# On the Growth of Dimension of Harmonic Spaces of Semipositive Line Bundles over Manifolds

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**Abstract** This thesis consists of six parts. In the first part, we give a general introduction of our topics.

In the second part, we study the harmonic space of line bundle valued forms over a covering manifold with a discrete group action, and obtain an asymptotic estimate for the von Neumann dimension of the space of harmonic $(n, q)$-forms with values in high tensor powers of a semipositive line bundle. In particular, we estimate the von Neumann dimension of the corresponding reduced $L^2$-Dolbeault cohomology group. The main tool is a local estimate of the pointwise norm of harmonic forms with values in semipositive line bundles over Hermitian manifolds.

In the third part, we study the holomorphic extension problem of smooth forms with values in holomorphic vector bundles from the boundary of a pseudo-concave domain, which is in a compact Hermitian manifold associated with a holomorphic line bundle. We prove the existence of the meromorphic extension of $\overline{\partial}_b$-closed $(n, q+1)$-forms with values in holomorphic vector bundles, when the domain is $q$-concave and the line bundle is semi-positive everywhere and positive at one point.

In the fourth part, we study the $L^2$ holomorphic functions on hyperconcave ends and prove that the dimension of the space of $L^2$ holomorphic functions on hyperconcave ends is infinite. The main tool is the construction of $L^2$-peak functions at boundary points by using the solution of $\overline{\partial}$-Neumann problem of Kohn and the compactification theorem of Marinescu-Dinh.

In the fifth part, we give a remark on the Bergman kernel of symmetric tensor power of trivial vector bundles on compact Hermitian manifold by the Theorem of Le Potier.

In the last part, we study the $\overline{\partial}$-equation on $\mathbb{C}^n$ with growing weights, and generalize a related result of Hedenmalm on $\mathbb{C}$.

## Kurzzusammenfassung

Die Doktorarbeit ist in sechs Teile unterteilt. Im ersten Teil wird eine Einführung in die behandelten Themen gegeben.

Im zweiten Teil wird der Raum der harmonischen Formen auf einem Linienbündel über einer Überdeckungsmannigfaltigkeit mit einer diskreten Gruppenwirkung untersucht und eine asymptotische Abschätzung für die von Neumann-Dimension des Raumes der harmonischen $(n, q)$-Formen mit Werten in den hohen Tensorprodukten eines semipositiven Linienbündels bewiesen. Insbesondere wird die von Neumann Dimension der entsprechenden reduzierten $L^2$-Dolbeault Kohomologiegruppe abgeschätzt. Das wichtigste Werkzeug dabei ist eine lokale Abschätzung der punktweisen Norm von harmonischen Formen mit Werten in semipositiven Linienbündeln über hermitischen Mannigfaltigkeiten.

Im dritten Teil wird das Problem der holomorphen Erweiterung von glatten Formen mit Werten in holomorphen Vektorbündeln vom Rand eines pseudo-konkaven

Gebiets behandelt, das in einer kompakten hermitische Mannigfaltigkeit liegt, die mit einem holomorphen Linienbündel assoziiert ist. Es wird die Existenz einer meromorphen Erweiterung von $\overline{\partial}_b$-geschlossenen $(n, q+1)$-Formen mit Werten in holomorphen Vektorbündeln bewiesen unter der Annahme, dass das Gebiet $q$-konkav und das Linienbündel überall semipositiv und positiv an einem Punkt ist.

Im vierten Teil werden die $L^2$ holomorphen Funktionen auf hyperkonkaven Enden studiert und bewiesen, dass die Dimension des Raumes der $L^2$ holomorphen Funktionen auf hyperkonkaven Enden unendlich ist. Der wichtigste Schritt ist die Konstruktion von $L^2$ maximierenden Funktionen in Randpunkten, wobei die Lösung des $\overline{\partial}$-Neumann Problems von Kohn und der Kompaktifizierungssatz von Marinescu-Dinh angewendet wird.

Im fünften Teil, wird der Bergman Kern eines symmetrischen Tensorprodukts von trivialen Vektorbündeln auf kompakten Hermitischen Mannigfaltigkeiten mit der Hilfe des Satzes von Le Potier betrachtet.

Im letzten Teil, wird die $\overline{\partial}$-Gleichung auf $\mathbb{C}^n$ mit wachsenden Gewichten studiert und eine verwandtes Resultat von Hedenmal auf $\mathbb{C}$ verallgemeinert.

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# Contents

# 1 Introduction

For a compact manifold $X$, the growth of the dimension of the Dolbeault cohomology $H^{0,q}(X, L^k \otimes E)$ as $k \to \infty$ is of fundamental importance in algebraic and complex geometry and is linked to the structure of the manifold, cf. [10, 11, 30]. If $(L, h^L)$ is positive, then $H^{0,q}(X, L^k \otimes E) = 0$ for $q \geq 1$ and $k$ large enough, by the Kodaira-Serre vanishing theorem [30, Theorem 1.5.6]. This reflects the fact that the remaining cohomology space $H^{0,0}(X, L^k \otimes E)$ is rich enough to provide a projective embedding of $X$, for large $k$.

Assume now $(L, h^L)$ is semipositive. The solution of the Grauert-Riemenschneider conjecture by Demailly [11] and Siu [40] shows that $\dim H^{0,q}(X, L^k \otimes E) = o(k^n)$ as $k \to \infty$ for $q \geq 1$. This can be used to show that $X$ is a Moishezon manifold, if $(L, h^L)$ is moreover positive at at least one point. Berndtsson [5] showed that we have actually $\dim H^{0,q}(X, L^k \otimes E) = O(k^{n-q})$ as $k \to \infty$ for $q \geq 1$. Note that the latter estimate can be proved by induction on the dimension if $X$ is projective, see [12, (6.7) Lemma]. For the Bergman kernel $B_k^0$ on $(n, 0)$-forms with values in a semipositive line bundle, it was shown by Hsiao-Marinescu [22, Theorem 1.7] that it has an asymptotic expansion on the set where the curvature is strictly positive.

The global information of complex manifolds associated with bundles, such as the dimension of cohomology spaces, can be deduced from the local behaviour of the Bergman kernels, see [30]. Let $(X, \omega)$ be a Hermitian manifold and $(L, h^L)$ and $(E, h^E)$ be Hermitian holomorphic line bundles over $X$. For the space $\mathcal{H}^{n,q}(X, L^k \otimes E)$ of harmonic $L^k \otimes E$-valued $(n, q)$-forms and an orthonormal basis $\{s_j^k\}_{j \geq 1}$, the Bergman density function is defined by

$$B_k^q(x) = \sum_{j \geq 1} |s_j^k(x)|^2_{h_k, \omega}, \ x \in X,$$

where $|\cdot|_{h_k,\omega}$ is the pointwise norm of a form. So the integration of this function on $X$ is exactly the dimension of the harmonic space, see [5] for the compact case.

For a general Hermitian manifold $(X, \omega)$ and a compact subset $K \subset X$. Suppose that $(L, h^L)$ is semipositive on a neighborhood of $K$. Then, see Theorem 2.1, we can show that there exists $C > 0$ depending on the compact set $K$, the metric $\omega$ and the bundles $(L, h^L)$ and $(E, h^E)$, such that for any $x \in K$, $k \geq 1$ and $q \geq 1$,

$$B_k^q(x) \leq C k^{n-q}. \tag{1.0.1}$$

The study of $L^2$ cohomology spaces on coverings of compact manifolds has also interesting applications, cf. [18, 29]. The results are similar to the case of compact manifolds, but we have to use the reduced $L^2$ cohomology groups and von Neumann dimension instead of the usual dimension. The estimate (1.0.1), see Theorem 2.2, can be used to obtain the following bounds for the von Neumann dimension of the harmonic spaces on covering manifolds with a semipositive line bundle $(L, h^L)$,

$$\dim_\Gamma \mathcal{H}^{n,q}(X, L^k \otimes E) \leq Ck^{n-q}\,, \; \dim_\Gamma \mathcal{H}^{0,q}(X, L^k \otimes E) \leq Ck^{n-q}\,. \tag{1.0.2}$$

The same estimate also holds for the reduced $L^2$-Dolbeault cohomology groups,

$$\dim_\Gamma \overline{H}^{0,q}_{(2)}(X, L^k \otimes E) \leq Ck^{n-q}. \tag{1.0.3}$$

In the situation of (1.0.2), see Theorem 2.2, if the invariant line bundle $(L, h^L)$ is positive, the Andreotti-Vesentini vanishing theorem [3] shows that $\overline{H}^{0,q}_{(2)}(X, L^k \otimes E) \cong \mathcal{H}^{0,q}(X, L^k \otimes E) = 0$ for $q \geq 1$ and $k$ large enough. The holomorphic Morse inequalities of Demailly [11] were generalized to coverings by Chiose-Marinescu-Todor [34, 41] (cf. also [30, (3.6.24)]) and yield in the conditions of (1.0.2) (see Theorem 2.2) that $\dim_\Gamma H^{0,q}_{(2)}(X, L^k \otimes E) = o(k^n)$ as $k \to \infty$ for $q \geq 1$. Hence (1.0.2) (see Theorem 2.2) generalizes [5] to covering manifolds and refines the estimates obtained in [34, 41]. Note also that the magnitude $k^{n-q}$ cannot be improved in general [5, Proposition 4.2].

As applications of the growth of dimension of harmonic spaces (or Dolbeault cohomology) of semipositive and positive line bundles over manifolds, see [30], we can prove some holomorphic or meromorphic extension results for differential forms from the boundary of domains satisfying certain convexity conditions.

For a bounded domain $M$ in $\mathbb{C}^n$, $n \geq 2$, with a smooth connected boundary $bM$ and a smooth function $f$ defined on $bM$, if $f$ is holomorphic in a neighbourhood of $bM$, then it can be extended to the whole $M$ by the theorem of Hartogs. The generalization to manifolds by Kohn-Rossi [28] shows that, if $M$ is a relatively compact domain with smooth boundary $bM$ in a complex manifold, and the Levi form on $bM$ has one positive eigenvalue everywhere, then every function on $bM$ which satisfies the tangential Cauchy-Riemann equations, i.e., $\overline{\partial}_b$-closed, has a holomorphic extension to all of $M$. In addition, they also proved a result on holomorphic extension of $\partial_b$-closed sections of vector bundles, when the Levi form on $bM$ has at least one positive eigenvalue.

By applying various estimates of the growth of the dimensions of harmonic spaces (or Dolbeault cohomology) associated to line bundles, and using the criterion of Kohn-Rossi [28] on the holomorphic extension of $\overline{\partial}_b$-closed forms, we can show the following holomorphic extension result, see Theorem 3.3.

Let $(X, \omega)$ be a $n$-dimensional compact Hermitian manifold. Let $(E, h^E)$ and $(L, h^L)$ be the holomorphic Hermitian vector bundles over $X$ and $\operatorname{rank}(L) = 1$. Let $M$ be a relatively compact domain in $X$ and the boundary $bM$ is smooth. Let $1 \leq q \leq n-3$. Assume $L$ is semi-positive on $X$ and positive at one point, and the Levi form of a defining function of $M$ has at least $n-q$ negative eigenvalues on $bM$. Then, there exists a non-zero holomorphic section $s \in H^0(X, L^{k_0})$ for some $k_0 \in \mathbb{N}$, such that for every $\overline{\partial}_b$-closed form $\sigma \in \Omega^{n,q+1}(bM, E)$, there exists a $\overline{\partial}$-closed extension $S$ of the $\overline{\partial}_b$-closed $s\sigma \in \Omega^{n,q+1}(bM, L^{k_0} \otimes E)$, i.e.,

$$S \in \Omega^{n,q+1}(\overline{M}, L^{k_0} \otimes E) \tag{1.0.4}$$

such that $\overline{\partial} S = 0$ on $M$ and $\mu(S|_{bM}) = \mu(s\sigma)$.

In particular, if $q = 1$, which is equivalent to say $M$ is a strictly pseudo-concave domain (also 1-concave manifold) in $X$ associated with the line bundle $L$, then, for each $2 \leq r \leq n-2$, we can extend $\overline{\partial}_b$-closed $(n, r)$-forms on $bM$, which are with values in $E$, to meromorphic (resp. holomorphic) forms on $M$ (resp. $M$ except a small set of zero points), see Remark 3.4.

Besides, we also study some related topics in several complex variables and complex geometry, such as the $L^2$-peak functions on hyperconcave ends.

The Levi problem is as follows, the strongly pseudo-convex domain is a domain of holomorphy, which was firstly proved in [15] by using sheaf theory. In [25],[27] and [26], Kohn provided a different proof. In fact, Kohn showed the existence and global regularity of the solution of $\overline{\partial}$-Neumann problem on a (relatively compact) strong pseudo-convex domain $\Omega$ in a complex manifold $M$. As an application, there exists a peak function for $\mathcal{O}(\Omega)$ at each boundary point of $\Omega$ and $\dim_{\mathbb{C}} \mathcal{O}(\Omega) = \infty$. So there does not exist holomorphic function extending peak functions crossing the boundary $b\Omega$ at any boundary point, which implies that $\Omega$ is a domain of holomorphy, see [14].

In order to extend these results to the case when $\Omega$ is not relatively compact in $M$, cf. [18], Gromov-Henkin-Shubin studied the regular covering $\Omega$ of a (relatively compact) strongly pseudo-convex domain. They showed that there exists a $L^2$-local peak functions for $\mathcal{O}(\Omega)$ at each boundary point and the von Neumann dimension $\dim_\Gamma L^2(\Omega) \cap \mathcal{O}(\Omega) = \infty$. In particular, if the discrete group is trivial, i.e., $\Gamma = \{e\}$, the $L^2$-local peak functions reduces to $L^2$-peak function and

$$\dim_{\mathbb{C}} L^2(\Omega) \cap \mathcal{O}(\Omega) = \infty.$$

We wish to extend the above result to a class of complex manifolds, namely, hyperconcave ends as follows (see [33]). A complex manifold $X$ with $\dim X \geq 2$ is called a hyperconcave end, if there exist $a \in \mathbb{R} \cup \{+\infty\}$ and a proper, smooth function $\varphi : X \to (-\infty, a)$, which is strictly plurisubharmonic on a set of the form $\{x \in X : \varphi(x) < b\}$ for some $b \leq a$. In Theorem 4.2, we show that, for $X_c = \{x \in X : \varphi(x) < c\}$ with $-\infty < c < b$, there exists $L^2$-peak functions for $\mathcal{O}(X_c)$ associated with some Hermitian metric $\Theta$, i.e., for every $x \in bX_c$, there exists a function

$$\mathit{\Phi}_x \in \mathcal{O}(X_c) \cap L^2(X_c, \Theta) \cap \mathcal{C}^\infty(\overline{X_c} \setminus \{x\}) \tag{1.0.5}$$

such that $\lim_{y \to x} |\mathit{\Phi}_x(y)| = +\infty$ for $y \in X_c$. And thus

$$\dim_{\mathbb{C}} L^2(X_c, \Theta) \cap \mathcal{O}(X_c) = \infty. \tag{1.0.6}$$

Note that, for a hyperconcave end, the existence of $\mathit{\Phi}_x \in \mathcal{O}(X_c) \cap \mathcal{C}^\infty(\overline{X_c} \setminus \{x\})$ with blowing up at $x$, was firstly established by Marinescu-Dinh [33]. So our result is a refinement and the only new feature is $\Phi_x \in L^2(X_c, \Theta)$. Our result also verifies on strongly pseudo-convex domains in normal Hermitian spaces of pure dimensional.

In additional, we give some remarks on Bergman kernels for high tensor powers of trivial vector bundles over compact manifolds, and the solution of $\overline{\partial}$-equations with growing weights on $\mathbb{C}^n$.

The organization of this thesis is as follows.

In Chapter 2, we study the harmonic space of line bundle valued forms over a covering manifold with a discrete group action, and obtain an asymptotic estimate for the von Neumann dimension of the space of harmonic $(n, q)$-forms with values in high tensor powers of a semipositive line bundle. In particular, we estimate the von Neumann dimension of the corresponding reduced $L^2$-Dolbeault cohomology group. The main tool is a local estimate of the pointwise norm of harmonic forms with valued in semipositive line bundles over Hermitian manifolds.

In Chapter 3, we study the holomorphic extension problem of smooth forms with values in holomorphic vector bundles from the boundary of a pseudo-concave domain, which is in a compact Hermitian manifold associated with a holomorphic line bundle. And we prove the existence of the meromorphic extension of $\overline{\partial}_b$-closed $(n, q+1)$-forms with values in holomorphic vector bundles, when the domain is $q$-concave and the line bundle is semi-positive everywhere and positive at one point.

In Chapter 4, we study the $L^2$ holomorphic functions on hyperconcave ends and prove that the dimension of the space of $L^2$ holomorphic functions on hyperconcave ends is infinite. The main tools is the construction of $L^2$-peak functions at boundary points by using Kohn's solution of $\overline{\partial}$-Neumann problem and the compactification theorem of Marinescu-Dinh.

In Chapter 5, we study the relation of $L^2$-orthonormal basis of the space of holomorphic sections of symmetric tensor power of a holomorphic vector bundle on a compact manifold and the space of holomorphic sections of the induced line bundle. And we get a formula on the Bergman kernel of symmetric tensor power of trivial vector bundles on compact Hermitian manifold by the Theorem of Le Potier.

In Chapter 6, we study the $\overline{\partial}$-equation on $\mathbb{C}^n$ with growing weights, and generalize a related result of Hedenmalm on $\mathbb{C}$. The method is analogue to Hedenmalm, which is essentially due to the classical works of Hörmander.

# 2 On the growth of von Neumann dimension of harmonic spaces of semipositive line bundles over covering manifolds

The purpose of this chapter is to study the growth of the von Neumann dimension of the space of harmonic forms with values in powers of an invariant semipositive line bundle over a Galois covering of a compact Hermitian manifold. The main technical tool will be an estimate of the Bergman kernel on a compact set of a Hermitian manifold, which generalizes a result of Berndtsson [5] for compact manifolds.

This chapter is organized in the following way. In Section 2.1, we state the main results of this chapter. In Section 2.2, we introduce the notations and recall the necessary facts. In Section 2.3, we prove some properties of harmonic line bundle valued forms, including $\partial\overline{\partial}$-formulas on non-compact manifolds and submeanvalue formulas, which imply Theorem 2.1. In Section 2.4, we prove our main results and corollaries, and explain that Theorem 2.1 implies Theorem 2.2.

## 2.1 The main results

Let $(X, \omega)$ be a Hermitian (paracompact) manifold of dimension $n$ and $(L, h^L)$ and $(E, h^E)$ be Hermitian holomorphic line bundles over $X$. For $k \in \mathbb{N}$ we form the Hermitian line bundles $L^k := L^{\otimes k}$ and $L^k \otimes E$, the latter endowed with the metric $h_k = (h^L)^{\otimes k} \otimes h^E$.

To the metrics $\omega$, $h^L$ and $h^E$ we associate the Kodaira Laplace operator $\square_k$ acting on forms with values in $L^k \otimes E$ and also $L^2$ spaces of forms with values in $L^k \otimes E$, and $\square_k$ has a (Gaffney) self-adjoint extension in the space of $L^2$-forms, denoted by the same symbol.

The space $\mathcal{H}^{p,q}(X, L^k \otimes E)$ of harmonic $L^k \otimes E$-valued $(p,q)$-forms is defined as the kernel of (the self-adjoint extension of) $\square_k$ acting on the $L^2$ space of $(p,q)$-forms.

In this chapter we mainly work with $(n,q)$-forms. Since $\mathcal{H}^{n,q}(X, L^k \otimes E)$ is separable, let $\{s_j^k\}_{j\geq 1}$ be an orthonormal basis and denote by $B_k^q$ the Bergman density function defined by

$$B_k^q(x) = \sum_{j\geq 1} |s_j^k(x)|^2_{h_k,\omega}\,, \; x \in X, \tag{2.1.1}$$

where $|\cdot|_{h_k,\omega}$ is the pointwise norm of a form. Definition (2.1.1) is independent of the choice of basis.

The first main result of this chapter is a uniform estimate of the Bergman density function for semipositive line bundles in a neighborhood of a compact subset of a Hermitian manifold.

**Theorem 2.1.** *Let $(X,\omega)$ be a Hermitian manifold and $(L,h^L)$ and $(E,h^E)$ be Hermitian holomorphic line bundles over $X$. Let $K\subset X$ be a compact subset and assume that $(L,h^L)$ is semipositive on a neighborhood of $K$.*

*Then there exists $C>0$ depending on the compact set $K$, the metric $\omega$ and the bundles $(L,h^L)$ and $(E,h^E)$, such that for any $x\in K$, $k\geq 1$ and $q\geq 1$,*

$$B_k^q(x)\leq Ck^{n-q}\,, \tag{2.1.2}$$

*where $B_k^q(x)$ is the Bergman kernel function* (2.1.1) *of harmonic $(n,q)$-forms with values in $L^k\otimes E$.*

For $X$ compact and $K=X$, Theorem 2.1 reduces to [5, Theorem 2.3]. Theorem 2.1 will be used to obtain the following bounds for the von Neumann dimension of the harmonic spaces on covering manifolds.

**Theorem 2.2.** *Let $(X,\omega)$ be a Hermitian manifold of dimension $n$ on which a discrete group $\Gamma$ acts holomorphically, freely and properly such that $\omega$ is a $\Gamma$-invariant Hermitian metric and the quotient $X/\Gamma$ is compact. Let $(L,h^L)$ and $(E,h^E)$ be two $\Gamma$-invariant holomorphic Hermitian line bundles on $X$. Assume $(L,h^L)$ is semipositive on $X$. Then there exists $C>0$ such that for any $q\geq 1$ and $k\geq 1$ we have*

$$\dim_\Gamma \mathcal{H}^{n,q}(X,L^k\otimes E)\leq Ck^{n-q}\,,\ \ \dim_\Gamma \mathcal{H}^{0,q}(X,L^k\otimes E)\leq Ck^{n-q}\,. \tag{2.1.3}$$

*The same estimate also holds for the reduced $L^2$-Dolbeault cohomology groups,*

$$\dim_\Gamma \overline{H}^{0,q}_{(2)}(X,L^k\otimes E)\leq Ck^{n-q}. \tag{2.1.4}$$

Note also that the magnitude $k^{n-q}$ in (2.1.3) cannot be improved in general [5, Proposition 4.2].

## 2.2 Preliminaries

We introduce here the notations and recall the necessary facts used in this chapter.

Let $(X,J)$ be a complex manifold with the complex structure $J$ and $\dim_{\mathbb{C}} X=n$. Let $g^{TX}$ be a Riemannian metric on the real tangent bundle $TX$ which is compatible with $J$. Explicitly, $J:TX\to TX$ is an automorphism such that $J^2=-\operatorname{Id}$ and

$$g^{TX}(U,V)=g^{TX}(JU,JV)$$

for any $U, V \in T_xX$, $x \in X$. We can extend the Riemannian metric $g^{TX}$ to a $\mathbb{C}$-bilinear form $\langle\cdot,\cdot\rangle^{\mathbb{C}}$ on the complexification of the real tangent bundle $TX \otimes_{\mathbb{R}} \mathbb{C}$ by

$$\langle aU, bV\rangle^{\mathbb{C}} := abg^{TX}(U,V)$$

for $a, b \in \mathbb{C}$ and $U, V \in TX$. And we can extend $J$ to a $\mathbb{C}$-linear map

$$J : TX \otimes_{\mathbb{R}} \mathbb{C} \to TX \otimes_{\mathbb{R}} \mathbb{C}$$

by $J(aU) = aJ(U)$ for $a \in \mathbb{C}$ and $U \in TX$. Thus we still have $J^2 = -\operatorname{Id}$ and $\langle\cdot,\cdot\rangle^{\mathbb{C}}$ is compatible with $J$ on $TX \otimes_{\mathbb{R}} \mathbb{C}$ by

$$\langle J(aU), J(bV)\rangle^{\mathbb{C}} = \langle aJU, bJV\rangle^{\mathbb{C}} = abg^{TX}(JU,JV) = abg^{TX}(U,V) = \langle aU, bV\rangle^{\mathbb{C}}.$$

Then $J$ induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$ where

$$T^{(1,0)}X = \{u \in TX \otimes_{\mathbb{R}} \mathbb{C} : Ju = \sqrt{-1}u\}, \quad T^{(0,1)}X = \{u \in TX \otimes_{\mathbb{R}} \mathbb{C} : Ju = -\sqrt{-1}u\},$$

and the $\mathbb{C}$-bilinear form $\langle\cdot,\cdot\rangle^{\mathbb{C}}$ vanishes on $T^{(1,0)}X \times T^{(1,0)}X$ and $T^{(0,1)}X \times T^{(0,1)}X$. Let $T^{(1,0)*}X$ and $T^{(0,1)*}X$ be the corresponding dual bundles. We denote the associated complex Hermitian vector bundle by

$$\wedge^{p,q}X := \wedge^p T^{(1,0)*}X \otimes_{\mathbb{C}} \wedge^q T^{(0,1)*}X.$$

We have a Hermitian inner product $h$ on $T^{(1,0)}X$ by

$$h(u,v) := \langle u, \overline{v}\rangle^{\mathbb{C}},$$

which induces the Hermitian inner product $h^{\wedge^{p,q}}$ on $\wedge^{p,q}X$. The non-degenerate skew-symmetric 2-form $\omega$ associated to $g^{TX}$ is defined by

$$\omega(U,V) := g^{TX}(JU,V)$$

for $U, V \in TX$, which is called the fundamental form. A complex manifold $(X,J)$ associated a compatible Riemannian metric $g^{TX}$ is called a Hermitian manifold, which is denoted by $(X,\omega)$. A Hermitian manifold $(X,\omega)$ is called complete, if all geodesics are defined for all time for the underlying Riemannian manifold. Every complex manifold has a compatible Riemannian metirc, thus it is a Hermitian manifold. We denote the volume form by $dv_X := \omega_n$, where $\omega_q := \frac{\omega^q}{q!}$ for $1 \leq q \leq n$. Suppose $\{\frac{\partial}{\partial z_i}\}_{i=1}^n$ is a local frame of $T^{(1,0)}X$ with dual frame $\{dz_i\}_{i=1}^n$, and $\{\frac{\partial}{\partial \overline{z_i}}\}_{i=1}^n$ is a local frame of $T^{(0,1)}X$ with dual frame $\{d\overline{z_i}\}_{i=1}^n$ respectively. The fundamental form is a real $(1,1)$-form and can be locally represented by

$$\omega = \sqrt{-1}\sum_{i,j=1}^n h_{ij} dz_i \wedge d\overline{z_i},$$

where $h_{ij} = h(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) = \langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}}\rangle^{\mathbb{C}}$ satisfy $h_{ij} = \overline{h_{ji}}$ and $h_{ii} > 0$ on $X$.

For differential $(p,q)$-forms on $X$, we have the Lefschetz operator

$$L := \omega \wedge \cdot$$

and its dual operator $\Lambda$ (the Hermitian metric adjoint of the operator exterior multiplication with $\omega$), that is,

$$\langle \Lambda\alpha, \beta \rangle_{h^{\wedge^{p,q}}} = \langle \alpha, L\beta \rangle_{h^{\wedge^{p,q}}}$$

where $h^{\wedge^{p,q}}$ is the Hermitian inner product on $\wedge^{p,q}X$.

### 2.2.1 Positive forms and the local representation of forms in $\Omega^{n,q}(X,F)$

Let $(X,\omega)$ be a Hermitian manifold of dimension $n$ and $(F,h^F)$ be a Hermitian holomorphic line bundles over $X$. Let $\Omega^{p,q}(X,F)$ be the space of smooth $(p,q)$-forms on $X$ with values in $F$ for $p,q \in \mathbb{N}$, i.e.,

$$\Omega^{p,q}(X,F) := \mathcal{C}^\infty(X, \wedge^{p,q}X \otimes F).$$

And we denote by $\Omega^{p,q}(X) := \Omega^{p,q}(X,\mathbb{C}) = \mathcal{C}^\infty(X,\wedge^{p,q}X)$ the space of smooth $(p,q)$-forms. The curvature of $(F,h^F)$ is defined by

$$R^F := \overline{\partial}\partial \log |s|^2_{h^F}$$

for any local holomorphic frame $s$ of $F$, and the Chern-Weil form of the first Chern character of $F$ is

$$c_1(F,h^F) = \frac{\sqrt{-1}}{2\pi} R^F, \tag{2.2.1}$$

which is a real $(1,1)$-form on $X$.

We will use several times the notion of positive $(p,p)$-form, for which we refer to [10, Chapter III, §1, (1.1) (1.2)(1.5)(1.7)]. Positivity is a property on the exterior algebra of complex vector spaces, and essentially the positivity of a differential form is pointwisely defined. Let $U$ be an open subset in $X$.

**Definition 2.3.** A differential $(1,1)$-form $u \in \Omega^{1,1}(U,\mathbb{C})$ is called a positive (resp. semi-positive) Hermitian $(1,1)$-form, if it can be represented locally by

$$u = \sqrt{-1} \sum_{i,j=1}^{n} u_{ij} dz_i \wedge d\overline{z}_j$$

such that the matrix $(u_{ij})_{n\times n}$ is a positive (resp. semi-positive) definite Hermitian matrix at each point of $U$.

By the definition, the fundamental form $\omega$ is a positive Hermitian $(1,1)$-form on $X$.

**Definition 2.4.** A Hermitian holomorphic line bundle $(F, h^F)$ is called positive (resp. semi-positive) on $U$, if the Chern-Weil form $c_1(F, h^F)$ is a positive (resp. semi-positive) Hermitian $(1,1)$-form on $U$. And we denote it by $F > 0$ or $c_1(F, h^F) > 0$ (resp. $F \geq 0$ or $c_1(F, h^F) \geq 0$) on $U$.

In general, we also can discuss the positivity for $(p,p)$-forms and the related propositions as follows, see [10, Chapter III, §1, (1.1)].

**Definition 2.5.** (cf. [10, Chapter III, §1, (1.1)])
A $(p,p)$-form $T \in \Omega^{p,p}(U, \mathbb{C})$ is called positive, if for any $\alpha_j \in \Omega^{1,0}(U, \mathbb{C})$, $1 \leq j \leq n-p$, there exists a non-negative function $\lambda \geq 0$ on $U$, such that

$$T \wedge (i\alpha_1 \wedge \overline{\alpha_1}) \wedge ... \wedge (i\alpha_{n-p} \wedge \overline{\alpha_{n-p}}) = \lambda \omega^n.$$

And we denoted it by $T \geq 0$. Also we say that $T_1 \geq T_2$, if $T_1 - T_2 \geq 0$.

By the definition, the volume form $dv_X := \omega_n$ is a positive $(n,n)$-form. And if $T \geq 0$ is a $(p,p)$-form and $u \geq 0$ is a $(1,1)$-form, then

$$T \wedge u \geq 0. \tag{2.2.2}$$

Since there exists (1,0)-forms $\beta_j$, $1 \leq j \leq r \leq n$, such that $u = \sqrt{-1}\sum_{j=1}^{r} \beta_j \wedge \overline{\beta_j}$ after diagonalizing $u$ at a fixed point.

The following proposition is from [10, Chapter III, §1, (1.2)(1.5)(1.7)].

**Proposition 2.6.**

*(1)* $i^{p^2}\beta \wedge \overline{\beta}$ *is positive for every* $\beta \in \Omega^{p,0}(U, \mathbb{C})$*;*

*(2) a* $(p,p)$*-form* $T \geq 0$ *implies* $T = \overline{T}$ *is a real form; and*

*(3) A* $(1,1)$*-form* $u$ *is positive if and only if* $u$ *is a semi-positive Hermitian* $(1,1)$*-form.*

*Proof.* Since the positivity of a $(p,p)$-form is pointwisely defined, we only need to consider the positivity at a point in $U$. We choose a local holomorphic coordinate chart $(z_1, ..., z_n)$ around $x \in U$ such that $\omega(x) = \frac{\sqrt{-1}}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z_j}$. Then

$$2^n\omega_n(x) = idz_1 \wedge d\overline{z}_1 \wedge ... \wedge idz_n \wedge d\overline{z}_n.$$

For any $\alpha_j \in \Omega^{1,0}(U, \mathbb{C})$, $1 \leq j \leq n-p$,

$$\begin{aligned}
& i^{p^2}\beta \wedge \overline{\beta} \wedge (i\alpha_1 \wedge \overline{\alpha_1}) \wedge ... \wedge (i\alpha_{n-p} \wedge \overline{\alpha_{n-p}}) \\
= \; & i^{p^2}\beta \wedge \overline{\beta} \wedge i^{(n-p)^2}\alpha_1 \wedge ... \wedge \alpha_{n-p} \wedge \overline{\alpha_1 \wedge ... \wedge \alpha_{n-p}} \\
= \; & i^{n^2}\beta \wedge \alpha_1 \wedge ... \wedge \alpha_{n-p} \wedge \overline{\beta \wedge \alpha_1 \wedge ... \wedge \alpha_{n-p}} \\
= \; & i^{n^2}\lambda(x)dz_1...dz_n \wedge \overline{\lambda(x)dz_1...dz_n} \\
= \; & |\lambda(x)|^2 idz_1 \wedge d\overline{z}_1 \wedge ... \wedge idz_n \wedge d\overline{z}_n \\
= \; & 2^n|\lambda(x)|^2\omega_n(x).
\end{aligned}$$

Then $i^{p^2}\beta \wedge \overline{\beta} \geq 0$ at $x \in U$, and thus (1) follows.

For any $\alpha_j \in \Omega^{1,0}(U, \mathbb{C}), 1 \leq j \leq n-p$, $T \geq 0$ implies that there exists a function $\lambda \geq 0$ on $U$ such that

$$T \wedge (i\alpha_1 \wedge \overline{\alpha_1}) \wedge ... \wedge (i\alpha_{n-p} \wedge \overline{\alpha_{n-p}}) = \lambda\omega^n.$$

Since $i\alpha_j \wedge \overline{\alpha_j}$ and $\lambda\omega_n$ are real forms,

$$(T - \overline{T}) \wedge (i\alpha_1 \wedge \overline{\alpha_1}) \wedge ... \wedge (i\alpha_{n-p} \wedge \overline{\alpha_{n-p}}) = 0.$$

Then $T = \overline{T}$ is a real form by the arbitrary choices of $\alpha_j$, and thus (2) follows.

Let $S$ be a 1-dimensional subspace of $T_x^{(1,0)}X$ and $\{\frac{\partial}{\partial z_i}\}_{i=1}^n$ be a basis of $T_x^{(1,0)}X$. By changing coordinates, we can assume $S = S_1 := \{k\frac{\partial}{\partial z_1} : k \in \mathbb{C}\}$. Let $u \in \Omega^{1,1}(U, \mathbb{C})$. Then the restriction $u(x)$ on $S$ is

$$u(x)|_S = \lambda_S(x) i dz_1 \wedge d\overline{z}_1 = 2\lambda_S(x)\omega|_S(x),$$

where $\lambda_S(x)$ is given by

$$\begin{aligned} u(x) \wedge i dz_2 \wedge d\overline{z}_2 \wedge ... \wedge i dz_n \wedge d\overline{z}_n &= \lambda_S(x) i dz_1 \wedge d\overline{z}_1 \wedge ... \wedge i dz_n \wedge d\overline{z}_n \\ &= 2^n \lambda_S(x)\omega_n(x). \end{aligned} \tag{2.2.3}$$

In particular, we consider 1-dimensional subspaces of $T_x^{(1,0)}X$ associated to $\xi \in \mathbb{C}^n \setminus \{0\}$, which are given by

$$S_\xi := \{t\sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j} : t \in \mathbb{C}\}.$$

Then

$$\begin{aligned} u(x)|_{S_\xi} &= i\sum u_{jk}(x) dz_j \wedge d\overline{z}_k|_{S_\xi} \\ &= \sum u_{jk}(x)\xi_j\overline{\xi_k} i dt \wedge d\overline{t} \\ &= 2\sum u_{jk}(x)\xi_j\overline{\xi_k}\omega|_{S_\xi(x)}. \end{aligned}$$

If a $(1,1)$-form $u \geq 0$ on $U$, then $\lambda_S(x) \geq 0$ for all 1-dimensional subspaces $S$ by (2.2.3). Then $\sum u_{jk}(x)\xi_j\overline{\xi_k} \geq 0$ for all $\xi \in \mathbb{C}^n \setminus \{0\}$. That is, a $(1,1)$-form $u \geq 0$ implies $u$ is a semi-positive Hermitian $(1,1)$-form on $U$.

Conversely, if $u$ is a semi-positive Hermitian $(1,1)$-form on $U$, then $\lambda_S(x) \geq 0$ for any 1-dimensional subspace $S$ in (2.2.3). Let $S_k := \{t\frac{\partial}{\partial z_k} : t \in \mathbb{C}\}$, $k = 1, ..., n$. For any $\alpha_j \in \Omega^{1,0}(U, \mathbb{C}), 1 \leq j \leq n-1$, there exist $\mu_k(x) \geq 0$, $k = 1, ..., n-1$, such that

$$(i\alpha_1 \wedge \overline{\alpha_1} \wedge ... \wedge i\alpha_{n-1} \wedge \overline{\alpha_{n-1}})(x) = \sum_{k=1}^n \mu_k(x) i dz_1 \wedge d\overline{z}_1 \wedge ... \widehat{i dz_k \wedge d\overline{z}_k} ... \wedge i dz_n \wedge d\overline{z}_n.$$

Then

$$\begin{aligned} & u(x) \wedge (i\alpha_1 \wedge \overline{\alpha_1} \wedge ... \wedge i\alpha_{n-1} \wedge \overline{\alpha_{n-1}})(x) \\ = & \sum_{k=1}^{n} \mu_k(x) u(x) i dz_1 \wedge d\overline{z_1} \wedge ... \widehat{i dz_k \wedge d\overline{z}_k} ... \wedge i dz_n \wedge d\overline{z}_n \\ = & \sum_{k=1}^{n} \mu_k(x) \lambda_{S_k(x)} 2^n \omega_n(x). \end{aligned}$$

Then $2^n \sum_{k=1}^{n} \mu_k(x)\lambda_{S_k(x)} \geq 0$ implies $u \geq 0$, and thus (3) follows. $\square$

Next we present the trivialization of holomorphic line bundles and some local formulas of smooth sections, which are quite useful in the following calculations.

Let $F \overset{\pi}{\to} X$ be a holomorphic line bundle on $X$. A trivialization of $F$ is given by an open covering $\{U_i\}_{i\in I}$ and biholomorphic maps

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C} \tag{2.2.4}$$

such that

$$\varphi_i : \pi^{-1}(x) \to \{x\} \times \mathbb{C} \simeq \mathbb{C} \tag{2.2.5}$$

is a $\mathbb{C}$-linear isomorphism. Then the transition maps, which are holomorphic non-zero functions, are given by

$$\begin{gathered} \varphi_{ij} : U_i \cap U_j \to \mathbb{C} \setminus \{0\}, \\ (x, \varphi_{ij}(x)\xi) := (\varphi_i \varphi_j^{-1})(x, \xi). \end{gathered} \tag{2.2.6}$$

for $x \in U_i \cap U_j$ and $\xi \in \mathbb{C}$.

In this trivialization, a section $s \in \mathcal{C}^\infty(X, F)$ can be locally represented by

$$\begin{aligned} s_i := \varphi_i s : \quad U_i & \longrightarrow U_i \times \mathbb{C}, \\ x & \longmapsto (x, s_i(x)). \end{aligned} \tag{2.2.7}$$

Here we identify $s_i(x)$ with a smooth function over $U_i$. Then on $U_i \cap U_j$ we have

$$\begin{gathered} \varphi_i^{-1} s_i = s = \varphi_j^{-1} s_j, \quad s_i = (\varphi_i \varphi_j^{-1}) s_j, \\ (x, s_i(x)) = (x, \varphi_{ij}(x) s_j(x)). \end{gathered} \tag{2.2.8}$$

Thus, we can describe a section by $s \simeq (s_i, \varphi_{ij})$ in the trivialization.

By (2.2.4) and (2.2.5), we obtain a holomorphic frame of $F$ over $U_i$ by

$$e_i : U_i \to F\,|_{U_i} := \pi^{-1}(U_i), \quad e_i(x) := \varphi_i^{-1}(x, 1), \tag{2.2.9}$$

then $\pi^{-1}(x) = \mathbb{C} e_i(x)$. And for $x \in U_i \cap U_j$, we have

$$\varphi_i e_i(x) = (x, 1) = \varphi_j e_j(x), \quad e_j(x) = (\varphi_j^{-1} \varphi_i) e_i(x). \tag{2.2.10}$$

Moreover, by (2.2.5), (2.2.6) and (2.2.9), it is clear that

$$\begin{aligned} \varphi_{ij}(x) e_i(x) & = \varphi_{ij} \varphi_i^{-1}(x, 1) = \varphi_i^{-1}(x, \varphi_{ij}(x)) = \varphi_i^{-1} \varphi_i \varphi_j^{-1}(x, 1) = \varphi_j^{-1}(x, 1) \\ & = e_j(x). \end{aligned} \tag{2.2.11}$$

**Proposition 2.7.** *Let* $s \simeq (s_i, \varphi_{ij}) \in \mathcal{C}^\infty(X, F)$ *under the trivialization. Then, the representation of* $s$*, given by*

$$s|_{U_i}(x) := s_i(x)e_i(x), \tag{2.2.12}$$

*is globally well-defined.*

*Proof.* For $x \in U_i$,

$$s(x) = \varphi_i^{-1}(x, s_i(x)) = s_i(x)\varphi_i^{-1}(x, 1) = s_i(x)e_i(x). \tag{2.2.13}$$

For $x \in U_i \cap U_j$, by (2.2.8) and (2.2.11),

$$s_i(x)e_i(x) = s_j(x)e_j(x). \tag{2.2.14}$$

□

**Proposition 2.8.** *Let* $h^F$ *be a Hermitian metric on* $F$*. Then, there exist*

$$\psi_i : U_i \to \mathbb{R} \tag{2.2.15}$$

*such that* $|e_i(x)|^2_{h^F} = e^{-\psi_i(x)}$*. Let* $s, t \in \mathcal{C}^\infty(X, F)$*. Then* $\langle s(x), t(x)\rangle_{h^F}$ *and* $|s(x)|^2_{h^F}$ *under the trivialization, given by*

$$\begin{aligned} \langle s(x), t(x)\rangle_{h^F}|_{U_i} &= s_i(x)\overline{t_i(x)}e^{-\psi_i(x)}, \quad \textit{and} \\ |s(x)|^2_{h^F}|_{U_i} &= |s_i(x)|^2 e^{-\psi_i(x)}, \end{aligned}$$

*are globally well-defined.*

*Proof.* We can define $\psi_i \in \mathcal{C}^\infty(U_i, \mathbb{R})$ by $e^{-\psi_i(x)} := |e_i(x)|^2_{h^F}$. Then for $x \in U_i$,

$$\langle s(x), t(x)\rangle_{h^F} = s_i(x)\overline{t_i(x)}|e_i(x)|^2_{h^F} = s_i(x)\overline{t_i(x)}e^{-\psi_i(x)}.$$

By (2.2.8) and (2.2.11), for $x \in U_i \cap U_j$, we see

$$s_i(x)\overline{t_i(x)}e^{-\psi_i(x)} = s_j(x)\overline{t_j(x)}e^{-\psi_j(x)} \tag{2.2.16}$$

□

In the same way, let $s \in \mathcal{C}^\infty(X, \wedge^{p,q}X \otimes F)$ be represented by

$$\begin{aligned} s_i := \varphi_i s : \quad U_i &\longrightarrow U_i \times \wedge^{p,q}(X), \\ x &\longmapsto (x, s_i(x)). \end{aligned} \tag{2.2.17}$$

Here we identify $s_i(x)$ with a smooth (p,q)-form over $U_i$. Then on $U_i \cap U_j$ we still have

$$\begin{aligned} \varphi_i^{-1}s_i = s = \varphi_j^{-1}s_j, \quad s_i = (\varphi_i\varphi_j^{-1})s_j, \\ (x, s_i(x)) = (x, \varphi_{ij}(x)s_j(x)) \end{aligned} \tag{2.2.18}$$

by $(x, s_i(x)) = \varphi_i s(x) = \varphi_i\varphi_j^{-1}\varphi_j s(x) = \varphi_i\varphi_j^{-1}(x, s_j(x)) = (x, \varphi_{ij}(x)s_j(x))$. As same as (2.2.12) and (2.2.15), we have the following propositions.

**Proposition 2.9.** *Let* $s \in \mathcal{C}^\infty(X, \wedge^{p,q}X \otimes F)$*. Then the representation of* $s$ *under the trivialization, given by*

$$s(x)|_{U_i} = s_i(x) \otimes e_i(x), \tag{2.2.19}$$

*is globally well-defined.*

*Proof.* For $x \in U_i$,

$$s(x) = \varphi_i^{-1}(x, s_i(x)) = s_i(x)\varphi_i^{-1}(x, 1) = s_i(x) \otimes e_i(x). \tag{2.2.20}$$

For $x \in U_i \cap U_j$, by (2.2.18) and (2.2.11),

$$s_i(x) \otimes e_i(x) = s_j(x) \otimes e_j(x). \tag{2.2.21}$$

□

**Proposition 2.10.** *Let* $h^F$ *be a Hermitian metric on* $F$ *and* $h_\omega$ *the induced Hermitian metric on* $\wedge^{p,q}X$ *by* $\omega$*. Let* $h = h_\omega \otimes h^F$*. Then, there exist*

$$\psi_i : U_i \to \mathbb{R} \tag{2.2.22}$$

*such that* $|e_i(x)|^2_{h^F} = e^{-\psi_i(x)}$*. Let* $s, t \in \mathcal{C}^\infty(X, \wedge^{p,q}X \otimes F)$*.* $\langle s(x), t(x)\rangle_h$ *and* $|s(x)|^2_h$ *under the trivialization, given by*

$$\begin{aligned} \langle s(x), t(x)\rangle_h|_{U_i} &= \langle s_i(x), t_i(x)\rangle_{h_\omega} e^{-\psi_i(x)}, \\ |s(x)|^2_h|_{U_i} &= |s_i(x)|^2_{h_\omega} e^{-\psi_i(x)}, \end{aligned}$$

*are globally well-defined.*

*Proof.* We can define $\psi_i \in \mathcal{C}^\infty(U_i, \mathbb{R})$ by $e^{-\psi_i(x)} := |e_i(x)|^2_{h^F}$. Then for $x \in U_i$,

$$\langle s(x), t(x)\rangle_h = \langle s_i(x), t_i(x)\rangle_{h_\omega} |e_i(x)|^2_{h^F} = \langle s_i(x), t_i(x)\rangle_{h_\omega} e^{-\psi_i(x)}.$$

For $x \in U_i \cap U_j$, by (2.2.18) and (2.2.11), we see

$$\langle s_i(x), t_i(x)\rangle_{h_\omega} |e_i(x)|^2_{h^F} = \langle s_j(x), t_j(x)\rangle_{h_\omega} |e_j(x)|^2_{h^F}. \tag{2.2.23}$$

□

**Proposition 2.11.** *Let* $\alpha, \beta$ *be differential forms with values in* $F$ *over* $X$*. Then* $\alpha \wedge \beta e^{-\psi}$*, given by*

$$\alpha \wedge \beta e^{-\psi}|_{U_i} := \alpha_i \wedge \beta_i e^{-\psi_i}, \tag{2.2.24}$$

*is a globally well-defined, scalar-valued differential form.*

*Proof.* We set $\alpha = \alpha_i \otimes e_i$, $\beta = \beta_i \otimes e_i$ on $U_i$. Then (2.2.24) follows by (2.2.18), (2.2.11) and (2.2.23). □

**Proposition 2.12.** *The curvature form* $\Theta_F$ *of a holomorphic Hermitian line bundle* $F$ *over* $X$ *can be represented by*

$$\Theta_F|_{U_i} := \sqrt{-1}R^F|_{U_i} = \sqrt{-1}\partial\overline{\partial}\psi_i. \tag{2.2.25}$$

*And the representation is globally well-defined.*

*Proof.* Let us choose a local frame $e_i$ as in (2.2.9). Then, by (2.2.1) and (2.2.22) over $U_i$,

$$\Theta_F = \sqrt{-1}R^F = -\sqrt{-1}\partial\overline{\partial}\log|e_i|^2_{h^F} = \sqrt{-1}\partial\overline{\partial}\psi_i. \tag{2.2.26}$$

On $U_i \cap U_j$, since the transition function $\varphi_{ji}$ is holomorphic as in (2.2.6), we see

$$\begin{aligned} \partial\overline{\partial}\log|e_i(x)|^2_{h^F} &= \partial\overline{\partial}\log|\varphi_{ji}(x)|^2|e_j(x)|^2_{h^F} \\ &= \partial\overline{\partial}\log|\varphi_{ji}(x)|^2 + \partial\overline{\partial}\log|e_j(x)|^2_{h^F} \\ &= \partial\overline{\partial}\log|e_j(x)|^2_{h^F}. \end{aligned} \tag{2.2.27}$$

□

**Definition 2.13.** Hodge star operator of the Hermitian manifold $(X, \omega)$ is defined by

$$\begin{aligned} \star : \Omega^{p,q}(U,\mathbb{C}) &\longrightarrow \Omega^{n-q,n-p}(U,\mathbb{C}) \quad \text{such that} \\ \beta \wedge \overline{\star\alpha} &= \langle \beta, \alpha\rangle_\omega \omega_n \end{aligned} \tag{2.2.28}$$

for any open set $U \subset X$ and any $\beta \in \Omega^{p,q}(U,\mathbb{C})$, where $\langle\cdot,\cdot\rangle_\omega = \langle\cdot,\cdot\rangle_{h_\omega}$ is the induced Hermitian metric on $\wedge^{p,q}X$.

Hodge star operator is well-defined, since the exterior product provides a non-degenerate pairing pointwisely. Essentially, Hodge star operator is defined on $\mathbb{C}$-vector spaces $\wedge^{p,q}_x X$ at each point $x \in U$, and $(\star\alpha)(x) = \star(\alpha(x))$ for $\alpha \in \Omega^{p,q}(U,\mathbb{C})$. Thus we can verify the following proposition under a local coordinate at point $x \in U$ such that $(T^{(1,0)*}_x X, h_\omega)$ is isometric to $\mathbb{C}^n$.

The following proposition is from [23, Proposition 1.2.20,1.2.24,1.2.31].

**Proposition 2.14.** *Let* $\alpha \in \Omega^{n,q}(U,\mathbb{C})$ *and* $\wedge^k_{\mathbb{C}}X := \oplus_{p+q=k} \wedge^{p,q} X$. *Then*

*(1)* $\star^2 = (-1)^{k(2n-k)}$ *on* $\wedge^k_{\mathbb{C}}X$ *;*

*(2)* $\alpha = C_{n-q}(\star\alpha) \wedge \omega_q$ *, where* $C_{n-q} := i^{(n-q)^2}$ *;*

*(3)* $\star^2\alpha = (-1)^{n-q}C_{n-q}(\star\alpha) \wedge \omega_q$ *;*

*(4)* $\alpha \wedge \overline{\star\alpha} = |\alpha|^2_\omega \omega_n$ *;*

*(5)* $|\star\alpha|_\omega = |\alpha|_\omega$.

*Proof.* For a fixed point $x \in U$, we can choose a local coordinate $(z_1, ..., z_n)$ around $x$, such that $\{dz_1, ..., dz_n\}$ forms an orthonormal basis of $T_x^{(1,0)*}X$ at $x$. Then

$$\omega_q(x) := \frac{\omega^q(x)}{q!} = \sum_{1 \le j_1 < j_2 < ... < j_q \le n} idz_{j_1} \wedge d\overline{z}_{j_1} \wedge ... \wedge idz_{j_q} \wedge d\overline{z}_{j_q}$$

for $1 \le q \le n$. By the notation of the ordered muti-indices $J$,

$$\omega_q(x) = \sum_{|J|=q} i^{q^2} dz_J \wedge d\overline{z}_J.$$

In particular, $\omega_n(x) = i^{n^2} dz \wedge d\overline{z}$, where $dz := dz_1 \wedge ... \wedge dz_n$ and $d\overline{z} = d\overline{z}_1 \wedge ... \wedge d\overline{z}_n$.

It is clear that $\star$ is $\mathbb{C}$-linear on $\wedge_x^{p,q}X$, then we only need to consider $dz_I \wedge d\overline{z}_J \in \wedge_x^{p,q}X$. Let us denote by $I^c, J^c$ the ordered complementary multi-indices of $I, J$. By the definition,

$$\overline{\star(dz_I \wedge d\overline{z}_J)} = \lambda dz_{I^c} \wedge d\overline{z}_{J^c}$$

where $\lambda \in \mathbb{C}$ is given by

$$dz_I \wedge d\overline{z}_J \wedge (\lambda dz_{I^c} \wedge d\overline{z}_{J^c}) = \omega_n(x).$$

Then

$$\star(dz_I \wedge d\overline{z}_J) = (-1)^{(n-q)(n-p)} \overline{\lambda} dz_{J^c} \wedge d\overline{z}_{I^c} \in \wedge_x^{n-q,n-p}X.$$

By the definition,

$$\overline{\star\star(dz_I \wedge d\overline{z}_J)} = \xi dz_J \wedge d\overline{z}_I$$

where $\xi \in \mathbb{C}$ is given by

$$\star(dz_I \wedge d\overline{z}_J) \wedge \xi dz_J \wedge d\overline{z}_I = \omega_n(x).$$

Then we have

$$\begin{aligned} (-1)^{(n-q)(n-p)} \overline{\lambda} dz_{J^c} \wedge d\overline{z}_{I^c} \wedge \xi dz_J \wedge d\overline{z}_I &= \omega_n(x) = \overline{\omega_n(x)} \\ &= \overline{dz_I \wedge d\overline{z_J} \wedge (\lambda dz_{I^c} \wedge d\overline{z}_{J^c})}. \end{aligned}$$

Then $\xi = (-1)^{(p+q)(2n-(p+q))+pq}$, that is, $\star^2(dz_I \wedge d\overline{z}_J) = (-1)^{(p+q)(2n-(p+q))} dz_I \wedge d\overline{z}_J$. Then (1) follows.

Let $dz \wedge d\overline{z}_J \in \wedge_x^{n,q}X$. By the definition,

$$\overline{\star(dz \wedge d\overline{z}_J)} = \tau d\overline{z}_{J^c}$$

where $\tau \in \mathbb{C}$ is given by

$$dz \wedge d\overline{z}_J \wedge \tau d\overline{z}_{J^c} = \omega_n(x) = i^{n^2} dz \wedge d\overline{z}.$$

Then

$$\tau d\overline{z}_J \wedge d\overline{z}_{J^c} = i^{n^2} d\overline{z}.$$

Then

$$\begin{aligned}
C_{n-q}(\star(dz\wedge d\overline{z}_J))\wedge\omega_q(x) &= i^{(n-q)^2}\overline{\tau}dz_{J^c}\wedge\omega_q(x)\\
&= i^{(n-q)^2}\overline{\tau}dz_{J^c}\wedge i^{q^2}dz_J\wedge d\overline{z}_J\\
&= i^{(n-q)^2+q^2}\overline{\tau}dz_J\wedge d\overline{z}_J\wedge dz_{J^c}\\
&= (-1)^{q(n-q)}i^{(n-q)^2+q^2}\overline{\tau}dz_J\wedge dz_{J^c}\wedge d\overline{z}_J\\
&= (-1)^{q(n-q)}i^{(n-q)^2+q^2}(-1)^{n^2}i^{n^2}dz\wedge d\overline{z}_J\\
&= dz\wedge d\overline{z}_J.
\end{aligned}$$

Then (2) follows.

From (1) and (2), (3) follows. And (4) is trivial. Finally, by (1) and (4), $|\star\alpha|^2_\omega\omega_n = |\alpha|^2_\omega\omega_n$ at $x$ implies (5). □

We can associate a scalar valued form to a form with value in line bundles by the local representation of sections as follows. Let $F$ be a holomorphic Hermitian bundle over a complex manifold $X$. Let $\{U_j\}$ be a covering of $X$ such that $F|_{U_j}$ is trivial. Let $\alpha\in\Omega^{n,q}(X,F)$. Then $\alpha|_{U_j}=\alpha_j\otimes e_j$ on $U_j$, where $\alpha_j$ is a scalar valued $(n,q)$-form and $e_j$ is a holomorphic section of $F|_{U_j}$ such that $|e_j|^2=e^{-\psi_j}$. Then

$$\gamma_j := \gamma_{\alpha_j} := \star\alpha_j$$

is a $(n-q,0)$-form over $U_j$. Furthermore, we have a form $\gamma_\alpha\in\Omega^{n-q,0}(X,F)$, given by

$$\gamma_\alpha|_{U_j} := \gamma_{\alpha_j}\otimes e_j = (\star\alpha_j)\otimes e_j, \tag{2.2.29}$$

globally well-defined by (2.2.18) and (2.2.11), where $\alpha_i$ verifies (2.2.28) and $e_i$ verifies (2.2.15). Thus, we can extend the notion $\star$ to each form $\alpha\in\Omega^{n,q}(X,F)$ by setting $\star(\alpha)=\gamma_\alpha\in\Omega^{n-q,0}(X,F)$.

**Definition 2.15.** Let $\alpha\in\Omega^{n,q}(X,F)$. The associated $(n-q,n-q)$-form $T_\alpha$ on $X$ is given by

$$T_\alpha|_{U_j} := C_{n-q}\gamma_j\wedge\overline{\gamma_j}e^{-\psi_j} \tag{2.2.30}$$

where $C_{n-q}:=i^{(n-q)^2}$. Also we denote it by $T:=T_\alpha$.

**Proposition 2.16.** *$T_\alpha$ is a globally well-defined, positive form on $X$, i.e., $T_\alpha\in\Omega^{n-q,n-q}(X)$ and $T_\alpha\geq 0$.*

*Proof.* By (2.2.18) and (2.2.11), for any $x\in U_i\cap U_j$,

$$e^{-\psi_j(x)} = |e_j(x)|^2 = |\varphi_{ij}(x)|^2|e_i(x)|^2 = |\varphi_{ij}(x)|^2e^{-\psi_i(x)}. \tag{2.2.31}$$

Thus

$$\gamma_i\wedge\overline{\gamma_i}e^{-\psi_i} = \gamma_{\varphi_{ij}\alpha_j}\wedge\overline{\gamma_{\varphi_{ij}\alpha_j}}\frac{1}{|\varphi_{ij}(x)|^2}e^{-\psi_j} = \frac{|\varphi_{ij}(x)|^2}{|\varphi_{ij}(x)|^2}\gamma_j\wedge\overline{\gamma_j}e^{-\psi_j} = \gamma_j\wedge\overline{\gamma_j}e^{-\psi_j}. \tag{2.2.32}$$

And Proposition 2.6(1) implies $T_\alpha\geq 0$. □

Note that we encode the curvature of $F$ to a local function $\psi$ and also encode line bundle valued form $\alpha$ to usual differential form $T$ for the purpose of further local calculations. Trivially, if $F = X \times \mathbb{C}$ with trivial metric, then the function $\psi = 0$ everywhere. Generally, for arbitrary $x \in X$, we can choose a trivialization of $F$ around $x \in U_i$ such that $\psi_i(x) = 0$, that is, $|e_i(x)|_{h^F} = 1$ and then $\langle \alpha(x), \beta(x) \rangle_h = \langle \alpha_i(x), \beta_i(x) \rangle_{h_\omega}$ for any $\alpha, \beta \in \Omega^{p,q}(X, F)$.

### 2.2.2 Reduced $L^2$-Dolbeault cohomology

Let $(X, \omega)$ be a Hermitian manifold and $(F, h^F)$ is a Hermitian holomorphic vector bundle on $X$. Let $\Omega^{p,q}(X, F) := \mathcal{C}^\infty(X, \wedge^p(T^{(1,0)*}X) \otimes \wedge^q(T^{(0,1)*}X) \otimes F)$ be the space of smooth $(p, q)$-forms with values in $F$ for $p, q \in \mathbb{N}$. Let $\Omega_0^{p,q}(X, F)$ be the subspace of $\Omega^{p,q}(X, F)$ consisting of elements with compact support.

The $L^2$-scalar product on $\Omega^{p,q}(X, F)$ is give by

$$\langle s_1, s_2 \rangle := \int_X \langle s_1(x), s_2(x) \rangle_{h^F,\omega} dv_X(x) \tag{2.2.33}$$

where $\langle , \rangle_{h^F,\omega}$ is the pointwise Hermitian inner product induced by $\omega$ and $h^F$. We set the $L^2$-norm by $\| \ \|_{L^2}^2 = \langle \ , \ \rangle$.

We denote by $L^2_{p,q}(X, F)$ the $L^2$ completion of $\Omega_0^{p,q}(X, F)$ with respect to $\| \ \|_{L_2}$. And we set $L^2_{p,\bullet}(X, F) = \oplus_{q=1}^n L^2_{p,q}(X, F)$.

Let $\alpha \in \Omega^{p,q}(X, \mathbb{C})$ and $s \in \mathcal{C}^\infty(X, F)$ such that $\alpha \wedge s \in \Omega^{p,q}(X, F)$. The Dolbeault operator $\overline{\partial}^F : \Omega_0^{p,q}(X, F) \to L^2_{p,q+1}(X, F)$ is given by

$$\overline{\partial}^F(\alpha \wedge s) = (\overline{\partial}\alpha) \wedge s + (-1)^{(p+q)} \alpha \wedge \overline{\partial}^F s.$$

In particular, $\overline{\partial}^F : \mathcal{C}_0^\infty(X, F) \to \Omega_0^{0,1}(X, F)$ is defined as follows. For $s \in \mathcal{C}_0^\infty(X, F)$, $s = \sum_l \phi_l \xi_l$, where $\xi_l$ is a local holomorphic frame of $F$ and $\phi_l$ are smooth functions, we set $\overline{\partial}^F s := \sum_l (\overline{\partial}\phi_l)\xi_l = \sum_l (\sum_j \dfrac{\partial \phi_l}{\partial \overline{z}_j} d\overline{z}_j)\xi_l$ in holomorphic coordinates $(z_1, z_2, ..., z_n)$.

We denote by $\overline{\partial}^{F*}$ the formal adjoint of $\overline{\partial}^F$, which is given by

$$\langle \overline{\partial}^F s_1, s_2 \rangle = \langle s_1, \overline{\partial}^{F*} s_2 \rangle$$

for $s_1 \in \Omega_0^{p,q}(X, F)$ and $s_2 \in \Omega_0^{p,q+1}(X, F)$.

For $s_1 \in L^2_{p,q}(X, F)$, we define $\overline{\partial}^F s_1$ in the current sense: $\langle \overline{\partial}^F s_1, s_2 \rangle = \langle s_1, \overline{\partial}^{F*} s_2 \rangle$ for $s_2 \in \Omega_0^{p,q+1}(X, F)$. Clearly, $\overline{\partial}^{F*} s_1$ in the current sense for $s_1 \in L^2_{p,q}(X, F)$ is similar.

The following lemma is from [30, Lemma 3.1.1].

**Lemma 2.17.** *(cf. [30, Lemma 3.1.1])*

*The operator* $\overline{\partial}^F_{\max}$ *defined by*

$$\begin{aligned} \mathrm{Dom}(\overline{\partial}^F_{\max}) &= \{s \in L^2_{p,\bullet}(X,F) : \overline{\partial}^F s \in L^2_{p,\bullet}(X,F)\}, \\ \overline{\partial}^F_{\max} : \mathrm{Dom}(\overline{\partial}^F_{\max}) &\to L^2_{p,\bullet}(X,F) \\ s &\mapsto \overline{\partial}^F_{\max} s = \overline{\partial}^F s \quad \textit{in the sense of currents} \end{aligned} \tag{2.2.34}$$

*is a densely defined, closed extension, called the maximal extension of* $\overline{\partial}^F$.

Furthermore, we define the Hilbert space adjoint $(\overline{\partial}^F_{\max})^*_H$ of $\overline{\partial}^F_{\max}$ by

$$\begin{aligned} &\mathrm{Dom}((\overline{\partial}^F_{\max})^*_H) \\ := \;& \{s \in L^2_{p,\bullet}(X,F) \mid \exists\, C > 0, \mid \langle \overline{\partial}^F_{\max} v, s\rangle \mid \leq C \parallel v \parallel^2 \text{ for } \forall v \in \mathrm{Dom}(\overline{\partial}^F_{\max})\} \\ = \;& \{s \in L^2_{p,\bullet}(X,F) \mid \exists! w \in L^2_{p,\bullet}(X,F),\ \langle \overline{\partial}^F_{\max} v, s\rangle = \langle v, w\rangle \text{ for } \forall v \in \mathrm{Dom}(\overline{\partial}^F_{\max})\}. \end{aligned} \tag{2.2.35}$$

**Definition 2.18.** The Kodaira Laplacian operator on $\Omega^{p,q}_0(X,F)$ is defined by

$$\Box^F = \overline{\partial}^{F*}\overline{\partial}^F + \overline{\partial}^{F*}\overline{\partial}^F \tag{2.2.36}$$

It is clear that $\Box^F$ is a densely defined, positive operator on $L^2_{p,q}(X,F)$, which is by $L^2_{p,q}(X,F) = \overline{\Omega^{p,q}_0(X,F)}$ in the $L^2$-norm and $\langle \Box^F s, s\rangle \geq 0$ for $s \in \Omega^{p,q}_0(X,F)$.

We describe now a self-adjoint extension of $\Box^F$ for $L^2$-cohomology, called the Gaffney extension. For simplifying the notations, we still denote the maximal extension $\overline{\partial}^F_{\max}$ by $\overline{\partial}^F$ and the Hilbert space adjoint $(\overline{\partial}^F_{\max})^*_H$ by $\overline{\partial}^{F*}$. Consider the complex of closed, densely defined operators

$$L^2_{p,q-1}(X,F) \xrightarrow{\overline{\partial}^F} L^2_{p,q}(X,F) \xrightarrow{\overline{\partial}^F} L^2_{p,q+1}(X,F) \tag{2.2.37}$$

Here $(\overline{\partial}^F)^2 = 0$ by $\langle (\overline{\partial}^F)^2 s, v\rangle = \langle s, (\overline{\partial}^{F*})^2 v\rangle = 0$ for any $v \in \Omega^{p,q+1}_0(X,F)$ and $s \in \mathrm{Dom}(\overline{\partial}^F) \cap L^2_{p,q-1}(X,F)$.

The following proposition is from [30, Propersition.3.1.2].

**Proposition 2.19.** *(cf. [30, Propersition.3.1.2])*

*The operator defined by*

$$\begin{aligned} \mathrm{Dom}(\Box^F) &= \{s \in \mathrm{Dom}(\overline{\partial}^F) \cap \mathrm{Dom}(\overline{\partial}^{F*}) : \Box^F s \in \mathrm{Dom}(\overline{\partial}^{F*}),\ \overline{\partial}^{F*} s \in \mathrm{Dom}(\Box^F)\}, \\ \Box^F s &= \overline{\partial}^{F*}\overline{\partial}^F s + \overline{\partial}^{F*}\overline{\partial}^F s \;\; \textit{for } s \in \mathrm{Dom}(\Box^F), \end{aligned} \tag{2.2.38}$$

*is a positive, self-adjoint extension of Kodaira Laplacian, called the Gaffney extension. The quadratic form associated to* $\Box^F$ *is the form* $Q$ *given by*

$$\begin{aligned} \mathrm{Dom}(Q) &= \mathrm{Dom}(\overline{\partial}^F) \cap \mathrm{Dom}(\overline{\partial}^{F*}), \\ Q(s_1, s_2) &= (\overline{\partial}^F s, \overline{\partial}^F s) + (\overline{\partial}^{F*} s, \overline{\partial}^{F*} s) \;\; \textit{for } s_1, s_2 \in \mathrm{Dom}(Q). \end{aligned} \tag{2.2.39}$$

**Remark 2.20.** (cf. [30, Propersition C.1.4])

The associated quadratic form $Q$ to $\square^F$ (a positive self-adjoint operator) satisfies that

$$\begin{aligned} & \mathrm{Dom}(\square^F) \\ = \quad & \{s \in \mathrm{Dom}(Q) : \exists v \in L^2_{p,q}(X,F),\ Q(s,w) = \langle v, w\rangle \ for\ any\ w \in \mathrm{Dom}(Q)\}, \\ \square^F s \quad & = v \quad for\ s \in \mathrm{Dom}(\square^F). \end{aligned} \tag{2.2.40}$$

Thus for $s_1 \in \mathrm{Dom}(\square^F) \subset \mathrm{Dom}(Q)$ and $s_2 \in \mathrm{Dom}(Q)$,

$$Q(s_1, s_2) = (\square^F s_1, s_2) = (\overline{\partial}^F s_1, \overline{\partial}^F s_2) + (\overline{\partial}^{F*} s_1, \overline{\partial}^{F*} s_2). \tag{2.2.41}$$

**Definition 2.21.** The space of harmonic forms $\mathcal{H}^{(p,q)}(X,F)$ is defined by

$$\mathcal{H}^{p,q}(X,F) := \mathrm{Ker}(\square^F) = \{s \in \mathrm{Dom}(\square^F) : \square^F s = 0\}. \tag{2.2.42}$$

The $q$-th reduced $L^2$-Dolbeault cohomology is defined by

$$\overline{H}^{0,q}_{(2)}(X,F) := \frac{\mathrm{Ker}(\overline{\partial}^F) \cap L^2_{0,q}(X,F)}{[\mathrm{Im}(\overline{\partial}^F) \cap L^2_{0,q}(X,F)]}, \tag{2.2.43}$$

where $[V]$ denotes the closure of the space $V$.

**Remark 2.22.** According to the general regularity theorem of differential operators (also see [30, Theorem A.3.4]), $s \in \mathcal{H}^{p,q}(X,F)$ implies $s \in \Omega^{p,q}(X,F)$. Thus (2.2.42) becomes

$$\mathcal{H}^{p,q}(X,F) = \{s \in \Omega^{p,q}(X,F) \cap \mathrm{Dom}(\square^F) : \square^F s = 0\} \subset \Omega^{p,q}(X,F) \cap L^2_{p,q}(X,F). \tag{2.2.44}$$

Since $\mathcal{H}^{n,q}(X, L^k \otimes E)$ is separable, let $\{s^k_j\}_{j\geq 1}$ be an orthonormal basis.

**Definition 2.23.** The Bergman density function $B^q_k$ is defined by

$$B^q_k(x) = \sum_{j=1}^{\infty} |s^k_j(x)|^2_{h_k,\omega}\,,\ x \in X,$$

where $|\cdot|_{h_k,\omega}$ is the pointwise norm of a form.

The Bergman kernel function defined in (2.1.1) is well-defined by an adaptation of [9, Lemma 3.1]. By weak Hodge decomposition, we have a canonical isomorphism as follows (see [30, (3.1.22)]).

**Proposition 2.24.**

$$\overline{H}^{0,q}_{(2)}(X,F) = \mathcal{H}^{0,q}(X,F) \tag{2.2.45}$$

*for any* $q \in \mathbb{N}$.

*Proof.* By (2.2.41), we see that

$$\mathcal{H}^{0,q}(X,F) = \mathrm{Ker}(\overline{\partial}^F) \cap \mathrm{Ker}(\overline{\partial}^{F*}). \tag{2.2.46}$$

Combining with the complex sequence (2.2.37) and the following

$$L^2_{0,q+1}(X,F) \xrightarrow{\overline{\partial}^{F*}} L^2_{0,q}(X,F) \xrightarrow{\overline{\partial}^{F*}} L^2_{0,q-1}(X,F), \tag{2.2.47}$$

we have

$$\begin{aligned} \mathrm{Im}(\overline{\partial}^F)^\perp &= \mathrm{Ker}(\overline{\partial}^{F*}) = \mathrm{Ker}(\overline{\partial}^{F*}) \cap L^2_{0,q}(X,F) \\ &= \mathrm{Ker}(\overline{\partial}^{F*}) \cap (\mathrm{Ker}(\overline{\partial}^F) \oplus \mathrm{Ker}(\overline{\partial}^F)^\perp) \\ &= (\mathrm{Ker}(\overline{\partial}^{F*}) \cap \mathrm{Ker}(\overline{\partial}^F)) \oplus (\mathrm{Ker}(\overline{\partial}^{F*}) \cap \mathrm{Ker}(\overline{\partial}^F)^\perp). \end{aligned} \tag{2.2.48}$$

Since $\mathrm{Ker}(\overline{\partial}^F)^\perp = [\mathrm{Im}(\overline{\partial}^{F*})]$, $(\overline{\partial}^F)^2 = 0$, then $[\mathrm{Im}(\overline{\partial}^{F*})] \subset \mathrm{Ker}(\overline{\partial}^{F*})$ and $\mathrm{Ker}(\overline{\partial}^{F*}) \cap \mathrm{Ker}(\overline{\partial}^F)^\perp = [\mathrm{Im}(\overline{\partial}^{F*})]$. Combining with (2.2.47) and (2.2.48), we see

$$\mathrm{Im}(\overline{\partial}^F)^\perp = \mathcal{H}^{(0,q)}(X,F) \oplus [\mathrm{Im}(\overline{\partial}^{F*})]. \tag{2.2.49}$$

Likewise, by $\mathrm{Im}(\overline{\partial}^F)^\perp = \mathrm{Ker}(\overline{\partial}^{F*})$,

$$\begin{aligned} \mathrm{Ker}(\overline{\partial}^F) &= \mathrm{Ker}(\overline{\partial}^F) \cap L^2_{0,q}(X,F) \\ &= (\mathrm{Ker}(\overline{\partial}^F) \cap \mathrm{Ker}(\overline{\partial}^{F*})) \oplus (\mathrm{Ker}(\overline{\partial}^F) \cap [\mathrm{Im}\overline{\partial}^F]) \\ &= \mathcal{H}^{(0,q)}(X,F) \oplus [\mathrm{Im}(\overline{\partial}^F)]. \end{aligned} \tag{2.2.50}$$

Form (2.2.49),

$$L^2_{0,q}(X,F) = \mathrm{Im}(\overline{\partial}^F)^\perp \oplus [\mathrm{Im}(\overline{\partial}^F)] = \mathcal{H}^{(0,q)}(X,F) \oplus [\mathrm{Im}(\overline{\partial}^{F*})] \oplus [\mathrm{Im}(\overline{\partial}^F)]. \tag{2.2.51}$$

From (2.2.50) and (2.2.51), we have

$$\mathcal{H}^{(0,q)}(X,F) = \frac{\mathrm{Ker}(\overline{\partial}^F)}{[\mathrm{Im}(\overline{\partial}^F)]} = \frac{\mathrm{Ker}(\overline{\partial}^F) \cap L^2_{0,q}(X,F)}{[\mathrm{Im}(\overline{\partial}^F)] \cap L^2_{0,q}(X,F)} = \overline{H}^{(0,q)}_{(2)}(X,F) \tag{2.2.52}$$

□

**Remark 2.25.** Similarly, we can define the maximal extension of $\overline{\partial}^{F*}$ and denote $(\overline{\partial}^{F*})_{\max}$. Now we have two type adjoint operator : $(\overline{\partial}^{F*})_{\max}$ and $(\overline{\partial}^F_{\max})^*_H$ induced by the initial differential operator $\overline{\partial}^F$ and $L^2$-scalar product (2.2.33). In general, they are not equal.

**Remark 2.26.** (cf. [30, Corollary 3.3.3]) If $g$ is a complete metric on $X$, then the two type adjoint operators are equal, i.e., $(\overline{\partial}^{F*})_{\max} = (\overline{\partial}^{F}_{\max})^*_H$, and the Gaffney extension and Fridrichs extension coincide for $\square^F$ by [30, Corollary 3.3.4]. In this case, if we denote the maximal extension $R_{\max}$ by $R$, where $R$ is $\overline{\partial}^F$ and $\overline{\partial}^{F*}$. Then we have for $s_1 \in \mathrm{Dom}(\overline{\partial}^F)$ and $s_2 \in \mathrm{Dom}(\overline{\partial}^{F*})$

$$\begin{aligned} \overline{\partial}^{F*} &= \overline{\partial}^{F*}_H, && (2.2.53)\\ \langle \overline{\partial}^F s_1, s_2\rangle &= \langle s_1, \overline{\partial}^{F*} s_2\rangle. && (2.2.54) \end{aligned}$$

In particular, we will see in the section 2.2.3 for a covering manifold $(X, \omega, \Gamma)$, the Riemannian metric $g$ is complete, which is from the compactness of $X/\Gamma$ and $g = \pi_\Gamma^* g^{T(X/\Gamma)}$.

### 2.2.3 Covering manifolds and von Neuman dimension ($\Gamma$-dimension)

Let $(X, J)$ be a (paracompact) complex manifold of dimension $n$ with a compatible Riemannian metric $g$. Let $\omega$ be the associated real $(1,1)$-form defined by $\omega(X, Y) = g(JX, Y)$ on $TX$. Then $(X, \omega)$ is a Hermitian manifold.

**Definition 2.27.** A group $\Gamma$ is called a discrete group acting holomorphically, freely and properly on $X$, if $\Gamma$ is equipped with the discrete topology such that
(1) the map $\Omega \times X \to X, (r, x) \mapsto r.x$ is holomorphic,
(2) $r.x = x$ for some $x \in X$ implies that $r = e$ the unit element of $\Gamma$, and
(3) the map $\Omega \times X \to X$ is proper.

**Definition 2.28.** $g$ (or $\omega$) is called $\Gamma$-equivariant, if the map $r : X \to X$ is an isometric with respect to $g$ for every $r \in \Gamma$ .

**Definition 2.29.** We say a Hermitian manifold $(X, \omega)$ is a covering manifold, if there exists a discrete group $\Gamma$ acting holomorpically, freely and properly on $X$ such that $\omega$ is $\Gamma$-equivariant and the quotient $X/\Gamma$ is compact.

In this section, $\Gamma$ is a discrete group acting holomorpically, freely and properly on a Hermitian manifold $(X, \omega)$ such that $g$ is $\Gamma$-equivariant and the quotient $X/\Gamma$ is compact. Let $X$ be paracompact so that $\Gamma$ will be countable. We denote the canonical projection by $\pi_\Gamma : X \to X/\Gamma$. Then $g$ is complete due to the compactness of $X/\Gamma$ and $g = \pi_\Gamma^* g^{T(X/\Gamma)}$.

**Definition 2.30.** An relatively compact open set U $\subset X$ is called a fundamental domain of the action $\Gamma$ on $X$, if the following conditions are satisfied:
(a) $X = \cup_{r\in\Gamma} r(\overline{U})$,
(b) $r_1(U) \cap r_2(U)$ is empty for $r_1, r_2 \in \Gamma, r_1 \neq r_2$, and
(c) $\overline{U} \setminus U$ has zero measure.

The fundamental domain exists, and we can construct one in the following way. Let $\{U_k\}$ be a finite cover of $X/\Gamma$ with open balls having the property that for each $k$, there exists an open set $\widetilde{U}_k \subset X$ such that $\pi_\Gamma : \widetilde{U}_k \to U_k$ is biholomorpic with inverse map $\phi_k : U_k \to \widetilde{U}_k$. Define $W_k = U_k \setminus (\cup_{j<k}\overline{U}_j \cap U_k)$. Then $U := \cup_k \phi_k(W_k)$ is a fundamental domain, see [30].

**Definition 2.31.** A holomrphic Hermitian vector bundle $(F, h^F)$ over $X$ is called $\Gamma$-invariant, if there is a map $r_F : F \to F$ associated to every $r : X \to X \in \Gamma$, which commutes with the fibre projection $\pi : F \to X$ (i.e., $r \circ \pi = \pi \circ r_F$), such that $h^F(v, w) = h^F(r_F v, r_F w)$ for any $v, w \in F$.

Next we introduce some definitions and propositions on $\Gamma$-dimension on covering manifolds, see [39] for details.

Let $\Gamma$ be a discrete group with the neutral element $e$. Let

$$L^2\Gamma := \left\{ f \mid f : \Gamma \to \mathbb{C}, \sum_{r\in\Gamma} |f(r)|^2 < \infty \right\}. \tag{2.2.55}$$

This is a Hilbert space with the scalar product

$$(f, g) := \sum_{r\in\Gamma} f(r)\overline{g(r)}, \quad \forall f, g \in L^2\Gamma. \tag{2.2.56}$$

It has an orthonormal basis $\{\delta_r | r \in \Gamma\}$, where

$$\delta_r(x) = \begin{cases} 1, \; if \; x = r \\ 0, \; if \; x \neq r \end{cases} \tag{2.2.57}$$

There are two natural unitary representations of $\Gamma$ in $L^2\Gamma$: Left regular representation $\Gamma \to U(L^2\Gamma)$, $\; r \mapsto L_r$ and Right regular representation $\Gamma \to U(L^2\Gamma)$, $\; r \mapsto R_r$, where $U(L^2\Gamma) = \{A \in \mathcal{L}(L^2\Gamma) : AA^* = A^*A = 1\}$ is the set of all unitary operator on $L^2\Gamma$, and

$$(L_r f)(x) = f(r^{-1}x), \quad (R_r f)(x) = f(xr), \quad r \in \Gamma, \; f \in L^2\Gamma. \tag{2.2.58}$$

By $(L_{r^{-1}} L_r f)(x) = (L_r f)(rx) = f(x)$, and

$$(L_r f, g) = \sum_{x\in\Gamma} f(r^{-1}x)\overline{g(x)} = \sum_{x\in\Gamma} f(x)\overline{g(rx)} = (f, L_{r^{-1}} g),$$

we obtain

$$L_r^* = (L_r)^{-1} = L_{r^{-1}}, \quad R_r^* = (R_r)^{-1} = R_{r^{-1}} \tag{2.2.59}$$

Let $\mathcal{L}_\Gamma$ (resp. $\mathcal{R}_\Gamma$) be the von Neumann algebra generated by $\{L_r | r \in \Gamma\}$ ( resp. $\{R_r | r \in \Gamma\}$). This is simply a weak closure of the set of all finite linear combinations of $L_r$ (resp. $R_r$).

The following lemma is from [39, 1.A.] and [13, Part I, ch.9].

**Lemma 2.32.** *(cf. [13, Part I, ch.9])*

$$\begin{aligned} \mathcal{R}_\Gamma &= \{B \in \mathcal{L}(L^2\Gamma) | BA = AB \ for \ A \in \mathcal{L}_\Gamma\}, \\ \mathcal{L}_\Gamma &= \{B \in \mathcal{L}(L^2\Gamma) | BA = AB \ for \ A \in \mathcal{R}_\Gamma\}. \end{aligned} \tag{2.2.60}$$

By the definition in (2.2.56), if $B = \sum_{r\in\Gamma} c_r R_r \in \mathcal{R}_\Gamma$, then $(B\delta_x, \delta_x) = c_e$ for any $x \in \Gamma$. We can introduce a trace

$$t_r A := (A\delta_e, \delta_e), \quad for \ A \in \mathcal{R}. \tag{2.2.61}$$

Consider the Hilbert space $(L^2\Gamma \otimes \mathcal{H}, (\cdot,\cdot))$ where $\mathcal{H}$ is a complex Hilbert space associated with an orthonormal basis $\{h_j\}_{j\in J}$, then $\{\delta_r \otimes h_j\}$ is an orthonormal basis of $L_2\Gamma \otimes \mathcal{H}$. Thus as before, we have two unitary representations: $\Gamma \to U(L^2\Gamma \otimes Id)$, $r \mapsto L_r \otimes Id$ and $\Gamma \to U(L^2\Gamma \otimes Id)$, $r \mapsto R_r \otimes Id$. Let $\mathcal{L}_\Gamma \otimes Id$ (resp. $\mathcal{R}_\Gamma \otimes Id$) be the von Neuman algebra generated by $\{L_r \otimes Id | r \in \Gamma\}$ (resp. $\{R_r \otimes Id | r \in \Gamma\}$).

According to Lemma 2.32, we define

$$\mathcal{A}_\Gamma := \mathcal{R}_\Gamma \otimes \mathcal{L}(\mathcal{H}) = \{A \in \mathcal{L}(L^2\Gamma \otimes \mathcal{H}) | AB = BA \ for \ B \in \mathcal{L}_\Gamma \otimes Id\}. \tag{2.2.62}$$

**Definition 2.33.**

$$Tr_\Gamma[A] := (t_r \otimes T_r)A \tag{2.2.63}$$

where $A \in \mathcal{A}_\Gamma$ and $T_r$ is the usual trace on $\mathcal{L}(\mathcal{H})$.

**Definition 2.34.** A subspace $V \subset L^2\Gamma \otimes \mathcal{H}$ is call a $\Gamma$-module, if $(L_r \otimes Id)V \subset V$ for all $r \in \Gamma$ (i.e $V$ is left $\Gamma$-invariant).

For example, $L^2\Gamma \otimes \mathcal{H}$ is a $\Gamma$-module trivially.

**Proposition 2.35.** *$V \subset L^2\Gamma \otimes \mathcal{H}$ is a $\Gamma$-module if and only if the orthogonal projection $P_V : L^2\Gamma \otimes \mathcal{H} \to V \in \mathcal{A}_\Gamma$*

*Proof.* Assume $P_V \in \mathcal{A}_\Gamma$, thus $P_V(L_r \otimes Id) = (L_r \otimes Id)P_V$ for any $r \in \Gamma$, and for any $v \in V$, $(L_r \otimes Id)v = (L_r \otimes Id)P_V v = P_V(L_r \otimes Id)v \in V$. That is, $(L_r \otimes Id)V \subset V$. Conversely, $P_V$ satisfies $P_V^2 = P_V$ and $P_V = P_V^*$, then for any $w \in L^2\Gamma \otimes \mathcal{H}$, it can be decomposed as $w = w_1 \oplus w_2$, where $w_1 \in V, P_V w_2 = 0$. By the assumption, for any $v \in V$, $(L_{r^{-1}} \otimes Id)v \in V$, thus $((L_r \otimes Id)w_2, v) = (w_2, (L_{r^{-1}} \otimes Id)v) = 0$. Hence $(L_r \otimes Id)w_2 \perp V$, $P_V(L_r \otimes Id)w_2 = 0$ and $P_V(L_r \otimes Id)w = P_V(L_r \otimes Id)w_1 = (L_r \otimes Id)w_1 = (L_r \otimes Id)P_V w_1 = (L_r \otimes Id)P_V w$. □

Now assume $P_V \in \mathcal{A}_\Gamma$ (i.e. $V$ is a $\Gamma$-module), let $\{s_k\}$ be an orthonormal basis of $V$ represented by

$$s_k = \sum_{x\in\Gamma, j\in J} s_k^{xj} \delta_x \otimes h_j, \tag{2.2.64}$$

where $\{\delta_x \otimes h_j\}$ is the orthonormal basis of $L^2\Gamma \otimes \mathcal{H}$ and $s_k^{xj} \in \mathbb{C}$. Consider the projection

$$\begin{aligned} V &\to \delta_e \otimes \mathcal{H} \simeq \mathcal{H} && (2.2.65)\\ s_k &\mapsto s_k(e) := (s_k, \delta_e \otimes h_j)\delta_e \otimes h_j = s_k^{ej}\delta_e \otimes h_j \simeq s_k^{ej} h_j =: s_k|_{\mathcal{H}} \end{aligned}$$

For example, later we will see, if $f \in V \subset L^2(X,F) \simeq L^2\Gamma \otimes L^2(U,F)$, then $f(e) \in L^2(X,F)$ can be consider as $f(e)(x) = f(x)$ when $x \in U$ and $f(e)(x) = 0$ when $x \in X - U$.

Hence it follows (2.2.64) that $P_V$ can be written by

$$\begin{aligned} L^2\Gamma \otimes \mathcal{H} &\to V && (2.2.66)\\ f \otimes h &\mapsto P_V(f \otimes h) = \sum_k (f \otimes h, s_k)s_k = \sum_k (f, \delta_y)(h, h_i)\overline{s_k^{yi}} s_k^{xj} \delta_x \otimes h_j. \end{aligned}$$

**Definition 2.36.** The $\Gamma$-dimension of a $\Gamma$-modula $V \subset L^2\Gamma \otimes \mathcal{H}$ is

$$\dim_\Gamma V := Tr_\Gamma[P_V] \tag{2.2.67}$$

Then, (2.2.63)-(2.2.67) imply a useful formula

$$\begin{aligned} \dim_\Gamma V &:= Tr_\Gamma[P_V] = (t_r \otimes T_r)P_V = \sum_j (P_V(\delta_e \otimes h_j), \delta_e \otimes h_j) = \sum_{j,k} |s_k^{ej}|^2 \\ &= \sum_k (s_k(e), s_k(e)). && (2.2.68) \end{aligned}$$

**Proposition 2.37.** *Assume* $\Gamma = \{e\}$ *is trivial. Then*
*(a) Any subspace* $V$ *of* $\mathcal{H}$ *is a* $\Gamma$*-module,*
*(b) the* $\Gamma$*-dimension of* $V$ *and the usual dimension coincide:*

$$\dim_\Gamma V = \dim_{\mathbb{C}} V. \tag{2.2.69}$$

*Proof.* $L^2\Gamma \otimes \mathcal{H} = \mathbb{C}\delta_e \otimes \mathcal{H} \simeq \mathcal{H}$, if $\Gamma = \{e\}$. $(L_e \otimes \mathrm{Id})V = V$, then $V$ is $\Gamma$-module. And by (2.2.65) and (2.2.3), we have $s_k(e) = s_k$ and thus $\dim_\Gamma V = \dim_{\mathbb{C}} V$. ☐

**Proposition 2.38.**

$$\dim_\Gamma L^2\Gamma \otimes \mathcal{H} = \dim_{\mathbb{C}} \mathcal{H} \tag{2.2.70}$$

*Proof.* $\{\delta_x \otimes h_k\}$ is the orthonormal basis of $L^2\Gamma \otimes \mathcal{H}$, then $(\delta_x \otimes h_k)(e) = \delta_e \otimes h_k$ when $x = e$, otherwise it is zero. Hence $\dim_\Gamma L^2\Gamma \otimes \mathcal{H} = \sum_k (\delta_e \otimes h_k, \delta_e \otimes h_k) = \sum_k (h_k, h_k) = \dim_{\mathbb{C}} \mathcal{H}$. ☐

As a special case, we set the above Hilbert space $\mathcal{H}$ to be $L^2(U,F)$, and focus on $L^2(X,F) \simeq L^2\Gamma \otimes L^2(U,F)$, where $U \subset X$ is the fundamental domain, $F$ is a $\Gamma$-invariant holomorphic Hermitian vector bundle, and the $L^2$-space $L^2(X,F)$ is given by $F, X$ in the usual way.

The following lemma is from [30, Lemma 3.6.2].

**Lemma 2.39.** *Let $V \subset L^2(X,F)$ be a $\Gamma$-modula, then*

$$\dim_\Gamma V = \sum_i \int_U |s_i(x)|^2 dv_X(x), \tag{2.2.71}$$

*where $\{s_i\}$ is an orthonormal basis of $V$. Moreover, here the domain $U$ can be replace by $\overline{U}$.*

*Proof.* $s_i \in V \subset L^2(X,F) \simeq L^2\Gamma \otimes L^2(U,F)$, then $s_i \simeq (s_i|_{rU})_{r\in\Gamma}$. And $s_i(e) = s_i|_U \in L^2(X,F)$ can be considered as $s_i(e)(x) = s_i(x)$ when $x \in U$ and $s_i(e)(x) = 0$ when $x \in X - U$. By (2.2.3), we have

$$\dim_\Gamma V = \sum_i (s_i(e), s_i(e)) = \sum_i \int_X |s_i(e)(x)|^2 dv_X(x) = \sum_i \int_U |s_i(x)|^2 dv_X(x). \tag{2.2.72}$$

Moreover, notice $s_i(e)(x) = 0$, when $x$ is in the boundary of $U$. □

Finally we combine these facts on $\Gamma$-dimension and reduced $L^2$-cohomology.

Let $(X, \omega)$ be a Hermitian manifold of dimension $n$ on which a discrete group $\Gamma$ acts holomorphically, freely and properly, such that $\omega$ is a $\Gamma$-invariant, the quotient $X = X$ is compact and $X$ is paracompact so that $\Gamma$ will be countable. Let $U \subset X$ be a fundamental domain such that $\overline{U}$ is compact. Moreover, suppose $(F, h^F)$ is a $\Gamma$-invariant holomorphic Hermitian vector bundle on $X$. Let $\Box := \Box^F$ be the Gaffney self-adjoint extension of the Kodaira Laplacian.

As in the proof of (2.2.71), let $L_r \otimes \mathrm{Id}$ be the left $\Gamma$-action on $L^2_{p,q}(X,F) \simeq L^2\Gamma \otimes L^2_{p,q}(U,F)$, then any $s \in L^2_{p,q}(X,F) \simeq L^2\Gamma \otimes L^2_{p,q}(U,F)$, then $s \simeq (s|_{rU})_{r\in\Gamma}$, and $s(r) = s|_{rU} \in L^2_{p,q}(X,F)$ can be considered as $s(r)(x) = s(x)$ when $x \in rU$ and $s(r)(x) = 0$ when $x \in X - rU$.

The following lemma is from [30, Lemma 3.6.3].

**Lemma 2.40.** *$\mathcal{H}^{(p,q)}(X,F)$ is a $\Gamma$-modula in $L^2_{p,q}(X,F)$.*

*Proof.* We only need to prove that $\Box^F s = 0$ implies $\Box^F(L_r \otimes \mathrm{Id})s = 0$. Assume $s = (s_{gU})_{g\in\Gamma} = \sum_{g\in\Gamma} \delta_g \otimes s_{gU} \in \mathrm{Ker}(\Box^F) = \mathcal{H}^{(0,q)}(X,F)$, then $0 = \Box^F s = (\Box^F s_{gU})_{g\in\Gamma}$, and $\Box^F s_{gU} = 0$ for any $g \in \Gamma$. By $(L_r \otimes \mathrm{Id})s = \sum_{g\in\Gamma} \delta_{rg} \otimes s_{rgU} = (s_{rgU})_{g\in\Gamma}$, we have $\Box^F(L_r \otimes \mathrm{Id})s = (\Box^F s_{rgU})_{g\in\Gamma} = 0$. □

**Lemma 2.41.**

$$\dim_\Gamma \mathcal{H}^{(p,q)}(X,F) = \sum_i \int_U |s_i(x)|^2 dv_X(x) \tag{2.2.73}$$

*where $\{s_i\}$ is an orthonormal basis of $\mathcal{H}^{(p,q)}(X,F)$ with respect to the scalar product in $L^2_{(p,q)}(X,F)$. In particular, $\dim_\Gamma \overline{H}^{(0,q)}_{(2)}(X,F) = \dim_\Gamma \mathcal{H}^{(0,q)}(X,F)$.*

*Proof.* Combining the (2.2.71), (2.2.45) and notice that $\mathcal{H}^{(p,q)}(X,F)$ is a $\Gamma$-modula in $L^2_{p,q}(X,F)$, we have

$$\dim_\Gamma \overline{H}^{(p,q)}_{(2)}(X,F) = \dim_\Gamma \mathcal{H}^{(p,q)}(X,F) = \sum_i \int_U |s_i(x)|^2 dv_X(x) \tag{2.2.74}$$

for $\{s_i\}$ is an orthonormal basis of $\mathcal{H}^{(p,q)}(X,F)$. □

**Remark 2.42.** In this chapter, we focus on the estimate of the right side of (2.2.73) when $p = n$.

## 2.3 Some properties of harmonic line bundle valued forms

In this section, we work under the following general setting, and later the covering manifold with a group action $\Gamma$ will be treated as a special case in the section 2.4.2.

Let $(X,\omega)$ be a Hermitian manifold of dimension $n$ and $(F,h^F)$ be a holomorphic Hermitian line bundle on $X$. For the Kodaira Laplacian $\Box := \Box^F$ we denote still by $\Box$ its (Gaffney) self-adjoint extension.

### 2.3.1 The $\partial\overline{\partial}$-Bochner formula for non-compact manifolds

By the local representation of forms in the section 2.2.1, we use the following notations instead of those in the section 2.2.2 as follows. Let $\alpha \in \Omega^{p,q}(X,F)$. Let $U$ be an open set such that $F|_U$ is trivial and let $e_F$ be a local holomorphic frame on $U$ and set $|e_F|^2_{h^F} = e^{-\psi}$. We can write $\alpha|_U = \xi \otimes e_F$ with $\xi \in \Omega^{n,q}(U,\mathbb{C})$.

For simplifying the notations, we still denote the maximal extension $\overline{\partial}^F_{\max}$ by $\overline{\partial}^F$ and the Hilbert space adjoint $(\overline{\partial}^F_{\max})^*_H$ by $\overline{\partial}^{F*}$. Moreover, we can rephrase

$$\overline{\partial} := \overline{\partial}^F \quad \text{on} \quad \mathrm{Dom}(\overline{\partial}^F) \cap \Omega^{p,q}(X,F) \tag{2.3.1}$$

$$\overline{\partial}^*_\psi := \overline{\partial}^{F*} \quad \text{on} \quad \mathrm{Dom}(\overline{\partial}^{F*}) \cap \Omega^{p,q}(X,F), \tag{2.3.2}$$

where $\psi$ is from the Hermitian metric on $F$ as above.

Then the Kodaira Laplacian becomes

$$\Box := \Box^F := \overline{\partial}\overline{\partial}^*_\psi + \overline{\partial}^*_\psi\overline{\partial} \quad \text{on} \quad \mathrm{Dom}(\Box^F) \cap \Omega^{p,q}(X,F). \tag{2.3.3}$$

Let $\{U_i\}$ be a covering of $X$ such that $F|_{U_i}$ is trivial. Let $s \in \Omega^{p,q}(X,F)$. Then $s|_{U_i} = s_i \otimes e_i$, where $s_i$ is a local $(p,q)$-form on $U_i$ and $e_i$ is a local holomorphic frame of $F|_{U_i}$. Then the operator $\overline{\partial}$ can be represented by

$$\overline{\partial}s|_{U_i} = (\overline{\partial}s_i) \otimes e_i, \tag{2.3.4}$$

which is globally well-defined by (2.2.18) and (2.2.11).

For $u \in \mathrm{Dom}(\overline{\partial}^F) \cap \Omega^{p,q}(X,F)$ and $v \in \mathrm{Dom}(\overline{\partial}^{F*}) \cap \Omega^{p,q}(X,F)$,

$$\langle \overline{\partial} u, v\rangle_{L^2} = \langle u, \overline{\partial}^*_\psi v\rangle_{L^2}, \quad \text{i.e.,} \quad \int_X \langle \overline{\partial} u, v\rangle_{h^F,\omega} dv_X = \int_X \langle u, \overline{\partial}^*_\psi v\rangle_{h^F,\omega} dv_X \tag{2.3.5}$$

And we can define a differential operator $\delta$ corresponding to $\overline{\partial}$, which is globally well defined by (2.2.18) and (2.2.11).

**Definition 2.43.** Let $\eta \in \Omega^{p,q}(X,F)$ with $\eta|_{U_i} = \eta_i \otimes e_i$ The differential operator $\delta$ is given by

$$\begin{array}{rcl} \delta & : & \Omega^{p,q}(X,F) \longrightarrow \Omega^{p+1,q}(X,F), \\ (\delta\eta)|_{U_i} & := & (\delta\eta)_i \otimes e_i := (\delta\eta_i) \otimes e_i := (e^{\psi_i}\partial(e^{-\psi_i}\eta_i)) \otimes e_i. \end{array} \tag{2.3.6}$$

Let $\eta \in \Omega^{p,q}(X,F)$ and $\xi \in \Omega^{r,s}(X,F)$. By (2.3.6) and (2.2.24), we have

$$\overline{\partial}(\eta \wedge \overline{\xi} e^{-\psi}) = \overline{\partial}\eta \wedge \overline{\xi} e^{-\psi} + (-1)^{\deg \eta} \eta \wedge \overline{\delta\xi} e^{-\psi}, \tag{2.3.7}$$

which indicates the relation between $\overline{\partial}$ and $\delta$, that is, locally

$$\overline{\partial}(\eta_i \wedge \overline{\xi_i} e^{-\psi_i}) = \overline{\partial}\eta_i \wedge \overline{\xi_i} e^{-\psi_i} + (-1)^{\deg \eta_i} \eta_i \wedge \overline{\delta\xi_i} e^{-\psi_i}.$$

Then we have Chern connection and the curvature with respect to the holomorphic Hermitian line bundle $F$ as follows.

$$\begin{array}{rcl} D & := & \delta + \overline{\partial}, \\ D^2 & = & \delta\overline{\partial} + \overline{\partial}\delta = \partial\overline{\partial}\psi = R^F \end{array} \tag{2.3.8}$$

over $\Omega^{p,q}(X,F)$. And they are also denoted by

$$\begin{array}{rcl} \nabla^F & := & (\nabla^F)^{1,0} + \overline{\partial} \\ R^F & = & (\nabla^F)^2. \end{array} \tag{2.3.9}$$

Then the following proposition indicates the relation between $\overline{\partial}^*_\psi$ and $\delta$.

**Proposition 2.44.** *Let $\alpha \in \mathrm{Dom}(\overline{\partial}^*_\psi) \cap \Omega^{n,q}(X,F)$. Then*

$$\begin{array}{rcl} \gamma_{\overline{\partial}^*_\psi \alpha} & = & (-1)^{n-q}\delta\gamma_\alpha, \\ \overline{\partial}^*_\psi \alpha & = & -\star(\delta\gamma_\alpha), \end{array} \tag{2.3.10}$$

*where $\star(\cdot) = \gamma_\cdot$ is defined by (2.2.29).*

*Proof.* For any $\eta \in \Omega_0^{n,q-1}(X,F) \subset \mathrm{Dom}(\overline{\partial}) \cap \Omega^{n,q-1}(X,F) \subset L^2_{n,q-1}(X,F)$, we have

$$\langle \overline{\partial}\eta, \alpha\rangle_{L^2} = \langle \eta, \overline{\partial}^*_\psi \alpha\rangle_{L^2}, \quad \text{i.e,} \quad \int_X \langle \overline{\partial}\eta, \alpha\rangle dv_X = \int_X \langle \eta, \overline{\partial}^*_\psi \alpha\rangle dv_X \tag{2.3.11}$$

by (2.3.5). According to Proposition 2.14(4), (2.2.24) and (2.2.22), we have

$$\begin{aligned} \langle \overline{\partial}\eta, \alpha\rangle \omega_n &= \overline{\partial}\eta \wedge \overline{\gamma_\alpha} e^{-\psi} \qquad \text{i.e.,} \\ \langle \overline{\partial}\eta_i, \alpha_i\rangle e^{-\psi_i}\omega_n &= \overline{\partial}\eta_i \wedge \overline{\gamma_{\alpha_i}} e^{-\psi_i}, \end{aligned} \tag{2.3.12}$$

where the first $\langle , \rangle$ is induced by $h^F$ and $\omega$, and the second is by $\omega$.

The left side of (2.3.11) equals, by (2.3.12) and (2.3.7),

$$\begin{aligned} \int_X \overline{\partial}\eta \wedge \overline{\gamma_\alpha} e^{-\psi} &= (-1)^{n-q} \int_X \eta \wedge \overline{\delta\gamma_\alpha} e^{-\psi} + \int_X \overline{\partial}(\eta \wedge \overline{\gamma_\alpha} e^{-\psi}) \\ &= (-1)^{n-q} \int_X \eta \wedge \overline{\delta\gamma_\alpha} e^{-\psi}, \end{aligned} \tag{2.3.13}$$

where the last equality is from Stokes' theorem. The right side of (2.3.11) is

$$\int_X \eta \wedge \overline{\gamma_{\overline{\partial}^*_\psi \alpha}} e^{-\psi}$$

by (2.3.12). By combining it with (2.3.13), we see

$$\begin{aligned} \langle \eta, \overline{\partial}^*_\psi \alpha\rangle_{L^2} &= \int_X \eta \wedge \overline{\gamma_{\overline{\partial}^*_\psi \alpha}} e^{-\psi} = (-1)^{n-q} \int_X \eta \wedge \overline{\delta\gamma_\alpha} e^{-\psi} \\ &= \int_X \eta \wedge \overline{\star(-\star\delta\gamma_\alpha)} e^{-\psi} \\ &= \langle \eta, -\star\delta\gamma_\alpha\rangle_{L^2}, \end{aligned} \tag{2.3.14}$$

where we use Proposition 2.14(1) acting on $(n-q+1,0)$-forms. So (2.3.14) leads to the second equality in (2.3.10) by the density. By acting $\gamma = \star$ to the both sides, then we obtain the first equality. □

Next we give a property of harmonic line bundle valued forms in our setting.

**Proposition 2.45.** *Let $\alpha \in \Omega^{n,q}(X, F)$.*

$$(-1)^{n-q}\gamma_{\alpha_i} \wedge \overline{\partial}\omega_q = -\overline{\partial}\gamma_{\alpha_i} \wedge \omega_q, \quad \textit{when} \quad \overline{\partial}\alpha = 0 \tag{2.3.15}$$

$$\delta\gamma_\alpha = 0, \quad \textit{when} \quad \overline{\partial}^*_\psi \alpha = 0 \tag{2.3.16}$$

*In particular, they both verify when $\Box\alpha = 0$.*

*Proof.* The first equation follows that $0 = \overline{\partial}\alpha = (\overline{\partial}\alpha_i) \otimes e_i = \overline{\partial}(C_{n-q}\gamma_i \wedge \omega_q) \otimes e_i$ by Proposition 2.14. And the second one is from (2.3.10). □

The following $\partial\overline{\partial}$-formula was obtained by B. Berndtsson in [5] and [6] for $\overline{\partial}$-closed, line bundle valued, $(n, q)$-forms over compact manifolds. We can rephrase it for $\Box$-closed, line bundle valued, $(n, q)$-forms over any compact subset of a Hermitian (possibly non-compact) manifold. The proof is analogue to [5, Proposition 2.2] and [6, Proposition 6.2].

Let $\alpha \in \mathcal{H}^{n,q}(X,F)$. We define now a positive $(n-q,n-q)$-form $T_\alpha$ on $X$ as before. Let $U$ be an open set such that $F|_U$ is trivial and let $e_F$ be a local holomorphic frame on $U$ and set $|e_F|^2_{h^F} = e^{-\psi}$. We write $\alpha|_U = \xi \otimes e_F$ with $\xi \in \Omega^{n,q}(U,\mathbb{C})$. The $(n-q,n-q)$ form $T_\alpha$ is defined locally by $T_\alpha|_U := i^{(n-q)^2}(\star\xi) \wedge \overline{(\star\xi)}e^{-\psi}$, where $\star : \Omega^{n,q}(U,\mathbb{C}) \to \Omega^{n-q,0}(U,\mathbb{C})$ is the Hodge star operator associated to the metric $\omega$ given by $\xi \wedge \overline{\star\xi} = |\xi|^2_\omega \omega_n$. It is easy to check that $T_\alpha$ is well defined globally.

We have a $F$-valued $(n-q,0)$ form $\gamma_\alpha \in \Omega^{n-q,0}(X,F)$ associated to $\alpha$ defined locally by $\gamma_\alpha|_U := (\star\xi) \otimes e_F$, which is also well defined globally. Let $L := \omega \wedge \cdot$ be the Lefschetz operator on $\Omega^{p,q}(X,F)$ and let $\Lambda$ be its dual operator defined by $\langle \Lambda \cdot, \cdot \rangle_{h^F,\omega} = \langle \cdot, L \cdot \rangle_{h^F,\omega}$.

The curvature form is given by $\Theta_F := \sqrt{-1}R^F = 2\pi c_1(F,h^F)$.

**Theorem 2.46.** *Let $(F,h^F)$ be a holomorphic Hermitian line bundle over a Hermitian manifold $(X,\omega)$. Assume $\alpha \in \mathcal{H}^{n,q}(X,F)$, $q \geq 1$, and $K \subset X$ is a compact subset. Then there exist non-negative constants $C_1$ and $C_2$ depending on $\omega$ and $K$, such that*

$$\begin{aligned} i\partial\overline{\partial}(T_\alpha \wedge \omega_{q-1}) &\geq (\langle \Theta_F \wedge \Lambda\alpha, \alpha\rangle_{h^F,\omega} - C_1|\alpha|^2_{h^F,\omega} + C_2|\overline{\partial}^F\gamma_\alpha|^2_{h^F,\omega})\omega_n \\ &\geq (\langle \Theta_F \wedge \Lambda\alpha, \alpha\rangle_{h^F,\omega} - C_1|\alpha|^2_{h^F,\omega})\omega_n \end{aligned} \tag{2.3.17}$$

*on $K$. Here $\langle \Theta_F \wedge \Lambda\alpha, \alpha\rangle\omega_n = \Theta_F \wedge T_\alpha \wedge \omega_{q-1}$ on $X$.*

*In particular, if $X$ is Kähler, then*

$$\begin{aligned} i\partial\overline{\partial}(T_\alpha \wedge \omega_{q-1}) &= (i\partial\overline{\partial}T_\alpha) \wedge \omega_{q-1} \\ &= (\langle \Theta_F \wedge \Lambda\alpha, \alpha\rangle_{h^F,\omega} + |\overline{\partial}^F\gamma_\alpha|^2_{h^F,\omega})\omega_n \\ &\geq \langle \Theta_F \wedge \Lambda\alpha, \alpha\rangle\omega_n \end{aligned} \tag{2.3.18}$$

*on $X$. Above $\langle\ ,\ \rangle_{h^F,\omega}$ and $|\ |^2_{h^F,\omega}$ denote the pointwise Hermitian metric and norm on $F$-valued differential forms induced by $\omega$ and $h^F$.*

*Proof.* First of all, we fix our notions for further arguments. Let $\alpha \in \mathcal{H}^{(n,q)}(X,F)$. Let $\{U_j\}$ be a covering of $X$ such that $F|_{U_j}$ is trivial. Then $\alpha|_{U_j} = \alpha_j \otimes e_j$ such that $|e_j|^2_{h^F} = e^{-\psi_j}$ and this representation is globally well defined under the trivialization of $F$. Let $C_q := i^{q^2}$ for $q \in \mathbb{N}$. Then $iC_q = (-1)^q C_{q-1}$ and $C_{q-1} = C_{q+1}$. Moreover, we have

$$T_\alpha|_{U_j} = C_{n-q}\gamma_j \wedge \overline{\gamma_j}e^{-\psi_j},$$

where the scalar valued $(n,q)$-form $\gamma_j = \star\alpha_j$. And we know $T := T_\alpha$ is a globally well defined $(n-q,n-q)$-form. Then, we can drop the subscription $j$ of $\gamma$ and $\psi$ in $T$, and denote it by

$$T = C_{n-q}\gamma \wedge \overline{\gamma}e^{-\psi},$$

since our following computation is independent of the choice of $U_j$.

Let $\alpha$, $\beta \in \Omega^{p,q}(X,F)$ such that $\alpha|_{U_j} = \alpha_j \otimes e_j$ and $\beta|_{U_j} = \beta_j \otimes e_j$ as before. Based on the trivialization of $F$, we denote by $\langle \cdot,\cdot \rangle_\psi$ the Hermitian metric $\langle\ ,\ \rangle_{h^F,\omega}$

on $\Omega^{p,q}(X, F)$ and by $\langle \cdot, \cdot \rangle$ the metric $\langle \ , \ \rangle_\omega$ on $\Omega^{p,q}(X, \mathbb{C})$ in this proof. Then, they can be linked by the following formula

$$\langle \alpha, \beta \rangle_\psi |_{U_j} = \langle \alpha_j, \beta_j \rangle e^{-\psi_j}. \tag{2.3.19}$$

Since our computation is independent of the choice of $U_j$, we can simply denote the formula (2.3.19) by dropping the subscription $j$ and $U_j$,

$$\langle \alpha, \beta \rangle_\psi = \langle \alpha, \beta \rangle e^{-\psi}. \tag{2.3.20}$$

Notice that $\langle \alpha, \beta \rangle_\psi \in \mathcal{C}^\infty(X, \mathbb{C})$ for given $\alpha$ and $\beta$, thus we can discuss its value at each point in $X$, in particular, the maximum and minimum of its absolute value on a compact subset $K$ later.

Then, in the spirit of our notions,

$$\begin{aligned} |\alpha|^2_\psi \omega_n &= \langle \alpha, \alpha \rangle e^{-\psi} \\ &= \alpha \wedge \overline{\star \alpha} e^{-\psi} \\ &= C_{n-q} \gamma \wedge \omega_q \wedge \overline{\gamma} e^{-\psi} \\ &= T_\alpha \wedge \omega_q. \end{aligned} \tag{2.3.21}$$

After fixing the notions, we wish to control the $F$-valued $(n, n)$-form $i\partial\overline{\partial}(T \wedge \omega_{q-1})$.

$$\begin{aligned} i\partial\overline{\partial}(T \wedge \omega_{q-1}) &= i\partial\overline{\partial}T \wedge \omega_{q-1} - i\overline{\partial}T \wedge \partial\omega_{q-1} + i\partial T \wedge \overline{\partial}\omega_{q-1} + T \wedge i\partial\overline{\partial}\omega_{q-1} \\ &=: \textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4}. \end{aligned} \tag{2.3.22}$$

Immediately, it follows that the second term conjugates to the third term, i.e,

$$\textcircled{2} = \overline{\textcircled{3}}.$$

Secondly, we estimate the term ①. By (2.3.6) and (2.3.7), we see

$$\begin{aligned} \overline{\partial}(\eta \wedge \overline{\xi} e^{-\psi}) &= \overline{\partial}\eta \wedge \overline{\xi} e^{-\psi} + (-1)^{\deg \eta} \eta \wedge \overline{\delta \xi} e^{-\psi}, \\ \partial(\eta \wedge \overline{\xi} e^{-\psi}) &= \partial\eta \wedge \overline{\xi} e^{-\psi} + (-1)^{\deg \eta} \eta \wedge \delta\overline{\xi} e^{-\psi}, \end{aligned} \tag{2.3.23}$$

where $\eta$, $\xi$ are scalar valued forms in our local representation.

Combining (2.3.23), (2.3.8) and $\delta\gamma = 0$ by (2.3.16), we have

$$\begin{aligned} \partial\overline{\partial}(\gamma \wedge \overline{\gamma} e^{-\psi}) &= \partial(\overline{\partial}\gamma \wedge \overline{\gamma} e^{-\psi}) \\ &= \partial(\overline{\gamma} \wedge \overline{\partial}\gamma e^{-\psi}) \\ &= \partial\overline{\gamma} \wedge \overline{\partial}\gamma e^{-\psi} + (-1)^{n-q} \overline{\gamma} \wedge \delta(\overline{\partial}\gamma) e^{-\psi} \\ &= (-1)^{n-q+1} \overline{\partial}\gamma \wedge \overline{\overline{\partial}\gamma} e^{-\psi} + \delta(\overline{\partial}\gamma) \wedge \overline{\gamma} e^{-\psi} \\ &= (-1)^{n-q+1} \overline{\partial}\gamma \wedge \overline{\overline{\partial}\gamma} e^{-\psi} + \partial\overline{\partial}\psi \wedge \gamma \wedge \overline{\gamma} e^{-\psi}. \end{aligned} \tag{2.3.24}$$

Then, by (2.3.24) and the definition of $T$,

$$\begin{aligned}
\text{①} &= iC_{n-q}\omega_{q-1}\wedge\partial\overline{\partial}(\gamma\wedge\overline{\gamma}e^{-\psi}) \\
&= i\partial\overline{\partial}\psi\wedge C_{n-q}\wedge\gamma\wedge\overline{\gamma}e^{-\psi}\omega_{q-1}+iC_{n-q}\omega_{q-1}\wedge(-1)^{n-q+1}\overline{\partial}\gamma\wedge\overline{\overline{\partial}\gamma}e^{-\psi} \\
&= \Theta_F\wedge T\wedge\omega_{q-1}+iC_{n-q}(-1)^{n-q+1}\overline{\partial}\gamma\wedge\overline{\overline{\partial}\gamma}\omega_{q-1}e^{-\psi} \\
&=: \text{ⓐ}+\text{ⓑ}.
\end{aligned} \tag{2.3.25}$$

We claim that

$$\text{ⓐ}=\langle\Theta_F\wedge\Lambda\alpha,\alpha\rangle_{\psi}\omega_n. \tag{2.3.26}$$

In fact, by [23, Propostion 1.2.31]:

$$\gamma=\star\alpha\in P^{n-q,0}:=\{\gamma\in\wedge^{n-q,0}:\Lambda\gamma=0\}$$

is a primitive $(n-q,0)$-form, implies

$$\star\, L\gamma=\star L\star\alpha=(-1)^{n-q}C_{n-q}\gamma\wedge\omega_{q-1}. \tag{2.3.27}$$

Combining (2.3.27) with Proposition 2.14(1), i.e, $\star^{-1}=(-1)^{(2n-k)k}\star$ on $\wedge^k_{\mathbb{C}}$, then

$$\star^{-1}L\star\alpha=C_{n-q}\gamma\wedge\omega_{q-1}. \tag{2.3.28}$$

We also have the dual Lefschetz operator

$$\Lambda=\star^{-1}L\star$$

by [23, Lemma 1.2.23], then (2.3.28) becomes

$$\Lambda\alpha=C_{n-q}\gamma\wedge\omega_{q-1}, \tag{2.3.29}$$

thus (2.3.24) implies

$$\text{ⓐ}=\langle i\partial\overline{\partial}\psi\wedge C_{n-q}\wedge\gamma\wedge\omega_{q-1},\alpha\rangle_{\psi}\omega_n=\langle\Theta_F\wedge\Lambda\alpha,\alpha\rangle_{\psi}\omega_n. \tag{2.3.30}$$

In order to estimate ⓑ, we need the following two facts in Hermitian and complex structure.

- Hodge-Riemann bilinear relation (cf. [23, Definition 1.2.35,Corllary 1.2.36]): Let $(V^{2n},\langle,\rangle,J)$ be an Euclidean vector space endowed with a compatible almost complex structure. Then the Hodge-Riemann pair $Q$ satisfies

  $$i^{p-q}Q(\chi,\overline{\chi})=(n-(p+q))!|\chi|^2\omega_n \tag{2.3.31}$$

  for $0\neq\chi\in P^{p,q}=\{\chi\in\wedge^{p,q}V^*:\omega^{n-(p+q)+1}\wedge\chi=0\}$ with $p+q\leq n$, where $Q(u,\overline{v}):=(-1)^{\frac{(p+q)(p+q-1)}{2}}u\wedge\overline{v}\wedge\omega^{n-(p+q)}$ for any $u,v\in\wedge^{p,q}V^*$.

- Lefschetz decomposition (cf. [23, Proposition 1.2.30]):

$$\begin{aligned}\wedge^k_{\mathbb{C}}V^* &= \oplus_{i\geq 0}L^i(P^{k-2i}_{\mathbb{C}}) \\ \wedge^{p,q}_{\mathbb{C}}V^* &= \oplus_{i\geq 0}L^i(P^{p-i,q-i}_{\mathbb{C}}),\end{aligned} \tag{2.3.32}$$

  where $\wedge^k_{\mathbb{C}} = \oplus_{p+q=k}\wedge^{p,q}$, $L^i = \omega^i$, $P^k_{\mathbb{C}} = \oplus_{p+q=k}P^{p,q}$, $P^{p,q} := \{\chi \in \wedge^{p,q} : \Lambda\chi = 0\}$ is the space of primitive forms, and $P^{p,q} = \{\chi \in \wedge^{p,q} : \omega^{n-(p+q)+1}\wedge\chi = 0\}$ when $p+q \leq n$.

Now we apply these facts to our case: $V = TX$ at points, $\wedge^{p,q}V^* = \wedge^p TX^{(1,0)*} \otimes \wedge^q TX^{(0,1)*}$, and $\overline{\partial}\gamma \in \wedge^{n-q,1}X \subset \wedge^{n-q+1}_{\mathbb{C}}X$. We obtain

$$\overline{\partial}\gamma = \chi_1 \oplus (\omega\wedge\chi_0) \tag{2.3.33}$$

where $\chi_1 \in P^{n-q,1} = \{\chi : \chi\wedge\omega^q = 0\}$, $\chi_0 \in P^{n-q-1,0} = \wedge^{n-q-1,0}X$. Note that if our manifold is Kähler, then $\overline{\partial}\gamma = \chi_1$. Here $\oplus$ is respect to $Q$, that is,

$$\begin{aligned}Q(\chi_1, \overline{\omega\wedge\chi_0}) &= (-1)^{\frac{(n-q+1)(n-q)}{2}}\chi_1\wedge\overline{\omega\wedge\chi_0}\wedge\omega^{q-1} \\ &= (-1)^{\frac{(n-q+1)(n-q)}{2}}\chi_1\wedge\omega^q\wedge\overline{\chi_0} \\ &= 0.\end{aligned} \tag{2.3.34}$$

And we also know, for $\chi_1 \neq 0$ in $P^{n-q,1}$,

$$i^{n-q-1}Q(\chi_1,\overline{\chi_1}) = (q-1)!|\chi_1|^2\omega_n > 0. \tag{2.3.35}$$

Consider a bilinear form on $(n-q,1)$ forms defined by

$$[\chi,\eta]\omega_n := iC_{n-q}(-1)^{n-q+1}\chi\wedge\overline{\eta}\wedge\omega_{q-1}, \tag{2.3.36}$$

then the relation between $[,]$ and $Q$ is given by

$$[\chi,\eta]\omega_n = \frac{i^{n-q-1}}{(q-1)!}Q(\chi,\overline{\eta}). \tag{2.3.37}$$

It is clear that $[\chi_1, \chi_0\wedge\omega]\omega = 0$ by (2.3.34), and notice (2.3.35) and (2.3.37), then

$$\begin{aligned}\textcircled{b} &= [\overline{\partial}\gamma,\overline{\partial}\gamma]e^{-\psi}\omega_n \\ &= [\chi_1,\chi_1]e^{-\psi}\omega_n + [\omega\wedge\chi_0,\omega\wedge\chi_0]e^{-\psi}\omega_n \\ &= |\chi_1|^2e^{-\psi}\omega_n + [\omega\wedge\chi_0,\omega\wedge\chi_0]e^{-\psi}\omega_n \\ &= |\chi_1|^2_\psi\omega_n + [\omega\wedge\chi_0,\omega\wedge\chi_0]e^{-\psi}\omega_n \\ &= |\overline{\partial}\gamma - \omega\wedge\chi_0|^2_\psi\omega_n + [\omega\wedge\chi_0,\omega\wedge\chi_0]e^{-\psi}\omega_n.\end{aligned} \tag{2.3.38}$$

We claim

$$[\omega\wedge\chi_0,\omega\wedge\chi_0]e^{-\psi}\omega_n \geq -c|\alpha|^2_\psi\omega_n \tag{2.3.39}$$

on $K$, where $c = c(\omega, K) \geq 0$. In fact, $\overline{\partial}\alpha = 0$ implies (2.3.15), and then

$$(\chi_0 \wedge \omega) \wedge \omega_q = (\overline{\partial}\gamma - \chi_1) \wedge \omega_q = \overline{\partial}\gamma \wedge \omega_q = (-1)^{n-q-1}\gamma \wedge \overline{\partial}\omega_q. \tag{2.3.40}$$

Note $\chi_0$ is of $(n-q-1, 0)$ form and (2.3.40), then

$$|\chi_0|_\psi \leq c_1|\gamma|_\psi = c_1|\star\alpha|_\psi = c_1|\alpha|_\psi, \tag{2.3.41}$$

where $c_1 = c_1(\omega, K) \leq 2^n \sup_K(\frac{|\overline{\partial}\omega_q|}{|\omega_{q+1}|})$. (Note: In Kähler case, $c_1$ is zero by $d\omega = \overline{\partial}\omega = \partial\omega = 0$) And by (2.3.36), we have

$$|[\chi_0 \wedge \omega, \chi_0 \wedge \omega]e^{-\psi}| \leq c_2|\chi_0 \wedge \omega|^2_\psi \leq c_3|\chi_0|^2_\psi \leq c_4|\alpha|^2_\psi \tag{2.3.42}$$

where $c_2 = c_2(\omega, K) = \sup_K \frac{|\omega_{q-1}|}{|\omega_n|}$, $c_3 = c_3(\omega, K) = c_2 \sup_K |\omega|^2$, and $c_4 = c_4(\omega, K) = {c_1}^2 c_3 \geq 0$, which leads to (2.3.39). Here the constant $c = c_4(\omega, K)$ is from the compact set $K \subset X$.

And we claim

$$|\overline{\partial}\gamma - \chi_0 \wedge \omega|^2_\psi \omega_n \geq c_6|\overline{\partial}\gamma|^2_\psi \omega_n - c_7|\alpha|^2_\psi \omega_n \tag{2.3.43}$$

on $K$, where $c_6 > 1/2$ is a constant, and $c_7 = c_7(\omega, K) \geq 0$. In fact, by (2.3.41), (2.3.42) and Young's inequality $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$,

$$\begin{aligned} |\overline{\partial}\gamma - \chi_0 \wedge \omega|^2_\psi &\geq ||\overline{\partial}\gamma|_\psi - |\chi_0 \wedge \omega|_\psi|^2 \\ &= |\overline{\partial}\gamma|^2_\psi + |\chi_0 \wedge \omega|^2_\psi - 2|\overline{\partial}\gamma|^2_\psi|\chi_0 \wedge \omega|^2_\psi \\ &\geq (1-\epsilon)|\overline{\partial}\gamma|^2_\psi + (1 - C_\epsilon)|\chi_0 \wedge \omega|^2_\psi \\ &\geq (1-\epsilon)|\overline{\partial}\gamma|^2_\psi + (1 - C_\epsilon)c_5|\alpha|^2_\psi, \end{aligned} \tag{2.3.44}$$

where small $\epsilon < \frac{1}{2} < 1$ and big $C_\epsilon > 1$ can be chosen, and $c_5 = c_5(\omega, K) = {c_1}^2 \sup_K |\omega|^2$. Then we set $c_6 = 1 - \epsilon > \frac{1}{2}$, and $c_7 = c_7(\omega, K) = (C_\epsilon - 1)c_5 \geq 0$, which lead to (2.3.43).

Hence (2.3.38), (2.3.39), (2.3.43) and (2.3.44) indicate

$$\text{ⓑ} \geq c_6|\overline{\partial}\gamma|^2_\psi \omega_n - (c_4 + c_7)|\alpha|^2_\psi \omega_n \tag{2.3.45}$$

on $K$. Combining (2.3.45) and (2.3.30), we get

$$\begin{aligned} \text{①} &= \text{ⓐ} + \text{ⓑ} \\ &\geq \langle \Theta_F \wedge \Lambda\alpha, \alpha \rangle_\psi \omega_n + c_6|\overline{\partial}\gamma|^2_\psi \omega_n - (c_4 + c_7)|\alpha|^2_\psi \omega_n. \end{aligned} \tag{2.3.46}$$

In particular, for the Kähler case (i.e., $d\omega = 0$), we see ⓑ $= |\overline{\partial}\gamma|^2_\psi \omega_n$ and ① $= \langle \Theta_F \wedge \Lambda\alpha, \alpha \rangle_\psi \omega_n + |\overline{\partial}\gamma|^2_\psi \omega_n$.

Thirdly, we estimate the term ② + ③ $= -i\overline{\partial}T \wedge \partial\omega_{q-1} + i\partial T \wedge \overline{\partial}\omega_{q-1}$.

$$\text{②} = -iC_{n-q}\overline{\partial}\gamma \wedge \overline{\gamma} \wedge \partial\omega_{q-1}e^{-\psi} \tag{2.3.47}$$

by $0 = \overline{\partial}\alpha = \overline{\partial}^*\alpha = -\star\delta\gamma$ and $\overline{\partial}(\gamma\wedge\overline{\gamma}e^{-\psi}) = \overline{\partial}\gamma\wedge\overline{\gamma}e^{-\psi} + (-1)^{deg\gamma}\gamma\wedge\overline{\delta\gamma}e^{-\psi}$. Then

$$|\text{②}| \leq c_8|\overline{\partial}\gamma|_\psi|\overline{\gamma}|_\psi = c_8|\overline{\partial}\gamma|_\psi|\alpha|_\psi \leq \epsilon|\overline{\partial}\gamma|^2_\psi + C_\epsilon|\alpha|^2_\psi, \tag{2.3.48}$$

where $c_8 = c_8(\omega, K) = \sup_K|\partial\omega_{q-1}| \geq 0$. Then,

$$\begin{aligned} \text{②} + \text{③} &= \text{②} + \overline{\text{②}} \\ &\geq -|\text{②} + \overline{\text{②}}|\omega_n \\ &\geq -2|\text{②}|\omega_n \\ &\geq -\epsilon|\overline{\partial}\gamma|^2_\psi\omega_n - C_\epsilon|\alpha|^2_\psi\omega_n \end{aligned} \tag{2.3.49}$$

where small $0 < \epsilon < \min\{\frac{1}{2}, c_6\}$ and $C_\epsilon > 1$ can be chosen such that they only depend on $(\omega, K)$.

Fourthly, let us consider the term $\text{④} = T\wedge i\partial\overline{\partial}\omega_q$.

$$\begin{aligned} |\text{④}| &\leq |T||i\partial\overline{\partial}\omega_q| \leq c_9|T| \\ &= c_9|C_{n-q}\gamma\wedge\overline{\gamma}e^{-\psi}| \leq c_9|\gamma|^2_\psi \\ &= c_9|\alpha|^2_\psi. \end{aligned} \tag{2.3.50}$$

Then

$$\text{④} \geq -|\text{④}|\omega_n = -c_9|\alpha|^2_\psi\omega_n \tag{2.3.51}$$

where $c_9 = c_9(\omega, K) = \sup_K|\partial\overline{\partial}\omega_q| \geq 0$.

Finally, ① + ② + ③ + ④ can be estimate by (2.3.51), (2.3.49) and (2.3.46), that is,

$$\begin{aligned} i\partial\overline{\partial}(T\wedge\omega_{q-1}) &\geq \langle\Theta_F\wedge\Lambda\alpha, \alpha\rangle_\psi\omega_n + (c_6-\epsilon)|\overline{\partial}\gamma|^2_\psi\omega_n - (c_4 + c_7 + C_\epsilon + c_9)|\alpha|^2_\psi\omega_n \\ &= \langle\Theta_F\wedge\Lambda\alpha, \alpha\rangle_\psi\omega_n + c_{10}|\overline{\partial}\gamma|^2_\psi\omega_n - c_{11}|\alpha|^2_\psi\omega_n \end{aligned} \tag{2.3.52}$$

where $c_{10} = c_{10}(\omega, K) = c_6 - \epsilon > 0$ and $c_{11} = c_{11}(\omega, K) = c_4 + c_7 + C_\epsilon + c_9 > 1$. Then (2.3.17) follows.

If $(X, \omega)$ is Kähler, the above $\partial\overline{\partial}$-inequality (2.3.52) reduces to

$$\begin{aligned} i\partial\overline{\partial}T\wedge\omega_{q-1} &= \langle\Theta_F\wedge\Lambda\alpha, \alpha\rangle_\psi\omega_n + |\overline{\partial}\gamma|^2_\psi\omega_n \\ &\geq \langle\Theta_F\wedge\Lambda\alpha, \alpha\rangle_\psi\omega_n \\ &= \Theta_F\wedge T\wedge\omega_{q-1}. \end{aligned} \tag{2.3.53}$$

In fact, $\overline{\partial}\omega_q = 0$ in (2.3.40) implies that (2.3.45) becomes $\text{ⓑ} = |\overline{\partial}\gamma|^2_\psi\omega_n$, then (2.3.46) becomes $\text{①} = \langle\Theta_F\wedge\Lambda\alpha, \alpha\rangle_\psi\omega_n + |\overline{\partial}\gamma|^2_\psi\omega_n$. By $\partial\omega_{q-1} = \overline{\partial}\omega_{q-1} = 0$, ② = ③ = ④ = 0. Then (2.3.18) follows. □

Based on the complete same argument of Theorem 2.46 and $\overline{\partial}$-closed case in [5, Proposition 2.2], we have the following equality for Kähler manifolds, which generalizes both the above formula (2.3.18) and the Kähler case of [5, Proposition 2.2]. The proof is analogue to Theorem 2.46, thus we omit it here. For compact Kähler manifolds, a general formula of this type can be found in [7].

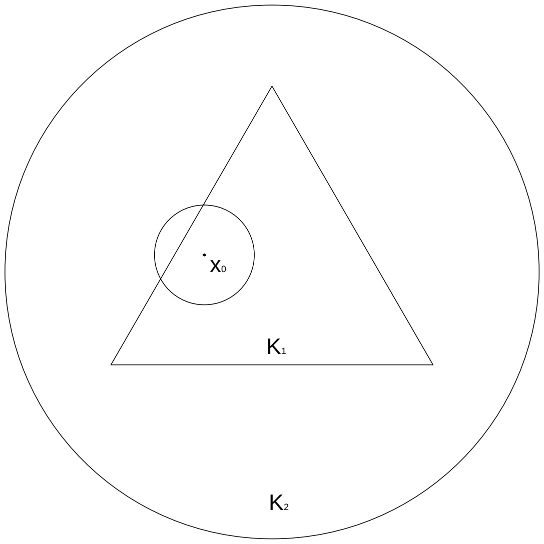

Figure 2.1: The holomorphic coordinate chart at $x_0$

**Corollary 2.47.** *Let $(F, h^F)$ be a holomorphic Hermitian line bundle over a Kähler manifold $(X, \omega)$. Assume $\alpha \in \Omega^{n,q}(X, F) \cap \mathrm{Dom}(\overline{\partial}^F) \cap \mathrm{Dom}(\overline{\partial}^{F*})$ such that $\overline{\partial}^F \alpha = 0$ and $q \geq 1$. Then,*

$$\begin{aligned} i\partial\overline{\partial}(T_\alpha \wedge \omega_{q-1}) &= (i\partial\overline{\partial}T_\alpha) \wedge \omega_{q-1} \\ &= -2Re\langle \overline{\partial}^F \overline{\partial}^{F*}\alpha, \alpha\rangle_{h^F,\omega}\omega_n + \langle 2\pi c_1(F, h^F) \wedge \Lambda\alpha, \alpha\rangle_{h^F,\omega}\omega_n \\ &\quad + |\overline{\partial}^{F*}\alpha|^2_{h^F,\omega}\omega_n + |\overline{\partial}^F \gamma_\alpha|^2_{h^F,\omega}\omega_n \end{aligned} \tag{2.3.54}$$

*where $\langle \cdot, \cdot \rangle_{h^F,\omega}$ and $| \cdot |^2_{h^F,\omega}$ are the pointwise Hermitian metric and norm on $F$-valued differential forms induced by $\omega$ and $h^F$.*

### 2.3.2 Submeanvalue formulas of harmonic forms in $\mathcal{H}^{n,q}(X, L^k \otimes E)$

Let $(L, h^L)$ and $(E, h^E)$ be Hermitian holomorphic line bundles over $X$. For any compact subset $K$ in $X$, the interior of $K$ is denoted by $\mathring{K}$. Let $K_1, K_2$ be compact subsets in $X$, such that $K_1 \subset \mathring{K}_2$. Then there exists a constant $c_0 = c_0(\omega, K_1, K_2) > 0$ such that for any $x_0 \in K_1$, the holomorphic coordinate around $x_0$ is $V \cong W \subset \mathbb{C}^n$, where

$$W := B(c_0) := \{z \in \mathbb{C}^n : |z| < c_0\}, \quad V := B(x_0, c_0) \subset \mathring{K}_2 \subset K_2,$$

$z(x_0) = 0$, and $\omega(z) = \sqrt{-1}\sum_{i,j} h_{ij}(z) dz_i \wedge d\overline{z}_j$ with $h_{ij}(0) = \frac{1}{2}\delta_{ij}$.

**Lemma 2.48.** *Let $(X, \omega)$ be a Hermitian manifold of dimension $n$ and $(L, h^L)$ and $(E, h^E)$ be Hermitian holomorphic line bundles over $X$. Let $K_1$ and $K_2$ be compact subsets in $X$ such that $K_1 \subset \overset{\circ}{K}_2$. Assume $L \geq 0$ in $\overset{\circ}{K}_2$ and $q \geq 1$. Then there exists a constant $C > 0$ depending on $\omega$, $K_1$, $K_2$ and $(E, h_E)$, such that*

$$\int_{|z|<r} |\alpha|^2_{h_k,\omega} dv_X \leq Cr^{2q} \int_X |\alpha|^2_{h_k,\omega} dv_X \tag{2.3.55}$$

*for any $\alpha \in \mathcal{H}^{n,q}(X, L^k \otimes E)$ and $0 < r < \frac{c_0}{2^n}$, where $|\cdot|^2_{h_k,\omega}$ is the pointwise Hermitian norm induced by $\omega$, $h^L$ and $h^E$.*

*Proof.* For simplifying notations, we denote by $\langle \cdot, \cdot \rangle_h$ and $|\cdot|_h$ the associated pointwise Hermitian metrics and norms here, and their meaning will be clear in the context. For $0 < t < c_0$, we define

$$\sigma(t) := \int_{|z|<t} |\alpha|^2_h \omega_n = \int_{|z|<t} T_\alpha \wedge \omega_q.$$

Assume $\|\alpha\|^2_{L^2} = \int_X |\alpha|^2_h \omega_n = 1$. Then this lemma says that: These exists a constant $C$, which is independent of the point $x_0$ and $k$ in $L^k \otimes E$, such that

$$\sigma(r) \leq Cr^{2q}, \tag{2.3.56}$$

when $0 < r < c_0/2^n$ (eventually we will use the special case $r = \frac{2}{\sqrt{k}}$ as $k \to \infty$).

From the Theorem 2.46 for $F = L^k \otimes E$, there exists $C_3 = C_3(\omega, K_2, E, h_E) \geq 0$ such that

$$\begin{aligned} \langle \Theta_F \wedge \Lambda\alpha, \alpha \rangle_h \omega_n &= \Theta_F \wedge T_\alpha \wedge \omega_{q-1} \\ &= (k\Theta_L + \Theta_E) \wedge T_\alpha \wedge \omega_{q-1} \\ &\geq \Theta_E \wedge T_\alpha \wedge \omega_{q-1} \\ &= \langle \Theta_E \wedge \Lambda\alpha, \alpha \rangle_h \omega_n \\ &\geq -C_3 |\alpha|^2_h \omega_n \end{aligned}$$

on $\overset{\circ}{K}_2$, since $L \geq 0$, $T_\alpha \geq 0$ and $\omega$ is positve Hermitian (1,1)-form on $\overset{\circ}{K}_2$. Thus over $\overset{\circ}{K}_2$, (2.3.17) becomes

$$i\partial\overline{\partial}(T_\alpha \wedge \omega_{q-1}) \geq -C_4 |\alpha|^2_h \omega_n \tag{2.3.57}$$

where $C_4 = C_4(\omega, K_2, E, h_E) \geq 0$. Then, it follows that

$$\begin{aligned} \int_{|z|<t} (t^2 - |z|^2) i\partial\overline{\partial}(T_\alpha \wedge \omega_{q-1}) &\geq \int_{|z|<t} -C_4(t^2 - |z|^2)|\alpha|^2_h \omega_n \\ &= -C_4 t^2 \sigma(t) + \int_{|z|<t} C_4 |z|^2 |\alpha|^2_h \omega_n \\ &\geq -C_4 t^2 \sigma(t). \end{aligned} \tag{2.3.58}$$

We denote the standard metric on $\mathbb{C}^n$ by

$$\beta := \frac{i}{2}\partial\overline{\partial}|z|^2 = \frac{i}{2}\sum_j dz_j \wedge d\overline{z_j}.$$

And we apply Stokes' formula to the left side of (2.3.58), that is,

$$\begin{aligned}
&\int_{|z|\leq t}(t^2-|z|^2)i\partial\overline{\partial}(T_\alpha\wedge\omega_{q-1})\\
= &\int_{|z|\leq t}\partial|z|^2\wedge i\overline{\partial}(T_\alpha\wedge\omega_{q-1}) + \int_{|z|\leq t}\partial[(t^2-|z|^2)i\overline{\partial}(T_\alpha\wedge\omega_{q-1})]\\
= &\int_{|z|\leq t}\partial|z|^2\wedge i\overline{\partial}(T_\alpha\wedge\omega_{q-1}) + \int_{|z|\leq t}d[(t^2-|z|^2)i\overline{\partial}(T_\alpha\wedge\omega_{q-1})]\\
= &\int_{|z|\leq t}\partial|z|^2\wedge i\overline{\partial}(T_\alpha\wedge\omega_{q-1}) + \int_{|z|=t}(t^2-|z|^2)i\overline{\partial}(T_\alpha\wedge\omega_{q-1})\\
= &\int_{|z|\leq t}\partial|z|^2\wedge i\overline{\partial}(T_\alpha\wedge\omega_{q-1}).
\end{aligned}$$

Then, by (2.3.58),

$$\begin{aligned}
&2\int_{|z|<t}T_\alpha\wedge\omega_{q-1}\wedge\beta\\
\leq\ &2\int_{|z|\leq t}T_\alpha\wedge\omega_{q-1}\wedge\beta\\
=\ &\int_{|z|\leq t}i\partial\overline{\partial}|z|^2\wedge T_\alpha\wedge\omega_{q-1}\\
=\ &-\int_{|z|\leq t}d[i\partial|z|^2\wedge T_\alpha\wedge\omega_{q-1}] - \int_{|z|\leq t}i\partial|z|^2\wedge d(T_\alpha\wedge\omega_{q-1})\\
=\ &-\int_{|z|=t}i\partial|z|^2\wedge T_\alpha\wedge\omega_{q-1} - \int_{|z|\leq t}i\partial|z|^2\wedge\overline{\partial}(T_\alpha\wedge\omega_{q-1})\\
=\ &\int_{|z|=t}-iT_\alpha\wedge\omega_{q-1}\wedge\partial|z|^2 - \int_{|z|\leq t}(t^2-|z|^2)i\partial\overline{\partial}(T_\alpha\wedge\omega_{q-1})\\
=\ &\int_{|z|=t}-iT_\alpha\wedge\omega_{q-1}\wedge\partial|z|^2 - \int_{|z|<t}(t^2-|z|^2)i\partial\overline{\partial}(T_\alpha\wedge\omega_{q-1})\\
\leq\ &\int_{|z|=t}-iT_\alpha\wedge\omega_{q-1}\wedge\partial|z|^2 + C_4t^2\sigma(t).
\end{aligned}\tag{2.3.59}$$

By the choice of holomorphic coordinates,

$$h_{ij}(z) = \frac{1}{2}\delta_{ij} + \mathcal{O}(|z|)\tag{2.3.60}$$

for any $z\in B(c_0)$. In particular, for $|z|=t$ with $0\leq t<c_0$, we can approximate the metric $\omega$ on $X$ by the standard one $\beta$ on $\mathbb{C}^n$ in the following sense

$$(1-R_1(t))\beta\leq\omega(z)\leq(1+R_1(t))\beta\tag{2.3.61}$$

by the smoothness of $\omega$, where $R_1(t) \geq 0$ and $R_1(t) = \mathcal{O}(t)$ as $t \to 0$. (Trivially if $X = \mathbb{C}^n$ then $R_1(t) = 0$). Hence

$$\begin{aligned} T_\alpha \wedge \omega_{q-1} \wedge \beta &\geq T_\alpha \wedge \omega_{q-1} \wedge (1 - R_2(t))\omega \qquad (2.3.62) \\ &= q(1 - R_2(t)))T_\alpha \wedge \omega_q \\ &= q(1 - R_2(t))|\alpha|_h^2 \omega_n \end{aligned}$$

where $R_2(t) \geq 0$ and $R_2(t) = \mathcal{O}(t)$.

Let $dS$ be the surface measure of $B(t)$. By $(1+R_1(t))^{-n} \leq \omega^n/\beta^n \leq (1+R_1(t))^n$, we have

$$\int_{|z|=t} -iT_\alpha \wedge \beta_{q-1} \wedge \partial|z|^2 \leq t \int_{|z|=t} |\alpha|_h^2 dS.$$

Then (2.3.61) implies

$$\begin{aligned} \int_{|z|=t} -iT_\alpha \wedge \omega_{q-1} \wedge \partial|z|^2 &\leq (1 + R_1(t))^{q-1} \int_{|z|=t} -iT_\alpha \wedge \beta_{q-1} \wedge \partial|z|^2 \\ &\leq t(1 + R_1(t))^{q-1} \int_{|z|=t} |\alpha|_h^2 dS \\ &\leq t(1 + R_1(t))^{n+q-1} \int_{|z|=t} |\alpha|_h^2 \frac{\omega^n}{\beta^n} dS \\ &= t(1 + R_3(t))\sigma'(t) \qquad (2.3.63) \end{aligned}$$

where $\sigma'(t) = \int_{|z|=t} |\alpha|_h^2 (\omega_n/\beta_n) dS$ by the definition of $\sigma$, $R_3(t) \geq 0$ and $R_3(t) = \mathcal{O}(t)$.

Combining (2.3.59),(2.3.62) and (2.3.63), we have

$$\begin{aligned} 2q(1 - R_2(t))\sigma(t) &\leq 2 \int_{|z|<t} T_\alpha \wedge \omega_{q-1} \wedge \beta \\ &\leq t(1 + R_3(t))\sigma'(t) + C_4 t^2 \sigma(t). \end{aligned}$$

Then, for any $0 \leq t < c_0$,

$$2q(1 - R_4(t))\sigma(t) \leq t\sigma'(t)$$

where $R_4(t) \geq 0$ and $R_4(t) = \mathcal{O}(t)$.

Substituting $s(t)^2 := \sigma(t) \geq 0$ and dividing by $2ts(t)$, we obtain

$$q\Big(\frac{1}{t} - R_5(t)\Big)s(t) \leq s'(t) \qquad (2.3.64)$$

for $q \geq 1$ and any $0 < t < c_0$, where $R_5(t) \geq 0$ and $R_5(t) = \mathcal{O}(1)$ for $0 \leq t < c_0$.

Now we only need to prove the statement as follows. There exists $C \geq 0$, such that for any $1 \leq q \leq n$ and $0 \leq t < \frac{c_0}{2^n}$,

$$s(t) \leq Ct^q, \qquad (2.3.65)$$

which is equivalent to (2.3.56).

Next we fix $q \geq 1$, so we only need to prove that $s(t) \leq Ct^m$ for any $0 \leq m \leq q$ and $0 \leq t \leq c_0/2^q$ by induction over $m$. Firstly, for $m = 0$,

$$s^2(t) := \sigma(t) := \int_{|z|<t} |\alpha|_h^2 \omega_n \leq 1$$

for $0 \leq t \leq c_0$. Secondly, assume there exists a constant $C_5 > 0$ such that

$$s(t) \leq C_5 t^m$$

for $0 \leq m < q$ and $0 \leq t \leq c_0/2^m$. Thirdly, in particular, we consider $1 \leq m+1 = q$ for $0 < t \leq c_0/2^{m+1}$, and thus (2.3.64) becomes

$$s'(t) - (m+1)\Big(\frac{1}{t} - R_5(t)\Big)s(t) \geq 0. \tag{2.3.66}$$

Let $\varPhi(t) := (m+1)(\log t - \int_0^t R_5(u)du)$ and $R_6(t) := t^{m+1}e^{-\varPhi(t)}$ for $0 < t \leq c_0/2^{m+1}$. Then $R_6(t) = e^{(m+1)\int_0^t R_5(u)du} \geq 1$. And according to (2.3.66), for $0 < t \leq c_0/2^{m+1}$,

$$\left(\frac{s(t)R_6(t)}{t^{m+1}}\right)' = \Big(s(t)e^{-\varPhi(t)}\Big)' \geq 0. \tag{2.3.67}$$

For any $0 < r \leq c_0/2^{m+1}$, by integration of (2.3.67) from $r$ to $c_0/2^{m+1}$, we have

$$\frac{s(\frac{c_0}{2^{m+1}})R_6(\frac{c_0}{2^{m+1}})}{(\frac{c_0}{2^{m+1}})^{m+1}} \geq \frac{s(r)R_6(r)}{r^{m+1}} \geq \frac{s(r)}{r^{m+1}}. \tag{2.3.68}$$

Finally, for the fixed $q \geq 1$ and any $0 < r \leq c_0/2^q$, we have

$$\frac{s(r)}{r^q} \quad\leq\quad C_5\frac{2^q}{c_0}R_6\Big(\frac{c_0}{2^q}\Big). \tag{2.3.69}$$

Let $q$ run over $\{1, ..., n\}$ in (2.3.69). Then there exists $C = C(\omega, K_1, K_2, E, h_E) \geq 0$ such that (2.3.65) verifies and also (2.3.56) and (2.3.55). $\square$

We will consider the following trivialization of holomorphic Hermitian line bundles in local charts. For any $x_0 \in K_1 \subset \overset{\circ}{K}_2$, we fix the holomorphic normal coordinate on $V \cong W \subset \mathbb{C}^n$ as before such that

$$\omega(x_0) = \beta := \frac{\sqrt{-1}}{2}\sum_{j=1}^n dz_j \wedge d\overline{z}_j,$$

which is the standard metric on $\mathbb{C}^n$. Let $L \geq 0$ on $\overset{\circ}{K}_2$. Then we can choose the trivialization of $L$ and $E$ over $V$ such that for any $z \in B(c_0)$, $|e_L(z)|^2_{h^L} = e^{-\phi(z)}$ and $|e_E(z)|^2_{h^E} = e^{-\varphi(z)}$ satisfying

$$\phi(z)=\sum_{i=1}^{n}\lambda_i|z_i|^2+\mathcal{O}(|z|^3),\quad \varphi(z)=\sum_{i=1}^{n}\mu_i|z_i|^2+\mathcal{O}(|z|^3) \tag{2.3.70}$$

and $\lambda_i=\lambda_i(x_0)\geq 0$. The induced Hermitian metric on $F:=L^k\otimes E$ is given by $|e_F(z)|^2_{h^F}=e^{-\psi(z)}$ with

$$\psi(z):=k\phi(z)+\varphi(z). \tag{2.3.71}$$

The quadratic part of $\phi$ is denoted by

$$\phi_0(z):=\sum_{i=1}^{n}\lambda_i|z_i|^2. \tag{2.3.72}$$

Assume $\alpha\in\Omega^{p,q}(X,F)$, then it has the form $\alpha=\xi\otimes e_F$ around $x_0\in K_1$ where $\xi=\sum f_{IJ}dz_I\wedge d\overline{z}_J$ is a local $(p,q)$-form and $f_{IJ}$ are smooth functions on $W\subset\mathbb{C}^n$. The scaled functions and sections with respect to $k\in\mathbb{N}$ are defined by

$$\psi^{(k)}(z):=\psi(z/\sqrt{k}),\quad e_F^{(k)}(z):=e_F(z/\sqrt{k})\,,\ \text{for } z\in\sqrt{k}W=B(\sqrt{k}c_0), \tag{2.3.73}$$

hence $|e_F^{(k)}|^2_{h^F}=e^{-\psi^{(k)}}$. The scaled forms are defined for $z\in\sqrt{k}W$ by

$$\begin{aligned}
&\omega^{(k)}(z):=\sqrt{-1}\sum h_{ij}^{(k)}(z)dz_j\wedge d\overline{z}_j:=\sqrt{-1}\sum h_{ij}(z/\sqrt{k})dz_j\wedge d\overline{z}_j,\\
&\xi^{(k)}(z):=f_{IJ}^{(k)}(z)dz_I\wedge d\overline{z}_J:=f_{IJ}(z/\sqrt{k})dz_I\wedge d\overline{z}_J,\\
&\alpha^{(k)}(z):=\xi^{(k)}(z)\otimes e_F^{(k)}(z)\,.
\end{aligned} \tag{2.3.74}$$

**Lemma 2.49.** *Let $(X,\omega)$ be a Hermitian manifold of dimension n and $(L,h^L)$ and $(E,h^E)$ be Hermitian holomorphic line bundles over X. Let $K_1$ and $K_2$ be compact subsets in X such that $K_1\subset\mathring{K}_2$. Assume $L\geq 0$ on $\mathring{K}_2$ and $q\geq 1$. Then there exists a constant $C>0$ depending on $\omega$, $K_1$, $K_2$, $(L,h^L)$ and $(E,h_E)$, such that*

$$|\alpha(x_0)|^2_{h_k,\omega}\leq Ck^n\int_{|z|<\frac{2}{\sqrt{k}}}|\alpha|^2_{h_k,\omega}dv_X. \tag{2.3.75}$$

*for any $x_0\in K_1$, $\alpha\in\mathcal{H}^{n,q}(X,L^k\otimes E)$ and k sufficiently large, where $|\cdot|^2_{h_k,\omega}$ is the pointwise Hermitian norm induced by $\omega$, $h^L$ and $h^E$.*

*Proof.* Denote a ball centred at the origin in $\mathbb{C}^n$ with radius $r>0$ by $B(r):=\{z\in\mathbb{C}^n:|z|<r\}$. Then, $B(r)$ is a subset of $W\cong V\subset X$ via the local chart, when $0<r\leq c_0$. Thus the right side of (2.3.75) is well defined, when $k$ is large enough. Let $r_k:=\frac{\log k}{\sqrt{k}}$ for $k\in\mathbb{N}$. Then $0\leq r_k\leq 1$ and $r_k\to 0$ as $k\to\infty$.

Under the local representation of forms valued in $F:=L^k\otimes E$ as in (2.3.3), (2.3.4), (2.3.6) and (2.3.10), the Kodaira Laplacian can be represented by $\square=\overline{\partial}\overline{\partial}^*_{\psi}+\overline{\partial}^*_{\psi}\overline{\partial}$ locally over $B(\frac{\log k}{\sqrt{k}})$ for $k$ large enough, where $\overline{\partial}^*_{\psi}=\overline{\partial}^{F*}$. Then the scaled Laplacian

$$\square^{(k)}:=\overline{\partial}\overline{\partial}^*_{\psi^{(k)}}+\overline{\partial}^*_{\psi^{(k)}}\overline{\partial} \tag{2.3.76}$$

is well defined on $B(\log k)$ for $k$ large enough.

Under our assumptions, we consider the harmonic $L^k \otimes E$-valued $(n,q)$-form $\alpha = \alpha\mid_{B(\frac{\log k}{\sqrt{k}})}$ on $B(\frac{\log k}{\sqrt{k}})$ for $k$ large enough, then the rescaled $\alpha^{(k)}$ is a $L^k \otimes E$-valued $(n,q)$-forms on $B(\log k)$ by (2.3.74).

We claim that

$$\Box^{(k)}\alpha^{(k)} = \frac{1}{k}(\Box\alpha)^{(k)} = 0 \tag{2.3.77}$$

over $B(\log k)$. That is, the scaled forms are still harmonic with respect to the scaled Laplacian. In fact, by the local representation $\alpha = \xi \otimes e_F$, and notice $\Box\alpha = 0$ globally, we only need to prove,

$$k\Box^{(k)}\alpha^{(k)} = (\Box\alpha)^{(k)}.$$

To see this, firstly we notice

$$\begin{aligned}
\overline{\partial}^*_\psi\alpha &= (-\star\delta\star\xi)\otimes e_F = (\star[(\partial\psi)\wedge(\star\xi)] - \star\partial\star\xi)\otimes e_F, &(2.3.78)\\
\overline{\partial}\overline{\partial}^*_\psi\alpha &= (\overline{\partial}\star[(\partial\psi)\wedge(\star\xi)] - \overline{\partial}\star\partial\star\xi)\otimes e_F,\\
\overline{\partial}^*_\psi\overline{\partial}\alpha &= (\star[(\partial\psi)\wedge(\star\overline{\partial}\xi)] - \star\partial\star\overline{\partial}\xi)\otimes e_F.
\end{aligned}$$

Then we can represent Laplacian by

$$\begin{aligned}
\Box\alpha &= (\overline{\partial}\star[(\partial\psi)\wedge(\star\xi)] + \star[(\partial\psi)\wedge(\star\overline{\partial}\xi)] + \triangle_{\overline{\partial}}\xi)\otimes e_F, &(2.3.79)\\
\Box^{(k)}\alpha^{(k)} &= (\overline{\partial}\star[(\partial\psi^{(k)})\wedge(\star\xi^{(k)})] + \star[(\partial\psi^{(k)})\wedge(\star\overline{\partial}\xi^{(k)})] + \triangle_{\overline{\partial}}\xi^{(k)})\otimes e_F^{(k)},
\end{aligned}$$

where $\triangle_{\overline{\partial}}\xi = -\overline{\partial}\star\partial\star\xi - \star\partial\star\overline{\partial}\xi$ and $\triangle_{\overline{\partial}}\xi^{(k)} = -\overline{\partial}\star\partial\star\xi^{(k)} - \star\partial\star\overline{\partial}\xi^{(k)}$. By definition of $\psi^{(k)}$ and Proposition 2.14(4) for scalar valued forms $\xi$ and $\eta$ as follows

$$\begin{aligned}
\xi^{(k)}\wedge\overline{\star\eta^{(k)}} &= \langle\xi^{(k)},\eta^{(k)}\rangle\omega_n^{(k)} = (\langle\xi,\eta\rangle\omega_n)^{(k)} = (\xi\wedge\overline{(\star\eta)})^{(k)}\\
&= \xi^{(k)}\wedge\overline{(\star\eta)^{(k)}},
\end{aligned}$$

then

$$\sqrt{k}\partial\psi^{(k)} = (\partial\psi)^{(k)}, \quad \sqrt{k}\overline{\partial}\psi^{(k)} = (\overline{\partial}\psi)^{(k)}, \quad \star\eta^{(k)} = (\star\eta)^{(k)}. \tag{2.3.80}$$

Now we consider the term $\overline{\partial}\star\partial\star\xi^{(k)}$ in $\triangle_{\overline{\partial}}\xi^{(k)}$ to prove $k\overline{\partial}\star\partial\star\xi^{(k)} = (\overline{\partial}\star\partial\star\xi)^{(k)}$, and thus $k\triangle_{\overline{\partial}}\xi^{(k)} = (\triangle_{\overline{\partial}}\xi)^{(k)}$. By (2.3.80),

$$\begin{aligned}
\overline{\partial}\star\partial\star\xi^{(k)}(z) &= \overline{\partial}\star\partial(\star\xi^{(k)})(z) = \overline{\partial}\star\partial(\star\xi)^{(k)}(z) &(2.3.81)\\
&= \frac{1}{\sqrt{k}}\overline{\partial}\star(\partial\star\xi)^{(k)}(z) = \frac{1}{\sqrt{k}}\overline{\partial}(\star\partial\star\xi)^{(k)}(z)\\
&= \frac{1}{k}(\overline{\partial}\star\partial\star\xi)^{(k)}(z).
\end{aligned}$$

By the same argument, (2.3.77) follows.

Next we introduce the following $L^2$-norms,

$$\begin{aligned} \|\cdot\|^2_{B(\frac{2}{\sqrt{k}})} &:= \int_{B(\frac{2}{\sqrt{k}})} |\cdot|^2_\omega e^{-\psi}\omega_n \\ \|\cdot\|^2_{\phi_0,B(2)} &:= \int_{B(2)} |\cdot|^2_\beta e^{-\phi_0}\beta_n. \end{aligned} \tag{2.3.82}$$

where Hermitian norm $|\cdot|_\omega$ is induced by $\omega$, and $|\cdot|_\beta$ is by $\beta$.

We claim that there exist $C(k) > 0$ bounded above and below for $k$ large enough (in fact, $C(k) \to 1$ as $k \to \infty$) such that

$$\|\alpha^{(k)}\|^2_{\phi_0,B(2)} = C(k)k^n\|\alpha\|^2_{B(\frac{2}{\sqrt{k}})}. \tag{2.3.83}$$

In fact, by (2.3.70)–(2.3.73), $\psi^{(k)}(z) - \phi_0(z) = \frac{\mathcal{O}(|z|^2)}{\sqrt{k}}$ and thus

$$\lim_{k\to\infty} \sup_{|z|<\log k} |\partial^N(\psi^{(k)} - \phi_0)(z)| = 0, \tag{2.3.84}$$

which means the scaled Hermitian metric on $L^k \otimes E$ convergences to a model metric on $B(\log k)$ with all derivatives. In particular, as $k \to \infty$, $\psi^{(k)}(z) \to \phi_0(z)$ uniformly over $B(\log k)$, and also $\omega^{(k)}(z) \to \beta$. Hence (2.3.83) follows by

$$\begin{aligned} \|\alpha^{(k)}\|^2_{\phi_0,B(2)} &= \int_{B(2)} |\xi^{(k)}(z)|^2_\beta e^{-\phi_0(z)}\beta_n, \\ k^n\|\alpha\|^2_{B(\frac{2}{\sqrt{k}})} &= k^n\int_{B(\frac{2}{\sqrt{k}})} |\xi(z)|^2_{\omega(z)} e^{-\psi(z)}\omega_n(z) \\ &= \int_{B(2)} |\xi^{(k)}(z)|^2_{\omega^{(k)}(z)} e^{-\psi^{(k)}(z)}\omega^{(k)}_n(z). \end{aligned} \tag{2.3.85}$$

Finally, we apply [4, Lemma 3.1] and identify $\alpha^{(k)}$ with a form in $L^2(\mathbb{C}^n, \phi_0)$ by extending with zero outside $B(\log k)$. Then there exists a constant $C_1 > 0$ independent of $k$ such that

$$\sup_{z\in B(1)} |\alpha^{(k)}(z)|^2_{\beta,\phi_0} \le C_1\|\alpha^{(k)}\|^2_{\phi_0,B(2)} \tag{2.3.86}$$

for $k$ large enough, where $|\cdot|^2_{\beta,\phi_0} := |\cdot|^2_\beta e^{-\phi_0}$. In fact, we also can obtain (2.3.86) by combining [35, Friedrichs' inequality 3.6.11] and [35, Sobolev lemma 3.5.12]. That is, there exists $C_k, C'_k > 0$ such that

$$\begin{aligned} \sup_{z\in B(1)} |\alpha^{(k)}(z)|_{\beta,\phi_0} &\le C_k\|\alpha^{(k)}\|_{2m,\phi_0,B(3/2)} \\ &\le C'_k(\|(\square^{(k)})^m\alpha^{(k)}\|_{\phi_0,B(2)} + \|\alpha^{(k)}\|_{\phi_0,B(2)}) \\ &= C'_k\|\alpha^{(k)}\|_{\phi_0,B(2)} \end{aligned} \tag{2.3.87}$$

where $2m > n$ and $\log k > 2$ such that $\alpha^{(k)}$ is harmonic on $B(2)$. Since $\square^{(k)}$ converges to $\square_{\phi_0}$ on the ball $B_2$ by (2.3.84),(2.3.76) and (2.3.79), here $C'_k$ can be chosen to be independent of $k$ and denote it by $C_1$. Thus we have (2.3.86).

Combining (2.3.83) and (2.3.86), we get

$$|\alpha(x_0)|^2_{h_k,\omega} = |\alpha^{(k)}(0)|^2_{\beta,\phi_0} \leq C_1\|\alpha^{(k)}\|^2_{\phi_0,B(2)} \leq 2C_1k^n\|\alpha\|^2_{B(\frac{2}{\sqrt{k}})} \tag{2.3.88}$$

for $k$ large enough. Notice that here $C_1$ works for all points sufficiently close to $x_0$ by continuity. That is, there exists a constant $C_1 > 0$ and a neighbourhood $B(x_0, \epsilon)$ of $x_0$, such that

$$|\alpha(x)|^2_{h_k,\omega} \leq 2C_1k^n\|\alpha\|^2_{B(\frac{2}{\sqrt{k}})}$$

for any $x \in B(x_0, \epsilon)$ and $k$ large enough. Since $K_1$ is compact, there exists a uniform constant $C > 0$ which works for all $x \in K_1$, and (2.3.75) follows. $\square$

To summarize, we have a local estimate of the pointwise norm of harmonic forms valued in semipositive line bundles, which is equivalent to Theorem 2.1. We define

$$S^q_k(x) := \sup\left\{\frac{|\alpha(x)|^2_{h_k,\omega}}{||\alpha||^2_{L^2}} : \alpha \in \mathcal{H}^{n,q}(X, L^k \otimes E)\right\},$$

where $(L, h^L)$ and $(E, h^E)$ are holomorphic Hermitian line bundles over a Hermitian manifold $(X, \omega)$ as before.

**Theorem 2.50.** *Let $(X, \omega)$ be a Hermitian manifold of dimension $n$ and $(L, h^L)$ and $(E, h^E)$ be Hermitian holomorphic line bundles over $X$. Let $K_1$ and $K_2$ be compact subsets in $X$ such that $K_1 \subset \overset{\circ}{K}_2$. Assume $L \geq 0$ on $\overset{\circ}{K}_2$ and $q \geq 1$. Then there exists $C > 0$ depending on $\omega$, $K_1$, $K_2$, $(L, h^L)$ and $(E, h^E)$ such that*

$$S^q_k(x) \leq Ck^{n-q}\,. \tag{2.3.89}$$

*for any $x \in K_1$ and $k \geq 1$.*

*Proof.* Combine (2.3.75) and the case $r = \frac{2}{\sqrt{k}}$ of (2.3.55). $\square$

**Remark 2.51.** In particular, when $X$ is compact without boundary, and $K_1 = K_2 = X$, then (2.3.89) implies the case $\lambda = 0$ in [5, Theorem 2.3].

## 2.4 Proof of the main results and applications

Let $(L, h^L)$ and $(E, h^E)$ are holomorphic Hermitian line bundles over a Hermitian manifold $(X, \omega)$ as before. Let $\{s^k_j\}_{j\geq 1}$ be an orthonormal basis of $\mathcal{H}^{n,q}(X, L^k \otimes E)$. Let $|\cdot|_{h_k,\omega}$ is the pointwise Hermitian norm of a form.

### 2.4.1 Proof of Theorem 2.1

At first, the following proposition is clear by the definitions of $S_k^q(x)$ and $B_k^q(x)$ in Theorem 2.50 and Theorem 2.1 for $x \in X$. Namely,

$$\begin{aligned} B_k^q(x) &:= \sum_{j\geq 1} |s_j^k(x)|^2_{h_k,\omega}, \\ S_k^q(x) &:= \sup\left\{ \frac{|\alpha(x)|^2_{h_k,\omega}}{||\alpha||^2_{L^2}} : \alpha \in \mathcal{H}^{n,q}(X, L^k \otimes E) \right\}. \end{aligned}$$

**Proposition 2.52.** *$S_k^q(x) \leq B_k^q(x) \leq C_n^q S_k^q(x)$ on $X$.*

*Proof.* For simplifying notions, we denote by $\{s_i\}$ an orthonormal basis of $\operatorname{Ker}\square = \mathcal{H}^{n,q}(X, L^k \otimes E)$, and thus $B(x) = \sum_i |s_i(x)|^2$ and $S(x) = \sup_{\alpha\in \operatorname{Ker}\square} \frac{|\alpha(x)|^2}{||\alpha||^2_{L^2}}$. Then we wish to show

$$S(x) \leq B(x) \leq C_n^q S(x). \tag{2.4.1}$$

For any $s \in \operatorname{Ker}\square$, there exists $a_i \in \mathbb{C}$ such that $s = \sum_i a_i s_i$. Fix a $x \in X$, we have

$$\begin{aligned} | s(x) |^2 &= | \sum_i a_i s_i(x) |^2 \leq (\sum_i | a_i || s_i(x) |)^2 \qquad (2.4.2) \\ &\leq (\sum_i | a_i |^2)(\sum_i | s_i(x) |^2) = \| s \|^2 B(x), \end{aligned}$$

which implies $S(x) \leq B(x)$.

We set $N := C_n^q = \operatorname{rank}(L^k \otimes E \otimes \wedge^{n,q} T^*X)$, then for any $s \in \mathcal{C}^\infty(L^k \otimes E \otimes \wedge^{n,q} T^*X)$ and a fixed $x \in X$, we have a local trivialization such that

$$L^k \otimes E \otimes \wedge^{n,q} T^*X \mid_V \cong V \times \mathbb{C}^N,$$

$$s(x) \cong (c^1(x), c^2(x), ..., c^N(x)) \in \mathbb{C}^N,$$

where $x \in V \subset X$. Then $|s(x)|^2 = \sum_{l=1}^N |c^l(x)|^2$. Moreover, we can assume $s_i(x) \cong (c_i^1(x), c_i^2(x), ..., c_i^N(x)) \in \mathbb{C}^N$ for the given orthonormal basis $\{s_i\}$, thus

$$B(x) = \sum_i |s_i(x)|^2 = \sum_i \sum_{l=1}^N |c_i^l(x)|^2. \tag{2.4.3}$$

For a fixed $l$, we have

$$\sum_i |c_i^l(x)|^2 = |\sum_i \overline{c_i^l(x)} c_i^l(x)| \leq (\sum_{k=1}^N |\sum_i \overline{c_i^l(x)} c_i^k(x)|^2)^{1/2} = |\sum_i \overline{c_i^l(x)} s_i(x)|.$$

And the definition of $S(x)$ indicates that $|\sum_i \overline{c_i^l(x)} s_i(x)| \leq (\sum_i |\overline{c_i^l(x)}|^2)^{1/2} S(x)^{1/2}$. Then $\sum_i |c_i^l(x)|^2 \leq S(x)$. And by (2.4.3), it follows that

$$B(x) = \sum_{l=1}^N \sum_i |c_i^l(x)|^2 \leq N S(x). \tag{2.4.4}$$

□

By this lemma and the submeanvalue formulas of harmonic forms in $\mathcal{H}^{n,q}(X, L^k \otimes E)$ in the section 2.3.2, we can prove the first main result immediately.

**Proof of Theorem 2.1**: Assume $U$ is a neighbourhood of $K$ such that $L$ is semipositive on $U$. Then we choose $K_2$ such that $K_1 \subset \overset{\circ}{K_2} \subset K_2 \subset U$, and apply Theorem 2.50 and Proposition 2.52. □

**Remark 2.53.** Let $(X, \omega)$ be a complete Hermitian manifold of dimension $n$ and $(L, h^L)$ and $(E, h^E)$ be Hermitian holomorphic line bundles over $X$. Suppose $X, L$ and $E$ have bounded geometry (i.e, $X$ has positive injectivity radius, and the curvature tensor of $X$ is uniformly bounded, as are all its covariant derivatives and the curvature tensors of $L, E$ also. For examples, any compact manifold has bounded geometry; the covering of a compact manifold has bounded geometry. Any noncompact manifold of bounded geometry has infinite volume). And we suppose $L \geq 0$ over $X$. By Theorem 2.1, there exists a constant $C > 0$ such that the Bergman kernel function $B_k^q(x) \leq Ck^{n-q}$ for any $x \in X$, $k \geq 1$ and $q \geq 1$.

## 2.4.2 Proof of Theorem 2.2

Now we can prove the second main results on $\Gamma$-dimension and covering manifolds.

**Proof of Theorem 2.2**: Under the assumption of $X$, $\Gamma$ and $X/\Gamma$, there exists an open fundamental domain $U \subset X$ of the action $\Gamma$ on $X$ such that the closure $\overline{U}$ is compact.

Since $L$ and $E$ are $\Gamma$-invariant holomorphic Hermitian line bundles over $X$, the induced Hermitian line bundle $F := L^k \otimes E$ is also $\Gamma$-invariant and holomorphic. Then the Kodaira Laplacian $\Box := \Box^F$ is $\Gamma$-invariant. Thus $\Box$ is essentially self-adjoint (see [30] Corollary 3.3.4), and we denote still by $\Box$ its self-adjoint extension, which commutes to the action of $\Gamma$. According to Lemma 2.40 (also see [30, Lemma C.3.1, Lemma 3.6.3]), the space of harmonic $F$-valued $(n, q)$-forms $\mathcal{H}^{n,q}(X, F)$ is a $\Gamma$-module on which $\Gamma$-dimension is well-defined. By (2.2.73) (also see [30, (3.6.11), (3.6.17)]), we have

$$\dim_\Gamma \mathcal{H}^{n,q}(X, F) = \sum_i \int_U |s_i(x)|^2_{h^F,\omega} dv_X(x), \tag{2.4.5}$$

where $\{s_i\}$ is an orthonormal basis of $\mathcal{H}^{n,q}(X, F)$ with respect to the scalar product in $L^2_{n,q}(X, F)$. Using Theorem 2.1, we have

$$B_k^q(x) := \sum_i |s_i(x)|^2_{h_k,\omega} \leq Ck^{n-q} \tag{2.4.6}$$

for any $x \in U$. Then integrating $B_k^q(x)$ over $U$ and combining these two formulas above we obtain $\dim_\Gamma \mathcal{H}^{n,q}(X, L^k \otimes E) \leq Ck^{n-q}$, that is the first asymptotic estimate in (2.1.3).

Let $\overline{H}^{0,q}_{(2)}(X, L^k \otimes E)$ be the reduced $L^2$-Dolbeault cohomology group, which is canonically isomorphic to $\mathcal{H}^{0,q}(X, L^k \otimes E)$ as $\Gamma$-modules by the weak Hodge decomposition, see (2.2.45), thus $\dim_\Gamma \overline{H}^{0,q}_{(2)}(X, L^k \otimes E) = \dim_\Gamma \mathcal{H}^{0,q}(X, L^k \otimes E)$. Substituting $E \bigotimes \Lambda^n(T^{(1,0)}X)$ for $E$ in the first estimate of (2.1.3), then the same asymptotic estimate also holds for the space of harmonic $L^k \otimes E$ valued $(0,q)$-forms and (2.1.4) follows. ☐

**Corollary 2.54.** *Let $(X, \omega)$ be a compact Hermitian manifold of dimension $n$ and $(E, h^E)$ be a semipositive holomorphic Hermitian vector bundle of rank $r$ (i.e., $L(E^*)^* \geq 0$). Then there exists $C > 0$ such that for any $q \geq 1$ and $k \geq 1$ we have*

$$\dim H^q(X, S^k(E)) \leqq Ck^{(n+r-1)-q} \tag{2.4.7}$$

*where $S^k(E)$ is the $k$-th symmetric tensor power of $E$.*

Proof. We assume $\Gamma$ and $E$ are trivial in (2.1.4), and notice the theorem of Le Potier (see [24, Chap.III §5 (5.7)]), which relates vector bundle cohomology to line bundle chohomology, then

$$\dim H^q(X, S^k(E)) = \dim H^q(P(E^*), (L(E^*)^*)^k) \leqq Ck^{(n+r-1)-q}, \tag{2.4.8}$$

where $P(E^*)$ is a compact manifold of dimension $n + r - 1$, called the projective bundle associated to $E^*$, and $L(E^*)^*$ is a semi-positive line bundle on $P(E^*)$, which are induced by $(X, \omega)$ and $(E, h^E)$. ☐

**Corollary 2.55.** *Let $(X, \omega)$ be a Hermitian manifold of dimension $n$ on which a discrete group $\Gamma$ acts holomorphically, freely and properly such that $\omega$ is a $\Gamma$-invariant Hermitian metric and the quotient $X/\Gamma$ is compact. Let $(L, h^L)$ be a $\Gamma$-invariant holomorphic Hermitian line bundle on $X$. Assume $(L, h^L)$ is semi-negative (i.e. $L^* \geq 0$). Then, there exists $C > 0$ such that for any $0 \leq q \leq n-1$ and $k \geq 1$ we have*

$$\dim_\Gamma \overline{H}^{0,q}_{(2)}(X, L^k) \leq Ck^q. \tag{2.4.9}$$

*In particular, for all $k \in \mathbb{N}$, $\dim_\Gamma \overline{H}^{0,0}_{(2)}(X, L^k) \leq C$.*

*Proof.* According to Serre duality (cf. [8, 3.15]) and Theorem 2.2, there exists $C > 0$ such that for any $q \leq n-1$ and $k \geq 1$ we have

$$\begin{aligned}
\dim_\Gamma \overline{H}^{0,q}_{(2)}(X, L^k) &= \dim_\Gamma \overline{H}^{n,n-q}_{(2)}(X, L^{*k}) \\
&= \dim_\Gamma \overline{H}^{0,n-q}_{(2)}(X, \Lambda^n(T^{*(1,0)}X) \otimes L^{*k}) \\
&\leq Ck^q.
\end{aligned}$$

☐

**Remark 2.56.** In the situation of Theorem 2.2, if $(L, h^L)$ is semi-positive and positive at some point,

$$\dim_\Gamma \overline{H}^{0,0}_{(2)}(X, L^k) \approx k^n \tag{2.4.10}$$

as $k \to +\infty$, see [41], [34] and [30]. This can also be obtained by using Theorem 2.2 and the asymptotic Hirzebruch-Riemann-Roch formula on covering manifolds.

# 3 On the holomorphic extension of forms with values in a holomorphic vector bundle from boundary of a pseudo-concave domain

We study the holomorphic extension problem of smooth forms with values in a vector bundle from the boundary of a pseudo-concave domain in a compact Hermitian manifold associated a line bundle. And we proved a result on the meromorphic extension of $(n, q+1)$-forms with values in a vector bundle, when the domain is $q$-concave and the line bundle is semi-positive and positive at one point.

This chapter is organized in the following way. In Section 3.1, we state the main result of this chapter. In Section 3.2, we introduce the notations and recall the necessary facts on the convexity. In Section 3.3, we prove the main result.

## 3.1 The main result: A $\overline{\partial}$-extension theorem

Let $(X, \omega)$ be a $n$-dimensional compact Hermitian manifold. Let $(E, h^E)$ and $(L, h^L)$ be the holomorphic Hermitian vector bundles over $X$ and $\operatorname{rank}(L) = 1$.

Let $\Omega^{p,q}$ be the sheaf of smooth $(p, q)$-forms on $X$. We also denote by $E$ (resp. $L$) the sheaf of smooth sections of $E$ (resp. $L$). Let $\mathcal{M}$ (resp. $\mathcal{O}$) be the sheaf of meromorphic (resp. holomorphic) functions on $X$. Let $\mathcal{M}(E) := \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(E)$ (resp. $\mathcal{O}(E)$) be the sheaf of meromorphic (resp. holomorphic) sections of $E$. We denote by $\Gamma(U, \mathcal{F})$ the space of sections of a sheaf $\mathcal{F}$ on a open subset $U \subset X$. Then $H^0(X, E) := \Gamma(X, \mathcal{O}(E))$ is the space of holomorphic sections over $X$.

Let $M$ be a relatively compact domain in $X$ and the boundary $bM$ is smooth. We denote its closure by $\overline{M} = M \cup bM$. Assume there exists a real smooth function $r$ on $X$ such that

$$M = \{x \in X : r(x) < 0\}, \quad bM = \{x \in X : r(x) = 0\}$$

and $dr(x) \neq 0$ for any $x \in bM$. We say that $r$ is a defining function of $M$. Let $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$ be the splitting of complex tangential bundle. The analytic tangent space to $bM$ at $x \in bM$ is given by

$$T_x^{(1,0)} bM := \{v \in T_x^{(1,0)} X : \partial r(v) = 0\}.$$

The definition does not depend on the choice of $r$. The Levi form of $r$ is the 2-form $\mathcal{L}_r \in \mathcal{C}^\infty(bM, T^{(1,0)*}bM \otimes T^{(0,1)*}bM)$ given by

$$\mathcal{L}_r(U, \overline{V}) := (\partial\overline{\partial}r)(U, \overline{V}) \tag{3.1.1}$$

for $U, V \in T_x^{(1,0)}bM,\ x \in bM$. The number of positive and negative eigenvalues of the Levi form is independent of the choice of the defining function (see [30, B.3]).

Let $\Omega^{p,q}(X, E)$ be the space of smooth $(p, q)$-forms with values in $E$, which is endowed with the pointwise Hermitian metric $\langle , \rangle_{h^E,\omega}$ induced by $\omega$ and $h^E$. Let $\Omega^{p,q}(bM, E)$ be the $(p, q)$-forms with values in $E$ over $bM$, i.e.,

$$\Omega^{p,q}(bM, E) := \Omega^{p,q}(X, E)|_{bM}.$$

**Definition 3.1.** A form $\alpha \in \Omega^{p,q}(bM, E)$ is called complex normal, if there exists $\psi \in \Omega^{p,q-1}(bM, E)$ such that

$$\alpha = \psi \wedge (\overline{\partial}r)|_{bM},$$

where $r$ is a defining function of $M$. We denote by $\mathcal{C}^{p,q}(bM, E)$ the subspace of $\Omega^{p,q}(bM, E)$ consisting of complex normal forms. A form $\beta \in \Omega^{p,q}(bM, E)$ is called complex tangential, if

$$\langle \alpha(x), \beta(x) \rangle_{h^E,\omega} = 0$$

for every $\alpha \in \mathcal{C}^{p,q}(bM, E)$ and every $x \in bM$. We denote by $\mathcal{D}^{p,q}(bM, E)$ the subspace of $\Omega^{p,q}(bM, E)$ consisting of complex tangential forms.

Thus we see

$$\Omega^{p,q}(bM, E) = \mathcal{C}^{p,q}(bM, E) \oplus \mathcal{D}^{p,q}(bM, E) \tag{3.1.2}$$

with respect to the pointwise Hermitian product $\langle \cdot, \cdot \rangle_{h^E,\omega}$. Moreover, we denote the projections by

$$\mu : \Omega^{p,q}(bM, E) \to \mathcal{D}^{p,q}(bM, E), \tag{3.1.3}$$

and by $\mu^\perp$ from $\Omega^{p,q}(bM, E)$ to $\mathcal{C}^{p,q}(bM, E)$.

**Definition 3.2.** We define a map

$$\overline{\partial}_b : \Omega^{p,q}(bM, E) \to \Omega^{p,q+1}(bM, E), \quad \overline{\partial}_b\sigma := \mu((\overline{\partial}\sigma')|_{bM}) \tag{3.1.4}$$

where $\sigma' \in \Omega^{p,q}(\overline{M}, E)$ and $\sigma'|_{bM} = \sigma$. We say $\sigma$ is $\overline{\partial}_b$-closed, if $\overline{\partial}_b\sigma = 0$. We say $\Sigma \in \Omega^{p,q}(\overline{M}, E)$ is a $\overline{\partial}$-closed extension of a $\overline{\partial}_b$-closed $\sigma$, if $\overline{\partial}\Sigma = 0$ on $M$ and

$$\mu(\Sigma|_{bM}) = \mu(\sigma),$$

i.e., $\Sigma$ is $\overline{\partial}$-closed and $\Sigma = \sigma$ in the complex tangential direction on $bM$ (see [28, 2.]).

**Our basic assumptions**: The triple $(X, L, M)$ satisfies:

(A) $L$ is semi-positive on $X$ and positive at one point;

(B) The Levi form $\mathcal{L}_r$ of a defining function of $M$ has at least $n-q$ negative eigenvalues on $bM$.

Our main result is the following extension theorem of $\overline{\partial}_b$-closed forms with values in a holomorphic vector bundle under our basis assumptions (A) and (B). The main ideas and techniques follow from the Kohn-Rossi extension theorem and its applications (see [28, 7.5. Theorem], [31, Lemma 2.5.]).

**Theorem 3.3.** *Let* $(X,\omega)$ *be a* $n$*-dimensional compact Hermitian manifold. Let* $(E,h^E)$ *and* $(L,h^L)$ *be holomorphic Hermitian vector bundles over* $X$ *and* $\operatorname{rank}(L)=1$*. Let* $M$ *be a relatively compact domain in* $X$ *and the boundary* $bM$ *is smooth. Let* $1\le q\le n-3$*. Assume* $L$ *is semi-positive on* $X$ *and positive at one point, and the Levi form of a defining function of* $M$ *has at least* $n-q$ *negative eigenvalues on* $bM$*.*

*Then, there exists a non-zero holomorphic section* $s\in H^0(X,L^{k_0})$ *for some* $k_0\in\mathbb{N}$*, such that for every* $\overline{\partial}_b$*-closed form* $\sigma\in\Omega^{n,q+1}(bM,E)$*, there exists a* $\overline{\partial}$*-closed extension* $S$ *of the* $\overline{\partial}_b$*-closed* $s\sigma\in\Omega^{n,q+1}(bM,L^{k_0}\otimes E)$*, i.e.,*

$$S\in\Omega^{n,q+1}(\overline{M},L^{k_0}\otimes E)$$

*such that* $\overline{\partial}S=0$ *and* $\mu(S|_{bM})=\mu(s\sigma)$*.*

**Remark 3.4.** As a consequence of Theorem 3.3, we obtain a result on meromorphic extensions as follows. Under the same assumption as above, there exists a non-zero holomorphic section $s\in H^0(X,L^{k_0})$ for some $k_0\in\mathbb{N}$, such that for every $\overline{\partial}_b$-closed form $\sigma\in\Omega^{n,q+1}(bM,E)$ and the $\overline{\partial}$-closed extension $S\in\Omega^{n,q+1}(\overline{M},L^{k_0}\otimes E)$ of $s\sigma$, and we can construct a section

$$\Sigma=s^{-1}\otimes S$$

of the sheaf $\mathcal{M}(L^{-k_0})\otimes L^{k_0}\otimes E\otimes\Omega^{n,q+1}$ on a neighbourhood of $\overline{M}$ satisfying

(i) The restriction of $\Sigma$ on $M$ is a meromorphic $(n,q+1)$-form with values in $E$, which is denoted by

$$\Sigma|_M\in\Gamma(M,\mathcal{M}(E)\otimes\Omega^{n,q+1});$$

(ii) $\Sigma$ is a meromorphic extension of $\sigma$ from the boundary $bM$ outside the zeros of $s$ in $bM$, i.e., $\Sigma$ is meromorphic on $M$ and $\mu(\Sigma|_{bM})=\mu(\sigma)$ on the set $\{x\in bM : s(x)\neq 0\}$;

(iii) $\Sigma$ is a holomorphic extension of $\sigma$ from the boundary $bM$ outside the zeros of $s$ in $\overline{M}$, i.e., $\Sigma$ is holomorphic on $M\setminus\{x\in M : s(x)=0\}$ and $\mu(\Sigma|_{bM})=\mu(\sigma)$ on the set $\{x\in bM : s(x)\neq 0\}$.

**Remark 3.5.** In particular, if $q=1$ in Theorem 3.3, $M$ is a strongly pseudo-concave domain (also 1-concave manifold) in $X$ associated with the line bundle $L$. It follows that, for each $2\le r\le n-2$, we can extend $\overline{\partial}_b$-closed $(n,r)$-forms on the boundary $bM$, which are with values in $E$, to meromorphic (resp. holomorphic) forms on $M$ (resp. $M$ except a small set of zero points) in the sense of Remark 3.4 (ii),(iii).

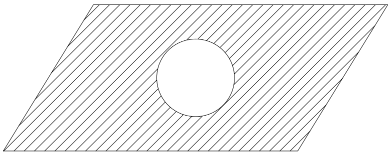

Figure 3.1: Strongly pseudo-concave domain

**Example 3.6.** For the projective space with the Fubini-Study metric $(\mathbb{CP}^n, \omega_{FS})$, the induced line bundle $\mathcal{O}(1)$ is positive everywhere. In general, if $X$ is Kähler and Moishezon (i.e., $X$ is a compact complex manifold which is embeddable in $\mathbb{CP}^n$, see [30]) with the induced metric $\omega := \omega_{FS}|_M$, the induced line bundle $L := \mathcal{O}(1)|_X$ is positive on $X$. Thus, we can extend $\overline{\partial}_b$-closed forms with values in $E$ on pseudo-concave domains $M \subset X$ by Theorem 3.3 and Remark 3.4.

## 3.2 Preliminary on the convexity of complex manifolds

Let $(X, \omega)$ be a $n$-dimensional Hermitian manifold and $(F, h^F)$ be a holomorphic Hermitian vector bundle over $X$. Let $M \subset\subset X$ be a relatively compact domain with a smooth boundary $bM$. We denote its closure by $\overline{M} = M \bigcup bM$. Let $\Omega^{p,q}(\overline{M}, F)$ be the space of $(p, q)$-forms with values in $F$ which are smooth up to and including $bM$, i.e., $\Omega^{p,q}(\overline{M}, F) := \Omega^{p,q}(X, F)|_{\overline{M}}$. We denote by $\langle \cdot, \cdot \rangle_{h^F,\omega}$ the pointwise Hermitian product induced by $\omega$ and $h^F$ on $\Omega^{p,q}(X, F)$. The $L^2$-scalar product on $\Omega^{p,q}(\overline{M}, F)$ is given by

$$\langle s_1, s_2 \rangle := \int_M \langle s_1(x), s_2(x) \rangle_{h^F,\omega} dV_M(x)$$

where $dV_M = \omega^n/n!$ is the volume form of $M$. We denote by $||\cdot||_{L^2}$ the corresponding $L^2$-norm and by $L^{p,q}(M, F)$ the $L^2$ completion of $\Omega^{p,q}(\overline{M}, F)$.

Let $\overline{\partial}^F : \mathrm{Dom}(\overline{\partial}^F) \to L^{p,q+1}(M, F)$ be the closure of the Cauchy-Riemann operator, whose graph is the closure of the graph of $\overline{\partial}^F$ on $\Omega^{p,q}(\overline{M}, F)$. Sometimes we will use $\overline{\partial}$ instead of $\overline{\partial}^F$ for simplifying notations. Let $\overline{\partial}^{F*} : \mathrm{Dom}(\overline{\partial}^{F*}) \to L^{p,q-1}(M, F)$ be the Hilbert space adjoint of $\overline{\partial}^F$ (see [28, (1.12)(1.13)]). Further we can define $\Box^F := \overline{\partial}^F \overline{\partial}^{F*} + \overline{\partial}^{F*} \overline{\partial}^F$ and

$$\mathrm{Dom}(\Box^F) = \{s \in \mathrm{Dom}(\overline{\partial}^F) \cap \mathrm{Dom}(\overline{\partial}^{F*}) : \overline{\partial}^F s \in \mathrm{Dom}(\overline{\partial}^{F*}),\ \overline{\partial}^{F*} s \in \mathrm{Dom}(\overline{\partial}^F)\}.$$

We denote by

$$\mathcal{H}^{p,q}(M, F) := \mathrm{Ker}(\Box^F) = \{s \in \mathrm{Dom}(\Box^F) : \Box^F s = 0\}$$

the space of harmonic $(p,q)$-forms with values in $F$, and by

$$H : L^{p,q}(M,F) \to \mathcal{H}^{p,q}(M,F)$$

the orthogonal projection. Finally, we denote by $H^q(M,F)$ the $q$-th cohomology group of the sheaf $\mathcal{O}(F)$, which is isomorphic to the Dolbeault cohomology

$$H^{0,q}(M,F) := \frac{\{s \in \Omega^{0,q}(M,F) : \overline{\partial}^F s = 0\}}{\overline{\partial}^F \Omega^{0,q-1}(M,F)}.$$

We recall some facts on the convexity of domains and manifolds (see [30, B.3]).

**Definition 3.7.** We say that $M$ satisfies the condition $Z(q)$ if the Levi form $\mathcal{L}_r$ has at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues at each point of $bM$ (see [14, P57]).

The number of positive and negative eigenvalues of the Levi form is independent of the choice of the defining function $r$ (see [30, Lemma B.3.8]).

**Definition 3.8.** A complex manifold $M$ of dimension $n$ is said to be $q$-concave if there exists a smooth function $\varphi : M \to \mathbb{R}$ such that $\{x \in M : \varphi(x) < c\}$ are relatively compact in $M$ for any $c < \sup \varphi$ and $i\partial\overline{\partial}\varphi$ has at least $n-q+1$ negative eigenvalues outside a compact subset of $M$; $\varphi$ is called an exhaustion function (see [32, Definition 4.1]).

We recall the following theorems which are fundamentally important for our proof.

**Theorem 3.9.** *(Kohn-Rossi [28, 5.11.Theorem])*
*If $M$ satisfies the condition $Z(q)$ and $q > 0$, then there exists a bounded operator (the Neumann operator) $\mathcal{N} : L^{p,q}(M,F) \to L^{p,q}(M,F)$ such that*

*(a) $\mathcal{N}L^{p,q}(M,F) \subset \mathrm{Dom}(\square^F)$ and $L^{p,q}(M,F) = \square^F \mathcal{N} L^{p,q}(M,F) + \mathcal{H}^{p,q}(M,F)$;*

*(b) $\mathcal{N}$ commutes with $\square^F$, $\overline{\partial}^F$, $\overline{\partial}^{F*}$, $H$;*

*(c) $\mathcal{N}(\Omega^{p,q}(\overline{M},F)) \subset \Omega^{p,q}(\overline{M},F)$ and $H(\Omega^{p,q}(\overline{M},F)) \subset \Omega^{p,q}(\overline{M},F)$;*

*(d) $\mathcal{H}^{p,q}(M,F)$ is finite dimensional.*

**Theorem 3.10.** *(Hörmander [14, (4.3.1) Theorem])*
*If $M$ satisfies the conditions $Z(q)$ and $Z(q+1)$, then*

$$H^q(M,F) \cong \mathcal{H}^{0,q}(M,F).$$

**Theorem 3.11.** *(Marinescu [32, Corollary 4.3.])*
*If $L$ and $F$ are holomorphic vector bundle of rank 1 and $r$ over the $n$-dimensional $q$-concave manifold $M$ ($n \geq 3$) and $L$ is semi-negative outside a compact set $K$, then*

$$\dim H^p(M, L^k \otimes F) \leq r\frac{k^n}{n!}\int_{M(p,h^L)} (-1)^q (\frac{i}{2\pi}c(L))^n + o(k^n)$$

*as $k \to \infty$ and $p \leq n-q-2$.*

**Theorem 3.12.** *(Kohn-Rossi [28, 5.13.Proposition])*
*Suppose $bM$ has property $Z(n-q-1)$ and $n-q-1>0$. Let $F$ be a holomorphic vector bundle over $X$. If $\phi_0 \in \Omega^{p,q}(bM,F)$, there exists a $\overline{\partial}$-closed extension $\varphi$ of $\varphi_0$ if and only if*

$$\overline{\partial}_b \varphi_0 = 0 \quad \textit{and} \quad \int_{bM} \theta \wedge \varphi_0 = 0, \quad \forall \theta \in \mathcal{H}^{n-p,n-q-1}(M,F^*).$$

## 3.3 Proof of the $\overline{\partial}$-extension theorem

Let $(X,\omega)$ be a $n$-dimensional compact Hermitian manifold. Let $(E,h^E)$ and $(L,h^L)$ be the holomorphic Hermitian vector bundles over $X$ and $\mathrm{rank}(L)=1$. Let $M$ be a relatively compact domain in $X$ and the boundary $bM$ is smooth. We denote by $L^k := L^{\otimes k}$ the $k$th tensor product of $L$ and by $E^*$ (resp. $L^{-1}$) the dual bundle of $E$ (resp. $L$).

The following proposition is from [30, Theorem 2.2.27 (2.2.43)].

**Proposition 3.13.** *Under our basic assumption (A), there exist $C_1,\ C_2>0$ such that for $k$ large enough,*

$$C_1 k^n \leq \dim H^0(X,L^k) \leq C_2 k^n.$$

**Proposition 3.14.** *Under our basic assumption (B), if $1 \leq q \leq n-2$, then*

*(1) $M$ satisfies the conditions $Z(n-q-2)$ and $Z(n-q-1)$; and*

*(2) $M$ is a $p$-concave manifold for each $p \geq q$.*

*Proof.* (1) follows from the definition of $Z(q)$ and the assumption (B). By [20, (6.7)] and the assumption (B), there exists $C>0$ and a compact subset $K \subset M$ such that the exhaustion function has the form

$$\varphi = e^{Cr} - 1$$

such that $i\partial\overline{\partial}\varphi$ has at least $n-q+1$ negative eigenvalues in $M \setminus K$. Then, (2) follows from the definition of $q$-concave manifold. □

**Proposition 3.15.** *Under our basis assumption (B), we have*

$$\dim \mathcal{H}^{0,n-q-2}(M,E^*) = \dim H^{n-q-2}(M,E^*) < \infty$$

*for $1 \leq q \leq n-2$.*

*Proof.* By the proposition 3.14 (1) and Theorem 3.10 for the conditions $Z(n-q-2)$ and $Z(n-q-1)$, the first equality follows. By the proposition 3.14 (2) and [36, Theorem 4.6(33)] for $q+1$-concave manifolds, the second inequality follows. □

**Proposition 3.16.** *Under our basic assumptions (A) and (B), then*

$$\dim \mathcal{H}^{0,n-q-2}(M, L^{-k} \otimes E^*) = \dim H^{n-q-2}(M, L^{-k} \otimes E^*) = o(k^n).$$

*as* $k \to \infty$ *and* $1 \leq q \leq n-2$.

*Proof.* By Proposition 3.14 (1) and Theorem 3.10 for the conditions $Z(n-q-2)$ and $Z(n-q-1)$, the first equality follows. By the assumption (A), the subset $M(n-q-2, h^{L^{-1}}) := \{x \in M : \ ic(L^{-1}, h^{L^{-1}})$ has $n-q-2$ negative eigenvalues and $q+2$ positive ones$\}$ is empty. Then, by Proposition 3.14 (2) and Theorem 3.11 for $q$-concave manifolds $M$ and the bundles $L^{-1}$ and $E^*$, the second equality follows. $\square$

**Proposition 3.17.** *Under our basic assumptions (A) and (B) and* $1 \leq q \leq n-2$, *then for the harmonic projection*

$$H : H^0(X, L^k) \times \mathcal{H}^{0,n-q-2}(M, L^{-k} \otimes E^*) \to \mathcal{H}^{0,n-q-2}(M, E^*),$$

*there exist* $k_0 \in \mathbb{N}$ *and a non-zero holomorphic section* $s \in H^0(X, L^{k_0})$ *such that*

$$H(s\theta) = 0$$

*for every* $\theta \in \mathcal{H}^{0,n-q-2}(M, L^{-k_0} \otimes E^*)$.

*Proof.* For simplifying of notations, we set $V := H^0(X, L^k)$, $U := \mathcal{H}^{0,n-q-2}(M, E^*)$ and

$$W := \mathcal{H}^{0,n-q-2}(M, L^{-k} \otimes E^*).$$

Moreover, we define a bilinear map

$$F(s, \alpha) := H(s\alpha)$$

for $s \in V$ and $\alpha \in W$. Then we obtain a linear map

$$G : V \to W^* \otimes U$$

by $G(s) := F(s, \cdot)$. Suppose the assertion would be false, that is, for any $k \in \mathbb{N}$ and any non-zero $s \in V$, there exists $\alpha_0 \in W$ such that $F(s, \alpha_0) \neq 0$. If $G(s_0) = 0$ for some $s_0 \in V$, then $F(s_0, \alpha) = 0$ for any $\alpha \in W$. Thus $s_0$ is zero, that is, $G$ is injective. And it follows that

$$\dim V \leq \dim(W^* \otimes U) = \dim W \times \dim U,$$

that is, for any $k \in \mathbb{N}$,

$$\dim H^0(X, L^k) \leq \dim \mathcal{H}^{0,n-q-2}(M, L^{-k} \otimes E^*) \times \dim \mathcal{H}^{0,n-q-2}(M, E^*).$$

And it follows that $k^n \leq o(k^n)$ for $k$ large enough by the propositions 3.13, 3.15 and 3.16, which can not hold. Thus, the assertion is true. $\square$

**Lemma 3.18.** *Under our basis assumptions (A) and (B) and* $1 \le q \le n-3$, *let* $s \in H^0(X, L^{k_0})$ *be the non-zero holomorphic section in the proposition 3.17 and* $\sigma \in \Omega^{n,q+1}(bM, E)$ *be* $\overline{\partial}_b$*-closed.*

*Then, there exists a* $\overline{\partial}$*-closed extension* $S$ *of the* $\overline{\partial}_b$*-closed* $s\sigma \in \Omega^{n,q+1}(bM, L^{k_0} \otimes E)$, *i.e.,* $S \in \Omega^{n,q+1}(\overline{M}, L^{k_0} \otimes E)$ *such that* $\overline{\partial} S = 0$ *and*

$$\mu(S|_{bM}) = \mu(s\sigma).$$

*Proof.* Let $\sigma' \in \Omega^{n,q+1}(\overline{M}, E)$ such that $\sigma'|_{bM} = \sigma$. Let $\theta \in \mathcal{H}^{0,n-q-2}(M, L^{-k_0} \otimes E^*)$. Thus

$$\theta \wedge (s\sigma') \in \Omega^{n,n-1}(\overline{M})$$

and $s\theta \in \Omega^{0,n-q-2}(\overline{M}, E^*)$ is $\overline{\partial}$-closed.

By Theorem 3.9 (a), (b) and $H(s\theta) = 0$, there exists $\xi \in \Omega^{0,n-q-3}(\overline{M}, E^*)$ such that

$$s\theta = \overline{\partial}\xi + H(s\theta) = \overline{\partial}\xi. \tag{3.3.1}$$

Then, by the Stokes' formula,

$$\begin{aligned} \int_{bM} \theta \wedge (s\sigma) &= \int_{bM} (s\theta) \wedge \sigma = \int_{bM} (\overline{\partial}\xi) \wedge \sigma = \int_M d(\overline{\partial}\xi \wedge \sigma') = \int_M \overline{\partial}(\overline{\partial}\xi \wedge \sigma') \\ &= (-1)^{n-q-2} \int_M \overline{\partial}\xi \wedge \overline{\partial}\sigma' = (-1)^{n-q-2} \int_M \overline{\partial}(\xi \wedge \overline{\partial}\sigma') \\ &= (-1)^{n-q-2} \int_{bM} \xi \wedge (\overline{\partial}\sigma'). \end{aligned} \tag{3.3.2}$$

By (3.1.2), (3.1.4) and $\overline{\partial}_b \sigma = 0$, we have

$$(\overline{\partial}\sigma')|_{bM} = \overline{\partial}_b\sigma + (\psi \wedge \overline{\partial} r)|_{bM} = (\psi \wedge \overline{\partial} r)|_{bM} \tag{3.3.3}$$

where $\psi \in \Omega^{n,q+1}(\overline{M}, E)$ and $r$ is the defining function of $M$. By the Stokes' formula and $bM = \{x \in X : r(x) = 0\}$, we have

$$\begin{aligned} \int_{bM} \xi \wedge (\overline{\partial}\sigma') &= \int_{bM} \xi \wedge \psi \wedge \overline{\partial} r = \int_M \overline{\partial}(\xi \wedge \psi \wedge \overline{\partial} r) = \int_M \overline{\partial}(\xi \wedge \psi) \wedge \overline{\partial} r \\ &= -\int_M \overline{\partial}(\overline{\partial}(\xi \wedge \psi) \wedge r) = -\int_{bM} \overline{\partial}(\xi \wedge \psi) \wedge r = 0. \end{aligned} \tag{3.3.4}$$

Then, by (3.3.2) and (3.3.4), it follows that

$$\int_{bM} \theta \wedge (s\sigma) = 0 \tag{3.3.5}$$

for the $\overline{\partial}_b$-closed form $s\sigma \in \Omega^{n,q+1}(bM, L^{k_0} \otimes E)$ and any $\theta \in \mathcal{H}^{0,n-q-2}(M, L^{-k_0} \otimes E^*)$.

Finally, there exists a $\overline{\partial}$-closed extension $S$ of $s\sigma$ by Theorem 3.12 for $Z(n-q-2)$ with $n-q-2 > 0$ in our case. □

Finally we can prove our main result and its remark as follows.

**Proof of Theorem 3.3**: It follows from Lemma 3.18 and Proposition 3.17. □

**Proof of Remark 3.4**: Let $s \in H^0(X, L^{k_0}) \setminus \{0\}$ be the non-zero holomorphic section given by Proposition 3.17. Let $\{U_\alpha\}$ be an open covering of $X$ such that $L^{k_0}|_{U_\alpha}$ is trivial. The holomorphic section $s$ has the form

$$s = f_\alpha e_\alpha$$

where $f_\alpha \in \mathcal{O}(U_\alpha)$ is a holomorphic function on $U_\alpha$ and $e_\alpha \in \Gamma(U_\alpha, \mathcal{O}(L^{k_0}))$ is a local holomorphic frame.

Let $L^{-k_0}$ be the dual bundle of $L^{k_0}$ and $t_\alpha \in \Gamma(U_\alpha, \mathcal{O}(L^{-k_0}))$ be the holomorphic frame which is dual to $e_\alpha$. For the meromorphic function $f_\alpha^{-1} \in \mathcal{M}(U_\alpha)$, we define locally

$$s^{-1} := f_\alpha^{-1} t_\alpha.$$

Thus, $s^{-1}$ is a meromorphic section on $X$, which is globally well defined and denoted by

$$s^{-1} \in \Gamma(X, \mathcal{M}(L^{-k_0})).$$

We denoted the $\overline{\partial}$-closed extension of the $\overline{\partial}_b$-closed $s\sigma$ from Lemma 3.18 by

$$S \in \Omega^{n,q+1}(\overline{M}, L^{k_0} \otimes E).$$

Finally, we can define a section on a neighbourhood of $\overline{M}$ by

$$\Sigma := s^{-1} \otimes S. \tag{3.3.6}$$

And its restriction on $M$ is a meromorphic $(n, q+1)$-form with values in $E$, which is denoted by

$$\Sigma|_M \in \Gamma(M, \mathcal{M}(E) \otimes \Omega^{n,q+1}).$$

Moreover, $\mu(\Sigma|_{bM}) = \mu(\sigma)$ at each point of the subset $\{x \in bM : s(x) \neq 0\}$, since

$$\begin{aligned}
\Sigma|_{bM} &= (s^{-1} \otimes S)|_{bM} \\
&= s^{-1}|_{bM} \otimes \big(\mu(S|_{bM}) \oplus \mu^\perp(S|_{bM})\big) \\
&= s^{-1}|_{bM} \otimes \big(\mu(s\sigma) \oplus \mu^\perp(S|_{bM})\big) \\
&= \mu(\sigma) \oplus \big(s^{-1}|_{bM} \otimes \mu^\perp(S|_{bM})\big) \\
&= \mu(\sigma) \oplus \mu^\perp(\Sigma|_{bM}).
\end{aligned} \tag{3.3.7}$$

□

# 4 The dimension of the space of $L^2$ holomorphic functions over hyperconcave ends

We study the $L^2$ holomorphic functions on hyperconcave ends with some Hermitian metrics and obtain that the dimension of the space of $L^2$ holomorphic functions on some domains in hyperconcave ends is infinite. The main tool is the existence of $L^2$-peak functions at boundary points by the classical solution of $\overline{\partial}$-Neumann problem on manifolds and the compactification theorem.

The organization of this chapter is as follows. In Section 4.1, we state our main results. In Section 4.2, we introduce the notations and recall the necessary facts. In Section 4.3, we construct $L^2$-peak functions on strongly pseudo-convex domain in normal Hermitian spaces of pure dimensional and apply it to hyperconcave ends.

## 4.1 The main result

Let $M$ be a complex manifold with dimension $n$. Let $\Omega$ be a relatively compact open subset of $M$ with smooth boundary $b\Omega$. We denote by $\mathcal{O}(\Omega)$ the space of all holomorphic functions on $\Omega$. A point $x \in b\Omega$ is called a peak point for $\mathcal{O}(\Omega)$ if there exists a function $f \in \mathcal{O}(\Omega)$ such that $f$ is unbounded on $\Omega$ but bounded outside $V \cap \Omega$ for any neighbourhood $V$ of $x$ in $\Omega$. And we say $f$ is a peak function for $\mathcal{O}(\Omega)$ at $x$.

In particular, we say $f$ is a $L^2$-peak function for $\mathcal{O}(\Omega)$, or a peak function for $L^2(\Omega, \omega) \cap \mathcal{O}(\Omega)$, if additionally $f \in L^2(\Omega, \omega) = \left\{ f : \Omega \to \mathbb{C} : \int_\Omega |f|^2 \omega^n < \infty \right\}$ for a

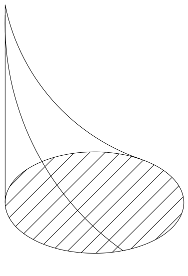

Figure 4.1: Peak function

Hermitian metric $\omega$ on $\Omega$. The existence of peak (resp. $L^2$-peak) function for $\mathcal{O}(\Omega)$ is obtained by Kohn [27] (resp. Gromov-Henkin-Shubin [18]) when $\Omega$ is strongly pseudo-convex. As applications, Levi problem can be solved immediately, see [25], [18].

In this chapter, we wish to extend their results to certain domains with singularities and smooth domains which are not relatively compact. Our main results are the existence of $L^2$-peak functions for strongly pseudo-convex domains in normal Hermitian spaces and hyperconcave ends as follows.

**Theorem 4.1.** *Let $X$ be a normal complex space of pure dimension $n \geq 2$. Let $D \subset\subset X$ be a strongly pseudo-convex domain with smooth boundary $bD$. Let $\omega$ be a Hermitian form on a neighbourhood of the closure $\overline{D}$.*

*Then, there exists a $L^2$-peak function for $\mathcal{O}(D)$ at each boundary point, i.e., for every $x \in bD$, there exists a function*

$$\Psi_x \in \mathcal{O}(D) \cap L^2(D_{reg}, \omega) \cap \mathcal{C}^\infty(\overline{D} \setminus \{x\}) \tag{4.1.1}$$

*such that* $\lim_{y \to x} |\Psi_x(y)| = +\infty$ *for $y \in D$. Thus*

$$\dim_{\mathbb{C}} L^2(D_{reg}, \omega) \cap \mathcal{O}(D) = \infty. \tag{4.1.2}$$

Let $(X, \varphi, a, b)$ be a hyperconcave end and $X_b := \{x \in X : \varphi(x) < b\}$, see Section 4.2. Let $\rho : X_b \cong \rho(X_b) \subset \widehat{X_b}$ be a biholomorphic map given by the compactification $\widehat{X_b}$ of $X_b$, where $\widehat{X_b}$ is a normal Stein spaces with at worst isolated singularities, see Theorem 4.8.

**Theorem 4.2.** *Let $(X, \varphi, a, b)$ be a hyperconcave end. Let $X_c = \{x \in X : \varphi(x) < c\}$ for $-\infty < c < b \leq a$. Let $\Theta$ be a Hermitian metric on $X_c$ such that there exists a Hermitian form $\omega$ on a neighbourhood of the closure of $\widehat{X_c}$ in $\widehat{X_b}$ with $\Theta \leq \rho^*\omega$ on $X_c$.*

*Then, there exists a $L^2$-peak function for $\mathcal{O}(X_c)$ at each boundary point, i.e., for every $x \in bX_c$, there exists a function*

$$\Phi_x \in \mathcal{O}(X_c) \cap L^2(X_c, \Theta) \cap \mathcal{C}^\infty(\overline{X_c} \setminus \{x\}) \tag{4.1.3}$$

*such that* $\lim_{y \to x} |\Phi_x(y)| = +\infty$ *for $y \in X_c$. Thus*

$$\dim_{\mathbb{C}} L^2(X_c, \Theta) \cap \mathcal{O}(X_c) = \infty. \tag{4.1.4}$$

**Remark 4.3.** For a hyperconcave end $(X, \varphi, a, b)$, the existence of $\Phi_x \in \mathcal{O}(X_c) \cap \mathcal{C}^\infty(\overline{X_c} \setminus \{x\})$ with blowing up at $x$, was firstly established by Marinescu-Dinh [33]. Theorem 4.2 is a refinement and the new feature is $\Phi_x \in L^2(X_c, \Theta)$.

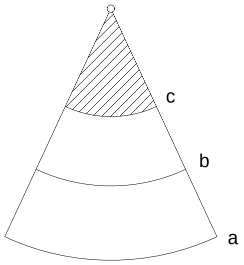

Figure 4.2: Hyperconcave end

## 4.2 Notions and preliminary

Let $X$ be a relatively compact open subset in a complex manifold with the smooth boundary $bX$. Assume $bX = bX_1 \cup bX_2$ and $bX_1 \cap bX_2 = \varnothing$. Moreover, we suppose $bX_2$ is strongly pseudo-convex and $bX_1$ is strongly pseudo-concave. We say that $X$ can be compactified, if there exists a compact manifold $\widehat{X}$ with smooth boundary $b\widehat{X}$ such that $X$ is biholomorphic to an open subset in $\widehat{X}$ and the image of $bX_2$ is exactly $b\widehat{X}$. We say that $X$ has a pseudoconcave hole at $bX_1$. For example, let $B(0,r)$ be an open ball of radius $r$ and center 0 in $\mathbb{C}^n, n \geq 2$. Let $X = B(0,2) \setminus \overline{B(0,1)}$ be an annulus, $bX_2 = bB(0,2)$ and $bX_1 = bB(0,1)$. Then, the compacification of $X$ is $B(0,2)$.

In general, any relatively compact domain $X$, which has a pseudoconcave hole at $bX_1$ as above, can be compactified when $\dim X \geq 3$ (see [38, Theorem 3, P245],[2, Proposition 3.2]). However, it is not true when $\dim X = 2$ (see a counterexample in [16],[2] and [38]). If we consider the following manifolds, which have a pseudoconcave hole at $-\infty$, the compactification result still holds when $\dim X = 2$ (see [33, Theorem 1.2]).

**Definition 4.4** ([33]). A complex manifold $X$ with $\dim X \geq 2$ is called a hyperconcave end, if there exist $a \in \mathbb{R} \cup \{+\infty\}$ and a proper, smooth function $\varphi : X \to (-\infty, a)$, which is strictly plurisubharmonic on a set of the form $\{x \in X : \varphi(x) < b\}$ for some $b \leq a$. We say $\varphi$ is the exhaustion function and set $X_r := \{x \in X : \varphi(x) < r\}$ for any $-\infty < r < a$. We denote by $(X, \varphi, a, b)$ the all dates of a hyperconcave end.

We say that a hyperconcave end can be compactified, if there exists a complex space $\widehat{X}$ such that $X$ is biholomorphic to an open subset in $\widehat{X}$ and $(\widehat{X} \setminus X) \cup \{\varphi \leq r\}$ is a compact subset in $\widehat{X}$ under the biholomorphic map for any $r < a$. We say $\widehat{X}$ the completion of $X$.

**Example 4.5.** The regular part of a variety with isolated singularities is a hyperconcave end. The complement of a compact completely pluripolar set (the set $\{\varphi = -\infty\}$ where $\varphi$ is a strongly plurisubharmonic function) in a complex manifold is a hyperconcave end, see [33].

**Example 4.6.** In the definition 4.4, if $a = +\infty$ and $\varphi$ is bounded from above, the manifold $X$ is called hyperconcave manifold (or hyper 1-concave). The regular part of a compact complex space with isolated singularities is a hyperconcave manifold. A complete Kähler manifold of finite volume and bounded negative sectional curvature is a hyperconcave manifold, see [30].

We introduce the notions on normal Stein spaces with isolated singularities of pure dimensional, see [17] and [30] for details.

A $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ is a pair $(X, \mathcal{O}_X)$ of a topological space $X$ and a subsheaf of rings $\mathcal{O}_X$ of the sheaf of continuous functions $\mathcal{C}_X$ such that $\mathcal{O}_X$ is sheaf of $\mathbb{C}$-algebra and each stalk $\mathcal{O}_{X,x}$ has unique maximal ideal $\boldsymbol{m}_{X,x}$ satisfying the quotient field $\mathcal{O}_{X,x}/\boldsymbol{m}_{X,x} \cong \mathbb{C}$. A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of $\mathbb{C}$-ringed spaces is a continuous map $f : X \to Y$ such that the induced map $\widetilde{f} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a $\mathbb{C}$-algebra morphism.

A reduced complex space $(X, \mathcal{O}_X)$ is a $\mathbb{C}$-ringed space which is Hausdroff and has the property that for each $x \in X$ there exist an open neighbourbood $U$ of $x$ and an analytic subset $A$ in $\mathbb{C}^N$ such that $(U, \mathcal{O}_U)$ is isomorphic to $(A, \mathcal{O}_A)$ as $\mathbb{C}$-ringed spaces, where $\mathcal{O}_U$ is the sheaf given by restriction of the sheaf $\mathcal{O}_X$ on $U$, and $\mathcal{O}_A$ is the sheaf of holomorphic function defined on open subset of $A$. We denote by $X_{reg}$ the set of regular points, i.e., $x \in X_{reg}$ if $A$ can be chosen to be an open subset in $\mathbb{C}^N$. The other part of $X$ are called singular and denoted by $X_{sing}$.

A Hermitian form on a reduced complex space $X$ is defined as a smooth $(1,1)$-form $\omega$ on $X$ such that for every point $x \in X$ there exists a local embedding $\tau : U \cong A \subset G \subset \mathbb{C}^N$ as above with $x \in U$ and a Hermitian form $\widetilde{\omega}$ on $G$ with $\omega = \tau^*\widetilde{\omega}$ on $U \cap X_{reg}$.

A normal complex space $(X, \mathcal{O}_X)$ is a reduced complex space such that the local ring $\mathcal{O}_{X,x}$ is a normal ring for every $x \in X$. For a normal space $X$, every holomorphic function on $X_{reg}$ extends uniquely to a holomorphic function on $X$.

**Definition 4.7.** A normal Hermitian space $(X, \omega)$ is a normal complex space $(X, \mathcal{O}_X)$ associated with a Hermitian form $\omega$ on $X$.

A normal space with pure dimensional is defined as follows. Let $X$ be a normal complex space. To each point $x \in X$ there exists a neighborhood $U$ and finitely many functions $f_1, ..., f_k \in \mathcal{O}(U)$ such that the set of common zeros of $f_1, ..., f_k$ in $U$ consists of $x$ only: $N(f_1, ..., f_k) = \{x\}$. Among all systems $f_1, ..., f_k$ with $N(f_1, ..., f_k) = \{x\}$, there exists one (defined in a suitable neighborhood) with minimal $k$. This minimal integer is called the (analytic) dimension of $X$ at $x$ and will denoted by $\dim_x X$. The global dimension of the space $X$ is defined by $\dim X := \sup_{x \in X} \dim_x X$. A normal complex space $X$ is called pure dimensional if $\dim_x X = \dim X$ for all $x \in X$, see [17, P.93]. For example, complex manifolds are normal complex spaces with pure dimensional. If the normal complex space $X$ is of pure dimensional $n$, then $\dim X = \dim_{\mathbb{C}} X_{reg} = n$, where the regular set $X_{reg}$ is a complex manifold of dimension $n$.

For a normal complex space $X$ of pure dimensional, the (relatively compact) strongly pseudo-convex domain $D$ with smooth boundary $bD$ in $X$ can be defined as same as for the case of complex manifolds, i.e., there exists a smooth function $\gamma$ on a neighbourhood $U \subset X$ of $bD$ such that $D \cap U = \{x \in U : \gamma(x) < 0\}$, $d\gamma \neq 0$ on $bD$ and the Levi form $\mathcal{L}_\gamma$ is positive definite on $T_x^{(1,0)} bD$ for all $x \in bD$, see (3.1.1).

A normal Stein space $X$ is a normal complex space satisfying holomorphically separable, regular and convex, see [30]. For a normal Stein space $X$ of pure dimensional with finitely many isolated singularities, the regular set $X_{reg}$ is a hyperconcave end with $a = b = +\infty$ in the definition 4.4. Conversely, Marinescu-Dinh [33] shows that the completion of a hyperconcave end can be chosen a normal Stein space with at worst isolated singularities by the compactification theorem as follows.

**Theorem 4.8** ([33],Theorem 1.2)**.** *Any hyperconcave end $X$ can be compactified. Moreover, if $\varphi$ is strictly plurisubharmonic on the whole $X$, the completion $\widehat{X}$ can be chosen a normal Stein space with at worst isolated singularities.*

## 4.3 $L^2$-peak functions on normal Hermitian spaces and hyperconcave ends

We study the existence of $L^2$-peak functions for strongly pseudo-convex domains, which are in normal Hermitian spaces of pure dimensional. As applications, we obtain the existence of $L^2$-peak functions on some domains in hyperconcave ends. The method is by using compactification theorem 4.8 and the existence of $L^2$-peak functions on strongly pseudo-convex domain in complex manifolds.

### 4.3.1 On normal Hermitian spaces of pure dimensional

Let $X$ be a normal complex space of pure dimension $n \geq 2$. Let $D \subset\subset X$ be a strongly pseudo-convex domain with smooth boundary $bD$. Let $\omega$ be a Hermitian form on $D$. We define the $L^2$ space of functions on $D_{reg}$ associated with $\omega$ by

$$L^2(D_{reg}, \omega) = \left\{ f : D_{reg} \to \mathbb{C} : \int_{D_{reg}} |f|^2 \omega^n < \infty \right\}. \tag{4.3.1}$$

We have the following result on $D$ by using resolution of singularities and the existence of $L^2$-peak functions on smooth strongly pseudo-convex domain.

**Theorem 4.9.** *Let $X$ be a normal complex space of pure dimension $n \geq 2$. Let $D \subset\subset X$ be a strongly pseudo-convex domain with smooth boundary $bD$. Let $\omega$ be a Hermitian form on a neighbourhood of the closure $\overline{D}$.*

*Then, there exists a $L^2$-peak function for $\mathcal{O}(D)$ at each boundary point, i.e., for every $x \in bD$, there exists a function*

$$\Psi_x \in \mathcal{O}(D) \cap L^2(D_{reg}, \omega) \cap \mathcal{C}^\infty(\overline{D} \setminus \{x\}) \tag{4.3.2}$$

*such that* $\lim_{y\to x}|\Psi_x(y)| = +\infty$ *for* $y \in D$*. Thus*

$$\dim_{\mathbb{C}} L^2(D_{reg}, \omega) \cap \mathcal{O}(D) = \infty. \tag{4.3.3}$$

*Proof.* Let $X'$ be a neighbourhood of $\overline{D}$ such that $D \subset\subset X' \subset\subset X$, $D_{sing} = X'_{sing}$, and the Hermitian form $\omega$ is well-defined on $X'$. By our assumptions, we can always choose such an open set $X'$. We have a resolution of singularities

$$\pi : M \to X' \tag{4.3.4}$$

such that $M$ is a complex connected manifold of dimension $n$ and $\pi$ is a proper holomorphic surjection. Moreover, we denote by $E = \pi^{-1}(X'_{sing})$ the exceptional set, and we can assume the restriction of $\pi$ on $M \setminus E$ is a biholomoprhic map

$$\pi : M \setminus E \cong X'_{reg}. \tag{4.3.5}$$

By restriction the biholomorphic map $\pi^{-1}$ to $D_{reg} \subset X'_{reg}$, we have

$$\pi : \Omega \setminus E \cong D_{reg}, \tag{4.3.6}$$

where $\Omega = \pi^{-1}(D) \subset M$ is a (relatively compact) strongly pseudo convex domain with smooth boundary $b\Omega$ in $M$. Note that $b\Omega \cap E$ is empty.

By the compactness of $\overline{\Omega}$, there exists a Hermitian metric $\theta$ on a neighbourhood $\Omega_1$ of $\overline{\Omega}$ such that $\overline{\Omega} \subset \Omega_1 \subset\subset M$ and

$$\theta \geq \pi^*\omega \quad \text{on} \quad \Omega_1 \setminus E. \tag{4.3.7}$$

Note $\theta$ is a Hermitian metric on $\Omega_1$ but not necessary on the whole $M$.

We consider $\Omega$ as a (relatively compact) strongly pseudo-convex domain in $\Omega_1$ with a Hermitan metric $\theta$ on $\Omega_1$. By applying Kohn's solution of $\overline{\partial}$-Neumann problem and its global regularity on $\Omega$ (see [18], [27]), there exists a $L^2$-peak function for $\mathcal{O}(\Omega)$ at each boundary point, i.e., for every boundary point $y \in b\Omega$,

$$h_y \in \mathcal{O}(\Omega) \cap L^2(\Omega, \theta) \cap \mathcal{C}^\infty(\overline{\Omega} \setminus \{y\}) \tag{4.3.8}$$

such that $\lim_{z\to y}|h_y(z)| = +\infty$ for $z \in \Omega$. Thus

$$\dim_{\mathbb{C}} L^2(\Omega, \theta) \cap \mathcal{O}(\Omega) = \infty. \tag{4.3.9}$$

Let $h_y|_{\Omega\setminus E}$ be the restriction of $h_y$ on $\Omega \setminus E$. Then, $h_y|_{\Omega\setminus E}$ is a $L^2$-peak function for $\mathcal{O}(\Omega \setminus E)$ at each point of $b\Omega$, i.e., (4.3.8) and (4.3.9) still verify on $\Omega \setminus E$ in stead of $\Omega$. By using (4.3.7) on $\Omega \setminus E \subset \Omega_1 \setminus E$, we have

$$L^2(\Omega \setminus E, \theta) \subset L^2(\Omega \setminus E, \pi^*\omega). \tag{4.3.10}$$

So it follows that

$$h_y|_{\Omega\setminus E} \in \mathcal{O}(\Omega \setminus E) \cap L^2(\Omega \setminus E, \pi^*\omega) \cap \mathcal{C}^\infty\left(((\Omega \setminus E) \cup b\Omega) \setminus \{y\}\right). \tag{4.3.11}$$

We can define a function $\Psi_x$ on $D_{reg}$ for every $x \in bD$ as follows. Let $y := \pi^{-1}(x) \in b\Omega$ and

$$\Psi_x := (h_y|_{\Omega\setminus E}) \circ \pi^{-1} : D_{reg} \to \mathbb{C}. \tag{4.3.12}$$

Then, by (4.3.11) and the biholomorphic map $\pi$ on $M \setminus E$,

$$\Psi_x \in \mathcal{O}(D_{reg}) \cap L^2(D_{reg}, \omega) \cap \mathcal{C}^\infty((D_{reg} \cup bD) \setminus \{x\}) \tag{4.3.13}$$

such that $\lim_{\xi\to x} |\Psi_x(\xi)| = +\infty$ for $\xi \in D$. By Riemann's second extension theorem on normal complex space [17], we have $\mathcal{O}(D_{reg}) = \mathcal{O}(D)$ and

$$\Psi_x \in \mathcal{O}(D) \cap L^2(D_{reg}, \omega) \cap \mathcal{C}^\infty(\overline{D} \setminus \{x\}). \tag{4.3.14}$$

Thus

$$\dim_{\mathbb{C}} L^2(D_{reg}, \omega) \cap \mathcal{O}(D) = \infty. \tag{4.3.15}$$

□

**Remark 4.10.** For a complex manifold $X$ or $\overline{D} \subset X_{reg}$, Theorem 4.9 reduces to the case of $\Gamma = \{e\}$ in [18, Theorem 0.2].

**Remark 4.11.** Suppose $X$ and $D$ are as same as in Theorem 4.9. Let $\pi$ be a resolution of singularities on a neighbourhood of $\overline{D}$. Let $\omega$ be a Hermitian form on $D$ such that there exists a Hermitian metric $\theta$ on a neighbourhood of the closure of $\pi^{-1}(D)$ satisfying $\theta \geq \pi^*\omega$ on $\pi^{-1}(D_{reg})$. The conclusion of Theorem 4.9 still verifies under these weaker hypothesises.

**Corollary 4.12.** *Let $(X, \omega)$ be a normal Hermitian space of pure dimension $n \geq 2$. Let $D \subset\subset X$ be a strongly pseudo-convex domain with smooth boundary $bD$.*

*Then, there exists a $L^2$-peak function for $\mathcal{O}(D)$ at each boundary point, i.e., for every $x \in bD$, there exists a function*

$$\Psi_x \in \mathcal{O}(D) \cap L^2(D_{reg}, \omega) \cap \mathcal{C}^\infty(\overline{D} \setminus \{x\}) \tag{4.3.16}$$

*such that* $\lim_{y\to x} |\Psi_x(y)| = +\infty$ *for* $y \in D$. *Thus*

$$\dim_{\mathbb{C}} L^2(D_{reg}, \omega) \cap \mathcal{O}(D) = \infty. \tag{4.3.17}$$

## 4.3.2 On hyperconcave ends

Let $(X, \varphi, a, b)$ be a hyperconcave end. Let $X_b := \{x \in X : \varphi(x) < b\}$ on which $\varphi$ is strictly plurisubharmonic. We set $X_c = \{x \in X : \varphi(x) < c\}$ for each $-\infty < c < b \leq a$.

Suppose $\widehat{X_b}$ is the completion of $X_b$ such that $\widehat{X_b}$ is a normal Stein space with isolated singularities due to Theorem 4.8. Then there exists a biholomorphic map

$$\rho : X_b \cong \rho(X_b) \subset \widehat{X_b}. \tag{4.3.18}$$

The restriction of $\rho$ on $X_c$ is given by

$$\rho : X_c \cong \rho(X_c) \subset \widehat{X_c} \tag{4.3.19}$$

where $\widehat{X_c} = \left(\widehat{X_b} \setminus \rho(X_b)\right) \cup \rho(X_c)$ is the completion of $X_c$. The closure of $\widehat{X_c}$ is $\left(\widehat{X_b} \setminus \rho(X_b)\right) \cup \rho(X_c \cup bX_c)$, which is compact in $\widehat{X_b}$. The boundary $b\widehat{X_c} = \rho(bX_c) = b\rho(X_c)$ is smooth and strongly pseudo-convex by the biholomorphic map $\rho$.

We have $\widehat{X_c}$ is a relatively compact strongly pseudo-convex domain with smooth boundary in $\widehat{X_b}$. Let $\Theta$ be a Hermitian metric on $X_c$ such that there exists a Hermitian form $\omega$ on a neighbourhood of the closure of $\widehat{X_c}$ with $\Theta \leq \rho^*\omega$ on $X_c$. By using Theorem 4.9 for $\widehat{X_c} \subset\subset \widehat{X_b}$ and $\omega$, for each $x \in b\widehat{X_c}$, there exists a function

$$\Psi_x \in \mathcal{O}\left(\widehat{X_c}\right) \cap L^2\left((\widehat{X_c})_{reg}, \omega\right) \cap \mathcal{C}^\infty\left(\overline{\widehat{X_c}} \setminus \{x\}\right) \tag{4.3.20}$$

such that $\lim_{y \to x} |\Psi_x(y)| = +\infty$ for $y \in \widehat{X_c}$. Thus

$$\dim_{\mathbb{C}} L^2\left((\widehat{X_c})_{reg}, \omega\right) \cap \mathcal{O}\left(\widehat{X_c}\right) = \infty. \tag{4.3.21}$$

The restriction of $\Psi_x$ on $\rho(X_c) \subset (\widehat{X_c})_{reg}$ is given by

$$\Psi_x|_{\rho(X_c)} \in \mathcal{O}(\rho(X_c)) \cap L^2\left(\rho(X_c), \omega\right) \cap \mathcal{C}^\infty\left(\overline{\rho(X_c)} \setminus \{x\}\right). \tag{4.3.22}$$

Finally, we can define a function $\Phi_p$ on $X_c$ for every point $p \in bX_c$ as follows. Let $x := \rho(p) \in \rho(bX_c) = b\widehat{X_c}$ and

$$\Phi_p := \left(\Psi_x|_{\rho(X_c)}\right) \circ \rho : X_c \to \mathbb{C}. \tag{4.3.23}$$

By $L^2(X_c, \rho^*\omega) \subset L^2\left(X_c, \Theta\right)$, it follows that

$$\Phi_p \in \mathcal{O}(X_c) \cap L^2(X_c, \Theta) \cap \mathcal{C}^\infty(\overline{X_c} \setminus \{p\}) \tag{4.3.24}$$

such that $\lim_{y \to p} |\Phi_p(y)| = +\infty$ for $y \in X_c$. Thus

$$\dim_{\mathbb{C}} L^2(X_c, \Theta) \cap \mathcal{O}(X_c) = \infty. \tag{4.3.25}$$

Thus we are lead to the following results on hyperconcave ends by the above argument.

**Theorem 4.13.** *Let $(X, \varphi, a, b)$ be a hyperconcave end. Let $X_c = \{x \in X : \varphi(x) < c\}$ for $-\infty < c < b \leq a$. Let $\Theta$ be a Hermitian metric on $X_c$ such that there exists a Hermitian form $\omega$ on a neighbourhood of the closure of the completion $\widehat{X_c}$ with $\Theta \leq \rho^*\omega$ on $X_c$.*

*Then, there exists a $L^2$-peak function for $\mathcal{O}(X_c)$ at each boundary point, i.e., for every $x \in bX_c$, there exists a function*

$$\Phi_x \in \mathcal{O}(X_c) \cap L^2(X_c, \Theta) \cap \mathcal{C}^\infty(\overline{X_c} \setminus \{x\}) \tag{4.3.26}$$

*such that* $\lim_{y \to x} |\Phi_x(y)| = +\infty$ *for $y \in X_c$. Thus*

$$\dim_{\mathbb{C}} L^2(X_c, \Theta) \cap \mathcal{O}(X_c) = \infty. \tag{4.3.27}$$

**Corollary 4.14.** *Let $(X, \varphi, a, b)$ be a hyperconcave end. Let $\Theta$ be a Hermitian metric on $X_b$ which can be extended to a Hermitian form on the completion $\widehat{X_b}$, i.e., there exists a Hermitian form $\omega$ on $\widehat{X_b}$ satisfying $\Theta = \rho^*\omega$ on $X_b$. Let $X_c = \{x \in X : \varphi(x) < c\}$ for $-\infty < c < b$.*

*Then, there exists a $L^2$-peak function for $\mathcal{O}(X_c)$ at each boundary point, i.e., for every $x \in bX_c$, there exists a function*

$$\varPhi_x \in \mathcal{O}(X_c) \cap L^2(X_c, \Theta) \cap \mathcal{C}^\infty(\overline{X_c} \setminus \{x\}) \tag{4.3.28}$$

*such that* $\lim_{y \to x} |\varPhi_x(y)| = +\infty$ *for $y \in X_c$. Thus*

$$\dim_{\mathbb{C}} L^2(X_c, \Theta) \cap \mathcal{O}(X_c) = \infty. \tag{4.3.29}$$

**Remark 4.15.** Another method to prove Theorem 4.2 associated with a complete metric is the classical construction of $L^2$-peak functions by studying the existence and global regularity of the solution of the $\overline{\partial}$-Neumann problem for $(0, 1)$-forms on domain $X_c$ with strongly pseudo-convex boundary $bX_c$ endowed with a complete metric in [33]. In fact, the existence and interior regularity of the solution was proved in [33]. Moreover, for each point $p \in bX_c$ there exists a function $g \in \mathcal{O}(X_c) \cap \mathcal{C}^\infty(\overline{X_c} \setminus \{p\})$ such that $\lim_{z \to p} |g(z)| = \infty$. But $g$ maybe not in $L^2(X_c)$. The difficulty is the boundary regularity of the solution due to Kohn, Folland-Kohn, see [25], [27] and [26]. Suppose we have the boundary regularity in the following sense,

$$\mathcal{N}(\mathrm{Im}\overline{\partial} \cap \Omega_0^{0,1}(U \cap \overline{X_c})) \subset \Omega^{0,1}(\overline{X_c}),$$

where $\mathcal{N}$ is the Neumann operator and $U$ is an arbitrary sufficient small holomorphic coordinate chart with $U \cap bX_c \neq \varnothing$. Then, one can directly construct a $L^2$ holomorphic function on $X_c$ which only blows up at any given boundary point.

# 5 A remark on Bergman kernel of symmetric tensor power of holomorphic vector bundles

Our purpose is to study the relation of $L^2$-orthonormal basis of the space of holomorphic sections of symmetric tensor power of a holomorphic vector bundle on a compact manifold and the space of holomorphic sections of the induced line bundle. Moreover, we obtain a formula on the induced Bergman kernels for trivial bundles.

Let $(X, \omega)$ be a compact Hermitian manifold of dimension $n$. Let $(E, h^E)$ be a holomorphic Hermitian vector bundle of rank $r$ on $X$. Let $E^*$ be the dual bundle of $E$. We denote by $P(E^*)$ the projective bundle associated to $E^*$, which is a compact Hermitian manifold of dimension $n+r-1$. Let $\mathcal{O}_{E^*}(-1)$ be the tautological line bundle on $P(E^*)$ and $\mathcal{O}_{E^*}(1)$ its dual bundle (see [42, 3.3.2]). Let $S^p(E)$ be the $p$-th symmetric tensor power of $E$ and $\mathcal{O}_{E^*}(p)$ the $p$-th tensor power of $\mathcal{O}_{E^*}(1)$.

We start from the theorem of Le Potier on compact Hermitian manifolds (see [24, Chap.III §5 (5.7)]), which implies that there exists an isomorphism between the spaces of holomorphic sections as follows

$$H^0(X, S^p(E)) \simeq H^0(P(E^*), \mathcal{O}_{E^*}(p)), \quad S \mapsto \widetilde{S}. \tag{5.0.1}$$

For any $x \in X$, we denote by $E_x$ the fibre of $E$ at $x$, which is a $\mathbb{C}$-linear vector space of dimension $r$. For $v \in E_x$, we denote its metric dual vector by $v^*$, which is given by $v^* = h^E(\cdot, v)$. And we denote the space of such dual vectors by $E_x^*$. Let $S^p(E_x^*)$ be the $p$-th symmetric tensor power of $E_x^*$ and $P(E_x^*)$ the projectlization of it. We denote the equivalent class of non-zero elements $v^*$ by $[v^*]$ in $P(E_x^*)$. Notice that $P(E_x^*)$ is isomorphic to $P(\mathbb{C}^r)$ as Hermitian vector spaces.

By our notations, suppose $v \in E_x \setminus \{0\}$ for $x \in X$, then

$$v^* \in E_x^*, \quad v^{*\otimes p} \in S^p(E_x^*), \quad [v^*] \in P(E_x^*).$$

And, by the definition of (5.0.1), we see

$$\widetilde{S}([v^*])(v^{*\otimes p}) = S(x)(v^{*\otimes p}) \in \mathbb{C},$$

where $\widetilde{S}([v^*]) \in \mathcal{O}_{E_x^*}(p)|_{[v^*]}$ acting on the $\mathbb{C}$-linear space of dimension 1, namely

$$\mathcal{O}_{E_x^*}(-1)^{\otimes p}|_{[v^*]} = \{\lambda v^{*\otimes p} : \lambda \in \mathbb{C}\},$$

and $S(x) \in S^p(E)|_x$ acting on $S^p(E_x)^* = S^p(E_x^*)$.

It follows that

$$\widetilde{S}([v^*]) = \langle S(x), v^{\otimes p}\rangle_h e_v^{\otimes p},$$

where $v \in E_x$ with $|v|_{h^E} = 1$ and $e_v^{\otimes p} \in \mathcal{O}_{E^*}(p)|_{[v^*]}$ such that $e_v^{\otimes p}(v^{*\otimes p}) = 1$. As a consequence, the following lemma is clear.

**Lemma 5.1.** *Let $(X, \omega)$ be a compact Hermitian manifold and $(E, h^E)$ the holomorphic Hermitian vector bundle. Suppose $x \in X$, $v \in E_x$ with unit norm, and $p \in \mathbb{N}$. For any $S, T \in H^0(X, S^p(E))$,*

$$\langle \widetilde{S}([v^*]), \widetilde{T}([v^*])\rangle_h = \langle S(x), v^{\otimes p}\rangle_h \langle v^{\otimes p}, T(x)\rangle_h, \tag{5.0.2}$$

*where $\langle , \rangle_h$ denote Hermitian metrics on the induced bundles by $(E, h^E)$ respectively.*

In the sequel, we always assume $(E, h^E)$ is trivial, i.e., $E = X \times \mathbb{C}^r$ and $h^E$ is the standard Hermitian product on $\mathbb{C}^r$,i.e., $h_x^E(z, w) = z\overline{w}$ for every $x \in X$ and $z, w \in \mathbb{C}^r$. In this case,

$$P(E^*) = X \times P(\mathbb{C}^r), \quad \mathcal{O}_{E^*}(-1) = X \times \mathcal{O}(-1),$$

and if $\pi : X \times P(\mathbb{C}^r) \to X$ is the natural projection, then the induced metric and volume form on $P(E^*)$ are given by

$$\begin{aligned} \omega_{P(E^*)} &= \pi^*(\omega_X) + \omega_{P(\mathbb{C}^r)} = \omega_X + \omega_{P(\mathbb{C}^r)}, \\ dV_{P(E^*)} &= dV_X \wedge dV_{P(\mathbb{C}^r)} \end{aligned} \tag{5.0.3}$$

**Proposition 5.2.** *Let $(X, \omega)$ be a compact Hermitian manifold of dimension $n$. Let $(E, h^E)$ be a holomorphic Hermitian vector bundle of rank $r$ on $X$. Suppose $E = X \times \mathbb{C}^r$ and $h^E$ the standard Hermitian product on $\mathbb{C}^r$. Let $\{S_i\}$ be an $L^2$-orthonormal basis of $H^0(X, S^p(E))$. Then*

$$\langle \widetilde{S}_i, \widetilde{S}_j\rangle_{L^2} = \frac{1}{(p+1)(p+2)...(p+r-1)} \langle S_i, S_j\rangle_{L^2}, \tag{5.0.4}$$

*where $\langle , \rangle_{L^2}$ are $L^2$ inner products on $H^0(X, S^p(E))$ and $H^0(P(E^*), \mathcal{O}_{E^*}(p))$ respectively. In particular, the above coefficient is one if $r = 1$.*

*Proof.* Given $x \in X$, we can assume $S_i(x) \neq 0$, then

$$\lambda_i(x) := |S_i(x)|_h > 0.$$

Choose an orthonormal basis of $S^p(E_x)$ with respect to the induced Hermitian metric $h$, such that

$$S_i(x) = \lambda_i(x) e_1^{\otimes p},$$

where $e_1 \in E_x$ with $|e_1|_{h_x^E} = 1$. By the definitions, we have

$$\langle \widetilde{S}_i, \widetilde{S}_j\rangle_{L^2} = \int_{P(E^*)} \langle \widetilde{S}_i([v^*]), \widetilde{S}_j([v^*])\rangle_h dV_{P(E^*)} = \int_X A(x) dV_X, \tag{5.0.5}$$

where

$$A(x) := \int_{P(E_x^*)} \langle \widetilde{S}_i([v^*]), \widetilde{S}_j([v^*])\rangle_h dV_{P(E_x^*)}$$

is the integration along fibres.

By (5.0.2),

$$A(x) = \int_{|v|_{h_x^E}=1, [v^*]\in P(E_x^*)} \langle S_i(x), v^{\otimes p}\rangle_h \langle v^{\otimes p}, S_j(x)\rangle_h dV_{P(E_x^*)}. \tag{5.0.6}$$

For any unit norm vector $v \in E_x$, by extending $e_1$ to an orthonormal basis $\{e_i\}_{i=1}^r$ of $E_x$, then

$$v = \sum_1^r v^i e_i \quad \text{and} \quad v^* = \sum_1^r \overline{v^i} e_i^* \in E_x^*,$$

and thus

$$\langle S_i(x), v^{\otimes p}\rangle_h = \langle \lambda_i(x) e_1^{\otimes p}, v^{\otimes p}\rangle_h = \lambda_i(x)\overline{v^1}^p. \tag{5.0.7}$$

Moreover, we have

$$S_j(x) = \sum_k b_k = \sum_k b_k^1 \otimes ... \otimes b_k^p \in S^p(E_x),$$

where

$$b_k^l = \sum_{i=1}^r b_k^{l,i} e_i \in E_x,$$

then

$$\langle v^{\otimes p}, S_j(x)\rangle_h = \sum_k \prod_{l=1}^p \langle v, b_k^l\rangle_h = \sum_k \prod_{l=1}^p (\sum_{i=1}^r v^i \overline{b_k^{l,i}}). \tag{5.0.8}$$

Next we consider $[v^*] \in P(E_x^*)$ with $|v|_{h_x^E} = |v^*|_h = 1$, then $v^* = \sum_1^r \overline{v^i} e_i^*$ such that $\sum_1^r |v^i|^2 = 1$. By (5.0.7), we can assume $\overline{v^1} \neq 0$, Then the integral area of $P(E_x^*)$ in (5.0.6) consists of $(v^1, u^2, ..., u^r) \in \mathbb{C}^r$ as follows:

$$\overline{v^1} \neq 0$$

and

$$u^j := \frac{\overline{v^j}}{\overline{v^1}}, \ j = 2, ..., r$$

such that

$$1 + |u^2|^2 + ... + |u^r|^2 = \frac{1}{|v^1|^2},$$

where $(u^2, ..., u^r) \in \mathbb{C}^{r-1}$. It follows that

$$dV_{P(E_x^*)} = \frac{\omega_{FS}^{r-1}}{(r-1)!} = \frac{dx_1 \wedge dy_1 \wedge ... \wedge dx_{r-1} \wedge dy_{r-1}}{\pi^{r-1}(1 + \sum_1^{r-1}(|x_j|^2 + |y_j|^2))^r}, \tag{5.0.9}$$

where

$$x_j+\sqrt{-1}y_j := u^{j+1} \text{ and } j=1,...,r-1.$$

By the definitions of $x_j+\sqrt{-1}y_j$, $u^j$ and $v^j$, combining (5.0.7) and (5.0.8), we see that for the fixed $x\in X$, $S_i$ and $S_j$,

$$\langle S_i(x),v^{\otimes p}\rangle_h\langle v^{\otimes p},S_j(x)\rangle_h=\lambda_i(x)\sum_k\frac{\prod_{l=1}^p(\overline{b_k^{l,1}}+\sum_{t=1}^{r-1}(x_t-\sqrt{-1}y_t)\overline{b_k^{l,t+1}})}{(1+\sum_{t=1}^{r-1}(|x_t|^2+|y_t|^2))^p}. \tag{5.0.10}$$

We substitute (5.0.9) and (5.0.10) into (5.0.6),

$$A(x)=\frac{\lambda_i(x)}{\pi^{r-1}}\sum_k\int_{\mathbb{R}^{2r-2}}\frac{\prod_{l=1}^p(\overline{b_k^{l,1}}+\sum_{t=1}^{r-1}(x_t-\sqrt{-1}y_t)\overline{b_k^{l,t+1}})}{(1+\sum_{t=1}^{r-1}(|x_t|^2+|y_t|^2))^{p+r}}dx_1\wedge dy_1...dx_{r-1}\wedge dy_{r-1}. \tag{5.0.11}$$

Changing the coordinates of $\mathbb{R}^{2r-2}$ by

$$\eta_te^{i\theta_t}=x_t+\sqrt{-1}y_t \text{ for } t=1,...,r-1,$$

then we see on the following integral area

$$\mathbb{R}^{2r-2}\simeq\{(\eta_1,\theta_1,...,\eta_{r-1},\theta_{r-1}):\eta_t\geq 0, 0\leq\theta_t\leq 2\pi, t=1,...,r-1\},$$

$$A(x)=\frac{\lambda_i(x)}{\pi^{r-1}}\sum_k\int\frac{\prod_{l=1}^p(\overline{b_k^{l,1}}+\sum_{t=1}^{r-1}(\eta_te^{-i\theta_t})\overline{b_k^{l,t+1}})}{(1+\sum_{t=1}^{r-1}\eta_t^2)^{p+r}}\eta_1d\eta_1\wedge d\theta_1\wedge...\wedge\eta_{r-1}d\eta_{r-1}\wedge d\theta_{r-1}. \tag{5.0.12}$$

By the fact that

$$\int_{0\leq\theta\leq 2\pi}e^{-i\theta}d\theta=0,$$

It follows that

$$\begin{aligned}
A(x) &= \frac{\lambda_i(x)}{\pi^{r-1}}\sum_k\int\frac{\prod_{l=1}^p\overline{b_k^{l,1}}}{(1+\sum_{t=1}^{r-1}\eta_t^2)^{p+r}}\eta_1d\eta_1\wedge d\theta_1\wedge...\wedge\eta_{r-1}d\eta_{r-1}\wedge d\theta_{r-1}\\
&= \frac{\lambda_i(x)}{\pi^{r-1}}(\sum_k\prod_{l=1}^p\overline{b_k^{l,1}})(2\pi)^{r-1}\int_{(\mathbb{R}^+)^n}\frac{\eta_1...\eta_{r-1}}{(1+\sum_{t=1}^{r-1}\eta_t^2)^{p+r}}d\eta_1\wedge...\wedge d\eta_{r-1}\\
&= \lambda_i(x)\langle e_1^{\otimes p},S_j(x)\rangle_h2^{r-1}\int_{(\mathbb{R}^+)^n}\frac{\eta_1...\eta_{r-1}}{(1+\sum_{t=1}^{r-1}\eta_t^2)^{p+r}}d\eta_1\wedge...\wedge d\eta_{r-1}\\
&= \langle S_i(x),S_j(x)\rangle_h2^{r-1}\int_{(\mathbb{R}^+)^n}\frac{\eta_1...\eta_{r-1}}{(1+\sum_{t=1}^{r-1}\eta_t^2)^{p+r}}d\eta_1\wedge...\wedge d\eta_{r-1}\\
&= \frac{\langle S_i(x),S_j(x)\rangle_h}{(p+r-1)(p+r-2)...(p+1)}.
\end{aligned} \tag{5.0.13}$$

Finally, we substitute it into (5.0.5), then (5.0.4) follows. □

Let $d := \dim H^0(X, S^p(E)) = \dim H^0(P(E^*), \mathcal{O}_{E^*}(p))$. Let $P_p^X(x)$ be the Bergman kernel associated to $S^p(E)$ given by

$$P_p^X(x) = \sum_{i=1}^{d} S_i(x) \otimes S_i(x)^*,$$

and $P_p^{P(E^*)}(\xi)$ the Bergman kernel associated to $\mathcal{O}_{E^*}(p)$ given by

$$P_p^{P(E^*)}(\xi) = \sum_{i=1}^{d} \langle T_i(\xi), T_i(\xi) \rangle_h,$$

where $\{T_i\}$ is a $L^2$ orthonormal basis of $H^0(P(E^*), \mathcal{O}_{E^*}(p))$. Then we have a corresponding relation between them as follows.

**Theorem 5.3.** *Let $(X, \omega)$ be a compact Hermitian manifold of dimension $n$. Let $(E, h^E)$ be a holomorphic Hermitian vector bundle of rank $r$ on $X$. Suppose $E = X \times \mathbb{C}^r$ and $h^E$ the standard Hermitian product on $\mathbb{C}^r$. Suppose $p \geq 2$. Let $v \in E_x$ be an unit norm vector at $x \in X$. Then*

$$\langle P_p^X(x) v^{\otimes p}, v^{\otimes p} \rangle_{h^{S^p(E)}} = \frac{P_p^{P(E^*)}([v^*])}{(p+r-1)(p+r-2)...(p+1)}. \tag{5.0.14}$$

*Proof.* By Proposition 5.2, $\{\sqrt{(p+r-1)(p+r-2)...(p+1)}\widetilde{S}_i\}$ is an orthonormal basis of $H^0(X, S^p(E))$. Suppose the dimension of this space is $d$, then

$$\begin{aligned}
\langle P_p^X(x) v^{\otimes p}, v^{\otimes p} \rangle_h &= \langle \sum_{i=1}^{d} S_i(x) \otimes S_i(x)^* v^{\otimes p}, v^{\otimes p} \rangle_h \\
&= \sum_{i=1}^{d} \langle S_i(x), v^{\otimes p} \rangle_h \langle v^{\otimes p}, S_i(x) \rangle_h \\
&= \sum_{i=1}^{d} \langle \widetilde{S}_i([v^*]), \widetilde{S}_i([v^*]) \rangle_h \\
&= \frac{P_p^{P(E^*)}([v^*])}{(p+r-1)(p+r-2)...(p+1)}.
\end{aligned}$$

□

**Remark 5.4.** For a general holomorphic Hermitian vector bundle $E$ on a compact Kähler manifold, one can refer to [42] for the construction of the metric $\omega_{P(E^*)}$ on the projectlization of the bundle $P(E^*)$. However, in general case, since the decomposition of $\omega_{P(E^*)}$ as (5.0.3) may not hold, we may not decompose the volume form of $P(E^*)$ to be the disjoint product of the volume form of $X$ and the volume form of $P(E_x^*) \cong P\mathbb{C}^{r-1}$ as simple as (5.0.3).

# 6 A generalization of Hedenmalm's solution of the $\overline{\partial}$-equation in $\mathbb{C}^n$

We generalize a result of Hedenmalm on the Hörmander's solution of the $\overline{\partial}$-equation in $\mathbb{C}$ with a growing weight to the case of $\mathbb{C}^n$.

## 6.1 Basic notations and the $\overline{\partial}$-equation with growing weights

Let $\mathbb{C}^n$ be the complex $n$-space and $(z_1, ..., z_n)$ the $n$-tuples of complex numbers. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by $(z_1, ..., z_n) = (x_1, y_1, ..., x_n, y_n)$. Here $z_j = x_j + iy_j$ for $j = 1, ..., n$. For $j = 1, ..., n$, we denote by

$$\overline{\partial}_j = \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}), \quad \partial_j = \frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}),$$

the complex differential operators. For a $\mathcal{C}^1$-smooth function $f \in \mathcal{C}^1(\mathbb{C}^n)$, we set

$$\overline{\partial} f = \sum_1^n (\overline{\partial}_j f) d\bar{z}_j.$$

We denote by $dA := dx_1 dy_1 ... dx_n dy_n$ the volume form on $\mathbb{C}^n$. And it is clear that the standard hermitian metric $\langle dz_i, dz_j \rangle = 2\delta_{ij}$ on $\mathbb{C}^n$ by $\langle dx_i, dx_j \rangle = \langle dy_i, dy_j \rangle = \delta_{ij}$ and $\langle dx_i, dy_j \rangle = 0$. For $j = 1, ..., n$, the Laplacian operator is given by

$$\Delta := \sum_{j=1}^n \Delta_j, \quad \Delta_j := 4\partial_j \overline{\partial}_j = (\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}).$$

Let $\mathcal{C}_c^\infty(\mathbb{C}^n)$ be the space of smooth functions on $\mathbb{C}^n$ with compact support and $\Omega_c^{0,1}(\mathbb{C}^n)$ the space of smooth $(0,1)$-forms on $\mathbb{C}^n$ with compact support. Let $\phi \in \mathcal{C}^2(\mathbb{C}^n, \mathbb{R})$ be a real valued $\mathcal{C}^2$-smooth function on $\mathbb{C}^n$. Let $L^2(\mathbb{C}^n, e^{\pm 2\phi})$ and $L^2_{(0,1)}(\mathbb{C}^n, e^{\pm 2\phi})$ be the $L^2$-completion of $\mathcal{C}_c^\infty(\mathbb{C}^n)$ and $\Omega_c^{0,1}(\mathbb{C}^n)$ with the indicated weights. The $L^2$-norms are given by $\|f\|^2_{L^2(\mathbb{C}^n, e^{\pm 2\phi})} = \int_{\mathbb{C}^n} |f|^2 e^{\pm 2\phi} dA$, and for each $f = \sum_1^n f_j d\bar{z}_j$ with $f_j \in L^2(\mathbb{C}^n, e^{\pm 2\phi})$, we see

$$\|f\|^2_{L^2_{(0,1)}(\mathbb{C}^n, e^{\pm 2\phi})} = 2\int_{\mathbb{C}^n} \sum_1^n |f_j|^2 e^{\pm 2\phi} dA = 2\sum_1^n \|f_j\|^2_{L^2(\mathbb{C}^n, e^{\pm 2\phi})}.$$

We denote by $\| \ \|_{L^2}$ and $\langle \ , \ \rangle_{L^2}$ the standard norm and inner product in the space $L^2(\mathbb{C}^n)$.

For $f \in L^2(\mathbb{C}^n, e^{\pm 2\phi})$, $\overline{\partial}_j f$ is defined in the sense of currents by

$$\langle \overline{\partial}_j f, g \rangle = \langle f, \overline{\partial}_j^* g \rangle_{L^2},$$

for $g \in \mathcal{C}_c^\infty(\mathbb{C}^n)$, where $\overline{\partial}_j^*$ is the formal adjoint of $\overline{\partial}_j$ in $L^2$-inner product. For $j = 1, ..., n$, we define two subspaces as follows

$$\begin{aligned} A_j^2(\mathbb{C}^n, e^{-2\phi}) &:= \{f \in L^2(\mathbb{C}^n, e^{-2\phi}) \mid \overline{\partial}_j f = 0 \text{ in the sense of currents}\}, \\ A_{(0,1)}^2(\mathbb{C}^n, e^{-2\phi}) &:= \{f \in L_{(0,1)}^2(\mathbb{C}^n, e^{-2\phi}) \mid \sum_1^n \overline{\partial}_j f_j = 0 \text{ in the sense of currents}\}. \end{aligned}$$

The following is our main result, which give a generalization of Hedenmalm's solution of the $\overline{\partial}$-equation in $\mathbb{C}^n$ with a growing weight (see [19]). Our proof is analogue to [19], which is essentially due to the works of Hörmander on $\overline{\partial}$-equations.

**Theorem 6.1.** *Let $\phi$ be a real-valued $\mathcal{C}^2$-smooth function on $\mathbb{C}^n$ with $\Delta\phi > 0$ everywhere. Suppose $f \in L_{(0,1)}^2(\mathbb{C}^n, e^{2\phi})$ with $f = \sum_1^n f_j d\overline{z}_j$ such that,*

$$\sum_1^n \int_{\mathbb{C}^n} f_j g_j dA = 0,$$

*for all $g \in A_{(0,1)}^2(\mathbb{C}^n, e^{-2\phi})$ with $g = \sum_1^n g_j d\overline{z}_j$. Then, there exists a solution to the $\overline{\partial}$-equation $\overline{\partial} u = f$ with*

$$\begin{aligned} &\|u\|_{L^2(\mathbb{C}^n, e^{2\phi}\Delta\phi)}^2 \le \|f\|_{L_{(0,1)}^2(\mathbb{C}^n, e^{2\phi})}^2, \\ \textit{i.e.}, \quad &\textstyle\int_{\mathbb{C}^n} |u|^2 e^{2\phi} \Delta\phi dA \le 2 \int_{\mathbb{C}^n} \sum_1^n |f_j|^2 e^{2\phi} dA. \end{aligned}$$

## 6.2 A norm identity and the solution of the $\overline{\partial}$-equation on $\mathbb{C}^n$

Let $\overline{\partial}_j = \frac{\partial}{\partial \overline{z}_j} : \mathcal{C}_c^\infty(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ and $\overline{\partial}_j^*$ be its formal adjoint. We still denote their maximal extensions by the same notations. And we can define $\partial_j$ and $\partial_j^*$ similarly. Then $\overline{\partial}_j^* = -\partial_j$ and $\partial_j^* = -\overline{\partial}_j$ on $\mathcal{C}_c^\infty(\mathbb{C}^n)$. For a function $F$, we let $M_F$ denote the operator of multiplication by $F$. Let $T_j : \mathcal{C}_c^\infty(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ be a differential operator given by

$$T_j := \overline{\partial}_j - M_{\overline{\partial}_j \phi}.$$

Then its formal adjoint is given by $T_j^* = -\partial_j - M_{\partial_j \phi}$. Moreover, we define the differential operator $T : \mathcal{C}_c^\infty(\mathbb{C}^n) \to L_{(0,1)}^2(\mathbb{C}^n)$ by

$$Tf := \sum_1^n (T_j f) d\overline{z}_j = \sum_1^n (\overline{\partial}_j f - f \overline{\partial}_j \phi) d\overline{z}_j = \overline{\partial} f - f \overline{\partial} \phi.$$

Then, its formal adjoint $T^* : \Omega_c^{0,1}(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ is given by $T^*f = 2\sum_1^n T_j^* f_j$ for $f = \sum_1^n f_j d\overline{z}_j$.

**Lemma 6.2.** *Let* $\phi : \mathbb{C}^n \to \mathbb{R}$ *be* $\mathcal{C}^2$*-smooth function. Let* $v \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ *and* $j = 1, ..., n$. *Then,*

$$\|\overline{\partial}_j v - v\overline{\partial}_j\phi\|_{L^2}^2 - \|\partial_j v + v\partial_j\phi\|_{L^2}^2 = \frac{1}{2}\int_{\mathbb{C}^n} |v|^2 \Delta_j\phi dA, \tag{6.2.1}$$

$$\sum_1^n \|T_j v\|_{L^2}^2 - \sum_1^n \|T_j^* v\|_{L^2}^2 = \frac{1}{2}\int_{\mathbb{C}^n} |v|^2 \Delta\phi dA, \tag{6.2.2}$$

$$\|Tv\|_{L^2_{(0,1)}(\mathbb{C}^n)}^2 \geq \|v\sqrt{\Delta\phi}\|_{L^2}^2, \quad \textit{when } \Delta\phi > 0. \tag{6.2.3}$$

*Proof.* Firstly, it is clear that $\overline{\partial}_j M_F = M_{\overline{\partial}_j F} + M_F\overline{\partial}_j$ and $\partial_j M_F = M_{\partial_j F} + M_F\partial_j$ on $\mathcal{C}^1$-smooth functions when $F$ is $\mathcal{C}^1$. Then

$$T_j^*T_j v - T_jT_j^* v = (\overline{\partial}_j - M_{\overline{\partial}_j\phi})^*(\overline{\partial}_j - M_{\overline{\partial}_j\phi})v - (\partial_j + M_{\partial_j\phi})^*(\partial_j + M_{\partial_j\phi})v = \frac{1}{2}M_{\Delta_j\phi}v.$$

By $\int_{\mathbb{C}^n} |v|^2\Delta_j\phi dA = \langle M_{\Delta_j\phi}v, v\rangle_{L^2}$ and

$$\|\overline{\partial}_j v - v\overline{\partial}_j\phi\|_{L^2}^2 = \langle T_j^*T_j v, v\rangle_{L^2}, \quad \|\partial_j v + v\partial_j\phi\|_{L^2}^2 = \langle T_jT_j^* v, v\rangle_{L^2},$$

(6.2.1) follows. And (6.2.2) follows by (6.2.1) and $\Delta = \sum_1^n \Delta_j$. Finally, we see (6.2.3) by (6.2.2) and $\|Tv\|_{L^2_{(0,1)}(\mathbb{C}^n)}^2 = \|\sum_1^n T_j v d\overline{z}_j\|_{L^2_{(0,1)}(\mathbb{C}^n)}^2 = 2\sum_1^n \|T_j v\|_{L^2}^2$. ☐

Let $\overline{T\mathcal{C}_c^\infty(\mathbb{C}^n)}$ be the $L^2_{(0,1)}(\mathbb{C}^n)$-closure of $T\mathcal{C}_c^\infty(\mathbb{C}^n)$.

**Lemma 6.3.** *Let* $h \in \overline{T\mathcal{C}_c^\infty(\mathbb{C}^n)} \subset L^2_{(0,1)}(\mathbb{C}^n)$ *and* $\Delta\phi > 0$, *then there exists a function* $v \in L^2(\mathbb{C}^n, \Delta\phi)$ *such that* $Tv = h$ *in the sense of currents and*

$$\|v\|_{L^2(\mathbb{C}^n,\Delta\phi)}^2 \leq \|h\|_{L^2_{(0,1)}(\mathbb{C}^n)}^2, \quad \textit{i.e.,} \int_{\mathbb{C}^n} |v|^2\Delta\phi dA \leq 2\int_{\mathbb{C}^n}\sum_1^n |h_j|^2 dA. \tag{6.2.4}$$

*Proof.* There exists a sequence $\{v_j\}_{j=1}^n \subset \mathcal{C}_c^\infty(\mathbb{C}^n)$ such that $h = \lim_{j\to\infty} Tv_j$ in $L^2_{(0,1)}(\mathbb{C}^n)$. By (6.2.3), the sequence $\{v_j\sqrt{\Delta\phi}\}$ converges in $L^2(\mathbb{C}^n)$. We set $u := \lim_{j\to\infty} v_j\sqrt{\Delta\phi}$ in $L^2(\mathbb{C}^n)$. We denote that $v := \frac{u}{\sqrt{\Delta\phi}}$ and it is clear that $v \in L^2(\mathbb{C}^n, \Delta\phi)$.

Firstly, we show that (6.2.4) verifies. By (6.2.1) and the definitions of $v$, $u$ and $h$, we have

$$\|v\|_{L^2(\mathbb{C}^n,\Delta\phi)}^2 = \|u\|_{L^2}^2 = \lim_{j\to\infty}\|v_j\sqrt{\Delta\phi}\|_{L^2}^2 \leq \lim_{j\to\infty}\|Tv_j\|_{L^2_{(0,1)}(\mathbb{C}^n)}^2 = \|h\|_{L^2_{(0,1)}(\mathbb{C}^n)}^2.$$

Secondly, for $f = \sum_1^n f_j d\overline{z}_j \in \Omega_c^{0,1}(\mathbb{C}^n)$, we claim

$$T^*f = 2\sum_1^n T_j^* f_j \in \mathcal{C}_c^\infty(\mathbb{C}^n). \tag{6.2.5}$$

In fact, by the definitions, $\langle T^*f, g\rangle_{L^2} = \langle f, Tg\rangle_{L^2_{(0,1)}}$ for each $g \in \mathcal{C}_c^\infty(\mathbb{C}^n)$. And $\sum_1^n T_j^* f_j \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ by $T_j^* f_j \in \mathcal{C}_c^\infty(\mathbb{C}^n)$. Then

$$\langle f, Tg\rangle_{L^2_{(0,1)}} = 2\sum_1^n \langle f_j, T_j g\rangle_{L^2} = 2\sum_1^n \langle T_j^* f_j, g\rangle_{L^2} = \langle 2\sum_1^n T_j^* f_j, g\rangle_{L^2}.$$

and (6.2.5) follows.

Finally, we show $Tv = h$ in the sense of currents. $Tv = h$ in the sense of currents is equivalent to

$$\begin{aligned} & \langle Tv, f\rangle = \langle h, f\rangle_{L^2_{(0,1)}}, \quad \forall f \in \Omega_c^{(0,1)}(\mathbb{C}^n) \\ \Leftrightarrow\ & \langle v, T^* f\rangle_{L^2} = \langle h, f\rangle_{L^2_{(0,1)}} \\ \Leftrightarrow\ & \langle \frac{u}{\sqrt{\Delta\phi}}, T^* f\rangle_{L^2} = \lim_{j\to\infty} \langle Tv_j, f\rangle_{L^2_{(0,1)}} = \lim_{j\to\infty} \langle v_j, T^* f\rangle_{L^2} \\ \Leftrightarrow\ & \lim_{j\to\infty} \langle \frac{u}{\sqrt{\Delta\phi}} - v_j, T^* f\rangle_{L^2} = 0. \end{aligned}$$

We notice that

$$\begin{aligned} |\langle \frac{u}{\sqrt{\Delta\phi}} - v_j, T^* f\rangle_{L^2}| &= |\langle u - v_j\sqrt{\Delta\phi}, \frac{T^* f}{\sqrt{\Delta\phi}}\rangle_{L^2}| \\ &\le \|u - v_j\sqrt{\Delta\phi}\|_{L^2} \|\frac{T^* f}{\sqrt{\Delta\phi}}\|_{L^2} \\ &\le C\|u - v_j\sqrt{\Delta\phi}\|_{L^2} \longrightarrow 0, \ j \to \infty. \end{aligned}$$

Then, $Tv = h$ follows. □

**Lemma 6.4.** *Let $h \in L^2_{(0,1)}(D) \ominus \operatorname{Ker} T^*$. Then there exists a solution to $Tv = h$ in the sense of currents with (6.2.4).*

*Proof.* Let $k \in L^2_{(0,1)}(D)$ and $k \in T\mathcal{C}_c^\infty(D)^\perp$, i.e., $\langle k, Tv\rangle_{L^2_{(0,1)}} = 0$, $\forall v \in \mathcal{C}_c^\infty(D)$. Then, the distribution theory gives $\langle T^* k, v\rangle = 0$, i.e., $T^* k = 0$ in the sense of currents. Then $k \in \operatorname{Ker} T^*$. It follows that $T\mathcal{C}_c^\infty(D)^\perp \subset \operatorname{Ker} T^*$ and

$$L^2_{(0,1)}(D) \ominus \operatorname{Ker} T^* = \operatorname{Ker} T^{*\perp} \subset (T\mathcal{C}_c^\infty(D)^\perp)^\perp = \overline{T\mathcal{C}_c^\infty(D)}.$$

Then, we have $h \in \overline{T\mathcal{C}_c^\infty(D)}$. By Lemme 6.3 for $h$, the assertion follows. □

**Lemma 6.5.** *Let $k \in L^2_{(0,1)}(\mathbb{C}^n)$ and $k = \sum_1^n k_j d\bar{z}_j$. Then $k \in \operatorname{Ker} T^*$ if and only if $\sum_1^n \overline{\partial}_j (e^\phi \overline{k_j}) = 0$ in the sense of currents.*

*Proof.* In the sense of currents, we see

$$\begin{aligned}
T^*k=0 \ &\Leftrightarrow\ \langle T^*k,v\rangle=\langle k,Tv\rangle_{L^2_{(0,1)}}=0,\ \forall v\in \mathcal{C}_c^\infty(\mathbb{C}^n)\\
&\Leftrightarrow\ \langle k,\sum_1^n (T_jv)d\overline{z}_j\rangle_{L^2_{(0,1)}}=2\sum_1^n\langle k_j,T_jv\rangle_{L^2}=0\\
&\Leftrightarrow\ \sum_1^n\langle T_j^*k_j,v\rangle=\langle\sum_1^n(-\partial_j-M_{\partial_j\phi})k_j,v\rangle=0\\
&\Leftrightarrow\ \langle -e^{-\phi}\sum_1^n\partial_j(e^\phi k_j),v\rangle=0\\
&\Leftrightarrow\ \sum_1^n\partial_j(e^\phi k_j)=0\\
&\Leftrightarrow\ \sum_1^n\overline{\partial}_j(e^\phi\overline{k}_j)=0.
\end{aligned}$$

□

**Proof of Theorem 6.1**: Let $h:=e^\phi f$. Then $h\in L^2_{(0,1)}(\mathbb{C}^n)$. We assume $h=\sum_1^n h_jd\overline{z}_j$, then $h_j=e^\phi f_j$ for $j=1,...,n$. For $g\in A^2_{(0,1)}(\mathbb{C}^n,e^{-2\phi})$ with $g=\sum_1^n g_jd\overline{z}_j$, we set $k_j:=e^{-\phi}\overline{g}_j$ and $k:=\sum_1^n k_jd\overline{z}_j$. By our assumption,

$$0=2\int_{\mathbb{C}^n}\sum_1^n f_jg_jdA=2\int_{\mathbb{C}^n}\sum_1^n h_j\overline{k}_jdA=\langle h,k\rangle_{L^2_{(0,1)}(\mathbb{C}^n)}.$$

As $g$ run over $A^2_{(0,1)}(\mathbb{C}^n,e^{-2\phi})$, $k$ run over all elements of $\operatorname{Ker}T^*$ by Lemma 6.5. We have $h\in\operatorname{Ker}T^{*\perp}=L^2_{(0,1)}(\mathbb{C}^n)\ominus\operatorname{Ker}T^*$. By the Lemma 6.4, there exists $v\in L^2(\mathbb{C}^n,\Delta\phi)$ such that $Tv=h$ and $\|v\|^2_{L^2(\mathbb{C}^n,\Delta\phi)}\le\|h\|^2_{L^2_{(0,1)}(\mathbb{C}^n)}$.

We set $u=e^{-\phi}v$ and notice $h=e^\phi f$. Then, in the sense of currents,

$$\begin{aligned}
e^\phi f &= T(e^\phi u)=\sum_1^n T_j(e^\phi u)d\overline{z}_j\\
&= \sum_1^n(e^\phi\overline{\partial}_jM_{e^{-\phi}})(e^\phi u)d\overline{z}_j=e^\phi\sum_1^n(\overline{\partial}_ju)d\overline{z}_j\\
&= e^\phi\overline{\partial}u.
\end{aligned}$$

Then $\overline{\partial}u=f$. And it follows that $\|u\|^2_{L^2(\mathbb{C}^n,e^{2\phi}\Delta\phi)}\le\|f\|^2_{L^2_{(0,1)}(\mathbb{C}^n,e^{2\phi})}$. □

**Remark 6.6.** Theorem 6.1 implies [19, Theorem 1.2] by choosing $n=1$. The following result is clear and the proof is analogue to above. Let $j$ be a fixed number in $\{1, ..., n\}$ and $\phi : \mathbb{C}^n \to \mathbb{R}$ is $\mathcal{C}^2$-smooth with $\Delta_j \phi > 0$. Suppose $f \in L^2(\mathbb{C}^n, e^{2\phi})$ with

$$\int_{\mathbb{C}^n} fg dA = 0, \quad \forall\ g \in A_j^2(\mathbb{C}^n, e^{-2\phi}).$$

Then there exists a solution to the $\overline{\partial}$-equation $\overline{\partial}_j u = f$ such that

$$\|u\|^2_{L^2(\mathbb{C}^n, e^{2\phi}\Delta_j \phi)} \leq 2\|f\|^2_{L^2(\mathbb{C}^n, e^{2\phi})}.$$

# Bibliography

[1] A. Andreotti and H. Grauert. Théorème de finitude pour la cohomologie des espaces complexes. *Bull. Soc. Math. France*, 90:193–259, 1962.

[2] A. Andreotti and Y.-T. Siu. Projective embedding of pseudoconcave spaces. *Ann. Scuola Norm. Sup. Pisa (3)*, 24:231–278, 1970.

[3] A. Andreotti and E. Vesentini. Carleman estimates for the Laplace-Beltrami equation on complex manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (25):81–130, 1965; Erratum: (27):153-155,1965.

[4] R. Berman. Bergman kernels and local holomorphic Morse inequalities. *Math. Z.*, 248(2):325–344, 2004.

[5] B. Berndtsson. An eigenvalue estimate for the $\overline{\partial}$-Laplacian. *J. Differential Geom.*, 60(2):295–313, 2002.

[6] B. Berndtsson. Bergman kernels related to Hermitian line bundles over compact complex manifolds. In *Explorations in complex and Riemannian geometry*, volume 332 of *Contemp. Math.*, pages 1–17. Amer. Math. Soc., Providence, RI, 2003.

[7] B. Berndtsson. An introduction to things $\overline{\partial}$. In *Analytic and algebraic geometry*, volume 17 of *IAS/Park City Math. Ser.*, pages 7–76. Amer. Math. Soc., Providence, RI, 2010.

[8] F. Campana and J.-P. Demailly. Cohomologie $L^2$ sur les revêtements d'une variété complexe compacte. *Ark. Mat.*, 39(2):263–282, 2001.

[9] D. Coman and G. Marinescu. Equidistribution results for singular metrics on line bundles. *Ann. Éc. Norm. Sup. (4)* 48(3):497-536, 2015.

[10] J.-P. Demailly. Complex analytic and differential geometry. *http://www-fourier.ujf-grenoble.fr/ demailly/manuscripts/agbook.pdf.*

[11] J.-P. Demailly. Champs magnétiques et inégalités de Morse pour la $d''$-cohomologie. *Ann. Inst. Fourier (Grenoble)*, 35(4):189–229, 1985.

[12] J.-P. Demailly. $L^2$ vanishing theorems for positive line bundles and adjunction theory. In *Transcendental methods in algebraic geometry (Cetraro, 1994)*, volume 1646 of *Lecture Notes in Math.*, pages 1–97. Springer, Berlin, 1996.

[13] J. Dixmier. *von Neumann algebras*, volume 27 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam-New York, 1981. With a preface by E. C. Lance, Translated from the second French edition by F. Jellett.

[14] G. B. Folland and J. J. Kohn. *The Neumann problem for the Cauchy-Riemann complex.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1972. Annals of Mathematics Studies, No. 75.

[15] H. Grauert. On Levi's problem and the imbedding of real-analytic manifolds. *Ann. of Math. (2)*, 68:460–472, 1958.

[16] H. Grauert. Theory of $q$-convexity and $q$-concavity. In *Several complex variables, VII*, volume 74 of *Encyclopaedia Math. Sci.*, pages 259–284. Springer, Berlin, 1994.

[17] H. Grauert and R. Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1984.

[18] M. Gromov, G. M. Henkin, and M. A. Shubin. Holomorphic $L^2$ functions on coverings of pseudoconvex manifolds. *Geom. Funct. Anal.*, 8(3):552–585, 1998.

[19] H. Hedenmalm. On Hörmander's solution of the $\overline{\partial}$-equation. I. *Math. Z.*, 281(1-2):349–355, 2015.

[20] G. M. Henkin and J. Leiterer. *Andreotti-Grauert theory by integral formulas*, volume 74 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 1988.

[21] L. Hörmander. $L^2$ estimates and existence theorems for the $\bar{\partial}$ operator. *Acta Math.*, 113:89–152, 1965.

[22] C.-Y. Hsiao and G. Marinescu. Asymptotics of spectral function of lower energy forms and Bergman kernel of semi-positive and big line bundles. *Comm. Anal. Geom.*, 22(1):1–108, 2014.

[23] D. Huybrechts. *Complex geometry: An introduction.* Universitext. Springer-Verlag, Berlin, 2005.

[24] S. Kobayashi. *Differential geometry of complex vector bundles*, volume 15 of *Publications of the Mathematical Society of Japan.* Princeton University Press, Princeton, NJ; Iwanami Shoten, Tokyo, 1987. Kanô Memorial Lectures, 5.

[25] J. J. Kohn. Harmonic integrals on strongly pseudo-convex manifolds. I. *Ann. of Math. (2)*, 78:112–148, 1963.

[26] J. J. Kohn. Regularity at the boundary of the $\bar{\partial}$-Neumann problem. *Proc. Nat. Acad. Sci. U.S.A.*, 49:206–213, 1963.

[27] J. J. Kohn. Harmonic integrals on strongly pseudo-convex manifolds. II. *Ann. of Math. (2)*, 79:450–472, 1964.

[28] J. J. Kohn and H. Rossi. On the extension of holomorphic functions from the boundary of a complex manifold. *Ann. of Math. (2)*, 81:451–472, 1965.

[29] J. Kollár. *Shafarevich maps and automorphic forms.* M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1995.

[30] X. Ma and G. Marinescu. *Holomorphic Morse inequalities and Bergman kernels*, volume 254 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 2007.

[31] G. Marinescu. *Some applications of the Kohn-Rossi extension theorems. arXiv:math/9705203.*

[32] G. Marinescu. Asymptotic Morse inequalities for pseudoconcave manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 23(1):27–55, 1996.

[33] G. Marinescu and T.-C. Dinh. On the compactification of hyperconcave ends and the theorems of Siu-Yau and Nadel. *Invent. Math.*, 164(2):233–248, 2006.

[34] G. Marinescu, R. Todor, and I. Chiose. $L^2$ holomorphic sections of bundles over weakly pseudoconvex coverings. *Geom. Dedicata*, 91:23–43, 2002.

[35] R. Narasimhan. *Analysis on real and complex manifolds.* Advanced Studies in Pure Mathematics, Vol. 1. Masson & Cie, Éditeurs, Paris; North-Holland Publishing Co., Amsterdam, 1968.

[36] T. Ohsawa. Isomorphism theorems for cohomology groups of weakly 1-complete manifolds. *Publ. Res. Inst. Math. Sci.*, 18(1):191–232, 1982.

[37] J. J. Perez. The Levi problem on strongly pseudoconvex $G$-bundles. *Ann. Global Anal. Geom.*, 37(1):1–20, 2010.

[38] H. Rossi. Attaching analytic spaces to an analytic space along a pseudoconcave boundary. In *Proc. Conf. Complex Analysis (Minneapolis, 1964)*, pages 242–256. Springer, Berlin, 1965.

[39] M. A. Shubin. $L^2$ Riemann-Roch theorem for elliptic operators. *Geom. Funct. Anal.*, 5(2):482–527, 1995.

[40] Y. T. Siu. A vanishing theorem for semipositive line bundles over non-Kähler manifolds. *J. Differential Geom.*, 19(2):431–452, 1984.

[41] R. Todor, I. Chiose, and G. Marinescu. Morse inequalities for covering manifolds. *Nagoya Math. J.*, 163:145–165, 2001.

[42] C. Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps.

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