Some results on orbifold quotients and related objects

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Zusammenfassung

Der Quotient eines endlichdimensionalen euklidischen Raums nach einer endlichen linearen Gruppe erbt verschiedene Strukturen vom ursprünglichen Raum, z.B. eine Topologie, eine Metrik und eine stückweise lineare Struktur. Die Frage, wann ein solcher Quotient eine Mannigfaltigkeit ist, führt auf das Studium von Gruppen, die von *Spiegelungen* und *Drehungen* erzeugt werden, d.h. von orthogonalen Transformationen deren Fixpunktunterraum *ein*- oder *zwei*dimensional ist. Wir klassifizieren derartige Gruppen und vervollständigen damit frühere Ergebnisse von M. A. Mikhaîlova aus den 70ern und 80ern. Wir zeigen ferner, dass eine endliche Gruppe genau dann von (Spiegelungen und) Drehungen erzeugt wird, wenn der zugehörige Quotient eine Lipschitz-, oder äquivalent, eine stückweise lineare Mannigfaltigkeit (mit Rand) ist. Für den Beweis dieser Aussage zeigen wir zudem, dass jede stückweise lineare Mannigfaltigkeit in Dimension kleiner gleich vier, auf der eine endliche Gruppe durch stückweise lineare Homöomorphismen wirkt, eine kompatible differenzierbare Struktur zulässt, bezüglich derer die Gruppe glatt wirkt. Dies löst eine Herausforderung von Thurston und bestätigt eine Vermutung von Kwasik und Lee. In der topologischen Kategorie liefert die binäre Ikosaedergruppe ein Gegenbeispiel zu der oben genannten Charakterisierung. Wir zeigen, dass dieses bis auf Produkte das einzige Gegenbeispiel ist. Insbesondere beantworten wir damit die Frage von Davis, wann der zugrundeliegende Raum einer Orbifaltigkeit eine topologische Mannigfaltigkeit ist.

Als Korollar unserer Ergebnisse verallgemeinern wir einen Fixpunktsatz von Steinberg über unitäre Reflektionsgruppen auf endliche von Spiegelungen und Drehungen erzeugte Gruppen. Als Anwendung davon beantworten wir eine Frage von Petrunin über Quotienten von Sphären.

Abstract

The quotient of a finite-dimensional Euclidean space by a finite linear group inherits different structures from the initial space, e.g. a topology, a metric and a piecewise linear structure. The question when such a quotient is a manifold leads to the study of finite groups generated by *reflections* and *rotations*, i.e. by orthogonal transformations whose fixed point subspace has codimension *one* or *two*. We classify such groups and thereby complete earlier results by M. A. Mikhaîlova from the 70s and 80s. Moreover, we show that a finite group is generated by (reflections and) rotations if and only if the corresponding quotient is a Lipschitz-, or equivalently, a piecewise linear manifold (with boundary). For the proof of this statement we show in addition that each piecewise linear manifold of dimension up to four on which a finite group acts by piecewise linear homeomorphisms admits a compatible smooth structure with respect to which the group acts smoothly. This solves a challenge by Thurston and confirms a conjecture by Kwasik and Lee. In the topological category a counterexample to the above mentioned characterization is given by the binary icosahedral group. We show that this is the only counterexample up to products. In particular, we answer the question by Davis of when the underlying space of an orbifold is a topological manifold.

As a corollary of our results we generalize a fixed point theorem by Steinberg on unitary reflection groups to finite groups generated by reflections and rotations. As an application thereof we answer a question by Petrunin on quotients of spheres.

Contents

Introduction and Summary

Скоро сказка сказывается,
да не скоро дело делается.

The quotient space $\mathbb{R}^n/G$ of $\mathbb{R}^n$ by a finite subgroup $G < \mathrm{O}_n$ of the orthogonal group inherits different structures from $\mathbb{R}^n$, e.g. a topology, a metric and a piecewise linear structure. The question when it is a manifold with respect to one of these structures arises naturally, for example in the theory of orbifolds as pointed out by Davis: It is equivalent to the question of when the underlying space of a smooth orbifold is a manifold [Dav11, p. 9].

If the quotient space $\mathbb{R}^n/G$ is a manifold (with boundary), then it is often true that the acting group G is generated by those $g \in G$ with $\mathrm{rank}(g - I) = 2$ ($\mathrm{rank}(g - I) \leq 2$). This conclusion holds for example in the piecewise linear category where it can be easily shown by induction (cf. Section 3.2) or by a holonomy argument (cf. [Pet15]). More generally, it holds in all cases in which in a simply connected n-dimensional space, the complement of the image of a $(n-3)$-dimensional space is simply connected: Suppose that $\mathbb{R}^n/G$ is a manifold and let $\tilde{G} = \langle g \in G | \mathrm{rank}(g - I) \leq 2 \rangle$. Then the restriction of the action $G/\tilde{G} \curvearrowright \mathbb{R}^n/\tilde{G}$ to its regular part is a free action whose corresponding quotient space is simply connected by assumption and this implies $\tilde{G} = G$. Similar conclusions have been drawn in the context of complex analytic geometry [Shv75, Shv91], algebraic geometry [KW82], symplectic geometry [Ver00] and piecewise smooth manifolds [Sty09].

We call an orthogonal transformation $g \in \mathrm{O}_n$ a *reflection* if $\mathrm{rank}(g - I) = 1$. We call it a *rotation* if $\mathrm{rank}(g - I) = 2$. Likewise, we say that a finite subgroup $G < \mathrm{O}_n$ is a *reflection-rotation group* if it is generated by reflection and rotations. We call it a *rotation group* if it is generated by rotations. Examples for reflection-rotation groups are real reflection groups, their orientation preserving subgroups and unitary reflection groups considered as real groups. In each of these cases the corresponding quotient space is a manifold. For a real reflection group $W < \mathrm{O}_n$ and its orientation preserving subgroups W^+ this follows from the fact that $\mathbb{R}^n/W$ is a cone over a spherical simplex (cf. Section 3.4.1). For a unitary reflection group $G < \mathrm{U}_n$ it is implied i.a. by Chevalley's theorem stating that the invariant ring $\mathbb{C}[z_1, \ldots, z_n]^G$ is a polynomial ring in n variables (cf. Section 3.4.2).

Motivated by such observations Mikhaîlova started to work on a classification of rotation groups in the 70*s*. In the subsequent years she published a series of papers in which the classifications of several subclasses of rotation groups are treated [Mea76], [Mik78], [Mik82]. Moreover, for each rotation group $G < \mathrm{SO}_n$ occurring in her papers she later either verified that the corresponding quotient space $\mathbb{R}^n/G$ is homeomorphic to $\mathbb{R}^n$ [Mik84] or at least described a method that could be used to do so. Only in a few cases additional arguments are needed for these methods to be applicable in rigorous proofs (cf. Chapter 3 for more details).

In *Chapter 1* of this thesis we present a complete classification of reflection-rotation groups. The classification contains both exceptional rotation groups and building blocks of infinite families of rotation groups that do not appear in [Mea76, Mik78, Mik82, Mik84] (cf. Introduction of Chapter 1). The results of Chapter 1 are contained in a joint paper with M. A. Mikhaîlova [LM15].

Based on the obtained classification we prove the following characterization in *Chapter 3*.

Theorem B. For a finite subgroup $G < \mathrm{O}_n$ the quotient space $\mathbb{R}^n/G$ is a piecewise linear manifold with boundary if and only if G is a reflection-rotation group. In this case $\mathbb{R}^n/G$ is either piecewise linear homeomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ and G contains a reflection, or $\mathbb{R}^n/G$ is piecewise linear homeomorphic to $\mathbb{R}^n$ and G does not contain a reflection.

As mentioned above, a partial result in this direction had been obtained in [Mik84] by Mikhaîlova before and led our way. We adapt some of the methods from [Mik84] as to also work in the piecewise linear category and describe new methods to prove the if direction of Theorem B by verifying its conclusion for all reflection-rotation groups. Thereby we avoid some difficulties and problems in the proofs from [Mik84] (cf. Introduction of Chapter 3).

For instance, as a tool used in the proof of Theorem B we show the following smoothing result in *Chapter 2*. Our proof solves a challenge posed by Thurston in dimension three [Thu97, p. 208] and the result confirms a conjecture by Kwasik and Lee in dimension four in a stronger form [KL88].

Theorem A. Let M be a piecewise linear manifold of dimension $n \leq 4$ on which a finite group G acts by piecewise linear homeomorphisms. Then M can be equivariantly smoothed, i.e. there exists a smooth structure on M compatible with its piecewise linear structure with respect to which the group G acts smoothly on M.

Here compatible means that there exists a triangulation of M as a piecewise linear manifold such that the restrictions of the identity map of M to the simplices of this triangulation are smooth and nondegenerate. In dimension $n > 4$ the statement of the theorem is false, even without the compatibility condition [KL88, Remark 3.9, p. 260] (cf. Section 2.3.7).

Notice that the only-if direction of Theorem A does not hold in the topological category. The quotient S^3/P of a 3-sphere by a realization of the binary icosahedral group $P < \mathrm{SO}_4$ is Poincaré's homology sphere and by Cannon's double suspension theorem its double suspension $\Sigma^2(S^3/P)$ is a topological 5-sphere [Can79] (cf. Section 4.2.9). Therefore, the quotient space $\mathbb{R} \times \mathbb{R}^4/P$ is homeomorphic to $\mathbb{R}^5$, though P is not a rotation group. In the following we refer to $P < \mathrm{SO}_4$ as a Poincaré group.

In *Chapter 4* we show, based on Theorem A, that a Poincaré group is the only counterexample to the converse of the only if direction of Theorem A in the topological category up to products. A key step in obtaining this result is to first prove an analogous version in the category of homology manifolds.

Theorem C. For a finite subgroup $G < \mathrm{O}_n$ the quotient space $\mathbb{R}^n/G$ is a homology manifold with boundary if and only if G has the form

$$G = G_{rr} \times P_1 \times \ldots \times P_k$$

for a reflection-rotation group G_{rr} and Poincaré groups $P_i < \mathrm{SO}_4$, $i = 1, \ldots, k$, such that the factors act in pairwise orthogonal spaces. In this case the boundary of $\mathbb{R}^n/G$ is nonempty if and only if G_{rr} contains a reflection.

Using this result we prove

Theorem D. For a finite subgroup $G < \mathrm{O}_n$ the quotient space $\mathbb{R}^n/G$ is a topological manifold with boundary if and only if G has the form

$$G = G_{rr} \times P_1 \times \ldots \times P_k$$

for a reflection-rotation group G_{rr} and Poincaré groups $P_i < \mathrm{SO}_4$, $i = 1, \ldots, k$, such that the factors act in pairwise orthogonal spaces and such that $n > 4$ if $k = 1$ and $n > 5$ if G contains in addition a reflection. In this case $\mathbb{R}^n/G$ is either homeomorphic to the half space $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ and G contains a reflection or $\mathbb{R}^n/G$ is homeomorphic to $\mathbb{R}^n$ and G does not contain a reflection.

In particular, the underlying space of a smooth orbifold is a topological manifold if and only if all local groups have a form as described in the theorem.

For a Riemannian orbifold it is also natural to ask Davis's question in the category of *Lipschitz manifolds*, i.e. to ask whether a given Riemannian orbifold is a Lipschitz manifold. Locally this version amounts to the question for which finite subgroups $G < \mathrm{O}_n$ the quotient space $\mathbb{R}^n/G$ with the quotient metric is locally bi-Lipschitz homeomorphic to $\mathbb{R}^n$. The answer to this question reads as follows.

Theorem E. For a finite subgroup $G < \mathrm{O}_n$ the quotient space $\mathbb{R}^n/G$ is a Lipschitz manifold with boundary if and only if G is a reflection-rotation group. In this case $\mathbb{R}^n/G$ is either bi-Lipschitz homeomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ and G contains a reflection, or $\mathbb{R}^n/G$ is bi-Lipschitz homeomorphic to $\mathbb{R}^n$ and G does not contain a reflection.

Isotropy groups of real reflection groups are generated by the reflections they contain [Hum90, Thm. 1.12 (c), p. 22]. More generally, the same statement is true for isotropy groups of unitary reflection groups due to a theorem of Steinberg [Ste64, Thm. 1.5, p. 394]. Independent proofs for Steinberg's theorem were given by Bourbaki [Bou68, Ch. V, Ex. 8] and Lehrer [Leh04]. Consequently, isotropy groups of reflection-rotation groups which are either unitary reflection groups considered as real groups or (orientation preserving subgroups of) real reflection groups are generated by the reflections and rotations they contain. As a corollary of our results we obtain the following more general version of these isotropy theorems.

Corollary F. Isotropy groups of reflection-rotation groups are generated by the reflections and rotations they contain, i.e. they are again reflection-rotation groups.

As an application of this corollary and our other results we answer a question by Petrunin in the following form.

Corollary G. For a finite subgroup $G < \mathrm{O}_n$ the quotient space S^{n-1}/G is a topological manifold if and only if G has the form

$$G = G_{rot} \times P_1 \times \ldots \times P_k$$

for a rotation group G_{rot} and Poincaré groups $P_i < \mathrm{SO}_4$, $i = 1, \ldots, k$, such that the factors act in pairwise orthogonal spaces and such that $n > 5$ if $k = 1$. In this case S^{n-1}/G is homeomorphic to S^{n-1}.

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Chapter 1

Classification of finite groups generated by reflections and rotations

Introduction

A *finite reflection group* is a finite group generated by *reflections* in a finite-dimensional Euclidean space, i.e. by orthogonal transformations of this space whose fixed point subspace has codimension *one*. Analogously, we say that a finite group is a *finite rotation group*, if it is generated by orthogonal *rotations* in a finite-dimensional Euclidean space, i.e. by orthogonal transformations of this space whose fixed point subspace has codimension two. A *finite reflection-rotation group* is then a finite group generated by reflections and rotations in a finite-dimensional Euclidean space. From now on the specification *finite* for reflection-rotation groups is understood.

Large classes of reflection-rotation groups are real reflection groups, their orientation preserving subgroups and unitary reflection groups considered as real groups. As explained in the introduction of this thesis, reflection-rotation groups naturally arise in the study of the quotient of a finite-dimensional Euclidean space by a finite orthogonal group. Motivated by such observations Mikhaîlova started to work on a classification of rotation groups in the 70s and published a series of papers on several subclasses of rotation groups in the subsequent years [Mea76, Mik78, Mik82]. In [Mik84] she even claimed to have obtained a complete classification of rotation groups. However, no proofs were provided for parts of the claimed classification result. Moreover, when examining the results we discovered both exceptional rotation groups and building blocks of infinite families of rotation groups that are not mentioned in [Mea76, Mik78, Mik82, Mik84]. The largest irreducible rotation group among them occurs in dimension 8 and is connected with some grading of the simple Lie algebra $\mathfrak{so}_8$ (cf. Theorem 1, (v), 3.). It is an extension of the alternating group on 8 letters by a nonabelian group of order 2^7 and contains many other exceptional rotation groups as subgroups (cf. Section 1.3.9). The other irreducible examples and the building blocks of the reducible examples appear as subgroups in the normalizers of reflection groups (cf. Theorem 1, the groups $W^*(\mathrm{A}_5)$ and $W^*(\mathrm{E}_6)$, and Theorem 3, (ii) for $k=2$, (vi), ($viii$), (ix), (xi) and (xvi), type A_5 and E_6). Other interesting reducible rotation groups that have not been studied in [Mik84] occur in the product of two copies of a reflection group W of type H_3 or H_4 due to the existence of outer automorphisms of W that map reflections onto reflections but cannot be realized through conjugation by elements in its normalizer (cf. Section 1.3.6).

In a joint paper with M. A. Mikhaîlova we have recently closed the gaps in the classification of rotation groups [LM15]. In that paper we moreover generalize the proofs as to also yield a complete classification of reflection-rotation groups. In this chapter we present the content of the paper [LM15]. The final publication is available at http://link.springer.com.

1.1 Notations

We denote the cyclic group of order n and the dihedral group of order $2n$ by $\mathfrak{C}_n$ and $\mathfrak{D}_n$, respectively. We denote the symmetric and alternating group on n letters by $\mathfrak{S}_n$ and $\mathfrak{A}_n$. For a finite field of order q we write $\mathbb{F}_q$. Classical Lie groups are denoted like SO_n and U_n. The classical groups over finite fields are denoted like $\mathrm{SL}_n(q) = \mathrm{SL}_n(\mathbb{F}_q)$ as in [CCN$^+$85]. For a finite subgroup $G < \mathrm{O}_n$ we denote its orientation preserving subgroup as $G^+ < \mathrm{SO}_n$. In particular, we write W^+ for the orientation preserving subgroup of a reflection group W of a certain type. A list of all groups we are going to introduce can be found in the appendix.

1.2 Classification strategy and results

Two reflections in a reflection group generate a dihedral group which is characterized by its order, or equivalently by the angle between the two corresponding reflection hyperplanes. In 1933 Coxeter classified reflection groups by determining the possible configurations of reflections in such a group [Cox34]. This information, i.e. the dihedral groups defined by pairs of certain generating reflections, is encoded in the corresponding Coxeter diagram. Similarly, two rotations in a rotation group generate a rotation group in dimension two, three or four and all groups that arise in this way are known explicitly. However, an approach to the classification of rotation groups similar to the one for reflection groups, albeit conceivable, seems to be unpractical. Instead, we follow an approach outlined in [Mik84] that has already been carried out partially. Classifications of several subclasses of rotation groups are treated in [Mea76, Mik78, Mik82]. From these papers and from results by Brauer, Huffman, Wales and others [Bra67, HW75, Huf75, Wol84] a complete classification of rotation groups can be obtained. We carry out this program and generalize the proofs as to also yield a complete classification of reflection-rotation groups.

A rotation group preserves the orientation. Conversely, all finite subgroups of SO_2 and SO_3 are rotation groups. The finite subgroups of SO_4 are listed in [DuV64] and the rotation groups among them can be singled out. The classifications of *irreducible* and *reducible* rotation groups have to be treated separately since a reducible rotation group does in general not split as a product of irreducible components. If the complexification of an irreducible rotation group is reducible then this group preserves a complex structure and is thus a unitary reflection group considered as a real group. Otherwise it is called *absolutely irreducible* and we make another case differentiation. Depending on whether there exists a decomposition of the underlying vector space into nontrivial subspaces that are interchanged by G, a so-called *system of imprimitivity*, or not, the group is either called *imprimitive* or *primitive*. For an imprimitive irreducible rotation group the subspaces forming a system of imprimitivity are either all one- or two-dimensional. In the first case the group is called *monomial.*

For a monomial group G we denote its diagonal subgroup, i.e. the set of all transformations that act trivially on its system of imprimitivity, by $D(G)$. Apart from the two families of orientation preserving subgroups of the reflection groups $W(\mathrm{BC}_n)$ and $W(\mathrm{D}_n)$, there are four monomial rotation groups M_5, M_6, M_7 and M_8 given as semidirect products of the diagonal subgroup of $W(\mathrm{D}_n)$ and a permutation group H, and two exceptional subgroups M_7^p and M_8^p of M_7 and M_8, respectively.

There is a class of imprimitive unitary reflection groups, denoted by $G(m,p,n) < \mathrm{U}_n$, which is defined to be the semidirect product of

$$A(m,p,n) := \left\{(\theta_1,\dots,\theta_n) \in \mu_m^n | (\theta_1 \dots \theta_n)^{m/p} = 1\right\}$$

with the symmetric group $\mathfrak{S}_n$, where $\mu_m < \mathbb{C}^*$ is the cyclic subgroup of m-th roots of unity and p is a factor of m. The only other imprimitive irreducible rotation groups are extensions of $G(m,1,n)$ and $G(2m,2,n)$ by a rotation r that conjugates two coordinates, i.e. $r(z_1,z_2,z_3\dots,z_n) = (\overline{z}_1,\overline{z}_2,z_3\dots,z_n)$. We denote these groups by $G^*(km,k,n)$, $k=1,2$.

Apart from the primitive rotation groups that are either orientation preserving subgroups of real reflection groups or unitary reflection groups considered as real groups, there are five primitive rotation groups W^* obtained by extending the orientation preserving subgroup W^+ of a real reflection group W by a normalizing rotation, two exceptional primitive rotation groups in dimensions five and six isomorphic to $\mathfrak{A}_5$ and $\mathrm{PSU}_2(7)$, respectively, and a primitive rotation group in dimension eight, which is generated by M_8 and another rotation (cf. Theorem 1, (v), (c)).

The rotation groups listed in Theorem 1, (v) are generated by rotations of order 2. A rotation group $G < \mathrm{SO}_n$ with this property defines a configuration $\mathfrak{P} = \{\sigma_i\}_{i\in I}$ of 2-planes in $\mathbb{R}^n$ given by the complements of the fixed point subspaces of the involutive rotations in G such that $r_\sigma(\mathfrak{P}) = \mathfrak{P}$ holds for all $\sigma \in \mathfrak{P}$ where r_σ is the rotation of order 2 defined by σ. We call such a configuration a *plane system* and denote the generated rotation group by $M(\mathfrak{P})$.

Theorem 1 ([LM15], Theorem 1)**.** *Every irreducible rotation group occurs, up to conjugation, in precisely one of the following cases*

(i) Orientation preserving subgroups W^+ *of irreducible real reflection groups* W *(cf. Section 1.3.1).*

(ii) Irreducible unitary reflection groups $G < \mathrm{U}_n$, $n \geq 2$, *that are not the complexification of a real reflection group, considered as real groups* $G < \mathrm{SO}_{2n}$ *(cf. Section 1.3.2).*

(iii) The imprimitive rotation groups $G^*(km,k,l) < \mathrm{SO}_n$ *for* $n = 2l > 4$, $k \in \{1,2\}$ *and* $km \geq 3$ *(cf. Section 1.3.5).*

(iv) The unique extensions W^* *of* W^+ *by a normalizing rotation for real reflection groups* W *of type* A_4, D_4, F_4, A_5 *and* E_6 *(cf. Section 1.3.1). These groups are primitive.*

(v) The following rotation groups which can be realized as $M(\mathfrak{P})$ *for a plane system* $\mathfrak{P}$ *of type* P_5, P_6, P_7, P_8, Q_7, Q_8, S_5, R_6 *or* T_8 *and which only contain rotations of order* 2*, namely*

   a) the monomial rotation groups M_5, M_6, M_7, M_8 *and* M_7^p *and* M_8^p *(cf. Section 1.3.4).*

b) the primitive rotation groups $R_5(\mathfrak{A}_5)$ and $R_6(\mathrm{PSL}_2(7))$ given as the image of the unique irreducible representations of $\mathfrak{A}_5$ in SO_5 and of $\mathrm{PSL}_2(7)$ in SO_6 (cf. Section 1.3.7).

c) a primitive rotation group in SO_8 isomorphic to an extension of $\mathfrak{A}_8$ by a nonabelian group of order 2^7 (cf. Section 1.3.8).

(vi) The remaining rotation groups in SO_4, i.e. an infinite family of imprimitive rotation groups described in [LM15, Prop. 35] (cf. Section 1.3.5) and 3 individual- and 6 infinite families of primitive rotation groups listed in [LM15, Table 1, Sect. 4.3] (cf. Section 1.3.3).

For a real reflection group W we denote by $W^\times$ its unique extension by a normalizing rotation, provided such exists, i.e. $W^\times = \langle W^*, W\rangle$. For a monomial rotation group M we denote by $M^\times$ its extension by a coordinate reflection. Finally, for an imprimitive rotation group of type $G(km,k,l)$ let $G^\times(km,k,l)$ be its extension by a reflection s of the form $s(z_1,\dots,z_l) = (\overline{z}_1, z_2, \dots, z_l)$.

Theorem 2 ([LM15], Theorem 2)**.** *Every irreducible reflection-rotation group either appears in Theorem 1 or it contains a reflection and occurs, up to conjugation, in one of the following cases*

(i) Irreducible real reflection groups W (cf. Section 1.3.1).

(ii) The groups $W^\times$ generated by a reflection group W of type A_4, D_4, F_4, A_5 or E_6 and a normalizing rotation (cf. Section 1.3.1).

(iii) The monomial groups $M^\times$ of type D_n, P_5, P_6, P_7, P_8, i.e. $M^\times(\mathrm{D}_n) := D(W(\mathrm{BC}_n)) \rtimes \mathfrak{A}_n$, $M_5^\times$, $M_6^\times$, $M_7^\times$ and $M_8^\times$ (cf. Section 1.3.4).

(iv) The imprimitive groups $G^\times(km,k,l) < \mathrm{SO}_n$ with $n = 2l$, $k = 1,2$ and $km \geq 3$ (cf. Section 1.3.5).

Let $G < \mathrm{O}_n$ be an arbitrary reflection-rotation group and let $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$ be a decomposition into irreducible components. For each $i \in I = \{1,\dots,k\}$ we denote the projection of G to $\mathrm{O}(V_i)$ by π_i and set $G_i = \pi_i(G)$. We distinguish two kinds of rotations in G (cf. [Mik82]).

Definition 1. A rotation $g \in G$ is called a rotation of the

(i) *first kind*, if for some $i_0 \in I$, $\pi_{i_0}(g)$ is a rotation in V_{i_0} and $\pi_i(g)$ is the identity on V_i for all $i \neq i_0$.

(ii) *second kind*, if for some $i_1, i_2 \in I$, $i_1 \neq i_2$, $\pi_{i_1}(g)$ and $\pi_{i_2}(g)$ are reflections in V_{i_1} and V_{i_2}, respectively, and $\pi_i(g)$ is the identity for all $i \neq i_1, i_2$.

Let H be the normal subgroup of G generated by rotations of the first kind, let F be the normal subgroup of G generated by reflections and rotations of the second kind and set $H_i = \pi_i(H)$ and $F_i = \pi_i(F)$. Then H_i is a rotation group, F_i is a reflection group and both are normal subgroups of G_i. In order to classify all reflection-rotation groups we first describe the possible triples (G_i, H_i, F_i) and then the ways how a reflection-rotation group can be recovered from a collection of such triples. Notice that G_i is generated by H_i and F_i. Hence, depending on whether F_i is trivial or not, G_i either appears in Theorem 1 or in Theorem 2. Recall that reflections $s_1,\dots,s_l$ whose corresponding fixed-point hyperplanes are the walls of a chamber of a reflection group W generate W, and that (W, S) is a Coxeter system for $S = \{s_1, \ldots, s_l\}$ (cf. [Hum90, p. 10, p. 23]). We refer to the reflections $s_1, \ldots, s_l$ as *simple reflections* (cf. [Hum90, p. 10]).

Theorem 3 ([LM15], Theorem 3)**.** *Let $G < \mathrm{O}_n$ be a reflection-rotation group. Then, for each i, either $G_i = H_i$ is an irreducible rotation group or F_i is nontrivial and a set of simple reflections generating F_i projects onto a set $\overline{S} \subset G_i/H_i$ for which $(G_i/H_i, \overline{S})$ is a Coxeter system. In the second case the quadruple $(G_i, H_i, F_i, \Gamma_i)$ occurs, up to conjugation, in one of the following cases where Γ_i denotes the Coxeter diagram of G_i/H_i.*

(i) $(M^\times, M, D(M^\times), \circ)$ *for* $M = M_5, M_6, M_7, M_8, M(\mathrm{D}_n) = W^+(\mathrm{D}_n)$.
(ii) $(G^\times(km, k, l), G^*(km, k, l), D(G^\times(km, k, l)), \circ)$ *for* $k = 1, 2$, $km \geq 3$ *and* $n = 2l$.
(iii) $(G^\times(2m, 1, l), G^*(2m, 2, l), D(G^\times(2m, 1, l)), \circ \quad \circ)$ *for* $m \geq 2$ *and* $n = 2l$.
(iv) $(W, \{e\}, W, \Gamma(W))$ *for any irreducible reflection group* W.
(v) $(W, W^+, W, \circ)$ *for any irreducible reflection group* W.
(vi) $(W(\mathrm{A}_3), W^+(\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1), W(\mathrm{A}_3), \circ - \circ)$
(vii) $(W(\mathrm{BC}_n), D(W^+(\mathrm{BC}_n)), W(\mathrm{BC}_n), \Gamma(\mathrm{A}_{n-1} \times \mathrm{A}_1) = \circ - \circ - \cdots \circ \quad \circ)$
(viii) $(W(\mathrm{BC}_n), W^+(\mathrm{D}_n), W(\mathrm{BC}_n), \circ \quad \circ)$
(ix) $(W(\mathrm{BC}_4), G^*(4, 2, 2), W(\mathrm{BC}_4), \circ - \circ \quad \circ)$
(x) $(W(\mathrm{D}_n), D(W(\mathrm{D}_n)), W(\mathrm{D}_n), \Gamma(\mathrm{A}_{n-1}) = \circ - \circ - \cdots \circ)$
(xi) $(W(\mathrm{D}_4), G^*(4, 2, 2), W(\mathrm{D}_4), \circ - \circ)$
(xii) $(W(\mathrm{I}_2(km)), W^+(\mathrm{I}_2(m)), W(\mathrm{I}_2(km)), \circ \overset{k}{-} \circ)$ *for* $m, k \geq 2$.
(xiii) $(W(\mathrm{F}_4), G^*(4, 2, 2), W(\mathrm{F}_4), \circ - \circ \quad \circ - \circ)$
(xiv) $(W(\mathrm{F}_4), W^+(\mathrm{D}_4), W(\mathrm{F}_4), \circ - \circ \quad \circ)$
(xv) $(W(\mathrm{F}_4), W^*(\mathrm{D}_4), W(\mathrm{F}_4), \circ \quad \circ)$
(xvi) $(W^\times, W^*, W, \circ)$ *for a reflection group* W *of type* A_4, D_4, F_4, A_5 *or* E_6.
(xvii) $(W^\times(\mathrm{D}_4), W^+(\mathrm{D}_4), W(\mathrm{D}_4), \circ - \circ)$ *(, but $H_i \neq F_i^+$, cf. [LM15, Prop. 55].)*

For each quadruple (G_{rr}, M, W, Γ) occurring in this list every reflection in G_{rr} is contained in W. The group W is reducible in the cases (i) to (iii), irreducible with $W = G_{rr}$ in the cases (iv) to (xv) and irreducible with $W \neq G_{rr}$ in the cases (xvi) and (xvii).

Remark 1. The preceding theorem is actually a classification of pairs $M \triangleleft G_{rr}$ where $G_{rr} < \mathrm{O}_n$ is an irreducible reflection-rotation group $G_{rr} < \mathrm{O}_n$ and $M \triangleleft G_{rr}$ a normal rotation group such that G_{rr} is generated by M and the reflections it contains.

Assume that the family of triples $\{(G_i, H_i, F_i)\}_{i \in I}$, $I = \{1, \ldots, k\}$, is induced by a reflection-rotation group. The reflections in $\tilde{G} = G_1/H_1 \times \cdots \times G_k/H_k$ are the cosets of the reflections in $F_1 \times \cdots \times F_k$. We call two such reflections $\overline{s}_1 \in G_i/H_i$ and $\overline{s}_2 \in G_j/H_j$ for $i \neq j$ *related*, if $s_1 \notin G$ and if there exists a rotation of the second kind $h \in G$ such that $s_1 = \pi_i(h)$ and $s_2 = \pi_j(h)$. Relatedness of reflections defines an equivalence relation on the set of reflections in $\tilde{G}$. This equivalence relation induces an equivalence relation on the set of irreducible components of $\tilde{G}$ and on the set of connected components of its Coxeter diagram such that equivalence classes of nontrivial irreducible components, i.e. of those whose Coxeter diagram is not an isolated vertex, consist of two isomorphic components that belong to different G_i/H_i and are isomorphic via an isomorphism induced by relatedness of reflections.

Conversely, given such data one first obtains an equivalence relation on the set of reflections contained in $G_1/H_1 \times \cdots \times G_k/H_k$ and then a reflection-rotation group $G < G_1 \times \cdots \times G_k$ generated by H, the rotations $s_1 s_2$ for reflections $s_1 \in F_i$ and $s_2 \in F_j$, $i \neq j$, whose cosets $\overline{s}_1$ and $\overline{s}_2$ are equivalent, and the reflections $s \in F_i$ whose cosets are not equivalent to any other coset of a reflection.

In fact, these assignments are inverse to each other.

Theorem 4 ([LM15], Theorem 4)**.** *Reflection-rotation groups are in one-to-one correspondence with families of triples occurring in Theorem 3,* $\{(G_i, H_i, F_i)\}_{i\in I}$*, with an equivalence relation on the set of irreducible components of* $\tilde{G} = G_1/H_1 \times \cdots \times G_k/H_k$ *such that*

(i) the elements of an equivalence class belong to pairwise different G_i/H_i*,*

(ii) each G_i/H_i *contains at most one trivial irreducible component that is not equivalent to another component,*

(iii) equivalence classes of nontrivial irreducible components contain precisely two isomorphic components

together with isomorphisms between the equivalent nontrivial irreducible components that map reflections onto reflections. A reflection-rotation group corresponding to such a set of data contains a reflection, if and only if there exists an equivalence class consisting of a single trivial component.

Notice that different isomorphisms between the irreducible components in general yield nonconjugate reflection-rotation groups (cf. Section 1.3.6).

1.3 Examples and properties

In this section we describe several classes of reflection-rotation groups and discuss some of their properties.

1.3.1 Real reflections groups

A real reflection group W is a finite subgroup of an orthogonal group O_n generated by reflections, i.e. by orthogonal transformations whose fixed point subspace has codimension one. Irreducible reflection groups are classified and the types of the occurring groups are denoted as A_n, BC_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4 and $\mathrm{I}_2(p)$ for $p \geq 3$ [Hum90]. Every reflection group splits as a direct product of irreducible reflection groups. Since the composition of two distinct reflections is a rotation and since all compositions of pairs of reflections in a reflection group W generate the orientation preserving subgroup W^+ of W, this subgroup W^+ is always a rotation group. There is another way to construct a rotation group from a reflection group. If there exists a rotation $h \in \mathrm{SO}_n \backslash W$ that normalizes W, then h also normalizes W^+ and the group $W^* = \langle W^+, h\rangle$ is again a rotation group. Now we specify the cases in which new examples arise this way.

Lemma 5. *Let* $W < \mathrm{O}_n$ *be a reflection group and suppose* $h \in \mathrm{SO}_n \backslash W$ *is a rotation that normalizes* W*. If* $\langle W, h\rangle$ *is not a reflection group, then there exists a chamber* C *of* W *such that* $hC = C$*.*

Proof. Since h normalizes W, it interchanges the chambers of W [Hum90, p. 23]. If $U = \mathrm{Fix}(h)$ intersects the interior of some chamber C of W, then we have $hC = C$. Otherwise U would be contained in a hyperplane corresponding to some reflection s in W. But then sh would be a reflection and thus $\langle W, h\rangle = \langle W, sh\rangle$ would be a reflection group. This contradicts our assumption and so the claim follows. □

Lemma 6. *Let $W < \mathrm{O}_n$ be an irreducible reflection group, let C be a chamber of W and suppose $h \in \mathrm{SO}_n$ is a rotation such that $hC = C$. Then W has type A_4, D_4, F_4, A_5 or E_6. Moreover, in the case of type A_4, F_4, A_5 and E_6 such a rotation h is unique. In the case of type D_4 there exist two such rotations which have order 3 and are inverse to each other.*

Proof. Because of $hC = C$, the rotation h permutes the walls of the chamber C and thus corresponds to an automorphism of the Coxeter diagram of W [Hum90, p. 29]. Since the fixed point subspace of h has codimension two, we conclude that only the types A_4, D_4, F_4, A_5 and E_6 can occur for W. The additional claims follow from the structure of the respective diagrams. □

Lemma 7. *Let $W < \mathrm{SO}_n$ be an irreducible reflection group. Then there exists a rotation $h \in \mathrm{SO}_n \backslash W$ that normalizes W and W^+ such that $W^* = \langle W^+, h\rangle$ is a rotation group which is not the orientation preserving subgroup of a reflection group if and only if W has type A_4, D_4, F_4, A_5 or E_6. In this case the extended group W^* is unique.*

Proof. The *only if* direction is clear by the preceding two lemmas. Conversely, suppose that W has type A_4, D_4, F_4, A_5 or E_6. In each case there exists a nontrivial automorphism of the Coxeter diagram of W. The vertices of this diagram correspond to a set Δ of outward normal vectors to the walls of a chamber of W and the diagram automorphism corresponds to a permutation of Δ [Hum90, p. 29]. Due to the fact that Δ is a basis of $\mathbb{R}^n$, this permutation can be extended to a linear transformation h of $\mathbb{R}^n$. Since h is induced by a diagram automorphism, it preserves the inner products of the vectors in Δ which are encoded in the Coxeter diagram of W. Hence the transformation h is orthogonal, i.e. we have $h \in \mathrm{O}_n$. Moreover, the structure of the Coxeter diagram of W implies that the fixed point subspace of h has codimension two and that the extension $W^* = \langle W^+, h\rangle$ obtained in this way is unique. Finally, it follows easily from the classification of reflection groups, e.g. by a counting argument, that W^* is not the orientation preserving subgroup of a reflection group in each of the cases A_4, D_4, F_4, A_5 and E_6. □

The next lemma will be needed later.

Lemma 8. *For $n \geq 5$ let $W < \mathrm{O}_n$ be an irreducible reflection group with orientation preserving subgroup W^+. Assume that $\langle W, -\mathrm{id}\rangle$ is not a reflection group. Then the group $G := \langle W^+, -\mathrm{id}\rangle$ is a rotation group different from W^+ if and only if W has type E_6.*

Proof. Assume that G is a rotation group different from W^+. Then there exists a rotation $h \in G\backslash W^+$ that normalizes W. It follows from Lemma 5, Lemma 6 and our assumption $n \geq 5$ that W has type A_5 or E_6. Since the inversion only preserves the orientation in even dimensions we conclude that W has type E_6.

Conversely, assume that W has type E_6 and let C be any chamber of W. Since the inversion interchanges the chambers of W and W acts transitively on them [Hum90, p. 10], there exists some $w \in W$ such that $-wC = C$. The fact that the inversion is not contained in $W(\mathrm{E}_6)$ implies that the transformation $-w$ is nontrivial and thus induces a nontrivial automorphism of the Coxeter diagram of W. It follows from the structure of this diagram that $-w$ is a rotation which is why G is a rotation group different from W^+ as claimed. □

Finally, we describe the groups $W^*(\mathrm{A}_5)$ and $W^*(\mathrm{E}_6)$ more explicitly.

Proposition 9. *The rotation groups $W^*(\mathrm{A}_5)$ and $W^*(\mathrm{E}_6)$ can be described as follows.*
(i) $W^*(\mathrm{A}_5) = \langle W^+(\mathrm{A}_5), -s\rangle \cong \mathfrak{S}_6$ *for any reflection $s \in W(\mathrm{A}_5)$.*
(ii) $W^*(\mathrm{E}_6) = \langle W^+(\mathrm{E}_6), -\mathrm{id}\rangle \cong \mathrm{PSU}_2(4) \times \mathbb{Z}_2$.

Proof. For (i) observe that $W(\mathrm{A}_5) \cong \mathfrak{S}_6$ has a trivial center and thus does not contain the inversion. It follows as in the proof of the preceding lemma that $W^\times(\mathrm{A}_5) = \langle W(\mathrm{A}_5), -\mathrm{id}\rangle$ and hence $W^*(\mathrm{A}_5) = \langle W^+(\mathrm{A}_5), -s\rangle \cong \mathfrak{S}_6$ as claimed. For (ii) see the proof of the preceding lemma. □

1.3.2 Unitary reflection groups

A unitary reflection group is a finite subgroup of some unitary group U_n generated by unitary reflections, i.e. by unitary transformations of finite order whose fixed point subspace has complex codimension one. A complete classification of such groups was first compiled by Shephard and Todd in 1954 [ST54] and is described in [LT09]. As in the real case, every unitary reflection group splits as a direct product of irreducible unitary reflection groups. The irreducible groups fall into two classes according to the following definition.

Definition 2. A finite subgroup $G < \mathrm{GL}(V)$ is called *imprimitive* if there exists a decomposition of the vector space V into a direct sum of proper subspaces $V_1, \dots, V_l$, a *system of imprimitivity*, such that for any $g \in G$ and for any $i \in \{1, \dots, l\}$ there exists a $j \in \{1, \dots, l\}$ such that $\rho(g)(V_i) = V_j$. Otherwise the subgroup is called *primitive*.

The imprimitive irreducible unitary reflection groups can be constructed as follows (cf. [LT09, Ch. 2, p. 25]). Let $\mu_m < \mathbb{C}^*$ be the cyclic subgroup of m-th roots of unity. For a factor p of m let

$$A(m,p,n) := \left\{(\theta_1, \dots, \theta_n) \in \mu_m^n | (\theta_1 \dots \theta_n)^{m/p} = 1\right\}$$

and let $G(m,p,n)$ be the semidirect product of $A(m,p,n)$ with the symmetric group $\mathfrak{S}_n$. Then the natural realization of $G(m,p,n)$ in U_n is an imprimitive unitary reflection group and every imprimitive irreducible unitary reflection group is of this form. The following proposition holds [LT09, Prop. 2.10, p. 26].

Proposition 10. *If $m > 1$, then $G(m,p,n)$ is an imprimitive irreducible unitary reflection group except when $(m,p,n) = (2,2,2)$ in which case $G(m,p,n)$ is not irreducible.*

A primitive unitary reflection group is either a cyclic group $\mu_n < \mathrm{U}_1$, a symmetric group in U_n given as a complexified real reflection group of type A_n, or one of 34 primitive unitary reflection groups in dimension at most 8 [LT09, p. 138]. Among the latter 34 groups 19 are two-dimensional. These groups decompose into 3 families according to whether their image in $\mathrm{PU}_2 \cong \mathrm{SO}_3$ is a tetrahedral, an octahedral or an icosahedral group [LT09, Ch. 6 and Appendix D, Table 1]. A collection $\mathfrak{L}$ of complex lines in $\mathbb{C}^n$ that is invariant under all reflections of order two defined by its lines is called a *line system* and determines a unitary reflection group $W(\mathfrak{L})$ [LT09, Ch. 7]. The remaining 15 groups arise in this way. The occurring line systems are denoted as $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$, $\mathcal{F}_4$ $\mathcal{H}_3$, $\mathcal{H}_4$, $\mathcal{J}_3^{(4)}$, $\mathcal{J}_3^{(5)}$, $\mathcal{K}_5$, $\mathcal{K}_6$, $\mathcal{L}_4$, $\mathcal{M}_3$, $\mathcal{N}_4$, $\mathcal{O}_4$ [LT09, Thm. 8.29, p. 152 and Appendix D, Table 2], among them the complexifications of the root systems of real reflection groups of type E_6, E_7, E_8, F_4, H_3 and H_4.

Clearly, a unitary reflection group $G < \mathrm{U}_n$ gives rise to a rotation group $G < \mathrm{SO}_{2n}$ when considered as a real group. Conversely, we have

Lemma 11. *An irreducible rotation group is a unitary reflection group considered as a real group, if and only if it is not absolutely irreducible.*

Proof. The complexification of a complex group considered as a real group is reducible. In fact, it commutes with the idempotent product of the two complex structures and thus leaves its nontrivial 1- and (-1)-eigenspace invariant. Conversely, let $G < \mathrm{SO}_n$ be an irreducible rotation group and suppose that G is not absolutely irreducible. Then its complexification splits into more than one irreducible component, i.e.

$$V^{\mathbb{C}} = V_1 \oplus \cdots \oplus V_k$$

for some $k \geq 2$. By considering the representations over the real numbers we recover two copies of the original representation and thus we have $k = 2$ and $V = V_1^{\mathbb{R}} = V_2^{\mathbb{R}}$ (since the isotypic components are uniquely determined by the representation). Hence, G is a unitary reflection group $G < \mathrm{U}_m$ with $m = \dim(V_1) = n/2$ considered as a real group. □

Moreover, we have

Lemma 12. *A rotation group $G < \mathrm{SO}_{2n}$ that is a unitary reflection group $G < \mathrm{U}_n$ considered as a real group is irreducible, if and only if $G < \mathrm{U}_n$ is irreducible as a complex group and not the complexification of a real reflection group.*

Proof. If a group $G < \mathrm{U}_n$ is irreducible over the real numbers, then it is also irreducible over the complex numbers and cannot be the complexification of a real group (cf. proof of Lemma 11). Conversely, assume that $G < \mathrm{U}_n$ is an irreducible unitary reflection group which becomes reducible after restricting the scalars to the real numbers and let

$$\mathbb{R}^{2n} = V_1 \oplus \cdots \oplus V_k$$

be a corresponding decomposition into irreducible real subspaces for some $k \geq 2$. Since the complex structure J is preserved by G, the complex subspaces $V_1 + JV_1$ and $V_1 \cap JV_1$ are invariant under the action of G and thus we have $\mathbb{R}^{2n} = V_1 \oplus JV_1$, as G is irreducible as a complex group by assumption. The fact that G and J commute moreover implies that the projection of G to $\mathrm{O}(V_1)$ is a real reflection group whose complexification is G. □

We have the following criterion for an irreducible rotation group not to be induced by a unitary reflection group. In particular, it applies to the groups W^+ and W^* for W not of type $\mathrm{I}_2(p)$.

Lemma 13. *Let $G < \mathrm{SO}_n$ be an irreducible rotation group that is normalized by a reflection s. If $n > 2$ then the group G is absolutely irreducible.*

Proof. Suppose that G is not absolutely irreducible. The group $G^\times = \langle G, s\rangle$ is absolutely irreducible since it contains a reflection. Hence, the reflection s permutes the irreducible components of the complexification of G. This implies $n = 2$ and thus the claim follows. $\square$

1.3.3 Rotation groups in low dimensions

All elements of SO_2 and SO_3 are rotations and thus every finite subgroup of SO_2 and SO_3 is a rotation group. This is not true for SO_4, but its finite subgroups and the rotation groups among them can still be described explicitly. We sketch this description here, a more detailed discussion can be found in [DuV64]. There are two-to-one covering maps of Lie groups $\varphi : \mathrm{SU}_2 \times \mathrm{SU}_2 \to \mathrm{SO}_4$ and $\psi : \mathrm{SU}_2 \to \mathrm{SO}_3$. Therefore, the finite subgroups of SO_4 can be determined based on the knowledge of the finite subgroups of SO_3. These are cyclic groups C_n of order n, dihedral groups D_n of order $2n$ and the symmetry groups of a tetrahedron, an octahedron and an icosahedron, which are isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$, respectively. Using the covering map ψ one finds that the finite subgroups of S^3 are cyclic groups $\mathbf{C}_n$ of order n, binary dihedral groups $\mathbf{D}_n$ of order $4n$ and binary tetrahedral, octahedral and icosahedral groups denoted by $\mathbf{T}$, $\mathbf{O}$ and $\mathbf{I}$, respectively, and we set $\mathbf{V} = \mathbf{D}_2$. Except for $\mathbf{C}_n$ with odd n, these are two-to-one preimages of respective subgroups of SO_3, i.e. subgroups of SU_2 of the form $\mathbf{C}_n$ with odd n are the only ones that do not contain the kernel of ψ. In the following we identify SU_2 with the unit quaternions in $\mathbb{H}$. Then the homomorphism φ is explicitly given by

$$\begin{array}{rccl} \varphi: & \mathrm{SU}_2 \times \mathrm{SU}_2 & \to & \mathrm{SO}_4 \\ & (l,r) & \mapsto & \varphi((l,r)) : q \mapsto lqr^{-1} \end{array}$$

where $\mathbb{R}^4$ is identified with the algebra of quaternions $\mathbb{H}$ and has kernel $\{\pm(1,1)\}$. The classification result reads as follows [DuV64, p. 54].

Proposition 14. *For every finite subgroup $G < \mathrm{SO}_4$ there are finite subgroups $\mathbf{L}, \mathbf{R} < \mathrm{SU}_2$ with $-1 \in \mathbf{L}, \mathbf{R}$ and normal subgroups $\mathbf{L}_K \lhd \mathbf{L}$ and $\mathbf{R}_K \lhd \mathbf{R}$ such that $\mathbf{L}/\mathbf{L}_K$ and $\mathbf{R}/\mathbf{R}_K$ are isomorphic via an isomorphism $\phi : \mathbf{L}/\mathbf{L}_K \to \mathbf{R}/\mathbf{R}_K$ for which*

$$G = \varphi(\{(l,r) \in \mathbf{L} \times \mathbf{R} | \phi(\pi_L(l)) = \pi_R(r)\})$$

holds, where $\pi_L : \mathbf{L} \to \mathbf{L}/\mathbf{L}_K$ and $\pi_R : \mathbf{R} \to \mathbf{R}/\mathbf{R}_K$ are the natural projections. In this case we write $G = (\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)_\phi$. Conversely, a set of data $(\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)_\phi$ with the above properties defines a finite subgroup G of SO_4 by the equation above.

Given a finite subgroup $G < \mathrm{SO}_4$, for $\mathbf{L} = \pi_1(\varphi^{-1}(G))$, $\mathbf{R} = \pi_2(\varphi^{-1}(G))$, $\mathbf{L}_K = \{l \in \mathbf{L} | \varphi((l,1)) \in G\}$ and $\mathbf{R}_K = \{r \in \mathbf{R} | \varphi((1,r)) \in G\}$ the quotient groups $\mathbf{L}/\mathbf{L}_K$ and $\mathbf{R}/\mathbf{R}_K$ are isomorphic and with the isomorphism ϕ induced by the relation $\varphi^{-1}(G) < \mathbf{L} \times \mathbf{R}$ we have

$G = (\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)_\phi$. In most cases the conjugacy class of $(\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)_\phi$ in SO_4 does not depend on the specific isomorphism ϕ. However, there are a few exceptions. Since the finite subgroups of SU_2 are invariant under conjugation [DuV64, p. 53], the groups $(\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)_\phi$ and $(\mathbf{R}/\mathbf{R}_K; \mathbf{L}/\mathbf{L}_K)_{\phi^{-1}}$ are conjugate in O_4. For a list of finite subgroups of SO_4 we refer to [DuV64, p. 57].

Elements of SO_4 of the form $\varphi(l, 1)$ and $\varphi(1, r)$ for $l, r \in \mathrm{SU}_2$ are called *left*- and *rightscrews*, respectively. They commute mutually and for $l, r \in \mathrm{SU}_2$ there exist $a, b \in \mathrm{SU}_2$ and $\alpha, \beta \in \mathbb{R}$ such that $a^{-1}la = \cos(\alpha) + \sin(\alpha)i$ and $brb^{-1} = \cos(\beta) + \sin(\beta)i$. Then, with respect to the basis $\mathbb{B} = \{ab, aib, ajb, akb\}$, we have

$$\varphi(l, 1)_\mathbb{B} = \begin{pmatrix} R(\alpha) & 0 \\ 0 & R(\alpha) \end{pmatrix}, \quad \varphi(1, r)_\mathbb{B} = \begin{pmatrix} R(\beta) & 0 \\ 0 & R(-\beta) \end{pmatrix}$$

and thus

$$\varphi(l, r)_\mathbb{B} = \begin{pmatrix} R(\alpha+\beta) & 0 \\ 0 & R(\alpha-\beta) \end{pmatrix}.$$

where $R(\alpha)$ is a rotation about the angle α. Consequently, $\varphi(l, r)$ is a rotation if and only if $\mathrm{Re}(l) = \mathrm{Re}(r) \notin \{\pm 1\}$. Using this observation it is possible to classify rotation groups in dimension 4. The primitive rotation groups among them are singled out in [Mea76]. The groups of the form $(\mathbf{C}_{km}/\mathbf{C}_m; \mathbf{R}/\mathbf{R}_K)$ listed in this paper under number 7.,...,11. preserve a complex structure and correspond to the primitive unitary reflection groups in dimension 2 (cf. Section 1.3.2). The groups in the list that come from real reflection groups are (cf. Section 1.3.1): $W^+(\mathrm{A}_4) = (\mathbf{I}/\mathbf{C}_1; \mathbf{I}/\mathbf{C}_1)^*$, $W^*(\mathrm{A}_4) = (\mathbf{I}/\mathbf{C}_2; \mathbf{I}/\mathbf{C}_2)^*$, $W^*(\mathrm{D}_4) = (\mathbf{T}/\mathbf{T}; \mathbf{T}/\mathbf{T})$, $W^+(\mathrm{F}_4) = (\mathbf{O}/\mathbf{T}; \mathbf{O}/\mathbf{T})$, $W^*(\mathrm{F}_4) = (\mathbf{O}/\mathbf{O}; \mathbf{O}/\mathbf{O})$ and $W^+(\mathrm{H}_4) = (\mathbf{I}/\mathbf{I}; \mathbf{I}/\mathbf{I})$. Here, the star $*$ indicates the choice of an outer automorphism (cf. Proposition 14). The remaining primitive rotation groups appearing in [Mea76] are listed in Table 1.1.

	rotation group	order
1.	$(\mathbf{D}_{3m}/\mathbf{D}_{3m}; \mathbf{T}/\mathbf{T})$	$144m$
2.	$(\mathbf{D}_m/\mathbf{D}_m; \mathbf{O}/\mathbf{O})$	$96m$
3.	$(\mathbf{D}_m/\mathbf{C}_{2m}; \mathbf{O}/\mathbf{T})$	$48m$
4.	$(\mathbf{D}_{2m}/\mathbf{D}_m; \mathbf{O}/\mathbf{T})$	$96m$
5.	$(\mathbf{D}_{3m}/\mathbf{C}_{2m}; \mathbf{O}/\mathbf{V})$	$48m$
6.	$(\mathbf{D}_m/\mathbf{D}_m; \mathbf{I}/\mathbf{I})$	$240m$
7.	$(\mathbf{T}/\mathbf{T}; \mathbf{O}/\mathbf{O})$	576
8.	$(\mathbf{T}/\mathbf{T}; \mathbf{I}/\mathbf{I})$	1440
9.	$(\mathbf{O}/\mathbf{O}; \mathbf{I}/\mathbf{I})$	2880

Table 1.1: Primitive rotation groups in O_4 that do not preserve a complex structure and are different from rotation groups of type W^+ and W^*.

1.3.4 Monomial reflection-rotation groups

An imprimitive linear group is called *monomial* if it admits a system of imprimitivity consisting of one-dimensional subspaces (cf. Introduction). Examples for monomial irreducible reflection-rotation groups are the reflection group of type BC_n and its orientation preserving subgroup. To construct other examples let $H < \mathfrak{S}_n$ be a permutation group generated by a set of double transpositions and 3-cycles, e.g. (cf. [Mik84, p. 104])

(i) $H = H_5 = \langle (1,2)(3,4), (2,3)(4,5) \rangle < \mathfrak{S}_5$, $H_5 \cong \mathfrak{D}_5$
(ii) $H = H_6 = \langle (1,2)(3,4), (1,5)(2,3), (1,6)(2,4) \rangle < \mathfrak{S}_6$, $H_6 \cong \mathfrak{A}_5$
(iii) $H = H_7 = \langle g_1, g_2, g_3 \rangle < \mathfrak{S}_7$, $H_7 \cong \mathrm{PSL}_2(7) \cong \mathrm{SL}_3(2)$,
(iv) $H = H_8 = \langle g_1, g_2, g_3, g_4 \rangle < \mathfrak{S}_8$, $H_8 \cong \mathrm{AG}_3(2) \cong \mathbb{Z}_2^3 \rtimes \mathrm{SL}_3(2)$
(v) $H = \mathfrak{A}_n < \mathfrak{S}_n$

where

$$g_1 = (1,2)(3,4),\ g_2 = (1,5)(2,6),\ g_3 = (1,3)(5,7),\ g_4 = (1,2)(7,8).$$

Regarding such a permutation group $H < \mathfrak{S}_n$ as a subgroup of SO_n yields a monomial rotation group, which is however not irreducible. Other examples of monomial reducible reflection-rotation groups are the diagonal subgroup $D(n) = D(W(\mathrm{BC}_n))$ of a reflection group of type BC_n and its orientation preserving subgroup $D^+(n) = D(W(\mathrm{D}_n))$. Both groups are normalized by $\mathfrak{S}_n < \mathrm{SO}_n$. Therefore, we obtain a class of examples defined as semidirect products of $D(n)$ and $D^+(n)$, respectively, with a permutation group $H < \mathfrak{S}_n$ as above. We define $M_n = D^+(n) \rtimes H_n$ for $n = 5, \ldots, 8$, $M_n^\times = D(n) \rtimes H_n$ for $n = 5, \ldots, 8$ and $M^\times(\mathrm{D}_n) = D(n) \rtimes \mathfrak{A}_n$. Moreover, we can define the following two exceptional examples of monomial irreducible rotation groups (cf. [Mik84, p. 104])

$$M_7^p = \langle g_1, g_2, g_3, g_5 \rangle < \mathrm{SO}_7,\ M_8^p = \langle g_1, g_2, g_3, g_4, g_5 \rangle < \mathrm{SO}_8,$$

with $g_5 = (1, \overline{2})(3, \overline{4})$ where we write $(i, \overline{j})$ for the linear transformation that maps the basis vectors e_i to $-e_j$ and $-e_j$ to e_i.

We record the following fact that can be checked by a computation. The groups $\mathrm{AG}_3(2)$ and M_7^p are isomorphic and the restriction of the permutation representation of $\mathrm{AG}_3(2)$ described in *(iv)* to $\mathbb{R}^7$ is equivalent to the natural representations of M_7^p on $\mathbb{R}^7$ (cf. [Mik84, p. 104, case II) and III)]).

1.3.5 Nonmonomial imprimitive reflection-rotation groups

The imprimitive unitary reflection groups $G(m,p,n)$ give rise to a family of imprimitive rotation groups (cf. Section 1.3.2). Related families of reflection-rotation groups can be constructed as follows. For a positive integer m and $k = 1, 2$ the groups $W^+(\mathrm{I}_2(m))$ and $W(\mathrm{I}_2(m))$ are normal subgroups of $W(\mathrm{I}_2(km))$ with abelian quotient. Hence,

$$A^*(km,k,n) = \{(g_1, \ldots, g_n) \in W(\mathrm{I}_2(km))^n | (g_1 \cdots g_n) \in W^+(\mathrm{I}_2(m))\}$$

and

$$A^\times(km,k,n) = \{(g_1, \ldots, g_n) \in W(\mathrm{I}_2(km))^n | (g_1 \cdots g_n) \in W(\mathrm{I}_2(m))\}$$

are groups. We define

$$G^*(km,k,n) = A^*(km,k,n) \rtimes \mathfrak{S}_n < \mathrm{SO}_{2n}$$

and

$$G^\times(km,k,l) = A^\times(km,k,l) \rtimes \mathfrak{S}_n < \mathrm{O}_{2n}$$

where the symmetric group $\mathfrak{S}_n$ permutes the components of $A^*(km, k, n)$ and $A^\times(km, k, n)$, respectively. Let s, r be the transformation of $\mathbb{C}^n$ defined by

$$s(z_1, \ldots, z_n) = (\overline{z}_1, z_2, \ldots, z_n), \; r(z_1, \ldots, z_l) = (\overline{z}_1, \overline{z}_2, z_3, \ldots, z_l)$$

Then we have

$$G^*(km, k, n) = \langle G(km, k, n), r\rangle \,, \; G^\times(km, k, n) = \langle G(km, k, n), s\rangle$$

where the complex groups on the right hand sides are regarded as real groups. In particular, the group $G^*(km, k, n)$ is an imprimitive irreducible rotation group and the group $G^\times(km, k, n)$ is an imprimitive irreducible reflection-rotation group for $km \geq 3$ and $k = 1, 2$.

In dimension four, other examples can be constructed in the following way. Let m and k be positive integers and let $\varphi : \mathfrak{D}_k \to \mathfrak{D}_k$ be an involutive automorphism of the dihedral group of order $2k$ that maps reflections onto reflections. The data

$$\{(W(\mathrm{I}_2(km)), W^+(\mathrm{I}_2(m)), W(\mathrm{I}_2(km))\}_{i\in\{1,2\}}$$

together with this automorphism defines a reducible rotation group D (cf. Theorem 4). Since φ has order 2, the rotation that interchanges the two irreducible componets of D normalizes D. We denote the rotation group generated by D and this normalizing rotation by $G^*(km, k, 2)_\varphi$.

1.3.6 Reducible reflection-rotation groups

We say that a reflection-rotation group G is *indecomposable* if it cannot be written as a product of subgroups that act in orthogonal spaces. Every reflection-rotation group splits as a product of indecomposable components. Basic examples for reducible but indecomposable rotation groups are $W^+(\mathrm{A}_1 \times \cdots \times \mathrm{A}_1)$ and the diagonal subgroup $\Delta(W \times W)$ of the product of two copies of an irreducible reflection group $W < \mathrm{O}_n$. The second example preserves a complex structure and coincides with the unitary reflection group of type W considered as a real group. More generally, for an automorphism $\varphi : W \to W$ that maps reflections onto reflections the group

$$\Delta_\varphi(W \times W) = \{(g, \varphi(g)) \in W \times W | g \in W\} < \mathrm{SO}_{2n}$$

is a rotation group. The groups $\Delta_\varphi(W \times W)$ and $\Delta(W \times W)$ are conjugate in SO_{2n}, if and only if the automorphism φ is realizable through conjugation by an element in O_n. This is possible if all labels of the Coxeter diagram of W lie in $\{2, 3, 4, 6\}$ [FH03, Cor. 19, p. 7]. However, reflection groups of type $\mathrm{I}_2(p)$, H_3 and H_4 admit automorphisms that map reflections onto reflections but cannot be realized through conjugation in O_n [Fra01, pp. 31-32]. The exceptional rotation groups arising in this way for W of type H_3 and H_4 do not preserve a complex structure (cf. Section 1.3.2) and are not considered in [Mik84] (the proof of [Mik84, Thm. 1.2, p. 102] does not work in general). General reducible but indecomposable reflection-rotation groups are extensions of the examples from this section by irreducible rotation groups we have described so far (cf. Section 1.6).

1.3.7 Exceptional primitive rotation groups

We have already seen a couple of primitive absolutely irreducible rotation groups. The rotation groups $W^+(\mathrm{A}_n)$, $W^+(\mathrm{E}_6)$, $W^+(\mathrm{E}_7)$, $W^+(\mathrm{E}_8)$, $W^*(\mathrm{A}_5)$ and $W^*(\mathrm{E}_6)$ belong to this class. In this section we describe two other examples.

Lemma 15. *There exists a primitive absolutely irreducible rotation group isomorphic to the alternating group* $\mathfrak{A}_5$*. We denote it as* $R_5(\mathfrak{A}_5) < \mathrm{SO}_5$*.*

Proof. We obtain a faithful linear representation of $\mathfrak{A}_5$ on $\mathbb{R}^5$ by restricting the nontrivial part of the permutation representation of $\mathfrak{S}_6$ to the image of an exceptional embedding $i : \mathfrak{A}_5 < \mathfrak{S}_5 \to \mathfrak{S}_6$. This is the unique absolutely irreducible representation of $\mathfrak{A}_5$ in dimension 5 [CCN+85, p. 2]. Since i maps double transpositions to double transpositions the corresponding linear group $R_5(\mathfrak{A}_5) < \mathrm{SO}_5$ is a rotation group isomorphic to $\mathfrak{A}_5$. The fact that $\mathfrak{A}_5$ is a simple group in combination with the results from Section 1.4.1 implies that $R_5(\mathfrak{A}_5)$ is primitive. □

Lemma 16. *There exists a primitive absolutely irreducible rotation group isomorphic to* $\mathrm{PSL}_2(7)$*. We denote it as* $R_6(\mathrm{PSL}_2(7)) < \mathrm{SO}_6$*. The group* $G = \langle R_6(\mathrm{PSL}_2(7)), -\mathrm{id}\rangle$ *is not a rotation group.*

Proof. The group $\mathrm{PSL}_2(7)$ has a unique faithful and absolutely irreducible representation in dimension 6 [CCN+85, p. 3] and we denote its image by $R_6(\mathrm{PSL}_2(7)) < \mathrm{SO}_6$. It can be obtained by restricting the natural representation of $\mathfrak{S}_7$ on $\mathbb{R}^6$ to a subgroup described in Section 1.3.4, (*iii*). This shows that $R_6(\mathrm{PSL}_2(7))$ is generated by rotations. The fact that $\mathrm{PSL}_2(7)$ is a simple group in combination with the results from Section 1.4.1 implies that $R_6(\mathrm{PSL}_2(7))$ is primitive.

Since the eigenvalues of a cycle $(1, \dots, k)$ regarded as a linear transformation are the k-th roots of unity, the maximal dimension of the -1-eigenspace of a permutation $\sigma \in \mathfrak{S}_7$ acting on $\mathbb{R}^6$ is 3. This shows that all rotations contained in G are also contained in $R_6(\mathrm{PSL}_2(7))$ and hence G is not a rotation group. □

1.3.8 A new primitive rotation group

The group $W(\mathrm{I}_2(4)) < \mathrm{O}_2$ is the image of the natural representation ρ of the dihedral group $\mathfrak{D}_4$ of order 8. Let H be the tensor product of 3 copies of $W(\mathrm{I}_2(4))$, i.e. $H = W(\mathrm{I}_2(4)) \otimes W(\mathrm{I}_2(4)) \otimes W(\mathrm{I}_2(4)) < \mathrm{SO}_8$. In other words the group H is the image of the representation $\rho\circ\pi_1 \otimes \rho\circ\pi_2 \otimes \rho\circ\pi_3$ of $\mathfrak{D}_4 \times \mathfrak{D}_4 \times \mathfrak{D}_4$ where π_i is the projection onto the ith factor. The group H is absolutely irreducible, has order 2^7 and its normalizer $N = N_{\mathrm{SO}_8}(H)$ contains rotations of order 2, e.g. the linear transformations that interchange two $W(\mathrm{I}_2(4))$ factors. We would like to classify primitive rotation groups G with $H < G < N$ as this problem occurs in our classification of rotation groups (cf. Proposition 45).

The images A and $N(A)$ in $\mathrm{Int}(\mathfrak{so}_8)$ of the groups H and N are members of a series of finite subgroups of $\mathrm{Int}(\mathfrak{so}_{2^m})$ studied in connection with gradings of simple Lie algebras. Namely let $H_m < \mathrm{SO}_{2^m}$ be the tensor product of m copies of $W(\mathrm{I}_2(4))$, i.e. $H = H_3$. Then the group $A_m = H_m/\{\pm 1\}$ is a so-called *Jordan subgroup* of $\mathrm{Int}(\mathfrak{so}_{2^m})$ [OVG94, Sect. 3.12]. It is a 2-elementary abelian group of order 2^{2m} and can be considered as a $2m$-dimensional vector space over $\mathbb{F}_2$. A quadratic form on a vector space over $\mathbb{F}_2$ is called nondegenerate, if the bilinear form $f(x,y) = Q(x+y)+Q(x)+Q(y)$ is nondegenerate. Its Witt index is defined to be the maximal dimension of a singular subspace, i.e. a subspace on which the quadratic form vanishes identically (cf. [Die63, Sect. I.16, p. 34]). It is known that the assignment $Q(x) = 0$ or $Q(x) = 1$ for $x = \{\pm h\}$ depending on whether $h^2 = 1$ or $h^2 = -1$ defines a nondegenerate quadratic form of Witt index m, that the natural action of $N(H_m)$ on A_m defines an isomorphism of the group $N(H_m)/H_m$ onto the orthogonal group $O(Q)$ [OVG94, Sect. 3.12, Example 4, p. 126], [Ale74, Ale92] and that the centralizer $C_{\mathrm{SO}_8(H_m)}$ of H_m in SO_8 coincides with its center, i.e. $C_{\mathrm{SO}_8(H_m)} = Z(H_m) = \{\pm 1\}$ [OVG94, Thm. 3.19, (1), p. 126]. The corresponding bilinear form f satisfies

$$[h_1, h_2] = (-1)^{f(\{\pm h_1\},\{\pm h_2\})}$$

and so its nondegeneracy amounts to the fact that $H_m = Z(H_m) = \{\pm 1\}$. A representative of a maximal singular subspace of $H/\{\pm 1\}$ is given by

$$(W(\mathrm{I}_2(2)) \otimes W(\mathrm{I}_2(2)) \otimes W(\mathrm{I}_2(2)))/\{\pm 1\}$$

where $W(\mathrm{I}_2(2)) < W(\mathrm{I}_2(4))$ is a Klein four-group. The preimage of an i-dimensional singular subspace of $H/\{\pm 1\}$ in H is an abelian normal subgroup of H of order 2^{i+1} with respect to which the space $\mathbb{R}^8$ decomposes into an orthogonal sum of 2^{3-i}-dimensional weight spaces. Denote the collection of 2^{3-i}-dimensional subspaces obtained in this way from the i-dimensional singular subspaces of $H/\{\pm 1\}$ by $K_{2^{3-i}}$ and the corresponding collections of involutions whose -1-eigenspaces are the subspaces from $K_{2^{3-i}}$ by $\mathfrak{R}_{2^{3-i}}$. Due to a Witt type theorem in characteristic 2 proved by C. Arf every isomorphism between singular subspaces of $H/\{\pm 1\}$ can be extended to an isometry of $H/\{\pm 1\}$ [Die63, Sect. I.16, p. 36; cf. Sect. I.11, p. 21]. In particular, flags of singular subspaces of $H/\{\pm 1\}$ with the same signature are $O(Q)$-equivalent. As a direct consequence we obtain

Lemma 17. *The group N acts transitively on flags of subspaces $U_1 < U_2 < U_4$ with $U_j \in K_j$, $j = 1, 2, 4$. For $i = 1, 2, 3$ the H-orbits in $K_{2^{3-i}}$ have order 2^i and are in one-to-one correspondence with the i-dimensional singular subspaces of $H/\{\pm 1\}$. In particular, every element of $K_{2^{3-i}}$ uniquely determines an i-dimensional singular subspace of $H/\{\pm 1\}$.*

Proof. Since the preimage of a singular subspace of $H/\{\pm 1\}$ in H is a normal abelian subgroup, its weight spaces are permuted by H. The group H being irreducible implies that these weight spaces are transitively permuted by H. We have already seen that the group N acts transitively on maximal flags of singular subspaces of $H/\{\pm 1\}$. Therefore, the transitivity of the action of N on flags of subspaces $U_1 < U_2 < U_4$ with $U_j \in K_j$, $j = 1, 2, 4$, follows from the fact that the actions of H_m on $\mathbb{R}^{2^m}$, $m = 1, 2, 3$, are irreducible (see the beginning of the section for the definition of H_m). □

The rotations in $\mathfrak{R}_2$ are natural candidates for rotations in N. It is straightforward to check that some rotation in $\mathfrak{R}_2$ normalizes the group H and thus all of them do by the preceding lemma. In fact, the two-dimensional subspace in K_2 obtained as the intersection of the four-dimensional fixed point subspaces of $r \otimes r \otimes \mathrm{id}$, $s_{e_1-e_2} \otimes s_{e_1+e_2} \otimes \mathrm{id} \in H$, where $r \in W(\mathrm{I}_2(4)) <$

O_2 is a rotation about $\pi/2$, $s_v \in W(\mathrm{I}_2(4)) < \mathrm{O}_2$ is a reflection in the hyperplane $v^\perp$ and $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$ for which the action of $W(\mathrm{I}_2(4))$ is monomial, defines a rotation that interchanges the first two factors of $H = W(\mathrm{I}_2(4)) \otimes W(\mathrm{I}_2(4)) \otimes W(\mathrm{I}_2(4))$ and hence normalizes H.

Definition 3. The group L is defined to be the rotation group generated by $\mathfrak{R}_2$.

Since every involution in $\mathfrak{R}_4$ is a product of two involutions in $\mathfrak{R}_2$ we have $H = \langle \mathfrak{R}_4 \rangle < L$. Our next aim is to show that the rotation group L is primitive. To this end we make the following observations.

For a bivector in $\bigwedge^2 \mathbb{F}_2^4$ we divide its exterior square computed over $\mathbb{Z}$ by 2 and consider the result modulo 2. This assignment defines a quadratic form Q' with the wedge product $\bigwedge : \bigwedge^2 \mathbb{F}_2^4 \times \bigwedge^2 \mathbb{F}_2^4 \to \mathbb{F}_2$ as associated bilinear form. Hence, the form Q' is nondegenerate and has Witt index 3, a three-dimensional singular subspace being $U \wedge \mathbb{F}_2^4$ for a one-dimensional subspace $U \subset \mathbb{F}_2^4$. It follows that there exists an isomorphism between $(H/\{\pm 1\}, Q)$ and $(\bigwedge^2 \mathbb{F}_2^4, Q')$ [Die63, Sect. I.16, p. 34]. We identify $H/\{\pm 1\}$ and $\bigwedge^2 \mathbb{F}^4$ via such an isomorphism. Under this identification elements $\{\pm h\} \in H/\{\pm 1\}$ with $Q(\{\pm h\}) = 0$ correspond to simple bivectors in $\bigwedge^2 \mathbb{F}^4$ and thus to two-dimensional subspaces of $\mathbb{F}^4$. The group $O(Q)$ is isomorphic to the symmetric group $\mathfrak{S}_8$ and the group $\mathrm{SL}_4(2)$ is isometric to the alternating group $\mathfrak{A}_8$ [CCN+85, p. 22]. Hence, we can consider the group $\mathrm{SL}_4(2)$ as an index 2 subgroup of $O(Q)$. Maximal singular subspaces of $H/\{\pm 1\}$ of the form $U \wedge \mathbb{F}_2^4$ for a one-dimensional subspace $U \subset \mathbb{F}_2^4$ are not $\mathrm{SL}_4(2)$-equivalent to maximal singular subspaces of the form $U \wedge U$ for a three-dimensional subspace $U \subset \mathbb{F}_2^4$. For, a subspace of the first type can be annihilate by wedging with a one-dimensional subspace of $\mathbb{F}_2^4$ whereas a subspace of the second type cannot. Hence, the set of maximal singular subspaces of $H/\{\pm 1\}$ decomposes into two $\mathrm{SL}_4(2)$-orbits represented by these two types of subspaces (recall that $O(Q)$ acts transitively on maximal singular subspaces). Each orbit conjoint with a trivial element inherits a vector space structure over $\mathbb{F}_2$ from $\mathbb{F}_2^4$. Three different maximal singular subspaces belong to a two-dimensional subspace of this vector space, if and only if they intersect in a one-dimensional singular subspace. Hence, $H/\{\pm 1\}$ contains 30 maximal singular subspaces. Since, every maximal singular subspace contains 7 one-dimensional (singular) subspaces and every one-dimensional singular subspace is contained in 3 maximal singular subspaces from each $\mathrm{SL}_4(2)$-orbit, we see that $H/\{\pm 1\}$ contains 35 one-dimensional singular subspaces. Moreover, every two-dimensional singular subspace is contained in precisely one maximal singular subspace from each $\mathrm{SL}_4(2)$-orbit.

A linear transformation f of a vector space V is called a *transvection*, if there exist $e \in V \backslash \{0\}$ and a nontrivial linear form α on V with $\alpha(e) = 0$ such that $f(v) = v + \alpha(v)e$ for all $v \in V$.

Lemma 18. *The rotations in* $\mathfrak{R}_2$ *project to transvections in* $\mathrm{SL}_4(2)$.

Proof. We identify $V = \mathbb{F}_2^4$ with the set of maximal singular subspaces of $H/\{\pm 1\}$ of the form $U \wedge \mathbb{F}_2^4$, $U \subset \mathbb{F}_2^4$ being one-dimensional, conjoint with a trivial element. Under these identifications an element $\{\pm h\} \in H/\{\pm 1\}$ with $Q(\{\pm h\}) = 0$ belongs to a maximal singular subspace W of $H/\{\pm 1\}$, if and only if $W \in V$ belongs to the two-dimensional subspace of V

defined by the bi-vector $\{\pm h\} \in \bigwedge^2 V$. For a rotation $r \in \mathfrak{R}_2$ let W_2 be the two-dimensional singular subspace of $H/\{\pm 1\}$ determined by r (cf. Lemma 17) and let W_3 be the unique maximal singular subspace of $H/\{\pm 1\}$ with $W_2 \subset W_3 \in V$. The union $U = \bigcup_{\sigma \in W_2 \setminus \{0\}} \sigma \subset V$, where the nontrivial elements $\sigma \in W_2$ are regarded as two-dimensional subspaces of V via the identification $H/\{\pm 1\} = \bigwedge^2 V$, is a three-dimensional subspace of V. Indeed, for $v_1 \in \sigma_1 \in W_2 \setminus \{0\}$ and $v_2 \in \sigma_2 \in W_2 \setminus \{0\}$ with $v_1 \wedge v_2 \notin W_2$ (otherwise $v_1 + v_2 \in v_1 \wedge v_2 \in W_2$) we have $(v_1 + v_2) \wedge (\sigma_1 + \sigma_2) = v_2 \wedge \sigma_1 + v_1 \wedge \sigma_2 = 0$ because of $\sigma_1 \wedge \sigma_2 = 0$ and thus $v_1 + v_2 \in \sigma_1 + \sigma_2 \subset U$. For a maximal singular subspace $W \in U$ there exists (by definition) some nontrivial $h \in H$ with $\{\pm h\} \in W \cap W_2$ (for $W \neq W_3$ take $\{\pm h\} = W \wedge W_3$). We can assume that $\mathrm{Fix}(h) \subset \mathrm{Fix}(r)$ (otherwise we take $-h$ instead of h). Therefore r leaves some weight space corresponding to W invariant. Since H acts transitively on the weight spaces corresponding to W, the rotation r permutes them and hence fixes W. This means that r acts trivially on U. In particular, it leaves the set of maximal singular subspaces of $H/\{\pm 1\}$ corresponding to V invariant and thus projects to $\mathrm{SL}_4(2)$. Let $W' \in V \setminus U$. If r would be the identity on V and thus on $H/\{\pm 1\} = \bigwedge^2 V$, we had $r \in C_{\mathrm{SO}_8(H)} = \{\pm 1\}$ (cf. [OVG94, Thm. 3.19, (1), p. 126]), a contradiction. Hence, $\overline{W} = W' + r(W')$ is nontrivial and lies in U since $r^2 = 1$ implies $r(\overline{W}) = \overline{W}$. Therefore, for any $W'' \in V \setminus U$ we have $W'' + W' \in U$ and thus $r(W'') = r(W'' + W') + r(W') = W'' + \overline{W}$. Consequently, the rotation r acts on V like the transvection defined by $e = \overline{W} \in U$ and the linear form α corresponding to the three-dimensional subspace $U \subset V$. □

Now we can show

Lemma 19. *The group L projects onto $\mathrm{SL}_4(2)$. The set $\mathfrak{R}_2$ has order 420.*

Proof. Since all transvections in $\mathrm{SL}_4(2)$ are conjugate and generate $\mathrm{SL}_4(2)$ (cf. [Die63, Sect. II.1, p. 37]) we see that $L = \langle \mathfrak{R}_2 \rangle$ maps onto $\mathrm{SL}_4(2)$ by the preceding lemma. The group $\mathrm{SL}_4(2)$ contains $(2^4 - 1)(2^3 - 1) = 105$ transvections and the H-orbit of any rotation in $\mathfrak{R}_2$ contains 4 rotations by Lemma 17. We conclude that the set $\mathfrak{R}_2$ has order 420. □

As a consequence we obtain

Proposition 20. *The rotation group $L < \mathrm{SO}_8$ is primitive.*

Proof. Since the group $\mathrm{SL}_4(2)$ is simple [CCN+85, p. 22], all normal subgroups of L are contained in H. Assume that L is an imprimitive group and let $\mathbb{R}^8 = V_1 \oplus \cdots \oplus V_k$, $k \in \{4, 8\}$, be a hypothetical decomposition into subspaces that are permuted by L. The diagonal subgroup $D = D(L)$ with respect to this system of imprimitivity is normal in L and satisfies $|D| \geq |L/k!|$, because L/D embeds into the symmetric group $\mathfrak{S}_k$. Because of $D(L) < H$ we must have $k = 8$ and thus $D(L)$ is abelian and has order at least 64. However, H does not contain abelian subgroups of index 2 and thus the claim follows. □

Our next aim is to show that every rotation in N is contained in $\mathfrak{R}_2$. To this end we first construct certain rotations in $\mathfrak{R}_2$ that are needed in the proof.

Lemma 21. *The span of any two distinct one-dimensional weight spaces corresponding to a maximal singular subspace of $H/\{\pm 1\}$ is contained in K_2, i.e. it occurs as a weight space of a two-dimensional singular subspace of $H/\{\pm 1\}$ and corresponds to a rotation in $\mathfrak{R}_2$.*

Proof. Let $H_0 < H$ be the subgroup corresponding to a maximal singular subspace of $H/\{\pm 1\}$ and let $v_1, v_2 \in \mathbb{R}^8$ be unit vectors spanning two different weight spaces of H_0. One only needs to find $h_1, h_2 \in H_0$ that project onto linearly independent elements in $H/\{\pm 1\}$ such that $v_1, v_2 \in \mathrm{Fix}(h_1) \cap \mathrm{Fix}(h_2)$. The subgroup H_{v_1} of H_0 that fixes v_1 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. A short case differentiation shows that it is always possible to find a subgroup $H_{v_2} < H_{v_1}$ that also fixes v_2. $\square$

Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ and let $\{\varepsilon_i | i \in \{1, \ldots, 8\}\} = \{e_i \otimes e_j \otimes e_k | i, j, k \in \{1, 2\}\}$ be the induced basis of $\mathbb{R}^8 = \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ ordered lexicographically. We set $V_1 := \langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle$, $\sigma_1 := \langle \varepsilon_1, \varepsilon_2 \rangle$ and $\sigma_2 := \langle \varepsilon_3, \varepsilon_4 \rangle$. Clearly, we have $V_1 \in K_4$ and $\sigma_1, \sigma_2 \in K_2$. For a reflection $s \in N_{\mathrm{O}_2}(W(\mathrm{I}_2(4))) \backslash W(\mathrm{I}_2(4))$ with $s(\sqrt{2} e_1) = e_1 + e_2$ we set $\nu_2 = \mathrm{id} \otimes s \otimes \mathrm{id} \in N$ and $\nu_3 = \mathrm{id} \otimes \mathrm{id} \otimes s \in N$. Then with

$$\alpha_1 := \nu_2(\varepsilon_1) = \varepsilon_1 + \varepsilon_3,\ \ \alpha_2 := \nu_2(\varepsilon_2) = \varepsilon_2 + \varepsilon_4,$$

and

$$\beta_1 := \nu_3(\varepsilon_1) = \varepsilon_1 + \varepsilon_2,\ \ \beta_2 := \nu_3(\varepsilon_5) = \varepsilon_5 + \varepsilon_6$$

we have $\langle \alpha_1 \rangle, \langle \alpha_2 \rangle, \langle \beta_1 \rangle, \langle \beta_2 \rangle \in K_1$ because of $\nu_2, \nu_3 \in N$, and $\sigma = \langle \alpha_1, \alpha_2 \rangle, \tau_1 = \langle \varepsilon_1, \varepsilon_5 \rangle, \tau_2 = \langle \beta_1, \beta_2 \rangle \in K_2$ by Lemma 21. Let $r, r_1, r_2 \in \mathfrak{R}_2$ be the rotations corresponding to σ, τ_1, and τ_2, respectively, and let $R = \langle r_1, r_2 \rangle < N$ be the group generated by r_1 and r_2. The group R is isomorphic to a dihedral group of order 8 and leaves σ_1 and σ_2 invariant. The rotation r interchanges σ_1 and σ_2 and thus so do the conjugates of r under the group R. Hence, there are at least 8 different rotations in $\mathfrak{R}_2$ that interchange σ_1 and σ_2.

Lemma 22. *Every rotation in N is contained in $\mathfrak{R}_2$.*

Proof. Let $g \in N$ be a rotation. Then there exists a one-dimensional singular subspace of $H/\{\pm 1\}$ spanned by some $\{\pm h\}$ with $g\{\pm h\}g^{-1} \neq \{\pm h\}$, because of $H = \langle \mathrm{R}_4 \rangle$ and $C_{\mathrm{SO}_8}(H) = \{\pm 1\}$. We set $h' = ghg^{-1}$. Since g is a rotation the intersection $\mathrm{Fix}(h) \cap \mathrm{Fix}(h')$ is nontrivial and so the fact that $h \neq h'$ implies that hh' has order 2, i.e. $Q(\{\pm hh'\}) = 0$. Hence, the group $H_0 = \langle h, h' \rangle$ projects onto a two-dimensional singular subspace of $H/\{\pm 1\}$ that is contained in a maximal singular subspace W. Two of the four weight spaces of H_0 are pointwise fixed by g, the other two are interchanged by g. In particular, the rotation g has order 2. The one-dimensional weight spaces defined by W are contained in the two-dimensional weight spaces of H_0. Since H acts transitively on the weight spaces corresponding to W and since g fixes one of them, it permutes the others. Due to the transitivity statement in Lemma 17 we can assume that $g\sigma_1 = \sigma_2$ and that the weight spaces corresponding to W are spanned by $\varepsilon_1, \ldots, \varepsilon_8$. There are only 8 rotations in SO_8 with these properties and we have seen above that all of them are contained in $\mathfrak{R}_2$. Hence the claim follows. $\square$

Now we show that the group L is the only primitive rotation group G with $H < G < L$. We need the following lemma. Recall that the action of L on $H/\{\pm 1\} = \bigwedge^2 \mathbb{F}_2^4$ descends to an action on $\mathbb{F}_2^4$.

Lemma 23. *Let $G < L$ be a rotation group and suppose that G leaves a symplectic form on $\mathbb{F}_2^4$ invariant. Then the group H is not contained in G.*

Proof. Suppose $G < L$ is a rotation group that leaves a symplectic form B on $\mathbb{F}_2^4$ invariant. The form B defines a nontrivial G-invariant linear form β on $\bigwedge^2 \mathbb{F}_2^4$. By duality with respect to the nondegenerate bilinear form $\bigwedge : \bigwedge^2 \mathbb{F}_2^4 \times \bigwedge^2 \mathbb{F}_2^4 \to \mathbb{F}_2$, the form β in turn gives rise to a nontrivial G-invariant bivector $b \in \bigwedge^2 \mathbb{F}_2^4$ that corresponds to a G-invariant coset $\{\pm h\} \in H/\{\pm 1\}$ for some nontrivial element $h \in H$. The two (possibly complex) four-dimensional eigenspaces of h corresponding to different eigenvalues cannot be permuted by a rotation. Hence, the group G, being generated by rotations, not only fixes $\{\pm h\}$ but also h. Since the center of H only consists of $\{\pm 1\}$, the group H cannot be completely contained in the group G. $\square$

Now we can prove

Lemma 24. *The only primitive rotation group G with $H < G < N$ is the group L.*

Proof. Let $G < N$ be a rotation group. By Lemma 22 we have $G < L$ and thus we can consider the action of G on $\mathbb{F}_2^4$. If there exists a one- or three-dimensional G-invariant subspace U of $\mathbb{F}_2^4$, then G leaves a maximal singular subspace of $H/\{\pm 1\} = \bigwedge^2 \mathbb{F}_2^4$ invariant (either $U \wedge \mathbb{F}_2^4$ or $U \wedge U$ depending on whether U is one- or three-dimensional). The corresponding collection of weight spaces defines a system of imprimitivity of G and thus G is imprimitive in this case. If there exists a two-dimensional invariant subspace of $\mathbb{F}_2^4$, then the group G fixes a one-dimensional singular subspace of $H/\{\pm 1\}$ spanned by some $\{\pm h\}$. Again, since the group G is generated by rotations it normalizes h (cf. proof of Lemma 23) and is thus reducible. Otherwise, the group G acts irreducibly on $\mathbb{F}_2^4$. Since its image in $\mathrm{SL}_4(2)$ is generated by transvections (cf. Lemma 22 and Lemma 19), it either preserves a symplectic form on $\mathbb{F}_2^4$ or we have $G = L$ [McL69, p. 108]. In the first case the group H is not contained in G by Lemma 23 and thus the claim follows. $\square$

Finally, we explain how the rotation group L is connected to a reflection group of type E_8. Recall that we set $V_1 := \langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle$. A computation shows that the subspaces in K_2 that lie in V_1 are given as follows

$$
\begin{array}{lllll}
\sigma_1 = \langle \alpha_1, \alpha_2 \rangle & \sigma_2 = \langle \alpha_1, \alpha_3 \rangle & \sigma_3 = \langle \alpha_1, \alpha_4 \rangle & \sigma_4 = \langle \alpha_2, \alpha_3 \rangle & \sigma_5 = \langle \alpha_2, \alpha_4 \rangle \\
\sigma_6 = \langle \alpha_3, \alpha_4 \rangle & \sigma_7 = \langle \alpha_{13}, \alpha_{15} \rangle & \sigma_8 = \langle \alpha_{13}, \alpha_{16} \rangle & \sigma_9 = \langle \alpha_{14}, \alpha_{15} \rangle & \sigma_{10} = \langle \alpha_{14}, \alpha_{16} \rangle \\
\sigma_{11} = \langle \alpha_{21}, \alpha_{23} \rangle & \sigma_{12} = \langle \alpha_{21}, \alpha_{24} \rangle & \sigma_{13} = \langle \alpha_{22}, \alpha_{23} \rangle & \sigma_{14} = \langle \alpha_{22}, \alpha_{24} \rangle & \sigma_{15} = \langle \alpha_{17}, \alpha_{19} \rangle \\
\sigma_{16} = \langle \alpha_{17}, \alpha_{20} \rangle & \sigma_{17} = \langle \alpha_{18}, \alpha_{19} \rangle & \sigma_{18} = \langle \alpha_{18}, \alpha_{20} \rangle & &
\end{array}
$$

and that their mutual intersections are spanned by the following vectors

$$
\begin{array}{lll}
\alpha_1 = \varepsilon_1, & \alpha_5 = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2, & \alpha_9 = (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2, \\
\alpha_2 = \varepsilon_2, & \alpha_6 = (\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2, & \alpha_{10} = (\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2, \\
\alpha_3 = \varepsilon_3, & \alpha_7 = (\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2, & \alpha_{11} = (\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4)/2, \\
\alpha_4 = \varepsilon_4, & \alpha_8 = (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)/2, & \alpha_{12} = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2,
\end{array}
$$

$$
\begin{array}{lll}
\alpha_{13} = \varepsilon_1 + \varepsilon_2, & \alpha_{17} = \varepsilon_1 + \varepsilon_4, & \alpha_{21} = \varepsilon_1 + \varepsilon_3, \\
\alpha_{14} = \varepsilon_1 - \varepsilon_2, & \alpha_{18} = \varepsilon_1 - \varepsilon_4, & \alpha_{22} = \varepsilon_1 - \varepsilon_3, \\
\alpha_{15} = \varepsilon_3 + \varepsilon_4, & \alpha_{19} = \varepsilon_2 + \varepsilon_3, & \alpha_{23} = \varepsilon_2 + \varepsilon_4, \\
\alpha_{16} = \varepsilon_3 - \varepsilon_4, & \alpha_{20} = \varepsilon_2 - \varepsilon_3, & \alpha_{24} = \varepsilon_2 - \varepsilon_4.
\end{array}
$$

Note that the vectors $\{\pm\alpha_i\}_{i=1,\dots,24}$ form a root system of type F_4 with 24 short roots, $\pm\alpha_1,\dots,\pm\alpha_{12}$, and 24 long roots, $\pm\alpha_{13},\dots,\pm\alpha_{24}$. Let R_1 and R_2 be given by the sets of vectors

$$\pm\varepsilon_i \pm \varepsilon_j \ (i<j),\ \frac{1}{2}\sum_{i=1}^{8} \pm\varepsilon_i \text{ (even number of + signs)}$$

and

$$\pm\varepsilon_i,\ (\pm\varepsilon_i \pm \varepsilon_{i+1} \pm \varepsilon_j \pm \varepsilon_{j+1})/2,\ i \neq j,\ i,j \in \{1,3,5,7\},$$
$$(\pm\varepsilon_i \pm \varepsilon_j \pm \varepsilon_k \pm \varepsilon_l)/2,\ i \in \{1,2\},\ j \in \{3,4\},\ k \in \{5,6\},\ l \in \{7,8\},$$
$$i+j+k+l \equiv 0 \bmod 2$$

respectively. Then R_1 and R_2 are root systems of type E_8 permuted by an involution of N. Moreover, the following holds

Lemma 25. *The elements of R_1 and R_2 span the subspaces in K_1 corresponding to the two orbits of the action of L. In particular, we have $L = \langle \mathfrak{R} \rangle < W(R_1) \cap W(R_2)$. A two-dimensional subspace belongs to K_2 if and only if its intersection with $R_1 \cup R_2$ is a root system of type* $\mathrm{I}_2(4)$*. A four-dimensional subspace belongs to K_4 if and only if its intersection with $R_1 \cup R_2$ is a root system of type* F_4*.*

Proof. We have $\alpha_{13} = \epsilon_1 + \epsilon_2 \in R_1$ and $\alpha_1 = \epsilon_1 \in R_2$, and $\langle \alpha_1 \rangle, \langle \alpha_{13} \rangle \in K_1$ lie in different orbits of the action of L. Hence, the first claim follows, since the set K_1 has order $8 \cdot 30 = 240$. The *only if* direction of the two other claims holds by transitivity (cf. Lemma 17) and our computation above. Suppose that V is a two-dimensional subspace such that $R = V \cap (R_1 \cup R_2)$ is a root system of type $\mathrm{I}_2(4)$. By transitivity we can assume that one short root in R is given by ε_1. From the list of roots in R_1 and R_2 we see that another short root in R must be of the form ε_i for some $i = 2,\dots,8$ and thus the second claim follows from Lemma 21. Now suppose that V is a four-dimensional subspace such that $R = V \cap (R_1 \cup R_2)$ is a root system of type F_4. Again by transitivity and the list of roots in R_1 and R_2 we can assume that 4 pairwise orthogonal short roots $\beta_1,\dots,\beta_4 \in R$ lie in $\{\varepsilon_i\}_{i=1,\dots,8}$. Moreover, using the rotations in $\mathfrak{R}_2$ that we have already described, it is easy to check that we can assume $\beta_i = \varepsilon_i$, $i = 1,\dots,4$ after further conjugations. But $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle \in K_4$ and thus the claim follows. ☐

Now we can show

Proposition 26. *The group L is the intersection of two reflection groups of type* E_8 *permuted by N. More precisely, $L = W(R_1) \cap W(R_2)$.*

Proof. Since $W(R_1) \cap W(R_2)$ leaves K_4 invariant, it normalizes H and thus we have $L < W(R_1) \cap W(R_2) < N$. An element $g \in N \backslash L$ satisfies $g(R_1) = R_2$ and is therefore not contained in $W(R_1) \cap W(R_2)$. Hence the claim follows as L has index 2 in N. ☐

1.3.9 Properties of exceptional rotation groups

The monomial rotation group M_8 contains the group H from the preceding section as a normal subgroup and is thus itself a subgroup of the rotation group L. In fact, every subgroup of L which is maximal among subgroups fixing one of the systems of imprimitivity of H described above is of this type. Therefore the group L also contains the rotation groups $R_6(\mathrm{PSL}_2(7))$, M_7, M_7^p and M_8^p as subgroups (cf. Section 1.3.4). In this section we record some of their properties that are related to properties of the corresponding quotient spaces.

Lemma 27. *The rotation groups listed in Theorem 1, (v) only contain rotations of order* 2.

Proof. For the groups $R_5(\mathfrak{A}_5)$, M_5 and M_6 the claim can be readily checked. For the group L it follows from Lemma 22 and thus it also holds for its subgroups $R_6(\mathrm{PSL}_2(7))$, M_7, M_8, M_7^p and M_8^p. $\square$

We denote the plane systems defined by M_5, M_6, M_7, M_8, M_7^p, M_8^p, $R_5(\mathfrak{A}_5)$, $R_6(\mathrm{PSL}_2(7))$ and L as P_5, P_6, P_7, P_8, Q_7, Q_8, R_5, S_6 and T_8 and the corresponding rotation group by M, e.g. $L = M(\mathrm{T}_8)$.

Lemma 28. *All isotropy groups of the rotation groups $R_5(\mathfrak{A}_5)$, $R_6(\mathrm{PSL}_2(7))$, M_7^p, M_8^p and L are rotation groups.*

Proof. Let G be one of the groups listed above. The claim follows if one can show that each element $g \in G$ can be written as a composition of rotations in G whose fixed point subspace contains the fixed point subspace of g. It suffices to check this property for one representative in each conjugacy class of G. For the listed groups this can be easily verified with a computer algebra system like GAP. $\square$

Lemma 29. *The rotation group $M(\mathrm{S}_6) = R_6(\mathrm{PSL}_2(7))$ of order $168 = 2^3 \cdot 3 \cdot 7$ contains a rotation group isomorphic to $\mathfrak{S}_4$.*

Proof. The double transpositions $(1,7)(3,5)$, $(1,5)(3,7)$ and $(1,4)(6,7)$ generate a subgroup of the rotation group $R_6(\mathrm{PSL}_2(7)) < \mathfrak{S}_7 < \mathrm{SO}_6$ (cf. Lemma 16) isomorphic to $\mathfrak{S}_4$. $\square$

Lemma 30. *The rotation group $M(\mathrm{Q}_7) = M_7^p$ of order $1344 = 2^6 \cdot 3 \cdot 7$ contains rotation groups of order $192 = 2^6 \cdot 3$ and $168 = 2^3 \cdot 3 \cdot 7$.*

Proof. The rotation group generated by $(1,\overline{3})(2,\overline{4})$, $(2,4)(5,7)$, $(2,3)(6,7)$ and $(3,\overline{4})(5,\overline{6})$ is a reducible subgroup of M_7^p (cf. Section 1.3.4) of type

$$\{(W(\mathrm{D}_4), D(W(\mathrm{D}_4)), W(\mathrm{D}_4), \Gamma(\mathrm{A}_3)), (W(\mathrm{A}_3), W^+(\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1), W(\mathrm{A}_3), \circ - \circ)\}.$$

Moreover, the rotation group $R_6(\mathrm{PSL}_2(7))$ of order 168 is contained in M_7^p (cf. Lemma 16 and Section 1.3.4). $\square$

Lemma 31. *The rotation group $M(\mathrm{Q}_8) = M_8^p$ of order $2^{10} \cdot 3 \cdot 7$ contains a reducible rotation group G of order $2^9 \cdot 3$ with $k = 2$ and*

$$(G_i, H_i, F_i, G_i/H_i) = (W(\mathrm{D}_4), D(W(\mathrm{D}_4)), W(\mathrm{D}_4), W(\mathrm{A}_3)),$$

$i = 1, 2$ *(cf. Theorem 3), which is normalized by an element h of order 2 that interchanges the irreducible components of G. Moreover, it contains the rotation group $R_6(\mathrm{PSL}_2(7))$ of order $2^3 \cdot 3 \cdot 7$.*

Proof. The rotations $(1, \overline{5})(4, \overline{8})$, $(1, 6)(3, 8)$, $(2, \overline{5})(3, \overline{8})$, $(3, 7)(4, 8)$ and $(3, 4)(5, 6)$ generate a subgroup G of $M_8^p < \mathfrak{S}_8$ (cf. Section 1.3.4) that leaves the subspace $\langle \varepsilon_1, \varepsilon_2, \varepsilon_5, \varepsilon_6 \rangle$ and its orthogonal complement invariant. In fact, it is a reducible rotation group of type $(G_i, H_i, F_i, G_i/H_i) = (W(\mathrm{D}_4), D(W(\mathrm{D}_4)), W(\mathrm{D}_4), W(\mathrm{A}_3))$, $i = 1, 2$. The involution $h = (1, 8)(2, 7)(3, 6)(4, 5)$ is contained in M_8^p, normalizes the group G and interchanges its two irreducible subspaces. The rotation group $R_6(\mathrm{PSL}_2(7))$ is contained in M_8^p as well (cf. Lemma 16 and Section 1.3.4). □

Lemma 32. *The rotation group $M(\mathrm{T}_8) = L$ of order $2^{13} \cdot 3^2 \cdot 5 \cdot 7$ contains the rotation group M_8 of order $2^{13} \cdot 3 \cdot 7$ and unitary reflection groups $W(\mathcal{F}_4)$ and $W(\mathcal{N}_4)$ of order $2^7 \cdot 3^2$ and $2^9 \cdot 3 \cdot 5$, respectively.*

Proof. It follows from the description of a *line system* of type $\mathcal{O}_4$ ([LT09, Sect. 6.2, p. 109], cf. Section 1.3.2) and the remark preceding Proposition 26 that L contains a unitary reflection group of type $W(\mathcal{O}_4)$, which itself contains unitary reflection groups of type $W(\mathcal{F}_4)$ and $W(\mathcal{N}_4)$ [LT09, Sect. 6.2, p. 109]. □

1.4 Irreducible rotation groups

In this section we prove the classification of irreducible rotation groups, i.e. Theorem 1. Let $G < \mathrm{SO}_n$ be an irreducible rotation group. If the complexification of G is reducible, then G is an irreducible unitary reflection group that is not the complexification of a real reflection group, considered as a real group by Lemma 11 and Lemma 12. Hence, we are in case (*i*) if $n = 2$ or in case (*ii*) or (*iii*) if $n > 2$ of Theorem 1. Otherwise G is absolutely irreducible. The classification of imprimitive absolutely irreducible rotation groups and primitive absolutely irreducible rotation groups in dimensions $n \geq 5$ is treated separately in the following two sections. Together with the classification in dimension $n \leq 4$ treated in Section 1.3.3, the results of these sections form a complete proof of Theorem 1.

1.4.1 Imprimitive rotation groups

For a finite imprimitive group G we can always assume that the subspaces $V_1, \ldots, V_l$ constituting a system of imprimitivity for G are orthogonal and that $G < \mathrm{SO}_n$. If G is moreover an irreducible rotation group, then it acts transitively on the set of these subspaces and thus all of them have the same dimension, either one or two. In the first case the group is called *monomial.* The classification of absolutely irreducible monomial and nonmonomial imprimitive rotation groups is treated separately in the following two paragraphs.

Monomial rotation groups. Assume that G is monomial. Since it is also orthogonal, each row and each column of any element of G contains precisely one element from $\{\pm 1\}$. In particular, G is contained in a reflection group of type BC_n. Therefore, we obtain a homomorphism from G to the symmetric group $\mathfrak{S}_n$ with the diagonal matrices D of G as kernel. Its image isomorphic to G/D is a transitive subgroup of $\mathfrak{S}_n$ generated by transpositions, double transpositions and 3-cycles. Such groups are classified in [Huf80, Thm. 2.1, p. 500].

Theorem 33. *Let H be a transitive permutation subgroup of $\mathfrak{S}_n$ generated by a set of transpositions, double transpositions and 3-cycles such that H does not admit a two-dimensional system of imprimitivity, i.e. a partition of $\{1,\dots,n\}$ into subsets of order two that are interchanged by H. Then, up to conjugation, H is one of the following groups.*

(i) $\mathfrak{S}_n$
(ii) $\mathfrak{A}_n$
(iii) $H_5 = \langle(1,2)(3,4),(2,3)(4,5)\rangle < \mathfrak{S}_5,\ H_5 \cong \mathfrak{D}_5$
(iv) $H_6 = \langle(1,2)(3,4),(2,3)(4,5),(3,4)(5,6)\rangle < \mathfrak{S}_6,\ H_6 \cong \mathfrak{A}_5$
(v) $H_7 = \langle(1,2)(3,4),(1,3)(5,6),(1,5)(2,7)\rangle < \mathfrak{S}_7,\ H_7 \cong \mathrm{PSL}_2(7) \cong \mathrm{SL}_3(2)$
(vi) $H_8 = \langle H_7,(5,6)(7,8)\rangle < \mathfrak{S}_8,\ H_8 \cong \mathrm{AG}_3(2) \cong \mathbb{Z}_2^3 \rtimes \mathrm{SL}_3(2)$.

For each permutation group H described above there exists a monomial rotation group $M < \mathrm{SO}_n$ whose diagonal subgroup $D = D^+(n)$ contains all linear transformations that change the sign of an even number of coordinates (cf. Section 1.3.4) such that $M/D \cong H$. Except in case (i) this is a semidirect product of the permutation group H with $D^+(n)$.

In [Huf80, Table I-III, p. 503] Huffman classifies irreducible monomial groups over the complex numbers that are generated by transformations with an eigenspace of codimension two. These tables contain all complexified monomial absolutely irreducible rotation groups. Together with [Mik78, p. 90] they imply the following result, where we write $(i,\bar{j})$ for the linear transformation that maps e_i to $-e_j$, $-e_j$ to e_i and all other standard basis vectors to itself.

Proposition 34. *Let $G < \mathrm{SO}_n$ be a monomial absolutely irreducible rotation group that does not admit a two-dimensional system of imprimitivity. Then, up to conjugation, G is one of the following groups*

(i) $M(\mathrm{P}_n) = M_n = D^+(n) \rtimes H_n$*, $n = 5,6,7,8$, for H_n as in Theorem 33.*
(ii) $M(\mathrm{Q}_7) = M_7^p < M_7$ *and* $M(\mathrm{Q}_8) = M_8^p < M_8$ *as in Section 1.3.4. These groups are extensions of* $\mathrm{PSL}_2(7)$ *by a group of order* 2^3 *and* 2^7*, respectively.*
(iii) An orientation preserving subgroup W^+ of a reflection group W of type BC_n *or* D_n*.*

For $n > 4$ this result follows from [Huf75, Table I-III, p. 503], since all other complexified real groups occurring in these tables are either reducible (Group 1, $e = g = \alpha = 1$, in Table I, Group 1, $e = g = \alpha = 1$, in Table II, the first groups for $G = \mathrm{AG}_3(2)$ and $G = \mathrm{PSL}_2(7)$ with $e = g = \alpha = 1$ in Table II, the second group for $G = \mathfrak{A}_5$ with $e = g = 1$, $\alpha = -1$ in Table III and Group $G = \mathfrak{D}_5$, $e = g = h = 1$ in Table III), conjugate to a group described in the proposition above (the conjugacy class in SO_8 of the second group for $G = \mathrm{AG}_3(2)$, $e = g = 1$, in Table III is independent of the choice of $\alpha \in \{\pm 1\}$) or conjugate to a primitive rotation group (Group 3, $e = g = 1$, $c = 1$ in Table II is conjugate to the primitive rotation group $R_5(\mathfrak{A}_5)$) (note that the conditions $f = 1$ and $e \in \{1,2\}$ must be satisfied in these tables in order for G to be an orientation preserving real group). For arbitrary n the result was independently obtained by working over the real numbers in [Mik78, p. 90]. In particular, for $n \leq 4$ a case differentiation shows that only the listed groups occur.

Two-dimensional system of imprimitivity. Now assume that the group G admits a two-dimensional system of imprimitivity $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_l$ where $n = 2l$. The block diagonal subgroup D is the kernel of the natural homomorphism $\phi : G \to \mathfrak{S}_l$. Since G is irreducible and generated by rotations, its image is a transitive subgroup of $\mathfrak{S}_l$ generated by transpositions and thus all of $\mathfrak{S}_l$ [LT09, Lem. 2.13, p. 28]. In particular, G contains transformations of type

$$t_i = \begin{pmatrix} I & & & & & \\ & \ddots & & & & \\ & & 0 & Q_i^{-1} & & \\ & & Q_i & 0 & & \\ & & & & \ddots & \\ & & & & & I \end{pmatrix} \quad \text{with } t_i V_{i-1} = V_i$$

for each $i = 2, \ldots, l$ such that $G = \langle D, t_2, \ldots, t_l \rangle$ [Huf75, p. 511]. Conjugating successively by the transformations $I \oplus Q_2^{-1} \oplus I \ldots \oplus I$, $I \oplus I \oplus (Q_3 Q_2)^{-1} \oplus I \ldots \oplus I$, ..., we can assume that $Q_i = I$, $i = 2, \ldots, l$. Each rotation in $g \in G$ is of one of the following four types (cf. [Mik82])

$$g_{|V_i} = Q, \; g_{|V_i^\perp} = \mathrm{id} \tag{1.1}$$

$$g_{|V_i \oplus V_j} = \begin{pmatrix} 0 & Q^{-1} \\ Q & 0 \end{pmatrix}, \; g_{|(V_i \oplus V_j)^\perp} = \mathrm{id} \tag{1.2}$$

$$g_{|V_i \oplus V_j} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}, \; g_{|(V_i \oplus V_j)^\perp} = \mathrm{id} \tag{1.3}$$

$$g_{|V_i \oplus V_j} = \begin{pmatrix} 0 & R^{-1} \\ R & 0 \end{pmatrix}, \; g_{|(V_i \oplus V_j)^\perp} = \mathrm{id} \tag{1.4}$$

for distinct $i, j \in \{1, \ldots, l\}$ and orthogonal matrices Q with determinant 1 and R, R_1 and R_2 with determinant -1, respectively. Note that if G contains a rotation of type (1.4), then it also contains a rotation of type (1.3). Therefore, if G does not contain a rotation of type (1.3), then it preserves the complex structure $J := J_0 \oplus \ldots \oplus J_0$, where

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In this case G is induced by a unitary reflection group of type $G(m', p', l)$ for some $p'|m'$ (cf. Section 1.3.2). Otherwise, each rotation in G can be written as a composition of rotations of type (1.3) and the t_i, $i = 2, \ldots, l$, i.e. G is generated by them. Moreover, since the t_i

normalize the set of rotations of type (1.3), the diagonal subgroup D is generated by these rotations. Let $G_i < \mathrm{O}(V_i)$ be the projection of G to $\mathrm{O}(V_i)$, let $H_i < \mathrm{SO}(V_i)$ be the subgroup of G generated by rotations of type (1.1) contained in G and set $H = H_1 \times \ldots \times H_l < G$. All G_i are conjugate and isomorphic to a dihedral group D_{km} of order $2km$ and all H_i are conjugate and isomorphic to a cyclic group C_m of order m. Assume that $Q \in G_1$ is a rotation of maximal order. Since the diagonal subgroup D is generated by rotations of type (1.3), there exists a rotation $Q' \in G_2$ such that

$$Q \oplus Q' \oplus I \oplus \ldots \oplus I \in G.$$

Let us now assume that $n > 4$. Then, because of $l \geq 3$, we also have

$$Q \oplus Q^{-1} \oplus I \oplus \ldots \oplus I \in G. \tag{1.5}$$

Since there exists a rotation $R_1 \oplus I \oplus R_2 \oplus I \oplus \ldots \oplus I \in G$ of type (1.3), we deduce that

$$QR_1Q^{-1} \oplus I \oplus R_2 \oplus \ldots \oplus I \in G$$

and thus

$$Q^2 \oplus I \oplus \ldots \oplus I = QR_1Q^{-1}R_1 \oplus I \oplus R_2^2 \oplus \ldots \oplus \in G. \tag{1.6}$$

This shows $k \in \{1, 2\}$, that the subgroup of G generated by the rotations of type (1.1) and (1.2) contained in G is given by $A^*(km, k, l)$ (cf. Section 1.3.5) and that the group G is generated by $G(km, k, l)$ and a transformation r that conjugates the first two coordinates, i.e. $r(z_1, z_2, z_3 \ldots, z_l) = (\overline{z}_1, \overline{z}_2, z_3 \ldots, z_l)$, where we identify $\mathbb{R}^{2l}$ with $\mathbb{C}^l$. Hence, we have proven the following proposition.

Proposition 35. *The imprimitive absolutely irreducible rotation groups $G < \mathrm{SO}_n$ for $n = 2l \geq 5$ that admit a two-dimensional system of imprimitivity are up to conjugation $G^*(km, k, l) = \langle G(km, k, l), \tau\rangle < \mathrm{SO}_n$ with $k = 1, 2$ and $km \geq 3$. The group $G^*(km, k, l)$ has order $2^{l-k}(km)^l l!$.*

For $n = 4$ there is no restriction on k and for a specific k there can be several geometrically inequivalent rotation groups (cf. Section 1.3.5). More precisely, we have

Proposition 36. *The imprimitive absolutely irreducible rotation groups $G < \mathrm{SO}_4$ that admit a two-dimensional system of imprimitivity are precisely the unique extensions of reducible rotation groups D defined by a set of data (cf. Theorem 4)*

$$(\{(W(\mathrm{I}_2(km)), W^+(\mathrm{I}_2(m)), W(\mathrm{I}_2(km))\}_{i\in\{1,2\}}, \varphi),$$

$km \geq 3$, where $\varphi : \mathfrak{D}_k \to \mathfrak{D}_k$ is an involutive automorphism of $\mathfrak{D}_k = W(\mathrm{I}_2(km)/W^+(\mathrm{I}_2(m))$ that maps reflections onto reflections, by a normalizing rotation that interchanges the two irreducible components of D. They are denoted as $G^*(km, k, 2)_\varphi$ (cf. Section 1.3.5).

Proof. Let $G < \mathrm{SO}_4$ be an imprimitive absolutely irreducible rotation group as considered above. Then we have $G = \langle D, t_2 \rangle$ where the block diagonal subgroup $D < G$ is a reducible rotation group described by a set of data

$$\{(W(\mathrm{I}_2(km)), W^+(\mathrm{I}_2(m)), W(\mathrm{I}_2(km))\}_{i \in \{1,2\}},$$

$\varphi : \mathfrak{D}_k \to \mathfrak{D}_k$ where φ is an automorphism of $\mathfrak{D}_k \cong W(\mathrm{I}_2(km))/W^+(\mathrm{I}_2(m))$ that maps reflections onto reflections. Since t_2 normalizes D, the automorphism φ has order 2. Any other rotation that normalizes D and interchanges its two irreducible components can be conjugated to t_2 by an element in the normalizer of D. $\square$

1.4.2 Primitive rotation groups

In this section we prove the classification of primitive absolutely irreducible rotation groups in dimension $n \geq 5$. The complexification of a primitive absolutely irreducible rotation group is irreducible but a priori not primitive (the complexification of $R_5(\mathfrak{A}_5)$ is monomial). However, we are going to show that it satisfies the following property if $n \geq 5$.

Definition 4. An irreducible complex representation $\rho : G \to \mathrm{GL}(V)$ is called *quasiprimitive* if for every normal subgroup N of G the restriction $\rho_{|N}$ splits into equivalent representations.

Indeed, we have

Lemma 37. *The complexified natural representation of a primitive absolutely irreducible rotation group $G < \mathrm{SO}_n$ with $n \geq 5$ is quasiprimitive.*

Proof. Let N be a normal subgroup of G and let $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$ be a decomposition into irreducible components with respect to the action of N. All of them are equivalent, because otherwise distinct isotypic components would define a nontrivial system of imprimitivity. Now the claim follows if we can show that the complexifications of the V_i split into subrepresentations all of which are equivalent. If this were not the case, by the Frobenius-Schur theorem [Wol84, Thm. 4.7.3, p. 153] we would have $V_i^{\mathbb{C}} = U_i \oplus U_i^*$ for equivalent irreducible representations U_i, $i = 1, \ldots, k$, which are inequivalent to U_i^* and accordingly

$$\mathbb{C}^n = \underbrace{U_1 \oplus \ldots \oplus U_k}_{=:\hat{U}} \oplus \underbrace{U_1^* \oplus \ldots \oplus U_k^*}_{=:\hat{U}^*}.$$

Then for any $g \in G$ we would either have $g\hat{U} = \hat{U}$ or $g\hat{U} = \hat{U}^*$. The second case yields a contradiction because G is generated by rotations and since $\dim \hat{U} > 2$ holds by assumption. But $g\hat{U} = \hat{U}$ for all $g \in G$ also yields a contradiction since we have assumed G to be absolutely irreducible. Consequently the complexified representation is quasiprimitive as claimed. $\square$

A nontrivial rotation group $G < \mathrm{SO}_n$ contains an element with eigenvalues $\xi, \bar{\xi}, 1, \ldots, 1$, where ξ is a nontrivial root of unity. We call such an element a special r-element if ξ is an r-th root of unity. On the assumptions of this section the complexification of G is quasiprimitive due to the preceding lemma. Finite quasiprimitive unimodular linear groups over the complex numbers in dimension higher than four that contain a special r-element are classified in [Bra67, HW75, Wa78, Huf75]. More precisely, in these papers possible quotient groups G/Z_1 are listed, where Z_1 is a subgroup of the center Z of G. According to [HW75, Thm. 1, p. 54] it is sufficient to consider the cases $r = 2$ and $r = 3$, since the existence of a special r-element for any prime $r = p > 3$ implies $n \leq 4$. The case $r = 3$ is treated in [Huf75, Thm. 2, p. 261] and the case $r = 2$ is treated in [Wa78, Thm. 1, p. 58] for $n \geq 6$ and in [Bra67, Thm. 9.A, p. 91], [Huf75, Table I-III] for $n = 5$.

Now we go through the cases and inspect which of the listed groups actually come from complexified primitive rotation groups, i.e. we examine the corresponding faithful complex representations. Such a representation can be excluded if it does not preserve the orientation or if it is not real meaning that it cannot be realized over the real numbers. The latter is in particular the case, if the restriction of the representation to a subgroup is not real or, by Schur's lemma, if the center Z of G has more than two elements. There is another convenient way to check whether an irreducible representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is real or not. The Schur indicator of such a representation is defined as

$$\mathrm{Ind}(\rho) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$$

and it takes values in $\{1, 0, -1\}$. Depending on its value the representation is said to be real, complex or quaternionic and only in the first case can it be realized over the real numbers [Ser77, p. 108]. Also note that if $\rho_i : G_i \to \mathrm{GL}(V_i)$, $i = 1, 2$, are irreducible complex representations, then their tensor product $\rho : G_1 \times G_2 \to \mathrm{GL}(V_1 \otimes V_2)$ is irreducible and its Schur indicator is given by $\mathrm{Ind}(\rho) = \mathrm{Ind}(\rho_1)\mathrm{Ind}(\rho_2)$. Once we have found a real and orientation preserving representation we check whether the corresponding linear group is actually generated by rotations.

If G is an absolutely irreducible rotation group its center is either trivial or $\{\pm 1\}$ by Schur's lemma and thus the same holds for Z_1. Therefore, in order to check if a given group $G_1 = G/Z_1$ comes from a rotation group we only have to examine the representations of G_1 and of its two-fold central extensions. We will often be in a situation where G_1 is a perfect group. In this case its two-fold central extensions can be described as follows. Recall that the quotient of a perfect group by its center is centerless due to Grün's Lemma [Ros94, p. 61].

Lemma 38. *Let G be a central extension of a perfect group G_1 by $Z_1 \cong \mathbb{Z}_2$. Then one of the following two cases holds*

(i) G is perfect and thus a perfect central extension of G_1.

(ii) $G \cong G_1 \times Z_1$

Moreover, if $Z(G) = Z_1$, then in the first case the center of G_1 is trivial by Grün's Lemma.

Proof. Let $\pi : G \to G_1$ be the natural projection. Since G_1 is perfect we have $\pi(G') = G_1$, where G' is the commutator subgroup of G. Therefore, the index of G' in G is either 1 or 2. If it is 1 we have $G = G'$ and we are in case (i). If it is 2 we have $G' \cap Z_1 = \{1\}$ and hence $G = G' \times Z_1$ with $G' \cong G_1$ and we are in case (ii). □

Likewise the following lemma follows.

Lemma 39. *Let G be a central extension of a group $G_1 = \langle P_1, a\rangle$ which is generated by a perfect group P_1 and an automorphism a of P_1 of order 2 by $Z_1 \cong \mathbb{Z}_2$. Let P be the preimage of P_1 in G and let $\tilde{a}$ be a preimage of a in G. Then we have $G = \langle P, \tilde{a}\rangle$ and one of the following two cases holds.*

(i) P is perfect and $Z_1 < P$.

(ii) $G \cong G_1 \times Z_1$

Note that if G_1 in Lemma 38 or P_1 in Lemma 39 is a simple group, then the irreducible representations of G_1 and G can be looked up in many cases in [CCN$^+$85].

We begin by inspecting the possible groups in dimension five. The only irreducible complex five-dimensional linear group generated by elements with codimension two fixed-point subspace that is monomial and quasiprimitive is described in [Huf75], Table II, Group 3, $e = g = c = 1$ (cf. [Huf75, Table I-III, p. 503] and note that the diagonal subgroup D of a monomial group G can only consists of homotheties in order for G to be quasiprimitive). This representation can also be realized over the real numbers and as such its image is the primitive absolutely irreducible rotation group $R_5(\mathfrak{A}_5)$ we have described in Lemma 15. All other complexified primitive absolutely irreducible rotation groups in dimension 5 must occur in the following list which we cite from [Bra67, Thm. 9.A, p. 91].

Theorem 40. *Let $\rho : G \to \mathrm{SL}_5(\mathbb{C})$ be a faithful and irreducible representation of a finite group G which is not monomial. Then one of the following cases holds.*

(A) $G/Z \cong \mathrm{PSL}_2(11)$.

(B) G/Z is a symmetric or alternating group on five or six letters.

(C) $G/Z \cong \mathrm{O}_5(3) \cong \mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$.

(D) G is a uniquely determined group of order $24 \cdot 5^4$ and has a nonabelian normal subgroup N of order 125 and exponent 5.

(E) G is a certain subgroup of the group in (D) that still contains N as a normal subgroup.

We can use this result to identify the primitive rotation groups in dimension five.

Proposition 41. *The primitive absolutely irreducible rotation groups $G < \mathrm{SO}_5$ are given, up to conjugation, as follows*

(i) The group $M(\mathrm{R}_5) = R_5(\mathfrak{A}_5)$ (cf. Lemma 15).

(ii) The orientation preserving subgroup W^+ of the reflection group W of type A_5.

(iii) The group $W^*(\mathrm{A}_5)$ isomorphic to $\mathfrak{S}_6$ (cf. Proposition 9, (i)).

Proof. We go through the cases listed in Theorem 40. We can assume that the center Z is trivial, since the dimension is odd and the orientation has to be preserved by G.

(A) All five-dimensional irreducible representations of $\mathrm{PSL}_2(11)$ have Schur indicator 0 [CCN$^+$85, p. 7] and thus this case can be excluded.

(B) The alternating group $\mathfrak{A}_5$ has one five-dimensional absolutely irreducible real representation [CCN$^+$85, p. 2], which is described in Lemma 15. This gives the rotation group in case (i).

The symmetric group $\mathfrak{S}_5$ has one five-dimensional absolutely irreducible real representation, which is induced by the exceptional embedding $i : \mathfrak{S}_5 \to \mathfrak{S}_6$ [CCN$^+$85, p. 2] that maps a transposition in $\mathfrak{S}_5$ to a triple transposition in $\mathfrak{S}_6$. Therefore, this representation does not preserve the orientation and is thus not generated by rotations. Hence, we can exclude this case.

The alternating group $\mathfrak{A}_6$ has two inequivalent five-dimensional absolutely irreducible real representations, but they only differ by an outer automorphism of $\mathfrak{A}_6$ and thus give rise to the same linear group, namely the orientation preserving subgroup of the reflection group of type A_5 [CCN+85, p. 5]. This gives the rotation group in case (*ii*).

The symmetric group $\mathfrak{S}_6$ has four inequivalent five-dimensional absolutely irreducible real representations, but for the same reason as above they only give rise to two different linear groups, the reflection group $W(\mathrm{A}_5)$ and the rotation group $W^*(\mathrm{A}_5)$ described in Proposition 9. This gives the rotation group in case (*iii*).

(C) All five-dimensional representations of $G \cong \mathrm{O}_5(3) \cong \mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$ have Schur indicator 0 [CCN+85, p. 27] and thus this case can be excluded.

(D) If G were a complexified rotation group, it would be quasiprimitive by Lemma 37 and thus the restriction of the representation to N would either split into five equivalent one-dimensional representations or it would be irreducible. The first case cannot occur since one-dimensional representations of N are not faithful. The second case cannot occur since the center of N is divisible by 5 which is why N does not have faithful absolutely irreducible real representations. Hence this case can be excluded.

(E) This case can be excluded by the same argument as in (D). $\square$

Next we treat the case where G contains a special 3-element and where $n \geq 6$. These groups are listed in [Huf75, Thm. 2] and in [Wa78, Thm. 1, case (A),(B),(H) cf. Rem. 1, p. 60]. We first cite the results from [Huf75] and [Wa78].

Theorem 42. *Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a faithful and quasiprimitive representation of a finite group G with $n \geq 6$ such that $\rho(G)$ contains a special 3-element. Then one of the following cases holds.*

(A) $G/Z = G_1$ where $G_1 \cong \mathfrak{A}_{n+1}$ or $G_1 \cong \mathfrak{S}_{n+1}$. All special elements lie in $\mathfrak{A}_{n+1}$ mod Z and $G = G_1 \times Z$ if $G_1 \cong \mathfrak{A}_{n+1}$, unless $n = 6$.

(B) $G/Z_1 = G_1$ with $G_1 \cong W(\mathrm{E}_n)$ or $G_1 \cong W^+(\mathrm{E}_n)$, $n = 6, 7, 8$ and $Z_1 < Z$.

(H) $n = 6$, $G/Z \cong \mathrm{PSU}_4(3)$ or an extension by an automorphism of order 2.

Proof. Only the claim on the special elements in case (A) is not explicitly proven in [Huf75] and [Wa78]. However, if there were a special 2-element not in $\mathfrak{A}_{n+1}$ mod Z, then the group would be listed in [Wa78, Table I, p. 63]. The only possibility is the first row, where the involution is a transposition $(1, 2)$ and the representation is the natural representation of the symmetric group. In particular, the involution is not a special 2-element as it is a reflection. For a general special r-element $g \in G_1$ we can assume that $r = 2^a 3^b$ by [HW75, Thm. 1, p. 54] and that $a < 2$ by [Huf75, Thm. 1, p. 261]. Because of $\mathfrak{S}_{n+1}/\mathfrak{A}_{n+1} \cong \mathbb{Z}_2$ all elements of odd order in G_1 are contained in $\mathfrak{A}_{n+1}$ and thus the special 3^b-element g^{2^a} is contained in $\mathfrak{A}_{n+1}$. The special element g^{3^b} of order 2^a for $a < 2$ is contained in $\mathfrak{A}_{n+1}$ by the argument above and thus so is g, since 2^a and 3^b are coprime. $\square$

Now we can identify the rotation groups appearing in Theorem 42.

Proposition 43. *The primitive absolutely irreducible rotation groups $G < \mathrm{SO}_n$ for $n \geq 6$ that contain a special 3-element are given, up to conjugation, as follows*

(i) The orientation preserving subgroups W^+ of reflection groups W of type A_n, E_6, E_7 and E_8.

(ii) The group $W^*(\mathrm{E}_6)$ (cf. Proposition 9, (ii)).

Proof. We go through the cases listed in Theorem 42.

(A) Since all special elements lie in $\mathfrak{A}_{n+1}$ mod Z we can assume that $G_1 = \mathfrak{A}_{n+1}$. For $n \geq 6$ the group $\mathfrak{A}_{n+1}$ has only one irreducible representation of dimension n namely the nontrivial subrepresentation of the permutation representation on $\mathbb{C}^n$ [Huf75, Lem. 4.1, p. 273]. This is a real representation and gives rise to the orientation preserving subgroup of the reflection group of type A_n, i.e. the rotation group in (i). The case $Z = \{\pm\mathrm{id}\}$, $G_1 \cong \mathfrak{A}_{n+1}$ and $G = G_1 \times Z$ does not give new examples due to Lemma 8. It remains to consider the case where G is a perfect central extension of $\mathfrak{A}_6$ by $Z \cong \mathbb{Z}_2$. Inspecting the character tables shows that there are no appropriate representations in this case [CCN$^+$85, p. 5].

(B) The group $W^+(\mathrm{E}_6)$ is isomorphic to the simple group $\mathrm{O}_5(3) \cong \mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$ [CCN$^+$85, p. 27]. It has only one absolutely irreducible real 6-dimensional representation namely its realization as the orientation preserving subgroup $W^+(\mathrm{E}_6)$ of the reflection group of type E_6. The double cover of $W^+(\mathrm{E}_6)$ does not have representations in dimension 6 [CCN$^+$85, p. 27]. According to Lemma 8, the group $W^*(\mathrm{E}_6) \cong W^+(\mathrm{E}_6) \times \mathbb{Z}_2$ is also a rotation group that occurs in this case. The group $W(\mathrm{E}_6)$ has two faithful absolutely irreducible real 6-dimensional representations, among them its standard representation, and they differ only by a sign on the complement of $W^+(\mathrm{E}_6)$ [CCN$^+$85, p. 27]. In particular, neither of them preserves the orientation. The double cover of $W(\mathrm{E}_6)$ does not have representations in dimension 6 [CCN$^+$85, p. 27].

The group $W^+(\mathrm{E}_7)$ is isomorphic to the simple group $\mathrm{PSp}_6(2)$ [CCN$^+$85, p. 46]. It has only one absolutely irreducible real 7-dimensional representation namely its realization as the orientation preserving subgroup $W^+(\mathrm{E}_7)$ of the reflection group of type E_7 [CCN$^+$85, p. 46]. Since the dimension is odd, the center must be trivial (the center of $W(\mathrm{E}_7)$ is not trivial, cf. [Hum90, p. 45]) and thus the rotation group $W^+(\mathrm{E}_7)$ is the only example that occurs in this case.

The group $W^+(\mathrm{E}_8)$ is a perfect central extension of the simple group $\mathrm{O}_8^+(2)$ by $\mathbb{Z}_2$ [CCN$^+$85, p. 85]. It has only one faithful absolutely irreducible real 8-dimensional representation namely its realization as the orientation preserving subgroup $W^+(\mathrm{E}_8)$ of the reflection group of type E_8 [CCN$^+$85, p. 85] and its image contains the negative unit [Hum90, p. 46]. For $Z = Z_1 = \{\pm 1\}$ and $G \neq W^+(\mathrm{E}_8) \times \mathbb{Z}_2$ the group G must be perfect by Lemma 38 and this contradicts Grün's lemma, stating that the quotient of a perfect group by its center is centerless (cf. [Ros94, p. 61]), since $W^+(\mathrm{E}_8)$ has a nontrivial center. The group $W(\mathrm{E}_8)$ has two faithful absolutely irreducible real 8-dimensional representations, but by the same reason as in (B) neither of them preserves the orientation [CCN$^+$85, p. 85]. Suppose the group $W(\mathrm{E}_8)$ had a perfect central extension G by $\mathbb{Z}_2$ with a suitable representation. Then the group P (in the notation of Lemma 39) would be a perfect central extension of $\mathrm{O}_8^+(2)$ by a group $Z(P)$ of order 4 containing Z_1 due to Lemma 39 and Grün's Lemma. Therefore, the restriction of the representation of G to P would be reducible. Since the representation of G is irreducible by assumption, the automorphism $\tilde{a}$ would permute two irreducible four-dimensional components of the representation of P. However, the linear group corresponding to such a representation cannot be generated by rotations. Hence, no further rotation groups occur in this case.

(H) There are no absolutely irreducible real 6-dimensional representations in this case [CCN+85, p. 53] and thus it can be excluded. $\square$

It remains to treat the cases where G contains special 2-elements but no special r-elements for $r \geq 3$. We first cite the result obtained in [Wa78, Thm. 1, p. 58].

Theorem 44. *Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a faithful and quasiprimitive representation of a finite group G with $n \geq 6$ such that $\rho(G)$ contains a special 2-element but no special r-element for $r > 2$. Then one of the following cases holds.*

(A) As in case (A) in Theorem 42.

(B) As in case (B) in Theorem 42.

(C) $G/Z = A \times K$ where K is a linear group generated projectively by reflections and $A \cong \mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$. Here, $\rho(G)$ is a subgroup of $Y \otimes Y_1$ where Y is a representation of degree 2 and Y_1 is a representation of K of degree $n/2$.

(D) $n = 6$ and $G/Z = G_1$ with $G_1 \cong \mathrm{PSL}_2(7)$ or an extension by an automorphism of order 2 to $G_1 \cong \mathrm{PGL}_2(7)$ and if $G_1 \cong \mathrm{PSL}_2(7)$ then $G = G_1 \times Z$.

(E) $n = 6, 7$ and $G/Z = G_1$ with $G_1 \cong \mathrm{PSU}_3(3)$ or an extension by an automorphism of order 2 to $G_1 \cong G_2(2)$. If $G_1 \cong U_3(3) \cong \mathrm{PSU}_3(3)$ then $G = G_1 \times Z$. There is a unique representation in dimension 6 and two representations in dimension 7. The 7-dimensional representations are not real and they do not extend to $G_2(2)$.

(F) $n = 6$ and $G/Z \cong \hat{J}_2$, a proper double cover of the Hall-Janko group of order 604800.

(G) $n = 6$ and $G/Z = G_1$ with $G_1 \cong \mathrm{PSL}_3(4)$ or an extension by an automorphism of order 2.

(H) As in case (H) in Theorem 42.

(I) $n = 6$ and $G/Z_1 = G_1 \cong \hat{\mathfrak{A}}_6$, the unique proper triple cover of $\mathfrak{A}_6$, or an extension by an automorphism of order 2. Here G_1 has a center of order 3 and $Z_1 < Z$.

(J) $n = 8$ and G contains a subgroup G_1 with $G = G_1 Z$, $G_1 \triangleright H$ where $H \cong Q_8 \circ \mathfrak{D}_4 \circ \mathfrak{D}_4$, $\mathfrak{D}_4 \circ \mathfrak{D}_4 \circ \mathfrak{D}_4$, or $\mathfrak{D}_4 \circ \mathfrak{D}_4 \circ \mathfrak{D}_4 \circ \mathbb{Z}_4$ and the restriction of the representation to H is the tensor product of faithful representations of the quaternion group Q_8 and the dihedral group $\mathfrak{D}_4$ with $|\mathfrak{D}_4| = 8$ on $\mathbb{C}^2$ and the cyclic group $\mathbb{Z}_4$ on $\mathbb{C}$ (cf. [Gor68, Theorems 2.7.1 and 2.7.2]). The quotient G_1/H is isomorphic to a subgroup of $\mathrm{O}_6^+(2) \cong \mathfrak{S}_8$, $\mathrm{O}_6^-(2)$ or $S_p(6,2)$ in the respective cases and $\rho_{|H}$ is irreducible.

(K) $n = 8$ and $G/Z \cong (\mathfrak{A}_5 \times \mathfrak{A}_5 \times \mathfrak{A}_5) \wr \mathfrak{S}_3$. G contains a normal subgroup $H \cong \mathrm{SL}_2(5) \circ \mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)$ and $\rho_{|H} = \rho_1 \otimes \rho_1 \otimes \rho_1$ for a two dimensional representation ρ_1 of $\mathrm{SL}_2(5)$.

(L) $n = 10$ and $G = G_1 \times Z$ with $G_1 = \mathrm{PSU}_5(2)$.

(M) $n = 6$ and $G = G_1 \circ Z$ where G_1 is a proper central extension of $\mathfrak{A}_7$ with a center of order 3.

Proposition 45. *The primitive absolutely irreducible rotation groups $G < \mathrm{SO}_n$ for $n \geq 6$ that contain no special r-element for $r \geq 3$ are given, up to conjugation, by $M(\mathrm{S}_6) = R_6(\mathrm{PSL}_2(7))$ (cf. Lemma 16) and $M(\mathrm{T}_8) = L$ (cf. Section 1.3.8).*

Proof. We go through the cases listed in Theorem 44. For (A), (B) and (H) the group contains special 3-elements (cf. [Wa78, *Remark* on p. 60]). These cases have already been treated in Proposition 43.

(C) Suppose that $A \otimes B \in \rho(G)$ is a rotation acting on $Y \otimes Y_1$. If λ_1, λ_2 denote the eigenvalues of A and $\mu_1, \ldots, \mu_m$, $m = n/2$, denote the eigenvalues of B, then the eigenvalues of $A \otimes B$ are $\lambda_1\mu_1, \ldots, \lambda_1\mu_m, \lambda_2\mu_1, \ldots, \lambda_2\mu_m$. The fact that $A \otimes B$ is a rotation and that $m \geq 3$ implies the existence of some μ_i with $\lambda_1\mu_i = \lambda_2\mu_i = 1$ by the pigeonhole principle and thus we have $\lambda_1 = \lambda_2$. It follows that all preimages in G of special r-elements in $\rho(G)$ are contained in K mod Z. In particular, $\rho(G)$ cannot be generated by special r-elements and hence this case can be excluded.

(D) The simple group $\mathrm{PSL}_2(7)$ has only one 6-dimensional absolutely irreducible real representation [CCN$^+$85, p. 3] which gives rise to the rotation group $R_6(\mathrm{PSL}_2(7))$. It extends to two 6-dimensional absolutely irreducible real representations of $\mathrm{PGL}_2(7)$. According to their character tables all special 2-elements lie in $\mathrm{PSL}_2(7)$ [CCN$^+$85, p. 3]. By Lemma 16, the group $\langle R_6(\mathrm{PSL}_2(7)), -1\rangle$ is not generated by rotations. Since there are no other appropriate representations [CCN$^+$85, p. 3] the rotation group $R_6(\mathrm{PSL}_2(7))$ is the only example that can occur in this case.

(E) None of the representations in question is real [CCN$^+$85, p. 14] and thus this case can be excluded.

(F) The group $\hat{J}_2$ does not have an absolutely irreducible real representations in dimension 6 [CCN$^+$85, p. 43]. The case $Z = \{\pm 1\}$ and $G \neq \hat{J}_2 \times Z$ cannot occur, since then G would be perfect by Lemma 38 contradicting Grün's lemma. Hence, no examples occur in this case.

(G) There are no 6-dimensional absolutely irreducible real representation in this case [CCN$^+$85, p. 23] and thus it can be excluded.

(I) For $G = G_1 \times Z_1$ we have $Z(G) \geq 3$, i.e. no rotation group can occur. The case $Z = Z_1 = \{\pm 1\}$ with $G \neq G_1 \times Z$ is impossible by Grün's lemma, since G_1 has a nontrivial center (cf. Lemma 38).

(J) The Schur indicators of the listed possible representations of H are -1, 1 and 0 and thus only the second case comes into question. In this case the representation is real and its image in SO_8 is given by the group $H < \mathrm{SO}_8$ described in Section 1.3.8. We have to look for primitive groups in the normalizer $N_{\mathrm{GL}_8(\mathbb{R})}(H)$ that are generated by pseudoreflections, contain the group H as a subgroup and only pseudoreflections of order 2. By Schur's lemma, it suffices to look for rotation groups in $N_{\mathrm{SO}_8}(H)$ with these properties. Therefore, according to Lemma 24 and Lemma 22 the group L defined in Section 1.3.8 is the only example that occurs in this case.

(K) There are two faithful representation of $\mathrm{SL}_2(5)$ in dimension two, both have Schur indicator -1 [CCN$^+$85, p. 2] and thus so has the irreducible representation $\rho_{|H}$. In particular, the representation of G is not real and hence no examples occur in this case.

(L) There are no absolutely irreducible real 10-dimensional representation in this case [CCN$^+$85, p. 72] and thus it can be excluded.

(M) Because of $|Z| \geq 3$ the representation of G cannot be real and thus this case can be excluded. □

We summarize what we have obtained in this section.

Proposition 46. *The primitive absolutely irreducible rotation groups $G < \mathrm{SO}_n$ for $n \geq 5$ are given, up to conjugation, as follows.*

(i) The orientation preserving subgroups W^+ of the reflection groups of type A_n, E_6, E_7 and E_8.

(ii) The group $M(\mathrm{R}_5) = R_5(\mathfrak{A}_5) < \mathrm{SO}_5$ (cf. Lemma 15).

(iii) The group $W^*(\mathrm{A}_5) < \mathrm{SO}_5$ (cf. Proposition 9, (i)).

(iv) The group $M(\mathrm{S}_6) = R_6(\mathrm{PSL}_2(7)) < \mathrm{SO}_6$ (cf. Lemma 16).

(v) The group $W^*(\mathrm{E}_6) < \mathrm{SO}_6$ (cf. Proposition 9).

(vi) The group $M(\mathrm{T}_8) = L < \mathrm{SO}_8$ (cf. Section 1.3.8).

1.5 Irreducible reflection-rotation groups

Let $G < \mathrm{O}_n$ be an irreducible reflection-rotation group. The case in which G is generated by rotations is subject of Theorem 1. So suppose that G contains a reflection. Let F be the normal subgroup of G generated by the reflections in G and let H be the normal subgroup of G generated by the rotations in G. Then H is the orientation preserving subgroup of G and it is absolutely irreducible for $n > 2$ by Lemma 13.

Proposition 47. *Let $G < \mathrm{O}_n$ be an irreducible reflection-rotation group that contains a reflection such that F is reducible. Then G is either one of the monomial groups $M_5^\times$, $M_6^\times$, $M_7^\times$, $M_8^\times$, $M^\times(\mathrm{D}_n)$ (cf. Section 1.3.4 and Table 1.5) or an imprimitive group $G^\times(km,k,l) < \mathrm{SO}_{2l}$ with $k = 1, 2$ and $km \geq 3$ (cf. Section 1.3.5 and Section 1.4.1).*

Proof. Observe that F is distinct from G and thus we have $n > 2$ and H is absolutely irreducible. Since G is irreducible, the group H permutes the irreducible components of F transitively. Therefore, the rotation group H is imprimitive with a system of imprimitivity given by the irreducible components of F which are all equivalent and either one- or two-dimensional. If they are one-dimensional, then H is a monomial group occurring in Proposition 34 that contains all transformations that change the sign of an even number of coordinates. Hence, the group G, being not a reflection group by assumption, is one of the listed monomial groups in this case.

In the second case, the Coxeter diagram of F is given by

$$\bullet_{s_1^{(1)}} \overset{m}{-} \bullet_{s_2^{(1)}} \quad \bullet_{s_1^{(2)}} \overset{m}{-} \bullet_{s_2^{(2)}} \cdots \bullet_{s_1^{(l)}} \overset{m}{-} \bullet_{s_2^{(l)}}$$

with $m > 2$ and $l > 1$. As in the proof of Proposition 35 we see that H acts like the symmetric group on the irreducible components of F. Since the orientation preserving subgroup of F is contained in H, this implies $G^*(m,1,l) < H$. Moreover, the fact that H normalizes F implies $H < G^*(2m,2,l)$. Therefore, H is an imprimitive rotation group of type $G^*(km,k,l)$ for $k = 1$ or $k = 2$ and $km \geq 3$ by Proposition 35 and Proposition 36. Accordingly, G is an imprimitive reflection-rotation group of type $G^\times(km,k,l) < \mathrm{SO}_{2l}$ for $k = 1$ or $k = 2$ and $km \geq 3$ in this case (cf. Section 1.3.5). □

Proposition 48. *Let G be an irreducible reflection-rotation group that contains a reflection such that F is irreducible and distinct from G. Then G is a group of type $W^\times$ generated by an* *irreducible reflection group W of type* A_4, D_4, F_4, A_5 *or* E_6 *and a normalizing rotation (cf. Lemma 7).*

Proof. By assumption there is a rotation $h \in G \backslash F$. According to Lemma 5 there exists a chamber of the reflection group F such that $hC = C$ and by Lemma 6 we deduce that F has type A_4, D_4, F_4, A_5 or E_6. Finally, the uniqueness statement of Lemma 7 implies that the group G is generated by F and h and is thus one of the listed groups. $\square$

1.6 General reflection-rotation groups

The structure of reducible rotation groups is described in [Mik82]. For a general reflection-rotation group $G < \mathrm{O}_n$ let $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$ be a decomposition into irreducible components. For each $i \in I$ we denote the projection of G to $\mathrm{O}(V_i)$ by π_i and set $G_i = \pi_i(G)$. Recall the following definition from the introduction.

Definition 5. A rotation $g \in G$ is called a rotation of the

(i) *first kind*, if for some $i_0 \in I$, $\pi_{i_0}(g)$ is a rotation in V_{i_0} and $\pi_i(g)$ is the identity on V_i for all $i \neq i_0$.

(ii) *second kind*, if for some $i_1, i_2 \in I$, $i_1 \neq i_2$, $\pi_{i_1}(g)$ and $\pi_{i_2}(g)$ are reflections in V_{i_1} and V_{i_2}, respectively, and $\pi_i(g)$ is the identity for all $i \neq i_1, i_2$.

Let H be the normal subgroup of G generated by rotations of the first kind, let F be the normal subgroup of G generated by reflections and rotations of the second kind and set $H_i = \pi_i(H)$ and $F_i = \pi_i(F)$. Then H_i is a rotation group, F_i is a reflection group, both are normal subgroups of G_i and G_i is generated by them. The triple (G_i, H_i, F_i) has an additional property that does not hold in general. It is described in Lemma 50 below.

Lemma 49. *For every reflection $s \in F_i$ there exists a reflection or a rotation of the second kind $g \in G$ such that $s = \pi_i(g)$.*

Proof. Let X_i be the set of reflections in F_i of the form $\pi_i(g)$ for some reflection or rotation of the second kind $g \in G$. Then X_i generates F_i and is invariant under conjugation by F_i. Therefore, every reflection in F_i is contained in X_i, i.e. is a reflection of the form $s = \pi_i(g)$ for some reflection or rotation of the second kind $g \in G$ [Hum90, Prop. 1.14, p. 24]. $\square$

Lemma 50. *Let $\tau \in F_i$ be a reflection and let $h \in H_i$. If $h\tau$ is a reflection, then it is contained in F_i.*

Proof. By Lemma 49 there exists a reflection or a rotation of the second kind $g \in G$ such that $\tau = \pi_i(g)$. Then hg is either a reflection or a rotation of the second kind in G and $h\tau = \pi_i(hg)$ is contained in F_i. $\square$

Lemma 51. *Let $s, \tau \in F_i$ be reflections conjugate under H_i, i.e. $\tau = hsh^{-1}$ for some $h \in H_i$. Then we have $s\tau \in H_i$.*

Proof. The claim follows from $s\tau = shsh^{-1} = (shs^{-1})h^{-1}$, since H_i is normal in G_i. $\square$

In the following all possible triples (G_i, F_i, H_i) are described. Later we will see from the classification that every reflection in G_i is contained in F_i and that we thus actually classify pairs $M \lhd G_{rr}$ with the properties stated in Remark 1. If F_i is trivial, then $G_i = H_i < G$ is an irreducible rotation group and splits off as a direct factor. Otherwise, G_i is one of the irreducible reflection-rotation groups we have described in the preceding section. Let $S = \{s_1, \dots, s_l\} < F_i$ be a set of simple reflections generating F_i [Hum90, p. 10]. We denote the image of a reflection $s \in G_i$ in G_i/H_i by $\overline{s}_i$. Since G_i is generated by F_i and H_i, the quotient group G_i/H_i is generated by the set $\overline{S}$ composed of the different cosets among the $\overline{s}_1, \dots, \overline{s}_l$. We have $G_i/H_i \cong F_i/\tilde{H}_i$ with $\tilde{H}_i = H_i \cap F_i$ and we will see that in each case $\tilde{H}_i$ is generated by the conjugates of elements of the form $(s_r s_s)^{\tilde{m}_{rs}}$ with $\tilde{m}_{rs} \leq m_{rs}$ where m_{rs} are the entries of the Coxeter matrix of F_i. It is then clear that $(G_i/H_i, \overline{S})$ is a Coxeter system with Coxeter matrix obtained by removing the redundant entries in $(\tilde{m}_{rs})$. We say that an element in a Coxeter group is a reflection, if it is conjugate to a generator or, equivalently, if its image under the geometric representation is a reflection [Hum90, p. 108]. It will then follow directly that the reflections in G_i/H_i are precisely the cosets of reflections in F_i (cf. Corollary 57).

For the proof of Theorem 3 three different cases are considered.

Proposition 52. *Assume that F_i is a nontrivial reducible reflection group. Then H_i is an imprimitive rotation group and a set of simple reflections generating F_i projects onto a set $\overline{S} \subset G_i/H_i$ for which $(G_i/H_i, \overline{S})$ is a Coxeter system of type* A_1 *or* $\mathrm{A}_1 \times \mathrm{A}_1$*. More precisely, the triple (G_i, F_i, H_i) occurs in one of the cases (i) to (iii) in Theorem 3.*

Proof. As in Proposition 47, the group H_i is an imprimitive rotation group with a system of imprimitivity given by the irreducible components of F_i, which are all equivalent and either one- or two-dimensional. If they are one-dimensional, then, given Lemma 51, it follows as in the proof of Proposition 47 that G_i is one of the monomial groups $M^\times$ listed in Proposition 47 whose orientation preserving subgroup is H_i. In particular, G_i/H_i is a Coxeter group of type A_1 and we are in case (i) of Theorem 3.

In the second case, F_i has the Coxeter diagram

$$\bullet_{s_1^{(1)}} \overset{m_0}{-} \bullet_{s_2^{(1)}} \quad \bullet_{s_1^{(2)}} \overset{m_0}{-} \bullet_{s_2^{(2)}} \cdots \bullet_{s_1^{(l)}} \overset{m_0}{-} \bullet_{s_2^{(l)}}$$

with $m_0 > 2$ and $l > 1$. As in Proposition 47 we see that $H_i < G^*(2m_0, 2, l)$ and that H_i acts on the irreducible components of F_i as the symmetric group $\mathfrak{S}_l$ (cf. Proposition 35 and Proposition 36). We can choose the generators of F_i such that all $s_1^{(j)}$ and all $s_2^{(j)}$ are conjugate among each other under H_i and thus identical in G_i/H_i by Lemma 51. Let k be the smallest positive integer such that $(s_1^{(j)} s_2^{(j)})^k \in H_i$. Then k divides m_0 and for $m = \frac{m_0}{k}$ we have $G^*(mk, k, l) < H_i$, because the rotations $s_1^{(j)} s_1^{(j')}$, $s_2^{(j)} s_2^{(j')}$, $j, j' = 1, \dots, l$, are contained in H_i (cf. Section 1.4.1). If $H_i = G^*(2m_0, 2, l)$, then $k = 1$ and $G = G^\times(2m_0, 2, l)$. Otherwise, we have $H_i < G^*(m_0, 1, l)$. For $k = 1$ this implies $H_i = G^*(m_0, 1, l)$ and $G = G^\times(m_0, 1, l)$. For $k \neq 1$ the relation

$$k = \mathrm{ord}(\overline{s}_1^{(1)} \overline{s}_2^{(1)}) = \mathrm{ord}(\overline{s}_1^{(1)} \overline{s}_2^{(2)}) \leq 2$$

shows that $k = 2$ and hence $H_i = G^*(m_0, 1, l)$ and $G = G^\times(m_0, 1, l)$. In each case the group $\tilde{H}_i = H_i \cap F_i$ is generated by the rotations $(s_1^{(j)} s_2^{(j)})^k$, $s_1^{(j)} s_1^{(j')}$, $s_2^{(j)} s_2^{(j')}$, $j, j' = 1, \dots, l$. Thus G_i/H_i is either a Coxeter group of type A_1 or $\mathrm{A}_1 \times \mathrm{A}_1$ depending on whether $k = 1$ or $k = 2$ and we are in case (ii) or (iii) of Theorem 3. □

For the other two cases we need the following two facts on reflection groups.

Lemma 53. *Let $s_1, \dots, s_l$ be simple reflections generating a reflection group W and let $M < W$ be a rotation group. Then every rotation $h \in M$ is conjugate to $(s_i s_j)^r$ for some $i, j \in \{1, \dots, l\}$ and some positive integer r. In particular, if $s_i s_j$ has prime order, then $h' = s_i s_j$ is a rotation contained in M.*

Proof. The linear fixed point subspace $U = \mathrm{Fix}(h)$ of h has codimension two and is contained in a hyperplane corresponding to a reflection $s \in W$, since W acts freely on its chambers [Hum90, p. 23]. The composition $s' = sh$ is another reflection in W whose linear fixed point subspace contains U. Let $s'' \in W$ be a reflection different from s with $U \subset \mathrm{Fix}(s'')$ such that s and s'' are faces of a common chamber. Then we have $h \in \langle ss'' \rangle$ and thus the claim follows, since all sets of generating simple reflections in a reflection group are conjugate to each other [Hum90, Thm. 1.4, p. 10]. □

Lemma 54. *Let $s_1, s_2, \tau \in W$ be reflections in a reflection group W and let $M \lhd W$ be a normal subgroup generated by rotations such that $\overline{s}_1 = \overline{s}_2 \in W/M$ and set $m = \mathrm{ord}(s_1\tau)$ and $n = \mathrm{ord}(s_2\tau)$. Then for $d = \gcd(m, n)$ the powers $(s_1\tau)^d$ and $(s_2\tau)^d$ are contained in M. In particular, $d = 1$ implies $\overline{s}_1 = \overline{s}_2 = \overline{\tau}$.*

Proof. Choose integers p, q such that $d = mp + nq$. Because of $\overline{s}_1 = \overline{s}_2$ we have

$$(\overline{s}_1\overline{\tau})^d = (\overline{s}_2\overline{\tau})^d = (\overline{s}_1\overline{\tau})^{mp}(\overline{s}_2\overline{\tau})^{nq} = e$$

and thus $(s_1\tau)^d, (s_2\tau)^d \in M$. □

Proposition 55. *Assume that $G_i = F_i$ is an irreducible reflection group. Then a set of simple reflections generating F_i projects onto a set $\overline{S} \subset G_i/H_i$ for which $(G_i/H_i, \overline{S})$ is a Coxeter system. More precisely, the quadruple $(G_i, F_i, H_i, \Gamma_i)$ occurs in one of the cases (iv) to (xii) in Theorem 3.*

Proof. Let $\{s_1, \dots, s_l\} < G_i$ be a set of simple reflections generating G_i and set $\overline{m}_{ij} = \mathrm{ord}(\overline{s}_i\overline{s}_j)$. According to Lemma 53 the group H_i is generated by conjugates of elements of the form $(s_r s_s)^{\tilde{m}_{rs}}$ with $\tilde{m}_{rs} \le m_{rs}$ and thus $(G_i/H_i, \overline{S})$ is a Coxeter system.

For trivial H_i the quotient G_i/H_i can be any irreducible Coxeter group and we are in case (iv) of Theorem 3. If all generators lie in the same coset of H_i, then H_i is the orientation preserving subgroup of the reflection group G_i by Lemma 51 and the quotient group $G_i/H_i \cong \mathbb{Z}_2$ is generated by the coset of a reflection in G_i. Hence we are case (v) of Theorem 3. If H_i is nontrivial, then Lemma 53 implies that $(s_i s_j)^r \in H_i$ for some pair of distinct generators s_i and s_j and some $r < \mathrm{ord}(s_i s_j)$. If we additionally assume that not all generators of G_i lie in the same coset of H_i, then Lemma 54 implies that only the types A_3, BC_n, D_n, $\mathrm{I}_2(m)$ and F_4

can occur for G_i. More precisely, the following cases can occur.
(A) $G_i = W(\mathrm{A}_3)$.

$$\bullet_{s_1} - \bullet_{s_2} - \bullet_{s_3}$$

We have $\overline{s}_1 = \overline{s}_3 \neq \overline{s}_2$. The group G_i is the symmetry group of a tetrahedron and $H_i = W^+(A_1 \times A_1 \times A_1)$ is its unique orientation preserving normal subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The quotient group G_i/H_i has the Coxeter diagram

$$\circ_{\overline{s}_1} - \circ_{\overline{s}_2}$$

(B) $G_i = W(\mathrm{BC}_l)$, $l \geq 3$.

$$\bullet_{s_1} - \bullet_{s_2} - \cdots \bullet_{s_{l-1}} = \bullet_{s_l}$$

In any case we have $\overline{m}_{l-1,l} = 2$ by Lemma 53 and Lemma 54 and thus H_i contains the diagonal subgroup of $W^+(\mathrm{BC}_l)$. If all generators of G_i lie in different cosets of H_i, then $H_i = D(W(\mathrm{D}_l))$ and the quotient group G_i/H_i has the Coxeter diagram

$$\circ_{\overline{s}_1} - \circ_{\overline{s}_2} - \cdots \circ_{\overline{s}_{l-1}} \quad \circ_{\overline{s}_l}$$

Otherwise, for $l \neq 4$ Lemma 54 implies $\overline{s}_1 = \ldots = \overline{s}_{l-1} \neq \overline{s}_l$. In this case we have $H_i = W^+(\mathrm{D}_l)$ and the quotient group G_i/H_i has the Coxeter diagram

$$\circ_{\overline{s}_1} \quad \circ_{\overline{s}_l}$$

For $l = 4$ we may also have $\overline{s}_1 = \overline{s}_3 \neq \overline{s}_2$ and $\overline{s}_1, \overline{s}_2 \neq \overline{s}_4$. In this case H_i is the preimage in $W^+(\mathrm{BC}_4) = (\mathbf{O}/\mathbf{V}; \mathbf{O}/\mathbf{V})$ of the normal subgroup of $\mathfrak{S}_4 \cong W^+(\mathrm{BC}_4)/D(W^+(\mathrm{BC}_4))$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (cf. case (A)). It is a monomial rotation group of type $G^*(4,2,2) = (\mathbf{V}/\mathbf{V}; \mathbf{V}/\mathbf{V})$. The quotient group G_i/H_i has the Coxeter diagram

$$\circ_{\overline{s}_1} - \circ_{\overline{s}_2} \quad \circ_{\overline{s}_4}$$

(C) $G_i = W(\mathrm{D}_l)$, $l \geq 4$.

$$\bullet_{s_1} - \bullet_{s_2} - \cdots \bullet_{s_{l-2}} < \begin{matrix} \bullet_{s_l} \\ \bullet_{s_{l-1}} \end{matrix}$$

In any case we have $\overline{s}_{l-1} = \overline{s}_l$ (perhaps after relabeling in the case $l = 4$) by Lemma 53 and Lemma 54 and thus $D(W(\mathrm{D}_l)) < H_i$. For $l \neq 4$ all other generators lie in different cosets of H_i. In this case we have $H_i = D(W(\mathrm{D}_l))$ and the Coxeter diagram of the quotient group G_i/H_i is

$$\circ_{\overline{s}_1} - \circ_{\overline{s}_2} - \cdots \circ_{\overline{s}_{l-2}} - \circ_{\overline{s}_{l-1}}$$

For $l = 4$ we may also have $\overline{s_1} = \overline{s_3} = \overline{s_4} \neq \overline{s_2}$. In this case $H_i = G^*(4,2,2) = (\mathbf{V}/\mathbf{V}; \mathbf{V}/\mathbf{V})$ (cf. case (B)) and the quotient group G_i/H_i has the Coxeter diagram

$$\circ_{\overline{s}_1} - \circ_{\overline{s}_2}$$

(D) $G_i = W(\mathrm{I}_2(m))$ for $m \geq 4$.

$$\bullet_{s_1} \overset{m}{-} \bullet_{s_2}$$

We have $\overline{s}_1 \neq \overline{s}_2$ and H_i is a cyclic group of order $\frac{m}{\overline{m}_{1,2}}$. Consequently, the quotient group G_i/H_i is a dihedral group of type $\mathrm{I}_2(\overline{m}_{1,2})$ with Coxeter diagram

$$\circ_{s_1} \overset{\overline{m}_{1,2}}{-} \circ_{s_2}$$

(E) $G_i = W(\mathrm{F}_4)$.

$$\bullet_{s_1} - \bullet_{s_2} = \bullet_{s_3} - \bullet_{s_4}$$

In any case we have $\overline{m}_{2,3} = 2$ by Lemma 53 and Lemma 54. If all generators lie in different cosets of H_i, then $H_i = G^*(4,2,2) = (\mathbf{V}/\mathbf{V};\mathbf{V}/\mathbf{V})$ and the quotient group G_i/H_i has the Coxeter diagram

$$\circ_{\overline{s}_1} - \circ_{\overline{s}_2} \quad \circ_{\overline{s}_3} - \circ_{\overline{s}_4}$$

If $\overline{s}_3 = \overline{s}_4$ and all other generators lie in different cosets of H_i, then $H_i = W^+(\mathrm{D}_4) = (\mathbf{T}/\mathbf{V};\mathbf{T}/\mathbf{V})$ and the quotient group G_i/H_i has the Coxeter diagram

$$\circ_{\overline{s}_1} - \circ_{\overline{s}_2} \quad \circ_{\overline{s}_3}$$

Finally, if $\overline{s}_1 = \overline{s}_2$, $\overline{s}_3 = \overline{s}_4$, then $H_i = W^*(\mathrm{D}_4) = (\mathbf{T}/\mathbf{T};\mathbf{T}/\mathbf{T})$ and the quotient group G_i/H_i has the Coxeter diagram

$$\circ_{\overline{s}_1} \quad \circ_{\overline{s}_3}$$

□

Proposition 56. *Assume that F_i is an irreducible reflection group different from G_i. Then a set of simple reflections generating F_i projects onto a set $\overline{S} \subset G_i/H_i$ for which $(G_i/H_i, \overline{S})$ is a Coxeter system of type A_1 or $\mathrm{A}_1 \times \mathrm{A}_1$. More precisely, the quadruple $(G_i, F_i, H_i, \Gamma_i)$ occurs in one of the cases (xvi) to (xvii) in Theorem 3.*

Proof. Let $h \in H_i \backslash F_i$ be a rotation. By Lemma 50 and the proof of Lemma 5 there exists a chamber of the reflection group F_i such that $hC = C$. By Lemma 6 we deduce that F_i has type A_4, D_4, F_4, A_5 or E_6. In the cases of A_4, F_4, A_5 and E_6 Lemma 51 and Lemma 54 imply that all generators of F_i lie in the same coset of H_i and thus we have $H_i = \langle F_i^+, h\rangle$ and $G_i/H_i = \mathbb{Z}_2$ in these cases. If F_i has type D_4, then h has order 3 and we have $\overline{s}_1 = \overline{s}_3 = \overline{s}_4$.

$$\bullet_{s_1} - \bullet_{s_2} < \begin{matrix} \bullet_{s_4} \\ \bullet_{s_3} \end{matrix}$$

If $\overline{s}_1 = \overline{s}_2$ also holds, then we have again $H_i = \langle F_i^+, h\rangle$ and $G_i/H_i = \mathbb{Z}_2$. Otherwise the group $\tilde{H}_i = H_i \cap F_i$ is generated by the conjugates of $s_1 s_3$ and $s_1 s_4$ and the group H_i is generated by $\tilde{H}_i$ and h. We have $H_i = W^+(\mathrm{D}_4)$ (, but $H_i \neq F_i^+$) and the quotient group $G_i/H_i = F_i/\tilde{H}_i$ has the Coxeter diagram

$$\circ_{\overline{s}_1} - \circ_{\overline{s}_2}$$

□

The preceding three propositions show that each triple (G_i, H_i, F_i) induced by a reflection-rotation groups occurs in one of the cases described in Theorem 3. Moreover, it is easy to check that each triple (G_{rr}, M, W) occurring in Theorem 3 satisfies the conclusion of this Theorem concerning the reflections in W and the properties described in Remark 1.

As a corollary we record

Corollary 57. *The reflections in $(G_i/H_i, \overline{S})$ are precisely the cosets of reflections in F_i.*

In order to describe the structure of the whole group G we need the following lemmas.

Lemma 58. *Let $\tau, \tau' \in G_i$ and $s \in G_j$ be reflections such that $\tau s \in G$. Then $\tau' s \in G$, if and only if $\overline{\tau} = \overline{\tau}'$.*

Proof. If $\tau' s \in G$ then $\tau' s s \tau = \tau' \tau$ is a rotation of the first kind in G and thus $\tau' \tau \in H_i$, i.e. $\overline{\tau} = \overline{\tau}'$. On the other hand $\overline{\tau} = \overline{\tau}'$ implies $\tau = h\tau'$ for some $h \in H_i$ and thus $\tau' s = h^{-1} \tau s \in G$. □

Lemma 59. *Let $s_1, s_2 \in F_i$, $s_3 \in F_j$ and $s_4 \in F_{j'}$, $i \neq j, j'$, be reflections such that $g = s_1 s_3, g' = s_2 s_4 \in G$. Then the following two implications hold.*

(i) $j = j' \Rightarrow \mathrm{ord}(\overline{s}_1 \overline{s}_2) = \mathrm{ord}(\overline{s}_3 \overline{s}_4)$.

(ii) $j \neq j' \Rightarrow \mathrm{ord}(\overline{s}_1 \overline{s}_2) \leq 2$.

Proof. (i) Assume that $j = j'$ and set $m = \mathrm{ord}(\overline{s}_1 \overline{s}_2)$ and $n = \mathrm{ord}(\overline{s}_3 \overline{s}_4)$. If $\overline{s}_1 = \overline{s}_2$ then Lemma 58 implies that $\overline{s}_3 = \overline{s}_4$ and thus $m = n = 1$. Otherwise, $(gg')^n = (s_1 s_2)^n (s_3 s_4)^n = (s_1 s_2)^n h$ for some $h \in H_j$ implies that $(s_1 s_2)^n \in G$ is a rotation of the first kind contained in H_i and therefore $m|n$. In the same way we obtain $n|m$ and thus $m = n$.

(ii) Since $(s_1 s_2)^2 = (gg')^2$ is a rotation of the first kind in G or trivial, we deduce that $\mathrm{ord}(\overline{s}_1 \overline{s}_2) \leq 2$. □

For reflections $s \in F_i$ and $\tau \in F_j$ we call $\overline{s}$ and $\overline{\tau}$ *related* if $s\tau \in G$ and $s \notin G$. Lemma 58 shows that this notion is well-defined. For a Coxeter group C we denote the set of reflections contained in C by $X(C)$ and we set $\tilde{G} = G_1/H_1 \times \cdots \times G_k/H_k$, $X = X(\tilde{G})$ and $X_i = X(G_i/H_i)$.

Lemma 60. *Relatedness of reflections defines an equivalence relation on the set X such that related reflections belong to different components.*

Proof. Let $\overline{s}_1 \in G_i/H_i$, $\overline{s}_2 \in G_j/H_j$ and $\overline{s}_3 \in G_l/H_l$ be reflections. If $\overline{s}_1$ and $\overline{s}_2$ are related as well as $\overline{s}_2$ and $\overline{s}_3$, then so are $\overline{s}_1$ and $\overline{s}_3$, because of $s_1 s_3 = (s_1 s_2)(s_2 s_3) \in G_i$. For $i = j$ the cosets $\overline{s}$ and $\overline{\tau}$ are related if and only if $\overline{s} = \overline{\tau}$. □

For $i, j \in I$, $i \neq j$, we define X_{ij} to be the set of reflections in G_i/H_i that are related to reflections in G_j/H_j. Let Γ_i be the Coxeter diagram of G_i/H_i and set $\Gamma = \bigcup \Gamma_i$. The vertices of Γ_i correspond to a set of simple reflections of G_i/H_i (cf. [Hum90, p. 29]).

Lemma 61. *A reflection $\overline{s}$ in G_i/H_i that is not related to any other reflection, corresponds to an isolated vertex of Γ_i.*

Proof. Suppose that $\overline{s} \in G_i/H_i$ is a reflection not related to any other reflection and that $\overline{\tau} \in G_i/H_i$ is another reflection with $\mathrm{ord}(\overline{s\tau}) \geq 3$. Then $\overline{\tau}$ is related to some reflection $\overline{\tau}' \in G_j/H_j$ for some $j \neq i$, because otherwise we would have $s\tau \in H_i$ by Lemma 49. This implies $\mathrm{ord}(\overline{s\tau}) \leq 2$ as in the proof of Lemma 59, (ii), and thus the claim follows by contradiction. □

Lemma 62. *Let M be a connected component of Γ_i and let $\mathcal{M}$ be the set of generators of G_i/H_i that correspond to the vertices of M. If there exists a reflection in $\mathcal{M}$ related to another reflection, then there exists some $j \neq i$ such that $\mathcal{M} \subset X_{ij}$. Moreover, $\mathcal{M} \subset X_{ij} \cap X_{ik}$ for some $k \neq i, j$ only if $M = \circ$.*

Proof. The first claim follows from Lemma 49 and the preceding lemma. Suppose we have distinct reflections $\overline{s}, \overline{\tau} \in \mathcal{M}$ with $\mathrm{ord}(\overline{s\tau}) \geq 3$. Again by Lemma 49 and the preceding lemma there are j, k such that $\overline{s} \in X_{ij}$ and $\overline{\tau} \in X_{ik}$. Lemma 59, (ii) implies that $j = k$. □

Due to this lemma the reflections related to the reflections of a nontrivial irreducible component of $\tilde{G}$ belong to a common G_i/H_i. The next proposition sharpens this statement.

Proposition 63. *The set of nontrivial irreducible components of $\tilde{G}$ decomposes into pairs of isomorphic constituents that belong to different G_i/H_i and for each such pair relatedness of reflections defines an isomorphism between its constituents that maps reflections onto related reflections.*

Proof. Let M be a nontrivial connected component of Γ_i and let $\mathcal{M}$ be the set of simple reflections corresponding to the vertices of M. According to Lemma 62 there exists a unique $j \neq i$ such that $\mathcal{M} \subset X_{ij}$. Define $\varphi : \mathcal{M} \to G_j/H_j$ by mapping a generator $\overline{s}_i \in \mathcal{M}$ to its related reflection in G_j/H_j. Due to Lemma 59, (i), this map can be extended to a homomorphism $\varphi : \langle \mathcal{M} \rangle \to G_j/H_j$. We claim that the image $\varphi(\overline{s})$ of any reflection $\overline{s} \in \langle \mathcal{M} \rangle$ is a reflection related to $\overline{s}$. Since $\overline{s}$ is conjugate to a reflection in M its image $\varphi(\overline{s})$ is a reflection in G_j/H_j and thus a coset of a reflection in F_j, say $\varphi(\overline{s}) = \overline{\tau}$ for some $\tau \in F_j$ (cf. Corollary 57). Write $\overline{s} = \overline{s}_{i_1} \cdots \overline{s}_{i_l}$ for generators $\overline{s}_{i_1}, \ldots, \overline{s}_{i_l} \in \mathcal{M}$ and let $\overline{\tau}_{i_j} = \varphi(\overline{s}_{i_j})$ be the related reflection. Then we have $\overline{\tau} = \overline{\tau}_{i_1} \cdots \overline{\tau}_{i_l}$. There exist $h_i \in H_i$ and $h_j \in H_j$ such that $s = h_i s_1 \cdots s_l$ and $\tau = h_j \tau_1 \cdots \tau_l$ and thus $s\tau = h_i h_j s_{i_1} \tau_{i_1} \cdots s_{i_l} \tau_{i_l} \in G$. Hence the reflections $\overline{s}$ and $\overline{\tau} = \varphi(\overline{s})$ are related.

The fact that φ maps reflections onto related reflections together with Lemma 59, (i) implies that $\varphi(\langle \mathcal{M} \rangle)$ is contained in an irreducible component of G_j/H_j (cf. the argument below). Let N be the connected component of Γ_j such that $\langle \varphi(\mathcal{M}) \rangle \subseteq \langle \mathcal{N} \rangle$ where $\mathcal{N}$ is the set of simple reflections corresponding to the vertices of N. Since N is connected, for $\overline{\tau}_0 \in \mathcal{N}$ there exist reflections $\overline{s}_k \in \mathcal{M}$ and $\overline{\tau}_0, \overline{\tau}_1, \ldots, \overline{\tau}_k \in \langle \mathcal{N} \rangle$ with $\overline{\tau}_k = \varphi(\overline{s}_k)$ and $\mathrm{ord}(\overline{\tau}_l \overline{\tau}_{l+1}) \geq 3$, $l = 0, \ldots, k-1$. Therefore, according to Lemma 49, Lemma 61 and Lemma 59, (ii), there are reflections $\overline{s}_0, \ldots, \overline{s}_{k-1} \in G_i/H_i$ such that $\overline{s}_l$ and $\overline{\tau}_l$ are related for $l = 0, \ldots, k-1$. Lemma 59, (i), implies that $\mathrm{ord}(\overline{s}_l \overline{s}_{l+1}) = \mathrm{ord}(\overline{\tau}_l \overline{\tau}_{l+1}) \geq 3$, $l = 0, \ldots, k-1$, and thus $\overline{s}_0, \ldots, \overline{s}_k \in \langle \mathcal{M} \rangle$. In particular, we have $\overline{\tau}_0 = \varphi(\overline{s}_0) \in \langle \varphi(\mathcal{M}) \rangle$ by what has been shown above and hence $\langle \varphi(\mathcal{M}) \rangle = \langle \mathcal{N} \rangle$, i.e. φ is an epimorphism between the irreducible component $\langle \mathcal{M} \rangle$ of G_i/H_i and the irreducible component $\langle \mathcal{N} \rangle$ of G_j/H_j. By the same argument there exists a homomorphism from $\langle\mathcal{N}\rangle$ to G_i/H_i which maps $\langle\mathcal{N}\rangle$ onto $\langle\mathcal{M}\rangle$. Therefore, $\langle\mathcal{M}\rangle$ and $\langle\mathcal{N}\rangle$ have the same cardinality and thus $\varphi : \langle\mathcal{M}\rangle \to \langle\mathcal{N}\rangle$ is an isomorphism of Coxeter groups. $\square$

Now we can prove Theorem 4.

Proof of Theorem 4. Let G be a reflection-rotation group and $\tilde{G}$ be given as above. According to what has been shown so far, relatedness of reflections induces an equivalence relation on the set of irreducible components of $\tilde{G}$ such that two related components belong to different G_i/H_i (cf. Lemma 60). By Lemma 49 each G_i/H_i contains at most one trivial irreducible component that is not related to another component. By the preceding proposition each equivalence class of a nontrivial irreducible component of $\tilde{G}$ contains precisely two isomorphic components and an isomorphism between them is induced by relatedness of reflections. Conversely, a family of possible triples $\{(G_i, H_i, F_i)\}_{i\in I}$ and an equivalence relation on the irreducible components of $\tilde{G} = G_1/H_1 \times \cdots \times G_k/H_k$ satisfying the conditions in Theorem 4 together with isomorphisms between the equivalent nontrivial irreducible components of $\tilde{G}$ that map reflections onto reflections defines a reflection-rotation group as described in the Introduction.

It remains to show that these assignments are inverse to each other. If we start with a reflection-rotation group G, assign to it a set of data as in the theorem and to this set of data another reflection-rotation group $\tilde{G}$, then $\tilde{G}$ is generated by the rotations in G and thus coincides with G. Suppose we start with a set of data as in the theorem, including a family $\{(G_i, H_i, F_i)\}_{i\in I}$ of triples occurring in Theorem 3, assign to it a reflection-rotation group G and to this reflection-rotation group another set of data including a family of triples $\{(\tilde{G}_i, \tilde{H}_i, \tilde{F}_i)\}_{i\in J}$. Then we clearly have $I = J = \{1, \ldots, k\}$ and $G_i = \tilde{G}_i$ for all $i \in I$. We also have $H_i < \tilde{H}_i$ and $F_i < \tilde{F}_i$. By construction (cf. condition (ii) in Theorem 4) the quotient $G/(H_1 \times \cdots \times H_k)$ does not contain nontrivial cosets of rotations of the first kind in G and thus $H_i = \tilde{H}_i$. Since each reflection in G_i is contained in F_i (cf. Theorem 3) $F_i = \tilde{F}_i$ holds as well. Now it is clear that the two sets of data coincide and thus the theorem is proven. $\square$

We record the following two corollaries. Recall that a reflection-rotation group is called *indecomposable* if it cannot be written as a product of nontrival subgroups that act in orthogonal spaces (cf. Section 1.3.6).

Corollary 64. *Let G be a reducible reflection-rotation group that only contains rotations of the first kind. Then G is a direct product of indecomposable rotation groups.*

Corollary 65. *For an indecomposable reflection-rotation group G that does not contain rotations of the first kind one of the following three cases holds*

(i) $k = 2$, $\dim V_1 = \dim V_2$ *and* $G_1 \cong G_2$ *for irreducible reflection groups* G_1, G_2.

(ii) $k > 2$, $\dim V_1 = \ldots = \dim V_k = 1$ *and G consists of all elements that change the sign of an even number of coordinates, i.e.* $G = W^+(\mathrm{A}_1 \times \cdots \times \mathrm{A}_1)$.

(iii) $G = W(\mathrm{A}_1)$.

Note that the group G in case (i) is only determind by G_1 *and* the choice of an isomorphism between G_1 and G_2 that maps reflections onto reflection (cf. Section 1.3.6).

Chapter 2

Equivariant smoothing of piecewise linear manifolds

2.1 Introduction

In this chapter we solve a challenge by Thurston on 3-manifolds which is helpful in the proof of Theorem B. We also prove the corresponding statement on 4-manifolds which confirms a conjecture by Kwasik and Lee in a stronger form. A piecewise linear- and a smooth structure on a manifold M are called *compatible* with each other, if there exists a triangulation of M as a piecewise linear manifold all of whose simplices are smoothly embedded with respect to the smooth structure. Due to a theorem by Whitehead every smooth manifold M admits a unique compatible piecewise linear structure [Whi40, Mun66]. An equivariant version of this result for smooth actions of finite groups on M holds by a theorem of Illman [Ill78]. Conversely, any piecewise linear manifold of dimension $n \leq 7$ admits a *smoothing*, i.e. a compatible smooth structure [HM74, KS77].

We will encounter an equivariant version of this smoothing problem in the following situation. Suppose we have finite groups $H \triangleleft G < \mathrm{O}_{n+1}$ and would like to understand the quotient space S^n/G. If we already know that S^n/H is a piecewise linear sphere, then we might try to study the piecewise linear action of G/H on S^n/H instead (cf. Chapter 3 for precise definitions). Moreover, if $n \leq 3$ and if S^n/H admits a compatible smooth structure with respect to which G/H acts smoothly, then this action can be smoothly conjugated to a linear action (cf. Section 2.2.4) and the problem of determining S^n/G be reduced to the easier problem of understanding the quotient of the linear action of G/H on S^n(cf. Section 3.3.1).

More generally, given a piecewise linear n-manifold M on which a finite group G acts by piecewise linear homeomorphisms one can ask if there exists an equivariant smoothing, i.e. a smoothing with respect to which G acts smoothly. It is always possible to find a piecewise linear triangulation of M with respect to which the group G acts simplicially (cf. Section 2.2.1). For $n = 1$ we can choose the lengths of the segments of such a triangulation so that G acts isometrically. In this way we obtain a desired smooth structure. The task of finding such in the case $n = 2$ appears in Thurston's book [Thu97, pp. 207-208] in a slightly modified form as a "problem" and in the case $n = 3$ as a "challenge". For $n = 4$ existence has been conjectured in a weaker form, i.e. without the compatibility condition, in [KL88] by Kwasik and Lee. In this chapter we show that for $n \leq 4$ equivariant smoothings always exist.

Theorem A. Every piecewise linear manifold M of dimension $n \leq 4$ on which a finite group

G acts by piecewise linear homeomorphisms admits an equivariant smoothing.

If there exists an equivariant smoothing, then there also exists an equivariant triangulation whose simplices are all smoothly embedded (cf. Proposition 66). In dimension higher than four the statement of the theorem is false, even without the compatibility condition (cf. [KL88] or Section 2.3.7 below).

In his book Thurston asks for so-called *canonical smoothings* of triangulated piecewise linear manifolds [Thu97, pp. 207-208], which are, in particular, preserved by simplicial isomorphisms. For $n = 3$ he remarks that one probably needs some "heavy machinery such as the uniformization theorem for Riemannian metrics on S^2, used with ingenuity" [Thu97, p. 208]. The uniformization theorem implies that smooth actions of finite groups on S^2 are smoothly conjugate to linear actions (cf. Section 2.2.4). The corresponding property for S^3 is assumed in [KL88] while formulating the above mentioned conjecture and has later been proven by Dinkelbach and Leeb [DL09] using Ricci flow techniques. Indeed, it turns out that the key ingredients for proving Theorem A are uniqueness of smoothings and the linearizability of finite smooth group actions on spheres up to dimension three. Using these tools we will be able to prove Theorem A.

2.2 Preliminaries and techniques

2.2.1 Piecewise linear spaces

In this section we prove a statement (Proposition 66) that enables us to reformulate Theorem A into a more workable version (cf. Section 2.3). First we remind of some concepts from piecewise linear topology. For more details we refer to [Hud69, RS72]. A subset $P \subset \mathbb{R}^n$ is called a *polyhedron* if, for every point $x \in P$, there exists a finite number a simplices contained in P such that their union is a neighbourhood of x in P. An open subset of a polyhedron is again a polyhedron. Every polyhedron P in $\mathbb{R}^n$ is the underlying space of some (locally finite) simplicial complex K in $\mathbb{R}^n$ [Hud69, Lem. 3.5]. Such a complex is called a *triangulation* of P. A continuous map $f : P \to Q$ between polyhedra $P \subset \mathbb{R}^n$, $Q \subset \mathbb{R}^m$ is called *piecewise linear* (*PL*), if its graph $\{(x, f(x))|x \in P\} \subset \mathbb{R}^{m+n}$ is a polyhedron. It is called *PL homeomorphism*, if it has in addition a PL inverse. This is the case if and only if there exist triangulations of P and Q with respect to which f is a simplicial isomorphism [Hud69, p. 84, Thm. 3.6.C]. A polyhedron P is called a *PL manifold* (*with boundary*) of dimension n, if every point $p \in P$ has an open neighbourhood in P that is PL homeomorphic to $\mathbb{R}^n$(or to $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$). If a simplicial complex K triangulates a polyhedron P, then P is a PL n-manifold, if and only if the link of every vertex of K is a PL $(n-1)$-sphere, i.e. PL homeomorphic to $\partial\Delta^n$. PL manifolds can also be defined as abstract spaces with a *PL structure* [Hud69, Ch. 3]. However, every such space can be realised as a polyhedron in some $\mathbb{R}^N$ [Hud69, Lem. 3.5, p. 80][Moi77, Thm. 7.1, p. 53]. The following statement is certainly known, but the author has not found a reference.

Proposition 66. *A piecewise linear manifold M on which a finite group G acts by piecewise linear homeomorphisms can be triangulated by a simplicial complex K such that G acts simplicially on $|K|$, i.e. it maps simplices linearly onto simplices.*

To prove it we need the notion of a (locally finite) *cell-complex* (cf. [RS72]). It can be defined like a simplicial complex, but it is not built up merely from simplices, but more generally from compact convex polyhedra, the so-called *cells* (cf. [RS72, pp. 14-15]). The linear image of a cell is again a cell and the intersection $K \cap L = \{A \cap B | A \in K, B \in L\}$ of two cell complexes K and L is again a cell complex. A *subdivision* of a cell complex K is a simplicial complex $\tilde{K}$ such that $|K| = |\tilde{K}|$ and such that every simplex of $\tilde{K}$ is contained in a cell of K. The *k-skeleton* $K_{(k)}$ of a cell complex K is the cell complex comprising all cells of K of dimension smaller or equal to k. We have

Lemma 67. *Any cell complex can be canonically subdivided into a simplicial complex.*

Proof. The 1-skeleton of K is already a simplicial complex. We successively subdivide the $2, \ldots, n$-skeleton of K. Assume that we have already subdivided the $2, \ldots, k$-skeletons of K. Then we *star* the $(k+1)$-cells of its $(k+1)$-skeleton at their barycenters, i.e. we replace each $(k+1)$-cell C by the simplicial complex obtained as the join of the boundary ∂C, which is already a simplicial complex, with the barycenter of C (cf. [RS72, p. 15]). Having replaced the n-cells we arrive at a simplicial complex that subdivides our initial cell complex. $\square$

With the same method one can prove

Lemma 68. *Let $\tilde{K}$ be a subcomplex of a cell complex K. Then any simplicial subdivision of $\tilde{K}$ can be extended to a simplicial subdivision of K.*

Now we can give a proof for Proposition 66.

Proof of Proposition 66. We first triangulate M by a simplicial complex $K \subset \mathbb{R}^N$. For each $g \in G$ we can choose a subdivision L_g of K such that g maps simplices of L_g linearly into simplices of K (cf. [Hud69, Thm. 3.6, B, p. 84]). Let L be the cell complex obtained by intersecting the cell complexes L_g. Then the restriction of each element $g \in G$ to each cell of L is linear. Therefore, the translates gL are again cell complexes. Hence, their intersection $\bigcap_{g \in G} gL$ is a cell complex on which G acts cellularly, i.e. it maps cells linearly onto cells. Now we apply Lemma 67 to this complex. By construction, the group G acts simplicially on the resulting simplicial complex. $\square$

Finally we fix some notations. Let K be a simplicial complex. We denote its first *barycentric subdivision* (cf. [Hat02, p. 119]) by $K^{(1)}$. The *support* $\mathrm{supp}_K(x)$ of a point x in K is defined as the smallest dimensional simplex of K that contains x. For a simplex $\sigma < K$ the *star* $\mathrm{star}_K(\sigma)$ is the smallest simplicial complex that contains all simplices of K that contain σ. The link of σ is defined as

$$\mathrm{lk}_K(\sigma) = \{\sigma' \in K | \sigma \cap \sigma' = \emptyset, \exists \tau \in K : \sigma, \sigma' < \tau\}.$$

We write $\mathrm{star}_K(\sigma)$ and $\mathrm{lk}_K(\sigma)$ interchangeably for the star and the link as a simplicial complex and as their underlying space. Also we sometimes omit the index K if its meaning is clear. For a topological space X we denote by $\overline{C}X$ its *closed cone* defined as $(X \times [0,1])/(X \times \{0\})$. For a compact simplicial complex K (in $\mathbb{R}^n$) its closed cone $\overline{C}K$ is again naturally a simplicial complex (in $\mathbb{R}^{n+1}$).

2.2.2 Piecewise differentiable maps and smoothings

The following definition is central for comparing piecewise linear and smooth spaces.

Definition 6. We call a map $f : P \to M$ from a polyhedron P to a smooth manifold with or without boundary M *piecewise differentiable* or *PD*, if there exists a triangulation K of P such that the restriction of f to each simplex is smooth. We call f a *PD homeomorphism (embedding)*, if it is moreover a homeomorphism (onto its image) and each simplex is smoothly embedded, i.e. for each simplex $\sigma \in K$ and each point $p \in \sigma$ the differential $(df_{|\sigma})_p$ is injective.

A smooth structure on a PL manifold with boundary M is called *compatible* with the PL structure of M, if the identity map from M as a PL manifold to M as a smooth manifold is a PD homeomorphism. A compatible smooth structure on M is called a *smoothing*. For the proof of our result we need the fact that smoothings in dimensions $n \leq 3$ are unique up to diffeomorphism [Thu97, Thm. 3.10.9, p. 202] (in fact, we only need this statement for S^n, $n \leq 3$, cf. Lemma 72). In the case $n = 3$ such a proof relies on the fact that every diffeomorphism of S^2 can be extended to a diffeomorphism of the corresponding unit ball (cf. [Thu97, Thm. 3.10.11, p. 202]).

2.2.3 Approximating PD maps by PL maps

In this section we explain how PD maps can be approximated by PL maps. This will be needed in the proof of Theorem A (cf. Lemma 73).

Definition 7. Two PD maps $f, \tilde{f} : P \to \mathbb{R}^n$ are called C^1 *δ-close*, if there exists a triangulation K of P such that both $f_{|\sigma}$ and $\tilde{f}_{|\sigma}$ are smooth and the values of $(f - \tilde{f})_{|\sigma}$ and their first derivatives are bounded by δ for every simplex $\sigma \in K$.

The following statement follows immediately from [Mun66, Thm. 8.8, p. 84, Thm. 8.4, p. 81].

Theorem 69. *Let $f : P \to M \subset \mathbb{R}^n$ be a PD homeomorphism from a compact polyhedron to a smooth connected submanifold M of $\mathbb{R}^n$. Then there exist some $\delta > 0$ such that every PD map $\tilde{f} : P \to M$ that is C^1 δ-close to f is also a PD homeomorphism.*

In order to approximate PD maps by PL maps we need the following concept (cf. [Mun66, p. 90]).

Definition 8. Let $\tilde{K}$ be a subdivision of K and let $f : K \to \mathbb{R}^n$ be a PD map. The *secant map* $L_{\tilde{K}} f : K \to \mathbb{R}^n$ is defined to be the map that is linear on the simplices of $\tilde{K}$ and coincides with f on the vertices of $\tilde{K}$.

By definition $L_{\tilde{K}} f$ is a PL map. For a finite simplicial complex K on which a finite group G acts simplicially, we would like to find G-subdivisions $\tilde{K}$ (i.e. subdivisions on which G acts simplicially) such that $L_{\tilde{K}} f$ becomes close to f in the C^1 sense. According to [Mun66, Lem. 9.3, p. 90], it is sufficient to find G-subdivisions $\tilde{K}$ of K whose simplices' diameters tend to zero while their thickness stay bounded from below. The thickness of a simplex is defined to be the ratio of the minimal distance of its barycenter to its boundary and its diameter. The existence of such subdivisions is proven in the following lemma.

Lemma 70. *Let K be a finite simplicial complex on which a finite group G acts simplicially. There is a $t_0 > 0$ such that K has arbitrarily fine G-subdivisions for which the minimal simplex thickness is at least t_0.*

Proof. For trivial G the claim is proven in [Mun66, Lem. 9.4, p. 92]. We slightly modify that proof so that the G-equivariance can be additionally guaranteed.

We can assume that K is a subcomplex of some standard simplex Δ^{p-1} of dimension $(p-1)$ having the standard basis vectors $\epsilon_1, \dots, \epsilon_p$ of $\mathbb{R}^p$ as vertices. Then the group G embeds into the linear symmetric group on p letters that permutes the basis vectors of $\mathbb{R}^p$. Let $i_0, i_1, \dots, i_p$ be integers and consider the unit cube

$$C(i_1, \dots, i_p) = \{x \in \mathbb{R}^p | i_j \leq x_j \leq i_j + 1, j = 1, \dots, p\}$$

in $\mathbb{R}^n$, where x_j denotes the jth coordinate of x with respect to $\epsilon_1, \dots, \epsilon_p$. Let J be the cell complex obtained by intersecting these unit cubes with the regions

$$R(i_0) = \{x \in \mathbb{R}^p | i_0 \leq x_1 + \dots + x_p \leq i_0 + 1\}.$$

The cell complex J has three properties that are important for us:

(i) Any cell of J is the image of one of the cells contained in the unit cube $C(0, \dots, 0)$ under a translation of $\mathbb{R}^p$.

(ii) The simplex Δ_m spanned by $m\epsilon_1, \dots, m\epsilon_p$ is the underlying space of a subcomplex of J

(iii) The group G acts cellularly on J, i.e. it maps cells linearly onto cells.

Now we subdivide J into a simplicial complex L as described in Lemma 67. It follows that conditions (i)-(iii) hold for the complex L as well. As a result, the simplices of L have s minimal thickness $t_0 > 0$ and a maximal diameter d.

The homothety of $\mathbb{R}^p$ which carries x into x/m does not change the thickness of any simplex, and it multiplies the diameters by $1/m$. Therefore the image of Δ_m under this transformation defines a G-subdivision of $\Delta_m = \Delta^{p-1}$, and thus of K, of thickness at least t_0 and diameter at most d/m. Since m is arbitrary, the lemma is proven. $\square$

As in [Mun66, Thm. 9.6, p. 94] we immediately obtain

Theorem 71. *Let K be a finite simplicial complex on which a finite group G acts simplicially and let $f : K \to \mathbb{R}^n$ be a PD map. Then for every $\delta > 0$ there exists a G-subdivision $\tilde{K}$ of K such that the secant map $L_{\tilde{K}} f$ is C^1 δ-close to f.*

2.2.4 Linearizing smooth actions of finite groups on spheres

Using *Ricci flow* techniques Dinkelbach and Leeb showed that any smooth action of a finite group G on S^3 is smoothly conjugate to an orthogonal action [DL09]. The same statement is true for smooth actions of finite groups on S^2, but in this case it follows more elementary by the *geometrization of spherical 2-orbifolds* [Dav11, Zim12] or by the *uniformization theorem*: Average an arbitrary Riemannian metric on S^2 to obtain a G-invariant Riemannian metric g

on S^2. The metric g determines a complex structure on S^2 (cf. [Che79]) with respect to which G acts biholomorphically. By the uniformization theorem there exists a biholomorphism to the Riemann sphere (cf. e.g. [For81, Thm. 27.9]) and thus a smooth function ϕ on S^2 such that $g_1 = e^{\phi} g$ has constant sectional curvature 1. This function satisfies the equation

$$2\Delta_g \phi + \mathrm{S}(g) = \mathrm{S}(g_1) e^{2\phi}$$

where $\mathrm{S}(g)$ is the curvature of g, $\mathrm{S}(g_1) = 1$ and Δ_g denotes the Laplace operator attached to g (cf. [BE87, II.3, p. 726]). Hence ϕ is unique by the maximum principle. Because of $\mathrm{S}(e^{(\phi \circ h)} g) = \mathrm{S}(h^* g_1) = \mathrm{S}(g_1) \circ h$ for each $h \in G$ the metric $e^{(\phi \circ h)} g$ has constant sectional curvature 1 on S^2 as well for each $h \in G$. This implies that ϕ is G-invariant by the uniqueness statement above. Hence, G acts isometrically with respect to g_1 and its action on S^2 can thus be smoothly conjugated to an orthogonal action on the standard unit sphere $S^2 \subset \mathbb{R}^3$.

2.2.5 Gluing smoothed PL manifolds

Let P_1 and P_2 be two PL manifolds with boundary endowed with smoothings. Suppose there exists a piecewise linear diffeomorphism $f : \partial P_1 \to \partial P_2$. Then there exists a smooth structure on $P_1 \cup_f P_2$ with respect to which P_1 and P_2 are smoothly embedded (cf. [Mil65, Thm. 4.1, p. 25; Remark, p. 24]). Moreover, using the methods described in Section 3.1.1 one can find triangulations of P_1 and P_2 whose simplices are smoothly embedded and with respect to which the map f is a simplicial isomorphism. Such triangulations give rise to a triangulation of $P_1 \cup_f P_2$ whose simplices are smoothly embedded. Hence, the smooth structure on $P_1 \cup_f P_2$ above in fact defines a smoothing.

2.3 Proof of Theorem A

Let K be a simplicial complex that is also a PL manifold of dimension $n \leq 4$ and let G be the group of all simplicial isomorphisms of K. We are going to show that the complex K admits a G-equivariant smooth structure and a subdivision all of whose simplices are smoothly embedded. In view of Proposition 66 on the existence of equivariant triangulations, this will imply the statement of Theorem A. Perhaps after taking the first barycentric subdivision of K we can assume that a simplex σ of K invariant under some $g \in G$ is pointwise fixed by g.

We endow K with an auxiliary polyhedral (length) metric such that all edges have unit length and such that all simplices of K are flat (cf. [BBI01]). For $n = 1$ we can isometrically identify the resulting metric space with a distance circle in $\mathbb{R}^2$ or a real line to obtain a desired equivariant smoothing. For $n \geq 2$ this strategy does not work. We can put a smooth structure on the complement K^* of the $(n-2)$-skeleton of K in K such that every isometry between a subset of K^* and a subset of $\mathbb{R}^n$ is smooth. However, in general it is not possible to extend it to a *compatible* smooth structure on K.

In order to extend the smoothing we have to change the smooth structure on K^* in a small neighbourhood of the $(n-2)$-skeleton. Let us begin with the simplest case.

2.3.1 Proof for 2-manifolds and strategy for higher dimensions

The canonical metric on K introduced above is the induced length metric of a canonical piecewise flat Riemannian metric on K. Suppose that K is 2-dimensional and that a vertex x of K is contained in n 2-simplices of K. Then we can embed $\mathrm{star}_K(x)$ as a regular n-gon of radius 1 into $\mathbb{R}^2$. Using the cone parameter of $\mathrm{star}_K(x) = \overline{C}\mathrm{lk}_K(x)$ and a smooth cut-off function, in a neighbourhood of x in K we can interpolate between the piecewise Riemannian metric induced from this embedding and the canonical piecewise Riemannian metric on K. Doing this for all vertices, we obtain a new equivariant piecewise Riemannian metric on K that coincides with the canonical piecewise Riemannian metric away from the vertices. Close to the vertices and away from the edges the metric defines an equivariant smoothing. Along the interior of the edges we can use the exponential map to define collars. These collars in turn define charts that extend the equivariant smoothing to all of K. In view of our proof in higher dimensions note that the same method works, if we start with a piecewise flat metric on K distinct from the canonical one.

What we essentially did is to first construct an equivariant *welding* of K via the metric, i.e. an equivariant and continuous choice of linearizations of the tangent spaces of K (cf. [Thu97, Def. 3.10.4]), and then an equivariant smoothing from it, thereby adopting the proof from the non-equivariant case (cf. [Thu97, Prop. 3.10.7]). In the non-equivariant case the actual difficulty it to construct a welding [Thu97, Prop. 3.10.7]. Up to dimension three linearizations at the vertices can be easily extended along the edges and higher dimensional faces to a welding of the whole complex [Thu97, Thm. 3.10.8]. In the equivariant case one could use the result in [Man71] (which yields equivariant linearizations at the vertices) to first construct an equivariant welding and then an equivariant smoothing from it. However, this approach does not easily generalize to dimension four, where additional issues arise when trying to extend the welding from the vertices over the 1-skeleton (cf. [Thu97, Challenge 3.10.17]) and where there is no analogue of [Man71] (cf. [Man71]). Instead, we start with the canonical smooth structure on K^*, the complement of the $(n-2)$-skeleton of K, and successively extend it, after modifications close the $(i+1)$-skeleton, to smoothings on the complements of neighbourhoods of the i-skeleton, $i = n-3, \dots, 0$, and finally to all of K. For the actual extension process we implement a hint by Thurston (cf. Introduction and [Thu97, Challenge 3.10.20]) in a way that generalizes to the four dimensional case. The main ingredients for a proof along these lines are provided in the next section.

2.3.2 Radially extending equivariant smoothings

We will need the following two lemmas to extend equivariant smoothings. The first lemma just formulates the uniqueness of smoothings and the linearizability of finite group actions on spheres in dimensions $n \leq 3$ in a suitable manner (cf. Section 2.2.4 and Section 2.2.2).

Lemma 72. *Let K be a triangulated PL n-sphere, $n \leq 3$, on which a finite group G acts simplicially. Suppose that K is equipped with a smoothing with respect to which G acts smoothly. Then there exists a group homomorphism $r : G \to O(n+1)$ and a G-equivariant PD homeomorphism $f : K \to S^n \subset \mathbb{R}^{n+1}$.*

The second lemma enables us to extend equivariant smoothings.

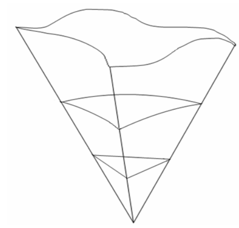

Figure 2.1: Image of a simplex of the triangulation of K under the map $\overline{F} : CK \to \mathbb{R}^{n+1}$ (cf. Lemma 73)

Lemma 73. *Let K be a triangulated PL n-sphere on which a finite group G acts simplicially. Suppose there exists a group homomorphism $r : G \to O(n+1)$ and a G-equivariant PD homeomorphism $f : K \to S^n \subset \mathbb{R}^{n+1}$. Then this PD homeomorphism can be extended to a G-equivariant PD homeomorphism $\overline{F} : \overline{C}K \to B^{n+1}$ from the closed cone of K to the unit $(n+1)$-ball in $\mathbb{R}^{n+1}$.*

Proof. We cannot simply extend the map f linearly to the origin, because then the restriction to a simplex could be degenerate at the cone point. However, we can isotopy f along the radial direction to a map that can be linearly extended to the cone point in a compatible way. We do this in two steps. In the first step we isotopy f such that the embedded simplices of K become spherical simplices. The second isotopy deforms the spherical simplices into flat simplices (cf. Figure 2.1). More precisely, let $p : \mathbb{R}^{n+1} \backslash \{0\} \to S^n$ be the radial projection and let $\theta : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function with $0 \le \theta(t) \le 1$, $\theta'(t) \le 0$, $\theta(t) = 1$ for $t < 1/3$ and $\theta(t) = 0$ for $2/3 < t$. According to Theorem 71 there is a G-equivariant subdivision $\tilde{K}$ of the complex K such that the map $L_{\tilde{K}} f : K \to \mathbb{R}^{n+1}$ is close to f in the C^1-sense and hence, the same is true for the map $\tilde{f} = p \circ L_{\tilde{K}} f : K \to S^n$. For a sufficiently good approximation the map

$$\begin{array}{rccc} F_t : & K & \to & S^n \\ & x & \mapsto & \frac{\theta_1(t)\tilde{f}(x) + (1-\theta_1(t))f(x)}{\|\theta_1(t)\tilde{f}(x) + (1-\theta_1(t))f(x)\|} \end{array}$$

with $\theta_1(t) = \theta(2t-1)$ is well-defined for $t > 1/2$. Moreover, if we choose a sequence of subdivisions such that $\tilde{f}$ converges to f in the C^1-sense, then F_t converges uniformly in t to f in the C^1-sense. Therefore, for a sufficiently fine subdivision the map

$$\begin{array}{rccc} F : & K \times (1/2, 1] & \to & \mathbb{R}^{n+1} \\ & (x,t) & \mapsto & t \cdot F_t(x) \end{array}$$

defines a G-equivariant PD embedding by Theorem 69 applied to the maps F_t. With $\theta_2(t) = \theta(2t)$ we set

$$\begin{array}{rccc} \mu : & K \times [0, 1/2] & \to & \mathbb{R} \\ & (x,t) & \mapsto & \theta_2(t)\frac{1}{\|\tilde{f}(x)\|} + (1 - \theta_2(t)) \end{array}$$

and define

$$\begin{array}{rccc} F : & K \times [0, 1/2] & \to & \mathbb{R}^{n+1} \\ & (x,t) & \mapsto & t\mu(x,t)\tilde{f}(x). \end{array}$$

Then the map $F : K \times [0,1] \to B^{n+1}$ descends to a G-equivariant PD homeomorphism $\overline{F} : \overline{C}K \to \mathbb{R}^{n+1}$. □

2.3.3 Product neighbourhoods

Before continuing the actual proof, we introduce some organizing notations. We denote the set of vertices of $K^{(1)}$, i.e. of the first barycentric subdivision of K, whose supporting simplex in K has dimension i by $v_i'(K)$. Moreover, we set $v(K) = v_0'(K)$ and $v'(K) = \bigcup_{i=0,\ldots,n} v_i'(K)$ where $n = \dim(K)$. Each $x \in v'(K)$ has an open neighbourhood $U_x \subset \mathrm{star}_{K^{(1)}}(x)$ that splits isometrically as a product $V_x \times S_x$ of connected open sets $V_x \subset \mathrm{supp}_K(x)$ and $S_x \subset \mathrm{supp}_K(x)^{\perp_x}$. Here $\mathrm{supp}_K(x)^{\perp_x}$ is the set of points $y \in \mathrm{star}_K(\mathrm{supp}_K(x))$ for which the straight line between x and y meets $\mathrm{supp}_K(x)$ orthogonally. Note that open sets $U_x \subset \mathrm{star}_{K^{(1)}}(x)$ and $U_y \subset \mathrm{star}_{K^{(1)}}(y)$ are disjoint for distinct $x, y \in v_i'(K)$.

Definition 9. A neighbourhood $U_x \subset \mathrm{star}_{K^{(1)}}(x)$ of $x \in v'(K)$ as above is called a *product neighbourhood*. We call it a *symmetric product neighbourhood* if U_x is in addition invariant under all simplicial isomorphisms of $\mathrm{star}_K(\mathrm{supp}_K(x))$ that leave $\mathrm{supp}_K(x)$ invariant. An open cover $\mathcal{U} = \{U_x\}_{x \in v'(K)}$ of K consisting of symmetric product neighbourhoods U_x of $x \in v'(K)$ is called a *symmetric product cover*, if for all $i = 0, \ldots, n$ and all $x, y \in v_i'(K)$ any simplicial isomorphism between $\mathrm{star}_K(\mathrm{supp}_K(x))$ and $\mathrm{star}_K(\mathrm{supp}_K(x))$ that maps $\mathrm{supp}_K(x)$ onto $\mathrm{supp}_K(x)$, maps U_x onto U_y.

Note that a symmetric product cover of K is in particular invariant under all simplicial isomorphisms of K. In order to have control on the sizes of product neighbourhoods of a symmetric product cover $\mathcal{U}$ we introduce its fineness $\mathrm{fin}(\mathcal{U})$ defined as

$$\mathrm{fin}(\mathcal{U}) := \max_{U_x = V_x \times S_x \in \mathcal{U}} \inf\{r > 0 | S_x \subset B_r(x)\}$$

and its cofineness $\mathrm{cofin}(\mathcal{U})$ defined as

$$\mathrm{cofin}(\mathcal{U}) := \max_{U_x = V_x \times S_x \in \mathcal{U}} \inf\{r > 0 | V_x \subset B_r(x)\}.$$

A symmetric product cover with small fineness has large cofiness and vice versa. Clearly, symmetric product covers with arbitrarily small (co)fineness exist.

2.3.4 Proof for 3-manifolds

Let $\mathcal{U}$ be a symmetric product cover of K with small fineness. For a point $x \in v_1'(K)$, lying on an edge of K, we set $S_x^* = S_x \backslash \{x\}$. The set $V_x \times S_x^*$ inherits a smoothing from K^* that respects the product structure and is invariant under all isometries in G that fix $\mathrm{supp}_K(x)$ pointwise. As in the 2-manifold case we obtain a smoothing of S_x invariant under these isometries that differs from the smoothing of S_x^* only in a small neighbourhood of x. We put the product smooth structure on $V_x \times S_x$. Working with representatives of G-orbits in $v_1'(K)$ we obtain a G-equivariant smoothing of $K_1^* = K \backslash \overline{N_{\varepsilon_1}(K_{(0)})}$, the complement of small closed balls around the vertices of K. Note that by sending $\mathrm{fin}(\mathcal{U})$ to zero, we can choose ε_1

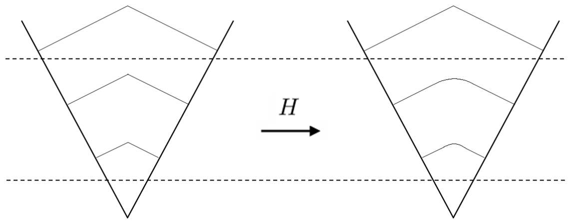

Figure 2.2: Two-dimensional sketch of the map H restricted to a simplex of K. Above the lower dotted line the smooth structure of K_1^* is defined (cf. proof). Above the upper dotted line the map H is the identity.

arbitrarily small. By construction, (open subsets of) intersections of simplices of K with K_1^* are smoothly embedded in K_1^*.

For a vertex $x \in v(K)$ of K let $B_x = \{y \in K | d(x, y) = \min_{z\in v(K)} d(y, z)\}$ be a Voronoi domain about x. It is a polyhedral 3-ball in $\mathrm{star}_K(x)$ invariant under all simplicial isomorphisms of $\mathrm{star}_K(x)$. Its bundary $P_x = \partial B_x$ projects homeomorphically to $\mathrm{lk}_K(x)$ with respect to the radial projections in $\mathrm{star}_K(x)$. In the present situation, in which all edges of K have unit length, we simply have $B_x = \mathrm{star}_{K^{(1)}}(x)$ and $P_x = \mathrm{lk}_{K^{(1)}}(x)$. We identify $\mathrm{star}_K(x)$ with a subset of the cone CP_x and work with cone coordinates $t \cdot v := (t, v) \in \mathbb{R}_{\geq 0} \times P_x$ to describe points in $\mathrm{star}_K(x)$.

We want to change the G-equivariant smoothing of K_1^* in neighbourhoods of the vertices of K such that for some small λ and each vertex x the polyhedron $\lambda \cdot P_x$ (multiplication with respect to the cone structure of CP_x) is a smooth submanifold. This would induce an equivariant smoothing of $\lambda \cdot P_x$ that could be extended to an equivariant smoothing of $\lambda \cdot B_x$ using Lemma 72 and Lemma 73. We could then glue together the smoothed balls $\lambda \cdot B_x$ and their complement in K_1^* as explained in Section 2.2.5 to obtain a smoothing of K. Moreover, by working with representatives of G-orbits in $v(K)$, we could guarantee that the obtained smoothing is equivariant.

We claim that there is some small λ with the following property. For each vertex x of K there exists an equivariant PD embedding

$$H : N_{2\varepsilon_1}(x)^C \cap \mathrm{star}_K(x) \to K_1^*$$

of the form $H(t, v) = (\varphi(t, v), v)$ that differs from the identity only for small t and away from the 1-skeleton of K such that $H(\lambda \cdot P_x) \subset K_1^*$ is a smooth submanifold. Using such PD embeddings as new charts alters the smoothing of K_1^* in a desired way so that our strategy above applies.

Close to an edge of K, say $\mathrm{supp}_K(x)$, $x \in v_1'(K)$, where H is supposed to be the identity, the condition that $H(\lambda \cdot P_x) \subset K_1^*$ is a smooth submanifold is automatically fulfilled. Indeed, by our choice of P_x, in these regions the polyhedron $t \cdot P_x \subset K_1^* \cap \mathrm{star}_K(x)$ factors through an S_x-slice with respect to the isometric splitting $U_x = V_x \times S_x \in \mathcal{U}$ and is thus a smooth submanifold of K_1^*. Away from the 1-skeleton of K the smooth structure on K_1^* is still the canonical smooth structure we started with. With respect to this smooth structure the construction of the map H is a matter of elementary calculus that can be performed simplex-wise (cf. Figure 2.2 for a 2-dimensional sketch of the construction of the map H and Section 2.3.6 for more details on the construction).

Note that due to the application of Lemma 73 and the gluing procedure, in neighbourhoods of vertices of K (open subsets of) the simplices of K are in general not smoothly embedded, only those of a subdivision. However, by choosing the fineness of the symmetric product cover $\mathcal{U}$ we started with sufficiently small, it can be arranged that these neighbourhoods are small.

2.3.5 Proof for 4-manifolds

The proof in the 4-dimensional case works along the same lines as in the 3-dimensional case. More care has to be taken only due to the simplicial subdivisions that had to be introduced in dimension three. Let $\mathcal{U}$ be a symmetric product cover of K with small fineness. As in the first step in the 3-dimensional case, from the canonical smoothing of K^* we obtain an equivariant smoothing of $K_1^* = K \backslash \overline{N_{\varepsilon_1}(K_{(1)})}$, the complement of a closed ε_1-neighbourhood of the 1-skeleton of K. The only difference is that in the present case a two-dimensional factor V_x splits off from the product neighbourhoods of $U_x = V_x \times S_x$, $x \in v_2'(K)$. Note that by sending $\mathrm{fin}(\mathcal{U})$ to zero, we can choose ε_1 arbitrarily small.

Now let $U_x = V_x \times S_x$, $x \in v_1'(x)$, be a product neighborhhood corresponding to an edge of K. The smoothing of K_1^* restricts to a product subset of $U_x = V_x \times S_x$ and respects the product structure. Treating the second factor S_x as in the 3-dimensional case and working with representatives of G-orbits in $v_1'(K)$ we obtain an equivariant smoothing of $K_2^* = K \backslash N_{\varepsilon_2}(K_{(0)})$, the complement of small balls $N_{\varepsilon_2}(K_{(0)})$ around the vertices of K. Note that by sending $\mathrm{fin}(\mathcal{U})$ and ε_1 to zero, we can choose ε_2 arbitrarily small. Also note that in a neighbourhood of the edges of K only simplices of a subdivsion of K are smoothly embedded in K_2^*. However, by choosing the fineness of our initial symmetric product cover $\mathcal{U}$ small, we can assume that this neighborhood is closely concentrated around the edges of K.

Finally, we claim that the smoothing can be extended to all of K, i.e. over neighbourhoods of the vertices of K, by the same method as in the three-dimensional case. More precisely, we claim that there is some λ such that for each vertex $x \in v(K)$ and $P_x = \mathrm{lk}_{K^{(1)}}(x)$ there exists an equivariant PD embedding

$$H : N_{2\varepsilon_2}(x)^C \cap \mathrm{star}_K(x) \to K_2^*$$

of the form $H(t, v) = (\varphi(t, v), v)$ that differs from the identity only for small t and away from the 1-skeleton of K such that $H(\lambda \cdot P_x) \subset K_2^*$ is a smooth submanifold. For details on the construction of this map we refer to the next section. Given such a map H, the proof can be concluded as in the three-dimensional case.

2.3.6 Construction of the map H

In the preceding two sections we have employed PD embeddings H on three occasions. In this section we describe their construction. We treat the case $n = 4$. The case $n = 3$ works analogously but more easily. One only has to note that in the case $n = 3$ the PD embeddings

H applied in Section 2.3.4 and Section 2.3.5 need to be constructed with respect to different polyhedral metrics.

Let $\Delta_4 = \Delta^4$ be a standard simplex with unit edge length and let $\Delta_3 = \Delta^3$ be a face of Δ_4. We regard Δ_4 as the subset $(\Delta_3 \times [0,1])/\sim$ of the cone $C\Delta_3$ with vertex x. Moreover, we suppose that $C\Delta_3$ is isometrically embedded in $\mathbb{R}^4$ such that the cone point is the origin. Let $\tilde{\mathcal{U}}$ be a symmetric product cover of Δ_3 of small cofineness (cf. Section 2.3.3). Let P be the simplicial complex $P = \mathrm{lk}_{\Delta_4^{(1)}}(x)$, which is the boundary of $\{y \in C\Delta_3 | d(0,y) = \min_{z\in v(\Delta_4)} d(y,z)\}$ in $C\Delta_3$. We identify P with Δ_3 via radial projection in $C\Delta_3$. In particular, the cover $\tilde{\mathcal{U}}$ gives rise to a cover of P that we also denote by $\tilde{\mathcal{U}}$. To describe points in $C\Delta_3$ we work with cone coordinates $(t,v) \in \mathbb{R}_{\geq 0} \times P$ corresponding to $t \cdot v \in CP = C\Delta_3$. In particular, for a subset $U \subset \Delta^3$ we write $CU := \mathbb{R} \cdot U \subset C\Delta_3$. Using a partition of unity it is easy to construct a PD map $\varphi_0 : \Delta_3 \to (0,1]$ such that the following properties hold

(i) φ_0 is equivariant with respect to all simplicial isomorphisms of Δ_3.

(ii) the restrictions of φ_0 to the stars $\mathrm{star}_{\Delta_3^{(1)}}(v)$, $v \in v(\Delta_3)$, are smooth.

(iii) $\varphi_0 \leq 1$, φ_0 is approximately constant and $\varphi_0(v) = 1$ for $v \in v(\Delta_3)$.

(iv) for $v \in \tilde{U} = \tilde{V} \times \tilde{S} \in \tilde{\mathcal{U}}$, the value of φ_0 only depends on the $\tilde{V}$-component of v.

(v) the subset $P' = \{\varphi_0(p) \cdot p | p \in P\} \subset \Delta_4$ is a smooth submanifold of Δ_4.

Note that in this situation the intersections of P' with the faces of $C\Delta_3$ are submanifolds by transversality. Also note that if P' is a smooth submanifold of Δ_4, then so is $\lambda P'$ for each $\lambda \in (0,1]$. Let ε_2 be as in the preceding section and let λ be such that $10\varepsilon_2 < \lambda < 1/100$. Given a function φ_0 as above, a PD embedding

$$h : C\Delta_3 \backslash N_{2\varepsilon_2} \to C\Delta_3$$

can be constructed as $h(t,v) = (\varphi(t,v), v)$ with $\varphi(t,v) = \theta(t)t + (1-\theta(t))t\varphi_0(v)$ where $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a smooth cutoff function with $0 \leq \theta(t) \leq 1$, $\theta'(t) \geq 0$, $\theta(t) = 1$ for $t > 1/10$ and $\theta(t) = 0$ for $t < 2\lambda$. In the situation of the preceding section, for a vertex x of K copies of h can be put together to define an embedding

$$H : D(H) := N_{2\varepsilon_2}(x)^C \cap \mathrm{star}_K(x) \to K_2^*.$$

We claim that this map has the desired properties if the cofineness of $\tilde{\mathcal{U}}$ and the fineness of the symmetric product cover $\mathcal{U}$ used in the preceding section are sufficiently small. The cover $\tilde{\mathcal{U}}$ induces a symmetric product cover of $\mathrm{lk}_K(x)$ that we also denote by $\tilde{\mathcal{U}}$. We go on to use cone coordinates to identify $\mathrm{star}_K(x)$ with a subset of $\mathrm{lk}_K(x) = C\mathrm{lk}_K(x)$.

First observe that H is PD: We can assume that $U_y \cap D(H) \subset C\tilde{U}_z \cap D(H)$ for all $z \in v(\mathrm{lk}_K(x))$ where $y \in v_1'(K)$ lies on the edge (x,z). On $C\tilde{U}_z \cap D(H)$, $z \in v(\mathrm{lk}_K(x))$, and sufficiently far apart from x the map H is the identity by (iii), (iv) and the choice of θ and thus H is trivially PD in these regions. Outside of them (open subsets in K_2^* of) the simplices of K are smoothly embedded in K_2^* by the above assumption and thus H is PD there too, since h is PD. It remains to show that $H(\lambda \cdot P_x) \subset K_2^*$ is a smooth submanifold where $P_x = \mathrm{lk}_{K^{(1)}}(x)$. To see this note that for sufficiently small $\mathrm{cofin}(\tilde{\mathcal{U}})$ and $\mathrm{fin}(\mathcal{U})$ we have $D(H) \cap U_y \cap C\tilde{U}_z = \emptyset$ for all $z \in v_i'(\mathrm{lk}_K(x))$, $i = 1, 2, 3$, and all $y \in v_j'(K) \cap P_x$ with $j \leq i$ and thus that the following holds due to our construction of the smooth structure on K_2^*. A point $p \in H(\lambda \cdot P_x) \cap C\tilde{U}_y$, $y \in v'(\mathrm{lk}_K(x))$, has an open neighbourhood U that splits isometrically $U = V \times S \subset C\tilde{U}_y \cap \mathrm{star}_K(x)$ as an open subset V of $C(\mathrm{supp}_{\mathrm{lk}(x)}(y)) \cap \mathrm{star}_K(x)$ and a submanifold S that is contained in $C(\mathrm{supp}_{\mathrm{lk}(x)}(y))^{\perp_z} \subset \mathrm{star}_K(x)$ for some $z \in C(\mathrm{supp}_{\mathrm{lk}(x)}(y)) \cap \mathrm{star}_K(x)$, such that the smooth structure on K_2^* restricts to the product smooth structure on $U = V \times S$ of the Euclidean smooth structure on V and the smooth structure on S. With respect to the splitting $U = V \times S$ a neighbourhood of p in $H(\lambda \cdot P_x)$ splits as a product of a smooth submanifold of V and an open subset of S by properties (iv), (v) and the choice of P and P_x. In particular, this neighbourhood is a smooth submanifold of U and thus of K_2^*. It follows that $H(\lambda \cdot P_x)$ is a smooth submanifold of K_2^* as claimed.

2.3.7 Higher dimensions

There exist piecewise linear actions of $\mathbb{Z}_2$ on a 5-dimensional piecewise linear sphere that cannot be equivariantly smoothed. One way to obtain such an example is as follows (cf. [KL88, p. 260]). The group $\mathbb{Z}_2$ admits a piecewise linear action on S^4 whose fixed point set is a knotted S^2, i.e. the fundamental group of its complement is distinct from $\mathbb{Z}$ [Rol76, p. 347]. By suspending this action one obtains a piecewise linear action of $\mathbb{Z}_2$ on S^5 with fixed point set S^3. However, this action cannot be equivariantly smoothed because its fixed point set $S^3 \subset S^5$ is not locally flat.

Chapter 3

Characterization of finite groups generated by reflections and rotations

In this chapter we prove Theorem B based on the classification result in [LM15] that we have described in Chapter 1. A partial result in this direction has already been obtained in [Mik84] by Mikhaîlova. In that paper the if direction of a topological version of Theorem B is verified for many rotation groups. In fact, in [Mik84] it is claimed that this implication holds for all rotation groups. However, several rotation groups are not mentioned in [Mik84] (cf. Introduction of Chapter 1), some cases are not explicitly treated and certain proofs lack arguments (cf. Section 3.4.7 and Section 3.4.4). All the proofs can be made rigorous though as we will shortly explain.

We adapt some of the methods from [Mik84] as to also work in the piecewise linear category and describe new methods to prove the if direction of Theorem B by verifying its conclusion for all reflection-rotation groups. For instance, we apply a result on equivariant smoothing of piecewise linear manifolds (cf. Chapter 2) and the equivariant Poincaré conjecture in dimension $n = 4$ (cf. Section 3.4.3) and the generalized Poincaré conjecture in dimensions $n > 5$ (cf. Section 3.3.3). With our methods we avoid the difficulties in [Mik84] alluded to above. In particular, in dimension up to four, our proof does not require the classification result. Finally, in the last section we suggest an alternative approach for proving Theorem B.

3.1 Preliminaries

3.1.1 PL spaces and PL structures

Recall the concepts from piecewise linear topology described in Section 2.2.1. We remind of the following concepts which have not been discussed in that section.

A *PL chart* (P, φ) for a topological space X is an embedding $\varphi : P \to X$ of a compact polyhedron P (i.e. a Euclidean polyhedron in terms of [Hud69]). Two PL charts (P, φ) and (Q, ϕ) are said to be *compatible* if $\varphi^{-1}(\phi(Q))$ is a compact polyhedron and $\phi^{-1} \circ \varphi : \varphi^{-1}(\phi(Q)) \to Q$ is piecewise linear. An *atlas* (*base of a PL structure* in terms of [Hud69]) on X is a family of compatible PL charts for X such that for each point $x \in X$ there is a chart (P, φ) for which $\varphi(P)$ is a topological neighborhood of x. A *PL structure* of X is a maximal atlas. A second-countable topological Hausdorff space X endowed with a PL structure is called a *PL space* (cf. [Hud69, p. 77]). A PL space X is called a *PL manifold* (*with boundary*) of dimension n, if for every point $p \in X$ there exists a chart $(P = \Delta^n, \varphi)$ of X such that $p \in \varphi(\mathrm{Int}(P))$ $(p \in \mathrm{Int}_X(\varphi(P)))$ (cf. [Hud69, p. 79]). In this case the boundary of X is defined in the usual way. Every PL space of dimension n can be triangulated by a locally finite simplicial complex $K \subset \mathbb{R}^{2n+1}$, i.e. it can be realized as a polyhedron [Hud69, Lem. 3.5, p. 80][Moi77, Thm. 7.1, p. 53]. Conversely, a locally finite simplicial complex K has a natural PL structure. It is a PL n-manifold (with boundary), if and only if the link of every vertex is a PL $(n-1)$-sphere (or a PL $(n-1)$-ball), i.e. it is PL homeomorphic to $\partial\Delta^n$ (or to Δ^{n-1}) (cf. [RS72, p. 24]). A map $f : X \to Y$ between PL spaces X and Y is called PL, if for any chart (P, φ) of X and any chart (Q, ϕ) of Y, the set $\varphi^{-1}f^{-1}\phi(Q)$ is either empty or a polyhedron contained in P and, if the latter, then $\phi^{-1}f\varphi : \varphi^{-1}f^{-1}\phi(Q) \to Q$ is a PL map. For f to be a PL map it suffices to check this condition for charts of bases of the PL structures on X and Y (cf. [Hud69, p. 83]). The map f is PL if and only if there exist triangulations K and L of X and Y such that each simplex of K is mapped linearly into a simplex of L (cf. [Hud69, p. 83],[RS72, p. 16] and [Hud69, p. 84, Thm. 3.6]). The map f is called a PL homeomorphisms, if it has in addition a PL inverse. In this case there exist triangulations K and L of X and Y with respect to which f is a simplicial isomorphism [Hud69, p. 84, Thm. 3.6.C].

The open cone CX of a compact PL space X inherits a natural PL structure from X. If X is embedded as a polyhedron P in some $\mathbb{R}^N$, then CX is PL homeomorphic to the internal open cone $CX = \mathbb{R}_{\geq 0} \cdot (P + e_{n+1}) \subset \mathbb{R}^{n+1}$. Here e_{n+1} denotes the last canonical basis vector of $\mathbb{R}^{n+1}$. The cone CX is PL homeomorphic to some $\mathbb{R}^n$ if and only if X is PL homeomorphic to the standard PL $(n-1)$-sphere $\partial\Delta^n$. To see the only-if direction of this statement one can triangulate X by a simplicial complex K and extend this triangulation to a triangulation L of CX such that the link of the cone point in L is K [RS72, proof of Prop. 2.9]. Then the statement follows from the remark above on links in simplical complexes that are PL manifolds.

3.1.2 PL quotients

Let $\sim$ be an equivalence relation on a PL space X. We would like to know whether $X/\sim$ is a PL space such that the projection map $q : X \to X/\sim$ is PL. More precisely, if there exists a PL space Y and a PL map $f : X \to Y$ that induces a homeomorphism $\overline{f} : X/\sim \to Y$ such that the projection map $q : X \to X/\sim$ is PL. This need not be the case. For instance, take a 2-simplex and collapse a side to a point. However, if such a pair (Y, f) exists, then the following universal property shows that Y is unique up to PL homeomorphism and can thus be considered the quotient of X with respect to $\sim$ in the PL category

Lemma 74. *If Y' is a PL space and $f' : X \to Y'$ a PL map such that $x \sim y$ for $x, y \in X$ implies $f'(x) = f'(y)$, then the unique map $g : Y \to Y'$ is PL.*

Proof. Let $x \in X$, $y = q(x)$ and $y' = f'(x)$. We choose charts (P, φ), (Q, ϕ) and (Q', ϕ') about x, y and y' that define topological neighborhoods of the respective points. Since q is open and since the image of a compact polyhedron under a PL map is a compact polyhedron [RS72, Cor. 2.5], we can assume that $q\varphi(P) = \phi(Q)$ and $P = \varphi^{-1}f'^{-1}\phi'(Q')$. In particular, we have $\phi^{-1}g^{-1}\phi'(Q') = Q$ and $\mathrm{graph}(\phi'^{-1}g\phi) = (\phi^{-1}q \times \phi'^{-1}f')(\varphi(P)) \subset Q \times Q'$ is a compact polyhedron, i.e. $\phi'^{-1}g\phi : Q \to Q'$ is a PL map. Now the general facts that a finite union $\bigcup_i P_i$ of compact polyhedra P_i is a compact polyhedron and that a map $f : \bigcup_i P_i \to Q$ is PL if all restrictions $f_{|P_i}$ are PL [RS72, p. 5, 1.5 (4)], implies that the map g is PL. □

A Euclidean vector space $\mathbb{R}^n$ carries a natural PL structure with respect to which it is a PL manifold and with respect to which O_n acts by PL homeomorphisms on it. In the following section we realize $\mathbb{R}^n/G$ as a simplicial complex and show that the projection from $\mathbb{R}^n$ to $\mathbb{R}^n/G$ with the induced PL structure is a PL map, i.e. that the quotient $\mathbb{R}^n/G$ is in a natural way a PL space.

3.1.3 Admissible triangulations

Let G be a finite group. A simplicial complex K is called a *G-complex*, if G acts simplicially on it. It is called a *regular* G-complex, if for each subgroup $H < G$ and each tuple of elements $g_0, g_1, \ldots, g_n \in H$ such that both of the sets $\{v_0, \ldots, v_n\}$ and $\{g_0v_0, \ldots, g_nv_n\}$ describe vertices of a simplex in K, there exists an element $g \in H$ such that $gv_i = g_iv_i$ for all i (cf. [Bre72, Ch. III, Def. 1.2, p. 116]). The second barycentric subdivision of a G-complex K is always regular and for a regular G-complex one can define in a natural way a simplicial complex K/G whose underlying space is homeomorphic to the topological quotient $|K|/G$ [Bre72, p. 117]. The vertices of K/G are the G-orbits of the vertices of K and a subset of these simplices forms a simplex if and only if there are representatives of these vertices in K that form a simplex in K.

For a finite subgroup $G < \mathrm{O}_n$ we call a triangulation K of $\mathbb{R}^n$ *admissible (for the action of G on $\mathbb{R}^n$)*, if K is a regular G-complex that contains the origin as a vertex. The following lemma shows that admissible triangulations always exist.

Lemma 75. *For a finite subgroup $G < \mathrm{O}_n$ there exists a triangulation K of $\mathbb{R}^n$ that is admissible for the action of G on $\mathbb{R}^n$.*

Proof. We start with any triangulation $\tilde{K}$ of $\mathbb{R}^n$ that contains the origin as a vertex and replace it by the common subdivision K of $\mathbb{R}^n$ of the triangulations $g\tilde{K}$ of $\mathbb{R}^n$, $g \in G$, as in Lemma 67. The resulting triangulation defines a G-complex. Upon passage to the second barycentric subdivision, we can assume that this G-complex is regular [Bre72, p. 117]. □

Let K be an admissible triangulation for the action of $G < \mathrm{O}_n$ on $\mathbb{R}^n$. Then $Y = K/G$ is a simplicial complex and the projection $K \to K/G$ maps simplices linearly onto simplices. In particular, Y is a PL space and the projection $K \to K/G$ is PL, i.e. $\mathbb{R}^n/G$ is in a natural way a PL space (cf. Section 3.1.2). The link of the origin in K is also a regular G-complex and its quotient by G is simplicially isomorphic to the link of the origin in K/G. Hence, the PL space $\mathbb{R}^n/G$ is a PL manifold (with boundary) if and only if this link in K/G is a PL $(n-1)$-sphere (or a PL $(n-1)$-ball) (cf. Section 3.1.1). Radially projecting the link of the origin in K to the unit sphere S^{n-1} defines a PL structure on S^{n-1} and induces a PL structure on S^{n-1}/G. We call triangulations and PL structures of S^{n-1} and S^{n-1}/G that arise in this way *admissible (for the action of G on S^{n-1})*. Two admissible PL structures on S^{n-1}/G need not be identical but are PL homeomorphic (cf. [RS72, pp. 20-21], "pseudoradial projection"). Hence, the question if $\mathbb{R}^n/G$ is a PL manifold is equivalent to the question if S^{n-1}/G is a PL sphere with respect to one and then any admissible PL structure. The following lemma gives a necessary condition on isotropy groups for this to hold.

Lemma 76. *Let $G < \mathrm{O}_n$ be a finite subgroup and suppose S^{n-1} is triangulated by a simplicial complex K in an admissible way for the action of G. Then S^{n-1}/G is a PL manifold (with boundary) with respect to the induced PL structure if and only if for every vertex v of K the quotient space T_vS^{n-1}/G_v is a PL manifold (with boundary). This is in particular the case if $\mathbb{R}^n/G$ is a PL manifold (with boundary).*

Proof. The quotient S^{n-1}/G is a PL manifold if and only if the link of every vertex $\overline{v}$ of K/G is a piecewise linear $(n-2)$-sphere (or a PL $(n-2)$-ball) (cf. Section 3.1.1). Let v be any vertex of K projecting to some $\overline{v}$. The action of the isotropy group G_v on the link $\mathrm{lk}_K(v)$ is regular and the corresponding quotient complex K/G_v is simplicially isomorphic to the link of $\overline{v}$ in K/G. We can assume that K is embedded as a G-invariant simplicial complex in $\mathbb{R}^n$ whose vertices lie on the unit sphere such that the triangulation is given by radial projection of K onto the unit sphere. Let the simplicial complex $\tilde{K}$ be the orthogonal projection of $\mathrm{lk}_K(v)$ onto $T_vS^{n-1} \subset \mathbb{R}^n$. The restriction of this projection to the vertices induces a simplicial isomorphism between $\tilde{K}$ and $\mathrm{lk}_K(v)$ which induces a simplicial isomorphism between $\mathrm{lk}_K(v)/G_v$ and $\tilde{K}/G_v$. The complex $\tilde{K}$ is a regular G_v-complex and maps homeomorphically onto the unit sphere in T_vS^{n-1} via the radial projection in T_vS^{n-1}. In particular, we have $T_vS^{n-1} = C\tilde{K}$ and $T_vS^{n-1}/G_v = (C\tilde{K})/G_v = C(\tilde{K}/G_v)$ as PL spaces and thus the claim follows (cf. Section 3.1.1). ☐

3.2 The only-if direction

Let us show by induction that a finite subgroup $G < \mathrm{O}_n$ is a reflection-rotation group, if the quotient space $\mathbb{R}^n/G$ is a PL manifold with boundary and that in this case G contains a reflection if and only if the boundary of $\mathbb{R}^n/G$ is nonempty. For $n \leq 2$ the claim is trivially true. Assume it holds for some $n \geq 2$ and let $G < \mathrm{O}_{n+1}$ be a finite subgroup such that $\mathbb{R}^{n+1}/G$ is a PL manifold with boundary. Then all isotropy groups G_v for $v \neq 0$ are reflection-rotation groups by Lemma 76 and the induction assumption. Let $G_{rr} \lhd G$ be the reflection-rotation group generated by all of them and let $v \in S^n$. Then the inclusions

$$G_v = (G_v)_{rr} \subseteq (G_{rr})_v \subseteq G_v$$

imply that $G_v = (G_{rr})_v$ for all $v \in S^n$. This means that the action of G/G_{rr} on S^n/G_{rr} is free. Because of $n \geq 2$ the quotient space S^n/G is simply connected by assumption and thus we conclude that G and G_{rr} coincide, i.e. that G is a reflection-rotation group. Since G_{rr} contains a reflection if and only if the boundary of S^n/G_{rr} is nonempty, it follows that also G contains a reflection if and only if the boundary of $\mathbb{R}^n/G$ is nonempty. Hence, the claim follows by induction.

3.3 Methods for the if direction

3.3.1 PL linearization principle

The idea of the PL linearization principle is to divide the determination of the PL quotient $\mathbb{R}^n/G$ for a finite subgroup $G < \mathrm{O}_n$ into several steps. Let $H \triangleleft G$ be a normal subgroup and assume there exists a PL homeomorphism $F : \mathbb{R}^n/H \to \mathbb{R}^n$ and a homomorphism $r : G \to \mathrm{O}_n$ with kernel H such that the left square in the following diagram commutes

$$\begin{tikzcd} G \times \mathbb{R}^n/H \arrow[r] \arrow[d, "r\times F"'] & \mathbb{R}^n/H \arrow[r, two heads] \arrow[d, "F"'] & \mathbb{R}^n/G \arrow[d, dashed, "\tilde{F}"'] \\ r(G) \times \mathbb{R}^n \arrow[r] & \mathbb{R}^n \arrow[r, two heads] & \mathbb{R}^n/r(G) \end{tikzcd}$$

Then we say that the PL linearization principle can be applied to the groups $H \triangleleft G$. In this case F induces a PL homeomorphism $\tilde{F} : \mathbb{R}^n/G \to \mathbb{R}^n/r(G)$ due to Lemma 74. This reduces the determination of $\mathbb{R}^n/G$ to the determination of $\mathbb{R}^n/r(G)$ and one might look for a suitable normal subgroup of $r(G)$ in order to apply the PL linearization principle again. If the PL linearization principle can be applied to $H \triangleleft G$ and to $\tilde{H} \triangleleft r(G)$, then it can also directly be applied to $r^{-1}(\tilde{H}) \triangleleft G$. As a direct consequence of the PL property we have

Lemma 77. *Suppose that the PL linearization principle can be applied to groups $H \triangleleft G$. If $g \in G$ is a reflection (rotation), then so is $r(g)$. In particular, if G is generated by reflections and rotations, then so is $r(G)$.*

The PL linearization principle can be established by describing a homeomorphism $f : S^{n-1}/H \to S^{n-1}$ and a homomorphism $r : G \to \mathrm{O}_n$ with kernel H such that the following square commutes and such that the PL structure on S^{n-1} induced by an admissible PL structure on S^{n-1}/H via f is admissible for the linearized action of $r(G)$ on S^{n-1}

$$\begin{tikzcd} G \times S^{n-1}/H \arrow[r] \arrow[d, "r\times f"'] & S^{n-1}/H \arrow[d, "f"'] \\ r(G) \times S^{n-1} \arrow[r] & S^{n-1} \end{tikzcd}$$

In fact, in this case we can take the open cone at each site and extend the PL homeomorphism f linearly to a PL homeomorphism $F : \mathbb{R}^n/H \to \mathbb{R}^n$ which makes the first diagram above commute.

3.3.2 Uniqueness of compatible PL structures.

A map $g : P \to M$ from a polyhedron P to a smooth manifold M is called *piecewise differentiable* or *PD*, if P admits a triangulation such that the restriction of f to each simplex in this triangulation is smooth. It is called *PD homeomorphism*, if it is in addition a homeomorphism and its restriction to each simplex has injective differential at each point. If g is a PD homeomorphism, then, due to a theorem by Whitehead [Whi40], the polyhedron P is a PL manifold.

Moreover, such a polyhedron exists and is unique up to PL homeomorphisms. According to a result of Illman these statements also hold equivariantly for a finite group acting smoothly on M [Ill78]. In particular, if M is a smooth manifold on which a finite group G acts smoothly and $g_i : P_i \to M$, $i = 1, 2$, are two PD homeomorphisms such that the induced actions of G on the polyhedra P_1 and P_2 are PL, then there exists a G-equivariant PL homeomorphism between P_1 and P_2.

Assume we have a PL $(n-1)$-sphere P_1 on which a finite group G acts by PL homeomorphisms, a PD homeomorphism $g : P_1 \to S^{n-1}$ and a group homomorphism $r : G \to \mathrm{O}_n$ such that the following diagram commutes

$$\begin{tikzcd} G \times P_1 \arrow[r] \arrow[d, "r\times g"'] & P_1 \arrow[d, "g"'] \\ r(G) \times S^{n-1} \arrow[r] & S^{n-1} \end{tikzcd}$$

Then, by Section 3.1.2 there exists an admissible triangulation of S^{n-1} by a polyhedron P_2 with respect to the action of $r(G)$ and this triangulation defines a PD homeomorphism from the polyhedron to S^{n-1}. Therefore P_1 and P_2 are G-equivariantly PL homeomorphic as explained above. In other words, we can replace g by another G-equivariant homeomorphism f which is in addition piecewise linear with respect to an admissible PL structure for the action of $r(G)$ on S^{n-1}.

3.3.3 Generalized Poincaré conjecture

The generalized Poincaré conjecture holds in the following version. Note that a closed simply connected topological manifold M with $H_*(M;\mathbb{Z}) = H_*(S^n;\mathbb{Z})$ is homotopy equivalent to an n-sphere (Proof: Any closed orientable manifold M^n admits a degree one map onto S^n. If M^n is a homology sphere this map induces an isomorphism on homology. If, in addition, M^n is simply connected, it induces isomorphisms on homotopy [Spa66, Thm. 7.5.9, p. 399] and is thus a homotopy equivalence [Hat02, Thm. 4.5, p. 346] due to theorems by Whitehead).

Theorem 78. *For $n \neq 4$ every closed simply connected PL manifold M with $H_*(M;\mathbb{Z}) = H_*(S^n;\mathbb{Z})$ is PL homeomorphic to a standard PL n-sphere.*

For $n = 1, 2$ the statement has long be known by the classification of manifolds in that dimensions. For $n \geq 6$ it follows from the PL h-cobordism theorem (c.f. [RS72, Thm. A, p. 17]). For $n \leq 5$ every PL manifold admits a compatible smooth structure [HM74, KS77] and thus, according to the uniqueness part of Whitehead's theorem [Whi40, Mun66], the statement can be reduced to the respective statement in the smooth category. For $n = 5$ the smooth version of the generalized Poincaré conjecture follows from the smooth h-cobordism theorem combined with the fact that every closed, smooth 5-manifold homotopy equivalent to S^5 bounds a smooth, compact, contractible 6-manifold [KM63, Wal62] (cf. [Mil65] for more details). Finally, for $n = 3$ the smooth Poincaré conjecture follows from Perelman's work [Per02, Per03a, Per03b] (cf. [MT07, KL06] for expositions of Perelman's work).

We would like to apply the statement of Theorem 78 in the following situation. For a rotation group $G < \mathrm{SO}_n$ the quotient S^{n-1}/G is simply connected unless $n \leq 2$ by the following lemma (for a proof see Appendix A.1 or [Arm68], where the result is proven in greater generality).

Lemma 79. *Let $G < \mathrm{SO}_{n+1}$ with $n \geq 2$ be a finite subgroup generated by elements that fix some point in S^n. Then the quotient space S^n/G is simply connected.*

Suppose S^{n-1} and S^{n-1}/G are equipped with PL structures that are admissible for the action of G (cf. Section 3.1.3). According to Lemma 76, in order to show that S^{n-1}/G is a closed PL manifold it suffices to check that for any point $p \in S^{n-1}$ the isotropy group G_p is a rotation group acting in $T_pS^{n-1} = \mathbb{R}^{n-1}$ such that the quotient space $\mathbb{R}^{n-1}/G_p$ is a PL manifold. The condition on the homology groups of S^{n-1}/G can be verified as follows. We choose an admissible triangulation K of S^{n-1} for the action of G and work with simplicial homology over $\mathbb{Z}$ (in the following the coefficient ring is omitted and understood to be $\mathbb{Z}$). Assume there exists a subgroup $H < G$ such that $H_i(S^{n-1}/H) = 0$ for some $i \in \{1, \dots, n-2\}$. For an i-cycle $c \in Z_i(K/G)$ there exists an i-cycle $c' \in Z_i(K/H)$ such that $\pi(c') = [G : H] \cdot c$ where $\pi_{G/H} : K/H \to K/G$ is the natural simplicial projection (e.g. $c' = \mu_{G/H}(c)$ in the notation of [Bre72, pp. 118-121]). The $(i+1)$-chain $a \in C_{i+1}(K/H)$ with $\partial a = c'$ satisfies $\partial \pi(a) = \pi(\partial a) = [G : H] \cdot c$ and thus $0 = (\pi_{G/H})_*(\mu_{G/H})_*([c]) = [G : H] \cdot [c]$ in $H_i(S^{n-1}/G)$ (the induced map $(\mu_{G/H})_* : H_*(K/G) \to H_*(K/H)$ is called *transfer*, cf. [Bre72, Ch. III. 2., pp. 118-121]). Hence, if there are subgroups H of G with coprime indices and $H_i(S^{n-1}/H) = 0$, then it follows that $H_i(S^{n-1}/G) = 0$. In particular, this conclusion holds if there are rotation subgroups H of G with coprime indices for which we already know that $\mathbb{R}^n/H$ is homeomorphic to $\mathbb{R}^n$, since this implies $H_*(S^{n-1}/H) = H_*(S^{n-1})$ (cf. [Hat02, p. 117]).

3.3.4 Chevalley's theorem

For a unitary reflection group $G < \mathrm{U}_n$ the following theorem due to Chevalley holds [LT09, Thm. 3.20, p. 48].

Theorem 80. *The algebra of invariants of a finite unitary reflection group $G < \mathrm{U}_n$ is generated by n algebraically independent homogenous polynomials.*

For n such generators $f_1, \dots, f_n$ of $\mathbb{C}[z_1, \dots, z_n]^G$ we will see in Section 3.4.2 that the map

$$f = [f_1, \dots, f_n] : \begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ v & \longmapsto & (f_1(v), \dots, f_n(v)) \end{array}$$

descends to a homeomorphism $\overline{f} : \mathbb{C}^n/G \to \mathbb{C}^n$ and, moreover, that $\mathbb{R}^{2n}/G$ is in fact PL homeomorphic to $\mathbb{R}^{2n}$.

3.3.5 The fundamental domain of a group

For a finite subgroup $G < \mathrm{O}_n$ there exists a vector $v_0 \in \mathbb{R}^n$ such that $gv_0 \neq v_0$ for all $g \in G \backslash \{e\}$. For such a vector v_0 the set

$$\Lambda = \bigcap_{g \in G} \{v \in \mathbb{R}^n | (v, v_0) \geq (v, gv_0)\}$$

is a fundamental domain for the group G, i.e. the translates $g\Lambda$ of Λ cover $\mathbb{R}^n$ and the union $\bigcup_{g\in G} g\mathring{\Lambda}$ is disjoint. It inherits a subspace topology and PL structure from $\mathbb{R}^n$ and the quotient space $\mathbb{R}^n/G$ with its quotient topology and PL structure (cf. Sections 3.1.2 and 3.1.3) can be obtained from Λ by identifying certain points on the boundary, namely those which belong to the same orbit of G.

3.3.6 Gluing construction

We need the following elementary gluing construction for PL balls. The lemma states that twisted PL spheres are standard PL spheres. Its proof is a direct consequence from the fact that a PL homeomorphism $f : \partial\Delta^n \to \partial\Delta^n$ can be linearly extended in a radial direction to a PL homeomorphism $\overline{f} : \Delta^n \to \Delta^n$ [RS72, Lem. 1.10, p. 8].

Lemma 81. *Suppose B_1^n and B_2^n are PL balls and $\varphi : \partial B_1^n \to \partial B_2^n$ is a PL homeomorphism. Then the space $B_1^n \cup_\varphi B_2^n$ obtained by gluing B_1^n and B_2^n together along their boundary via φ is a PL n-sphere.*

3.3.7 Collapsing

Let K be a simplicial complex and let $\sigma, \tau \in K$ be simplices such that
(i) $\tau < \sigma$, i.e. τ is a proper face of σ,
(ii) σ is a maximal simplex in K and τ is not contained in any other maximal simplex of K,
then τ is called a *free face* of K. A *simplicial collaps* of K is the removal of all simplices ρ of K with $\tau \leq \rho \leq \sigma$. We say that K collapses onto a subcomplex L of K if there exists a finite sequence of collapses leading from K to L. A simplicial complex that collapses onto a point is called *collapsible*. Being collapsible to a subcomplex is a PL property, i.e. it does not depend on a specific triangulation (cf. [RS72, p. 39]). In our proof we will apply the following characterization (cf. [RS72, Cor. 3.28, p. 41])

Lemma 82. *A collapsible PL n-manifold (with or without boundary) is a PL n-ball.*

In order to be able to apply this characterization, we will need the following lemma.

Lemma 83. *Let $p : K \to \tilde{K}$ be a simplicial surjection between finite simplicial complexes K and $\tilde{K}$ that maps simplices of K homeomorphically onto simplices of $\tilde{K}$. Suppose further that L is a subcomplex of K such that p restricts to a bijection $p : K\backslash L \to \tilde{K}\backslash p(L)$. If K collapses onto L, then $\tilde{K}$ collapses onto $p(L)$*

Proof. Let $\tau < \sigma \in K$ with $\tau, \sigma \notin L$ and suppose that $\tau < \sigma$ defines a simplicial collaps of K. By assumption on p we have $p(\tau), p(\sigma) \notin p(L)$ and thus the claim follows inductively, if we can show that $p(\tau) < p(\sigma) \in \tilde{K}$ defines a simplicial collapse of $\tilde{K}$. But again, by assumption on p, and because of $\tau, \sigma \notin L$ it is clear that $p(\sigma)$ is a maximal simplex of $\tilde{K}$ and that $p(\tau)$ is not contained in any other maximal simplex of $\tilde{K}$. $\square$

3.4 Proof of Theorem B

In this section we prove the if direction of our main result by verifying its conclusion for all reflection-rotation groups. The proof is structured as follows. For each reflection-rotation group G we either prove the conclusion of our main result directly or we reduce such a proof to the respective claim on reflection-rotation groups of lower order via the PL linearization principle. In doing this we will need to show that for each pair $M \lhd G_{rr}$ of an irreducible reflection-rotation G_{rr} that contains a reflection and a proper nontrivial normal rotation group $M \lhd G_{rr}$ such that G_{rr} is generated by the reflections it contains and by M, there exists a nontrivial rotation group $H \lhd M$ normalized by G_{rr} such that the PL linearization principle can be applied to the groups $H \lhd G_{rr}$ (cf. Section 3.4.4). All such pairs $M \lhd G_{rr}$ are listed in Theorem 3. In each case we will either show this property directly or reduce it to proving our main result for reflection-rotation groups of order less than G_{rr}. Once we have treated all the cases, the if direction of our main result follows by induction. References to the sections in which the respective cases are treated can be found in the appendix.

3.4.1 Real reflection groups

The fundamental domain Λ of a reflection group $W < \mathrm{O}_n$ acting on S^{n-1} is a spherical simplex [Cox34, Thm. 4, p. 595]. Let W^+ be the orientation preserving subgroup of W and, if there exists some rotation $h \in \mathrm{O}_n \backslash W$ that normalizes W, set $W^\times = \langle W, h\rangle$ and $W^* = \langle W^+, h\rangle$. Choose an admissible triangulation for the action of W (and hence of W^+) on S^{n-1} that refines the triangulation of S^{n-1} by the fundamental domains of W. Then the quotient space S^{n-1}/W is a PL ball, namely the fundamental domain Λ of W, and the quotient space S^{n-1}/W^+ can be obtained by gluing together two copies of Λ along their boundary, i.e. the resulting space is a PL sphere by Lemma 81. Moreover, a coset $\overline{s}$ of a reflection $s \in W$ interchanges the two copies. Therefore the PL linearization principle can be applied to the groups $W^+ \lhd W$. If h exists, then its action on S^{n-1}/W^+ commutes with the action of a reflection $s \in W$ on S^{n-1}/W^+, since h normalizes W by assumption. In particular, the quotient space S^{n-1}/W^* can be realized as the suspension of $\partial\Lambda/\overline{h}$ whose cone points are interchanged by $\overline{s}$. Therefore, it is clear that S^{n-1}/W^* is a PL sphere, that $S^{n-1}/W^\times$ is a PL ball and that the PL linearization principle can be applied to the groups $W^+ \lhd W^\times$ and $W^* \lhd W^\times$. Hence, we have proven

Lemma 84. *In the notation used above our main theorem holds for groups of type W, W^+, W^*, $W^\times$ and the PL linearization principle can be applied to $W^+ \lhd W$, $W^+ \lhd W^\times$ and $W^* \lhd W^\times$. In particular, it can be applied to $W^+(\mathrm{D}_n) \lhd W(\mathrm{BC}_n)$.*

3.4.2 Reflection groups induced by unitary reflection groups

For a unitary reflection group $G < \mathrm{U}_n$ we choose n algebraically independent homogenous generators $f_1, \ldots, f_n \in \mathbb{C}[z_1, \ldots, z_n]^G$ given by Chevalley's theorem (cf. Section 3.3.4). The continuous map

$$\begin{array}{rccc} f = [f_1, \ldots, f_n]: & \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ & v & \longmapsto & (f_1(v), \ldots, f_n(v)) \end{array}$$

descends to a continuous map $\overline{f} : \mathbb{C}^n/G \to \mathbb{C}^n$. The map $\overline{f}$ is injective, since the algebra of invariants of G separates its orbits [LT09, Thm. 3.5, p. 41], and also onto [LT09, Thm. 3.15, p. 45]. Moreover, since $\mathbb{C}[z_1, \ldots, z_n]$ is a finitely generated $\mathbb{C}[z_1, \ldots, z_n]^G$-module [Sta79, Thm. 1.3, p. 478], the map f is a finite and hence proper morphism of complex affine varieties [EGA61, 6.1.11, 5.5.3]. Therefore, the map f is also proper with respect to the usual topology in the sense of [Bou71, Ch. 1, §10, no. 1, Def. 1] by [SGA71, Ch. XII., Prop. 3.2]. In particular, the map f is closed. Consequently $\overline{f}$ is a homeomorphism and thus $\mathbb{R}^{2n}/G$ and $\mathbb{R}^{2n}$ are homeomorphic where G is regarded as a real rotation group.

The continuity of $\overline{f}^{-1}$ can alternatively be shown by induction as follows. According to a theorem by Steinberg isotropy groups of unitary reflection groups are again unitary reflection groups [Ste64, Thm. 1.5, p. 394] (cf. [LT09, Thm. 9.44, p. 186] and [Leh04]). Hence, it follows by induction that $\mathbb{C}^n/G - \{\overline{0}\}$ is a topological manifold, where $\overline{0}$ is the coset of $0 \in \mathbb{C}^n$ in $\mathbb{C}^n/G$. Then the domain invariance theorem [Hat02, Thm. 2B.3] implies that the restriction $\overline{f} : \mathbb{C}^n/G - \{\overline{0}\} \to \mathbb{C}^n - \{0\}$ is a homeomorphism. In particular, the complement in $\mathbb{C}^n/G - \{\overline{0}\}$ of the preimage S of the unit sphere in $\mathbb{C}^n$ has two components. Since S is compact and does not contain $\overline{0}$, it has strictly positive distance from $\overline{0}$ and $\mathbb{C}^n/G - S$ has two components as well. By continuity of $\overline{f}$ the component of $\overline{0}$ maps onto the interior of the unit sphere in $\mathbb{C}^n$. Moreover, since $\overline{f}$ is unbounded (e.g. by Liouville's theorem) and S is bounded in $\mathbb{C}^n/G$, it follows that the preimage B in $\mathbb{C}^n/G$ of the unit ball in $\mathbb{C}^n$ is bounded and hence compact. For, since B is path connected there would otherwise exist a path $\gamma : [0, 1) \to \mathbb{C}^n/G - S$ with $\gamma(0) = \overline{0}$ such that $\overline{f}(\gamma(t))$ tends to infinity as t goes to 1, contradicting the intermediate value theorem. The preimage B being compact in turn implies continuity of $\overline{f}^{-1}$ in 0. A sequence (z_n) in $\mathbb{C}^n/G$ for which $f(z_n)$ converges to 0 must have a convergent subsequence converging to some z by compactness of B. Continuity of f implies $z = \overline{0}$. If (z_n) did not converge to $\overline{0}$, it had another accumulation point $z' \neq \overline{0}$ with $\overline{f}(z') = 0$ contradicting the fact that f is a bijection. Hence, in any case we see that $\overline{f} : \mathbb{C}^n/G \to \mathbb{C}^n$ defines a homeomorphism.

Finally, since isotropy groups of unitary reflection groups are again unitary reflection groups by Steinbergs theorem, it follows by induction as explained in Section 3.3.3 that the PL quotient $\mathbb{R}^{2n}/G$ for a unitary reflection group G regarded as a rotation group is PL homeomorphic to $\mathbb{R}^{2n}$, i.e. we have

Lemma 85. *Our main theorem holds for unitary reflection groups considered as real groups.*

3.4.3 Reflection-rotation groups in dimension up to four

Groups up to dimension three can be easily treated by hand. For instance, all finite subgroups of SO_2 and SO_3 are orientation preserving subgroups of reflection groups which have been treated in Section 3.4.1. In dimension four a classification based proof becomes rather long-winded and cumbersome (cf. [Mik84, §3]).

The following approach dispenses with the classification in dimensions up to four. The proof is based on induction on the dimension. For simplicity let us suppose that we have already treated the cases $n < 4$, i.e. we formulate the proof for $n = 4$. The same arguments apply in dimenions $n < 4$. All isotropy groups of a rotation group $G < \mathrm{SO}_4$ are again rotation groups. Therefore, the quotient S^3/G is a closed simply connected PL manifold with respect to a PL structure induced by K/G where K is an admissible triangulation for the action of G on S^3 (cf. Lemma 76). Hence, by the PL version of the Poincaré conjecture (cf. Section 3.3.3), the PL quotient S^3/G is PL homeomorphic to the standard PL 3-sphere and $\mathbb{R}^4/G$ is PL homeomorphic to $\mathbb{R}^4$, i.e. we have

Lemma 86. *Our main theorem holds for rotation groups in dimension up to four.*

Now let $G < \mathrm{O}_4$ be a finite group and suppose that $H \triangleleft G$ is a rotation group (again, the case of lower dimensions works analogous). We endow S^3 with an admissible PL structure for the action of G (and hence H). According to Theorem A the quotient S^3/H admits a smooth structure such that the identity map of S^3/H is a PD homeomorphism and such that G/H acts smoothly on it. Since smoothings of PL manifolds in dimension three are unique up to diffeomorphism [Thu97, Thm. 3.10.9], this action is smoothly conjugate to a smooth action of G/H on the standard sphere S^3 and by [DL09, Thm. E] it is thus smoothly conjugate to a linear action on S^3. Therefore, we have a PD homeomorphism $g : S^3/H \to S^3$ and a homomorphism $r : G \to \mathrm{SO}_4$ such that the following diagram commutes

$$\begin{tikzcd} G \times S^3/H \arrow[r] \arrow[d, "r\times g"'] & S^3/H \arrow[d, "g"] \\ r(G) \times S^3 \arrow[r] & S^3 \end{tikzcd}$$

According to Section 3.3.2 we can replace g by a PD homeomorphism f such that the induced PL structure on S^3 is admissible with respect to the action of $r(G)$. Therefore, the PL linearization principle can be applied to the groups $H \triangleleft G$ (cf. Section 3.3.1). In particular, taking G as a reflection-rotation group and H as its orientation preserving subgroup proves our main theorem for all reflection-rotation groups up to dimension four. Summarizing we have

Lemma 87. *Our main theorem holds for reflection-rotation groups in dimension up to four. If $H < \mathrm{O}_n$, $n \leq 4$, is a rotation group normalized by a finite group $G < \mathrm{O}_n$, then the PL linearization principle can be applied to the groups $H \triangleleft G$.*

Note that in principle the usage of the Poincaré conjecture can be avoided by applying other means such as the algebra of polynomial invariants or explicit constructions of fundamental domains (cf. [Mik84] and Section 3.4.7 for illustrations of these methods). However, proofs along such lines are cumbersome and so we do not refrain from using the Poincaré conjecture as a convenient tool.

3.4.4 Reducible reflection-rotation groups

Now let $G < \mathrm{O}_n$ be a reducible reflection-rotation group and let $\mathbb{R}^n = V_1 + \ldots + V_k$ be a decomposition into irreducible components. Let $H_i \triangleleft G$ be the normal subgroup generated by rotations that only act in V_i (i.e. by rotations of the first kind in V_i in terms Chapter 1) and let G_i be the projection of G to the i-th factor. We can assume that $H_i \neq G_i$ because otherwise H_i splits of as a direct factor. The pairs $H_i \triangleleft G_i$ that occur in this way are classified in Theorem 3. It is shown that this classification amounts to a classification of pairs $M \lhd G_{rr}$ of an irreducible reflection-rotation group G_{rr} that contains a reflection and a proper normal subgroup $M \lhd G_{rr}$ generated by rotations such that G_{rr} is generated by its reflections and by M (cf. the remark following Theorem 3). All such pairs are listed in Table 1.7 in the appendix.

Suppose there is some $i \in \{1, \dots, k\}$ and some nontrivial rotation group $H < H_i$ normalized by G_i such that the PL linearization principle can be applied to $H \lhd G_i$. Then the PL linearization principle can also be applied to $H \lhd G$. Hence, the following lemma holds (cf. Section 3.3.1).

Lemma 88. *Let G be a reducible reflection-rotation group and suppose that some H_i is nontrivial. Suppose further that our main theorem holds for all reflection-rotation groups of smaller order than G. If for each pair $M \lhd G_{rr}$ occurring in Theorem 3 there exists some nontrivial rotation group $H \lhd M$ normalized by G_{rr} such that the PL linearization principle can be applied to the groups $H \lhd G_{rr}$, then our main theorem holds for the group G.*

Note that it is necessary to verify the assumption on the pair $M \lhd G_{rr}$ in each case of Theorem 3. In fact, given such a pair $M \lhd G_{rr}$ a reducible rotation group G can be constructed with two irreducible components and with $H_1 = H_2 = M$, $G_1 = G_2 = G_{rr}$ (cf. Chapter 1). In the course of the proof, in each case of Theorem 3 we either verify the condition on $M \lhd G_{rr}$ directly or reduce such a proof to showing our main result for rotation groups of order less than G_{rr}. A reference to the lemma in which we do this for a specific pair $M \lhd G_{rr}$ can also be found in Theorem 3.

The only case in which we cannot apply Lemma 88 is, in the notation above, when all the H_i are trivial, i.e. when there are no rotations in G that act in a single V_i factor. Suppose this is the case. If the group G does not split as a product of nontrivial factors of lower order, it is either a reflection group of type A_1, a rotation group of type $W^+(\mathrm{A}_1 \times \dots \times \mathrm{A}_1)$ or a rotation group of the form

$$\Delta_\varphi(W \times W) := \{(g, \varphi(g)) \in W \times W | g \in W\} < \mathrm{O}_m \times \mathrm{O}_m$$

for some reflection group $W < \mathrm{O}_m$ and some isomorphism $\varphi : W \to W$ that maps reflections onto reflections [LM15, Thm. 4, Cor. 64]. The first two cases are treated in Lemma 84. For $m < 3$ the third case is treated in Lemma 87. If all labels of the Coxeter graph of W lie in $\{3, 4, 6\}$, then every automorphism of W that maps reflections onto reflections can be realized through conjugation by an orthogonal transformation in the normalizer of W in O_m [FH03, Cor. 19, p. 7]. In this case the quotient $\mathbb{R}^{2m}/\Delta_\varphi(W \times W)$ is PL homeomorphic to $\mathbb{R}^{2m}/\Delta_{\mathrm{id}}(W \times W)$. Hence this case is subject of Lemma 85, since $\Delta = \Delta_{\mathrm{id}}(W \times W) < W \times W$ preserves the complex structure

$$J = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix} .$$

and can thus be regarded as a unitary reflection group acting on $\mathbb{C}^m$. The only remaining cases are $W = W(\mathrm{H}_3)$ and $W = W(\mathrm{H}_4)$ (cf. Theorem 3) and indeed, in these cases there exist outer automorphisms of W that map reflections onto reflections but cannot be realized through conjugation in O_m (cf. [Fra01, pp. 31-32]). Note that the argument in [Mik84] breaks down for groups $\Delta_\varphi(W \times W)$ for which φ cannot be realized through conjugation since the proof of [Mik84, Thm. 1.2] does not work in this case. Summarizing, we have

Lemma 89. *In the notation above, let G be a reducible reflection-rotation group such that all the H_i are trivial. Suppose that our main theorem holds for all reflection-rotation groups of smaller order than G. If G is different from $\Delta_\varphi(W \times W)$ for W of type* H_3 *and* H_4*, then our main theorem holds for the group G.*

The two exceptional cases excluded in the lemma are treated in Section 3.4.7.

3.4.5 Monomial reflection-rotation groups

Let $D(n)$ be the diagonal subgroup of O_n and let $D^+(n)$ be its orientation preserving subgroup. For a permutation group $H < \mathfrak{S}_n$ consider the monomial groups $M = D^+(n) \rtimes H < \mathrm{SO}_n$ and $M^\times = D(n) \rtimes H < \mathrm{O}_n$. The subgroup $H < M^\times$ leaves the spherical simplex $\Lambda = \{x \in S^{n-1} | x_i \geq 0, i = 1, \ldots, n\}$ invariant and acts on its boundary $\partial\Lambda$. Choose an admissible triangulation for the action of $M^\times$ on S^{n-1} that refines the triangulation of S^{n-1} by the $D(n)$ translates of Λ. The quotient $S^{n-1}/D^+(n)$ of S^{n-1} by the diagonal subgroup $D^+(n)$ can be obtained by gluing together two copies of Λ along their boundaries. It can be realized as the suspension of $\partial\Lambda$ whose cone points are interchanged by the reflections in $D(n)$. Therefore the PL linearization principle can be applied to the groups $D^+(n) \lhd M^\times$. In particular, we have (note that $D(W^+(\mathrm{BC}_n)) = D(W^+(\mathrm{D}_n)) = D^+(n)$)

Lemma 90. *The PL linearization principle can be applied to the groups $D(W^+(\mathrm{BC}_n)) \lhd W(\mathrm{BC}_n)$ and thus also to the groups $D(W^+(\mathrm{D}_n)) \lhd W(\mathrm{D}_n)$.*

Note that with respect to the constructed linearization the reflection in $M^\times/D^+(n)$ acts in a 1-dimensional subspace orthogonal to a subspace $\mathbb{R}^{n-1}$ in which $H \cong M/D^+(n)$ acts. Now suppose that M is a rotation group. Then the linearization of H acts as a rotation group (cf. Section 3.3.1). With these observations we obtain

Lemma 91. *In the notation above, suppose that M is a rotation group and that our main theorem holds for all reflection-rotation groups of smaller order than M. Then our main theorem holds for the reflection-rotation groups M and $M^\times$ and the PL linearization principle can be applied to the pairs $M \lhd M^\times$. In particular, this applies in the cases of $M = M_5, M_6, M_7, M_8, M(\mathrm{D}_n)$ in Theorem 1, (v), (a).*

The exceptional monomial rotation groups $M(\mathrm{Q}_7) = M_7^p < \mathrm{SO}_7$ and $M(\mathrm{Q}_8) = M_8^p < \mathrm{SO}_8$ (cf. Theorem 1, (v), (a)) are treated in Section 3.4.7.

3.4.6 Imprimitive reflection-rotation groups

In this section we treat the irreducible imprimitive rotation groups

$$G^*(kq, k, n) = \langle G(kq, k, n), \tau \rangle < \mathrm{SO}_{2n}$$

where $n > 2$, $q \in \mathbb{N}$, $k = 1, 2$, $kq \geq 3$ and where τ is a rotation that conjugates the first two coordinates, i.e.

$$\tau(z_1, z_2, z_3 \ldots, z_n) = (\overline{z}_1, \overline{z}_2, z_3 \ldots, z_n),$$

and the corresponding irreducible imprimitive reflection-rotation groups

$$G^\times(kq, k, n) = \langle G(kq, k, n), s \rangle < \mathrm{SO}_{2n},$$

where s is a reflection that conjugates the first coordinate, i.e.

$$s(z_1, z_2, z_3 \ldots, z_n) = (\overline{z}_1, z_2, \ldots, z_n)$$

(cf. [LM15] for more details on the constructions of these groups). Let $\mathbb{R}^{2n} = V_1 + \cdots + V_n$ be a decomposition into components of a system of imprimitivity of $G^\times(kq, k, n)$ (and hence of $G^*(kq, k, n)$), i.e. a decomposition into subspaces that are permuted by the group. Let $H \triangleleft G^\times(kq, k, n)$ be the normal subgroup generated by rotations in $G^\times(kq, k, n)$ that only act in one of the factors V_i, $i = 1, \ldots, n$. The projection H_i of H to $\mathrm{O}(V_i)$ is a cyclic group of order q and the group H splits as a product of these projections, i.e. $H = H_1 \times \cdots \times H_n$. Because of $k \in \{1, 2\}$ and $kq \geq 3$, the group H is nontrivial. Due to Lemma 87 the PL linearization principle can be applied to the groups $H \triangleleft G^\times(kq, k, n)$. Hence, we have

Lemma 92. *Let G be a reflection-rotation group of type $G^\times(kq, k, n)$ or $G^*(kq, k, n)$ with $n > 2$, $k = 1, 2$, $kq \geq 3$ and suppose that our main theorem holds for reflection-rotation groups of smaller order than G. Then our main result holds for the group G.*

Moreover, we see

Lemma 93. *Let $M \triangleleft G_{rr}$ be a pair occurring in Theorem 3 of type $G^*(kq, k, n) \triangleleft G^\times(kq, k, n)$ or $G^*(2q, 2, n) \triangleleft G^\times(2q, 1, n)$, $n > 2$, $k = 1, 2$, $kq \geq 3$. Then there exists a nontrivial rotation group $H < M$ normalized by G_{rr} such that the PL linearization principle can be applied to the groups $H \triangleleft G_{rr}$.*

Observe that by now we have verified the conclusion of the preceding lemma for all pairs of groups $M \triangleleft G_{rr}$ occurring in Theorem 3, and hence established the respective condition in Lemma 88 on reducible reflection-rotation groups.

3.4.7 Exceptional rotation groups

The only indecomposable rotation groups for which we have not verified the conclusion of our main result yet are the exceptional irreducible rotation groups $M(\mathrm{R}_5)$, $M(\mathrm{S}_6)$, $M(\mathrm{Q}_7)$, $M(\mathrm{Q}_8)$ and $M(\mathrm{T}_8)$ and the exceptional reducible rotation groups $\Delta_\varphi(W \times W)$ for W of type H_3 and H_4 (cf. Section 3.4.4 and [LM15, Sect. 4.6]). The proofs in [Mik84] in the cases of $M(\mathrm{R}_5)$, $M(\mathrm{S}_6)$, $M(\mathrm{Q}_7)$, $M(\mathrm{Q}_8)$ in principle work [Mik84, II)-IV) in Thm. 1.4, p. 105] but lack some arguments. This manifests in the fact that isotropy groups, which determine the local structure of the respective quotient, are not examined. The cases of $M(\mathrm{T}_8)$ and $\Delta_\varphi(W \times W)$ for W of type H_3 and H_4 are not considered in [Mik84].

In the cases $n > 5$ we will make use of the PL version of the generalized Poincaré conjecture (cf. Theorem 78). For $n = 5$ this tool is not available, which is why we have to perform a computation by hand in the case of $M(\mathrm{R}_5)$. Our arguments turn the approach in [Mik84, p. 102] to this case into a rigorous proof.

Lemma 94. *For $G = M(\mathrm{R}_5) < \mathrm{SO}_5$ the PL quotient $\mathbb{R}^5/G$ is PL homeomorphic to $\mathbb{R}^5$.*

Proof. The outline of the proof is as follows. First we construct a fundamental domain Λ on S^4 and choose an admissible triangulation that refines the tesselation of S^4 by the translates of Λ. With respect to the induced PL structure Λ is a PL 4-ball. We can choose a PL collar A of $\partial\Lambda$ in Λ that collapses onto $\partial\Lambda$ [RS72, Cor. 3.17, Cor. 3.30] (cf. Section 3.3.7). The closure of the complement of A in Λ, say B, is a PL 4-ball. We set $Q = \partial\Lambda/\sim$ and $N = A/\sim$ where $\sim$ denotes the equivalence relation induced by G. Then we can recover S^4/G by gluing together N and B along the PL 3-sphere ∂B which is not affected by $\sim$. In particular, we see that S^4/G is a PL 4-sphere, if we can show that N is a PL 4-ball. Due to Section 3.4.3 and the fact that all isotropy groups of G are rotation groups [LM15, Lem. 27], we already know that N is a PL 4-manifold with boundary (cf. Lemma 76). Moreover, since our triangulation is admissible, the projection $A \to N$ is simplicial and maps simplices homeomorphically onto simplices (cf. Section 3.1.3). Hence, N collapses onto Q by Lemma 83. Therefore, according to Lemma 82 it is sufficient to show that Q is collapsible in order to prove the lemma.

The group G is isomorphic to the alternating group $\mathfrak{A}_5$ and a specific set of generators in $\mathfrak{A}_6$ is given by $(12)(34), (15)(23), (16)(24)$ (cf. [Mik84, p. 102]) where G is regarded as the restriction of the permutation action of $\mathfrak{S}_6$ on $\mathbb{R}^6$ to the subspace $\mathbb{R}^5 = \{(x_1, \ldots, x_6) \in \mathbb{R}^6 | x_1 + \ldots + x_6 = 0\}$ of $\mathbb{R}^6$. A fundamental domain Λ for the action of G on S^4 is constructed in [Mik84, p. 103] as follows: For $v_0 = \{-1, -1, -1, 0, 1, 2\}$ we have $gv_0 \neq v_0$ for all $g \in G$ and thus

$$\Lambda = \bigcap_{g \in G} \{v \in S^4 | (v, v_0) \geq (v, gv_0)\}$$

is a fundamental domain for the action of G on $S^4 \subseteq \mathbb{R}^5 \subseteq \mathbb{R}^6$. It can be described by the 8 inequalities

$$\begin{aligned} -x_3 + x_4 &\geq 0, & -x_2 + x_6 &\geq 0, \\ -x_4 + x_5 &\geq 0, & -x_1 - 2x_2 + x_4 + 2x_5 &\geq 0, \\ -x_5 + x_6 &\geq 0, & -2x_1 - x_2 + x_4 + 2x_5 &\geq 0, \\ -x_1 + x_5 &\geq 0, & -2x_2 - x_3 + x_4 + 2x_5 &\geq 0 \end{aligned}$$

and has vertices

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{30}}(-5, 1, 1, 1, 1, 1), & v_2 &= \frac{1}{\sqrt{30}}(1, -5, 1, 1, 1, 1), \\ v_3 &= \frac{1}{\sqrt{30}}(1, 1, -5, 1, 1, 1), & v_4 &= \frac{1}{\sqrt{30}}(-1, -1, -1, -1, -1, 5), \\ v_5 &= \frac{1}{\sqrt{6}}(1, -1, -1, -1, 1, 1), & v_6 &= \frac{1}{\sqrt{84}}(1, 1, -5, -5, 4, 4), \\ v_7 &= \frac{1}{\sqrt{6}}(-1, 1, -1, -1, 1, 1), & v_8 &= \frac{1}{\sqrt{84}}(-5, 4, -5, 1, 1, 4). \end{aligned}$$

Let P_i be the boundary of the half-space in $\mathbb{R}^6$ determined by the ith inequality above. The faces of the fundamental domain are the following three-dimensional polytopes: In the plane P_1 the pentagonal pyramid $v_1v_2v_4v_5v_6v_7$ with vertex v_4; in P_2 the double pyramid $v_1v_2v_3v_4v_8$ with vertices v_2 and v_8; in P_3 the pentagonal pyramid $v_1v_2v_3v_5v_6v_7$ with vertex v_3; in P_4 the simplex $v_2v_3v_4v_5$; in P_5 the simplex $v_1v_3v_7v_8$; in P_6 the double pyramid $v_3v_4v_6v_7v_8$ with vertices v_6 and v_8; in P_7 the simplex $v_3v_4v_5v_6$ and in P_8 the simplex $v_1v_4v_7v_8$. The boundary of the fundamental domain is illustrated in Figure 3.1.

The 59 elements $g \in G - \{\mathrm{id}\}$ induce the following identifications on $\partial\Lambda$

$$
\begin{aligned}
(12)(34) \text{ in } P_1 &: v_1 \rightleftharpoons v_2, v_4 \rightleftharpoons v_4, v_6 \rightleftharpoons v_6, v_5 \rightleftharpoons v_7,\\
(13)(45) \text{ in } P_2 &: v_1 \rightleftharpoons v_3, v_2 \rightleftharpoons v_2, v_4 \rightleftharpoons v_4, v_8 \rightleftharpoons v_8,\\
(12)(56) \text{ in } P_3 &: v_1 \rightleftharpoons v_2, v_3 \rightleftharpoons v_3, v_5 \rightleftharpoons v_7, v_6 \rightleftharpoons v_6,\\
(15)(23) \text{ in } P_4 &: v_2 \rightleftharpoons v_3, v_4 \rightleftharpoons v_4, v_5 \rightleftharpoons v_5,\\
(13)(26) \text{ in } P_5 &: v_1 \rightleftharpoons v_3, v_7 \rightleftharpoons v_7, v_8 \rightleftharpoons v_8,\\
(14)(25) \text{ in } P_6 &: v_3 \rightleftharpoons v_3, v_4 \rightleftharpoons v_4, v_6 \rightleftharpoons v_8, v_7 \rightleftharpoons v_7,\\
(13425) &: v_1 \rightharpoonup v_3, v_4 \rightleftharpoons v_4, v_7 \rightharpoonup v_5, v_8 \rightharpoonup v_6,\\
(15243) &: v_1 \leftharpoondown v_3, v_4 \rightleftharpoons v_4, v_7 \leftharpoondown v_5, v_8 \leftharpoondown v_6,
\end{aligned}
$$

$$
\begin{aligned}
(3,4)(5,6) &: v_{(1,2,5,6,7)} \rightleftharpoons v_{(1,2,5,6,7)}\\
(2,3)(4,6) &: v_1 \rightleftharpoons v_1, \rightharpoonup v_2 \rightharpoonup v_3, v_3 \rightharpoonup v_2,\\
(2,6)(4,5) &: v_1 \rightleftharpoons v_1, v_3 \rightleftharpoons v_3, v_8 \rightleftharpoons v_8,\\
(1,3,2)(4,6,5) &: v_1 \rightharpoonup v_3, v_2 \rightharpoonup v_1, v_3 \rightharpoonup v_2,\\
(1,2,3)(4,5,6) &: v_1 \leftharpoondown v_3, v_2 \leftharpoondown v_1, v_3 \leftharpoondown v_2,\\
(1,4,3)(2,5,6) &: v_3 \rightharpoonup v_1, v_6 \rightharpoonup v_8, v_7 \rightleftharpoons v_7,\\
(1,3,4)(2,6,5) &: v_3 \leftharpoondown v_1, v_6 \leftharpoondown v_8, v_7 \leftharpoondown v_7,\\
(1,2,3,5,4) &: v_1 \rightharpoonup v_2, v_2 \rightharpoonup v_3, v_4 \rightleftharpoons v_4,
\end{aligned}
$$

$$
\begin{aligned}
(1,5)(4,6) &: v_2 \rightleftharpoons v_2, v_3 \rightleftharpoons v_3,\\
(2,4)(3,5) &: v_1 \rightleftharpoons v_1, v_4 \rightleftharpoons v_4,\\
(1,5,6)(2,3,4) &: v_2 \rightharpoonup v_3, v_5 \rightleftharpoons v_5,\\
(1,6,5)(2,4,3) &: v_3 \rightharpoonup v_2, v_5 \rightleftharpoons v_5,\\
(2,3,6,5,4) &: v_1 \rightleftharpoons v_1, v_2 \rightharpoonup v_3,\\
(2,4,5,6,3) &: v_1 \rightleftharpoons v_1, v_2 \leftharpoondown v_3,\\
(1,3,5,6,4) &: v_1 \rightharpoonup v_3, v_2 \rightleftharpoons v_2,\\
(1,4,6,5,3) &: v_1 \leftharpoondown v_3, v_2 \rightleftharpoons v_2,
\end{aligned}
$$

$$
\begin{aligned}
(1,4,5,3,2) &: v_1 \leftharpoondown v_2, v_2 \leftharpoondown v_3, v_4 \rightleftharpoons v_4,\\
(1,5,6,2,4) &: v_3 \rightleftharpoons v_3, v_5 \rightharpoonup v_7, v_6 \rightharpoonup v_8,\\
(1,4,2,6,5) &: v_3 \rightleftharpoons v_3, v_5 \leftharpoondown v_7, v_6 \leftharpoondown v_8,\\
(1,6,5,2,3) &: v_2 \rightharpoonup v_3, v_3 \rightharpoonup v_1, v_5 \rightharpoonup v_7,\\
(1,3,2,5,6) &: v_2 \leftharpoondown v_3, v_3 \leftharpoondown v_1, v_5 \leftharpoondown v_7,\\
(1,6)(2,4) &: v_3 \rightleftharpoons v_3, v_5 \rightleftharpoons v_5,
\end{aligned}
$$

$$
\begin{aligned}
(1,3,6,4,2) &: v_1 \rightharpoonup v_3, v_2 \rightharpoonup v_1,\\
(1,2,4,6,3) &: v_1 \leftharpoondown v_3, v_2 \leftharpoondown v_1,\\
(1,6,2,3,4) &: v_2 \rightharpoonup v_3, v_5 \rightharpoonup v_7,\\
(1,4,3,2,6) &: v_2 \leftharpoondown v_3, v_5 \leftharpoondown v_7,\\
(1,2,5,4,6) &: v_1 \rightharpoonup v_2, v_3 \rightleftharpoons v_3,\\
(1,6,4,5,2) &: v_1 \leftharpoondown v_2, v_3 \rightleftharpoons v_3
\end{aligned}
$$

$$
\begin{aligned}
(1,4)(3,6) :& v_2 \rightleftharpoons v_2, \\
(1,6)(3,5) :& v_2 \rightleftharpoons v_2, \\
(2,5)(3,6) :& v_1 \rightleftharpoons v_1, \\
(1,3,6)(2,4,5) :& v_1 \rightharpoonup v_3, \\
(1,6,3)(2,5,4) :& v_1 \leftharpoondown v_3, \\
(1,3,5)(2,4,6) :& v_1 \rightharpoonup v_3, \\
(1,5,3)(2,6,4) :& v_1 \leftharpoondown v_3, \\
(1,4,6)(2,3,5) :& v_2 \rightharpoonup v_3, \\
(1,6,4)(2,5,3) :& v_2 \leftharpoondown v_3, \\
(1,2,6)(3,4,5) :& v_1 \rightharpoonup v_2, \\
(1,6,2)(3,5,4) :& v_1 \leftharpoondown v_2,
\end{aligned}
\qquad
\begin{aligned}
(1,5,4)(2,6,3) :& v_3 \rightharpoonup v_2, \\
(1,4,5)(2,3,6) :& v_3 \leftharpoondown v_2, \\
(1,4,2)(3,5,6) :& v_2 \rightharpoonup v_1, \\
(1,2,4)(3,6,5) :& v_2 \leftharpoondown v_1, \\
(1,2,5)(3,6,4) :& v_1 \rightharpoonup v_2, \\
(1,5,2)(3,4,6) :& v_1 \leftharpoondown v_2, \\
(2,5,3,4,6) :& v_1 \rightleftharpoons v_1, \\
(2,6,4,3,5) :& v_1 \rightleftharpoons v_1, \\
(1,5,4,3,6) :& v_2 \rightleftharpoons v_2, \\
(1,6,3,4,5) :& v_2 \rightleftharpoons v_2, \\
(1,2,6,3,5) :& v_1 \rightharpoonup v_2, \\
(1,5,3,6,2) :& v_1 \leftharpoondown v_2,
\end{aligned}
$$

The first six of them correspond to "pasting in half" the faces of Λ lying in the planes $P_1, \dots, P_6$. Since points in the interior of such a face are not identified with points outside the interior of this face, we see that the images of these faces in the quotient Q can be collapsed to the images of their boundary in Q. In particular, we see that Q collapses onto the images of the faces $v_3v_4v_5v_6$ and $v_1v_4v_7v_8$ of Λ. Since these faces are identified with each other by (13425), we see that Q collapses onto the image of $v_3v_4v_5v_6$ in Q. Examining the list of generating identifications shows that this image is a 3-simplex itself. Hence, Q is collapsible and thus the claim follows by the remarks above.

□

In principle, the proofs in [Mik84] in the cases $M(\mathrm{S}_6)$, $M(\mathrm{Q}_7)$ and $M(\mathrm{Q}_8)$ can be made rigorous in the same way. However, in order to avoid long computations with fundamental domains and identifications, we provide the following alternative argument.

Since $n > 5$ in each of the remaining cases, it is sufficient to show that S^{n-1}/G is a simply connected piecewise linear manifold with $H_*(S^{n-1}/G) = H_*(S^{n-1})$ in order to prove that $\mathbb{R}^n/G$ is piecewise linear homeomorphic to $\mathbb{R}^n$ (cf. Section 3.3.3, Theorem 78). According to Lemma 28 (cf. [LM15, Lem. 27]) all isotropy groups of the remaining irreducible exceptional rotation groups $M(\mathrm{S}_6)$, $M(\mathrm{Q}_7)$, $M(\mathrm{Q}_8)$ and $M(\mathrm{T}_8)$ are rotation groups. The same statement is true for reducible rotation groups of type $\Delta_\varphi(W \times W)$ (cf. Section 3.4.4), since isotropy groups of real reflection groups are generated by the reflections they contain [Hum90, Thm. 1.12 (c), p. 22] (apply this result twice). Therefore, the proof that S^{n-1}/G is a piecewise linear manifold, is reduced to proving the conclusion of Theorem B for rotation groups of lower order than G. For a rotation group G the quotient S^{n-1}/G is always simply connected for $n > 2$ by Lemma 79. In particular, we have $H_1(S^{n-1}/G) = \pi_1(S^{n-1}/G)_{ab} = 0$ [Hat02, Thm. 2A.1]. By Poincaré duality and the universal coefficient theorem, it is sufficient to show that $H_2(S^{n-1}/G) = 0$ or $H_2(S^{n-1}/G) = H_3(S^{n-1}/G) = 0$, respectively, depending on whether

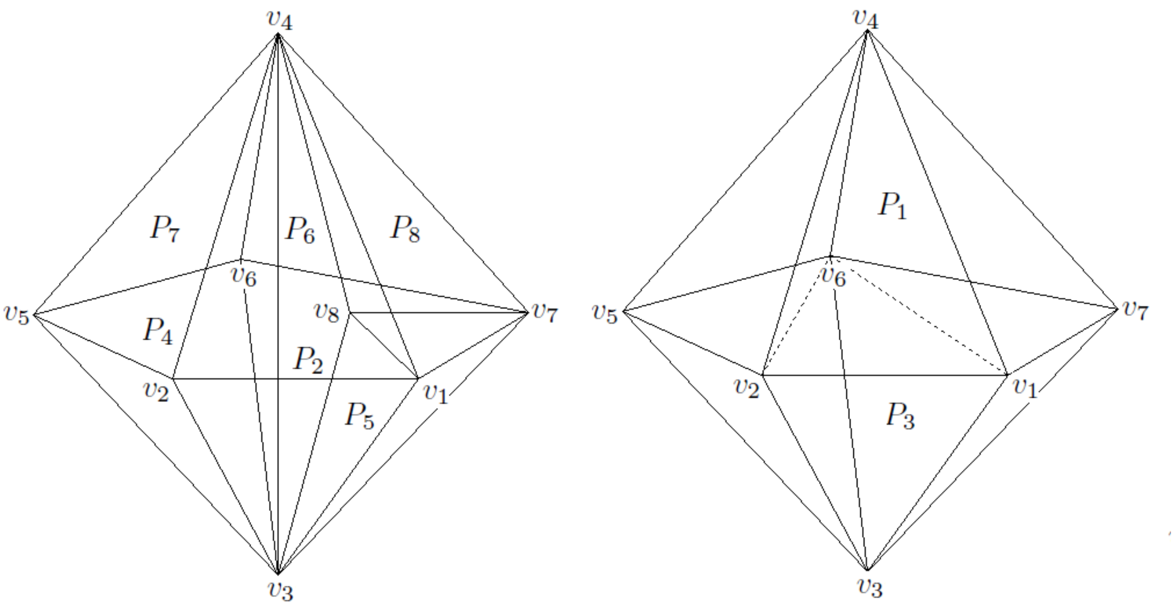

Figure 3.1: Boundary of a fundamental domain Λ for the action of $M(\mathrm{S}_5) = R_5(\mathfrak{A}_5)$ on S^4 cut into two pieces.

$n \leq 6$ or $6 < n \leq 8$ [Hat02, Thm. 3.2, Thm. 3.30], in order to prove $H_*(S^{n-1}/G) = H_*(S^{n-1})$. In Section 3.3.3 we have seen that the existence of subgroups $H < G$ with coprime indices and $H_i(S^{n-1}/H) = 0$ implies $H_i(S^{n-1}/G) = 0$. This is in particular the case for rotation groups H for which we have already shown that $\mathbb{R}^n/H$ is homeomorphic to $\mathbb{R}^n$ (cf. the long exact sequence in [Hat02, p. 117]). Hence, in order to treat the remaining cases, it is sufficient to find suitable subgroups.

Lemma 95. *If our main result holds for all rotation groups of smaller order than $M(\mathrm{S}_6)$, then it also holds for $M(\mathrm{S}_6)$.*

Proof. The group $M(\mathrm{S}_6) \cong \mathrm{PSL}_2(7)$ has order $2^3 \cdot 3 \cdot 7$ and thus contains a subgroup H of order 7. Since $M(\mathrm{S}_6)$ can be realized as the isomorphic image of a permutation group in $S_7 < \mathrm{SO}_7$ to SO_6 (cf. [LM15, Lem. 16] or Section 1.3.7), this subgroup of order 7 is generated by a 7-cycle in S_7 and acts thus freely on the unit sphere $S^5 \subset \mathbb{R}^6$. The facts that $H_2(\mathbb{Z}_7) = 0$ [Bro94, (3.1), p. 35] and that $H_2(S^5/H) = H_2(H)$ for groups acting freely on S^5 [Bro94, p. 20] imply that $H_2(S^5/H) = 0$. A rotation group of order $2^3 \cdot 3$ contained in $M(\mathrm{S}_6)$ is described in Lemma 29 (cf. [LM15, Lem. 28]). Hence, the claim follows by the remarks above. $\square$

The reflection groups $W(\mathrm{H}_3)$ and $W(\mathrm{H}_4)$ properly contain reflection groups with coprime indices (cf. [DPR13, Table 8, Table 9]) and thus the rotation groups $\Delta_\varphi(W \times W)$ for W of type H_3 and H_4 properly contain rotation groups with coprime indices. The rotation groups $M(\mathrm{Q}_7)$ and $M(\mathrm{T}_8)$ also properly contain rotation groups with coprime indices by Lemma 32 and Lemma 30 (cf. [LM15, Lem. 29, Lem. 31]). By the remarks above, we obtain

Lemma 96. *Let G be a rotation group of type $M(\mathrm{Q}_7)$, $M(\mathrm{T}_8)$ or $\Delta_\varphi(W \times W)$ for W of type H_3 or H_4. If our main result holds for all rotation groups of smaller orden than G, then it also holds for G.*

The group $M(\mathrm{Q}_7)$ does not contain rotation groups with coprime indices. However, recall the statement of Lemma 31 about the rotation group $M(\mathrm{Q}_8)$ (cf. [LM15, Lem. 30]).

Lemma 97. *The rotation group $M(\mathrm{Q}_8)$ of order $21504 = 2^{10}\cdot 3\cdot 7$ contains a reducible rotation group G of order $1536 = 2^9 \cdot 3$ with $k = 2$ and*

$$(G_i, H_i, F_i, G_i/H_i) = (W(\mathrm{D}_4), D(W^+(\mathrm{D}_4)), W(\mathrm{D}_4), W(\mathrm{A}_3)),$$

$i = 1, 2$, which is normalized by an element τ of order two that interchanges the two components of G. Moreover, it contains the rotation group $R_6(\mathrm{PSL}_2(7))$ of order $168 = 2^3 \cdot 3 \cdot 7$.

We have

Lemma 98. *For $N = \langle G, \tau\rangle$ as in the preceding lemma we have $\mathbb{R}^8/N \cong C(S^3 * \mathbb{RP}^3)$ and hence $H_*(S^7/N) = H_*(\Sigma^4(\mathbb{RP}^3))$. In particular, $H_2(S^7/N) = H_3(S^7/N) = 0$.*

Proof. Let $\mathbb{R}^8 = V_1 + V_2$ be a decomposition into irreducible components with respect to G. We can assume that V_1 and V_2 are two identical copies of $\mathbb{R}^4$ such that the projections of G to V_1 and V_2 coincide. Since τ normalizes G it has the form

$$\tau = \begin{pmatrix} 0 & h \\ h^{-1} & 0 \end{pmatrix}$$

for some $h \in N_{\mathrm{O}_4}(W(\mathrm{D}_4))$. After conjugation we can assume that $h = \mathrm{id}$ and thus we can apply the linearization principle to the groups $H_1 \times H_2 \lhd N$. It then remains to show that $\mathbb{R}^8/\overline{N} \cong C(S^3 * \mathbb{RP}^3)$ for $\overline{N} = \langle \overline{G}, \sigma\rangle$ with $\sigma = \overline{\tau}$ and $\overline{G} = G_\varphi = \{(g, \varphi(g))| g \in W(\mathrm{A}_3)\} \subset W(\mathrm{A}_3) \times W(\mathrm{A}_3) < \mathrm{O}_4 \times \mathrm{O}_4$ for some $\varphi \in \mathrm{Aut}(W(\mathrm{A}_3))$. Since the symmetric group $\mathfrak{S}_4$ has no outer automorphisms we can assume that $\varphi = \mathrm{id}$ after conjugation. Then σ has the form

$$\sigma = \begin{pmatrix} 0 & g_0 \\ g_0^{-1} & 0 \end{pmatrix} < \mathrm{SO}_8$$

for some $g_0 \in W(\mathrm{A}_3) < \mathrm{O}_4$. Now $\sigma \in N(\overline{G})$ implies $g_0^2 \in C_{\mathrm{O}_4}(W(\mathrm{A}_3)) = \{\pm\mathrm{id}_3\} \times \{\pm\mathrm{id}_1\} < \mathrm{O}_3 \times \mathrm{O}_1$. Because of $g_0 \in W(\mathrm{A}_3)$ and since $W(\mathrm{A}_3)$ has a trivial center, we have $g_0^2 = 1$, i.e. $g_0 = g_0^{-1}$. If we identify $\mathbb{R}^8$ with $\mathbb{C}^4$ then $\overline{G}$ can be regarded as a unitary reflection group and the action of σ is given by $\sigma((z_1, z_2, z_3, z_4)) = i \cdot g_0(\overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4)$ where g_0 permutes the coordinates. Let s_i be the elementary symmetric polynomial of degree i in the z_j, $i, j = 1, 2, 3, 4$. Then the map

$$\begin{array}{rccc} f: & \mathbb{C}^4/\overline{G} & \to & \mathbb{C}^4 \\ & (z_1, z_2, z_3, z_4) & \mapsto & (s_1, s_2, s_3, s_4) \end{array}$$

defines a homeomorphism as explained in Section 3.3.4. Since the s_i are invariant under coordinate permutations, the induced action of $\overline{N}/\overline{G}$ on $\mathbb{C}^4$ is given by

$$\overline{\sigma}(s_1, s_2, s_3, s_4) = (i \cdot \overline{s}_1, -\overline{s}_2, -i \cdot \overline{s}_3, -\overline{s}_4).$$

It follows that $\mathbb{R}^8/N \cong C(S^3 * \mathbb{RP}^3)$. In particular, we have $H_2(S^7/N) = H_3(S^7/N) = 0$ by the long exact sequence in [Hat02, p. 117]. □

With the preceding two lemmas and the remarks above we obtain

Lemma 99. *If our main result holds for all rotation groups of smaller order than $M(\mathrm{Q}_8)$, then it also holds for $M(\mathrm{Q}_8)$.*

Since we have by now treated all cases, Theorem B follows by induction.

3.5 Towards a classification free proof

Our proof of the if direction of Theorem B relies on the classification of reflection-rotation groups. As a corollary of Theorem B reflection-rotation groups share the following properties.

(i) Isotropy groups of reflection-rotation groups are generated by the reflections and rotations they contain.

(ii) For a rotation group $G < \mathrm{SO}_n$ we have

$$H_*(S^{n-1}/G;\mathbb{Z}) = H_*(S^{n-1};\mathbb{Z}).$$

(iii) For a reflection-rotation group $G < \mathrm{O}_n$ that contains a reflection we have

$$H_*(S^{n-1}/G;\mathbb{Z}) = H_*(\{*\};\mathbb{Z}).$$

Conversely, properties (i), (ii) and (iii) together with the PL h-cobordism theorem and some extra work in low dimensions imply the if direction of Theorem B by induction as explained in Section 3.3.3. Hence, in order to essentially dispense with the classification of reflection-rotation groups in the proof of our result one might first try to find conceptual proofs for properties (i), (ii) and (iii) that do not depend on a classification of reflection-rotation groups.

Chapter 4

When is the underlying space of an orbifold a manifold?

In this chapter we answer the question posed by Davis "When is the underlying space $|\mathcal{O}|$ of a smooth orbifold $\mathcal{O}$ a topological manifold?" [Dav11, p. 9] and other variants of it. For instance, for Riemannian orbifolds it is natural to ask Davis's question in the category of *Lipschitz manifolds*, i.e. to ask whether a given Riemannian orbifold is a Lipschitz manifold. Moreover, in order to answer the original version of the question in the category of *topological manifolds*, it turns out to be useful to first look at the analogous question in the category of *homology manifolds*. In addition, it makes sense to admit *manifolds with boundary* in the formulation of Davis's question and we also investigate this possibility.

4.1 Reformulation and strategy

Riemannian orbifolds can be defined as follows.

Definition 10. A *Riemannian orbifold* of dimension n is a length space $\mathcal{O}$ such that for each point $x \in \mathcal{O}$ there exists an open neighborhood U of x in $\mathcal{O}$ and a Riemannian manifold M of dimension n together with a finite group G acting by isometries on M such that U and M/G are isometric.

Every Riemannian orbifold admits a unique compatible smooth orbifold structure (cf. Appendix A.2) and every paracompact smooth orbifold admits a compatible Riemannian structure (cf. [BH99, Ch. III.1]). Hence, in view of Davis's question, we can work with Riemannian orbifolds. The question whether a given Riemannian orbifold is a manifold only depends on its local structure. For a Riemannian orbifold $\mathcal{O}$ the isotropy group $G_p < \mathrm{O}(T_pM)$ of an inverse image $p \in M$ of a point $x \in \mathcal{O}$ is unique up to conjugation. It is called the *local group* of $\mathcal{O}$ at x and is denoted by $\mathcal{O}_x$. The local group $\mathcal{O}_x$ of a point $x \in \mathcal{O}$ determines the topology and geometry of $\mathcal{O}$ in a neighborhood of x in the following way.

Lemma 100. *For an n-dimensional Riemannian orbifold $\mathcal{O}$ and a point $x \in \mathcal{O}$ there exists a neighborhood of x in $\mathcal{O}$ that is locally bi-Lipschitz homeomorphic to $\mathbb{R}^n/G_x$.*

Proof. Let p, U, M and G be given as above. For sufficiently small balls $B_r(0) \subset T_pM$ and $B_r(p) \subset M$ the exponential map $\exp_p : B_r(0) \to B_r(p)$ defines a G_p-equivariant diffeomorphism. We can compose it with a G_p-equivariant diffeomorphism between $T_pM = \mathbb{R}^n$ and

$B_r(0)$ to obtain a G_p-equivariant locally bi-Lipschitz homeomorphism between $\mathbb{R}^n$ and $B_r(p)$. This homeomorphism descends to a locally bi-Lipschitz homeomorphism between the quotient spaces $\mathbb{R}^n/G_p$ and $B_r(p)/G_p$. The latter is isometric to a neighborhood of x in $\mathcal{O}$. □

According to this lemma, Davis's question and its variants can be reformulated as follows.

Reformulation. For which finite subgroups $G < \mathrm{O}_n$ is the quotient space $\mathbb{R}^n/G$ a topological- or Lipschitz manifold, possibly with boundary?

In Chapter 3 we have already answered this question in the piecewise linear category. The quotient $\mathbb{R}^n/G$ is a piecewise linear manifold with boundary, if and only if G is a reflection-rotation group and the boundary of $\mathbb{R}^n/G$ for such a group is nonempty, if and only if G contains a reflection. In particular, the quotient of $\mathbb{R}^n$ by a reflection-rotation group $G < \mathrm{O}_n$ is a Lipschitz manifold with boundary. We show that the converse also holds. We will also see that the *binary icosahedral group* has a representation in dimension 4 whose image, which we call a *Poinaré group*, yields another example in the topological- but not in the Lipschitz category. We show that it is the only additional example in the topological category up to products.

The proof in the topological category is divided into three steps. In the first step we observe that if $\mathbb{R}^n/G$ is a homology manifold for a finite subgroup $G < \mathrm{O}_n$, then strata of $\mathbb{R}^n/G$ that are not contained in the closure of any higher dimensional singular stratum either have codimension two or codimension four and that the corresponding local groups are either cyclic groups or Poincaré groups. In this way we obtain a "sufficiently large" normal subgroup $\tilde{G}$ of G generated by rotations and Poincaré groups. A key ingredient in this step is a theorem due to Zassenhaus that characterizes certain representations of the binary icosahedral group. In the second step we use an elementary fact about spherical triangles and the specific geometric structure of the 600-cell, i.e. the orbit of one point under the action of the Poinaré group on S^3, to show that the rotation group and all Poincaré groups generating $\tilde{G}$ act in pairwise orthogonal spaces. In the last step we show by induction that $G/\tilde{G} \curvearrowright S^{n-1}/\tilde{G}$ is a free action on a homology sphere. The algebraic information on $G/\tilde{G}$ obtained in this way suffices to identify $G/\tilde{G}$ as a trivial group. In the Lipschitz category we apply a result by Siebenmann and Sullivan. The case of manifolds with boundary is reduced to the manifold case upon passage to double covers that turn out to be orbifold covers. At the end of the chapter we use our results to generalize a fixed point theorem by Steinberg and answer a question by Petrunin on quotients of spheres.

4.2 Methods and preliminaries

4.2.1 Triangulations

Let $G < \mathrm{O}_n$ be a finite subgroup. Recall from Section 3.1.3 that a triangulation $t : K \to \mathbb{R}^n$ is called *admissible for the action of G on $\mathbb{R}^n$*, if K is a regular G-complex that contains the origin as a vertex, that such a triangulation always exists and that it defines a simplicial complex K/G that triangulates $\mathbb{R}^n/G$. Also recall that triangulations of S^{n-1} that occur as radial projections of the link of the origin of an admissible triangulation $t : K \to \mathbb{R}^n$ are also called *admissible for the action of* G *on* S^{n-1}. Since admissible triangulations always exist, we can both work with simplicial homology and with singular homology.

4.2.2 Homology manifolds

Homology manifolds are generalizations of topological manifolds. They are easier to recognize and to work with than topological manifolds, since their definition is based on homological properties and does not involve homeomorphisms that are usually difficult to handle (cf. [Can78]). In order to simplify our proofs we make the following modified definition.

Definition 11. We say that a Hausdorff space X is a *homology* n*-manifold*, if all its local homology groups coincide with the local homology groups of $\mathbb{R}^n$, i.e. if for all $x \in X$

$$H_i(X, X - \{x\}) = \begin{cases} 0, & \text{for } i \neq n \\ \mathbb{Z}, & \text{for } i = n \end{cases}$$

holds.

Remark 2. All spaces occurring in this paper are finite-dimensional simplicial complexes (cf. Section 4.2.1). Every such space is a finite-dimensional absolute neighborhood retract [Lef42, Application 18.4, p. 61]. Therefore, our modification of other stricter definitions of homology manifolds (cf. [Can78], [Wei02]) does not make any difference for the formulation of our results (cf. Theorem C).

For a topological space X and a subspace $Y \subset X$ we define the *double of* X *along* Y to be

$$2_Y X = X \times \{0, 1\}/\sim \;\text{ where } (y, 0) \sim (y, 1) \text{ for all } y \in Y$$

endowed with the quotient topology and we simply denote it by $2X$ if the meaning of the subspace is clear. In order to deal with Davis's question for manifolds with boundary, we define homology manifolds with boundary in the following way.

Definition 12. We say that a Hausdorff space X is a *homology* $(n+1)$*-manifold with boundary*, if it can be decomposed into a nonempty set of interior points $\mathring{X}$ and a set of boundary points ∂X such that its double $2X$ along its boundary is a homology $(n+1)$-manifold, its boundary ∂X is either empty or a homology n-manifold and the local homology groups at boundary points coincide with those of $0 \in \partial(\mathbb{R}^n \times \mathbb{R}_{\leq 0})$, i.e. for all $x \in \partial X$ and all $i \geq 0$ we have $H_i(X, X - \{x\}) = 0$.

Remark 3. If the space X in the definition of a homology manifold with boundary is sufficiently nice, then its boundary and its double are automatically homology manifolds. This is for example the case if X is a PL space (cf. [Mau80, p. 510], [Mit90, Prop. 5.4.11, p. 188]) and so it holds for all spaces we are working with in this thesis (cf. Section 4.2.1).

The open cone of a topological space X is defined to be $CX = (X \times [0, 1))/(X \times \{0\})$. Homology manifolds share the following properties.

Lemma 101. *For Hausdorff spaces* X *and* Y *the following statements hold for integers* $n \geq 0$.

(i) X *and* Y *are homology manifolds, if and only if* $X \times Y$ *is a homology manifold.*

(ii) If X *is a homology manifold, then* $X \times Y$ *is a homology manifold with boundary, if and only if* Y *is a homology manifold with boundary. In this case we have* $\partial(X \times Y) = X \times \partial Y$*.*

(iii) CX *is a homology* $(n+1)$*-manifold if and only if* X *is a homology* n*-manifold and* $H_*(X) = H_*(S^n)$*.*

(iv) CX *is a homology* $(n+2)$*-manifold with nonempty boundary, if and only if* X *is a homology* $(n+1)$*-manifold with nonempty boundary and* $H_*(X) = H_*(\{*\})$*,* $H_*(\partial X) = H_*(S^n)$*. In this case we have* $\partial(CX) = C(\partial X)$*.*

Proof. (i). For open subsets $A \subset X$ and $B \subset Y$ in Hausdorff spaces X and Y the following Künneth formula holds [Dol80, Cor. VI.12.10, p. 181]

$$\begin{aligned} 0 \to \bigoplus_{i+j=k} H_i(X,A) \otimes H_j(Y,B) \to H_k(X \times Y, A \times Y \cup X \times B) \\ \to \bigoplus_{i+j=k-1} H_i(X,A) * H_j(Y,B) \to 0 \end{aligned}$$

where $M * N := \mathrm{Tor}_1^{\mathbb{Z}}(M,N)$. Assume that X and Y are homology manifolds. Then $H_i(X, X - \{x\}) * H_i(Y, Y - \{y\})$ for all $x \in X$, $y \in Y$ and $i, j \in \mathbb{N}_0$ and so $X \times Y$ is a homology manifold as well by the Künneth formula above. Conversely, assume that $X \times Y$ is a homology n-manifold. Let $x \in X$, $y \in Y$ and $M_i = H_i(X, X - \{x\})$ and $N_i = H_i(Y, Y - \{y\})$. From the Künneth formula above we obtain

$$0 \to \bigoplus_{p=0}^{n} M_p \otimes N_{n-p} \to \mathbb{Z} \to \bigoplus_{q=0}^{n-1} M_p * N_{n-p} \to 0$$

and

$$M_p \otimes N_q = 0 \text{ for } p + q \neq n$$

Hence, the product $\bigoplus_{p=0}^{n} M_p \otimes N_{n-p}$ is isomorphic to $\mathbb{Z}$, since $\bigoplus_{q=0}^{n-1} M_p * N_{n-p}$ is a torsion module. Therefore, there exists $m \in \{0, \ldots, n\}$ such that $M_m \otimes N_{n-m} \cong \mathbb{Z}$ and $M_p \otimes N_{n-p} = 0$ for all $p \neq m$. By the classification of abelian groups this implies $M_m \cong N_{n-m} \cong \mathbb{Z}$, $M_p = 0$ for $p \neq m$ and $N_q = 0$ for $q \neq n - m$, i.e. X and Y are homology manifolds.

(ii). As in (i) one shows that Y has the local homology groups of a homology manifold with boundary, if and only if $X \times Y$ has the local homology groups of a homology manifold with boundary and that in this case $\partial(X \times Y) = X \times \partial Y$ holds. The additional conditions on the boundary and the double in the definition of a homology manifold with boundary then follows immediately from (i).

(iii),(iv). Statements (i) and (ii) show that for $x_0 = \overline{X \times \{0\}} \in CX = (X \times [0,1))/(X \times \{0\})$ the space X is a homology n-manifold with boundary ∂X, if and only if $Y = CX - \{x_0\}$ is a homology $(n+1)$-manifold with boundary $\partial Y = C(\partial X) - \{x_0\}$. The long exact sequence of homology groups [Hat02, p. 117]

$$\ldots \to H_n(Y) \to H_n(CX) \to H_n(CX, Y) \to H_{n-1}(Y) \to \ldots \to H_0(CX, Y) \to 0.$$

together with the facts $H_*(Y) = H_*(X \times (0,1)) = H_*(X)$ and $H_*(CX) = H_*(\{*\})$ yields

$$H_i(CX, CX - \{x_0\}) = \begin{cases} 0, & \text{for } i = 0,1 \\ H_{i-1}(X), & \text{for } i > 1 \end{cases}$$

and this implies (*iii*). If CX is a homology $(n+2)$-manifold with nonempty boundary, then so is $Y = CX - \{x_0\}$, since the boundary contains more than one point because of $n \geq 0$. Now (*ii*) implies that X is a homology $n+1$ manifold with nonempty boundary and that $\partial(CX) = C(\partial X)$. Conversely, if X is a homology manifold with nonempty boundary, then CX has the correct local homolgy groups by our computations above and $\partial(CX) = C(\partial X)$. Now the claim follows by (*iii*). □

4.2.3 Finite groups with periodic cohomology

A *complete resolution* of a finite group G is an acyclic complex $F = (F_i)_{i \in \mathbb{Z}}$ of projective $\mathbb{Z}G$-modules, together with a homomorphism $\epsilon : F_0 \to \mathbb{Z}$ such that the complex

$$\ldots \to F_2 \to F_1 \to F_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is acyclic. The *Tate homology* of G with coefficients in a G-module M is defined as

$$\widehat{H}_i(G, M) = H_*(F \otimes_{\mathbb{Z}G} M)$$

where F is a complete resolution of G regarded as a right-module via the anti-automorphism $g \mapsto g^{-1}$ of G. The *Tate cohomology* $\widehat{H}^*(G, M)$ of G with coefficients M is given by $\widehat{H}^i = \widehat{H}_{-i-1}$. The usual homology and cohomology of G with coefficients in M can be expressed in terms of the corresponding Tate homology and cohomology as $H_i(G, M) = \widehat{H}_i(G, M)$ and $H^i(G, M) = \widehat{H}^i(G, M)$ for $i > 0$ (cf. [Bro94, p. 134]). If G has periodic Tate cohomology groups, we simply say that G has *periodic cohomology.* It turns out that a finite group has periodic cohomology, if and only if every Sylow subgroup has periodic cohomology [Bro94, Thm. VI.9.4, p. 156]. Examples for finite groups with periodic cohomology are the finite subgroups of SU_2, i.e. cyclic groups and binary dihedral, -tetrahedral, -octahedral and -icosahedral groups (cf. Section 1.3.3), since they act freely on S^3 (cf. Proposition 107). The binary dihedral groups have order $4n$ and admit the representations $\langle x, y | x^{2n} = 1, y^2 = x^n, y^{-1}xy = x^{-1} \rangle$. In particular for n being a power of 2 they are also referred to as *generalized quaternion groups.* The following characterization in particular shows that cyclic groups and generalized quaternion groups are the only p-groups with periodic cohomology (see also [Bro94, Prop. VI.9.3, p. 156]).

Theorem 102 ([Bro94], Thm. VI.9.5, p. 157)**.** *The following conditions are equivalent for a finite group G.*

(i) G has periodic cohomology.

(ii) Every abelian subgroup of G is cyclic.

(iii) The Sylow subgroups of G are cyclic or generalized quaternion groups.

For our purpose the following class of examples is important.

Example. Let p be a prime and let $G = \mathrm{SL}_2(p)$, the group of 2×2 matrices of determinant 1 over the prime field of order p. Then G has periodic cohomology [Bro94, p. 157].

Finite groups with periodic cohomology have been classified by Suzuki.

Theorem 103 ([Suz55])**.** *Let G be a finite group such that*

(i) Sylow subgroups of odd order are cyclic.

(ii) the 2-Sylow subgroup is a generalized quaternion group.

Then G contains a normal subgroup $\tilde{G}$ such that $[G : \tilde{G}] \leq 2$ and $\tilde{G} = Z \times \mathrm{SL}_2(p)$ for some prime p and some solvable group Z whose Sylow subgroups are all cyclic.

In the case where all Sylow subgroups are cyclic the following result due to Burnside holds. A proof can also be found in [Wol84, Lem. 5.4.3, p. 163].

Theorem 104 ([Bur11], Ch. IX.128, p. 163)**.** *If all Sylow subgroups of a finite group G are cyclic, then G is solvable.*

From the preceding two results we deduce the following proposition.

Proposition 105. *A finite and nontrivial perfect group G with periodic cohomology is isomorphic to $\mathrm{SL}_2(p)$ for some prime $p > 3$.*

Proof. According to Theorem 102, all Sylow subgroups of G are either cyclic or generalized quaternion groups. The case where all Sylow subgoups are cyclic cannot occur by Theorem 104, since G is perfect. Otherwise, we are in the situation of Theorem 103 and thus G contains a normal subgroup $\tilde{G}$ such that $[G : \tilde{G}] \leq 2$ and $\tilde{G} = Z \times \mathrm{SL}_2(p)$ for some prime p and some solvable group Z. The fact that $[G : \tilde{G}] \leq 2$ implies $[G, G] \subset \tilde{G}$ and hence $G = \tilde{G}$, since G is perfect by assumption. Consequently, Z must be trivial. Now the claim follows, as $\mathrm{SL}_2(p)$, for p prime, is perfect if and only if $p > 3$ [Ros94, p. 61]. $\square$

In [Ade94, Cor. 6.18, p. 151] it is shown that $H^*(\mathrm{SL}_2(p); \mathbb{Z}_2) = \mathbb{Z}_2[e_4] \otimes E(b_3)$, a polynomial algebra on a four-dimensional generator tensored with an exterior algebra on a three-dimensional generator. In particular, we have $H^3(\mathrm{SL}_2(p); \mathbb{Z}_2) = \mathbb{Z}_2$. Using this we obtain

Proposition 106. *For every prime $p > 3$ we have $H_3(\mathrm{SL}_2(p); \mathbb{Z}) \neq 0$.*

Proof. Because of $H^3(\mathrm{SL}_2(p); \mathbb{Z}_2) = \mathbb{Z}_2$ and $H_2(\mathrm{SL}_2(p); \mathbb{Z}) = 0$ [Sch07, p. 119] the short exact sequence of the universal coefficient theorem reads (cf. [Bro94, p. 8])

$$0 \to H^3(\mathrm{SL}_2(p); \mathbb{Z}_2) \to \mathrm{Hom}(H_3(\mathrm{SL}_2(p); \mathbb{Z}), \mathbb{Z}_2) \to 0$$

and thus we have $H_3(\mathrm{SL}_2(p); \mathbb{Z}) \neq 0$ as claimed. $\square$

4.2.4 Free actions on homology spheres

The following proposition generalizes a method explained in [Bro94, p. 20].

Proposition 107. *Let $G \curvearrowright K$ be a regular simplicial (cf. Section 3.1.2), free action of a finite group G on an n-dimensional simplicial complex K with $H_*(K) = H_*(S^n)$ such that the induced action on homology is trivial. Then we have*

$$H_i(G) = H_i(K/G), \text{ for } 0 < i < n$$

and G has periodic cohomology.

Proof. Let $C_* = C_*(K)$ be the chain complex of the simplicial complex K. We have an exact sequence of G-modules

$$0 \to \mathbb{Z} \to C_n \to \ldots \to C_1 \to C_0 \to \mathbb{Z} \to 0,$$

where each C_i is free, the G-action on $\mathbb{Z}$ is trivial and the map $\mathbb{Z} \to C_n$ sends $1 \in \mathbb{Z}$ to a generator of the cycle subgroup of C_n. Note that the map $\mathbb{Z} \to C_n$ is G-equivariant since the action of G on $H_n(K) \cong \mathbb{Z}$ is trivial by assumption. Hence, we can splice an infinite number of copies of this sequence to obtain a periodic complete resolution of G. Consequently, G has periodic cohomology (cf. Section 4.2.3). The claim on $H_i(G)$ follows because of $C_*(K/G) = \mathbb{Z} \otimes_{\mathbb{Z}G} C_*(K)$ (cf. [Bro94, Prop. 2.4, p. 20]). $\square$

We will apply this proposition in the following situation. Assume we have a finite group $G < \mathrm{O}_{n+1}$ and a normal subgroup $\tilde{G} \lhd G$ such that both $\mathbb{R}^{n+1}/G$ and $\mathbb{R}^{n+1}/\tilde{G}$ are homology manifolds and such that the action $G/\tilde{G} \curvearrowright S^n/\tilde{G}$ is free. An admissible triangulation of S^n for the action of G induces a triangulation of $S^n/\tilde{G}$ such that the action of $G/\tilde{G}$ is simplicial. Moreover, according to Lemma 101, S^n/G and $S^n/\tilde{G}$ have the homology groups of an n-sphere. In particular, we have $H_n(S^n/G) = \mathbb{Z}$ and thus the following lemma implies $G < \mathrm{SO}_{n+1}$.

Lemma 108. *Let $G < \mathrm{O}_{n+1}$ be a finite subgroup and assume that $H_n(S^n/G) = \mathbb{Z}$. Then G preserves the orientation, i.e. $G < \mathrm{SO}_{n+1}$.*

Proof. Since $H_n(S^n/G)$ is nontrivial, there is a nontrivial cycle c in $C_n(S^n/G)$. This cycle gives rise to a cycle $c' \in C_n(S^n)$ such that $\pi(c') = |G| \cdot c$ where $\pi : S^n \to S^n/G$ is the natural projection (e.g. $c' = \mu_G(c)$ in the notation of [Bre72, pp. 118-121]). Hence, $\pi_* : H_n(S^n) \to H_n(S^n/G)$ is nontrivial. The commutativity of the diagram

$$\begin{tikzcd}
H_n(S^n) \arrow[rr, "g_*=\deg(g)=\det(g)"] \arrow[dr, "\pi_*"'] & & H_n(S^n) \arrow[dl, "\pi_*"] \\
& H_n(S^n/G) &
\end{tikzcd}$$

for all $g \in G$ implies $\det(g) = 1$ for all $g \in G$ and thus $G < \mathrm{SO}_{n+1}$. $\square$

As in the lemma, it follows that the induced map on homology $p_* : H_n(S^n) \to H_n(S^n/\tilde{G})$ is nontrivial. Therefore, the commutativity of the following diagram

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "g_*"] \arrow[d, "p_*"'] & H_n(S^n) \arrow[d, "p_*"] \\
H_n(S^n/\tilde{G}) \arrow[r, "\bar{g}_*"] & H_n(S^n/\tilde{G})
\end{tikzcd}$$

for all $g \in G$ shows that $G/\tilde{G}$ acts trivially on $H_*(S^n/\tilde{G})$. Hence, all assumptions of Proposition 107 are fulfilled for the action of $G/\tilde{G}$ on $S^n/\tilde{G}$ and thus $G/\tilde{G}$ has periodic cohomology and

$$H_i(G/\tilde{G}) = H_i((S^n/\tilde{G})/(G/\tilde{G})) = H_i(S^n/G) = H_i(S^n) = 0$$

holds for $0 < i < n$.

4.2.5 Poincaré groups

Let $I < \mathrm{SO}_3$ be the orientation preserving symmetry group of a centered icosahedron in $\mathbb{R}^3$. The preimage of this subgroup under the canonical Lie group covering map $\varphi : \mathrm{SU}_2 \to \mathrm{SO}_3$ is called the *binary icosahedral group* and we denote it by $\mathbf{I}$. Since φ is two-to-one, we have $|\mathbf{I}| = 2|I| = 120$. Moreover, $\mathbf{I}$ is isomorphic to $\mathrm{SL}_2(5)$ [Wol84, p. 196] and thus perfect. The Lie group SU_2 can be identified with the unit quaternions $S^3 \subset \mathbb{H}$. Under this identification a binary icosahedral group in SU_2 is given by the union of the 24 Hurwitz units

$$\{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$$

together with all 96 unit quaternions obtained from $\frac{1}{2}(\pm i \pm \tau^{-1} j \pm \tau k)$ by an even permutation of coordinates, where τ is the golden ratio [DuV64]. From the natural representation of SU_2 on $\mathbb{C}^2$ we obtain a faithful and irreducible representation of the binary icosahedral group on $\mathbb{R}^4$ which is equivalent to the representation obtained by left multiplication on $\mathbb{H}$. Note that there are two dual representations of $\mathbf{I}$ on $\mathbb{C}^2$ obtained in this way and that these differ by an outer automorphism of $\mathbf{I}$ [Wol84, p. 202]. However, the resulting representations on $\mathbb{R}^4$ are equivalent. We refer to this realization of the binary icosahedral group in SO_4 as a *Poincaré group* and denote it by P. Consider the free action $P \curvearrowright S^3$. Since $\mathbf{I}$ is a perfect group, we have $H_1(S^3/P) = \pi_1(S^3/P)_{ab} = P_{ab} = 0$ [Hat02, Thm. 2A.1] and thus Poincaré duality and the universal coefficient theorem [Hat02, Thm. 3.2, Thm. 3.30] imply that

$$H_2(S^3/P) = H_1(S^3/P) = 0,$$

i.e. the quotient manifold S^3/P is a homology sphere. In fact, it is Poincaré's homology sphere (cf. [Rol76, Ch. 9.D]).

4.2.6 Finite subgroups of SO_4

We remind of the statement of Proposition 14.

Proposition 109. *For every finite subgroup $G < \mathrm{SO}_4$ there are finite subgroups $\mathbf{L}, \mathbf{R} < \mathrm{SU}_2$ with $-1 \in \mathbf{L}, \mathbf{R}$ and normal subgroups $\mathbf{L}_K \triangleleft \mathbf{L}$ and $\mathbf{R}_K \triangleleft \mathbf{R}$ such that $\mathbf{L}/\mathbf{L}_K$ and $\mathbf{R}/\mathbf{R}_K$ are isomorphic via an isomorphism $\phi : \mathbf{L}/\mathbf{L}_K \to \mathbf{R}/\mathbf{R}_K$ for which*

$$G = \varphi(\{(l, r) \in \mathbf{L} \times \mathbf{R} | \phi(\pi_L(l)) = \pi_R(r)\})$$

holds, where $\pi_L : \mathbf{L} \to \mathbf{L}/\mathbf{L}_K$ and $\pi_R : \mathbf{R} \to \mathbf{R}/\mathbf{R}_K$ are the natural projections. In this case we write $G = (\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)_\phi$. Conversely, a set of data $(\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)_\phi$ with the above properties defines a finite subgroup G of SO_4 by the equation above.

Given a finite subgroup $G < \mathrm{SO}_4$, for $\mathbf{L} = \pi_1(\varphi^{-1}(G))$, $\mathbf{R} = \pi_2(\varphi^{-1}(G))$, $\mathbf{L}_K = \{l \in \mathbf{L} | \varphi((l,1)) \in G\}$ and $\mathbf{R}_K = \{r \in \mathbf{R} | \varphi((1,r)) \in G\}$ the quotient groups $\mathbf{L}/\mathbf{L}_K$ and $\mathbf{R}/\mathbf{R}_K$ are isomorphic and with the isomorphism ϕ induced by the relation $\varphi^{-1}(G) < \mathbf{L} \times \mathbf{R}$ we have $G = (\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)_\phi$. The group

$$[\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K]_\phi := \{(l,r) \in \mathbf{L} \times \mathbf{R} | \phi([l]) = [r]\} < \mathbf{L} \times \mathbf{R}$$

is mapped two-to-one onto $\Gamma = (\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)_\phi$ via φ and we have

$$|\Gamma| = \frac{1}{2}|\mathbf{L}||\mathbf{R}_K| = \frac{1}{2}|\mathbf{R}||\mathbf{L}_K|.$$

For a normal subgroup $\tilde{\Gamma} = (\tilde{\mathbf{L}}/\tilde{\mathbf{L}}_K; \tilde{\mathbf{R}}/\tilde{\mathbf{R}}_K)_\phi$ of Γ we also have

$$\Gamma/\tilde{\Gamma} \cong [\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K]_\phi / [\tilde{\mathbf{L}}/\tilde{\mathbf{L}}_K; \tilde{\mathbf{R}}/\tilde{\mathbf{R}}_K]_\phi$$

and, in particular,

$$(\mathbf{L}/\mathbf{L}; \mathbf{R}/\mathbf{R})/(\tilde{\mathbf{L}}/\tilde{\mathbf{L}}; \tilde{\mathbf{R}}/\tilde{\mathbf{R}}) \cong \mathbf{L}/\tilde{\mathbf{L}} \times \mathbf{R}/\tilde{\mathbf{R}}.$$

4.2.7 Characterization of Poincaré groups

In the notation of Section 1.3.3 a Poincaré group is given by $P = (\mathbf{C}_2/\mathbf{C}_2; \mathbf{I}/\mathbf{I})$. In particular, a Poincaré group does not contain any rotation. The quotient group P/P_{rot} is thus isomorphic to $\mathrm{SL}_2(p)$ for some prime, namely for $p = 5$ (cf. Section 4.2.5). The next lemma shows that Poincaré groups are the only subgroups of SO_4 with this property.

Lemma 110. *Let $G < \mathrm{SO}_4$ be a finite subgroup and let $G_{rot} \triangleleft G$ be the normal subgroup generated by the rotations contained in G. If G/G_{rot} is isomorphic to $\mathrm{SL}_2(p)$ for some prime $p > 3$, then $p = 5$ and G is a Poincaré group $P < \mathrm{SO}_4$.*

Proof. Assume that $G = (\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)_\phi$ is a finite subgroup of SO_4 such that the quotient group G/G_{rot} is isomorphic to $\mathrm{SL}_2(p)$ for some prime $p > 3$. The group G_{rot} can be represented as $G_{rot} = (\tilde{\mathbf{L}}/\tilde{\mathbf{L}}_K; \tilde{\mathbf{R}}/\tilde{\mathbf{R}}_K)_{\tilde{\phi}}$ for normal subgroups $\tilde{\mathbf{L}} \triangleleft \mathbf{L}$ and $\tilde{\mathbf{R}} \triangleleft \mathbf{R}$ with $-1 \in \tilde{\mathbf{L}}, \tilde{\mathbf{R}}$. Since $\mathrm{SL}_2(p)$ is a perfect group for every prime $p > 3$ [Ros94, p. 61], for all $k > 0$ we have

$$G/G_{rot} \cong \Gamma/\tilde{\Gamma} = D^k(\Gamma/\tilde{\Gamma}) \cong D^k(\Gamma)/(D^k(\Gamma) \cap \tilde{\Gamma})$$

where $\Gamma = [\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K]_\phi$ and $\tilde{\Gamma} = [\tilde{\mathbf{L}}/\tilde{\mathbf{L}}_K; \tilde{\mathbf{R}}/\tilde{\mathbf{R}}_K]_{\tilde{\phi}}$ (for the notation see the preceding section) and where D^k denotes the kth interated commutator subgroup operator. Since $\mathfrak{C}_n$, $\mathfrak{D}_n$, $\mathfrak{S}_4$, $\mathfrak{A}_4$ and $\mathbf{C}_2$ are solvable, so are $\mathbf{C}_n$, $\mathbf{D}_n$, $\mathbf{T}$ and $\mathbf{O}$. Hence, we have, perhaps after interchanging the factors, $\mathbf{R} = \mathbf{I}$. For, otherwise $D^k(\Gamma)$ would be trivial for sufficiently large k. The only normal subgroups of $\mathbf{I}$ are the trivial subgroup $\mathbf{C}_1$, $\mathbf{I}$ itself and its center $\mathbf{C}_2$. In the case $\mathbf{R}_K \neq \mathbf{I}$ we would either have $G = (\mathbf{I}/\mathbf{C}_2; \mathbf{I}/\mathbf{C}_2)$ or $G = (\mathbf{I}/\mathbf{C}_1; \mathbf{I}/\mathbf{C}_1)$. A group $(\mathbf{I}/\mathbf{C}_2; \mathbf{I}/\mathbf{C}_2)$ has order 120 and is the direct product of a group $(\mathbf{I}/\mathbf{C}_1; \mathbf{I}/\mathbf{C}_1)$ with the group $\{\pm \mathrm{id}\}$. In particular, neither of them can be isomorphic to $\mathrm{SL}_2(5)$. Hence, because of $\mathrm{SL}_2(p) > 120$ for $p > 5$, this case cannot occur and thus we have $\mathbf{R}_K = \mathbf{I}$. The only remaining possibilities for G are $(\mathbf{C}_{2n}/\mathbf{C}_{2n}; \mathbf{I}/\mathbf{I})$, $(\mathbf{D}_n/\mathbf{D}_n; \mathbf{I}/\mathbf{I})$, $(\mathbf{T}/\mathbf{T}; \mathbf{I}/\mathbf{I})$, $(\mathbf{O}/\mathbf{O}; \mathbf{I}/\mathbf{I})$ and $(\mathbf{I}/\mathbf{I}; \mathbf{I}/\mathbf{I})$.

In the case $\tilde{\mathbf{R}} = \tilde{\mathbf{R}}_K = \mathbf{I}$ the quotient group G/G_{rot} would be either cyclic, dihedral or $|G/\tilde{G}| < 120$ would hold. Hence, this case cannot occur. In the cases $\tilde{\mathbf{R}} = \mathbf{I}$ and $\tilde{\mathbf{R}}_K = \mathbf{C}_i$, $i = 1, 2$, we would have $G = (\mathbf{I}/\mathbf{I}; \mathbf{I}/\mathbf{I})$ and $G_{rot} = (\mathbf{I}/\mathbf{C}_i; \mathbf{I}/\mathbf{C}_i)$ contradicting $G_{rot} \triangleleft G$, since $\mathbf{I}$ has conjugacy classes with more than two elements.

Consequently, we must have $\tilde{\mathbf{R}} = \mathbf{C}_2$. Transformations of the form $\varphi((l, \pm 1))$ are never rotations and thus we conclude that $G_{rot} = (\mathbf{C}_2/\mathbf{C}_1; \mathbf{C}_2/\mathbf{C}_1) = \{\mathrm{id}\}$. Since $(\mathbf{I}/\mathbf{I}; \mathbf{I}/\mathbf{I})$ contains rotations, the only remaining possible perfect group G listed above is $G = (\mathbf{C}_2/\mathbf{C}_2; \mathbf{I}/\mathbf{I})$, a Poincaré group isomorphic to $\mathrm{SL}_2(5)$. □ □

In the same way we obtain

Lemma 111. *A finite subgroup $G < \mathrm{SO}_4$ isomorphic to $\mathrm{SL}_2(5)$ is a Poincaré group.*

4.2.8 Characterization of the binary icosahedral group

In [Wol84] linear and free actions of finite groups on spheres are classified. The quotients of these actions are precisely the spherical space forms. In this context the following result due to Zassenhaus holds.

Theorem 112 ([Wol84], Thm. 6.2.1, p. 181)**.** *Let G be a nontrivial perfect finite group. If G has an irreducible complex fixed point free representation π, i.e. $\pi(g)$ has all eigenvalues different from 1 for all $g \in G$, then π has rank 2 and $G = \mathbf{I}$, i.e. π is equivalent to one of the two possible representations induced by embeddings $\mathbf{I} \subset \mathrm{SU}_2$ (cf. [Wol84, p. 202]).*

This theorem should be seen in the context of Theorem 105. Indeed, by assumption G is a perfect group with periodic cohomology [Wol84, Thm. 5.3.1, Thm. 5.3.2, p. 160] and thus has to be $\mathrm{SL}_2(p)$ for some prime $p > 3$ by that theorem. Among these groups, $\mathrm{SL}_2(5)$ is the only group that admits a free and linear action on a sphere [Bro94, p. 158].

4.2.9 The double suspension theorem

The following result bases on work by Edwards and Cannon and was first obtained in full generality by Cannon.

Theorem 113 ([Can79])**.** *The double suspension $\Sigma^2 X$ of a homology n-sphere X is a topological $(n+2)$-sphere.*

The double suspension of X is homeomorphic to the join of X with S^1, i.e. $\Sigma^2 X \cong X * S^1$. Hence, an identification $\Sigma^2 X \cong S^5$ induces an embedding $i : S^1 \hookrightarrow S^5$. Since $S^5 \backslash i(S^1)$ and X are homotopy equivalent, the space obtained from S^5 after removing $i(S^1)$ is not simply connected anymore. Hence, by transversality the embedding i can neither be smooth nor piecewise linear, but has to be “wild” in a sense [Can78].

Now let $P \curvearrowright \mathbb{R}^4$ be the action of a Poincaré group $P < \mathrm{SO}_4$ described in Section 4.2.5. The quotient of this action is not a topological manifold: Since S^3/P is not simply connected, the origin of $\mathbb{R}^4/P$ does not have a pointed simply connected neighborhood. However, the quotient of the action $P \curvearrowright \mathbb{R}^5$ is homeomorphic to $\mathbb{R}^5$ where we have extended the action $P \curvearrowright \mathbb{R}^4$ trivially to the fifth factor. Indeed, we have

$$\begin{aligned}\mathbb{R}^5/P &\cong \mathbb{R}^4/P \times \mathbb{R} \cong (CS^3)/P \times \mathbb{R} \cong C(S^3/P) \times CS^0 \cong C(S^3/P * S^0) \\ &\cong C\Sigma(S^3/P) \cong \Sigma^2(S^3/P)\backslash\{*\} \cong S^5\backslash\{*\} \cong \mathbb{R}^5\end{aligned}$$

by the double suspension theorem and the property $C(X_1 * X_2) \cong CX_1 \times CX_2$ of the spherical join [BH99, Prop. I.5.15, p. 64]. For a deeper reason for this equation and the double suspension theorem in general we refer to [Dav86, Can79] and to [Can78] for a readable overview. This example illustrates a difficulty one encounters trying to prove the *only if* direction of Theorem D. The open cone CX of the topological space $X = \Sigma(S^3/P)$ is homeomorphic to $\mathbb{R}^5$ whereas X itself is not even a topological manifold.

4.2.10 Simplicial Lipschitz manifolds

We say that a metric space X is a *Lipschitz n-manifold* if each point has a neighborhood that is bi-Lipschitz homoemorphic to some neighborhood in $\mathbb{R}^n$. For a finite subgroup $G < \mathrm{O}_n$ the quotient $\mathbb{R}^n/G$ inherits a metric from $\mathbb{R}^n$, the so-called *quotient metric*, where the distance between two points in $\mathbb{R}^n/G$ is defined to be the distance of the corresponding orbits in $\mathbb{R}^n$. With respect to this metric $\mathbb{R}^n/G$ is a *length space*, i.e. the distance between two points is the infimum of the lengths of all rectifiable paths connecting these points (cf. [BBI01]). We would like to know when it is a Lipschitz manifold. Since $\mathbb{R}^n/G$ can be triangulated by a simplicial complex K/G, where K is a simplicial complex triangulating $\mathbb{R}^n$ on which G acts simplicially and regularly (cf. Lemma 75), the metric on $\mathbb{R}^n/G$ can be recovered from the flat metrics on the simplices of K/G as an induced length metric. Therefore, $\mathbb{R}^n/G$ is a Lipschitz manifold with boundary if it is a PL manifold with boundary. In particular, $\mathbb{R}^n/G$ is a Lipschitz manifold with boundary if G is a reflection-rotation group by Theorem B.

Siebenmann and Sullivan established the following necessary and sufficient condition for a simplicial complex to be a Lipschitz manifold.

Theorem 114 ([SS77], Thm. 1, p. 504; Thm. 2, p. 506 in combination with Remark (i), p. 507)**.** *A locally finite simplicial complex K with a length metric induced by flat metrics on its simplices is a Lipschitz manifold, if and only if the link of every simplex of K is a homotopy sphere and a Lipschitz manifold with respect to its induced length metric.*

According to this result the same argument as in the PL category shows that G is a rotation group, if $\mathbb{R}^n/G$ is a Lipschitz manifold (cf. Section 3.2). First, the fact that the link of the origin in the triangulation K of $\mathbb{R}^n$ with its induced length metric is again a Lipschitz manifold implies by induction that all proper isotropy groups in G are rotation groups (cf. proof of lemma 76). Then the simply connectedness of this link implies that G is generated by its isotropy groups and thus is a rotation group itself. Necessary conditions for the case that $\mathbb{R}^n/G$ is a Lipschitz manifold with nonempty boundary will be deduced in the next section.

4.3 Necessary conditions for manifolds without boundary

In this section we establish necessary conditions on a finite group $G < \mathrm{O}_n$ in order for $\mathbb{R}^n/G$ to be a topological and thus a homology manifold. We will first treat the case in which $\mathbb{R}^n/G$ does not have a boundary. The case for manifolds with boundary is treated in Section 4.4. In the following we introduce some concepts in order to organize the further argument.

4.3.1 Subgroup and subspace systems

To a finite subgroup $G < \mathrm{O}_n$ we associate a system of subgroups $\mathfrak{F}$ of G and a system of linear subspaces $\mathfrak{L}$ of $\mathbb{R}^n$ as follows.

Definition 13. Let $\mathfrak{F}$ be the system of subgroups of G and let $\mathfrak{L}$ be the system of linear subspaces of $\mathbb{R}^n$ defined by

(i) $\tilde{G} \in \mathfrak{F}$ for $\tilde{G} < G$, if and only if there exists a linear subspace L of $\mathbb{R}^n$ such that $\tilde{G} = F(L) := \{g \in G | \forall x \in L : gx = x\}$.

(ii) $L \in \mathfrak{L}$ for a linear subspace L of $\mathbb{R}^n$, if and only if there exists a subgroup $\tilde{G} < G$ such that $L = \mathrm{Fix}(\tilde{G}) = \{x \in \mathbb{R}^n | \forall g \in G : gx = x\}$.

Inclusion induces partial orders on $\mathfrak{L}$ and $\mathfrak{F}$. The group G acts by translation on $\mathfrak{L}$ and by conjugation on $\mathfrak{F}$. The correspondence

$$\begin{array}{cccc} & \mathfrak{F} & \overset{1:1}{\longleftrightarrow} & \mathfrak{L} \\ L: & \tilde{G} & \longmapsto & \mathrm{Fix}(\tilde{G}) \\ F: & F(L) & \longleftarrow & L, \end{array}$$

is order-reversing, one-to-one and G-equivariant. Note that $F(L)$ is the maximal subgroup of G that fixes L pointwise. The equivalence classes under the action of G are in one-to-one correspondence to the strata of $\mathbb{R}^n/G$ and the type of a stratum is given by the corresponding subgroup in $\mathfrak{F}$. We say that a subgroup $\tilde{G} \in \mathfrak{F}$ is *minimal* if it is a nontrivial minimal subgroup in $\mathfrak{F}$ with respect to inclusion. Corresponding subspaces are called *maximal* subspaces in $\mathfrak{L}$. Minimal subgroups and maximal subspaces correspond to strata of $\mathbb{R}^n/G$ that are not contained in the closure of any higher dimensional singular stratum. Moreover, we introduce the following notations.

Definition 14. Define $\mathfrak{L}_{\max} = \{L \in \mathfrak{L} | L \text{ maximal}\}$, $\mathfrak{F}_{\min} = \{\tilde{G} \in \mathfrak{F} | \tilde{G} \text{ minimal}\}$ and $G_{\min} = \langle \mathfrak{F}_{\min} \rangle$.

Since $\mathfrak{F}_{\min}$ is closed under the action of G, the subgroup $G_{\min}$ is normal in G. We need the following two lemmas.

Lemma 115. *For a finite subgroup $G < \mathrm{O}_n$ and a point $x \in S^{n-1}$ we have $(G_x)_{\min} \subseteq (G_{\min})_x$.*

Proof. Maximal subspaces of the action $G_x \curvearrowright \langle x \rangle^{\perp}$ are in one-to-one correspondence to maximal subspaces of the action $G \curvearrowright \mathbb{R}^n$ that contain x and thus the same holds for the corresponding minimal subgroups. Therefore, we have

$$\begin{aligned}(G_x)_{\min} &= \Big\langle \mathfrak{F}_{\min}(G_x \curvearrowright \langle x\rangle^{\perp})\Big\rangle \\ &= \Big\langle \{\tilde{G} \in \mathfrak{F}_{\min}(G \curvearrowright \mathbb{R}^n) | \tilde{G} \subset G_x\}\Big\rangle \\ &\subseteq (G_{\min})_x.\end{aligned}$$

□

The next lemma is just a reformulation of definitions.

Lemma 116. *Let $G < \mathrm{O}_n$ be a finite subgroup and let $\tilde{G} \in \mathfrak{F}$ be nontrivial. Then the action $\tilde{G} \curvearrowright S^{d-1} \subset \mathbb{R}^d = L(\tilde{G})^{\perp}$ is free if and only if $\tilde{G}$ is minimal.*

4.3.2 Homology manifold criterion

In this section we deduce necessary conditions on minimal subgroups in $\mathfrak{F}$ in order for $\mathbb{R}^n/G$ to be a homology manifold. Let $G < \mathrm{O}_n$ be a finite subgroup and assume that $\mathbb{R}^n/G$ is a homology manifold. Then, according to Lemma 101, S^{n-1}/G has the homology groups of a sphere and is again a homology manifold. The first condition is global and implies $G < \mathrm{SO}_n$ by Lemma 108. The second condition is local and implies that each point in $x = \pi(p) \in S^{n-1}/G$ has a neighborhood which is a homology manifold. A suitable neighborhood is homeomorphic to $T_pS^{n-1}/G_p \cong \mathbb{R}^{n-1}/G_p$ via the exponential map. Hence, we conclude that T_pS^{n-1}/G_p is a homology manifold for all $p \in S^{n-1}$. Proceeding iteratively we find that for each $\tilde{G} \in \mathfrak{F}$ the quotient space of the action $\tilde{G} \curvearrowright \mathbb{R}^d = F(\tilde{G})^{\perp}$ is a homology manifold. Moreover, if $\tilde{G}$ is a minimal subgroup of G, then $S^{d-1}/\tilde{G}$ is a homology sphere, since in this case the action $\tilde{G} \curvearrowright S^{d-1}$ is free by Lemma 116. The next proposition shows that only very special minimal subgroups can occur.

Proposition 117. *Assume that the quotient $\mathbb{R}^n/G$ is a homology manifold for a finite subgroup $G < \mathrm{SO}_n$. Then, for a minimal subgroup $\tilde{G} \in \mathfrak{F}_{\min}$ with $d = \mathrm{codim} L(\tilde{G})$, one of the following two cases holds.*

(i) $d = 2$ and $\tilde{G} = C_k < \mathrm{SO}_2$, a cyclic group for some $k \geq 2$.

(ii) $d = 4$ and $\tilde{G} = P < \mathrm{SO}_4$, a Poincaré group.

Proof. Let $\tilde{G} \in \mathfrak{F}_{\min}$ be a minimal subgroup and set $d = \mathrm{codim} L(\tilde{G})$. We have seen that the action $\tilde{G} \curvearrowright S^{d-1}$ is free and that $S^{d-1}/\tilde{G}$ is a homology sphere. First, assume that $d = 2$. Then $\tilde{G}$ is a cyclic group $C_k < \mathrm{SO}_2$ for some $k \geq 2$. Now suppose that $d \geq 3$. Because of $\tilde{G}/[\tilde{G},\tilde{G}] = \pi_1(S^{d-1}/\tilde{G})_{\mathrm{ab}} = H_1(S^{d-1}/\tilde{G}) = 0$, $\tilde{G}$ is a perfect group in this case. Since the action $\tilde{G} \curvearrowright S^{d-1}$ is free, the complexification of the action $\tilde{G} \curvearrowright \mathbb{R}^d$ is fixed point free and thus, so are its irreducible components. Hence, according to Theorem 112, we have $\tilde{G} \cong \mathbf{I}$ and all irreducible components are two-dimensional. Now, $H_i(\mathbf{I}) = H_i(S^{d-1}/\tilde{G}) = 0$ for $i = 1, \dots, d-2$ by Proposition 107 and $H_3(S^{d-1}/\tilde{G}) = H_3(\mathbf{I}) \neq 0$ by Proposition 107 and Proposition 106 imply $d \in \{2, 4\}$. Since $\mathbf{I}$ admits no faithful representation on $\mathbb{R}^2$, we conclude that $d = 4$ and that $\tilde{G} = P < \mathrm{SO}_4$ is a Poincaré group by Lemma 111. □

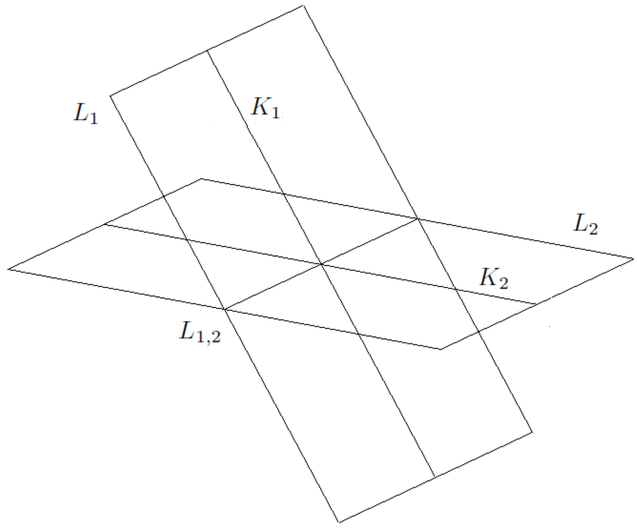

Figure 4.1: Intersection of two fixed point subspaces.

Since minimal subgroups always exist, we have identified $G_{\min}$ as a nontrivial normal subgroup of G generated by all rotations contained in G and certain Poincaré groups $P_1, \ldots, P_k < G$, i.e. we have $G_{\min} = \langle G_{rot}, P_1, \ldots, P_k \rangle$ where G_{rot} is the maximal rotation group contained in G.

4.3.3 Orthogonal splitting

In this section we show that the subgroups $G_{rot}, P_1, \ldots, P_k < G_{\min}$ act in orthogonal spaces. Set $\mathfrak{L}_P = \{L \in \mathfrak{L} | \mathrm{codim}(L) = 4\} = \{F(P_i) | i = 1, \ldots, k\}$. We claim that $\mathbb{R}^n$ splits as an orthogonal sum of subspaces

$$\mathbb{R}^n = V_0 \oplus V_{rot} \oplus \bigoplus_{L \in \mathfrak{L}_P} L^\perp$$

where V_{rot} is the span of the orthogonal complements of all maximal subspaces of codimension two and where $V_0 = \mathrm{Fix}(G_{\min})$. This would imply a splitting of $G_{\min}$ into direct factors

$$G_{\min} = G_{rot} \times P_1 \times \ldots \times P_k.$$

We begin by showing that the Poincaré groups act in pairwise orthogonal spaces. The claim that the rotation group G_{rot} acts in a space orthogonal to all of them can then be reduced to this case (cf. Lemma 121). The idea of the proof is to reach a contradiction to the finiteness of $\mathfrak{L}_P$ on the assumption that there are two Poincaré groups that do not act in orthogonal spaces. To this end we define a "distance" function $D : \mathfrak{L}_P \times \mathfrak{L}_P \to \mathbb{R}$ as follows. For $L_1, L_2 \in \mathfrak{L}_P$ we set $K_i = (L_1 \cap L_2)^{\perp_{L_i}}$, $i = 1, 2$ (here $\perp_{L_i}$ denotes the orthogonal complement in L_i). Then L_1 and L_2 decompose into orthogonal sums

$$L_1 = L_1 \cap L_2 \oplus K_1 \text{ and } L_2 = L_1 \cap L_2 \oplus K_2.$$

We define the distance between L_1 and L_2, composed of a "rough" distance and a "fine" distance, by

$$D(L_1, L_2) = \mathrm{dim} K_1 + d(K_1, K_2)$$

where

$$d(K_1, K_2) = \begin{cases} 0, & \text{for } \dim K_1 \cdot \dim K_2 = 0 \\ \frac{2}{\pi} \min_{v_1 \in K_1 \setminus \{0\}, v_2 \in K_2 \setminus \{0\}} \angle(v_1, v_2), & \text{else} \end{cases}$$

and $\angle(v_1, v_2)$ denotes the angle between the vectors v_1 and v_2. Note that $\dim K_1 = \dim K_2 \in \{0, 1, 2, 3, 4\}$ and $d(K_1, K_2) \in (0, 1]$. We record some properties of D in the following lemma.

Lemma 118. *The distance D defined on $\mathfrak{L}_P$ satisfies the following properties for subspaces $L_1, L_2 \in \mathfrak{L}_P$.*
(i) $0 \leq D(L_1, L_2) \leq 5$.
(ii) $D(L_1, L_2) = 0 \Leftrightarrow L_1 = L_2$.
(iii) $D(L_1, L_2) = 5 \Leftrightarrow K_1 \perp K_2$ *and* $\mathrm{dim} K_1 = \mathrm{dim} K_2 = 4 \Leftrightarrow L_1^{\perp} \perp L_2^{\perp}$.

To pursue our strategy we need the following elementary geometric lemma.

Lemma 119. *Let V be a Euclidean vector space with a proper subspace W and let $U = W^{\perp}$ be the orthogonal complement of W in V. Let $v = w + u$ be the orthogonal decomposition with respect to W and U of a normalized vector $v \in V$ with $u \neq 0$. Let $\phi \in \mathrm{O}(V)$ be such that $\phi_{|W} = \mathrm{id}_W$. Then $\alpha := \angle(u, \phi(u)) \leq 60°$ implies*

$$\gamma := \angle(v, \phi(v)) < \angle(v, W) =: \beta.$$

Proof. For $w = 0$ or $\phi = \mathrm{id}$ the claim is trivially true. Otherwise, the angles α, β and γ appear in a spherical triangle with vertices $\hat{w} = \frac{w}{\|w\|}$, v and $\phi(v)$ as depicted in Figure 4.2. Then the claim follows by comparing this spherical triangle with an isosceles triangle in $\mathbb{R}^2$ with opening angle $\alpha \leq 60°$ and leg length β. $\square$

In order to apply this lemma, we need to understand the orbit geometry of the action $P \curvearrowright S^3$. First note that all orbits are isometric, since the canonical metric on S^3 is S^3-bi-invariant. So let $x \in S^3$ be an arbitrary point and let $X = Px$ be its orbit under the action of P. Since P is finite, the sets $X_\alpha = \{y \in X | \angle(x, y) = \alpha\} \subset X$ are empty except for a finite number of values of α. The points of X_α lie in the intersection of S^3 with a hyperplane of $\mathbb{R}^4$. This intersection is a point for $\alpha \in \{0, \pi\}$ and a two-dimensional sphere for $\alpha = (0, \pi)$. From the explicit coordinate representation of P one can read off the structure of

α	$\lvert X_\alpha \rvert$	X_α
$0°, 180°$	1	point
$36°, 144°$	12	icosahedron
$60°, 120°$	20	dodecahedron
$72°, 108°$	12	icosahedron
$90°$	30	icosidodecahedron

Table 4.1: Structure of the 600-cell.

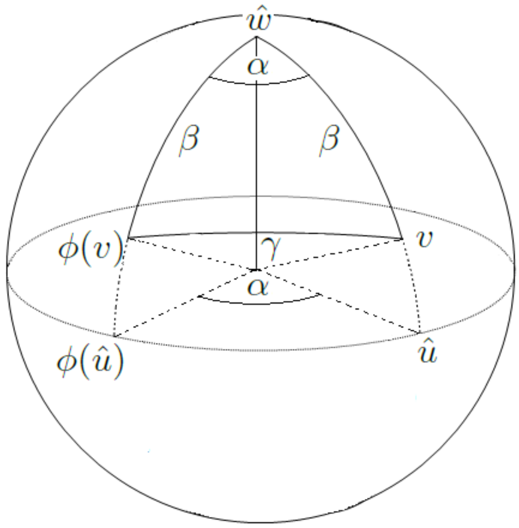

Figure 4.2: In an isosceles spherical triangle with opening angle $\alpha \leq 60°$, leg length β and base length γ we have $\gamma < \beta$. In the figure we have set $\hat{u} = \frac{u}{\|u\|}$ and $\hat{w} = \frac{w}{\|w\|}$.

$X = \bigcup_{\alpha \in [0,\pi]} X_\alpha$. It is summarized in Table 4.1 (cf. [BP98, p. 98]). The last column specifies the geometric structure of the points in X_α. For our proof the fact that $\bigcup_{\alpha \in (0,60°]} X_\alpha$ is not contained in a three-dimensional affine subspace of $\mathbb{R}^4$ is crucial as we will see in the following lemma.

Lemma 120. *For distinct $L_1, L_2 \in \mathfrak{L}_P$ we have $D(L_1, L_2) = 5$, i.e. $L_1^\perp \perp L_2^\perp$ holds by Lemma 118 and thus the corresponding Poincaré groups act in orthogonal spaces.*

Proof. Suppose we have distinct $L_1, L_2 \in \mathfrak{L}_P$ with $D(L_1, L_2) < 5$. Since L_1 and L_2 are distinct, we then also have $0 < D(L_1, L_2)$. We construct a contradiction to the finiteness of $\mathfrak{L}_P$ by showing that for every such pair there is another pair $L_1', L_2' \in \mathfrak{L}_P$ such that

$$0 < D(L_1', L_2') < D(L_1, L_2) < 5.$$

Let $v_1 \in K_1 = (L_1 \cap L_2)^{\perp_{L_1}}$ and $v_2 \in K_2 = (L_1 \cap L_2)^{\perp_{L_2}}$ with $\|v_1\| = \|v_2\| = 1$ such that $\angle(v_1, v_2)$ is minimal. We decompose v_2 into orthogonal components with respect to L_1 and $L_1^\perp$, i.e. $v_2 = w + u$ where $w := \mathrm{Pr}_{L_1} v_2 \in L_1$ and $u := \mathrm{Pr}_{L_1^\perp} v_2 \in L_1^\perp$. Since L_1 and L_2 are distinct, u is nontrivial and we can define $S := \{g \in P_1 | 0 < \angle(gu, u) \leq 60°\} \subset P_1$. Because of

$$gv_2 = g(w + u) = w + gu$$

and the geometric structure of the orbit $P_1 u$ described above, the smallest affine subspace of V that contains Sv_2 is four-dimensional. Moreover, it is not a linear subspace if $\dim K_1 = \dim K_2 = 4$, since v_1 and v_2 are not orthogonal and thus w is not trivial in this case. In particular, there is a $g \in S$ such that $gv_2 \notin K_2$, since K_2 is a linear subspace of dimension less or equal to four. Since the orthogonal transformation g fixes $L_1 \cap L_2$ pointwise, we have

$gv_2 \perp L_1 \cap L_2$. Now, $gv_2 \notin K_2$ and $gv_2 \perp L_1 \cap L_2$ imply $gv_2 \notin L_2$. Therefore, $L_2' := gL_2 \in \mathfrak{L}_P$ is distinct from L_2. We claim that $D(L_2, L_2') < D(L_1, L_2)$ holds. Since g fixes $L_1 \cap L_2$, we have $L_1 \cap L_2 \subseteq L_2 \cap L_2'$. In the case that $L_2 \cap L_2'$ is strictly larger than $L_1 \cap L_2$ the rough distance decreases and so we have $D(L_2, L_2') < D(L_1, L_2)$. In the case $L_1 \cap L_2 = L_2 \cap L_2'$ we have $\dim K_1 = \dim K_2 = \dim K_3$ where $K_2' := (L_2 \cap L_2')^{\perp_{L_2'}} = gK_2$. In this case it remains to show that $d(K_2, K_2') < d(K_1, K_2)$ in order to prove the claim. Indeed, Lemma 119 applied to $W = L_1$, $U = L_1^{\perp}$, $v = v_2 = w + u$ and $\phi = g$ yields

$$d(K_2, K_2') \leq \frac{2}{\pi} \angle(v_2, gv_2) < \frac{2}{\pi} \angle(v_1, v_2) = d(K_1, K_2).$$

Consequently, in any case we obtain

$$0 < D(L_2, L_2') < D(L_1, L_2) < 5,$$

and thus the desired contradiction. □

We have shown that all Poincaré groups act in orthogonal spaces. We claim that G_{rot} acts in a space orthogonal to all of them. To this end we choose two maximal subspaces, say $L_2, L_4 \in \mathfrak{L}_{\max}$ with codim $L_2 = 2$ and codim $L_4 = 4$, and denote the corresponding minimal subgroups of G by C and P. The claim follows from the subsequent lemma.

Lemma 121. *The subgroups C and P of G act in orthogonal spaces, i.e. $L_2^{\perp} \perp L_4^{\perp}$.*

Proof. For any nontrivial $r \in C$ the intersection $L_4^{\perp} \cap r L_4^{\perp}$ is nontrivial because of $\dim(L_4^{\perp}) = 4$ and $\mathrm{codim}(\mathrm{Fix}(r)) = 2$. Hence, according to Lemma 120 we can assume that C leaves $L_4^{\perp}$ invariant. Uniqueness of isotypic components with respect to the action of C yields orthogonal decompositions

$$L_4^{\perp} = L_2^{\perp} \cap L_4^{\perp} \oplus L_2 \cap L_4^{\perp} \text{ and } L_4 = L_2^{\perp} \cap L_4 \oplus L_2 \cap L_4$$

and thus we have an orthogonal decomposition

$$L_2^{\perp} = L_2^{\perp} \cap L_4^{\perp} \oplus L_2^{\perp} \cap L_4.$$

By the maximality assumption on L_4 the subspace $L_2^{\perp} \cap L_4$ is nontrivial. It remains to exclude the case that $L_2^{\perp} \cap L_4^{\perp}$ is one-dimensional. In this case C would be reducible and of order 2 and the only rotation $r \in C$ would be a product of a reflection s_1 that inverts $L_2^{\perp} \cap L_4^{\perp}$ and another reflection s_2 that fixes $L_4^{\perp}$ pointwise. Since the action of P on $L_4^{\perp}$ is irreducible there would exist some $g \in P$ with $gs_1g^{-1} \neq s_1$. Then $r' = rgrg^{-1} = s_1gs_1g^{-1}$ would be a rotation in G with $L_4 \subset \mathrm{Fix}(r')$ contradicting the maximality of L_4. Hence, we have $L_2^{\perp} \perp L_4^{\perp}$ and the lemma is proven. □

The preceding lemma completes the proof on the claimed orthogonal splitting.

4.3.4 Free actions on homology spheres

We are still assuming that $G < \mathrm{O}_n$ is a finite subgroup for which $\mathbb{R}^n/G$ is a homology manifold. In the preceding sections we have seen that the normal subgroup $G_{\min}$ of G is generated by the largest rotation group contained in G and certain Poincaré groups and that it splits as a direct product of these. In this section we show that $G = G_{\min}$.

Lemma 122. *Let $G \curvearrowright X$ be a group action and let $\tilde{G} \triangleleft G$ be a normal subgroup. If $G_x = \tilde{G}_x$ for all $x \in X$, then the action*

$$G/\tilde{G} \curvearrowright X/\tilde{G}$$

is free.

Proof. Assume there are $g \in G$, $\tilde{g} \in \tilde{G}$ and $x \in X$ such that $gx = \tilde{g}x$. Then $(\tilde{g})^{-1}g \in G_x = \tilde{G}_x \subset \tilde{G}$ and so $g \in \tilde{G}$. □

Lemma 123. *For a product $G = G_{rot} \times P_1 \times \ldots \times P_k < \mathrm{SO}_n$ of a rotation group G_{rot} and Poincaré groups $P_i < \mathrm{SO}_4$, $i = 1, \ldots, k$ that act in orthogonal spaces the quotient space $\mathbb{R}^n/G$ is a homology manifold and for $n > 2$ the quotient space S^{n-1}/G is simply connected unless $n = 4$ and $k = 1$.*

Proof. The quotient space $\mathbb{R}^n/G$ is a homology manifold by Theorem B and the fact that S^3/P is a homology sphere in combination with Lemma 101. For $k = 0$ the space S^{n-1}/G is simply connected by Theorem B (or Lemma A.1). In the case $k > 1$ and in the case $n > 4$ and $k = 1$ the space S^{n-1}/G can be written as the join of two topological spaces one of which is path-connected [BH99, Prop. I.5.15, p. 64] and is thus simply connected. □

Now we are ready for the last step of our argument. Recall the remark from the end of Section 4.2.9: There are topological spaces X such that CX is homeomorphic to $\mathbb{R}^n$ whereas X is not even a topological manifold. Hence, the induction argument in the following lemma would not work, if the assumption "homology" were replaced by "topological". In particular, we cannot assume that the quotient space S^{n-1}/G is simply connected, an assumption that would simplify the proof considerably.

Lemma 124. *Let $G < \mathrm{SO}_n$ be a finite subgroup such that $\mathbb{R}^n/G$ is a homology manifold. Then we have $G = G_{\min}$.*

Proof. The proof is by induction on n. For $n = 2$ and $n = 3$ all finite subgroups of SO_n are rotation groups and so nothing has to be shown. Now let $n \geq 4$ be fixed and assume that the claim holds for $n - 1$. We are going to show that it also holds for n. For $x \in S^{n-1}$ the quotient space $\langle x \rangle^{\perp}/G_x$ is a homology manifold and so we have $G_x = (G_x)_{\min}$ by induction. Lemma 115 implies

$$G_x = (G_x)_{\min} \subseteq (G_{\min})_x \subseteq G_x$$

and thus $G_x = (G_{\min})_x$ for all $x \in S^{n-1}$. Therefore, the action $G/G_{\min} \curvearrowright S^{n-1}/G_{\min}$ is free by Lemma 122. Since $\mathbb{R}^n/G$ is a homology manifold by assumption and $\mathbb{R}^n/G_{\min}$ is a homology manifold by Lemma 123, it follows from Section 4.2.4 that $G/G_{\min}$ has periodic cohomology and that

$$H_i(G/G_{\min}) = H_i(S^{n-1}/G) = 0, \text{ for } 0 < i < n-1. \tag{4.1}$$

In particular, $G/G_{\min}$ is a perfect group, because of $H_1(G/G_{\min}) = 0$. Now Proposition 105 implies that $G/G_{\min}$ is either trivial or isomorphic to $\mathrm{SL}_2(p)$ for some prime $p > 3$. For $n > 4$ the second case cannot occur due to Proposition 106 stating that $H_3(\mathrm{SL}_2(p)) \neq 0$. For $n = 4$ and $G_{\min} = P$ the action of G is free and thus we have $G = G_{\min}$. For $n = 4$ and $G_{\min} = G_{rot}$ Lemma 110 implies $G = P$ and $G_{\min} = G_{rot} = \{1\}$, a contradiction. Consequently, in any case we have $G = G_{\min}$ and so the claim follows by induction. □

Summarizing, we have shown (cf. Lemma 101)

Theorem 125 (Theorem C for manifolds without boundary)**.** *For a finite subgroup $G < \mathrm{O}_n$ the quotient $\mathbb{R}^n/G$ is a homology manifold, if and only if G has the form*

$$G = G_{rot} \times P_1 \times \ldots \times P_k$$

for a rotation group G_{rot} and Poincaré groups $P_i < \mathrm{SO}_4$, $i = 1, \ldots, k$, such that the factors act in pairwise orthogonal spaces.

4.4 Necessary conditions for manifolds with boundary

4.4.1 Metric constructions

For a metric space (X, d) with $\mathrm{diam}(X) \leq \pi$ a metric d_c on the open cone CX can be defined as follows (cf. [BBI01, Def. 3.6.12., Prop. 3.6.13]). For $q, p \in CX$ with $p = (x, t)$ and $q = (y, s)$ set

$$d_c(p, q) = \sqrt{t^2 + s^2 - 2ts\cos(d(x, y))}.$$

Note that if X is the unit sphere in $\mathbb{R}^n$ with its induced length metric, then (CX, d_c) is naturally isometric to $\mathbb{R}^n$. As a direct consequence we obtain

Lemma 126. *Let $G < \mathrm{O}_n$ be a finite subgroup. Then the natural map between $\mathbb{R}^n/G$ and $C(S^{n-1}/G)$ is an isometry.*

Moreover, if (X, d) is a length space, then so is (CX, d_c) [BBI01, Thm. 3.6.17, p. 93]. This implies

Lemma 127. *Let (X, d) be a length space, $Y \subset X$ a subspace and suppose that* $\mathrm{diam}(2_Y X) \leq \pi$*. Then the natural map between $C(2_Y X)$ and $2_{CY} CX$ is an isometry.*

Proof. Since the reflection of $2_Y X$ at Y is an isometry, for any path of length l connecting two points x and y lying in a common half of $2_Y X$ there exists another path of length l connecting x and y that lies completely in the half of x and y. Hence, the restriction of the metric of $2_Y X$ to each of its two halves is a length metric and thus coincides with d. Now by construction $C(2_Y X)$ decomposes into two halves isometric to CX (with respect to the restricted metrics) that are glued together along CY. The claim follows, since $C(2_Y X)$ and $2_{CY} CX$ are length spaces □

4.4.2 Orbifold cover

In order to prove the boundary versions of Theorem C, D and E we first prove a lemma based on the following concept (cf. Lemma 134).

Definition 15. A *covering orbifold* of a Riemannian orbifold $\mathcal{O}$ is a Riemannian orbifold $\mathcal{O}'$ together with a surjective map $\varphi : \mathcal{O}' \to \mathcal{O}$ such that each point $x \in \mathcal{O}$ has a neighborhood U isometric to some M/G (cf. Definition 10) for which each connected component U_i of $\varphi^{-1}(U)$ is isometric to M/G_i for some subgroup $G_i < G$. The isometries must respect the projections φ and $M/G_i \to M/G$.

Now we can show

Lemma 128. *Let $G < \mathrm{O}_n$ be a finite subgroup with orientation preserving subgroup G^+ and assume that $\mathbb{R}^n/G$ is a homology manifold with nonempty boundary. Then G contains a reflection and there exists an isometry φ from the double $2(\mathbb{R}^n/G)$ with its induced length metric to $\mathbb{R}^n/G^+$ such that $p_0 = p_1 \circ \varphi$ where p_0 and p_1 are the natural projections from $2(\mathbb{R}^n/G)$ and $\mathbb{R}^n/G^+$ to $\mathbb{R}^n/G$.*

Proof. The proof is by induction on n. For $n = 1, 2$ the claim is clear. Assume it holds for some fixed $n > 1$ and let $G < \mathrm{O}_{n+1}$ be a finite subgroup such that $\mathbb{R}^{n+1}/G$ is a homology manifold with nonempty boundary. Then S^n/G is also a homology manifold with nonempty boundary by Lemma 101. For a point $x \in S^n$ whose coset lies in the boundary of S^n/G the quotient space T_xS^n/G_x is a homology manifold with nonempty boundary. Therefore, it follows by induction that $G_x \subset G$ contains a reflection and that there exists an isometry $\tilde{\theta} : 2(T_xS^n/G_x) \to T_xS^n/G_x^+$ with the property stated in the lemma. Using the exponential map, we obtain an equivariant bijection $\theta : B_r(x_0) \to B_r(x_1)$ between small balls $B_r(x_0)$ and $B_r(x_1)$ about the cosets x_0 and x_1 of x in $2(S^n/G_x)$ and S^n/G_x^+, respectively. By construction the map θ descends to an isometry between the quotients of $B_r(x_0)$ and $B_r(x_1)$ by the respective reflection. Since the metrics on $2(S^n/G_x)$ and S^n/G_x^+ are length metrics, this implies that the restriction $\theta : B_{r/4}(x_0) \to B_{r/4}(x_1)$ is an isometry (cf. proof of Lemma 127). In particular, we see that $2(S^n/G)$ is a Riemannian orbifold and that the natural projection $p_0 : 2(S^n/G) \to S^n/G$ is a covering of Riemannian orbifolds. By the assumption $n > 1$ the sphere S^n is simply connected. Therefore there exists an index 2 subgroup $\tilde{G} < G$ and an isometry $\varphi : 2(S^n/G) \to S^n/\tilde{G}$ with $p_0 = p_1 \circ \varphi$ where $p_1 : S^n/\tilde{G} \to S^n/G$ is the natural projection (cf. [Thu79, Ch. 13, p. 305] and Appendix, Lemma 134 for more details). Moreover, since $2(S^n/G) = S^n/\tilde{G}$ has the integral homology of a sphere (cf. Definition 12 and Lemma 101), the subgroup $\tilde{G}$ preserves the orientation by Lemma 108. The fact that both G^+ and $\tilde{G}$ are orientation preserving subgroups of index 2 in G implies $G^+ = \tilde{G}$. The linear extension $\varphi : 2(\mathbb{R}^{n+1}/G) \to \mathbb{R}^{n+1}/G^+$ of $\varphi : 2(S^n/G) \to S^n/G^+$ is an isometry that satisfies the desired property (note that $C(2(S^n/G)) = 2(C(S^n/G)) = 2((CS^n)/G)$ as metric spaces by Lemma 126 and Lemma 127). $\square$

Now we are in the position to prove Theorem C in its general form.

Proof of Theorem C. The if direction follows from Lemma 101, (ii), and Theorem B. Conversely, assume that $G < \mathrm{O}_n$ is a finite subgroup such that $\mathbb{R}^n/G$ is a homology manifold with boundary. According to Lemma 128, our Definition 12 (cf. the subsequent remark) and Theorem 125, the orientation preserving subgroup G^+ of G is a product of a rotation group and a certain number of Poincaré groups. So we are done, if G itself preserves the orientation. Otherwise G contains a reflection s by Lemma 128 which normalizes G^+. This reflection can only act in one of the factors of G^+. Therefore the claim follows, if we can show that $\mathbb{R}^4/\tilde{P}$ is not a homology manifold with boundary for $\tilde{P} = \langle P, s\rangle$ where s is one of the existing reflections in the normalizer of P in O_4. The coset of s in $\tilde{P}/P$ acts as an orientation reversing isometry on S^3/P. Hence, its fixed point subspace is a disjoint union of points and embedded surfaces. If $\mathbb{R}^4/\tilde{P}$ were a homology manifold, then only a single embedded sphere could occur and S^3/P would be the double of $S^3/\tilde{P}$ along this sphere by Lemma 128. In this case P would be a free product of isomorphic groups due to the theorem of Seifert and van Kampen on fundamental groups [Hat02, Thm. 1.20., p. 43] ($S^3/\tilde{P}$ is a smooth manifold with boundary which thus admits a collar). This is a contradiction, since P is neither trivial nor infinite and thus the theorem is proven. □

4.5 Proof for topological and Lipschitz manifolds

Now we are able to characterize finite subgroups $G < \mathrm{O}_n$ for which the quotient space $\mathbb{R}^n/G$ is a topological manifold. Recall that $\mathbb{R}^n/G$ is a topological manifold if $G < \mathrm{SO}_n$ is a reflection-rotation group by Theorem B.

Proof of Theorem D for manifolds without boundary. In Theorem 125 we have seen that G has a form as described in the Theorem D if the quotient space $\mathbb{R}^n/G$ is a topological manifold without boundary. Moreover, the additional condition $n > 4$ if $k = 1$ must also hold, since $\mathbb{R}^4/P$ is not a topological manifold.

Conversely, assume that G has a form as described in the theorem. In the case $n > 4k$ we can successively apply Theorem B and then k-times the double suspension theorem to show that $\mathbb{R}^n/G$ is homeomorphic to $\mathbb{R}^n$. This is possible, since after dividing out the rotation group there is always a trivial fifth factor available not involving any action. In the same way the case $n = 4k$ can be reduced to the case $n = 8$ and $G = P_1 \times P_2$. In this case the claim follows if we can show that $X := S^3/P_1 * S^3/P_2$ is a topological sphere, because of $\mathbb{R}^8/G \cong C(S^3/P_1 * S^3/P_2)$. To this end we assume that S^3/P_1 and S^3/P_2 are triangulated by simplicial complexes K_1 and K_2 induced by admissible triangulations of S^3 (cf. Section 3.1.1). Then $K = K_1 * K_2$ is again a simplicial complex and it triangulates X. Since the action $P \curvearrowright S^3$ is free, all the links of K_1 and K_2 are topological 2-spheres. Hence, for a vertex $x \in K$ we have

$$|\mathrm{lk}(x)| \cong S^2 * S^3/P \cong \Sigma^3(S^3/P) \cong S^6$$

by the double suspension theorem and thus X is a topological manifold. Since $\mathbb{R}^4/P$ is a homology manifold, so is $\mathbb{R}^8/G$ and thus X has the homology groups of S^7 by Lemma 101. Moreover, X is simply connected as the join of two path-connected spaces. Consequently, X is a simply connected homology sphere and as such a topological 7-sphere by the generalized Poincaré conjecture (cf. Section 3.3.3). This completes the proof by the remarks above. □

The general case can now be proven as follows.

Proof of Theorem D. According to Theorem B and what has been shown above the quotient space $\mathbb{R}^n/G$ is a topological manifold with boundary for all groups described in Theorem C. Conversely, suppose $G < \mathrm{O}_n$ is a finite subgroup for which $\mathbb{R}^n/G$ is a topological manifold with boundary. Then G has the form $G = G_{rr} \times P_1 \times \ldots \times P_k$ by Theorem C with $n > 4$ for $k = 1$ (cf. proof above). Moreover, the additional condition $n > 5$ for $k = 1$ in the case that G contains a reflection also holds, since $\mathbb{R}^4/P$, which would have to be the boundary of $\mathbb{R}_{\geq 0} \times \mathbb{R}^4/P$ by homological reasons (cf. Lemma 101), is not a topological manifold. $\square$

For the proof in the Lipschitz category note that if a metric space is a Lipschitz manifold with boundary, then its double with its induced length metric is a Lipschitz manifold without boundary.

Proof of Theorem E. The if direction is a direct consequence of Theorem B. The only-if direction in the case in which the boundary is empty has been treated in Section 4.2.10. So assume that $G < \mathrm{O}_n$ is a finite subgroup for which $\mathbb{R}^n/G$ is a Lipschitz manifold with nonempty boundary. Then its double coincides with $\mathbb{R}^n/G^+$ by Lemma 128 and is thus a Lipschitz manifold without boundary. Therefore the orientation preserving subgroup G^+ of G is a rotation group by what we have already shown. Since Lemma 128 also guarantees the existence of a reflection in G, the proof of Theorem E is now complete. $\square$

4.6 Generalized fixed point theorem and applications

Isotropy groups of real reflection groups are generated by the reflections they contain [Hum90, Thm. 1.12 (c), p. 22]. More generally, the same statement is true for isotropy groups of unitary reflection groups due to a theorem of Steinberg [Ste64, Thm. 1.5, p. 394]. In particular, isotropy groups of rotation groups which are either unitary reflection groups considered as real groups or orientation preserving subgroups of real reflection groups are generated by the rotations they contain. In the following we generalize this statement.

Lemma 129. *A rotation group does not contain Poincaré groups as minimal subgroups.*

Proof. Let $G < \mathrm{SO}_n$ be a rotation group and assume that it contains a Poincaré group $P < G$. Then P splits of as a direct factor by Section 4.3.3 and thus G would not be generated by rotations, a contradiction. $\square$

Now we can prove a general fixed point theorem for reflection-rotation groups.

Theorem 130. *Isotropy groups of reflection-rotation groups are again reflection-rotation groups, i.e. they are generated by the reflections and rotations they contain.*

Proof. Let $G < \mathrm{O}_n$ be a reflection-rotation group and let $x \in \mathbb{R}^n$. We have already seen that the quotient space $\mathbb{R}^n/G$ is a topological manifold with boundary. If the coset of x lies in the interior of $\mathbb{R}^n/G$, then we have $G_x = (G_x)_{\min}$ by Lemma 124. Since minimal subgroups of G_x are also minimal subgroups of G and since G contains no Poincaré subgroups as minimal subgroups by Lemma 129, neither does G_x. Consequently, $G_x = (G_x)_{\min}$ is a rotation group due to Proposition 117 in this case. If the coset of x lies on the boundary of $\mathbb{R}^n/G$, then G_x

contains a reflection and its orientation presererving subgroup is a rotation group by Lemma 128 and by what we have shown a few lines above. Hence, in any case G_x is a reflection-rotation group. □

Note that this theorem could have been more easily deduced from Theorem B. However, having deduced it in this way allows for an alternative approach to proving our main results. First, one would need to show the if direction of Theorem B only in the topological category. Compared to the proof in Chapter 3 such a proof could for instance dispense with Steinberg's fixedpoint theorem (cf. Section 3.4.2), the theorems of Whitehead and Illman (cf. Section 3.3.2) and the result from Chapter 2 (instead one could use [KL88] in Section 3.4.3). Then an inductive argument based on the preceding theorem, the topological version of the if direction of Theorem B and the generalized Poincaré conjecture in the PL category (cf. Section 3.5) would reduce the claim to a few cases in low dimensions where the latter tool is not available (cf. Section 3.3.3).

4.6.1 Sphere version of Theorem D

As an application we prove an analogue of Theorem D for spheres. This answers a question posed by Petrunin [Pet12].

Corollary 131. *For a finite subgroup $G < \mathrm{O}_n$ the quotient space S^{n-1}/G is a topological manifold if and only if G has the form*

$$G = G_{rot} \times P_1 \times \ldots \times P_k$$

for a rotation group G_{rot} and Poincaré groups $P_i < \mathrm{SO}_4$, $i = 1, \ldots, k$, such that the factors act in pairwise orthogonal spaces and such that $n > 5$ if $k = 1$. In this case S^{n-1}/G is homeomorphic to S^{n-1}.

Proof. By Proposition 125 G must have a form as described in the theorem, if the quotient space S^{n-1}/G is a topological- and thus a homology manifold. The additional condition $n > 5$ for $k = 1$ must also hold, since S^3/P is not simply connected and $\Sigma(S^3/P)$ is not a topological manifold.

Conversely, assume that G has a form as described in the theorem with $n > 5$ if $k = 1$. By [BH99, Prop. I.5.15, p. 64] and Theorem B (links in PL manifolds are PL spheres, cf. [RS72, p. 24]) the following spaces are homeomorphic

$$\begin{aligned} S^{n-1}/G &\cong S^{n-4k-1}/G_{rot} * S^3/P_1 * \cdots * S^3/P_k \\ &\cong S^{n-4k-1} * S^3/P_1 * \cdots * S^3/P_k \end{aligned}$$

Hence, the claim follows directly from the double suspension theorem in the case $n > 4k+1$ and as in the proof of Theorem D in Section 4.5 otherwise. □

Appendix

A.1 Quotients by groups with large isotropy

Lemma 132. *Let $G < \mathrm{SO}_{n+1}$ with $n \geq 2$ be a finite subgroup generated by elements that fix some point in S^n. Then the quotient space S^n/G is simply connected.*

Proof. Let $p : S^n \to S^n/G$ be the natural projection. Since S^n/G is connected, locally contractible and hence locally pathwise-connected and semilocally simply connected, there exists a universal covering $\varphi : \tilde{X} \to S^n/G$ [Hat02, p. 68]. For $n \geq 2$ the sphere S^n is simply connected and thus the map p can be lifted to $\tilde{X}$ [Hat02, Prop. 1.33, p. 61]

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\varphi"] \\ S^n \arrow[ur, dashed, "\overline{p}"] \arrow[r, "p"'] & X = S^n/G. \end{tikzcd}$$

Since φ is a local homeomorphism and p is an open map, the lift $\overline{p}$ is open, too. Therefore, S^n being compact and $\tilde{X}$ being Hausdorff as a cover of a Hausdorff space, the image of $\overline{p}$ is open and closed in $\tilde{X}$ and thus $\overline{p}$ is onto, since $\tilde{X}$ is connected. Let $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$ be such that $\varphi(\tilde{x}_1) = \varphi(\tilde{x}_2)$. The claim follows if we can show that $\tilde{x}_1 = \tilde{x}_2$. By surjectivity of $\overline{p}$ there are points $y_1, y_2 \in S^n$ with $\overline{p}(y_i) = \tilde{x}_i$, $i = 1, 2$. Because of $p(y_1) = p(y_2)$ there exists a $g \in G$ such that $y_1 = gy_2$. Assume that g fixes some point $y \in S^n$, i.e. $gy = y$, and choose a path $\gamma : [0, 1] \to S^n$ with $\gamma(0) = y$ and $\gamma(1) = y_1$. Then $\gamma' = g\gamma$ is a path from y to y_2. The fact that $(\overline{p} \circ \gamma)(0) = (\overline{p} \circ \gamma')(0) = \overline{p}(y)$ and $\varphi \circ \overline{p} \circ \gamma = \varphi \circ \overline{p} \circ \gamma'$ implies

$$\tilde{x}_1 = (\overline{p} \circ \gamma)(1) = (\overline{p} \circ \gamma')(1) = \overline{p}(y) = \tilde{x}_2$$

by the uniqueness of the lift of $p \circ \gamma$ to $\tilde{X}$ with prescribed initial point $\overline{p}(y)$. In general, we write $g = g_1 \dots g_k$ for elements $g_i \in G$ that fix some point in S^n. Applying the argument above successively we obtain $\overline{p}(g_i \dots g_k y_2) = \overline{p}(g_{i-1} \dots g_k y_2)$, $i = 1, \dots, k$, and thus finally $\tilde{x}_1 = \overline{p}(gy_2) = \overline{p}(y_2) = \tilde{x}_2$. $\square$

A.2 Orbifolds

The notion of an orbifold was introduced by Satake [Sat56, Sat57] under the name of *V-manifold*. This concept was rediscovered in the 70s by Thurston [Thu79] who introduced the term “orbifold”. It can be defined as follows (cf. [BH99, Dav11]).

Definition 16. A *smooth orbifold structure of dimension* n on a Hausdorff topological space X is given by the following data:

(i) An open cover $(U_i)_{i \in I}$ of X indexed by a set I.

(ii) For each $i \in I$ a finite subgroup G_i of the group of diffeomorphisms of a simply connected n-manifold M_i and a continuous map $p_i : M_i \to U_i$ such that p_i induces a homeomorphism from M_i/G_i onto U_i.

(iii) For all $x_i \in M_i$ and $x_j \in M_j$ such that $p_i(x_i) = p_j(x_j)$, there is a diffeomorphism h from an open neighborhood W of x_i to a neighborhood of x_j such that $p_j \circ h = p_i|_W$. Such a map h is called a *change of charts* and it is well-defined up to composition with an element of G_j (cf. [BH99, Ch. III 1.5 (1), p. 588]). In particular, if $i = j$, then h is the restriction of an element of G_i.

The orbifold structure on X is said to be *Riemannian*, if each M_i is a Riemannian manifold and the changes of charts are isometries. The collection $\{(M_i, G_i, U_i, p_i)\}_{i \in I}$ is called an *orbifold atlas* on X.

Definition 17. An *n-dimensional smooth orbifold* $\mathcal{O}$ is a Hausdorff topological space X together with a smooth orbifold structure. An *n-dimensional Riemannian orbifold* $\mathcal{O}$ is a second-countable Hausdorff topological space X together with a Riemannian orbifold structure. In this case the Riemannian structure induces a quotient metric on X compatible with its topology (cf. [BH99, Ch. III 1.1, p. 586]).

Every paracompact smooth orbifold in this sense admits a compatible Riemannian orbifold structure (cf. [BH99, Ch. III 1.5, p. 588]). Alternatively to the definition above, Riemannian orbifolds can be defined as follows.

Definition 18. An *n-dimensional Riemannian orbifold* $\mathcal{O}$ is a length space such that for each point $p \in \mathcal{O}$, there exists a neighborhood U of p in $\mathcal{O}$, an n-dimensional Riemannian manifold M and a finite group Γ acting by isometries on M such that U and M/Γ are isometric.

In order to show the equivalence of the two definitions above, we need the following technique. Let M and $\overline{M}$ be Riemannian manifolds, fix points $p \in M$, $\overline{p} \in \overline{M}$ and choose a linear isometry $I : T_pM \to T_{\overline{p}}\overline{M}$. A broken geodesic is a continuous curve $\gamma : [0,1] \to M$ such that there exist $0 = t_0 < t_1 \cdots < t_{n-1} < t_n < 1$ for which the restrictions $\gamma_{|[t_i,t_{i+1}]}$ are smooth geodesics. Set

$$\gamma_i = \gamma_{|[0,t_i]}$$

and define $v_i \in T_{\gamma(t_i)}M$ by

$$\gamma_{|[t_i,t_{i+1}]} = t \mapsto \exp_{\gamma(t_i)}((t - t_i)v_i).$$

A correspondence between broken geodesics γ and $\overline{\gamma}$ emanating from p and $\overline{p}$ can be defined as follows. Let γ be a broken geodesic in M starting at p. We set $\overline{\gamma}(0) = \overline{p}$ and $\overline{\gamma}_0 = \overline{\gamma}_{|[0,t_0]}$. Assume that $\overline{\gamma}_i$ is already defined. Then we set

$$\overline{\gamma}_{|[t_i,t_{i+1}]}(t) = \exp_{\overline{\gamma}_i(t_i)}((t - t_i)(P_{\overline{\gamma}_i} \circ I \circ P_{\gamma_i}^{-1})(v_i))$$

and $\overline{\gamma}_{i+1} = \overline{\gamma}_{|[0,t_i]}$ if the exponential map is defined where P_γ denotes the parallel transport along γ. If the exponential map is defined in each step, this construction yields a broken geodesic $\overline{\gamma} = \overline{\gamma}_n : [0,1] \to \overline{M}$. In this case we set $I_{\gamma(t)} = P_{\overline{\gamma}_{[0,t]}} \circ I \circ P^{-1}_{\gamma_{[0,t]}} : T_{\gamma(t)}M \to T_{\overline{\gamma}(t)}\overline{M}$. The following statement is known, but the author could not find a reference.

Lemma 133. *Let M and $\overline{M}$ be simply connected Riemannian manifolds on which finite groups Γ and $\overline{\Gamma}$, respectively, act isometrically. Suppose that M/Γ and $\overline{M}/\overline{\Gamma}$ are isometric. Then there exists a homomorphism $\varphi : \Gamma \to \overline{\Gamma}$ and a φ-equivariant isometry $\Phi : M \to \overline{M}$.*

Proof. Clearly M and $\overline{M}$ have the same dimension n. The proof is by induction on n. For $n = 1$ the statement is true. Suppose that $n > 1$ and that the statement holds in all dimensions lower than n. Let X be a metric space isometric to M/Γ and $\overline{M}/\overline{\Gamma}$ and let $\pi : M \to X$ and $\overline{\pi} : \overline{M} \to X$ be projections. Let $p \in M$ and $\overline{p} \in \overline{M}$ be points in the regular parts (i.e. with trivial isotropy groups) and $\pi(p) = \overline{\pi}(\overline{p})$. Choose a linear isometry $I : T_pM \to T_{\overline{p}}\overline{M}$ such that $d_p\pi = d_{\overline{p}}\overline{\pi} \circ I$. Then there exists some $r > 0$ such that

$$\left(\pi \circ \exp_p\right)_{|B_r(0)} = \left(\overline{\pi} \circ \exp_{\overline{p}} \circ I\right)_{|B_r(0)}$$

and for sufficiently small r the map

$$\phi := \exp_{\overline{p}} \circ I \circ \exp_p^{-1} : B_r(p) \to B_r(\overline{p})$$

is an isometry with $d_p\phi = I$ (cf. [DoC95, Prop. 2.9, p. 65]). We are going to show that for every broken geodesic $\gamma : [0,1] \to M$ with $\gamma(0) = p$ the broken geodesic $\overline{\gamma} : [0,1] \to \overline{M}$ (see above) is defined, that we have $\pi(\gamma(t)) = \overline{\pi}(\overline{\gamma}(t))$ for all $t \in [0,1]$ and that there exists a locally defined isometry $\phi : B_r(\gamma(1)) \to B_r(\overline{\gamma}(1))$ with $d_{\gamma(1)}\phi = I_{\gamma(1)}$ (see above for the notation). Then the proof of the Cartan-Ambrose-Hicks Theorem [CE08, Thm. 1.42, p. 32] implies that for all broken geodesics γ_0 and γ_1 emanating from p such that $\gamma_0(l_0) = \gamma_1(l_1)$ we have $\overline{\gamma_0}(l_0) = \overline{\gamma_1}(l_1)$ and that the map $\Phi : M \to \overline{M}$ defined by $\gamma(l) \mapsto \overline{\gamma}(l)$ is a local isometry and hence an isometry as we can define its inverse in the same way. Since the map Φ is moreover an orbit equivalence, the orbits of Γ and $\Phi^{-1}(\overline{\Gamma}) = \Phi^{-1}\overline{\Gamma}\Phi$ on M coincide. Therefore, for every $g \in \Gamma$ there exists some $\overline{g} \in \overline{\Gamma}$ such that g and $\overline{g}$ coincide in a neighborhood of p and hence everywhere, since an isometry of a connected manifold is determined by its value and differential at one point. This shows that $\Gamma = \Phi^{-1}(\overline{\Gamma})$ as subgroups of the isometry group of M. In other words, conjugation by Φ induces an isomorphism $\varphi : \Gamma \to \overline{\Gamma}$ with respect to which $\Phi : M \to \overline{M}$ is equivariant.

So let $\gamma : [0,1] \to M$ be a broken geodesic with $\gamma(0) = p$ and let $l \in [0,1]$ be the supremum of all $t \in [0,1]$ for which $\overline{\gamma}$ can be defined on $[0,t]$ such that $\pi(\gamma(s)) = \overline{\pi}(\overline{\gamma}(s))$ for all $s \in [0,t]$ and such that there exists an locally defined isometry $\phi_{\gamma(t)} : B_r(\gamma(t)) \to B_r(\overline{\gamma}(t))$ with $d_{\gamma(t)}\phi_{\gamma(t)} = I_{\gamma(t)}$. In order to prove $l = 1$ and therewith the lemma, it clearly suffices to show that the supremum is attained. Since M/G and $\overline{M}/\overline{G}$ are isometric and $\pi(\gamma(t)) = \overline{\pi}(\overline{\gamma}(t))$ for all $t < l$, the path $\overline{\gamma}$ can be defined on $[0,l]$ such that $\pi(\gamma(t)) = \overline{\pi}(\overline{\gamma}(t))$ for all $t \in [0,l]$. For sufficiently small $r > 0$ the exponential maps $\exp_{\gamma(l)} : B_r(0) \to M$ and $\exp_{\overline{\gamma}(l)} : B_r(0) \to M$ are defined and diffeomorphisms onto their images. By assumption, for sufficiently small $r > 0$ the quotient spaces $S_r(\gamma(l))/\Gamma_{\gamma(l)}$ and $S_r(\overline{\gamma}(l))/\overline{\Gamma}_{\overline{\gamma}(l)}$ of the radius r distance spheres at $\gamma(l)$

and at $\overline{\gamma}(l)$ in M and $\overline{M}$ by the isotropy groups $\Gamma_{\gamma(l)}$ and $\overline{\Gamma}_{\overline{\gamma}(l)}$ are isometric. For $n \geq 3$ it follows by induction that there exists an isometry $\hat{\phi} : S_r(\gamma(l)) \to S_r(\overline{\gamma}(l))$ and an isomorphism $\varphi_0 : \Gamma_{\gamma(l)} \to \overline{\Gamma}_{\overline{\gamma}(l)}$ with respect to which $\hat{\phi}$ is equivariant. The same conclusion also holds for $n = 2$. In this case the groups $\Gamma_{\gamma(l)}$ and $\overline{\Gamma}_{\overline{\gamma}(l)}$ are determined by the geometry of $S_r(\gamma(l))/\Gamma_{\gamma(l)}$ and $S_r(\overline{\gamma}(l))/\overline{\Gamma}_{\overline{\gamma}(l)}$ in the limit $r \to 0$. The radial extension $\phi : B_r(\gamma(l)) \to B_r(\overline{\gamma}(l))$ is equivariant with respect to φ_0 and, by assumption on the quotient spaces, a Riemannian isometry on the regular part. Since the groups Γ, $\overline{\Gamma}$ are finite this implies that ϕ is an isometry with respect to the induced length metric and thus a smooth isometry everywhere [Hel01, Thm. 11.1]. Let $s \in [0, l)$ be such that $\gamma(s) \in B_r(\gamma(l))$. Composing ϕ and φ_0 with (conjugation by) an element of $\overline{\Gamma}_{\overline{\gamma}(l)}$, we can assume that ϕ and $\phi_{\gamma(s)}$ coincide in a neighborhood of $\gamma(s)$. This implies $d_{\gamma(l)}\phi = I_{\gamma(l)}$ and $l = 1$ and thus the lemma is proven. □

As an immediate consequence of this lemma we see that the two definitions for (paracompact) Riemannian orbifolds given above are equivalent. Finally we explain in more detail a statement that has been applied in Section 4.4.2. Recall the definition of a Riemannian covering orbifold.

Definition 19. A *covering orbifold* of a Riemannian orbifold $\mathcal{O}$ is a Riemannian orbifold $\mathcal{O}'$ together with a surjective map $\varphi : \mathcal{O}' \to \mathcal{O}$ such that each point $x \in \mathcal{O}$ has a neighborhood U isometric to some M/G (cf. Definition 10) for which each connected component U_i of $\varphi^{-1}(U)$ is isometric to M/G_i for some subgroup $G_i < G$ such that the isometries respect the projections, i.e. the following diagram commutes

$$\begin{tikzcd} U_i \arrow[r, "\sim"] \arrow[d, "\varphi"'] & M/G_i \arrow[d] \\ U \arrow[r, "\sim"] & M/G \end{tikzcd}$$

Now we explain the statement in question.

Lemma 134. *Let M be a simply connected complete Riemannian manifold, let G be a finite group acting by isometries on M and let $q_1 : M' \to M/G$ be a covering of connected Riemannian orbifolds. Then there exists a finite subgroup G' of G and an isometry $\varphi : M/G' \to M'$ such that $q_2 = q_1 \circ \varphi$ where $q_2 : M/G' \to M/G$ is the natural projection.*

Proof. We first observe that the projection $p : M \to M/G$ lifts to a covering map $q : M \to M'$ of orbifolds (cf. [Thu79, Def. 13.2.2.]). As a simply connected manifold M is the universal covering orbifold of M/G and thus the map q exists [Thu79, Ch. 13, p. 305]. The map $q : M_{\text{reg}} \to M'_{\text{reg}}$ is a covering in the usual sense and its group of deck transformations $G' < G$ acts transitively on its fibers (cf. [Hat02, Prop. 1.39., p. 71]). Since the restrictions of q_1 and p to the regular parts are local isometries, so is q. Therefore, the metric completion $\varphi : M/G' \to M'$ of the induced isometry $\varphi : M_{\text{reg}}/G' \to M'_{\text{reg}}$ is an isometry and thus the metric completion $q : M \to M'$ of $q : M_{\text{reg}} \to M'_{\text{reg}}$ is a covering of Riemannian orbifolds. □

A.3 Classification of reflection-rotation groups

We summarize the classification of irreducible reflection-rotation groups (cf. [LM15]). There are 29 primitive rotation groups that are a unitary reflection group considered as a real group. Among them 19 groups occur in dimension 4 and are listed under number $4-22$ in [LT09, Ch. 6 and Appendix D, Table 1]. The remaining 10 groups are generated by unitary reflections of order 2 and are denoted as $W(\mathcal{J}_3^{(4)})$, $W(\mathcal{J}_3^{(5)})$, $W(\mathcal{K}_5)$, $W(\mathcal{K}_6)$, $W(\mathcal{L}_4)$, $W(\mathcal{M}_3)$, $W(\mathcal{M}_3)$, $W(\mathcal{N}_4)$, $W(\mathcal{O}_4)$ [LT09, Ch. 6 and Appendix D, Table 2] (cf. Section 1.3.2). All other primitive irreducible rotation groups are absolutely irreducible and are listed in Table 1.3.

symbol	meaning
$\mathfrak{C}_n$, $\mathfrak{D}_n$	Cyclic and dihedral group of order n and $2n$, respectively
$\mathfrak{S}_n$, $\mathfrak{A}_n$	Symmetric and alternating group on n letters.
$\mathbf{G}$	Preimage of a group $G < \mathrm{SO}_3$ under the covering $\psi : \mathrm{SU}_2 \to \mathrm{SO}_3$
	(cf. Section 1.3.3 for the meaning of $(\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)$
W^+	Orientation preserving subgroup of a real reflection group W.
W^*	Unique extension of W^+ by a normalizing rotation.
$W^\times$	Unique extension of W by a normalizing rotation.
$\mathfrak{P}$	Plane system (cf. Introduction of Chapter 1 or [LM15, Sect. 4.9]).
$M(\mathfrak{P})$	Rotation group generated by rotations corresponding to the planes of $\mathfrak{P}$.
$M^\times$	Unique extension of a rotation group M by a normalizing reflection.
$R_n(G)$	Image of the unique irreducible representation of G in SO_n.
L	Rotation group in the normalizer of
	$N_{\mathrm{SO}_8}(W(\mathrm{I}_2(4)) \otimes W(\mathrm{I}_2(4)) \otimes W(\mathrm{I}_2(4)))$ (cf. Section 1.3.8).
R_1, R_2	Root systems of type E_8 (cf. Section 1.3.8).
$D(G)$	Diagonal subgroup of a monomial group G.
$D(n)$	$D(W(\mathrm{BC}_n))$
$D^+(n)$	$D(W^+(\mathrm{BC}_n))$

Table 1.2: List of notations (cf. [LM15]).

The imprimitive irreducible rotation groups that preserve a complex structure are induced by unitary reflection groups of type $G(m, p, n)$ (cf. Section 1.3.2). All other imprimitive irreducible rotation groups are absolutely irreducible and are listed in Table 1.4. The groups $G^*(km, k, n)$, $k = 1, 2$, are extensions of $G(km, k, n)$ by a normalizing rotation τ of the form $\tau(z_1, z_2, z_3 \ldots, z_l) = (\overline{z}_1, \overline{z}_2, z_3 \ldots, z_l)$ (cf. Section 1.3.5). The groups $G^*(km, k, 2)_\varphi$ are described in Section 1.3.5.

All irreducible reflection-rotation groups that contain a reflection are listed in Table 1.5. The groups $G^\times(km, k, l)$, $k = 1, 2$, are generated by $G(km, k, l)$ and a reflection of type $s(z_1, \ldots, z_l) = (\overline{z}_1, z_2, \ldots, z_l)$.

	G	description	order
1.	$W^+(\mathrm{A}_n)$		$(n+1)!/2$
2.	$W^+(\mathrm{H}_3)$		$2^2 \cdot 3 \cdot 5 = 60$
3.		$(\mathbf{D}_{3m}/\mathbf{D}_{3m}; \mathbf{T}/\mathbf{T})$	$144m$
4.		$(\mathbf{D}_m/\mathbf{D}_m; \mathbf{O}/\mathbf{O})$	$96m$
5.		$(\mathbf{D}_{3m}/\mathbf{C}_{2m}; \mathbf{O}/\mathbf{V})$	$48m$
6.		$(\mathbf{D}_m/\mathbf{C}_{2m}; \mathbf{O}/\mathbf{T})$	$48m$
7.		$(\mathbf{D}_{2m}/\mathbf{D}_m; \mathbf{O}/\mathbf{T})$	$96m$
8.		$(\mathbf{D}_m/\mathbf{D}_m; \mathbf{I}/\mathbf{I})$	$240m$
9.		$(\mathbf{T}/\mathbf{T}; \mathbf{O}/\mathbf{O})$	$2^6 \cdot 3^2 = 576$
10.		$(\mathbf{T}/\mathbf{T}; \mathbf{I}/\mathbf{I})$	$2^5 \cdot 3^2 \cdot 5 = 1440$
11.		$(\mathbf{O}/\mathbf{O}; \mathbf{I}/\mathbf{I})$	$2^6 \cdot 3^2 \cdot 5 = 2880$
12.	$W^+(\mathrm{A}_4)$	$(\mathbf{I}/\mathbf{C}_1; \mathbf{I}/\mathbf{C}_1)^*$	$2^2 \cdot 3 \cdot 5 = 60$
13.	$W^*(\mathrm{A}_4)$	$(\mathbf{I}/\mathbf{C}_2; \mathbf{I}/\mathbf{C}_2)^*$	$2^3 \cdot 3 \cdot 5 = 120$
14.	$W^*(\mathrm{D}_4)$	$(\mathbf{T}/\mathbf{T}; \mathbf{T}/\mathbf{T})$	$2^5 \cdot 3^2 = 288$
15.	$W^+(\mathrm{F}_4)$	$(\mathbf{O}/\mathbf{T}; \mathbf{O}/\mathbf{T})$	$2^6 \cdot 3^2 = 576$
16.	$W^*(\mathrm{F}_4)$	$(\mathbf{O}/\mathbf{O}; \mathbf{O}/\mathbf{O})$	$2^7 \cdot 3^2 = 1152$
17.	$W^+(\mathrm{H}_4)$	$(\mathbf{I}/\mathbf{I}; \mathbf{I}/\mathbf{I})$	$2^5 \cdot 3^2 \cdot 5^2 = 7200$
18.	$M(\mathrm{R}_5)$	$R_5(\mathfrak{A}_5)$	$2^2 \cdot 3 \cdot 5 = 60$
19.	$W^*(\mathrm{A}_5)$		$2^3 \cdot 3 \cdot 5^2 = 720$
20.	$M(\mathrm{S}_6)$	$R_6(\mathrm{PSL}_2(7))$	$2^3 \cdot 3 \cdot 7 = 168$
21.	$W^+(\mathrm{E}_6)$		$2^6 \cdot 3^4 \cdot 5 = 25920$
22.	$W^*(\mathrm{E}_6)$		$2^7 \cdot 3^4 \cdot 5 = 51840$
23.	$W^+(\mathrm{E}_7)$		$2^9 \cdot 3^4 \cdot 5 \cdot 7 = 1451520$
24.	$M(\mathrm{T}_8)$	$L = W(R_1) \cap W(R_2)$	$2^{13} \cdot 3^2 \cdot 5 \cdot 7 = 2580480$
25.	$W^+(\mathrm{E}_8)$		$2^{13} \cdot 3^5 \cdot 5^2 \cdot 7 = 348364800$

Table 1.3: Primitive absolutely irreducible rotation groups (cf. [LM15]). See Table 1.2 for unknown notations.

	G	description	order
1.	$G^*(km,k,2)_\varphi$	cf. Section 1.3.5	$4km^2$
2.	$G^*(km,k,l)$	$\langle G(km,k,l), \tau\rangle$, $k=1,2$, $l>2$, $km \geq 3$	$2^{l-1}k^{l-1}m^l l!$
3.	$W^+(\mathrm{BC}_n)$		$2^{n-1}n!$
4.	$W^+(\mathrm{D}_n)$		$2^{n-2}n!$
5.	$M(\mathrm{P}_5)$	$M_5 = D^+(5) \rtimes H_5$	$2^4 \cdot \lvert H_5\rvert = 160$
6.	$M(\mathrm{P}_6)$	$M_6 = D^+(6) \rtimes H_6$	$2^5 \cdot \lvert H_6\rvert = 1920$
7.	$M(\mathrm{Q}_7)$	$M_7^p = \langle g_5, H_7\rangle < M_7$, $\lvert D(M_7^p)\rvert = 2^3$	$2^3 \cdot \lvert H_7\rvert = 1344$
8.	$M(\mathrm{P}_7)$	$M_7 = D^+(7) \rtimes H_7$	$2^6 \cdot \lvert H_7\rvert = 10752$
9.	$M(\mathrm{Q}_8)$	$M_8^p = \langle g_5, H_8\rangle < M_8$, $\lvert D(M_8^p)\rvert = 2^4$	$2^4 \cdot \lvert H_8\rvert = 21504$
10.	$M(\mathrm{P}_8)$	$M_8 = D^+(8) \rtimes H_8$	$2^7 \cdot \lvert H_8\rvert = 172032$

Table 1.4: Imprimitive absolutely irreducible rotation groups (cf. [LM15]). See Table 1.6 and Table 1.2 for unknown notations.

	G	description	order
1.	W	any irreducible reflection group	
2.	$G^{\times}(km,k,l)$	$\langle G(km,k,l),s\rangle$, $k=1,2$, $l\geq 2$, $km\geq 3$	$2^l k^{l-1} m^l l!$
3.	$M^{\times}(\mathrm{D}_n)$	$D(n)\rtimes\mathfrak{A}_n$	$2^{n-1}n!$
4.	$W^{\times}(\mathrm{A}_4)$		$2^4\cdot 3\cdot 5=240$
5.	$W^{\times}(\mathrm{D}_4)$		$2^6\cdot 3^2=576$
6.	$W^{\times}(\mathrm{F}_4)$		$2^8\cdot 3^2=2304$
7.	$M^{\times}(\mathrm{P}_5)$	$M_5^{\times}=D(5)\rtimes H_5$	$2^5\cdot\lvert H_5\rvert=320$
8.	$W^{\times}(\mathrm{A}_5)$		$2^5\cdot 3^2\cdot 5=1440$
9.	$M^{\times}(\mathrm{P}_6)$	$M_6^{\times}=D(6)\rtimes H_6$	$2^6\cdot\lvert H_6\rvert=3840$
10.	$W^{\times}(\mathrm{E}_6)$		$2^8\cdot 3^4\cdot 5=103680$
11.	$M^{\times}(\mathrm{P}_7)$	$M_7^{\times}=D(7)\rtimes H_7$	$2^7\cdot\lvert H_7\rvert=21504$
12.	$M^{\times}(\mathrm{P}_8)$	$M_8^{\times}=D(8)\rtimes H_8$	$2^8\cdot\lvert H_8\rvert=344064$

Table 1.5: Irreducible reflection-rotation groups that contain a reflection (cf. [LM15]). See Table 1.6 and Table 1.2 for unknown notations.

symbol	meaning
H_5	$\langle(1,2)(3,4),(2,3)(4,5)\rangle<\mathfrak{S}_5$, $H_5\cong\mathfrak{D}_5$
H_6	$\langle(1,2)(3,4),(1,5)(2,3),(1,6)(2,4)\rangle<\mathfrak{S}_6$, $H_6\cong\mathfrak{A}_5$
H_7	$\langle g_1,g_2,g_3\rangle<\mathfrak{S}_7$, $H_7\cong\mathrm{PSL}_2(7)\cong\mathrm{SL}_3(2)$
H_8	$\langle g_1,g_2,g_3,g_4\rangle<\mathfrak{S}_8$, $H_8\cong\mathrm{AG}_3(2)\cong\mathbb{Z}_2^3\rtimes\mathrm{SL}_3(2)$.
g_i	$g_1=(1,2)(3,4),\ g_2=(1,5)(2,6),\ g_3=(1,3)(5,7),\ g_4=(1,2)(7,8)\ g_5=(1,\overline{2})(3,\overline{4})$
$(i,\overline{j})$	Linear transformation that maps e_i to $-e_j$, $-e_j$ to e_i and e_k to e_k for $k\neq i,j$, where $e_1,\dots,e_n$ are standard basis vectors.

Table 1.6: Explanation of symbols appearing in Table 1.4 and Table 1.5.

	(G_{rr}, M, W, Γ)	Section
(i)	$(M^{\times}, M, D(M^{\times}), \circ)$ for $M = M_5, M_6, M_7, M_8, M(\mathrm{D}_n) = W^{+}(\mathrm{D}_n)$.	3.4.5
(ii)	$(G^{\times}(km,k,l), G^{*}(km,k,l), D(G^{\times}(km,k,l)), \circ)$ for $km \geq 3$ and $n = 2l$.	3.4.6
(iii)	$(G^{\times}(2m,1,l), G^{*}(2m,2,l), D(G^{\times}(2m,1,l)), \circ \quad \circ)$ for $m \geq 2$ and $n = 2l$.	3.4.6
(iv)	$(W, \{e\}, W, \Gamma(W))$ for any irreducible reflection group W.	-
(v)	$(W, W^{+}, W, \circ)$ for any irreducible reflection group W.	3.4.1
(vi)	$(W(\mathrm{A}_3), W^{+}(\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1), W(\mathrm{A}_3), \circ - \circ)$	3.4.3
(vii)	$(W(\mathrm{BC}_n), D(W^{+}(\mathrm{BC}_n)), W(\mathrm{BC}_n), \Gamma(\mathrm{A}_{n-1} \times \mathrm{A}_1) = \circ - \circ - \cdots \circ \quad \circ)$	3.4.5
$(viii)$	$(W(\mathrm{BC}_n), W^{+}(\mathrm{D}_n), W(\mathrm{BC}_n), \circ \quad \circ)$	3.4.1
(ix)	$(W(\mathrm{BC}_4), G^{*}(4,2,2), W(\mathrm{BC}_4), \circ - \circ \quad \circ)$	3.4.3
(x)	$(W(\mathrm{D}_n), D(W(\mathrm{D}_n)), W(\mathrm{D}_n), \Gamma(\mathrm{A}_{n-1}) = \circ - \circ - \cdots \circ)$	3.4.5
(xi)	$(W(\mathrm{D}_4), G^{*}(4,2,2), W(\mathrm{D}_4), \circ - \circ)$	3.4.3
(xii)	$(W(\mathrm{I}_2(km)), W^{+}(\mathrm{I}_2(m)), W(\mathrm{I}_2(km)), \circ \overset{k}{-} \circ)$ for $m, k \geq 2$.	3.4.3
(xii)	$(W(\mathrm{F}_4), G^{*}(4,2,2), W(\mathrm{F}_4), \circ - \circ \quad \circ - \circ)$	3.4.3
(xiv)	$(W(\mathrm{F}_4), W^{+}(\mathrm{D}_4), W(\mathrm{F}_4), \circ - \circ \quad \circ)$	3.4.3
(xv)	$(W(\mathrm{F}_4), W^{*}(\mathrm{D}_4), W(\mathrm{F}_4), \circ \quad \circ)$	3.4.3
(xvi)	$(W^{\times}, W^{*}, W, \circ)$ for a reflection group W of type A_4, D_4, F_4, A_5 or E_6.	3.4.1
$(xvii)$	$(W^{\times}(\mathrm{D}_4), W^{+}(\mathrm{D}_4), W(\mathrm{D}_4), \circ - \circ)$ (, but $H_i \neq F_i^{+}$, cf. [LM15, Prop. 55].)	3.4.3

Table 1.7: List of all triples (G_{rr}, M, W) occurring in the classification of reducible reflection-rotation groups, Theorem 3 (cf. [LM15]). The last column specifies the section in which the PL linearization principle is established for the respective pair $M \triangleleft G_{rr}$.

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Erklärung

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Christian Lange

Teilpublikationen

(1) Christian Lange and Marina A. Mikhaîlova, *Classification of finite groups generated by reflections and rotations*, Transformation Groups, (2016), DOI 10.1007/s00031-016-9385-6.

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