The symplectic plactic monoid, words, MV cycles, and non-Levi branchings

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Para Mamá, Papá y Camila

Abstract

We study cells in generalised Bott-Samelson varieties for types A_n and C_n. These cells are parametrised by certain galleries in the affine building. In type C_n we define a set of *readable galleries* - we show that the closure in the affine Grassmannian of the image of the cell associated to a gallery in this set is an MV cycle. This then defines a map from the set of readable galleries to the set of MV cycles, which we show to be a morphism of crystals. In type A_n we show that the existing map between all galleries and MV cycles is a crystal morphism. This builds on results of Gaussent-Littelmann, Baumann-Gaussent, and Gaussent-Nguyen-Littelmann. In the last chapter we prove for some cases a conjecture of Naito-Sagaki on the branching of representations from $\mathfrak{sl}(2n, \mathbb{C})$ to $\mathfrak{sp}(2n, \mathbb{C})$.

Kurzusammenfassung

In dieser Arbeit untersuchen wir Zellen in verallgemeinerten Bott-Samelson Varietäten in Typ A_n und C_n. Diese Zellen werden von bestimmten Galerien im affinem Gebäude parametrisiert. Für den Typ C_n definieren wir eine Menge von *lesbaren* Galerien - wir zeigen, dass der Abschluss des Bildes einer Zelle assoziiert zu eine Galerie aus diese Menge einen MV Zykel in der affiner Grassmanschen bildet. Auf diese Weise bekommen wir eine Abbildung von der Menge der lesbaren Galerien in die Menge der MV Zykel und beweisen, dass diese Abbildung ein Morphsiums von Kristallen ist. Im Typ A_n zeigen wir, dass die bereits bekannte Abbildung von allen Galerien in die MV Zykel auch ein Kristallmorphismus ist. Dies baut auf Resultate von Gaussent-Littelmann, Baumann-Gaussent und Gaussent-Nguyen-Littelmann auf. Im letzten Kapitel beweisen wir in gewissen Spezialfällen eine Vermutung von Naito-Sagaki über das Zerlegungsverhalten von Einschränkungen von $\mathfrak{sl}(2n, \mathbb{C})$-Darstellungen auf $\mathfrak{sp}(2n, \mathbb{C})$.

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Köln, August 2015 — Jacinta Perez Gavilan Torres

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Introduction

Summary and main results

There is a beautiful description of Bott-Samelson resolutions of singularities of Schubert varieties as sets of galleries of a certain fixed type in the associated dual building, originally introduced by Contou-Carrere in [**3**] for buildings of spherical type, and generalised to buildings of affine type by Gaussent-Littelmann in [**6**]. Generalised Schubert varieties are parametrised by dominant weights, and the type associated to a Bott-Samelson resolution corresponds to a piecewise linear path in the weight lattice contained in the fundamental alcove. There is a natural action of root operators on the set of galleries given by Littelmann, originally for paths, in [**20**]. The induced crystal structure gives rise to the concept of plactic equivalence. Two galleries are *plactic equivalent* if and only if there is a crystal isomorphism between the connected components in which they lie that maps one of them to the other. The *plactic monoid* ([**21**]) is defined to be the set of equivalence classes with multiplication given by concatenation.

For classical groups of types A_n, B_n and C_n, a type in the above sense corresponds to the shape of a key - a class of combinatorial objects that generalise Young tableaux in the sense that their shape is not necessarily the shape of a partition. For type C_n, Lecouvey has described the corresponding 'symplectic plactic monoid' in an explicit way ([**18**]). In this case we call galleries 'symplectic keys,' and we define a certain subset of these, which we call *readable* keys, and which are discussed below. This is relevant to this thesis in the following setting.

Let G be a complex reductive group and $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$ a choice of maximal torus T and Borel subgroup B of G containing it. Let λ be a dominant integral coweight with respect to this choice, and let $\mathrm{X}_\lambda \subset \mathcal{G}$ be the corresponding generalised Schubert variety contained in the affine Grassmannian $\mathcal{G}$ associated to G. Let $\mathrm{G}^\vee$ be the group that is Langlands dual to G. Given a type $\underline{i}$ denote by $\Sigma_{\underline{i}} \xrightarrow{\pi_{\underline{i}}} \mathrm{X}_\lambda$ the Bott-Samelson resolution of this type. There is a one-to-one correspondence between T-fixed points in $\Sigma_{\underline{i}}$ and galleries of type $\underline{i}$. There is an induced $\mathbb{C}^\times$-action on the variety $\Sigma_{\underline{i}}$ with the same fixed points as the T-action, which allows one to consider a Białynicki-Birula cell C_η for any gallery η of type $\underline{i}$. The following theorem is the main result that we show in this thesis.

Theorem A. ([**29**], cf. Proposition 3.10, Proposition 3.24, Theorem 3.26 in Chapter 3) Let $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$, and let γ and ν be two plactic equivalent readable keys of types $\underline{i}_\gamma$ and $\underline{i}_\nu$ respectively. Then $\overline{\pi_{\underline{i}_\nu}(\mathrm{C}_\nu)} = \overline{\pi_{\underline{i}_\gamma}(\mathrm{C}_\gamma)}$ as sets in the affine Grassmannian $\mathcal{G}$.

By results of Gaussent-Littelmann ([**6**]) this implies that the image closure $\overline{\pi(\mathrm{C}_\nu)}$ is always a Mirković-Vilonen cycle as long as ν is a readable key. The set of Mirković-Vilonen cycles introduced by Mirković and Vilonen in [**22**] has been given a crystal structure by Braverman and Gaitsgory in [**2**]. Using results obtained by Baumann and Gaussent in [**1**] one deduces that the map established by the above theorem is in fact compatible with this crystal structure.

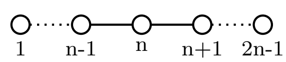
Theorem B. (cf. Theorem 3.26) Given a readable gallery ν, the image closure $\overline{\pi(\mathrm{C}_\nu)}$ is a Mirković-Vilonen cycle. Moreover, the map defined by $\nu \mapsto \overline{\pi(\mathrm{C}_\nu)}$ from the set of readable galleries to the set of all Mirković-Vilonen cycles is a surjective morphism of crystals.

For $\mathrm{G}^\vee = \mathrm{SL}(n,\mathbb{C})$ the fact that word reading is a crystal morphism for all keys together with results obtained by Gaussent-Littelmann-Nguyen in [**8**] enables one to prove the following result.

Theorem C. ([**30**], cf. Theorem 2.21) The map from the set of all keys to the set of Mirković-Vilonen cycles established in [**8**] is a surjective morphism of crystals for $\mathrm{G}^\vee = \mathrm{SL}(n,\mathbb{C})$.

We also compute the fibers of this map in terms of the Littelmann path model (in both cases). See the next section of this introduction for a more detailed description of this.

The last chapter of this thesis is almost independent from the first three. Chapter 4 should be seen as the beginning of a new project. Let us switch to Lie algebras here, and consider the complex semi-simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(2n,\mathbb{C})$. The automorphism of the Dynkin diagram



that sends node i to node $2n-i$ induces an automorphism of $\mathfrak{g}$ which has as fixed point set a Lie sub algebra $\hat{\mathfrak{g}}$ isomorphic to $\mathfrak{sp}(2n,\mathbb{C})$. If $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra of diagonal matrices, then $\hat{\mathfrak{h}} = \mathfrak{h} \cap \hat{\mathfrak{g}}$ is a Cartan sub algebra of $\hat{\mathfrak{g}}$. Given a path π we may define a new path $\mathrm{res}(\pi)$ which may or may not be dominant for $\hat{\mathfrak{g}}$. In this Chapter we show the following conjecture by Naito-Sagaki [**23**], for $n = 2$ and for $\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$ and arbitrary n. To do so we use combinatorics of Littlewood-Richardson and Sundaram symplectic tableaux.

Conjecture. Let λ be a dominant integral weight and $\mathrm{L}(\lambda)$ the associated irreducible representation of $\mathfrak{sl}(n,\mathbb{C})$. Let $\mathrm{domres}(\lambda)$ be the set of restricted paths $\mathrm{res}(\pi)$ where π is the path that corresponds to a semi-standard Young tableau of shape λ, which stay dominant for $\hat{\mathfrak{g}}$. Then

$$\mathrm{res}\,^{\mathfrak{g}}_{\hat{\mathfrak{g}}}(\mathrm{L}(\lambda)) = \bigoplus_{\delta \in \mathrm{domres}(\lambda)} \mathrm{L}(\mu_\delta).$$

Theorem D. The above conjecture is true for $n = 2$ and for $\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$ and arbitrary n.

Motivational background

The reader is perhaps familiar with some highest weight theory - given a complex connected semi-simple algebraic group G, once a choice of a maximal torus $\mathrm{T} \subset \mathrm{G}$ is made (for example if $\mathrm{G} = \mathrm{SL}(n, \mathbb{C})$ then a possible choice for T is the set of all diagonal matrices in G), then any irreducible representation L can be decomposed as a direct sum

$$\mathrm{L} = \bigoplus_{\chi} \mathrm{L}_{\chi} \tag{1}$$

of spaces L_χ on which the torus T acts by a character $\chi : \mathrm{T} \to \mathbb{C}^\times$. The choice of a Borel subgroup $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$ (for example, upper diagonal matrices in $\mathrm{GL}(n, \mathbb{C})$) determines a choice of positive roots Φ^+ (recall that roots arise as the characters when considering the adjoint representation). This choice of positive roots determines a partial order on this set of characters (they are also known as *weights*) that is defined by

$$\lambda \geq \mu \iff \lambda - \mu = \sum_{\alpha \in \Phi^+} a_\alpha \alpha$$

for $a_\alpha \in \mathbb{Z}_{\geq 0}$ and for characters $\lambda, \mu \in \mathrm{X} := \mathrm{Hom}(\mathrm{T}, \mathbb{C}^\times)$. With respect to this partial order, there is always one weight (cf. (1)) that is the largest, or the *highest*. This weight determines the irreducible representation up to isomorphism. The representation L is then denoted by $\mathrm{L}(\lambda)$, where λ indicates the highest weight.

To $\mathrm{L}(\lambda)$ is associated the *crystal graph* $\mathrm{B}(\lambda)$, which is a "combinatorial model" of $\mathrm{L}(\lambda)$ in the following sense:

- The vertices of $\mathrm{B}(\lambda)$ correspond to basis vectors for an associated representation $\widetilde{\mathrm{L}}(\lambda)$ of the quantum group $\mathrm{U}_q(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.
- The (coloured) arrows mimic the action of the Chevalley generators e_{α_i} and f_{α_i} of $\mathfrak{g}$ (these are tagged by *simple roots* $\Delta = \{\alpha_1, \cdots, \alpha_k\} \subset \Phi^+, i \in \{1, \cdots, k\}$).

Sometimes a particular construction of this crystal leads to a better understanding of certain aspects of the representation $\mathrm{L}(\lambda)$. The first chapters of this thesis are part of a project whose aim is to study the relationship between two constructions of this crystal for the groups $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SP}(2n, \mathbb{C})$.

One is the Littelmann path model, which is purely combinatorial. To construct $\mathrm{B}(\lambda)$, one first considers a path $\pi : [0, 1] \to \mathrm{X} \otimes_{\mathbb{Z}} \mathbb{R}$ with image contained in the dominant Weyl chamber (usually called *dominant*), with starting point $\pi(0) = 0$ the origin and endpoint $\pi(1) = \lambda$ the highest weight. This may be any such path, but in this thesis we will only consider paths whose image is contained in the *one-skeleton* of the associated affine Coxeter complex, which has all of its faces contained in

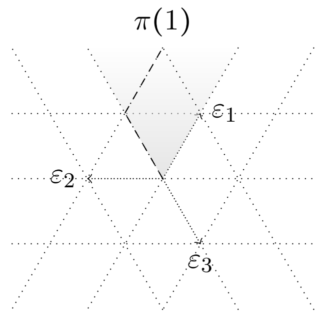


FIGURE 1. A path π for $G = SL(3,\mathbb{C})$. The one-skeleton is denoted by grey dotted lines.

$X \otimes_{\mathbb{Z}} \mathbb{R}$. Recall that this affine Coxeter complex is associated to affine Weyl group W^{aff} that is generated by the affine reflections $s_{\alpha^\vee,n}$ (which act on the space $X \otimes_{\mathbb{Z}} \mathbb{R}$) with respect to the hyperplanes

$$H_{(\alpha^\vee,n)} = \{x \in X \otimes_{\mathbb{Z}} \mathbb{R} : \langle v, \alpha^\vee \rangle = n\},$$

for coroots $\alpha \in \Phi^\vee$, and where $\langle -,- \rangle$ is the natural pairing between X and the cocharacter lattice $X^\vee$, which contains the set of coroots $\Phi^\vee$. The simplices, or *faces*, of this complex are intersections of the form

$$F = \bigcap_{(\alpha,n)\in\Phi^{e_{\alpha,n}}\otimes\mathbb{Z}} H^{e_{\alpha,n}}_{(\alpha^\vee)}$$

where $e_{\alpha,n} \in \{\varnothing, +, -\}$,

$$\begin{aligned}
H^+_{(\alpha^\vee,n)} &= \{x \in X \otimes_{\mathbb{Z}} \mathbb{R} : \langle v, \alpha^\vee \rangle \geq n\}, \\
H^-_{(\alpha^\vee,n)} &= \{x \in X \otimes_{\mathbb{Z}} \mathbb{R} : \langle v, \alpha^\vee \rangle \leq n\}, \text{ and} \\
H^\varnothing_{(\alpha^\vee,n)} &= H_{(\alpha^\vee,n)}.
\end{aligned}$$

The one-skeleton is the union of all one dimensional faces.

To construct the crystal $B(\lambda)$ one needs to apply the root operators f_{α_i} successively. These are defined via certain folding operations on paths, which are obtained by considering certain positive or negative crossings, with respect to the upper or lower half spaces $H^\pm_{(\alpha^\vee,n)}$. See Chapter 1, Section 2 for a formal definition of a crystal and for the definition of the root operators. The vertices of $B(\lambda)$ are given by the paths $f_{\alpha_{i_1}} \cdots f_{\alpha_{i_r}} \pi$ and the coloured arrows join two paths which can be obtained from each other by applying a root operator.

$$\pi' \xrightarrow{f_{\alpha_i}} f_{\alpha_i}\pi'$$

The resulting crystal is always isomorphic to $\mathrm{B}(\lambda)$ provided the original path π is dominant [**20**].

Using his model, Littelmann was able to prove formulas for the decomposition into irreducible summands of the tensor product $\mathrm{L}(\lambda) \otimes \mathrm{L}(\nu)$ of two irreducible representations $\mathrm{L}(\lambda)$ and $\mathrm{L}(\nu)$ as well as of the restriction of $\mathrm{L}(\lambda)$ to a Levi subgroup of G, both in terms of paths [**20**], [**19**] .

Let $\mathcal{O} = \mathbb{C}[[t]]$ and $\mathcal{K} = \mathbb{C}((t))$. The other construction of $\mathrm{B}(\lambda)$ that is relevant to this thesis is given by a certain set of Mirković-Vilonen (or MV) cycles, which arise in the context of the Geometric Satake equivalence [**22**], an equivalence of monoidal categories between the category $\mathrm{Perv}_{\mathrm{G}(\mathcal{O})}(\mathcal{G})$ of $\mathrm{G}(\mathcal{O})$-equivariant perverse sheaves on the affine Grassmannian $\mathcal{G} = \mathrm{G}(\mathcal{K})/\mathrm{G}(\mathcal{O})$ associated to G, and $\mathrm{rep}(\mathrm{G}^\vee)$, the category of finite-dimensional representations of $\mathrm{G}^\vee$, the Langlands dual of G. Here $\mathrm{G}(\mathcal{R})$ denotes the $\mathcal{R}$-points of G for some $\mathbb{C}$-algebra $\mathcal{R}$. Recall that reductive groups are determined by their root data - if $(\mathrm{X}, \mathrm{X}^\vee, \Phi, \Phi^\vee)$ is the root datum associated to G, then $\mathrm{G}^\vee$ is defined by the dual root datum $(\mathrm{X}^\vee, \mathrm{X}, \Phi^\vee, \Phi)$. The coweight lattice $\mathrm{X}^\vee$ can be identified with the quotient $\mathrm{T}(\mathcal{K})/\mathrm{T}(\mathcal{O})$ (for example, $\mathbb{C}^\times(\mathcal{K})/\mathbb{C}^\times(\mathcal{O}) \cong \mathbb{Z}$), which is embedded into $\mathcal{G}$. The affine Grassmannian is a projective ind-variety, which means that it is the direct limit of projective varieties, where all the maps are closed immersions [**15**].

The $\mathrm{G}(\mathcal{O})$-orbits in $\mathcal{G}$ are parametrised by $\lambda \in \mathrm{X}^{\vee,+}$, just as are the irreducible representations of $\mathrm{G}^\vee$. Each orbit $\mathrm{G}(\mathcal{O})\lambda$ is finite-dimensional of dimension $\langle \rho, 2\lambda \rangle$, where ρ is the half sum of all positive roots. Its closure

$$\mathrm{X}_\lambda = \overline{\mathrm{G}(\mathcal{O})\lambda}$$

is a projective algebraic variety. The equivariant intersection cohomology complexes IC^λ over X_λ are the simple objects in the semi-simple category $\mathrm{Perv}_{\mathrm{G}(\mathcal{O})}(\mathcal{G})$. The equivalence is defined by the hypercohomology functor $\mathrm{H}^* : \mathrm{Perv}_{\mathrm{G}(\mathcal{O})}(\mathcal{G}) \to \mathrm{Mod}_{\mathbb{C}}$, which assigns to IC^λ a space which is then identified with $\mathrm{L}(\lambda)$. The grading is defined by considering the Iwasawa decomposition

$$\mathcal{G} = \bigcup_{\nu \in \mathrm{X}^\vee} \mathrm{U}(\mathcal{K})\nu$$

of $\mathcal{G}$ into $\mathrm{U}(\mathcal{K})$-orbits (U is the unipotent radical of B - for example, if B is the set of all upper-triangular matrices in $\mathrm{GL}(n, \mathbb{C})$, then U consists of upper-triangular matrices with 1's on the diagonal) which are parametrised by the whole lattice $\mathrm{X}^\vee$. The intersection

$$\mathrm{U}(\mathcal{K})\mu \cap \mathrm{G}(\mathcal{O})\lambda \tag{2}$$

is non-empty if $\mu \in \mathrm{X}_\lambda$, which means that it is Weyl group conjugate to a dominant coweight $\eta \leq \lambda$ ([**22**], Theorem 3.2). The geometric Satake equivalence implies that the number of irreducible components of the closure

$$\overline{\mathrm{U}(\mathcal{K})\mu \cap \mathrm{G}(\mathcal{O})\lambda}$$

is $\dim(\mathrm{L}(\lambda)_\mu)$ ([**22**], Corollary 7.4). Let $\mathcal{Z}(\lambda)_\mu$ be the set consisting of these irreducible components, and let

$$\mathcal{Z}(\lambda) = \bigcup_{\mu \leq \lambda} \mathcal{Z}(\lambda)_\mu.$$

This set was endowed with the structure of a crystal in [**2**], using a certain decomposition with respect to a parabolic subgroup, which relates the aforementioned orbits in $\mathcal{G}$ with the orbits for a smaller affine Grassmannian associated to a Levi subgroup of G (see Section 3.2 of [**2**]).

There is a connection between these two constructions given by the affine building $\mathcal{J}^{aff}$ associated to G. The building [**24**] $\mathcal{J}^{aff}$ is a polysimplicial complex, which in particular means that it is a union of Coxeter complexes, called *apartments*, each of which is isomorphic to the Coxeter complex associated to the affine Weyl group W^{aff}. Paths $\pi : [0,1] \to \mathrm{X}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ contained in the one-skeleton are interpreted as *combinatorial one-skeleton* galleries, which are sequences of vertices and edges in the *standard apartment* of $\mathcal{J}^{aff}$. For simplicity, we call them simply "galleries". The explicit connection is given by a Bott-Samelson type desingularization

$$\Sigma_{\underline{\lambda}} \xrightarrow{\pi} \mathrm{X}_\lambda$$

of the *generalised Schubert variety* X_λ, which depends on a weight decomposition $\underline{\lambda}$ of λ - this determines a combinatorial gallery! For example, for $n \geq 3$, the decomposition

$$\lambda = \omega_1 + 2\omega_2 = \omega_2 + \omega_1 + \omega_2$$

(where ω_i denotes the $i-th$ fundamental coweight) defines the gallery $\gamma_{\underline{\lambda}}$ defined by the following sequence of vertices (see Chapter 1, Section 1.3)

$$(0, \omega_2, \omega_1 + \omega_2, \omega_1 + 2\omega_2).$$

The group $\mathrm{G}(\mathcal{K})$ acts on the affine building $\mathcal{J}^{aff}$ - we may consider the stabiliser P_F of a face F. For this example the Bott-Samelson variety $\Sigma_{\underline{\lambda}}$ is defined, as a set, by the following quotient

$$\Sigma_{\underline{\lambda}} = \mathrm{P}_0 \times \mathrm{P}_{\omega_2} \times \mathrm{P}_{\omega_1+\omega_2} / \mathrm{P}_{[o,\omega_2]} \times \mathrm{P}_{[\omega_2,\omega_1+\omega_2]} \times \mathrm{P}_{[\omega_1+\omega_2,\omega_1+2\omega_2]}$$

where (see Definition 1.4 for a general definition of $\Sigma_{\underline{\lambda}}$)

$$(p_0, p_1, p_2) \cdot (q_0, q_1, q_2) = (q_0 p_0, p_0^{-1} q_1 p_1, p_1^{-1} q_2 p_2).$$

The reason that $\Sigma_{\underline{\lambda}}$ is a projective algebraic variety is that $\mathrm{G}(\mathcal{O})$ has the structure of a pro-algebraic group. In this case this means in particular that the $\mathrm{G}(\mathcal{O})$-orbits in $\mathcal{J}^{aff}$ are also finite-dimensional. There is a $\mathbb{C}^\times$-action on $\Sigma_{\underline{\lambda}}$ that is induced by left multiplication by T on the first coordinate. With respect to this action, combinatorial galleries of a certain *type* (see Definition 1.4) are identified with $\mathbb{C}^\times$-fixed points in $\Sigma_{\underline{\lambda}}$ (see Chapter 1, Section 1.4). Thus, to each such gallery δ corresponds a Białynicki-Birula cell

$$\mathrm{C}_\delta = \{p \in \Sigma_{\underline{\lambda}} : \lim_{t\to 0} t \cdot p = \delta\}$$

of dimension

$$\dim(\mathrm{C}_\delta) \le \langle \rho, \lambda + \mu_\delta \rangle,$$

where μ_δ is the last vertex of δ. The project began in [**6**] for galleries of alcoves, (and subsequently in [**7**] for one-skeleton galleries) where Gaussent-Littelmann showed that if the cell C_δ has maximal dimension $\langle \rho, \lambda + \mu_\delta \rangle$ then $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle $\mathcal{Z}(\lambda)_{\mu_\delta} \subset \mathcal{Z}(\lambda)$. This assignment was then shown to be a crystal isomorphism [**1**] by Baumann-Gaussent. The galleries δ whose cell C_δ has maximal dimension are called *LS galleries*, which is short for "Lakshmibai Seshadri" galleries. These were the first galleries to be considered (interpreted as paths) in [**17**] and [**19**] . In this thesis we will work with more combinatorial definitions of LS galleries (cf. Chapters 1-3).

Main results in Chapters 1-3

What if the gallery δ is not LS?

In [**8**] Gaussent-Nguyen-Littelmann solved this problem for $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$, using their previous results and the combinatorics of Young tableaux. They showed that in this case $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle for any combinatorial gallery δ. In Chapter 2, Theorem 2.21 we extend this theorem and show that for $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$, the map

$$\begin{aligned} \Gamma &\to \bigoplus_{\lambda \in \mathrm{X}^{\vee,+}} \mathcal{Z}(\lambda) \\ \delta &\mapsto \overline{\pi(\mathrm{C}_\delta)} \end{aligned}$$

(we denote the set of all combinatorial galleries by Γ) is a surjective morphism of crystals, which restricts to an isomorphism on each connected component of Γ (cf. Theorem C). We also compute its fibres in terms of the Littelmann path model - the number of connected components in the pre-image of $\mathcal{Z}(\lambda)$ is the number of dominant combinatorial galleries in Γ with endpoint the coweight λ.

In general (see Definition 1.4), the Bott-Samelson variety $\Sigma_{\gamma^f} \xrightarrow{\pi} \mathcal{G}$ is defined for any combinatorial gallery γ of the same *type* as a combinatorial gallery γ^f contained in the fundamental alcove (cf. Lemma 1.3). In Chapter 3 we consider $\mathrm{G}^\vee = \mathrm{SP}(2n, \mathbb{C})$. Here there exist combinatorial galleries δ for which the closure $\overline{\pi(\mathrm{C}_\delta)}$ is not an MV cycle - see Chapter 3, Section 5. We define, however, a subset $\Gamma^{\mathrm{R}} \subset \Gamma$ of *readable* galleries - let $\delta \in \Gamma^{\mathrm{R}}$ be a readable gallery, and let δ^+ be the *highest weight vertex* of the crystal component that δ lies in. Then we show (cf. Theorem B, Theorem 3.26) that $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in $\mathcal{Z}(\mu_{\delta^+})$ and that

$$\Gamma^{\mathrm{R}} \longrightarrow \bigoplus_{\delta\in\Gamma^{\mathrm{R}}} \mathcal{Z}(\mu_{\delta^+})$$
$$\delta \mapsto \overline{\pi(\mathrm{C}_\delta)}$$

is a surjective morphism of crystals. As in the case of $\mathrm{SL}(n,\mathbb{C})$, we compute its fibers in terms of the Littelmann path model.

Methods. As was briefly mentioned in the last section, in [**8**], Gaussent-Nguyen-Littelmann use the combinatorics of Young tableaux; they relate them to the affine Grassmannian using the well-known Chevalley relations (see Chapter 1, (4)). We build on their methods for $\mathrm{G}^\vee = \mathrm{SL}(n,\mathbb{C})$, and for $\mathrm{G}^\vee = \mathrm{SP}(2n,\mathbb{C})$ which are very similar in spirit. Two galleries γ and δ are *equivalent* if they lie in the same place of the connected crystal they belong to. For example, all dominant galleries with the same endpoint are equivalent. Further, any gallery is equivalent to an LS gallery. The method is then to show that if two readable galleries γ and δ are equivalent, then

$$\overline{\pi(\mathrm{C}_\gamma)} = \overline{\pi(\mathrm{C}_\delta)}.$$

This is where combinatorics of tableaux and words come into play. The main point is to view galleries or paths as tableaux, or more generally keys or readable keys, for $\mathrm{G}^\vee = \mathrm{SL}(n,\mathbb{C})$ and $\mathrm{G}^\vee = \mathrm{SP}(n,\mathbb{C})$, respectively. To see how, see Chapter 2, Section 2.2 and Chapter 3, Section 1.3. To each key or readable key δ is associated a word $w(\delta)$, which is also regarded as a gallery in a natural way. Equivalence of these *word galleries* is expressed very precisely in these cases via the *plactic monoid*

$$\mathcal{P}_n = \mathcal{W}_n / \sim$$

and the *symplectic plactic monoid*

$$\mathcal{P}_{\mathcal{C}_n} = \mathcal{W}_{\mathcal{C}_n} / \sim$$

which are quotients on the word monoids on the alphabets $\mathcal{A}_n = \{1, \cdots, n\}$ and $\mathcal{C}_n = \{1, \cdots, n, \bar{n}, \cdots, \bar{1}\}$, by the relations given in Definitions 2.6 and 3.18, respectively. These monoids are formed exactly by the equivalence classes of all keys / readable keys in the sense that any word in these alphabets is in the same equivalence class as the word of an LS key.

The technique is to use the results by Gaussent-Littelmann and Baumann-Gaussent and to follow the following steps, which are carried out in Chapter 2 and Chapter 3, in order to show Theorem 2.21 and Theorem 3.26:

(1) Show that $\overline{\pi(\mathrm{C}_\delta)} = \overline{\pi'(\mathrm{C}_{w(\delta)})}$.
(2) Show that if γ and δ are equivalent, then $w(\gamma)$ and $w(\delta)$ are in the same conjugacy class in their corresponding plactic/ symplectic plactic monoid.

(3) Show that if two words w_1 and w_2 are in the same conjugacy class, then $\overline{\pi(\mathrm{C}_{w_1})} = \overline{\pi'(\mathrm{C}_{w_2})}$.

Organization

This thesis is organised in five chapters.

In **Chapter 1**, preliminary information on the affine Grassmannian and the affine building is provided, with some examples and some proofs. In **Chapter 2** we treat the case $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$ and its combinatorics. In **Chapter 3** we do the same for $\mathrm{G}^\vee = \mathrm{SP}(2n, \mathbb{C})$ - this is the longest chapter of this thesis, as the proofs in Section 4 are quite involved. In **Chapter 4** we introduce the Naito-Sagaki conjecture together with the proof for the cases we have already mentioned, and in **Chapter 5** (the appendix) we provide a technical result that we need in Chapter 3, and we fix a small mistake from [**8**].

CHAPTER 1

Buildings and the affine Grassmannian

1. Preliminaries

Let G be the complex connected reductive algebraic group associated to a root datum $(\mathrm{X}, \mathrm{X}^\vee, \Phi, \Phi^\vee)$. We denote its Langlands dual by $\mathrm{G}^\vee$ - it is the complex connected reductive algebraic group with root datum $(\mathrm{X}^\vee, \mathrm{X}, \Phi^\vee, \Phi)$. Let $\mathrm{T} \subset \mathrm{G}$ be a maximal torus of G with character group $\mathrm{X} = \mathrm{Hom}(\mathrm{T}, \mathbb{C}^\times)$ and cocharacter group $\mathrm{X}^\vee = \mathrm{Hom}(\mathbb{C}^\times, \mathrm{T})$. We identify the Weyl group W with the quotient $\mathrm{N}_\mathrm{G}(\mathrm{T})/\mathrm{T}$, where $\mathrm{N}_\mathrm{G}(\mathrm{T})$ is the normaliser of T in G, and will make abuse of notation by denoting a representative in $\mathrm{N}_\mathrm{G}(\mathrm{T})$ of an element $w \in \mathrm{W}$ in the Weyl group by the same symbol that we use to denote the element itself. We fix a choice of positive roots $\Phi^+ \subset \Phi$ (this determines a set $\Phi^{\vee,+}$ of positive coroots), and denote the dominance order on X and $\mathrm{X}^\vee$ determined by this choice by '$\leq$'. Namely:

$$\mu \leq \lambda \iff \begin{cases} \lambda - \mu \in \mathbb{Z}\Phi^+ & \text{if } \lambda \in \mathrm{X} \text{ and } \mu \in \mathrm{X} \\ \lambda - \mu \in \mathbb{Z}\Phi^{\vee,+} & \text{if } \lambda \in \mathrm{X}^\vee \text{ and } \mu \in \mathrm{X}^\vee. \end{cases}$$

Let $\Delta = \{\alpha_i\}_{i \in \{1,\cdots,n\}} \subset \Phi^+$ be the set of simple roots of Φ^+. Then the set $\Delta^\vee = \{\alpha_i^\vee\}_{i\in\{1,\cdots,n\}}$ of all coroots of elements of Δ forms a basis of the root system $\Phi^\vee$. Let $\langle -, - \rangle$ be the non-degenerate pairing between X and $\mathrm{X}^\vee$, and denote half the sum of positive roots (respectively coroots) by ρ (respectively by $\rho^\vee$). If $\lambda = \sum_{\alpha\in\Delta} n_\alpha \alpha$ is a sum of positive roots (respectively $\lambda = \sum_{\alpha^\vee \in \Delta^\vee} n_\alpha \alpha^\vee$) then $\langle \lambda, \rho^\vee \rangle = \sum_{\alpha\in\Delta} n_\alpha$ (respectively $\langle \rho, \lambda \rangle = \sum_{\alpha^\vee\in\Delta^\vee} n_\alpha$).

Let $\mathrm{B} \subset \mathrm{G}$ be the Borel subgroup of G containing T that is determined by the choice of positive roots Φ^+, and let $\mathrm{U} \subset \mathrm{B}$ be its unipotent radical. The group U is generated by the elements $\mathrm{U}_\alpha(b)$ for $b \in \mathbb{C}$ and $\alpha \in \Phi^+$, where for each root α, U_α is the one-parameter group that it determines. For each cocharacter $\lambda \in \mathrm{X}^\vee$ and each non-zero complex number $a \in \mathbb{C}^\times$ we denote by a^λ its image $\lambda(a) \in \mathrm{T}$.

The following identities hold in G (See §6 in [**26**]):

(1) For any $\lambda \in \mathrm{X}^\vee, a \in \mathbb{C}^\times, b \in \mathbb{C}, \alpha \in \Phi$,

$$a^\lambda \mathrm{U}_\alpha(b) = \mathrm{U}_\alpha(a^{\langle \alpha, \lambda \rangle} b) a^\lambda \tag{3}$$

(2) (Chevalley's commutator formula) Given linearly independent roots $\alpha, \beta \in \Phi$ and integers i and j such that $i\alpha + j\beta$, there exist non-zero numbers $c_{\alpha,\beta}^{i,j}$ such that, for all $a, b \in \mathbb{C}$:

$$\mathrm{U}_\alpha(b)^{-1}\mathrm{U}_\beta(a)^{-1}\mathrm{U}_\alpha(b)\mathrm{U}_\beta(a) = \prod_{i,j\in\mathbb{Z}^{>0}} \mathrm{U}_{i\alpha+j\beta}(c_{\alpha,\beta}^{i,j}(-a)^i b^j) \tag{4}$$

where the product is taken in some fixed order. The $c^{ij}_{\alpha,\beta}{}'s$ are integers depending on α, β, and on the chosen order in the product.

1.1. Affine Grassmannians. Let $\mathcal{O} = \mathbb{C}[[t]]$ denote the ring of complex formal power series and let $\mathcal{K} = \mathbb{C}((t))$ denote its field of fractions; it is the field of complex Laurent power series. For any $\mathbb{C}$-algebra $\mathcal{R}$, denote by $\mathrm{G}(\mathcal{R})$ the set of $\mathcal{R}$-valued points. The set

$$\mathcal{G} = \mathrm{G}(\mathcal{K})/\mathrm{G}(\mathcal{O})$$

is called the **affine Grassmannian** associated to G. We will denote the class in $\mathcal{G}$ of an element $g \in \mathrm{G}(\mathcal{K})$ by $[g]$. A coweight $\lambda : \mathbb{C}^{\times} \to \mathrm{T} \subset \mathrm{G}$ determines a point $t^{\lambda} \in \mathrm{G}(\mathcal{K})$ and hence a class $[t^{\lambda}] \in \mathcal{G}$. This map is injective, and we may therefore consider $\mathrm{X}^{\vee}$ as a subset of $\mathcal{G}$.

$\mathrm{G}(\mathcal{O})$-orbits in $\mathcal{G}$ are determined by the Cartan decomposition:

$$\mathcal{G} = \bigsqcup_{\lambda \in \mathrm{X}^{\vee,+}} \mathrm{G}(\mathcal{O})[t^{\lambda}].$$

Each $\mathrm{G}(\mathcal{O})$-orbit has the structure of an algebraic variety induced from the pro-group structure of $\mathrm{G}(\mathcal{O})$ (this is built in to the definition of a pro-group - see Definitions 4.4.1 in [**15**]) and it is known that for $\lambda \in \mathrm{X}^{\vee,+}$:

$$\overline{\mathrm{G}(\mathcal{O})[t^{\lambda}]} = \bigsqcup_{\mu \in \mathrm{X}^{\vee,+}, \mu \leq \lambda} \mathrm{G}(\mathcal{O})[t^{\mu}].$$

We call the closure $\overline{\mathrm{G}(\mathcal{O})[t^{\lambda}]}$ a **generalised Schubert variety** and we denote it by X_{λ}. This variety is usually singular. In Section 1.4 of this chapter we will review certain resolutions of singularities of it.

The $\mathrm{U}(\mathcal{K})$-orbits are given by the Iwasawa decomposition:

$$\mathcal{G} = \bigsqcup_{\lambda \in \mathrm{X}^{\vee}} \mathrm{U}(\mathcal{K})[t^{\lambda}].$$

These orbits are ind-varieties, and their closures can be described as follows (see [**22**], Proposition 3.1 a.):

$$\overline{\mathrm{U}(\mathcal{K})[t^{\lambda}]} = \bigcup_{\mu \leq \lambda} \mathrm{U}(\mathcal{K})[t^{\mu}]$$

for any $\lambda \in \mathrm{X}$.

1.2. MV Cycles and Crystals. Let $\lambda \in \mathrm{X}^{\vee,+}$ and $\mu \in \mathrm{X}^{\vee}$ be a dominant integral coweight and any coweight, respectively. Then by Theorem 3.2 a in [**22**], the intersection

$$\mathrm{U}(\mathcal{K})[t^{\mu}] \cap \mathrm{G}(\mathcal{O})[t^{\lambda}]$$

is non-empty if and only if $\mu \leq \lambda$; in that case its closure is pure dimensional of dimension $\langle \rho, \lambda + \mu \rangle$ and has the same number of irreducible components as the dimension of the μ-weight space $\mathrm{L}(\lambda)_{\mu}$ of the irreducible representation $\mathrm{L}(\lambda)$ of $\mathrm{G}^{\vee}$ of highest weight λ (this is Corollary 7.4 in [**22**]). Note that this makes sense because $\mathrm{X}^\vee$ may be identified with the character group of a maximal torus of $\mathrm{G}^\vee$. Explicitly, $\mathrm{X}^\vee \cong \mathrm{Hom}(\mathrm{T}^\vee, \mathbb{C}^\times)$, where $\mathrm{T}^\vee$ is the Langlands dual of T, which is a maximal torus of $\mathrm{G}^\vee$ (see Section 7 in [**22**]).

We denote the set of all irreducible components of a given topological space Y by Irr(Y). Consider the sets

$$\mathcal{Z}(\lambda)_\mu = \mathrm{Irr}(\overline{\mathrm{U}(\mathcal{K})[t^\mu] \cap \mathrm{G}(\mathcal{O})[t^\lambda]}) \text{ and}$$
$$\mathcal{Z}(\lambda) = \bigsqcup_{\mu \in \mathrm{X}^\vee} \mathcal{Z}(\lambda)_\mu.$$

The elements of these sets are called **MV cycles**. In [**2**], Section 3.3, Braverman and Gaitsgory have endowed the set $\mathcal{Z}(\lambda)$ with a crystal structure and have shown the existence of an isomorphism of crystals $\mathrm{B}(\lambda) \xrightarrow{\sim} \mathcal{Z}(\lambda)$. We will not use the definition of this crystal structure, but we denote by $\tilde{f}_i$ (respectively $\tilde{e}_i$) the corresponding root operators for $i \in \{1, \cdots, n\}$, where n is the rank of the root system Φ.

1.3. Galleries in the Affine Building. Let $\mathcal{J}^{aff}$ be the affine building associated to G and $\mathcal{K}$. It is a union of simplicial complexes called *apartments*, each of which is isomorphic to the Coxeter complex of the same type as the extended Dynkin diagram associated to G. See Chapters 9 and 10 in [**24**] for detailed information on affine buildings. The affine Grassmannian $\mathcal{G}$ can be $\mathrm{G}(\mathcal{K})$-equivariantly embedded into the building $\mathcal{J}^{aff}$, which also carries a $\mathrm{G}(\mathcal{K})$-action. Denote by Φ^{aff} the set of real affine roots associated to Φ; we identify it with the set

$$\Phi^{aff} = \Phi \times \mathbb{Z}.$$

Let $\mathbb{A} = \mathrm{X}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. For each $(\alpha, n) \in \Phi^{\mathrm{aff}}$, consider the associated hyperplane

$$\mathrm{H}_{(\alpha,n)} = \{x \in \mathbb{A} : \langle \alpha, x \rangle = n\}$$

and the positive, respectively negative half spaces

$$\mathrm{H}^+_{(\alpha,n)} = \{x \in \mathbb{A} : \langle \alpha, x \rangle \geq n\}$$
$$\mathrm{H}^-_{(\alpha,n)} = \{x \in \mathbb{A} : \langle \alpha, x \rangle \leq n\}.$$

Denote by W^{aff} the affine Weyl group generated by all the affine reflections $s_{(\alpha,n)}$ with respect to the affine hyperplanes $\mathrm{H}_{(\alpha,n)}$. We have an embedding $\mathrm{W} \hookrightarrow \mathrm{W}^{aff}$ given by $s_\alpha \mapsto s_{(\alpha,0)}$. The **dominant Weyl chamber** is the set

$$\mathrm{C}^+ = \{x \in \mathbb{A} : \langle \alpha, x \rangle \ > 0 \ \ \forall \alpha \in \Delta\}$$

and the **fundamental alcove** is in turn

$$\Delta^{\mathrm{f}} = \{x \in \mathrm{C}^+ : \langle \alpha, x \rangle \ < 1 \ \ \forall \alpha \in \Phi^+\}.$$

There is a unique apartment in the affine building $\mathcal{J}^{aff}$ that contains the image of the set of coweights $\mathrm{X}^\vee \subset \mathcal{G}$ under the embedding $\mathcal{G} \hookrightarrow \mathcal{J}^{aff}$. This apartment is isomorphic to the affine Coxeter complex associated to W^{aff}; its faces are given by all possible intersections of the hyperplanes $H_{(\alpha,n)}$ and their associated (closed) positive and negative half-spaces $H^{\pm}_{(\alpha,n)}$. It is called the **standard apartment** in the affine building $\mathcal{J}^{aff}$. The action of W^{aff} on the affine building $\mathcal{J}^{aff}$ coincides, when restricted to the standard apartment, with the one induced by the natural action of W^{aff} on $\mathbb{A}$; the fundamental alcove is a fundamental domain for the latter.

To each real affine root $(\alpha, n) \in \Phi^{aff}$ is attached the one-parameter additive **root subgroup** $U_{(\alpha,n)}$ of $G(\mathcal{K})$ defined by $b \mapsto U_\alpha(bt^n)$ for $b \in \mathbb{C}$. Let $\lambda \in X^\vee$ and $b \in \mathbb{C}$. Then identity (3) implies that:

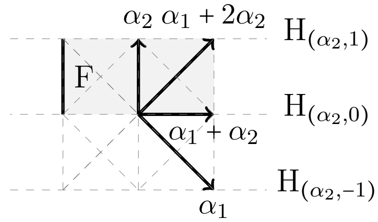
$$U_{(\alpha,n)}(b)[t^\lambda] = [U_\alpha(bt^n)t^\lambda] = [t^\lambda U_\alpha(bt^{n-\langle\alpha,\lambda\rangle})], \tag{5}$$

and

$$[t^\lambda U_\alpha(bt^{n-\ \alpha,\lambda\rangle})] = [t^\lambda] \text{ if and only if } U_\alpha(bt^{n-\langle\alpha,\lambda\rangle}) \in G(\mathcal{O}),$$

or, equivalently, $\langle\alpha,\lambda\rangle \leq n$. Hence, the root subgroup $U_{(\alpha,n)}$ stabilises the point $[t^\lambda] \in \mathcal{G} \hookrightarrow \mathcal{J}^{aff}$ if and only if $\lambda \in H^-_{(\alpha,n)}$. For each face F in the standard apartment, denote by P_F, U_F and W^{aff}_F its stabilizer in $G(\mathcal{K}), U(\mathcal{K})$ and W^{aff} respectively. These subgroups are generated by the torus T and the root subgroups $U_{(\alpha,n)}$ such that $F \subset H^-_{(\alpha,n)}$, the root subgroups $U_{(\alpha,n)} \subset P_F$ such that $\alpha \in \Phi^+$, and those affine reflections $s_{(\alpha,n)} \in W^{aff}$ such that $F \subset H_{(\alpha,n)}$, respectively. See [**6**], Section 3.3, Example 3, and Proposition 5.1 (ii) in [**1**].

Example 1.1. Let $G^\vee = SP(4,\mathbb{C})$, then $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$. In the picture below the shaded region is the upper halfspace $H^+_{(\alpha_2,0)}$. Let F be the face in the standard apartment that joins the vertices $-(\alpha_1+\alpha_2)$ and $-\alpha_1$.



The subgroup P_F is generated by the root subgroups associated to the following real roots

$$(\alpha_1, n) \; n \geq -1$$
$$(\alpha_2, n) \; n \geq 1$$
$$(\alpha_1+\alpha_2, n) \; n \geq -1$$
$$(\alpha_1+2\alpha_2, n) \; n \geq 0$$
$$(-\alpha_1, n) \; n \geq 2$$
$$(-\alpha_2, n) \; n \geq 0$$
$$(-(\alpha_1+\alpha_2), n) \; n \geq 1$$
$$(-(\alpha_1+2\alpha_2, n)) \; n \geq 1$$

The stabiliser $\mathrm{U_F}$ is generated by the root subgroups associated to those previously stated roots (α, n) such that $\alpha \in \Phi^+$ is a positive root, and

$$\mathrm{W}_{\mathrm{F}}^{aff} = \{s_{(\alpha_1+\alpha_2,-1)}, 1\}.$$

A **gallery** is a sequence of faces in the affine building $\mathscr{J}^{aff}$

$$\gamma = (\mathrm{V}_0 = 0, \mathrm{E}_0, \mathrm{V}_1, \cdots, \mathrm{E}_k, \mathrm{V}_{k+1}) \tag{6}$$

such that:

1. For each $i \in \{1, \cdots, k\}$, $\mathrm{V}_i \subset \mathrm{E}_i \supset \mathrm{V}_{i+1}$.
2. Each face labelled V_i has dimension zero (is a **vertex**) and each face labelled E_i has dimension one (is an **edge**). In particular, each face in the sequence γ is contained in the one-skeleton of the standard apartment.
3. The last vertex V_{k+1} is a **special vertex**: its stabiliser in the affine Weyl group W^{aff} is isomorphic to the finite Weyl group W associated to G.

Denote all the set of all galleries in the affine building by Σ. If in addition each face in the sequence belongs to the standard apartment, then γ is called a **combinatorial gallery**. We will denote the set of all combinatorial galleries in the affine building by Γ. In this case, the third condition is equivalent to requiring the last vertex V_{k+1} to be a coweight. From now on, if γ is a combinatorial gallery we will denote the coweight corresponding to its final vertex by μ_γ in order to distinguish it from the vertex.

Remark 1.2. The galleries we defined above are actually called *one-skeleton* galleries in the literature [**7**]. The word 'gallery' was originally used to describe a more general class of face sequences but since we only work with one-skeleton galleries in this paper, we have chosen to leave the word 'one-skeleton' out.

1.4. Bott-Samelson varieties. Let γ be a combinatorial gallery (notation as above). The following lemma can be obtained from Lemma 4.8 and Definition 4.6 in [**7**], but we provide a proof nevertheless.

Lemma 1.3. For each $j \in \{1, \cdots, k\}$ there exist elements $w_j \in \mathrm{W}_{\mathrm{V}_i^f}^{aff}$ and a unique combinatorial gallery

$$\gamma^f = (\mathrm{V}_0^f, \mathrm{E}_0^f, \mathrm{V}_1^f, \cdots, \mathrm{V}_{k+1}^f)$$

with each one of its faces is contained in the fundamental alcove such that

$$w_0 \cdots w_r \mathrm{E}_r^f = \mathrm{E}_r.$$

Proof. By induction on j. Let $w_0 \in \mathrm{W}^{aff}_{\mathrm{V}_0}$ be an element in W^{aff} such that $w_0^{-1}\mathrm{E}_0$ is contained in the fundamental alcove. Define $\mathrm{E}_0^f := w_0^{-1}\mathrm{E}_0$. Now let $j \le k$; for $i \in \{0, \cdots, j-1\}$ let w_i, E_i^f, and V_i^f be the Weyl group elements, respectively the faces of the fundamental alcove such that $(\mathrm{V}_0^f = 0, \mathrm{E}_0^f, \mathrm{V}_1^f, \cdots, \mathrm{E}_{j-1}^f, \mathrm{V}_j^f)$ is a combinatorial gallery and such that $w_0 \cdots w_i \mathrm{E}_i^f = \mathrm{E}_i$. Define w_j to be the unique element in $\mathrm{W}^{aff}_{\mathrm{V}_r^f}$ such that

$$w_j^{-1} w_{j-1}^{-1} \cdots w_0^{-1} \mathrm{E}_j =: \mathrm{E}_j^f$$

is contained in the fundamental alcove. This completes the proof by induction. □

If two galleries have the same associated gallery we say that the two galleries have **the same type**. We will denote the set of all the combinatorial galleries that have the same type as a given combinatorial gallery γ by $\Gamma(\gamma)$. The map

$$\mathrm{W}^{aff}_{\mathrm{V}_0} \times \cdots \times \mathrm{W}^{aff}_{\mathrm{V}_k} \to \Gamma(\gamma) \tag{7}$$

$$(w_0, \cdots, w_k) \mapsto (\mathrm{V}_0, w_0\mathrm{E}_0, w_0\mathrm{V}_1, w_0 w_1 \mathrm{E}_1, \cdots, w_0 \cdots w_k \mathrm{V}_{k+1}) \tag{8}$$

induces a bijection between $\Gamma(\gamma)$ and the set $\prod_{i=0}^r \mathrm{W}^{aff}_{\mathrm{V}_i}/\mathrm{W}^{aff}_{\mathrm{E}_i}$. This implies that the set $\Gamma(\gamma)$ it is in particular finite. For a proof see Lemma 4.8 in [**7**].

Definition 1.4. The **Bott-Samelson variety** of type γ^f is the quotient of

$$\mathrm{G}(\mathcal{O}) \times \mathrm{P}_{\mathrm{V}_1^f} \times \cdots \times \mathrm{P}_{\mathrm{V}_k^f}$$

by the following left action of $\mathrm{P}_{\mathrm{E}_0^f} \times \cdots \times \mathrm{P}_{\mathrm{E}_k^f}$:

$$(p_0, p_1, \cdots, p_k) \cdot (q_0, \cdots, q_k) = (q_0 p_0, p_0^{-1} q_1 p_1, \cdots, p_{k-1}^{-1} q_k p_k).$$

We will denote it by Σ_{γ^f}.

The pro-group structure of the groups $\mathrm{P}_{\mathrm{V}_i^f}, \mathrm{P}_{\mathrm{E}_i^f}$ assures that Σ_{γ^f} is in fact a smooth variety. To each point $(g_0, \cdots, g_k) \in \mathrm{G}(\mathcal{O}) \times \mathrm{P}_{\mathrm{V}_1^f} \times \cdots \times \mathrm{P}_{\mathrm{V}_k^f}$ one can associate a gallery

$$(\mathrm{V}_0^f, g_0\mathrm{E}_0^f, g_0\mathrm{V}_1^f, g_0 g_1 \mathrm{V}_2^f, \cdots, g_0 \cdots g_k \mathrm{V}_{k+1}^f).$$

This induces a well defined injective map $i : \Sigma_{\gamma^f} \hookrightarrow \Sigma$; we call the elements in the image $i(\Sigma_{\gamma^f})$ **galleries of type** γ^f. With respect to this identification, T-fixed points in Σ_{γ^f} are in natural bijection with the set $\Gamma(\gamma^f)$ of combinatorial galleries of type γ^f.

Let $\omega \in \mathbb{A}$ be a fundamental coweight. We consider the following combinatorial gallery, which starts at 0 and ends at ω. Let $\mathrm{V}_1^\omega, \cdots, \mathrm{V}_k^\omega$ be the vertices in the standard apartment that lie on the open line segment joining 0 and ω, numbered such that V_{i+1}^ω lies on the open line segment joining V_i^ω and ω. We denote by E_i^ω denote the face contained in $\mathbb{A}$ that contains the vertices V_i^ω and V_{i+1}^ω. The gallery

$$\gamma_\omega = (0 = \mathrm{V}_0^\omega, \mathrm{E}_0^\omega, \mathrm{V}_1^\omega, \mathrm{E}_1^\omega, \cdots, \mathrm{E}_k^\omega, \mathrm{V}_{k+1}^\omega = \omega)$$

is called a **fundamental gallery**. Galleries of the same type as a fundamental gallery γ_ω will be called **galleries of fundamental type** ω.

Now let $\lambda \in X^{\vee,+}$ be a dominant integral coweight and γ_λ a gallery with endpoint the coweight λ and such that it is a concatenation of fundamental galleries, where concatenation of two combinatorial galleries $\gamma_1 * \gamma_2$ is defined by translating the second one to the endpoint of the first one. Then the map

$$\begin{aligned} \Sigma_{\gamma_\lambda^f} &\xrightarrow{\pi} X_\lambda \\ [g_0, \cdots, g_r] &\to g_0 \cdots g_r [t^{\mu_{\gamma^f}}] \end{aligned} \tag{9}$$

is a resolution of singularities of the generalised Schubert variety X_λ.

REMARK 1.5. That the above map is in fact a resolution of singularities is due to the fact that a gallery such as the one considered is minimal (see [**7**], Section 5 and Section 4.3, Proposition 3). This resembles the condition for usual Bott-Samelson varieties associated to a reduced expression: see [**6**], Section 9, Proposition 7.

REMARK 1.6. The map (9) makes sense for any combinatorial gallery γ : in this generality one has a map $\Sigma_{\gamma^f} \xrightarrow{\pi} \mathcal{G}, g_0, \cdots, g_r[t^{\mu_\gamma}]$

$$\begin{aligned} \Sigma_{\gamma^f} &\xrightarrow{\pi} \mathcal{G} \\ [g_0, \cdots, g_r] &\to g_0 \cdots g_r [t^{\mu_{\gamma^f}}] \end{aligned} \tag{10}$$

REMARK 1.7. Note that it follows from the definitions that if γ, ν are two galleries of the same type as δ, respectively η, then $\gamma * \nu$ is of the same type as $\delta * \eta$. Actually, if $\gamma = \gamma_1 * \cdots * \gamma_r$ then $\Gamma(\gamma) = \{\delta_1 * \cdots * \delta_r : \delta_i \in \Gamma(\gamma_i)\}$.

1.5. Cells and positive crossings. Let $r_\infty : \mathcal{J}^a \to \mathbb{A}$ be the retraction at infinity (see Definition 8 in [**6**]). It extends to a map $r_{\gamma^f} : \Sigma_{\gamma^f} \to \Gamma(\gamma^f)$. The cell $C_\delta = r_{\gamma^f}^{-1}(\delta)$ (for $\delta \in \Gamma(\gamma^f)$) is explicitly described in [**6**] and [**7**] by Gaussent-Littelmann and in [**1**] by Baumann-Gaussent. In this subsection we recollect their results - we will need them later. Their results are formulated in terms of galleries of the same type as a gallery γ_λ that is a concatenation of fundamental galleries; we formulate them for any combinatorial gallery. The proofs remain the same but we provide them nevertheless for the comfort of the reader.

First consider the subgroup $U(\mathcal{K})$ of $G(\mathcal{K})$. It is generated by the elements $U_{(\alpha,n)}(a)$ for $\alpha \in \Phi^+$ a positive root, $a \in \mathbb{C}$, and $n \in \mathbb{Z}$. Let $V \subset E$ be a vertex and an edge (respectively) in the standard apartment, the vertex contained in the edge. Consider the subset of affine roots

$$\Phi^+_{(V,E)} = \{(\alpha, n) \in \Phi^{aff} \,|\, \alpha \in \Phi^+, V \in H_{(\alpha,n)}, E_i \notin H^-_{(\alpha,n)}\}$$

and let $\mathbb{U}_{(V,E)}$ denote the subgroup of $U(\mathcal{K})$ generated by $U_{(\alpha,n)}(a)$ for all $(\alpha, n) \in \Phi^+_{(V,E)}$ and $a \in \mathbb{C}$. The following proposition will be very useful in Chapters 2 and 3. It is stated and proven as Proposition 5.1 (ii) in [**1**].

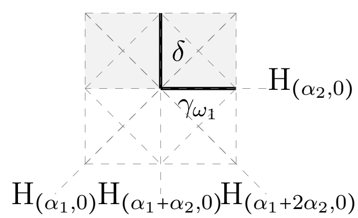
Proposition 1.8. Let $\mathrm{V} \subset \mathrm{E}$ be a vertex and an edge in the standard apartment as above. Then $\mathbb{U}_{(\mathrm{V},\mathrm{E})}$ is a set of representatives for the right cosets of U_E in U_V. Furthermore for any total order on the set $\Phi^+_{(\mathrm{V},\mathrm{E})}$, the map

$$\begin{aligned} \mathbb{C}^{|\Phi^+_{(\mathrm{V},\mathrm{E})}|} &\xrightarrow{\varphi} \mathbb{U}_{(\mathrm{V},\mathrm{E})} \\ (a_\beta)_{\beta\in\Phi^+_{(\mathrm{V},\mathrm{E})}} &\mapsto \prod_{\beta\in\Phi^+_{(\mathrm{V},\mathrm{E})}} \mathrm{U}_\beta(a_\beta) \end{aligned}$$

is a bijection. The product on the right hand side is taken in the chosen order.

Now let γ be a combinatorial gallery with notation as in (6). For each $i \in \{1, \cdots, k\}$, let $\mathbb{U}^\gamma_i = \mathbb{U}_{(\mathrm{V}_i,\mathrm{E}_i)}$. For later use we fix the notation $\Phi^\gamma_i := \Phi^+_{\mathrm{V}_i,\mathrm{E}_i}$.

Example 1.9. Let $\mathrm{G}^\vee = \mathrm{SP}(4,\mathbb{C})$ as in Example 1.1 and consider the fundamental gallery γ_{ω_1}. Then $\mathbb{U}^{\gamma_{\omega_1}}_0$ is generated by the root subgroups associated to the real roots $(\alpha_1, 0), (\alpha_1 + \alpha_2, 0), (\alpha_1 + 2\alpha_2, 0)$. If δ is the gallery with one edge and endpoint α_2, then $\mathbb{U}^\delta_0$ is generated by the groups associated to the roots $(\alpha_2, 0)$ and $(\alpha_1 + 2\alpha_2, 0)$.



Now write $\delta = (\mathrm{V}_0, \mathrm{E}_0, \cdots, \mathrm{E}_k, \mathrm{V}_{k+1}) \in \Gamma(\gamma^f)$ in terms of Definition 1.4 and Lemma 1.3 as $\delta = [\delta_0, \cdots, \delta_k]$. This means that $\delta_i \in \mathrm{W}^{aff}_{\mathrm{V}^f_i}$ and $\delta_0 \cdots \delta_j \mathrm{E}^f_j = \mathrm{E}_j$. A beautiful exposition of the following description of the cell C_δ can be found in [**7**].

Theorem 1.10. Let $\delta \in \Gamma(\gamma^f)$ as above. The map

$$\begin{aligned} \mathbb{U}^\delta := \mathbb{U}^\delta_0 \times \mathbb{U}^\delta_1 \times \cdots \times \mathbb{U}^\delta_k &\xrightarrow{\varphi} \Sigma_{\gamma^\mathrm{f}} \\ (u_0, \cdots, u_k) &\mapsto [u_0\delta_0, \delta_0^{-1}u_1\delta_0\delta_1, \cdots, (\delta_0 \cdots \delta_{k-1})^{-1}u_k\delta_0 \cdots \delta_k] \end{aligned}$$

is injective and has image C_δ.

Proof. Let

$$\tilde{\mathbb{U}} = \mathrm{U}_{\mathrm{V}_0} \times \cdots \mathrm{U}_{\mathrm{V}_k} / \mathrm{U}_{\mathrm{E}_0} \times \cdots \mathrm{U}_{\mathrm{E}_k}$$

where

$$(b_0, \cdots, b_k) \cdot (v_0, \cdots, v_k) = (v_0 b_0, b_0^{-1} v_1 b_1, \cdots, b_{k-1}^{-1} v_k b_k).$$

The map

$$(v_0, \cdots, v_k) \mapsto [v_1, \cdots, v_k]$$

defines a bijection $\phi : \mathrm{U} \to \tilde{\mathbb{U}}$. Indeed, by Proposition 4.17 (2) in [**7**], $\mathbb{U}^\delta_i$ is a set of representatives for right cosets of $\mathrm{U}_{\mathrm{E}_i}$ in $\mathrm{U}_{\mathrm{V}_i}$, and hence for $[a_0, \cdots, a_k] \in \tilde{\mathbb{U}}$ there is a unique $(v_0, \cdots, v_k) \in \mathbb{U}^\delta$ such that (for some $b_j \in \mathrm{U}_{\mathrm{E}_j}$) $v_0 b_0 = a_0, v_j b_j = b_{j-1} a_j$, i.e. $\phi((v_0, \cdots, v_k)) = [a_0, \cdots, a_k]$. We use this bijection and consider instead the map

$$\tilde{\varphi} = \varphi \circ \phi^{-1} : \tilde{\mathbb{U}} \to \Sigma_{\gamma^\mathrm{f}}.$$

Fix $[v_0, \cdots, v_k] \in \tilde{\mathbb{U}}$. The map $\tilde{\varphi}$ is well defined because $(\delta_0 \cdots \delta_{j-1})^{-1} v_j (\delta_0 \cdots \delta_j) \in \mathrm{P}_{\mathrm{V}_j^f}$ and if $b_j \in \mathrm{U}_{\mathrm{E}_j}$ then $(\delta_0 \cdots \delta_j)^{-1} b_j (\delta_0 \cdots \delta_j) \in \mathrm{U}_{\mathrm{E}_j^f}$. Since by Proposition 1 in [**6**] the fibers of r_∞ are $\mathrm{U}(\mathcal{K})$-orbits, an element $p = [p_0, \cdots, p_k] \in \Sigma_{\gamma^f}$ belongs to C_δ if and only if there exist elements $u_0, \cdots, u_k \in \mathrm{U}(\mathcal{K})$ such that

$$p_0 \cdots p_j \mathrm{E}_j^f = u_j \mathrm{E}_j \text{ and} \tag{11}$$

$$u_{j-1} \mathrm{V}_j = u_j \mathrm{V}_j. \tag{12}$$

Define

$$u_0 = v_0 \text{ and } u_j = v_0 \cdots v_j.$$

Then conditions (11) and (12) above hold for

$$p_j = (\delta_0 \cdots \delta_{j-1})^{-1} v_j (\delta_0 \cdots \delta_j).$$

Hence the image of the map is contained in the cell C_δ. For the other inclusion, define $v_j = u_{j-1}^{-1} u_j$ (see Proposition 4.19 in [**7**]). To show injectivity assume $\tilde{\varphi}([v_0, \cdots, v_k]) = \tilde{\varphi}([v_0', \cdots, v_k'])$. Then there exist elements $b_j \in \mathrm{U}_{\mathrm{E}_j}$ such that $v_0 \cdots v_j = v_0' \cdots v_j' b_j$, and this implies injectivity. □

The following corollary can be found in [**8**] as Corollary 3 for $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$. Note that in particular it implies that $u\pi(\mathrm{C}_\delta) = \pi(\mathrm{C}_\delta)$ for all $u \in \mathrm{U}_{\mathrm{V}_0}$.

Corollary 1.11. The following equality holds.

$$\pi(\mathrm{C}_\delta) = \mathbb{U}_0^\delta \cdots \mathbb{U}_k^\delta [t^{\mu_\delta}] = \mathrm{U}_{\mathrm{V}_0} \cdots \mathrm{U}_{\mathrm{V}_k} [t^{\mu_\delta}]$$

Proof. By Theorem 1.10 the image of the map

$$\begin{aligned} \mathrm{U}_{\mathrm{V}_0} \times \cdots \times \mathrm{U}_{\mathrm{V}_r} &\to \Sigma_{\gamma^f} \\ (u_0, \cdots, u_r) &\mapsto [u_0 \delta_0, \delta_0^{-1} u_1 \delta_0 \delta_1, \cdots, \delta_0 \cdots \delta_{r-1}^{-1} u_r \delta_0 \cdots \delta_r] \end{aligned}$$

is the cell C_δ. The corollary follows since
$\delta_0 \cdots \delta_j \mu_{\gamma^f} = \mu_\delta$. □

2. Crystal structure on combinatorial galleries, the Littelmann path model, and Lakshmibai Seshadri galleries

Let $\lambda \in \mathrm{X}^{+,\vee}$ be a dominant integral coweight and let $\mathrm{L}(\lambda)$ be the corresponding simple module of $\mathrm{G}^\vee$. To $\mathrm{L}(\lambda)$ is associated a certain graph $\mathrm{B}(\lambda)$ that is its "combinatorial model". It is a connected *highest weight* crystal, which means that there exists $b_\lambda \in \mathrm{B}(\lambda)$ such that $e_{\alpha_i} b_\lambda = 0$ for all $i \in \{1, \cdots, n-1\}$. The crystal $\mathrm{B}(\lambda)$ also has the characterising property that

$$\dim(\mathrm{L}(\lambda)_\mu) = \#\{b \in \mathrm{B}(\lambda) : \mathrm{wt}(b) = \mu\}.$$

See below for the definitions. After recalling the notion of a crystal we review the crystal structure on the set of all combinatorial galleries Γ.

2.1. Crystals. A **crystal** is a set B together with maps

$$e_{\alpha_i}, f_{\alpha_i} : \mathrm{B} \to \mathrm{B} \cup \{0\} (\text{the } \textbf{root operators}),$$
$$\mathrm{wt} : \mathrm{B} \to \mathrm{X}^\vee (\text{ for } i \in \{1, \cdots, n\})$$

such that for every $b, b' \in \mathrm{B}$ and $i \in \{1, \cdots, n-1\}, b' = e_{\alpha_i}(b)$ if and only if $b = f_{\alpha_i}(b')$, and, in this case, setting

$$\epsilon_i(b'') = \max\{n : e^n_{\alpha_i}(b) \neq 0\}$$

and

$$\phi_i(b'') = \max\{n : f^n_{\alpha_i}(b'') \neq 0\}$$

for any $b'' \in \mathrm{B}$, the following properties are satisfied.

(1) $\mathrm{wt}(b') = \mathrm{wt}(b) + \alpha_i^\vee$
(2) $\phi(b) = \epsilon_i(b) + \langle \alpha_i, \mathrm{wt}(b) \rangle$

A crystal is in particular a graph, which we may decompose into the disjoint union of its connected components. Each element $b \in \mathrm{B}$ lies in a unique connected component which we will denote by $\mathrm{Conn}(b)$. A **crystal morphism** is a map $\mathrm{F} : \mathrm{B} \to \mathrm{B}'$ between the underlying sets of two crystals B and B′ such that $\mathrm{wt}(\mathrm{F}(b)) = \mathrm{wt}(b)$ and such that it commutes with the action of the root operators. A crystal morphism is an isomorphism if it is bijective.

2.2. Crystal structure on combinatorial galleries.

Definition 1.12. For each $i \in \{1, \cdots, n\}$ and each simple root α_i, we recall the definition of the root operators f_{α_i} and e_{α_i} on the set of combinatorial galleries Γ and endow the set of combinatorial galleries with a crystal structure. We follow Section 6 in [**6**] and Section 1 in [**2**]. We refer the reader to [**12**] for a detailed account of the theory of crystals.

Let $\gamma = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \cdots, \mathrm{E}_k, \mathrm{V}_{k+1})$ be a combinatorial gallery. Define $\mathrm{wt}(\gamma) = \mu_\gamma$. Let $m_{\alpha_i} = m \in \mathbb{Z}$ be minimal such that $\mathrm{V}_r \in \mathrm{H}_{(\alpha_i, m)}$ for $r \in \{1, \cdots, k+1\}$. Note that $m \leq 0$.

Definition of $\mathbf{f}_{\alpha_i}$: Suppose $\langle \alpha_i, \mu_\gamma \rangle \geq m+1$. Let j be maximal such that $\mathrm{V}_j \in \mathrm{H}_{(\alpha_i, m)}$ and let $j < r \leq k+1$ be minimal such that $\mathrm{V}_r \in \mathrm{H}_{(\alpha_i, m+1)}$. Let

$$\mathrm{E}'_i = \begin{cases} \mathrm{E}_i & \text{if } i < j \\ s_{(\alpha_i, m)}(\mathrm{E}_i) & \text{if } j \leq i < r \\ t_{-\alpha_i^\vee}(\mathrm{E}_i) & \text{if } i \geq r \end{cases}$$

and define

$$f_{\alpha_i}(\gamma) = (\mathrm{V}'_0, \mathrm{E}'_0, \mathrm{V}'_1, \mathrm{E}'_1, \cdots, \mathrm{E}'_r, \mathrm{V}'_{k+1}).$$

If $\langle \alpha_i, \mu_\gamma \rangle < m+1$, then $f_{\alpha_i}(\gamma) = 0$.

Definition of $\mathbf{e}_{\alpha_i}$: Suppose that $m \leq -1$. Let r be minimal such that the vertex $\mathrm{V}_r \in \mathrm{H}_{(\alpha_i, m)}$ and let $0 \leq j < r$ maximal such that $\mathrm{V}_j \in \mathrm{H}_{(\alpha_i, m+1)}$. Let

$$\mathrm{E}'_i = \begin{cases} \mathrm{E}_i & \text{if } i < j \\ s_{(\alpha_i, m+1)}(\mathrm{E}_i) & \text{if } j \le i < r \\ t_{\alpha_i^\vee}(\mathrm{E}_i) & \text{if } i \ge r \end{cases}$$

and define

$$e_{\alpha_i}(\gamma) = (\mathrm{V}'_0, \mathrm{E}'_0, \mathrm{V}'_1, \mathrm{E}'_1, \cdots, \mathrm{E}'_r, \mathrm{V}'_{k+1}).$$

If $m = 0$ then $e_{\alpha_i}(\gamma) := 0$.

REMARK 1.13. It follows from the definitions that the maps $e_{\alpha_i}, f_{\alpha_i}$ and wt define a crystal structure on Γ. Note as well that if γ is a combinatorial gallery then $f_{\alpha_i}(\gamma)$ and $e_{\alpha_i}(\gamma)$ are combinatorial galleries of the same type (as long as they are not zero). We say that the root operators are type preserving. See also [**6**], Lemma 6.

2.3. The Littelmann path model and Lakshmibai Seshadri galleries. Readable galleries. Let γ be a combinatorial gallery that has each one of its faces contained in the fundamental chamber. We call such galleries **dominant**. We will denote the set of all dominant combinatorial galleries by Γ^{dom}, respectively $\Gamma^{\mathrm{dom}}(\gamma)$ the set of all dominant galleries of the same type as a given gallery γ. By succesively applying the root operators to γ one produces a crystal $\mathbb{P}(\gamma)$. It is known as the *Littelmann path model* of the representation $\mathrm{L}(\lambda)$ of $\mathrm{G}^\vee$ (by considering galleries as piecewise linear paths in $\mathrm{X}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ - see 2.3.2); it is isomorphic to the connected crystal $\mathrm{B}(\mu_\gamma)$ (Theorem 7.1 in [**20**]). We say that a combinatorial gallery γ is a **Littelmann gallery** if there exist indices $i_1, \cdots, i_r$ such that $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma) = \gamma^+$ is a dominant gallery. If $\mu_{\gamma^+} = \mu_{\delta^+}$ and $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\gamma) = \gamma^+; e_{\alpha_{i_1}} \cdots e_{\alpha_{i_r}}(\delta) = \delta^+$ for two Littelmann galleries γ and δ we say that they are **equivalent**.

2.3.1. *LS galleries.* Let $\lambda = \omega_{i_1} + \cdots + \omega_{i_{k+1}} \in \mathrm{X}^{\vee,+}$ be a dominant coweight and $\gamma_{\underline{\lambda}} = \gamma_{\omega_{i_1}} * \cdots * \gamma_{\omega_{i_{k+1}}}$ the concatenation of the fundamental galleries $\gamma_{\omega_{i_j}}$ in the chosen order of the indices i_j. (Note that this is less general than the situation in the preceding paragraph.) The set $\mathbb{P}(\gamma_{\underline{\lambda}})$ defined above is the set of **combinatorial LS galleries** (short for Lakshmibai Seshadri galleries) of same type as $\gamma_{\underline{\lambda}}$; we will denote it by $\Gamma^{\mathrm{LS}}(\gamma_{\underline{\lambda}})$. Littelmann galleries generalise LS galleries enormously. In particular, all LS galleries are 'Littelmann' - see [**20**], Section 4. Moreover, the set $\Gamma^{\mathrm{LS}}(\gamma_{\underline{\lambda}})$ is stable under the root operators and crystal isomorphic to $\mathrm{B}(\lambda)$. It was proven by Gaussent-Littelmann in [**6**] that the resolution in (9) induces a bijection $\Gamma^{\mathrm{LS}}(\gamma_\lambda) \cong \mathcal{Z}(\lambda)$ which was shown to be a crystal isomorphism in [**1**] by Baumann-Gaussent. See Definition 18 in [**6**] for a geometric definition of LS galleries, and Definition 23 in [**6**] for an equivalent combinatorial characterisation that for one skeleton galleries agrees with the original definition by Lakshmibai, Musili, and Seshadri (see for example [**16**]) in the context of standard monomial theory. In Chapters 2 and 3 we recall a combinatorial characterisation of LS galleries of fundamental type in the cases $\mathrm{G}^\vee = \mathrm{SL}(n, \mathbb{C})$ and $\mathrm{G}^\vee = \mathrm{SP}(2n, \mathbb{C})$ respectively. We therefore omit the more general original definition of LS galleries.

2.3.2. *Relationship between galleries and paths.* Galleries can be considered as paths $\pi : [0,1] \to \mathrm{X}^\vee \otimes_\mathbb{Z} \mathbb{R}$ in the following way. For each fundamental coweight ω there is an associated path

$$\begin{aligned}\pi_\omega : [0,1] &\to \mathrm{X}^\vee \otimes_\mathbb{Z} \mathbb{R}\\ t &\mapsto t\omega.\end{aligned}$$

In this way fundamental galleries γ_ω are considered as paths π_ω. Given two paths $\pi_1, \pi_2 : [0,1] \to \mathrm{X}^\vee \otimes_\mathbb{Z} \mathbb{R}$ their concatenation $\pi_1 * \pi_2$ is defined as

$$\pi_1 * \pi_2(t) = \begin{cases} \pi_1(2t) & \text{if } 0 \leq t \leq 1/2 \\ \pi_1(1) + \pi_2(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

More generally, a combinatorial gallery

$$\gamma = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \cdots, \mathrm{E}_k, \mathrm{V}_{k+1})$$

is considered as the path

$$\pi_\gamma := \gamma_{\mathrm{V}_1} * \cdots * \gamma_{\mathrm{V}_{k+1}},$$

where

$$\begin{aligned}\gamma_{\mathrm{V}_0} : [0,1] &\to \mathrm{X}^\vee \otimes_\mathbb{Z} \mathbb{R}\\ t &\mapsto t(\mathrm{V}_0),\end{aligned}$$

and for $0 < i \leq k+1$:

$$\begin{aligned}\gamma_{\mathrm{V}_i} : [0,1] &\to \mathrm{X}^\vee \otimes_\mathbb{Z} \mathbb{R}\\ t &\mapsto t(\mathrm{V}_i - \mathrm{V}_{i-1}),\end{aligned}$$

where the vertices V_i are considered as points in the real space $\mathrm{X}^\vee \otimes_\mathbb{Z} \mathbb{R}$. We will use paths explicitly in Chapter 4. in particular we will refer to dominant paths - a path $\pi : [0,1] \to \mathrm{X}^\vee \otimes_\mathbb{Z} \mathbb{R}$ is **dominant** if its image $\pi([0,1])$ is contained in the dominant Weyl chamber. Note that if a combinatorial gallery γ is dominant then π_γ is dominant as well.

We finish this chapter with a question. Let γ be any combinatorial gallery with each one of its edges contained in the fundamental chamber. Then, as noted in Remark 1.6, the map

$$\begin{aligned}\Sigma_{\gamma^f} &\to \mathcal{G}\\ [g_0, \cdots, g_r] &\mapsto g_0 \cdots g_r [t^{\mu_\gamma^f}]\end{aligned}$$

is still defined.

Question. Does this map induce a crystal isomorphism $\mathbb{P}(\gamma) \cong \mathcal{Z}(\mu_\gamma)$?

This question is answered positively in [**8**] and Chapter 2 (which follows [**30**]) for $G^\vee = SL(n,\mathbb{C})$, and in Chapter 3 (which follows [**29**]) for $G^\vee = SP(2n,\mathbb{C})$ and γ a *readable* gallery.

Definition 1.14. A **readable** gallery is a concatenation of its **parts**: LS galleries of fundamental type and galleries of the form $(V_0, E_0, V_1, E_1, V_2)$ (we call them **zero lumps**) such that both edges E_0 and E_1 are contained in the dominant chamber and such that the endpoint $V_2 = 0$ is equal to zero. We denote the set of all readable galleries by Γ^R, and if a combinatorial gallery γ is fixed, by $\Gamma(\gamma)^{\mathrm{R}}$ the set of all readable galleries of same type as γ.

For $G^\vee = SL(n,\mathbb{C})$ all galleries are readable; this is due to the well known fact that in this case fundamental coweights are all minuscule. In Chapter 2 we will therefore work with the set of all galleries. In Chapter 3 we will describe readable galleries explicitly for $G^\vee = SP(2n,\mathbb{C})$ and show that they are Littelmann galleries. They are also more general than galleries of type γ_λ for a gallery γ_λ that is a concatenation of fundamental galleries - this means they belong to a larger class of galleries, but not that they contain $\Gamma(\gamma_\lambda)$.

CHAPTER 2

Word reading is a crystal morphism

In this chapter we will consider $G^\vee = \mathrm{SL}(n, \mathbb{C})$. First, in Section 1 we recall the combinatorics of keys and words in the alphabet $\mathcal{A}_n = \{1, \cdots, n\}$ and the plactic monoid associated to $\mathrm{SL}(n, \mathbb{C})$. Then we will review how one may associate a gallery to a given key and show that keys of a given *shape* are in one to one correspondence with a set of galleries of the same *type*.

1. Keys, their words, and the plactic monoid

In this section we recall basic definitions of keys, their words, and the *plactic monoid* associated to $\mathrm{SL}(n, \mathbb{C})$. One subtle thing to note in this section is the restriction on the lengths of the columns of a key - we also define semistandard Young tableaux taking this restriction into account. The 'usual' plactic monoid that is usually considered is associated to the representation theory of $\mathrm{GL}(n, \mathbb{C})$ and is defined by the *Knuth relations*; keys are usually concatenations of columns of length at most n. We include an extra relation, and consider concatenations of columns of length at most $n-1$. That said, this section is purely combinatorial.

1.1. Keys and their words. A **shape** is a finite sequence of positive integers $\underline{d} = (d_1, \cdots, d_{k+1})$ such that $d_s \leq n-1$ for all $s \in \{1, \cdots, k+1\}$. An arrangement of boxes of shape $\underline{d}$ is an arrangement of r columns of boxes such that column s (read from right to left) has d_s boxes.

EXAMPLE 2.1.

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline \multicolumn{1}{c|}{} & & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array}$$

An arrangement of boxes of shape (1,1,2,1).

A **key** of shape $\underline{d}$ is a filling of an arrangement of boxes of the given shape with letters from the ordered alphabet $\mathcal{A}_n = \{1, \cdots, n : 1 < \cdots < n\}$ such that entries are strictly increasing along each column of boxes. We will denote the set of keys of shape $\underline{d}$ by $\mathbf{\Gamma}(\underline{d})$, and the set of all keys by $\mathbf{\Gamma}$.

EXAMPLE 2.2.

$$\begin{array}{|c|c|c|c|} \hline 3 & 1 & 5 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array}$$

A key of shape (1,1,2,1).

REMARK 2.3. The reason for restricting the length of the columns of a shape is that for $\mathrm{SL}(n, \mathbb{C})$ there are only $n-1$ fundamental coweights. This will be clear after the next section.

Let $\mathcal{W}_n$ denote the word monoid on $\mathcal{A}_n$. To a word $w = a_1 \cdots a_k$ is associated the key $\mathscr{K}_w = \boxed{a_k}\boxed{\cdots}\boxed{a_1}$. The **word** of a key $\mathscr{K}$ of shape $(m), m \in \mathbb{Z}^{>0}$ is the word in $\mathcal{W}_n$ that corresponds to reading the entries of $\mathscr{K}$ from top to bottom and writing them down from left to right. The word of an arbitrary key $\mathscr{T}$, which we denote by $w(\mathscr{T})$, is the concatenation of the words of each of its columns read from right to left.

EXAMPLE 2.4. The keys $\mathscr{K} = \begin{array}{|c|c|c|c|c|}\hline 3 & 2 & 1 & 5 & 2 \\ \hline\end{array}$ and $\mathscr{T} = \begin{array}{|c|c|c|c|}\hline 3 & 1 & 5 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{2}{c}{} \\ \cline{2-2}\end{array}$ both have word $25123 = w(\mathscr{T}) = w(\mathscr{K})$.

1.2. The plactic monoid. We say that a key of shape $\underline{d} = (d_1, \cdots, d_{k+1})$ is a **semi-standard Young tableau** if $d_1 \leq \cdots \leq d_{k+1}$ and if the entries are weakly increasing from left to right in rows. We will denote the set of all semi-standard Young tableaux of shape $\underline{d}$ by $\boldsymbol{\Gamma}(\underline{d})^{\mathrm{SSYT}}$.

EXAMPLE 2.5. The gallery

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

is a semi-standard Young tableau. Note that both keys considered in Example 2.4 are not.

DEFINITION 2.6. The *plactic monoid* is the quotient $\mathcal{P}_n = \mathcal{W}_n / \sim$ of $\mathcal{W}_n$ by the ideal $\sim$ generated by the following relations.

a. For $x \leq y < z$, $y\ x\ z = y\ z\ x$.
b. For $x < y \leq z$, $x\ z\ y = x\ y\ z$.
c. $1 \cdots n = \varnothing$, where $\varnothing$ denotes the trivial word.

If two words have equal classes in the plactic monoid, we say they are **plactic equivalent**. If two words have equal classes in the quotient $\mathcal{W}_n / \sim_{\mathrm{K}}$ by the ideal $\sim_{\mathrm{K}}$ generated by relations a. and b. above, they are usually said to be **Knuth equivalent**.

Theorem 2.7 below is generally well known (originally Theorem 6 in [**14**]) and similar to Theorem 1 in [**8**], but (as already mentioned previously) there is an extra restriction on the length of the longest column of the keys that we consider.

THEOREM 2.7. Given any key $\mathscr{T}$ there exists a unique semi-standard Young tableau $\mathscr{T}_{SS}$ such that $w(\mathscr{T})$ is plactic equivalent to $w(\mathscr{T}_{SS})$.

PROOF. Let $\mathscr{T}$ be a key and let w be a representative of minimal length of the class in $\mathcal{P}_n$ of its word $w(\mathscr{T})$. Let $\mathscr{T}_{SS}$ be the semistandard Young tableau obtained by applying Robinson-Schensted-Knuth insertion (see for example [**25**], second definition in Part I) to w read from right to left (the reason for this is that we want to keep the word reading convention of [**8**]). Now we use a result of C. Schensted (Theorem 2 in [**25**], the general version from Part II): the number of rows (or the length of the longest column) of $\boldsymbol{\gamma}_{\mathbf{SS}}$ equals the length of the longest decreasing subsequence of w (read from right to left!). Since w is a minimal length representative, we claim that it cannot have an decreasing subsequence of length n. Indeed, for any $i \leq n$, the relations a. and b. above imply that for $j \leq i, 1 \cdots ij$ and $j 1 \cdots i$ are plactic equivalent. The claim from the previous sentence then follows by induction. Hence, by Schensted's result, $\mathscr{T}_{SS}$ has columns of length at most $n-1$. By Theorem 6 in [**14**], $\mathscr{T}_{SS}$ is the unique semistandard Young tableau such that its word $w(\mathscr{T}_{SS})$ is Knuth equivalent to w, which is plactic equivalent to $w(\mathscr{T})$ by definition. $\square$

EXAMPLE 2.8. For $n=3$, the galleries $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline\end{array}$, and $\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 1 & 3 & 2 & 1 \\ \hline\end{array}$ are all equivalent to the semi-standard Young tableau $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$.

2. The gallery associated to a key

2.1. Weights, coweights, and notation. In this section we recall some basic facts and establish some notation. First consider the group $\mathrm{GL}(n,\mathbb{C})$ of invertible $n\times n$ matrices, and in it the maximal torus $\mathrm{T}_{\mathrm{GL}}(n,\mathbb{C})$ of diagonal matrices. Then maximal tori for $\mathrm{SL}(n,\mathbb{C}) = [\mathrm{GL}(n,\mathbb{C}),\mathrm{GL}(n,\mathbb{C})]$ and $\mathrm{PSL}(n,\mathbb{C}) = \mathrm{GL}(n,\mathbb{C})/\mathbb{C}^\times \mathrm{Id}$ are given by

$$\mathrm{T}_{\mathrm{SL}}(n,\mathbb{C}) = \mathrm{T}_{\mathrm{GL}}(n,\mathbb{C}) \cap \mathrm{SL}(n,\mathbb{C})$$

and

$$\mathrm{T}_{\mathrm{PSL}}(n,\mathbb{C}) = \mathrm{can}(\mathrm{T}_{\mathrm{GL}}(n,\mathbb{C}))$$

respectively, where

$$\mathrm{can} : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{PSL}(n,\mathbb{C})$$

is the canonical map.

Consider $\mathbb{R}^n$ with inner product $(-,-)$ and orthonormal basis $\{\varepsilon_1,\cdots,\varepsilon_n\}$. We make the following identifications:

$$\begin{aligned}
\mathrm{X} = \mathrm{X}(\mathrm{T}_{\mathrm{PSL}}(n,\mathbb{C}))) &= \mathrm{Hom}(\mathrm{T}_{\mathrm{PSL}}(n,\mathbb{C}),\mathbb{C}^\times) \cong \mathrm{Hom}(\mathbb{C}^\times,\mathrm{T}_{\mathrm{SL}}(n,\mathbb{C})) \\
&= \mathrm{X}^\vee(\mathrm{T}_{\mathrm{SL}}(n,\mathbb{C}))) = \{a_1\varepsilon_1 + \cdots + a_n\varepsilon_n : a_i \in \mathbb{Z}; \sum_{i=1}^n a_i = 0\} = \mathbb{Z}\Phi \cong \mathbb{Z}\Phi^\vee, \\
\mathrm{X}^\vee = \mathrm{X}^\vee(\mathrm{T}_{\mathrm{PSL}}(n,\mathbb{C}))) &= \mathrm{Hom}(\mathbb{C}^\times,\mathrm{T}_{\mathrm{PSL}}(n,\mathbb{C})) \cong \mathrm{Hom}(\mathrm{T}_{\mathrm{SL}}(n,\mathbb{C}),\mathbb{C}^\times) \\
&= \mathrm{X}(\mathrm{T}_{\mathrm{SL}}(n,\mathbb{C})) = \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i / \langle \sum_{i=1}^n \varepsilon_i \rangle.
\end{aligned}$$

where Φ and $\Phi^\vee$ are the sets of roots and coroots, respectively. The inner product $(-,-)$ restricts to the pairing between X and $\mathrm{X}^\vee$. In particular the root data

$$(\mathrm{X}(\mathrm{T}_{\mathrm{SL}}(n,\mathbb{C}))), \mathrm{X}^\vee(\mathrm{T}_{\mathrm{SL}}(n,\mathbb{C}))), \Phi, \Phi^\vee)$$

associated to $(\mathrm{SL}(n,\mathbb{C}), \mathrm{T}_{\mathrm{SL}}(n,\mathbb{C})))$ is dual to that

$$(\mathrm{X}(\mathrm{T}_{\mathrm{PSL}}(n,\mathbb{C}))), \mathrm{X}^\vee(\mathrm{T}_{\mathrm{PSL}}(n,\mathbb{C}))), \Phi^\vee, \Phi)$$

of $(\mathrm{PSL}(n,\mathbb{C}), \mathrm{T}_{\mathrm{PSL}}(n,\mathbb{C})))$. We choose the set of simple roots

$$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} : 1 \le i < n\},$$

which in this case coincides with the corresponding set of simple coroots $\alpha_i^\vee = \alpha_i \in \Delta^\vee$. We write Φ^+ and $\Phi^{\vee,+}$ for the corresponding sets of positive roots and coroots, respectively. The corresponding i-th fundamental coweight is $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$, for

$i \in \{1, \cdots, n-1\}$. The dominant Weyl chamber is identified with the intersection $\bigcap_{\alpha_i \in \Delta} \mathrm{H}_{\alpha_i,0}$. Also,

$$\mathbb{A} \cong \{w \in \mathbb{R}^n : (w, e_1 + \cdots + e_n) = 0\} \cong \mathbb{R}^n / \mathbb{R}(e_1 + \cdots + e_n).$$

We will use the following notation, especially in Chapter 4: $\mathrm{P}_{A_{n-1}} = \mathrm{X}^\vee, \mathrm{P}^+_{A_{n-1}} = \mathrm{X}^{\vee,+} = \bigoplus_{i=1}^{n-1} \mathbb{Z}\omega_i$, and $\mathbb{V}_{A_{n-1}} = \mathrm{X}^\vee \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{A}$ (This last notation is introduced to distinguish between type A_{n-1} and type C_n, the last of which is dealt with in Chapter 3).

2.2. The gallery associated to a key. Let $\mathscr{T} = \mathrm{C}_{k+1} \cdots \mathrm{C}_1$ be a key with columns C_i, each one of which is made up of entries $1 \leq l_i^1 < \cdots l_i^{r_i} \leq n$ Define

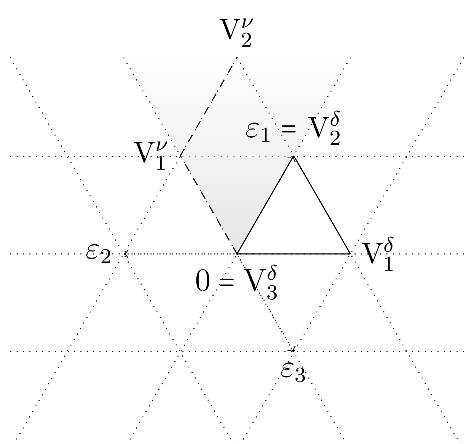
$$\begin{aligned}
&\mathrm{V}_0^{\mathscr{T}} = 0, \mathrm{V}_1^{\mathscr{T}} := \sum_{j=1}^{r_1} \varepsilon_{l_1^j}, \mathrm{E}_1^{\mathscr{T}} := \{t\mathrm{V}_1 : t \in [0,1]\}, \text{ and, recursively} \\
&\mathrm{V}_{i+1}^{\mathscr{T}} = \mathrm{V}_i^{\mathscr{T}} + \sum_{j=1}^{r_i} \varepsilon_{l_j^i}, \\
&\mathrm{E}_{j+1}^{\mathscr{T}} = \{\mathrm{V}_j^{\mathscr{T}} + t(\mathrm{V}_{j+1}^{\mathscr{T}} - \mathrm{V}_j^{\mathscr{T}}) : t \in [0,1]\}.
\end{aligned}$$

The points $\mathrm{V}_j^{\mathscr{T}}$ are in fact vertices of the standard apartment $\mathbb{A}$ because $(\alpha, \mathrm{V}_j^{\mathscr{T}}) \in \mathbb{Z}$ is an integer for every root $\alpha \in \Phi$. The line segments $\mathrm{E}_j^{\mathscr{T}}$ are edges in the standard apartment because there are no hyperplanes that intersect $\mathrm{E}_i^{\mathscr{T}}$ transversally and because they are contained in every hyperplane that contains both vertices $\mathrm{V}_j^{\mathscr{T}}$ and $\mathrm{V}_{j+1}^{\mathscr{T}}$. The gallery

$$\gamma_{\mathscr{T}} = (\mathrm{V}_0^{\mathscr{T}}, \cdots, \mathrm{V}_{k+1}^{\mathscr{T}})$$

is the **gallery associated to the key** $\mathscr{T}$.

EXAMPLE 2.9. Let $n = 3$. In the picture below (the shaded region is the dominant Weyl chamber), we see the galleries $\gamma_{\mathscr{T}}$ and $\gamma_{\mathscr{K}}$ associated to the keys $\mathscr{T} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline & 2 \\ \hline\end{array}$ and $\mathscr{K} = \begin{array}{|c|c|c|}\hline 2 & 3 & 1 \\ \hline\end{array}$, respectively. Note that the gallery $\gamma_{\mathscr{T}}$ is dominant while $\gamma_{\mathscr{K}}$ is not.



2.3. Type and shape. The following proposition is a generalisation of Proposition 4.12 in [**7**]. The proof in this case is the same, but we provide it anyway for the comfort of the reader.

PROPOSITION 2.10. The map that asigns the gallery $\gamma_{\mathscr{T}}$ to the key $\mathscr{T}$ induces a bijection

$$\begin{aligned} \mathbf{\Gamma}(\underline{d}) &\xleftrightarrow{1:1} \Gamma(\gamma_{\omega_{d_1}} * \cdots * \gamma_{\omega_{d_{k+1}}}) \\ \mathscr{T} &\mapsto \gamma_{\mathscr{T}} \end{aligned}$$

for all shapes $\underline{d} = (d_1, \cdots, d_{k+1})$. If $d_1 \leq \cdots \leq d_{k+1}$ The set of galleries associated to semi-standard Young tableaux of shape $\underline{d}$ coincides with the set of LS galleries of the same type as $\gamma_{\omega_{d_1}} * \cdots * \gamma_{\omega_{d_{k+1}}}$.

PROOF. By Remark 1.7 it is enough to show the statement in Proposition 2.10 for $\underline{d} = (d)$ for some $d \leq n-1$. In that case $\mathbf{\Gamma}(\underline{d})$ consists of columns of length d with entries $1 \leq l_1 < \cdots < l_d \leq n$. On the other hand, the set $\Gamma(\gamma_{\omega_d})$ consists of galleries of the form $\gamma = (0, \mathrm{E}_0, \mathrm{V}_1)$, where $\mathrm{E}_0 = \{t\mathrm{V}_1 : t \in [0,1]\}$ and $\mathrm{V}_1 = w(\omega_d) = w(\varepsilon_1 + \cdots + \varepsilon_d) = \varepsilon_{w(1)} + \cdots + \varepsilon_{w(d)}$ for some element $w \in \mathrm{S}_n = \mathrm{W} = \mathrm{W}_0^{aff}$. Since there is a bijection

$$\{w(1), \cdots, w(d) : w \in \mathrm{S}_n\} \xleftrightarrow{1:1} \{1 \leq l_1 < \cdots < l_d \leq n : l_i \in \mathbb{Z}^{>0}\}$$

the statement follows. For the statement about LS galleries, see Proposition 18 in [**7**].

$\square$

2.4. Crystal structure on the set of keys. The previous identification allows one to describe in this case the crystal structure on galleries using only the set of keys. The definition we provide here is a straightforward generalization of the crystal operators on Young tableaux given in [**12**] and is the translation of the crystal operators on galleries as defined in Chapter 1.

Let $\mathscr{T}$ be a key of shape $\underline{d} = \{d_1, \cdots, d_r\}$. Define

$$\mathrm{wt}(\mathscr{T}) = \sum_{i \in w(\mathscr{T})} \varepsilon_i,$$

where the word $w(\mathscr{T})$ is regarded as a set with possible repetitions. In Example 2.9, $\mathrm{wt}(\mathscr{T}) = 2\varepsilon_1 + \varepsilon_2$ and $\mathrm{wt}(\mathscr{K}) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$. The action of the root operator f_{α_i} (respectively e_{α_i}) on $\mathscr{T}$ is defined as follows.

a. Tag the columns of $\mathscr{T}$ with a sign $\sigma \in \{+, -, \varnothing\}$ (the resulting sequence of tags is sometimes called the *i-signature* of $\mathscr{T}$). If both i and $i+1$ appear in the given column or if they do not appear in the column, then the column is tagged with a $(\varnothing)$. If only i appears, it is tagged with a $(+)$, and if only $i+1$ appears, with a $(-)$.
b. Ignore the $(\varnothing)$-tagged columns to produce a sub-key, and then ignore all pairs of consecutive columns tagged $(-\ +)$, and get another sub-gallery.

Continue this process, recursively obtaining sub-keys, until a final sub-key is produced with tags of the form

$$(+)^s(-)^r$$

To apply the operator f_{α_i} (respectively e_{α_i}), modify the column corresponding to the right most $(+)$ (respectively left most $(-)$) in the final sub-key tags, and replace the entry i with $i+1$ (respectively $i+1$ with i). If $s=0$ (resp. $r=0$), then $f_{\alpha_i}(\mathscr{T})=0$ (resp $e_{\alpha_i}(\mathscr{T})=0$).

PROPOSITION 2.11. The previous definition is compatible with the crystal structure on galleries. Explicitly, for all $i \leq n-1$, $\gamma_{f_{\alpha_i}(\mathscr{T})} = f_{\alpha_i}(\gamma_{\mathscr{T}}), \gamma_{e_{\alpha_i}(\mathscr{T})} = e_{\alpha_i}(\gamma_{\mathscr{T}})$, and $\mathrm{wt}(\gamma_{\mathscr{T}}) = \mathrm{wt}(\mathscr{T})$.

PROOF. The proof follows directly from the definitions. $\square$

EXAMPLE 2.12. Let $n \geq 5$. To apply the crystal operator f_{α_2} to

$$\mathscr{T} = \begin{array}{|c|c|c|c|}\hline 3 & 1 & 5 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array},$$

one obtains that the corresponding taggings of the columns read from left to right are $- + \varnothing +$. The first sub-key obtained is

$$\begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array},$$

which is tagged by $- + +$. The next sub-key is then $\boxed{2}$, hence

$$f_{\alpha_2}(\mathscr{T}) = \begin{array}{|c|c|c|c|}\hline 3 & 1 & 5 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}.$$

We also obtain that $f_{\alpha_1}(\mathscr{T}) = 0$.

3. Words and the Littelmann path model

3.1. Words. The following proposition is very important for our purposes. It is well known for semistandard Young tableaux (see for example [**12**], Section 5.3). Let $\underline{d} = (d_1, \cdots, d_r)$ be a shape, $l_d = \sum_{j=1}^{r} d_j$ the number of boxes in the arrangement of boxes of shape $\underline{d}$ and $\underline{l_d} = \underbrace{(1, \cdots, 1)}_{l_d\text{–}times}$.

PROPOSITION 2.13. The map

$$\begin{aligned} \mathbf{\Gamma}(\underline{d}) &\longrightarrow \mathbf{\Gamma}(\underline{l_d}) \\ \mathscr{T} &\longmapsto \mathscr{K}_{w(\mathscr{T})} \end{aligned}$$

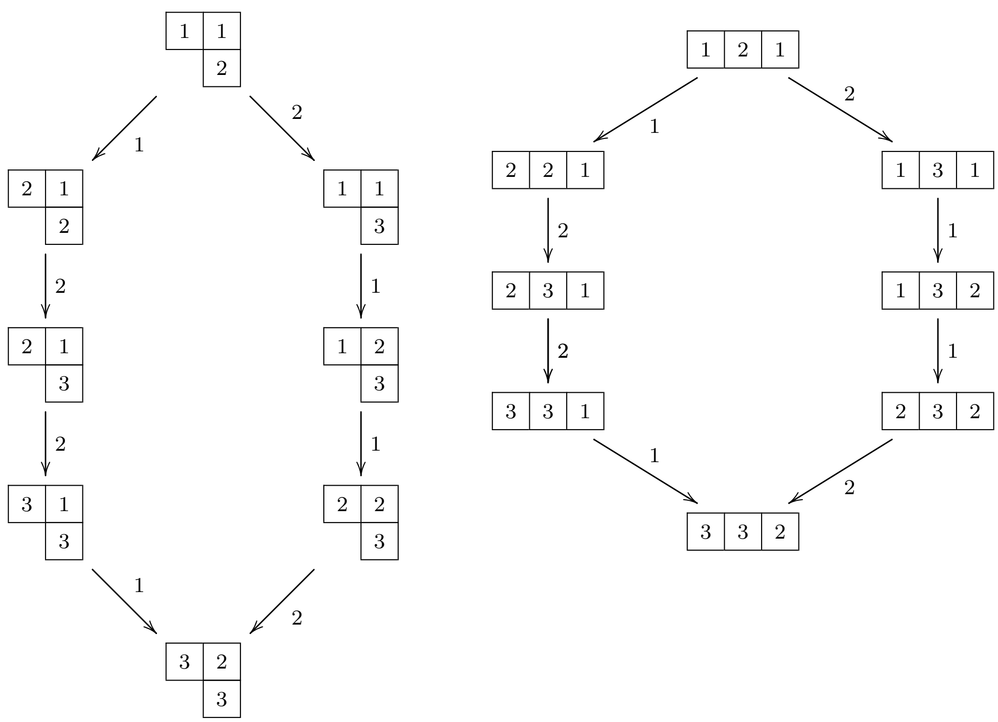
is a crystal morphism.

PROOF. First note that since the weight of a key only depends on the entries of its boxes, $\mathrm{wt}(\mathscr{T}) = \mathrm{wt}(\mathscr{K}_{w(\mathscr{T})})$. If two single column keys $\mathscr{T}_1, \mathscr{T}_2$ are labelled by $(+)$ and $(-)$ respectively, then the word associated to their concatenation $\gamma_2 * \gamma_1$ is in turn labelled by $(- +)$. If the key $\mathscr{T}$ is not labelled, then $\mathscr{K}_{w(\mathscr{T})}$ is labelled either by $(- +)$ or by $\varnothing$. It is therefore enough to show that for any $i \in \{1, \cdots, n-1\}$

and any key $\mathscr{T}$ of shape (m), $f_{\alpha_i}(\mathscr{K}_{w(\mathscr{T})}) = \mathscr{T}_{w(f_{\alpha_i}(\mathscr{T}))}$. This is shown in [**12**], Section 5.3, Proposition 5.1. We give a proof nevertheless, for the comfort of the reader.

Let $\mathscr{T}$ be a column key of shape (m) with entries $1 \leq a_1 < \cdots < a_m \leq n$ and $i \in \{1, \cdots, n-1\}$. If $\mathscr{T}$ is labelled by $(\varnothing)$ or by $(-)$ then $f_{\alpha_i}(\mathscr{K}_{w(\mathscr{T})}) = \mathscr{K}_{w(f_{\alpha_i}(\mathscr{T}))} = 0$. If $\mathscr{T}$ is labelled by $(+)$, then, for some $k \in \{1, \cdots, r\}$, $a_k = i$ and since the column is labelled by only a $(+)$, $a_{k+1} > a_k + 1$. Hence, $f_{\alpha_i}(\mathscr{T})$ is obtained from $\mathscr{T}$ by replacing $i = a_k$ by $i + 1$, with no need of reordering the entries, and therefore $f_{\alpha_i}(\mathscr{K}_{w(\mathscr{T})}) = \mathscr{K}_{w(f_{\alpha_i}(\mathscr{T}))}$. □

Example 2.14. A connected crystal of keys of shape $(2, 1)$ and the crystal formed by its word-readings, regarded as galleries, in the case $n = 3$. Both crystals are isomorphic to the crystal $\mathrm{B}(\omega_1 + \omega_2)$ associated to the simple module $\mathrm{L}(\omega_1 + \omega_2)$ for $\mathrm{SL}(3, \mathbb{C})$.



3.2. The Littelmann path model. Proposition 2.13 allows an enhanced version of Theorem 7.1 in [**20**] which we state in Theorem 2.16 (it is well-known but the author has not found an explicit reference). To prove it we need the following lemma which characterizes dominant galleries as highest weight vertices.

Lemma 2.15. A key $\mathscr{T} \in \mathbf{\Gamma}$ is dominant if and only if $e_{\alpha_i}(\mathscr{T}) = 0$ for all $i \in \{1, \cdots, n-1\}$.

Proof. Let $\mathscr{T} \in \mathbf{\Gamma}$ be a key. First notice the following two things.

1. Since entries are strictly increasing in columns, the key $\mathscr{T}$ is dominant if and only if $\mathscr{K}_{w(\mathscr{T})}$ is dominant.
2. For a word $w \in \mathcal{W}_n$, the condition $e_{\alpha_i}(\mathscr{K}_w) = 0$ for all $i \in \{1, \cdots, n-1\}$ means that to the right of each $i+1$ in γ_w is at least one i which has not

been cancelled out in the tagging and subword-producing process. This is equivalent to $\mathscr{K}_w$ being dominant.

Now assume that $e_{\alpha_i}(\mathscr{T}) = 0$ for all $i \in \{1,\cdots,n\}$. By Proposition 2.13 this is equivalent to $e_{\alpha i}(\mathscr{K}_{w(\mathscr{T})}) = 0$ for all $i \in \{1,\cdots,n\}$, which by 2. above is equivalent to $\mathscr{K}_{w(\mathscr{T})}$ being dominant, which is in turn equivalent to $\mathscr{T}$ being dominant by 1. above.

□

THEOREM 2.16 (The type A path model). The connected components of Γ are all of the form $\mathrm{Conn}(\delta) \cong \mathrm{B}(\mathrm{wt}(\delta))$ for a dominant gallery δ.

PROOF. By Theorem 7.1 in [**20**] it is enough to show that for every gallery ν there is a dominant gallery $\delta \in \mathrm{Conn}(\nu)$ that belongs to the same connected component as ν. To see this consider a key $\mathscr{T} \in \mathbf{\Gamma}(\underline{d})$ of shape $\underline{d}$. Its word, seen as the gallery $\mathscr{K}_{w(\mathscr{T})}$, lies in the crystal $\mathbf{\Gamma}(\underline{l_d})$. As is explained in Section 13 of [**21**], this is the crystal B_M associated to the representation $\mathrm{M} := \mathrm{L}(\omega_1)^{\otimes l(w(\mathscr{T}))}$, where $l(w(\mathscr{T}))$ is the length of the word $w(\mathscr{T})$. The representation M is semisimple, hence $\mathscr{K}_{w(\mathscr{T})}$ lies in a connected component $\mathrm{Conn}(\gamma_{w(\mathscr{T})}) \cong \mathrm{B}(\lambda)$ isomorphic to the crystal associated to a simple module $\mathrm{L}(\lambda)$ of highest weight $\lambda \in \mathrm{X}^+$, with highest vertex $b_\lambda \in \mathbf{\Gamma}(\underline{l_d})$. Proposition 2.13 implies that $\mathrm{Conn}(\mathscr{T}) \cong \mathrm{Conn}(\mathscr{K}_{w(\mathscr{T})})$ - hence there exists a key $\mathscr{P} \in \mathbf{\Gamma}(\underline{d})$ such that $\mathscr{K}_{w(\mathscr{P})} = b_\lambda$. In particular, since $\mathscr{K}_{w(\mathscr{P})}$ is a highest weight vertex, by Lemma 2.15 it is dominant, hence by 1. in the proof of Lemma 2.15, so is $\mathscr{P}$. □

3.3. Equivalence of galleries and plactic equivalence.

LEMMA 2.17. Two galleries $\gamma_{\mathscr{T}}$ and $\gamma_{\mathscr{K}}$ are equivalent if and only if the words $w(\mathscr{T})$ and $w(\mathscr{K})$ are plactic equivalent.

PROOF. Let $\gamma_{\mathscr{T}}$ and $\gamma_{\mathscr{K}}$ be two such galleries, and assume that the words $w(\mathscr{T})$ and $w(\mathscr{K})$ are plactic equivalent. Then by Main Theorem C b. in [**21**] this is equivalent to $\gamma_{w(\mathscr{T})} \sim \gamma_{w(\mathscr{K})}$. Proposition 2.13 implies that word reading induces isomorphisms of crystals $\mathrm{Conn}(\nu) \xrightarrow{\sim} w(\mathrm{Conn}(\nu)), \nu \mapsto \gamma_{w(\nu)}$ for any gallery ν, where $w(\mathrm{Conn}(\nu))$ is the crystal of all words of elements in $\mathrm{Conn}(\nu)$. This concludes the proof. □

REMARK 2.18. Lemma 2.17 implies that our definition of equivalence of galleries coincides with Definition 5 in [**8**] (after adding the relation $1\cdots n = \emptyset$).

REMARK 2.19. The crystal structure we have defined coincides with the usual crystal structure on the set of semi-standard Young tableaux (see [**9**], section 7.4).

4. Galleries and MV cycles

To each shape $\underline{d} = (d_1,\cdots,d_r)$ we assign the dominant integral coweight $\lambda_{\underline{d}} = \omega_{d_1} + \cdots + \omega_{d_r} \in \mathrm{X}^{\vee,+}$. By Proposition 2.10, to each shape $\underline{d}$ we may denote (for this section!) the corresponding Bott-Samelson variety $\Sigma_{\underline{d}} \xrightarrow{\pi_{\underline{d}}} \mathrm{X}_{\lambda_{\underline{d}}}$. Also, for each $\mathscr{T} \in \mathbf{\Gamma}$, we denote its shape by $\underline{d}(\mathscr{T})$. The following theorem is the combination of Theorem 2 in [**6**] and Section 6 in [**7**] for part a., and Theorem 25 in [**1**] for part b..

THEOREM 2.20. Let $\underline{d} = (d_1, \cdots, d_r)$ be a shape such that $d_1 \leq \cdots \leq d_r$ and consider the desingularization $\pi_{\underline{d}} : \Sigma_{\underline{d}} \to \mathrm{X}_{\lambda_{\underline{d}}}$.

a. If $\delta \in \mathbf{\Gamma}(\underline{d})^{\mathrm{SSYT}}$ is a semi-standard Young tableau, the closure $\overline{\pi_{\underline{d}}(\mathrm{C}_\delta)}$ is an MV cycle in $\mathcal{Z}(\lambda_{\underline{d}})$. This induces a bijection $\mathbf{\Gamma}(\underline{d})^{\mathrm{SSYT}} \xrightarrow{\varphi_{\underline{d}}} \mathcal{Z}(\lambda_{\underline{d}})$.
b. The bijection $\varphi_{\underline{d}}$ is a morphism of crystals.

Let $\underline{d}$ be a shape. For $\lambda \in \mathrm{X}^{\vee,+}$, let

$$n_{\underline{d}}^{\lambda} = \#\{\gamma \in \Gamma(\underline{d})^{\mathrm{dom}} : \lambda_{\underline{d}(\gamma)} = \lambda\}$$

and let

$$\mathrm{X}_{\underline{d}}^{\vee,+} = \{\lambda \in \mathrm{X}^{+} : n_{\underline{d}}^{\lambda} \neq 0\}.$$

Here $\Gamma(\underline{d})^{\mathrm{dom}}$ is the set of all dominant galleries of shape $\underline{d}$. Fix $\lambda = \lambda_1\omega_1 + \cdots + \lambda_{n-1}\omega_{n-1}$ and $\mathrm{Z} \in \mathcal{Z}(\lambda)_\mu$ for some $\mu \leq \lambda$. By Theorem 2.20 there exists a unique semi-standard Young tableau $\mathscr{T}_{\mu,\mathrm{Z}}^{\lambda} \in \mathbf{\Gamma}(\underline{\lambda})^{\mathrm{SSYT}}$ of shape $\underline{\lambda} = (d_1^{\lambda}, \cdots, d_{k_\lambda}^{\lambda})$, where $k_\lambda = \sum_{i=1}^{n-1} \lambda_i$ and $d_j^{\lambda} = i$ for $\lambda_{i-1} < j \leq \lambda_i, \lambda_0 = 0$, such that $\varphi_{\underline{\lambda}}(\mathscr{T}_{\mu,\mathrm{Z}}^{\lambda}) = \mathrm{Z}$.

THEOREM 2.21. a. The map

$$\begin{aligned} \mathbf{\Gamma}(\underline{d}) &\xrightarrow{\varphi_{\underline{d}}} \bigoplus_{\lambda \in \mathrm{X}_{\underline{d}}^{\vee,+}} \mathcal{Z}(\lambda) \\ \mathscr{T} &\longmapsto \overline{\pi_{\underline{d}}(\mathrm{C}_{\gamma_{\mathscr{T}}})} \end{aligned}$$

is a well-defined surjective morphism of crystals.
b. If C is a connected component of $\mathbf{\Gamma}(\underline{d})$, the restriction $\varphi_{\underline{d}}|_{\mathrm{C}}$ is an isomorphism onto its image.
c. The number of connected components C of $\mathbf{\Gamma}(\underline{d})$ such that $\varphi_{\underline{d}}(\mathrm{C}) = \mathcal{Z}(\lambda)$ (for $\lambda \in \mathrm{X}_{\underline{d}}^{\vee,+}$) is equal to $n_{\underline{d}}^{\lambda}$.
d. The fibre $\varphi_{\underline{d}}^{-1}(\mathrm{Z})$ is given by

$$\varphi_{\underline{d}}^{-1}(\mathrm{Z}) = \{\mathscr{T} \in \mathbf{\Gamma}(\underline{d}) : \varphi_{\underline{d}}(\mathscr{T}) = \mathrm{Z}\} = \{\mathscr{T} \in \mathbf{\Gamma}(\underline{d}) : w(\mathscr{T}) \sim w(\mathscr{T}_{\mu,\mathrm{Z}}^{\lambda})\}.$$

We consider the direct sum $\bigoplus_{\lambda \in \mathrm{X}_{\underline{d}}^{\vee,+}} \mathcal{Z}(\lambda)$ in the category of crystals, regarding the sets $\mathcal{Z}(\lambda)$ as abstract crystals.

PROOF. Let $\underline{d}$ be a shape and $\mathscr{T} \in \mathbf{\Gamma}(\underline{d})$ as in the statement of the Theorem. By Lemma 2.7 there exists a unique semi-standard Young tableau $\mathscr{T}_{SS}$ such that $w(\mathscr{T}) \sim w(\mathscr{T}_{SS})$. By Theorem 5.1 b. (Theorem 2 b. in [**8**] up to a small correction, see the Appendix) and Lemma 2.17,

$$\overline{\pi_{\underline{d}}(\mathrm{C}_{\mathscr{T}})} = \overline{\pi_{\underline{d}(\mathscr{T}_{SS})}(\mathrm{C}_{\mathscr{T}_{SS}})}. \tag{13}$$

Now let r be a root operator. By definition of equivalence of galleries, Lemma 2.17 $\gamma_{r(\mathscr{T})} \sim \gamma_{r(\mathscr{T}_{SS})}$. Note also that $\underline{d}(r(\mathscr{T})) = \underline{d}$ and $\underline{d}(r(\mathscr{T}_{SS})) = \underline{d}(\mathscr{T}_{SS})$. Lemma 2 and Theorem 5.1 b. again imply

$$\overline{\pi_{\underline{d}}(\mathrm{C}_{\gamma_{r(\mathscr{T})}})} = \overline{\pi_{\underline{d}(\delta_{SS})}(\mathrm{C}_{\gamma_{r(\mathscr{T}_{SS})}})}.$$

Theorem 2.20 b. says that

$$\overline{\pi_{\underline{d}(\gamma_{\mathscr{T}_{SS}})}(\mathrm{C}_{\gamma_{r(\mathscr{T}_{SS})}})} = \tilde{r}(\overline{\pi_{\underline{d}(\gamma_{\mathscr{T}_{SS}})}(\mathrm{C}_{\gamma_{\mathscr{T}_{SS}}})}),$$

and since (13) implies $\tilde{r}(\overline{\pi_{\underline{d}}(\mathrm{C}_{\mathscr{T}})}) = \tilde{r}(\overline{\pi_{\underline{d}(\mathscr{T}_{SS})}(\mathrm{C}_{\mathscr{T}_{SS}})})$, the proof of part a. of Theorem 3.26 is complete.

Parts b., c., and d. are a direct consequence of Theorem 2.16: Indeed, since the action of the root operators does not affect the shape of a key, Theorem 2.16 implies that the set $\mathbf{\Gamma}(\underline{d})$ is a disjoint union $\mathbf{\Gamma}(\underline{d}) = \bigsqcup_{\mathscr{Q}\in\mathbf{\Gamma}(\underline{d})^{\mathrm{dom}}} \mathrm{Conn}(\mathscr{Q})$. The above argument and Theorem 2.20 imply that $\varphi_{\underline{d}}(\mathrm{Conn}(\mathscr{Q})) = \mathcal{Z}(\mathrm{wt}(\mathscr{Q}))$ for $\mathscr{Q} \in \mathbf{\Gamma}(\underline{d})^{\mathrm{dom}}$ and that $\varphi_{\underline{d}}$ is a crystal isomorphism onto its image when restricted to $\mathrm{Conn}(\mathscr{Q})$. $\square$

CHAPTER 3

The symplectic plactic monoid and MV cycles

In this Chapter we study the case $G^\vee = SP(2n, \mathbb{C})$.

1. Symplectic keys, words, and the symplectic plactic monoid

1.1. Symplectic keys and words. A **symplectic shape**

$$\underline{d} = (d_1, \cdots, d_{k+1})$$

is a sequence of natural numbers $d_i \leq n$. This defines an arrangement of boxes just as in Chapter 2. Consider the ordered alphabet

$$\mathcal{C}_n = \{1 < 2 < \cdots < n-1 < n < \overline{n} < \cdots < \overline{1}\}.$$

A **symplectic key** of shape $\underline{d}$ is a filling of an arrangement of boxes of symplectic shape $\underline{d}$ with letters of the alphabet $\mathcal{C}_n$ in such a way that the entries are strictly increasing along each column.

EXAMPLE 3.1. A symplectic key, for $n \geq 5$, of symplectic shape $(1, 3, 2, 1)$.

$$\begin{array}{|c|c|c|c|} \hline \overline{1} & 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 5 & 2 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \overline{2} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$$

We denote the word monoid on $\mathcal{C}_n$ by $\mathcal{W}_{\mathcal{C}_n}$. To a word $w = w_1 \cdots w_k$ in $\mathcal{W}_{\mathcal{C}_n}$, just as in Chapter 2, we associate a symplectic key $\mathscr{K}_w$ that consists of only one row of length k, and with the boxes filled in from right to left with the letters of w read in turn from left to right. For example, the word 12 corresponds to the key $\begin{array}{|c|c|} \hline 2 & 1 \\ \hline \end{array}$. Denote the set of all keys associated to words by $\Gamma(\text{wor})$.

1.2. Weights and coweights. Consider $\mathbb{R}^n$ with canonical basis $\{\varepsilon_1, \cdots, \varepsilon_n\}$ and standard inner product $\langle -, - \rangle$ (in particular $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$). From now on we consider the root datum $(X, \Phi, X^\vee, \Phi^\vee)$ that is defined by:

$$\Phi = \{\pm\varepsilon_i, \varepsilon_i \pm \varepsilon_j\}_{i,j \in \{1, \cdots, n\}}$$
$$\Phi^\vee = \{\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}\}$$
$$X := \{v \in \mathbb{R}^n : \langle v, \alpha^\vee \rangle \in \mathbb{Z}\}$$
$$X^\vee = \{v \in \mathbb{R}^n : \langle \alpha, v \rangle \in \mathbb{Z}\}.$$

Indeed the sets X and $X^\vee$ are free abelian groups which form a root datum together with the pairing $\langle -, - \rangle$ between them and the subsets $\Phi \subset X$ and $\Phi^\vee \subset X^\vee$. We choose a basis $\Delta \subset \Phi$ given by

$$\Delta = \{\alpha_i := \varepsilon_i - \varepsilon_{i+1}; \alpha_n := \varepsilon_n : i \in \{1, \cdots, n-1\}\},$$

hence the set

$$\Delta^\vee = \{\alpha_i^\vee := \varepsilon_i - \varepsilon_{i+1}, \alpha_n^\vee := 2\varepsilon_n : i \in \{1, \cdots, n-1\}\}$$

is a basis for $\Phi^\vee$. Then $\mathrm{X}^\vee$ has a $\mathbb{Z}$-basis given by $\{\omega_i\}_{i\in\{1,\cdots,n\}}$, where

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \;\; 1 \leq i \leq n.$$

Then $\mathrm{G} = \mathrm{SO}(2n+1, \mathbb{C})$ and $\mathrm{G}^\vee = \mathrm{Sp}(2n, \mathbb{C})$. Since here we will be focusing on the representation theory of $\mathrm{Sp}(2n, \mathbb{C})$, we will fix the notation $\mathrm{P}_{C_n} = \mathrm{X}^\vee = \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i, \mathrm{P}_{C_n}^+ = \mathrm{X}^{\vee,+}$ and $\mathbb{V}_{C_n} = \mathrm{X}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ We will use this notation in Chapter 4 (cf.).

1.3. Symplectic keys associated to readable galleries. The aim of this section is to assign a symplectic key to every readable gallery.

1.4. Readable blocks. For a subset $\mathrm{X} \subseteq \mathcal{C}_n$, we denote the corresponding subset of barred elements by $\overline{\mathrm{X}} := \{\bar{x} : x \in \mathrm{X}\}$, where, for i unbarred, $\bar{\bar{i}} = i$.

Definition 3.2. Let $\mathscr{T}$ be a symplectic key. We call $\mathscr{T}$ an **LS block** if the arrangement of boxes associated to its type consists of only one box or if there exist positive integers k, r, s such that $2k + r + s \leq n$, and disjoint sets of positive integers

$$\begin{aligned}
\mathrm{A} &= \{a_i : 1 \leq i \leq r, a_1 < \cdots < a_r\} \\
\mathrm{B} &= \{b_i : 1 \leq i \leq s, b_1 < \cdots < b_s\} \\
\mathrm{Z} &= \{z_i : 1 \leq i \leq k, z_1 < \cdots < z_k\} \\
\mathrm{T} &= \{t_i : 1 \leq i \leq k, t_1 < \cdots < t_k\}
\end{aligned}$$

such that $\mathscr{T}$ consists of two columns: the rightmost one (respectively the leftmost one) is the column with entries the ordered elements of the set $\{\overline{\mathrm{T}}, \mathrm{Z}, \mathrm{A}, \overline{\mathrm{B}}\}$ (respectively $\{\overline{\mathrm{Z}}, \mathrm{T}, \mathrm{A}, \overline{\mathrm{B}}\}$), and such that the elements of T are uniquely characterised by the properties

$$t_k = \max\{t \in \mathcal{C}_n : t < z_k, t \notin \mathrm{Z} \cup \mathrm{A} \cup \mathrm{B}\} \tag{14}$$

$$t_{j-1} = \max\{t \in \mathcal{C}_n : t < \min(z_{j-1}, t_j), t \notin \mathrm{Z} \cup \mathrm{A} \cup \mathrm{B}\} \text{ for } j \leq k. \tag{15}$$

We say that $\mathscr{T}$ is a **zero block** if there exists a non-zero integer k such that $\mathscr{T}$ consists of two columns, both of k boxes; the right-most one is filled in with the ordered letters $1 < \cdots < k$ and the left-most one, with $\bar{k} < \cdots < \bar{1}$. A symplectic key is called a **readable block** if it is either an LS block or a zero block. A **readable key** is a concatenation of readable blocks. Now assume that $\underline{d} = (d_1, \cdots, d_{k+1})$ is such that $d_1 \leq \cdots \leq d_{k+1}$. A symplectic key of shape $\underline{d}$ is called an **LS symplectic key** if the entries are weakly increasing in rows and if it is a concatenation of LS blocks. We denote the set of LS symplectic keys of shape $\underline{d}$ as $\mathbf{\Gamma}(\underline{d})^{\mathrm{LS}}$.

Example 3.3. The symplectic key $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & \bar{1} \\ \hline\end{array}$ is an LS block, with

$$\mathrm{A} = \mathrm{B} = \varnothing, \mathrm{Z} = \{2\}, \mathrm{T} = \{1\}.$$

The symplectic key $\begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline 2 & \bar{1} \\ \hline\end{array}$ is not an LS block. The symplectic key $\begin{array}{|c|c|}\hline \bar{2} & 1 \\ \hline \bar{1} & 2 \\ \hline\end{array}$ is a zero block.

Remark 3.4. A pair of columns that form an LS block is sometimes called a pair of admissible columns. The original definition of admissible columns was given by DeConcini in [**4**], using a slightly diferent convention than Kashiwara and Nakayima's (which is the one we use here). The map, given by Lecouvey, that translates the two can be found in [**18**] at the end of Section 2.2.

To a readable block $\mathscr{T}$ we assign a gallery $\gamma_{\mathscr{T}}$ as follows. If $\mathscr{T}$ consists of only one box filled in with the letter $l \in \mathcal{C}_n$, then we define $\mathrm{V}_0^{\mathscr{T}}, \mathrm{V}_1^{\mathscr{T}} = \varepsilon_l$, $\mathrm{E}_0^{\mathscr{T}} := \{t\mathrm{V}_1^{\mathscr{T}}, t \in [0,1]\}$, and

$$\gamma_{\mathscr{T}} := \{\mathrm{V}_0^{\mathscr{T}}, \mathrm{E}_0^{\mathscr{T}}, \mathrm{V}_1^{\mathscr{T}}\}.$$

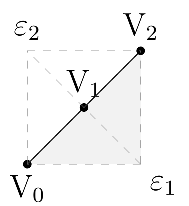
If not, then its columns are filled with the letters $l_1^1 < \cdots < l_d^1$ and $l_1^2 < \cdots < l_d^2$ respectively. We then define

$$\begin{aligned}
\mathrm{V}_1^{\mathscr{T}} &= \frac{1}{2}(\varepsilon_{l_1^1} + \cdots + \varepsilon_{l_d^1}) \\
\mathrm{V}_2^{\mathscr{T}} &= \varepsilon_{l_1^1} + \cdots + \varepsilon_{l_d^1} + \varepsilon_{l_1^2} + \cdots + \varepsilon_{l_d^2} \\
\mathrm{E}_1^{\mathscr{T}} &= \text{line segment joining } \mathrm{V}_1 \text{ and } \mathrm{V}_2.
\end{aligned}$$

and

$$\gamma_{\mathscr{T}} = (\mathrm{V}_0^{\mathscr{T}}, \mathrm{E}_0^{\mathscr{T}}, \mathrm{V}_1^{\mathscr{T}}, \mathrm{E}_1^{\mathscr{T}}).$$

Example 3.5. Let $n = 2$ and $\gamma = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2)$ where $\mathrm{V}_0 = 0, \mathrm{V}_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2), \mathrm{V}_2 = \varepsilon_1 + \varepsilon_2$ and the edges are the line segments joining the vertices in order. Below is a picture of γ and of its associated key K_γ.



1	1
2	2

$\gamma_{\mathscr{K}} = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2)$; $\mathscr{K}$

To a readable key we associate the concatenation of the galleries of each of the readable blocks that it is a concatenation of (from right to left). Given a symplectic shape $\underline{d}$, we will denote the set of all readable keys of shape $\underline{d}$ by $\mathbf{\Gamma}(\underline{d})^{\mathrm{R}}$. (This set may be empty.) Let $\underline{d}$ be a shape such that $\mathbf{\Gamma}(\underline{d})^{\mathrm{R}} \neq 0$. Then it must have the form

$$\underline{d} = (\underline{d}^{l_1}, \cdots, \underline{d}^{l_m})$$

where $\underline{d}^{l_i} = l_i, l_i$ for $l_i \geq 2$ and $\underline{d}^{l_i} = 1$ if $l_i = 1$. For instance, in Example 3.3, all symplectic keys have shape $(2,2)$. To such a shape $\underline{d}$ we associate the dominant coweight

$$\lambda_{\underline{d}} = \omega_{l_1} + \cdots + \omega_{l_m}.$$

For example, to the shape $(2,2)$ is associated the coweight ω_2. The following proposition follows from Lemma 2 in [**7**].

Proposition 3.6. The map

$$\bigcup_{\underline{d}} \mathbf{\Gamma}^{\mathrm{R}}(\underline{d}) \xrightarrow{1:1} \Gamma^{\mathrm{R}}$$
$$\mathscr{T} \mapsto \gamma_{\mathscr{T}}$$

is well defined and is a bijection. Moreover, if $d_1 \leq \cdots \leq d_{k+1}$ then this map induces a bijection

$$\mathbf{\Gamma}^{\mathrm{LS}}(\underline{d}) \xleftrightarrow{1:1} \Gamma^{\mathrm{LS}}(\gamma_{\omega_{l_1}} * \cdots * \gamma_{\omega_{l_m}}).$$

Remark 3.7. Zero lumps are not necessarily of fundamental type: this follows from Lemma 2 in [**7**] for a zero lump with k uneven in the above description. This is why readable galleries are not necessarily of the same type as a concatenation of fundamental galleries. This also means that there can be two readable keys of the same shape but such that their associated galleries are not of the same type! For example, take $n > 3$. Then the key $\mathscr{T} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}$ is LS and $\gamma_{\mathscr{T}}$ is of fundamental type γ_{ω_3}. The key $\mathscr{K} = \begin{array}{|c|c|}\hline \bar{1} & 1 \\ \hline \bar{2} & 2 \\ \hline \bar{3} & 3 \\ \hline \end{array}$ is a zero block. Its associated gallery, $\gamma_{\mathscr{K}}$, is not of fundamental type. For this reason we cannot write $\Sigma_{\underline{d}}$ for a Bott-Samelson variety as we were able to do in Chapter 2.

2. The word of a readable gallery

The **word** of a block $\mathscr{B} = \mathrm{C}_l\mathrm{C}_r$ (C_l is the left column; C_r the right) is obtained by reading first the unbarred entries in C_r and then the barred entries in C_l. We denote it by $w(\mathscr{B}) \in \mathcal{W}_{\mathcal{C}_n}$. For an LS block this is the word of the associated single admissible column defined by Kashiwara and Nakashima - see [**18**], Example 2.2.6.

Definition 3.8. Let $\gamma_{\mathscr{K}}$ be a readable gallery associated to the key $\mathscr{K}$, which we may write as a concatenation of blocks

$$\mathscr{K} = \mathscr{B}_1 \cdots \mathscr{B}_k.$$

The **word** of $\gamma_{\mathscr{K}}$ (or of $\mathscr{K}$) is $w(\mathscr{B}_k)\cdots w(\mathscr{B}_1)$. We denote it by $w(\gamma_{\mathscr{K}})$ (or $w(\mathscr{K})$).

Example 3.9. Let

$$\mathscr{B}_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & \bar{1} \\ \hline \end{array}, \ \mathscr{B}_2 = \begin{array}{|c|}\hline 1 \\ \hline \end{array},$$

and

$$\mathscr{K} = \mathscr{B}_1\mathscr{B}_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline \bar{2} & \bar{1} & \\ \cline{1-2} \end{array}.$$

Then $w(\mathscr{B}_1) = 2\bar{2}, w(\mathscr{B}_2) = 1$, and $w(\mathscr{K}) = 12\bar{2}$.

We have the following result about words of readable galleries, which we prove in Section 4. We will use it in Section 3.26. It is in this sense that such galleries are called *readable*.

Proposition 3.10. Let γ and ν be combinatorial galleries and $\mathscr{K}$ be a readable key. Then

$$\overline{\pi(\mathrm{C}_{\gamma*\gamma_{w(\mathscr{K})}*\nu})} = \overline{\pi'(\mathrm{C}_{\gamma*\gamma_{\mathscr{K}}*\nu})}.$$

2.1. Word galleries. Just as in Chapter 2, we associate a (readable!) gallery γ_w of the same type as $\underbrace{\gamma_{\omega_1} * \cdots * \gamma_{\omega_1}}_{m\ times}$ to a word $w \in \mathcal{W}_{\mathcal{C}_n}$ of length m - it is the gallery $\gamma_{\mathscr{K}_w}$ associated to the readable key $\mathscr{K}_w$. We denote the set of word galleries in this case by $\Gamma_{\mathcal{W}_{\mathcal{C}_n}}$. Below we recall the crystal structure on the set $\mathcal{W}_{\mathcal{C}_n}$ as described by Kashiwara and Nakashima in [**13**], Proposition 2.1.1. The set of words $\mathcal{W}_{\mathcal{C}_n}$, just like the set $\mathcal{W}_n$, is in one-to-one correspondence with the set of vertices of the crystal of the representation $\bigotimes_{l\in\mathbb{Z}^{\geq 0}} \mathrm{V}_n^{\otimes l}$, where V_n is the natural representation $\mathrm{L}(\omega_1)$ and hence inherits its crystal structure. Proposition 3.12 says that this structure is compatible with the crystal structure defined on galleries in Chapter 1.

Definition 3.11. Let $w \in \mathcal{C}_n$ be a word and $i \in \{1, \cdots, n\}$. To apply the root operators e_{α_i} and f_{α_i} to w one first assigns to w a word consisting of letters in the alphabet $\{+, -, \varnothing\}$. The word will be obtained from w by replacing every occurence of i or $\overline{i+1}$ by $(+)$, every occurence of $i+1$ or $\overline{i}$ by $(-)$ and all other letters by $\varnothing$. This word $s(w)$, just as the one in Chapter 2, is sometimes called the *i-signature* of w. Erase all symbols $\varnothing$ and then all subwords of the form $+-$. Repeat this process until the i-signature $s(w)$ of w has been reduced to a word of the form

$$s(w)' = (-)^r(+)^s.$$

To apply f_{α_i} (respectively e_{α_i}) to w, change the letter whose tag corresponds to the rightmost $(-)$ (respectively to the leftmost $(+)$) from $i+1$ to i and from $\overline{i}$ to $\overline{i+1}$ (repectively from i to $i+1$ and from $\overline{i+1}$ to $\overline{i}$).

Proposition 3.12. The crystal structure on words from Definition 3.11 coincides with the one induced from Definition 1.12.

For a proof, see Section 13 of [**21**]. It also follows directly from the definitions.

2.2. Word Reading is a Crystal Morphism. This subsection is the 'symplectic' version of Proposition 2.13 in Chapter 2. Since the root operators are type preserving (see 1.12), the set of words $\mathcal{W}_{\mathcal{C}_n}$ is naturally endowed with a crystal structure. The following proposition will be useful in Section 3.26. This result was shown for LS blocks by Kashiwara and Nakashima in [**13**], Proposition 4.3.2. They show that word reading induces an isomorphism of crystals from $\mathrm{B}(\omega_k)$ onto the subcrystal of $\bigotimes_{l\in\mathbb{Z}_{\geq 1}} \mathrm{B}(\omega_1)^{\otimes l}$ generated by the tensor product $\boxed{\mathrm{k}} \otimes \cdots \otimes \boxed{1}$. We show that for readable galleries the proof is reduced to this case.

Proposition 3.13. The map

$$\Gamma^{\mathrm{R}} \xrightarrow{w} \Gamma_{\mathcal{W}_{\mathcal{C}_n}}$$
$$\gamma_{\mathscr{K}} \mapsto \gamma_{w(\mathscr{K})}$$

is a crystal morphism.

PROOF. Let γ be a readable gallery and let

$$\gamma_{\mathscr{B}} = (\mathrm{V}_0^{\mathscr{B}}, \mathrm{E}_0^{\mathscr{B}}, \mathrm{V}_1^{\mathscr{B}}, \mathrm{E}_1^{\mathscr{B}}, \mathrm{V}_2^{\mathscr{B}})$$

be one of its parts, associated to the readable block $\mathscr{B}$; we write

$$\gamma_{\mathscr{K}_{w(\mathscr{B})}} = (\mathrm{V}_0^{\mathscr{K}_{w(\mathscr{B})}}, \mathrm{E}_0^{\mathscr{K}_{w(\mathscr{B})}}, \cdots, \mathrm{V}_{r+s}^{\mathscr{K}_{w(\mathscr{B})}}).$$

If

$$w(\mathscr{B}) = g_1 \cdots g_s \overline{h_r} \cdots \overline{h_1}$$

for g_i and h_i unbarred, then $\mathrm{V}_j^{\mathscr{K}_{w(\mathscr{B})}} = \sum_{i=1}^{j} \varepsilon_{x_i}$, where $x_i = g_i$ for $1 \leq i \leq s$ and $x_{s+i} = \bar{h}_i$ for $1 \leq i \leq r$. Let

$$h(j) = \langle \alpha, \mathrm{V}_j^{\mathscr{B}} \rangle$$
$$h'(j) = \langle \alpha, \mathrm{V}_j^{\mathscr{K}_{w(\mathscr{B})}} \rangle.$$

Then there exist $d_1 \leq s, s < d_2 \leq s + r$ such that

$$h'(j) = \begin{cases} h(0) & \text{for } 0 \leq j < d_1 \\ h(1) & \text{for } d_1 \leq j < d_2 \\ h(2) & \text{for } d_2 \leq j \leq r + s + 1 \end{cases}.$$

From this we conclude that it is enough to consider readable blocks. As mentioned previously, this was shown in [**13**] for LS blocks. Hence let $\mathscr{L}$ be a zero lump; it has word $w(\mathscr{L}) = 1 \cdots k \bar{k} \cdots \bar{1}$ and let α_i be a simple root. Then, since the galleries associated to $\mathscr{L}$ and $w(\mathscr{L})$ are both dominant, $f_{\alpha_i}(\mathscr{L}) = e_{\alpha_i}(\mathscr{L}) = f_{\alpha_i}(w(\mathscr{L})) = e_{\alpha_i}(w(\mathscr{L})) = 0$. □

EXAMPLE 3.14. Let $n = 2$ and $\mathscr{B}$ be the readable block $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \bar{2} & \bar{1} \\ \hline \end{array}$. Then $w(\mathscr{B}) = 2\bar{2}$. To calculate $f_{\alpha_1}(\mathscr{B})$, note that $m_{\alpha_1} = -1$, j = 1, r = 2, hence $f_{\alpha_1}(\mathscr{B}) = \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \bar{1} & \bar{1} \\ \hline \end{array}$. Similarly, $f_{\alpha_1}(w(\mathscr{B})) = 2\bar{1} = w(f_1(\mathscr{B}))$.

2.3. Readable galleries are Littelmann galleries. We begin with a lemma.

LEMMA 3.15. Let $\gamma_{\mathscr{K}}$ be a readable gallery. Then $\gamma_{\mathscr{K}}$ is dominant if and only if $\gamma_{w(\mathscr{K})}$ is dominant.

PROOF. Since the entries in the columns symplectic keys are strictly increasing, it follows from the definition of word reading that if γ is a dominant readable gallery then $w(\gamma)$ is also dominant. Now let γ be a non-dominant readable gallery. Then there is a readable block $\mathscr{B} = \mathrm{C}_l\mathrm{C}_r$ such that $\gamma = \eta_1 * \gamma_{\mathscr{B}} * \eta_2$ with η_1 dominant and $\eta_1 * \gamma_{\mathscr{B}}$ not dominant. This block can't be a zero lump (they are dominant) - so it must be LS. Let A, B, Z, T be the sets described in Definition 3.2. The entries of C_r are the letters in $\mathrm{A} \cup \mathrm{Z} \cup \overline{\mathrm{B}} \cup \overline{\mathrm{T}}$ and the entries of C_l are the letters in $\mathrm{A} \cup \mathrm{T} \cup \overline{\mathrm{B}} \cup \overline{\mathrm{Z}}$. Now, $\mu_{\eta_1 * \gamma_{\mathscr{B}}}$ may be dominant or not. If it is, then, since $\mu_{\gamma_{w(\eta_1 * \gamma_{\mathscr{B}})}} = \mu_{\mu_{\eta_1 * \gamma_{\mathscr{B}}}}$, the word gallery $\gamma_{w(\eta_1 * \gamma_{\mathscr{B}})}$ is not dominant, and this implies that $\gamma_{w\mathscr{K}}$ is not dominant either. Now assume that

$$\mu_{\eta_1 * \gamma_{\mathscr{B}}} = \mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a - \sum_{b \in \mathrm{B}} \varepsilon_b$$

is dominant, but that the gallery $\eta_1 * \gamma_{\mathscr{B}}$ is not. The last three vertices of this gallery are

$$\mathrm{V}_{l-1} = \mu_{\eta_1} \in \mathrm{C}^+ \tag{16}$$

$$\mathrm{V}_l = \mu_{\eta_1} + \frac{1}{2}\Big(\sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z - \sum_{b \in \mathrm{B}} \varepsilon_b - \sum_{t \in \mathrm{T}} \varepsilon_t\Big) \notin \mathrm{C}^+ \tag{17}$$

$$\mathrm{V}_{l+1} = \mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a - \sum_{b \in \mathrm{B}} \varepsilon_b \in \mathrm{C}^+. \tag{18}$$

Let $d_1 < \cdots < d_{r+k}$ be the ordered elements of $\mathrm{A} \cup \mathrm{Z}$ and let $f_1 < \cdots < f_{s+k}$ be the ordered elements of $\mathrm{B} \cup \mathrm{Z}$. We have

$$w(\mathscr{B}) = d_1 \cdots d_{r+k} \bar{f}_{s+k} \cdots \bar{f}_1.$$

We claim that the weight

$$\mu_{\eta_1} + \sum_{i=1}^{r+k} \varepsilon_{d_i} = \mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z,$$

which is the endpoint of $\eta * \gamma_{d_1 \cdots d_{r+k}}$ and therefore a vertex of $\eta * \gamma_{w(\mathscr{B})}$, is not dominant. To see this, assume otherwise:

$$\mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z \in \mathrm{C}^+.$$

Since the dominant Weyl chamber C^+ is convex, this means that the line segment that joins μ_{η_1} and $\mu_{\eta_1} + \sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z$ is contained in C^+, in particular the point

$$\mu_{\eta_1} + \frac{1}{2}\Big(\sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z\Big) \in \mathrm{C}^+$$

belongs to the dominant Weyl chamber. The dominant Weyl chamber has, in this case, the following description in the coordinates $\varepsilon_1, \cdots, \varepsilon_n$:

$$\mathrm{C}^+ = \Big\{\sum_{i=1}^{n} p_i \varepsilon_i : p_i \in \mathbb{R}_{\geq 0} \ \&\ p_1 \geq \cdots \geq p_n\Big\}.$$

Write

$$\mu_{\eta_1} = \sum_{i=1}^{n} q_i \varepsilon_i$$

We will now show that $\mu_{\eta_1} + \frac{1}{2}\big(\sum_{a \in \mathrm{A}} \varepsilon_a + \sum_{z \in \mathrm{Z}} \varepsilon_z - \sum_{b \in \mathrm{B}} \varepsilon_b - \sum_{t \in \mathrm{T}} \varepsilon_t\big) \in \mathrm{C}^+$. This would contradict our assumption and therefore complete the proof.

For every $i \in \{1, \cdots, r\}$, we have $t_i < z_i < j$ for every $j \in \{1, \cdots, n\}$ such that $t_i < j$. Since $\mu_{\eta_1} + \frac{1}{2}(\sum_{a\in A} \varepsilon_a + \sum_{z\in Z} \varepsilon_z) \in C^+$, we know therefore that

$$q_j \leq q_{z_i} + \frac{1}{2} \leq q_{t_i},$$

which implies, since $q_{t_i} \in \mathbb{Z}$, that

$$q_j \leq q_{z_i} + \frac{1}{2} \leq q_{t_i} - \frac{1}{2}.$$

Now let $b \in B$, and let $j \in \{1, \cdots, n\}$ such that $b < j$. Since (cf. (18)),

$$\mu_{\eta_1} + \sum_{a\in A} \varepsilon_a - \sum_{b\in B} \varepsilon_b \in C^+,$$

if $j \in (Z \cup T)^c$, then this implies

$$q_j \leq q_b - \frac{1}{2}.$$

If $j \in Z \cup T$ then, as before, by the definition of an LS block we may assume that $j = t \in T$. But this means $q_t \leq q_b$, therefore $q_t - \frac{1}{2} \leq q_b - \frac{1}{2}$. All of these arguments imply

$$\mu_{\eta_1} + \frac{1}{2}(\sum_{a\in A} \varepsilon_a + \sum_{z\in Z} \varepsilon_z - \sum_{b\in B} \varepsilon_b - \sum_{t\in T} \varepsilon_t) \in C^+,$$

which contradicts (17).

$\square$

As in Chapter 2 we have the following lemma.

LEMMA 3.16. A readable gallery ν is dominant if and only if $e_{\alpha_i}(\nu) = 0$ for all $i \in \{1, \cdots, n\}$.

PROOF. Notice that for a word $w \in \mathcal{W}_{\mathcal{C}_n}$ and α_i a simple root, $e_{\alpha_i}(\mathscr{K}_w) = 0$ means that to the right of each $i+1$ in $\mathscr{K}_w$ there is at least one i which has not been cancelled out in the tagging and subword extraction process described in Definition 3.11. This is equivalent to the gallery γ_w being dominant. Lemma 3.15 and Proposition 3.13 imply the desired result. $\square$

PROPOSITION 3.17. Every readable gallery is a Littelmann gallery.

PROOF. Let V_n be the vector representation of $SP_{2n}(\mathbb{C})$. Then the crystal of words $\mathcal{W}_{\mathcal{C}_n}$ is isomorphic to the crystal associated to $T(V_n) = \bigoplus_{l\in\mathbb{Z}_{\geq 1}} V_n^{\otimes l}$, see for example Section 2.1 in [**18**]. Now let γ be any readable gallery. Then there exist indices $i_1, \cdots, i_r$ such that $e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma_{w(\gamma)})$ is a highest weight vertex, hence dominant, by Lemma 3.16. Since word reading is a morphism of crystals by Proposition 3.13, $\gamma_{w(e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma))} = e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma_{w(\gamma)})$. It follows from Lemma 3.15 that $e_{\alpha_{i_r}} \cdots e_{\alpha_{i_1}}(\gamma)$ is dominant. $\square$

DEFINITION 3.18. The **symplectic plactic monoid** $\mathcal{P}_{\mathcal{C}_n}$ is the quotient of the word monoid $\mathcal{W}_{\mathcal{C}_n}$ by the ideal generated by the following relations

(R1) For $z \neq \overline{x}$:

$$y\ x\ z \equiv y\ z\ x \quad \text{for } x \leq y < z$$
$$x\ z\ y \equiv z\ x\ y \quad \text{for } x < y \leq z$$

(R2) For $1 < x \leq n$ and $x \leq y \leq \bar{x}$:

$$y\ \overline{x-1}\ x-1 \equiv y\ x\ \bar{x}$$
$$\overline{x-1}\ x-1\ y \equiv x\ \bar{x}\ y$$

(R3)

$$a_1 \cdots a_r\ z\ \ \bar{z}\ \bar{b}_s \cdots \bar{b}_1 \equiv a_1 \cdots a_r\ \bar{b}_s \cdots \bar{b}_1$$

for $a_i, b_i \in \{1, \cdots, n\}, i \in \{1, \cdots, \max\{\mathrm{s}, \mathrm{r}\}\}$, such that $a_1 < \cdots < a_r, b_1 < \cdots < b_s$, and such that the left hand side of the above expression is not the word of an LS block.

If two words $w_1, w_2 \in \mathcal{W}_{\mathcal{C}_n}$ are representatives of the same class in $\mathcal{W}_{\mathcal{C}_n}$ we say they are **symplectic plactic equivalent**.

Example 3.19.

$$12\bar{2}\bar{1} \sim 1\bar{1} \sim \varnothing$$
$$112 \sim 121$$

Remark 3.20. Relations (R1) are the Knuth relations in type A, while relation (R3) may be understood as the general relation which specialises to $1\bar{1} \cong \varnothing$. Note that the gallery γ_w associated to $w = 1\bar{1}$ is a zero lump. This definition of the symplectic plactic monoid is the same as Definition 3.1.1 in [**18**] except for relation (R3). The equivalence between the relation (R3) above and the one in [**18**] is given in the Appendix.

The following Theorem is due to Lecouvey and it is proven in [**18**].

Theorem 3.21. Two words $w_1, w_2 \in \mathcal{W}_{\mathcal{C}_n}$ are symplectic plactic equivalent if and only if their associated galleries γ_{w_1} and γ_{w_2} are equivalent.

Theorem 3.21 implies the following Proposition (cf. Theorem 2.16 and Lemma 2.17 in Chapter 2).

Proposition 3.22. Two readable galleries γ and ν are equivalent if and only if the words $w(\gamma)$ and $w(\nu)$ are symplectic plactic equivalent.

The following theorem is originally due to Kashiwara and Nakashima (see [**13**]). For this particular formulation, see Proposition 3.1.2 in [**18**].

Theorem 3.23. For each word w in $\mathcal{W}_{\mathcal{C}_n}$ there exists a unique symplectic LS key $\mathscr{T}$ such that $w \sim w(\mathscr{T})$.

Proposition 3.24. Let γ and ν be combinatorial galleries and let $w_1 \in \mathcal{W}_{\mathcal{C}_n}$ be two plactic equivalent words. Then

$$\overline{\pi(\mathrm{C}_{\gamma * \gamma_{w_1} * \nu})} = \overline{\pi(\mathrm{C}_{\gamma * \gamma_{w_2} * \nu})}$$

3. Readable Galleries and MV cyles

As in the previous chapter, we have the following result (which holds in higher generality) by Gaussent-Littelmann (a. is an instance of Theorem C in [**6**]) and Baumann-Gaussent (b. is an instance of Theorem 5.8 in [**1**]).

THEOREM 3.25. Let $\underline{d} = (d_1, \cdots, d_r)$ be a symplectic shape such that $\mathbf{\Gamma}(\underline{d})^{\mathrm{LS}} \neq \varnothing$ and consider the desingularization $\pi : \Sigma_{\underline{d}} \to \mathrm{X}_{\lambda_{\underline{d}}}$.

a. If $\delta \in \mathbf{\Gamma}(\underline{d})^{\mathrm{LS}}$ is a symplectic LS key, the closure $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in $\mathcal{Z}(\lambda_{\underline{d}})$. This induces a bijection $\mathbf{\Gamma}(\underline{d})^{\mathrm{LS}} \xrightarrow{\varphi_{\underline{d}}} \mathcal{Z}(\lambda_{\underline{d}})$.
b. The bijection $\varphi_{\underline{d}}$ is an isomorphism of crystals.

Given a readable gallery γ and a dominant coweight $\lambda \in \mathrm{X}^{\vee,+}$, let

$$n^{\lambda}_{\gamma^f} = \#\{\nu \in \Gamma^{\mathrm{dom}} \cap \Gamma(\gamma^f) : \mu_\nu = \lambda\},$$

and let

$$\mathrm{X}^{\vee,+}_{\gamma^f} = \{\lambda \in \mathrm{X}^{\vee,+} : n^{\lambda}_{\gamma^f} \neq 0\}.$$

THEOREM 3.26. Let $\delta \in \Gamma(\gamma^f)^{\mathrm{R}}$ be a readable gallery, and (Σ_{γ^f}, π) the corresponding Bott-Samelson variety together with its map π to the affine Grassmannian as in (1.6). Let δ^+ be the dominant readable gallery that is the highest weight vertex in $\mathrm{Conn}(\delta)$. Then

a. $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in $\mathcal{Z}(\mu_{\delta^+})_{\mu_\delta}$.
b. The map

$$\begin{gathered}\Gamma(\gamma^f)^{\mathrm{R}} \xrightarrow{\varphi_{\gamma^f}} \bigoplus_{\delta \in \Gamma(\gamma^f)^{\mathrm{R}}} \mathcal{Z}(\mu_{\delta^+}) \\ \delta \mapsto \overline{\pi(\mathrm{C}_\delta)}\end{gathered}$$

   is a surjective morphism of crystals. The direct sum on the right-hand side is a direct sum of abstract crystals.
c. If C is a connected component of $\Gamma(\gamma^f)^{\mathrm{R}}$, then $\varphi|_{\mathrm{C}}$ is an isomorphism onto its image.
d. The number of connected components C of $\Gamma^{\mathrm{R}}(\gamma^f)$ such that $\varphi_{\gamma^f}(\mathrm{C}) = \mathcal{Z}(\lambda)$ is equal to $n^{\lambda}_{\gamma^f}$.
e. Given an MV cycle $\mathrm{Z} \in \mathcal{Z}(\lambda)_\mu$, the fibre $\varphi^{-1}_{\gamma^f}(\mathrm{Z})$ is given by

$$\varphi^{-1}_{\gamma^f}(\mathrm{Z}) = \{\delta \in \Gamma^{\mathrm{R}}(\gamma^f) : \varphi_{\gamma^f}(\delta) = \mathrm{Z}\} = \{\delta \in \Gamma^{\mathrm{R}}(\gamma^f) : \gamma \sim \gamma^{\lambda}_{\mu,\mathrm{Z}}\}$$

   where $\gamma^{\lambda}_{\mu,\mathrm{Z}}$ is the unique LS key which exists by Theorem 3.25.

PROOF. Let δ be a readable gallery. Then by Theorem 3.23 there exists a (unique) LS key ν such that $\delta \sim \nu$. By Proposition 3.22, the words $w(\delta)$ and $w(\nu)$ are plactic equivalent. Propositions 3.24 and 3.10 together with Theorem 3.21 then imply that

$$\overline{\pi(\mathrm{C}_\delta)} = \overline{\pi(\mathrm{C}_\nu)},$$

which, by Theorem 3.25 implies that $\overline{\pi(\mathrm{C}_\delta)}$ is an MV cycle in $\mathcal{Z}(\mu_{\delta^+})_{\mu_\delta}$. Proposition 3.10, Proposition 3.24, Theorem 3.25, Theorem 3.21, and Proposition 3.17, the rest of the proof of a., b., and c. is identical to the proof of Theorem 2.21. Points d. and e. follow from Theorem 7.1 in [**20**], which asserts that if γ is a dominant gallery, then $\mathrm{Conn}(\gamma)$ is isomorphic to the crystal $\mathrm{B}(\mu_\gamma)$, and from Proposition 3.17.

□

4. Counting Positive Crossings

In this section we provide proofs of Propositions 3.10 and 3.24. We begin with analysing the *tail* of a gallery in 4.1. We show Proposition 3.27: it will be an important tool in our proofs. In 4.2 we calculate an example in which it can be seen how to apply it. Then in 4.3 we prove Proposition 3.10 and in 4.4 we prove Proposition 3.24. We also wish to establish some notation that we will use throughout. Recall our convention $\varepsilon_{\bar{l}} := -\varepsilon_l$ for $l \in \mathcal{C}_n$ unbarred. We will write, for $l, s, d, m \in \mathcal{C}_n$, $c^{i,j}_{ls,dm}$ for the constant $c^{i,j}_{\varepsilon_l+\varepsilon_s,\varepsilon_d+\varepsilon_m}$ in Chevalley's commutator formula (4), and $c^{i,j}_{l,dm}, c^{i,j}_{ls,d}$ for $c^{i,j}_{\varepsilon_l,\varepsilon_d+\varepsilon_m}, c^{i,j}_{\varepsilon_l+\varepsilon_s,\varepsilon_d}$ respectively.

4.1. Truncated Images and Tails. Let γ be a combinatorial gallery with notation as in (6) with endpoint the coweight μ_γ and let $r \le k+1$ such that V_r is a special vertex; we denote it by $\mu_r \in \mathrm{X}^\vee$. By Corollary 1.11 we know that the image $\pi(\mathrm{C}_\gamma)$ is stable under U_0.

Proposition 3.27. The **r-truncated image** of γ

$$\mathrm{T}^{\ge r}_\gamma = \mathbb{U}^\gamma_r \mathbb{U}^\gamma_{r+1} \cdots \mathbb{U}^\gamma_k [t^{\mu_\gamma}]$$

is U_{μ_r}-stable, i.e. for any $u \in \mathrm{U}_{\mu_r}$, $u\mathrm{T}^{\ge r}_\gamma = \mathrm{T}^{\ge r}_\gamma$.

Proof. By (5), we know that $t^{\mu_r}\mathrm{U}_0 t^{-\mu_r} = \mathrm{U}_{\mu_r}$. On the other hand, we may also consider the **r-truncated gallery**

$$\gamma^{\ge r} = (\mathrm{V}'_0, \mathrm{E}'_0, \cdots, \mathrm{V}'_{k-r+1}),$$

which is the combinatorial gallery obtained from the sequence

$$(\mathrm{V}_r, \mathrm{E}_r, \mathrm{V}_{r+1}, \cdots, \mathrm{E}_k, \mathrm{V}_{k+1})$$

by translating it to the origin. Since V_r is a special vertex, we also have $t^{\mu_r}\mathbb{U}^{\gamma^{\ge r}}_i t^{-\mu_r} = \mathbb{U}^\gamma_{i+r}$. This gallery has endpoint $\mu_\gamma - \mu_r$ and is in turn a T-fixed point of a Bott-Samelson variety (Σ, π'). Let $u \in \mathrm{U}_{\mu_r}$ and $u' = t^{-\mu_r} u t^{\mu_r} \in \mathrm{U}_0$. Then:

$$\begin{aligned} u\mathrm{T}^{\ge r}_\gamma &= u\mathbb{U}^\gamma_r \mathbb{U}^\gamma_{r+1} \cdots \mathbb{U}^\gamma_k [t^{\mu_\gamma}] \\ &= t^{\mu_r} u' \mathbb{U}^{\gamma^{\ge r}}_0 \cdots \mathbb{U}^{\gamma^{\ge r}}_{k-r} [t^{\mu_\gamma - \mu_r}] \\ \text{(by Corollary 1.11)} &= t^{\mu_r} \mathbb{U}^{\gamma^{\ge r}}_0 \cdots \mathbb{U}^{\gamma^{\ge r}}_{k-r} [t^{\mu_\gamma - \mu_r}] = \mathrm{T}^{\ge r}_\gamma \end{aligned}$$

□

For later use let us fix the notation

$$\mathrm{T}^{<r}_\gamma = \mathbb{U}_{\mathrm{V}_0} \cdots \mathbb{U}_{\mathrm{V}_{r-1}};$$

one may then write

$$\pi(\mathrm{C}_\gamma) = \mathrm{T}^{<r}_\gamma \mathrm{T}^{\ge r}_\gamma.$$

REMARK 3.28. This Proposition is proven for $\mathrm{SL}_n(\mathbb{C})$ in [**8**], Proposition 3. The proof we have provided is exactly the same, except for the restriction of only being able to truncate at special vertices.

4.2. Example. Let $n = 2$. Consider the symplectic keys

$$\mathscr{K}_1 = \begin{array}{|c|c|c|}\hline 1 & 1 & \bar{1} \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$$

$$\mathscr{K}_2 = \begin{array}{|c|c|c|}\hline 2 & 1 & 2 \\ \hline & \bar{2} & \bar{1} \\ \cline{2-3}\end{array}$$

and their words

$$w(\mathscr{K}_1) = \bar{1}12$$
$$w(\mathscr{K}_2) = 2\bar{2}2.$$

Note that

$$\gamma_{\omega_1} * \gamma_{\omega_2} \sim \gamma_{\omega_2} * \gamma_{\omega_1}$$

since both $\gamma_{\omega_1} * \gamma_{\omega_2}$ and $\gamma_{\omega_2} * \gamma_{\omega_1}$ are contained in the fundamental chamber and have the same endpoint $\omega_1 + \omega_2$; one checks that

$$f_{\alpha_1} f_{\alpha_2} f_{\alpha_1}(\gamma_{\omega_1} * \gamma_{\omega_2}) = \gamma_{\mathscr{K}_1}$$

and

$$f_{\alpha_1} f_{\alpha_2} f_{\alpha_1}(\gamma_{\omega_2} * \gamma_{\omega_1}) = \gamma_{\mathscr{K}_2}.$$

Therefore $\gamma_{\mathscr{K}_1} \sim \gamma_{\mathscr{K}_2}$. Lemma 3.15 and Proposition 3.13 then imply that $\gamma_{w(\mathscr{K}_1)} \sim \gamma_{w(\mathscr{K}_2)}$ (or it can also be checked directly using relation R2 in Theorem 3.21 with $y = x = 2$). Now consider combinatorial galleries γ and ν. The galleries $\gamma * \gamma_{\mathscr{K}_1} * \nu$ and $\gamma * \gamma_{\mathscr{K}_2} * \nu$ are T-fixed points in the Bott-Samelson varieties $(\Sigma_{(\gamma*\gamma_{\mathscr{K}_1}*\nu)^f}, \pi)$ respectively $(\Sigma_{(\gamma*\gamma_{\mathscr{K}_2}*\nu)^f}, \pi')$. The galleries $\gamma_{w(\mathscr{K}_1)}$ and $\gamma_{w(\mathscr{K}_1)}$ that correspond to their words are T-fixed points in

$$(\Sigma_{(\gamma*\gamma_{\omega_1}*\gamma_{\omega_1}*\gamma_{\omega_1}*\nu)^f}, \pi'').$$

We show that

$$\overline{\pi(\mathrm{C}_{\gamma*\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\gamma*w(\mathscr{K}_1)}*\nu})} = \overline{\pi'(\mathrm{C}_{\gamma*\gamma_{w(\mathscr{K}_2)}*\nu})}.$$

We use the same notation as in (6) for γ. Since for any combinatorial gallery $\eta, (\alpha, n) \in \Phi^{\gamma*\eta}_{k+1}$ if and only if $(\alpha, n - \langle\alpha, \mu_\gamma\rangle) \in \Phi^{\gamma}_0$, we may assume that $\gamma = \varnothing$. Since $\gamma_{\mathscr{K}_1}, \gamma_{\mathscr{K}_2}, \gamma_{w(\mathscr{K}_1)}$ and $\gamma_{w(\mathscr{K}_2)}$ have the same endpoint ε_2, this also implies that $\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_1}*\nu} = \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_2}*\nu} = \mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_2)}*\nu} = \mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_1)}*\nu}$. By Proposition 1.8, for $a', b', c', d' \in \mathbb{C}$

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1*\nu}}) = \mathrm{U}_{(\varepsilon_1,-1)}(a')\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}(b')\mathrm{U}_{(\varepsilon_2,0)}(c')\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(d')\mathrm{T}^2_{\gamma_{\mathscr{K}_1}*\nu}.$$

By Chevalley's commutator formula (4) and applying Proposition 3.27 to $\mathrm{U}_{(\varepsilon_1-\varepsilon_2,-1)}(e) \in \mathrm{U}_{\varepsilon_2}$,

$\pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_1)*\nu}}) =$
$\mathrm{U}_{(\varepsilon_1,-1)}(a)\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}(b)\mathrm{U}_{(\varepsilon_1-\varepsilon_2,-1)}(e)\mathrm{U}_{(\varepsilon_2,0)}(c)\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(d)\mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_1)}*\nu}$
$= \mathrm{U}_{(\varepsilon_1,-1)}(a + c^{1,1}_{1\bar{2},2}(-e)c)\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}(b + c^{1,2}_{1\bar{2},2}(-e)c^2)\mathrm{U}_{(\varepsilon_2,0)}(c)\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(d)\mathrm{U}_{(\varepsilon_1-\varepsilon_2,-1)}(e)\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_1}*\nu}$
$= \mathrm{U}_{(\varepsilon_1,-1)}(a + c^{1,1}_{1\bar{2},2}(-e)c)\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}(b + c^{1,2}_{1\bar{2},2}(-e)c^2)\mathrm{U}_{(\varepsilon_2,0)}(c)\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(d)\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_1}*\nu}$
$\subset \pi(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$

for $a, b, c, d, e \in \mathbb{C}$. Choosing $a = a', b = b', c = c', d = d', e = 0$, we have

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1}}) \subset \pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_1)}}).$$

Hence in this case $\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1}}) = \pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_1)}})$. Similarly, for $\{a'', b'', c'', d'', e''\} \subset \mathbb{C}$,

$\pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_2)*\nu}}) = \mathrm{U}_{(\varepsilon_2,0)}(a'')\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(b'')\mathrm{U}_{(\varepsilon_1-\varepsilon_2,-1)}(e'')\mathrm{U}_{(\varepsilon_2,0)}(c'')\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(d'')\mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_2)}*\nu} =$
$\mathrm{U}_{(\varepsilon_1,-1)}(c^{1\bar{2},2}_{1,1}(-e'')c'')\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}(c^{1\bar{2},2}_{1,2}(-e'')c''^2)\mathrm{U}_{(\varepsilon_2,0)}(a'' + c'')\mathrm{U}_{(\varepsilon_1+\varepsilon_2,0)}(b'' + d'')\mathrm{T}^{\geq 3}_{\gamma_{w(\mathscr{K}_2)}*\nu}$
$\subset \pi(\mathrm{C}_{\gamma_{\mathscr{K}_1*\nu}})$.

Hence the open subset of $\pi(\mathrm{C}_{\gamma_{\mathscr{K}_1*\nu}})$ given by $a \neq 0, b \neq 0, c \neq 0, d \neq 0$ is contained in $\pi''(\mathrm{C}_{\gamma_{w(\mathscr{K}_2)*\nu}})$.

4.3. Proof of Proposition 3.10. We want to show that if γ and ν are combinatorial galleries and $\mathscr{K}$ is a readable block,

$$\overline{\pi(\mathrm{C}_{\gamma*\gamma_{\mathscr{K}}*\nu})} = \overline{\pi'(\mathrm{C}_{\gamma*\gamma_{w(\mathscr{K})}*\nu})}.$$

Proof. We assume $\gamma = \varnothing$; we may do so by the argument given at the beginning of Example 4.2. Let $\mathscr{K}$ be an LS block and let $\mathrm{A} = \{a_1, \cdots, a_r\}, \mathrm{B} = \{b_1, \cdots b_s\}, \mathrm{Z} = \{z_1, \cdots, z_k\}$ and $\mathrm{T} = \{t_1, \cdots, t_k\}$ be the subsets of $\{1, \cdots, n\}$ from Definition 3.2 that determine $\mathscr{K}$. We will use the notation $d_1 < \cdots < d_{r+k}$ to denote the ordered elements of $\mathrm{Z} \cup \mathrm{A}$, and $f_1 < \cdots < f_{s+k}$ the ordered elements of $\mathrm{B} \cup \mathrm{Z}$. We also write

$$\gamma_{\mathscr{K}} = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2).$$

The proof is divided into Lemmas 3.29 and 3.30 below.

Lemma 3.29. Let ν be a combinatorial gallery and $\mathscr{K}$ be a readable block. Then

$$\overline{\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu})} \subseteq \overline{\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})}.$$

We first need the following claim.

Claim 1.

$$\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu}) \subset \mathrm{U}_0 \mathbb{P}'''_{\bar{f}_{k+s}} \cdots \mathbb{P}'''_{\bar{f}_1} \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}$$

where

$$\mathbb{P}'''_{\bar{b}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T};\\ l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(k_{l\bar{b}}) \prod_{t\in\mathrm{T}^{<b}} \mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(k_{t\bar{b}}) \prod_{a\in\mathrm{A}^{<b}} \mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(k_{a\bar{b}}) \text{ for } b \in \mathrm{B} \tag{19}$$

$$\mathbb{P}'''_{\bar{z}} = \prod_{\substack{l \notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T};\\ l<z}} \mathrm{U}_{(\varepsilon_l-\varepsilon_z,-1)}(k_{l\bar{z}}) \prod_{t\in\mathrm{T}^{<z}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}}) \prod_{b\in\mathrm{B}^{<z}} \mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(k_{b\bar{z}}) \text{ for } z \in \mathrm{Z} \tag{20}$$

PROOF OF CLAIM 1. The points of $\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})^*\nu}})$ are of the form

$$\mathbb{P}_{d_1}\cdots\mathbb{P}_{d_{r+k}}\mathbb{P}_{\bar{f}_{k+s}}\cdots\mathbb{P}_{\bar{f}_1}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})^*\nu}} \tag{21}$$

where

$$\mathbb{P}_d = \mathrm{U}_{(\varepsilon_d,0)}(g_d)\prod_{d<l\leq n}\mathrm{U}_{(\varepsilon_d-\varepsilon_l,0)}(g_{d\bar{l}})\prod_{l\notin(\mathrm{Z}\cup\mathrm{A})^{<d}}\mathrm{U}_{(\varepsilon_d+\varepsilon_l,0)}(g_{dl})\prod_{l\in(\mathrm{Z}\cup\mathrm{A})^{<d}}\mathrm{U}_{(\varepsilon_d+\varepsilon_l,1)}(g^1_{dl}) \tag{22}$$

$$\mathbb{P}_{\bar{b}} = \mathrm{S}_{\bar{b}}\overbrace{\prod_{\substack{l\notin\mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T};\\ l<b}}\mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(g_{l\bar{b}})\prod_{t\in\mathrm{T}^{<b}}\mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(g_{t\bar{b}})\prod_{a\in\mathrm{A}^{<b}}\mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(g_{a\bar{b}})}^{=\mathbb{P}^{iv}_{\bar{b}}} \tag{23}$$

$$\mathrm{S}_{\bar{b}} = \prod_{b'\in\mathrm{B}^{<b}}\mathrm{U}_{(\varepsilon_{b'}-\varepsilon_b,0)}(g_{b'\bar{b}})\prod_{z\in\mathrm{Z}^{<b}}\mathrm{U}_{(\varepsilon_z-\varepsilon_b,1)}(g^1_{z\bar{b}})\in\mathrm{U}_0 \tag{24}$$

$$\mathbb{P}_{\bar{z}} = \mathrm{J}_{\bar{z}}\underbrace{\prod_{\substack{l\notin\mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T};\\ l<z}}\mathrm{U}_{(\varepsilon_l-\varepsilon_z,-1)}(g_{l\bar{z}})\prod_{t\in\mathrm{T}^{<z}}\mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(g_{t\bar{z}})\prod_{b\in\mathrm{B}^{<z}}\mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(g_{b\bar{z}})}_{=\mathbb{P}^{iv}_{\bar{z}}} \tag{25}$$

$$\mathrm{J}_{\bar{z}} = \prod_{a\in\mathrm{A}^{<z}}\mathrm{U}_{(\varepsilon_a-\varepsilon_z,0)}(g_{a\bar{z}})\prod_{z'\in\mathrm{Z}^{<z}}\mathrm{U}_{(\varepsilon_{z'}-\varepsilon_z,0)}(g_{z'\bar{z}})\in\mathrm{U}_0. \tag{26}$$

for $d\in\mathrm{A}\cup\mathrm{Z}$, $z\in\mathrm{Z}$, and $b\in\mathrm{B}$. All the terms in $\mathrm{J}_{\bar{z}}$ commute with $\mathbb{P}^{iv}_{z'}$ for $z'\in\mathrm{Z}^{>z}$ and with $\mathbb{P}^{iv}_{\bar{b}}$ for $b\in\mathrm{B}^{>z}$. All the terms in $\mathrm{S}_{\bar{b}}$ commute with $\mathbb{P}^{iv}_{\bar{b}'}$ for $b'\in\mathrm{B}^{>b}$. For $z'>b$ it commutes with all terms of $\mathbb{P}^{iv}_{\bar{z}'}$ except for the term $\mathrm{U}_{(\varepsilon_b-\varepsilon_{z'},-1)}(g_{b\bar{z}'})$. However, commuting $\mathrm{S}_{\bar{b}}$ with this term (using Chevalley's commutator formula 4) produces terms $\mathrm{U}_{(\varepsilon_z-\varepsilon_{z'},0)}(*)$ and $\mathrm{U}_{(\varepsilon_{b'}-\varepsilon_{z'},-1)}(*)$. Out of these terms, $\mathrm{U}_{(\varepsilon_z-\varepsilon_{z'},0)}(*)$ commutes with $\mathbb{P}^{iv}_{z'}$ for $z'\in\mathrm{Z}^{>z}$ and with $\mathbb{P}^{iv}_{\bar{b}}$ for $b\in\mathrm{B}^{>z}$, and $\mathrm{U}_{(\varepsilon_{b'}-\varepsilon_{z'},-1)}(*)$ is a term of the form of those appearing in $\mathbb{P}^{iv}_{\bar{z}}$. Therefore (and since the the terms that appear in $\mathbb{P}^{iv}_{\bar{b}}$ and $\mathbb{P}^{iv}_{\bar{z}}$ are the same as $\mathbb{P}''_{\bar{b}}$ and $\mathbb{P}''_{\bar{z}}$ respectively) concludes the proof of Claim 1. □

CLAIM 2. There is a dense subset of $\mathbb{P}'''_{\bar{f}_{k+s}}\cdots\mathbb{P}'''_{\bar{f}_1}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})^*\nu}}$ that is contained in the subset

$$\mathbb{P}_{\mathrm{T,B}}\mathbb{P}_{\mathscr{K},\bar{f}_s}\cdots\mathbb{P}_{\mathscr{K},\bar{f}_s}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})^*\nu}}\subset\overline{\pi(\mathrm{C}_{\gamma_{\mathscr{K}^*\nu}})},$$

where

$$\mathbb{P}_{\mathrm{T,B}} = \prod_{\substack{l\notin\mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T},\\ t\in\mathrm{T} l<t}}\mathrm{U}_{(\varepsilon_l-\varepsilon_t,0)}(v_{l\bar{t}})\prod_{\substack{l\notin\mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T},\\ b\in\mathrm{B},l<b}}\mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(v_{l\bar{b}})\in\mathrm{U}_{\mathrm{V}_0}$$

$$\mathbb{P}_{\mathscr{K},\bar{b}} = \prod_{\substack{b\in\mathrm{B};\\ t\in\mathrm{T}^{<b}}}\mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(v_{t\bar{b}})\prod_{a\in\mathrm{A}^{<b}}\mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(v_{a\bar{b}})\in\mathrm{U}_{\mathrm{V}_1}$$

$$\mathbb{P}_{\mathscr{K},\bar{z}} = \prod_{t\in\mathrm{T}^{<z}}\mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}})\prod_{b\in\mathrm{B}^{<z}}\mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(v_{b\bar{z}})\in\mathrm{U}_{\mathrm{V}_1},$$

for $v_{ij}\in\mathbb{C}, b\in\mathrm{B}$ and $z\in\mathrm{Z}$. (It is indeed a subset by Corollary 1.11.)

Note that $\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} = \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}$ and that

$$u = \prod_{\substack{l\notin \mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T},\\ t\in\mathrm{T} l<t}} \mathrm{U}_{(\varepsilon_l-\varepsilon_t,0)}(v_{l\bar{t}}) \in \mathrm{U}_{\mu_{\gamma_{\mathscr{K}}}}.$$

We have the following equalities

$$\begin{aligned}&\mathbb{P}_{\mathrm{T},\mathrm{B}}\mathbb{P}_{\mathscr{K},\bar{f}_s}\cdots\mathbb{P}_{\mathscr{K},\bar{f}_s}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} =\\ &\mathbb{P}''_{\bar{f}_s}\cdots\mathbb{P}''_{\bar{f}_s}u\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu} =\\ &\mathbb{P}''_{\bar{f}_s}\cdots\mathbb{P}''_{\bar{f}_s}\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}\end{aligned}$$

where, for $z\in\mathrm{Z}$ and $b\in\mathrm{B}$:

$$\mathbb{P}''_{\bar{b}} = \prod_{\substack{l\notin\mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T};\\ l<b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_b,0)}(\xi_{l\bar{b}})\prod_{t\in\mathrm{T}^{<b}}\mathrm{U}_{(\varepsilon_t-\varepsilon_b,0)}(\xi_{t\bar{b}})\prod_{a\in\mathrm{A}^{<b}}\mathrm{U}_{(\varepsilon_a-\varepsilon_b,1)}(\xi_{a\bar{b}}) \text{ c.f (19)}$$

$$\mathbb{P}''_{\bar{z}} = \prod_{\substack{l\notin\mathrm{Z}\cup\mathrm{A}\cup\mathrm{B}\cup\mathrm{T};\\ l<z}} \mathrm{U}_{(\varepsilon_l-\varepsilon_z,-1)}(\xi_{l\bar{z}})\prod_{t\in\mathrm{T}^{<z}}\mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(\xi_{t\bar{z}})\prod_{b\in\mathrm{B}^{<z}}\mathrm{U}_{(\varepsilon_b-\varepsilon_z,-1)}(\xi_{b\bar{z}}) \text{ c.f (20)}$$

$$\xi_{l\bar{b}} = v_{l\bar{b}} + \sum_{\substack{l<t<b,\\ t\in\mathrm{T}}} c^{1,1}_{s\bar{t},t\bar{b}}(-v_{l\bar{t}})v_{t\bar{b}}$$

$$\xi_{l\bar{z}} = \rho_{l\bar{z}} + \sum_{z'\in\mathrm{Z}} c^{1,1}_{l\bar{z}',z'\bar{z}}(-\rho_{l\bar{z}'})v_{z'\bar{z}} + \sum_{\substack{l<b<z,\\ b\in\mathrm{B}}} c^{1,1}_{l\bar{b},b\bar{z}}(-\xi_{l\bar{b}})v_{b\bar{z}}$$

$$\rho_{l\bar{z}} = \sum_{\substack{l<t<z,\\ t\in\mathrm{T}}} c^{1,1}_{l\bar{t},t\bar{z}}(-v_{l\bar{t}})v_{t\bar{z}} \text{ (for } z\in\mathrm{Z})$$

$$\xi_{t\bar{z}} = v_{t\bar{z}}$$

$$\xi_{b\bar{z}} = v_{b\bar{z}}$$

$$\xi_{t\bar{b}} = v_{t\bar{b}}.$$

To prove Claim 2 we must set open conditions on the parameters k_{ij} such that the system of equations defined by $v_{ij} = \xi_{ij}$ has a solution in the variables v_{ij}. Setting $v_{t\bar{z}} := k_{t\bar{z}}$ and $v_{b\bar{z}} := k_{b\bar{z}}$ this is reduced to setting conditions on the k_{ij} so that the following system can be solved:

$$k_{l\bar{b}} = v_{l\bar{b}} + \sum_{l<t<b,t\in\mathrm{T}} c^{1,1}_{l\bar{t},t\bar{b}}(-v_{l\bar{t}})k_{t\bar{b}} \tag{27}$$

$$k_{l\bar{z}} = \rho_{l\bar{z}} - \sum_{l<b<z,b\in\mathrm{B}} c^{1,1}_{l\bar{b},b\bar{z}}(v_{l\bar{b}} + \sum_{\substack{l<t<b,\\ t\in\mathrm{T}}} c^{1,1}_{l\bar{t},t\bar{b}}(-v_{l\bar{t}})k_{t\bar{b}})k_{b\bar{z}} \tag{28}$$

$$\rho_{l\bar{z}} = \sum_{l<t<z,t\in\mathrm{T}} c^{1,1}_{l\bar{t},t\bar{z}}(-v_{l\bar{t}})k_{t\bar{z}}. \tag{29}$$

Lines (27) and (28) above define a linear system of as many equations as variables: the variables are $\{v_{l\bar{b}}\}_{l\notin\mathrm{A}\cup\mathrm{B}\cup\mathrm{T};b\in\mathrm{B}^{>l}} \cup \{v_{l\bar{t}}\}_{l\notin\mathrm{A}\cup\mathrm{B}\cup\mathrm{Z}\cup\mathrm{T};t\in\mathrm{T}^{>l}}$, there is one equation for each $l\bar{b}, l\notin \mathrm{A}\cup\mathrm{B}\cup\mathrm{T}; b\in\mathrm{B}^{>l}$, for each $l\bar{z}, l\notin\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}; z\in\mathrm{Z}^{>l}$, and note that by definition of an LS block the sets $\{l\bar{z}, l\notin\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}; z\in\mathrm{Z}^{>l}\}$ and $\{l\bar{t}, s\notin\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}; b\in\mathrm{B}^{>l}\}$ have the same cardinality (t_i is the maximal element of the set $\{l\notin\mathrm{A}\cup\mathrm{B}\cup\mathrm{T}, s<t_{i+1}, s<z_i\}$). Therefore the system has a solution as long as the matrix of coefficients has non-zero determinant, which imposes open conditions on the $k'_{ij}s$. Hence Claim 2 is proven. Now, to finish the proof of Lemma 3.29, note that if the $k'_{ij}s$ satisfy the open conditions established by Claim

2, then

$$\mathbb{P}'''_{\bar{f}_{k+s}}\cdots\mathbb{P}'''_{\bar{f}_1}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathrm{K})}*\nu}\subseteq\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu}),$$

and therefore Proposition 3.27 implies that

$$\mathrm{U}_0\mathbb{P}'''_{\bar{f}_{k+s}}\cdots\mathbb{P}'''_{\bar{f}_1}\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathrm{K})}*\nu}\subseteq\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu}),$$

which implies Lemma 3.29. Now we show the second contention towards Proposition 3.10.

LEMMA 3.30. Let ν be a combinatorial gallery and $\mathscr{K}$ be an LS block. Then

$$\overline{\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})}\subseteq\overline{\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu})}$$

Recall that

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})=\mathbb{U}_0^{\gamma_{\mathscr{K}}*\nu}\mathbb{U}_1^{\gamma_{\mathscr{K}}*\nu}\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}.$$

Notice that $\mathbb{U}_0^{\gamma_{\mathscr{K}}*\nu}\subset\mathrm{U}_0$ and that all generators of $\mathbb{U}_1^{\gamma_{\mathscr{K}}*\nu}$ also belong to U_0 except for those of the form $\mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}})$ or $\mathrm{U}_{(\varepsilon_t+\varepsilon_{t'},-1)}(v_{tt'})$ for $t,t'\in\mathrm{T},z\in\mathrm{Z}^{>t}$, and $v_{t\bar{z}},v_{tt'}\in\mathbb{C}$. Hence, since, again, $\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}=\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}$ all elements of $\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu})$ belong to

$$\mathrm{U}_0\prod_{\substack{t\in\mathrm{T}\\ z\in\mathrm{Z}^{>t}}}\mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}})\prod_{t,t'\in\mathrm{T}}\mathrm{U}_{(\varepsilon_t+\varepsilon_{t'},-1)}(v_{tt'})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}.\tag{30}$$

Now consider

$$\prod_{t'\in\mathrm{T},z\in\mathrm{Z}}\mathrm{U}_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'})\prod_{t\in\mathrm{T},z\in\mathrm{Z}^{>t}}\mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}$$

which is a subset of $\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu})$ (by Proposition 3.27) because

$$\prod_{t\in\mathrm{T},z\in\mathrm{Z}}\mathrm{U}_{(\varepsilon_z+\varepsilon_t,0)}(k_{zt})\in\mathrm{U}_0\text{ and}$$

$$\prod_{\substack{t\in\mathrm{T}\\ z\in\mathrm{Z}^{>t}}}\mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}\subset\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu}).$$

We have

$$\prod_{t'\in\mathrm{T},z\in\mathrm{Z}}\mathrm{U}_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'})\prod_{t\in\mathrm{T},z\in\mathrm{Z}^{>t}}\mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}=\tag{31}$$

$$\prod_{\substack{t,t'\in\mathrm{T}\\ t\neq t'}}\mathrm{U}_{(\varepsilon_t+\varepsilon_{t'},-1)}(\xi_{tt'})\prod_{t\in\mathrm{T},z\in\mathrm{Z}^{>t}}\mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}})\prod_{t'\in\mathrm{T},z\in\mathrm{Z}}\mathrm{U}_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}\tag{32}$$

$$\prod_{\substack{t,t'\in\mathrm{T}\\ t\neq t'}}\mathrm{U}_{(\varepsilon_t+\varepsilon_{t'},-1)}(\xi_{tt'})\prod_{t\in\mathrm{T},z\in\mathrm{Z}^{>t}}\mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}})\mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu}\tag{33}$$

where

$$\xi_{tt'}=\sum_{z\in\mathrm{Z}^{>t'}}c^{1,1}_{zt,t'\bar{z}}(-k_{zt})k_{t'\bar{z}}+\sum_{z\in\mathrm{Z}^{>t}}c^{1,1}_{zt',t\bar{z}}(-k_{zt'})k_{t\bar{z}}.\tag{34}$$

The equality between (31) and (32) is due to Chevalley's commutator formula (4) and the equality between (32) and (33) is obtained by using Proposition 3.27 and $\mathrm{U}_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'})\in\mathrm{U}_{\mu_{\gamma_{\mathscr{K}}}}$. Now fix an element in (30). Setting $k_{t\bar{z}}=v_{t\bar{z}}$ defines the linear equations

$$v_{tt'}=\sum_{z\in\mathrm{Z}^{>t'}}c^{1,1}_{zt,t'\bar{z}}(-k_{zt})v_{t'\bar{z}}+\sum_{z\in\mathrm{Z}^{>t}}c^{1,1}_{zt',t\bar{z}}(-k_{zt'})v_{t\bar{z}}$$

in the variables k_{zt}, for $z \in \mathrm{Z}$ and $t \in \mathrm{T}$. There are more variables than equations: for each equation indexed by a non ordered pair (t_i, t_j) there are the variables v_{zt_i} and $v_{z't_j}$ for $z > t'$ and $z' > t$ (which always exist by definition of an LS block); hence the system has solutions as long as the matrix of coefficients has non-zero determinants. This imposes an open condition on the parameters $v_{t\bar{z}}$. Hence for such $v_{t\bar{z}}, v_{tt'}, k_{t\bar{z}} = v_{t\bar{z}}$, and solutions k_{ij}, for the latter equations we have

$$\begin{aligned}&\prod_{\substack{t\in \mathrm{T}\\ z\in \mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}}) \prod_{t,t'\in \mathrm{T}} \mathrm{U}_{(\varepsilon_t+\varepsilon_{t'},-1)}(v_{tt'}) \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} = \\ &\prod_{\substack{t'\in \mathrm{T},\\ z\in \mathrm{Z}}} \mathrm{U}_{(\varepsilon_z+\varepsilon_{t'},0)}(k_{zt'}) \prod_{\substack{t\in \mathrm{T},\\ z\in \mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(k_{t\bar{z}}) \mathrm{T}^{\geq 2k+r+s}_{\gamma_{w(\mathscr{K})}*\nu} \subset \pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu});\end{aligned}$$

Proposition 3.27 then implies

$$\mathrm{U}_0 \prod_{\substack{t\in \mathrm{T}\\ z\in \mathrm{Z}^{>t}}} \mathrm{U}_{(\varepsilon_t-\varepsilon_z,-1)}(v_{t\bar{z}}) \prod_{t,t'\in \mathrm{T}} \mathrm{U}_{(\varepsilon_t+\varepsilon_{t'},-1)}(v_{tt'}) \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu} \subset \pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu});$$

this completes the proof of Lemma 3.30 and hence of Proposition 3.10.

□

Now let $\mathscr{K}$ be a zero lump. This means there exists $k > 1$ such that the right (respectively left) column of $\mathscr{K}$ has as entries the integers $1 < \cdots < k$ (respectively $\bar{k} < \cdots < \bar{1}$); its word is therefore $w(\mathscr{K}) = 1 \cdots k\bar{k} \cdots \bar{1}$. This means, in particular, that the truncated images $\mathrm{T}^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu} = \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}$ are stabilised by U_0, by Proposition 3.27. We have

$$\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu}) = \mathbb{U}_0^{\gamma_{w(\mathscr{K})}*\nu} \cdots \mathbb{U}_{2k-1}^{\gamma_{w(\mathscr{K})}*\nu} \mathrm{T}^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu}$$

by Theorem 1.10. Clearly all the subgroups $\mathbb{U}_l^{\gamma_{w(\mathscr{K})}*\nu} \subset \mathrm{U}_0$ for $1 \leq l \leq k$. For $0 \leq j \leq k-1$, the generators of $\mathbb{U}_{k+j}^{\gamma_{w(\mathscr{K})}*\nu}$ are all of the form $\mathrm{U}_{(\varepsilon_s-\varepsilon_{k-j}, n_{k-j})}$ for $l < k-j$. In particular the gallery $\gamma_{1\cdots k\bar{k}\cdots\overline{k-j-1}}$ has crossed the hyperplanes $\mathrm{H}_{(\varepsilon_s-\varepsilon_{k-j},m)}$ once positively at $m = 0$ and once negatively at $m = 1$, which means that $n_{k-j} = 0$, $\mathrm{U}_{(\varepsilon_s-\varepsilon_{k-j},n_{k-j})}(a) = \mathrm{U}_{(\varepsilon_s-\varepsilon_{k-j},0)}(a) \in \mathrm{U}_0$, for all $a \in \mathbb{C}$. Hence

$$\begin{aligned}\pi'(\mathrm{C}_{\gamma_{w(\mathscr{K})}*\nu}) &= \mathbb{U}_0^{\gamma_{w(\mathscr{K})}*\nu} \cdots \mathbb{U}_{2k-1}^{\gamma_{w(\mathscr{K})}*\nu} \mathrm{T}^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu} \\ &= \mathrm{T}^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu} \\ &= \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}.\end{aligned}$$

In

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu}) = \mathbb{U}_0^{\gamma_{\mathscr{K}}*\nu} \mathbb{U}_1^{\gamma_{\mathscr{K}}*\nu} \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu}$$

we have $\mathbb{U}_1^{\gamma_{\mathscr{K}}*\nu} = \{\mathrm{Id}\}$ and $\mathbb{U}_0^{\gamma_{\mathscr{K}}*\nu} \subset \mathrm{U}_0$, therefore

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}}*\nu}) = \mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}}*\nu} = \mathrm{T}^{\geq 2k}_{\gamma_{w(\mathscr{K})}*\nu}$$

since $\mu_{\gamma_{\mathscr{K}}} = \mu_{\gamma_{w(\mathscr{K})}}$.

4.4. Proof of Proposition 3.24.

Proof of Proposition 3.24. Let ν be a combinatorial gallery.

Relation R1. For $z \neq \overline{x}$:

$$a)\ y\ x\ z \equiv y\ z\ x \quad \text{for} \quad x \leq y < z$$
$$b)\ x\ z\ y \equiv z\ x\ y \quad \text{for} \quad x < y \leq z$$

LEMMA 3.31. Let $w_1 = y\ x\ z$ and $w_2 = y\ z\ x$, $w_3 = x\ z\ y$, and $w_4 = z\ x\ y$ for $z \neq \bar{x}$. Then

$$a)\overline{\pi(\mathrm{C}_{\gamma_{w_1} * \nu})} = \overline{\pi(\mathrm{C}_{\gamma_{w_2} * \nu})}$$
$$b)\overline{\pi(\mathrm{C}_{\gamma_{w_3} * \nu})} = \overline{\pi(\mathrm{C}_{\gamma_{w_4} * \nu})}$$

PROOF. For the proof we recall the notation $\varepsilon_{\bar{a}} = -\varepsilon_a$ and $\bar{\bar{\imath}} = i$ for any $i \in \{1, \cdots, n\}$. Also note that $\mathrm{T}^{\geq 3}_{\gamma_{w_i} * \nu}$ all coincide for $i \in \{1,2,3,4\}$; we will denote them by T^w. We divide the proof of Lemma 3.31 in three cases.

Case 1: $x < y < z$

CLAIM 3. If $z \neq \bar{y}$ and $y \neq \bar{x}$:

i. $\pi(\mathrm{C}_{\gamma_{w_1} * \nu}) = \mathrm{U}_0 \mathrm{U}_{(\varepsilon_x - \varepsilon_y, -1)}(v_{x\bar{y}}) \mathrm{T}^w$
ii. $\pi(\mathrm{C}_{\gamma_{w_2} * \nu}) = \mathrm{U}_0 \mathrm{U}_{(\varepsilon_x - \varepsilon_y, -1)}(v_{x\bar{y}}) \mathrm{U}_{(\varepsilon_x - \varepsilon_z, -1)}(v_{x\bar{z}}) \mathrm{T}^w$
iii. $\pi(\mathrm{C}_{\gamma_{w_3} * \nu}) = \mathrm{U}_0 \mathrm{U}_{(\varepsilon_y - \varepsilon_z, -1)}(v_{y\bar{z}}) \mathrm{T}^w$
iv. $\pi(\mathrm{C}_{\gamma_{w_4} * \nu}) = \mathrm{U}_0 \mathrm{U}_{(\varepsilon_x - \varepsilon_z, -1)}(v_{x\bar{z}}) \mathrm{U}_{(\varepsilon_y - \varepsilon_z, -1)}(v_{y\bar{z}}) \mathrm{T}^w$.

PROOF OF CLAIM 3. We first remark that, regardless whether x, y, and z are barred or unbarred, the roots $\varepsilon_x - \varepsilon_z, \varepsilon_y - \varepsilon_z$, and $\varepsilon_x - \varepsilon_y$ are always positive. Now we recall the notation from Theorem 1.10:

$$\pi(\mathrm{C}_{\gamma_{w_1} * \nu}) = \mathbb{U}_0^{\gamma_{w_i} * \nu} \mathbb{U}_1^{\gamma_{w_i} * \nu} \mathbb{U}_2^{\gamma_{w_i} * \nu} \mathrm{T}^w$$

Assume that $z \neq \bar{y}$ and $y \neq \bar{x}$.

i. We have $\mathrm{U}_{(\varepsilon_x - \varepsilon_y, -1)}(v_{x\bar{y}}) \in \mathbb{U}_1^{\gamma_{w_1} * \nu}$ for any $v_{x\bar{y}} \in \mathbb{C}$, hence
$$\mathrm{U}_0 \mathrm{U}_{(\varepsilon_x - \varepsilon_y, -1)}(v_{l\bar{y}}) \mathrm{T}^w \subseteq \pi(\mathrm{C}_{\gamma_{w_1} * \nu}).$$
Out of all generators of $\mathbb{U}_i^{\gamma_{w_1} * \nu}$ for $i \in \{0,1,2\}$, the only one that does not belong to U_0 is of the form $\mathrm{U}_{(\varepsilon_x - \varepsilon_y, -1)}(v_{x\bar{y}}) \in \mathbb{U}_1^{\gamma_{w_1} * \nu}$, and the ones from $\mathbb{U}_2^{\gamma_{w_1} * \nu}$ that do not commute with it are those of the form $\mathrm{U}_{(\varepsilon_y + \varepsilon_z, 1)}(a)$, but in that case Chevalley's commutator formula produces a term $\mathrm{U}_{(\varepsilon_x + \varepsilon_z, 0)}(c^{1,1}_{x\bar{y},yz}(-v_{x\bar{y}})a) \in \mathrm{U}_0$. This implies the other inclusion, together with Proposition 1.8, which allows us to write down the generators of each $\mathbb{U}_i^{\gamma_{w_1} * \nu}$ in any order.
ii. The only generators of $\mathbb{U}_i^{\gamma_{w_2} * \nu}$ for $i \in \{0,1,2\}$ that do not belong to U_0 are those of the form $\mathrm{U}_{(\varepsilon_x - \varepsilon_y, -1)}(v_{x\bar{y}}) \in \mathbb{U}_2^{\gamma_{w_2} * \nu}$ and $\mathrm{U}_{(\varepsilon_x - \varepsilon_z, -1)}(v_{x\bar{z}}) \in \mathbb{U}_2^{\gamma_{w_2} * \nu}$. The equality follows by Proposition 1.8, Theorem 1.10, and Proposition 3.27.
iii. All the generators of $\mathbb{U}_0^{\gamma_{w_3} * \nu}$ and $\mathbb{U}_1^{\gamma_{w_3} * \nu}$ belong to U_0, and the only generators of $\mathbb{U}_2^{\gamma_{w_3} * \nu}$ that do not are $\mathrm{U}_{(\varepsilon_y - \varepsilon_z, -1)}$. Thus Claim 3 follows by Proposition 3.27 and Theorem 1.10.
iv. As in the previous cases, we have
$$\pi(\mathrm{C}_{\gamma_{w_4 * \nu}}) = \mathbb{U}_0^{\gamma_{w_4} * \nu} \mathbb{U}_1^{\gamma_{w_4} * \nu} \mathbb{U}_2^{\gamma_{w_4} * \nu} \mathrm{T}^w,$$
and $\mathbb{U}_0^{\gamma_{w_4} * \nu} \subset \mathrm{U}_0$. All generators of $\mathbb{U}_1^{\gamma_{w_4} * \nu}$ and respectively $\mathbb{U}_2^{\gamma_{w_4} * \nu}$ belong to U_0 except for $\mathrm{U}_{(\varepsilon_x - \varepsilon_z, -1)}(a) \in \mathbb{U}_1^{\gamma_{w_4} * \nu}$ and $\mathrm{U}_{(\varepsilon_y - \varepsilon_z, -1)}(b) \in \mathbb{U}_2^{\gamma_{w_4} * \nu}$, respectively, for

$\{a, b\} \subset \mathbb{C}$. To prove this part of Claim 3 we observe that $\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(a)$ commutes with all generators of $\mathbb{U}_2^{\gamma_{w_4}*\nu}$ except for $\mathrm{U}_{(\varepsilon_z+\varepsilon_y,1)}(d)$, with $d \in \mathbb{C}$. However, commuting the latter two terms produces elements $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(c^{1,1}_{x\bar{z},zy}(-a)d) \in \mathrm{U}_0$. Therefore

$$\pi(\mathrm{C}_{\gamma_{w_4}*\nu}) \subseteq \mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w,$$

and the other inclusion is clear by Proposition 3.27 and the above discussion. This finishes the proof of Claim 3.

□

Now we make use of Claim 3 to prove Lemma 3.31 in this case, assuming $z \neq \bar{y}$ and $y \neq \bar{x}$. For both a) and b) Claim 3 immediately implies

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) \subseteq \pi(\mathrm{C}_{\gamma_{w_2}*\nu}) \text{ and}$$
$$\pi(\mathrm{C}_{\gamma_{w_3}*\nu}) \subseteq \pi(\mathrm{C}_{\gamma_{w_4}*\nu}).$$

Next we will show

$$\overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})} \subseteq \overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})}.$$

For this, let $v_{y\bar{z}} \in \mathbb{C}$ and $v_{x\bar{y}} \in \mathbb{C}$ with $v_{x\bar{y}} \neq 0$. Then since $\mathrm{U}_{(\varepsilon_y-\varepsilon_z,0)}(v_{y\bar{z}}) \in \mathrm{U}_{\mu_w} \cap \mathrm{U}_0$ for any $v_{y\bar{z}} \in \mathbb{C}$, Lemma 3.31 , Chevalley's commutator formula, and Proposition 3.27 imply

$$\begin{aligned}\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) \supset \mathrm{U}_{(\varepsilon_y-\varepsilon_z,0)}(v_{y\bar{z}})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{T}^w =\\ \mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(c^{1,1}_{y\bar{z},v\bar{y}}(-v_{y\bar{z}})v_{x\bar{y}})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,0)}(v_{y\bar{z}})\mathrm{T}^w =\\ \mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(c^{1,1}_{y\bar{z},v\bar{y}}(-v_{y\bar{z}})v_{x\bar{y}})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{T}^w\end{aligned}$$

Therefore

$$\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{T}^w \subset \pi(\mathrm{C}_{\gamma_{w_1}*\nu})$$

as long as $v_{x\bar{y}} \neq 0$, since in that case $c^{1,1}_{y\bar{z},v\bar{y}}(-v_{y\bar{z}})v_{x\bar{y}} = v_{x\bar{z}}$ has a solution in $v_{y\bar{z}}$. Hence Proposition 3.27 implies

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{T}^w \subset \pi(\mathrm{C}_{\gamma_{w_1}*\nu}).$$

Claim 3 (i. and ii.) then implies that a dense subset of $\pi(\mathrm{C}_{\gamma_{w_2}*\nu})$ is contained in $\pi(\mathrm{C}_{\gamma_{w_1}*\nu})$, which implies Lemma 3.31 , a) in this case. To finish the proof of Lemma 3.31 b), let $v_{x\bar{y}} \in \mathbb{C}$ and $v_{y\bar{z}} \in \mathbb{C}$ with $v_{y\bar{z}} \neq 0$. Then, just as for a)

$$\pi(\mathrm{C}_{\gamma_{w_3}*\nu}) \supset \mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w = \tag{35}$$
$$\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(c^{1,1}_{x\bar{y},y\bar{z}}(-v_{x\bar{y}})v_{y\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(v_{y\bar{z}})\mathrm{T}^w = \tag{36}$$
$$\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(c^{1,1}_{x\bar{y},y\bar{z}}(-v_{x\bar{y}})v_{y\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w. \tag{37}$$

Therefore the elements of the set

$$\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w$$

such that $v_{y\bar{z}} \neq 0$ are contained in (37). By Claim 3 (iii. and iv) and Proposition 3.27 there is a dense subset of

$$\pi(\mathrm{C}_{\gamma_{w_4}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_z,-1)}(v_{x\bar{z}})\mathrm{U}_{(\varepsilon_y-\varepsilon_z,-1)}(v_{y\bar{z}})\mathrm{T}^w$$

that is contained in $\pi(\mathrm{C}_{\gamma_{w_3}*\nu})$.

The cases $z = \bar{y}$ and $y = \bar{x}$ are missing so far. (Note that $z \neq \bar{x}$ is not allowed. Also note that and that if $y = \bar{x}$ then x must be unbarred and if $z = \bar{y}$ then y must be unbarred.)

Case 1.1 $z = \bar{y}$

a. We first show that

$$\overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})} \subseteq \overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})}. \tag{38}$$

All of the generators of $\mathbb{U}_1^{\gamma_{w_1}*\nu}$ belong to U_0 except for $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})$, for $v_{x\bar{y}} \in \mathbb{C}$. The generators of $\mathbb{U}_1^{\gamma_{w_1}*\nu}$ are $\mathrm{U}_{(\varepsilon_l-\varepsilon_y,-1)}(v_{l\bar{y}})$ for $l \neq x$ and $v_{l\bar{y}} \in \mathbb{C}$, and $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,0)}(v_{x\bar{y}})$ for $v_{x\bar{y}} \in \mathbb{C}$. This last term commutes with $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})$. Therefore, by parallel arguments to those given in the proof of Claim 3,

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\prod_{\substack{l<y\\ l\neq x}}\mathrm{U}_{(\varepsilon_l-\varepsilon_y,-1)}(v_{l\bar{y}})\mathrm{T}^w$$

All terms in the product $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\prod_{\substack{l<y\\ l\neq x}}\mathrm{U}_{(\varepsilon_l-\varepsilon_y,-1)}(v_{l\bar{y}})$ are at the same time generators of $\mathbb{U}_1^{\gamma_{w_2}}$ as well, therefore, by Proposition 3.27,

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) \subseteq \pi(\mathrm{C}_{\gamma_{w_2}*\nu}),$$

as wanted. Next we would like to show

$$\overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})} \subseteq \overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})}. \tag{39}$$

To do so we will make use of Proposition 3.10. Let

$$\mathscr{K}_1 = \begin{array}{|c|c|c|}\hline \mathrm{x} & \mathrm{x} & \mathrm{y} \\ \hline \bar{\mathrm{y}} & \bar{\mathrm{y}} & \\ \cline{1-2}\end{array}$$

and

$$\mathscr{K}_2 = \begin{array}{|c|c|c|}\hline \mathrm{x} & \mathrm{y\text{-}1} & \mathrm{y} \\ \hline & \bar{\mathrm{y}} & \overline{\mathrm{y\text{-}1}} \\ \cline{2-3}\end{array}.$$

Then we have $w_1 = y\ x\ \bar{y} = w(\mathscr{K}_1)$ and $w_2 = y\ \bar{y}\ x = w(\mathscr{K}_2)$. By Proposition 3.10 it then suffices to show

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}})}.$$

First assume $y-1 \neq x$. Note that in this case $\mathbb{U}_1^{\gamma_{\mathscr{K}_2}*\nu}$ is generated by terms $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)$ with $a \in \mathbb{C}$, and all generators of $\mathbb{U}_0^{\gamma_{\mathscr{K}_2}*\nu}$ and $\mathbb{U}_2^{\gamma_{\mathscr{K}_2}*\nu}$ belong to U_0. Out of these, the only ones in $\mathbb{U}_2^{\gamma_{\mathscr{K}_2}*\nu}$ that do not commute with with $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)$ are $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(b)$ and $\mathrm{U}_{(\varepsilon_x-\varepsilon_{y-1},0)}(d)$. Then for every element in $\pi(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$ there is a $u \in \mathrm{U}_0$ such that it belongs to

$$\begin{aligned}&u\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)\overbrace{\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(b)\mathrm{U}_{(\varepsilon_x-\varepsilon_{y-1},0)}(d)}^{=:u'}\mathrm{T}^w = \\ &uu'\mathrm{U}_{(\varepsilon_{y-1}+\varepsilon_x,-1)}(c^{1,1}_{y-1\bar{y},xy}(-a)b)\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(c^{1,1}_{y-1\bar{y},x\overline{y-1}}(-a)d)\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)\mathrm{T}^w.\end{aligned}$$

Fix such u, a, b, and d such that $abd \neq 0$. Such elements form a dense subset of $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$. We will show

$$\mathrm{U}_{(\varepsilon_{y-1}+\varepsilon_x,-1)}(c^{1,1}_{y-1\bar{y},xy}(-a)b)\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(c^{1,1}_{y-1\bar{y},x\overline{y-1}}(-a)d)\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)\mathrm{T}^w \subset \pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$$

If this is true, then (39) is then implied by Proposition 3.27 applied to $u\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(b)\mathrm{U}_{(\varepsilon_x-\varepsilon_{y-1},0)}(d) \in \mathrm{U}_0$.

First note that for all $\{a_{x\bar{y}}, a_{y-1\bar{y}}, a_{yy-1}\} \subset \mathbb{C}$, $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}})$ and $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1y})$ belong to $\mathbb{U}_1^{\gamma_{\mathscr{K}_1}*\nu}$, and $v := \mathrm{U}_{(\varepsilon_y+\varepsilon_{y-1},0)}(a_{yy-1}) \in \mathrm{U}_{\varepsilon_x} \cap \mathrm{U}_0$ stabilises the truncated image T^w as well as the whole image $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$. Therefore all elements of

$$v^{-1}\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}})\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1\bar{y}})v\mathrm{T}^w = \mathrm{U}_{(\varepsilon_x+\varepsilon_{y-1},-1)}(c^{1,1}_{x\bar{y},yy-1}(-a_{x\bar{y}})a_{yy-1})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}})\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1\bar{y}})\mathrm{T}^w$$

belong to $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ and since $abd \neq 0$ we may find $a_{x\bar{y}}, a_{y-1\bar{y}}$, and a_{yy-1} such that

$$\begin{aligned} a_{x\bar{y}} &= c^{1,1}_{y-1\bar{y},x\overline{y-1}}(-a)d, \\ c^{1,1}_{x\bar{y},yy-1}(-a_{x\bar{y}})a_{yy-1} &= c^{1,1}_{y-1\bar{y},xy}(-a)b, \text{ and} \\ a_{y-1\bar{y}} &= a. \end{aligned}$$

This concludes the proof if $y \neq x-1$. Now assume that $y = x-1$. In this case all generators of $\mathbb{U}_2^{\gamma_{\mathrm{K}_2}*\nu}$ commute with $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a_{y-1\bar{y}})$, and therefore all elements in $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$ belong to

$$u\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a)\mathrm{T}^w$$

for some $u \in \mathrm{U}_0$ and $a \in \mathbb{C}$ - but $\mathrm{U}_{(\varepsilon_{y-1}-\varepsilon_y,-1)}(a) \in \mathbb{U}_1^{\gamma_{\mathscr{K}_1}*\nu}$, which implies (39) by applying Proposition 3.27 to $u \in \mathrm{U}_0$.

b. We now have

$$w_3 = x\ \bar{y}\ y = w(\mathscr{K}_3) \text{ and } w_4 = \bar{y}\ x\ y = w(\mathscr{K}_4),$$

where

$$\mathscr{K}_3 = \begin{array}{|c|c|c|}\hline \mathrm{y} & \mathrm{x} & \mathrm{x} \\ \hline & \bar{\mathrm{y}} & \bar{\mathrm{y}} \\ \hline \end{array}$$

and

$$\mathscr{K}_4 = \begin{array}{|c|c|c|}\hline \mathrm{x} & \mathrm{x} & \bar{\mathrm{y}} \\ \hline \mathrm{y} & \mathrm{y} & \\ \hline \end{array}.$$

We want to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}.$$

First $\mathbb{U}_0^{\gamma_{\mathscr{K}_3}*\nu}$ and $\mathbb{U}_1^{\gamma_{\mathscr{K}_3}*\nu}$ are both contained in U_0. The generators of $\mathbb{U}_2^{\gamma_{\mathscr{K}_3}*\nu}$ that do not belong to U_0 are $\mathrm{U}_{(\varepsilon_y,-1)}(\alpha_y), \mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl})$, and $\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}})$ for $\{\alpha_y, \beta_{yl}, \gamma_{y\bar{s}}\} \subset \mathbb{C}$ and $l \leq n, l \neq x, y < s \leq n$. All of these are also generators of $\mathbb{U}_1^{\gamma_{\mathscr{K}_4}*\nu}$, hence by Proposition 3.27 and Theorem 1.10 we have

$$\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) \subset \pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}).$$

The discussion above also implies that

$$\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_y,-1)}(\alpha_y)\prod_{\substack{l\leq n\\ l\neq x}}\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl})\prod_{y<s\leq n}\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}})\mathrm{T}^w \tag{40}$$

There is one more generator of $\mathbb{U}_1^{\gamma_{\mathscr{K}_4}*\nu}$, not mentioned above, which is $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,-1)}(d_{xy})$. Since all generators of $\mathbb{U}_2^{\gamma_{\mathscr{K}_4}*\nu}$(which are $\mathrm{U}_{(\varepsilon_x+\varepsilon_y,0)}(d') \in \mathrm{U}_0$ for $d' \in \mathbb{C}$) commute with those of $\mathbb{U}_1^{\gamma_{\mathscr{K}_3}*\nu}$, we have by Proposition 3.27:

$$\begin{aligned}&\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}) =\\ &\mathrm{U}_0\mathrm{U}_{(\varepsilon_x+\varepsilon_y,-1)}(d_{xy})\mathrm{U}_{(\varepsilon_y,-1)}(a_y)\prod_{\substack{l\leq n\\ l\neq x}}\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(b_{yl})\prod_{\substack{s\leq n\\ s>y}}\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(c_{y\bar{s}})\mathrm{T}^w\end{aligned}$$

We now would like to show

$$\overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})} \subset \overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})}.$$

To do this we will see that for complex numbers $a_y, b_{yl}, c_{y\bar{s}}$, and d_{xy}, with $a_y \neq 0$,

$$\mathrm{U}_{(\varepsilon_x+\varepsilon_y,-1)}(d_{xy})\mathrm{U}_{(\varepsilon_y,-1)}(a_y)\prod_{\substack{l\leq n\\ l\neq x}}\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(b_{yl})\prod_{\substack{s\leq n\\ s>y}}\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(c_{y\bar{s}})\mathrm{T}^w \tag{41}$$

$$\subset \pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}). \tag{42}$$

By (40) we conclude that for any complex numbers $\alpha_y, \beta_{yl}, \gamma_{y\bar{s}}$, and δ the following set is contained in $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$

$$\begin{aligned}&v^{-1}\mathrm{U}_{(\varepsilon_x-\varepsilon_y,1)}(\delta)\mathrm{U}_{(\varepsilon_y,-1)}(\alpha_y)\prod_{\substack{l\leq n\\ l\neq x}}\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl})\prod_{\substack{s\leq n\\ s>y}}\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}})\mathrm{T}^w =\\ &v^{-1}v\mathrm{U}_{(\varepsilon_x+\varepsilon_y,-1)}(\rho_{xy})\mathrm{U}_{(\varepsilon_y,-1)}(\alpha_y)\prod_{\substack{l\leq n\\ l\neq x}}\mathrm{U}_{(\varepsilon_y+\varepsilon_l,-1)}(\beta_{yl})\prod_{\substack{s\leq n\\ s>y}}\mathrm{U}_{(\varepsilon_y-\varepsilon_s,-1)}(\gamma_{y\bar{s}})\mathrm{T}^w\end{aligned}$$

where

$$\begin{aligned}&v =\\ &\mathrm{U}_{(\varepsilon_x,0)}(c^{1,1}_{x\bar{y},y}(-\delta)\alpha_y)\prod_{\substack{l\leq n\\ l\neq x}}\mathrm{U}_{(\varepsilon_x+\varepsilon_l,0)}(c^{1,1}_{x\bar{y},yl}(-\delta)\beta_{yl})\prod_{\substack{s\leq n\\ s>y}}\mathrm{U}_{(\varepsilon_x-\varepsilon_s,0)}(c^{1,1}_{x\bar{y},y\bar{s}}(-\delta)\gamma_{y\bar{s}})\\ &\rho_{xy} = c^{1,2}_{x\bar{y},y}(-\delta)\alpha_y^2,\end{aligned}$$

and where the latter equality is obtained by applying Chevalley's commutator formula and Proposition 3.27 applied to $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,1)}(\delta)$, which stabilises the truncated image T^w. We will have shown our claim in (41) if we find complex numbers $\alpha_y, \beta_{yl}, \gamma_{y\bar{s}}$, and δ such that

$$\begin{aligned}c^{1,2}_{x\bar{y},y}(-\delta)\alpha_y^2 &= d_{xy}\\ \alpha_y &= a_y\\ \beta_{yl} &= b_{yl},\end{aligned}$$

which we may obtain since $a_y \neq 0$. This concludes the proof in case $z = \bar{y}$.

Case 1.2 $y = \bar{x}$. This means that x is necessarily unbarred and therefore $z = \bar{b}$ for some $b < x$.

a. As before, we will use Proposition 3.10. We have

$$w_1 = \bar{x}\ x\ \bar{b} = w(\mathscr{K}_1) \text{ and}$$
$$w_2 = \bar{x}\ \bar{b}\ x = w(\mathscr{K}_2),$$

where

$$\mathscr{K}_1 = \begin{array}{|c|c|c|} \hline \mathrm{x} & \mathrm{x} & \bar{\mathrm{x}} \\ \hline \bar{\mathrm{b}} & \bar{\mathrm{b}} & \\ \cline{1-2} \end{array}$$

and

$$\mathscr{K}_2 = \begin{array}{|c|c|c|} \hline \mathrm{x} & \bar{\mathrm{x}} & \bar{\mathrm{x}} \\ \hline & \bar{\mathrm{b}} & \bar{\mathrm{b}} \\ \cline{2-3} \end{array}.$$

First we show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1} * \nu})} \subseteq \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2} * \nu})}. \tag{43}$$

To do this, we claim that

$$\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1} * \nu}) = \mathrm{U}_0 \mathrm{U}_{(\varepsilon_x, -1)}(a_x) \prod_{\substack{s \in \mathcal{C}_n \neq b \\ \varepsilon_x + \varepsilon_s \in \Phi^+}} \mathrm{U}_{(\varepsilon_x + \varepsilon_s, -1)}(a_{xs}) \mathrm{T}^w. \tag{44}$$

Indeed, $\mathrm{U}_{(\varepsilon_x,-1)}(a_x)$ and $\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})$ for $s \in \mathcal{C}_n$ and $s \neq b$ are the generators of $\mathbb{U}_1^{\gamma_{\mathscr{K}_1} * \nu}$ that do not belong to U_0, and $\mathbb{U}_2^{\gamma_{\mathscr{K}_1} * \nu}$ is the identity, because $\varepsilon_x - \varepsilon_b$ is not a positive root. Therefore (44) follows by Proposition 3.27. The aforementioned terms are also generators (but not all!) of $\mathbb{U}_2^{\gamma_{\mathscr{K}_2} * \nu}$, therefore (43) follows. Now we show

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2} * \nu})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1} * \nu})}. \tag{45}$$

To do this, let us first analyse the image

$$\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2} * \nu}) = \mathbb{U}_0^{\gamma_{\mathscr{K}_2} * \nu} \mathbb{U}_1^{\gamma_{\mathscr{K}_2} * \nu} \mathbb{U}_2^{\gamma_{\mathscr{K}_2} * \nu} \mathrm{T}^w.$$

In this case $\mathbb{U}_0^{\gamma_{\mathscr{K}_2} * \nu} \subset \mathrm{U}_0$ and $\mathbb{U}_1^{\gamma_{\mathscr{K}_2} * \nu}$ is the identity, because $-(\varepsilon_x + \varepsilon_b)$ is not a positive root. The generators of $\mathbb{U}_2^{\gamma_{\mathscr{K}_2} * \nu}$ are $\mathrm{U}_{(\varepsilon_x,-1)}(\alpha_x), \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(\alpha_{xs})$ and $\mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}(\alpha_{xb})$ for $s \in \mathcal{C}_n$ such that $s \neq b$ and complex numbers α_x, α_{xs}, and α_{xb}. Therefore

$$\pi(\mathrm{C}_{\gamma_{\mathscr{K}_2} * \nu}) = \mathrm{U}_0 \mathrm{U}_{(\varepsilon_x,-1)}(\alpha_x) \prod_{\substack{s \neq b \\ \varepsilon_x + \varepsilon_s \in \Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(\alpha_{xs}) \mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}(\alpha_{xb}) \mathrm{T}^w. \tag{46}$$

Let us fix complex numbers α_x, α_{xs}, and α_{xb}, such that $\alpha_x \neq 0$. We will show that (cf. (44))

$$\mathrm{U}_{(\varepsilon_x,-1)}(\alpha_x) \prod_{\substack{s \neq b \\ \varepsilon_x + \varepsilon_s \in \Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(\alpha_{xs}) \mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}(\alpha_{xb}) \mathrm{T}^w \subset \pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1} * \nu}) \tag{47}$$

To do this we will use Corollary 1.11, which says, in particular, that, if we write

$$\gamma_{\mathscr{K}_1} = (\mathrm{V}_0, \mathrm{E}_0, \mathrm{V}_1, \mathrm{E}_1, \mathrm{V}_2, \mathrm{E}_2, \mathrm{V}_3),$$

then

$$\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}}) \supset \mathrm{U}_{\mathrm{V}_0} \mathrm{U}_{\mathrm{V}_1} \mathrm{U}_{\mathrm{V}_2} \mathrm{T}^w.$$

Therefore, since $u := \mathrm{U}_{(\varepsilon_b - \varepsilon_x, 0)}(a) \in \mathrm{U}_{\mathrm{V}_2} \cap \mathrm{U}_0$ for all $a \in \mathbb{C}$, and since $\mathrm{U}_{(\varepsilon_x,-1)}(a_x)$ and $\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})$, for $s \in \mathcal{C}_n$ and $s \neq b$ are the generators of $\mathbb{U}_1^{\gamma_{\mathscr{K}_1} * \nu} \subset \mathrm{U}_{\mathrm{V}_1}$ for any complex numbers a_{xs} and a_x we have (using, again, Proposition 3.27 applied to $u \in \mathrm{U}_0$ and $v \in \mathrm{U}_{\mathrm{V}_3}$ (V_3 stabilises the truncated image T^w; see below for a definition of v)):

$$\begin{aligned}
&\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}) \supset \\
&u^{-1}\mathrm{U}_{(\varepsilon_x,-1)}(a_x) \prod_{\substack{s\neq b \\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})u\mathrm{T}^w = \\
&u^{-1}u\mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}(c^{2,1}_{x,b\bar{x}}(a_x^2)b)\mathrm{U}_{(\varepsilon_x,-1)}(a_x) \prod_{\substack{s\neq b \\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})v\mathrm{T}^w = \\
&\mathrm{U}_{(\varepsilon_x+\varepsilon_b,-2)}(c^{2,1}_{x,b\bar{x}}(a_x^2)b)\mathrm{U}_{(\varepsilon_x,-1)}(a_x) \prod_{\substack{s\neq b \\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})\mathrm{T}^w.
\end{aligned}$$

where

$$v = \mathrm{U}_{(\varepsilon_b,-1)}(c^{1,1}_{x,b\bar{x}}(-a_x)b) \prod_{\substack{s\neq b \\ \varepsilon_x+\varepsilon_s\in\Phi^+}} \mathrm{U}_{(\varepsilon_b+\varepsilon_s,-1)}(c^{1,1}_{x,bs}(-a_{xs})b) \in \mathrm{U}_{\mathrm{V}_3}.$$

In order to show (47) it suffices to find complex numbers a_x, a_{xs}, and b such that

$$\begin{aligned}
c^{2,1}_{x,b\bar{x}}(a_x^2)b &= \alpha_{xb} \\
a_x &= \alpha_x \\
a_{xs} &= \alpha_{xs},
\end{aligned}$$

and we may do this, since $\alpha_x \neq 0$.

b. We will again use Proposition 3.10. We have

$$w_3 = x\ \bar{b}\ \bar{x} = w(\mathscr{K}_3) \text{ and } w_4 = \bar{b}\ x\ \bar{x} = w(\mathscr{K}_4),$$

where

$$\mathscr{K}_3 = \begin{array}{|c|c|c|} \hline \bar{\mathrm{x}} & \mathrm{x} & \mathrm{x} \\ \hline & \bar{\mathrm{b}} & \bar{\mathrm{b}} \\ \hline \end{array}$$

and

$$\mathscr{K}_4 = \begin{array}{|c|c|c|} \hline \mathrm{x\text{-}1} & \mathrm{x} & \bar{\mathrm{b}} \\ \hline \bar{\mathrm{x}} & \overline{\mathrm{x\text{-}1}} & \\ \hline \end{array}.$$

By Proposition 3.10 it is enough to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}. \tag{48}$$

We analyse both images $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ and $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$ separately and then show (48). First , since $\mathbb{U}_0^{\gamma_{\mathscr{K}_3}*\nu} \subset \mathrm{U}_0$ and $\mathbb{U}_1^{\gamma_{\mathscr{K}_3}*\nu}$ is the identity (this is because $\varepsilon_x - \varepsilon_b$ is not a positive root), we have

$$\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) = \mathrm{U}_0\prod_{\substack{l<x \\ l\neq b}}\mathrm{U}_{(\varepsilon_l-\varepsilon_x,-1)}(a_{l\bar{x}})\mathrm{U}_{(\varepsilon_b-\varepsilon_x,-2)}(a_{b\bar{x}})\mathrm{T}^w. \tag{49}$$

Now, $\mathbb{U}_2^{\gamma_{\mathscr{K}_4}*\nu}$ is generated by elements $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(\alpha_{x-1x})$, for $\alpha_{x-1x} \in \mathbb{C}$, and $\mathbb{U}_1^{\gamma_{\mathscr{K}_4}*\nu}$ is generated by $\mathrm{U}_{(\varepsilon_b-\varepsilon_{x-1},-1)}(\alpha_{b\overline{x-1}})$ for $\alpha_{b\overline{x-1}} \in \mathbb{C}$, by $\mathrm{U}_{(\varepsilon_l-\varepsilon_{x-1},0)}(\alpha_{l\overline{x-1}})$ for $l < x-1$ and $\alpha_{l\overline{x-1}} \in \mathbb{C}$ (this last element stabilises the truncated image T^w) , and by other elements of U_0. Therefore

$$\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}) \tag{50}$$

$$= \mathrm{U}_0 \prod_{\substack{l<x\\ l\neq b}} \mathrm{U}_{(\varepsilon_l-\varepsilon_{x-1},0)}(\alpha_{l\overline{x-1}})\mathrm{U}_{(\varepsilon_b-\varepsilon_{x-1},-1)}(\alpha_{b\overline{x-1}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(\alpha_{x-1\bar{x}})\mathrm{T}^w \tag{51}$$

$$= \mathrm{U}_0 \prod_{\substack{l<x\\ l\neq b, l\neq x-1}} \mathrm{U}_{(\varepsilon_l-\varepsilon_x,-1)}(\xi_{l\bar{x}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(\alpha_{x-1\bar{x}})\mathrm{U}_{(\varepsilon_b-\varepsilon_x,-2)}(\xi_{b\bar{x}})\mathrm{T}^w, \text{ where} \tag{52}$$

$$\xi_{b\bar{x}} = c^{1,1}_{b\overline{x-1},x-1\bar{x}}(-\alpha_{b\overline{x-1}}\alpha_{x-1\bar{x}})$$
$$\xi_{l\bar{x}} = c^{1,1}_{l\overline{x-1},x-1\bar{x}}(-\alpha_{l\overline{x-1}}\alpha_{x-1\bar{x}})$$

and where the equality between (51) and (52) arises by using (4) and Proposition 3.27 applied to $\mathrm{U}_{(\varepsilon_l-\varepsilon_{x-1},0)}(\alpha_{l\overline{x-1}})\mathrm{U}_{(\varepsilon_b-\varepsilon_{x-1},-1)}(\alpha_{b\overline{x-1}}) \in \mathrm{U}_{\mu_{\gamma_{\mathscr{K}_4}}}$. The sets displayed in (49) and (52) are equal as long as all the parameters are non-zero.

Case 2: $x = y < z, z \neq \bar{x}$
In this case we have $w_1 = y\ y\ z$ and $w_2 = y\ z\ y$. We want to look at

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) = \mathbb{U}_0^{\gamma_{w_1}*\nu}\mathbb{U}_1^{\gamma_{w_1}*\nu}\mathbb{U}_2^{\gamma_{w_1}*\nu}\mathrm{T}^w$$
$$\pi(\mathrm{C}_{\gamma_{w_2}*\nu}) = \mathbb{U}_0^{\gamma_{w_2}*\nu}\mathbb{U}_1^{\gamma_{w_2}*\nu}\mathbb{U}_2^{\gamma_{w_2}*\nu}\mathrm{T}^w$$

In this case all generators of $\mathbb{U}_i^{\gamma_{w_1}*\nu}$ and of $\mathbb{U}_i^{\gamma_{w_2}*\nu}$ belong to U_0 for $i \in \{1,2,3\}$. Therefore Proposition 3.27 implies in this case that

$$\pi(\mathrm{C}_{\gamma_{w_1}*\nu}) = \mathrm{U}_0\mathrm{T}^w = \pi(\mathrm{C}_{\gamma_{w_2}*\nu}),$$

which concludes the proof.

Case 3: $x < y = z, z \neq \bar{x}$
For this case it will be convenient to use Proposition 3.10. Let

$$\mathscr{K}_1 = \begin{array}{|c|c|}\hline \mathrm{y} & \mathrm{x} \\ \hline \multicolumn{1}{c|}{} & \mathrm{y} \\ \cline{2-2}\end{array}$$

and

$$\mathscr{K}_2 = \begin{array}{|c|c|}\hline \mathrm{x} & \mathrm{y} \\ \hline \mathrm{y} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

It is then enough to show (by Proposition 3.10) that

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})},$$

since

$$w_1 = x\ y\ y = w(\mathscr{K}_1) \text{ and}$$
$$w_2 = y\ x\ y = w(\mathscr{K}_2).$$

However, this case is now the same as the previous one: all generators of $\mathbb{U}_i^{\gamma_{\mathscr{K}_1}*\nu}$ and $\mathbb{U}_i^{\gamma_{\mathscr{K}_2}*\nu}$ belong to U_0, therefore, as before,

$$\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}) = \mathrm{U}_0\mathrm{T}^w = \pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}).$$

With this case we conclude the proof of Lemma 3.31. $\square$

Relation R2. For $1 < x \leq n$ and $x \leq y \leq \bar{x}$:

a. $y\ \overline{x-1}\ x-1 \equiv y\ x\ \bar{x}$ and

b. $\overline{x-1}\ x-1\ y \equiv x\ \bar{x}\ y.$

LEMMA 3.32. Let

$$\begin{aligned} w_1 &= y\ \overline{x-1}\ x-1 \\ w_2 &= y\ x\ \bar{x} \\ w_3 &= \overline{x-1}\ x-1\ y \\ w_4 &= x\ \bar{x}\ y \end{aligned}$$

for $z \neq \bar{x}$. Then

$$\begin{aligned} a)\overline{\pi(\mathrm{C}_{\gamma_{w_1}*\nu})} &= \overline{\pi(\mathrm{C}_{\gamma_{w_2}*\nu})} \\ b)\overline{\pi(\mathrm{C}_{\gamma_{w_3}*\nu})} &= \overline{\pi(\mathrm{C}_{\gamma_{w_4}*\nu})} \end{aligned}$$

PROOF. As usual, the proof is divided in some cases: we first consider the case where $y \notin \{x, \bar{x}\}$ and then we analyse $y = x$ and $y = \bar{x}$ separately.

Case 1 $y \notin \{x, \bar{x}\}$

a) We will use Proposition 3.10. Note that

$$w_1 = y\ \overline{x-1}\ x-1 = w\left(\begin{array}{|c|c|c|} \hline \text{x-1} & \text{y} & \text{y} \\ \hline \multicolumn{1}{c|}{} & \overline{\text{x-1}} & \overline{\text{x-1}} \\ \cline{2-3} \end{array}\right)$$

and

$$w_2 = y\ x\ \bar{x} = w\left(\begin{array}{|c|c|c|} \hline \text{x-1} & \text{x} & \text{y} \\ \hline \bar{x} & \overline{\text{x-1}} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\right).$$

Hence by Proposition 3.10, to show Lemma 3.32 a) it is enough to show that

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})},$$

where

$$\mathscr{K}_2 = \begin{array}{|c|c|c|} \hline \text{x-1} & \text{x} & \text{y} \\ \hline \bar{\text{x}} & \overline{\text{x-1}} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \text{ and } \mathscr{K}_1 = \begin{array}{|c|c|c|} \hline \text{x-1} & \text{y} & \text{y} \\ \hline \multicolumn{1}{c|}{} & \overline{\text{x-1}} & \overline{\text{x-1}} \\ \cline{2-3} \end{array}.$$

First we check

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})}.$$

Clearly $\mathbb{U}_0^{\gamma_{\mathscr{K}_2}*\nu} \subset \mathrm{U}_0$; the only generators of $\mathbb{U}_1^{\gamma_{\mathscr{K}_2}*\nu}$ that do not belong to U_0 are those of the form $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a), a \in \mathbb{C}$, and those in $\mathbb{U}_2^{\gamma_{\mathscr{K}_2}*\nu}$ are $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)$, for $b \in \mathbb{C}$. This means that every element in $\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}$ belongs to

$$u\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)\mathrm{T}^w$$

for some $u \in \mathrm{U}_0$. Both $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a)$ and $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)$ belong to $\mathrm{U}_{\varepsilon_y-\varepsilon_{x-1}}$, and this implies the contention by Proposition 3.27 and Corollary 1.11. Now we want to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} \subseteq \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}.$$

By Theorem 1.10, all elements of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ belong to

$$u\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(v_{x-1\bar{y}})\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\prod_{\substack{l\geq x\\ l\neq y}}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})\prod_{s\neq y}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{T}^w \tag{53}$$

for $u\in\mathrm{U}_0$ and $v_{x-1j}\in\mathbb{C}$. This is because both $\mathbb{U}_0^{\gamma_{\mathscr{K}_1}*\nu}$ and $\mathbb{U}_1^{\gamma_{\mathscr{K}_1}*\nu}$ are contained in U_0. Fix such an element such that $v_{x-1\bar{x}}\neq 0$. We know that $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\in\mathbb{U}_2^{\gamma_{\mathscr{K}_2}*\nu}$ and that for any $a_{x\bar{y}}\in\mathbb{C},\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(a_{x\bar{y}})\in\mathrm{U}_{\varepsilon_y}$; this means that these elements stabilise both the truncated images $\mathrm{T}^{\geq 3}_{\gamma_{\mathscr{K}_2}*\nu}$ and $\mathrm{T}^{\geq 1}_{\gamma_{\mathscr{K}_2}*\nu}$. Hence the elements in

$$\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{T}^w= \tag{54}$$

$$\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(v_{x\bar{y}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(c^{1,1}_{x-1\bar{x},x\bar{y}}(-v_{x-1\bar{x}})a_{x\bar{y}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w \tag{55}$$

all belong to $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$; more precisely to $\mathbb{U}_2^{\gamma_{\mathscr{K}_1}*\nu}\mathrm{T}^w\subset\mathrm{T}^{\geq 1}_{\gamma_{\mathscr{K}_1}*\nu}$, hence by Proposition 3.27, we may multiply by $\mathrm{U}_{(\varepsilon_x-\varepsilon_y,-1)}(-v_{x\bar{y}})$ on the left of line (55) and the product still belongs to $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$, hence

$$\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(c^{1,1}_{x-1\bar{x},x\bar{y}}(-v_{x-1\bar{x}})a_{x\bar{y}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w\subset\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}).$$

Now consider the product

$$u=\mathrm{U}_{(\varepsilon_y+\varepsilon_x,1)}(a_{yx})\mathrm{U}_{(\varepsilon_x,0)}(a_x)\prod_{\substack{l>x\\ l\neq y}}\mathrm{U}_{(\varepsilon_x-\varepsilon_l,0)}(a_{x\bar{l}})\prod_{s\neq y}\mathrm{U}_{(\varepsilon_x+\varepsilon_s,0)}(a_{xs})\in\mathrm{U}_{\varepsilon_y}\cap\mathrm{U}_0.$$

Proposition 3.27 then implies that

$$\begin{aligned}
&\pi(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})\supset\\
&u^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(c^{1,1}_{x-1\bar{x},x\bar{y}}(-v_{x-1\bar{x}})a_{x\bar{y}})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})u\mathrm{T}^w=\\
&\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1},-1)}(\rho_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(\rho_{x-1y})\\
&\prod_{\substack{l>x\\ l\neq y}}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\rho_{x-1l})\prod_{s\neq y}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\rho_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w\\
&\rho_{x-1x}=c^{1,2}_{x-1\bar{x},x}(-v_{x-1\bar{x}})a_x^2-c^{1,1}_{x-1y,yx}c^{1,1}_{x-1\bar{x},x\bar{y}}(v_{x-1\bar{x}})a_{x\bar{y}}a_{yx}\\
&\rho_{x-1j}=c^{1,1}_{x-1\bar{x},xj}(-v_{x-1\bar{x}})a_{xj}j\neq y,j\in\{\bar{l}:l>x\}\cup\{s:\varepsilon_{x-1}+\varepsilon_s\in\Phi^+\}\\
&\rho_{x-1}=c^{1,1}_{x-1\bar{x},x}(-v_{x-1\bar{x}})a_x
\end{aligned}$$

The system of equations defined by $v_{x-1}=\rho_{x-1},v_{x-1j}=\rho_{x-1j}$ has indeed solutions (the variables are $a_x,a_{yx},a_{x\bar{l}}$, and a_{xs}) since $v_{x-1,x}\neq 0$! This means that for such solutions (cf. (53))

$$\begin{aligned}
&\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(v_{x-1\bar{y}})\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\prod_{\substack{l\geq x\\ l\neq y}}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})\prod_{s\neq y}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{T}^w=\\
&\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\rho_{x-1x})\mathrm{U}_{(\varepsilon_{x-1},-1)}(\rho_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_y,-2)}(\rho_{x-1y})\\
&\prod_{\substack{l>x\\ l\neq y}}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\rho_{x-1l})\prod_{s\neq y}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\rho_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\mathrm{T}^w\subset\pi(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})
\end{aligned}$$

and so by Proposition 3.27 we get that all elements in (53) belong to $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$. All such elements of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ form a dense open subset. This finishes the proof in this case.

b) Let

$$\mathscr{K}_3 = \begin{array}{|c|c|c|}\hline \text{x-1} & \text{x-1} & \overline{\text{x-1}} \\ \hline \text{y} & \text{y} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

and

$$\mathscr{K}_4 = \begin{array}{|c|c|c|}\hline \text{y} & \text{x-1} & \text{x} \\ \hline \multicolumn{1}{c|}{} & \bar{\text{x}} & \overline{\text{x-1}} \\ \cline{2-3} \end{array}.$$

Then $w_3 = \overline{x-1}\ x-1\ y = w(\mathscr{K}_3)$ and $w_4 = x\ \bar{x}\ y = w(\mathscr{K}_4)$. As in a), by Proposition 3.10, it is enough to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}.$$

To show

$$\overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})} \subset \overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})},$$

note first that the only generator of $\mathbb{U}_i^{\gamma_{\mathscr{K}_4}*\nu}$ that does not belong to U_0 is

$$\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a) \in \mathbb{U}_1^{\gamma_{\mathscr{K}_4}*\nu}, \text{ for } a \in \mathbb{C}.$$

Of $\mathbb{U}_2^{\gamma_{\mathscr{K}_4}*\nu}$, the only generators that do not commute with $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)$ are $\mathrm{U}_{(\varepsilon_y+\varepsilon_x,0)}(b)$, with $b \in \mathbb{C}$. Then Chevalley's commutator formula (4) implies that all elements of $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$ belong to the set

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(c^{1,1}_{x-1\bar{x},xy}(-a)b)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)\mathrm{T}^w. \tag{56}$$

Since both $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(c^{1,1}_{x-1\bar{x},xy}(-a)b)$ and $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)$ belong to $\mathbb{U}_1^{\gamma_{\mathscr{K}_3}*\nu}$, the desired contention follows by Proposition 3.27. Now we show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}})} \subset \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}})}. \tag{57}$$

The proof is similar to that of a), but there are some subtle differences. First we look at the image $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$. Out of all the generators of $\mathbb{U}_i^{\gamma_{\mathscr{K}_3}*\nu}$, the only ones that do not belong to U_0 belong to $\mathbb{U}_1^{\gamma_{\mathscr{K}_3}*\nu}$: $\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x), \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(v_{x-1s})$, and $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(v_{x-1l})$ for $l \neq x-1, s > x, s \neq y$, and complex numbers v_{x-1}, v_{x-1s}, and v_{x-1l}. The group $\mathbb{U}_2^{\gamma_{\mathscr{K}_3}*\nu}$ has as generators (only) the terms $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_y,0)}(a)$, and these commute with all the latter terms. Therefore all elements of $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ belong to

$$u\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)\prod_{\substack{s>x-1\\ s\neq y}}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(v_{x-1s})\prod_{l\neq x-1}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(v_{x-1l})\mathrm{T}^w \tag{58}$$

for some $u \in \mathrm{U}_0$. Fix such a u, and assume $v_{x-1\bar{x}} \neq 0$ and $v_{x-1y} \neq 0$. Such elements as (58) form a dense open subset of $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$. Now, for all complex numbers a, a_{xy}, and $a_{x\bar{y}}$ we have $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a) \in \mathbb{U}_1^{\gamma_{\mathscr{K}_4}*\nu}$,

$U_{(\varepsilon_x+\varepsilon_y,0)}(a_{xy}) \in \mathbb{U}_1^{\gamma_{\mathscr{K}_4}*\nu}$, and $U_{(\varepsilon_x-\varepsilon_y,0)}(a_{x\bar{y}}) \in U_0$, which stabilises the truncated image $T^{\geq 2}_{\gamma_{\mathscr{K}_4}*\nu}$. Therefore, setting $c = U_{(\varepsilon_x+\varepsilon_y,0)}(a_{xy})U_{(\varepsilon_x-\varepsilon_y,0)}(a_{x\bar{y}}) \in U_0$, all elements in

$$
\begin{aligned}
&c^{-1}U_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)cT^w = \\
&U_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\varrho_{x-1x})U_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\varrho_{x-1y})U_{(\varepsilon_{x-1}-\varepsilon_y,-1)}(c^{1,1}_{x-1x,x\bar{y}}(-a)a_{x\bar{y}})U_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)T^w = \\
&U_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\varrho_{x-1x})U_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\varrho_{x-1y})U_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)U_{(\varepsilon_{x-1}-\varepsilon_y,-1)}(c^{1,1}_{x-1x,x\bar{y}}(-a)a_{x\bar{y}})T^w = \\
&U_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\varrho_{x-1x})U_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\varrho_{x-1y})U_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)T^w
\end{aligned}
$$

belong to $\pi''''(C_{\gamma_{\mathscr{K}_4}*\nu})$, where

$$
\begin{aligned}
\varrho_{x-1x} &= c^{1,1}_{x-1y,x\bar{y}}c^{1,1}_{x-1\bar{x},xy}aa_{xy}a_{x\bar{y}} \\
\varrho_{x-1y} &= c^{1,1}_{x-1\bar{x},xy}(-a)a_{xy},
\end{aligned}
$$

and where the last equality holds because $U_{(\varepsilon_{x-1}-\varepsilon_y,-1)}(c^{1,1}_{x-1x,x\bar{y}}(-a)a_{x\bar{y}}) \in U_{\varepsilon_y}$, and all elements of the latter stabilise the truncated image T^w by Proposition 3.27. Now let

$$
c' = U_{(\varepsilon_x,0)}(a_x)\prod_{\substack{s>x\\ s\neq y}}U_{(\varepsilon_x-\varepsilon_s,0)}(a_{x\bar{s}})\prod_{\substack{l\neq x-1\\ l\neq y}}U_{(\varepsilon_x+\varepsilon_l,0)}(a_{xl}) \in U_{\varepsilon_y}\cap U_0
$$

for $a_x, a_{x\bar{s}}$, and a_{xl} complex numbers; by Proposition 3.27 this element stabilises the truncated image T^w and the image $\pi''''(C_{\gamma_{\mathscr{K}_4}*\nu})$. Therefore

$$
\begin{aligned}
&\pi''''(C_{\gamma_{\mathscr{K}_4}}) \supset && (59)\\
&c'^{-1}U_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\varrho_{x-1x})U_{(\varepsilon_{x-1}+\varepsilon_y,-1)}(\varrho_{x-1y})U_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)c'T^w = && (60)\\
&U_{(\varepsilon_{x-1},-1)}(\varrho_x)\prod_{\substack{s>x-1\\ s\neq y\\ s\neq x}}U_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(\varrho_{x-1s})U_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(a)U_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\varrho'_{x-1x}) && (61)\\
&\prod_{l\notin\{x-1,x\}}U_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(\varrho_{x-1l})T^w && (62)
\end{aligned}
$$

where

$$
\begin{aligned}
\varrho_{x-1} &= c^{1,1}_{x-1x,x}(-a)a_x \\
\varrho'_{x-1x} &= \varrho_{x-1x} + c^{1,2}_{x-1x,x}(-a)a_x^2 \\
\varrho_{x-1l} &= c^{1,1}_{x-1\bar{x},xl}(-a)a_{xl} \\
\varrho_{x-1\bar{s}} &= c^{1,1}_{x-1\bar{x},x\bar{s}}(-a)a_{x\bar{s}}.
\end{aligned}
$$

We want to show that $U_{(\varepsilon_{x-1},-1)}(v_{x-1})\prod_{\substack{s>x-1\\ s\neq y}}U_{(\varepsilon_{x-1}-\varepsilon_s,-1)}(v_{x-1s})\prod_{l\neq x-1}U_{(\varepsilon_{x-1}+\varepsilon_l,-1)}(v_{x-1l})T^w$ is equal to the product in the last lines (61) and (62) above (cf. (58)), for some $a_x, a_{xl}, a_{x\bar{s}}$. This determines a system of equations

$$
\begin{aligned}
v_{x-1\bar{x}} &= a\\
v_{x-1x} &= c^{1,1}_{x-1y,x\bar{y}}c^{1,1}_{x-1\bar{x},xy}aa_{xy}a_{x\bar{y}} + c^{1,2}_{x-1x,x}(-a)a_x^2\\
v_{x-1} &= c^{1,1}_{x-1x,x}(-a)a_x\\
v_{x-1\bar{s}} &= c^{1,1}_{x-1\bar{x},x\bar{s}}(-a)a_{x\bar{s}}\\
v_{x-1l} &= c^{1,1}_{x-1\bar{x},xl}(-a)a_{xl}\\
v_{x-1y} &= c^{1,1}_{x-1\bar{x},xy}(-a)a_{xy}.
\end{aligned}
$$

which can always be solved since $v_{x-1y} \neq 0$ and $v_{x\overline{x-1}} \neq 0$. This completes the proof of b) in this case!

Case 2 $y = x$

a) As in Case 1, we will make use of Proposition 3.10. Let

$$\mathscr{K}_1 = \begin{array}{|c|c|c|}\hline \text{x-1} & \text{x} & \text{x}\\ \hline & \overline{\text{x-1}} & \overline{\text{x-1}}\\ \hline\end{array}$$

and

$$\mathscr{K}_2 = \begin{array}{|c|c|c|}\hline \text{x-1} & \text{x} & \text{x}\\ \hline \bar{\text{x}} & \overline{\text{x-1}} & \\ \hline\end{array}.$$

Then

$$
\begin{aligned}
w_1 &= x\ \overline{x-1}\ x-1 = w(\mathscr{K}_1) \text{ and}\\
w_2 &= x\ x\ \bar{x} = w(\mathscr{K}_2).
\end{aligned}
$$

By Proposition 3.10 it is enough to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}.$$

First we show

$$\overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})} \subseteq \overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})}. \tag{63}$$

Since $\mathbb{U}_2^{\gamma_{\mathscr{K}_2}*\nu}$ is generated by elements of the form $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a), a \in \mathbb{C}$ and the generators of $\mathbb{U}_i^{\gamma_{\mathscr{K}_2}*\nu}$ belong to U_0 for $i \in \{1,2\}$, all elements of $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$ are of the form

$$u\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)\mathrm{T}^w$$

for some $u \in \mathrm{U}_0$. Since $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a) \in \mathbb{U}_2^{\gamma_{\mathscr{K}_1}*\nu}$, (63) follows by applying Proposition 3.27 to u. To finish the proof in this case it remains to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} \subseteq \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}. \tag{64}$$

The generators of $\mathbb{U}_i^{\gamma_{\mathscr{K}_1}*\nu}$ belong to U_0 for $i \in \{0,1\}$, and the generators of $\mathbb{U}_2^{\gamma_{\mathscr{K}_2}*\nu}$ that do not are $\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x), \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}}), \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})$, and $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(v_{x-1\bar{x}})$, for $n \geq l > x, s \notin \{x, x-1\}$, and complex numbers $v_x, v_{x-1\bar{l}}, v_{x-1s}$, and $v_{x-1\bar{x}}$. Therefore all elements of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ belong to

$$u\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(v_{x-1\bar{x}})\mathrm{T}^w.$$

Fix such $u \in \mathrm{U}_0$ and $v_x, v_{x-1\bar{l}}, v_{x-1s}, v_{x-1\bar{x}}$ complex numbers such that $v_{x-1\bar{x}} \neq 0$. We know that for any $a \in \mathbb{C}, \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a) \in \mathbb{U}_{\gamma_{\mathscr{K}_2}*\nu}$; let

$$q = \mathrm{U}_{(\varepsilon_x,1)}(a_x)\prod_{s>x}\mathrm{U}_{(\varepsilon_x-\varepsilon_s,1)}(a_{x\bar{s}})\prod_{l\neq x}\mathrm{U}_{(\varepsilon_x+\varepsilon_l,1)}(a_{xl}) \in \mathrm{U}_{(\varepsilon_x)} \cap \mathrm{U}_0$$

for any complex numbers $a_x, a_{x\bar{s}}, a_{xl}$. Then by Proposition 3.27,

$$q^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)q\mathrm{T}^w \subset \pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}). \tag{65}$$

As in the previous cases, we want to find $a, a_x, a_{x\bar{s}}, a_{xl}$ such that

$$t\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_x)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(v_{x-1\bar{x}})\mathrm{T}^w$$

equals (65), for some $t \in \mathrm{U}_0$. But

$$q^{-1}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)q\mathrm{T}^w =$$
$$t^{-1}\mathrm{U}_{(\varepsilon_{x-1},-1)}(\varrho_x)\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\varrho_{x-1\bar{l}})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\varrho_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-2)}(a)\mathrm{T}^w$$

where

$$\begin{aligned} t^{-1} &= \mathrm{U}_{(\varepsilon_x+\varepsilon_{x-1},0)}(c^{1,2}_{x-1\bar{x},x})(-a)a_x^2 \in \mathrm{U}_0 \\ \varrho_x &= c^{1,1}_{x-1\bar{x},x}(-a)a_x \\ \varrho_{x-1\bar{l}} &= c^{1,1}_{x-1\bar{x},x\bar{l}}(-a)a_{x\bar{l}} \\ \varrho_{x-1s} &= c^{1,1}_{x-1\bar{x},xs}(-a)a_{xs}. \end{aligned}$$

The system

$$\begin{aligned} v_{x-1\bar{x}} &= a \\ v_{x-1\bar{l}} &= \varrho_{x-1\bar{l}} \\ v_{x-1s} &= \varrho_{x-1s} \end{aligned}$$

always has a solution since $v_{x-1\bar{x}} \neq 0$. This concludes the proof.

b) Let

$$\mathscr{K}_3 = \begin{array}{|c|c|c|}\hline \text{x-1} & \text{x-1} & \overline{\text{x-1}} \\ \hline \text{x} & \text{x} & \\ \cline{1-2}\end{array}$$

and

$$\mathscr{K}_4 = \begin{array}{|c|c|c|}\hline \text{x} & \text{x-1} & \text{x} \\ \hline & \bar{\text{x}} & \overline{\text{x-1}} \\ \cline{2-3}\end{array}.$$

Then

$$w_3 = \overline{x-1}\ x-1\ x = w(\mathscr{K}_3) \text{ and } w_4 = x\ \bar{x}\ x = w(\mathscr{K}_4).$$

By Proposition 3.10 it is enough to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}.$$

To do this we will describe a common dense subset of $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ and $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$.

Consider first

$$\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) = \mathbb{U}_0^{\gamma_{\mathscr{K}_3}*\nu}\mathbb{U}_1^{\gamma_{\mathscr{K}_3}*\nu}\mathbb{U}_2^{\gamma_{\mathscr{K}_3}*\nu}\mathrm{T}^w.$$

We have $\mathbb{U}_0^{\gamma_{\mathscr{K}_3}*\nu} \subset \mathrm{U}_0$ and also $\mathbb{U}_2^{\gamma_{\mathscr{K}_3}*\nu} \subset \mathrm{U}_0$, since it is generated by the terms $\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,0)}(d), d \in \mathbb{C}$. These commute with all generators of $\mathbb{U}_1^{\gamma_{\mathscr{K}_3}*\nu}$, out of which $\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1}), \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})$, and $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})$, (for $s \leq n, s \neq x-1, l > x$, and v_{x-1}, v_{x-1s} and $v_{x-1\bar{l}}$ complex numbers) do not belong to U_0. Therefore $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ coincides with

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\prod_{\substack{s\leq n\\ s\neq x-1}}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\prod_{x<l\leq n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1\bar{l}})\mathrm{T}^w \tag{66}$$

for complex numbers v_{x-1}, v_{x-1s} and $v_{x-1\bar{l}}$. Now we look at elements of

$$\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}) = \mathbb{U}_0^{\gamma_{\mathscr{K}_4}*\nu}\mathbb{U}_1^{\gamma_{\mathscr{K}_4}*\nu}\mathbb{U}_2^{\gamma_{\mathscr{K}_4}*\nu}\mathrm{T}^w.$$

Both $\mathbb{U}_0^{\gamma_{\mathscr{K}_4}*\nu}$ and $\mathbb{U}_2^{\gamma_{\mathscr{K}_4}*\nu}$ are contained in U_0, and $\mathbb{U}_1^{\gamma_{\mathscr{K}_4}*\nu}$ is generated by the elements $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(d)$, which belong to $\mathrm{U}_{\varepsilon_x}$ and therefore stabilise the truncated image T^w by Proposition 3.27. Now, by Proposition 1.8, we may write any element k of $\mathbb{U}_2^{\gamma_{\mathscr{K}_4}*\nu}$ as

$$k = \mathrm{U}_{(\varepsilon_x,0)}(k_x)\prod_{x<l\leq n}\mathrm{U}_{(\varepsilon_x-\varepsilon_l,0)}(k_{x\bar{l}})\prod_{s\leq n s\neq x}\mathrm{U}_{(\varepsilon_x+\varepsilon_s,0)}(k_{xs}) \in \mathrm{U}_0$$

for some complex numbers $k_x, k_{x\bar{l}}$, and k_{xs}. Theorem 1.10 and Proposition 3.27 imply that

$$\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(d)k\mathrm{T}^w \tag{67}$$

$$= \mathrm{U}_0 k\mathrm{U}_{(\varepsilon_{x-1},-1)}(\sigma_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-1)}(\sigma_{x-1x})\prod_{x<l\leq n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\sigma_{x-1\bar{l}}) \tag{68}$$

$$\prod_{s\leq n s\neq x}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,0)}(\sigma_{x-1s})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(d)\mathrm{T}^w \tag{69}$$

for $k \in \mathbb{U}_2^{\gamma_{\mathscr{K}_4}*\nu}$ and $d \in \mathbb{C}$, where

$$\begin{aligned}
\sigma_{x-1} &= c^{1,1}_{x-1\bar{x},x}(-d)k_x\\
\sigma_{x-1x} &= c^{1,2}_{x-1\bar{x},x}(-d)k_x^2\\
\sigma_{x-1\bar{l}} &= c^{1,1}_{x-1\bar{x},x\bar{l}}(-d)k_{x\bar{l}}\\
\sigma_{x-1s} &= c^{1,1}_{x-1\bar{x},xs}(-d)k_{xs}
\end{aligned}$$

This set (67) is clearly contained in (66). Moreover, the system

$$\begin{aligned}
v_{x-1} &= \sigma_{x-1}\\
v_{x-1x} &= \sigma_{x-1x}\\
v_{x-1\bar{l}} &= \sigma_{x-1\bar{l}}\\
v_{x-1s} &= \sigma_{x-1s}
\end{aligned}$$

has solutions for $d, k_x, k_{x\bar{l}}$, and k_{xs} as long as $\{v_{x-1}, v_{x-1x}, v_{x-1\bar{l}}, v_{x-1s}\} \subset \mathbb{C}^\times$. Proposition 3.27 then implies that a dense subset of $\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})$ is contained in $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})$, which finishes the proof in this case.

Case 3 $y = \bar{x}$

a) Let

$$\mathscr{K}_1 = \begin{array}{|c|c|c|}\hline x-1 & \bar{x} & \bar{x}\\ \hline \multicolumn{1}{c|}{} & x-1 & x-1\\ \cline{2-3}\end{array}$$

and

$$\mathscr{K}_2 = \begin{array}{|c|c|c|}\hline x-1 & x & \bar{x}\\ \hline \bar{x} & \overline{x-1} & \multicolumn{1}{c}{}\\ \cline{1-2}\end{array}.$$

Then

$$w_1 = \bar{x}\ \overline{x-1}\ x-1 = w(\mathscr{K}_1) \text{ and } w_2 = \bar{x}\ x\ \bar{x} = w(\mathscr{K}_2)$$

By Proposition 3.10 it is enough to show

$$\overline{\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})} = \overline{\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})}$$

In this case we have $\mathbb{U}_0^{\gamma_{\mathscr{K}_1}*\nu} = 1 = \mathbb{U}_0^{\gamma_{\mathscr{K}_1}*\nu}$; Proposition 1.8 and Theorem 1.10 then say

$$\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}) = \tag{70}$$
$$\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(v_{x-1x})\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(v_{x-1x}) \tag{71}$$
$$\prod_{x<l\le n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1l})\prod_{\substack{s\le n\\ s\ne x-1\\ s\ne x}}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(v_{x-1s})\mathrm{T}^w \tag{72}$$

for complex numbers $v_{x-1x}, v_{x-1}, v_{x-1x}, v_{x-1l}$, and v_{x-1s}. Fix such complex numbers. Now we look at $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}})$. We have that $\mathbb{U}_0^{\gamma_{\mathscr{K}_2}*\nu}$ and $\mathbb{U}_2^{\gamma_{\mathscr{K}_2}*\nu}$ are both contained in U_0, and the latter is generated by elements $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(a), a\in\mathbb{C}$. Out of the generators of $\mathbb{U}_1^{\gamma_{\mathrm{K}_2}*\nu}$, the ones that do not belong to U_0 are $\mathrm{U}_{(\varepsilon_x,-1)}(a_x), \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})$, and $\mathrm{U}_{(\varepsilon_x-\varepsilon_l,-1)}(a_{x\bar{l}})$. Therefore, if

$$\mathrm{A} = \mathrm{U}_{(\varepsilon_x,-1)}(a_x)\mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{xs})\mathrm{U}_{(\varepsilon_x-\varepsilon_l,-1)}(a_{x\bar{l}})\in\mathrm{U}_{\varepsilon_{\bar{x}}},$$

we conclude that

$$\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu}) = \tag{73}$$
$$\mathrm{U}_0\mathrm{A}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(a)\mathrm{T}^w = \tag{74}$$
$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(a)\mathrm{U}_{(\varepsilon_{x-1},-1)}(\xi_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(\xi_{x-1x})\prod_{x<l\le n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\xi_{x-1l})\prod_{\substack{s\le n\\ s\ne x-1\\ s\ne x}}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(\xi_{x-1s})\mathrm{A}\mathrm{T}^w = \tag{75}$$
$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1},-1)}(\xi_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(\xi_{x-1x})\prod_{x<l\le n}\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\xi_{x-1l})\prod_{\substack{s\le n\\ s\ne x-1\\ s\ne x}}\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(\xi_{x-1s})\mathrm{T}^w \tag{76}$$

where

$$\xi_{x-1} = c^{1,1}_{x,x-1\bar{x}}(-a_x)a$$
$$\xi_{x-1x} = c^{2,1}_{x,x-1\bar{x}}(a_x^2)a$$
$$\xi_{x-1\bar{l}} = c^{1,1}_{x\bar{l},x-1\bar{x}}(-a_{x\bar{l}})a$$
$$\xi_{x-1s} = c^{1,1}_{xs,x-1\bar{x}}(-a_{xs})a.$$

Therefore it follows directly that in fact

$$\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})\subseteq\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu}).$$

Now, the system of equations

$$v_{x-1} = \xi_{x-1}$$
$$v_{x-1x} = \xi_{x-1x}$$
$$v_{x-1\bar{l}} = \xi_{x-1\bar{l}}$$
$$v_{x-1s} = \xi_{x-1s}$$

has solutions as long as $\{v_{x-1}, v_{x-1x}, v_{x-1\bar{l}}, v_{x-1s}\} \subset \mathbb{C}^\times$. For such a set of solutions we conclude

$$\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(v_{x-1x}) \prod_{x<l\leq n} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(v_{x-1l}) \prod_{\substack{s\leq n \\ s\neq x-1 \\ s\neq x}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(v_{x-1s}) =$$

$$\mathrm{U}_{(\varepsilon_{x-1},-1)}(\xi_{x-1})\mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_x,-2)}(\xi_{x-1x}) \prod_{x<l\leq n} \mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_l,-1)}(\xi_{x-1l}) \prod_{\substack{s\leq n \\ s\neq x-1 \\ s\neq x}} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s)}(\xi_{x-1s})$$

and therefore we conclude by Proposition 3.27 (applied to $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,0)}(v_{x-1x})$ in (71)) that a dense subset of $\pi'(\mathrm{C}_{\gamma_{\mathscr{K}_1}*\nu})$ is contained in $\pi''(\mathrm{C}_{\gamma_{\mathscr{K}_2}*\nu})$ (cf. (71), (76)).

b) Let

$$\mathscr{K}_3 = \begin{array}{|c|c|c|} \hline \text{x-1} & \text{x-1} & \overline{\text{x-1}} \\ \hline \bar{\text{x}} & \bar{\text{x}} & \\ \cline{1-2} \end{array}$$

and

$$\mathscr{K}_4 = \begin{array}{|c|c|c|} \hline \bar{\text{x}} & \text{x-1} & \text{x} \\ \hline & \bar{\text{x}} & \overline{\text{x-1}} \\ \cline{2-3} \end{array}.$$

Then

$$w_3 = \overline{x-1}\ x-1\ \bar{x} = w(\mathscr{K}_3)$$
$$w_4 = x\ \bar{x}\ \bar{x} = w(\mathscr{K}_4)$$

By Proposition 3.10 it is enough to show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}})} = \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}})}.$$

First we claim

$$\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu}) \subseteq \pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}).$$

This is easy. Note that the terms $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b), b \in \mathbb{C}$ generate both $\mathbb{U}_1^{\gamma_{\mathscr{K}_4}*\nu}$ and are contained in $\mathbb{U}_1^{\gamma_{\mathscr{K}_3}*\nu}$. Also, the terms $\mathrm{U}_{(\varepsilon_l-\varepsilon_x,0)}$, which generate $\mathbb{U}_2^{\gamma_{\mathscr{K}_4}*\nu}$, commute with $\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)$. Therefore

$$\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}}) = \mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)\mathrm{T}^w \subseteq \pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}}),$$

where the last contention follows by Proposition 3.27. Now we will show

$$\overline{\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu})} \subseteq \overline{\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}*\nu})}.$$

We claim that

$$\pi'''(\mathrm{C}_{\gamma_{\mathscr{K}_3}*\nu}) = \tag{77}$$

$$\mathrm{U}_0\mathrm{U}_{(\varepsilon_{x-1},-1)}(v_{x-1})\mathrm{U}_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}}) \prod_{s\neq x\varepsilon_s+\varepsilon_{x-1}\in\Phi^+} \mathrm{U}_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s})\mathrm{T}^w \tag{78}$$

for complex numbers $v_{x-1}, v_{x-1\bar{x}}$, and v_{x-1s}. Let us fix such complex numbers. Let

$$D = \mathrm{U}_{(\varepsilon_x,0)}(a_x) \prod_{s\neq x\varepsilon_s+\varepsilon_{x-1}\in\Phi^+} \mathrm{U}_{(\varepsilon_x+\varepsilon_s,-1)}(a_{x-1s}) \in \mathrm{U}_0.$$

Then by the usual arguments (note that U_0 stabilises both the image $\pi''''(\mathrm{C}_{\gamma_{\mathscr{K}_4}})$ and the truncated image $\mathrm{T}^{\geq 2}_{\gamma_{\mathscr{K}_4}*\nu}$).

$$D^{-1}U_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)DT^w \subset \pi''''(C_{\gamma_{\mathscr{K}_4}})$$

and

$$D^{-1}U_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)DT^w =$$
$$U_{(\varepsilon_{x-1},-1)}(\rho_{x-1})U_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)\prod_{s\neq x\varepsilon_s+\varepsilon_{x-1}\in\Phi^+}U_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(\rho_{x-1s})U_{(\varepsilon_x+\varepsilon_{x-1},-1)}(\rho_{xx-1})$$

where

$$\rho_{x-1} = c^{1,1}_{x-1\bar{x},x}(-b)a_x$$
$$\rho_{x-1x} = c^{2,1}_{x-1\bar{x},x}(-b)a_x^2$$
$$\rho_{x-1s} = c^{1,1}_{x-1\bar{x},xs}(-b)a_{xs}.$$

As usual by requiring that $v_{x-1}, v_{x-1\bar{x}}, v_{x-1x}$, and ρ_{x-1s} be non-zero we may find suitable complex numbers b, a_x, a_{xs} such that

$$U_{(\varepsilon_{x-1},-1)}(v_{x-1})U_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(v_{x-1\bar{x}})\prod_{s\neq x\varepsilon_s+\varepsilon_{x-1}\in\Phi^+}U_{(\varepsilon_{x-1}+\varepsilon_s,-1)}(v_{x-1s}) =$$
$$D^{-1}U_{(\varepsilon_{x-1}-\varepsilon_x,-1)}(b)DT^w.$$

Therefore Proposition 3.27 (cf. (78)) implies that a dense open subset of $\pi'''(C_{\gamma_{\mathscr{K}_3}*\nu})$ is contained in $\pi''''(C_{\gamma_{\mathscr{K}_4}*\nu})$.

□

Claim 4. (R3) Let $w \in \mathcal{W}_{\mathcal{C}_n}$ be a word and w_1 that is not the word of an LS block, and such that it has the form $w_1 = a_1\cdots a_r z\bar{z}\overline{b_s}\cdots\overline{b_1}$, and let $w_2 = a_1\cdots a_r\overline{b_s}\cdots\overline{b_1}$ with $a_1 < \cdots a_r < z > b_s > \cdots > b_1$. Then $\overline{\pi(C_{\gamma_{w_1 w}})} = \overline{\pi'(C_{\gamma_{w_2 w}})}$.

Proof of Claim 4. Let $A = \{a_1, \cdots, a_r\}$. We have

$$\pi(C_{\gamma_{w_1 w}}) = \mathbb{P}_{a_1}\cdots\mathbb{P}_{a_r}\mathbb{P}_z\mathbb{P}_{\bar{z}}\mathbb{P}_{\overline{b_s}}\cdots\mathbb{P}_{\overline{b_1}}T^{\geq r+s+2}_{\gamma_{w_1 w}}$$

where

$$\mathbb{P}_z = U_{(\varepsilon_z,0)}(v_z)\prod_{l>z}U_{(\varepsilon_z-\varepsilon_l,0)}(v_{z\bar{l}})\prod_{l\notin A}U_{(\varepsilon_z+\varepsilon_l,0)}(v_{zl})\prod_{a_i\in A}U_{(\varepsilon_z+\varepsilon_{a_i},1)}(v_{za_i}),$$
$$\mathbb{P}_{\bar{z}} = \prod_{a_i\in A}U_{(\varepsilon_{a_i}-\varepsilon_z,0)}(v_{a_i\bar{z}})$$

and note that $\mu_{\gamma_{w_1}} = \mu_{\gamma_{w_2}} = \sum_{i\in I_r}\varepsilon_{a_i} - \sum_{j\in I_s}\varepsilon_{b_j}$. The terms that appear in $\mathbb{P}_z$ all stabilise $\mu_{\gamma_{w_1}}$ and commute with $\mathbb{P}_{\overline{b_j}}$, while the terms in $\mathbb{P}_{\bar{z}}$ all appear in $\mathbb{P}_{a_i}$ and commute with $\mathbb{P}_{a_l}$ for $l > i$. This concludes the proof of the claim with the usual arguments. □

□

5. Non-examples for non-readable galleries

Let $n = 2$ and $\lambda = \varepsilon_1 + \varepsilon_2$, and $(\Sigma_{\gamma_\lambda}, \pi)$ the corresponding Bott-Samelson as in (9). Let γ be the gallery corresponding to the block

1	$\bar{2}$
2	$\bar{1}$

.

Then points in $\pi(C_\gamma)$ are of the form

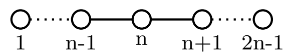
$$\mathrm{U}_{(\varepsilon_1+\varepsilon_2,-1)}(b)[t^0]$$

for $b \in \mathbb{C}$, hence form an affine set of dimension 1. We claim that the set $\mathrm{Z} = \overline{\pi(\mathrm{C}_\gamma)}$ cannot be an MV cycle in $\mathcal{Z}(\mu)$ for any dominant coweight μ. First note that for any $u \in \mathrm{U}(\mathcal{K})$ a necessary condition for ut^0 to lie in the closure $\overline{\mathrm{U}(\mathcal{K})t^\nu \cap G(\mathcal{O})t^\mu}$ is that $0 \leq \nu$, since it would in particular imply that $ut^0 \in \overline{\mathrm{U}(\mathcal{K})t^\nu}$. Also note that it is necesary for $\nu \leq \mu$ in order for the set $\mathcal{Z}(\mu)_\nu$ not to be empty. Any MV cycle in $\mathcal{Z}(\mu)_\nu$ has dimension $\langle \rho, \mu + \nu \rangle$, and the only possibility for the latter to be equal to 1 (since $\mu + \nu$ is a sum of positive coroots) is for either $\mu = 0$ and $\nu = \alpha_i^\vee$, or $\nu = 0$ and $\mu = \alpha_i^\vee$, for some $i \in \mathrm{I}$, and both options are impossible: the first contradicts $\nu \leq \mu$, and the second contradicts the dominance of μ. Note that γ is not a Littelmann gallery.

CHAPTER 4

Some branching properties of tableaux under restriction: The Naito-Sagaki conjecture

The automorphism of the type A_{2n-1} Dyinkin diagram



that sends node i to node $2n-i$ for $i \in [1, 2n-1]$ induces a Lie algebra automorphism $\sigma : \mathfrak{sl}(2n, \mathbb{C}) \to \mathfrak{sl}(2n, \mathbb{C})$. The fixed point set of σ is a Lie algebra $\hat{\mathfrak{g}}$ isomorphic to $\mathfrak{sp}(2n, \mathbb{C})$. In this chapter we will present a conjecture by Naito and Sagaki [**23**] on the decomposition into irreducible summands of $\operatorname{res}^{\mathfrak{g}}_{\hat{\mathfrak{g}}}(\mathrm{L}(\lambda))$ of a simple module $\mathrm{L}(\lambda)$ for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{C})$ using LS paths associated to semi-standard Young tableaux. We prove it for $n = 2$ and for several other cases.

1. Notation

Let $\mathfrak{h} \subset \mathfrak{sl}(2n, \mathbb{C})$ be the Cartan sub-algebra of diagonal matrices. We write $\mathfrak{sl}(2n, \mathbb{C}) = \langle x_i, y_i, h_i \rangle_{i \in \{1, \cdots, 2n-1\}}$ where $h_i = \mathrm{E}_{ii} - \mathrm{E}_{i+1,i+1}$ and where x_i and y_i are the Chevalley generators corresponding to the simple root $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. We identify $\mathfrak{h}^*$ with the vector space $\mathbb{V}_{A_{2n-1}}$ defined in Chapter 2.13 by identifying ε_i with the linear map $\mathfrak{h}^* \to \mathbb{C}$ defined by $\operatorname{diag}(a_1, \cdots, a_{2n}) \mapsto a_i$. The automorphism σ is given by

$$\begin{aligned}
\sigma(x_i) &= x_{2n-i}, \\
\sigma(y_i) &= y_{2n-i}, \text{ and} \\
\sigma(h_i) &= h_{2n-i}
\end{aligned}$$

The fixed point set $\hat{\mathfrak{g}}$ is generated as a Lie algebra by $\langle \hat{x}_i, \hat{y}_i, \hat{h}_i \rangle_{i \in \{1, \cdots, n\}}$ (see Proposition 7.9 in [**11**]), where

$$\hat{x}_i = \begin{cases} x_i + x_{2n-i} & \text{if } i \in [0, n) \cup (n, 2n-1] \\ x_n & \text{if } i = n \end{cases}$$

$$\hat{y}_i = \begin{cases} y_i + y_{2n-i} & \text{if } i \in [0, n) \cup (n, 2n-1] \\ y_n & \text{if } i = n \end{cases}$$

$$\hat{h}_i = \begin{cases} h_i + h_{\alpha_{2n-i}} & \text{if } i \in [0, n) \cup (n, 2n-1] \\ h_n & \text{if } i = n. \end{cases}$$

This Lie algebra is isomorphic to $\mathfrak{sp}(2n, \mathbb{C})$ (see Proposition 7.9 in [**11**]) and $\hat{\mathfrak{h}} = \bigoplus_{i=1}^{n} \hat{h}_i = \mathfrak{h} \cap \hat{\mathfrak{g}} \subset \mathfrak{h}$ is a Cartan sub algebra. We identify $\mathbb{V}_{C_n}$ with the real sub vector space of $\hat{\mathfrak{h}}^*$ spanned by P_{C_n}, and $\mathbb{V}_{A_{2n-1}}$ with the sub vector space of $\mathfrak{h}^*$ spanned by $\mathrm{P}_{A_{2n-1}}$.

2. Restricted paths

The map

$$\begin{aligned} \mathfrak{h}^* &\to \hat{\mathfrak{h}}^* \\ \varphi &\mapsto \varphi|_{\hat{\mathfrak{h}}} \end{aligned}$$

induces a map $\mathrm{res}' : \mathrm{P}_{A_{2n}} \to \mathrm{P}_{C_n}$. Given a path $\pi : [0,1] \to \mathbb{V}_{A_{2n}}$, we define a restricted path $\mathrm{res}(\pi)$ by

$$\begin{aligned} \mathrm{res}(\pi) : [0,1] &\to \mathbb{V}_{C_n} \\ t &\mapsto \mathrm{res}'(\pi(t)). \end{aligned}$$

2.1. Restriction of paths that come from words. Let $w = w_1 \cdots w_k \in \mathcal{W}_{2n}$ be a word. Consider its associated key γ_w and the corresponding path π_{γ_w}. We have

$$\pi_{\gamma_w} = \pi_{\gamma_{w_k}} * \cdots * \pi_{\gamma_{w_1}}.$$

Note that for $w_i \in \mathcal{A}_n$, the path $\pi_{\gamma_{w_i}} : [0,1] \to \mathbb{V}_{A_{2n-1}}$ is given by $t \mapsto t\varepsilon_{w_i}$. Also, in general, for paths $\pi_1, \cdots, \pi_k$, we have

$$\mathrm{res}(\pi_1 * \cdots * \pi_k) = \mathrm{res}(\pi_1) * \cdots * \mathrm{res}(\pi_k).$$

Set

$$\hat{\varepsilon}_i = \mathrm{res}(\varepsilon_i) \text{ for } i \in \{1, \cdots, 2n\}.$$

Then, for $i \in \{1, \cdots, 2n\}$ and $j \in \{1, \cdots, n\}$ we have

$$\hat{\varepsilon}_i(\hat{h}_j) = \begin{cases} 1 & \text{if } i \in \{j, 2n-j\} \\ -1 & \text{if } i \in \{j+1, 2n-j+1\} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\hat{\varepsilon}_i = -\hat{\varepsilon}_{2n-i+1}$, which means we can describe $\mathrm{res}(\pi_{\gamma_w})$ in the following easy way: First obtain from w a word $\mathrm{res}(w)$ in the alphabet $\mathcal{C}_n$ by replacing a letter w_i in w with $\overline{2n - w_i + 1}$ if $n < w_i \le 2n$. All other letters stay the same. Then

$$\mathrm{res}(\pi_{\gamma_w}) = \pi_{\gamma_{\mathrm{res}(w)}}.$$

Example 4.1. Let $n = 2$ and $w = 121223341$. Then $\mathrm{res}(w) = 12122\bar{2}\bar{2}\bar{1}1$.

3. The Naito-Sagaki conjecture

Let $\lambda \in \mathrm{P}_{A_{2n-1}}$ be dominant and let $\mathrm{L}(\lambda)$ be the associated simple module. Recall the set $\mathbf{\Gamma}(\underline{d}_\lambda)^{\mathrm{SSYT}}$ of semi-standard Young tableaux of shape $\underline{d}_\lambda$. Let

$$\mathrm{domres}(\lambda) = \{\delta \in \mathbf{\Gamma}(\underline{d}_\lambda)^{\mathrm{SSYT}} : \mathrm{res}(\pi_{\gamma_{w(\delta)}}) \text{ is dominant }\}$$

be the subset of $\mathbf{\Gamma}(\underline{d}_\lambda)^{\mathrm{SSYT}}$, elements of which are those semi-standard Young tableaux whose words are dominant under restriction (considered as paths), and for $\nu \le \lambda$, let

$$\mathrm{domres}(\lambda, \nu) = \{\delta \in \mathrm{domres}(\lambda) : \mathrm{wt}(\delta) = \nu\}.$$

EXAMPLE 4.2. Let $n = 2$ and $\lambda = \omega_1 + \omega_2$. Then

$$\mathrm{domres}(\lambda) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\}$$

$$\mathrm{domres}(\lambda, \lambda) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\}$$

$$\mathrm{domres}(\lambda, \omega_1) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\}.$$

To avoid confusion we will denote elements of $\mathrm{P}_{A_{2n-1}}$ by λ and elements of P_{C_n} by $\hat{\lambda}$; in particular, the fundamental (co)weights in $\mathrm{P}_{A_{2n-1}}$ will be denoted by $\{\omega_1, \cdots, \omega_{2n-1}\}$, and the fundamental (co)weights in P_{C_n} by $\{\hat{\omega}_1, \cdots, \hat{\omega}_n\}$. Also, for

$$\lambda = a_1\omega_1 + \cdots + a_n\omega_n$$

we write

$$\hat{\lambda} = a_1\hat{\omega}_1 + \cdots + a_n\hat{\omega}_n.$$

The following conjecture by Naito and Sagaki is stated in [**23**]. It is shown for (co)weights of the form $\lambda = a\omega_1 + \omega_k$ and $\lambda = a\omega_k$, $a \in \mathbb{Z}^{>0}$.

CONJECTURE 4.3. [**23**] Let $\lambda \in \mathrm{P}^+_{A_{2n-1}}$ be dominant, and let $\mathrm{L}(\lambda)$ be the associated simple module for $\mathfrak{sl}(2n, \mathbb{C})$. Then

$$\mathrm{res}\,^{\mathfrak{g}}_{\mathfrak{g}}(\mathrm{L}(\lambda)) = \bigoplus_{\delta \in \mathrm{domres}(\lambda)} \mathrm{L}(\hat{\mathrm{wt}}(\delta))$$

THEOREM 4.4 ([**23**]). Conjecture 4.3 is true for $\lambda = a\omega_1 + \omega_k$ and $\lambda = a\omega_k$, $a \in \mathbb{Z}^{\geq 0}$.

EXAMPLE 4.5. Let $n = 2$ and $\lambda = \omega_1 + \omega_2$ as in Example 4.2. Then

$$\mathrm{res}\,^{\mathfrak{g}}_{\mathfrak{g}}(\mathrm{L}(\lambda)) = \mathrm{L}(\hat{\omega}_1) \oplus \mathrm{L}(\hat{\omega}_1 + \hat{\omega}_2).$$

REMARK 4.6. Conjecture 4.3 is stated in [**23**] for $\mathrm{L}(\lambda)$ a representation of $\mathfrak{gl}(2n, \mathbb{C})$ for λ non-negative and dominant. However, the representation of $\mathfrak{gl}(2n, \mathbb{C})$ induced by an irreducible representation of $\mathfrak{sl}(2n, \mathbb{C})$ has the same highest weight and restricts back to itself. See §15.3 in [**5**].

4. Littlewood-Richardson tableaux and n-symplectic Sundaram tableaux: branching

DEFINITION 4.7. Let λ and ν be two dominant (co)weights in $P^+_{A_{2n}}$ such that $\underline{d}_\nu \subset \underline{d}_\lambda$ (this means that one shape is contained in the other when aligned with respect to their top left corners. In Example 4.11 we see that $\underline{d}_\nu \subset \underline{d}_\lambda$). A tableau of skew shape λ/ν is a filling of an arrangement of boxes T of shape λ leaving the boxes that belong to $\nu \subset \lambda$ blank, with the others having entries in the alphabet $\mathcal{A}_{2n}$ and such that the entries are strictly increasing in the columns. The word $w(\mathrm{T})$ of T is obtained just as for keys, reading from right to left and from top to bottom, ignoring the blank boxes.

A shape $\underline{d} = (d_1, \cdots, d_k)$ is **even** if d_i is an even number for all $i \in \{1, \cdots, k\}$. Also, for a shape $\underline{d}$ define

$$l(\underline{d}) = \max\{d_i : 1 \leq i \leq k\}$$

to be the length of the longest column of the associated arrangement of boxes.

Definition 4.8. Let $\lambda \in P^+_{A_{2n-1}}$ and let $\nu, \eta \in P^+_{A_{2n-1}}$ be (co)weights whose shapes $\underline{d}_\nu$ and $\underline{d}_\eta$ are contained in the shape $\underline{d}_\lambda$ of λ, and such that $\underline{d}_\eta$ is even. A **Littlewood-Richardson (n-symplectic Sundaram)** tableau of skew shape λ/ν and weight $\underline{d}_\eta$ is a tableau of skew shape λ/ν that is semi-standard, and has a dominant word of weight η (and $2i+1$ does not appear strictly below row $n+i$ for $i \in \{0, 1, \cdots, \frac{1}{2}l(\underline{d}_\eta)\}$). Here a word $w \in \mathcal{W}_{2n}$ is dominant if the gallery γ_w is dominant. We will denote them by $\mathrm{LR}(\lambda/\nu, \eta)$ ($\mathrm{LRS}(\lambda/\nu, \eta)$).

Remark 4.9. Note that if $l(\lambda) \leq n$ (such (co)weights are called **stable**) then $\mathrm{LRS}(\lambda/\nu, \eta) = \mathrm{LR}(\lambda/\nu, \eta)$.

Remark 4.10. If λ is stable and T is a Littlewood-Richardson tableau of skew shape λ/ν then its entries belong to the set $\{1, \cdots, n\}$. This is because if, say, k appears in row l_k of T, then, since the word of T is dominant, a $k-1$ must appear either directly above k in the same column, or in a column to the right, and since T is semi-standard, it appears in at most row $l_k - 1$.

Example 4.11. The tableau $\mathrm{L} = \begin{array}{|c|c|c|}\hline \; & 1 & 1 \\ \hline \; & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$ is a Littlewood-Richardson tableau of skew shape λ/ν and weight η for $\lambda = \omega_1 + \omega_2 + \omega_3, \nu = \omega_2$, and $\eta = 2\omega_2$ and the tableau $\mathrm{T} = \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline 1 \\ \hline \end{array}$ is a Littlewood-Richardson tableau of skew shape λ'/ν' and weight η' for $\lambda' = \omega_3, \nu' = \omega_2$, and $\eta' = \omega_1$. Notice that L is 2-symplectic Sundaram while T is not.

Definition 4.12. The Littlewood-Richardson coefficient is defined as the number $c^\lambda_{\nu,\eta} \in \mathbb{Z}^{\geq 0}$ such that

$$\mathrm{L}(\nu) \otimes \mathrm{L}(\eta) = \bigoplus_{\nu \leq \lambda} c^\lambda_{\nu,\eta} \mathrm{L}(\lambda)$$

where $\mathrm{L}(\lambda), \mathrm{L}(\nu)$, and $\mathrm{L}(\eta)$ are all representations of $\mathfrak{sl}(2n, \mathbb{C})$.

Theorem 4.13 below is known as the **Littlewood-Richardson rule**. It was first stated in 1943 by Littlewood and Richardson

Theorem 4.13. [**10**] The Littlewood-Richardson coefficients are obtained by counting Littlewood-Richardson tableaux:

$$c^\lambda_{\nu,\eta} = |\,\mathrm{LR}(\lambda/\nu, \eta)|.$$

Remark 4.14. Theorem 4.13 implies that $c^\lambda_{\nu,\eta} = c^\lambda_{\eta,\nu}$.

We will use the notation $c^\lambda_{\nu,\eta}(\mathrm{S}) = |\,\mathrm{LRS}(\lambda/\nu, \eta)\,|$. The following theorem was proven by Sundaram in Chapter IV of her PhD thesis [**27**]. See also Corollary 3.2 of [**28**]. For stable (co)weights it was proven by Littlewood in and is known as the Littlewood branching rule.

Theorem 4.15. [**27**] Let $\lambda \in \mathrm{P}^+_{A_{2n-1}}$ be dominant. Then

$$\mathrm{res}\,^{\mathfrak{g}}_{\mathfrak{g}}(\mathrm{L}(\lambda)) = \bigoplus_{\substack{\underline{d}_\nu \subset \underline{d}_\lambda \\ l(\underline{d}_\nu) \leq n}} \mathrm{N}_{\lambda,\nu} \mathrm{L}(\nu)$$

where

$$\mathrm{N}_{\lambda,\nu} = \sum_{\underline{d}_\eta \text{ even}} c^\lambda_{\nu,\eta}(\mathrm{S}).$$

5. Proof of the Naito-Sagaki conjecture for $n = 2$ and $\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$.

As its title suggests, in this section we give a proof of Conjecture 4.3 in the cases $n = 2$ and $\lambda = a_1\omega_1 a_2 + \omega_2 + a_3\omega_3$, for all n. We will do so using Theorem 4.15 from Section 4. The following construction should provide some insight. Given a tableau $\mathrm{T} \in \mathrm{domres}(\lambda)$ we will construct a (co)weight η_{T} with even shape $\underline{d}_{\eta_{\mathrm{T}}}$. To do this, first replace, in T, all letters $w > n$ by $\overline{2n - w + 1}$. The word of the resulting symplectic key, which we denote by res(T), is res(w(T)). Now, in each column of res(T), replace an entry w by a blank square if $\overline{w}$ appears in the same column as w. Count the number of blank squares in each column, and order these numbers to obtain a shape

$$\underline{d}_{\eta_{\mathrm{T}}} = (\eta_1, \cdots, \eta_k),$$

where $\eta_1 \leq \eta_2 \cdots \leq \eta_k$, and η_i is the number of blank squares obtained in some column. The shape $\underline{d}_{\eta_{\mathrm{T}}}$ automatically determines a dominant weight $\eta_{\mathrm{T}} = \sum_{i=1}^{2n-1} a_i\omega_i$, where $a_i = \sum_{\eta_j = i} \eta_j$.

Example 4.16. Let $n = 2$ and

$$\mathrm{T} = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\ .$$

Then

$$\mathrm{res}(\mathrm{T}) = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \overline{2} & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\ , \quad \eta_{\mathrm{T}} = (2), \text{ and } \underline{d}_{\eta_{\mathrm{T}}} = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$$

where the shape $\underline{d}_{\eta_{\mathrm{T}}}$ is obtained by replacing $\begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline\end{array}$ in res(T) by blank squares.

Lemma 4.17. Let T be as above. Then η_{T} is even.

Remark 4.18. Lemma 4.17 is only true for T a semi-standard Young tableau. Consider for example $n = 2$ and the key $\mathscr{T} = \begin{array}{|c|c|c|}\hline 1 & 4 & 1 \\ \hline\end{array}$. Then $\mathrm{res}(\mathscr{T}) = \begin{array}{|c|c|c|}\hline 1 & \overline{1} & 1 \\ \hline\end{array}$ is dominant, however, the shape $\eta_{\mathscr{T}} = (1, 1)$, which is not even.

Proof. We will call a column standard if its entries are consecutive integers, starting with 1. The proof is by induction on the number of right-most aligned consecutive standard columns in T, counted from right to left. For example: the tableau $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$ has two right-most aligned consecutive standard columns, the tableau $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$ has one, and $\mathrm{T} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ has none. Let D be the first column (counted from right to left) that is not standard. Then there exists $s > 0$ such that the first s boxes of D are filled in with the numbers i such that $i \leq s$, and its $s+1$-th box is filled in with $\overline{l}$ for some $l \leq n$. Since res(T) has a dominant word, it must even hold that $l \leq s$. The same holds for the rest of the entries in D, which are barred since entries are strictly increasing. The boxes in D with barred entries together with the boxes in D that have as entries their non-barred versions (they all exist, since the word of T is dominant) make up one of the columns of $\underline{d}_{\eta}$. This column has an even number of boxes. Let us now ignore these entries. Let C be the closest column to the left of D that is not fundamental. For the induction step, we construct a new semi-standard Young tableau in which C is the first column to not be fundamental. Let L be the semi-standard Young tableau made up of the first right-most aligned standard columns of T. Since the word of res(T) is dominant, we may use the non-ignored entries in D to construct a new tableau L′ from L. We do this by adding one of these boxes either at the end of L or at the end of a column, in such a way that the resulting arrangement is still a semi-standard Young tableau with standard columns only. For example, if $n = 3$ and $\mathrm{T} = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\ \hline 2&2&2&2\\ \cline{1-4} 3&3&3\\ \cline{1-3} 4&5\\ \cline{1-2} 6\\ \cline{1-1}\end{array}$, then $\mathrm{res(T)} = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\ \hline 2&2&2&2\\ \cline{1-4} 3&3&3\\ \cline{1-3} \bar{3}&\bar{2}\\ \cline{1-2} \bar{1}\\ \cline{1-1}\end{array}$ has a dominant word. Also, $\mathrm{L} = \begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2&2\\ \cline{1-2} 3\\ \cline{1-1}\end{array}$, $\mathrm{D} = \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline 5\\ \hline\end{array}$, $\mathrm{C} = \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline 4\\ \hline 6\\ \hline\end{array}$, $\mathrm{L}' = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 2&2\\ \cline{1-2} 3&3\\ \cline{1-2}\end{array}$, and the ignored boxes in D are $\boxed{2}$ and $\boxed{\bar{2}}$. Define a new tableau res(T)′ by concatenating all the columns in res(T), from left to right and up to D, with L′. It follows from the construction that res(T)′ has a dominant word and is semi-standard. In the previous example, this new tableau is $\mathrm{res(T)}' = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\ \hline 2&2&2\\ \cline{1-3} 3&3&3\\ \cline{1-3} \bar{3}\\ \cline{1-1} \bar{1}\\ \cline{1-1}\end{array}$. We may then apply the induction hypothesis. This concludes the proof. □

LEMMA 4.19. If

$$\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$$

and ν and η are dominant weights in $\mathrm{P}_{A_{2n-1}}$ such that $\underline{d}_\eta$ is even and both $\underline{d}_\eta$ and $\underline{d}_\nu$ are contained in $\underline{d}_\lambda$, then

$$c^\lambda_{\nu,\eta}(\mathrm{S}) = c^\lambda_{\nu,\eta}.$$

PROOF. Assume that T is a Littlewood-Richardson tableau of skew shape λ/ν and weight η that is not Sundaram. This means that there is at least a "1" in the third row. Since T is semi-standard, all the "1"'s in the third row must appear left-most and all next to one another. But since $\underline{d}_\eta$ is even, for each of these 1's there must exist a "2" that appears before it , in the word reading order. But this means, since the word is dominant, that there must have appeared a 1 before this "2". This contradicts the evenness of $\underline{d}_\eta$! □

THEOREM 4.20. The Naito-Sagaki conjecture is true for $n = 2$ and for any n if $\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$.

PROOF. Fix λ and ν as in Lemma 4.19 above. Then, for all η such that $\underline{d}_\eta$ is even, $c^\lambda_{\nu,\eta}(\mathrm{S}) = c^\lambda_{\eta,\nu}$, by Lemma 4.19 and Remark 4.14. The following Claim therefore proves the theorem in this case.

CLAIM 5. There is a bijection

$$\mathrm{domres}(\lambda,\nu) \overset{1:1}{\longleftrightarrow} \bigcup_{\substack{\underline{d}_\eta \subseteq \underline{d}_\lambda;\\ \underline{d}_\eta \text{ even}}} \mathrm{LR}(\lambda/\eta,\nu)$$

Let $\mathrm{T} \in \mathrm{domres}(\lambda, \nu)$, and set

$$b_1 = \# \text{ columns in T of the form } \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 2n \\ \hline \end{array},$$

$$b_2 = \# \text{ columns in T of the form } \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 2n\text{-}1 \\ \hline \end{array},$$

$$b_3 = \# \text{ columns in T of the form } \begin{array}{|c|}\hline 1 \\ \hline 2n \\ \hline \end{array} \text{ and}$$

$$b_4 = \# \text{ columns in T of the form } \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}.$$

If $n = 2$ then $b_2 = b_4$. It follows from semi-standardness and dominance of $\mathrm{res}(w(\mathrm{T}))$ that these are the only possible columns aside from columns of the form $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$ and the single box columns $\boxed{1}$. Note that since $\mathrm{res}(w(\mathrm{T}))$ is dominant, the following condition holds

$$b_1 \leq \lambda_1 - \lambda_2. \tag{79}$$

Actually (79) is equivalent to the dominance of $\mathrm{res}(w(\mathrm{T}))$, once the b_i are set. We assign to T a Littlewood-Richardson tableau $\varphi(\mathrm{T}) \in \mathrm{LR}(\lambda/\eta_{\mathrm{T}}, \nu)$. By Lemma 4.17, η_{T} is even. Write

$$\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \lambda_3\varepsilon_3.$$

Note that η_{T} has $b = b_1 + b_2 + b_3$ columns, all of length 2. Fill in the first $\lambda_1 - b$ right-most boxes in the first row with a "1" , and the first $\lambda_2 - b$ right-most boxes in the second row with a "2". If $n \neq 2$ fill in the first b_4 rightmost entries of the third row with a "3". Then fill in the next rightmost b_1 entries in the third row with a "2", and the remaining entries with a "1". The resulting tableau $\varphi(\mathrm{T})$ is a Littlewood-Richardson tableau by construction. Now we will show that any element in $\bigcup_{\substack{\eta \leq \lambda; \\ \eta \text{ even}}} \mathrm{LR}(\lambda/\eta, \nu)$ can be obtained in this way. Let $\eta \in \mathrm{P}^+_{A_{2n-1}}$ have an even shape $\underline{d}_\eta \subset \underline{d}_\lambda$ (this means $\underline{d}_\eta$ consists of size 2 columns) and let $\mathrm{L} \in \mathrm{LR}(\lambda/\eta, \nu)$. Set

$$\begin{aligned} l_1 &= \# \text{ of 1's in L} \\ l_2 &= \# \text{ of 2's in L, and} \\ b &= \# \text{ of columns of } \eta. \end{aligned}$$

Note that this information determines L together with λ. In view of the previous construction, we would like to find non-negative integers b_1, b_2, b_3 and b_4 such that

$$l_1 = \lambda_1 - b_1 - b_3 \tag{80}$$

$$l_2 = \lambda_2 - b_3 - b_2 \tag{81}$$

$$b = b_1 + b_2 + b_3 \tag{82}$$

$$b_4 = \# \text{ of 3's in L} \tag{83}$$

Since L has a dominant word, we have

$$l_2 - (\lambda_2 - b) \leq \lambda_1 - \lambda_2. \tag{84}$$

Substituting (81) and (82) in (84) we get precisely (79), so if we find solutions b_1, b_2, b_3, the resulting tableau will automatically belong to $\mathrm{domres}(\lambda, \nu)$.

Claim 6. The system determined by (80), (81), and (82), has integer solutions (possibly zero) b_1, b_2 and b_3 if and only if

$$b \geq \lambda_1 - l_1 \tag{85}$$

$$b \geq \lambda_2 - l_2 \tag{86}$$

$$\lambda_1 + \lambda_2 \geq b + l_1 + l_2. \tag{87}$$

It follows from the definitions that these conditions are satisfied by all elements of

$$\bigcup_{\substack{\underline{d}_\eta \subset \underline{d}_\lambda; \\ \underline{d}_\eta \text{ even}}} \mathrm{LR}(\lambda/\eta, \nu).$$

To conclude we give a proof of Claim 6.

Proof of Claim 87. We only need to solve the system of equations determined by (80), (81), and (82). From (80) and (82) we have

$$b_1 = \lambda_1 - l_1 - b_3 \tag{88}$$

$$b_2 = b - \lambda_1 + l_1 - \lambda_1 \tag{89}$$

Therefore $b_2 \geq 0$ if and only (85) holds. Substituting (89) into (81) we get

$$b_3 = \lambda_1 + \lambda_2 - l_1 - l_2 - b \tag{90}$$

Hence $b_3 \geq 0$ if and only if (87) holds. Now substitute (90) into (88) and get

$$b_1 = l_2 - \lambda_2 + b, \tag{91}$$

and hence $b_1 \geq 0$ if and only if (86) holds. This concludes the proof of Claim 6. $\square$

$\square$

CHAPTER 5

Appendix

1. Appendix to Chapter 2

Here we state Theorem 2 in [**8**] with a small correction, which we prove. What is missing in the formulation given in [**8**] is the relation $1\cdots n = \varnothing$. The proof we provide shows the failure of Theorem 5.1 without it.

THEOREM 5.1. Let γ be a gallery of type $\underline{d}$, and let γ_{SS} be the unique semistandard Young tableau such that the words $w(\gamma)$ and $w(\gamma_{SS})$ are plactic equivalent. Let $\underline{c}$ be the shape of γ_{SS}. Consider the Schubert varieties $\mathrm{X}_{\lambda_{\underline{c}}} \subset \mathrm{X}_{\lambda_{\underline{d}}}$ and the desingularizations $\pi_{\underline{d}} : \Sigma_{\underline{d}} \to \mathrm{X}_{\lambda_{\underline{d}}}$ and $\pi_{\underline{c}} : \Sigma_{\underline{c}} \to \mathrm{X}_{\lambda_{\underline{c}}}$.

a. The closure $\overline{\pi_{\underline{d}}(\mathrm{C}_\gamma)} \subset \mathrm{X}_{\lambda_{\underline{d}}}$ is an MV cycle in $\mathcal{Z}(\lambda_{\underline{c}})$.

b. Let γ' be a second gallery of type $\underline{d}'$. Then $\gamma \sim \gamma'$ if and only if $\overline{\pi_{\underline{d}}(\mathrm{C}_\gamma)} = \overline{\pi_{\underline{d'}}(\mathrm{C}_{\gamma'})}$.

PROOF OF THEOREM 5.1. The only thing missing in the proof in [**8**] is: Let ν and δ be galleries, let $\underline{b}$ be the shape of $\nu * \delta$, and $\underline{a}$ be the shape of $\nu * \gamma_{1\cdots n} * \delta$. Then

$$\overline{\pi_{\underline{a}}(\mathrm{C}_{\nu*\gamma_{1\cdots n}*\delta})} = \overline{\pi_{\underline{b}}(\mathrm{C}_{\nu*\delta})}.$$

By Proposition 3 in [**8**], and by the argument given at the beginning of Example 4.2 it is enough to assume that ν and δ are both trivial. This means $\Sigma_{\underline{b}}$ and $\mathrm{X}_{\lambda_{\underline{b}}}$ are both just a point (and $\mathrm{X}_{\lambda_{\underline{b}}} = 1$). Note that $(\alpha, m) \in \Phi_i^{\gamma_{1\cdots n}}$ means $m = 0$. We also have the relation

$$t^\lambda \mathrm{U}_{\alpha,m}(h) t^{-\lambda} = \mathrm{U}_{\alpha,m+(\lambda,\alpha)}$$

(see (5)) for $\lambda \in \mathrm{X}, \alpha \in \Phi$ and $n \in \mathbb{Z}$. This means that

$$\mathbb{U}_i^{\gamma_{1\cdots n}} \subset \mathrm{SL}_n(\mathcal{O}).$$

Therefore Theorem 1.10 implies

$$\pi_{\underline{a}}(\mathrm{C}_{\gamma_{1\cdots n}}) \subset \mathrm{SL}_n(\mathcal{O})[t^0] = [t^0] = 1.$$

$\square$

2. Appendix to Chapter 3

Here we show that relation (R3) in Theorem 3.21 is equivalent to relation R_3 in [**18**], Definition 3.1. For a word $w \in \mathcal{W}_{\mathcal{C}_n}$ and $m \leq n$ define $\mathrm{N}(w,m) = |\{x \in w : x \leq m \text{ or } \overline{m} \leq x\}|$. Lecouvey's relation R_3 is: "Let w be a word that is not the word of an LS block and such that each strict subword is. Let z be the lowest unbarred letter such that the pair $(z,\overline{z})$ occurs in w and $\mathrm{N}(w,z) = z+1$. Then $w \cong w'$, where w' is the subword obtained by erasing the pair $(z,\overline{z})$ in w." The following Lemma is a translation between R_3 and (R3).

Lemma 5.2. Let w be a word that is not the word of an LS block and such that each strict subword is. Then $w = a_1 \cdots a_r z\overline{z}\overline{b_s} \cdots \overline{b_1}$ for $a_i.b_i$ unbarred and $a_1 < \cdots a_r, b_1 < \cdots, b_s$.

Proof. By Remark 2.2.2 in [**18**], w is the word of an LS block if and only if $\mathrm{N}(w,m) \leq m$ for all $m \leq n$. Let w be as in the statement of Lemma 5.2. Then there exists in w a pair $(z,\overline{z})$ such that $\mathrm{N}(w,z) > z$. Let z be minimal with this property. In particular $\mathrm{N}(w,z) = z+1$ since if w'' is the word obtained from w by erasing z, then $z \geq \mathrm{N}(w'',z) = \mathrm{N}(w,z)-1$. We claim that z is the largest unbarred letter to appear in w. If there was a larger letter y then $\mathrm{N}(w''',z) = \mathrm{N}(w,z) = z+1$ where w''' denotes the word obtained from w by deleting y. This is impossible since by assumption w''' is the word of an LS block. Likewise $\overline{z}$ is the smallest unbarred letter to appear in w. The $a_i's$ and $b_i's$ are then those from Definition 3.2 for the word obtained from w by deleting $z, \overline{z}$ from it. $\square$

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Erklärung gemäß §4 der Promotionsordung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Peter Littelmann betreut worden.

Nachfolgend genannte Teilpublikationen liegen vor:

(1) Jacinta Torres, *Word reading is a crystal morphism*, preprint arXiv:1407.4625. Erscheint in Transformation Groups.
(2) Jacinta Torres, *The symplectic plactic monoid, crystals and MV cycles*, preprint arXiv:1407.5015.