\pi e^{in\tau_0}\hat{p}_n = \int_0^{2\pi} p(t)e^{in(\tau_0 - t)}\,dt = \sum_{j=0}^{2n-1}(-1)^j\left[\psi(\tau_0 + \tfrac{j\pi}{n}) - i\varphi(\tau_0 + \tfrac{j\pi}{n})\right].$$

□

C The hull of an almost periodic function

Consider an almost periodic function $u \in C(\mathbb{R})$. We write $u_\tau(t)$ for the translation $u(\tau + t)$. The hull of u is given by

$$\mathcal{H}_u = \overline{\{u_\tau : \tau \in \mathbb{R}\}},$$

where the closure is taken with respect to uniform convergence. On $\mathcal{H}_u$ we define the group operation $*$ as follows. For $v, w \in \mathcal{H}_u$ with $v = \lim_{n\to\infty} u_{\tau_n^v}$ and $w = \lim_{n\to\infty} u_{\tau_n^w}$, let

$$v * w = \lim_{n\to\infty} u_{\tau_n^v + \tau_n^w}, \quad -v = \lim_{n\to\infty} u_{-\tau_n^v}.$$

These limits exist and define continuous operations.

Lemma C.1. *Let $u \in C(\mathbb{R})$ be almost periodic. If the sequences $(u_{\tau_n}), (u_{s_n})$ are uniformly convergent, then $(u_{\tau_n - s_n})$ is uniformly convergent as well.*

Proof. Let $\varepsilon > 0$ be given. Since $(u_{\tau_n}), (u_{s_n})$ are Cauchy sequences, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$ we have

$$|u_{\tau_n}(-s_n + t) - u_{\tau_m}(-s_n + t)| < \frac{\varepsilon}{2}$$

and

$$|u_{s_n}(\tau_m - s_n - s_m + t) - u_{s_m}(\tau_m - s_n - s_m + t)| < \frac{\varepsilon}{2},$$

where $t \in \mathbb{R}$ is arbitrary. Together this yields

$$|u(\tau_n - s_n + t) - u(\tau_m - s_m + t)| < \varepsilon.$$

for all $n, m \geq N$ and $t \in \mathbb{R}$, and thus proves the assertion. □

The continuity of both operations can be shown by a similar argument. Therefore, the hull becomes a commutative topological group with neutral element u.

If $u \in C(\mathbb{R})$ is an almost periodic function representable over (Ω, ψ), then also any element of its hull is a.p. and representable over (Ω, ψ). In fact, we have the following.

Lemma C.2. *Given $U \in C(\Omega)$ and $u(t) = U(\psi(t))$, let $v \in \mathcal{H}_u$. Then, there is $\omega \in \Omega$ such that $v(t) = U(\omega + \psi(t))$.*

Proof. Let $(\tau_n)_{n\in\mathbb{N}}$ be such that $\lim_{n\to\infty} u_{\tau_n} = v$. Since Ω is compact, there is $\omega \in \Omega$ and a subsequence $(\tau_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \psi(\tau_{n_k}) = \omega$. But this implies

$$v(t) = \lim_{k\to\infty} u_{\tau_{n_k}}(t) = \lim_{k\to\infty} U(\psi(\tau_{n_k}) + \psi(t)) = U(\omega + \psi(t)).$$

□

In particular, this implies that if $u \in C^k(\mathbb{R})$ then also its hull $\mathcal{H}_u$ consists of functions in $C^k(\mathbb{R})$.

Notation index

Symbol	Meaning
$\mathbb{N}, \mathbb{N}_0$	the positive and non-negative integer numbers
$\mathbb{S}^1$	the circle $\mathbb{R}/2\pi\mathbb{Z}$
M_v	the upper plane $\mathbb{R} \times [v, \infty)$
$\mathbb{M}_v$	the cylinder $\mathbb{S}^1 \times [v, \infty)$
G_ρ	the ρ-neighborhood of a domain $G \subset \mathbb{R}^d$
$\mathbb{T}^N$	the N-torus, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$
(Ω, ψ)	a compact group Ω together with a homomorphism $\psi : \mathbb{R} \to \Omega$ (see Section 4.1.1)
λ, λ^2	the Lebesgue measure of $\mathbb{R}$ and $\mathbb{R}^2$, respectively
μ_Ω	the Haar-measure of Ω (see Section 4.1.3)
∂_x	partial derivative with respect to a variable x
$\partial^{\boldsymbol{\nu}}$	mixed derivative $\partial_1^{\nu_1} \cdots \partial_d^{\nu_d}$, where $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ denotes a multi-index and ∂_i is the derivative with respect to the i-th coordinate
∂_ψ	derivative along the flow (see Section 4.1.2)
Df	Jacobian matrix of a function f
$p(\boldsymbol{\nu}, r)$	set of all ordered partitions of a multi-index $\boldsymbol{\nu}$ into r non-zero multi-indices (see Lemma 5.2)
$\mathcal{H}_u$	the hull of an a.p. function u (see Appendix C)
$C^k(G)$	space of k times continuously differentiable functions $f : G \to \mathbb{R}$; if $G \in \{\mathbb{S}^1, \mathbb{M}_v\}$ we refer to functions periodic in the respective coordinate
C_b^k	space of functions in C^k with bounded derivatives up to order k

Symbol	**Meaning**
$C^{k,\beta}$	space of functions in C^k such that the k-th order derivatives are Hölder continuous with exponent β
$C^k_\psi(\Omega)$	space of functions with continuous derivatives along the flow up to order k; $C^k_\psi(\Omega \times G)$ with $G \subset \mathbb{R}^d$ is defined accordingly
$\mathcal{F}^k(r)$	space of functions $F(\tau, v)$ in $C^k(M_{v_*})$ for some $v_* > 0$ such that $\partial^{\boldsymbol{\nu}} F = \mathcal{O}(v^{-r-\nu_2})$; $\mathcal{F}^k_{\mathrm{u}}(r)$ denotes the subspace where $v^r F(\tau, v)$ converges uniformly as $v \to \infty$
$\mathcal{H}^k_{\rho,\sigma}, \tilde{\mathcal{H}}^k_{\rho,\sigma}$	classes of Hamiltonians $H(x,t,\varepsilon)$ defined on $G_\rho \times \mathbb{T} \times [0,\sigma]$ and with bounded derivatives in x up to order k (see Definition 5.5)
$\mathcal{M}^k_{\rho,\sigma}$	class of functions $l(x,\varepsilon)$ with bounded continuous derivatives (see Definition 5.14)
$\lvert\cdot\rvert$	depending on the context, the maximum norm on $\mathbb{R}, \mathbb{R}^d$ or the length $\lvert\boldsymbol{\nu}\rvert = \lvert\nu_1\rvert + \ldots + \lvert\nu_d\rvert$ of a multi-index $\boldsymbol{\nu} \in \mathbb{N}_0^d$
$\lVert\cdot\rVert_\infty$	the supremum norm
$\lVert\cdot\rVert_{C^k}$	the norm $\lVert f\rVert_{C^k} = \max_{\lvert\boldsymbol{\nu}\rvert \le k} \lVert\partial^{\boldsymbol{\nu}} f\rVert_\infty$
$\lVert\cdot\rVert_{k,\rho,\sigma}$	the norm $\lVert H\rVert_{k,\rho,\sigma} = \sup_{\varepsilon\in[0,\sigma]} \sup_{t\in\mathbb{R}} \lVert H(\cdot,t,\varepsilon)\rVert_{C^k(G_\rho)}$
$\lVert\cdot\rVert^*_{k,\rho,\sigma}$	the norm $\lVert l\rVert^*_{k,\rho,\sigma} = \lVert l\rVert_{k,\rho,\sigma} + \sup_{\varepsilon\in(0,\sigma]} \lVert\partial_\varepsilon l(\cdot,\varepsilon)\rVert_{C^k(G_\rho)}$

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