Harmonic
and
locally harmonic
Maaß forms

Inaugural-Dissertation

zur

Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Fakultät

der Universität zu Köln

vorgelegt von

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Köln, 2023

Die vorliegende Arbeit wurde von der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln als Dissertation angenommen.

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Tag der mündlichen Prüfung:

15. Februar 2023

Für meinen Bruder Jan

Abstract

This thesis contains the results of seven research papers on various types of harmonic and locally harmonic Maaß forms. To this end, this thesis is divided into two parts. The first part deals with harmonic Maaß forms and variants thereof, while the second part is devoted to constructions of new locally harmonic Maaß forms.

The first chapter is divided into three parts. The first part provides an overall introduction to holomorphic and nonholomorphic modular forms, and summarizes some previous results in the theory of both classes of forms. The second part collects and presents the main results of this thesis, while the content of the third part is a brief discussion of some of the results from the second half of this thesis.

The second chapter introduces the basic theory of harmonic Maaß forms, constructs a new example of a polar harmonic Maaß form of weight $\frac{3}{2}$ on the level of Fourier expansions, and presents a connection to Hurwitz class numbers as well as a p-adic property ($p > 2$ prime) of our new polar form. The construction is based on a technique by Zagier and Zwegers, and adapts a paper by Mertens, Ono, and Rolen.

The third chapter generalizes the construction from the second chapter producing polar harmonic Maaß forms of non-positive integral weights. The Fourier coefficients of the holomorphic part of these forms are given by certain twisted divisor sums. As outlined in the introduction of the second chapter, these new examples thus are certain nonholomorphic "Eisenstein series", whose holomorphic parts resemble the Fourier expansion of the usual holomorphic Eisenstein series in a rough sense.

The fourth chapter explores the connection of (higher depth) harmonic Maaß forms to the area of hypergeometric q-series and their combinatorial interpretation. This goes back to Ramanujan's mock theta functions and Zwegers' modular completions of them in depth 1. We define mock theta functions of depth 2, and provide three natural examples. This chapter concludes the first part of this thesis.

The fifth chapter is the first one of the second part of this thesis. It introduces the language of integral binary quadratic forms, and utilizes them to construct elliptic, parabolic and hyperbolic nonholomorphic Eisenstein series of even weights $k \geq 2$. This combines Zagier's $f_{k,D}$-function with a modular integral by Parson and Hecke's trick to include weight 2. We summarize the analytic continuation in weight $k = 2$ to the spectral parameter $s = 0$, in the parabolic and elliptic case, which is known to yield a harmonic Maaß form resp. polar harmonic Maaß form. We then complete the picture by proving the existence of the analytic continuation to $s = 0$ (with $k = 2$) in the hyperbolic case by calculating the Fourier expansion. This yields an explicit formula of the continuation as

well, and we infer that the continuation coincides with a locally harmonic Maaß form if the imaginary part of the input variable in the complex upper half plane is sufficiently large.

The sixth chapter continues the work of the fifth chapter by connecting the hyperbolic Eisenstein series of positive even weights defined there to a locally harmonic Maaß form on all of $\mathbb{H}$ except an explicit exceptional set of measure zero intrinsic to such forms. This leads to the new concept of local cusp forms. In addition, we provide a second independent perspective on our locally harmonic Maaß form of positive weight by rewriting them as twisted traces of cycle integrals of one of Petersson's two-variable Poincaré series $\mathbb{P}_{2k+2}$. Löbrich and Schwagenscheidt obtained the archetypal example $\mathcal{F}_{1-k,D}$ of a negative weight locally harmonic Maaß form introduced by Bringmann, Kane, and Kohnen by investigating the very same twisted trace of $\mathbb{P}_{2k}$ with respect to the other variable.

The seventh chapter constructs bimodular completions of 30 year old functions defined by Knopp, which he obtained as lifts of Zagier's $f_{k,D}$-function under the Bol operator. In the course of inspecting some properties of our completions, we rediscover the local cusp forms of positive even weights from the sixth chapter. We define a negative weight lift of these forms under both the Bol operator and the shadow operator along the lines of $\mathcal{F}_{1-k,D}$. Our new negative weight form is a locally harmonic Maaß form with continuously, but not differentially removable singularities. Lastly, we show that this negative weight locally harmonic Maaß form is an output of a certain scalar valued theta lift by modifying a theta lift yielding $\mathcal{F}_{1-k,D}$.

The eight chapter utilizes the machinery of regularized theta lifts to establish so called Eichler–Selberg type relations for a wide class of weakly holomorphic modular forms, as well as demonstrating an alternative approach to local weak (and locally harmonic) Maaß forms via theta lifts. This idea goes back to the PhD thesis by Hövel in weight 0, who utilized a Siegel theta kernel in a vector valued setting. We modify his theta lift by including a Maaß raising operator yielding other weights than 0. The weight now is determined by the parameters of the lattice and the degrees of a homogeneous polynomial inside the theta kernel. The overall construction relies on foundational work by Borcherds and by Bruinier, and the evaluation of the theta lift follows a method by Bruinier, Ehlen, and Yang.

Acknowledgements

First and foremost, I would like to thank my advisor Kathrin Bringmann for her genuine interest to my research and my academic career. This thesis would not have been possible without her time investment, continuous effort, and answers to my questions.

I would also like to thank Larry Rolen and Joshua Males, who supported me and my research since my early days in Cologne. I owe both of them many insights and a lot of advice. In addition to that, Larry invited me for a research visit to Vanderbilt University in August 2022, for which I thank him as well.

I am also grateful to Sander Zwegers for being my secondary examiner for this thesis as well as to Ken Ono for the opportunitiy to visit him at the University of Virginia in September 2022.

Moreover, I would not be in this position without Özlem Imamoḡlu, who advised both my Bachelors and Masters thesis at ETH Zurich introducing me to the world of holomorphic modular forms and classical Maaß wave forms.

Furthermore, I would not be in this position without my relatives and friends. I am very grateful to everyone individually for their continuous support, help and guidance during all the years.

Finally, I would also like to thank my office mate Caner Nazaroglu as well as both number theory groups in Cologne for the encouraging atmosphere and many fruitful conversations during my stay there.

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Chapter I

Introduction and Statement of Objectives

This thesis consists of seven research articles [MMR22, Mon21, MMR21, Mon22a, Mon22b, BM22, MM21], whose unifying theme is the construction of various new Maaß forms in different frameworks. We recall the basic definitions and results of the theory, collect the main results of this thesis, and discuss some of them at the end of this chapter.

I.1 Definitions and previous results

In this preliminary section, we summarize some background with a focus on motivating and contextualizing the constructions in the main chapters of this thesis. A good exposition on modular forms can be found in [BvdGHZ08, Iwa97, Kob93, Ser73], and we follow [BFOR17] to present the theory of harmonic Maaß forms. The subsection on locally harmonic Maaß forms is mainly based on the paper [BKK15].

I.1.1 Holomorphic modular forms

Holomorphic modular forms are ubiquitous objects in (analytic) number theory and in many other areas of pure mathematics, as they are equipped with a lot of internal symmetries and therefore offer a rich theory with many applications. To motivate their structure, we let

$$\mathbb{H} := \{\tau = u + iv \in \mathbb{C} \colon v > 0\}$$

be the *complex upper half plane*, and $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ be the *modular group*. The group Γ is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and acts on $\mathbb{H}$ by *fractional linear transformations*

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}, \qquad \tau \in \mathbb{H}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

because

$$\mathrm{Im}\,(\gamma\tau) = \frac{\det(\gamma)v}{|c\tau + d|^2} = \frac{v}{|c\tau + d|^2} > 0.$$

Now, one might ask for functions $f\colon \mathbb{H} \to \mathbb{C}$ "respecting" the group action of Γ, namely

$$f(\gamma\tau) = j(\gamma, \tau)f(\tau)$$

for every $\tau \in \mathbb{H}$ and some function $j(\gamma, \tau)$. To motivate a choice for $j(\gamma, \tau)$, we consider γ_1, $\gamma_2 \in \Gamma$. Our condition on f becomes

$$j(\gamma_1\gamma_2, \tau)f(\tau) = f(\gamma_1\gamma_2\tau) = f(\gamma_1(\gamma_2\tau)) = j(\gamma_1, \gamma_2\tau)f(\gamma_2\tau) = j(\gamma_1, \gamma_2\tau)j(\gamma_2, \tau)f(\tau).$$

In other words, the function $j(\gamma, \tau)$ has to satisfy the cocycle condition

$$j(\gamma_1\gamma_2, \tau) = j(\gamma_1, \gamma_2\tau)j(\gamma_2, \tau),$$

and one can verify that

$$j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) := c\tau + d \tag{I.1}$$

is indeed such a cocycle. Clearly, we have the freedom to introduce powers $k \in \mathbb{Z}$ of $j(\gamma, \tau)$ without changing the previous discussion. We further impose an analytic condition and a growth condition on f as follows.

Definition I.1.1.

(1) A *modular form of weight* $k \in \mathbb{Z}$ is a function $f\colon \mathbb{H} \to \mathbb{C}$, which satisfies the following three conditions:

   (i) We have $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for every $\tau \in \mathbb{H}$ and every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.
   (ii) The function f is holomorphic on $\mathbb{H}$.
   (iii) The function f is holomorphic at $i\infty$.

(2) We call f a *cusp form* if f is a modular form that vanishes at $i\infty$.

Condition (iii) requires some explanation. Firstly, one requires that $i\infty$ is at most an isolated and non-essential singularity of f. Secondly, conditions (i) and (ii) imply that every modular form f has a Fourier expansion of the shape

$$f(\tau) = \sum_{n \gg -\infty} a_f(n)q^n := \sum_{n \geq n_f} a_f(n)q^n, \qquad q := e^{2\pi i \tau},$$

for some $n_f \in \mathbb{Z}$. Thirdly, condition (iii) now translates to $n_f \geq 0$, so that f is regular "at $q = 0$". We note that f is a cusp form if and only if $n_f > 0$.

The notion of modular forms has been generalized in various directions in the literature. One may consider half integral weights $k \in \frac{1}{2}\mathbb{Z}$, or congruence subgroups of Γ, such as the *Hecke congruence subgroup*

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \,(\mathrm{mod}\ N) \right\},$$

where $N \in \mathbb{N}$ is called the *level* of a modular form. Both is motivated by the properties of *Jacobi's theta function*

$$\vartheta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2 \sum_{n \geq 1} q^{n^2},$$

and its transformation law is captured by invariance with respect to the *slash operator*

$$\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)(\tau) := \begin{cases} \frac{f(\gamma\tau)}{(c\tau+d)^k} & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{2k} \frac{f(\gamma\tau)}{(c\tau+d)^k} & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases} \quad \varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \,(\mathrm{mod}\ 4), \\ i & \text{if } d \equiv 3 \,(\mathrm{mod}\ 4), \end{cases} \quad \text{(I.2)}$$

where $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol, and we stipulate to take the principal branch of the complex square root throughout. In this notation, we have

$$\left(\vartheta|_{\frac{1}{2}} \gamma \right)(\tau) = \vartheta(\tau)$$

for every $\gamma \in \Gamma_0(4)$. Equivalently, the space of modular forms of weight $\kappa + \frac{1}{2}$, $\kappa \in \mathbb{Z}$, and level 4 may be defined as the space of functions being holomorphic on $\mathbb{H}$ as well as at the cusps of $\Gamma_0(4)$, and transforming under the action of $\Gamma_0(4)$ like $\vartheta(\tau)^{2\kappa+1}$.

Further generalizations are

- *weakly holomorphic modular forms*, namely modular forms, which are permitted to have poles at the cusps,
- *meromorphic modular forms* (permitting poles on $\mathbb{H}$ and possibly at $i\infty$),
- quasimodular forms (relaxing the transformation law in a prescribed manner to include derivatives of modular forms),
- or Jacobi forms (introducing a second "elliptic" variable)

for instance.

The fruitfulness of modular forms stems from the fact that their Fourier coefficients often encode arithmetic, geometric, or combinatorial quantities. A classical example is a short proof of Lagrange's four squares theorem by producing exact formulas for representations of natural number as sums of four squares in terms of simple divisor sums. Additionally, (weighted) traces of non-zero discriminants are the Fourier coefficients of a modular form, and the generating function of the number of partitions of $n \in \mathbb{N}_0$ is a modular form of weight $-\frac{1}{2}$ up to the factor $q^{\frac{1}{24}}$. Consequently, modular objects are often investigated via their Fourier expansions.

I.1.2   Harmonic Maaß forms

Another direction of generalizing modular forms is to replace complex analyticity (holomorphicity) by real analyticity, but retaining the modular transformation law. This leads to the notion of weak Maaß forms. To describe them, we define the *weight k hyperbolic Laplace operator* by

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right) = -4v^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \overline{\tau}} + 2ikv \frac{\partial}{\partial \overline{\tau}},$$

which reduces to the usual hyperbolic Laplace operator in weight $k = 0$. We note that an eigenfunction of Δ_k is real-analytic, because Δ_k is an elliptic differential operator (see [Rud91, Theorem 8.12] and the paragraph afterwards for example). Moreover, we relax the growth condition of modular forms to include weakly holomorphic modular forms into the theory of Maaß forms, and to permit a power of v in the constant term of the Fourier expansion later. We alert the reader to the fact that there exist various conventions regarding the terminology of Maaß forms in the literature, and we deviate from the convention used in [BFOR17] here.

Definition I.1.2. Let $k \in \frac{1}{2}\mathbb{Z}$, and choose $N \in \mathbb{N}$ such that $4 \mid N$ whenever $k \notin \mathbb{Z}$. Let $f \colon \mathbb{H} \to \mathbb{C}$ be smooth.

(1) The function f is a *weight k weak Maaß form* for $\Gamma_0(N)$ if it satisfies the following three properties:

   (i) We have $(f|_k\gamma)(\tau) = f(\tau)$ for all $\gamma \in \Gamma_0(N)$ and all $\tau \in \mathbb{H}$.
   (ii) The function f is an eigenfunction of Δ_k.
   (iii) There exists some $\varepsilon > 0$ such that $f \in O(e^{\varepsilon v})$ as $v \to \infty$. An analogous condition is required at the other cusps of $\Gamma_0(N)$.

(2) A *harmonic Maaß form* is a weak Maaß form with eigenvalue 0 under Δ_k.
(3) A *weak (resp. harmonic) Maaß form with cuspidal shadow* is a weak (resp. harmonic) Maaß form f such that there exists a polynomial $P_f \in \mathbb{C}[q^{-1}]$ (the *principal part of f*) satisfying $f(\tau) - P_f(\tau) = O(e^{-\delta v})$ as $v \to \infty$ for some $\delta > 0$. We require an analogous condition at all other cusps of $\Gamma_0(N)$.

Harmonic Maaß forms are often inspected via their behaviour under certain differential operators. Bruinier and Funke [BF04] defined the *shadow operator*

$$\xi_k := 2iv^k \overline{\frac{\partial}{\partial \overline{\tau}}} = iv^k \overline{\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right)},$$

which yields the splitting $\Delta_k = -\xi_{2-k} \circ \xi_k$. In turn, this splitting implies that the Fourier expansion of a harmonic Maaß form f of weight $2 - k \neq 1$ splits into a *holomorphic* and a *nonholomorpic part*

$$f^+(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n, \quad f^-(\tau) = c_f^-(0) v^{1-k} + \sum_{\substack{n \ll \infty \\ n \neq 0}} c_f^-(n) \Gamma(1 - k, -4\pi nv) q^n, \tag{I.3}$$

$$f(\tau) = f^+(\tau) + f^-(\tau).$$

Here,

$$\Gamma(s, z) := \int_z^\infty t^{s-1} e^{-t} \mathrm{d}t, \qquad \mathrm{Re}(s) > 0,$$

is the *incomplete Gamma function*. The function $s \mapsto \Gamma(s, z)$ is multi-valued, but restricting to principal values yields the single-valued principal branch of this function. Provided that $z \neq 0$, each branch of $s \mapsto \Gamma(s, z)$ admits an analytic continuation to all $s \in \mathbb{C}$ with removable singularities at non-positive integers (see Section II.2 for the functional equation). We refer the reader to [BDE17, Section 2.2] for more details on this, and to [GR07, item 8.3357] (or [BCLO10, §8.11]) for the asymptotic expansion of $\Gamma(s, \cdot)$.

One can verify that the shadow operator intertwines with the slash operator by changing the weight from $2 - k$ to k. In other words, if f is a harmonic Maaß form of weight $2 - k$ then $g_1 := \xi_{2-k}(f)$ is a weakly holomorphic modular form of weight k. In addition, Bruinier and Funke [BF04] proved that the shadow operator is surjective onto the space of weakly holomorphic modular forms of weight k.

If we are in the situation of integral weight $k \geq 2$ then the shadow operator has a holomorphic companion given by the *Bol operator* [BOR08]

$$\mathbb{D}^{k-1} := \left(\frac{1}{2\pi i} \frac{\partial}{\partial \tau} \right)^{k-1} = \left(q \frac{\partial}{\partial q} \right)^{k-1}.$$

For such k, the Bol operator intertwines with the slash operator by changing the weight from $2 - k$ to k as well, and hence $g_2 := \mathbb{D}^{k-1}(f)$ is a weakly holomorphic modular form of weight k whenever f is a harmonic Maaß form of weight $2 - k$. Moreover, the Bol operator is surjective onto the space of weakly holomorphic modular forms of weight k as well.

An inverse of ξ_{2-k} (resp. $\mathbb{D}^{k-1}$) is given by the nonholomorphic (resp. holomorphic) Eichler [Eic57] integral of g_1 (resp. g_2). Both Eichler integrals can be defined on the level of Fourier expansions straightforwardly. However, each preimage is not unique, since we may add any weakly holomorphic modular form of weight $2 - k$ to the nonholomorphic Eichler integral of g_1 for instance. If g_1 (resp. g_2) is a cusp form then its *holomorphic*

and *nonholomorphic Eichler integral* have a representation given by

$$
\begin{aligned}
\mathcal{E}_{g_2}(\tau) &:= -\frac{(2\pi i)^{k-1}}{(k-2)!} \int_\tau^{i\infty} g_2(w)(\tau - w)^{k-2} \mathrm{d}w, \\
g_1^*(\tau) &:= (2i)^{1-k} \int_{-\overline{\tau}}^{i\infty} \overline{g_1(-\overline{w})}(w + \tau)^{k-2} \mathrm{d}w.
\end{aligned}
\tag{I.4}
$$

Conversely, two cusp forms determine a harmonic Maaß form with cuspidal shadow. However, both cusp forms have to be different, otherwise the resulting harmonic Maaß form with cuspidal shadow vanishes identically[1]. This leads to the "lifting problem": Given either g_1 or g_2, find the other cusp form such that both together give rise to a non-trivial harmonic Maaß form with cuspidal shadow via their respective Eichler integrals.

I.1.3 Locally harmonic Maaß forms

In 2012, Bringmann, Kane, and Kohnen [BKK15] attacked the aforementioned lifting problem from a different perspective. They investigated the highly influential weight $2k$ cusp form

$$
f_{k,D}(\tau) := \sum_{\substack{a,b,c \in \mathbb{Z} \\ b^2 - 4ac = D}} \frac{1}{(a\tau^2 + b\tau + c)^k}
\tag{I.5}
$$

defined by Zagier [Zag75], where $k > 2$ is an even integer and $D > 0$ is a discriminant.

For example, the $f_{k,D}$-function generates the theta kernel function of the Shimura [Shi73] and Shintani [Shi75] lifts by work of Kohnen [Koh85] and of Kohnen and Zagier [KZ81], see Remark VI.4 as well. Additionally, the even periods of $f_{k,D}$ are rational by further work of Kohnen and Zagier [KZ84]. As shown by Katok [Kat85], the $f_{k,D}$-function is a hyperbolic Poincaré series, namely an infinite sum of terms of the shape $\tau^{-k}|_{2k}\gamma$ with γ running over an explicit (shifted) set of hyperbolic cosets, see [IO09] for more details on such constructions, and [BKK15, Section 3] for a summary of Katok's work. Hence certain collections of $f_{k,D}$ with respect to D span the space of weight $2k$ cusp forms, because τ^{-k} is the constant term in the hyperbolic expansion of a cusp form.

Bringmann, Kane, and Kohnen now asked for a lift of $f_{k,D}$ under both differential operators ξ_{2-2k} and $\mathbb{D}^{k-1}$ in a local sense, which imitates the construction of $f_{k,D}$. This question translated to a differential equation in terms of τ^{-k}, because both operators

---

[1]An example of a harmonic Maaß form with non-cuspidal shadow, which maps to the same non-trivial function under both differential operators up to normalization is given in [BFOR17, Theorem 6.15].

intertwine with the slash operator as discussed above. Assuming that D is not a square, they solved this differential equation, and the solution is given by the function

$$\mathcal{F}_{1-k,D}(\tau) := \frac{1}{2} \sum_{\substack{a,b,c \in \mathbb{Z} \\ b^2 - 4ac = D}} \operatorname{sgn}\left(a|\tau|^2 + bu + c\right) \left(a\tau^2 + b\tau + c\right)^{k-1}$$

$$\times \beta\left(\frac{Dv^2}{|a\tau^2 + b\tau + c|^2}; k - \frac{1}{2}, \frac{1}{2}\right), \quad \text{(I.6)}$$

which indeed has a representation as a certain hyperbolic Poincaré series like $f_{k,D}$. Here,

$$\beta(x; r, s) := \int_0^x t^{r-1}(1-t)^{s-1}\mathrm{d}t, \qquad x \in (0, 1], \qquad r, s > 0,$$

denotes the *incomplete β-function*. Due to the presence of the sign-function and for convergence reasons, $\mathcal{F}_{1-k,D}$ has jumping singularities (certain discontinuities, see Section VII.2 for a definition) on the *exceptional set*

$$E_D := \left\{\tau \in \mathbb{H} : \exists a, b, c \in \mathbb{Z} \text{ such that } b^2 - 4ac = D \text{ and } a|\tau|^2 + bu + c = 0\right\} \quad \text{(I.7)}$$

of measure zero. Geometrically, E_D is the union of all *Heegner geodesics*

$$S_{[a,b,c]} = \{\tau \in \mathbb{H} : [a, b, c]_\tau = 0\}, \qquad [a, b, c]_\tau := \frac{1}{v}\left(a\,|\tau|^2 + bu + c\right), \quad \text{(I.8)}$$

namely all semicircles in $\mathbb{H}$ centered on $\mathbb{R}$ and indexed by integral binary quadratic forms $[a, b, c]$ of some non-square discriminant $D > 0$.

The function $\mathcal{F}_{1-k,D}$ is modular of weight $2 - 2k$, is harmonic with respect to Δ_{2-2k} outside E_D, satisfies a limit condition on E_D, and is of at most polynomial growth as $v \to \infty$. Altogether, this suggests that $\mathcal{F}_{1-k,D}$ is a new type of automorphic form called a *locally harmonic Maaß form with exceptional set*[2] E_D. Returning to the lifting problem, Bringmann, Kane, and Kohnen proved that

$$\mathcal{F}_{1-k,D}(\tau) = P_\mathcal{C}(\tau) - \frac{D^{k-\frac{1}{2}}(2k-2)!}{(4\pi)^{2k-1}}\mathcal{E}_{f_{k,D}}(\tau) + D^{k-\frac{1}{2}}f_{k,D}^*(\tau), \quad \text{(I.9)}$$

where

$$P_\mathcal{C}(\tau) := c_\infty + (-1)^k 2^{3-2k}\binom{2k-2}{k-1}\pi \sum_{\substack{a,b,c \in \mathbb{Z} \\ b^2 - 4ac = D \\ a < 0 < a|\tau|^2 + bu + c}}\left(a\tau^2 + b\tau + c\right)^{k-1}$$

---

[2]Other exceptional sets of measure 0 are possible as well, but are not studied in the literature yet. A full definition of a locally harmonic Maaß form can be found in [BKK15, Section 2], and is recalled in Subsections VI.2.7, VII.2.3.

is a so-called *local polynomial*, whose shape depends on the connected component $\mathcal{C}$ of $\mathbb{H} \setminus E_D$ in which τ is located. More precisely, it captures the jumping singularities of $\mathcal{F}_{1-k,D}$, and the sum selects precisely those quadratic forms for which τ is in the bounded component of the associated Heegner geodesic. Note that there are finitely many such quadratic forms, and there are none if $v > \frac{\sqrt{D}}{2}$ (see [BKK15, Lemma 5.1 (1), (7.6)]). The term c_∞ is an explicit global constant defined in equation (VII.23) (renormalizing [BKK15, (4.2), (7.3)]). One may view equation (I.9) as a local generalization of the Fourier expansion (I.3) with the local polynomial $P_\mathcal{C}$ serving as a "constant term of higher degree".

Zagier's $f_{k,D}$-function as well as the local polynomials $P_\mathcal{C}$ are connected to central L-values of cusp forms of certain levels N. The former is due to Kohnen and Zagier [KZ81] for odd and squarefree level, while the latter is understood only in the special case of a one-dimensional space of cusp forms. In this special case, the idea of Ehlen, Guerzhoy, Kane, and Rolen [EGKR20] is to generalize work of Kohnen [Koh85] to other levels, and the splitting (I.9) to weight 0. In essence, the authors of [EGKR20] translate the vanishing of a central L-value to the modularity of $P_\mathcal{C}$, and we refer to their paper for more details.

I.2 Statement of objectives

We conclude this chapter by collecting the main results of this thesis grouped by the chapters in which they appear. We omit intermediate results and corollaries.

I.2.1 Polar harmonic Maaß forms and holomorphic projection

In this first main chapter, we construct a family of harmonic Maaß forms of weight $\frac{3}{2}$, which are permitted to have poles on $\mathbb{H}$. To describe them, we require some notation. Let ψ, χ be two Dirichlet characters of moduli M_ψ, M_χ, and define

$$D_n := \left\{ d \mid n : 1 \le d \le \frac{n}{d} \text{ and } d \equiv \frac{n}{d} \pmod{2} \right\},$$

$$\sigma_2^{\mathrm{sm}}(n) := \sum_{d \in D_n} \chi\left(\frac{\frac{n}{d} - d}{2} \right) \psi\left(\frac{\frac{n}{d} + d}{2} \right) d^2.$$

Letting θ_ψ be Shimura's theta-function associated to ψ (see Sections II.1, III.1), and $\mathbb{1}$ be the trivial character, we further define the functions

$$\mathcal{F}^+(\tau) := \frac{1}{\theta_\psi(\tau)} \begin{cases} \sum_{n \ge 1} \sigma_2^{\mathrm{sm}}(n) q^n & \text{if } \chi \ne \mathbb{1}, \\ \frac{1}{2} \sum_{n \ge 1} \psi(n) n^2 q^{n^2} + \sum_{n \ge 1} \sigma_2^{\mathrm{sm}}(n) q^n & \text{if } \chi = \mathbb{1}, \end{cases}$$

and

$$\mathcal{F}^-(\tau) := \frac{i}{\pi\sqrt{2}} \int_{-\overline{\tau}}^{i\infty} \frac{\theta_\chi(w)}{(-i(w+\tau))^{\frac{3}{2}}} \,\mathrm{d}w.$$

Then, we have the following result.

Theorem I.2.1 (Theorems II.1.1, II.1.3)**.** *Suppose that ψ is odd and χ is even. Then the function $\mathcal{F}^+ + \mathcal{F}^-$ is a polar harmonic Maaß form of weight $\frac{3}{2}$ on $\Gamma_0(4M_\chi^2) \cap \Gamma_0(4M_\psi^2)$ with Nebentypus $\overline{\chi} \cdot (\psi \cdot \chi_{-4})^{-1}$.*

The proof works on the level of Fourier expansions, and utilizes a method by Zagier and Zwegers based on the technique of holomorphic projection. The construction adapts the paper [MOR21], and offers a connection to Hurwitz class numbers in a special case (see Corollary II.1.6). Moreover, we inspect the iterated action of the U-operator

$$\left(\sum_{n \gg -\infty} \alpha(n) q^n \right) \Big| \, U(p) := \sum_{n \gg -\infty} \alpha(pn) q^n,$$

and find the following p-adic property.

Theorem I.2.2 (Theorem II.1.8)**.** *Let a, b, $p \in \mathbb{N}$ and suppose that p is an odd prime. Then we have*

$$\left(\theta_\psi \left(p^{2a} \tau \right) \mathcal{F}^+(\tau) \right) \Big| \, U\left(p^b \right) \equiv 0 \quad \left(\mathrm{mod}\ p^{\min(a,b)} \right).$$

This parallels [MOR21, Theorem 1.4]. It would be interesting to relate other specializations of σ_2^{sm} to arithmetic or combinatorial quantities.

I.2.2 Multidimensional small divisor functions

This second main chapter extends the work of the first chapter obtaining polar harmonic Maaß forms of non-positive integral weight. Let $\ell \in \mathbb{N}$, and ψ, χ be two Dirichlet characters of moduli M_ψ, M_χ as in the previous chapter. Let $\lambda_\psi := \frac{1-\psi(-1)}{2}$, and $P_\ell \in \mathbb{Q}(X, Y)$. We define

$$\mathcal{D}_{\boldsymbol{n}} := \left\{ \boldsymbol{d} \in \mathbb{N}^\ell : d_j \mid n_j,\ 1 \le d_j \le \frac{n_j}{d_j},\ \text{and } d_j \equiv \frac{n_j}{d_j} \ (\mathrm{mod}\ 2) \text{ for every } 1 \le j \le \ell \right\},$$

the generalized coefficients,

$$\sigma_\ell^{\mathrm{sm}}(\boldsymbol{n}) := \sum_{\boldsymbol{d} \in \mathcal{D}_{\boldsymbol{n}}} \left(\prod_{j=1}^\ell \chi\left(\frac{\frac{n_j}{d_j} - d_j}{2} \right), \psi\left(\frac{\frac{n_j}{d_j} + d_j}{2} \right) \left(\frac{\frac{n_j}{d_j} - d_j}{2} \right)^{\lambda_\chi} \left(\frac{\frac{n_j}{d_j} + d_j}{2} \right)^{\lambda_\psi} \right)$$

$$\times P_\ell \left(\left\| \left(\frac{n_j}{d_j} \right)_{1 \leq j \leq \ell} \right\|^2, \|\boldsymbol{d}\|^2 \right),$$

and the functions

$$f_\ell^-(\tau) := \frac{1}{\Gamma(1 - k_{f_\ell})} \sum_{\boldsymbol{m} \in \mathbb{N}^\ell} \chi(\boldsymbol{m}!)(\boldsymbol{m}!)^{\lambda_\chi} \|\boldsymbol{m}\|^{2(k_{f_\ell}-1)} \Gamma(1 - k_{f_\ell}, 4\pi\|\boldsymbol{m}\|^2 v) q^{-\|\boldsymbol{m}\|^2},$$

$$f_\ell^+(\tau) := \frac{1}{\theta_\psi(\tau)^\ell} \sum_{\boldsymbol{n} \in \mathbb{N}^\ell} \sigma_\ell^{\mathrm{sm}}(\boldsymbol{n}) q^{|\boldsymbol{n}|}, \qquad f_\ell(\tau) := (f_\ell^+ + f_\ell^-)(\tau), \qquad k_{f_\ell} := 2 - \frac{\ell}{2}.$$

Then, we have the following result.

Theorem I.2.3 (Theorem III.2.1). *Let ψ be an odd Dirichlet character, χ be an even and non-trivial Dirichlet character. Let $\ell \in 2\mathbb{N} + 2$. Define P_ℓ as indicated in Corollay III.3.2, obtaining the corresponding small divisor function $\sigma_\ell^{\mathrm{sm}}$. Then the resulting function f_ℓ is a polar harmonic Maaß form of weight $k_{f_\ell} \in -\mathbb{N}_0$ on $\Gamma_0(4M_\chi^2) \cap \Gamma_0(4M_\psi^2)$ with Nebentypus $\overline{\chi} \cdot (\psi \cdot \chi_{-4})^{-1}$. Its shadow is given by a non-zero constant multiple of $\theta_{\overline{\chi}}^\ell$.*

The proof is along the same lines as the proof of Theorem I.2.1. We offer some numerical examples of polynomials P_ℓ in Section III.4.2.

I.2.3   Higher depth mock theta functions and q-hypergeometric series

The third main chapter is devoted to the interplay between the theory of harmonic Maaß forms and so called q-hypergeometric series extending foundational work by Zwegers [Zwe02] and Bringmann and Ono [BO06, BO10a]. For $n \in \mathbb{N} \cup \{0, \infty\}$, we let

$$(a)_n := (a;q)_n := \prod_{j=0}^{n-1} \left(1 - aq^j \right)$$

be the usual q-*Pochhammer symbol*. Zwegers investigated Ramanujan's mock theta functions, which are certain holomorphic but non-modular hypergeometric q-series. Three examples of them are given by

$$\nu(q) := \sum_{n \geq 0} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}, \qquad \phi(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q^2;q^2)_n}, \qquad \rho(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(q;q^2)_n}.$$

Their modular properties remained mysterious for 80 years until Zwegers [Zwe02] discovered their nonholomorphic modular completions in 2002. In today's terminology, Ramanujan's mock theta functions are holomorphic parts of weight $\frac{1}{2}$ harmonic Maaß forms. In other words, mock modular forms such as Ramanujan's mock theta functions are preimages of (weakly) holomorphic modular forms under the shadow operator $\xi_{\frac{1}{2}}$. Following unpublished work by Zagier and Zwegers, work of Alexandrov, Banerjee, Manschot and Pioline [ABMP18], and of Nazaroglu [Naz18], the idea is to extend the search for preimages under the shadow operator inductively, which leads to the notion of higher depth (mixed) mock modular forms (see Section IV.1 and [BFOR17, Definition 13.2]).

Many of the (depth one) mock theta functions are specializations of

$$\mathcal{R}(\alpha, \beta; q) := \sum_{n \geq 0} \frac{(\alpha\beta)^n q^{n^2}}{(\alpha q; q)_n (\beta q; q)_n},$$

up to the additon of a modular form. We let $\zeta := e^{2\pi i z}$ with $z \in \mathbb{C}$, and let

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{(q; q)_m}{(q; q)_{m-n}(q; q)_n}$$

be the *q-binomial coefficient.* We define the functions

$$f_1(z, \tau) := (1 + \nu(q)) \left(1 + \frac{\zeta}{(1 - \zeta)(1 + q)} \mathcal{R}\left(\zeta, -q; q^2\right) \right),$$

$$f_2(z, \tau) := \phi(q) \left(1 + \frac{\zeta}{(1 - \zeta)(1 + q^2)} \mathcal{R}\left(\zeta, -q^2; q^2\right) \right),$$

$$f_3(z, \tau) := \rho(q) \left(1 + \frac{\zeta}{(1 - \zeta)(1 - q)} \mathcal{R}\left(\zeta, q; q^2\right) \right),$$

and refer to Definition IV.2.4 for the notion of higher depth mock theta functions.

Theorem I.2.4 (Theorems IV.1.1, IV.1.2). *Let ζ be a root of unity. Then the functions f_j for $j \in \{1, 2, 3\}$ are each mock theta functions of depth two with a natural modular completion provided in Section IV.2.3. Furthermore, we have the following representations as double-sum q-series:*

(1) *The function f_1 can be written as*

$$f_1(z, \tau) = \left(1 + q^{-1} \right) \sum_{m,n \geq 0} (-1)^n \, q^{2n^2} \zeta^{n+m} \begin{bmatrix} m + n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n}}{\zeta}; q^2 \right)_m}{\left(1 + q^{2n-1} \right) \left(-q; q^2 \right)_{m+2n}}.$$

(2) *The function f_2 can be written as*

$$f_2(z,\tau) = 2 \sum_{m,n \geq 0} (-1)^n \, q^{2n^2+n} \zeta^{n+m} \begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n+1}}{\zeta}; q^2\right)_m}{(1+q^{2n})\,(-q^2; q^2)_{m+2n}}.$$

(3) *The function f_3 can be written as*

$$f_3(z,\tau) = \sum_{m,n \geq 0} (-1)^n \, q^{2n^2+n} \zeta^{n+m} \begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n+1}}{\zeta}; q^2\right)_m \left(1-q^{-1}\right)}{(1-q^{2n-1})\,(q; q^2)_{m+2n}}.$$

The proof uses the known modular completions of ν, ϕ, ρ, and $\mathcal{R}$ as well as a product formula by Srivastava [Sri87].

I.2.4 Eisenstein series of even weight $k \geq 2$ and integral binary quadratic forms

The fourth main chapter introduces the language of integral binary quadratic forms, which enables us to construct and inspect new modular objects. We let $\mathcal{Q}_D$ be the set of integral binary quadratic forms

$$Q(x,y) = [a,b,c](x,y) = ax^2 + bxy + cy^2$$

of discriminant $D := b^2 - 4ac \in \mathbb{Z}$. The modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$ by linear subsitution, that is

$$(Q \circ \left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{smallmatrix}\right))(x,y) := Q\left(\alpha_{11}x + \alpha_{12}y, \alpha_{21}x + \alpha_{22}y\right).$$

If we evaluate at $(x,y) = (\tau, 1)$, we obtain (recall equation (I.1))

$$(Q \circ \gamma)(\tau, 1) = j(\gamma, \tau)^2 Q(\gamma\tau, 1)$$

for every $\gamma \in \Gamma$. Using the usual averaging technique and assuming absolute convergence hence yields modular objects such as Zagier's $f_{k,D}$-function (see equation (I.5)).

Suppose that D is not a square, and let

$$\mathrm{sgn}([a,b,c]) := \mathrm{sgn}(a), \qquad \mathrm{Re}(s) > 1 - \frac{k}{2}, \qquad k \in 2\mathbb{N},$$

d be a fundamental discriminant dividing D, and χ_d be the extend level 1 genus character [GKZ87, Proposition 1]. Utilizing Hecke's trick and an idea of Parson [Par93], Zagier's construction from equation (I.5) extends to the function

$$\mathcal{E}_{k,D}(\tau, s) := \sum_{Q \in \mathcal{Q}_D} \chi_d(Q) \frac{\mathrm{sgn}(Q)^{\frac{k}{2}} v^s}{Q(\tau, 1)^{\frac{k}{2}} |Q(\tau, 1)|^s}.$$

Depending on the sign of D, $\mathcal{E}_{k,D}(\tau,s)$ is an elliptic ($D < 0$), parabolic ($D = 0$), or hyperbolic ($D > 0$) Eisenstein series with respect to τ. It is natural to investigate wether $\mathcal{E}_{k,D}$ posseses an analytic continuation to $s = 0$ in weight $k = 2$. If $D \leq 0$ then the continuation of $\mathcal{E}_{2,D}$ to $s = 0$ exists and is given by a harmonic ($D = 0$) resp. polar harmonic ($D < 0$) Maaß form essentially[3]. The main result of Chapter V completes the picture by adding the hyperbolic case. We recall equations (I.7), (I.8), and let Γ_Q be the stabilizer of Q giving rise to the projection $\Gamma_Q \backslash S_Q$ of S_Q into a fundamental domain for Γ. Additionally, we let

$$E_2^*(\tau) := 1 - 24 \sum_{n \geq 1} \sum_{d \mid n} d q^n - \frac{3}{\pi v}$$

be the completed Eisenstein series of weight 2 (see [BFOR17, Section 6.1.1] for example), let j be the Hauptmodul for Γ, and j_m be the Duke–Jenkins [DJ08] basis of weight 0 (see Section VIII.1 for their description as well).

Theorem I.2.5 (Theorem V.1.1). *Suppose that $D > 0$ is not a square, and $k = 2$. Then, the function $\mathcal{E}_{2,D}(\tau,s)$ can be analytically continued to $s = 0$ and the continuation is given by*

$$\lim_{s \to 0} \mathcal{E}_{2,D}(\tau,s) = \frac{-2}{D^{\frac{1}{2}}} \sum_{m \geq 0} \sum_{Q \in \mathcal{Q}_D/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} (j_m(w) - E_2^*(\tau)) \frac{|\mathrm{d}w|}{\mathrm{Im}(w)} \, q^m$$

for any $\tau \in \mathbb{H}$. Furthermore, if v is sufficiently large, that is τ is located above the net of geodesics E_D, then we have

$$\lim_{s \to 0} \mathcal{E}_{2,D}(\tau,s) = \frac{-2}{D^{\frac{1}{2}}} \sum_{Q \in \mathcal{Q}_D/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} \left(\frac{\mathbb{D}(j)(\tau)}{j(w) - j(\tau)} - E_2^*(\tau) \right) \frac{|\mathrm{d}w|}{\mathrm{Im}(w)}.$$

This result follows calculating the Fourier expansion of $\mathcal{E}_{k,D}$ using work of Zagier [Zag75] as well as of Duke, Imamoḡlu, and Tóth [DIT11]. As a byproduct, we obtain the Fourier expansion of $\mathcal{E}_{k,D}$ for any even weight $k \geq 4$ at $s = 0$.

Theorem I.2.6 (Theorem V.1.2). *Suppose that $D > 0$ is not a square, and $k \geq 4$ is even. Moreover, let $G_m(\tau,s)$ be Niebur's Poincaré series[4]. Then, we have*

$$\mathcal{E}_{k,D}(\tau,0) = \frac{(-1)^{\frac{k}{2}} 2\pi^{\frac{k}{2}}}{D^{\frac{k}{4}} \Gamma\left(\frac{k}{4}\right)^2} \sum_{m \geq 1} m^{\frac{k}{2}-1} \sum_{Q \in \mathcal{Q}_D/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} G_{-m}\left(w, \frac{k}{2}\right) \frac{|\mathrm{d}w|}{\mathrm{Im}(w)} \, q^m.$$

These results are connected to further work by Duke, Imamoḡlu, and Tóth as well, compare [DIT10, (8), (16)].

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[3]We summarize the elliptic case, and a slightly more special variant of the elliptic case in Chapter V.

[4]See equation (V.1), Definition VI.2.4. Up to normalization, also see [BFOR17, equation (13.4)].

I.2.5 Locally harmonic Maaß forms of positive even weight

This fifth main chapter continues the work of Chapter V. We observe that the analytic continuation of $\mathcal{E}_{2,D}(\tau, s)$ to $s = 0$ from Chapter V coincides with a locally harmonic Maaß form whenever τ is located above the net of geodesics E_D. Being more precise, this form is given by

$$\tau \mapsto \sum_{Q \in \mathcal{Q}_D/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} \left(\frac{\mathbb{D}(j)(\tau)}{j(w) - j(\tau)} - E_2^*(\tau) \right) \frac{|\mathrm{d}w|}{\mathrm{Im}(w)}, \qquad \tau \in \mathbb{H} \setminus E_D,$$

and its locality is caused by the Γ-orbits of the integration variable inside the cycle integral. Consequently, this twisted trace of cycle integrals has the exceptional set E_D as a function of τ again.

A natural question now is to ask for the obstructions of $\lim_{s \to 0} \mathcal{E}_{2,D}(\tau, s)$ and $\mathcal{E}_{k,D}(\tau, 0)$ ($k > 2$) to coincide with these twisted traces of cycle integrals in the other connected components of $\mathbb{H} \setminus E_D$. We show that this question boils down to relate the sign functions $\mathrm{sgn}(Q)$, $\mathrm{sgn}(Q_\tau)$, and $\mathbb{1}_Q(\tau)$ to each other. Here, $\mathbb{1}_Q$ is the characteristic function of the connected component enclosed by the Heegner geodesic S_Q and the real interval with endpoints given by the two zeros of Q (see [Sch18, Corollary 5.3.5], [Mat20a, p. 8], Section VI.2). Explicity, we find that (recall equation (I.8), compare Lemma VI.2.1)

$$\mathrm{sgn}(Q_\tau) = \mathrm{sgn}(Q)(1 - 2\mathbb{1}_Q(\tau)),$$

and consequently define

$$\widehat{\mathcal{E}}_{k,D}(\tau, s) := \sum_{Q \in \mathcal{Q}_D} \chi_d(Q) \frac{\mathrm{sgn}(Q_\tau)^{\frac{k}{2}} v^s}{Q(\tau, 1)^{\frac{k}{2}} |Q(\tau, 1)|^s} \tag{I.10}$$

under the same assumptions on D, k, d and s as in the previous section. We further recall a Poincaré series of Petersson [Pet50] (recall equation (I.2))

$$\begin{aligned}
\mathbb{P}_k(z_1, z_2) &:= \mathrm{Im}(z_2)^{k-1} \sum_{\gamma \in \Gamma} \left(\frac{1}{(z_1 - z_2)(z_1 - \overline{z_2})^{k-1}} \right) \Big|_{k, z_1} \gamma \\
&= \mathrm{Im}(z_2)^{k-1} \sum_{\gamma \in \Gamma} \left(\frac{1}{(z_1 - z_2)(z_1 - \overline{z_2})^{k-1}} \right) \Big|_{2-k, z_2} \gamma.
\end{aligned} \tag{I.11}$$

Here, the second subscript of the slash-operator indicates on which variable it acts. By work of Bringmann and Kane [BK20], $\mathbb{P}_k(z_1, \cdot)$ is a polar harmonic Maaß form of weight $2 - k$, while Petersson proved in [Pet50] that $\mathbb{P}_k(\cdot, z_2)$ is a meromorphic modular form of weight k without a pole at the cusp.

We call a function $f \colon \mathbb{H} \to \mathbb{C}$ a *local cusp form of weight k* if it behaves like a cusp form of weight k outside E_D, but has jumping singularities on E_D additionally, compare Sections VI.2, VII.1. Then, we offer the following result.

Theorem I.2.7 (Theorem VI.1.1). *Let $0 < k \equiv 2 \pmod 4$, and $\tau \in \mathbb{H} \setminus E_D$. Let $D > 0$ be a non-square discriminant, and d be a positive fundamental discriminant dividing D.*

(1) *The function $\widehat{\mathcal{E}}_{2,D}(\tau, 0)$ is a locally harmonic Maaß form of weight 2 for Γ with exceptional set E_D as a function of τ.*

(2) *If $2 < k \equiv 2 \pmod 4$ then $\widehat{\mathcal{E}}_{k,D}(\tau, 0)$ is a local cusp form of weight k for Γ with exceptional set E_D as a function of τ.*

(3) *Moreover, we have the alternative representation*

$$\widehat{\mathcal{E}}_{k,D}(\tau, 0)$$
$$= \sum_{Q \in \mathcal{Q}_D / \Gamma} \chi_d(Q) \begin{cases} \frac{-2}{D^{\frac{1}{2}}} \int_{\Gamma_Q \backslash S_Q} \left(\frac{\mathbb{D}(j)(\tau)}{j(z) - j(\tau)} - E_2^*(\tau) \right) Q(z,1)^{-1} \mathrm{d}z & \text{if } k = 2, \\ \frac{(-1)^k \Gamma(k) C_1(k) C_2(k)}{2^{\frac{k}{2} - 2} \Gamma\left(\frac{k}{4}\right)^2} \int_{\Gamma_Q \backslash S_Q} \mathbb{P}_k(\tau, z) Q(z,1)^{-\frac{k}{2}} \mathrm{d}z & \text{if } k > 2, \end{cases}$$

where $C_1(k)$ and $C_2(k)$ are explicit constants provided in Section VI.4.

As discussed at the end of the previos section (and proven in [BKK15, Lemma 5.1 (1)]), we have $\mathbb{1}_Q(\tau) \neq 0$ for only finitely many quadratic forms Q and for any fixed $\tau \in \mathbb{H} \setminus E_D$. In other words, the part of $\widehat{\mathcal{E}}_{k,D}(\tau, 0)$ containing $\mathbb{1}_Q$ yields the local part of our local cusp forms, which is now a local rational function instead of a local polynomial due to the opposite sign of the weight. This parallels Knopp's [Kno78, Kno81] extension of period polynomials to rational period functions. And indeed, local polynomials and period polynomials are closely related (see [BKK15, Section 8]).

I.2.6 A modular framework of functions of Knopp and indefinite binary quadratic forms

In the sixth main chapter, we inspect a 30-year old function introduced by Knopp [Kno90] in the course of constructing a term-by-term lift of Zagier's $f_{k,D}$-function (recall equation (I.5)) under the Bol operator, which is made precise in Proposition VII.3.1. However, averaging his result over $Q \in \mathcal{Q}_D$ would yield a divergent series, and hence Knopp changed the sign of k in his lift to enforce convergence. This leads to the function

$$\psi_{k+1,D}(\tau) := \sum_{Q \in \mathcal{Q}_D} \frac{\mathrm{Log}\left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right)}{Q(\tau, 1)^{k+1}}, \qquad \alpha_{[a,b,c]}^{\pm} := \frac{-b \pm \sqrt{D}}{2a} \in \mathbb{R},$$

where $k \in 2\mathbb{N}$, $D > 0$ is a non-square discriminant, and Log denotes the principal branch of the complex logarithm. Although the Bol operator intertwines with the slash operator, the function $\psi_{k+1,D}$ is not modular due to Knopp's sign change of k. Instead,

the obstruction towards modularity of $\psi_{k+1,D}$ (see Proposition VII.3.1 (3), correcting a typo in [Kno90, (4.6)]) satisfies the so-called period relations, which characterize period polynomials in the space $\mathbb{C}[X]$, see [Kno90, (2.2), p. 334]. However, this obstruction is neither a polynomial nor a rational function.

The first main result of this chapter provides a completion of $\psi_{k+1,D}$ to a bimodular form[5], which is explicitly given by

$$\Omega_{k+1,D}(\tau, w) := \sum_{Q \in \mathcal{Q}_D} \frac{\mathrm{Log}\left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+}\right) - \mathrm{Log}\left(\frac{w - \alpha_Q^-}{w - \alpha_Q^+}\right) + \pi i\, \mathrm{sgn}(Q) + 2i \arctan\left(\frac{Q_w}{\sqrt{D}}\right)}{Q(\tau, 1)^{k+1}},$$

where $w \in \mathbb{H}^- := \{z \in \mathbb{C} \colon \mathrm{Im}(z) < 0\}$. We prove the following properties of $\Omega_{k+1,D}$.

Theorem I.2.8 (Theorem VII.1.1). *Let $\tau \in \mathbb{H}$, $w \in \mathbb{H}^-$.*

(1) *The function $\Omega_{k+1,D}$ is bimodular of weight $(2k+2, 0)$ that is*

$$\Omega_{k+1,D}(\tau + 1, w + 1) = \Omega_{k+1,D}(\tau, w), \quad \Omega_{k+1,D}\left(-\frac{1}{\tau}, -\frac{1}{w}\right) = \tau^{2k+2}\Omega_{k+1,D}(\tau, w).$$

(2) *We have $\lim_{w \to -i\infty} \Omega_{k+1,D}(\tau, w) = \psi_{k+1,D}(\tau)$.*
(3) *We have $\lim_{\tau \to i\infty} \Omega_{k+1,D}(\tau, w) = 0$.*
(4) *The functions $\Omega_{k+1,D}$ are holomorphic with respect to τ and antiholomorphic with respect to w.*
(5) *We have that $\Omega_{k+1,D}(\tau, \overline{\tau}) = 0$.*

In the course of proving part (5), we encounter the local cusp forms (see equations (VII.7), (VII.20))

$$g_{k+1,D}(\tau) := \widehat{\mathcal{E}}_{2k+2,D}(\tau, 0) = \sum_{Q \in \mathcal{Q}_D} \frac{\mathrm{sgn}\left(Q_\tau\right)}{Q(\tau, 1)^{k+1}} \tag{I.12}$$

from Chapter VI, where we choose $d = 1$ in equation (I.10). We observe that $g_{k+1,D}$ is an "odd analogue" of $f_{k,D}$, which motivates to investigate $g_{k+1,D}$ along the lines of [BKK15]. To this end, we define the "even analogue" (see Section VII.1)

$$\mathcal{G}_{-k,D}(\tau) := \frac{1}{2} \sum_{Q \in \mathcal{Q}_D} Q(\tau, 1)^k \beta\left(\frac{Dv^2}{|Q(\tau, 1)|^2}; k + \frac{1}{2}, \frac{1}{2}\right), \tag{I.13}$$

---

[5]This terminology was introduced by Stienstra and Zagier [SZ06]. A definition can be found in [BKMN21, Footnote 7].

of $\mathcal{F}_{1-k,D}$ from equation (I.6). Like $\mathcal{F}_{1-k,D}$, the function $\mathcal{G}_{-k,D}$ is well-defined for $\tau \in \mathbb{H} \setminus E_D$ (see Proposition VII.5.1). We show that both Eichler integrals (recall equation (I.4)) of $g_{k+1,D}$ exist on its exceptional set E_D (recall equation (I.7), compare Proposition VII.4.4). Then, we offer the following properties of $\mathcal{G}_{-k,D}$.

Theorem I.2.9 (Theorem VII.1.2).

(1) *The function $\mathcal{G}_{-k,D}$ is a locally harmonic Maaß form of weight $-2k$ with continuously (however not differentially) removable singularities on E_D.*

(2) *If $\tau \in \mathbb{H} \setminus E_D$, then we have, with c_∞ defined in equation (VII.23),*

$$\mathcal{G}_{-k,D}(\tau) = c_\infty - \frac{D^{k+\frac{1}{2}}(2k)!}{(4\pi)^{2k+1}}\mathcal{E}_{g_{k+1,D}}(\tau) + D^{k+\frac{1}{2}}g^*_{k+1,D}(\tau).$$

The last main result of Chapter VII realizes $\mathcal{G}_{-k,D}$ as an output of a scalar-valued theta lift by modifying a construction of Bringmann, Kane, and Viazovska [BKV13]. We define our theta lift $\mathfrak{L}^*_{-k}$ as in Section VII.1, and the Maaß–Poincaré series $\mathcal{P}_{\frac{1}{2}-k,m}$ as in equation (VII.13)). Then, we have the following result.

Theorem I.2.10 (Theorem VII.1.3). *Let $\tau \in \mathbb{H} \setminus E_D$. We have*

$$\mathfrak{L}^*_{-k}\left(\mathcal{P}_{\frac{1}{2}-k,D}\right)(\tau) = \frac{D^{\frac{1}{4}-\frac{k}{2}}k!}{3\Gamma\left(k+\frac{1}{2}\right)(4\pi)^{\frac{k}{2}+\frac{1}{4}}}\mathcal{G}_{-k,D}(\tau).$$

We describe and discuss the picture arising from some of the results from Chapters V to VII in Section I.3.

I.2.7 Local weak Maaß forms and Eichler–Selberg type relations for negative weight vector-valued mock modular forms

The content of the seventh and last main chapter is twofold. On one hand, we prove the existence of so called Eichler–Selberg type relations in a broader context. More precisely, we extend earlier results by Mertens [Mer16] in weights $\frac{1}{2}$, $\frac{3}{2}$ and for scalar-valued functions to weights greater than two and for vector-valued functions. On the other hand, we inspect the connection of local weak Maaß forms (see Definition VIII.2.2) to theta lifts. This is motivated by work of Hövel [Höv12], who constructed such forms in weight 0 using regularized theta lifts in a vector-valued framework. His approach is independent from the scalar-valued and negative weight case of Bringmann, Kane, and Kohnen [BKK15]. Both results appeared around the same time as well.

Let L be an even lattice of signature (r,s) with quadratic form Q and dual lattice L'. We denote by $\mathrm{Gr}(L)$ the Grassmannian of r-dimensional subspaces of $L \otimes \mathbb{R}$, and by

$\boldsymbol{\lambda_z}$, $\boldsymbol{\lambda_{z^\perp}}$ the orthogonal projections of $\boldsymbol{\lambda} \in L + \mu$ onto the linear subspaces spanned by $\boldsymbol{z}$, $\boldsymbol{z}^\perp$ (the orthogonal complement of $\boldsymbol{z}$ with respect to $(\cdot, \cdot)_Q$) respectively. To obtain different weights while preserving modularity of the theta kernel, we let $p_r \colon \mathbb{R}^{r,0} \to \mathbb{C}$, and $p_s \colon \mathbb{R}^{0,s} \to \mathbb{C}$ be spherical polynomials, which are assumed to be homogeneous of degree d^+, $d^- \in \mathbb{N}_0$ respectively. Put $p_\otimes := p_r \otimes p_s$, and let $\psi \colon L \otimes \mathbb{R} \to \mathbb{R}^{r,s}$ be an isometry. We take the "points" $\boldsymbol{z} := \psi^{-1}(\mathbb{R}^{r,0}) \in Z$, $\boldsymbol{z}^\perp = \psi^{-1}(\mathbb{R}^{0,s})$, where $Z \subseteq \mathrm{Gr}(L)$ is the set of all such subspaces on which Q is positive definite. Letting $\mathfrak{e}_\mu$, $\mu \in {}^{L'}\!/L$, be the standard basis of $\mathbb{C}\big[{}^{L'}\!/L\big]$, we are now in position to define the *theta kernel* by

$$\Theta_L(\tau, \psi, p_\otimes) := v^{\frac{s}{2}+d^-} \sum_{\mu \in L'/L} \sum_{\boldsymbol{\lambda} \in L+\mu} p_\otimes(\psi(\boldsymbol{\lambda}))\, e^{2\pi i\left(Q(\boldsymbol{\lambda_z})\tau + Q(\boldsymbol{\lambda_{z^\perp}})\bar{\tau}\right)} \mathfrak{e}_\mu.$$

Following Borcherds [Bor98], Θ_L is modular of weight $k := \frac{r-s}{2} + d^+ - d^-$ with respect to the Weil representation ρ_L for $\mathrm{Mp}_2(\mathbb{Z})$ (see Section VIII.2). If R_κ^ℓ denotes the usual Maaß raising operator (see Subsections VI.2.5, VII.2.2, VIII.2.3,), and f is a weight $k - 2j < 0$ harmonic Maaß form with cuspidal shadow, $j \in \mathbb{N}_0$, for $\mathrm{SL}_2(\mathbb{Z})$, then we consider the *theta lift*

$$\Psi_j^{\mathrm{reg}}(f, \boldsymbol{z}) := \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle R_{k-2j}^j(f)(\tau), \overline{\Theta_L(\tau, \psi, p_\otimes)} \right\rangle v^k \mathrm{d}\mu(\tau).$$

Here, $\mathcal{F}$ is a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, $\int^{\mathrm{reg}}$ indicates Borcherds' regularization of the integral (see Section VII.2), $\mathrm{d}\mu(\tau)$ is the usual hyperbolic measure, and the dependence of the right hand side on $\boldsymbol{z}$ is captured by ψ.

Following Bruinier and Schwagenscheidt [BS21], we say that $\boldsymbol{w} \in \mathrm{Gr}(L)$ is a *special point* if $\boldsymbol{w} \in L \otimes \mathbb{Q}$. Moreover, we let $\mathscr{G}_P^+$, $\Lambda_L(\psi, p_\otimes, j)$ be as in Subsection VIII.3.3, Θ_{N^-} be as in Subsection VIII.2.7, and refer to Subsection VIII.2.6 for further notation.

Theorem I.2.11 (Theorem VIII.3.4)**.** *Let L be an even lattice of signature (r, s), $\boldsymbol{w}$ be a special point defined by the isometry ψ, and $p_\otimes$ be as before. Let $j \in \mathbb{N}$ and k be such that $2j + 2 - k > 2$. Then the function $\big[\mathscr{G}_P^+(\tau), \Theta_{N^-}(\tau, p_s)\big]_j^L - \Lambda_L(\psi, p_\otimes, j)$ is a holomorphic vector-valued modular form of weight $2j + 2 - k$ for ρ_L.*

Moving to vector-valued local weak Maaß forms (see Defintion VIII.2.2), we let $F_{m,k-2j,\mathfrak{s}}$ be the vector-valued Maaß–Poincaré series of weight $k - 2j$, index $m \in \mathbb{N}\backslash\{Q(\mu)\}$, and spectral parameter $\mathfrak{s}$ (see Subsection VIII.2.5).

Theorem I.2.12 (Theorem VIII.1.2)**.** *Suppose that L is an even isotropic lattice of signature $(2, s)$. Then the lift $\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{z})$ is a local weak Maaß form on $\mathrm{Gr}(L)$ with eigenvalue $\left(\mathfrak{s} - \frac{k}{2}\right)\left(1 - \mathfrak{s} - \frac{k}{2}\right)$ under the Laplace–Beltrami operator on $\mathrm{Gr}(L)$ (see Subsection VIII.2.2).*

The last main result of this chapter deals with an explicit scalar-valued example of the previous theorem. Namely, we restrict ourselves to signature $(1, 2)$, in which case there is a correspondence between the vector-valued framework and the usual scalar-valued setting on $\mathbb{H}$ by virtue of a result of Eichler and Zagier [EZ85, Theorem 5.4]. We choose $\mathcal{H}_\ell f_{-2\ell,N}$ as input, where $\mathcal{H}_\ell$ refers to the usual Cohen–Eisenstein series, and $f_{-2\ell,N}$ is the N-th Duke–Jenkins [DJ08] basis form of weight -2ℓ (both recalled in Section VIII.1).

Theorem I.2.13 (Theorem VIII.1.3). *The function $\Psi_j^{\mathrm{reg}}(f_{-2\ell,N}\mathcal{H}_\ell, z)$ is a local weak Maaß form on $\mathbb{H}$ for every $j \in \mathbb{N}$, $\ell \in \mathbb{N}\backslash\{1\}$, and $-m \leq N \in \mathbb{N}$ with exceptional set given by the net of Heegner geodesics $\bigcup_{D=1}^{N} E_D$, and eigenvalue $(1 - k + j)(-j) = j(j - \ell - \frac{3}{2})$ under $\Delta_{-\ell-\frac{1}{2}}$.*

At the very end of Chapter VIII, we sketch the steps to obtain Eichler–Selberg type relations for the function $f_{-2\ell,N}\mathcal{H}_\ell$. This concludes Chapter VIII, and this thesis.

I.3 Synopsis of Chapters V, VI, VII and a brief discussion

Let $k \in 2\mathbb{N}$, and $D > 0$ be a non-square discriminant. We recall the definitions of $f_{k,D}$, $\mathcal{F}_{1-k,D}$, $g_{k+1,D}$, and $\mathcal{G}_{-k,D}$ from equations (I.5), (I.6), (I.12), and (I.13). We further recall Petersson's Poincaré series $\mathbb{P}_k$ from equation (I.11), and the Maaß–Poincaré series $\mathcal{P}_{\frac{1}{2}-k,m}$ from equation (VII.13).

Note that $\mathcal{P}_{\frac{3}{2}-k,m}$ is of negative weight, and becomes the (weakly) holomorphic Poincaré series of exponential type[6] after mapping $k \mapsto 2 - k$. According to[7] [BKV13, (2.12)], both Poincaré series are related by

$$\xi_{\frac{3}{2}-k}\left(\mathcal{P}_{\frac{3}{2}-k,D}\right) = \left(k - \frac{1}{2}\right)\mathcal{P}_{k+\frac{1}{2},D}.$$

Furthermore, Bringmann, Kane and Viazovska [BKV13] constructed certain (scalar-valued) theta lifts, which lift the Poincaré series $\mathcal{P}_{\frac{3}{2}-k,D}$ (resp. $\mathcal{P}_{k+\frac{1}{2},D}$) to the functions $\mathcal{F}_{1-k,D}$ (resp. $f_{k,D}$) after specializing to a certain spectral parameter there. As discussed above, Theorem I.2.10 follows by a modification of their theta lift with a slightly modified theta kernel.

In addition, Löbrich and Schwagenscheidt [LS22c, Theorem 4.2] connected $\mathbb{P}_{2k}$ from equation (I.11) and $\mathcal{F}_{1-k,D}$ from equation (I.6). Explicitly, their result implies that

$$\mathcal{F}_{1-k,D}(\tau) = (-1)^k \binom{2k-2}{k-1} \sum_{Q \in \mathcal{Q}_{D}/\Gamma} \int_{\Gamma_Q \backslash S_Q} \mathbb{P}_{2k}(z, \tau) Q(z, 1)^{k-1} \mathrm{d}z$$

---

[6]See Definition VI.2.4 and Remark VI.2.6.

[7]A proof can be found in [BFOR17, Theorem 6.11] up to normalization and sign convention of m (see Remark VI.2.6).

after summing over $Q \in \mathcal{Q}_D/\Gamma$ there. This should be compared with Theorem I.2.7 noting that we exchanged z and τ inside Petersson's Poincaré series and switched between $2k$ and $2k + 2$.

We collect these results as well as [BKK15, Theorem 7.1] (which is equation (I.9)), and some results from Chapters V to VII in the following diagram:

$$\mathcal{P}_{\frac{3}{2}-k,D} \text{ resp. } \mathcal{P}_{\frac{1}{2}-k,D}$$
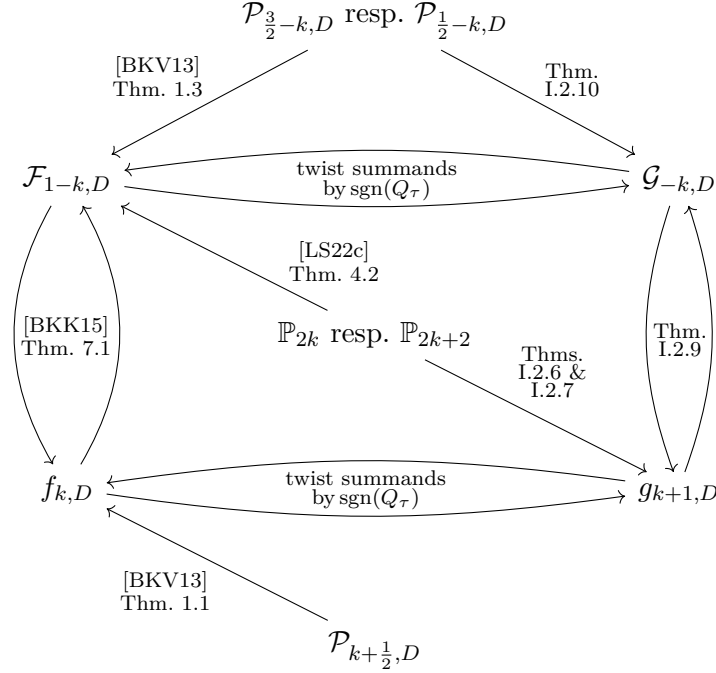
Figure I.1: Synopsis of Chapters V, VI, VII

On one hand, it would be interesting to realize $g_{k+1,D}$ as a theta lift of some Poincaré series as well. However, one can not simply twist the theta kernel used in [BKV13, Theorem 1.1] by $\operatorname{sgn}(Q_\tau)$, because this would yield a function with singularities in a dense subset of $\mathbb{H}$.

On the other hand, the diagram suggests that there might exist a connection between $f_{k,D}$ and $\mathcal{G}_{-k,D}$ evolving from a two variable automorphic form $\mathscr{P}_k$, where $\mathscr{P}_k$ parallels the role of $\mathbb{P}_k$ (see Remark VI.4 as well). This connection could be realized as a twisted trace of cycle integrals again, which is motivated by [Koh85, Proposition 7], or in terms of some other "bridging concept". If such a connection exists, one could ask naturally for the interplay between $\mathbb{P}_k$ and $\mathscr{P}_k$, which would shed some light on a theory behind locally harmonic Maaß forms from a more general perspective.

Part A

Harmonic Maaß forms

Chapter II

Polar harmonic Maaß forms and holomorphic projection

This chapter is based on a paper [MMR22] of the same title published in *The International Journal of Number Theory.* This is joint work with Dr. Joshua Males and Prof. Dr. Larry Rolen.

II.1   Introduction and statement of results

A recent paper by Mertens, Ono, and Rolen [MOR21] defined and investigated a new type of mock modular form. Their construction is motivated by work of Hecke [Hec27], whose results imply that the functions

$$\frac{1}{2}L\left(1-k,\phi\right) + \sum_{n\geq 1}\left(\sum_{d|n}\phi(d)d^{k-1}\right)q^n, \qquad \sum_{n\geq 1}\left(\sum_{d|n}\phi\left(\frac{n}{d}\right)d^{k-1}\right)q^n$$

are holomorphic weight k modular forms on $\Gamma_0(N)$ with Nebentypus ϕ if $k > 2$, where ϕ is any primitive Dirichlet character of modulus N satisfying $\phi(-1) = (-1)^k$. Here and throughout, we let $\tau = u + iv \in \mathbb{H}$ and $q := e^{2\pi i\tau}$. The notation $L(s,\phi)$ refers to the Dirichlet L-function of ϕ. Mertens, Ono, and Rolen focussed on the case $k = 2$, and to this end defined a different class of twisted and restricted versions of classical divisor sums

$$\sigma_{k-1}(n) := \sum_{d|n}d^{k-1}.$$

Since $\sigma_{k-1}(n)$ is a Fourier coefficient of the classical holomorphic Eisenstein series E_k for even $k \geq 2$, they called these "mock modular Eisenstein series". Following the setting of [MOR21], let ψ be any non-trivial Dirichlet character of modulus $M_\psi > 1$, define the set of admissible "small" divisors

$$D_n := \left\{d \mid n \,:\, 1 \leq d \leq \frac{n}{d} \text{ and } d \equiv \frac{n}{d} \,(\mathrm{mod}\ 2)\right\},$$

and the *small divisor function*

$$\sigma_{1,\psi}^{\mathrm{sm}}(n) := \sum_{d \in D_n} \psi\left(\frac{\left(\frac{n}{d}\right)^2 - d^2}{4}\right) d.$$

In addition, let

$$\lambda_\psi := \frac{1 - \psi(-1)}{2} \in \{0, 1\},$$

depending on the parity of ψ, let χ_{-4} be the unique odd character of modulus 4, and we recall Shimura's theta function

$$\theta_\psi(\tau) := \frac{1}{2} \sum_{n \in \mathbb{Z}} \psi(n) n^{\lambda_\psi} q^{n^2}.$$

Then the main result of [MOR21] states that the function

$$\mathcal{E}^+(\tau) := \frac{1}{\theta_\psi(\tau)} \left(\alpha_\psi E_2(\tau) + \sum_{n \geq 1} \sigma_{1,\psi}^{\mathrm{sm}}(n) q^n \right)$$

can be completed to a polar harmonic Maaß form of weight $\frac{3}{2} - \lambda_\psi$ on $\Gamma_0(4M_\psi^2)$ with Nebentypus $\overline{\psi} \cdot \chi_{-4}^{\lambda_\psi}$, where α_ψ is an implicit constant to ensure a certain growth condition. That is, it has the transformation and analytic properties of a harmonic Maaß form, but it may have poles on the upper half plane arising from $\theta_\psi(\tau)$ in the denominator. Similar ideas have been utilized for specific examples before by Andrews, Rhoades, and Zwegers [ARZ13], and by Bringmann, Kane, and Zwegers [BKZ14] for instance. Additionally, the authors of [MOR21] presented some special choices of ψ where their polar harmonic Maaß forms happen to have no poles on $\mathbb{H}$, and offer a *p*-adic property of $\mathcal{E}^+$ for primes $p > 3$.

A natural question is whether there are more classes of small divisor functions for which a similar phenomenon to the setting of [MOR21] appears. Our two main results give such generalizations. We let χ be a second Dirichlet character of modulus M_χ and define our small divisor function by

$$\sigma_{2,\chi}^{\mathrm{sm}}(n) := \sum_{d \in D_n} \chi\left(\frac{\frac{n}{d} - d}{2}\right) \psi\left(\frac{\frac{n}{d} + d}{2}\right) d^2.$$

We stipulate that ψ is odd and fixed throughout, and thus omit the dependency of $\sigma_{2,\chi}^{\mathrm{sm}}$ on ψ. We moreover define $\mathcal{F}(\tau) := \mathcal{F}^+(\tau) + \mathcal{F}^-(\tau)$, where

$$\mathcal{F}^+(\tau) := \frac{1}{\theta_\psi(\tau)} \sum_{n \geq 1} \sigma_{2,\chi}^{\mathrm{sm}}(n) q^n, \qquad \mathcal{F}^-(\tau) := \frac{i}{\pi\sqrt{2}} \int_{-\overline{\tau}}^{i\infty} \frac{\theta_\chi(w)}{(-i(w+\tau))^{\frac{3}{2}}} \mathrm{d}w.$$

We obtain the following result.

Theorem II.1.1. *If χ is even and non-trivial then the function $\mathcal{F}$ is a polar harmonic Maaß form of weight $\frac{3}{2}$ on $\Gamma_0\big(\operatorname{lcm}\big(4M_\chi^2, 4M_\psi^2\big)\big)$ with Nebentypus $\overline{\chi} \cdot (\psi \cdot \chi_{-4})^{-1}$. Its shadow is given by $\frac{1}{2\pi}\theta_{\overline{\chi}}$.*

Including the possibility of $\chi = \mathbb{1}$, where a constant term arises, requires adjustments either in the holomorphic part $\mathcal{F}^+$ or in the nonholomorphic part $\mathcal{F}^-$. On one hand, we may subtract the arising constant term from $\mathcal{F}^-$ again. The shadow of $\mathcal{F}$ would be given by the partial theta function

$$\frac{1}{2\pi}\sum_{n\geq 1}q^{n^2} = \frac{1}{2\pi}\theta_{\mathbb{1}}(\tau) - \frac{1}{4\pi}.$$

In [BR16], it is proved that all one-dimensional partial theta functions are (strong) quantum modular forms, which were first introduced by Zagier [Zag10]. Furthermore, such partial theta functions are related to Appell–Lerch sums of level 2, to meromorphic Jacobi forms, and closely to false theta functions as well. An exposition on the former two connections was given by Bringmann, Zwegers, and Rolen in [BRZ16]. To state their results, let $\zeta := e^{2\pi i z}$, and

$$\vartheta(z;\tau) := \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}\zeta^n = -i\zeta^{-\frac{1}{2}}q^{\frac{1}{8}}\prod_{j=0}^{\infty}\left(1 - q^{j+1}\right)\left(1 - \zeta q^j\right)\left(1 - \zeta^{-1}q^{j+1}\right)$$

be the standard Jacobi theta function of index and weight $\frac{1}{2}$. The second equality is the Jacobi triple product identity, from which we deduce that $\vartheta(z;\tau)$ has zeros precisely in $\mathbb{Z}\tau + \mathbb{Z}$ as a function of z and all zeros are simple. Moreover, letting $\ell \in \mathbb{N}$, the Appell–Lerch sum of level ℓ is defined by

$$A_\ell(w,z;\tau) := e^{\pi i \ell w}\sum_{n\in\mathbb{Z}}\frac{(-1)^{\ell n}q^{\ell\frac{n(n+1)}{2}}e^{2\pi i n z}}{1 - e^{2\pi i w}q^n}.$$

The results of [BRZ16] can be specialized to our setting and read as follows.

Proposition II.1.2 ([BRZ16, Corollaries 1.2 and 1.5])**.**

(1) *We have*

$$\sum_{n\geq 1}q^{n^2} = -\int_{-\frac{1}{2}}^{\frac{1}{2}}e^{2\pi i(2t)}A_2\left(t - \frac{\tau}{2}, 0; \tau\right)\mathrm{d}t.$$

(2) *Let $f_n(\tau)$ be defined by the expansion [BRZ16, equation (2.5)]*

$$\frac{1}{\vartheta(z;\tau)^2} =: \sum_{n\geq -2}f_n(\tau)(2\pi i z)^n.$$

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Then it holds that

$$f_{-1}(\tau) \sum_{n \geq 1} q^{n^2} + 2f_{-2}(\tau) \sum_{n \geq 1} nq^{n^2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{2\pi i(2t)}}{\vartheta \left(t - \frac{\tau}{2}; \tau\right)^2} \mathrm{d}t,$$

and the functions $f_{-1}(\tau)$, $f_{-2}(\tau)$ both are known to be quasimodular forms (compare [EZ85, Theorem 3.2], [BFOR17, Corollary 2.36], [BRZ16, p. 8]).

Zwegers provided the nonholomorphic completion of A_ℓ to a two-variable Jacobi form of weight 1 and matrix index $\left(\begin{smallmatrix} -\ell & 1 \\ 1 & 0 \end{smallmatrix}\right)$, see [Zwe19, Theorem 4]. Moreover, a third perspective arises from the close relation between partial and false theta functions. Bringmann and Nazaroglu described the completion of false theta functions to functions with certain Jacobi transformation properties in a recent paper [BN19]. In particular, their result includes the completion for

$$\sum_{n \in \mathbb{Z}} \mathrm{sgn}(n) q^{\frac{n^2}{2}} e^{2\pi i n z},$$

where $\mathrm{sgn}(0) := 0$.

On the other hand, we may compensate for the additional constant term by adjusting the holomorphic part $\mathcal{F}^+$. Being more precise, we define $\mathcal{G}(\tau) := \mathcal{G}^+(\tau) + \mathcal{G}^-(\tau)$, where

$$\mathcal{G}^+(\tau) := \frac{1}{\theta_\psi(\tau)} \left(\frac{1}{2} \sum_{n \geq 1} \psi(n) n^2 q^{n^2} + \sum_{n \geq 1} \sigma_{2,\mathbb{1}}^{\mathrm{sm}}(n) q^n \right),$$

$$\mathcal{G}^-(\tau) := \frac{i}{\pi\sqrt{2}} \int_{-\overline{\tau}}^{i\infty} \frac{\theta_{\mathbb{1}}(w)}{(-i(w+\tau))^{\frac{3}{2}}} \mathrm{d}w.$$

It turns out that the strategy of the proof of Theorem II.1.1 still applies, enabling us to complete the picture (regarding even χ) by the following result.

Theorem II.1.3. *The function $\mathcal{G}$ is a polar harmonic Maaß form of weight $\frac{3}{2}$ on $\Gamma_0\left(4M_\psi^2\right)$ with Nebentypus $(\psi \cdot \chi_{-4})^{-1}$. Its shadow is given by $\frac{1}{2\pi}\theta_{\mathbb{1}}$.*

Rouse and Webb showed in [RW15] that the only modular forms on $\Gamma_0(N)$ with integer Fourier coefficients and no zeros on $\mathbb{H}$ are eta quotients. In addition, Mersmann and Lemke-Oliver completed the classification of theta functions which may be written as such an eta quotient. We cite their results in the formulation of [MOR21, Theorem 1.2].

Theorem II.1.4 ([LO13, Mer91])**.** *The only nontrivial primitive characters ψ for which θ_ψ is an eta quotient are contained in the set of Kronecker characters*

$$\Psi := \left\{ \left(\frac{D}{\cdot}\right) : D \in \{-8, -4, -3, 2, 12, 24\} \right\}.$$

Combining Theorems II.1.1, II.1.3, and II.1.4 immediately yields the following corollary. (Recall that there is no odd character of modulus 2.)

Corollary II.1.5. *If $\psi \in \Psi \setminus \{\left(\frac{2}{\cdot}\right)\}$ is odd then $\mathcal{F}^+$ and $\mathcal{G}^+$ are mock theta functions.*

The result of Theorem II.1.4 motivates us to investigate the odd choices $\psi \in \Psi \setminus \{\left(\frac{2}{\cdot}\right)\}$ in greater detail. To this end, let $H(n)$ be the Hurwitz class number, counting the weighted number of classes of positive definite binary quadratic forms of discriminant $-n$. Phrased in todays terminology, Zagier discovered in [Zag75] that

$$\mathcal{H}^+(\tau) := -\frac{1}{12} + \sum_{n \geq 1} H(n)q^n.$$

can be completed to a harmonic Maaß form $\mathcal{H}(\tau) := \mathcal{H}^+(\tau) - \frac{1}{4}\mathcal{G}^-(\tau)$ of weight $\frac{3}{2}$ on $\Gamma_0(4)$. To see this, one may rewrite $\mathcal{G}^-$ as in Lemma II.4.1 below, and compare [BFOR17, Theorem 6.3] for instance. The function $\mathcal{H}$ is then often called *Zagier's weight $\frac{3}{2}$ nonholomorphic Eisenstein series*. A standard computation using the Sturm bound yields the following.

Corollary II.1.6. *Let $\psi = \chi_{-4}$. Then we have*

$$\sum_{n \geq 1} \sigma_{2,\mathbb{1}}^{sm}(8n)q^{8n} = -4\theta_\psi(\tau) \sum_{n \geq 1} H(8n-1)q^{8n-1}.$$

Or in other words, by definition of $\psi = \chi_{-4}$ and θ_ψ,

$$\sigma_{2,\mathbb{1}}^{sm}(8n) = 4 \sum_{\substack{j \geq 1 \\ (2j-1)^2 < 8n}} (-1)^j (2j-1) H\left(8n - (2j-1)^2\right).$$

In addition, we note that both the coefficients of $\theta_\psi \mathcal{H}^+$ and the values of $\sigma_{2,\mathbb{1}}^{sm}$ grow at most polynomially. Based on the observations from the preceeding discussion we inquire the following.

Question. *For every $\psi \in \Psi \setminus \{\left(\frac{2}{\cdot}\right)\}$, do there exist numbers $0 \neq C, t \in \mathbb{Q}$, such that*

$$C \cdot \mathcal{G}^+(t\tau)$$

generates a linear combination of Hurwitz class numbers on some arithmetic progression?

In the course of proving Theorem II.1.1 and Theorem II.1.3 we compute the holomorphic projection of a product similar to the structure of mixed harmonic Maaß forms. However, we just rely on translation invariance and impose suitable growth conditions

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on the coefficients to ensure convergence. The resulting expression can be written in terms of a particular class of Jacobi polynomials $\mathcal{P}_r^{(a,b)}$ (sometimes also referred to as *hypergeometric polynomials.*). For $r \in \mathbb{N}_0$, these polynomials are defined by

$$
\begin{aligned}
\mathcal{P}_r^{(a,b)}(z) &:= \frac{\Gamma(a+r+1)}{r!\,\Gamma(a+b+r+1)} \sum_{j=0}^{r} \binom{r}{j} \frac{\Gamma(a+b+r+j+1)}{\Gamma(a+j+1)} \left(\frac{z-1}{2}\right)^j \\
&= \frac{\Gamma(a+r+1)}{r!\,\Gamma(a+1)} {}_2F_1\left(-r, a+b+r+1, a+1, \frac{1-z}{2}\right),
\end{aligned}
\tag{II.1}
$$

which can be found in [GR07, item 8.962] for example. Here, ${}_2F_1$ denotes the usual Gauß hypergeometric function. Then we have the following result.

Proposition II.1.7. *Let $k_f \in \mathbb{R} \setminus \mathbb{N}$, $k_g \in \mathbb{R} \setminus (-\mathbb{N})$, such that $\kappa := k_f + k_g \in \mathbb{N}_{\geq 2}$. Let $\alpha(m)$, $\beta(n)$ be two complex sequences, and define*[1]

$$
f(\tau) := \sum_{m \geq 1} \alpha(m) m^{k_f-1} \Gamma(1-k_f, 4\pi m v) q^{-m}, \qquad g(\tau) := \sum_{n \geq 1} \beta(n) q^n.
$$

Suppose that

   (i) *the function $(fg)(r+iv)$ grows at most polynomially as $v \searrow 0$, where $r \in \mathbb{Q}$, and that*
   (ii) *the function $(fg)(iv)$ grows at most polynomially as $v \nearrow \infty$.*

Then the weight κ holomorphic projection of fg is given by

$$
\pi_\kappa(fg)(\tau)
$$
$$
= -\Gamma(1-k_f) \sum_{m \geq 1} \sum_{n-m \geq 1} \alpha(m)\beta(n) \left(n^{k_f-1} \mathcal{P}_{\kappa-2}^{(1-k_f, 1-\kappa)}\left(1 - 2\frac{m}{n}\right) - m^{k_f-1} \right) q^{n-m}.
$$

We provide two proofs of this result, which correspond to either definition of the Jacobi polynomials. The first one relies on identities of the Gauß hypergeometric function, while the second one relies on two Lemmas from [Mer16].

Remarks.

   (1) Assuming the framework of mixed harmonic Maaß forms, some variants of this result appear in [IRR14, Theorem 3.5], and [Mer16, Theorem 4.6]. Moreover, if $k_g = 2 - k_f$, then $\mathcal{P}_0^{a,b}(z) = 1$. Therefore, working with regularized holomorphic projection, our result includes [MOR21, Proposition 2.1].

---

[1] The function $\Gamma(s, z)$ denotes the incomplete Gamma function, which will be introduced in Section II.2.

(2) Note that the summation conditions imply $-1 < 1 - 2\frac{m}{n} < 1$. The asymptotic behavior of the Jacobi polynomials inside $(-1, 1)$ is well known and can be found in [GR07, item 8.965] for instance.

(3) One may choose various other special values of half integral k_f, k_g, which simplify the Jacobi polynomial and then the whole factor

$$n^{k_f - 1} \mathcal{P}_{\kappa-2}^{(1-k_f, 1-\kappa)} \left(1 - 2\frac{m}{n} \right) - m^{k_f - 1}.$$

This idea leads to other choices of polynomials $P\left(\frac{n}{d}, d\right) \in \mathbb{Q}[X, Y]$ than d^2 in the definition of $\sigma_{2,\chi}^{\mathrm{sm}}$ such that

$$\pi_\kappa \left(\sum_{n \geq 1} \sum_{d \in D_n} \chi \left(\frac{\frac{n}{d} - d}{2} \right) \psi \left(\frac{\frac{n}{d} + d}{2} \right) P\left(\frac{n}{d}, d \right) q^n \right.$$

$$\left. + \sum_{n \geq 1} \sum_{m \geq 1} \alpha\left(m^2\right) \beta\left(n^2\right) m^{2(k_f - 1)} \Gamma(1 - k_f, 4\pi m^2 v) q^{n^2 - m^2} \right) = 0.$$

We demonstrate during the proofs of Theorem II.1.1 and II.1.3 how to rewrite the corresponding generating function

$$\sum_{n \geq 1} \sum_{d \in D_n} \chi \left(\frac{\frac{n}{d} - d}{2} \right) \psi \left(\frac{\frac{n}{d} + d}{2} \right) P\left(\frac{n}{d}, d \right)$$

to obtain a choice of $P\left(\frac{n}{d}, d\right) \in \mathbb{Q}[X, Y]$, which matches the factor involving the Jacobi polynomial.

A second application of Proposition II.1.7 arises from p-adic properties of $\mathcal{F}^+$ and $\mathcal{G}^+$. Mertens, Ono and Rolen proved such a property for their mock modular Eisenstein series $\mathcal{E}^+$, see [MOR21, Theorem 1.4]. More precisely, the idea is to inspect the iterated action of the U-operator

$$\left(\sum_{n \gg -\infty} \alpha(n) q^n \right) \bigg| \, U(p) := \sum_{n \gg -\infty} \alpha(pn) q^n$$

on $\theta_\psi\left(p^{2a}\tau\right) \mathcal{E}^+(\tau)$ for every $a \in \mathbb{N}$ and $p > 3$ prime. (The notation $\sum_{n \gg -\infty}$ is explained in Lemma II.2.7.) Then they showed that this is congruent to some meromorphic modular form of weight 2. In our case Theorems II.1.1 and II.1.3 imply that the products $\theta_\psi(\tau)\mathcal{F}(\tau)$ and $\theta_\psi(\tau)\mathcal{G}(\tau)$ are modular of weight 3 with Nebentypus $\overline{\chi}$ or trivial Nebentypus respectively. Therefore, we find a different result.

Theorem II.1.8. *Let a, b, $p \in \mathbb{N}$ and suppose that p is an odd prime. Then we have*

$$\left(\theta_\psi \left(p^{2a} \tau \right) \mathcal{F}^+(\tau) \right) \Big| U \left(p^b \right) \equiv 0 \quad \left(\mathrm{mod}\ p^{\min(a,b)} \right),$$

and

$$\left(\theta_\psi \left(p^{2a} \tau \right) \mathcal{G}^+(\tau) \right) \Big| U \left(p^b \right) \equiv 0 \quad \left(\mathrm{mod}\ p^{\min(a,b)} \right).$$

The third remark on page 3 in [MOR21] states that "the generating function of $\sigma_{1,\psi}^{\mathrm{sm}}$ can be given in terms of Appell–Lerch sums as studied by Zwegers" in [Zwe02, Zwe19] (see also [Mer14b, Lemma 2]). This remark applies verbatim to the generating function of $\sigma_{2,\chi}^{\mathrm{sm}}$ as well, and we present a strategy which applies to both generating functions. Let $D_z := \frac{1}{2\pi i} \frac{\partial}{\partial z}$, giving

$$(D_z^j A_\ell)(w, z, \tau) = e^{\pi i \ell w} \sum_{n \in \mathbb{Z}} n^j \frac{(-1)^{\ell n} q^{\ell \frac{n(n+1)}{2}} e^{2\pi i n z}}{1 - e^{2\pi i w} q^n}.$$

for every integer $j \geq 0$. Then we have the following result.

Proposition II.1.9. *Suppose that χ is non-trivial and even. Additionally assume $M_\psi \mid M_\chi$. Then we have that*

$$\sum_{n \geq 1} \sigma_{2,\chi}^{sm}(n) q^n = \frac{1}{2} \sum_{b=1}^{M_\chi - 1} \chi(b)$$

$$\times \sum_{c=0}^{M_\chi - 1} \psi(b+c) q^{c(c+2b-M_\chi)} \left(M_\chi D_z + c \right)^2 A_1 \left(2M_\chi c \tau, z, 2M_\chi^2 \tau \right) \Big|_{z = (2(b+c) - M_\chi) M_\chi \tau + \frac{1}{2}}.$$

Lastly, it is also likely that the function $\sigma_{2,\chi}^{\mathrm{sm}}$ can be viewed as a Siegel theta lift in the following way. A recent paper of Bruinier and Schwagenscheidt [BS20] investigated the Siegel theta lift on Lorentzian lattices, and its connection to coefficients of mock theta functions. In an isotropic lattice of signature $(1,1)$, for example, they obtained a formula for the Siegel theta lift of a particular weakly holomorphic modular form evaluated at a certain point in the Grassmanian in terms of the sum

$$\sum_{\substack{r \in \mathbb{Z} \\ r \equiv 1 \,(\mathrm{mod}\ 2)}} H \left(4m - r^2 \right).$$

Furthermore, in [ANBMS21], Alfes-Neumann, Bringmann, Males, and Schwagenscheidt considered the same lift on a lattice of signature $(1,2)$, but with the inclusion of an

iterated Maaß raising operator acting on the weakly holomorphic modular form. There, we obtained an expression of the form (see [ANBMS21, Example 1.2])

$$\sum_{\substack{n,m\in\mathbb{Z}\\ n\equiv D \pmod 2}} \left(4D - 10n^2 - 10m^2\right) H\left(D - n^2 - m^2\right).$$

The quadratic form in the Hurwitz class number is explained by the signature being $(1,2)$ in this case, and the addition of the polynomial is a consequence of the iterated Maaß raising operators.

In view of Corollary II.1.6, our smallest divisor function seems to lie at the interface of these two situations, i.e. it is natural to expect that it can be realized as a theta lift in signature $(1,1)$ where the lift includes iterated Maaß raising operators. Perhaps this phenomenon can also be extended to further classes of small divisor functions - the theta lift construction allows one to choose many examples of mock theta functions of appropriate weight, not just the generating function for Hurwitz class numbers.

II.2 Preliminaries

II.2.1 Growth conditions and modular forms

To describe our various modular objects, we first require some terminology on growth conditions, which can be phrased at all other cusps via suitable scaling matrices.

Definition II.2.1. Let

$$f(\tau) := \sum_{n\in\mathbb{Z}} c_f(n)q^n$$

with some complex coefficients $c_f(n)$. Then we say that

(1) the function f is *holomorphic* at $i\infty$ if $c_f(n) = 0$ for every $n < 0$,

(2) the function f is of *moderate growth* at $i\infty$ if $f \in O\left(v^m\right)$ as $v \to \infty$ for some $m \in \mathbb{N}$. In other words f grows at most polynomially,

(3) the function f is of *governable growth* at $i\infty$ if there exists $P_f \in \mathbb{C}\left[q^{-1}\right]$ such that for some $\delta > 0$ we have

$$f(\tau) - P_f(\tau) \in O\left(e^{-\delta v}\right), \quad v \to \infty.$$

The polynomial P_f is called the *principal part* of f. Equivalently, f is permitted to have a pole at $i\infty$.

(4) The function f is of *linear exponential growth* if $f(\tau) \in O(e^{\delta v})$ as $v \to \infty$ for some $\delta > 0$.

To define the slash-operator, we take the principal branch of the holomorphic square root throughout.

Definition II.2.2. Let $k \in \frac{1}{2}\mathbb{Z}$, ϕ be a Dirichlet character, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z})$. Then the *slash operator* is defined as

$$
(f|_k\gamma)(\tau) := \begin{cases} \phi(d)^{-1}(c\tau + d)^{-k}f(\gamma\tau) & \text{if } k \in \mathbb{Z}, \\ \phi(d)^{-1}\left(\frac{c}{d}\right)\varepsilon_d^{2k}(c\tau + d)^{-k}f(\gamma\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases}
$$

where $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol, and

$$
\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4} \end{cases}
$$

for odd integers d.

For the sake of completeness, we define the variants of modular forms appearing in this chapter.

Definition II.2.3. Let $f \colon \mathbb{H} \to \mathbb{C}$ be a function, $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup, ϕ be a Dirichlet character, and $k \in \frac{1}{2}\mathbb{Z}$. Then we say that

(1) The function f is a *modular form* of weight k on Γ with Nebentypus ϕ if

  (i) for every $\gamma \in \Gamma$ and every $\tau \in \mathbb{H}$ we have $(f|_k\gamma)(\tau) = f(\tau)$,
  (ii) f is holomorphic on $\mathbb{H}$,
  (iii) f is holomorphic at every cusp.

  We denote the vector space of functions satisfying these conditions by $M_k(\Gamma, \phi)$.
(2) If in addition f vanishes at every cusp, then we call f a *cusp form*. The subspace of cusp forms is denoted by $S_k(\Gamma, \phi)$.
(3) If f satisfies the conditions (i) and (ii) from (1) and is allowed to have a pole at one or more cusps, then we call f a *weakly holomorphic modular form* of weight k on Γ with Nebentypus ϕ. The vector space of such functions is denoted by $M_k^!(\Gamma, \phi)$.

Furthermore, we recall the following fact which we require throughout.

Lemma II.2.4. *The following are true*

$$
\theta_\psi \in \begin{cases} M_{\frac{1}{2}}(\Gamma_0(4M_\psi^2), \psi) & \text{if } \lambda_\psi = 0, \\ S_{\frac{3}{2}}(\Gamma_0(4M_\psi^2), \psi \cdot \chi_{-4}) & \text{if } \lambda_\psi = 1. \end{cases}
$$

A proof can be found in [Iwa97, Theorem 10.10] for instance. Finally, Corollary II.1.6 follows directly from the following result.

Lemma II.2.5 (Sturm bound)**.** *Let* $f, g \in M_k(\Gamma_0(N), \phi)$, $k > 1$, *and*

$$m \coloneqq [\mathrm{SL}_2(\mathbb{Z}) \colon \Gamma_0(N)] = N \prod_{p | N} \left(1 + \frac{1}{p} \right).$$

Define $B \coloneqq \left\lfloor \frac{km}{12} \right\rfloor$, *and denote by* $c_f(n)$, $c_g(n)$ *the coefficients of the q-expansions of* f *and* g *respectively. If* $c_f(n) = c_g(n)$ *for all* $n \leq B$ *then* $f = g$.

A proof can be found in [Ste07, Corollary 9.20].

II.2.2 Harmonic Maaß forms and shadows

We define our main objects of interest.

Definition II.2.6. Let $k \in \frac{1}{2}\mathbb{Z}$, and choose $N \in \mathbb{N}$ such that $4 \mid N$ whenever $k \notin \mathbb{Z}$. Let ϕ be a Dirichlet character of modulus N.

(1) A weight k *harmonic Maaß form* on a subgroup $\Gamma_0(N)$ with Nebentypus ϕ is any smooth function $f \colon \mathbb{H} \to \mathbb{C}$ satisfying the following three properties:

   (i) For all $\gamma \in \Gamma_0(N)$ and all $\tau \in \mathbb{H}$ we have $(f|_k \gamma)(\tau) = f(\tau)$.
   (ii) The function f is harmonic with respect to the weight k hyperbolic Laplacian on $\mathbb{H}$, explicitly

$$0 = \Delta_k(f) \coloneqq \left(-v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right) \right)(f).$$

   (iii) The function f has at most linear exponential growth at all cusps.

   We denote the vector space of such functions by $H_k^!(\Gamma_0(N), \phi)$.
(2) If we restrict the growth condition (iii) to governable growth then the vector space of such forms is denoted by $H_k^{\mathrm{cusp}}(\Gamma_0(N), \phi)$.
(3) A polar harmonic Maaß form is a harmonic Maaß form with isolated poles on the upper half plane.

Remark. If we restrict the growth condition (iii) to moderate growth at all cusps, and allow arbitrary eigenvalues in (ii), then f is a classical *Maaß wave form*.

During the following summary, we may assume that ϕ is trivial for simplicity, since the generalization to a nontrivial Nebentypus is immediate.

Bruinier and Funke observed in [BF04] that the Fourier expansion of a harmonic Maaß form[2] naturally splits into two parts. One of them involves the incomplete Gamma

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[2]Be aware that their terminology refers to our harmonic Maaß forms as "harmonic *weak* Maaß forms" instead.

function

$$\Gamma(s, z) := \int_z^\infty t^{s-1} e^{-t} \mathrm{d}t,$$

defined for $\mathrm{Re}(s) > 0$ and $z \in \mathbb{C}$. Specializing to principal values as a function of s, which yields the single-valued principal branch, it can be analytically continued in s via the functional equation

$$\Gamma(s+1, z) = s\Gamma(s, z) + z^s e^{-z},$$

provided that $z \neq 0$. As a function of the second argument, it has the asymptotic behavior

$$\Gamma(s, v) \sim v^{s-1} e^{-v}, \quad |v| \to \infty$$

for $v \in \mathbb{R}$ (see the paragraph following equation (I.3) for more details and some references). We state their result in the formulation of [BFOR17, Lemma 4.3].

Lemma II.2.7. *Let $k \in \frac{1}{2}\mathbb{Z} \setminus \{1\}$ and $f \in H_k^!(\Gamma_0(N))$. Then f has a Fourier expansion of the shape*

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n + c_f^-(0) v^{1-k} + \sum_{\substack{n \ll \infty \\ n \neq 0}} c_f^-(n) \Gamma(1-k, -4\pi nv) q^n.$$

In particular, if $f \in H_k^{\mathrm{cusp}}(\Gamma_0(N))$ then f has a Fourier expansion of the shape

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n < 0} c_f^-(n) \Gamma(1-k, -4\pi nv) q^n.$$

The notation $\sum_{n \gg -\infty}$ abbreviates $\sum_{n \geq m_f}$ for some $m_f \in \mathbb{Z}$. The notation $\sum_{n \ll \infty}$ is defined analogously, and similar expansions hold at the other cusps.

We follow the following terminology from [BFOR17, Definition 4.4].

Definition II.2.8. We refer to the functions

$$f^+(\tau) := \sum_{n \gg -\infty} c_f^+(n) q^n, \qquad f^-(\tau) := c_f^-(0) v^{1-k} + \sum_{\substack{n \ll \infty \\ n \neq 0}} c_f^-(n) \Gamma(1-k, -4\pi nv) q^n$$

as the *holomorphic part* of f and to f^- as its *nonholomorphic part*.

In the same paper, Bruinier and Funke introduced the operator

$$\xi_k := 2iv^k \overline{\frac{\partial}{\partial \overline{\tau}}} = iv^k \overline{\left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)}.$$

We summarize its relevant properties, see [BFOR17, Section 5] for example.

Lemma II.2.9. *Let f be a smooth function on $\mathbb{H}$. Then the ξ-operator satisfies the following properties.*

(1) *We have $\xi_k(f) = 0$ if and only if f is holomorphic.*

(2) *The slash operator intertwines with ξ_k, that is we have*

$$\xi_k (f|_k \gamma) = (\xi_k(f))|_{2-k} \gamma$$

for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ if $k \in \mathbb{Z}$ or $\gamma \in \Gamma_0(4)$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ respectively.

(3) *The kernel of ξ_k restricted to $H_k^{\mathrm{cusp}}(\Gamma_0(N))$ or $H_k^!(\Gamma_0(N))$ is precisely the space $M_k^!(\Gamma_0(N))$ in both cases.*

(4) *Let $f \in H_k^!(\Gamma_0(N))$. Assuming the notation of Lemma II.2.7 we have*

$$\begin{aligned} \xi_k(f)(\tau) &= \xi_k (f^-)(\tau) \\ &= (1-k)c_f^-(0) - (4\pi)^{1-k} \sum_{n \gg -\infty} \overline{c_f^-(-n)} n^{1-k} q^n \in M_{2-k}^! (\Gamma_0(N)), \end{aligned}$$

and in particular if $f \in H_k^{\mathrm{cusp}}(\Gamma_0(N))$ then

$$\xi_k(f)(\tau) = -(4\pi)^{1-k} \sum_{n \geq 1} \overline{c_f^-(-n)} n^{1-k} q^n \in S_{2-k} (\Gamma_0(N)).$$

In addition, we have $\xi_k \colon H_k^!(\Gamma_0(N)) \twoheadrightarrow M_{2-k}^!(\Gamma_0(N))$.

The first item is simply a reformulation of the Cauchy–Riemann equations and the second one is induced by the corresponding well known property for the Maaß lowering operator

$$L_k := v^2 \frac{\partial}{\partial \overline{\tau}} = \frac{1}{2} v^2 \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

We fix some more terminology, following [BFOR17, Definition 5.16] and the second remark afterwards.

Definition II.2.10.

(1) A function f is called a *mock modular form* if f is the holomorphic part of a harmonic Maaß form for which f^- is nontrivial.

(2) If $f \in H_k^!(\Gamma_0(N))$ then we refer to the form $\xi_k(f)$ as the *shadow* of f^+.

(3) In particular, f is called a *mock theta function* if f is a mock modular form of weight $\frac{1}{2}$ or $\frac{3}{2}$, whose shadow is a linear combination of unary theta functions.

Moreover, we study the following objects, which were introduced first in [DMZ12, Section 7.3]. However, we follow the definition given in [BFOR17, Section 13.2].

Definition II.2.11.

(1) A *mixed harmonic Maaß form* of weight (k_1, k_2) is a function h of the shape

$$h(\tau) = \sum_{j=1}^{n} f_j(\tau)g_j(\tau),$$

where $f_j \in H_{k_1}^!$ and $g_j \in M_{k_2}^!$ for every j.

(2) Analogously, a *mixed mock modular form* of weight (k_1, k_2) is a function h of the shape

$$h(\tau) = \sum_{j=1}^{n} f_j(\tau)g_j(\tau),$$

where each f_j is a mock modular form of weight k_1 and $g_j \in M_{k_2}^!$ for every j.

We extend the last result of Lemma II.2.9 to mixed harmonic Maaß forms. It suffices to consider products involving the nonholomorphic part of a mixed harmonic Maaß form.

Lemma II.2.12. *Let k_f, $k_g \in \mathbb{R}$, $\kappa := k_f + k_g$, and let $\alpha(m)$, $\beta(n)$ be two complex sequences such that*

$$f(\tau) := \sum_{m \geq 1} \alpha(m)m^{k_f-1}\Gamma(1 - k_f, 4\pi mv)q^{-m}, \qquad g(\tau) := \sum_{n \geq 1} \beta(n)q^n$$

both converge absolutely. Then

$$\xi_\kappa\left(fg\right)(\tau) = -(4\pi)^{1-k_f}v^{k_g} \sum_{m \geq 1} \overline{\alpha(m)}q^m \sum_{n \geq 1} \overline{\beta(n)q^n}.$$

Proof. We have

$$(fg)(\tau) = \sum_{m \geq 1}\sum_{n \geq 1} \alpha(m)m^{k_f-1}\Gamma(1 - k_f, 4\pi mv)\beta(n)q^{n-m}$$

and

$$\frac{\partial}{\partial v}\Gamma(a, v) = -v^{a-1}e^{-v}.$$

We compute that

$$
\xi_\kappa \left(\Gamma(1 - k_f, 4\pi m v) q^{n-m} \right) = i v^\kappa \left\{ \Gamma(1 - k_f, 4\pi m v) \overline{2\pi i (n - m) q^{n-m}} \right.
$$
$$
\left. -i \left[-(4\pi m v)^{-k_f} e^{-4\pi m v} 4\pi m \overline{q^{n-m}} + \Gamma(1 - k_f, 4\pi m v)(-2\pi(n - m)) \overline{q^{n-m}} \right] \right\}
$$
$$
= -v^{k_g} (4\pi m)^{1-k_f} e^{-4\pi m v} \overline{q^{n-m}}
$$
$$
= -v^{k_g} (4\pi m)^{1-k_f} q^m e^{-2\pi i n \overline{\tau}},
$$

and infer that

$$
\xi_\kappa \left(fg \right)(\tau) = -(4\pi)^{1-k_f} v^{k_g} \sum_{m \geq 1} \sum_{n \geq 1} \overline{\alpha(m)\beta(n)} q^m \overline{q^n},
$$

as claimed. $\qquad \square$

II.2.3 Holomorphic projection

We introduce the holomorphic projection operator. Its origin lies in the search for an operator which preserves the (regularized) Petersson inner product. Although this can be derived implicitly from the Riesz representation theorem, an explicit description comes in handy quite often.

Definition II.2.13. Let $f \colon \mathbb{H} \to \mathbb{C}$ be a translation invariant function and $k \in \mathbb{N}_{\geq 2}$. If f has at most moderate growth towards the cusps, then the weight k *holomorphic projection* of f is defined by (see [BFOR17, equation (10.3)])

$$
\pi_k(f)(\tau) := \frac{(k-1)(2i)^k}{4\pi} \int_{\mathbb{H}} \frac{f(x + iy) y^k}{(\tau - x + iy)^k} \frac{\mathrm{d}x\mathrm{d}y}{y^2},
$$

whenever the integral converges absolutely.

Furthermore, we require the following result.

Lemma II.2.14 (Lipschitz summation formula)**.** *For any $r \in \mathbb{N}_{\geq 2}$ we have that*

$$
\sum_{j \in \mathbb{Z}} \frac{1}{(w + j)^r} = \frac{(-2\pi i)^r}{(r-1)!} \sum_{j \geq 1} j^{r-1} e^{2\pi i j w}.
$$

A short proof is due to Zagier and can be found in [BvdGHZ08, p. 16]. We summarize two further properties of the holomorphic projection operator, both of which are proven in [BKZ14, Section 3] for instance.

Lemma II.2.15. *Let $f \colon \mathbb{H} \to \mathbb{C}$ be a translation invariant function of moderate growth such that the integral defining $\pi_k f$ converges absolutely, and $k \in \mathbb{N}_{\geq 2}$. Then π_k enjoys the following properties.*

(1) *If f is holomorphic then $\pi_k(f) = f$.*
(2) *If f is modular with some Nebentypus ϕ (but not necessarily holomorphic), then*

$$\pi_k(f) \in \begin{cases} M_k\left(\Gamma_0\left(N\right), \phi\right) & \text{if } k \in \mathbb{N}_{\geq 3}, \\ M_2\left(\Gamma_0\left(N\right), \phi\right) \oplus M_0\left(\Gamma_0\left(N\right), \phi\right) \cdot E_2 & \text{if } k = 2. \end{cases}$$

Therefore, the slash operator and π_k commute if $k \geq 3$.

II.3 Two proofs of Proposition II.1.7

During the first proof of Proposition II.1.7, we appeal to the following results.

Lemma II.3.1.

(1) *If $\mathrm{Re}(b), \mathrm{Re}(a + b) > 0$ and $\mathrm{Re}(c + s) > 0$ then*

$$\int_0^\infty \Gamma(a, cz) z^{b-1} e^{-sz} \mathrm{d}z = \frac{c^a \Gamma(a+b)}{b(c+s)^{a+b}}\ {}_2F_1\left(1, a + b, b + 1; \frac{s}{s+c}\right).$$

(2) *The hypergeometric function ${}_2F_1$ satisfies*

$${}_2F_1(a, b, c; z) = (1 - z)^{c-a-b}\ {}_2F_1(c - a, c - b, c; z),$$

and

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} {}_2F_1(a, b, a + b - c + 1; 1 - z)$$
$$+ (1 - z)^{c-a-b} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c - a, c - b, c - a - b + 1; 1 - z).$$

The first identity is [GR07, item 6.455]. Both hypergeometric transformations can be found in [GR07, page 1008].

First proof of Proposition II.1.7. The q-expansion of $(fg)(\tau)$ is given by

$$(fg)(\tau) = \sum_{m \geq 1} \sum_{n \geq 1} \alpha(m)\beta(n) m^{k_f - 1} \Gamma(1 - k_f, 4\pi m v) q^{n-m}.$$

We see that fg is translation invariant, and recall that it has moderate growth towards all cusps by assumption, so $\pi_\kappa(fg)$ exists. Hence, we need to calculate

$$\pi_\kappa\left(fg\right)(\tau) = \frac{(\kappa-1)(2i)^\kappa}{4\pi} \int_{\mathbb{H}} \frac{(fg)\,(x+iy)\,y^\kappa}{(\tau-x+iy)^\kappa} \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

The integral converges since $\kappa - 2 \geq 0$, and converges absolutely if $\kappa > 2$. Using the translation invariance of fg, we rewrite the integral over $\mathbb{H}$ as

$$\int_{\mathbb{H}} \frac{(fg)\,(x+iy)\,y^\kappa}{(\tau-x+iy)^\kappa} \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int_0^\infty \int_0^1 (fg)\,(x+iy)y^{\kappa-2} \sum_{j\in\mathbb{Z}} \frac{1}{(\tau-x+iy+j)^\kappa} \mathrm{d}x\mathrm{d}y,$$

and consequently

$$\pi_\kappa\left(fg\right)(\tau) = \frac{(\kappa-1)(2i)^\kappa}{4\pi} \sum_{m\geq 1}\sum_{n\geq 1} \alpha(m)m^{k_f-1}\beta(n)$$

$$\times \int_0^\infty \int_0^1 \Gamma(1-k_f,4\pi my)y^{\kappa-2} \sum_{j\in\mathbb{Z}} \frac{1}{(\tau-x+iy+j)^\kappa} e^{2\pi i(n-m)(x+iy)}\mathrm{d}x\mathrm{d}y.$$

By the Lipschitz summation formula and then Lemma II.3.1 (1), we infer that

$$\pi_\kappa\left(fg\right)(\tau) = \frac{(\kappa-1)(2i)^\kappa}{4\pi} \frac{(-2\pi i)^\kappa}{(\kappa-1)!} \sum_{m\geq 1}\sum_{n\geq 1} \alpha(m)m^{k_f-1}\beta(n)$$

$$\times \int_0^\infty \Gamma(1-k_f,4\pi my)y^{\kappa-2} \int_0^1 \sum_{j\geq 1} j^{\kappa-1} e^{2\pi ij(\tau-x+iy)} e^{2\pi i(n-m)(x+iy)} \mathrm{d}x\mathrm{d}y$$

$$= \frac{(\kappa-1)(2i)^\kappa}{4\pi} \frac{(-2\pi i)^\kappa}{(\kappa-1)!} \sum_{m\geq 1}\sum_{n-m\geq 1} \alpha(m)m^{k_f-1}\beta(n)(n-m)^{\kappa-1}$$

$$\times \int_0^\infty \Gamma(1-k_f,4\pi my)y^{\kappa-2} e^{-4\pi(n-m)y}\mathrm{d}y\ q^{n-m}$$

$$= \frac{\Gamma(k_g)}{(\kappa-1)!} \sum_{m\geq 1}\sum_{n\geq m+1} \alpha(m)\beta(n)\frac{(n-m)^{\kappa-1}}{n^{k_g}}\ {}_2F_1\left(1,k_g,\kappa;1-\frac{m}{n}\right)q^{n-m}.$$

Finally, we apply the hypergeometric transformations from Lemma II.3.1 (2). Explicitly,

$${}_2F_1\left(1,k_g,\kappa;1-\frac{m}{n}\right) = \frac{(\kappa-1)}{k_f-1}\ {}_2F_1\left(1,k_g,2-k_f;\frac{m}{n}\right)$$

$$+ \frac{\Gamma(\kappa)\Gamma(1-k_f)}{\Gamma(k_g)}\left(1-\frac{m}{n}\right)^{1-\kappa}\left(\frac{m}{n}\right)^{k_f-1},$$

and

$$
{}_2F_1\left(1, k_g, 2 - k_f; \frac{m}{n}\right) = \left(1 - \frac{m}{n}\right)^{1-\kappa} {}_2F_1\left(1 - k_f, 2 - \kappa, 2 - k_f; \frac{m}{n}\right).
$$

Thus, we arrive at

$$
\begin{aligned}
&\frac{\Gamma(k_g)}{(\kappa - 1)!} \frac{(n - m)^{\kappa - 1}}{n^{k_g}} \, {}_2F_1\left(1, k_g, \kappa; 1 - \frac{m}{n}\right) \\
&= \frac{\Gamma(k_g)}{(\kappa - 1)!} \left(n^{k_f - 1} \frac{(\kappa - 1)}{k_f - 1} \, {}_2F_1\left(1 - k_f, 2 - \kappa, 2 - k_f; \frac{m}{n}\right) + \frac{\Gamma(\kappa)\Gamma(1 - k_f)}{\Gamma(k_g)} m^{k_f - 1}\right) \\
&= -\Gamma(1 - k_f) \left(n^{k_f - 1} \frac{\Gamma(k_g)}{(\kappa - 2)! \, \Gamma(2 - k_f)} \, {}_2F_1\left(2 - \kappa, 1 - k_f, 2 - k_f, \frac{m}{n}\right) - m^{k_f - 1}\right) \\
&= -\Gamma(1 - k_f) \left(n^{k_f - 1} \mathcal{P}_{\kappa - 2}^{(1 - k_f, 1 - \kappa)}\left(1 - 2\frac{m}{n}\right) - m^{k_f - 1}\right),
\end{aligned}
$$

and ultimately obtain

$$
\begin{aligned}
&\pi_\kappa\left(fg\right)(\tau) \\
&= -\Gamma(1 - k_f) \sum_{m \geq 1} \sum_{n - m \geq 1} \alpha(m)\beta(n) \left(n^{k_f - 1} \mathcal{P}_{\kappa - 2}^{(1 - k_f, 1 - \kappa)}\left(1 - 2\frac{m}{n}\right) - m^{k_f - 1}\right) q^{n - m},
\end{aligned}
$$

as desired. $\qquad\square$

The second proof of Proposition II.1.7 emphasizes that the hypergeometric function specializes to some polynomial (the Jacobi polynomial), and it requires the following identities.

Lemma II.3.2 ([Mer16, Lemmas 4.7, 5.1]). *Define the homogeneous polynomial*

$$
P_{a,b}(X, Y) := \sum_{j=0}^{a-2} \binom{j + b - 2}{j} X^j (X + Y)^{a - j - 2} \in \mathbb{C}[X, Y].
$$

of degree $a - 2$.

(1) *Then we have*

$$
\begin{aligned}
&\int_0^\infty \Gamma(1 - k_f, 4\pi m y) y^{\kappa - 2} e^{-4\pi r y} \mathrm{d}y \\
&= -(4\pi)^{1-\kappa} m^{1 - k_f} \frac{\Gamma(1 - k_f)(\kappa - 2)!}{r^{\kappa - 1}} \left((r + m)^{1 - k_g} P_{\kappa, 2 - k_f}(r, m) - m^{k_f - 1}\right).
\end{aligned}
$$

(2) *If $b \neq 1, 2$, then*

$$P_{a,b}(X, Y) = \sum_{j=0}^{a-2} \binom{a+b-3}{a-2-j}\binom{j+b-2}{j}(X+Y)^{a-2-j}(-Y)^j.$$

The first relies on the fact that $\kappa = k_f + k_g$ is an integer. Full proofs of both items can be found in Mertens' thesis [Mer14a, Lemmas V.1.7, V.1.8].

Second proof of Proposition II.1.7. One copies the first proof until the application of the Lipschitz summation formula, which produced the expression

$$\pi_\kappa (fg)(\tau) = \frac{(\kappa-1)(2i)^\kappa}{4\pi} \frac{(-2\pi i)^\kappa}{(\kappa-1)!} \sum_{m \geq 1} \sum_{n-m \geq 1} \alpha(m)m^{k_f-1}\beta(n)(n-m)^{\kappa-1}$$
$$\times \int_0^\infty \Gamma(1-k_f, 4\pi m y)y^{\kappa-2}e^{-4\pi(n-m)y}\mathrm{d}y \; q^{n-m}.$$

Next, one proceeds by writing

$$\pi_\kappa (fg)(\tau)$$
$$= -\Gamma(1-k_f) \sum_{m \geq 1} \sum_{n-m \geq 1} \alpha(m)\beta(n)\left(n^{1-k_g}P_{\kappa,2-k_f}(n-m, m) - m^{k_f-1}\right)q^{n-m},$$

according to the first item of the previous lemma, and then writing

$$n^{1-k_g}P_{\kappa,2-k_f}(n-m, m) = \sum_{j=0}^{\kappa-2}\binom{k_g-1}{\kappa-2-j}\binom{j-k_f}{j}n^{k_f-1-j}(-m)^j$$
$$= \frac{\Gamma(k_g)n^{k_f-1}}{(\kappa-2)!\,\Gamma(1-k_f)}\sum_{j=0}^{\kappa-2}\binom{\kappa-2}{j}\frac{1}{j+1-k_f}\left(-\frac{m}{n}\right)^j = n^{k_f-1}\mathcal{P}_{\kappa-2}^{(1-k_f, 1-\kappa)}\left(1-2\frac{m}{n}\right),$$

by virtue of the second item of the previous lemma. Summing up, one obtains

$$\pi_\kappa (fg)(\tau)$$
$$= -\Gamma(1-k_f) \sum_{m \geq 1} \sum_{n-m \geq 1} \alpha(m)\beta(n)\left(n^{k_f-1}\mathcal{P}_{\kappa-2}^{(1-k_f, 1-\kappa)}\left(1-2\frac{m}{n}\right) - m^{k_f-1}\right)q^{n-m},$$

as claimed. □

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II.4 Proof of Theorem II.1.1 and Theorem II.1.3

We collect the results needed to prove Theorem II.1.1 and Theorem II.1.3. We begin by rewriting the definitions of $\mathcal{F}^-$ and $\mathcal{G}^-$.

Lemma II.4.1. *We have*

$$\mathcal{F}^-(\tau) = \frac{2}{\Gamma\left(-\frac{1}{2}\right)} \sum_{m \geq 1} \chi(m) m \Gamma\left(-\frac{1}{2}, 4\pi m^2 v\right) q^{-m^2},$$

and

$$\mathcal{G}^-(\tau) = \frac{2}{\Gamma\left(-\frac{1}{2}\right)} \sum_{m \geq 1} m \Gamma\left(-\frac{1}{2}, 4\pi m^2 v\right) q^{-m^2} - \frac{1}{2\pi v^{\frac{1}{2}}}.$$

Proof. We compute

$$-(2\pi)^{-\frac{1}{2}} i \int_{-\overline{\tau}}^{i\infty} \frac{\theta_\chi(w)}{(-i(w+\tau))^{\frac{3}{2}}} \mathrm{d}w = -(2\pi)^{-\frac{1}{2}} i \sum_{m \geq 1} \chi(m) \int_{2iv}^{i\infty} \frac{e^{2\pi i m^2(z-\tau)}}{(-iz)^{\frac{3}{2}}} \mathrm{d}z$$

$$= (2\pi)^{-\frac{1}{2}} \sum_{m \geq 1} \chi(m) \left(\int_{2v}^{\infty} x^{-\frac{3}{2}} e^{-2\pi m^2 x} \mathrm{d}x\right) q^{-m^2}$$

$$= \sum_{m \geq 1} \chi(m) m \left(\int_{4\pi m^2 v}^{\infty} t^{-\frac{1}{2}-1} e^{-t} \mathrm{d}t\right) q^{-m^2},$$

and the first claim follows directly, since $\frac{2}{\Gamma\left(-\frac{1}{2}\right)} = -\frac{1}{\sqrt{\pi}}$. To prove the second claim, it remains to separate the constant term of $\theta_\mathbb{1}$, and next to calculate

$$\frac{i}{\pi\sqrt{2}} \int_{-\overline{\tau}}^{i\infty} \frac{\frac{1}{2}}{(-i(w+\tau))^{\frac{3}{2}}} \mathrm{d}w = -\frac{1}{2\pi\sqrt{2}} \int_{2v}^{\infty} t^{-\frac{3}{2}} \mathrm{d}t = -\frac{1}{2\pi v^{\frac{1}{2}}},$$

as asserted. $\qquad \square$

In addition, we have the following immediate corollary of Proposition II.1.7.

Corollary II.4.2. *Let $f(\tau) := f^+(\tau) + f^-(\tau)$ be the splitting of f into its holomorphic and nonholomorphic part. Assume the notation and hypotheses as in Proposition II.1.7. Then*

$$\pi_\kappa(fg)(\tau) = \left(f^+ g\right)(\tau)$$
$$- \Gamma(1-k_f) \sum_{m \geq 1} \sum_{n-m \geq 1} \alpha(m)\beta(n) \left(n^{k_f-1} \mathcal{P}_{\kappa-2}^{(1-k_f, 1-\kappa)}\left(1 - 2\frac{m}{n}\right) - m^{k_f-1}\right) q^{n-m}.$$

Moreover, we combine the properties of the slash-operator, the shadow operator, and of the holomorphic projection operator. This yields a third preparatory result.

Proposition II.4.3 ([MOR21, Proposition 2.3])**.** *Let $f \colon \mathbb{H} \to \mathbb{C}$ be a translation invariant function such that $|f(\tau)|v^\delta$ is bounded on $\mathbb{H}$ for some $\delta > 0$. If the weight k holomorphic projection of f vanishes identically for some $k > \delta + 1$ and $\xi_k(f)$ is modular of weight $2 - k$ for some subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$, then f is modular of weight k for Γ.*

Proof. This is a straightforward adaption of [BKZ14, Proposition 3.5]. Let $\gamma \in \Gamma$. Then the modularity of $\xi_k f$ implies that

$$\xi_k \left(f|_k \gamma - f \right) = \xi_k(f)|_{2-k} \gamma - \xi_k(f) = 0.$$

Hence, $f|_k \gamma - f$ is holomorphic. This yields

$$f|_k \gamma - f = \pi_\kappa \left(f|_k \gamma - f \right) = \pi_\kappa \left(f \right)|_k \gamma - \pi_\kappa \left(f \right),$$

and by assumption the right hand side vanishes. This proves the claim. $\square$

Remark. The subtle growth conditions are required to include the case π_2, and are clearly satisfied if we deal with higher weight holomorphic projections, in which case the integral defining π_k converges absolutely.

Proof of Theorem II.1.1. We need to check the three conditions required by the definition of a harmonic Maaß form.

(1) Growth conditions: Recall that θ_ψ is a cusp form, namely it decays exponentially towards all cusps. In turn, the function $\mathcal{F}^+$ admits at most linear exponential growth towards all cusps. Note that in particular $i\infty$ is a removable singularity of $\mathcal{F}^+$, since $i\infty$ is a simple zero of both θ_ψ and $\theta_\psi \mathcal{F}^+$. To inspect the nonholomorphic part, we have (see [GR07, item 8.3357])

$$\Gamma \left(-\frac{1}{2}, 4\pi m^2 y \right) q^{-m^2} \sim \left(4\pi m^2 v \right)^{-\frac{3}{2}} e^{-2\pi m^2 v}, \quad v \to \infty,$$

and hence the function $\mathcal{F}^-$ decays exponentially towards the cusp $i\infty$. By the transformation properties of θ_χ under the full modular group $\mathrm{SL}_2(\mathbb{Z})$, we deduce that $\mathcal{F}^-$ is of moderate growth towards all cusps. This establishes the growth condition required by Definition II.2.6.

As pointed out after Corollary II.1.6, the Fourier coefficients of $\theta_\psi \mathcal{F}^+$ at $i\infty$ are of moderate growth, wherefore the growth of the function $\theta_\psi \mathcal{F}^+$ towards any cusp is moderate. One may see this by choosing suitable scaling matrices, whose action yields additional polynomial factors in τ. Consequently, the growth of $\theta_\psi \mathcal{F}$ is moderate. This justifies the existence of $\pi_3 \left(\theta_\psi \mathcal{F} \right)$ as well as the application of Proposition II.4.3 to $\theta_\psi \mathcal{F}^+$ during the upcoming item.

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(2) The transformation law: First, we compute

$$n^{\frac{1}{2}}\mathcal{P}_1^{(-\frac{1}{2},-2)}\left(1-2\frac{m}{n}\right)-m^{\frac{1}{2}}=n^{\frac{1}{2}}\left(\frac{1}{2}+\frac{m}{2n}\right)-m^{\frac{1}{2}}=\frac{\left(m^{\frac{1}{2}}-n^{\frac{1}{2}}\right)^2}{2n^{\frac{1}{2}}}.$$

Comparing our initial setting with Proposition II.1.7 and Lemma II.2.12, we need to switch to squares above. By virtue of Lemma II.4.1 and the definition of θ_ψ, we have the coefficients

$$\alpha\left(m^2\right)=\frac{2}{\Gamma\left(-\frac{1}{2}\right)}\chi(m),\qquad \beta\left(n^2\right)=\psi(n)n.$$

We rewrite the generating function of $\sigma_{2,\chi}^{\mathrm{sm}}$ as

$$\left(\mathcal{F}^+\theta_\psi\right)(\tau)=\sum_{n\geq 1}\sigma_{2,\chi}^{\mathrm{sm}}(n)q^n=\sum_{n\geq 1}\sum_{d\in D_n}\chi\left(\frac{\frac{n}{d}-d}{2}\right)\psi\left(\frac{\frac{n}{d}+d}{2}\right)d^2q^n$$

$$=\sum_{d\geq 1}\sum_{\substack{j\geq d\\ j\equiv d\,(\mathrm{mod}\,2)}}\chi\left(\frac{j-d}{2}\right)\psi\left(\frac{j+d}{2}\right)d^2q^{dj}$$

$$=\sum_{m\geq 1}\sum_{n-m\geq 1}\chi(m)\psi(n)\left(n-m\right)^2q^{n^2-m^2},$$

and apply Corollary II.4.2 to obtain

$$\pi_3\left(\mathcal{F}\theta_\psi\right)(\tau)=\left(\mathcal{F}^+\theta_\psi\right)(\tau)-\sum_{m\geq 1}\sum_{n-m\geq 1}\chi(m)\psi(n)\left(m-n\right)^2q^{n^2-m^2}=0.$$

Furthermore, we apply Lemma II.2.12, obtaining

$$\xi_3\left(\mathcal{F}\theta_\psi\right)(\tau)=-(4\pi)^{-\frac{1}{2}}\frac{2}{\Gamma\left(-\frac{1}{2}\right)}v^{\frac{3}{2}}\sum_{m\geq 1}\overline{\chi(m)}q^{m^2}\sum_{n\geq 1}\overline{\psi(n)n}q^{n^2}$$

$$=\frac{1}{2\pi}v^{\frac{3}{2}}\theta_{\overline{\chi}}(\tau)\frac{|\theta_\psi(\tau)|^2}{\theta_\psi(\tau)}.$$

We observe that $\xi_3\left(\mathcal{F}\theta_\psi\right)(\tau)$ is modular of weight -1 for $\Gamma_0(4M_\chi^2)\cap\Gamma_0(4M_\psi^2)=\Gamma_0\left(\mathrm{lcm}\left(4M_\chi^2,4M_\psi^2\right)\right)$ with Nebentypus $\overline{\chi}\cdot\psi^{-1}\cdot\chi_{-4}^{-1}$. Indeed, for any $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0\left(\mathrm{lcm}\left(4M_\chi^2,4M_\psi^2\right)\right)$ we have

$$\xi_3\left(\mathcal{F}\theta_\psi\right)(\gamma\tau)=\frac{1}{2\pi}\frac{v^{\frac{3}{2}}}{|c\tau+d|^3}\overline{\chi(d)}(c\tau+d)^{\frac{1}{2}}\theta_{\overline{\chi}}(\tau)\frac{\left|\psi(d)\chi_{-4}(d)(c\tau+d)^{\frac{3}{2}}\theta_\psi(\tau)\right|^2}{\psi(d)\chi_{-4}(d)(c\tau+d)^{\frac{3}{2}}\theta_\psi(\tau)}$$

$$=\overline{\chi(d)}\psi(d)^{-1}\chi_{-4}^{-1}(d)(c\tau+d)^{-1}\xi_3\left(\mathcal{F}\theta_\psi\right)(\tau).$$

Finally, Proposition II.4.3 applies directly, because the growth conditions are met thanks to absolute convergence. We deduce that $\mathcal{F}\theta_\psi$ is modular of weight 3 with respect to the same data, as desired.

(3) Harmonicity: Clearly, $\mathcal{F}^+$ is holomorphic away from the zeros of θ_ψ. Hence, the Cauchy–Riemann equations imply $\Delta_{\frac{3}{2}}(\mathcal{F}^+) = 0$ directly. The computation for $\mathcal{F}^-$ is standard, so we only sketch its results. It holds that

$$\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\Gamma\left(-\frac{1}{2}, 4\pi m^2 v\right) q^{-m^2} = -\frac{i}{2}\frac{e^{-2\pi m^2(iu+v)}}{\pi^{\frac{1}{2}} m v^{\frac{3}{2}}},$$

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\Gamma\left(-\frac{1}{2}, 4\pi m^2 v\right) q^{-m^2} = \frac{3}{4}\frac{e^{-2\pi m^2(iu+v)}}{\pi^{\frac{1}{2}} m v^{\frac{5}{2}}},$$

$$\Delta_{\frac{3}{2}}(\mathcal{F}^-)(\tau) = -v^2\left(\frac{3}{4}\frac{e^{-2\pi m^2(iu+v)}}{\pi^{\frac{1}{2}} m v^{\frac{5}{2}}}\right) + \frac{3i}{2}v\left(-\frac{i}{2}\frac{e^{-2\pi m^2(iu+v)}}{\pi^{\frac{1}{2}} m v^{\frac{3}{2}}}\right) = 0,$$

and thus $\Delta_{\frac{3}{2}}(\mathcal{F}) = 0$ away from the zeros of θ_ψ.

Altogether, this completes the proof, since the shadow is a byproduct of the second item. $\qquad\square$

We move to the proof of Theorem II.1.3.

Proof of Theorem II.1.3. The proof of Theorem II.1.3 uses the same ideas as the proof of Theorem II.1.1, so we just emphasize the differences. Recall from Lemma II.4.1 that

$$\mathcal{G}^-(\tau) = \frac{2}{\Gamma\left(-\frac{1}{2}\right)} \sum_{m \geq 1} m\Gamma\left(-\frac{1}{2}, 4\pi m^2 v\right) q^{-m^2} - \frac{1}{2\pi v^{\frac{1}{2}}}.$$

Therefore, to compute $\pi_3(\theta_\psi \mathcal{G}^-)$, it suffices to deal with the second term. We see that $-\frac{\theta_\psi(\tau)}{2\pi v^{\frac{1}{2}}}$ is translation invariant, vanishes at $i\infty$, and has a removable singularity at all other cusps inspecting the order of vanishing as $v \searrow 0$. Hence the integral defining its weight 3 holomorphic projection exists and converges absolutely. The computation begins exactly as in the proof of Proposition II.1.7 and we employ the Lipschitz summation formula. This yields

$$\pi_3\left(-\frac{\theta_\psi(\tau)}{2\pi v^{\frac{1}{2}}}\right) = -\frac{2(2i)^3}{8\pi^2}\sum_{n \geq 1}\psi(n)n\sum_{j \in \mathbb{Z}}\int_0^\infty\int_0^1 \frac{y^{\frac{1}{2}}e^{2\pi i n^2(x+iy)}}{(\tau - x + iy + j)^3}\mathrm{d}x\mathrm{d}y$$

$$= -8\pi\sum_{n \geq 1}\psi(n)n\sum_{j \geq 1}j^2\int_0^\infty\int_0^1 y^{\frac{1}{2}}e^{2\pi i\left(n^2(x+iy)+j(\tau-x+iy)\right)}\mathrm{d}x\mathrm{d}y$$

$$= -8\pi\sum_{n \geq 1}\psi(n)n^5\left(\int_0^\infty y^{\frac{1}{2}}e^{-4\pi n^2 y}\mathrm{d}y\right)q^{n^2} = -\frac{1}{2}\sum_{n \geq 1}\psi(n)n^2 q^{n^2},$$

which also appeared in [Mer16, Lemma 4.4] in the framework of mixed harmonic Maaß forms. Furthermore, it follows by Corollary II.4.2 (with identical parameters of the Jacobi polynomial as in the proof of Theorem II.1.1) that

$$\pi_3\left(\mathcal{G}\theta_\psi\right)(\tau) = \left(\mathcal{G}^+\theta_\psi\right)(\tau) - \sum_{m\geq 1}\sum_{n-m\geq 1}\psi(n)\left(m-n\right)^2 q^{n^2-m^2} - \frac{1}{2}\sum_{n\geq 1}\psi(n)n^2 q^{n^2}.$$

In addition, we note that

$$\sum_{n\geq 1}\sigma_{2,\mathbb{1}}^{\mathrm{sm}}(n)q^n = \sum_{d\geq 1}\sum_{\substack{j\geq d \\ j\equiv d\,(\mathrm{mod}\,2)}}\psi\left(\frac{j+d}{2}\right)d^2 q^{jd} = \sum_{m\geq 1}\sum_{n\geq m+1}\psi(n)(n-m)^2 q^{n^2-m^2},$$

where we substituted $d = n - m$, $j = n + m$ in the last equation. Collecting these observations and inserting the definition of $\mathcal{G}^+$, we obtain

$$\pi_3\left(\mathcal{G}\theta_\psi\right)(\tau) = 0.$$

Moreover,

$$\xi_3\left(\mathcal{G}\theta_\psi\right)(\tau) = \frac{1}{2\pi}v^{\frac{3}{2}}\overline{\theta_\psi(\tau)}\sum_{m\geq 1}q^{m^2} + \frac{1}{4\pi}v^{\frac{3}{2}}\overline{\theta_\psi(\tau)} = \frac{1}{4\pi}v^{\frac{3}{2}}\frac{\left|\theta_\psi(\tau)\right|^2}{\theta_\psi(\tau)}\sum_{m\in\mathbb{Z}}q^{m^2},$$

which is modular of weight -1 on $\Gamma_0\left(4M_\psi^2\right)\cap\Gamma_0(4) = \Gamma_0\left(\mathrm{lcm}\left(4, 4M_\psi^2\right)\right) = \Gamma_0\left(4M_\psi^2\right)$ with Nebentypus $\left(\psi\cdot\chi_{-4}\right)^{-1}$ by the same argument as in the proof of Theorem II.1.3. This establishes weight 3 modularity of $\mathcal{G}\theta_\psi$ via Proposition II.4.3 again.

Additionally, we clearly have

$$\Delta_{\frac{3}{2}}\left(-\frac{1}{2\pi v^{\frac{1}{2}}}\right) = 0,$$

and hence harmonicity is preserved. Finally, the growth properties of $\mathcal{G}^+$ towards all cusps agree verbatim with $\mathcal{F}^+$. Summing up, this establishes the Theorem. □

II.5 Proof of Theorem II.1.8

We move to the proof of Theorem II.1.8. To this end, we adapt the proof of [MOR21, Theorem 1.4].

Proof of Theorem II.1.8. We prove the first claim. On one hand, by the same computation as during the proof of Theorem II.1.1 we infer

$$\pi_3 \left(\theta_\psi \left(p^{2a} \tau \right) \mathcal{F}(\tau) \right) = \theta_\psi \left(p^{2a} \tau \right) \mathcal{F}^+(\tau) + \sum_{r \geq 1} \left(\sum_{\substack{m,n \geq 1 \\ (p^a n)^2 - m^2 = r}} \chi(m) \psi(n) \left(m - p^a n \right)^2 \right) q^r$$

$$=: g(\tau).$$

Invoking Theorem II.1.1 and the properties of holomorphic projection we deduce that g is a modular form of weight 3 on $\Gamma_0 \left(\operatorname{lcm} \left(4M_\psi^2 p^{2a}, 4M_\chi^2 \right) \right)$ with Nebentypus $\overline{\chi}$. However, clearly $-I \in \Gamma_0(N)$ for every level N and hence g vanishes identically (recall that χ is assumed to be even throughout).

On the other hand, the inner sum can be rewritten as a sum over small divisors of r. To this end, the set of admissible small divisors is given by

$$D_r(p) := \left\{ d \mid r \; : \; 1 \leq d \leq \frac{r}{d}, \quad d \equiv \frac{r}{d} \pmod{2}, \quad d + \frac{r}{d} \equiv 0 \pmod{2p^a} \right\},$$

and exactly as in the proof of Theorem II.1.1 we see that

$$\sum_{\substack{m,n \geq 1 \\ (p^a n)^2 - m^2 = r}} \chi(m) \psi(n) \left(m - p^a n \right)^2 = \sum_{d \in D_r(p)} \chi \left(\frac{\frac{n}{d} - d}{2} \right) \psi \left(\frac{\frac{n}{d} + d}{2p^a} \right) d^2.$$

If we apply the operator $U(p^b)$ to g then we need to replace r by $p^b r$ everywhere above. This produces the condition

$$d + \frac{p^b r}{d} \equiv 0 \pmod{2p^a}$$

in the set of admissible small divisors, which eventually forces

$$d \equiv 0 \quad \left(\operatorname{mod} p^{\min(a,b)} \right),$$

since d is a divisor of r. Combining, we arrive at

$$0 = g(\tau) = g(\tau) \Big| \, U \left(p^b \right) \equiv \left(\theta_\psi \left(p^{2a} \tau \right) \mathcal{F}^+(\tau) \right) \Big| \, U \left(p^b \right) \quad \left(\operatorname{mod} p^{\min(a,b)} \right),$$

as claimed.

The proof of the second claim is completely analogous. The character χ is trivial, $M_\chi = 1$, and one can remove the condition $d \equiv \frac{r}{d} \pmod{2}$ from the definition of the set of admissible small divisors. However, this does not affect the rest of the proof and we provided the necessary computations during the proof of Theorem II.1.3 essentially. $\quad \square$

II.6 Proof of Proposition II.1.9

We conclude this chapter with the proof of Proposition II.1.9.

Proof of Proposition II.1.9: The first step is to apply the geometric series. We compute

$$\sum_{n\geq 1} \sigma_{2,\chi}^{\mathrm{sm}}(n)q^n = \sum_{m\geq 1} \sum_{n-m\geq 1} \chi(m)\psi(n)\,(n-m)^2\,q^{n^2-m^2}$$

$$= \sum_{m\geq 1} \sum_{s\geq 1} \chi(m)\psi(m+s)s^2 q^{s^2+2ms}$$

$$= \sum_{s\geq 1} \sum_{a\geq 0} \sum_{b=0}^{M_\chi-1} \chi(b)\psi(s+b)s^2 q^{s^2+2(aM_\chi+b)s} - \chi(0)\psi(s+b)s^2 q^{s^2}$$

$$= \sum_{b=1}^{M_\chi-1} \chi(b) \sum_{s\geq 1} \psi(s+b)s^2 \frac{q^{s^2+2bs}}{1-q^{2M_\chi s}},$$

where we have used the assumption $M_\psi \mid M_\chi$ after the substitution $m = aM_\chi + b$ and the assumption that χ is non-trivial in the last equation.

The second step is to convert the sum in s to a sum over $\mathbb{Z}$ instead of $\mathbb{N}$. Note that

$$2\sum_{s\geq 1} \psi(s+b)s^2 \frac{q^{s^2+2bs}}{1-q^{2M_\chi s}} = \sum_{s\geq 1} \psi(s+b)s^2 \frac{q^{s^2+2bs}}{1-q^{2M_\chi s}} + \sum_{s\leq -1} \psi(-s+b)s^2 \frac{q^{s^2-2bs}}{1-q^{-2M_\chi s}}$$

$$= \sum_{s\geq 1} \psi(s+b)s^2 \frac{q^{s^2+2bs}}{1-q^{2M_\chi s}} + \sum_{s\leq -1} \psi(s-b)s^2 \frac{q^{s^2+2M_\chi s-2bs}}{1-q^{2M_\chi s}},$$

using that ψ is odd. The key observation is

$$\sum_{b=1}^{M_\chi-1} \chi(b) \sum_{s\leq -1} \psi(s-b)s^2 \frac{q^{s^2+2M_\chi s-2bs}}{1-q^{2M_\chi s}} = \sum_{b=1}^{M_\chi-1} \chi(b) \sum_{s\leq -1} \psi(s+b)s^2 \frac{q^{s^2+2bs}}{1-q^{2M_\chi s}}.$$

by mapping $b \mapsto M_\chi - b$ and using that χ is even. Thus, we have

$$\sum_{n\geq 1} \sigma_{2,\chi}^{\mathrm{sm}}(n)q^n = \frac{1}{2} \sum_{b=1}^{M_\chi-1} \chi(b) \sum_{s\in\mathbb{Z}} \psi(s+b)s^2 \frac{q^{s^2+2bs}}{1-q^{2M_\chi s}},$$

since the constant term in s vanishes as well.

The third step is to substitute $s = nM_\chi + c$ to isolate ψ from its s-dependency (recall $M_\psi \mid M_\chi$), getting

$$\sum_{n\geq 1} \sigma_{2,\chi}^{\mathrm{sm}}(n)q^n = \frac{1}{2} \sum_{b=1}^{M_\chi-1} \chi(b) \sum_{c=0}^{M_\chi-1} \psi(b+c) \sum_{n\in\mathbb{Z}} (nM_\chi + c)^2 \frac{q^{(nM_\chi+c)^2+2b(nM_\chi+c)}}{1-q^{2M_\chi(nM_\chi+c)}},$$

from which we read off the claim. $\qquad\qquad\Box$

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Chapter III

Multidimensional small divisor functions

This chapter is based on a paper [Mon21] of the same title published in *Integers*.

III.1 Introduction - One-dimensional case

In a recent paper [MOR21], Mertens, Ono, and Rolen defined and investigated a new type of mock modular form, whose coefficients are given by a small divisor function. We summarize their approach. As usual, we let $\tau = u + iv \in \mathbb{H}$ and $q := e^{2\pi i \tau}$. Let $P_\ell\left(\frac{n}{d}, d\right) \in \mathbb{Q}[X, Y]$, and ψ, χ be Dirichlet characters of moduli M_ψ, M_χ respectively. We denote by χ_{-4} the unique odd Dirichlet character of modulus 4, and we define

$$D_n := \left\{ d \mid n \ : \ 1 \leq d \leq \frac{n}{d} \ \text{and} \ d \equiv \frac{n}{d} \ (\mathrm{mod} \ 2) \right\},$$

$$\sigma_\ell^{\mathrm{sm}}(n) := \sum_{d \in D_n} \chi\left(\frac{\frac{n}{d} - d}{2}\right) \psi\left(\frac{\frac{n}{d} + d}{2}\right) P_\ell\left(\frac{n}{d}, d\right).$$

Additionally, we require *Shimura's theta-function*

$$\theta_\psi(\tau) := \frac{1}{2} \sum_{n \in \mathbb{Z}} \psi(n) n^{\lambda_\psi} q^{n^2}, \qquad \lambda_\psi := \frac{1 - \psi(-1)}{2},$$

and recall that (see Lemma II.2.4 including a reference)

$$\theta_\psi \in \begin{cases} M_{\frac{1}{2}}\left(\Gamma_0\left(4M_\psi^2\right), \psi\right) & \text{if } \lambda_\psi = 0, \\ S_{\frac{3}{2}}\left(\Gamma_0\left(4M_\psi^2\right), \psi \cdot \chi_{-4}\right) & \text{if } \lambda_\psi = 1. \end{cases} \tag{III.1}$$

Furthermore, we recall the definition of a harmonic Maaß form[1]. An exposition on the theory of harmonic Maaß forms can be found in [BFOR17], and the required facts for this chapter are summarized in Section II.2.

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[1] Be aware that there is no overall convention which terminology encodes which growth condition.

Definition III.1.1. Let $k \in \frac{1}{2}\mathbb{Z}$, and choose $N \in \mathbb{N}$ such that $4 \mid N$ whenever $k \notin \mathbb{Z}$. Let ϕ be a Dirichlet character of modulus N.

(1) A *weight k harmonic Maaß form on a subgroup $\Gamma_0(N)$ with Nebentypus ϕ* is any smooth function $f \colon \mathbb{H} \to \mathbb{C}$ satisfying the following three properties:

  (i) For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N)$ and all $\tau \in \mathbb{H}$ we have

$$f(\tau) = (f|_k \gamma)(\tau) := \begin{cases} \phi(d)^{-1}(c\tau + d)^{-k} f(\gamma\tau) & \text{if } k \in \mathbb{Z}, \\ \phi(d)^{-1} \left(\frac{c}{d} \right) \varepsilon_d^{2k} (c\tau + d)^{-k} f(\gamma\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

  where $\left(\frac{c}{d} \right)$ denotes the extended Legendre symbol, and

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

  (ii) The function f satisfies

$$0 = \Delta_k(f) := \left(-v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \right)(f).$$

  (iii) The function f has at most linear exponential growth at all cusps.

(2) A *polar harmonic Maaß form* is a harmonic Maaß form with isolated poles on $\mathbb{H}$.

Let $\mathbb{1}$ be the trivial character. Then the main result of [MOR21] reads as follows.

Theorem III.1.2 ([MOR21, Theorem 1.1]). *Suppose that $\psi = \chi \neq \mathbb{1}$, and that $P_1\left(\frac{n}{d}, d \right) = d$. Denote the corresponding small divisor function by σ_1^{sm}, and by E_2 the Eisenstein series*

$$E_2(\tau) := 1 - 24 \sum_{n \geq 1} \sum_{d \mid n} d q^n.$$

Define

$$\mathcal{E}^+(\tau) := \frac{1}{\theta_\psi(\tau)} \left(\alpha_\psi E_2(\tau) + \sum_{n \geq 1} \sigma_1^{\mathrm{sm}}(n) q^n \right),$$

$$\mathcal{E}^-(\tau) := (-1)^{\lambda_\psi} \frac{(2\pi)^{\lambda_\psi - \frac{1}{2}} i}{8\Gamma\left(\frac{1}{2} + \lambda_\psi \right)} \int_{-\overline{\tau}}^{i\infty} \frac{\theta_{\overline{\psi}}(w)}{(-i(w+\tau))^{\frac{3}{2} - \lambda_\psi}} \mathrm{d}w,$$

where α_ψ is an implicit constant depending only on ψ to ensure a certain growth condition. Then the function $\mathcal{E}^+ + \mathcal{E}^-$ is a polar harmonic Maaß form of weight $\frac{3}{2} - \lambda_\psi$ on $\Gamma_0\left(4M_\psi^2 \right)$ with Nebentypus $\overline{\psi} \cdot \chi_{-4}^{\lambda_\psi}$.

In analogy to the classical divisor sums $\sigma_k(n)$, Mertens, Ono, and Rolen called their function $\mathcal{E}^+$ a *mock modular Eisenstein series with Nebentypus*. Furthermore, they related their result to partition functions for special choices of ψ, and proved a *p*-adic property of $\mathcal{E}^+$, compare [MOR21, Corollary 1.3, Theorem 1.4].

In Theorem II.1.1 and Theorem II.1.3 of Chapter II, we discovered the polar harmonic Maaß forms $\mathcal{F}$ and $\mathcal{G}$ adapting the construction from [MOR21]. Moreover, if $\psi = \chi_{-4}$, $\chi = \mathbb{1}$, then we related the holomorphic part $\mathcal{G}^+$ of $\mathcal{G}$ to Hurwitz class numbers, and proved a *p*-adic property of $\mathcal{F}^+$, $\mathcal{G}^+$ as well, compare Corollary II.1.6 and Theorem II.1.8.

The proof of Theorems III.1.2, II.1.1 and Theorem II.1.3 is performed in three main steps. To describe them, we let

$$\Gamma(s, z) := \int_z^\infty t^{s-1} e^{-t} \mathrm{d}t,$$

be the *incomplete Gamma function*, which is defined for $\mathrm{Re}(s) > 0$ and $z \in \mathbb{C}$. We refer to the paragraph following equation (I.3) (and to Section II.2) for more details and some references. As in Chapter II, we let

$$\xi_\kappa := 2iv^\kappa \overline{\frac{\partial}{\partial \overline{\tau}}} = iv^\kappa \overline{\left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)}$$

be the *Bruinier–Funke operator of weight κ*, and

$$\pi_\kappa(f)(\tau) := \frac{(\kappa - 1)(2i)^\kappa}{4\pi} \int_{\mathbb{H}} \frac{f(x + iy) y^k}{(\tau - x + iy)^\kappa} \frac{\mathrm{d}x\mathrm{d}y}{y^2},$$

be the *weight κ holomorphic projection operator*, whenever f is translation invariant, and the integral converges absolutely. Moreover, we let

$$g(\tau) := \sum_{n \geq 1} \beta(n) q^n, \qquad f^+(\tau) := \frac{1}{g(\tau)} \sum_{n \geq 1} \sigma_\ell^{\mathrm{sm}}(n) q^n,$$

$$f^-(\tau) := \sum_{m \geq 1} \alpha(m) m^{k_f - 1} \Gamma(1 - k_f, 4\pi mv) q^{-m}, \qquad f(\tau) := (f^+ + f^-)(\tau).$$

Then we proceed as follows.

(I) Show that

$$\pi_\kappa(fg)(\tau) = 0.$$

To this end, we rewrite the definition of the given nonholomorphic part (see Lemma II.4.1 for instance). Next, we recall the *Jacobi polynomial $\mathcal{P}_r^{(a,b)}$ of degree r*

and parameter a, b from equation II.1 (see Section III.4.1 as well), and utilize Proposition II.1.7 (resp. Corollary II.4.2) from the previous chapter. In addition, the holomorphic part f^+g has to be rewritten as well, see the proof of Theorem II.1.1 in Section II.4.

(II) We compute

$$\xi_\kappa \left(fg\right)\left(\tau\right) = -(4\pi)^{1-k_f} v^{k_g} \left(\sum_{m \geq 1} \overline{\alpha(m)} q^m\right) \overline{g(\tau)},$$

and choose the coefficients $\alpha\left(m\right)$, $\beta\left(n\right)$, such that this function is modular of weight $2 - \kappa$.

(III) Conclude that fg is modular of weight κ by [MOR21, Proposition 2.3] (which is provided in Proposition II.4.3 as well). Lastly, verify harmonicity and the growth property towards the cusps required by the definition of a harmonic Maaß form.

Finally, we mention one remark following Proposition II.1.7, which states that there are more choices of half integral parameters k_f, k_g, which lead to other choices of polynomials $P_\ell\left(\frac{n}{d}, d\right)$ in the definition of $\sigma_\ell^{\mathrm{sm}}$, such that step (I) above works.

We refer to the first two sections of Chapter II for more details, and for overall preliminaries introducing the aforementioned objects together with their key properties.

III.2   Statement of the result

We arrive at the following result by combining the lemmas from the Section III.3 as outlined during Section III.1. The functions $\sigma_\ell^{\mathrm{sm}}$ and f_ℓ are defined at the beginning of Section III.3.

Theorem III.2.1. *Let ψ be an odd Dirichlet character, χ be an even and non-trivial Dirichlet character. Let $\ell \in 2\mathbb{N} + 2$. Define P_ℓ as indicated in Corollay III.3.2, obtaining the corresponding small divisor function $\sigma_\ell^{\mathrm{sm}}$. Then the resulting function f_ℓ is a polar harmonic Maaß form of weight $2 - \frac{\ell}{2} \in -\mathbb{N}_0$ on $\Gamma_0\left(4M_\chi^2\right) \cap \Gamma_0\left(4M_\psi^2\right)$ with Nebentypus $\overline{\chi} \cdot (\psi \cdot \chi_{-4})^{-1}$. Its shadow $\xi_{2-\frac{\ell}{2}}\left(f_\ell\right)$ is given by a non-zero constant multiple of $\theta_{\overline{\chi}}^\ell$.*

In other words, the technique presented in Chapter II, [MOR21] applies straightforward in higher even dimensions, except for dimension two. We plan to find and investigate applications of f_ℓ to other areas of number theory, such as combinatorics, as in the one-dimensional case [MOR21, Corollary 1.3].

III.3 Multidimensional Case

We fix $\ell \in \mathbb{N}$ throughout. Let $\boldsymbol{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$. We recall the usual multi-index conventions

$$\boldsymbol{n}! := n_1 n_2 \cdots n_\ell, \quad |\boldsymbol{n}| := n_1 + \ldots + n_\ell, \quad \|\boldsymbol{n}\| := \sqrt{n_1^2 + \ldots + n_\ell^2}.$$

We let $\psi \neq \mathbb{1}$, and consider

$$\theta_\psi(\tau)^\ell = \sum_{\boldsymbol{n} \in \mathbb{N}^\ell} \psi(\boldsymbol{n}!) (\boldsymbol{n}!)^{\lambda_\psi} q^{\|\boldsymbol{n}\|^2}.$$

Moreover, we relax our assumption to $P_\ell \in \mathbb{Q}(X, Y)$, and we let

$$\mathcal{D}_{\boldsymbol{n}} = \left\{ \boldsymbol{d} \in \mathbb{N}^\ell : d_j \mid n_j, \ 1 \leq d_j \leq \frac{n_j}{d_j}, \text{ and } d_j \equiv \frac{n_j}{d_j} \pmod 2 \text{ for every } 1 \leq j \leq \ell \right\},$$

as well as

$$\sigma_\ell^{\mathrm{sm}}(\boldsymbol{n}) := \sum_{\boldsymbol{d} \in \mathcal{D}_{\boldsymbol{n}}} \left(\prod_{j=1}^\ell \chi\left(\frac{\frac{n_j}{d_j} - d_j}{2} \right) \psi\left(\frac{\frac{n_j}{d_j} + d_j}{2} \right) \left(\frac{\frac{n_j}{d_j} - d_j}{2} \right)^{\lambda_\chi} \left(\frac{\frac{n_j}{d_j} + d_j}{2} \right)^{\lambda_\psi} \right)$$

$$\times P_\ell \left(\left\| \left(\frac{n_j}{d_j} \right)_{1 \leq j \leq \ell} \right\|^2, \|\boldsymbol{d}\|^2 \right).$$

Consequently,

$$f_\ell^+(\tau) := \frac{1}{\theta_\psi(\tau)^\ell} \sum_{\boldsymbol{n} \in \mathbb{N}^\ell} \sigma_\ell^{\mathrm{sm}}(\boldsymbol{n}) q^{|\boldsymbol{n}|},$$

$$f_\ell^-(\tau) := \frac{1}{\Gamma(1 - k_{f_\ell})} \sum_{\boldsymbol{m} \in \mathbb{N}^\ell} \chi(\boldsymbol{m}!) (\boldsymbol{m}!)^{\lambda_\chi} \|\boldsymbol{m}\|^{2(k_{f_\ell}-1)} \Gamma(1 - k_{f_\ell}, 4\pi\|\boldsymbol{m}\|^2 v) q^{-\|\boldsymbol{m}\|^2},$$

$$f_\ell(\tau) := (f_\ell^+ + f_\ell^-)(\tau).$$

We insert this setting into the constructive method described in the first section, and devote a subsection to each step.

III.3.1 First step

We verify that the first step continues to hold due to exactly the same proofs as in Section II.3. We have to be careful regarding the summation conditions, which are

determined one step after the application of the Lipschitz summantion formula. Explicitly, we obtain

$$\pi_\kappa \left(f_\ell^- \theta_\psi^\ell \right)(\tau) = -\sum_{r \geq 1} \sum_{\substack{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^\ell \\ \|\boldsymbol{n}\|^2 - \|\boldsymbol{m}\|^2 = r}} \chi\left(\boldsymbol{m}!\right)\left(\boldsymbol{m}!\right)^{\lambda_\chi} \psi\left(\boldsymbol{n}!\right)\left(\boldsymbol{n}!\right)^{\lambda_\psi}$$

$$\times \left(\|\boldsymbol{n}\|^{2(k_{f_\ell}-1)} \mathcal{P}_{\kappa-2}^{(1-k_{f_\ell},1-\kappa)}\left(1 - 2\frac{\|\boldsymbol{m}\|^2}{\|\boldsymbol{n}\|^2}\right) - \|\boldsymbol{m}\|^{2(k_{f_\ell}-1)} \right) q^r.$$

To match this expression with $f_\ell^+ g$, we rewrite the small divisor function. We substitute

$$\boldsymbol{a} := \left(\frac{\frac{n_1}{d_1} + d_1}{2}, \dots, \frac{\frac{n_\ell}{d_\ell} + d_\ell}{2} \right), \qquad \boldsymbol{b} := \left(\frac{\frac{n_1}{d_1} - d_1}{2}, \dots, \frac{\frac{n_\ell}{d_\ell} - d_\ell}{2} \right),$$

from which we deduce

$$\boldsymbol{d} = \boldsymbol{a} - \boldsymbol{b}, \qquad \boldsymbol{a} + \boldsymbol{b} = \left(\frac{n_j}{d_j} \right)_{1 \leq j \leq \ell}, \qquad |\boldsymbol{n}| = \|\boldsymbol{a}\|^2 - \|\boldsymbol{b}\|^2.$$

Thus,

$$\left(f_\ell^+ \theta_\psi^\ell \right)(\tau) = \sum_{\boldsymbol{b} \in \mathbb{N}^\ell} \sum_{\boldsymbol{a} - \boldsymbol{b} \in \mathbb{N}^\ell} \chi\left(\boldsymbol{b}!\right)\left(\boldsymbol{b}!\right)^{\lambda_\chi} \psi\left(\boldsymbol{a}!\right)\left(\boldsymbol{a}!\right)^{\lambda_\psi} \; P_\ell\left(\|\boldsymbol{a} + \boldsymbol{b}\|, \|\boldsymbol{a} - \boldsymbol{b}\|\right) q^{\|\boldsymbol{a}\|^2 - \|\boldsymbol{b}\|^2}.$$

We transform the summation condition.

Lemma III.3.1. *We have*

$$\left(f_\ell^+ \theta_\psi^\ell \right)(\tau) = \sum_{r \geq 1} \sum_{\substack{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^\ell \\ \|\boldsymbol{n}\|^2 - \|\boldsymbol{m}\|^2 = r}} \chi\left(\boldsymbol{m}!\right)\left(\boldsymbol{m}!\right)^{\lambda_\chi} \psi\left(\boldsymbol{n}!\right)\left(\boldsymbol{n}!\right)^{\lambda_\psi} \; P_\ell\left(\|\boldsymbol{m} + \boldsymbol{n}\|, \|\boldsymbol{m} - \boldsymbol{n}\|\right) q^r.$$

Proof. Note that if $\boldsymbol{a} - \boldsymbol{b} \in \mathbb{N}^\ell$, then

$$\|\boldsymbol{a}\|^2 - \|\boldsymbol{b}\|^2 = \sum_{j=1}^\ell (a_j + b_j)(a_j - b_j) \geq 1.$$

Conversely, suppose $\|\boldsymbol{a}\|^2 - \|\boldsymbol{b}\|^2 \geq 1$. Recall that $n_j = (a_j + b_j)(a_j - b_j) \in \mathbb{N}$ for every $1 \leq j \leq \ell$ by definition of f^+, and $a_j + b_j$ is always positive. Thus, $(a_j - b_j) \geq 1$ for every $1 \leq j \leq \ell$, which proves the lemma. $\qquad \square$

Hence, we achieve the following result by virtue of Proposition II.1.7.

Corollary III.3.2. *If P_ℓ is defined by the condition*

$$\|\boldsymbol{b}\|^{2(k_{f_\ell}-1)} \mathcal{P}_{\kappa-2}^{(1-k_{f_\ell},1-\kappa)}\left(1 - 2\frac{\|\boldsymbol{a}\|^2}{\|\boldsymbol{b}\|^2}\right) - \|\boldsymbol{a}\|^{2(k_{f_\ell}-1)} = P_\ell\left(\|\boldsymbol{a} + \boldsymbol{b}\|, \|\boldsymbol{a} - \boldsymbol{b}\|\right),$$

then we have $\pi_\kappa\left(f_\ell \theta_\psi^\ell\right)(\tau) = 0.$

III.3.2 Second step

We summarize the result of a standard calculation.

Lemma III.3.3. *We have*

$$\xi_\kappa\left(f_\ell\theta_\psi^\ell\right)(\tau) = -\frac{(4\pi)^{1-k_{f_\ell}}}{\Gamma\left(1-k_{f_\ell}\right)}v^{k_{\theta_\psi^\ell}}\theta_{\overline{\chi}}(\tau)^\ell\frac{|\theta_\psi(\tau)|^{2\ell}}{\theta_\psi(\tau)^\ell}$$

away from the zeros of θ_ψ.

Proof. By definition and linearity of ξ_κ, it holds that

$$\xi_\kappa\left(f_\ell^-\theta_\psi^\ell\right)(\tau) = \xi_\kappa\left(f_\ell^-\right)(\tau)\cdot\overline{\theta_\psi(\tau)^\ell} + \overline{f_\ell^-(\tau)}\cdot\xi_\kappa\left(\theta_\psi^\ell\right)(\tau)$$
$$= \xi_\kappa\left(f_\ell^-\right)(\tau)\cdot\overline{\theta_\psi(\tau)^\ell},$$

where the last step used that θ_ψ^ℓ is holomorphic. Next, one computes[2]

$$\xi_\kappa\left(f_\ell^-\right)(\tau) = -\frac{(4\pi)^{1-k_{f_\ell}}}{\Gamma\left(1-k_{f_\ell}\right)}v^{k_{\theta_\psi^\ell}}\sum_{\boldsymbol{m}\in\mathbb{N}^\ell}\overline{\chi\left(\boldsymbol{m}!\right)}\left(\boldsymbol{m}!\right)^{\lambda_\chi}q^{\|\boldsymbol{m}\|^2},$$

from which we infer the claim. □

Combining the previous result with the modularity of Shimura's theta function (see equation (III.1)), and the fact that

$$\text{Im}\left(\gamma\tau\right) = \frac{v}{|c\tau+d|^2}$$

for every $\gamma = \left(\begin{smallmatrix}a & b\\c & d\end{smallmatrix}\right)\in\text{SL}_2(\mathbb{Z})$ and every $\tau\in\mathbb{H}$, we obtain the following corollary.

Corollary III.3.4. *If $\chi\neq\mathbb{1}$ then $\xi_\kappa(f_\ell\theta_\psi^\ell)$ is modular of weight*

$$\ell\left(\frac{1}{2}+\lambda_{\overline{\chi}}\right) - \ell\left(\frac{1}{2}+\lambda_\psi\right)$$

on $\Gamma_0(4M_\chi^2)\cap\Gamma_0(4M_\psi^2)$ with Nebentypus $\overline{\chi}\cdot(\psi\cdot\chi_{-4})^{-1}$.

Thus, we stipulate ψ to be odd, and χ to be even and non-trivial, getting

$$\kappa = 2 - (-\ell)\in\mathbb{N}_{\geq 2}, \qquad k_{f_\ell} = 2 - \frac{\ell}{2},$$

as desired.

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[2]Compare the proof of Lemma II.2.12 for some intermediate steps.

III.3.3 Third step

We verify the two remaining conditions of a polar harmonic Maaß form.

Lemma III.3.5. *Let $\tau \in \mathbb{H}$ with $\theta_\psi(\tau) \neq 0$. Then, the function $f_\ell = f_\ell^+ + f_\ell^-$ satisfies*

$$0 = \Delta_{k_{f_\ell}}(f_\ell)(\tau),$$

and has the required growth property of a polar harmonic Maaß form.

Proof. The first assertion follows by construction of f_ℓ. Since θ_ψ^ℓ is of exponential decay towards all cusps, the function f_ℓ^+ admits at most linear exponential growth towards all cusps. In particular, the cusp $i\infty$ is a removable singularity of f^+, because both numerator and denominator vanish at $i\infty$ of order ℓ. In addition, the function f_ℓ^- decays exponentially towards $i\infty$, since the incomplete Gamma function does (and it dominates the powers of q). The transformation behaviour of θ_χ under the full modular group $\mathrm{SL}_2(\mathbb{Z})$ implies that f_ℓ^- is of at most moderate growth towards all cusps. Indeed, choosing suitable scaling matrices yields additional factors of polynomial growth inside the Fourier expansion of f_ℓ^-. This establishes the second assertion. $\square$

III.3.4 Conclusion

We justify the application of Proposition II.1.7, which proves Theorem III.2.1.

Proof of Theorem III.2.1. By definition, the Fourier coefficients of $\theta_\psi^\ell f_\ell^+$ expanded at $i\infty$ are of moderate growth, whence the growth of $\theta_\psi^\ell f_\ell^+$ towards any cusp has to be moderate. Consequently, the growth of $\theta_\psi^\ell f_\ell$ towards any cusp is moderate according to the proof of Lemma III.3.5. Thus, the assumptions in Proposition II.1.7 are satisfied by $\theta_\psi^\ell f_\ell$. Performing the outlined steps concludes the proof of Theorem III.2.1. $\square$

III.4 Numerical examples

III.4.1 An interlude on Jacobi polynomials

The Jacobi polynomials $\mathcal{P}_r^{(a,b)}$ admit a representation in terms of of Gauß' hypergeometric function $_2F_1$, see equation (II.1). This yields many identities between Jacobi polynomials of "neighboring" degree r and parameters a, b, that is $r \in \{r-1, r, r+1\}$ and analogously for a, b. For instance, one could use Gauß contiguous relations, to obtain such identities. In particular, this leads to a recursive characterization of the Jacobi polynomials. More precisely, we have

$$\mathcal{P}_0^{(a,b)}(z) = 1, \qquad \mathcal{P}_1^{(a,b)}(z) = \frac{1}{2}\left(a - b + (a+b+2)z\right),$$

and

$$c_1(j)\mathcal{P}_{j+1}^{(a,b)}(z) = (c_2(j) + c_3(j)z)\,\mathcal{P}_j^{(a,b)}(z) - c_4(j)\mathcal{P}_{j-1}^{(a,b)}(z),$$

where

$$c_1(j) = 2(j+1)(j+a+b+1)(2j+a+b), \qquad c_2(j) = (2j+a+b+1)\left(a^2 - b^2\right),$$
$$c_3(j) = (2j+a+b)(2j+a+b+1)(2j+a+b+2),$$
$$c_4(j) = 2(j+a)(j+b)(2j+a+b+2).$$

III.4.2 Explicit examples

Note that the parallelogram law and the fact $|n| = \|\boldsymbol{a}+\boldsymbol{b}\|\|\boldsymbol{a}-\boldsymbol{b}\|$ yield

$$\|\boldsymbol{a}\|^2 = \frac{\|\boldsymbol{a}+\boldsymbol{b}\|^2 + \|\boldsymbol{a}-\boldsymbol{b}\|^2}{4} + \frac{\|\boldsymbol{a}+\boldsymbol{b}\|\|\boldsymbol{a}-\boldsymbol{b}\|}{2},$$
$$\|\boldsymbol{b}\|^2 = \frac{\|\boldsymbol{a}+\boldsymbol{b}\|^2 + \|\boldsymbol{a}-\boldsymbol{b}\|^2}{4} - \frac{\|\boldsymbol{a}+\boldsymbol{b}\|\|\boldsymbol{a}-\boldsymbol{b}\|}{2}.$$

Higher even dimensions

The case $\ell = 2$ has to be excluded since $k_{f_\ell} \neq 1$. On one hand, if $\ell = 4$ for instance, we have

$$\kappa = 6, \qquad k_{f_4} = 0, \qquad \frac{\mathcal{P}_4^{(1,-5)}\left(1 - 2\frac{\|\boldsymbol{a}\|^2}{\|\boldsymbol{b}\|^2}\right)}{\|\boldsymbol{b}\|^2} - \frac{1}{\|\boldsymbol{a}\|^2} = \frac{(\|\boldsymbol{a}\|^2 - \|\boldsymbol{b}\|^2)^5}{\|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^{10}},$$

and thus, we choose the function P_4 as

$$P_4\left(\|\boldsymbol{a}+\boldsymbol{b}\|, \|\boldsymbol{a}-\boldsymbol{b}\|\right)$$
$$= \frac{\|\boldsymbol{a}-\boldsymbol{b}\|^5\|\boldsymbol{a}+\boldsymbol{b}\|^5}{\left(\frac{\|\boldsymbol{a}+\boldsymbol{b}\|^2+\|\boldsymbol{a}-\boldsymbol{b}\|^2}{4} + \frac{\|\boldsymbol{a}+\boldsymbol{b}\|\|\boldsymbol{a}-\boldsymbol{b}\|}{2}\right)\left(\frac{\|\boldsymbol{a}+\boldsymbol{b}\|^2+\|\boldsymbol{a}-\boldsymbol{b}\|^2}{4} - \frac{\|\boldsymbol{a}+\boldsymbol{b}\|\|\boldsymbol{a}-\boldsymbol{b}\|}{2}\right)^5}.$$

Similarly, we compute (with $x := \|\boldsymbol{a}\|$, $y := \|\boldsymbol{b}\|$)

$$y^{-4}\,\mathcal{P}_6^{(2,-7)}\left(1 - 2\frac{x^2}{y^2}\right) - x^{-4} = \frac{(x^2-y^2)^7}{x^4y^{16}}\left(7x^2 + y^2\right),$$

$$y^{-6}\,\mathcal{P}_8^{(3,-9)}\left(1 - 2\frac{x^2}{y^2}\right) - x^{-6} = \frac{(x^2-y^2)^9}{x^6y^{22}}\left(45x^4 + 9x^2y^2 + y^4\right),$$

$$y^{-8}\,\mathcal{P}_{10}^{(4,-11)}\left(1 - 2\frac{x^2}{y^2}\right) - x^{-8} = \frac{(x^2-y^2)^{11}}{x^8y^{28}}\left(286x^6 + 66x^4y^2 + 11x^2y^4 + y^6\right),$$

from which we read off the corresponding definitions of P_ℓ.

Because of the aforementioned recursive nature of the Jacobi polynomials, the indicated pattern continues to hold for every even dimension $\ell \in 2\mathbb{N} + 2$ by induction.

Higher odd dimensions

On the other hand, the case of dimension $\ell \in 2\mathbb{N}_{\geq 2} - 1$ produces more complicated functions P_ℓ. For example, if $\ell = 3$ we have $\kappa = 5$, $k_{f_3} = \frac{1}{2}$, and

$$\frac{\mathcal{P}_3^{\left(\frac{1}{2}, -4\right)}\left(1 - 2\frac{\|\boldsymbol{a}\|^2}{\|\boldsymbol{b}\|^2}\right)}{\|\boldsymbol{b}\|} - \frac{1}{\|\boldsymbol{a}\|}$$
$$= -\frac{(\|\boldsymbol{a}\| - \|\boldsymbol{b}\|)^4 \left(5\|\boldsymbol{a}\|^3 + 20\|\boldsymbol{a}\|^2\|\boldsymbol{b}\| + 29\|\boldsymbol{a}\|\|\boldsymbol{b}\|^2 + 16\|\boldsymbol{b}\|^3\right)}{16\|\boldsymbol{a}\|\|\boldsymbol{b}\|^7}.$$

If $\ell = 5$, we have $\kappa = 7$, $k_{f_5} = -\frac{1}{2}$, and

$$\frac{\mathcal{P}_5^{\left(\frac{3}{2}, -6\right)}\left(1 - 2\frac{\|\boldsymbol{a}\|^2}{\|\boldsymbol{b}\|^2}\right)}{\|\boldsymbol{b}\|^3} - \frac{1}{\|\boldsymbol{a}\|^3} = \frac{1}{256\|\boldsymbol{a}\|^3\|\boldsymbol{b}\|^{13}}\left(-693\|\boldsymbol{a}\|^{13} + 4095\|\boldsymbol{a}\|^{11}\|\boldsymbol{b}\|^2\right.$$
$$\left. -10010\|\boldsymbol{a}\|^9\|\boldsymbol{b}\|^4 + 12870\|\boldsymbol{a}\|^7\|\boldsymbol{b}\|^6 - 9009\|\boldsymbol{a}\|^5\|\boldsymbol{b}\|^8 + 3003\|\boldsymbol{a}\|^3\|\boldsymbol{b}\|^{10} - 256\|\boldsymbol{b}\|^{13}\right).$$

We observe that we are left with odd powers of $\|\boldsymbol{a}\|$, $\|\boldsymbol{b}\|$ in both odd-dimensional cases. If we keep the dependence of P_ℓ on $\|\boldsymbol{a} \pm \boldsymbol{b}\|$, which ultimately justifies the terminology "divisor function", then odd powers obstruct a definition of P_ℓ via the parallelogram law in these cases of ℓ. Once more, an inductive argument via the recursive characterization of the Jacobi polynomials extends this phenomenon to all odd dimensions $\ell \in 2\mathbb{N} + 1$.

III.5   References

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Chapter IV

Higher depth mock theta functions and q-hypergeometric series

This chapter is based on a paper [MMR21] of the same title published in *Forum Mathematicum*. This is joint work with Dr. Joshua Males and Prof. Dr. Larry Rolen.

IV.1 Introduction and statement of results

The study of mock theta functions goes back to Ramanujan, who gave the first examples in his enigmatic last letter to Hardy one hundred years ago. In 2002, Zwegers [Zwe02] achieved a major breakthrough by providing nonholomorphic completions of Ramanujan's classical mock theta functions to modular objects. More precisely and in todays terminology, he recognized Ramanujan's mock theta functions as holomorphic parts of so-called harmonic Maaß forms of weight $\frac{1}{2}$ with shadow given by weight $\frac{3}{2}$ unary theta functions. More generally, his thesis provides a "Maaß–Jacobi form", which is roughly speaking a nonholomorphic generalization of classical Jacobi forms. We refer to [BFOR17, Chapter 8] for a discussion of this perspective.

Since then, there has been an enormous amount of interest in, and new results related to, mock theta functions and harmonic Maaß forms[1]. For example, the landmark paper of Bringmann and Ono [BO10a] showed a deep connection between the ranks of partitions, mock theta functions, and harmonic Maaß forms. To describe this, we define

$$\mathcal{R}(\alpha, \beta; q) := \sum_{n \geq 0} \frac{(\alpha\beta)^n q^{n^2}}{(\alpha q; q)_n (\beta q; q)_n},$$

where for $n \in \mathbb{N} \cup \{0, \infty\}$

$$(a)_n := (a; q)_n := \prod_{j=0}^{n-1} \left(1 - aq^j\right)$$

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[1]A definition can be found in each Chapter I, II, III, or in [BFOR17, Chapter 4].

is the usual *q-Pochhammer symbol* and $q := e^{2\pi i \tau}$ with $\tau \in \mathbb{H}$ throughout. The function $\mathcal{R}$ is the three-variable generalization of the partition rank generating function $\mathcal{R}(w, w^{-1}; q)$ and was studied by Folsom [Fol16]. In particular, $\mathcal{R}$ is in essence a universal mock theta function in the sense of Gordon and McIntosh [GM12]. That is, many of Ramanujan's original mock theta functions may be written as specializations of $\mathcal{R}$, up to the addition of a modular form - see [Fol16, p. 490], and also [BFR12, Theorem 3.1].

Returning to the discussion of the results by Bringmann and Ono [BO10a], they showed that $\mathcal{R}(\zeta, \zeta^{-1}; q)$ evaluated at an odd order root of unity $\zeta \neq 1$ is a mock modular form of weight $\frac{1}{2}$ with a certain shadow. The knowledge of the modularity behaviour then allows one to obtain deep arithmetical information on ranks of partitions, including on their asymptotics and exact formulae [Bri09, BO06], as well as congruences that are satisfied, in analogy to the famed Ramanujan congruences. Afterwards, Zagier [Zag09] provided a new proof of Bringmann and Ono's result on the rank generating function in an expository paper, and his proof applies to an even order root of unity $\zeta \neq 1$ as well.

Consequently, the investigation of q-hypergeometric series became a leitmotif in the area of combinatorics as well as in the area of mock modular forms. In many cases, the focus of active research in both of these fields originates in the investigation of some peculiar explicit examples, which shed the first light on a new phenomenon or object. To name one such example, Lovejoy and Osburn [LO13a, LO17] offered a new perspective on some of the classical mock theta functions. In short, they discovered four examples of double sum q-series which are also mock theta functions. In other words, their work can be regarded as the observation that some (if not all) mock theta functions are double sum q-series, which happen to collapse to single sums. Two such functions are $\mathcal{M}_{10}$ and $\mathcal{M}_{17}$ in [LO17], explicitly given by

$$\sum_{n \geq 1} \sum_{n \geq j \geq 1} \frac{(-1)^j (-1)_{2n} q^{j^2+j+n}}{(q^2; q^2)_{n-j} (q^2; q^2)_{j-1} (1 - q^{4j-2})},$$
$$\sum_{n \geq 1} \sum_{n \geq j \geq 1} \frac{(-1)^j (-1)_{2n} q^{j^2-j+n}}{(q^2; q^2)_{n-j} (q^2; q^2)_{j-1} (1 - q^{4j-2})}.$$

(IV.1)

Both are expressible in terms of Zwegers' μ-function (defined below) and ratios of classical theta functions, see [LO17, Theorems 1.6, 1.7], and consequently are mock theta functions.

A second example of a new phenomenon or object is the notion of higher depth mock modular forms, which arose parallel to the work of Lovejoy and Osburn. Roughly speaking, classical mock modular forms can be viewed as preimages of (weakly) holomorphic modular forms under the differential operator

$$\xi_k := 2iv^k \overline{\frac{\partial}{\partial \overline{\tau}}}, \qquad \tau = u + iv,$$

introduced by Bruinier and Funke [BF04], and which is surjective on the space of weight k harmonic Maaß forms. Since the kernel of ξ_k contains precisely the holomorphic functions, this idea generalizes directly to mixed mock modular forms, which were first defined in [DMZ12] (see [BFOR17, Definition 13.2] for a definition as well). Consequently, weakly holomorphic modular forms can be regarded as mock modular forms of depth zero. Now, one can define mixed mock modular forms of depth d inductively as preimages of mixed mock modular forms of depth $d-1$ under ξ_k (see Section IV.2.2 for a precise definition). In other words, resembling the extension of holomorphic modular forms to holomorphic quasimodular forms, higher depth mock modular forms extend the scope of admissible images under ξ_k in a natural fashion. Recently, such forms were connected to black holes by Alexandrov and Pioline [AP20], to the Gromov–Witten theory of elliptic orbifolds [BKR18], and to indefinite theta functions on arbitrary lattices of signature $(r, n-r)$ - see [ABMP18] for the $r = 2$ case and [Naz18] for general r, each of which are generalizations of Zwegers' groundbreaking thesis [Zwe02] where $r = 1$. There are also further applications after relaxing to the notion of higher depth quantum modular forms [BKM19a, BKM19b, Mal20].

The q-hypergeometric structure of examples of mock theta functions is also crucial to applications in geometry and topology. For instance, Nahm's Conjecture asserts that certain q-hypergeometric series are modular if and only if some associated elements of a K-theoretic group (the Bloch group) is torsion. Zagier brilliantly proved this in the case of rank 1 [Zag07]. While it turned out to not be true in higher rank cases (as shown by Vlasenko and Zwegers [VZ11]), Calegari, Garoufalidis, and Zagier [CGZ17] later showed that one direction of it was true in general. Their proof was closely related to objects in knot theory. Indeed, there are procedures whereby knot diagrams produce q-hypergeometric series of a similar shape as Ramanujan's mock theta functions (see, e.g., [GL15]). The proofs of the above cases of Nahm's Conjecture relied on the same sort of asymptotic analysis near roots of unity that Ramanujan employed to discover the mock theta functions. It could have been in families like the one Zagier studied in [Zag07] that some examples were mock modular, and not just modular, but they didn't happen to turn up there. However, such series in more general contexts may play a role in more exotic modular constructions, motivating studies such as that in this chapter[2]. The asymptotics of these knot q-series are closely connected to quantum modular forms and the important Volume Conjecture which now seems to be heavily tied to modularity-type properties [Zag10].

To our knowledge, all higher depth mock modular forms beyond the original sorts considered in Zwegers' thesis are constructed as indefinite theta functions, but not via other means which were historically important in the development of the original mock

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[2]Further related examples are discussed in a recent preprint by Wang [Wan22].

modular forms. In light of these observations from combinatorics and topology, it is natural to ask the following.

Question. *Are there interesting q-hypergeometric depth ≥ 2 mock modular forms?*

This chapter aims to answer this question and to propose a new structure inductively extending Ramanujan's original mock theta functions to a set of spaces of distinguished higher depth mock modular forms.

Classical mock theta functions are special mock modular forms of depth one, whose image under the ξ-operator is a linear combination of unary theta functions, wherefore their weight is either $\frac{1}{2}$ or $\frac{3}{2}$. In this chapter, we focus on a new class of objects closely related to higher depth mixed mock modular forms, which we call higher depth mock theta functions. We define them explicitly in Section IV.2.2, and we construct the first examples of depth two mock theta functions. Being more precise, we focus on three of Ramanujan's order three mock theta functions throughout, given by

$$\nu(q) := \sum_{n \geq 0} \frac{q^{n(n+1)}}{(-q; q^2)_{n+1}}, \qquad \phi(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q^2; q^2)_n}, \qquad \rho(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(q; q^2)_n}.$$

Note that we also cover the cases of the order three mock theta functions ω and ψ implicitly[3] by virtue of their simple relationships to those that appear here. Next, we multiply each of the three mock theta functions by a certain specialization of $\mathcal{R}$.

Our main result shows that these first examples of higher depth mock theta functions arise as double-sum q-hypergeometric functions. Throughout we let $\zeta := e^{2\pi i z}$ with $z \in \mathbb{C}$, and let

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{(q; q)_m}{(q; q)_{m-n}(q; q)_n}$$

be the q-binomial coefficient. We define

$$f_1(z, \tau) := (1 + \nu(q)) \left(1 + \frac{\zeta}{(1 - \zeta)(1 + q)} \mathcal{R}\left(\zeta, -q; q^2 \right) \right),$$

$$f_2(z, \tau) := \phi(q) \left(1 + \frac{\zeta}{(1 - \zeta)(1 + q^2)} \mathcal{R}\left(\zeta, -q^2; q^2 \right) \right),$$

$$f_3(z, \tau) := \rho(q) \left(1 + \frac{\zeta}{(1 - \zeta)(1 - q)} \mathcal{R}\left(\zeta, q; q^2 \right) \right),$$

and have the following result.

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[3]See [BFOR17, Appendix A.2], for instance

Theorem IV.1.1. *Let ζ be a root of unity. Then the functions f_j for $j \in \{1, 2, 3\}$ are each mock theta functions of depth two. Furthermore, we have the following representations as double-sum q-series:*

(1) *The function f_1 can be written as*

$$f_1(z, \tau) = \left(1 + q^{-1}\right) \sum_{m,n \geq 0} (-1)^n \, q^{2n^2} \zeta^{n+m} \begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n}}{\zeta}; q^2\right)_m}{(1 + q^{2n-1})(-q; q^2)_{m+2n}}.$$

(2) *The function f_2 can be written as*

$$f_2(z, \tau) = 2 \sum_{m,n \geq 0} (-1)^n \, q^{2n^2+n} \zeta^{n+m} \begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n+1}}{\zeta}; q^2\right)_m}{(1 + q^{2n})(-q^2; q^2)_{m+2n}}.$$

(3) *The function f_3 can be written as*

$$f_3(z, \tau) = \sum_{m,n \geq 0} (-1)^n \, q^{2n^2+n} \zeta^{n+m} \begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n+1}}{\zeta}; q^2\right)_m (1 - q^{-1})}{(1 - q^{2n-1})(q; q^2)_{m+2n}}.$$

Remarks.

(1) We emphasize that the q-series on the right-hand side of Theorem IV.1.1 may be viewed combinatorially as counting certain families of partitions, in analogy with the depth one case. For succinctness, we do not provide explicit details here.

(2) We highlight that our results and the examples (IV.1) of Lovejoy, Osburn have a similar shape. Further evidence of this connection is found in the fact that Zwegers' μ-function essentially provides the completion of both the double-sum q-series in equation (IV.1) and in Theorem IV.1.1.

Theorem IV.1.2. *Each of the functions f_j has a natural modular completion (see Section IV.2.3).*

Invoking different product formulae for single-sum q-hypergeometric functions may yield a broader set of higher depth mock theta functions, as studied by Lovejoy and Osburn [LO13b] and by Lovejoy [Lov14, Lov22]. In analogy to the depth one case, it is clear that there is a combinatorial interpretation of the q-hypergeometric series described in Theorem IV.1.1. Furthermore, the modularity properties yield asymptotics for the coefficients using standard techniques, and we offer the following general question.

Question. *What are the applications of higher depth mock theta functions?*

IV.2 Preliminaries

In this section we collect some preliminary results and definitions pertinent to the rest of the chapter.

IV.2.1 q-hypergeometric series

We utilize the *q-hypergeometric series*

$$_r\phi_s\left(a_1,\ldots,a_r;b_1,\ldots,b_s;q;z\right) := \sum_{n\geq 0} \frac{(a_1;q)_n\cdots(a_r;q)_n}{(b_1;q)_n\cdots(b_s;q)_n}\frac{z^n}{(q;q)_n}\left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{s-r+1},$$

where r, $s \in \mathbb{N}_0$, and $a_1,\ldots,a_r$, $b_1,\ldots,b_s$, and q are complex parameters, and z is a complex variable. Assuming that no factor $(a_j;q)_n$, $(b_\ell;q)_n$ vanishes, convergence of $_r\phi_s$ as a function of z is discussed by Oshima [Osh17]. Summarizing his exposition, if $0 < |q| < 1$, its radius of convergence equals

$$\begin{cases} \infty & \text{if } r \leq s, \\ 1 & \text{if } r = s+1. \end{cases}$$

If $|q| > 1$ and $a_1\cdots a_r b_1\cdots b_s \neq 0$, then its radius of convergence equals $\left|\frac{b_1\cdots b_s q}{a_1\cdots a_r}\right|$. The case that $|q| = 1$ is discussed in [Osh17, Theorem 1.1].

The function $_r\phi_s$ enables us to state a central formula in our work.

Lemma IV.2.1 ([Sri87, equation (2.10)])**.** *We have*

$$_1\phi_1\left(\lambda;\mu;q;-z\right)\,_2\phi_1\left(\lambda,0;\mu;q;\zeta\right)$$

$$= \sum_{m,n\geq 0} q^{n(n-1)}\frac{(\lambda;q)_{m+n}\left(-q^n\frac{z}{\zeta};q\right)_m\left(\frac{\mu}{\lambda};q\right)_n}{(\mu;q)_{m+2n}(\mu;q)_n}\cdot\frac{\zeta^m}{(q;q)_m}\cdot\frac{(-\lambda z\zeta)^n}{(q;q)_n}. \quad \text{(IV.2)}$$

In addition, we use Fine's function [Fin88]

$$F(a,b;t;q) := \,_2\phi_1(aq,q;bq;q;t) = \sum_{n\geq 0}\frac{(aq;q)_n}{(bq;q)_n}t^n, \qquad |q| < 1.$$

Due to its representation as a specialization of $_2\phi_1$, we see that F converges inside the unit disc as a function of t. However, the function F as a function of t admits a meromorphic extension outside the unit disc with simple poles at most at $t = q^{-n}$, $n \geq 0$, provided that $b \neq q^{-\ell}$, $\ell \geq 1$, according to [Fin88, p. 2].

The function F has the following transformation properties (among others).

Lemma IV.2.2 ([Fin88, equations (4.4), (6.3), (12.3)]). *It holds that*

$$F(a, b; t; q) = \frac{b}{b - at} + \frac{(b - a)t}{(1 - bq)(b - at)} F(a, bq; t; q), \tag{IV.3}$$

$$F(a, b; t; q) = \frac{1 - b}{1 - t} F\left(\frac{at}{b}, t; b; q\right). \tag{IV.4}$$

$$(1 - t)F(0, b; t; q) = \sum_{n \geq 0} \frac{(bt)^n q^{n^2}}{(bq; q)_n (tq; q)_n}. \tag{IV.5}$$

IV.2.2 Higher depth mock modular forms

Higher depth (mixed) mock modular forms arose from talks of Zagier and Zwegers, work of Alexandrov, Banerjee, Manschot and Pioline [ABMP18], and of Nazaroglu [Naz18]. In the following, we specialize [BFOR17, Definition 13.2] to our framework.

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, and let $M_k(\Gamma)$ be the space of holomorphic modular forms of weight k on Γ. We say that a function $f \colon \mathbb{H} \to \mathbb{C}$ transforms like a modular form of weight k on Γ, if for all $\gamma \in \Gamma$ and all $\tau \in \mathbb{H}$ we have

$$f(\tau) = \begin{cases} (c\tau + d)^{-k} f(\gamma\tau) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{2k} (c\tau + d)^{-k} f(\gamma\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

where $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol, and

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

for odd integers d.

Definition IV.2.3. A *modular completion* of a function $f \colon \mathbb{H} \to \mathbb{C}$ on Γ is a function $g \colon \mathbb{H} \to \mathbb{C}$, such that $f + g$ transforms like a modular form of some weight on Γ.

Note that a modular completion is not unique. Indeed, one may add a modular form of suitable weight (or a more general automorphic function) to such a completion. Thus, we emphasize that a *natural* modular completion g should provide new insight on the obstruction towards modularity of the initial function f.

We observe that mock modular forms of depth one are precisely the mixed mock modular forms. In a similar fashion, we now define higher depth mock theta functions.

Definition IV.2.4. Let $\Theta_{\frac{1}{2}}(\Gamma)$, $\Theta_{\frac{3}{2}}(\Gamma)$ be the space of unary theta functions of weight $\frac{1}{2}$ or $\frac{3}{2}$ on Γ, respectively. In addition, let $\mathbb{M}_k^0(\Gamma) := M_k(\Gamma)$. For $d > 0$, the space $\mathbb{M}_k^d(\Gamma)$ of *mock theta functions of depth d and weight k on Γ* is the space of real-analytic functions on $\mathbb{H}$ that

(1) that admit a modular completion of weight k on Γ,

(2) have images under ξ_k that are contained in the space[4]

$$\left(\Theta_{\frac{1}{2}}(\Gamma) \otimes \mathbb{M}_{k-\frac{1}{2}}^{d-1}(\Gamma) \right) \oplus \left(\Theta_{\frac{3}{2}}(\Gamma) \otimes \mathbb{M}_{k-\frac{3}{2}}^{d-1}(\Gamma) \right),$$

(3) are of at most linear exponential growth towards the cusps of Γ.

Following Zagier [Zag09], we observe once more that mock theta functions of depth one are precisely the classical mock theta functions multiplied by modular forms.

IV.2.3   Modular completions

In this section we collect the modular completions of ν, ϕ, ρ, and $\mathcal{R}$. Suppose that $z_1, z_2 \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$. Zwegers [Zwe02] defined his μ-function by

$$\mu\left(e^{2\pi i z_1}, e^{2\pi i z_2}; \tau \right) :=$$

$$\frac{e^{\pi i z_1}}{-i q^{\frac{1}{8}} e^{-\pi i z_2} (q;q)_\infty \left(e^{2\pi i z_2}; q \right)_\infty \left(e^{-2\pi i z_2} q; q \right)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}} e^{2\pi i n z_2}}{1 - e^{2\pi i z_1} q^n},$$

where we used the Jacobi triple product identity. To describe a natural modular completion of μ, we recall the *error function*

$$E(z) := 2 \int_0^z e^{-\pi t^2} \mathrm{d}t$$

for $z \in \mathbb{R}$, along with

$$R(z_1; \tau) := \sum_{n \in \frac{1}{2} + \mathbb{Z}} \left(\mathrm{sgn}(n) - E\left(\left(n + \frac{\mathrm{Im}(z_3)}{v} \right) \sqrt{2v} \right) \right) (-1)^{n - \frac{1}{2}} e^{-\pi i n^2 \tau - 2\pi i n z_1}.$$

Following Choi [Cho11], we additionally consider

$$\mathcal{U}(\alpha, \beta; q) := \sum_{n \geq 1} \left(\alpha^{-1}; q \right)_n \left(\beta^{-1}; q \right)_n q^n,$$

and

$$\mathcal{M}(z_1, z_2, \tau)$$
$$:= i q^{\frac{1}{8}} \left(1 - e^{2\pi i z_1} \right) e^{\pi i (z_2 - z_1)} \left(e^{2\pi i (\tau - z_1)}; q \right)_\infty \left(e^{-2\pi i z_2}; q \right)_\infty \mu\left(e^{2\pi i z_1}, e^{2\pi i z_2}; \tau \right).$$

---

[4]The symbol $\otimes$ refers to the usual tensor product of vector spaces.

Lemma IV.2.5. *The function* $\mathcal{R}(e^{2\pi i z_1}, e^{2\pi i z_2}; q)$ *admits a completion given by*

$$
\begin{aligned}
\mathcal{C}(e^{2\pi i z_1}, e^{2\pi i z_2}; q) := -\frac{q^{\frac{1}{8}}}{2} \left(1 - e^{2\pi i z_1}\right) e^{\pi i (z_2 - z_1)} \\
\times \left(e^{2\pi i (\tau - z_1)}; q\right)_\infty \left(e^{-2\pi i z_2}; q\right)_\infty R(z_1 - z_2; \tau) + \mathcal{U}\left(e^{2\pi i z_1}, e^{2\pi i z_2}; q\right).
\end{aligned}
$$

Proof. We follow an idea of Folsom, Ono, and Rhoades [FOR13, Section 3], which we recall here for convenience. Ramanujan's identity [AB09, p. 67, entry 3.4.7] can be rewritten as

$$
\mathcal{M}(z_1, z_2, \tau) = \mathcal{R}\left(e^{2\pi i z_1}, e^{2\pi i z_2}; q\right) + \mathcal{U}\left(e^{2\pi i z_1}, e^{2\pi i z_2}; q\right),
$$

compare [Cho11, Theorem 4] as well. By Zwegers' thesis [Zwe02, Theorem 1.11], the modular completion of $\mathcal{M}(z_1, z_2; \tau)$ is given by

$$
C(z_1, z_2; \tau) := -\frac{q^{\frac{1}{8}}}{2} \left(1 - e^{2\pi i z_1}\right) e^{\pi i (z_2 - z_1)} \left(e^{2\pi i (\tau - z_1)}; q\right)_\infty \left(e^{-2\pi i z_2}; q\right)_\infty R(z_1 - z_2; \tau).
$$

This in turn produces the modular completion of $\mathcal{R}$ as $C(z_1, z_2; \tau) + \mathcal{U}(e^{2\pi i z_1}, e^{2\pi i z_2}; q)$. $\quad\square$

The modular completions of Ramanujan's mock theta functions are known by Zwegers' thesis [Zwe02], and their representations in terms of μ (see [BFOR17]Appendix A.2, for example). For convenience, we recall the modular completions here without proof.

Lemma IV.2.6. *We have the following modular completions.*

(1) *The mock theta function* $\nu(q)$ *admits a completion given by*

$$
-q^{-\frac{1}{2}} R\left(2\tau; 12\tau\right).
$$

(2) *The mock theta function* $\phi(q)$ *admits a completion given by*

$$
-e^{\frac{\pi i}{8}} q^{-\frac{1}{8}} R\left(-\tau; 3\tau + \frac{1}{2}\right).
$$

(3) *The mock theta function* $\rho(q)$ *admits a completion given by*

$$
-\frac{1}{2} q^{-\frac{3}{4}} R\left(\tau; 6\tau\right).
$$

IV.3 Proof of Theorems IV.1.1 and IV.1.2

We prepare the application of equation (IV.2) with a lemma. First note that it is easy to show that

$$1 + \nu(q) = {}_1\phi_1\left(q^2; -q; q^2; -1\right),\tag{IV.6}$$

$$\phi(q) = {}_1\phi_1\left(q^2; -q^2; q^2; -q\right),\tag{IV.7}$$

$$\rho(q) = {}_1\phi_1\left(q^2; q; q^2; -q\right).\tag{IV.8}$$

Then we prove the following.

Lemma IV.3.1. *We have the identities*

$$
{}_2\phi_1\left(q^2, 0; -q; q^2; \zeta\right) = 1 + \frac{\zeta}{(1-\zeta)(1+q)}\mathcal{R}\left(\zeta, -q; q^2\right),
$$

$$
{}_2\phi_1\left(q^2, 0; -q^2; q^2; \zeta\right) = 1 + \frac{\zeta}{(1-\zeta)\left(1+q^2\right)}\mathcal{R}\left(\zeta, -q^2; q^2\right),
$$

$$
{}_2\phi_1\left(q^2, 0; q; q^2; \zeta\right) = 1 + \frac{\zeta}{(1-\zeta)(1-q)}\mathcal{R}\left(\zeta, q; q^2\right).
$$

Proof. To verify the first equation, we first rewrite the left hand side in terms of Fine's function F, namely

$$
{}_2\phi_1\left(q^2, 0; -q; q^2; \zeta\right) = F\left(0, -q^{-1}; \zeta; q^2\right).
$$

Then by equation (IV.3) we find that

$$
F\left(0, -q^{-1}; \zeta; q^2\right) = 1 + \frac{\zeta}{1+q^2}F\left(0, -q; \zeta; q^2\right).
$$

Next, by equation (IV.4), we obtain

$$
F\left(0, \zeta; -q; q^2\right) = \frac{1-\zeta}{1+q}F\left(0, -q; \zeta; q^2\right).
$$

Thus,

$$
F\left(0, -1; \zeta; q^2\right) = 1 + \frac{\zeta}{1-\zeta}F\left(0, \zeta; -q; q^2\right).
$$

Using equation (IV.5), we arrive at

$$
F\left(0, \zeta; -q; q^2\right) = \frac{1}{1+q}\left(\sum_{n\geq 0}\frac{(-q\zeta)^n q^{2n^2}}{(\zeta q^2; q^2)_n(-q^3; q^2)_n}\right) = \frac{1}{1+q}\mathcal{R}\left(\zeta, -q; q^2\right).
$$

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Hence,

$$_2\phi_1\left(q^2, 0; -q; q^2; \zeta\right) = 1 + \frac{\zeta}{(1-\zeta)(1+q)}\mathcal{R}\left(\zeta, -q; q^2\right),$$

as claimed.

To verify the second equation, we proceed analogously. Explicitly, we begin with

$$_2\phi_1\left(q^2, 0; -q^2; q^2; \zeta\right) = F\left(0, -1; \zeta; q^2\right).$$

Then, by equation (IV.3) we find that

$$F\left(0, -1; \zeta; q^2\right) = 1 + \frac{\zeta}{1+q^2}F\left(0, -q^2; \zeta; q^2\right).$$

Next, by equation (IV.4), we obtain

$$F\left(0, \zeta; -q^2; q^2\right) = \frac{1-\zeta}{1+q^2}F\left(0, -q^2; \zeta; q^2\right).$$

Thus,

$$F\left(0, -1; \zeta; q^2\right) = 1 + \frac{\zeta}{1-\zeta}F\left(0, \zeta; -q^2; q^2\right).$$

Using equation (IV.5) to inspect

$$F\left(0, \zeta; -q^2; q^2\right) = \frac{1}{1+q^2}\left(\sum_{n\geq 0}\frac{(-q^2\zeta)^n q^{2n^2}}{(\zeta q^2; q^2)_n(-q^4; q^2)_n}\right) = \frac{1}{1+q^2}\mathcal{R}\left(\zeta, -q^2; q^2\right).$$

This proves the second equation.

For the final equality in the lemma, we see that

$$_2\phi_1\left(q^2, 0; q; q^2; \zeta\right) = F\left(0, q^{-1}; \zeta; q^2\right).$$

By equation (IV.3) this is

$$F\left(0, q^{-1}; \zeta; q^2\right) = 1 + \frac{\zeta}{1-q}F\left(0, q; \zeta; q^2\right).$$

Using equation (IV.4), we then have that

$$F\left(0, \zeta; q; q^2\right) = \frac{1-\zeta}{1-q}F\left(0, q; \zeta; q^2\right).$$

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We obtain

$$F\left(0, q^{-1}; \zeta; q^2\right) = 1 + \frac{\zeta}{1-\zeta} F\left(0, \zeta; q; q^2\right).$$

Inspecting the final term more closely with equation (IV.5), we find that

$$F\left(0, \zeta; q; q^2\right) = \frac{1}{1-q}\left(\sum_{n\geq 0} \frac{(\zeta q)^n q^{2n^2}}{(\zeta q^2; q^2)_n (q^3; q^2)_n}\right) = \frac{1}{1-q}\mathcal{R}\left(\zeta, q; q^2\right).$$

Combining these yields the third equation, and thus the lemma is proven. $\qquad\square$

We now are able to prove our main theorems.

Proof of Theorem IV.1.1. We utilize equation (IV.2) to prove the representations of f_1, f_2, and f_3 as double-sum q-series. We begin with the first case. By Lemma IV.3.1 and equation (IV.6) the left-hand side is $_1\phi_1(q^2; -q; q^2; -1)_2\phi_1(q^2, 0; -q; q^2; \zeta)$. We compute by equation (IV.2) that

$$_1\phi_1\left(q^2; -q; q^2; -1\right)_2\phi_1\left(q^2, 0; -q; q^2; \zeta\right)$$

$$= \sum_{m,n\geq 0} q^{2n(n-1)} \frac{(q^2; q^2)_{m+n}\left(-\frac{q^{2n}}{\zeta}; q^2\right)_m (-q^{-1}, q^2)_n}{(-q; q^2)_{m+2n}(-q; q^2)_n} \cdot \frac{\zeta^m}{(q^2; q^2)_m} \cdot \frac{\left(-q^2\zeta\right)^n}{(q^2; q^2)_n}$$

$$= \sum_{m,n\geq 0} (-1)^n q^{2n^2} \zeta^{n+m} \begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n}}{\zeta}; q^2\right)_m (-q^{-1}; q^2)_n}{(-q; q^2)_n (-q; q^2)_{m+2n}}$$

$$= \sum_{m,n\geq 0} (-1)^n q^{2n^2} \zeta^{n+m} \begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n}}{\zeta}; q^2\right)_m (1+q^{-1})}{(1+q^{2n-1})(-q; q^2)_{m+2n}}.$$

Next, consider the second case. By Lemma IV.3.1 and equation (IV.7), the left-hand side is $_1\phi_1(q^2; -q^2; q^2; -q)_2\phi_1(q^2, 0; -q^2; q^2; \zeta)$. Utilizing equation (IV.2), we obtain

$$_1\phi_1\left(q^2; -q^2; q^2; -q\right)_2\phi_1\left(q^2, 0; -q^2; q^2; \zeta\right)$$

$$= \sum_{m,n\geq 0} q^{2n(n-1)} \frac{(q^2; q^2)_{m+n}\left(-\frac{q^{2n+1}}{\zeta}; q^2\right)_m (-1; q^2)_n}{(-q^2; q^2)_{m+2n}(-q^2; q^2)_n} \cdot \frac{\zeta^m}{(q^2; q^2)_m} \cdot \frac{\left(-q^3\zeta\right)^n}{(q^2; q^2)_n}$$

$$= \sum_{m,n\geq 0} (-1)^n q^{2n^2+n} \zeta^{n+m} \begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n+1}}{\zeta}; q^2\right)_m (-1; q^2)_n}{(-q^2; q^2)_n (-q^2; q^2)_{m+2n}}$$

$$= 2\sum_{m,n\geq 0} (-1)^n q^{2n^2+n} \zeta^{n+m} \begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n+1}}{\zeta}; q^2\right)_m}{(1+q^{2n})(-q^2; q^2)_{m+2n}}.$$

Finally, we prove the third case. By Lemma IV.3.1 and equation (IV.8) the left-hand side is $_1\phi_1(q^2;q;q^2;-q)_2\phi_1(q^2,0;q;q^2;\zeta)$. Utilizing equation (IV.2), we obtain

$$
_1\phi_1\left(q^2;q;q^2;-q\right){}_2\phi_1\left(q^2,0;q;q^2;\zeta\right)
$$

$$
= \sum_{m,n\geq 0} q^{2n(n-1)} \frac{(q^2;q^2)_{m+n}\left(-\frac{q^{2n+1}}{\zeta};q^2\right)_m (q^{-1};q^2)_n}{(q;q^2)_{m+2n}(q;q^2)_n} \cdot \frac{\zeta^m}{(q^2;q^2)_m} \cdot \frac{(-q^3\zeta)^n}{(q^2;q^2)_n}
$$

$$
= \sum_{m,n\geq 0} (-1)^n q^{2n^2+n}\zeta^{n+m}\begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n+1}}{\zeta};q^2\right)_m (q^{-1};q^2)_n}{(q;q^2)_n (q;q^2)_{m+2n}}
$$

$$
= \sum_{m,n\geq 0} (-1)^n q^{2n^2+n}\zeta^{n+m}\begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2} \frac{\left(-\frac{q^{2n+1}}{\zeta};q^2\right)_m (1-q^{-1})}{(1-q^{2n-1})(q;q^2)_{m+2n}}.
$$

This proves Theorem IV.1.1. $\qquad\square$

Finally, we conclude with the proof of Theorem IV.1.2.

Proof of Theorem IV.1.2. Combining Lemmas IV.2.5 and IV.2.6 immediately yields modular completions of f_1, f_2, and f_3 in the obvious fashion. For instance, since

$$
f_1(\tau) = 1 + \nu(q) + \frac{\zeta}{(1-\zeta)(1+q)}\mathcal{R}\left(\zeta,-q;q^2\right) + \nu(q)\frac{\zeta}{(1-\zeta)(1+q)}\mathcal{R}\left(\zeta,-q;q^2\right),
$$

a natural modular completion of f_1 is given by

$$
-1 - q^{-\frac{1}{2}}R(2\tau;12\tau) + \frac{\zeta}{(1-\zeta)(1+q)}\mathcal{C}\left(\zeta,-q;q^2\right)
$$

$$
- q^{-\frac{1}{2}}R(2\tau;12\tau)\frac{\zeta}{(1-\zeta)(1+q)}\mathcal{C}\left(\zeta,-q;q^2\right).
$$

The cases of f_2 and f_3 are completely analogous. This proves that these functions are indeed mock theta functions of depth two. $\qquad\square$

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Part B

Locally harmonic Maaß forms

Chapter V

Eisenstein series of even weight $k \geq 2$ and integral binary quadratic forms

This chapter is based on a paper [Mon22a] of the same title published in *Proceedings of the American Mathematical Society*.

V.1   Introduction and statement of results

Integral binary quadratic forms play a decisive role in the construction of many modular objects, mainly to investigate various classes of theta functions. However, they can also be utilized to construct another prominent class of modular objects, namely families of Eisenstein series. We will define Eisenstein series associated to some $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \setminus \{\pm\mathbb{1}\}$, and call them elliptic, parabolic or hyperbolic respectively corresponding to the motion γ induces on the upper half plane $\mathbb{H}$. Although such constructions go back to Petersson [Pet44] essentially, and the analytic continuation of the classical parabolic Eisenstein series was established by Selberg [Sel56] and Roelcke [Roe66, Roe67] some years later, similar results remained elusive in the other two cases.

A first breakthrough was made in weight 0 some years ago, which completes the picture regarding analytic continuation to $s = 1$, and was established by Jorgenson, Kramer, von Pippich, Schwagenscheidt, and Völz, compare [JKvP10, Theorem 4.2], [vP16, Section 4], [vPSV21, Theorem 1.2], [Mat20, Appendix B]. Hence, it seems natural to ask whether similar results hold in weight 2 or higher, overleaping the "point of symmetry" $k = 1$. In a recent preprint [Mat20], Matsusaka investigated parabolic, elliptic, and hyperbolic Eisenstein series in weight 2. In a second breakthrough, Bringmann and Kane [BK16] provided the analytic continuation of Petersson's weight 2 elliptic Poincaré series to $s = 0$, which enabled Matsusaka [Mat20, Theorem 2.3] to extend this result to the weight 2 elliptic Eisenstein series.

Consequently, we focus on the case of hyperbolic Eisenstein series. If $k = 2$, Schwagenscheidt [Sch18, Remark 5.4.6] argued towards existence of the analytic continuation

to $s = 0$, and Matsusaka [Mat20, equation (2.12)] conjectured its shape. We extend Matsusaka's setting to general even weight $k \geq 2$, and embed his Eisenstein series into a framework based on discriminants of integral binary quadratic forms. This enables us to prove Matsusaka's conjecture for any positive non-square discriminant in weight 2 by computing the Fourier expansion of our hyperbolic Eisenstein series. To this end, we adapt Zagier's method [Zag75, Section 2], and appeal to results of Duke, Imamoḡlu, and Tóth [DIT11].

We stipulate $\tau = u + iv \in \mathbb{H}$ throughout, and introduce all involved objects and terminology during sections V.2 to V.4 in detail.

Theorem V.1.1. *Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and primitive. Then the function $\mathcal{E}_{2,\gamma}(\tau, s)$ can be analytically continued to $s = 0$ and the continuation is given by*

$$
\lim_{s \to 0} \mathcal{E}_{2,\gamma}(\tau, s)
$$
$$
= \frac{-2}{\Delta(\gamma)^{\frac{1}{2}}} \sum_{m \geq 0} \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} j_m(w) \frac{|\mathrm{d}w|}{\mathrm{Im}(w)} \, q^m - \frac{-2}{\Delta(\gamma)^{\frac{1}{2}}} \mathrm{tr}_{d,\Delta(\gamma)}(1) E_2^*(\tau)
$$

for any $\tau \in \mathbb{H}$. Here, $\mathrm{tr}_{d,\Delta(\gamma)}(1)$ is a twisted trace of cycle integrals given by

$$
\mathrm{tr}_{d,\Delta(\gamma)}(1) := \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} \frac{|\mathrm{d}w|}{\mathrm{Im}(w)}.
$$

Furthermore, if v is sufficiently large, that is τ is located above the net of geodesics $\bigcup_{Q \in \mathcal{Q}_{\Delta(\gamma)}} S_Q$, then we have

$$
\lim_{s \to 0} \mathcal{E}_{2,\gamma}(\tau, s) = \frac{-2}{\Delta(\gamma)^{\frac{1}{2}}} \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} \left(\frac{\frac{1}{2\pi i} \frac{\partial j}{\partial \tau}(\tau)}{j(w) - j(\tau)} - E_2^*(\tau) \right) \frac{|\mathrm{d}w|}{\mathrm{Im}(w)}.
$$

Remarks.

(1) The function $\mathcal{E}_{2,\gamma}(\tau, s)$ is a twisted trace of individual hyperbolic Eisenstein series, which we denote by $E_{2,\gamma}(\tau, s)$.

(2) We will indicate below that the analytic continuation of the weight 2 parabolic / elliptic Eisenstein series to $s = 0$ is a harmonic / polar harmonic Maaß form. Such forms generalize the notion of classical holomorphic modular forms by relaxing analytical and growth conditions to a nonholomorphic setting. Theorem V.1.1 completes the picture in the sense that the resulting cycle integral is a locally harmonic Maaß form of weight 2 in τ with $\mathrm{Im}(\tau)$ sufficiently large. These objects were introduced by Bringmann, Kane, and Kohnen in [BKK15] for weights $2 - 2\ell$,

$\ell \in \mathbb{N}_{\geq 2}$, (see also [BFOR17, Section 13.4]), and independently by Hövel [Höv12] for weight 0 in his PhD. thesis. Roughly speaking, such a form is a harmonic Maaß form that is permitted to have singularities on the net of geodesics $\bigcup_{Q \in \mathcal{Q}_D} S_Q$. The singularities occur due to the presence of a sign-function in [BKK15], and are called "jumping singularities" (see Section VII.2 for a definition).

As a byproduct of our approach, we obtain the expansion of $\mathcal{E}_{k,\gamma}(\tau, 0)$ for every even weight $k \geq 4$. This was known by Parson [Par93, Theorem 3.1] without the twisting. In particular, if $k \geq 4$ satisfies $k \equiv 0 \pmod 4$, then the hyperbolic Eisenstein series $E_{k,\gamma}(\tau, 0)$ is a holomorphic cusp form. In this case, the Fourier expansion of the twisted traces of hyperbolic Eisenstein series of weight $4 \mid k > 2$ was already established by Gross, Kohnen, and Zagier [GKZ87, p. 517].

Theorem V.1.2. *Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and primitive, and suppose $k \geq 4$ is even. Moreover, let $G_m(\tau, s)$ be the Niebur Poincaré series defined in equation* (V.1) *and Definition VI.2.4 below. Then, we have*

$$\mathcal{E}_{k,\gamma}(\tau, 0) = \frac{(-1)^{\frac{k}{2}} 2\pi^{\frac{k}{2}}}{\Delta(\gamma)^{\frac{k}{4}} \Gamma\left(\frac{k}{4}\right)^2} \sum_{m \geq 1} m^{\frac{k}{2}-1} \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} G_{-m}\left(w, \frac{k}{2}\right) \frac{|\mathrm{d}w|}{\mathrm{Im}(w)} \; q^m.$$

We devote Section V.5 to the development of both theorems.

V.2 Preliminaries

Let us summarize some general framework first, more details can be found for example in [Iwa97, Chapter 2] regarding hyperbolic geometry, and in [Zag81, § 8] regarding integral binary quadratic forms.

V.2.1 Fractional linear transformations

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) =: \Gamma$. The group Γ acts on $\mathbb{H} \cup \mathbb{R} \cup \{i\infty\}$ by Möbius transformations. We write $j(\gamma, \tau) := c\tau + d$ for the usual modular multiplier, and summarize some standard facts on the classification of motions.

(1) An element $\gamma \in \Gamma \setminus \{\pm \mathbb{1}\}$ is called *parabolic* if $|\mathrm{tr}\,(\gamma)| = 2$. We have a unique fixed point $\mathfrak{a}_\gamma$ of γ, called a *cusp*, and located in $\mathbb{Q} \cup \{i\infty\}$. The stabilizer of each cusp is conjugate to the stabilizer of $i\infty$, which is generated by $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ up to sign. In other words $\gamma = \pm \sigma_{\mathfrak{a}_\gamma} T^n \sigma_{\mathfrak{a}_\gamma}^{-1}$ for some $n \in \mathbb{Z} \setminus \{0\}$, where $\sigma_{\mathfrak{a}_\gamma}$ is a scaling matrix of the cusp, namely it satisfies $\sigma_{\mathfrak{a}_\gamma} \infty = \mathfrak{a}_\gamma$. Points are moved by γ along horocycles, that are circles in $\mathbb{H}$ tangent to $\mathbb{R}$.

(2) An element $\gamma \in \Gamma$ is called *elliptic* if $|\mathrm{tr}\,(\gamma)| < 2$. Recall that any elliptic fixed point w_γ is Γ-equivalent to either i or $\omega := e^{\frac{\pi i}{3}}$. Letting $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $U := TS = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we see that $\Gamma_i = \{\mathbb{1}, S, S^2, S^3\}$, $\Gamma_\omega = \{\mathbb{1}, U, \ldots, U^5\}$. Points are moved by γ along circles centered at w_γ.

(3) An element $\gamma \in \Gamma$ is called *hyperbolic* if $|\mathrm{tr}\,(\gamma)| > 2$. Recall that γ has precisely two different fixed points w_γ, w'_γ, located on the real axis. Writing $\Gamma_{w_\gamma, w'_\gamma} = \pm\langle \eta_{w_\gamma, w'_\gamma} \rangle$, there exists a scaling matrix $\sigma_{w_\gamma, w'_\gamma} \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma_{w_\gamma, w'_\gamma} 0 = w_\gamma$, $\sigma_{w_\gamma, w'_\gamma} \infty = w'_\gamma$, and $\sigma_{w_\gamma, w'_\gamma}^{-1} \eta_{w_\gamma, w'_\gamma} \sigma_{w_\gamma, w'_\gamma} = \pm\left(\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix}\right)$ for some $y \in \mathbb{R}_{>0}$. Points are moved by γ along hypercycles, that are lines and circle arcs intersecting $\mathbb{R}$ at non-perpendicular angles.

V.2.2 Integral binary quadratic forms

Let Q be an integral binary quadratic form, and the terminology "quadratic form" abbreviates such forms throughout. The group Γ acts on the set of quadratic forms by $(Q \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(x, y) := Q(ax + by, cx + dy)$, and this induces an equivalence relation, which we denote by $\sim$. Moreover, the actions of Γ on $\mathbb{H}$ and on quadratic forms are compatible, in the sense that $(Q \circ \gamma)(\tau, 1) = j(\gamma, \tau)^2 Q(\gamma\tau, 1)$. Sometimes, we abbreviate $[a, b, c] := ax^2 + bxy + cy^2$, and we denote its discriminant $b^2 - 4ac$ by $\Delta([a, b, c])$. One can check that the discriminant is invariant under $\sim$. For every $D \in \mathbb{Z}$, we let $\mathcal{Q}_D := \{Q \colon \Delta(Q) = D\}$ be the set of all quadratic forms with discriminant D. If $D \neq 0$ the set $\mathcal{Q}_D/\Gamma$ is finite, and its cardinality is called the class number $h(D)$. If $D \equiv 0 \pmod 4$ or $D \equiv 1 \pmod 4$, then $\mathcal{Q}_D/\Gamma$ is non-empty.

V.2.3 Heegner geodesics

Let $Q \neq 0$ be a quadratic form. If $\Delta(Q) > 0$ then we associate to Q the *Heegner geodesic* $S_Q := \{\tau \in \mathbb{H} : a|\tau|^2 + b\,\mathrm{Re}(\tau) + c = 0\}$, which joins the two distinct zeros of $Q(\tau, 1)$. If $a = 0$, then the second point is given by $-\frac{c}{b}$.

V.2.4 Quadratic forms associated to $\gamma \in \Gamma$

In addition, we define $Q_\gamma(x, y)$ to be the quadratic form $cx^2 + (d-a)xy - by^2$ associated to $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. We set $\Delta(\gamma) := \Delta(Q_\gamma) = \mathrm{tr}\,(\gamma)^2 - 4$, and observe that the sign of $\Delta(\gamma)$ depends precisely on hyperbolicity, parabolicity, or ellipticity of γ respectively. Futhermore, we note that $Q_{-\gamma}(x, y) = Q_{\gamma^{-1}}(x, y) = -Q_\gamma(x, y)$. Hence, we invoke a sign-function on quadratic forms. Namely, we define

$$\mathrm{sgn}\,([a, b, c]) := \begin{cases} \mathrm{sgn}(a) & \text{if } a \neq 0, \\ \mathrm{sgn}(c) & \text{if } a = 0. \end{cases}$$

This will cause a difference in the case of positive discriminant only.

Lemma V.2.1. *Suppose $\Delta(Q) \leq 0$. Then $Q \sim -Q$ implies $Q = 0$.*

See [Völ18, p. 21] (final paragraph) as well.

V.2.5 Genus characters

This subsection follows the exposition given by Gross, Kohnen, and Zagier in [GKZ87, p. 508]. Let $Q = [a, b, c]$ be a quadratic form. We observe that $\sim$ preserves $\gcd(a, b, c)$ as well. We would like to define a Γ-invariant function on $\mathcal{Q}_D$ (assume $D \equiv 0 \pmod 4$ or $D \equiv 1 \pmod 4$). If $D \neq 0$, let d be a fundamental discriminant dividing D, and let $\left(\frac{d}{\cdot}\right)$ be the Kronecker symbol. In addition, an integer n is represented by Q if there exist x, $y \in \mathbb{Z}$, such that $Q(x, y) = n$. This established, we define

$$\chi_d([a, b, c]) := \begin{cases} \left(\dfrac{d}{n}\right) & \text{if } \gcd(a, b, c, d) = 1, [a, b, c] \text{ represents } n, \gcd(d, n) = 1, \\ 0 & \text{if } \gcd(a, b, c, d) > 1. \end{cases}$$

One can verify that such an integer n always exists, and the definition is independent from its choice. Since equivalent quadratic forms represent the same integers, this function is indeed invariant under $\sim$. The choice $d = 1$ yields the trivial character. The definition of $\chi_d([a, b, c])$ extends to $D = 0$ by choosing $d = 0$ in this case, compare the proof of Lemma V.3.5. Additional properties of χ_d are summarized in [GKZ87, Propositions 1, 2].

V.3 Construction of Eisenstein series

V.3.1 Eisenstein series associated to a quadratic form

This construction is based on the following two observations, and follows [Mat20].

Lemma V.3.1. *Let $\gamma \in \Gamma \setminus \{\pm \mathbb{1}\}$, and Q_γ be the associated quadratic form to γ.*

(1) *The zeros of $Q_\gamma(\tau, 1)$ are precisely the fixed points of γ in $\mathbb{H} \cup \mathbb{R}$.*
(2) *The equivalence class of $Q_\gamma(\tau, 1)$ is precisely the set $\{Q_{M^{-1}\gamma M}(\tau, 1) \colon M \in \Gamma\}$.*

We observe that division by $Q_\gamma(\cdot, 1)$ and averaging over equivalence classes of $Q_\gamma(\cdot, 1)$ modulo its zeros provides a function of weight 2. Consequently, we define the following functions.

Definition V.3.2. Let $\gamma \in \Gamma \setminus \{\pm\mathbb{1}\}$, $k \in 2\mathbb{N}$, $\tau \in \mathbb{H}$, $\mathrm{Re}(s) > 1 - \frac{k}{2}$, and $w(\gamma)$ be the set of fixed points of γ. Then we define

$$E_{k,Q_0}(\tau, s) := \sum_{Q \sim Q_0} \frac{\mathrm{sgn}\,(Q)^{\frac{k}{2}} v^s}{Q(\tau, 1)^{\frac{k}{2}} |Q(\tau, 1)|^s},$$

$$E_{k,\gamma}(\tau, s) := E_{k,Q_\gamma}(\tau, s) = \sum_{M \in \Gamma_{w(\gamma)} \backslash \Gamma} \frac{\mathrm{sgn}\,(Q_{M^{-1}\gamma M})^{\frac{k}{2}} v^s}{Q_{M^{-1}\gamma M}(\tau, 1)^{\frac{k}{2}} \left|Q_{M^{-1}\gamma M}(\tau, 1)\right|^s}.$$

We establish convergence.

Lemma V.3.3. *For every $k \in 2\mathbb{N}$ the series defining $E_{k,Q}(\tau, s)$ converges absolutely and locally uniformly for $\tau \in \mathbb{H}$ and $\mathrm{Re}(s) > 1 - \frac{k}{2}$.*

Proof. This follows by results of Petersson [Pet48, Satz 1, Satz 4, Satz 6]. $\qquad\square$

However, $E_{k,\gamma}$ is not modular yet, because the sign-function is not invariant under equivalence of quadratic forms. The circumvention of this obstruction depends on the motion γ induces.

V.3.2 Eisenstein series associated to a given discriminant

Let $D \equiv 0 \,(\mathrm{mod}\ 4)$ or $D \equiv 1 \,(\mathrm{mod}\ 4)$. If $D \neq 0$, we let d be the positive fundamental discriminant dividing D, else we set $d = 0$. We average over $\mathcal{Q}_D$. Henceforth, we twist the average by a genus character, and split the sum into equivalence classes (recall that such a character descends to $\mathcal{Q}_D/\Gamma$).

Definition V.3.4. Let $\gamma \in \Gamma \setminus \{\pm\mathbb{1}\}$, $k \in 2\mathbb{N}$, $\tau \in \mathbb{H}$, $\mathrm{Re}(s) > 1 - \frac{k}{2}$. Then we define

$$\mathcal{E}_{k,D}(\tau, s) := \sum_{0 \neq Q \in \mathcal{Q}_D/\Gamma} \chi_d\,(Q)\, E_{k,Q}(\tau, s),$$

$$\mathcal{E}_{k,\gamma}(\tau, s) := \mathcal{E}_{k,\Delta(\gamma)}(\tau, s) = \sum_{0 \neq Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d\,(Q)\, E_{k,Q}(\tau, s).$$

We establish convergence again.

Lemma V.3.5. *For every $k \in 2\mathbb{N}$, $\tau \in \mathbb{H}$, and $\mathrm{Re}(s) > 1 - \frac{k}{2}$ the series defining $\mathcal{E}_{k,D}(\tau, s)$ converges absolutely and locally uniformly.*

Proof. If $D = 0$, then $\chi_0(Q) = 0$ except if Q is primitive, and represents ± 1. Thus, we reduce to the quadratic forms $[\pm c^2, 2cd, \pm d^2]$ for any coprime pair $(c, d) \in \mathbb{Z}^2$. But such a quadratic form is equivalent to either $[-1, 0, 0]$ or $[1, 0, 0]$. If $D \neq 0$ the class number $h(D)$ is finite. This proves the claim. $\qquad\square$

V.4 Parabolic and elliptic Eisenstein series

V.4.1 Parabolic case

It suffices to study the case $\gamma = T^n$, $n \neq 0$. Let $M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma$. We compute

$$M^{-1} T^n M = \begin{pmatrix} 1 + cdn & d^2 n \\ -c^2 n & 1 - cdn \end{pmatrix}, \qquad Q_{M^{-1} T^n M}(\tau, 1) = -n j(\gamma, \tau)^2.$$

Hence, using $\operatorname{sgn}(n) = \frac{n}{|n|}$, we recover the usual real-analytic Eisenstein series

$$E_{k,\gamma}(\tau, s) = E_{k,T^n}(\tau, s) = \frac{1}{|n|^{s + \frac{k}{2}}} \sum_{M \in \Gamma_\infty \backslash \Gamma} \frac{\operatorname{Im}(M\tau)^s}{j(M, \tau)^k}.$$

For any parabolic motion $\gamma \neq \mathbb{1}$, we infer that

$$\mathcal{E}_{k,\gamma}(\tau, s) = 2 \sum_{M \in \Gamma_\infty \backslash \Gamma} \frac{\operatorname{Im}(M\tau)^s}{j(M, \tau)^k}.$$

Modularity

Clearly $\mathcal{E}_{k,\gamma}$ is modular of weight k according to the cocycle property of the modular multiplier, that is $j(M_1 M_2, \tau) = j(M_1, M_2 \tau) j(M_2, \tau)$ for every M_1, $M_2 \in \Gamma$ and every $\tau \in \mathbb{H}$.

Analytic continuation

It is a classical fact that $\mathcal{E}_{k,\gamma}$ can be continued meromorphically to the whole s-plane, see [Sel56, p. 76-79], [Roe67, p. 293]. Be aware of the fact that Roelcke [Roe66, equation (1.6)] uses the automorphy factor $\left(\frac{j(\gamma,\tau)}{|j(\gamma,\tau)|}\right)^{-k}$, whence his initial domain of convergence is $\operatorname{Re}(s) > 1$ for every $k \in \mathbb{R}$.

If $k > 2$, then we may simply insert $s = 0$, and obtain the classical holomorphic modular Eisenstein series

$$\mathcal{E}_{k,\gamma}(\tau, 0) = \sum_{\gcd(c,d)=1} \frac{1}{(c\tau + d)^k}.$$

If $k = 2$, then we utilize Hecke's trick (see Zagier [BvdGHZ08, p. 19-20]) and the Fourier expansion of $E_{2,T^{\pm 1}}(\tau, s)$ (see Iwaniec [Iwa97, p. 51]), to achieve

$$\lim_{s \searrow 0} \mathcal{E}_{k,\gamma}(\tau, s) = 2E_2^*(\tau) := 2\left(E_2(\tau) - \frac{3}{\pi v}\right) := 2\left(1 - 24 \sum_{n \geq 1} \sum_{d|n} dq^n - \frac{3}{\pi v}\right).$$

This is the holomorphic Eisenstein series of weight 2 completed by adding $-\frac{3}{\pi v}$, and thus a harmonic Maaß form of weight 2. A good exposition on the theory as well as on the applications of harmonic Maaß forms can be found in [BFOR17].

V.4.2 Elliptic case

Recall that any elliptic motion is conjugate to either S or U, so it suffices to deal with those two cases, up to a change of sign and class numbers. Those cases correspond to discriminants -4 and -3 respectively, and both class numbers are equal to 1. Reduced primitive representatives are $[1, 0, 1]$, $[1, 1, 1]$, and the genus character of both forms equals 1. Hence, it suffices to investigate[1] $E_{k,S}$ and $E_{k,U}$. To this end, we define the following function.

Definition V.4.1. Let $\tau, z \in \mathbb{H}$ be Γ-inequivalent, and $\operatorname{Re}(s) > 1 - \frac{k}{2}$, $k \in 2\mathbb{N}$. Then, let

$$E_k\left(\tau, z, s\right) := \sum_{M \in \Gamma} \frac{\operatorname{Im}(z)^{s+\frac{k}{2}} \operatorname{Im}(M\tau)^s}{j\left(M, \tau\right)^k \left(M\tau - z\right)^{\frac{k}{2}} \left(M\tau - \overline{z}\right)^{\frac{k}{2}} \left|\left(M\tau - z\right)\left(M\tau - \overline{z}\right)\right|^s}.$$

Modularity

The function E_k enjoys the following properties.

Lemma V.4.2.

(1) *If $\gamma = S$ or $\gamma = U$, then $\operatorname{sgn}\left(Q_{M^{-1}\gamma M}\right) = 1$ for any $M \in \Gamma$, and*

$$E_{k,\gamma}(\tau, s) = \frac{\operatorname{Im}(w_\gamma)^{-s-\frac{k}{2}}}{|\Gamma_{w_\gamma}|} E_k\left(\tau, w_\gamma, s\right).$$

in both cases.

(2) *For any $M \in \Gamma$ we have*

$$E_k\left(M\tau, z, s\right) = j(M, \tau)^k E_k\left(\tau, z, s\right), \quad E_k\left(\tau, Mz, s\right) = E_k\left(\tau, z, s\right).$$

Proof. Both items can be checked by computation, and we provide the main steps.

(1) Suppose $\gamma = S$ (resp. $\gamma = U$) and $w_\gamma = i$ (resp. $w_\gamma = \omega$). Letting $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we compute

$$Q_{M^{-1}SM}(\tau, 1) = \left(a^2 + c^2\right)\tau^2 + 2\left(ab + cd\right)\tau + b^2 + d^2,$$

$$Q_{M^{-1}UM}(\tau, 1) = \left(a^2 + c^2 - ac\right)\tau^2 + \left(2ab + 2cd - ad - bc\right)\tau + b^2 + d^2 - bd,$$

---

[1]Note that the two cases do not cover the more general case of $E_{k,Q}(\tau, s)$ with $\Delta(Q) < 0$. However, one may reuse the function $E_k(\tau, z, s)$ to deal with this case.

which implies the first claim, and additionally

$$j(M,\tau)^2 \left(M\tau - w_\gamma\right)\left(M\tau - \overline{w_\gamma}\right) = Q_{M^{-1}\gamma M}(\tau, 1)$$

in both cases. The second claim follows directly.

(2) One checks the following two identities. For every M_1, $M_2 \in \Gamma$ and every $\tau, z \in \mathbb{H}$ we have

$$j(M_1,\tau)j(M_1,\overline{\tau}) = |j(M_1,\tau)|^2, \quad j(M_2,z)(\tau - M_2 z) = j\left(M_2^{-1},\tau\right)\left(M_2^{-1}\tau - z\right).$$

Modularity in τ follows directly by the cocycle property of the modular multiplier. To show modularity in z we substitute $M_1 = M_3 M_2$, and see that

$$\left(\left|j(M_3,z)\right|^{2s+k}\left|j(M_3,M_2\tau)\right|^{2s}j(M_3 M_2,\tau)^k\right.$$
$$\left.\times \frac{j(M_3^{-1},M_3 M_2\tau)^k}{j(M_3,z)^{\frac{k}{2}}j(M_3,\overline{z})^{\frac{k}{2}}}\left|\frac{j(M_3^{-1},M_3 M_2\tau)^2}{j(M_3,z)j(M_3,\overline{z})}\right|^s\right)^{-1} = \frac{1}{j(M_2,\tau)^k}.$$

This proves the second item. $\qquad\square$

Analytic continuation in weight 2

To describe the analytic continuation we recall one of Petersson's Poincaré series.

Definition V.4.3. Let $\tau, z \in \mathbb{H}$ be Γ-inequivalent, and $\mathrm{Re}(s) > 1 - \frac{k}{2}$, $k \in 2\mathbb{N}$. Then, let

$$P_k\left(\tau,z,s\right) := \sum_{M\in\Gamma} \frac{\mathrm{Im}(z)^{s+\frac{k}{2}}}{j(M,\tau)^k \left|j(M,\tau)\right|^{2s}\left(M\tau - z\right)^{\frac{k}{2}}\left(M\tau - \overline{z}\right)^{\frac{k}{2}}\left|M\tau - \overline{z}\right|^{2s}}.$$

The series P_k enjoys the following transformation properties.

Lemma V.4.4. *Let $M \in \Gamma$. Then*

$$\mathrm{Im}(M\tau)^s P_k\left(M\tau,z,s\right) = j(M,\tau)^k v^s P_k\left(\tau,z,s\right), \quad P_k\left(\tau,Mz,s\right) = P_k\left(\tau,z,s\right).$$

Proof. This follows by the same argument as in the case of $E_k\left(\tau,z,s\right)$. $\qquad\square$

The analytic continuation of P_2 to $s = 0$ was established by Petersson [Pet44], and to $\mathrm{Re}(s) > -\frac{1}{4}$ by Bringmann and Kane [BK16, Theorem 3.1]. To describe it, let $j(\tau)$ be Klein's modular invariant for Γ. Then Asai, Kaneko, and Ninomiya discovered in [AKN97, Theorem 3] the Fourier expansion

$$\frac{\frac{1}{2\pi i}\frac{\partial j}{\partial\tau}(\tau)}{j(w) - j(\tau)} = \sum_{m\geq 0} j_m(w)q^m, \qquad \mathrm{Im}(\tau) > \mathrm{Im}(w),$$

where $j_m(w)$ is the unique element in $\mathbb{C}\left[j(w)\right]$ of the shape

$$j_m(w) = e^{-2\pi i m w} + O\left(e^{2\pi i w}\right).$$

In [BKLOR18], the authors proved that the functions $j_m(w)$ form a Hecke system, namely if T_m denotes the normalized Hecke operator, then

$$j_0(w) = 1, \qquad j_1(w) = j(w) - 744, \qquad j_m(w) = T_m\left(j_1\right)(w).$$

Afterwards, they simplified the expressions from [BK16, Theorem 3.1], based on earlier work of Duke, Imamoḡlu, and Tóth [DIT11, Theorem 5], and proved that

$$\lim_{s \searrow 0} P_2(\tau, z, s) = -2\pi \left(\frac{\frac{1}{2\pi i}\frac{\partial j}{\partial \tau}(\tau)}{j(z) - j(\tau)} - E_2^*(\tau) \right).$$

Matsusaka [Mat20, Theorem 2.3] extended the latter result to $E_2(\tau, z, s)$, especially

$$\lim_{s \searrow 0} E_2(\tau, z, s) = -2\pi \left(\frac{\frac{1}{2\pi i}\frac{\partial j}{\partial \tau}(\tau)}{j(z) - j(\tau)} - E_2^*(\tau) \right),$$

which in turn provides the analytic continuation of $E_{2,\gamma}(\tau, s)$ in the elliptic case.

Remark. Note that the analytic continuation is a polar harmonic Maaß form of weight 2 on Γ in τ. The poles are located on Γz. Such forms satisfy all conditions of an ordinary harmonic Maaß form, but are permitted to have poles in $\mathbb{H}$. See [BFOR17, Section 13.3] for more details.

V.5   Hyperbolic Eisenstein series

Let $\gamma \in \Gamma$ be hyperbolic. Thus $\Delta(\gamma) > 0$, and $\Delta(\gamma)$ is not a square. The two fixed points w_γ, w_γ' of γ are real quadratic irrationals, which are Galois conjugate to each other. The geodesic S_{Q_γ} is an arc in $\mathbb{H}$ connecting w_γ and w_γ' (equivalently, the two zeros of $Q_\gamma(\tau, 1)$), which is perpendicular to $\mathbb{R}$.

V.5.1   Fourier expansion in general

We suppose in addition that γ is primitive throughout, that is the stabilizer $\Gamma_{w_\gamma, w_\gamma'}$ is infinite cyclic, and generated by γ.

Fourier expansion of $E_{k,\gamma}$

We appeal to Zagier's method [Zag75, Section 2]. We recall the double coset decomposition $\Gamma_{w_\gamma,w'_\gamma}\backslash\Gamma/\Gamma_\infty$, and unfold

$$E_{k,\gamma}(\tau,s) = \sum_{M_1 \in \Gamma_{w_\gamma,w'_\gamma}\backslash\Gamma/\Gamma_\infty} \sum_{M_2 \in \langle T\rangle} \frac{\mathrm{sgn}\,(Q_{(M_1M_2)^{-1}\gamma(M_1M_2)})^{\frac{k}{2}}\,\mathrm{Im}(M_1M_2\tau)^s}{j(M_1M_2,\tau)^k Q_\gamma(M_1M_2\tau,1)^{\frac{k}{2}}\,|Q_\gamma(M_1M_2\tau,1)|^s}.$$

We observe that the innermost sum is one-periodic, and hence has a Fourier expansion

$$E_{k,\gamma}(u+iv,s) = \sum_{a\in\mathbb{Z}\backslash\{0\}} \sum_{\substack{b\,(\mathrm{mod}\,2a)\\ \left[a,b,\frac{b^2-\Delta(\gamma)}{4a}\right]\sim Q_\gamma}} \sum_{m\in\mathbb{Z}} c_m(v,s)e^{2\pi imu}$$

with coefficients

$$c_m(v,s)$$
$$= \int_{-\infty}^\infty \frac{v^s\,\mathrm{sgn}\,(a)^{\frac{k}{2}}e^{-2\pi imx}}{\left(a(x+iv)^2 + b(x+iv) + \frac{b^2-\Delta(\gamma)}{4a}\right)^{\frac{k}{2}}\left|a(x+iv)^2 + b(x+iv) + \frac{b^2-\Delta(\gamma)}{4a}\right|^s}\mathrm{d}x.$$

We abbreviate

$$\lambda := \frac{\sqrt{\Delta(\gamma)}}{2\,|a|} > 0,$$

and substitute $x + iv =: it - \frac{b}{2a}$. We infer

$$c_m(v,s) = \frac{-iv^s e^{2\pi im\left(\frac{b}{2a}+iv\right)}}{(-1)^{\frac{k}{2}}\,|a|^{\frac{k}{2}+s}} \int_{v-i\infty}^{v+i\infty} \frac{e^{2\pi mt}}{(t^2+\lambda^2)^{\frac{k}{2}}\,|t^2+\lambda^2|^s}\mathrm{d}t.$$

Splitting the integral at $v \pm i\lambda$ and majorizing each, we obtain the following result.

Lemma V.5.1. *If $\mathrm{Re}(s) > 1 - \frac{k}{2}$, then the integral expression defining $c_m(v,s)$ converges absolutely.*

Fourier expansion of $\mathcal{E}_{k,\gamma}$

Now, we turn our interest to the Fourier expansion of $\mathcal{E}_{k,\gamma}$. We let d be the positive fundamental discriminant dividing $\Delta(\gamma)$, and in addition, we let

$$W_Q(m,a) := \sum_{\substack{b\,(\mathrm{mod}\,2a)\\ \left[a,b,\frac{b^2-\Delta(\gamma)}{4a}\right]\sim Q}} e^{\pi im\frac{b}{a}}, \qquad a\in\mathbb{Z}\setminus\{0\}, \quad m\in\mathbb{Z}, \quad Q\in\mathcal{Q}_{\Delta(\gamma)},$$

which is a so called *quadratic Weyl sum*. Now, the additional averaging over $\mathcal{Q}_{\Delta(\gamma)}/\Gamma$ comes in handy.

Lemma V.5.2. *Let $m \in \mathbb{Z}$. Then we have*

$$2 \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \sum_{a \geq 1} W_Q(m, a) = \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \sum_{a \in \mathbb{Z} \setminus \{0\}} W_Q(m, a)$$

$$= \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \sum_{a \in \mathbb{Z} \setminus \{0\}} W_Q(-m, a).$$

Proof. We prove the first equality, and observe that $a \mapsto -a$ yields

$$\sum_{\substack{b \,(\mathrm{mod}\,{-2a}) \\ \left[-a, b, \frac{b^2 - \Delta(\gamma)}{-4a}\right] \sim Q}} e^{\pi i m \frac{b}{-a}} = \sum_{\substack{b \,(\mathrm{mod}\,{2a}) \\ \left[-a, b, \frac{b^2 - \Delta(\gamma)}{-4a}\right] \sim Q}} e^{\pi i m \frac{-b}{a}} = \sum_{\substack{b \,(\mathrm{mod}\,{2a}) \\ \left[-a, -b, \frac{b^2 - \Delta(\gamma)}{-4a}\right] \sim Q}} e^{\pi i m \frac{b}{a}}$$

by reordering summands. We compute

$$\Delta\left(\left[-a, -b, \frac{b^2 - \Delta(\gamma)}{-4a}\right]\right) = (-b)^2 - 4(-a)\frac{b^2 - \Delta(\gamma)}{-4a} = \Delta(\gamma).$$

Furthermore, we have $\chi_d(-Q) = \mathrm{sgn}\,(d)\chi_d(Q)$. Indeed, suppose that Q represents some n, and $\gcd(d, n) = 1$. Then $-Q$ represents $-n$, and $\gcd(d, -n) = 1$. This enables us to write $\left(\frac{d}{-n}\right) = \left(\frac{d}{-1}\right)\left(\frac{d}{n}\right)$, and $\left(\frac{d}{-1}\right) = \mathrm{sgn}(d) = 1$. In other words, changing the sign of Q permutes quadratic forms of discriminant $\Delta(\gamma)$ up to equivalence. The second equality follows analogously. $\qquad\square$

We deduce that ($\lambda = \frac{\sqrt{\Delta(\gamma)}}{2a}$)

$$\mathcal{E}_{k,\gamma}(\tau, s)$$
$$= \frac{-2iv^s}{(-1)^{\frac{k}{2}}} \sum_{m \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \sum_{a \geq 1} \frac{W_Q(m, a)}{a^{\frac{k}{2}+s}} \int_{v-i\infty}^{v+i\infty} \frac{e^{2\pi m t}}{(t^2 + \lambda^2)^{\frac{k}{2}} |t^2 + \lambda^2|^s} \mathrm{d}t \; q^m.$$

We re-establish convergence.

Lemma V.5.3. *Suppose* $\mathrm{Re}(s) > 1 - \frac{k}{2}$. *Then the Fourier expansion of* $\mathcal{E}_{k,\gamma}(\tau, s)$ *converges absolutely.*

Before the proof, we rewrite the Fourier expansion of $\mathcal{E}_{k,\gamma}$. To this end, we define d' by $0 < \Delta(\gamma) = dd'$ with d fundamental, and recall the *Salié sum*

$$T_m(d, d', c) := \sum_{\substack{b \,(\mathrm{mod}\,c) \\ b^2 \equiv dd' \,(\mathrm{mod}\,c)}} \chi_d\left(\left[\frac{c}{4}, b, \frac{b^2 - dd'}{c}\right]\right) e^{2\pi i \left(\frac{2mb}{c}\right)}.$$

We observe that

$$T_m(d, d', 4a)$$

$$= 2 \sum_{\substack{b \,(\mathrm{mod}\; 2a) \\ b^2 \equiv dd' \,(\mathrm{mod}\; 4a)}} \chi_d\left(\left[a, b, \frac{b^2 - dd'}{4a}\right]\right) e^{\pi i\left(\frac{mb}{a}\right)} = 2 \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) W_Q(m, a).$$

Thus, we arrive at

$$\mathcal{E}_{k,\gamma}(\tau, s) = \frac{-iv^s}{(-1)^{\frac{k}{2}}} \sum_{m \in \mathbb{Z}} \sum_{a \geq 1} \frac{T_m(d, d', 4a)}{a^{\frac{k}{2}+s}} \int_{v-i\infty}^{v+i\infty} \frac{e^{2\pi mt}}{(t^2 + \lambda^2)^{\frac{k}{2}} |t^2 + \lambda^2|^s} \mathrm{d}t \; q^m.$$

Proof of Lemma V.5.3. We utilize the Weil bound

$$T_m(d, d', 4a) \ll \gcd\left(d', m^2 d, 4a\right)^{\frac{1}{2}} (4a)^\varepsilon$$

for any $\varepsilon > 0$, compare [AD20, p. 1545] for instance. Hence, we have

$$\left| \sum_{a \geq 1} \frac{T_m(d, d', 4a)}{a^{\frac{k}{2}+s}} \right| \ll \sum_{a \geq 1} \frac{a^\varepsilon}{a^{\frac{k}{2}+\mathrm{Re}(s)}}$$

by bounding $\gcd\left(d', m^2 d, 4a\right) \ll 1$ (see [DIT11, p. 965] as well), which converges absolutely for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - \frac{k}{2}$. We conclude by Lemma V.5.1. $\qquad\square$

V.5.2 Evaluation at $s = 0$

Let

$$J_\mu(x) := \left(\frac{x}{2}\right)^\mu \sum_{j \geq 0} \frac{(-1)^j}{j!\, \Gamma(\mu + j + 1)} \left(\frac{x}{2}\right)^{2j}$$

be the usual *J-Bessel function*. We compute the inverse Laplace transform from above.

Lemma V.5.4. *Let $\rho > 0$, $m \in \mathbb{Z}$. Then*

$$\frac{1}{2\pi i} \int_{v-i\infty}^{v+i\infty} \frac{e^{2\pi mt}}{(t^2 + \lambda^2)^\rho} \mathrm{d}t = \begin{cases} \frac{\sqrt{\pi}}{\Gamma(\rho)} \left(\frac{\pi m}{\lambda}\right)^{\rho - \frac{1}{2}} J_{\rho - \frac{1}{2}}(2\pi \lambda m) & \text{if } m > 0, \\ 0 & \text{if } m \leq 0. \end{cases}$$

Proof. The case $m > 0$ follows directly by item [AS72, item 29.3.57]. If $m \leq 0$, we see that the poles of the integrand are on the imaginary axis, to the left of the contour of integration. Hence, we may deform the contour to the right up to $i\infty$ without including any poles, see [Koh85, p. 249]. Since $m \leq 0$, the integrand is holomorphic at $i\infty$ as well, and the claim follows by Cauchy's Theorem. $\qquad\square$

Next, we invoke the *I-Bessel function* $I_\mu(x) := i^{-\mu} J_\mu(ix)$ to define the auxiliary function

$$\phi_m(y,s) := \begin{cases} y^s & \text{if } m = 0, \\ 2\pi \sqrt{|m|\,y}\, I_{s-\frac{1}{2}}\left(2\pi\,|m|\,y\right) & \text{if } m \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Averaging this function gives rise to the Niebur Poincaré series [Neu73, Nie73]

$$G_m(\tau,s) := \sum_{M \in \Gamma_\infty \backslash \Gamma} \phi_m\left(\text{Im}(M\tau), s\right) e^{2\pi i m \,\text{Re}(M\tau)}, \quad \text{Re}(s) > 1. \tag{V.1}$$

The analytic properties of G_m can be easily derived from its Fourier expansion, see [DIT11, pp. 969-970] for instance. In particular, $G_m(\cdot, s)$ is invariant under the action of Γ on $\mathbb{H}$. Moreover, recall the notation Γ_Q for the stabilizer of the two zeros of Q. The main ingredient is the following result due to Duke, Imamoḡlu, and Tóth.

Lemma V.5.5 ([DIT11, Proposititon 4][2].). *Let* $\text{Re}(\rho) > 1$, $m \in \mathbb{Z}$, $\Delta(\gamma) = dd' > 0$ *with* $d > 0$ *fundamental, and* $d \neq d'$. *Then, we have*

$$\frac{\Gamma(\rho)}{2^\rho \Gamma\left(\frac{\rho}{2}\right)^2} \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} G_m(w, \rho) \frac{|dw|}{\text{Im}(w)}$$

$$= \begin{cases} \sqrt{2}\pi\,|m|^{\frac{1}{2}}\,\Delta(\gamma)^{\frac{1}{4}} \sum_{0 < c \equiv 0 \,(\text{mod } 4)} \frac{T_m(d,d',c)}{c^{\frac{1}{2}}} J_{\rho-\frac{1}{2}}\left(\frac{4\pi\sqrt{m^2\Delta(\gamma)}}{c}\right) & \text{if } m \neq 0, \\ 2^{\rho-1}\Delta(\gamma)^{\frac{\rho}{2}} \sum_{0 < c \equiv 0 \,(\text{mod } 4)} \frac{T_0(d,d',c)}{c^\rho} & \text{if } m = 0. \end{cases}$$

Remark. Note that Duke, Imamoḡlu, and Tóth [DIT11], and Matsusaka [Mat20, p. 10] use different notations regarding the cycle integral. This is caused by a different choice of generators of Γ_Q. Let $Q = [a, b, c]$ be a given primitive quadratic form, and let $t, u \in \mathbb{N}$ be the smallest solutions to Pell's equation $t^2 - \Delta(Q)u^2 = 4$. Then, the authors of [DIT11] employed the generator $\eta_Q = \pm \begin{pmatrix} \frac{t+bu}{2} & cu \\ -au & \frac{t-bu}{2} \end{pmatrix}$, while Matsusaka works with $\Gamma_{Q_\gamma} = \pm \langle \gamma \rangle$. The associated quadratic form Q_{η_Q} to η_Q is given by $[-au, -bu, -cu] = -uQ$.

Recall that $T_m(d, d', c) = T_{-m}(d, d', c)$ by Lemma V.5.2. We deduce from Lemma V.5.4 and V.5.5 that the Fourier coefficients corresponding to $m \neq 0$ are all regular at $s = 0$, and vanish for every $m < 0$. To inspect the coefficient corresponding to $m = 0$, we separate the cases $k \geq 4$ even and $k = 2$.

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[2]See also [DIT16, Lemma 4], [AD20, equation (2-7)]

The case $k \geq 4$ even

If $k \geq 4$ is even, then $G_0(\tau, \rho)$ is regular at $\rho = \frac{k}{2}$. Hence, the Fourier coefficient corresponding to $m = 0$ vanishes at $s = 0$ by Lemma V.5.4. In other words, $\mathcal{E}_{k,\gamma}(\tau, 0)$ is holomorphic and vanishes at the cusp. Thus,

$$\mathcal{E}_{k,\gamma}(\tau, 0) = \frac{(-1)^{\frac{k}{2}} 2^{\frac{k+3}{2}} \pi^{\frac{k}{2}+1}}{\Delta(\gamma)^{\frac{k-1}{4}} \Gamma\left(\frac{k}{2}\right)} \sum_{m \geq 1} m^{\frac{k-1}{2}} \sum_{a \geq 1} \frac{T_m(d, d', 4a)}{2\sqrt{a}} J_{\frac{k-1}{2}}\left(\frac{\pi m \sqrt{\Delta(\gamma)}}{a}\right) q^m,$$

and by Lemma V.5.5, we ultimately obtain that

$$\mathcal{E}_{k,\gamma}(\tau, 0) = \frac{(-1)^{\frac{k}{2}} 2\pi^{\frac{k}{2}}}{\Delta(\gamma)^{\frac{k}{4}} \Gamma\left(\frac{k}{4}\right)^2} \sum_{m \geq 1} m^{\frac{k}{2}-1} \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} G_{-m}\left(w, \frac{k}{2}\right) \frac{|dw|}{\operatorname{Im}(w)} q^m.$$

This proves Theorem V.1.2.

The case $k = 2$

We first suppose that $m \geq 1$. Then we define for $\operatorname{Re}(s) > 1$

$$j_m(\tau, s) := G_{-m}(\tau, s) - \frac{2m^{1-s}\sigma_{2s-1}(m)}{\pi^{-s-\frac{1}{2}}\Gamma\left(s + \frac{1}{2}\right)\zeta(2s-1)} G_0(\tau, s),$$

which has an analytic continuation up to $\operatorname{Re}(s) > \frac{1}{2}$ (see [DIT11, p. 970]). On one hand, the left hand side specializes at $s = 1$ to (see [DIT11, equation (4.11)])

$$j_m(\tau, 1) = j_m(\tau) = q^{-m} + O(q),$$

which we encountered during the weight 2 elliptic case already. On the other hand,

$$\lim_{s \to 1}(s-1)G_0(\tau, s) = \frac{3}{\pi}, \qquad \lim_{s \to 1}(s-1)\zeta(2s-1) = \frac{1}{2},$$

from which we infer (see [AD20, p. 1545] as well)

$$\lim_{s \to 1} \frac{2m^{1-s}\sigma_{2s-1}(m)}{\pi^{-s-\frac{1}{2}}\Gamma\left(s + \frac{1}{2}\right)\zeta(2s-1)} G_0(\tau, s) = 24\sigma_1(m).$$

Combining, we arrive at the Fourier coefficients

$$\frac{-2}{\Delta(\gamma)^{\frac{1}{2}}} \sum_{m \geq 1} \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} (j_m(w) + 24\sigma_1(m)) \frac{|dw|}{\operatorname{Im}(w)} q^m.$$

Secondly, we consider the case $m = 0$, namely the Fourier coefficient

$$i \sum_{a \geq 1} \frac{T_0(d, d', 4a)}{a^{s+1}} \int_{v-i\infty}^{v+i\infty} \frac{v^s}{(t^2 + \lambda^2) |t^2 + \lambda^2|^s} dt.$$

By Lemma V.5.5, the pole of

$$\sum_{a \geq 1} \frac{T_0(d, d', 4a)}{a^\rho} = \frac{2\Gamma(\rho)}{\Delta(\gamma)^{\frac{\rho}{2}} \Gamma\left(\frac{\rho}{2}\right)^2} \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} G_0(w, \rho) \frac{|dw|}{\operatorname{Im}(w)}.$$

at $\rho = 1$ is simple, while

$$f(\rho) := \int_{v-i\infty}^{v+i\infty} \frac{iv^{\rho-1}}{(t^2 + \lambda^2) |t^2 + \lambda^2|^{\rho-1}} dt$$

has a zero at $\rho = 1$ by Lemma V.5.4. We perform a Taylor expansion of f around 1, and note that only the term $(\rho - 1)\frac{df}{d\rho}\big|_{\rho=1}$ survives in the limit $\rho \to 1$. We compute

$$\frac{df}{d\rho}\Big|_{\rho=1} = i \int_{v-i\infty}^{v+i\infty} \frac{v^{\rho-1} \log\left(\frac{v}{|t^2+\lambda^2|}\right)}{|t^2 + \lambda^2|^{\rho-1} (t^2 + \lambda^2)}\Big|_{\rho=1} dt = i \int_{v-i\infty}^{v+i\infty} \frac{\log(v) - \log\left(|t^2 + \lambda^2|\right)}{t^2 + \lambda^2} dt$$

$$= -i \int_{v-i\infty}^{v+i\infty} \frac{\log\left(|t^2 + \lambda^2|\right)}{t^2 + \lambda^2} dt = \int_{-\infty}^{\infty} \frac{\log\left(|(v + it)^2 + \lambda^2|\right)}{(v + it)^2 + \lambda^2} dt.$$

We expand the integrand around $\lambda = 0$, which yields

$$\frac{df}{d\rho}\Big|_{\rho=1} = \left(-\frac{1}{v} \arctan\left(\frac{t}{v}\right) + \frac{\log\left(v^2 + t^2\right) + 1}{t - iv}\right)\Big|_{t=-\infty}^{t=\infty} + O\left(\lambda^2\right) = -\frac{\pi}{v} + O\left(\lambda^2\right).$$

Recalling the definition of λ, we express the error as $O(\frac{1}{a^2})$. Thus, the additional sums over a caused by the expansion with respect to λ are all regular at $\rho = 1$ due to the proof of Lemma V.5.3. Hence, letting $\rho \to 1$ annihilates all error terms. Invoking Lemma V.5.5 and the residue of $G_0(w, \rho)$ at $\rho = 1$ once more, we obtain

$$\frac{-2}{\Delta(\gamma)^{\frac{1}{2}}} \sum_{Q \in \mathcal{Q}_{\Delta(\gamma)}/\Gamma} \chi_d(Q) \int_{\Gamma_Q \backslash S_Q} \frac{3}{\pi v} \frac{|dw|}{\operatorname{Im}(w)}.$$

In conclusion, we have shown that the analytic continuation of $\mathcal{E}_{2,\gamma}(\tau, s)$ to $s = 0$ exists, and indeed equals the shape which Matsusaka conjectured in [Mat20, equation (2.12)] for an individual hyperbolic Eisenstein series $E_{k,\gamma}(\tau, s)$. This proves Theorem V.1.1.

Remark. In [DIT10, equation (16)], Duke, Imamoḡlu, and Tóth related Parson's Poincaré series [Par93] with the generating function $F(z, Q)$ of cycle integrals of functions $f_{k,m}$, where the functions $f_{k,m}$ generalize the functions j_m to any even weight k (compare [DIT10, Theorem 1, equation (8)]. Since we realized the Fourier coefficients as cycle integrals of $G_{-m}\left(w, \frac{k}{2}\right)$, there might be a relation between them.

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Chapter VI

Locally harmonic Maaß forms of positive even weight

This chapter is based on a paper [Mon22b] of the same title accepted for publication in *Israel Journal of Mathematics.*

VI.1 Introduction and statement of results

In 1975, Zagier [Zag75] defined the function

$$f_{k,D}(\tau) := \sum_{Q \in \mathcal{Q}_D} \frac{1}{Q(\tau,1)^k}, \qquad \tau = u + iv \in \mathbb{H} := \{ z \in \mathbb{C} \colon \operatorname{Im}(z) > 0 \} \,,$$

to investigate the Doi–Naganuma lift. Here and troughout, $\mathcal{Q}_D$ is the set of all integral binary quadratic forms of discriminant $D \in \mathbb{Z}$, and $k \geq 2$. On one hand, if $D > 0$, Zagier proved that they define holomorphic cusp forms of weight $2k$ for $\Gamma := \operatorname{SL}_2(\mathbb{Z})$, and computed their Fourier expansions. On the other hand, if $D < 0$, Bengoechea [Ben13] proved that these are meromorphic cusp forms with respect to the same data, namely meromorphic modular forms which decay like cusp forms towards $i\infty$. The poles are precisely the CM points (sometimes called Heegner points instead) of discriminant D, and of order k.

Parson [Par93, Theorem 3.1] investigated Zagier's $f_{k,D}$-function based on an individual equivalence class $[Q_0]_{\sim} \in \mathcal{Q}_D/\Gamma$ of indefinite integral binary quadratic forms, and twisted it by a sign function. More precisely, she defined

$$f_{k,Q_0}(\tau) := \sum_{Q \sim Q_0} \frac{\operatorname{sgn}(Q)}{Q(\tau,1)^k}, \qquad \operatorname{sgn}(Q) = \operatorname{sgn}([a,b,c]) := \begin{cases} \operatorname{sgn}(a) & \text{if } a \neq 0, \\ \operatorname{sgn}(c) & \text{if } a = 0. \end{cases}$$

Due to the presence of the sign function, we have a non-zero error to modularity, which is a finite sum, and explicitly given by (see Lemma VII.3.3 as well)

$$F_{k,Q_0}(\tau) := f_{k,Q_0}(\tau) - \tau^{-2k} f_{k,Q_0}\left(-\frac{1}{\tau}\right) = 2 \sum_{\substack{[a,b,c]=Q \sim Q_0 \\ \operatorname{sgn}(ac)<0}} \frac{\operatorname{sgn}(Q)}{Q(\tau,1)^k}.$$

In other words, the function f_{k,Q_0} is a modular integral of weight $2k$ for the rational period function $F_{k,Q_0}(\tau)$. We refer the reader to the work of Knopp [Kno90] for more details.

In Chapter V, we investigated a certain class of Eisenstein series

$$\mathcal{E}_{k,D}(\tau,s) := \sum_{0 \neq Q_0 \in \mathcal{Q}_D/\Gamma} \chi_d(Q_0) \sum_{Q \sim Q_0} \frac{\operatorname{sgn}(Q)^{\frac{k}{2}} \operatorname{Im}(\tau)^s}{Q(\tau,1)^{\frac{k}{2}} |Q(\tau,1)|^s}, \qquad \text{(VI.1)}$$

for any $k \in 2\mathbb{N}$, which arises by applying Hecke's trick to Parson's construction. The function χ_d is a genus character (defined in Section VI.2). By results of Petersson [Pet48, Satz 1, Satz 4, Satz 6], the sum converges absolutely for any $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1 - \frac{k}{2}$. Like in the case of $f_{k,D}$, the behaviour of $\mathcal{E}_{k,D}(\tau,s)$ is dictated by the sign of D, and consequently we distinguish between hyperbolic ($D > 0$), parabolic ($D = 0$), and elliptic ($D < 0$) Eisenstein series. This terminology comes from the fact that one can associate a quadratic form to any $\gamma \in \Gamma \setminus \{\pm \mathbb{1}\}$[1], and the sign of its discriminant depends precisely on hyperbolicity, parabolicity, or ellipticity of γ. Although we focus on the case of weights $k \in 2\mathbb{N}$, one may also consider different weights. For instance, all three types of Eisenstein series were studied by Jorgenson, Kramer, von Pippich, Schwagenscheidt, and Völz for weight $k = 0$, see [JKvP10, Theorem 4.2], [vP16, Section 4], [vPSV21, Theorem 1.2].

Chapter V as well as the present one are devoted to the hyperbolic case. Letting $D > 0$ be a non-square discriminant, and d be a positive fundamental discriminant dividing D, we computed the Fourier expansion of hyperbolic Eisenstein series for any integral weight $k \in 2\mathbb{N}$ at $s = 0$ to prove a conjecture of Matsusaka [Mat20b, equation (2.12)] about their analytic continuation in weight 2. This computation extends earlier work by Gross, Kohnen, and Zagier [GKZ87, p. 517], who dealt with weights $4 | k > 2$ not involving the sign function. In turn, the computation for weights $k \in 2\mathbb{N}$ relies mainly on results of Duke, Imamoḡlu, and Tóth [DIT11] after appealing to Zagiers work [Zag75, Appendix 2] on the Fourier expansion of his aforementioned function. Furthermore, we computed the analytic continuation $\mathcal{E}_{2,D}(\tau,0)$ explicitly. Up to the addition of the completed Eisenstein series E_2^* and some constants, it agrees with another modular integral with rational period function, which was studied by Duke, Imamoḡlu, and Tóth in [DIT10].

In addition, one can inspect the automorphic object arising from the analytic continuation to $s = 0$. On one hand, the parabolic and elliptic (twisted) Eisenstein series extend to an ordinary and a polar harmonic Maaß form respectively in weight 2. (We define all occuring types of Maaß forms in Section VI.2.) While the parabolic case is known by Roelcke [Roe66, Roe67] and Selberg [Sel56], the elliptic case was proven by Matsusaka in [Mat20b, Theorem 2.3] by combining results of Bringmann and Kane

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[1]Explicitly given by $Q_\gamma(x,y) := cx^2 + (d-a)xy - by^2$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

[BK16] and of Bringmann, Kane, Löbrich, Ono, and Rolen [BKLOR18]. On the other hand, the hyperbolic Eisenstein series $\mathcal{E}_{2,D}(\tau, 0)$ (with D, d as above) coincides with a locally harmonic Maaß form for any τ with sufficiently large imaginary part. This raises the natural question towards the obstruction of $\mathcal{E}_{k,D}(\tau, s)$ to coincide with a local automorphic form, whenever the imaginary part of τ is not sufficiently large. To this end, we relate $\mathcal{E}_{k,D}(\tau, s)$ to the completed hyperbolic Eisenstein series

$$\widehat{\mathcal{E}}_{k,D}(\tau, s) := \sum_{0 \neq Q_0 \in \mathcal{Q}_D/\Gamma} \chi_d(Q_0) \sum_{Q \sim Q_0} \frac{\operatorname{sgn}(Q_\tau)^{\frac{k}{2}} \operatorname{Im}(\tau)^s}{Q(\tau, 1)^{\frac{k}{2}} |Q(\tau, 1)|^s}, \tag{VI.2}$$

$$Q_\tau = [a, b, c]_\tau := \frac{1}{v} \left(a |\tau|^2 + bu + c \right),$$

outside the net of Heegner geodesics

$$E_D := \bigcup_{[a,b,c]=Q \in \mathcal{Q}_D} \left\{ \tau \in \mathbb{H} : a |\tau|^2 + bu + c = 0 \right\}$$

by adding a correction term to $\mathcal{E}_{k,D}(\tau)$ (see equation (VI.4)). A possible connection of our correction term to quantum modular forms (introduced by Zagier [Zag10]) is stated in Section VI.3.

In particular, the function $\widehat{\mathcal{E}}_{k,D}(\tau, s)$ is modular of weight k outside E_D. To describe the result, we let $\mathcal{C}_\kappa(h, Q)$ be the weight κ cycle integral of h associated to Q (defined in equation (VI.3)), where h is modular of weight κ. Moreover, we let $\mathbb{P}_k(z_1, z_2)$ be a Poincaré series due to Petersson [Pet48] (see Definition VI.2.4), whose properties are collected in Lemma VI.2.5 below. We refer the reader to Subsection VI.2.7 for definitions of our local automorphic forms.

Theorem VI.1.1. *Let $0 < k \equiv 2 \pmod 4$, and $\tau \in \mathbb{H} \setminus E_D$. Let $D > 0$ be a non-square discriminant, and d be a positive fundamental discriminant dividing D.*

(1) *The function $\widehat{\mathcal{E}}_{2,D}(\tau, 0)$ is a locally harmonic Maaß form of weight 2 for Γ with exceptional set E_D as a function of τ.*
(2) *If $2 < k \equiv 2 \pmod 4$ then $\widehat{\mathcal{E}}_{k,D}(\tau, 0)$ is a local cusp form of weight k for Γ with exceptional set E_D as a function of τ.*
(3) *Moreover, we have the alternative representation*

$$\widehat{\mathcal{E}}_{k,D}(\tau, 0) = \sum_{Q \in \mathcal{Q}_D/\Gamma} \chi_d(Q) \begin{cases} \frac{-2}{D} \mathcal{C}_0 \left(\frac{\frac{1}{2\pi i} \frac{\partial j}{\partial \tau}(\tau)}{j(\cdot) - j(\tau)} - E_2^*(\tau), Q \right) & \text{if } k = 2, \\ C(k, D) \mathcal{C}_{2-k} \left(\mathbb{P}_k(\tau, \cdot), Q \right) & \text{if } k > 2, \end{cases}$$

where $C(k, D)$ is an explicit constant provided in equation (VI.8).

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Remarks.

(1) The cycle integral $\mathcal{C}_{2k}(\mathbb{P}_{2k}(\cdot,\tau),Q)$ was computed by Löbrich, Schwagenscheidt [LS22c]. Let $Q_0 \in \mathcal{Q}_D$, and $\mathcal{F}_{1-k,Q_0}$ be the locally harmonic Maaß form introduced by Bringmann, Kane, and Kohnen [BKK15] (see Section VI.2.7) with summation restricted to quadratic forms equivalent to Q_0 under Γ. Then [LS22c, Theorem 4.2] states that

$$\mathcal{F}_{1-k,Q_0}(\tau) = (-1)^k \binom{2k-2}{k-1} D^{\frac{k}{2}-\frac{1}{2}} \mathcal{C}_{2k}\left(\mathbb{P}_{2k}(\cdot,\tau),Q_0\right).$$

In other words, a cycle integral of $\mathbb{P}_k$ in either of its arguments yields a local automorphic form in the other argument.

(2) A natural splitting of $z_2 \mapsto \mathbb{P}_k(z_1,z_2)$ into meromorphic and non-meromorphic parts is due to Bringmann and Kane [BK20, equation (3.6)].

(3) Choosing $d = 1$, the function $\widehat{\mathcal{E}}_{2\kappa+2,D}(\tau,0)$, $\kappa \in 2\mathbb{N}$, also appears in equation (VII.7), and further properties of it are stated in Section VII.4. In particular, $\widehat{\mathcal{E}}_{2\kappa+2,D}(\tau,0)$ can be "lifted" to a locally harmonic Maaß form of weight -2κ, whose properties are discussed in Theorem VII.1.2.

As an application of Theorem VI.1.1, we would like to highlight a possible connection to twisted central L-values. This goes back to Kohnen [Koh85, Proposition 7, Corollary 3], who established an identity between the Petersson inner product of a cusp form with Zagiers $f_{k,D}$-function, and such L-values for positive even weights. More recently, Kohnen's work was utilized by Ehlen, Guerzhoy, Kane, and Rolen [EGKR20, Theorem 1.1] to prove a criterion on the vanishing of certain twisted L-values under some technical conditions. Their argument relies on the theory of locally harmonic Maaß forms, and in particular on the connection of the $f_{k,D}$-function to the locally harmonic Maaß form $\mathcal{F}_{1-k,D}$, see Section VI.2.7. (In addition, the theory of theta lifts comes in handy to ensure existence of an analytic continuation of $\mathcal{F}_{1-k,D}$ to the case $k = 1$.) Being more precise, the form $\mathcal{F}_{1-k,D}$ splits into three components (see [BKK15, Theorem 7.1]). Two of them are a holomorphic and a nonholomorphic Eichler integral of the $f_{k,D}$-function, while the third component is a so called local polynomial, which captures the behaviour of $\mathcal{F}_{1-k,D}$ between different connected components of $\mathbb{H} \setminus E_D$. The idea in [EGKR20] now is to formulate a condition on the local polynomial of $\mathcal{F}_{1-k,D}$, evaluated at two special points on the real axis, and relate this conditions to the twisted central L-values via the work of Kohnen, and of Bringmann, Kane, and Kohnen. Since the function $\widehat{\mathcal{E}}_{k,D}(\tau,0)$ is a twisted version of the function $f_{\frac{k}{2},D}$, and since we found a connection of $\widehat{\mathcal{E}}_{k,D}(\tau,0)$ to a locally harmonic Maaß form (resp. local cusp form), we expect that $\widehat{\mathcal{E}}_{k,D}(\tau,0)$ may serve as a "building block" to detect the vanishing of suitable twisted L-values as well. This inspection is motivated by our remarks following Theorem VI.1.1.

VI.2 Preliminaries

We let $q \coloneqq e^{2\pi i \tau}$ troughout.

VI.2.1 Integral binary quadratic forms

Let Q be an integral binary quadratic form, and we abbreviate such forms by the terminology "quadratic form" throughout. We call a quadratic form *primitive* if its coefficients are coprime. The full modular group Γ acts on the set of quadratic forms by letting

$$(Q \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)) (x, y) \coloneqq Q(ax + by, cx + dy),$$

and this action induces an equivalence relation, which we denote by $\sim$. Moreover, the action of Γ on $\mathbb{H}$ by fractional linear transformations is compatible with the action of Γ on the set of quadratic forms, in the sense that

$$(Q \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)) (\tau, 1) = (c\tau + d)^2 Q(\gamma\tau, 1).$$

A quadratic form Q may be written as $[a, b, c]$, and we denote its discriminant by $D \coloneqq b^2 - 4ac$. One can check that equivalent quadratic forms have the same discriminant. The set $\mathcal{Q}_D/\Gamma$ is finite, whenever $D \neq 0$, and its cardinality is called the class number $h(D)$. If $D \equiv 0 \pmod 4$ or $D \equiv 1 \pmod 4$, then $\mathcal{Q}_D/\Gamma$ is non-empty. To simplify notation, we identify an equivalence class in $\mathcal{Q}_D/\Gamma$ with any representative of it throughout. A good reference on this is Zagier's book [Zag81].

VI.2.2 Genus characters

We follow the exposition given by Gross, Kohnen, and Zagier in [GKZ87, p. 508]. Let $Q = [a, b, c]$ be a quadratic form, and observe that $\gcd(a, b, c)$ is invariant under $\sim$ as well. For any $D \neq 0$, let d be a fundamental discriminant dividing D, and stipulate $d = 0$ if $D = 0$. We say that an integer n is represented by Q if there exist $x, y \in \mathbb{Z}$, such that $Q(x, y) = n$, and recall the the Kronecker symbol $\left(\frac{d}{\cdot} \right)$. This established, an extended genus character associated to D is given by

$$\chi_d([a, b, c]) \coloneqq \begin{cases} \left(\frac{d}{n} \right) & \text{if } \gcd(a, b, c, d) = 1, [a, b, c] \text{ represents } n, \gcd(d, n) = 1, \\ 0 & \text{if } \gcd(a, b, c, d) > 1. \end{cases}$$

One can check that such an integer n always exists, and that the definition is independent from its choice. Since equivalent quadratic forms represent the same integers, a genus character descends to $\mathcal{Q}_D/\Gamma$. If $d = 1$, the character is trivial, and if $d = 0$, we have

$\chi_0(Q) = 0$ except Q is primitive, and represents ± 1. In the latter case, we note that such a quadratic form is equivalent to either $[-1, 0, 0]$ or $[1, 0, 0]$. Lastly, we have

$$\chi_d(-Q) = \operatorname{sgn}(d)\chi_d(Q)$$

for every $d \neq 0$, linking the two choices $\pm d$. We refer the reader to [GKZ87, Proposition 1 and 2] regarding additional properties of χ_d.

VI.2.3 Heegner geodesics

Once more, let $Q = [a, b, c] \in Q_D$, and suppose that $D > 0$. Hence, Q is indefinite, and $Q(\tau, 1)$ has the two distinct zeros

$$\frac{-b - D^{\frac{1}{2}}}{2a}, \quad \frac{-b + D^{\frac{1}{2}}}{2a} \in \mathbb{R} \cup \{\infty\}.$$

If $a = 0$, then the second zero is given by $-\frac{c}{b}$. We associate to Q the Heegner geodesic

$$S_Q := \left\{\tau \in \mathbb{H} : a\left|\tau\right|^2 + bu + c = 0\right\},$$

which connects the two zeros of $Q(\tau, 1)$. On one hand, if D is a square and $a \neq 0$, then both zeros are rational. In other words, one zero of $Q(\tau, 1)$ is Γ-equivalent to ∞, and S_Q is a straight line in $\mathbb{H}$, perpendicular to $\mathbb{R}$, based on the second zero. Moreover, the stabilizer

$$\Gamma_Q := \{\gamma \in \Gamma : Q \circ \gamma = Q\}$$

is trivial in this case. On the other hand, if $D > 0$ is not a square and $a \neq 0$, then both zeros of $Q(\tau, 1)$ are real quadratic irrationals, which are Galois conjugate to each other. The geodesic S_Q is an arc in $\mathbb{H}$, which is perpendicular to $\mathbb{R}$, and S_Q is preserved by Γ_Q.

We stipulate that D is a positive non-square discriminant. We obtain infinitely many connected components on $\mathbb{H}$, and finitely many such components in a fundamental domain for Γ, because the class number of D is finite. Since D is not a square, each geodesic S_Q divides $\mathbb{H}$ into a bounded and an unbounded component, and we denote the bounded component ("interior") of $\mathbb{H} \setminus S_Q$ by A_Q. Moreover, there is precisely one unbounded connected component in a fundamental domain for Γ, to which we refer as the region "above" the net of geodesics.

Furthermore, we introduce the characteristic funtion

$$\mathbb{1}_Q(\tau) := \begin{cases} 1 & \text{if } \tau \in A_Q, \\ 0 & \text{if } \tau \notin A_Q, \end{cases}$$

whenever $\tau \in \mathbb{H} \setminus E_D$. Variants of $\mathbb{1}_Q$ appear in [Sch18, Corollary 5.3.5], and in [Mat20a, p. 8]. We collect the properties of our sign functions.

Lemma VI.2.1.

(1) *For every $\gamma \in \Gamma$, we have*
$$Q_{\gamma\tau} = (Q \circ \gamma)_\tau \,.$$

(2) *We have that $\tau \in A_Q$ if and only if*
$$\mathrm{sgn}(Q)\,\mathrm{sgn}\,(Q_\tau) < 0.$$

(3) *If $\tau \in \mathbb{H} \setminus E_D$ then the sign functions $\mathrm{sgn}(Q)$, $\mathrm{sgn}\,(Q_\tau)$, and $\mathbb{1}_Q(\tau)$ are related by*
$$\mathrm{sgn}\,(Q_\tau) = \mathrm{sgn}(Q)\,(1 - 2\mathbb{1}_Q(\tau))\,.$$

Proof. It suffices to check the first item for the two generators
$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
of Γ. Indeed, we calculate that

$$\begin{aligned}
Q_{S\tau} &= \frac{a\,|S\tau|^2 + b\,\mathrm{Re}\,(S\tau) + c}{\mathrm{Im}\,(S\tau)} = \frac{a\,\left|-\frac{1}{\tau}\right|^2 - b\frac{u}{|\tau|^2} + c}{\frac{v}{|\tau|^2}} = \frac{c\,|\tau|^2 - bu + a}{v} = [c, -b, a]_\tau \\
&= (Q \circ S)_\tau \,,
\end{aligned}$$

and

$$\begin{aligned}
Q_{T\tau} &= \frac{a\,|\tau + 1|^2 + b\,\mathrm{Re}(\tau + 1) + c}{\mathrm{Im}(\tau + 1)} = \frac{a\,((u+1)^2 + v^2) + b(u+1) + c}{v} \\
&= [a, 2a + b, a + b + c]_\tau = (Q \circ T)_\tau \,.
\end{aligned}$$

The second item is stated as a sentence directly in front of [LS22c, Lemma 4.4], and follows by [BKK15, equations (5.1), (7.12)]. The third item follows by a case by case analysis using the second item. Indeed, suppose that $\mathrm{sgn}(Q) = 1$. Then the second item implies that

$$\mathrm{sgn}\,(Q_\tau) = \begin{cases} -1 & \text{if } \tau \in A_Q, \\ +1 & \text{if } \tau \notin A_Q, \end{cases}$$

and this coincides with $\mathrm{sgn}(Q)(1 - 2\mathbb{1}_Q(\tau))$. The case $\mathrm{sgn}(Q) = -1$ follows in the same manner. $\qquad\square$

VI.2.4 Cycle integrals

Let $Q = [a, b, c] \in \mathcal{Q}_D$ with $D > 0$ not a square. If Q is primitive, and t, $r \in \mathbb{N}$ are the smallest solutions to Pell's equation $t^2 - Dr^2 = 4$, the stabilizer Γ_Q is generated by

$$\pm \begin{pmatrix} \frac{t+br}{2} & cr \\ -ar & \frac{t-br}{2} \end{pmatrix}.$$

If Q is not primitive, one may divide its coefficients by $\gcd(a, b, c)$ to obtain a generator.

The weight k cycle integral of a smooth function h, which transforms like a modular form of weight k, is defined as[2]

$$\mathcal{C}_k(h, Q) := D^{\frac{1}{2} - \frac{k}{4}} \int_{\Gamma_Q \backslash S_Q} h(z) Q(z, 1)^{\frac{k}{2} - 1} \mathrm{d}z. \tag{VI.3}$$

The integral is oriented counterclockwise if $\mathrm{sgn}(Q) > 0$, and clockwise if $\mathrm{sgn}(Q) < 0$.

We collect the properties of cycle integrals in the following lemma, which can be proven by calculation, and the fact that Γ_Q only depends on the equivalence class of Q.

Lemma VI.2.2. *Let $f \colon \mathbb{H} \to \mathbb{C}$ be smooth, and suppose that f is modular of weight k. Let Q be a quadratic form of positive, non-square discriminant. Then the weight k cycle integral $\mathcal{C}_k(f, Q)$ is a class invariant, namely it depends only on the equivalence class of Q under $\sim$. Additionally, the weight k cycle integral $\mathcal{C}_k(f, Q)$ is invariant under modular substitutions of the integration variable.*

Hence, $\Gamma_Q \backslash S_Q$ projects to a circle in a fundamental domain of Γ. The beautiful article [DIT11] due to Duke, Imamoḡlu, and Tóth provides a good reference on Heegner geodesics as well as on cycle integrals.

VI.2.5 Maaß forms and modular forms

We recall the definition of various classes of Maaß forms appearing in this chapter. The *slash operator* is given by

$$\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(\tau) := \begin{cases} (c\tau + d)^{-k} f(\gamma \tau) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{2k} (c\tau + d)^{-k} f(\gamma \tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

where $\left(\frac{c}{d}\right)$ denotes the Kronecker symbol, and

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \ (\mathrm{mod}\ 4), \\ i & \text{if } d \equiv 3 \ (\mathrm{mod}\ 4), \end{cases}$$

for odd integers d.

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[2]The normalization by $D^{\frac{1}{2} - \frac{k}{4}}$ is ommitted by some authors.

Definition VI.2.3. Let $k \in \frac{1}{2}\mathbb{Z}$, choose $N \in \mathbb{N}$ such that $4 \mid N$ whenever $k \notin \mathbb{Z}$, and let $f \colon \mathbb{H} \to \mathbb{C}$ be smooth.

(1) We say that f is a *weight k harmonic Maaß form* for $\Gamma_0(N)$, if f satisfies the following three properties:

   (i) For all $\gamma \in \Gamma_0(N)$ and all $\tau \in \mathbb{H}$ we have $(f|_k \gamma)(\tau) = f(\tau)$.
   (ii) The function f is harmonic with respect to the weight k hyperbolic Laplacian on $\mathbb{H}$, that is

$$0 = \Delta_k(f) := \left(-v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \right)(f).$$

   (iii) The function f is of at most linear exponential growth towards all cusps of $\Gamma_0(N)$.

(2) A *polar harmonic Maaß form* is a harmonic Maaß form, which is permitted to posses isolated poles on the upper half plane.

(3) A *weak Maaß form* satisfies conditions (i) and (iii) of a harmonic Maaß form, but is allowed to have an arbitrary eigenvalue under Δ_k.

To study his forms [Maa49], Hans Maaß introduced the *Maaß lowering* and *raising operators*[3] [Maa52]

$$L_k := -2iv^2 \frac{\partial}{\partial \bar{\tau}} = iv^2 \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right), \quad R_k := 2i \frac{\partial}{\partial \tau} + \frac{k}{v},$$

which decreases or increases the weight of a weak Maaß form by 2, and increases the eigenvalue under the hyperbolic Laplace operator by $2 - k$ or k respectively. A proof can be found in [BFOR17, Lemma 5.2] for instance. For any $n \in \mathbb{N}_0$, we let

$$L_\kappa^0 := \mathrm{id}, \qquad L_\kappa^n := L_{\kappa-2n+2} \circ \ldots \circ L_{\kappa-2} \circ L_\kappa,$$
$$R_\kappa^0 := \mathrm{id}, \qquad R_\kappa^n := R_{\kappa+2n-2} \circ \ldots \circ R_{\kappa+2} \circ R_\kappa$$

be the iterated Maaß lowering and raising operators respectively.

Bruinier and Funke [BF04] introduced the *shadow operator*

$$\xi_k := 2iv^k \overline{\frac{\partial}{\partial \bar{\tau}}} = iv^k \overline{\left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)}$$

to study harmonic Maaß forms. They proved that the Fourier expansion of a harmonic Maaß form splits naturally into a holomorphic and a nonholomorphic part.

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[3] Be aware that some authors shift their dependence on k, such as Maaß himself.

We define $M_k^!$ as the space of weakly holomorphic modular forms of weight k, and it turns out that $M_k^!$ is precisely kernel of ξ_k restricted to weight k harmonic Maaß forms. Analogously, a meromorphic modular form of weight k can be regarded as an element of the kernel of ξ_k restricted to weight k polar harmonic Maaß forms. The space of holomorphic modular forms of weight k is denoted by $M_k \subseteq M_k^!$. More details on various Maaß forms and their properties can be found in [BFOR17] for instance.

VI.2.6 Poincaré series

A first class of examples of (weakly) holomorphic modular forms, and of Maaß forms is given by constructing suitable Poincaré series. Such functions arise by averaging a specific auxiliary function ("seed"). Various seeds then lead to various examples of Poincaré series.

Definition VI.2.4.

(1) For any $m \in \mathbb{Z}$, and any $\kappa \in \mathbb{N}_{>2}$, let

$$P_{\kappa,m}(\tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} q^m \big|_\kappa \gamma.$$

be the weight κ *Poincaré series of exponential type.*

(2) Let $M_{\mu,\nu}$ be the usual M-Whittaker function, $m \in \mathbb{Z} \setminus \{0\}$, and define the seed

$$g_m(\tau, s) := \frac{\Gamma(s)}{\Gamma(2s)} M_{0,s-\frac{1}{2}}(4\pi |m| y) e^{2\pi i m u}.$$

Then the *Niebur Poincaré series* [Nie73, Neu73] is given by

$$G_m(\tau, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} g_m(\tau, s) \big|_0 \gamma, \qquad \mathrm{Re}(s) > 1.$$

(3) More generally, define the seed

$$\varphi_{\kappa,m}(\tau) := \frac{(-\operatorname{sgn}(m))^{1-\kappa} (4\pi |m| v)^{-\frac{\kappa}{2}}}{\Gamma(2-\kappa)} M_{\operatorname{sgn}(m\kappa)\frac{\kappa}{2}, \frac{1-\kappa}{2}}(4\pi |m| v) e^{2\pi \operatorname{sgn}(\kappa) m u}$$

for any $m \in \mathbb{Z} \setminus \{0\}$, and $\kappa \in -\frac{1}{2}\mathbb{N}$. We require the *Maaß–Poincaré series* of negative integral weight $\kappa \in -\mathbb{N}$, which are defined as

$$\Phi_{\kappa,m}(\tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi_{\kappa,m}(\tau) \big|_\kappa \gamma.$$

(4) We encounter one of Petersson's Poincaré series [Pet50], namely let $\cdot|_{k,z_1}$ be the weight k-operator acting on z_1, and let $k \in \mathbb{N}_{>2}$. Then we define

$$\mathbb{P}_k(z_1, z_2) := \operatorname{Im}(z_2)^{k-1} \sum_{\gamma \in \Gamma} \left(\frac{1}{(z_1 - z_2)(z_1 - \overline{z_2})^{k-1}} \right) \Big|_{k,z_1} \gamma$$

$$= \operatorname{Im}(z_2)^{k-1} \sum_{\gamma \in \Gamma} \left(\frac{1}{(z_1 - z_2)(z_1 - \overline{z_2})^{k-1}} \right) \Big|_{2-k,z_2} \gamma.$$

Remarks.

(1) Note that the functions $\Phi_{\kappa,m}$ (taken from [BFOR17, Definition 6.10]) and $\mathcal{P}_{\kappa,m}$ from equation (VII.13) (taken from [BKV13, Section 2]) are normalized differently, and use opposite signs of the index $m \in \mathbb{Z}$. Up to these conventions, the Maaß–Poincaré series $\Phi_{\kappa,m}$ becomes the Poincaré series of exponential type $P_{\kappa,m}$ in the case of weight $\kappa > 2$ (by [BCLO10, item 13.18.2] for example).
(2) Following Hecke's trick, the function $\Phi_{\kappa,m}$ admits an analytic continuation to $\kappa = 0$ by introducing a spectral parameter s in the summation, and this continuation coincides with $G_m(\tau, s)$. We refer the reader to [FO08] (and [BFOR17, p. 97]) for more details.

We summarize their properties.

Lemma VI.2.5.

(1) *The function $P_{k,m}$ is a holomorphic cusp form for any $m > 0$, and a weakly holomorphic modular form for any $m < 0$.*
(2) *The function $G_m(\tau, s)$ is a weak Maaß form of weight 0 and eigenvalue $s(1-s)$ with respect to τ.*
(3) *The function $\Phi_{\kappa,m}(\tau)$ is a harmonic Maaß form of weight κ. It decays like a cusp form towards all cusps inequivalent to $i\infty$, and the principal part at the cusp $i\infty$ is given by $\varphi_{\kappa,m}(\tau) q^m$.*
(4) *The function $\mathbb{P}_k(z_1, z_2)$ is a polar harmonic Maaß form of weight $2-k$ in z_2, and a meromorphic modular form of weight k without a pole at the cusp in z_1. Moreover, the singularities of $\mathbb{P}_k(z_1, z_2)$ as a function of either argument are the Γ-orbits of the other argument.*

Proof. To check the claimed growth conditions, one has to compute the Fourier expansions and investigate the constant term in each expansion. We provide references for each item.

(1) This can be found in [BFOR17, Theorems 6.8, 6.9] for example.
(2) This is computed in [Fay77, Theorem 3.4] (see [Gol79, equation (1.13)], [DIT11, p. 19] as well).

(3) This can be found in [BFOR17, pp. 97]. The projection to Kohnen's plus space was calculated in [BO07, Theorem 2.1].

(4) The statement in z_1 is due to Petersson [Pet50], see [BK20, Proposition 3.3] as well. The statement in z_2 is proven in [BK20, Proposition 3.2]. The claim dealing with the singularities of $\mathbb{P}_k$ follows by its definition.

Modularity is obvious, and the analycicity condition is straightforward to check due to absolute convergence of each series. $\qquad\square$

We refer the reader to the exposition in [BK20] for more details on $\mathbb{P}_k$ and related functions.

VI.2.7   Locally harmonic Maaß forms and local cusp forms

In [BKK15], Bringmann, Kane, and Kohnen introduced locally harmonic Maaß forms for $k > 1$, which were independently investigated for $k = 1$ (i. e. weight 0) by Hövel [Höv12] in his PhD thesis as well. We follow [BKK15] here.

Definition VI.2.6. A *locally harmonic Maaß form* of weight k for Γ with exceptional set $X \subsetneq \mathbb{H}$ is a function $f \colon \mathbb{H} \to \mathbb{C}$, which satisfies the following properties:

(1) For all $\gamma \in \Gamma$ and all $\tau \in \mathbb{H}$ we have $(f|_k \gamma)(\tau) = f(\tau)$.
(2) For every $\tau \in \mathbb{H} \setminus X$, there exists a neighborhood of τ, in which f is real-analytic and $\Delta_k(f) = 0$.
(3) For every $\tau \in X$, we have

$$f(\tau) = \frac{1}{2} \lim_{\varepsilon \searrow 0} \left(f(\tau + i\varepsilon) + f(\tau - i\varepsilon) \right).$$

(4) The function f exhibits at most polynomial growth towards the cusp $i\infty$, namely $f \in O(v^\delta)$ for some $\delta > 0$.

The points in the exceptional set X are called *jump singularities* due to a wall-crosing behaviour between any two connected components of $\mathbb{H} \setminus X$ (see Section VII.2 for a definition as well). This definition is motivated by the peculiar first example

$$\mathcal{F}_{1-k,D}(\tau) \coloneqq \frac{1}{2} \sum_{Q \in \mathcal{Q}_D} \operatorname{sgn}(Q_\tau) \, Q(\tau, 1)^{k-1} \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; k - \frac{1}{2}, \frac{1}{2} \right),$$

where $D > 0$ is a non-square discriminant, and $\beta(x; r, s)$ refers to the incomplete β-function (see [BCLO10, item 8.17.1] for example). We observe that "locality" is caused precisely by the presence of the sign function in the definition of $\mathcal{F}_{1-k,D}$, and indeed Bringmann, Kane, and Kohnen proved that $\mathcal{F}_{1-k,D}$ satisfies their definition with weight $2 - 2k \in -2\mathbb{N}$ and exeptional set E_D.

Definition VI.2.7. A *local cusp form* of weight k for Γ with exceptional set $X \subsetneq \mathbb{H}$ is a function $f \colon \mathbb{H} \to \mathbb{C}$, which satisfies the following properties:

(1) For all $\gamma \in \Gamma$ and all $\tau \in \mathbb{H}$ we have $(f|_k \gamma)(\tau) = f(\tau)$.

(2) For every $\tau \in \mathbb{H} \setminus X$, there exists a neighborhood of τ, in which f is holomorphic.

(3) For every $\tau \in X$, we have $f(\tau) = \frac{1}{2} \lim_{\varepsilon \searrow 0} (f(\tau + i\varepsilon) + f(\tau - i\varepsilon))$.

(4) The function f vanishes as $\tau \to i\infty$.

Altogether, this motivates the definition and inspection of $\widehat{\mathcal{E}}_{k,D}(\tau, s)$.

VI.2.8 The functions E_2^*, j, and j_m

The holomorphic Eisenstein series are given by

$$E_k(\tau) := P_{k,0}(\tau) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sum_{\ell \mid n} \ell^{k-1} q^n,$$

where B_k is the k-th Bernoulli number. If $k \geq 4$ is even then $E_k \in M_k(\Gamma)$, and E_2 is quasimodular. We define

$$E_2^*(\tau) := E_2(\tau) - \frac{3}{\pi v},$$

and observe that E_2^* is a harmonic Maaß form of weight 2 for Γ (see [BFOR17, Lemma 6.2]). The *modular invariant* for Γ is the function

$$j(\tau) := \frac{E_4(\tau)^3}{\Delta(\tau)} \in M_0^!(\Gamma),$$

where

$$\Delta(\tau) := q \prod_{n \geq 1} (1 - q^n)^{24} = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} \in S_{12}(\Gamma)$$

is the normalized *modular discriminant function*. We have

$$\frac{1}{2\pi i} \frac{\partial j}{\partial \tau}(\tau) = -\frac{E_4(\tau)^2 E_6(\tau)}{\Delta(\tau)} \in M_2^!(\Gamma),$$

which can be verified by Ramanujan's differential system [BvdGHZ08, Proposition 15]

$$\frac{1}{2\pi i} \frac{\partial E_2}{\partial \tau} = \frac{E_2^2 - E_4}{12}, \qquad \frac{1}{2\pi i} \frac{\partial E_4}{\partial \tau} = \frac{E_2 E_4 - E_6}{3}, \qquad \frac{1}{2\pi i} \frac{\partial E_6}{\partial \tau} = \frac{E_2 E_6 - E_4^2}{2}.$$

As an intermediate result, one can check that

$$\frac{1}{2\pi i}\frac{\partial \Delta}{\partial \tau} = E_2(\tau)\Delta(\tau).$$

For every $m \geq 0$, let $j_m(\tau)$ be the unique function in the space $M_0^!(\Gamma)$ having a Fourier expansion of the form $q^{-m} + O(q)$. For instance, we have

$$j_0(\tau) = 1, \qquad j_1(\tau) = j(\tau) - 744, \qquad j_2(\tau) = j(\tau)^2 - 1488j(\tau) + 159768,$$

and the set $\{j_m \colon m \geq 0\}$ is a basis for $M_0^!$. This was proven by Asai, Kaneko, and Ninomiya [AKN97], and they additionally established the expansion

$$\frac{\frac{1}{2\pi i}\frac{\partial j}{\partial \tau}(\tau)}{j(w) - j(\tau)} = \sum_{m\geq 0} j_m(w)q^m, \qquad \mathrm{Im}(\tau) > \mathrm{Im}(w).$$

Alternatively, the functions j_m can be constructed following [BKLOR18]. More precisely, the authors proved that the functions j_m form a Hecke system, that is if T_m denotes the normalized Hecke operator, then define j_0, j_1 as above, and extend inductively by

$$j_m = T_m(j_1).$$

VI.3 Hyperbolic Eisenstein series

Let $D > 0$ be a non-square discriminant, d be a positive fundamental discriminant dividing D, and $k \in 2\mathbb{N}$. We recall the definition of our two hyperbolic Eisenstein series from equations (VI.1), (VI.2), and the fact that both converge absolutely for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - \frac{k}{2}$.

Remark. Let d_{hyp} be the hyperbolic distance (see [Iwa02, p. 7] for example), and $Q \in \mathcal{Q}_D$. Then, we have

$$\frac{|Q(\tau, 1)|}{v} = D^{\frac{1}{2}}\cosh\left(d_{\mathrm{hyp}}(\tau, S_Q)\right).$$

A proof of this idendity can be found in [Völ18, Lemma 2.5.4]. Note that $z \in S_Q$ if and only if $d_{\mathrm{hyp}}(z, S_Q) = 0$.

By Theorem V.1.1, the function $\mathcal{E}_{2,D}$ possesses an analytic continuation to $s = 0$. Along the lines of Lemma VI.2.1 (3), we define

$$\widetilde{\mathcal{E}}_{k,D}(\tau, s) := \sum_{Q_0 \in \mathcal{Q}_D/\Gamma} \chi_d(Q_0) \sum_{Q \sim Q_0} \frac{\mathrm{sgn}(Q)^{\frac{k}{2}}\mathbb{1}_Q(\tau)v^s}{Q(\tau,1)^{\frac{k}{2}}|Q(\tau,1)|^s}.$$

Remark. In [Zag10], Zagier introduced the notion of quantum modular forms, and discusses various examples. In addition to his paper, we refer the reader to [BFOR17, Chapter 21] for a discussion and more recent connections of such forms to the theory of modular forms as well as of Maaß forms. In particular, Zagier's second example involves the quantum modular form

$$\sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D \\ a<0}} \max\{Q(x,1),0\}^5 = \sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D \\ a<0<Q(x,1)}} Q(x,1)^5, \qquad x\in\mathbb{Q},$$

which appears also in his earlier paper [Zag99]. Recall that we have $\mathbb{1}_Q(\tau)=1$ if and only if $\operatorname{sgn}(Q)\operatorname{sgn}(vQ_\tau)=-1$. As the zeros of $Q(\tau,1)$ are quadratic irrationals, the limit $\lim_{\tau\to x}\frac{1}{Q(\tau,1)}$ exists for every $x\in\mathbb{Q}$. Furthermore, we note that

$$\lim_{\tau\to x}(vQ_\tau) = \lim_{\tau\to x}\left(a\,|\tau|^2 + bu + c\right) = Q(x,1).$$

Altogether, this suggests that there might be a connection of the rational function (letting $d=1$ here)

$$x \mapsto \lim_{\tau\to x}\widetilde{\mathcal{E}}_{2k-2,D}(\tau,0) = \sum_{\substack{Q\in\mathcal{Q}_D \\ \operatorname{sgn}(Q)\operatorname{sgn}(Q(x,1))=-1}} \frac{\operatorname{sgn}(Q)^{k-1}}{Q(x,1)^{k-1}} = -2\sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D \\ a<0<Q(x,1)}} \frac{1}{Q(x,1)^{k-1}}$$

to quantum modular forms for certain weights k.

We combine Lemma VI.2.1 with Theorem V.1.1.

Proposition VI.3.1. *Assume that $0 < k \equiv 2 \pmod 4$, $\tau\in\mathbb{H}\setminus E_D$, and $\operatorname{Re}(s) > 1 - \frac{k}{2}$.*

(1) *The function $\widehat{\mathcal{E}}_{k,D}(\tau,s)$ is modular of weight k, and we have*

$$\widehat{\mathcal{E}}_{k,D}(\tau,s) = \mathcal{E}_{k,D}(\tau,s) - 2\widetilde{\mathcal{E}}_{k,D}(\tau,s). \tag{VI.4}$$

(2) *The function $\widehat{\mathcal{E}}_{2,D}(\tau,s)$ has an analytic continuation to $s=0$.*
(3) *The identity* (VI.4) *holds in the case that $k=2$ and $s=0$ as well.*

Proof. The first item is a direct consequence of Lemma VI.2.1. Thus, it suffices to show that $\widetilde{\mathcal{E}}_{2,D}(\tau,s)$ has an analytic continuation to $s=0$ to prove the second item. To this end, we observe that $\widetilde{\mathcal{E}}_{k,D}(\tau,s)$ vanishes above the net of geodesics E_D, and coincides locally with $\mathcal{E}_{k,D}(\tau,s)$ up to some non-zero constant in any bounded connected component of $\mathbb{H}\setminus E_D$. Hence, one may obtain a Fourier expansion of $\widetilde{\mathcal{E}}_{2,D}$ locally by Theorem V.1.1. (The computation was presented in Chapter V). This establishes the existence of $\widehat{\mathcal{E}}_{2,D}(\tau,0)$ via the identity (VI.4) from the first item, and in addition proves the third item by uniqueness of the limit. $\qquad\square$

Moreover, we recall the Fourier expansion of $\mathcal{E}_{k,D}(\tau,0)$ for higher weights from Theorem V.1.2. Since $\mathcal{E}_{k,D}$ converges absolutely on $\mathbb{H}$ at $s=0$ for any $k \geq 4$ even, we may rearrange its Fourier expansion, and study the integrand

$$f(w,\tau) := \sum_{m \geq 1} m^{\frac{k}{2}-1} G_{-m}\left(w, \frac{k}{2}\right) q^m, \quad w \in E_D, \quad \tau \in \mathbb{H},$$

inside the cycle integral. In other words, we may rewrite the Fourier expansion from Theorem V.1.2 as

$$\mathcal{E}_{k,D}(\tau,0) = \frac{(-1)^{\frac{k}{2}} 2\pi^{\frac{k}{2}}}{D^{\frac{k+2}{4}} \Gamma\left(\frac{k}{4}\right)^2} \sum_{Q \in \mathcal{Q}_D/\Gamma} \chi_d(Q) \mathcal{C}_0(f(\cdot,\tau),Q).$$

We obtained an alternative representation of the Fourier expansion of $\mathcal{E}_{2,D}(\tau,0)$ already if τ is located in the unbounded component of a fundamental domain for Γ. The main ingredient to prove the second claim of Theorem VI.1.1 is to find such an representation in the case of higher weights under the same assumption on τ.

Proposition VI.3.2. *Let $2 < k \equiv 2 \pmod 4$, let $D > 0$ be a non-square discriminant, and d be a positive fundamental discriminant dividing D. Suppose that v is sufficiently large, that is τ is located above the net of geodesics E_D. Then $\mathcal{E}_{k,D}(\tau,0)$ coincides with the function*

$$\sum_{Q \in \mathcal{Q}_D/\Gamma} \chi_d(Q) \mathcal{C}_{2-k}\left(\mathbb{P}_k(\tau,\cdot),Q\right)$$

up to an explicit non-zero constant, which is provided in equation (VI.8).

Remark. Let $W_{\mu,\nu}$ be the usual W-Whittaker function. Inserting the Fourier expansion of G_{-m}, next comparing with the Fourier expansion of $P_{k,m}$ (see the proof of Lemma VI.2.5 for a list of references), and rearranging further, one obtains

$$f(w,\tau) = \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma(k)} \sum_{m \geq 1} m^{\frac{k}{2}-1} M_{0,\frac{k}{2}-\frac{1}{2}}(4\pi |m| \operatorname{Im}(w)) e^{-2\pi i m \operatorname{Re}(w)} q^m$$

$$+ \frac{2^{2-k} \pi^{-\frac{k}{2}} \Gamma(k)}{(k-1)\Gamma\left(\frac{k}{2}\right)} \sin\left(\frac{\pi}{2}(1-k)\right) \operatorname{Im}(w)^{1-\frac{k}{2}} \left(E_k(\tau)-1\right)$$

$$+ i^{-k} \sum_{n \neq 0} |n|^{\frac{k-1}{2}} W_{0,\frac{k}{2}-\frac{1}{2}}(4\pi |n| \operatorname{Im}(w)) \left(P_{k,n}(\tau)-q^n\right) e^{-2\pi i n \operatorname{Re}(w)}.$$

However, we may not split the final sum involving $P_{k,n}(\tau)-q^n$ into two separate sums over n, since the resulting expressions would not converge with respect to τ. This emphasizes the error to modularity of $\mathcal{E}_{k,D}$ from a different viewpoint.

VI.4 Proof of Theorem VI.1.1

We begin with the proof of Proposition VI.3.2. To this end, we write $w = x+iy \in \Gamma_Q \backslash S_Q$ for the integration variable of the cycle integral, and collect three intermediate results first. In case of ambiguity, we specify the variable a Maaß operator shall act on by an additional subscript next to the weight. The first step is the following relation between G_{-m} and $\Phi_{2-k,-m}$.

Lemma VI.4.1. *We have*

$$L_0^{\frac{k}{2}-1}\left(G_{-m}\right)\left(w, \frac{k}{2}\right) = \frac{C_1(k)\Gamma(k)}{(8\pi\,|m|)^{\frac{k}{2}-1}}\Phi_{2-k,-m}(w), \quad C_1(k) := \prod_{j=0}^{\frac{k-4}{2}}(k+2j).$$

Proof. By absolute convergence of the series defining G_{-m}, we may differentiate the seed directly. We calculate that

$$L_0^{\frac{\ell}{2}+1}\left(M_{0,\frac{k}{2}-\frac{1}{2}}(4\pi\,|m|\,y)e^{-2\pi imx}\right) = \prod_{j=0}^{\frac{\ell}{2}}(k+2j)\left(\frac{y}{2}\right)^{\frac{\ell}{2}+1}M_{\frac{\ell}{2}+1,\frac{k}{2}-\frac{1}{2}}(4\pi\,|m|\,y)e^{-2\pi imx}$$

for every $\ell \in 2\mathbb{N}_0$. We compare this with the definition of the seed $\varphi_{\kappa,m}$, and choose $\ell = k - 4$. This yields the claim. $\square$

The second step is to connect this result to the Fourier expansion of $\mathcal{E}_{k,D}(\tau, 0)$. Thus, we need an identity involving (iterated) Maaß operators and cycle integrals. This was performed by Alfes-Neumann and Schwagenscheidt [ANS20], generalizing earlier results of Bringmann, Guerzhoy, and Kane [BGK14, BGK15]. To simplify the notation, we omit the weights of the cycle integrals temporarily.

Lemma VI.4.2 ([ANS20, Theorem 1.1]). *Let $h: \mathbb{H} \to \mathbb{C}$ be a smooth function, which transforms like a modular form of weight $2 - 2\kappa \in 2\mathbb{Z}$ for Γ. Then we have the identity*

$$\mathcal{C}(L_{2-2\kappa}(h), Q) = \mathcal{C}(R_{2-2\kappa}(h), Q) = \overline{\mathcal{C}(\xi_{2-2\kappa}(h), Q)}.$$

Moreover, if h is a weak Maaß form of weight $2 - 2\kappa$ with eigenvalue λ, then we have

$$\mathcal{C}\left(R_{2-2\kappa}^{\kappa-\ell}(h), Q\right) = ((\kappa+\ell)(\kappa-\ell-1) - \lambda)\,\mathcal{C}\left(R_{2-2\kappa}^{\kappa-\ell-2}(h), Q\right), \; \textit{if } \ell \leq \kappa - 2, \quad \text{(VI.5)}$$

$$\mathcal{C}\left(L_{2-2\kappa}^{-\kappa-\ell+2}(h), Q\right) = ((\kappa+\ell)(\kappa-\ell-1) - \lambda)\,\mathcal{C}\left(L_{2-2\kappa}^{-\kappa-\ell}(h), Q\right), \; \textit{if } \ell \leq -\kappa. \quad \text{(VI.6)}$$

Note that the conditions on ℓ in equations (VI.5), (VI.6) include the cases $R_{2-2\kappa}^0$ and $L_{2-2\kappa}^0$. Thus, we may insert a suitable chain of raising or lowering operators in our cycle integrals and compensate for that by factors in κ, ℓ from equations (VI.5), (VI.6).

The third step is to utilize an identity due to Bringmann and Kane [BK20].

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Lemma VI.4.3 ([BK20, equations (3.10), (3.11)]). *We have*

$$\sum_{m \geq 1} \Phi_{2-k,-m}(w)q^m = \frac{i}{2\pi}(2i)^{k-1}\mathbb{P}_k(\tau, w),$$

whenever

$$\text{Im}(\tau) > \max\left(\text{Im}(w), \frac{1}{\text{Im}(w)}\right).$$

Remark. By work of Kohnen [Koh85] and of Kohnen and Zagier [KZ81], the Shimura [Shi73] and Shintani [Shi75] lifts between integral and half integral weight cusp forms both admit a representation as a (scalar-valued) theta lift with kernel function ($k \in \mathbb{N}_{>2}$)

$$\Omega(z, \tau) := \sum_{D > 0} D^{k-\frac{1}{2}} f_{k,D}(z)q^D. \tag{VI.7}$$

In particular, Ω is a weight $2k$ cusp form of level 1 with respect to z, and a weight $k + \frac{1}{2}$ cusp form of level 4 in the so-called Kohnen's plus space (see Section VII.2) with respect to τ by virtue of a result of Vignéras [Vig77]. Moreover, Katok [Kat85] proved that $f_{k,D}$ can be written as a hyperbolic Poincaré series[4]. The cited result by Bringmann and Kane from Lemma VI.4.3 now can be viewed as a natural "parabolic analogue" of the "hyperbolic summation formula" (VI.7) provided that $\text{Im}(\tau) > \max\left(\text{Im}(w), \frac{1}{\text{Im}(w)}\right)$, because $\Phi_{2-k,-m}$ is constructed as a certain parabolic Poincaré series.

Now, we are in position to prove Proposition VI.3.2.

Proof of Proposition VI.3.2. Since τ is assumed to be located above the net of geodesics, the assumption from Lemma VI.4.3 is satisfied for every $w \in E_D$. ($\text{Im}(w)$ is bounded from below and above.) In addition, $\mathbb{P}_k$ has no poles for such τ and w.

We invoke Lemma VI.4.2, and employ equation (VI.6) backwards and iteratively to the integrand

$$f(w, \tau) = \sum_{m \geq 1} m^{\frac{k}{2}-1} G_{-m}\left(w, \frac{k}{2}\right)q^m,$$

from the Fourier expansion of $\mathcal{E}_{k,D}$. Here, we keep τ fixed, and take $\kappa = 1$, $\lambda = \frac{k}{2}\left(1 - \frac{k}{2}\right)$, and $\ell = -1, -3, \ldots, -\frac{k}{2} + 2$ using that $k \equiv 2 \pmod 4$. This produces the constant

$$C_2(k) := \prod_{\substack{\ell = -\frac{k}{2}+2 \\ \ell \text{ odd}}}^{-1} \frac{1}{(1+\ell)(-\ell) - \frac{k}{2}\left(1 - \frac{k}{2}\right)}.$$

---

[4]Her result is summarized in [BKK15, equation (3.6)] as well. An excellent exposition on various types of Poincaré series, including the hyperbolic ones, can be found in a paper by Imamoḡlu and O'Sullivan [IO09].

To indicate the steps, we keep the constants until the last equation. Combining, we have

$$
\begin{aligned}
\mathcal{E}_{k,D}(\tau,0) &= \frac{(-1)^{\frac{k}{2}}2\pi^{\frac{k}{2}}}{D^{\frac{k}{4}}\Gamma\left(\frac{k}{4}\right)^2} \sum_{Q\in\mathcal{Q}_D/\Gamma} \chi_d(Q)\mathcal{C}_0\left(L_{0,\cdot}^0 f(\cdot,\tau),Q\right) \\
&= \frac{(-1)^{\frac{k}{2}}2\pi^{\frac{k}{2}}C_2(k)}{D^{\frac{k}{4}}\Gamma\left(\frac{k}{4}\right)^2} \sum_{Q\in\mathcal{Q}_D/\Gamma} \chi_d(Q)\mathcal{C}_{2-k}\left(L_{0,\cdot}^{\frac{k}{2}-1} f(\cdot,\tau),Q\right) \\
&= \frac{(-1)^{\frac{k}{2}}2\pi^{\frac{k}{2}}C_2(k)}{D^{\frac{k}{4}}\Gamma\left(\frac{k}{4}\right)^2}\frac{C_1(k)\Gamma(k)}{(8\pi)^{\frac{k}{2}-1}} \sum_{Q\in\mathcal{Q}_D/\Gamma} \chi_d(Q)\mathcal{C}_{2-k}\left(\sum_{m\geq 1}\Phi_{2-k,-m}(\cdot)q^m,Q\right) \\
&= \frac{(-1)^{\frac{k}{2}}2\pi^{\frac{k}{2}}C_2(k)}{D^{\frac{k}{4}}\Gamma\left(\frac{k}{4}\right)^2}\frac{C_1(k)\Gamma(k)}{(8\pi)^{\frac{k}{2}-1}}\frac{i}{2\pi}(2i)^{k-1} \sum_{Q\in\mathcal{Q}_D/\Gamma} \chi_d(Q)\mathcal{C}_{2-k}\left(\mathbb{P}_k(\tau,\cdot),Q\right).
\end{aligned}
$$

The constant in front of the final sum simplifies to

$$
C(k,D) := \frac{(-1)^k\Gamma(k)}{2^{\frac{k}{2}-2}D^{\frac{k}{4}}\Gamma\left(\frac{k}{4}\right)^2}C_1(k)C_2(k). \tag{VI.8}
$$

This establishes the propostition. $\qquad\square$

We conclude this section and the chapter with the proof of Theorem VI.1.1.

Proof of Theorem VI.1.1.

(1) The case $k = 2$ was shown in Chapter V in the unbounded component of $\mathbb{H}\setminus E_D$ for $\mathcal{E}_{2,D}(\tau,0)$. Since $\widehat{\mathcal{E}}_{k,D}(\tau,0) = \mathcal{E}_{k,D}(\tau,0)$ in the unbounded component by definition of $\mathbb{1}_Q$, the result of Chapter V extends to $\widehat{\mathcal{E}}_{k,D}(\tau,0)$ in the unbounded component of $\mathbb{H}\setminus E_D$ directly. Now, we can use modularity of $\widehat{\mathcal{E}}_{k,D}(\tau,0)$ to obtain the claim in the other connected components of $\mathbb{H}\setminus E_D$.

(2) Suppose that $2 < k \equiv 2 \pmod 4$. Modularity follows by Lemma VI.2.1 (1), and $\widehat{\mathcal{E}}_{k,D}(\tau,0)$ is holomorphic outside E_D. The limit condition on E_D can be verified as in the proof of Proposition VII.4.2, which adapts [BKK15, Proposition 5.2]. The vanishing at $i\infty$ either follows by $\mathrm{sgn}(Q_\tau) = \mathrm{sgn}(Q)$ in the unbounded component and cuspidality of $f_{k,D}$, or by the Fourier expansions of $\mathcal{E}_{k,D}(\tau,0)$ and $\widetilde{\mathcal{E}}_{k,D}(\tau,0)$.

(3) We prove the explicit representation of $\widehat{\mathcal{E}}_{k,D}(\tau,0)$ outside E_D. If τ is located above the net of geodesics E_D, we have $\widehat{\mathcal{E}}_{k,D}(\tau,0) = \mathcal{E}_{k,D}(\tau,0)$. We apply Propostion VI.3.2, and obtain the claimed representation of $\widehat{\mathcal{E}}_{k,D}$ above the net of geodesics. Finally, the representation extends to every connected component of $\mathbb{H}\setminus E_D$ by virtue of weight k modularity of both sides of the claimed identity.

$\qquad\square$

VI.5   References

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Chapter VII

A modular framework of functions of Knopp and indefinite binary quadratic forms

This chapter is based on a preprint [BM22] of the same title submitted for publication. This is joint work with Prof. Dr. Kathrin Bringmann.

VII.1   Introduction and statement of results

Throughout the chapter $D > 0$ is a non-square discriminant, $k \in 2\mathbb{N}$, $\mathcal{Q}_d$ denotes the set of integral binary quadratic forms $Q = [a, b, c]$ of discriminant $d \in \mathbb{Z}$, and $\mathbb{H}$ is the complex upper half-plane. In 1975, Zagier [Zag75] introduced the functions[1]

$$f_{\kappa,D}(\tau) := \sum_{Q \in \mathcal{Q}_D} \frac{1}{Q(\tau,1)^\kappa}, \qquad \tau \in \mathbb{H},$$

and proved that they are cusp forms if $\kappa > 1$ (if $\kappa = 1$, one may use Hecke's trick, see [Koh85, p. 239]). To name a few prominent applications of the $f_{\kappa,D}$'s, they are coefficients of the holomorphic kernel function of the Shimura [Shi73] and Shintani [Shi75] lifts due to [Koh85] (see Remark VI.4 as well), and they are closely related to central L-values by [KZ81]. Their even periods are rational according to [KZ84].

Over 30 years ago, Knopp [Kno90, equation (4.5)] found a term-by-term preimage of each $f_{\kappa,D}$ under the *Bol operator* $\mathbb{D}^{2k-1}$ [BOR08], where $\mathbb{D} := \frac{1}{2\pi i} \frac{\partial}{\partial \tau}$ (compare Proposition VII.3.1 (2)). To ensure convergence after summing over $Q \in \mathcal{Q}_D$, he changed the sign of k in his result afterwards, which lead to (here and throughout Log denotes the prinicpal branch of the complex logarithm)

$$\psi_{k+1,D}(\tau) := \sum_{Q \in \mathcal{Q}_D} \frac{\mathrm{Log}\left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+}\right)}{Q(\tau,1)^{k+1}}, \qquad \alpha_{[a,b,c]}^\pm := \frac{-b \pm \sqrt{D}}{2a} \in \mathbb{R}. \qquad \text{(VII.1)}$$

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[1] We define $f_{\kappa,D}$ in Zagier's original normalization, which differs from the normalization used in [BKK15] for instance.

He furthermore stated that $\psi_{k+1,D}(\tau + 1) = \psi_{k+1,D}(\tau)$, and the behaviour of $\psi_{k+1,D}$ under modular inversion[2] (see [Kno90, equation (4.6)]). Correcting a typo there, we find that (see Proposition VII.3.1 (3))

$$\tau^{-2k-2}\psi_{k+1,D}\left(-\frac{1}{\tau}\right) - \psi_{k+1,D}(\tau)$$

$$= \sum_{Q \in \mathcal{Q}_D} \frac{\log\left|\frac{\alpha_Q^+}{\alpha_Q^-}\right|}{Q(\tau,1)^{k+1}} - 2\pi i \sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D \\ a<0<c}} \frac{1}{Q(\tau,1)^{k+1}}. \quad \text{(VII.2)}$$

On the one hand, we observe that $\psi_{k+1,D}$ is holomorphic and vanishes at $i\infty$ (this follows by Proposition VII.3.1 (1) and (VII.18)). On the other hand, $\psi_{k+1,D}$ itself is not modular. Hence, it is natural to "complete" $\psi_{k+1,D}$. Setting $\mathbb{H}^- := -\mathbb{H}$ throughout, completions of $\psi_{k+1,D}$ are bimodular forms[3] $\Omega_{k+1,D}$ of weight $(2k+2,0)$ defined on $\mathbb{H} \times \mathbb{H}^-$ such that

$$\lim_{w \to -i\infty} \Omega_{k+1,D}(\tau,w) = \psi_{k+1,D}(\tau). \quad \text{(VII.3)}$$

Here we construct such completions explicitly. Firstly, we note that the final sum appearing in (VII.2) is finite, because $b^2 + 4|ac| = D > 0$ has only finitely many integral solutions. This leads to Knopp's modular integrals with rational period functions [Kno78]. Roughly speaking, period polynomials describe the obstruction of modularity of Eichler integrals [Eic57] (defined in (VII.8)) of cusp forms, and Knopp generalized this notion to rational functions instead of polynomials. Such functions are called *modular integrals*. Parson [Par93] defined such modular integrals by

$$\varphi_{k+1,D}(\tau) := \frac{1}{2}\sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q)}{Q(\tau,1)^{k+1}} = \sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D \\ a>0}} \frac{1}{Q(\tau,1)^{k+1}}, \quad \text{(VII.4)}$$

$$\text{sgn}\left([a,b,c]\right) := \text{sgn}(a)$$

and we recall her result on the $\varphi_{k+1,D}$'s in Lemma VII.3.3. Secondly, we define

$$Q_w := \frac{1}{\text{Im}(w)}\left(a|w|^2 + b\,\text{Re}(w) + c\right), \quad S_Q := \{\tau \in \mathbb{H} : Q_\tau = 0\}, \quad E_D := \bigcup_{Q \in \mathcal{Q}_D} S_Q,$$

for $w \in \mathbb{C} \setminus \mathbb{R}$, $Q \in \mathcal{Q}_D$, as well as the functions

$$\rho_{k+1,D}(\tau,w) := \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log}\left(\frac{w-\alpha_Q^-}{w-\alpha_Q^+}\right)}{Q(\tau,1)^{k+1}}, \quad \lambda_{k+1,D}(\tau,w) := 2i\sum_{Q \in \mathcal{Q}_D} \frac{\arctan\left(\frac{Q_w}{\sqrt{D}}\right)}{Q(\tau,1)^{k+1}}. \quad \text{(VII.5)}$$

---

[2]We alert the reader to the fact that Knopp used the older convention $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

[3]We slightly modify the initial definition by Stienstra and Zagier [SZ06] to include the domain $\mathbb{H} \times \mathbb{H}^-$.

for $w \in \mathbb{H}^-$. We refer to Propositions VII.3.2 and VII.3.4 for some of their properties. Thirdly, we define

$$\Omega_{k+1,D}(\tau, w) := \psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau, w) + 2\pi i \varphi_{k+1,D}(\tau) + \lambda_{k+1,D}(\tau, w), \qquad \text{(VII.6)}$$

on $\mathbb{H} \times \mathbb{H}^-$ and have the following results.

Theorem VII.1.1. *Let $\tau \in \mathbb{H}$, $w \in \mathbb{H}^-$.*

(1) *The functions $\Omega_{k+1,D}$ are bimodular of weight $(2k+2, 0)$ that is*

$$\Omega_{k+1,D}(\tau+1, w+1) = \Omega_{k+1,D}(\tau, w), \quad \Omega_{k+1,D}\left(-\frac{1}{\tau}, -\frac{1}{w}\right) = \tau^{2k+2}\Omega_{k+1,D}(\tau, w).$$

(2) *Condition (VII.3) holds.*
(3) *We have*

$$\lim_{\tau \to i\infty} \Omega_{k+1,D}(\tau, w) = 0.$$

(4) *The functions $\Omega_{k+1,D}$ are holomorphic with respect to τ and antiholomorphic with respect to w.*
(5) *We have that*

$$\Omega_{k+1,D}(\tau, \overline{\tau}) = 0.$$

Remark. During the proof of Theorem VII.1.1 (5), we show that

$$\text{Log}\left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+}\right) - \text{Log}\left(\frac{\overline{\tau} - \alpha_Q^-}{\overline{\tau} - \alpha_Q^+}\right) + \pi i \, \text{sgn}(Q) + 2i \arctan\left(\frac{Q_{\overline{\tau}}}{\sqrt{D}}\right) = 0.$$

This implies that the $\Omega_{k+1,D}$'s have representations on $\mathbb{H} \times \mathbb{H}$ as well, and these representations coincide with the functions

$$\omega_{k+1,D}(\tau, z) := \psi_{k+1,D}(\tau) - \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log}\left(\frac{z - \alpha_Q^-}{z - \alpha_Q^+}\right)}{Q(\tau, 1)^{k+1}}, \qquad (\tau, z) \in \mathbb{H} \times \mathbb{H}.$$

The $\omega_{k+1,D}$'s satisfy

$$\omega_{k+1,D}(\tau+1, z+1) = \omega_{k+1,D}(\tau, z), \qquad \omega_{k+1,D}\left(-\frac{1}{\tau}, -\frac{1}{z}\right) = \tau^{2k+2}\omega_{k+1,D}(\tau, z),$$

$$\lim_{z \to i\infty} \omega_{k+1,D}(\tau, z) = \psi_{k+1,D}(\tau).$$

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In the course of proving Theorem VII.1.1 (5), we encounter the functions (see (VII.20))

$$g_{k+1,D}(\tau) := \sum_{Q \in \mathcal{Q}_D} \frac{\mathrm{sgn}(Q_\tau)}{Q(\tau,1)^{k+1}}, \qquad\qquad\qquad \text{(VII.7)}$$

which are *local cusp forms*. That is, they behave like cusp forms of weight $2k+2$ outside E_D, however, in addition, have jumping singularities[4] on E_D; see Definition VI.2.7 and Proposition VII.4.1 for details. By Theorem VI.1.1, the functions $g_{k+1,D}$ can be written in terms of traces of cycle integrals.

Next, we construct negative weight analogues $\mathcal{G}_{-k,D}$ of the $g_{k+1,D}$'s along the lines of [BKK15]. This is natural, because the $g_{k+1,D}$'s are "odd" positive weight analogues of the $f_{\kappa,D}$'s, and the $f_{\kappa,D}$'s recently motivated the definition of new automorphic objects by Bringmann, Kane, and Kohnen [BKK15]. To be more precise, we let $\beta(x;s,w) :=$ $\int_0^x t^{s-1}(1-t)^{w-1} \mathrm{d}t$, $x \in (0,1]$, $\mathrm{Re}(s)$, $\mathrm{Re}(w) > 0$, be the *incomplete β-function*, $\tau = u+iv$ throughout, and we define

$$\mathcal{G}_{-k,D}(\tau) := \frac{1}{2} \sum_{Q \in \mathcal{Q}_D} Q(\tau,1)^k \beta\left(\frac{Dv^2}{|Q(\tau,1)|^2}; k+\frac{1}{2}, \frac{1}{2}\right), \qquad \tau \in \mathbb{H} \setminus E_D.$$

In the spirit of Knopp's initial preimage of $f_{k,D}$ under the Bol operator (without an additional sign change of k), it turns out that $\mathcal{G}_{-k,D}$ is a preimage of $g_{k+1,D}$ under the Bol operator as well as the *shadow operator* $\xi_\kappa := 2iv^\kappa \overline{\frac{\partial}{\partial \overline{\tau}}}$ due to Bruinier and Funke [BF04] (up to constants). Such a behaviour is impossible in the situation of a (globally defined) non-trivial harmonic Maaß form with cuspidal shadow[5]. If f is a cusp form of weight $2k+2$, then preimages under $\mathbb{D}^{2k+1}$ and ξ_{-2k}, respectively, are provided by the holomorphic and nonholomorphic *Eichler integrals* (see (VII.24))

$$
\begin{aligned}
\mathcal{E}_f(\tau) &:= -\frac{(2\pi i)^{2k+1}}{(2k)!} \int_\tau^{i\infty} f(w)(\tau - w)^{2k} \mathrm{d}w, \\
f^*(\tau) &:= (2i)^{-2k-1} \int_{-\overline{\tau}}^{i\infty} \overline{f(-\overline{w})}(w + \tau)^{2k} \mathrm{d}w.
\end{aligned}
\qquad \text{(VII.8)}
$$

To be able to insert the local cusp forms $g_{k+1,D}$ into each integral in (VII.8), we work in a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$, in which we have just finitely many equivalence classes of geodesics S_Q. Integrating piecewise, both Eichler integrals of $g_{k+1,D}$ are well-defined on $\mathbb{H} \setminus E_D$. In addition we ensure in Proposition VII.4.4 that both Eichler integrals of $g_{k+1,D}$ exist on E_D. This established, we prove the following properties of $\mathcal{G}_{-k,D}$. We refer the reader to Subsection VII.2.3 for definitions.

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[4]We explain this terminology in Section VII.2.

[5]One may overcome this by weakening the growth condition in Definition VII.2.6 to linear exponential growth. See [BFOR17, Theorem 6.15] for an example of such a harmonic Maaß form.

Theorem VII.1.2.

(1) *The functions $\mathcal{G}_{-k,D}$ are locally harmonic Maaß forms of weight $-2k$ with continuously (however not differentially) removable singularities on E_D.*

(2) *If $\tau \in \mathbb{H} \setminus E_D$, then we have, with c_∞ defined in equation* (VII.23),

$$\mathcal{G}_{-k,D}(\tau) = c_\infty - \frac{D^{k+\frac{1}{2}}(2k)!}{(4\pi)^{2k+1}} \mathcal{E}_{g_{k+1,D}}(\tau) + D^{k+\frac{1}{2}} g^*_{k+1,D}(\tau).$$

The functions $\mathcal{G}_{-k,D}$ are outputs of a theta lift. To motivate this, we parallel a construction of Bringmann, Kane, and Viazovska [BKV13]. We employ Borcherds [Bor98] regularization of the Petersson inner product $\langle \cdot, \cdot \rangle^{\mathrm{reg}}$ for $\Gamma_0(4)$ (see Section VII.2), and define the theta kernel[6] ($z = x + iy$, $w \in \mathbb{H}^-$)

$$\theta^*_{-k}(\tau, z) := y^{k+1} \sum_{d \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_d} |Q_\tau| \, Q(\tau, 1)^k e^{-\frac{4\pi |Q(\tau,1)|^2 y}{v^2}} e^{-2\pi i dz}. \tag{VII.9}$$

The function θ^*_{-k} transform like modular forms of weight $\frac{1}{2} - k$ in z, and of weight $-2k$ in τ, see Lemma VII.2.8. Thus, they give rise to the theta lift (F a weight $\frac{1}{2} - k$ harmonic Maaß form with cuspidal shadow)

$$\mathfrak{L}^*_{-k}(F)(\tau) := \langle F, \theta^*_{-k}(-\overline{\tau}, \cdot) \rangle^{\mathrm{reg}}.$$

It suffices to compute $\mathfrak{L}^*_{-k}$ on the Maaß–Poincaré series $\mathcal{P}_{\frac{1}{2}-k,m}$ (defined in equation (VII.13)) as they generate the space of harmonic Maaß forms with cuspidal shadows.

Theorem VII.1.3. *Let $\tau \in \mathbb{H} \setminus E_D$. We have, with Γ the usual Γ-function*

$$\mathfrak{L}^*_{-k}\left(\mathcal{P}_{\frac{1}{2}-k,D}\right)(\tau) = \frac{D^{\frac{1}{4}-\frac{k}{2}} k!}{3\Gamma\left(k + \frac{1}{2}\right)(4\pi)^{\frac{k}{2}+\frac{1}{4}}} \mathcal{G}_{-k,D}(\tau).$$

VII.2   Preliminaries

VII.2.1   Integral binary quadratic forms and Heegner geodesics

The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_d$ by ($\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$)

$$Q \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix}(x,y) := Q(ax+by, cx+dy).$$

---

[6]We use the variable τ for integral weight, and z for half-integral weight. This is opposite to the notation in [BKV13].

The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ is compatible with the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{Q}_d$, in the sense that[7]

$$(Q \circ \gamma)(\tau, 1) = j(\gamma, \tau)^2 Q(\gamma\tau, 1), \qquad j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) := c\tau + d \qquad \text{(VII.10)}$$

Since $D > 0$ is not a square, the two roots $\alpha_Q^{\pm}$ of $Q \in \mathcal{Q}_D$ are real-quadratic and connected by the Heegner geodesic S_Q. We orientate S_Q counterclockwise (resp. clockwise) if $\mathrm{sgn}(Q) > 0$ (resp. $\mathrm{sgn}(Q) < 0$). The orientation of S_Q in turn determines the sign one catches if τ jumps across S_Q. More precisely, one has $\mathrm{sgn}(Q)\,\mathrm{sgn}(Q_\tau) < 0$ if and only if τ lies in the bounded component of $\mathbb{H} \setminus S_Q$. The unbounded connected component of $\mathbb{H} \setminus E_D$ is the unique such component containing $i\infty$ on its boundary. We refer the reader to the beautiful article by Duke, Imamoḡlu, and Tóth [DIT11, Section 4] for more on Heegner geodesics.

We next collect some results, which we utilize throughout. The following lemma is straightforward.

Lemma VII.2.1. *For $Q \in \mathcal{Q}_d$, $d \in \mathbb{Z}$, we have*

$$dv^2 + Q_\tau^2 v^2 = |Q(\tau, 1)|^2.$$

To determine the weights of our functions, the following lemma is useful.

Lemma VII.2.2. *For every $Q \in \mathcal{Q}_D$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have*

$$(Q \circ \gamma)_\tau = Q_{\gamma\tau}, \qquad \frac{\mathrm{Im}(\gamma\tau)}{|Q(\gamma\tau, 1)|} = \frac{v}{|(Q \circ \gamma)(\tau, 1)|}.$$

We also require the following elementary lemma.

Lemma VII.2.3. *Let $U \subseteq \mathbb{C}$ be open. Assume that $f : U \to \mathbb{C}$ is real-differentiable and satisfies $\overline{f(\overline{\tau})} = f(\tau)$. Then*

$$\overline{\frac{\partial}{\partial \overline{\tau}} f\,(\overline{\tau})} = \frac{\partial}{\partial \tau} f(\tau).$$

The following differentiation rules are obtained by a direct calculation.

Lemma VII.2.4. *Let $Q \in \mathcal{Q}_D$.*

(1) *We have*

$$v^2 \frac{\partial}{\partial \tau} Q_{-\overline{\tau}} = \frac{i}{2} Q(-\overline{\tau}, 1), \qquad v^2 \frac{\partial}{\partial \tau} Q_\tau = \frac{i}{2} Q(\overline{\tau}, 1), \qquad v^2 \frac{\partial}{\partial \tau} \frac{Q(\tau, 1)}{v^2} = iQ_\tau.$$

---

[7]A good reference is for example Zagier's book [Zag81, § 8].

(2) *We have*

$$\frac{\partial}{\partial\bar{\tau}}\frac{v^2}{Q(\bar{\tau},1)} = \frac{iv^2 Q_\tau}{Q(\bar{\tau},1)^2}, \qquad 2iv^2 \frac{\partial}{\partial\bar{\tau}}Q_\tau = Q(\tau,1), \qquad iv^2 \frac{\partial}{\partial\bar{\tau}}\frac{Q(\bar{\tau},1)}{v^2} = Q_\tau.$$

Letting $Q'(\tau,1) := \frac{\partial}{\partial\tau}Q(\tau,1)$, the following lemma can be verified by direct calculation.

Lemma VII.2.5. *Let $Q \in \mathcal{Q}_D$ and $\tau \in \mathbb{H}$. We have*

$$Q_\tau v + ivQ'(\tau,1) = Q(\tau,1), \qquad Q'(\tau,1)^2 - 2Q''(\tau,1)Q(\tau,1) = D.$$

VII.2.2 Maaß forms and modular forms

We collect the definitions of various automorphic objects appearing in this chapter; see [BFOR17] for more details on various types of harmonic Maaß forms. Let $\kappa \in \frac{1}{2}\mathbb{Z}$, and $N := 1$ if $\kappa \in \mathbb{Z}$ and $N := 4$ if $\kappa \notin \mathbb{Z}$. The *slash operator* is defined as $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)\right)$

$$f(\tau)\big|_\kappa \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} (c\tau + d)^{-\kappa} f(\gamma\tau) & \text{if } \kappa \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)\varepsilon_d^{2\kappa}(c\tau+d)^{-\kappa}f(\gamma\tau) & \text{if } \kappa \in \frac{1}{2}+\mathbb{Z}, \end{cases}$$

where $\left(\frac{c}{d}\right)$ is the extended Legendre symbol, and for d odd $\varepsilon_d := 1$ if $d \equiv 1 \pmod 4$ and $\varepsilon_d := i$ if $d \equiv 3 \pmod 4$. The *weight κ hyperbolic Laplace operator* is given as

$$\Delta_\kappa := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + i\kappa v \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

We require various classes of modular objects.

Definition VII.2.6. Let $f : \mathbb{H} \to \mathbb{C}$ be a real-analytic function.

(1) We call f a *(holomorphic) modular form* of weight κ for $\Gamma_0(N)$, if f satisfies the following:

   (i) We have $f|_\kappa\gamma = f$ for all $\gamma \in \Gamma_0(N)$.
   (ii) The function f is holomorphic on $\mathbb{H}$.
   (iii) The function f is holomorphic at the cusps of $\Gamma_0(N)$.

(2) We call f a *cusp form* of weight κ for $\Gamma_0(N)$, if f is a modular form that vanishes at all cusps of $\Gamma_0(N)$.

(3) We call f a *weight κ harmonic Maaß form with cuspidal shadow* for $\Gamma_0(N)$, if f satisfies the following:

   (i) For every $\gamma \in \Gamma_0(N)$ and every $\tau \in \mathbb{H}$ we have that $f|_\kappa\gamma = f$.
   (ii) The function f has eigenvalue 0 under Δ_κ.

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(iii) There exists a polynomial $P_f \in \mathbb{C}[q^{-1}]$ (the principal part of f) such that

$$f(\tau) - P_f(\tau) = O\left(e^{-\delta v}\right)$$

as $v \to \infty$ for some $\delta > 0$, and we require analogous conditions at all other cusps of $\Gamma_0(N)$.

Forms in *Kohnen's plus space* have the additional property that their Fourier expansion is supported on indices n satisfying $(-1)^{\kappa - \frac{1}{2}} n \equiv 0, 1 \pmod 4$ with $\kappa \in \mathbb{Z} + \frac{1}{2}$.

We remark that Δ_κ splits as

$$\Delta_\kappa = -\xi_{2-\kappa} \circ \xi_\kappa, \tag{VII.11}$$

which in turn implies that a harmonic Maaß form with cuspidal shadow naturally splits into a holomorphic and a nonholomorphic part. The operator ξ_κ annihilates the holomorphic part, while the Bol operator $\mathbb{D}^{1-\kappa}$, $\kappa \in -\mathbb{N}_0$, annihilates the nonholomorphic part (since our growth condition rules out a nonholomorphic constant term in the Fourier expansion). Letting $\ell \in \mathbb{N}$, the Bol operator can be written in terms of the *iterated Maaß raising operator*

$$(-4\pi)^{\ell-1} \mathbb{D}^{\ell-1} = R_{2-\ell}^{\ell-1} := R_{\ell-2} \circ \ldots \circ R_{2-\ell+2} \circ R_{2-\ell}, \tag{VII.12}$$

$$R_{2-\ell}^0 := \mathrm{id}, \qquad R_\kappa := 2i\frac{\partial}{\partial \tau} + \frac{\kappa}{v}.$$

This identity is called *Bol's identity*, a proof can for example be found in [BFOR17, Lemma 5.3].

VII.2.3 Locally harmonic Maaß forms

In [BKK15], so-called locally harmonic Maaß forms, were introduced (for negative weights). See also [Höv12] for the case of weight 0.

Definition VII.2.7 ([BKK15, Section 2])**.** A function $f : \mathbb{H} \to \mathbb{C}$ is called a *locally harmonic Maaß form of weight κ* with exceptional set E_D, if it obeys the following four conditions:

(1) For every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have $f|_\kappa \gamma = f$.
(2) For all $\tau \in \mathbb{H} \setminus E_D$, there exists a neighborhood of τ, in which f is real-analytic and in which we have $\Delta_\kappa(f) = 0$.
(3) For every $\tau \in E_D$, we have that $f(\tau) = \frac{1}{2} \lim_{\varepsilon \to 0^+} (f(\tau + i\varepsilon) + f(\tau - i\varepsilon))$.
(4) The function f exhibits at most polynomial growth towards $i\infty$.

We say that a function $f : \mathbb{H} \setminus E_D \to \mathbb{C}$ has *jumping singularities* on E_D if

$$\lim_{\varepsilon \to 0^+} (f(\tau + i\varepsilon) - f(\tau - i\varepsilon)) \in \mathbb{C} \setminus \{0\}$$

for $\tau \in E_D$. Note that this limit depends on the geodesic S_Q on which τ is located. If

$$\lim_{\varepsilon \to 0^+} (f(\tau + i\varepsilon) - f(\tau - i\varepsilon)) = 0$$

for all $\tau \in E_D$, then we say that f has *continuously removable singularities* on E_D.

VII.2.4 A theta lift and Poincaré series

A fundamental domain of $\Gamma_0(4)$ truncated at height $T > 0$ is given by

$$\mathbb{F}_T(4) := \bigcup_{\gamma \in \Gamma_0(4) \backslash \mathrm{SL}_2(\mathbb{Z})} \gamma \mathbb{F}_T,$$

where

$$\mathbb{F}_T := \left\{ z \in \mathbb{H} : |x| \leq \tfrac{1}{2}, |z| \geq 1, y \leq T \right\}.$$

Let f and g satisfy weight κ modularity for $\Gamma_0(4)$ and let $d\mu(\tau) := \frac{du\,dv}{v^2}$. Borcherds regularization of the Petersson inner product of f and g is given by

$$\langle f, g \rangle^{\mathrm{reg}} := \int_{\Gamma_0(4) \backslash \mathbb{H}}^{\mathrm{reg}} f(w)\overline{g(w)} \operatorname{Im}(w)^\kappa d\mu(w) := \lim_{T \to \infty} \int_{\mathbb{F}_T(4)} f(w)\overline{g(w)} \operatorname{Im}(w)^\kappa d\mu(w),$$

whenever the limit exists. Although the definition of $\mathbb{F}_T(4)$ depends on a set of representatives of $\Gamma_0(4) \backslash \mathrm{SL}_2(\mathbb{Z})$, the inner product is independent of this choice.

We next define the Poincaré series appearing in Theorem VII.1.3, and follow the exposition from [BKV13, Section 2][8]. We let $\cdot |\operatorname{pr}$ denote the projection operator into Kohnen's plus space (see e.g. [BFOR17, equation (6.12)]). We furthermore let $M_{\mu,\nu}$ be the usual M-Whittaker function, and define

$$\mathcal{M}_{\kappa,s}(t) := |t|^{-\frac{\kappa}{2}} M_{\operatorname{sgn}(t)\frac{\kappa}{2}, s-\frac{1}{2}}(|t|).$$

We then define the *Maaß–Poincaré series* of weights $\kappa \in -\mathbb{N} + \frac{1}{2}$ and indices $m \in \mathbb{N}$ projected into Kohnen's plus space as (see [BKV13, equation (2.12)]), with $\Gamma_\infty := \left\{ \pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) : n \in \mathbb{Z} \right\}$,

$$\mathcal{P}_{\kappa,m}(z) := \frac{(4\pi m)^{\frac{\kappa}{2}}}{\Gamma(2-\kappa)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \left(\mathcal{M}_{\kappa, 1-\frac{\kappa}{2}}(-4\pi my)e^{-2\pi imx} \right) \Big|_\kappa \gamma \Big| \operatorname{pr}. \qquad \text{(VII.13)}$$

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[8]We remind the reader of Remark VI.2.6.

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The functions $\mathcal{P}_{\kappa,m}$ converge absolutely and are harmonic Maaß forms with cuspidal shadow, see [BFOR17, Theorem 6.11].

We lastly summarize the transformation behaviour of the theta kernel from equation (VII.9).

Lemma VII.2.8.

  (1) *The function $z \mapsto \theta^*_{-k}(\tau, z)$ is modular of weight $\frac{1}{2} - k$ and is in Kohnen's plus space.*

  (2) *The function $\tau \mapsto \theta^*_{-k}(\tau, z)$ is modular of weight $-2k$.*

Proof.

  (1) This follows by a result of Vignéras [Vig77]. The application of her result in this case can be found in [BKZ14, Section 2]. This part is also contained in [BKV13, Section 2] (up to a local sign factor).

  (2) This follows by Lemma VII.2.2 and equation (VII.10).

<div align="right">□</div>

VII.3   Proof of Theorem VII.1.1

VII.3.1   Knopp's claims on $\psi_{k+1,D}$

We now discuss the initial claims of Knopp on $\psi_{k+1,D}$.

Proposition VII.3.1.

  (1) *The functions $\psi_{k+1,D}$ converge absolutely on $\mathbb{H}$ and uniformly towards $i\infty$.*

  (2) *For $n \in \mathbb{N}$, we have*

$$\mathbb{D}^{2n-1}\left(\mathrm{Log}\left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+}\right) Q(\tau,1)^{n-1}\right) = -i(2\pi)^{2n-1}(n-1)!^2 D^{n-\frac{1}{2}}\frac{1}{Q(\tau,1)^n}.$$

  (3) *The functions $\psi_{k+1,D}$ satisfy $\psi_{k+1,D}(\tau + 1) = \psi_{k+1,D}(\tau)$ and* (VII.2).

Proof.

  (1) Let $Q = [a, b, c]$ and suppose that $v > 1$. Since $\alpha_Q^\pm \in \mathbb{R}$ are the zeros of Q, we have $Q(\tau, 1) = a(\tau - \alpha_Q^+)(\tau - \alpha_Q^-)$ and $v > 1$ implies that $\left|\tau - \alpha_Q^+\right| > 1$. Using $|a| \geq 1$ gives

$$\left|\mathrm{Log}\left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+}\right)\right| \leq \left|\log\left(\frac{|Q(\tau,1)|}{|a|\left|\left(\tau - \alpha_Q^+\right)\right|^2}\right)\right| + \pi \leq |\log|Q(\tau,1)|| + \pi,$$

and (1) thus follows by the properties of $f_{\kappa,D}$ for $\kappa > 1$ (see [Zag75]).

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(2) We proceed by induction on n. The claims for $n = 1$ and $n = 2$ follow by computing

$$\frac{\partial}{\partial \tau} \text{Log}\left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+}\right) = -\frac{\sqrt{D}}{Q(\tau, 1)}, \tag{VII.14}$$

$$\frac{\partial^3}{\partial \tau^3}\left(\text{Log}\left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+}\right) Q(\tau, 1)\right) = \frac{D^{\frac{3}{2}}}{Q(\tau, 1)^2},$$

utilizing Lemma VII.2.5 for $n = 2$. To proceed with the induction step, we define for $n \in \mathbb{N}$

$$\mathfrak{f}_n(\tau) := \text{Log}\left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+}\right) Q(\tau, 1)^{n-1}, \qquad c_n := (-1)^n (n-1)!^2.$$

Since Q is a polynomal of degree 2, we have, using the Leibniz rule,

$$\frac{\partial^{2n+1}}{\partial \tau^{2n+1}} \mathfrak{f}_{n+1}(\tau) = \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} \left(\mathfrak{f}_n(\tau) Q(\tau, 1)\right)$$
$$= \mathfrak{f}_n^{(2n+1)}(\tau) Q(\tau, 1) + (2n+1)\mathfrak{f}_n^{(2n)}(\tau) Q'(\tau, 1) + (2n+1)n\mathfrak{f}_n^{(2n-1)} Q''(\tau, 1).$$

To apply the induction hypothesis, we write $\mathfrak{f}_n^{(2n)}(\tau) = \frac{\partial}{\partial \tau} \mathfrak{f}_n^{(2n-1)}(\tau)$. Combining with the second identity of Lemma VII.2.5 then yields

$$\frac{\partial^{2n+1}}{\partial \tau^{2n+1}} \mathfrak{f}_{n+1}(\tau) = -\frac{n^2 c_n D^{n+\frac{1}{2}}}{Q(\tau, 1)^{n+1}}.$$

Simplifying gives the claim.

(3) Translation invariance of $\psi_{k+1,D}$ follows immediately from equation (VII.10) and the fact that

$$[a, b, c] \circ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = [a, -2a + b, a - b + c].$$

Again using (VII.10) and the fact that

$$[a, b, c] \circ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = [c, -b, a],$$

we obtain that

$$\tau^{-2k-2} \psi_{k+1,D}\left(-\frac{1}{\tau}\right) - \psi_{k+1,D}(\tau) = \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log}\left(\frac{-\frac{1}{\tau} - \frac{b-\sqrt{D}}{2c}}{-\frac{1}{\tau} - \frac{b+\sqrt{D}}{2c}}\right) - \text{Log}\left(\frac{\tau - \frac{-b-\sqrt{D}}{2a}}{\tau - \frac{-b+\sqrt{D}}{2a}}\right)}{Q(\tau, 1)^{k+1}}.$$

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Next, we recall that for $z, w \in \mathbb{C} \setminus \mathbb{R}$

$$\text{Log}(z) - \text{Log}(w) = \text{Log}\left(\frac{z}{w}\right) + i\left(\text{Arg}(z) - \text{Arg}(w) - \text{Arg}\left(\frac{z}{w}\right)\right). \quad \text{(VII.15)}$$

Choosing $z = \frac{-\frac{1}{\tau} - \frac{b - \sqrt{D}}{2c}}{-\frac{1}{\tau} - \frac{b + \sqrt{D}}{2c}}$, $w = \frac{\tau - \frac{-b - \sqrt{D}}{2a}}{\tau - \frac{-b + \sqrt{D}}{2a}}$ yields

$$\frac{z}{w} = \frac{\left(-\frac{1}{\tau} - \frac{b - \sqrt{D}}{2c}\right)\left(\tau - \frac{-b + \sqrt{D}}{2a}\right)}{\left(-\frac{1}{\tau} - \frac{b + \sqrt{D}}{2c}\right)\left(\tau - \frac{-b - \sqrt{D}}{2a}\right)} = \frac{\alpha_Q^+}{\alpha_Q^-} = \text{sgn}(ac)\left|\frac{\alpha_Q^+}{\alpha_Q^-}\right|.$$

Hence $\text{Arg}(z) = \text{Arg}(\text{sgn}(ac)w)$ and thus $\text{Arg}(z) - \text{Arg}(w) - \text{Arg}(\frac{z}{w})$ vanishes whenever $\text{sgn}(ac) = 1$. Therefore the corresponding terms do not contribute to $\text{Arg}(z) - \text{Arg}(w) - \text{Arg}(\frac{z}{w})$. If $\text{sgn}(ac) = -1$, we extend Log by its principal value $\text{Log}(x) = \log|x| + \pi i$ for $x \in \mathbb{R}^-$. Then we use that

$$\text{Arg}(-w) - \text{Arg}(w) = -\text{sgn}(\text{Im}(w))\pi, \quad \text{(VII.16)}$$

and $\text{Arg}(\frac{z}{w}) = \pi$. Hence, $\text{Arg}(z) - \text{Arg}(w) - \text{Arg}(\frac{z}{w})$ vanishes if $\text{sgn}(ac) = -1$ and $\text{Im}(w) < 0$. To determine the sign of $\text{Im}(w)$, we calculate that

$$\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} = \frac{\alpha_Q^+ \alpha_Q^- - \left(\alpha_Q^+ + \alpha_Q^-\right)u + u^2 + v^2}{\left|\tau - \alpha_Q^+\right|^2} - \frac{i\left(\alpha_Q^+ - \alpha_Q^-\right)v}{\left|\tau - \alpha_Q^+\right|^2}$$

$$= \frac{1}{\left|\tau - \alpha_Q^+\right|^2}\left(\frac{vQ_\tau}{a} - i\frac{\sqrt{D}}{a}v\right). \quad \text{(VII.17)}$$

Consequently, we have $\text{Im}(w) > 0$ if and only if $a < 0$. We conclude by (VII.15) and (VII.16) that

$$\text{Arg}(z) - \text{Arg}(w) - \text{Arg}\left(\frac{z}{w}\right) = \begin{cases} -2\pi & \text{if } a < 0 < c, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\tau^{-2k-2}\psi_{k+1,D}\left(-\frac{1}{\tau}\right) - \psi_{k+1,D}(\tau) = \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log}\left(\frac{\alpha_Q^+}{\alpha_Q^-}\right)}{Q(\tau,1)^{k+1}} - 2\pi i \sum_{\substack{Q \in \mathcal{Q}_D \\ a < 0 < c}} \frac{1}{Q(\tau,1)^{k+1}}.$$

By mapping $Q \mapsto -Q$, we arrive at

$$\sum_{Q \in \mathcal{Q}_D} \frac{\text{Log}\left(\frac{\alpha_Q^+}{\alpha_Q^-}\right)}{Q(\tau, 1)^{k+1}} = \sum_{Q \in \mathcal{Q}_D} \frac{\log \left|\frac{\alpha_Q^+}{\alpha_Q^-}\right|}{Q(\tau, 1)^{k+1}} + \pi i \sum_{\substack{Q=[a,b,c] \in \mathcal{Q}_D \\ \text{sgn}(ac)=-1}} \frac{1}{Q(\tau, 1)^{k+1}}$$

$$= \sum_{Q \in \mathcal{Q}_D} \frac{\log \left|\frac{\alpha_Q^+}{\alpha_Q^-}\right|}{Q(\tau, 1)^{k+1}}.$$

$\square$

Remark. By (VII.17) the branch cut of $\text{Log}(\frac{w-\alpha_Q^-}{w-\alpha_Q^+})$ is the interval $[\alpha_Q^-, \alpha_Q^+]$ or $[\alpha_Q^+, \alpha_Q^-]$.

VII.3.2 The functions $\rho_{k+1,D}$, $\varphi_{k+1,D}$, and $\lambda_{k+1,D}$

Adapting the proof of Proposition VII.3.1 (1), (3) we deduce the following results.

Proposition VII.3.2. *Let $\tau \in \mathbb{H}$, $w \in \mathbb{H}^-$.*

(1) *The functions $\rho_{k+1,D}$ converge absolutely on $\mathbb{H} \times \mathbb{H}^-$ and uniformly as $\tau \to i\infty$ resp. $w \to -i\infty$.*

(2) *We have*

$$\lim_{w \to -i\infty} \rho_{k+1,D}(\tau, w) = 0, \qquad \lim_{\tau \to i\infty} \rho_{k+1,D}(\tau, w) = 0.$$

(3) *The functions $\rho_{k+1,D}$ satisfy*

$$\rho_{k+1,D}(\tau + 1, w + 1) = \rho_{k+1,D}(\tau, w),$$

and

$$\tau^{-2k-2}\rho_{k+1,D}\left(-\frac{1}{\tau}, -\frac{1}{w}\right) - \rho_{k+1,D}(\tau, w)$$

$$= \sum_{Q \in \mathcal{Q}_D} \frac{\log \left|\frac{\alpha_Q^+}{\alpha_Q^-}\right|}{Q(\tau, 1)^{k+1}} + 2\pi i \sum_{\substack{Q=[a,b,c] \in \mathcal{Q}_D \\ a<0<c}} \frac{1}{Q(\tau, 1)^{k+1}}.$$

We next cite Parson's [Par93] result on her modular integral $\varphi_{k+1,D}$.

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Lemma VII.3.3 ([Par93, Theorem 3.1]). *The functions $\varphi_{k+1,D}$ satisfy*

$$\varphi_{k+1,D}(\tau + 1) = \varphi_{k+1,D}(\tau),$$

and

$$\tau^{-2k-2}\varphi_{k+1,D}\left(-\frac{1}{\tau}\right) - \varphi_{k+1,D}(\tau)$$

$$= -\sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D \\ ac<0}} \frac{\operatorname{sgn}(Q)}{Q(\tau,1)^{k+1}} = 2\sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D \\ a<0<c}} \frac{1}{Q(\tau,1)^{k+1}}.$$

Furthermore, we have

$$\lim_{\tau\to i\infty}\varphi_{k+1,D}(\tau) = 0.$$

We continue with some properties of $\lambda_{k+1,D}$.

Proposition VII.3.4. *Let $\tau \in \mathbb{H}$, $w \in \mathbb{H}^-$.*

(1) *The functions $\lambda_{k+1,D}$ converge absolutely on $\mathbb{H} \times \mathbb{H}^-$, and uniformly as $\tau \to i\infty$ resp. $w \to -i\infty$.*

(2) *The funtions $\lambda_{k+1,D}$ are bimodular of weight $(2k+2, 0)$, that is*

$$\lambda_{k+1,D}\left(\tau + 1, w + 1\right) = \lambda_{k+1,D}\left(\tau, w\right), \quad \lambda_{k+1,D}\left(-\frac{1}{\tau}, -\frac{1}{w}\right) = \tau^{2k+2}\lambda_{k+1,D}\left(\tau, w\right).$$

(3) *We have*

$$\lim_{w\to -i\infty}\lambda_{k+1,D}(\tau,w) = -2\pi i\varphi_{k+1,D}(\tau), \qquad \lim_{\tau\to i\infty}\lambda_{k+1,D}(\tau,w) = 0.$$

(4) *We have*

$$\lambda_{k+1,D}(\tau,w) = \sum_{Q\in\mathcal{Q}_D}\frac{\operatorname{Log}\left(\frac{1+i\frac{Q_w}{\sqrt{D}}}{1-i\frac{Q_w}{\sqrt{D}}}\right)}{Q(\tau,1)^{k+1}} = \sum_{Q\in\mathcal{Q}_D}\frac{\operatorname{Log}\left(-\frac{\frac{Q_w}{\sqrt{D}}-i}{\frac{Q_w}{\sqrt{D}}+i}\right)}{Q(\tau,1)^{k+1}}.$$

Proof.

(1) By the definition of $\lambda_{k+1,D}$ in (VII.5), we have

$$|\lambda_{k+1,D}(\tau)| \le 2\sum_{Q\in\mathcal{Q}_D}\frac{\left|\arctan\left(\frac{Q_w}{\sqrt{D}}\right)\right|}{|Q(\tau,1)|^{k+1}} \le \pi\sum_{Q\in\mathcal{Q}_D}\frac{1}{|Q(\tau,1)|^{k+1}}.$$

The claim follows by the absolute convergence of the $f_{\kappa,D}$'s on $\mathbb{H}$.

(2) Bimodularity is a direct consequence of Lemma VII.2.2 and equation (VII.10).

(3) The assumption that D is not a square guarantees that the sum defining $\lambda_{k+1,D}$ runs over quadratic forms $Q = [a, b, c]$ with $ac \neq 0$. To prove the first assertion, we observe that

$$\frac{Q_w}{\sqrt{D}} \asymp a \operatorname{Im}(w)$$

as $\operatorname{Im}(w) \to -\infty$, and hence

$$\lim_{w \to -i\infty} \arctan\left(\frac{Q_w}{\sqrt{D}}\right) = -\frac{\pi}{2} \operatorname{sgn}(Q).$$

The first claim follows by the definition of $\varphi_{k+1,D}$ in (VII.4). As $a \neq 0$, we have $\frac{1}{|Q(\tau,1)|^{k+1}} \to 0$ for $\tau \to i\infty$. The second claim follows by (1).

(4) The claim follows by rewriting the arctangent in (VII.5).

$\square$

VII.3.3 Proof of Theorem VII.1.1

We conclude this section with the proof of Theorem VII.1.1.

Proof of Theorem VII.1.1.

(1) This follows by combining Propositions VII.3.1 (3), VII.3.2 (3), and VII.3.4 (2) with Lemma VII.3.3.

(2) This follows by combining (VII.6) with Propositions VII.3.2 (2) and VII.3.4 (2).

(3) Proposition VII.3.1 (1) along with

$$\lim_{\tau \to i\infty} \operatorname{Log}\left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+}\right) = 0 \tag{VII.18}$$

implies that $\psi_{k+1,D}$ is cuspidal. By Propositions VII.3.2 (2), VII.3.4 (3), and Lemma VII.3.3, every function defining $\Omega_{k+1,D}$ in (VII.6) is cuspidal (with respect to τ).

(4) As each function defining $\Omega_{k+1,D}$ in (VII.6) is holomorphic as a function of τ, we obtain the assertion with respect to τ directly. To verify that $\Omega_{k+1,D}$ is anti-holomorphic as a function of w, we compute by Lemmas VII.2.1 and VII.2.4 (1) that

$$2i \frac{\partial}{\partial w} \arctan\left(\frac{Q_w}{\sqrt{D}}\right) = -\frac{\sqrt{D}}{Q(w,1)}.$$

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By (VII.14), we deduce that

$$2i\frac{\partial}{\partial w}\arctan\left(\frac{Q_w}{\sqrt{D}}\right) = \frac{\partial}{\partial w}\mathrm{Log}\left(\frac{w-\alpha_Q^-}{w-\alpha_Q^+}\right).$$

By (VII.5) and (VII.6), we conclude that

$$\frac{\partial}{\partial w}\Omega_{k+1,D}(\tau,w) = 0.$$

(5) We first inspect the functions $\psi_{k+1,D} - \rho_{k+1,D}$. By definitions (VII.1) and (VII.5) we have

$$\psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau,\overline{\tau}) = \sum_{Q\in\mathcal{Q}_D}\frac{\mathrm{Log}\left(\frac{\tau-\alpha_Q^-}{\tau-\alpha_Q^+}\right) - \mathrm{Log}\left(\frac{\overline{\tau}-\alpha_Q^-}{\overline{\tau}-\alpha_Q^+}\right)}{Q(\tau,1)^{k+1}}.$$

We note that

$$\mathrm{Log}\left(\frac{\tau-\alpha_Q^-}{\tau-\alpha_Q^+}\right) - \mathrm{Log}\left(\frac{\overline{\tau}-\alpha_Q^-}{\overline{\tau}-\alpha_Q^+}\right) \equiv \mathrm{Log}\left(\frac{\left(\tau-\alpha_Q^-\right)\left(\overline{\tau}-\alpha_Q^+\right)}{\left(\tau-\alpha_Q^+\right)\left(\overline{\tau}-\alpha_Q^-\right)}\right) \pmod{2\pi i},$$

and we determine the multiple of $2\pi i$ now. From (VII.17), we deduce that

$$\frac{\left(\tau-\alpha_Q^-\right)\left(\overline{\tau}-\alpha_Q^+\right)}{\left(\tau-\alpha_Q^+\right)\left(\overline{\tau}-\alpha_Q^-\right)} = \frac{\frac{Q_\tau}{\sqrt{D}}-i}{\frac{Q_\tau}{\sqrt{D}}+i}.$$

We use (VII.15) and hence need to compute

$$\mathrm{Log}\left(\frac{\tau-\alpha_Q^-}{\tau-\alpha_Q^+}\right) - \mathrm{Log}\left(\frac{\overline{\tau}-\alpha_Q^-}{\overline{\tau}-\alpha_Q^+}\right) - \mathrm{Log}\left(\frac{\left(\tau-\alpha_Q^-\right)\left(\overline{\tau}-\alpha_Q^+\right)}{\left(\tau-\alpha_Q^+\right)\left(\overline{\tau}-\alpha_Q^-\right)}\right)$$
$$= i\left(\mathrm{Arg}\left(\frac{\tau-\alpha_Q^-}{\tau-\alpha_Q^+}\right) - \mathrm{Arg}\left(\frac{\overline{\tau}-\alpha_Q^-}{\overline{\tau}-\alpha_Q^+}\right) - \mathrm{Arg}\left(\frac{\left(\tau-\alpha_Q^-\right)\left(\overline{\tau}-\alpha_Q^+\right)}{\left(\tau-\alpha_Q^+\right)\left(\overline{\tau}-\alpha_Q^-\right)}\right)\right). \tag{VII.19}$$

Note that for $z\in\mathbb{C}\setminus\mathbb{R}$

$$\mathrm{Arg}(z) - \mathrm{Arg}\left(\overline{z}\right) - \mathrm{Arg}\left(\frac{z}{\overline{z}}\right) = \pi\left(1-\mathrm{sgn}\left(\mathrm{Re}(z)\right)\right)\mathrm{sgn}\left(\mathrm{Im}(z)\right).$$

We use this for $z = \frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+}$. By (VII.17), (VII.19) thus becomes $\pi i(\text{sgn}(Q_\tau) - \text{sgn}(Q))$. We infer that

$$\psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau, \overline{\tau}) = \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log}\left(\frac{\frac{Q_\tau}{\sqrt{D}} - i}{\frac{Q_\tau}{\sqrt{D}} + i}\right)}{Q(\tau, 1)^{k+1}} + \pi i \sum_{\mathcal{Q} \in \mathcal{Q}_D} \frac{\text{sgn}\left(Q_\tau\right) - \text{sgn}(Q)}{Q(\tau, 1)^{k+1}}.$$

Combining with (VII.4) gives

$$\psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau, \overline{\tau}) + \varphi_{k+1,D}(\tau, \overline{\tau})$$

$$= \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log}\left(\frac{\frac{Q_\tau}{\sqrt{D}} - i}{\frac{Q_\tau}{\sqrt{D}} + i}\right)}{Q(\tau, 1)^{k+1}} + \pi i \sum_{\mathcal{Q} \in \mathcal{Q}_D} \frac{\text{sgn}\left(Q_\tau\right)}{Q(\tau, 1)^{k+1}}, \quad \text{(VII.20)}$$

which is modular of weight $2k + 2$ by (VII.10) and Lemma VII.2.2. To finish the proof, we inspect $\lambda_{k+1,D}(\tau, \overline{\tau})$. Combining $Q_{\overline{\tau}} = -Q_\tau$ with Proposition VII.3.4 (4) yields

$$\lambda_{k+1,D}(\tau, \overline{\tau}) = - \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log}\left(-\frac{\frac{Q_\tau}{\sqrt{D}} - i}{\frac{Q_\tau}{\sqrt{D}} + i}\right)}{Q(\tau, 1)^{k+1}}.$$

By (VII.16), we obtain

$$\text{Log}\left(\frac{\frac{Q_\tau}{\sqrt{D}} - i}{\frac{Q_\tau}{\sqrt{D}} + i}\right) - \text{Log}\left(-\frac{\frac{Q_\tau}{\sqrt{D}} - i}{\frac{Q_\tau}{\sqrt{D}} + i}\right) = -\pi i \, \text{sgn}(Q_\tau),$$

from which we conclude the claim using (VII.6).

$\square$

VII.4   The function $g_{k+1,D}$

VII.4.1   Local cusp forms

Recall the definition of $g_{k+1,D}$ in (VII.7) and Definition VI.2.7.

Remark. Let $d(z, w)$ denote the hyperbolic distance between $z, w \in \mathbb{C}$ with $y \, \text{Im}(w) > 0$. Since $D > 0$, we have (with $\tau_{[a,b,c]} := -\frac{b}{2a} + \frac{i}{2|a|}\sqrt{D}$) $\frac{Q_\tau}{\sqrt{D}} = \cosh(d(\tau, \tau_Q))$. This yields an alternative representation of $g_{k+1,D}$ as well as of $\lambda_{k+1,D}$.

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We next prove our claim for $g_{k+1,D}$.

Proposition VII.4.1. *The functions $g_{k+1,D}$ are local cusp forms.*

Proof. We observe that the $g_{k+1,D}$'s converge absolutely on $\mathbb{H}$ utilizing absolute convergence of the $f_{\kappa,D}$'s. We directly deduce that the $g_{k+1,D}$'s are holomorphic. Using Lemma VII.2.2 and (VII.10) shows that the $g_{k+1,D}$'s are modular of weight $2k + 2$. If $v > \frac{\sqrt{D}}{2}$, then $\mathrm{sgn}(Q_\tau) = 1$. Thus, cuspidality of the $g_{k+1,D}$'s follows by cuspidality of the $f_{\kappa,D}$'s for $\kappa > 1$. The local behaviour and the jumping singularities are dictated by $\mathrm{sgn}(Q_\tau)$. $\qquad\square$

VII.4.2 The local behaviour of $g_{k+1,D}$

We next provide the behaviour of $g_{k+1,D}$ on E_D.

Proposition VII.4.2. *If $\tau \in E_D$, then we have that*

$$\lim_{\varepsilon \to 0^+} (g_{k+1,D}(\tau + i\varepsilon) - g_{k+1,D}(\tau - i\varepsilon)) = 2 \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_\tau = 0}} \frac{\mathrm{sgn}(Q)}{Q(\tau,1)^{k+1}}.$$

Remark. The sum on the right-hand side is finite by [BKK15, Lemma 5.1 (1)].

Proof of Proposition VII.4.2. We adapt the proof of [BKK15, Proposition 5.2]. We write

$$g_{k+1,D}(\tau \pm i\varepsilon) = \left(\sum_{\substack{Q \in \mathcal{Q}_D \\ Q_\tau = 0}} + \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_\tau \neq 0}} \right) \frac{\mathrm{sgn}(Q_{\tau \pm i\varepsilon})}{Q(\tau \pm i\varepsilon, 1)^{k+1}}.$$

Absolute convergence of Zagier's $f_{k,D}$ function implies that $g_{k+1,D}$ converges absolutely on $\mathbb{H}$ and uniformly towards $i\infty$, which permits us to interchange the sums with the limit, and argue termwise in the following. If $Q_\tau \neq 0$, then $\tau \pm i\varepsilon$ are in the same connected component of $\mathbb{H} \setminus E_D$ for $\varepsilon > 0$ sufficiently small. Combining with [BKK15, equation (5.4)], we deduce that for $\varepsilon > 0$ sufficiently small

$$\mathrm{sgn}([a,b,c]_{\tau+i\varepsilon}) = \mathrm{sgn}([a,b,c]_{\tau-i\varepsilon}) = \delta \, \mathrm{sgn}(a),$$

where

$$\delta := \mathrm{sgn}\left(\left| \tau + i\varepsilon + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2\,|a|} \right) = \mathrm{sgn}\left(\left| \tau - i\varepsilon + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2\,|a|} \right) = \pm 1.$$

Thus

$$\lim_{\varepsilon \to 0^+} \left(\frac{\mathrm{sgn}(Q_{\tau+i\varepsilon})}{Q(\tau+i\varepsilon,1)^{k+1}} - \frac{\mathrm{sgn}(Q_{\tau-i\varepsilon})}{Q(\tau-i\varepsilon,1)^{k+1}} \right)$$
$$= \lim_{\varepsilon \to 0^+} \delta \left(\frac{\mathrm{sgn}(Q)}{Q(\tau+i\varepsilon,1)^{k+1}} - \frac{\mathrm{sgn}(Q)}{Q(\tau-i\varepsilon,1)^{k+1}} \right) = 0.$$

If $Q_\tau = 0$, then $\tau \pm i\varepsilon$ are in different connected components of $\mathbb{H} \setminus E_D$ for all $\varepsilon > 0$. This is justified by [BKK15, equation (5.6)], namely

$$\left| \tau - i\varepsilon + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|} < \left| \tau + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|} = 0 < \left| \tau + i\varepsilon + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|}$$

for every $\varepsilon > 0$. Combining with [BKK15, equation (5.4)] implies that $\mathrm{sgn}(Q_{\tau\pm i\varepsilon}) = \pm \mathrm{sgn}(Q)$, and consequently

$$\lim_{\varepsilon \to 0^+} \left(\frac{\mathrm{sgn}(Q_{\tau+i\varepsilon})}{Q(\tau+i\varepsilon,1)^{k+1}} - \frac{\mathrm{sgn}(Q_{\tau-i\varepsilon})}{Q(\tau-i\varepsilon,1)^{k+1}} \right) = 2 \frac{\mathrm{sgn}(Q)}{Q(\tau,1)^{k+1}}.$$

$\square$

We next inspect the sum appearing in Proposition VII.4.2.

Lemma VII.4.3. *The sum*

$$\sum_{\substack{Q \in \mathcal{Q}_D \\ Q_\tau = 0}} \frac{\mathrm{sgn}(Q)}{Q(\tau,1)^{k+1}}$$

does not vanish identically on E_D.

Proof. Let $\tau \in E_D$. Then we have $\tau \in S_{\mathfrak{Q}}$ for some $\mathfrak{Q} \in \mathcal{Q}_D$. On the one hand, the sum in the lemma has a pole of order $k+1 > 0$ at $\alpha_{\mathfrak{Q}}^\pm$, and hence both limits

$$\lim_{\substack{\tau \to \alpha_{\mathfrak{Q}}^\pm \\ \tau \in S_{\mathfrak{Q}}}} \left| \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_\tau = 0}} \frac{\mathrm{sgn}(Q)}{Q(\tau,1)^{k+1}} \right|$$

tend towards ∞. On the other hand, the sum is continuous on $S_{\mathfrak{Q}}$, and the contribution from the terms corresponding to $Q \neq \mathfrak{Q}$ is finite at $\alpha_{\mathfrak{Q}}^\pm$. $\square$

VII.4.3 The local behaviour of $\mathcal{E}_{g_{k+1,D}}$ and $g_{k+1,D}^*$

We next prove that the Eichler integrals of $g_{k+1,D}$ exist on E_D.

Proposition VII.4.4. *Let $\tau \in E_D$. Then we have*

$$\lim_{\varepsilon \to 0^+} \left(\mathcal{E}_{g_{k+1,D}}(\tau + i\varepsilon) - \mathcal{E}_{g_{k+1,D}}(\tau - i\varepsilon) \right)$$

$$= -\frac{2(2\pi i)^{2k+1}}{(2k)!} \int_0^{i\infty} \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_{\tau+w}=0}} \frac{\mathrm{sgn}(Q)}{Q(\tau+w,1)^{k+1}} w^{2k} \mathrm{d}w,$$

and

$$\lim_{\varepsilon \to 0^+} \left(g_{k+1,D}^*(\tau + i\varepsilon) - g_{k+1,D}^*(\tau - i\varepsilon) \right)$$

$$= -\frac{2}{(2i)^{2k+1}} \int_{2iv}^{i\infty} \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_{\tau-w}=0}} \frac{\mathrm{sgn}(Q)}{Q(\tau-w,1)^{k+1}} w^{2k} \mathrm{d}w.$$

Remark. As remarked after Proposition VII.4.2, the sums inside the integrals on the right-hand sides of Propostion VII.4.4 are finite. They can be written as integrals over a bounded domain, because the integrands vanish as soon as[9] $\mathrm{Im}(\tau+w) > \frac{\sqrt{D}}{2}$ or $\tau - w$ moves out of $\mathbb{H}$. If $\tau \pm w \notin E_D$, then the sums are in fact empty. Hence, the integrals on the right-hand side of Propostion VII.4.4 exist.

Proof of Proposition VII.4.4. As $\tau \pm i\varepsilon \notin E_D$ for every $\varepsilon > 0$, we utilize (VII.7). Changing variables gives

$$\lim_{\varepsilon \to 0^+} \left(\mathcal{E}_{g_{k+1,D}}(\tau + i\varepsilon) - \mathcal{E}_{g_{k+1,D}}(\tau - i\varepsilon) \right)$$

$$= -\frac{(2\pi i)^{2k+1}}{(2k)!} \lim_{\varepsilon \to 0^+} \int_0^{i\infty} \sum_{Q \in \mathcal{Q}_D} \left(\frac{\mathrm{sgn}(Q_{\tau+i\varepsilon+w})}{Q(\tau+i\varepsilon+w,1)^{k+1}} - \frac{\mathrm{sgn}(Q_{\tau-i\varepsilon+w})}{Q(\tau-i\varepsilon+w,1)^{k+1}} \right) w^{2k} \mathrm{d}w,$$

and

$$\lim_{\varepsilon \to 0^+} \left(g_{k+1,D}^*(\tau + i\varepsilon) - g_{k+1,D}^*(\tau - i\varepsilon) \right)$$

$$= \frac{1}{(2i)^{2k+1}} \lim_{\varepsilon \to 0^+} \left(-\int_{2i(v+\varepsilon)}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \frac{\mathrm{sgn}(Q_{\tau-w+i\varepsilon})}{Q(\tau-w+i\varepsilon,1)^{k+1}} w^{2k} \mathrm{d}w \right.$$

$$\left. + \int_{2i(v-\varepsilon)}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \frac{\mathrm{sgn}(Q_{\tau-w-i\varepsilon})}{Q(\tau-w-i\varepsilon,1)^{k+1}} w^{2k} \mathrm{d}w \right),$$

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[9]If $\mathrm{Im}(\tau+w) > \frac{\sqrt{D}}{2}$, then $\tau + w$ lies in the unbounded component of $\mathbb{H} \setminus E_D$.

where we use that $Q_{\bar{z}} = -Q_z$ in the case of $g_{k+1,D}^*$.

We consider the holomorphic Eichler integral first, and justify interchanging the limits $\varepsilon \to 0^+$ with the holomorphic Eichler integral. By (VII.7), $g_{k+1,D}$ vanishes at $i\infty$, and converges uniformly towards $i\infty$ as the sign-function is bounded (using that $f_{\kappa,D}$ converges uniformly towards $i\infty$ for $\kappa > 1$). By virtue of modularity of $g_{k+1,D}$, both assertions hold towards 0 as well. In other words, the integral converges uniformly, and this permits the exchange of the limit $\varepsilon \to 0^+$ with the integral. We infer that

$$\lim_{\varepsilon \to 0^+} \left(\mathcal{E}_{g_{k+1,D}}(\tau + i\varepsilon) - \mathcal{E}_{g_{k+1,D}}(\tau - i\varepsilon) \right)$$
$$= -\frac{(2\pi i)^{2k+1}}{(2k)!} \int_0^{i\infty} \lim_{\varepsilon \to 0^+} \left(g_{k+1,D}(\tau + w + i\varepsilon) - g_{k+1,D}(\tau + w - i\varepsilon) \right) w^{2k} \mathrm{d}w.$$

If $\tau + w \notin E_D$, then the limit inside the integral vanishes, because $\tau + w + i\varepsilon$ and $\tau + w - i\varepsilon$ are in the same connected component for ε sufficiently small. If $\tau + w \in E_D$, then we apply Proposition VII.4.2 to obtain

$$\lim_{\varepsilon \to 0^+} \left(\mathcal{E}_{g_{k+1,D}}(\tau + i\varepsilon) - \mathcal{E}_{g_{k+1,D}}(\tau - i\varepsilon) \right)$$
$$= -\frac{2(2\pi i)^{2k+1}}{(2k)!} \int_0^{i\infty} \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_{\tau+w} = 0}} \frac{\mathrm{sgn}(Q)}{Q(\tau + w, 1)^{k+1}} w^{2k} \mathrm{d}w.$$

Now, we treat the nonholomorphic Eichler integrals, and first split one of them as

$$\int_{2i(v-\varepsilon)}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \frac{\mathrm{sgn}\left(Q_{\tau-w-i\varepsilon} \right)}{Q(\tau - w - i\varepsilon, 1)^{k+1}} w^{2k} \mathrm{d}w$$
$$= \left(\int_{2i(v-\varepsilon)}^{2i(v+\varepsilon)} + \int_{2i(v+\varepsilon)}^{i\infty} \right) \sum_{Q \in \mathcal{Q}_D} \frac{\mathrm{sgn}\left(Q_{\tau-w-i\varepsilon} \right)}{Q(\tau - w - i\varepsilon, 1)^{k+1}} w^{2k} \mathrm{d}w.$$

We note that

$$\lim_{\varepsilon \to 0^+} \int_{2i(v-\varepsilon)}^{2i(v+\varepsilon)} \frac{\mathrm{sgn}\left(Q_{\tau-w-i\varepsilon} \right)}{Q(\tau - w - i\varepsilon, 1)^{k+1}} w^{2k} \mathrm{d}w = 0,$$

because the integrand is bounded in the domain of integration, which has measure 0 as $\varepsilon \to 0^+$. Hence, it remains to consider the integral from $2i(v + \varepsilon)$ to $i\infty$. If $\tau - w \notin E_D$, then we have

$$\lim_{\varepsilon \to 0^+} \int_{2i(v+\varepsilon)}^{i\infty} \left(-\frac{\mathrm{sgn}\left(Q_{\tau-w+i\varepsilon} \right)}{Q(\tau - w + i\varepsilon, 1)^{k+1}} + \frac{\mathrm{sgn}\left(Q_{\tau-w-i\varepsilon} \right)}{Q(\tau - w - i\varepsilon, 1)^{k+1}} \right) w^{2k} \mathrm{d}w = 0,$$

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as in the previous case, because $\tau - w \pm i\varepsilon$ are in the same connected component for ε sufficiently small. If $\tau - w \in E_D$, then we obtain

$$\lim_{\varepsilon \to 0^+} \int_{2i(v+\varepsilon)}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \left(-\frac{\operatorname{sgn}(Q_{\tau-w+i\varepsilon})}{Q(\tau - w + i\varepsilon, 1)^{k+1}} + \frac{\operatorname{sgn}(Q_{\tau-w-i\varepsilon})}{Q(\tau - w - i\varepsilon, 1)^{k+1}} \right) w^{2k} \mathrm{d}w$$

$$= -2 \int_{2iv}^{i\infty} \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_{\tau-w}=0}} \frac{\operatorname{sgn}(Q)}{Q(\tau - w, 1)^{k+1}} w^{2k} \mathrm{d}w$$

by Proposition VII.4.2 exactly as in the previous case. $\qquad \square$

VII.5 The function $\mathcal{G}_{-k,D}$ and the proof of Thm. VII.1.2

VII.5.1 Convergence of $\mathcal{G}_{-k,D}$

We first establish convergence of $\mathcal{G}_{-k,D}$.

Proposition VII.5.1. *The sum defining $\mathcal{G}_{-k,D}$ converges compactly for every $\tau \in \mathbb{H} \backslash E_D$, and does not converge on E_D.*

Proof. If $\tau \in \mathbb{H} \setminus E_D$, then $\operatorname{sgn}(Q_\tau) = \pm 1$ and thus the claim follows directly by [BKK15, Proposition 4.1] after summing over all narrow equivalence classes there. (The class number of positive discriminants is finite.) If $\tau \in E_D$, then the incomplete β-function reduces to a constant depending only on k according to Lemma VII.2.1. Hence, the sum defining $\mathcal{G}_{-k,D}$ does not converge on E_D as the sum is infinite and $\beta(1; k + \frac{1}{2}, \frac{1}{2}) \neq 0$. $\quad \square$

VII.5.2 Behaviour of $\mathcal{G}_{-k,D}$ under differentiation

We inspect the behaviour of $\mathcal{G}_{-k,D}$ under differential operators.

Proposition VII.5.2. *Let $\tau \in \mathbb{H} \setminus E_D$.*

(1) *We have*

$$\xi_{-2k}(\mathcal{G}_{-k,D}) = D^{k+\frac{1}{2}} g_{k+1,D}.$$

(2) *We have*

$$\mathbb{D}^{2k+1}(\mathcal{G}_{-k,D}) = -\frac{D^{k+\frac{1}{2}}(2k)!}{(4\pi)^{2k+1}} g_{k+1,D}.$$

(3) *We have*

$$\Delta_{-2k}(\mathcal{G}_{-k,D}) = 0.$$

Define

$$g_n^{[1]}(\tau) := Q(\tau,1)^n \beta\left(\frac{Dv^2}{|Q(\tau,1)|^2}; n + \frac{1}{2}, \frac{1}{2}\right), \qquad n \in \mathbb{N}_0.$$

The proof of Proposition VII.5.2 is based on the following three technical lemmas.

Lemma VII.5.3. *We have for $n \in \mathbb{N}_0$*

$$g_{n+1}^{[1]}(\tau) = \frac{n + \frac{1}{2}}{n+1} Q(\tau,1) g_n^{[1]}(\tau) - \frac{D^{n+\frac{1}{2}}}{n+1} \frac{v^{2n+2} |Q_\tau|}{Q(\overline{\tau},1)^{n+1}}.$$

Proof. By [BCLO10, item 8.17.20], we have that

$$\frac{\beta(x; a, b)}{\beta(1; a, b)} = \frac{\beta(x; a+1, b)}{\beta(1; a+1, b)} + \frac{x^a(1-x)^b}{a\beta(1; a, b)}.$$

This gives that

$$\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2}; n + \frac{3}{2}, \frac{1}{2}\right)$$
$$= \frac{\beta\left(1; n + \frac{3}{2}, \frac{1}{2}\right)}{\beta\left(1; n + \frac{1}{2}, \frac{1}{2}\right)} \left(\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2}; n + \frac{1}{2}, \frac{1}{2}\right) - \frac{\left(\frac{Dv^2}{|Q(\tau,1)|^2}\right)^{n+\frac{1}{2}} \left(1 - \frac{Dv^2}{|Q(\tau,1)|^2}\right)^{\frac{1}{2}}}{n + \frac{1}{2}}\right).$$

Using Lemma VII.2.1, we compute

$$\left(\frac{Dv^2}{|Q(\tau,1)|^2}\right)^{n+\frac{1}{2}} \left(1 - \frac{Dv^2}{|Q(\tau,1)|^2}\right)^{\frac{1}{2}} = \frac{D^{n+\frac{1}{2}} v^{2n+2} |Q_\tau|}{|Q(\tau,1)|^{2n+2}},$$

and since $\frac{\beta(1; n + \frac{3}{2}, \frac{1}{2})}{\beta(1; n + \frac{1}{2}, \frac{1}{2})} = \frac{n + \frac{1}{2}}{n+1}$, we obtain the claim. $\qquad\square$

Lemma VII.5.3 motivates to define the auxiliary function

$$g_n^{[2]}(\tau) := \frac{D^{n-\frac{1}{2}} v^{2n} |Q_\tau|}{Q(\overline{\tau},1)^n}.$$

The second technical lemma treats the image of $g_{n+1}^{[2]}$ under differentiation.

Lemma VII.5.4. *We have for* $n \in \mathbb{N}$

$$\frac{\partial^{2n+1}}{\partial \tau^{2n+1}} g_n^{[2]}(\tau) = 0.$$

Proof. We prove the claim by induction. If $n = 1$, then the claim follows by applying Lemma VII.2.4 (1) three times.

For the induction step, Lemma VII.2.4 (1) yields that

$$\frac{\partial}{\partial \tau} \left(v^{\ell+2} Q_\tau \right) = -\frac{i}{2} \ell v^{\ell+1} Q_\tau + \frac{i}{2} v^\ell Q \left(\overline{\tau}, 1 \right)$$

for every $\ell \in \mathbb{N}_0$. Noting that $\frac{\partial^{\ell+2}}{\partial \tau^{\ell+2}} (v^\ell Q(\overline{\tau}, 1)) = 0$, we obtain

$$\frac{\partial^{\ell+2}}{\partial \tau^{\ell+2}} \left(v^{\ell+1} Q_\tau \right) = -\frac{i}{2} (\ell + 1) \frac{\partial^{\ell+1}}{\partial \tau^{\ell+1}} \left(v^\ell Q_\tau \right).$$

Consequently, we find that

$$\frac{\partial^{2n+3}}{\partial \tau^{2n+3}} g_{n+1}^{[2]}(\tau) = -\frac{D(2n+2)(2n+1)}{4Q\left(\overline{\tau}, 1\right)} \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} g_n^{[2]}(\tau).$$

The right-hand side vanishes by the induction hypothesis, as desired. $\square$

The third lemma contains the main technical claim.

Lemma VII.5.5. *We have for* $n \in \mathbb{N}_0$

$$\frac{\partial^{2n+1}}{\partial \tau^{2n+1}} g_n^{[1]}(\tau) = \frac{i(-1)^{n+1} D^{n+\frac{1}{2}} (2n)! \operatorname{sgn}(Q_\tau)}{2^{2n} Q(\tau, 1)^{n+1}}.$$

Proof. We prove the lemma by induction.
Step 1: The case $n = 0$
We apply the Fundamental Theorem of Calculus, Lemma VII.2.1, and Lemma VII.2.4, yielding

$$\frac{\partial}{\partial \tau} \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) = -\frac{iD^{n+\frac{1}{2}} v^{2n} \operatorname{sgn}(Q_\tau)}{|Q(\tau, 1)|^{2n} Q(\tau, 1)} \quad \text{(VII.21)}$$

for every $n \in \mathbb{N}_0$. In particular, this proves the desired identity for $n = 0$.
Step 2: The case $n = 1$
Using (VII.21) and the first identity of Lemma VII.2.5, we compute that

$$R_{-2n} \left(g_n^{[1]}(\tau) \right) = -2n Q_\tau \frac{g_n^{[1]}(\tau)}{Q(\tau, 1)} + \frac{2D^{n+\frac{1}{2}} v^{2n} \operatorname{sgn}(Q_\tau)}{Q\left(\overline{\tau}, 1\right)^n Q(\tau, 1)}.$$

Lemma VII.5.3 with $n \mapsto n - 1$ gives

$$\frac{g_n^{[1]}(\tau)}{Q(\tau,1)} = \frac{n - \frac{1}{2}}{n} g_{n-1}^{[1]}(\tau) - \frac{D^{n-\frac{1}{2}}}{n} \frac{v^{2n} \operatorname{sgn}(Q_\tau) Q_\tau}{Q(\overline{\tau},1)^n Q(\tau,1)}.$$

Plugging into the previous equation and applying Lemma VII.2.1 yields

$$R_{-2n}\left(g_n^{[1]}(\tau)\right) = -(2n-1)Q_\tau g_{n-1}^{[1]}(\tau) + \frac{2D^{n-\frac{1}{2}} v^{2n-2} \operatorname{sgn}(Q_\tau)}{Q(\overline{\tau},1)^{n-1}}.$$

We compute

$$R_{2-2n}\left(\frac{2D^{n-\frac{1}{2}} v^{2n-2} \operatorname{sgn}(Q_\tau)}{Q(\overline{\tau},1)^{n-1}}\right) = 0,$$

$$R_{2-2n}\left(Q_\tau g_{n-1}^{[1]}(\tau)\right) = Q_\tau R_{2-2n}\left(g_{n-1}^{[1]}(\tau)\right) - g_{n-1}^{[1]}(\tau)\frac{Q(\overline{\tau},1)}{v^2}$$

by Lemma VII.2.4 (1). We infer that

$$R_{2-2n} \circ R_{-2n}\left(g_n^{[1]}(\tau)\right) = -(2n-1)\left(Q_\tau R_{2-2n}\left(g_{n-1}^{[1]}(\tau)\right) - g_{n-1}^{[1]}(\tau)\frac{Q(\overline{\tau},1)}{v^2}\right).$$

Now, we suppose that $n = 1$. Then the previous equation gives

$$R_0 \circ R_{-2}\left(g_1^{[1]}(\tau)\right) = -Q_\tau R_0\left(g_0^{[1]}(\tau)\right) + g_0^{[1]}(\tau)\frac{Q(\overline{\tau},1)}{v^2}.$$

We then compute, using (VII.21)

$$R_0\left(g_0^{[1]}(\tau)\right) = 2i\frac{\partial}{\partial\tau}\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2}; \frac{1}{2}, \frac{1}{2}\right) = \frac{2D^{\frac{1}{2}} \operatorname{sgn}(Q_\tau)}{Q(\tau,1)}.$$

Combining this with the previous equation we obtain

$$R_2 \circ R_0 \circ R_{-2}\left(g_1^{[1]}(\tau)\right) = R_2\left(-2Q_\tau \frac{D^{\frac{1}{2}} \operatorname{sgn}(Q_\tau)}{Q(\tau,1)} + \beta\left(\frac{Dv^2}{|Q(\tau,1)|^2}; \frac{1}{2}, \frac{1}{2}\right)\frac{Q(\overline{\tau},1)}{v^2}\right).$$

By Lemma VII.2.4 (1) and (VII.21), we calculate that

$$\frac{\partial}{\partial\tau}\left(-2Q_\tau \frac{D^{\frac{1}{2}} \operatorname{sgn}(Q_\tau)}{Q(\tau,1)} + \beta\left(\frac{Dv^2}{|Q(\tau,1)|^2}; \frac{1}{2}, \frac{1}{2}\right)\frac{Q(\overline{\tau},1)}{v^2}\right)$$

$$= -\frac{iQ(\overline{\tau},1)}{v^2}\frac{D^{\frac{1}{2}} \operatorname{sgn}(Q_\tau)}{Q(\tau,1)} + 2Q_\tau \frac{D^{\frac{1}{2}} \operatorname{sgn}(Q_\tau)}{Q(\tau,1)^2}Q'(\tau,1) - \frac{iD^{\frac{1}{2}} \operatorname{sgn}(Q_\tau)}{Q(\tau,1)}\frac{Q(\overline{\tau},1)}{v^2}$$

$$+ \frac{i}{v^3}\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2}; \frac{1}{2}, \frac{1}{2}\right)Q(\overline{\tau},1).$$

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Hence, by Lemma VII.2.1 and the first identity of Lemma VII.2.5

$$R_2 \circ R_0 \circ R_{-2}\left(g_1^{[1]}(\tau)\right) = \frac{4D^{\frac{3}{2}}\operatorname{sgn}(Q_\tau)}{Q(\tau,1)^2}.$$

We can directly conclude the claim using Bol's identity (VII.12).

Step 3: Application of Lemmas VII.5.3 and VII.5.4 and reducing to $2n+2$ derivatives

Employing Lemma VII.5.3 and Lemma VII.5.4 with $n \mapsto n+1$ yields

$$\frac{\partial^{2n+3}}{\partial\tau^{2n+3}}g_{n+1}^{[1]}(\tau) = \frac{n+\frac{1}{2}}{n+1}\frac{\partial^{2n+3}}{\partial\tau^{2n+3}}\left(Q(\tau,1)g_n^{[1]}(\tau)\right). \tag{VII.22}$$

By equation (VII.21), we compute that

$$\frac{\partial}{\partial\tau}\left(Q(\tau,1)g_n^{[1]}(\tau)\right)$$
$$= (n+1)Q(\tau,1)^nQ'(\tau,1)\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2};n+\frac{1}{2},\frac{1}{2}\right) - \frac{iD^{n+\frac{1}{2}}v^{2n}\operatorname{sgn}(Q_\tau)}{Q(\bar\tau,1)^n}.$$

We observe that the final term gets annihilated by differentiating $2n+1$ times and thus

$$\frac{n+\frac{1}{2}}{n+1}\frac{\partial^{2n+3}}{\partial\tau^{2n+3}}\left(Q(\tau,1)g_n^{[1]}(\tau)\right)$$
$$= \left(n+\frac{1}{2}\right)\frac{\partial^{2n+2}}{\partial\tau^{2n+2}}\left(Q(\tau,1)^nQ'(\tau,1)\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2};n+\frac{1}{2},\frac{1}{2}\right)\right).$$

Step 4: Reducing to $2n+1$ derivatives

By equation (VII.21), we furthermore calculate that

$$\frac{\partial}{\partial\tau}\left(Q(\tau,1)^nQ'(\tau,1)\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2};n+\frac{1}{2},\frac{1}{2}\right)\right)$$
$$= nQ(\tau,1)^{n-1}Q'(\tau,1)^2\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2};n+\frac{1}{2},\frac{1}{2}\right)$$
$$+ Q(\tau,1)^nQ''(\tau,1)\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2};n+\frac{1}{2},\frac{1}{2}\right) - iQ(\tau,1)^nQ'(\tau,1)\frac{D^{n+\frac{1}{2}}v^{2n}\operatorname{sgn}(Q_\tau)}{|Q(\tau,1)|^{2n}Q(\tau,1)}.$$

By the first identity of Lemma VII.2.5, the final term may be rewritten as

$$-iQ(\tau,1)^nQ'(\tau,1)\frac{D^{n+\frac{1}{2}}v^{2n}\operatorname{sgn}(Q_\tau)}{|Q(\tau,1)|^{2n}Q(\tau,1)} = -\frac{D^{n+\frac{1}{2}}v^{2n-1}\operatorname{sgn}(Q_\tau)}{Q(\bar\tau,1)^n} + \frac{D^{n+\frac{1}{2}}v^{2n}|Q_\tau|}{Q(\bar\tau,1)^nQ(\tau,1)}.$$

Again the final term gets annihilated upon differentiating $2n+1$ times. Consequently, we obtain, by the second identity of Lemma VII.2.5,

$$\frac{n+\frac{1}{2}}{n+1}\frac{\partial^{2n+3}}{\partial\tau^{2n+3}}\left(Q(\tau,1)g_n^{[1]}(\tau)\right)$$
$$=\left(n+\frac{1}{2}\right)\frac{\partial^{2n+1}}{\partial\tau^{2n+1}}\left(DnQ(\tau,1)^{n-1}\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2};n+\frac{1}{2},\frac{1}{2}\right)\right.$$
$$\left.+(2n+1)Q(\tau,1)^nQ''(\tau,1)\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2};n+\frac{1}{2},\frac{1}{2}\right)+\frac{D^{n+\frac{1}{2}}v^{2n}|Q_\tau|}{Q\left(\overline{\tau},1\right)^nQ(\tau,1)}\right)$$
$$=\left(n+\frac{1}{2}\right)\frac{\partial^{2n+1}}{\partial\tau^{2n+1}}\left(Dn\frac{g_n^{[1]}(\tau)}{Q(\tau,1)}+(2n+1)Q''(\tau,1)g_n^{[1]}(\tau)+\frac{Dg_n^{[2]}(\tau)}{Q(\tau,1)}\right).$$

Step 5: Application of the induction hypothesis
We use Lemma VII.5.3 with $n\mapsto n-1$, to obtain

$$\frac{g_n^{[1]}(\tau)}{Q(\tau,1)}=\frac{n-\frac{1}{2}}{n}g_{n-1}^{[1]}(\tau)-\frac{g_n^{[2]}(\tau)}{nQ(\tau,1)},$$

and hence, using step 4,

$$\frac{n+\frac{1}{2}}{n+1}\frac{\partial^{2n+3}}{\partial\tau^{2n+3}}\left(Q(\tau,1)g_n^{[1]}(\tau)\right)$$
$$=\left(n+\frac{1}{2}\right)\frac{\partial^{2n+1}}{\partial\tau^{2n+1}}\left(D\left(n-\frac{1}{2}\right)g_{n-1}^{[1]}(\tau)+(2n+1)Q''(\tau,1)g_n^{[1]}(\tau)\right).$$

The induction hypothesis for n and $n-1$, and the fact that $Q''(\tau,1)$ is independent of τ, then gives

$$\frac{n+\frac{1}{2}}{n+1}\frac{\partial^{2n+3}}{\partial\tau^{2n+3}}\left(Q(\tau,1)g_n^{[1]}(\tau)\right)$$
$$=\frac{(-1)^niD^{n+\frac{1}{2}}\left(n+\frac{1}{2}\right)(2n)!\operatorname{sgn}(Q_\tau)}{4^n}\left(\frac{1}{n}\frac{\partial^2}{\partial\tau^2}\frac{1}{Q(\tau,1)^n}-(2n+1)\frac{Q''(\tau,1)}{Q(\tau,1)^{n+1}}\right).$$

Step 6: Simplifying the expressions
Using the second identity of Lemma VII.2.5, we compute

$$\frac{1}{n}\frac{\partial^2}{\partial\tau^2}\frac{1}{Q(\tau,1)^n}-(2n+1)\frac{Q''(\tau,1)}{Q(\tau,1)^{n+1}}=\frac{D(n+1)}{Q(\tau,1)^{n+2}}.$$

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Inserting this into the result from step 5 yields

$$\frac{n+\frac{1}{2}}{n+1}\frac{\partial^{2n+3}}{\partial\tau^{2n+3}}\left(Q(\tau,1)g_n^{[1]}(\tau)\right)=\frac{(-1)^n i\left(n+\frac{1}{2}\right)(n+1)(2n)!D^{n+\frac{3}{2}}\operatorname{sgn}(Q_\tau)}{4^n Q(\tau,1)^{n+2}}.$$

By equation (VII.22), we ultimately arrive at the claim of the lemma (with $n\mapsto n+1$). $\square$

We are now ready to prove Proposition VII.5.2.

Proof of Proposition VII.5.2.

(1) By Lemma VII.2.3 and equation (VII.21), we obtain

$$\frac{\partial}{\partial\overline{\tau}}\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2};k+\frac{1}{2},\frac{1}{2}\right)=\frac{iD^{k+\frac{1}{2}}v^{2k}\operatorname{sgn}(Q_\tau)}{|Q(\tau,1)|^{2k}\,Q(\overline{\tau},1)}.$$

This implies the claim.

(2) Lemma VII.5.5 implies that

$$\frac{1}{2}\mathbb{D}^{2k+1}\left(g_k^{[1]}(\tau)\right)=-\frac{D^{k+\frac{1}{2}}(2k)!}{(4\pi)^{2k+1}}\frac{\operatorname{sgn}(Q_\tau)}{Q(\tau,1)^{k+1}},$$

from which we deduce the claim by (VII.7).

(3) The claim follows directly from (VII.11) along with part (1) and (VII.7).

$\square$

VII.5.3 Further properties of $\mathcal{G}_{-k,D}$ and the proof of Theorem VII.1.2

We prove the local behaviour of $\mathcal{G}_{-k,D}$ first. Similar as in the proof of Proposition VII.4.2, we obtain.

Proposition VII.5.6. *Let $\tau\in E_D$.*

(1) *We have*
$$\lim_{\varepsilon\to 0^+}\left(\mathcal{G}_{-k,D}(\tau+i\varepsilon)-\mathcal{G}_{-k,D}(\tau-i\varepsilon)\right)=0.$$

(2) *We have*
$$\frac{1}{2}\lim_{\varepsilon\to 0^+}\left(\mathcal{G}_{-k,D}(\tau+i\varepsilon)+\mathcal{G}_{-k,D}(\tau-i\varepsilon)\right)=\mathcal{G}_{-k,D}(\tau).$$

(3) *We have*
$$\lim_{\varepsilon\to 0^+}\left(\frac{\partial}{\partial\overline{\tau}}\mathcal{G}_{-k,D}(\tau+i\varepsilon)-\frac{\partial}{\partial\overline{\tau}}\mathcal{G}_{-k,D}(\tau-i\varepsilon)\right)=iD^{k+\frac{1}{2}}v^{2k}\sum_{\substack{Q\in\mathcal{Q}_D\\ Q_\tau=0}}\frac{\operatorname{sgn}(Q)}{Q(\overline{\tau},1)^{k+1}}.$$

Secondly, we require the constant from [BKK15, equation (4.2), (7.3)] (using a different normalization)

$$c_\infty := \frac{\pi D^{k+\frac{1}{2}}}{2^{2k}(2k+1)} \sum_{a \geq 1} \sum_{\substack{0 \leq b < 2a \\ b^2 \equiv D \, (\mathrm{mod} \, 4a)}} \frac{1}{a^{k+1}}, \tag{VII.23}$$

which can be evaluated using a result of Zagier [Zag77, Proposition 3].

As a third ingredient, we have, for every $\tau \in \mathbb{H} \setminus E_D$,

$$
\begin{aligned}
\xi_{-2k}\left(g_{k+1,D}^*(\tau)\right) &= g_{k+1,D}(\tau), & \mathbb{D}^{2k+1}\left(g_{k+1,D}^*(\tau)\right) &= 0, \\
\xi_{-2k}\left(\mathcal{E}_{g_{k+1,D}}(\tau)\right) &= 0, & \mathbb{D}^{2k+1}\left(\mathcal{E}_{g_{k+1,D}}(\tau)\right) &= g_{k+1,D}(\tau).
\end{aligned}
\tag{VII.24}
$$

The third claim follows by holomorphicity of $\mathcal{E}_{g_{k+1,D}}$, while the second claim holds as $g_{k+1,D}^*$ (as a function of τ) is a polynomial of degree at most $2k$ by (VII.8). The first and fourth claim follow by a standard calculation using the integral representations from (VII.8) directly. Now, we are ready to prove Theorem VII.1.2.

Proof of Theorem VII.1.2. We prove part (2) first, and use it to prove part (1) afterwards.

(2) We define

$$f(\tau) := \mathcal{G}_{-k,D}(\tau) + \frac{D^{k+\frac{1}{2}}(2k)!}{(4\pi)^{2k+1}} \mathcal{E}_{g_{k+1,D}}(\tau) - D^{k+\frac{1}{2}} g_{k+1,D}^*(\tau).$$

Combining Proposition VII.5.2 with (VII.24), we deduce that

$$\xi_{-2k}(f) = \mathbb{D}^{2k+1}(f) = 0.$$

Hence, f is a polynomial in τ of degree at most $2k$. By Proposition VII.5.6 (1), $\mathcal{G}_{-k,D}$ has no jumps on E_D. Thus, we may freely select an arbitrary connected component of $\mathbb{H} \setminus E_D$ to compute f. Choosing the connected component of $\mathbb{H} \setminus E_D$ containing $i\infty$, we are in the same situation as in the induction start during the proof of [BKK15, Theorem 7.1]. In other words, the function f is in fact constant, and this constant was computed in [BKK15, Lemma 7.3]. We infer that f coincides with c_∞.

(1) We verify the four conditions in Definition VII.2.7.

   (i) Modularity of weight $-2k$ follows by Lemma VII.2.2 and equation (VII.10).
   (ii) Local harmonicity with respect to Δ_{-2k} outside E_D is Proposition VII.5.2 (3).
   (iii) The required behaviour on E_D is given in Proposition VII.5.6 (2).

(iv) The function $\mathcal{G}_{-k,D}$ is of at most polynomial growth towards $i\infty$ by virtue of its splitting in Theorem VII.1.2 (2). Being more precise, $g_{k+1,D}$ admits a Fourier expansion of the shape $\sum_{n\geq 1} c(n)e^{2\pi i n\tau}$, where the Fourier coefficients $c(n)$ depend on the connected component of $\mathbb{H} \setminus E_D$ in which τ is located. Letting $\Gamma(s,y)$ denote the incomplete Γ-function, we obtain for $v \gg 1$

$$\mathcal{E}_{g_{k+1,D}}(\tau) = \sum_{n\geq 1} \frac{c(n)}{n^{2k+1}} e^{2\pi i n\tau},$$

$$g_{k+1,D}^*(\tau) = \sum_{n\geq 1} \frac{c(n)}{(4\pi n)^{2k+1}} \Gamma(2k+1, 4\pi n v) e^{-2\pi i n\tau}.$$

We observe that the holomorphic Eichler integral vanishes as $\tau \to i\infty$, and the same holds for the nonholomorphic Eichler integral due to [BCLO10, § 8.11 (i)]. This proves that

$$\lim_{\tau \to i\infty} \mathcal{G}_{-k,D}(\tau) = c_\infty.$$

Proposition VII.5.6 (1) yields that the singularities of $\mathcal{G}_{-k,D}$ on E_D are continuously removable. Combining Proposition VII.5.6 (3) with Lemmas VII.2.3 and VII.4.3 shows that $\mathcal{G}_{-k,D}$ has no differentiable continuation to E_D. This completes the proof.

$\square$

VII.6   Proof of Theorem VII.1.3

We finish this chapter with the proof of Theorem VII.1.3.

Proof of Theorem VII.1.3. We follow [BKV13, Sections 4, 5] and shift $k \mapsto k+1$ in the calculations there. The treatment of Borcherds regularization can be adapted from [BKV13, Section 4] to our case straightforwardly. This implies that the integral over the unbounded region of $\mathbb{F}_T(4)$ vanishes as $T \to \infty$, while the truncated integral over the bounded region converges to the usual Petersson inner product as $T \to \infty$. We may use the usual unfolding argument and the computation of the integral over the real part both exactly as in [BKV13, Section 4]. Combining, this yields

$$\mathfrak{L}_{-k}^* \left(\mathcal{P}_{\frac{1}{2}-k,D} \right)(\tau) = \frac{1}{6\Gamma\left(k + \frac{3}{2}\right)(4\pi D)^{\frac{k}{2}+\frac{1}{4}}} \sum_{Q\in\mathcal{Q}_D} |Q_\tau| \, Q(\tau,1)^k I\left(\frac{Dv^2}{|Q(\tau,1)|^2}\right),$$

where (compare [BKV13, equation (5.2)])

$$I(t) := \int_0^\infty \mathcal{M}_{\frac{1}{2}-k, \frac{k}{2}+\frac{3}{4}}(-x) e^{\frac{x}{2}-\frac{x}{t}} x^{-\frac{1}{2}} \mathrm{d}x.$$

The evaluation is permitted, since $\tau \notin E_D$ gives $\frac{Dv^2}{|Q(\tau,1)|^2} < 1$ by Lemma VII.2.1, so the series on the right-hand side converges. This can be seen directly after rewriting I in the upcoming sentence, and comparing with Proposition VII.5.1. The integral I can be evaluated mutans mutandis as in [BKV13], giving

$$I(t) = k! \left(k + \frac{1}{2}\right)(1-t)^{-\frac{1}{2}} t^{\frac{1}{2}} \beta\left(t; k + \frac{1}{2}, \frac{1}{2}\right).$$

We conclude the theorem by Lemma VII.2.1. $\qquad\square$

VII.7 References

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Chapter VIII

Local weak Maaß forms and Eichler–Selberg type relations for negative weight vector-valued mock modular forms

This chapter is based on a preprint [MM21] recommended for publication in *Pacific Journal of Mathematics.* This is joint work with Dr. Joshua Males.

VIII.1 Introduction and statement of results

Theta lifts have a storied history in the literature, receiving a vast amount of attention in the past few decades with applications throughout mathematics. In this chapter, we are concerned with generalizations of the Siegel theta lift originally studied by Borcherds in the celebrated paper [Bor98]. The classical Siegel lift maps half-integral weight modular forms to those of integral weight, and has seen a wide number of important applications. For example, in arithmetic geometry [BZ22, ES18], deep results in number theory [BO10b], fundamental work of Bruinier and Funke [BF04], among many others.

More recently, Bruinier and Schwagenscheidt [BS21] investigated the Siegel theta lift on Lorentzian lattices (that is, even lattices of signature $(1, n)$), and in doing so provided a construction of recurrence relations for mock modular forms of weight $\frac{3}{2}$, as well as commenting as to how one could provide a similar structure for those of weight $\frac{1}{2}$, thereby including Ramanujan's classical mock theta functions.

In the last few years, several authors have also considered so-called "higher" Siegel theta lifts of the shape ($k := \frac{1-n}{2}$, $j \in \mathbb{N}_0$)

$$\int_{\mathcal{F}}^{\mathrm{reg}} \left\langle R_{k-2j}^j(f), \overline{\Theta_L(\tau, \boldsymbol{z})} \right\rangle v^k \mathrm{d}\mu(\tau)$$

where $R_\kappa^n := R_{n-2} \circ R_{n-4} \circ \cdots \circ R_\kappa$ is an iterated version of the Maaß raising operator $R_\kappa := 2i\frac{\partial}{\partial \tau} + \frac{\kappa}{v}$, f is weight $k - 2j$ harmonic Maaß form with cuspidal shadow, and Θ_L is

the standard Siegel theta function associated to an even lattice L of signature $(1, n)$. Here and throughout, $\tau = u + iv \in \mathbb{H}$ and $\boldsymbol{z} \in \mathrm{Gr}(L)$, the Grassmanian of L. Furthermore, $\langle \cdot, \cdot \rangle$ denotes the natural bilinear pairing. For example, they were considered by Bruinier and Ono (for $k = 0$, $j = 1$) in the influential work [BO13], by Bruinier, Ehlen, and Yang in in the breakthrough paper [BEY21] in relation to the Gross–Zagier conjecture, and by Alfes-Neumann, Bringmann, Males, and Schwagenscheidt in [ANBMS21] for $n = 2$ and generic j.

In [Mer14], Mertens investigated the classical Hurwitz class numbers, denoted by $H(n)$ for $n \in \mathbb{N}$. Using techniques in (scalar-valued) mock modular forms, he gave an infinite family of class number relations for odd n, two of which are

$$\sum_{s \in \mathbb{Z}} H\left(n - s^2\right) + \lambda_1(n) = \frac{1}{3}\sigma_1(n), \qquad \sum_{s \in \mathbb{Z}} \left(4s^2 - n\right) H\left(n - s^2\right) + \lambda_3(n) = 0, \quad \text{(VIII.1)}$$

where $\lambda_k(n) := \frac{1}{2}\sum_{d|n} \min(d, \frac{n}{d})^k$ and σ_k is the usual k-th power divisor function. Because of their close similarity to the classical formula of Kronecker [Kro60] and Hurwitz [Hur85]

$$\sum_{s \in \mathbb{Z}} H\left(n - s^2\right) - 2\lambda_1(n) = 2\sigma_1(n),$$

and those arising from the Eichler–Selberg trace formula, Mertens referred to the relationships (VIII.1) as *Eichler–Selberg type relations*. More generally, let $[\cdot, \cdot]_\nu$ denote the ν-th Rankin–Cohen bracket (see Section VIII.2). In general, the Rankin–Cohen bracket $[f, g]$ is a mixed mock modular form of degree ν. It is of inherent interest to determine its natural completion, say Λ, to a holomorphic modular form. Then following Mertens [Mer16], we say that a (mock-) modular form f satisfies an Eichler–Selberg type relation if there exists some holomorphic modular form g and some form Λ such that

$$[f, g]_\nu + \Lambda$$

is a holomorphic modular form. In the influential paper [Mer16], Mertens showed the beautiful result that all mock-modular forms of weight $\frac{3}{2}$ with holomorphic shadow satisfy Eichler–Selberg type relations, using the powerful theory of holomorphic projection and the Serre–Stark theorem stating that unary theta series form a basis for the spaces of holomorphic modular forms of the dual weight[1] $\frac{1}{2}$. In particular, Mertens explicitly describes the form Λ which completes the Rankin–Cohen brackets.

Following previous examples, to demonstrate the statement, let $\mathcal{H}$ denote the generating function of Hurwitz class numbers, let $\vartheta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}$, where $q^n = e^{2\pi i n \tau}$

---

[1]Mertens also provided results for mock theta functions in weight $\frac{1}{2}$, but since there is no analogue of Serre–Stark in the dual weight $\frac{3}{2}$ this is a real restriction.

throughout. Then Mertens' results show that [Mer16, pp. 377]

$$[\mathcal{H}, \vartheta]_\nu + 2^{-2\nu-1}\binom{2\nu}{\nu}\left(2\sum_{r\geq 1}\sum_{\substack{m^2-n^2=r\\m,n\geq 1}}(m-n)^{2\nu-1}\,q^r + \sum_{r\geq 1}r^{2\nu+1}q^r\right)$$

is a holomorphic modular form of weight $2\nu+2$ for all $\nu\geq 1$, and a quasimodular form of weight 2 if $\nu=0$.

In [Mal22], Males combined techniques of [ANBMS21, BS21] during a further investigation of the higher Siegel lift on Lorentzian lattices. This lift was shown to be central in producing certain Eichler–Selberg type relations in the vector-valued case, providing an analogue of the scalar-valued weight $\frac{3}{2}$ case of Mertens. We remark that the shape of the form Λ in the case of signature $(1,1)$ is very close to that of Mertens (see [Mal22, Theorem 1.1]), though we do not recall it here to save on complicated definitions in the introduction.

In the current chapter, we develop the theory for even generic signature (r,s) lattices L and more general modified Siegel theta functions as in Borcherds [Bor98], and consider the lift

$$\Psi_j^{\mathrm{reg}}(f, \boldsymbol{z}) := \int_{\mathcal{F}}^{\mathrm{reg}}\left\langle R_{k-2j}^j(f)(\tau), \overline{\Theta_L(\tau, \psi, p_\otimes)}\right\rangle v^k \mathrm{d}\mu(\tau),$$

where Θ_L is a modified Siegel theta function as in Borcherds [Bor98], essentially obtained by including a certain polynomial $p_\otimes$ in the summand of the usual vector-valued Siegel theta function. We require $p_\otimes$ to be homogenous and spherical of degree $d^+ \in \mathbb{N}_0$ in the first r variables, and $d^- \in \mathbb{N}_0$ in the last s variables (see Section VIII.2.7 for precise definitions). Here, ψ is an isometry which in turn defines $\boldsymbol{z}$ - see Section VIII.2.7. Modifying the theta function in this way preserves modular properties of Θ_L, while allowing us to obtain different weights of output functions. Furthermore, since the case $j=0$ is well-understood in the literature, we assume throughout that $j>0$. We remark that the signature $(1,2)$ with $j=0$ case has also been studied in [Cra15, CF21].

In particular, we evaluate the higher lift in the now-standard ways of unfolding in Theorem VIII.3.2, as well as recognizing it as a constant term in the Fourier expansion of the Rankin–Cohen bracket of a holomorphic modular form and a theta function (up to a boundary integral that vanishes for a certain class of input functions) in Theorem VIII.3.3. For the second of these theorems, we use that at special points $\boldsymbol{w}$, one may define positve- and negative-definite sublattices $P := L \cap \boldsymbol{w}$ and $N := L \cap \boldsymbol{w}^\perp$. In the simplest case, which we assume for the introduction, we have that $L = P \oplus N$. Then the theta series splits as $\Theta_L = \Theta_P \otimes \Theta_N$, where Θ_P is a positive definite theta series, and Θ_N a negative definite one. Then we let $\mathcal{G}_P^+$ be the holomorphic part of a preimage of Θ_P

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under $\xi_\kappa := 2iv^\kappa \overline{\frac{\partial}{\partial \bar\tau}}$. For the sake of simplicity, we assume that $\mathcal{G}_P^+ + g$ in the statement of Theorem VIII.1.1 is bounded at $i\infty$ in the introduction; we overcome this assumption in Theorem VIII.3.4 and offer a precise relation there. Following the ideas of [Mal22], by comparing these two evaluations of our lift and invoking Serre duality, we obtain the following theorem.

Theorem VIII.1.1. *Let L be an even lattice of signature (r, s), with associated Weil representation ρ_L. Let g be any holomorphic vector-valued modular form of weight $2 - \left(\frac{r}{2} + d^+\right)$ for ρ_L. Suppose that $\mathcal{G}_P^+ + g$ is bounded at $i\infty$. Then $\mathcal{G}_P^+ + g$ satisfies an explicit Eichler–Selberg type relation. In particular, the form Λ is explicitly determined.*

The concept of so-called locally harmonic Maaß forms was introduced by Bringmann, Kane, and Kohnen in [BKK15]. These are functions that behave like classical harmonic Maaß forms, except for an exceptional set of density zero, where they have jump singularities. Since their inception, locally harmonic Maaß forms have seen applications throughout number theory, for example in relation to central values of L-functions of elliptic curves [EGKR20], as well as traces of cycle integrals and periods of meromorphic modular forms [ANBMS21,LS22b] among many others. Examples of such locally harmonic Maaß forms are usually achieved in the literature via a similar theta lift machinery to that studied here. In addition to the direction of Theorem VIII.1.1, we also discuss the action of the Laplace–Beltrami operator on the lift Ψ_j^{reg} in Theorem VIII.4.2. In doing so, we prove the following theorem, thereby providing an infinite family of local weak Maaß forms (and locally harmonic Maaß forms) in signatures $(2, s)$. To state the result, we let $F_{m,k-2j,\mathfrak{s}}$ be a Maaß–Poincaré series as defined in Section VIII.2.5.

Theorem VIII.1.2. *Let L be an even isotropic lattice of signature $(2, s)$. Then the lift $\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{z})$ is a local weak Maaß form on $\mathrm{Gr}(L)$ with eigenvalue $(\mathfrak{s} - \frac{k}{2})(1 - \mathfrak{s} - \frac{k}{2})$ under the Laplace–Beltrami operator.*

We provide an example of an input function to our lift. To this end, we specialize our setting to signature $(1, 2)$, in which case vector-valued modular forms can be identified with the usual scalar-valued framework on the complex upper half plane, and in particular $\mathrm{Gr}(L) \cong \mathbb{H}$. (We explain the required choices in Section VIII.5.) In 1975, Cohen [Coh75] defined the generalized class numbers

$$H(\ell - 1, |D|) := \begin{cases} 0 & \text{if } D \neq 0, 1 \ (\mathrm{mod} \ 4), \\ \zeta(3 - 2\ell) & \text{if } D = 0, \\ L\left(2 - \ell, \left(\frac{D_0}{\cdot}\right)\right) \sum_{d|j} \mu(d) \left(\frac{D_0}{d}\right) d^{\ell-2} \sigma_{2\ell-3}\left(\frac{j}{d}\right) & \text{else,} \end{cases}$$

where $D = D_0 j^2$, as well as their generating functions

$$\mathcal{H}_\ell(\tau) := \sum_{n \geq 0} H(\ell, n) q^n, \qquad \ell \in \mathbb{N} \setminus \{1\}.$$

Here, ζ refers to the Riemann zeta function, $L(s, \chi)$ to the Dirichlet L-function twisted by a Dirichlet character χ, and μ is the Möbius function. The functions $\mathcal{H}_\ell$ are known as *Cohen–Eisenstein series* today, and can be viewed as half integral weight analogues of the classical integral weight Eisenstein series. Note that the numbers $H(2, n)$ are precisely the Hurwitz class numbers introduced above, and $\mathcal{H}_2 = \mathcal{H}$. Cohen proved that $\mathcal{H}_\ell \in M_{\ell - \frac{1}{2}}(\Gamma_0(4))$, the space of scalar-valued modular forms of weight $\frac{1}{2}$ on the usual congruence subgroup $\Gamma_0(4)$, and the coefficients satisfy Kohnen's plus space condition by definition. We refer the reader to [BFOR17, equations (2.13), (2.14), (2.15), Corollary 2.25] for more details on the Cohen–Eisenstein series.

However, evaluating our lift requires negative weight, and a non-constant principal part of the input function. To overcome both obstructions, we let

$$f_{-2\ell, N}(\tau) = q^{-N} + \sum_{n > m} c_{-2\ell}(N, n) q^n, \quad N \geq -m,$$

$$m := \begin{cases} \lfloor \frac{-2\ell}{12} \rfloor - 1 & \text{if } -2\ell \equiv 2 \pmod{12}, \\ \lfloor \frac{-2\ell}{12} \rfloor & \text{else,} \end{cases}$$

be the unique weakly holomorphic modular form of weight -2ℓ for $\mathrm{SL}_2(\mathbb{Z})$ with such a Fourier expansion. An explicit description of $f_{-2\ell, N}$ was given by Duke and Jenkins [DJ08], and by Duke, Imamoḡlu, and Tóth [DIT10, Theorem 1]. Our machinery now enables us to obtain Eichler–Selberg type relations for the weakly holomorphic function $f_{-2\ell, N}(\tau) \mathcal{H}_\ell(\tau)$ along the lines of [Coh75, Section 6], as well as the following variant of Theorem VIII.1.2.

Theorem VIII.1.3. *The lift* $\Psi_j^{\mathrm{reg}}(f_{-2\ell, N} \mathcal{H}_\ell, z)$ *is a local weak Maaß form on* $\mathbb{H}$ *for every* $j \in \mathbb{N}$, $\ell \in \mathbb{N} \setminus \{1\}$, *and* $-m \leq N \in \mathbb{N}$ *with exceptional set given by the net of Heegner geodesics*

$$\bigcup_{D=1}^{N} \left\{ z = x + iy \in \mathbb{H} \colon \exists a, b, c \in \mathbb{Z}, \ b^2 - 4ac = D, \ a\,|z|^2 + bx + c = 0 \right\},$$

and eigenvalue $(1 - k + j)(-j) = j(j - \ell - \frac{3}{2})$ *under* $\Delta_{-\ell - \frac{1}{2}}$.

Remarks.

(1) Theorem VIII.1.3 generalizes immediately to any weakly holomorphic modular form g. The exceptional set is given by the union of geodesics of discriminant $D > 0$, for which the coefficient of g at q^{-D} is non-zero.

(2) Recently, Wagner [Wag18] constructed a pullback of $\mathcal{H}_\ell$ under the ξ-operator, namely a harmonic Maaß form $\mathscr{H}_\ell$ of weight $-\ell + \frac{1}{2}$ on $\Gamma_0(4)$ that satisfies

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$\xi_{\frac{1}{2}-\ell}(\mathscr{H}_\ell) = \mathcal{H}_{\ell+2}$. An explicit definition of $\mathscr{H}_\ell$ can be found in [Wag18, equations (1.5), (1.6)]. However, $\mathscr{H}_\ell$ is a harmonic Maaß form with non-cuspidal shadow, and we restrict ourselves to a more restricitve growth condition in Section VIII.2.3 to ensure convergence of our lift. It would be interesting to investigate different regularizations of our lift, and in particular lift the function $\mathscr{H}_\ell$.

VIII.2 Preliminaries

We summarize some facts, which we require throughout.

VIII.2.1 The Weil representation

We recall that $j\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau\right) = c\tau + d$, and define the *metaplectic double cover*

$$\widetilde{\Gamma} := \mathrm{Mp}_2(\mathbb{Z}) := \left\{ (\gamma, \phi) \colon \gamma \in \mathrm{SL}_2(\mathbb{Z}), \ \phi \colon \mathbb{H} \to \mathbb{C} \text{ holomorphic}, \ \phi^2(\tau) = j(\gamma, \tau) \right\},$$

of $\mathrm{SL}_2(\mathbb{Z})$, which is generated by the pairs

$$\widetilde{T} := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \qquad \widetilde{S} := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right),$$

where we fix a suitable branch of the complex square root throughout. Furthermore, we define $\widetilde{\Gamma}_\infty$ as the subgroup generated by $\widetilde{T}$.

We let L be an even lattice of signature (r, s), and Q be a quadratic form on L with associated bilinear form $(\cdot, \cdot)_Q$. Moreover, we denote the dual lattice of L by L', and the group ring of L'/L by $\mathbb{C}[L'/L]$. The group ring $\mathbb{C}[L'/L]$ has a standard basis, whose elements will be called $\mathfrak{e}_\mu$ for $\mu \in L'/L$. We recall that there is a natural bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}[L'/L]$ defined by $\langle \mathfrak{e}_\mu, \mathfrak{e}_\nu \rangle = \delta_{\mu,\nu}$.

Equipped with this structure, the *Weil representation* ρ_L of $\widetilde{\Gamma}$ associated to L is defined on the generators by

$$\rho_L\left(\widetilde{T}\right)(\mathfrak{e}_\mu) := e(Q(\mu))\mathfrak{e}_\mu, \qquad \rho_L\left(\widetilde{S}\right)(\mathfrak{e}_\mu) := \frac{e\left(\frac{1}{8}(s-r)\right)}{\sqrt{|L'/L|}} \sum_{\nu \in L'/L} e(-(\nu,\mu)_Q)\mathfrak{e}_\nu,$$

where we stipulate $e(x) := e^{2\pi i x}$ throughout. We let $L^- := (L, -Q)$, and call ρ_{L^-} the *dual Weil representation* of L.

VIII.2.2 The generalized upper half plane and the invariant Laplacian

We follow the exposition in [Bru02, Subsections 3.2, 4.1], and let the signature of L be $(2, s)$ here. We assume that L is isotropic, i.e., that it contains a non-trivial vector $\boldsymbol{x}$ of norm 0, and by rescaling we may assume that it is primitive, that is if $\boldsymbol{x} = c\boldsymbol{y}$ for some $y \in L$ and $c \in \mathbb{Z}$ then $c = \pm 1$. Note that for $s \geq 3$ all lattices contain such an isotropic vector (see [Bor98, Section 8]).

Let $\boldsymbol{z} \in L$ be a primitive norm 0 vector, and $\boldsymbol{z}' \in L'$ with $(\boldsymbol{z}, \boldsymbol{z}')_Q = 1$. Let $K := L \cap \boldsymbol{z}^\perp \cap \boldsymbol{z}'^\perp$. Let $\boldsymbol{d} \in K$ be a primitive norm 0 vector, and $\boldsymbol{d}' \in K'$ with $(\boldsymbol{d}, \boldsymbol{d}')_Q = 1$. It follows that $D := K \cap \boldsymbol{d}^\perp \cap \boldsymbol{d}'^\perp$ is a negative definite lattice, and we write

$$\boldsymbol{Z} = (\boldsymbol{d}' - Q(\boldsymbol{d}')\boldsymbol{d}) \, z_1 + z_2\boldsymbol{d} + z_3 d_3 + \ldots + z_\ell d_\ell =: (z_1, z_2, \ldots, z_\ell) \in K \otimes \mathbb{C},$$

since $z_3 d_3 + \ldots + z_\ell d_\ell \in D \otimes \mathbb{C}$. Each z_j has a real part x_j and a imaginary part y_j, and we note that

$$Q(\boldsymbol{Y}) := Q(y_1, \ldots, y_\ell) = y_1 y_2 - y_3^2 - y_4^2 - \ldots - y_\ell^2.$$

This gives rise to the generalized upper half plane

$$\mathbb{H}_\ell := \{\boldsymbol{Z} \in K \otimes \mathbb{C} \colon y_1 > 0, Q(\boldsymbol{Y}) > 0\} \cong \mathrm{Gr}(L).$$

Letting

$$\partial_\mu := \frac{\partial}{\partial z_\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_\mu} - i\frac{\partial}{\partial y_\mu}\right), \qquad \overline{\partial}_\mu := \frac{\partial}{\partial \overline{z}_\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_\mu} + i\frac{\partial}{\partial y_\mu}\right),$$

it can be shown that the invariant Laplacian on $\mathbb{H}_\ell$ has the coordinate representation [Nak82]

$$\Omega := \sum_{\mu,\nu=1}^{\ell} y_\mu y \nu \partial_\mu \overline{\partial}_\nu - Q(\boldsymbol{Y})\left(\partial_1 \overline{\partial}_2 + \overline{\partial}_1 \partial_2 - \frac{1}{2}\sum_{\mu=3}^{\ell} \partial_\mu \overline{\partial}_\mu\right).$$

VIII.2.3 Maaß forms

Let $\kappa \in \frac{1}{2}\mathbb{Z}$, $(\gamma, \phi) \in \widetilde{\Gamma}$, and consider a function $f \colon \mathbb{H} \to \mathbb{C}[L'/L]$. The modular transformation in this setting is captured by the *slash-operator*

$$f\mid_{\kappa,\rho_L} (\gamma, \phi)(\tau) := \phi(\tau)^{-2\kappa} \rho_L^{-1}(\gamma, \phi) f(\gamma\tau),$$

which leads to vector-valued Maaß forms as follows [BF04].

Definition VIII.2.1. Let $f\colon \mathbb{H} \to \mathbb{C}\big[L'/L\big]$ be smooth. Then f is a *weight κ weak Maaß form with cuspidal shadow* with respect to ρ_L if it satisfies the following three conditions.

(1) We have $f\mid_{\kappa,\rho_L} (\gamma,\phi)(\tau) = f(\tau)$ for every $\tau \in \mathbb{H}$ and every $(\gamma,\phi) \in \widetilde{\Gamma}$.

(2) The function f is an eigenfunction of the weight κ hyperbolic Laplace operator, which is explicitly given by

$$\Delta_\kappa := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + i\kappa v \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right).$$

(3) There exists a polynomial[2] in q denoted by $P_f\colon \{0 < |w| < 1\} \to \mathbb{C}[L'/L]$ such that $f(\tau) - P_f(q) \in O\left(e^{-\varepsilon v}\right)$ as $v \to \infty$ for some $\varepsilon > 0$.

We call f a *harmonic Maaß form with cuspidal shadow* if the eigenvalue equals 0.

We write $H_{\kappa,L}^{\mathrm{cusp}}$ for the vector space of weight κ harmonic Maaß forms with cuspidal shadows with respect to ρ_L, and $M_{\kappa,L}^{!} \subseteq H_{\kappa,L}^{\mathrm{cusp}}$ for the subspace of weakly holomorphic vector-valued modular forms. The subspace $S_{\kappa,L}^{!} \subseteq M_{\kappa,L}^{!}$ collects all forms that vanish at all cusps, and such forms are referred to as weakly holomorphic cusp forms.

Bruinier and Funke [BF04] proved that a weight $\kappa \neq 1$ harmonic Maaß form with cuspidal shadow f decomposes as a sum $f = f^+ + f^-$ of a holomorphic and a nonholomorphic part, whose Fourier expansions are of the shape

$$f^+(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c_f^+(\mu,n) q^n \mathfrak{e}_\mu, \quad f^-(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n < 0}} c_f^-(\mu,n) \Gamma\left(1 - \kappa, 4\pi |n| v\right) q^n \mathfrak{e}_\mu,$$

where $\Gamma(t,x) := \int_x^\infty u^{t-1} e^{-u} \mathrm{d}u$, $x > 0$, denotes the *incomplete Gamma function* (see the paragraph following equation (I.3) and Section II.2 for more details and references).

Harmonic Maaß forms with cuspidal shadow can be inspected via the action of various differential operators. We require the antiholomorphic operator

$$\xi_\kappa := 2iv^\kappa \overline{\frac{\partial}{\partial \overline{\tau}}},$$

as well as the *Maaß raising* and *lowering operators*

$$R_\kappa := 2i\frac{\partial}{\partial \tau} + \frac{\kappa}{v}, \qquad L_\kappa := -2iv^2 \frac{\partial}{\partial \overline{\tau}}.$$

The operator ξ_κ defines a surjective map from $H_{\kappa,L}^{\mathrm{cusp}}$ to $S_{2-\kappa,L^-}^{!}$ [BF04]. In particular, it intertwines with the slash operator introduced above, and the space $M_{\kappa,L}^{!}$ is precisely the kernel of ξ_κ when restricted to $H_{\kappa,L}^{\mathrm{cusp}}$.

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[2]Such a polynomial is called the principal part of f.

The operators R_κ and L_κ increase and decrease the weight κ by 2 respectively, but do not preserve the eigenvalue under Δ_κ. For any $n \in \mathbb{N}_0$, we let

$$R_\kappa^0 := \mathrm{id}, \qquad R_\kappa^n := R_{\kappa+2n-2} \circ \ldots \circ R_{\kappa+2} \circ R_\kappa,$$
$$L_\kappa^0 := \mathrm{id}, \qquad L_\kappa^n := L_{\kappa-2n+2} \circ \ldots \circ L_{\kappa-2} \circ L_\kappa$$

be the *iterated Maaß raising* and *lowering operators*, which increase or decrease the weight κ by $2n$.

Remark. If one relaxes the growth condition (3) to linear exponential growth, that is $f(\tau) \in O(e^{\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$, then f^- is permitted to have an additional (constant) term of the form $c_f^-(\mu, 0) v^{1-\kappa} \mathfrak{e}_\mu$. In this case, ξ_κ maps such a form to a weakly holomorphic modular form instead of a weakly holomorphic cusp form.

VIII.2.4 Local Maaß forms

Locally harmonic Maaß forms were introduced by Bringmann, Kane, and Kohnen [BKK15] for negative weights, and independently by Hövel [Höv12] for weight 0. We generalize the exposition by Bringmann, Kane, and Kohnen here, and provide a definition in our setting on Grassmannians and for arbitrary eigenvalues.

Definition VIII.2.2. A *local weak Maaß form* of weight κ with closed exceptional set $X \subsetneq \mathbb{H}_\ell$ of measure zero is a function $f \colon \mathbb{H}_\ell \to \mathbb{C}[L'/L]$, which satisfies the following four properties:

(1) For all $(\gamma, \phi) \in \widetilde{\Gamma}$ and all $\boldsymbol{Z} \in \mathbb{H}_\ell$ it holds that $f \mid_{\kappa, \rho_L} (\gamma, \phi)(\boldsymbol{Z}) = f(\boldsymbol{Z})$.
(2) For every $\boldsymbol{Z} \in \mathbb{H}_\ell \setminus X$, there exists a neighborhood of $\boldsymbol{Z}$, in which f is real-analytic and an eigenfunction of Ω.
(3) We have

$$f(\boldsymbol{Z}) = \frac{1}{2} \lim_{\varepsilon \searrow 0} \left(f\left(\boldsymbol{Z} + (i\varepsilon, 0, \ldots, 0)^T\right) + f\left(\boldsymbol{Z} - (i\varepsilon, 0, \ldots, 0)^T\right) \right)$$

for every $\boldsymbol{Z} \in X$.
(4) The function f is of at most polynomial growth towards all cusps.

Paralleling the definition of harmonic Maaß forms, we call a local weak Maaß form *locally harmonic* if the eigenvalue from the second condition is 0.

VIII.2.5   Poincaré series

Weakly holomorphic Poincaré series

Following Knopp and Mason [KM04, Section 3], we let $m \in \mathbb{Z}$, $\kappa \in \frac{1}{2}\mathbb{N}$ satisfying $\kappa > 2$, $\mu \in {}^{L'}/_L$, and define

$$\mathbb{F}_{\mu,m,\kappa}(\tau) := \frac{1}{2} \sum_{(\gamma,\phi) \in \widetilde{\Gamma}_\infty \backslash \widetilde{\Gamma}} \left(e((m+1)\tau)\mathfrak{e}_\mu \right) |_{\kappa,\rho_L} (\gamma, \phi).$$

The authors of [KM04] proved that $\mathbb{F}_{\mu,m,\kappa}$ converges absolutely, and that it defines a weakly holomorphic modular form of weight κ for ρ_L. In addition, they computed the Fourier expansion of $\mathbb{F}_{\mu,m,\kappa}$, which is of the shape

$$\mathbb{F}_{\mu,m,\kappa}(\tau) = \sum_{\nu \in L'/L} \left(\delta_{\mu,\nu} q^{m+1} + \sum_{n \geq 0} c(n) q^{n+1} \right) \mathfrak{e}_\nu.$$

The Fourier coefficients $c(n)$ can be found in [KM04, Theorem 3.2] explicitly.

Maaß–Poincaré series

We recall an important example of harmonic Maaß forms with cuspidal shadows. To this end, let $\kappa \in -\frac{1}{2}\mathbb{N}$, let $M_{\mu,\nu}$ be the usual M-Whittaker function (see [BCLO10, § 13.14]), and define the auxiliary function

$$\mathcal{M}_{\kappa,\mathfrak{s}}(y) := |y|^{-\frac{\kappa}{2}} M_{\mathrm{sgn}(y)\frac{\kappa}{2},\mathfrak{s}-\frac{1}{2}}(|y|), \qquad y \in \mathbb{R} \setminus \{0\}.$$

We average $\mathcal{M}_\kappa$ over $\widetilde{\Gamma}$ as usual with respect to the parameters $\mu \in {}^{L'}/_L$, $m \in \mathbb{N} \setminus \{Q(\mu)\}$, and κ, $\mathfrak{s}$. This yields the vector-valued Maaß–Poincaré series [Bru02]

$$F_{\mu,m,\kappa,\mathfrak{s}}(\tau) := \frac{1}{2\Gamma(2\mathfrak{s})} \sum_{(\gamma,\phi) \in \widetilde{\Gamma}_\infty \backslash \widetilde{\Gamma}} \left(\mathcal{M}_{\kappa,\mathfrak{s}}(4\pi m v) e(-mu)\mathfrak{e}_\mu \right) |_{\kappa,\rho_L} (\gamma, \phi).$$

By our choice of parameters and taking cosets, the series converges absolutely. The eigenvalue under Δ_κ is given by $(\mathfrak{s} - \frac{\kappa}{2})(1 - \mathfrak{s} - \frac{\kappa}{2})$. Hence if $\mathfrak{s} = \frac{\kappa}{2}$ or $\mathfrak{s} = 1 - \frac{\kappa}{2}$, then we have $F_{\mu,m,\kappa,\mathfrak{s}} \in H_{\kappa,L}^{\mathrm{cusp}}$. The principal part of $F_{\mu,m,\kappa,\mathfrak{s}}$ is given by $e(-m\tau)(\mathfrak{e}_\mu + \mathfrak{e}_{-\mu})$ in this case, and $\xi_\kappa(F_{\mu,-m,\kappa,\mathfrak{s}})$ is a weight $2 - \kappa$ cusp form.

Furthermore, the Maaß–Poincaré series have the following useful property thanks to their simple principal part.

Lemma VIII.2.3. *Let $f \in H_{\kappa,L}^{\mathrm{cusp}}$ with $\kappa \in -\frac{1}{2}\mathbb{N}$, and principal part*

$$P_f(\tau) = \sum_{\mu \in L'/L} \sum_{n<0} c_f^+(\mu, n) e(n\tau) \mathfrak{e}_\mu \in \mathbb{C}\left[L'/L\right]\left[e(-\tau)\right].$$

Then, we have

$$f(\tau) = \frac{1}{2} \sum_{\mu \in L'/L} \sum_{m>0} c_f^+(\mu, -m) F_{\mu, m, \kappa, 1-\frac{\kappa}{2}}(\tau).$$

Additionally, we require the following computational lemma, which is taken from [ANBMS21, Lemma 2.1], and follows inductively from [BEY21, Proposition 3.4].

Lemma VIII.2.4. *For any $n \in \mathbb{N}_0$ it holds that*

$$R_\kappa^n\left(F_{\mu, m, \kappa, \mathfrak{s}}\right)(\tau) = (4\pi m)^n \frac{\Gamma\left(\mathfrak{s} + n + \frac{\kappa}{2}\right)}{\Gamma\left(\mathfrak{s} + \frac{\kappa}{2}\right)} F_{\mu, m, \kappa+2n, \mathfrak{s}}(\tau).$$

VIII.2.6 Restriction, trace maps, and Rankin–Cohen brackets

As before, we fix an even lattice L. We let $A_{\kappa,L}$ be the space of smooth functions $f \colon \mathbb{H} \to \mathbb{C}\left[L'/L\right]$, which are invariant under the weight κ slash operator with respect to the representation ρ_L. Moreover, let $K \subseteq L$ be a finite index sublattice. Hence, we have $L' \subseteq K'$, and thus $L/K \subseteq L'/K \subseteq K'/K$. This induces a map $L'/K \to L'/L$, given by $\mu \mapsto \bar{\mu}$. If $\mu \in K'/K$, $f \in A_{\kappa,L}$, $g \in A_{\kappa,K}$, and μ is a fixed preimage of $\bar{\mu}$ in L'/K, we define

$$(f_K)_\mu := \begin{cases} f_{\bar{\mu}} & \text{if } \mu \in L'/K, \\ 0 & \text{if } \mu \notin L'/K, \end{cases} \qquad \left(g^L\right)_{\bar{\mu}} = \sum_{\alpha \in L/K} g_{\alpha+\mu},$$

The following lemma may be found in [BY09, Section 3].

Lemma VIII.2.5. *In the notation above, there are two natural maps*

$$\mathrm{res}_{L/K} \colon A_{\kappa,L} \to A_{\kappa,K}, \quad \mathrm{tr}_{L/K} \colon A_{\kappa,K} \to A_{\kappa,L},$$
$$f \mapsto f_K \qquad\qquad\qquad g \mapsto g^L$$

satisfying

$$\left\langle f, \overline{g}^L \right\rangle = \langle f_K, \overline{g} \rangle$$

for any $f \in A_{\kappa,L}$, $g \in A_{\kappa,K}$.

Let $\kappa, \ell \in \frac{1}{2}\mathbb{Z}$, $f \in A_{\kappa,K}$, $g \in A_{\ell,L}$. Writing

$$f = \sum_\mu f_\mu \mathfrak{e}_\mu, \qquad g = \sum_\nu g_\nu \mathfrak{e}_\nu,$$

and letting $n \in \mathbb{N}_0$, we define the tensor product of f and g as well as the *n-th Rankin–Cohen bracket* of f and g as

$$f \otimes g := \sum_{\mu,\nu} f_\mu g_\nu \mathfrak{e}_{\mu+\nu} \in A_{\kappa+\ell,K\oplus L},$$

$$[f,g]_n := \frac{1}{(2\pi i)^n} \sum_{\substack{r,s\geq 0 \\ r+s=n}} \frac{(-1)^r \Gamma(\kappa+n)\Gamma(\ell+n)}{\Gamma(s+1)\Gamma(\kappa+n-s)\Gamma(r+1)\Gamma(\ell+n-r)} f^{(r)} \otimes g^{(s)},$$

where $f^{(r)}$ and $g^{(s)}$ are usual higher derivatives of f and g. Then we have the following vector-valued analogue of [BEY21, Proposition 3.6].

Lemma VIII.2.6. *Let $f \in H_{\kappa,L_1}^{\mathrm{cusp}}$ and $g \in H_{\ell,L_2}^{\mathrm{cusp}}$. For $n \in \mathbb{N}_0$ it holds that*

$$(-4\pi)^n L_{\kappa+\ell+2n}\left([f,g]_n\right) = \frac{\Gamma(\kappa+n)}{n!\,\Gamma(\kappa)} L_\kappa(f) \otimes R_\ell^n(g) + (-1)^n \frac{\Gamma(\ell+n)}{n!\,\Gamma(\ell)} R_\kappa^n(f) \otimes L_\ell(g).$$

Finally, we have the following lemma, which can be verified straightforwardly (see [ANBMS21, Proof of Theorem 4.1]).

Lemma VIII.2.7. *Let h be a smooth function, g be holomorphic, and $\kappa, \ell \in \mathbb{R}$. Then it holds that*

$$R_{\ell-\kappa}(v^\kappa \overline{g} \otimes h) = v^k \overline{g} \otimes R_\ell(h).$$

VIII.2.7 Theta functions and special points

We fix an even lattice L of signature (r,s), and extend the quadratic form on L to $L \otimes \mathbb{R}$ in the natural way. We denote the orthogonal projection of $\boldsymbol{\lambda} \in L + \mu$ onto the linear subspaces spanned by $\boldsymbol{z}$ and its orthogonal complement with respect to $(\cdot,\cdot)_Q$ by $\lambda_{\boldsymbol{z}}$ and $\lambda_{\boldsymbol{z}^\perp}$ respectively. In other words, we have

$$L \otimes \mathbb{R} = \boldsymbol{z} \oplus \boldsymbol{z}^\perp, \qquad \boldsymbol{\lambda} = \boldsymbol{\lambda}_{\boldsymbol{z}} + \boldsymbol{\lambda}_{\boldsymbol{z}^\perp}.$$

Let $\mathrm{Gr}(L)$ be the Grassmannian of r-dimensional subspaces of $L \otimes \mathbb{R}$. Let $Z \subseteq \mathrm{Gr}(L)$ be the set of all such subspaces on which Q is positive definite. One can endow Z with the structure of a smooth manifold.

Let $p_r\colon \mathbb{R}^{r,0} \to \mathbb{C}$, and $p_s\colon \mathbb{R}^{0,s} \to \mathbb{C}$ be spherical polynomials, which are homogeneous of degree d^+, $d^- \in \mathbb{N}_0$ respectively. Define

$$p_\otimes := p_r \otimes p_s,$$

and let $\psi\colon L \otimes \mathbb{R} \to \mathbb{R}^{r,s}$ be an isometry. We set

$$\boldsymbol{z} := \psi^{-1}(\mathbb{R}^{r,0}) \in Z, \qquad \boldsymbol{z}^\perp = \psi^{-1}(\mathbb{R}^{0,s}).$$

For a positive-definite lattice (K, Q) of rank n, and a homogeneous spherical polynomial p of degree d, we define the usual theta function

$$\Theta_K(\tau, \psi_K, p_\otimes) := \sum_{\boldsymbol{\lambda} \in K'} p_\otimes(\psi_K(\boldsymbol{\lambda})) e\left(Q(\boldsymbol{\lambda})\tau\right),$$

where ψ_K is the isometry associated to K. It is a holomorphic modular form of weight $\frac{n}{2} + d$ for ρ_K. If the isometry is trivial, we write $\Theta_K(\tau, p_\otimes)$.

Following Borcherds [Bor98] and Hövel [Höv12], we define the general Siegel theta function as follows.[3]

Definition VIII.2.8. Let $\tau \in \mathbb{H}$ and assume the notation above. Then we put

$$\Theta_L(\tau, \psi, p_\otimes) := v^{\frac{s}{2}+d^-} \sum_{\mu \in L'/L} \sum_{\boldsymbol{\lambda} \in L+\mu} p_\otimes(\psi(\boldsymbol{\lambda})) e\left(Q(\boldsymbol{\lambda}_{\boldsymbol{z}})\tau + Q(\boldsymbol{\lambda}_{\boldsymbol{z}^\perp})\overline{\tau}\right) \mathfrak{e}_\mu.$$

One can check that the function Θ_L converges absolutely on $\mathbb{H} \times Z$. The following result is [Höv12, Satz 1.55], which follows directly from [Bor98, Theorem 4.1].

Lemma VIII.2.9. *Let* $(\gamma, \phi) \in \widetilde{\Gamma}$. *Then we have*

$$\Theta_L(\gamma\tau, \psi, p_\otimes) = \phi(\tau)^{r+2d^+ - (s+2d^-)} \rho_L(\gamma, \phi) \Theta_L(\tau, \psi, p_\otimes).$$

Thus, we define

$$k := \frac{r-s}{2} + d^+ - d^-.$$

The following terminology is borrowed from [BS21].

Definition VIII.2.10. An element $\boldsymbol{w} \in \mathrm{Gr}(L)$ is called a *special point* if it is defined over $\mathbb{Q}$ that is $\boldsymbol{w} \in L \otimes \mathbb{Q}$.

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[3]In fact, Borcherds considered a slightly more general theta function, where the polynomial p does not necessarily vanish under Δ_κ. For us however, this more general case would not yield spherical theta functions as we desire.

We observe that if $\boldsymbol{w}$ is a special point, then $\boldsymbol{w}^{\perp}$ is a special point as well. This yields the splitting

$$L \otimes \mathbb{Q} = \boldsymbol{w} \oplus \boldsymbol{w}^{\perp},$$

which in turn yields the positive and negative definite lattices

$$P := L \cap \boldsymbol{w}, \qquad N := L \cap \boldsymbol{w}^{\perp}.$$

Clearly, $P \oplus N$ is a sublattice of L of finite index, and according to Lemma VIII.2.5, the theta functions associated to both lattices are related by

$$\Theta_L = (\Theta_{P \oplus N})^L.$$

We identify $\mathbb{C}\big[(P \oplus N)'/(P \oplus N)\big]$ with $\mathbb{C}\big[P'/P\big] \otimes \mathbb{C}\big[N'/N\big]$, and let ψ_P, ψ_N be the restrictions of ψ onto P, N, respectively. Consequently, we have the splitting

$$\Theta_{P \oplus N}(\tau, \psi, p_{\otimes}) = \Theta_P(\tau, \psi_P, p_r) \otimes v^{\frac{s}{2}+d^-} \overline{\Theta_{N^-}(\tau, \psi_N, p_s)}$$

at a special point $\boldsymbol{w}$, which can be verified straightforwardly. Furthermore, we observe that $\Theta_P(\tau, \psi_P, p_r)$ is holomorphic and of weight $\frac{r}{2} + d^+$ as a function of τ, while $v^{\frac{s}{2}+d^-} \overline{\Theta_{N^-}(\tau, \psi_N, p_s)}$ is of weight $-\frac{s}{2} - d^-$ with respect to τ.

VIII.2.8 Serre duality

The following result can be found in [LS22a, Proposition 2.5] for instance.

Proposition VIII.2.11 (Serre duality)**.** *Let L be an even lattice, and $\kappa \in \frac{1}{2}\mathbb{Z}$. Assume that*

$$g(\tau) = \sum_{h \in L'/L} \sum_{n \geq 0} c_g(h, n) e(n\tau) \mathfrak{e}_h$$

is bounded at the cusp $i\infty$. Then g is a holomorphic modular form of weight κ for the Weil representation ρ_L if and only if we have

$$\sum_{h \in L'/L} \sum_{n \geq 0} c_g(h, n) c_f(h, -n) = 0$$

for every weakly holomorphic modular form f of weight $2 - \kappa$ for $\overline{\rho}_L$.

VIII.3 The theta lift

We consider the theta lift $\Psi_j^{\mathrm{reg}}(f, \boldsymbol{z})$ and evaluate it in two different ways. Using Serre duality goes back to Borcherds [Bor99].

VIII.3.1 Evaluation in terms of $_2F_1$

We begin by evaluating the higher modified lift as a series involving Gauss hypergeometric functions as follows.

Evaluating the theta lift of Maaß–Poincaré series for general spectral parameters

Let $\mathfrak{s} \in \mathbb{C}$ be such that

$$F_{m,\kappa,\mathfrak{s}}(\tau) := \sum_{\mu \in L'/L} F_{\mu,m,\kappa,\mathfrak{s}}(\tau)$$

converges absolutely, that is $\operatorname{Re}(\mathfrak{s}) > 1 - \frac{\kappa}{2}$.

Theorem VIII.3.1. *We have*

$$\Psi_j^{\mathrm{reg}}\left(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{z}\right) = (4\pi m)^{j+1-k-\frac{s}{2}-d^-} \frac{\Gamma\left(\mathfrak{s}+\frac{k}{2}\right)\Gamma\left(\frac{k+s}{2}+d^--1+\mathfrak{s}\right)}{2\Gamma(2-k+2j)\Gamma\left(\mathfrak{s}+\frac{k}{2}-j\right)} \sum_{\mu \in L'/L} \sum_{\substack{\boldsymbol{\lambda} \in L+\mu \\ Q(\boldsymbol{\lambda})=-m}}$$

$$\times \overline{p_\otimes(\psi(\boldsymbol{\lambda}))} \left(\frac{Q(\boldsymbol{\lambda})}{Q\left(\boldsymbol{\lambda}_{\boldsymbol{z}^\perp}\right)}\right)^{\frac{k+s}{2}+d^--1+\mathfrak{s}} {}_2F_1\left(k+\mathfrak{s}, \frac{k+s}{2}+d^--1+\mathfrak{s}; 2\mathfrak{s}; \frac{Q(\boldsymbol{\lambda})}{Q\left(\boldsymbol{\lambda}_{\boldsymbol{z}^\perp}\right)}\right).$$

Remark. Choosing the homogeneous polynomial in the theta kernel function to be the constant function 1 and computing the action of R_{k-2j}^j on $F_{m,k-2j,\mathfrak{s}}$ by Lemma VIII.2.4, this result becomes [Bru02, Theorem 2.14].

Proof. We summarize the argument from [Bru02, Theorem 2.14] for convenience of the reader. We need to evaluate

$$\Psi_j^{\mathrm{reg}}\left(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{z}\right) = \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle R_{k-2j}^j(F_{m,k-2j,\mathfrak{s}})(\tau), \overline{\Theta_L(\tau,\psi,p_\otimes)} \right\rangle v^k \mathrm{d}\mu(\tau).$$

Consequently, we compute the action of the raising operator first, and have

$$\Psi_j^{\mathrm{reg}}\left(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{z}\right) = (4\pi m)^j \frac{\Gamma\left(\mathfrak{s}+\frac{k}{2}\right)}{\Gamma\left(\mathfrak{s}+\frac{k}{2}-j\right)} \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle (F_{m,k,\mathfrak{s}})(\tau), \overline{\Theta_L(\tau,\psi,p_\otimes)} \right\rangle v^k \mathrm{d}\mu(\tau)$$

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by Lemma VIII.2.4. Secondly, we insert the definitions of both functions, and unfold the integral, obtaining

$$
\Psi_j^{\mathrm{reg}}\left(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{z}\right) = \frac{(4\pi m)^j \Gamma\left(\mathfrak{s} + \frac{k}{2}\right)}{2\Gamma(2 - k + 2j)\Gamma\left(\mathfrak{s} + \frac{k}{2} - j\right)} \sum_{\mu \in L'/L} \sum_{\lambda \in L + \mu} \overline{p_\otimes(\psi(\boldsymbol{\lambda}))}
$$
$$
\times \int_0^1 \int_0^\infty (4\pi m v)^{-\frac{k}{2}} M_{-\frac{k}{2}, \mathfrak{s} - \frac{1}{2}}(4\pi m v) e(-mu) \overline{e\left(Q(\boldsymbol{\lambda_z})\tau + Q(\boldsymbol{\lambda_{z^\perp}})\overline{\tau}\right)} v^{\frac{s}{2} + d^- + k - 2} \mathrm{d}v \mathrm{d}u.
$$

Third, we compute the integral over u using that $\overline{e(w)} = e(-\overline{w})$, and that

$$
\int_0^1 e(-mu) e\left(-Q(\boldsymbol{\lambda_z})u - Q(\boldsymbol{\lambda_{z^\perp}})u\right) \mathrm{d}u = \begin{cases} 1 & \text{if } Q(\boldsymbol{\lambda_z}) + Q(\boldsymbol{\lambda_{z^\perp}}) = -m, \\ 0 & \text{else.} \end{cases}
$$

Hence, we obtain

$$
\Psi_j^{\mathrm{reg}}\left(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{z}\right) = \frac{(4\pi m)^{j - \frac{k}{2}} \Gamma\left(\mathfrak{s} + \frac{k}{2}\right)}{2\Gamma(2 - k + 2j)\Gamma\left(\mathfrak{s} + \frac{k}{2} - j\right)} \sum_{\mu \in L'/L} \sum_{\substack{\boldsymbol{\lambda} \in L + \mu \\ Q(\boldsymbol{\lambda}) = -m}} \overline{p_\otimes(\psi(\boldsymbol{\lambda}))}
$$
$$
\times \int_0^\infty M_{-\frac{k}{2}, \mathfrak{s} - \frac{1}{2}}(4\pi m v) e^{-2\pi v \left(Q(\boldsymbol{\lambda_z}) - Q(\boldsymbol{\lambda_{z^\perp}})\right)} v^{\frac{s+k}{2} + d^- - 2} \mathrm{d}v.
$$

The integral is a Laplace transform. Using that $\frac{m}{2m} + \frac{Q(\boldsymbol{\lambda_z}) - Q(\boldsymbol{\lambda_{z^\perp}})}{2m} = \frac{Q(\boldsymbol{\lambda_{z^\perp}})}{Q(\boldsymbol{\lambda})}$ along with [BCLO10, item 13.23.1], it evaluates

$$
\int_0^\infty M_{-\frac{k}{2}, \mathfrak{s} - \frac{1}{2}}(4\pi m v) e^{-2\pi v \left(Q(\boldsymbol{\lambda_z}) - Q(\boldsymbol{\lambda_{z^\perp}})\right)} v^{\frac{k+s}{2} + d^- - 2} \mathrm{d}v
$$
$$
= \frac{(4\pi m)^{1 - \frac{k+s}{2} - d^-} \Gamma\left(\frac{k+s}{2} + d^- - 1 + \mathfrak{s}\right)}{\left(\frac{Q(\boldsymbol{\lambda_z}) - Q(\boldsymbol{\lambda_{z^\perp}})}{2m} + \frac{1}{2}\right)^{\frac{k+s}{2} + d^- - 1 + \mathfrak{s}}}
$$
$$
\times\ {}_2F_1\left(k + \mathfrak{s}, \frac{k+s}{2} + d^- - 1 + \mathfrak{s}; 2\mathfrak{s}; \frac{1}{\frac{1}{2} + \frac{Q(\boldsymbol{\lambda_z}) - Q(\boldsymbol{\lambda_{z^\perp}})}{2m}}\right).
$$

We recall $Q(\boldsymbol{\lambda}) = Q(\boldsymbol{\lambda_z}) + Q(\boldsymbol{\lambda_{z^\perp}}) = -m$, and rewrite the argument of the hypergeometric function to

$$
\frac{m}{2m} + \frac{Q(\boldsymbol{\lambda_z}) - Q(\boldsymbol{\lambda_{z^\perp}})}{2m} = \frac{Q\left(\boldsymbol{\lambda_{z^\perp}}\right)}{Q(\boldsymbol{\lambda})}.
$$

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Thus, we arrive at

$$\Psi_j^{\text{reg}}\left(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{z}\right) = (4\pi m)^{j+1-k-\frac{s}{2}-d^-}\frac{\Gamma\left(\mathfrak{s}+\frac{k}{2}\right)\Gamma\left(\frac{k+s}{2}+d^--1+\mathfrak{s}\right)}{2\Gamma(2-k+2j)\Gamma\left(\mathfrak{s}+\frac{k}{2}-j\right)}\sum_{\mu\in L'/L}\sum_{\substack{\boldsymbol{\lambda}\in L+\mu \\ Q(\boldsymbol{\lambda})=-m}}$$

$$\times\overline{p_\otimes(\psi(\boldsymbol{\lambda}))}\left(\frac{Q(\boldsymbol{\lambda})}{Q\left(\boldsymbol{\lambda}_{\boldsymbol{z}^\perp}\right)}\right)^{\frac{k+s}{2}+d^--1+\mathfrak{s}}{}_2F_1\left(k+\mathfrak{s},\frac{k+s}{2}+d^--1+\mathfrak{s};2\mathfrak{s};\frac{Q(\boldsymbol{\lambda})}{Q\left(\boldsymbol{\lambda}_{\boldsymbol{z}^\perp}\right)}\right),$$

as claimed. $\qquad\square$

Combining the previous result with Lemma VIII.2.3 yields the following consequence.

Corollary VIII.3.2. *Let $j \in \mathbb{N}_0$, and $f \in H_{k-2j,L}^{\text{cusp}}$. Assume that $k - 2j < 0$. Then we have*

$$\Psi_j^{\text{reg}}\left(f, \boldsymbol{z}\right) = \frac{(4\pi)^{j+1-k-\frac{s}{2}-d^-}\,j!\,\Gamma\left(\frac{s}{2}+d^-+j\right)}{4\Gamma(2-k+2j)}\sum_{\substack{\boldsymbol{\lambda}\in L' \\ Q(\boldsymbol{\lambda})<0}}c_f^+(\boldsymbol{\lambda}, Q(\boldsymbol{\lambda}))\overline{p_\otimes(\psi(\boldsymbol{\lambda}))}$$

$$\times\frac{|Q(\boldsymbol{\lambda})|^{2j+1-k}}{|Q(\boldsymbol{\lambda}_{\boldsymbol{z}^\perp})|^{\frac{s}{2}+j+d^-}}{}_2F_1\left(1+j,\frac{s}{2}+d^-+j;2-k+2j;\frac{Q(\boldsymbol{\lambda})}{Q(\boldsymbol{\lambda}_{\boldsymbol{z}^\perp})}\right).$$

Proof. Since the weight of f is negative, we have

$$f(\tau) = \frac{1}{2}\sum_{h\in L'/L}\sum_{m\geq 0}c_f^+(h,-m)F_{h,m,k-2j,1-\frac{k}{2}+j}(\tau)$$

according to Lemma VIII.2.3, and we observe that the term corresponding to $m = 0$ will vanish due to $c_f^+(h,0) = 0$ by our more restrictive growth condition on Maaß forms. Consequently, we have

$$\Psi_j^{\text{reg}}\left(f, \boldsymbol{z}\right) = \frac{1}{2}\sum_{\mu\in L'/L}\sum_{m>0}c_f^+(\mu,-m)\Psi_j^{\text{reg}}\left(F_{\mu,m,k-2j,1-\frac{k}{2}+j}, \boldsymbol{z}\right).$$

We insert the spectral parameter $\mathfrak{s} = 1 - \frac{k-2j}{2}$ into Theorem VIII.3.1, which yields the claim. $\qquad\square$

VIII.3.2 Evaluation in terms of the constant term in a Fourier expansion

We let

$$\text{CT}\left(\sum_{n\gg-\infty}a(n)q^n\right) := a(0),$$

and determine the lift as such a constant term in a Fourier expansion plus a certain boundary integral that vanishes for a certain class of input function.

Theorem VIII.3.3. *Let $f \in H_{k-2j,L}^{\mathrm{cusp}}$ and w be a special point, and $\mathcal{G}_P^+$ be the holomorphic part of a preimage of Θ_P under $\xi_{2-(\frac{r}{2}+d^+)}$. Then we have*

$$\Psi_j^{\mathrm{reg}}(f, \boldsymbol{w}) = \frac{j!(4\pi)^j \Gamma\left(2 - \frac{r}{2} - d^+\right)}{\Gamma\left(2 - \frac{r}{2} - d^+ + j\right)} \left(\mathrm{CT}\left(\left\langle f_{P\oplus N}(\tau), \left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau) \right]_j \right\rangle \right) \right.$$
$$\left. - \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle L_{k-2j}\left(f_{P\oplus N}\right)(\tau), \left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau) \right]_j \right\rangle v^{-2} \mathrm{d}\tau \right).$$

Remark. In general, the coefficients of $\mathcal{G}_P^+$ are expected to be transcendental. However, in weight $\frac{1}{2}$ and $\frac{3}{2}$ the function $\mathcal{G}_P^+$ may be chosen to have rational coefficients - a situation which is expected to also hold for ξ-preimages of CM modular forms. It is therefore expected that one obtains rationality (up to powers of π) of the modified higher lift only in these cases, and stipulating that f is weakly holomorphic meaning that the final integral vanishes.

By a slight abuse of notation, we write $\Theta_L(\tau, \boldsymbol{w}, p_\otimes)$ for the theta function evaluated at an isometry ψ that produces a special point $\boldsymbol{w}$.

Proof of Theorem VIII.3.3. We restrict to special points $\boldsymbol{w} \in \mathrm{Gr}(L)$. This enables us to write

$$\left\langle R_{k-2j}^j(f)(\tau), \overline{\Theta_L(\tau, \boldsymbol{w}, p_\otimes)} \right\rangle = \left\langle R_{k-2j}^j(f_{P\oplus N})(\tau), \overline{\Theta_{P\oplus N}(\tau, \boldsymbol{w}, p_\otimes)} \right\rangle.$$

Next, we use that the raising and lowering operator are adjoint to each other (see [Bru02, Lemma 4.2]), which gives

$$\Psi_j^{\mathrm{reg}}(f, \boldsymbol{w}) = \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle f_{P\oplus N}(\tau), L_k^{j-1}\left(\overline{\Theta_{P\oplus N}(\tau, \boldsymbol{w}, p_\otimes)} \right) \right\rangle v^{k-2} \mathrm{d}\tau.$$

We observe that the boundary terms disappear in the same fashion as during the proof of [Bru02, Lemma 4.4]. Next, we rewrite

$$\Psi_j^{\mathrm{reg}}(f, \boldsymbol{w}) = (-1)^j \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle f_{P\oplus N}(\tau), R_{-k}^j\left(\overline{\Theta_{P\oplus N}(\tau, \boldsymbol{w}, p_\otimes)} v^k \right) \right\rangle v^{-2} \mathrm{d}\tau,$$

and recall that

$$\Theta_{P\oplus N}(\tau, \boldsymbol{w}, p_\otimes) = \Theta_P(\tau, p_r) \otimes v^{\frac{s}{2}+d^-} \overline{\Theta_{N^-}(\tau, p_s)} = v^{\frac{s}{2}+d^-} \Theta_P(\tau, p_r) \otimes \overline{\Theta_{N^-}(\tau, p_s)}.$$

Consequently, we obtain

$$R_{-k}^j \left(\overline{\Theta_{P \oplus N}(\tau, \boldsymbol{w}, p_\otimes)} v^k \right) = R_{-k}^j \left(v^{k+\frac{s}{2}+d^-} \overline{\Theta_P(\tau, p_r)} \otimes \Theta_{N^-}(\tau, p_s) \right)$$
$$= v^{k+\frac{s}{2}+d^-} \overline{\Theta_P(\tau, p_r)} \otimes \left(R_{\frac{s}{2}+d^-}^j (\Theta_{N^-})(\tau, p_s) \right),$$

by Lemma VIII.2.7. In particular we note that $v^{k+\frac{s}{2}+d^-} \overline{\Theta_P(\tau, p_r)}$ has weight $-k-\frac{s}{2}-d^- = -\frac{r}{2}-d^+$.

We choose a preimage $\mathcal{G}_P$ of $\Theta_P(\tau, p_r)$ under $\xi_{2-(\frac{r}{2}+d^+)}$, namely

$$\Theta_P(\tau, p_r) = \xi_{2-\frac{r}{2}-d^+} (\mathcal{G}_P)(\tau) = v^{-\frac{r}{2}-d^+} \overline{L_{2-\frac{r}{2}-d^+} (\mathcal{G}_P)(\tau)},$$

which yields

$$R_{-k}^j \left(\overline{\Theta_{P \oplus N}(\tau, \boldsymbol{w}, p_\otimes)} v^k \right) = L_{2-\frac{r}{2}-d^+} (\mathcal{G}_P)(\tau) \otimes R_{\frac{s}{2}+d^-}^j (\Theta_{N^-})(\tau, p_s).$$

We apply the computation of the Rankin–Cohen brackets given in Lemma VIII.2.6 noting that $L_\ell(\Theta_{N^-}) = 0$, and that it suffices to deal with the holomorphic part $\mathcal{G}_P^+$ of $\mathcal{G}_P$ (both by virtue of holomorphicity in computing the Rankin–Cohen bracket). Thus,

$$R_{-k}^j \left(\overline{\Theta_{P \oplus N}(\tau, \boldsymbol{w}, p_\otimes)} v^k \right)$$
$$= \frac{j!(-4\pi)^j \Gamma(2-k)}{\Gamma(2-k+j)} v^{-\frac{s}{2}-d^-} L_{2-k+\frac{s}{2}+d^-+2j} \left(\left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau, p_s) \right]_j \right).$$

Hence, the theta lift becomes

$$\Psi_j^{\mathrm{reg}}(f, \boldsymbol{w})$$
$$= \frac{j!(4\pi)^j \Gamma\left(2-\frac{r}{2}-d^+\right)}{\Gamma\left(2-\frac{r}{2}-d^++j\right)} \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle f_{P \oplus N}(\tau), L_{2-k+2j} \left(\left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau, p_s) \right]_j \right) \right\rangle v^{-2} \mathrm{d}\tau.$$

The last step is to apply Stokes' Theorem, compare the proof of [Bru02, Lemma 4.2] for example, which yields

$$\Psi_j^{\mathrm{reg}}(f, \boldsymbol{w})$$
$$= \frac{j!(4\pi)^j \Gamma\left(2-\frac{r}{2}-d^+\right)}{\Gamma\left(2-\frac{r}{2}-d^++j\right)} \left(\lim_{T \to \infty} \int_{iT}^{1+iT} \left\langle f_{P \oplus N}(\tau), \left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau, p_s) \right]_j \right\rangle v^{-2} d\tau \right.$$
$$\left. - \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle L_{k-2j}(f_{P \oplus N})(\tau), \left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau, p_s) \right]_j \right\rangle v^{-2} \mathrm{d}\tau \right),$$

utilizing again that boundary terms vanish. We observe that the left integral can be regarded as the Fourier coefficient of index 0 in the Fourier expansion of the integrand, see the bottom of page 14 in [BS21]. This proves the claim. $\qquad \square$

We end this section by noting that to obtain recurrence relations, as in [BS21], one would need to compute the Fourier expansion of the lift. In general, this is a lengthy but straightforward process, and since we do not require it in this chapter we omit the details. In essence, one follows the calculations of Borcherds [Bor98] by using Lemma VIII.2.4. A resulting technicality is to then take care of the different spectral parameter. One may overcome this by relating the coefficients of Maaß–Poincaré series to those with other spectral parameters, again using the action of the iterated Maaß raising operator as in Lemma VIII.2.4.

VIII.3.3   Proof of Theorem VIII.1.1

We now prove a refined version of Theorem VIII.1.1. To this end, we define the function

$$
\Lambda_L(\psi, p_\otimes, j) := \frac{(4\pi)^{1 - \frac{r}{2} - d^+} \Gamma\left(\frac{s}{2} + j + d^-\right) \Gamma\left(2 - \frac{r}{2} - d^+ + j\right)}{4\Gamma(2 - k + 2j)\Gamma\left(2 - \frac{r}{2} - d^+\right)}
$$
$$
\times \sum_{\substack{m \geq 1 \\ \boldsymbol{\lambda} \in L' \\ Q(\boldsymbol{\lambda}) = -m}} \overline{p_\otimes(\psi(\boldsymbol{\lambda}))} \frac{|Q(\boldsymbol{\lambda})|^{2j+1-k}}{|Q(\boldsymbol{\lambda}_{\boldsymbol{z}^\perp})|^{\frac{s}{2} + j + d^-}} {}_2F_1\left(1 + j, \frac{s}{2} + j + d^-; 2 - k + 2j; \frac{Q(\boldsymbol{\lambda})}{Q(\boldsymbol{\lambda}_{\boldsymbol{z}^\perp})}\right) q^m
$$

for $j > 0$. We write

$$
\mathcal{G}_P^+(\tau) = \sum_{\mu \in L'/L} \sum_{n \gg -\infty} a(n) q^n \mathfrak{e}_\mu,
$$

and furthermore define

$$
\mathscr{G}_P^+(\tau) := \mathcal{G}_P^+(\tau) - \sum_{\mu \in L'/L} \sum_{n < 0} a(n) \mathbb{F}_{\mu, n-1, 2j+2-k}(\tau).
$$

Since one may add any weakly holomorphic modular form of appropriate weight for ρ_L to $\mathcal{G}_P^+$, Theorem VIII.1.1 follows directly from the following result (noting that the linear combination of Maaß–Poincaré series may change).

Theorem VIII.3.4. *Let L be an even lattice of signature (r, s), let p be as before, and w be a special point defined by the isometry ψ. Let $j > 0$ and k be such that $2j + 2 - k > 2$. Then the function*

$$
\left[\mathscr{G}_P^+(\tau), \Theta_{N^-}(\tau, p_s)\right]_j^L - \Lambda_L(\psi, p_\otimes, j)
$$

is a holomorphic vector-valued modular form of weight $2j + 2 - k$ for ρ_L.

Remarks.

(1) This provides the general vector-valued analogue, assuming that the lattice is chosen such that $2j + 2 - k > 2$, of Mertens' scalar-valued results in weight $\frac{1}{2}$ and $\frac{3}{2}$ [Mer16].

(2) Note that the slight correction of $\mathcal{G}_P^+$ by Poincaré series was missing in [Mal22].

(3) In certain cases the hypergeometric function may be simplified (for example, the $n = 1$ case as in [BS21, Mal22], which yields a form very similar to Mertens' scalar-valued result). It appears to be possible that one should be able to prove the same results via holomorphic projection acting on vector-valued modular forms (see [IRR14]) in much the same way as Mertens' original scalar-valued proofs in [Mer16].

Proof of Theorem VIII.3.4. Let f be a weakly holomorphic form of weight $k - 2j$ with Fourier coefficients c_f^+. By construction, the form $\mathcal{G}_P^+$ is holomorphic at $i\infty$, and hence

$$\mathrm{CT}\left(\left\langle f_{P \oplus N}(\tau), \left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau, p_s)\right]_j^L \right\rangle\right)$$

contains only the Fourier coefficients of non-positive index of f. We note that $L_{k-2j}(f) = 0$, and subtract the resulting expressions of the lift from Corollary VIII.3.2 and Theorem VIII.3.3. We obtain

$$0 = \mathrm{CT}\left(\left\langle f_{P \oplus N}(\tau), \left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau, p_s)\right]_j^L \right\rangle\right)$$

$$- \frac{(4\pi)^{1-\frac{r}{2}-d^+}\Gamma\left(\frac{s}{2}+j+d^-\right)\Gamma\left(2-\frac{r}{2}-d^++j\right)}{4\Gamma(2-k+2j)\Gamma\left(2-\frac{r}{2}-d^+\right)} \sum_{\substack{m \geq 1 \\ \boldsymbol{\lambda} \in L' \\ Q(\boldsymbol{\lambda})=-m}} c_f^+(\boldsymbol{\lambda}, -m)\overline{p_\otimes(\psi(\boldsymbol{\lambda}))}$$

$$\times \frac{|Q(\boldsymbol{\lambda})|^{2j+1-k}}{|Q(\boldsymbol{\lambda}_{\boldsymbol{z}^\perp})|^{\frac{s}{2}+j+d^-}} {}_2F_1\left(1+j, \frac{s}{2}+j+d^-; 2-k+2j; \frac{Q(\boldsymbol{\lambda})}{Q(\boldsymbol{\lambda}_{\boldsymbol{z}^\perp})}\right).$$

The Rankin–Cohen bracket is bilinear, and a linear combination of vector-valued Poincaré series is modular itself. We apply Proposition VIII.2.11 and the claim follows. $\square$

In a similar way to [Mer16, Corollary 5.4], we obtain the following structural corollary by rewriting Theorem VIII.3.4, keeping the same notation as throughout this chapter.

Corollary VIII.3.5. *Let θ denote the space generated by all Θ_{N^-} functions of weight $\frac{s}{2} + d^-$ for ρ_{N^-}. Then the equivalence classes $\Lambda_L(\psi, p_\otimes, j) + M_{2j+2-k,L}^!$ generate the $\mathbb{C}$-vector space*

$$\left[\mathcal{M}_{2j+2-k,P}^{\mathrm{mock}}, \theta\right]_j^L \Big/ {}_{M_{2j+2-k,L}^!}.$$

VIII.4 The action of the Laplace–Beltrami operator

In this section, we prove Theorem VIII.1.2. To this end, we compute the action of the Laplace–Beltrami operator on the lift, and show that for certain spectral parameters, we obtain a local weak Maaß form. We recall that the signature of L is assumed to be $(2, s)$ here. Moreover, we observe that our Siegel theta function Θ_L and the Siegel theta function inspected by Bruinier depend in the same way on $\boldsymbol{Z}$, and thus the following result applies.

Proposition VIII.4.1 ([Bru02, Proposition 4.5]). *The Siegel theta function $\Theta_L(\tau, \boldsymbol{Z}, p_\otimes)$ considered as a function on $\mathbb{H} \times \mathbb{H}_\ell$ satisfies the differential equation*

$$\Omega\left(\Theta_L(\tau, \boldsymbol{Z}, p_\otimes)v^{\frac{\ell}{2}}\right) = -\frac{1}{2}\Delta_k\left(\Theta_L(\tau, \boldsymbol{Z}, p_\otimes)v^{\frac{\ell}{2}}\right).$$

Our next step is to inspect the action of Ω on our theta lift. By Lemma VIII.2.3 it suffices to investigate

$$\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{Z}) = \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle R_{k-2j}^j(F_{m,k-2j,\mathfrak{s}})(\tau), \overline{\Theta_L(\tau, \boldsymbol{Z}, p_\otimes)} \right\rangle v^k \mathrm{d}\mu(\tau).$$

Let

$$H(m) := \bigcup_{\mu \in L'/L} \bigcup_{\substack{\boldsymbol{\lambda} \in \mu + L \\ Q(\boldsymbol{\lambda}) = -m}} \boldsymbol{\lambda}^\perp \subseteq \mathrm{Gr}(L),$$

which collects the singularities of $\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{Z})$ as a function of $\boldsymbol{Z}$. We apply the previous proposition to our theta lift, which yields a variant of [Bru02, Theorem 4.6].

Theorem VIII.4.2. *Let $\boldsymbol{Z} \in \mathbb{H}_\ell \setminus H(m)$, and $\mathrm{Re}(\mathfrak{s}) > 1 - \frac{k}{2}$. Then it holds that*

$$\Omega\left(\Psi_j^{\mathrm{reg}}\right)(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{Z}) = \left(\mathfrak{s} - \frac{k}{2}\right)\left(1 - \mathfrak{s} - \frac{k}{2}\right)\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{Z}).$$

Proof. First, we note that

$$\Omega\left(\Psi_j^{\mathrm{reg}}\right)(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{Z}) = \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle R_{k-2j}^j(F_{m,k-2j,\mathfrak{s}})(\tau), \Omega\left(\overline{\Theta_L(\tau, \boldsymbol{Z}, p_\otimes)}v^{\frac{\ell}{2}}\right) \right\rangle v^{k-\frac{\ell}{2}} \mathrm{d}\mu(\tau),$$

because all partial derivatives with respect to $\boldsymbol{Z}$ converge locally uniformly in $\boldsymbol{Z}$ as $T \to \infty$ (see [Bru02, p. 99]). By the previous proposition, we infer that

$$\begin{aligned}
&\Omega\left(\Psi_j^{\mathrm{reg}}\right)(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{Z}) \\
&\qquad = -\frac{1}{2}\int_{\mathcal{F}}^{\mathrm{reg}} \left\langle R_{k-2j}^j(F_{m,k-2j,\mathfrak{s}})(\tau), \Delta_k\left(\overline{\Theta_L(\tau, \boldsymbol{Z}, p_\otimes)}v^{\frac{\ell}{2}}\right) \right\rangle v^{k-\frac{\ell}{2}} \mathrm{d}\mu(\tau).
\end{aligned}$$

By the splitting $\Delta_k = R_{k-2}L_k$, and the adjointness of both operators (see [Bru02, Lemmas 4.2 to 4.4]), we obtain

$$\Omega\left(\Psi_j^{\text{reg}}\right)(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{Z})$$
$$= -\frac{1}{2}\int_{\mathcal{F}}^{\text{reg}}\left\langle \Delta_k\left(R_{k-2j}^j(F_{m,k-2j,\mathfrak{s}})\right)(\tau), \overline{\Theta_L(\tau, \boldsymbol{Z}, p_\otimes)}v^{\frac{\ell}{2}}\right\rangle v^{k-\frac{\ell}{2}}\,\mathrm{d}\mu(\tau).$$

Lastly, we observe that Δ_k and R_{k-2j}^j commute by virtue of Lemma VIII.2.4. Namely, we have

$$\Delta_k\left(R_{k-2j}^j(F_{m,k-2j,\mathfrak{s}})\right)(\tau) = \left(\mathfrak{s} - \frac{k}{2}\right)\left(1 - \mathfrak{s} - \frac{k}{2}\right)R_{k-2j}^j(F_{m,k-2j,\mathfrak{s}})(\tau),$$

and this establishes the claim by rewriting

$$\left\langle R_{k-2j}^j(F_{m,k-2j,\mathfrak{s}})(\tau), \overline{\Theta_L(\tau, \boldsymbol{Z}, p_\otimes)}v^{\frac{\ell}{2}}\right\rangle v^{k-\frac{\ell}{2}} = \left\langle R_{k-2j}^j(F_{m,k-2j,\mathfrak{s}})(\tau), \overline{\Theta_L(\tau, \boldsymbol{Z}, p_\otimes)}\right\rangle v^k$$

again. $\qquad\square$

We end this section by proving Theorem VIII.1.2.

Proof of Theorem VIII.1.2. By Theorem VIII.4.2, the lift is an eigenfunction of the Laplace Beltrami operator with the quoted eigenvalue. Since $\Psi_j^{\text{reg}}(F_{m,k-2j,\mathfrak{s}}, \boldsymbol{Z})$ is an eigenfunction of an elliptic differential operator, it is real-analytic in $\mathrm{Gr}(L)$ outside of $H(m)$. The other conditions for the lift to be a vector-valued local weak Maaß form can be easily seen by applying the proof of [BKV13, Theorem 1.1] *mutatis mutandis*. When $\mathfrak{s} = \frac{k}{2}$ or $\mathfrak{s} = \frac{k}{2} - 1$ we obtain locally harmonic Maaß forms. $\qquad\square$

VIII.5 Cohen–Eisenstein series

VIII.5.1 Proof of Theorem VIII.1.3

We specialize the framework from Section VIII.2 following [BS21, Section 4.4] (or [Sch18, Section 2.2]). We fix the signature $(1, 2)$ and the rational quadratic space

$$V := \left\{X = \left(\begin{smallmatrix} x_2 & x_1 \\ x_3 & -x_2 \end{smallmatrix}\right) \in \mathbb{Q}^{2\times 2}\right\},$$

with quadratic form $Q(X) = \det(X)$. The Grassmannian of positive lines in $V \otimes \mathbb{R}$ can be identified with $\mathbb{H}$ via

$$\lambda(x + iy) = \frac{1}{\sqrt{2}y}\left(\begin{smallmatrix} -x & x^2+y^2 \\ -1 & x \end{smallmatrix}\right).$$

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We choose the lattice

$$L := \left\{ \begin{pmatrix} b & c \\ -a & -b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\},$$

with dual lattice

$$L' = \left\{ \begin{pmatrix} \frac{b}{2} & c \\ -a & -\frac{b}{2} \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

We observe that L' can be identified with the set of integral binary quadratic forms of discriminant $\det \begin{pmatrix} \frac{b}{2} & c \\ -a & -\frac{b}{2} \end{pmatrix} = -\frac{1}{4}(b^2 - 4ac)$. Furthermore, $L'/L \cong \mathbb{Z}/2\mathbb{Z}$ with quadratic form $x \mapsto -\frac{1}{4}x^2$.

According to [BS21, p. 22], it holds that

$$Q\left(\begin{pmatrix} \frac{b}{2} & c \\ -a & -\frac{b}{2} \end{pmatrix}_{x+iy} \right) = \frac{1}{4y^2} \left(a(x^2 + y^2) + bx + c \right)^2,$$

$$Q\left(\begin{pmatrix} \frac{b}{2} & c \\ -a & -\frac{b}{2} \end{pmatrix}_{(x+iy)^\perp} \right) = -\frac{1}{4y^2} \left| [a, b, c](x + iy, 1) \right|^2.$$

We remark that both are invariant under modular substutions. By a result from Eichler and Zagier [EZ85, Theorem 5.4], the space of vector-valued modular forms of weight k for ρ_L is isomorphic to the space $M_k^+(\Gamma_0(4))$ of scalar-valued modular forms satisfying the Kohnen plus space condition via the map

$$f_0(\tau)\mathfrak{e}_0 + f_1(\tau)\mathfrak{e}_1 \mapsto f_0(4\tau) + f_1(4\tau).$$

This enables us to use scalar-valued forms as inputs for our theta lift.

Proof of Theorem VIII.1.3. As outlined between Theorems VIII.1.2 and VIII.1.3, the function $f := f_{-2\ell, N}\mathcal{H}_\ell$ is of weight $-\ell - \frac{1}{2} < 0$ for $\Gamma_0(4)$, has non-constant principal part at the cusp $i\infty$, and its image under ξ is trivial, hence in particular cuspidal. This enables us to apply Corollary VIII.3.2 to f. To this end, we have the parameters

$$k = -\frac{1}{2} + d^+ + d^-, \qquad k - 2j = -\ell - \frac{1}{2}, \qquad j = \frac{\ell + d^+ + d^-}{2},$$

and the hypergeometric function from Theorem VIII.3.1 becomes

$$_2F_1\left(\frac{\ell + 2 + d^+ + d^-}{2}, \frac{\ell + 2 + d^+ + 3d^-}{2}, \frac{5}{2} + \ell, \frac{4my^2}{|[a, b, c](z, 1)|^2} \right).$$

Inspecting the parameters, we have the condition $\ell + d^+ + d^- \in 2\mathbb{N}$ by $j \in \mathbb{N}$, and combining with d^+, $d^- \in \mathbb{N}_0$, $\ell \in \mathbb{N} \setminus \{1\}$, the smallest possible values are $(\ell, d^+, d^-) = (2, 0, 0)$, $(2, 2, 0)$, $(2, 1, 1)$, $(2, 0, 2)$. For example, the corresponding hypergeometric functions for the cases $(\ell, d^+, d^-) = (2, 0, 0)$, $(2, 1, 1)$ are

$$_2F_1\left(2, 2, \frac{9}{2}, z\right) = -\frac{35(11z - 15)}{12z^3} - \frac{35\left(2z^2 - 7z + 5\right)\arcsin\left(\sqrt{z}\right)}{4z^{\frac{7}{2}}\sqrt{1-z}},$$

$$_2F_1\left(3, 4, \frac{9}{2}, z\right) = -\frac{35(8z^2 - 26z + 15)}{128z^3(z-1)^2} + \frac{105\left(8z^2 - 12z + 5\right)\arcsin\left(\sqrt{z}\right)}{128z^{\frac{7}{2}}\sqrt{1-z}(z-1)^2},$$

and the other cases are of similar shape. Analogous expressions can be obtained for higher integer parameters via Gauß' contiguous relations for the hypergeometric function, which can be found in [BCLO10, § 15.5 (ii)] for instance.

We infer a local behaviour as sketched between Theorems VIII.1.1 and VIII.1.2 by virtue of $(4m = D = b^2 - 4ac)$

$$\arcsin\left(\frac{\sqrt{D}y}{|az^2 + bz + c|}\right) = \arctan\left|\frac{\sqrt{D}y}{a|z|^2 + bx + c}\right|,$$

which in turn follows by

$$\left(b^2 - 4ac\right)y^2 + \left(a|z|^2 + bx + c\right)^2 = \left|az^2 + bz + c\right|^2,$$

compare [BKK15, Section 3] for both identities. The denominator $a|z|^2 + bx + c$ vanishes if and only if z is located on the Heegner geodesic associated to $Q = [a, b, c]$. Since the principal part of f is given by

$$\sum_{n=0}^{N} H(\ell, n)q^{n-N} + O\left(q^{m+1}\right), \qquad m = \begin{cases} \lfloor\frac{-2\ell}{12}\rfloor - 1 & \text{if } -2\ell \equiv 2 \,(\text{mod } 12), \\ \lfloor\frac{-2\ell}{12}\rfloor & \text{else}, \end{cases}$$

we conclude that f has the exceptional set

$$\bigcup_{D=1}^{N} \left\{z = x + iy \in \mathbb{H} \colon \exists a, b, c \in \mathbb{Z}, \ b^2 - 4ac = D, \ a|z|^2 + bx + c = 0\right\}.$$

In other words, the exceptional set of f is a finite union of nets of Heegner geodesics. Furthermore, we recall that the spectral parameter in Corollary VIII.3.2 is $\mathfrak{s} = 1 - \frac{k-2j}{2}$, and hence the eigenvalue under $\Delta_{-\ell-\frac{1}{2}}$ is

$$\left(\mathfrak{s} - \frac{k}{2}\right)\left(1 - \mathfrak{s} - \frac{k}{2}\right) = (1 - k + j)(-j) = j\left(j - \ell - \frac{3}{2}\right).$$

This proves the Theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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VIII.5.2 Eichler–Selberg type relations for Cohen–Eisenstein series

Eichler–Selberg type relations for Cohen–Eisenstein series could be obtained as follows. On one hand, the input function $f(\tau) = f_{-2\ell,N}(\tau)\mathcal{H}_\ell(\tau)$ is weakly holomorphic, thus we do not need to deal with the additional term

$$\int_{\mathcal{F}}^{\mathrm{reg}} \left\langle L_{k-2j}\left(f_{P\oplus N}\right)(\tau), \left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau)\right]_j \right\rangle v^{-2}\mathrm{d}\tau$$

arising from Theorem VIII.3.3. On the other hand, the function Λ_L from Section VIII.3.3 simplifies to

$$\Lambda_L(\psi, p_\otimes, j) = \frac{4^{3d^-}\pi^{\frac{1}{2}-d^+}\Gamma\left(j+1+d^-\right)\Gamma\left(\frac{3}{2}-d^++j\right)}{\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-d^+\right)} \sum_{D\geq 1}\sum_{Q\in\mathcal{Q}_D} \overline{p_\otimes(\psi(Q))}$$

$$\times \frac{D^{\ell+\frac{3}{2}}y^{2+2j+2d^-}}{|Q(z,1)|^{2+2j+2d^-}}{}_2F_1\left(\frac{\ell+2+d^++d^-}{2}, \frac{\ell+2+d^++3d^-}{2}, \frac{5}{2}+\ell, \frac{Dy^2}{|Q(z,1)|^2}\right)q^D,$$

where $\mathcal{Q}_D$ denotes the set of integral binary quadratic forms of discriminant D. After evaluating the hypergeometric function as in the previous proof, one may follow our proof of Theorem VIII.3.4, namely subtract the two evaluations of $\Psi_j^{\mathrm{reg}}(f,z)$ from each other, and apply Serre duality to the resulting expression. Computing the principal part of $\mathcal{G}_P^+$ in addition, this yields the desired result. However, we do not pursue this here explicitly as the resulting expression is rather lengthy.

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Erklärung zur Dissertation

(Gemäß §7 Absatz (8) Satz 1 der Promotionsordnung vom 12. März 2020)

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne die Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Werken dem Wortlaut oder dem Sinn nach entnommen wurden, sind als solche kenntlich gemacht. Ich versichere an Eides statt, dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen und eingebundenen Artikeln und Manuskripten – noch nicht veröffentlicht worden ist sowie, dass ich eine Veröffentlichung der Dissertation vor Abschluss der Promotion nicht ohne Genehmigung des Promotionsausschusses vornehmen werde. Die Bestimmungen dieser Ordnung sind mir bekannt. Darüber hinaus erkläre ich hiermit, dass ich die Ordnung zur Sicherung guter wissenschaftlicher Praxis und zum Umgang mit wissenschaftlichem Fehlverhalten der Universität zu Köln gelesen und sie bei der Durchführung der Dissertation zugrundeliegenden Arbeiten und der schriftlich verfassten Dissertation beachtet habe und verpflichte mich hiermit, die dort genannten Vorgaben bei allen wissenschaftlichen Tätigkeiten zu beachten und umzusetzen. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.

Teilpublikationen:

1. J. Males, A. Mono, and L. Rolen, *Polar harmonic Maaß forms and holomorphic projection*, Int. J. Number Theory **18** (2022), no. 9, 1975–2004
2. A. Mono, *Multidimensional small divisor functions*, Integers **21** (2021), Paper No. A104
3. J. Males, A. Mono, and L. Rolen, *Higher depth mock theta functions and q-hypergeometric series*, Forum Math. **33** (2021), no. 4, 857—866
4. A. Mono, *Eisenstein series of even weight $k \geq 2$ and integral binary quadratic forms*, Proc. Amer. Math. Soc. **150** (2022) no. 5, 1889–1902
5. A. Mono, *Locally harmonic Maaß forms of positive even weight* (2022), `http://arxiv.org/abs/2104.03127`. Accepted for publication in Israel J. Math.
6. K. Bringmann, and A. Mono, *A modular framework of functions of Knopp and indefinite binary quadratic forms*, preprint (2022), `http://arxiv.org/abs/2208.01451`. Submitted for publication

7. J. Males, and A. Mono, *Eichler–Selberg type relations for negative weight vector-valued mock modular forms*, preprint (2021), `http://arxiv.org/abs/2108.13198`. Recommended for publication in Pacific J. Math.

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General

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Education

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03/2018–08/2019	**Master of Science ETH in Mathematics (with distinction)**, *ETH Zurich*, Switzerland Advisor: Özlem Imamoḡlu. Thesis titled *Spectral Theory of Automorphic Forms on the Hyperbolic Plane.*
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06/2013	**Abitur (with distinction)**, *Musterschule*, Frankfurt am Main, Germany

Employment

02/2020–present	**Scientific assistant**, *University of Cologne*, Germany
06/2016–01/2019	**Teaching assistant**, *ETH Zurich*
08/2013–08/2014	**Federal Volunteers Service**, *Arbeiterwohlfahrt Langen (Hessen)*, Germany

Research interests

My field is analytic number theory with a focus on automorphic forms and objects. Recently, I studied various types of Maaß forms (for instance local Maaß forms), and related topics such as central L-values.

Publications

Peer reviewed publications

2021   A. Mono, *Locally harmonic Maaß forms of positive even weight*, accepted for publication in Israel J. Math.
arxiv.org:2104.03127

2021   J. Males, A. Mono, and L. Rolen, *Higher depth mock theta functions and q-hypergeometric series*, Forum Math. **33** (2021), no. 4, 857—866

2020   A. Mono, *Eisenstein series of even weight $k \geq 2$ and integral binary quadratic forms*, Proc. Amer. Math. Soc. **150** (2022) no. 5, 1889–1902

2020   A. Mono, *Multidimensional small divisor functions*, Integers **21** (2021), Paper No. A104

2020   J. Males, A. Mono, and L. Rolen, *Polar harmonic Maaß forms and holomorphic projection*, Int. J. Number Theory **18** (2022), no. 9, 1975–2004

Preprints

2023   A. Mono, and B. V. Pandey, *Linear congruence relations for exponents of Borcherds products*, submitted for publication
arxiv.org:2301.11184

2022   K. Bringmann, and A. Mono, *A modular framework of functions of Knopp and indefinite binary quadratic forms*, submitted for publication
arxiv.org:2208.01451

2021   J. Males, and A. Mono, *Eichler–Selberg type relations for negative weight vector-valued mock modular forms*, recommended for publication in Pacific J. Math.
arxiv.org:2108.13198

2020   A. Mono, *On a conjecture of Kaneko and Koike*
arxiv.org:2005.06882

Professional Service

10/2022   **Co-organisator**, *Early Number Theory Researchers (ENTR) Workshop*, Joint workshop by TU Darmstadt, University of Bielefeld and University of Cologne
Link to webpage

04/2022–
07/2022   **Co-organisator**, *Early Number Theory Researchers (ENTR) Seminar*, Joint online seminar by TU Darmstadt and University of Cologne
Link to webpage

10/2021–
02/2023   **Mentor**, *University of Cologne*
Individual and periodical supervision of a very talented teenager in Analysis I, II and III including exam preparation for each course.

2020–present	**Referee**
	○ Archiv der Mathematik (two times)
	○ Research in the Mathematical Sciences
	○ Research in Number Theory (three times)
	○ The Ramanujan Journal
03/2021–04/2021	**Course instructor**, *Junior Uni*, University of Cologne
	Outreach-style programme on integer partitions for 30 schoolchildren joint with Joshua Males.

Grants and Awards

03/2022	**Talk award**, 34*th Automorphic Forms Workshop*, Brigham Young University, Provo, Utah, USA
	Awarded a small amount of prize money for an outstanding talk (see list of talks below).
11/2020	**Outreach grant**, *Junior Uni*, University of Cologne
	A small grant awarded to run an outreach programme on integer partitions for 30 schoolchildren joint with Joshua Males during spring 2021.
12/2017	**Teaching assistant award**, *Mathematicians and Physicists Association*, ETH Zurich
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Co-Advising

Masters

| 2020 | Lukas Gebert, *Nonholomorphic Eisenstein series* |

Bachelors

| 2022–2023 | Jakob Sauer, *Zagier's weight 3/2 mock modular Eisenstein series* |
| 2021 | Hei Loi Lin, *Supercongruences* |

Teaching

2021	**Lecturer, Coordinator, and Teaching assistant**, *University of Cologne*
	○ Spring semester 2021: Linear Algebra II (undergraduate course)
2020–2022	**Coordinator and Teaching assistant**, *University of Cologne*
	○ Fall semester 2022: Theta functions (undergraduate and graduate seminar)
	○ Spring semester 2022: Elementary Number Theory (undergraduate and graduate course)
	○ Fall semester 2021: Algebra (undergraduate and graduate course)
	○ Fall semester 2020: Linear Algebra I (undergraduate course)

- ○ Spring semester 2020: Modular forms (undergraduate and graduate seminar)

2020, 2022 **Teaching assistant**, *University of Cologne*
  - ○ Fall semester 2022: Algebra (undergraduate and graduate course)
  - ○ Spring semester 2020: Complex Analysis (undergraduate and graduate course)

2016–2019 **Teaching assistant**, *ETH Zurich*
  - ○ Fall semester 2018: Complex Analysis (undergraduate course)
  - ○ Spring semester 2018: Measure and Integration (undergraduate course)
  - ○ Fall semester 2017: Complex Analysis (undergraduate course)
  - ○ Spring semester 2017: Analysis II (undergraduate course)
  - ○ Fall semester 2016: Analysis I (undergraduate course)

2016 **Course instructor**, *ETH Zurich*
  Exam preparation course in Analysis I and Analysis II (undergraduate courses)

Talks

09/2022 Ramanujan-Serre Seminar, *In person talk*, University of Virginia
Link to details

07/2022 Ramanujan and Euler: Partitions, mock theta functions, and *q*-series, *Online conference and summer school*
Link to details

05/2022 100 Years of Mock Theta Functions, *Speedtalk*, *In person conference by Vanderbilt University*, Nashville, Tennessee, USA
Link to details

03/2022 Young Scholars in the Analytic Theory of Numbers and Automorphic Forms (Fantasy), *In person conference by University of Bonn*, Germany
Link to details

03/2022 34th Automorphic Forms Workshop, *Online workshop by Brigham Young University*, Provo, Utah, USA
Link to details

11/2021 Japan Europe Number Theory Exchange seminar (JENTE seminar), *Joint online seminar by Nagoya University and University of Copenhagen*
Link to details

06/2021 CDE seminar, *Early career online seminar on automorphic forms joining Cologne, Darmstadt, and ETH Zürich*

04/2021 International Seminar on Automorphic Forms, *Joint online seminar by ETH Zurich and TU Darmstadt*
Link to details

02/2021	Number Theory Lunch Seminar, *Max Planck Institute for Mathematics*, Bonn Link to details
02/2021	Oberseminar Geometrische Analysis und Zahlentheorie, *University of Paderborn* Link to details
01/2021	PhD Seminar Geometrische Analysis und Zahlentheorie, *University of Paderborn* Link to details

Conferences, Workshops, Research Visits

09/2022	University of Virginia, Charlottesville, Virginia, USA, research visit
08/2022	Vanderbilt University, Nashville, Tennessee, USA, research visit
06/2022	Universities of Aachen, Bonn, Köln, Lille, Siegen, 59*th ABKLS seminar*, Conference Link to details
05/2022	Vanderbilt University, 100 *Years of Mock Theta Functions*, Nashville, Tennessee, USA, Conference Link to details
03/2022	University of Bonn, *Young Scholars in the Analytic Theory of Numbers and Automorphic Forms (Fantasy)*, Conference Link to details
03/2022	Brigham Young University, 34*th Automorphic Forms Workshop* Link to details
10/2021	Politecnico di Torino and Università degli Studi di Torino, 5*th number theory meeting*, Conference Link to details
05/2021	Isaac Newton Institute for Mathematical Sciences, *New connections in number theory and physics*, Workshop Link to details

Skills

Languages	○ German (native)
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Programming	○ C++ (basic)
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Cologne, January 30, 2023